THE RESPONSE OF NONLINEAR SYSTEMS TO STOCHASTIC EXCITATION

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ABSTRACT

The response of a dynamical system modelled by differential equations with white noise as the forcing term may be represented by a Markov process with incremental moments simply related to the differential equation. The structure of such Markov processes is completely characterized by a transition probability density function which satisfies a partial differential equation known as the Fokker-Planck equation. Sufficient conditions for the uniqueness and convergence of the transition probability density function to the steady-state are established.

Exact solutions for the transition probability density function are known only for linear stochastic differential equations and certain special first order nonlinear systems. Exact solutions for the steady-state density are known for special nonlinear systems. Eigenfunction expansions are shown to provide a convenient vehicle for obtaining approximate solutions for first order systems and for self-excited oscillators. The first term in an asymptotic expansion of the transition probability density function is found for self-excited oscillators.

A class of first passage problems for oscillators, which includes the zero crossing problem, is formulated.

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PART I

THE MARKOVIAN NATURE OF STOCHASTIC PROCESSES DEFINED BY DIFFERENTIAL EQUATIONS INVOLVING WHITE NOISE

Ordinary differential equations serve as mathematical models for many dynamical systems. One is often concerned with the response of these dynamical systems to external disturbances. The response is then represented by the solution of an ordinary differential equation, for example

$$\frac{d^2x}{dt^2} + f\left(x, \frac{dx}{dt}\right)\frac{dx}{dt} + g(x) = u(t), \tag{1.1}$$

where u(t) represents the external disturbance.

In some applications, the external disturbance cannot be successfully modelled by a single time function, for only the statistical nature of the disturbance is known. One possible approach for the determination of the response in this circumstance is to find the solutions of the differential equations with many of the possible time functions which might represent the disturbance. It is clear, even from this suggestion, that the response of the system can no longer be thought of as a single solution of an ordinary differential equation. The response would be better thought of as, for example, the average solution, in some sense.

In this way, one is led to consider the problem of determining the statistical nature of the solutions of differential equations in

which the time functions representing the external disturbance are only know statistically. Of course, for a specific choice of u(t), there will be but one solution x(t). Such specific realizations are called sample functions since they are samples from a large collection of disturbing functions and corresponding solutions. The entire collection of possible disturbing functions and solutions are called stochastic processes.

In this thesis, we shall be concerned with stochastic differential equations involving a special type of stochastic disturbing function. The stochastic processes representing the external disturbance will be taken to be white noise, which, formally, has the properties of the derivative of a Wiener process.

There are several reasons for this choice. In many systems, the disturbing forces change much more rapidly than do the system variables. White noise is a mathematical limiting case in which the characteristic time for a change in the disturbing force as compared to the characteristic time for a change in the system variables tends to zero. As compared to a real disturbing force of this type, white noise differs in its effect only in the changes of the system variables for very short periods of time, so that after a period of time comparable to the characteristic time associated with a real disturbing force, the effects of the real and idealized, i.e., white noise, disturbing forces are very nearly the same.

The reason for adopting the idealized model for the disturbing force is, of course, that computations are much easier for white noise. In fact, choosing white noise allows a rather complete theory as we shall see in the discussions which follow.

In other applications, the disturbing force itself can not justifiably be modelled as white noise, but can be represented as the response of a secondary system to white noise. That is, the stochastic process representing the disturbance in a given dynamical system can often itself be represented as a stochastic process related to white noise by a differential equation. For example, u(t) as defined by

$$\frac{du}{dt} + \beta u = n(t) \tag{1.2}$$

with n(t) white noise, may serve as the disturbance for a dynamical system represented by equation (1.1). We can then think of the desired response as the response represented by an enlarged system of differential equations, for example, equations (1.1) and (1.2) together, with the disturbing force being white noise. u(t) is now thought of as an additional system variable. This procedure of enlarging the set of differential equations expands the possibility of applications to a wide range of problems.

Wiener Process

A Wiener process, w(t), is characterized by the following properties:

i) if $t_1 < \ldots < t_n$, the differences

$$w(t_2) - w(t_1),\ldots,w(t_n) - w(t_{n-1})$$

are mutually independent;

ii) $w(t) - w(s)$ is normally distributed (Gaussian), with[1]

$$E[w(t) - w(s)] = 0$$

$$E[(w(t) - w(s))^2] = \sigma^2|t-s| \ .$$

Formally, one can express these properties in terms of differentials:

i) $dw(t_i)$, $i = 1,\ldots,n$, are mutually independent

ii) $E[dw(t)] = 0$

$$E[(dw(t))^2] = \sigma^2 dt.$$

It is also useful to note that formal differentiation leads to

$$E[dw(t)\ dw(s)] = \sigma^2\delta(t-s)\ dt\ ds$$

where $\delta(t-s)$ is the dirac delta function. The formal properties of white noise, defined by

$$n(t) = \frac{dw}{dt} \ ,$$

are then

i) $n(t_i)$, $i = 1,\ldots,n$, are mutually independent;

ii) $n(t)$ is normally distributed (Gaussian) with

$$E[n(t)] = 0$$

$$E[n(t)\ n(s)] = \sigma^2\delta(t-s)$$

[1] $E[\cdot]$ denotes an expectation.

Any ordinary differential equation of n^{th} order can be written as a system of n first order differential equations. For the problems considered in this thesis, the reduction to a system of first order differential equations will be of the form

$$\frac{dx}{dt}_j = a_j(\underline{x},t) + b_j^k(\underline{x},t)\frac{dw_k(t)}{dt}$$

where $\underline{x} = (x_1,\ldots,x_n)$, $x_j = \frac{d^{j-1}x}{dt^{j-1}}$, and the $w_k(t)$ are independent Wiener processes. For simplicity in the following discussion, an equation of first order involving only one dependent variable will be considered. An alternate expression for a first order differential equation is a relation between differentials:

$$dx = a(x,t)\,dt + b(x,t)\,dw(t). \tag{1.3}$$

At this point, we require an interpretation of this relation between (stochastic) differentials which accords with the physical phenomena being studied. If b(x,t) is a constant, there is general agreement on an interpretation. First, if there were no stochastic disturbances, b = 0, the system would evolve deterministically, and in this case

$$\lim_{\Delta t\to 0}\left\{\frac{1}{\Delta t}[x(t+\Delta t) - x(t)]\right\} = a(x(t),t) \quad .$$

Often, the effect of the stochastic disturbance is to cause the system to wander randomly away from the deterministic path in such a way that

the average behavior is the deterministic path. We can represent this by the statement

$$\lim_{\Delta t \to 0} E\left\{\frac{1}{\Delta t}\left[x(t+\Delta t) - x(t)\right]\right\} = a(x(t),t) \tag{1.4}$$

differing from the previous statement concerning the deterministic behavior only in that an expected value has been introduced. The effect of the stochastic disturbance can be seen by considering higher order incremental moments - - - equation (1.4) expresses the first incremental moment. Using the properties of the Wiener process discussed above, we see that

$$\lim_{\Delta t \to 0} E\left\{\frac{1}{\Delta t}\left[x(t+\Delta t) - x(t)\right]^2\right\} = b(x(t),t) \tag{1.5}$$

if we take $E[(dw(t))^2] = dt$. Due to the Gaussian nature of the Wiener process, we can also see that

$$\lim_{\Delta t \to 0} E\left\{\frac{1}{\Delta t}\left[x(t+\Delta t) - x(t)\right]^k\right\} = 0 \tag{1.6}$$

for $k>2$. Equation (1.5) has the interpretation that in a small interval of time Δt, the sample paths will spread out around the deterministic path by an amount characteristically about $b\Delta t$.

When $b(x,t)$ depends on x, there is some question as to whether the sole effect of the stochastic disturbance is to cause the spreading out; the stochastic disturbance may also affect the average behavior as expressed by the first incremental moment. The resolution of this point depends ultimately upon a consideration of the physical phenomena being studied. A discussion of this point has been taken up by

Caughey and Gray [1]. Ito [2] has chosen the interpretation expressed by equations (1.4)-(1.6) when b(x,t) depends on x. Stratonovich [3] has offered an alternate interpretation in which the stochastic disturbance does affect the first incremental moment.

We shall not have to distinguish between the two in the applications in this thesis since we shall always start with a system in which b is constant. In this case, the stochastic disturbance does not affect the first incremental moment, even in the interpretation adopted by Stratonovich. We shall proceed, however, by using Ito's interpretation, for this approach has certain technical advantages.

Problems of existence and uniqueness of solutions are basic to the study of systems of ordinary differential equations. In the study of stochastic differential equations (or equations relating stochastic differentials) there is the analogous problem of establishing the existence and uniqueness of a stochastic process with specified incremental moments. Ito [2] designed a stochastic calculus which allows one, with appropriate conditions on the incremental moments, to do just that.

We shall only summarize the results. (This material is also available in Doob [4, pp. 273-291].) First, the relation between stochastic differentials, equation (1.3), is converted to an integral equation

$$(1.7) \qquad x(t) - x(t_o) = \int_{t_o}^{t} a[x(s),s]ds + \int_{t_o}^{t} b[x(s),s]dw(s) \quad .$$

In fact, Ito regards the relation between stochastic differentials as a shorthand notation for the integral equation. His stochastic calculus is based on a definition of the stochastic integrals which appear in this integral equation which leads to the incremental moment properties expressed by equations (1.4)-(1.6). If we make the following hypotheses:[2]

1) There is a constant K for which

$$|a(t,\xi)| \leq K(1+\xi^2)^{1/2}$$

$$0 \leq b(t,\xi) \leq K(1+\xi^2)^{1/2} \quad .$$

2) $a(\cdot,\cdot)$ and $b(\cdot,\cdot)$ satisfy a uniform Lipschitz condition in ξ

$$|a(t,\xi_1) - a(t,\xi_2)| \leq K|\xi_2-\xi_1|$$

$$|b(t,\xi_1) - b(t_1\xi_2)| \leq K|\xi_2-\xi_1|$$

where K is independent of t and ξ; then there is a stochastic process, x(t), with the following properties:

1) The x(t) sample functions are almost all[3] continuous.

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[2]There is the further technical requirement that a, b are Baire functions of the pair (t,ξ) for $t_o \leq t \leq T$, $-\infty < \xi < \infty$. This is obviously no restriction in applications.

[3]"almost all" means with probability 1. The mathematically rigorous definition of a stochastic process requires the use of an underlying sample space with probabilities defined on certain subsets of this space. It is with respect to this space to which "with probability 1" refers. A discussion of the definition of a stochastic process can be found in Doob [4, pp. 46-50]. Since reservations such as "with probability 1" in no way affect any statistics with which we will be concerned in our applications, they will be omitted in the remainder of this thesis.

2) For each $t\varepsilon(t_o,T)$, $x(t) - x(t_o)$ is independent of the aggregate of differences $\{y(T) - y(s),\ s>t\}$.

3) For each $t\varepsilon(t_o,T)$, equation (1.4) is true.

From the proof of these facts, which proceeds by the method of successive approximations familiar in the theory of ordinary differential equations, it is obvious that x(t) is a Markov process. (The explicit mathematical expression of the Markov property will be discussed in Part II.) This latter fact is the most important feature of the development. The generalization to systems of stochastic differential equations is immediate and implies that n^{th} order differential equations with disturbing functions represented as white noise are n-component Markov processes, provided the incremental moments have the required properties.

The Markovian property of these systems has the interpretation that the statistics of the future states of the system depend only on the present state and the stochastic process, w(t), in the intervening time interval. This is analogous to the situation for ordinary differential equations.

The difficulties in extending the method employed by Ito to establish existence and uniqueness for other than these quasi-linear systems are related to the distinction between establishing local and global existence for ordinary differential equations. In the theory of ordinary differential equations the method of successive approxi-

mations will establish local or global existence according to whether one can impose a global Lipschitz condition or one has only a local Lipschitz condition. It is inherent in the nature of the stochastic processes we are considering that the system variables have finite probability of exceeding any fixed bound so that we must always concern ourselves with a global-type existence when dealing with stochastic differential equations with white noise disturbance. It is because quasi-linear stochastic differential equations admit a global Lipschitz condition that one is able to establish existence and uniqueness.

For the most part, this thesis deals with systems whose incremental moments do not satisfy the requirements for the development above, so that we are not assured that a Markov process exists with the given incremental moments. On the other hand, if there is a solution of the integral equation (1.7), it will be a Markov process with the proper incremental moments since the Markov property depends only on the independence of increments of the Wiener process. Furthermore, the strong analogy between these Markov processes and solutions of ordinary differential equations provides intuitive ground for adopting a Markov process as a representation for the response of the system.

To sum up, the response of a dynamical system which is modelled as a set of ordinary differential equations with white noise disturbance (as, for example, equation (1.3)) shall be represented by a

Markov process whose incremental time behavior is directly related to the differential equations (as (1.4)-(1.6) are related to equation (1.3)). We shall show in Parts II and III how this representation can be used to obtain a statistical description, that is, the response, of the dynamical system. In Part IV, we will solve some specific problems and present some general methods of solution.

PART II

THE STATISTICAL DESCRIPTION OF A CONTINUOUS PARAMETER MARKOV PROCESS

Stochastic Processes

We have seen that the response of a dynamical system to stochastic excitation cannot be represented as a single time function as is the case for systems modelled by a set of ordinary differential equations with deterministic inputs. In this section we shall take up a discussion of how the response may be represented.

As we have noted, for each sample function representing the disturbance there is a corresponding sample function representing the response of the system. For a specified collection of disturbance sample functions there is a corresponding collection of response sample functions. We might ask ourselves what we would like to know about such a collection of response sample functions.

One type of question involves statements about the values of the sample functions over a time interval, e.g., how many sample functions have values exceeding a certain bound? Such a question would be appropriate in the analysis of the failure of vibrating structures.

A second type of question involves values of the sample functions at discrete instants, e.g., what is the expected value (average) of the sample functions at some specific time?

We desire, then, a statistical description of the response stochastic process which will provide answers to these two general types of questions.

First, we shall introduce the notation to be used. An n-component Markov process takes on values in an n-dimensional space which we shall call the phase space, and we shall write $\underline{x} = (x_1,\ldots,x_n)\varepsilon R$. For our purposes, this usually will be n-dimensional Euclidian space. Sets in phase space shall be denoted by Γ. Then the conditional probability of the system occupying sets $\Gamma_1,\ldots,\Gamma_m$ at successive time instants $t_1,\ldots,t_m$, given that the system occupied the states $\underline{\xi}_1,\ldots,\underline{\xi}_p$, at successive time instants $s_1,\ldots,s_p$ shall be denoted by

$$P(\Gamma_1,t_1;\ldots;\Gamma_m,t_m|\underline{\xi}_1,s_1;\ldots;\underline{\xi}_p,s_p) \ .$$

If P has a density so that

$$P(\Gamma_1,t_1;\ldots;\Gamma_m,t_m|\underline{\xi}_1,s_1;\ldots;\underline{\xi}_p,s_p) =$$

$$\int_{\Gamma_1} d\underline{x}_1 \cdots \int_{\Gamma_m} d\underline{x}_m p(\underline{x}_1,t_1;\ldots;\underline{x}_m,t_m|\underline{\xi}_1,s_1;\ldots;\underline{\xi}_p,s_p)$$

p shall be termed the probability density function. We shall almost always assume that P has a density and shall work with the probability density function, p. Distribution functions, P, and their densities, p, without conditions shall be written as

$$P(\Gamma_1,t_1;\ldots;\Gamma_m,t_m)$$

and

$$p(\underline{x}_1,t_1;\ldots;\underline{x}_m,t_m) \ .$$

It is evident that if we could specify a consistent set of joint distribution functions

$$P(\Gamma_1,t_1),\ P(\Gamma_1,t_1;\Gamma_2,t_2),\ldots$$

called the first distribution, second joint distribution, etc., we would be able to answer any question of the second general type mentioned above. Kolmogorov [5, p. 29] has shown that the consistency conditions

$$P(\Gamma_1,t_1;\ldots;\Gamma_m,t_m) = P(\Gamma_{i_1},t_{i_1};\ldots;\Gamma_{i_m},t_{i_m})$$

$$P(\Gamma_1,t_1;\ldots;\Gamma_k,t_k;R,t_{k+1};\ldots;R,t_m) = P(\Gamma_1,t_1;\ldots;\Gamma_k,t_k)$$

for an aribtrary permutation $(i_1,\ldots,i_m)$, and any finite m, are necessary and sufficient to determine a valid distribution function. Furthermore, he has shown by means of his Extension Theorem [5, p. 17] that this distribution function can be extended uniquely to assign probabilities to events such as $|x(t)| \leq M$, $0 \leq t \leq T$, that is, the distribution function can be used to answer questions of the first general type mentioned above.

Transition Probability Density Function

Thus we see that a stochastic process can be completely characterized by a collection of consistent, finite joint distribution functions.

Our discussion of stochastic differential equations with white noise inputs has shown that their "solutions" may be regarded as

continuous parameter Markov processes. At that time we gave only a rough statement of the Markov property. To be precise, a stochastic process is Markovian if for $t_1<\ldots<t_m<\ldots<t_{m+n}$

$$P(\Gamma_1,t_{m+1};\ldots;\Gamma_n,t_{m+n}|x_1,t_1;\ldots;x_m,t_m) = P(\Gamma_1,t_{m+1};\ldots;\Gamma_n,t_{m+n}|x_m,t_m) \ .$$

Actually, this is but one possible formulation of the Markov property. Doob [4, p. 80] presents several equivalent forms; we have merely chosen one that is convenient for our work.

Now it is generally true that

$$\begin{aligned} p(\underline{x}_1,t_1;\ldots;\underline{x}_n,t_n) &= p(\underline{x}_2,t_2;\ldots;\underline{x}_n,t_n|\underline{x}_1,t_1)\ p(\underline{x}_1,t_1) \\ &= p(x_3,t_3;\ldots;\underline{x}_n,t_n|\underline{x}_1,t_1;\underline{x}_2t_2) \\ &\qquad p(\underline{x}_2,t_2|\underline{x}_1,t_1)\ p(\underline{x}_1,t_1) \\ &= p(\underline{x}_n,t_n|\underline{x}_1,t_1;\ldots;\underline{x}_{n-1},t_{n-1})\ \ldots \\ &\qquad p(\underline{x}_2,t_2|\underline{x}_1,t_1)\ p(\underline{x}_1,t_1) \ . \end{aligned}$$

When we take $t_1<t_2<\ldots<t_n$, the Markovian property allows considerable simplification:

$$p(\underline{x}_1,t_1;\ldots;\underline{x}_n,t_n) = p(\underline{x}_n,t_n|\underline{x}_{n-1},t_{n-1})\ldots p(\underline{x}_2,t_2|\underline{x}_1,t_1)\ p(\underline{x}_1,t_1) \ .$$

The conditional probability density function, $p(\underline{x},t|\underline{\xi},\tau)$, is, for Markov processes, called the transition probability density function because of the way the joint probability density function of order n

can be built up from it and an initial distribution. The transition probability density function gives the density of probability of transitions from a point in phase space, $\underline{\xi}$, at time τ to a point in phase space, $\underline{x}$, at time t.

Chapman-Kolmogorov Equation

Using the Markov property, we also have

$$p(\underline{x}_2,t_2;\underline{x}_3,t_3|\underline{x}_1,t_1) = p(\underline{x}_3,t_3|\underline{x}_2,t_2)\ p(\underline{x}_2,t_2|\underline{x}_1,t_1), \qquad t_1<t_2<t_3 \ .$$

Integrating over $\underline{x}_2$, this becomes

$$p(\underline{x}_3,t_3|\underline{x}_1,t_1) = \int_R p(\underline{x}_3,t_3|\underline{x}_2,t_2)\ p(\underline{x}_2,t_2|\underline{x}_1,t_1)\ d\underline{x}_2 \ ,$$

the reduction on the left hand side being a consequence of the necessity of Kolmogorov's consistency condition. This equation is known as the Chapman-Kolmogorov or Smoluchowski equation.

Further simplification results if the system is stationary so that

$$p(\underline{x}_2,t_2|\underline{x}_1,t_1) = p(\underline{x}_2,t_2-t_1|\underline{x}_1,0) \ .$$

In this case, we would simply write $p(\underline{x}_2,t|\underline{x}_1)$ for the transition probability density function. For example, the Chapman-Kolomogorov equation becomes

$$p(\underline{x}_3,t+\tau|\underline{x}_1) = \int_R p(\underline{x}_3,t|\underline{x}_2)\ p(\underline{x}_2,\tau|\underline{x}_1)\ d\underline{x}_2 \ . \tag{2.1}$$

The joint probability density functions are then functions only of the time differences,

$$p(\underline{x}_1,t_1;\ldots;\underline{x}_n,t_n) = p(\underline{x}_1,t_1;\underline{x}_2,t_2-t_1;\ldots;\underline{x}_n,t_n-t_{n-1}) \quad ,$$

and fixed initial time, since then,

$$p(\underline{x}_1,t_1;\ldots;\underline{x}_n,t_n) = p(\underline{x}_1,t_1) \sum_{j=2}^{n} p(\underline{x}_j,t_j-t_{j-1}|\underline{x}_{j-1}) \quad .$$

If the limit

$$\lim_{t\to\infty} p(\underline{x},t|\underline{\xi}) = p_s(\underline{x})$$

exists, independent of $\underline{\xi}$, $p_s(\underline{x})$ is termed the steady-state distribution.

Construction of a Markov Process

Suppose we are given a stationary transition probability density function, $p(\underline{x},\tau|\underline{\xi})$, which satisfies the Chapman-Kolmogorov equation, and an initial probability density, $f(\underline{\xi})$. Then

$$(2.2) \quad \int f(\underline{x}_1)d\underline{x}_1 \int_{A_m} d\underline{x}_2\, p(\underline{x}_2,\tau_2|\underline{x}_1)\ldots \int d\underline{x}_m\, p(\underline{x}_m,\tau_m|\underline{x}_{m-1})$$

defines a probability on m-dimensional sets A_m in the space of points $(\underline{x}_1,\ldots,\underline{x}_m)$. If

$$(2.3) \qquad t_k = \sum_{j=1}^{k} \tau_j, \qquad \tau_j > 0, \qquad k=1,\ldots,m,$$

so $t_1<\ldots<t_m$, then the random variables $x(t_1),\ldots,x(t_m)$, are assigned a joint distribution determined by equation (2.2).

The first condition of Kolmogorov consistency is satisfied trivially, since for any permutation, the m^{th} joint distribution is given by expression (2.2), where we assume t_k to be defined by equation (2.3). The second condition holds because the transition probability density function satisfies the Chapman-Kolmogorov equation.

It is easy to see that this sequence of random variables is a Markov process, if we set

$$p(\Gamma,t|\underline{\xi}_1,t_1;\ldots;\underline{\xi}_m,t_m) = p(\Gamma,t|\underline{\xi}_m,t_m)$$

when $t_1<\ldots<t_m<t$.

Moments, Autocorrelation Function, Spectral Density Function

It is clear that the set of joint distribution functions generated by a transition probability and an initial distribution suffice to answer any questions which can be described in terms of values taken by the stochastic process at discrete instants of time. We shall now consider a few statistics of this type which are of some importance.

Of obvious interest are the moments of the transition probability density function, defined by

$$m_{j_1,\ldots,j_n}(t|\underline{\xi}) = \int_R x_1^{j_1}\ldots x_n^{j_n}\, p(\underline{x},t|\underline{\xi})\, d\underline{x}$$

with the first order moments

$$m_k(t|\underline{\xi}) = \int_R x_k \; p(\underline{x},t|\underline{\xi}) \; d\underline{x}$$

being of special interest. Moments of the n^{th} joint distribution function can be similarly defined, but only certain moments are usually considered.

For a dynamical system governed by a set of linear stochastic differential equations with white noise input, the transition probability density function is Gaussian. This derives from the fact that the Wiener process is Gaussian and a linear transformation of a Gaussian stochastic process is also a Gaussian stochastic process[4].

The transition probability density function of a Gaussian process is completely characterized by its first and second moments. The first moments, $m_j(t|\xi)$, as defined above, are the components of the mean vector; and the second (central) moment matrix, with elements

$$K_{jk}(t|\underline{\xi}) \equiv \int_R [x_j - m_j(t|\underline{\xi})] \; [x_k - m_k(t|\underline{\xi})] \; p(\underline{x},t|\underline{\xi}) \; d\underline{x}$$

is called the correlation matrix.

We have, in fact[5], for $\underline{x} = (x_1,\ldots,x_n)$,

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[4] Parzen, E. [6, p. 90].

[5] Parzen, E. [6, p. 88].

$$p(\underline{x},t|\underline{\xi}) = \frac{1}{(2\pi)^{n/2}} \frac{1}{|K|^{1/2}} \exp\left\{-\frac{1}{2}\sum_{j,k=1}^{n}(x_j-m_j(t|\underline{\xi}))K^{jk}(x_k-m_k(t|\underline{\xi}))\right\}$$

where $|K|$ is the determinant of the correlation matrix with elements $K_{jk}(t|\underline{\xi})$ and K^{jk} are the elements of the inverse of the correlation matrix.

Also of particular interest for linear equations is the matrix of second order moments for the second joint probability density function,

$$R_{jk}(t,t+\tau|\underline{\xi}) = \int\int_{-\infty}^{\infty} [x_j-m_j(t|\underline{\xi})]\ [y_k-m_k(t|\underline{\xi})]\ p(\underline{x},t,\underline{y},t+\tau|\underline{\xi})\ d\underline{x}\ d\underline{y}$$

$$= \int\int_{-\infty}^{\infty} [x_j-m_j(t|\underline{\xi})](y_k-m_k(t|\underline{\xi})]\ p(\underline{y},\tau|\underline{x})\ p(\underline{x},t|\underline{\xi})\ d\underline{x}\ d\underline{y},$$

where, in obtaining the last line, we have used the Markov property and stationarity. To be more precise, we are interested in

$$R_{jk}(\tau) = \lim_{t\to\infty} R_{jk}(t,t+\tau|\underline{\xi}) \tag{2.4}$$

when this limit exists, independent of $\underline{\xi}$. Usually, one has

$$\lim_{t\to\infty} m_j(t|\underline{\xi}) = 0,$$

and then one has simply

$$R_{jk}(\tau) = \int_{-\infty}^{\infty} x_j y_k \; p(\underline{y},\tau|\underline{x}) \; p_s(\underline{x}) \; d\underline{x} \; d\underline{y},$$

when the steady-state probability density function exists, independent of $\underline{\xi}$.

$R_{11}(\tau)$ is commonly termed the autocorrelation function. The spectral density function, $\phi(\omega)$, is related to the autocorrelation by the Wiener-Khintchine relations:

$$\phi(\omega) = \frac{2}{\pi} \int_0^{\infty} R_{11}(\tau) \cos \omega\tau \; d\tau \; , \tag{2.5}$$

$$R_{11}(\tau) = \int_0^{\infty} \phi(\omega) \cos \omega\tau \; d\omega \; .$$

These functions are important in Harmonic analysis and prediction theory for linear systems[6].

For nonlinear stochastic differential equations, the structure of the transition probability density function is not so simple, nor can the spectral density function play the important role that it does in Harmonic analysis and prediction theory for linear systems. Nonetheless, interest in these statistics persists due to the familiarity of those in applications with properties of these statistics for linear systems.

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[6] See, for example, Davenport and Root [7].

Even in such cases as it is possible to determine the transition probability density function, the first and second order statistics discussed here express the response of a nonlinear system in terms which allow more direct interpretation. Higher order moments could also be defined, but they are of little interest.

First Passage Time Problems

As we have suggested, in applications one is often interested in determining the first time a system variable, such as a displacement in a mechanical system, exceeds a specified value. This problem, and related ones, are termed first passage time problems.

The distribution of first passage times can be related to the transition probability density function. For this purpose, let $f(\Gamma,\underline{\sigma},t|\underline{\xi})d\underline{\sigma}\ dt$ denote the probability that a system, initially at state $\underline{\xi}$, enters the closed region Γ through the element of surface $\underline{\sigma}\ d\underline{\sigma}$ for the first time in the time interval t to t+dt. If the system occupies a state $\underline{x}\varepsilon\Gamma$ at time t, it must have entered Γ through some element of the surface, say at $\underline{\sigma}$, at some time s, $0 \leq s \leq t$ for the first time. This can be expressed by

$$p(\underline{x},t|\underline{\xi}) = \int_{\partial\Gamma}\int_{0}^{t} f(\Gamma,\underline{\sigma},s|\underline{\xi})\ p(\underline{x},t-s|\underline{\sigma})\ d\underline{\sigma}\ ds, \tag{2.6}$$

where $\partial\Gamma$ denotes the boundary of Γ. This formulation only makes sense if we restrict $\xi\notin\Gamma$ and require Γ to be connected.

For one-dimensional Markov processes, we can reduce the integral equation to

$$p(x,t|\xi) = \int_0^t f(y,t|\xi)\ p(x,t-s|y)\ ds, \tag{2.7}$$

where either $\xi < y < x$ or $x < y < \xi$, for in this case we can take Γ to be a semi-infinite interval of the real line and the boundary reduces to a single point.

Summary

The discussions of the first two parts have led us to the following. The response of a dynamical system governed by stochastic differential equations with white noise disturbance functions is to be represented by a continuous parameter Markov process with incremental moments simply related to the stochastic differential equations. In turn, a continuous parameter Markov process is characterized by a transition probability density function in that all questions concerning these processes can be expressed in terms of the transition probability density function.

The third part of this thesis is concerned with the partial differential equations satisfied by the transition probability density function. It will be seen that these partial differential equations are simply related to the incremental moments of the Markov process.

PART III
THE FOKKER-PLANCK EQUATION

The Backward and Forward (Fokker-Planck) Equations

If we are given a set of n first order stochastic differential equations of the form

$$dx_i(t) = a_i(\underline{x},t) + c_i^j(\underline{x},t)\, dw_j(t), \qquad i=1,\ldots,n, j = 1,\ldots,m ,$$

in which $\underline{x} = (x_1,\ldots,x_n)$ and the $w_j(t)$ are independent Wiener processes with unit incremental variances, the incremental moments of $\underline{x}$ are easily seen to be[7]

$$\lim_{\Delta t\to 0} E\left\{\frac{1}{\Delta t}\,[x_i(t+\Delta t)-x_i(t)\,|\underline{x}(t)]\right\} = a_i(\underline{x}(t),t) \tag{3.1}$$

$$\lim_{\Delta t\to 0} E\left\{\frac{1}{\Delta t}\,[x_i(t+\Delta t)-x_i(t)][x_j(t+\Delta t)-x_j(t)]\,|\underline{x}(t)\right\} = c_i^k(\underline{x}(t),t)c_j^k(\underline{x}(t),t) \tag{3.2}$$

by a straightforward generalization of our discussion of the one-dimensional equation in Part I. For convenience, we define

$$b_{ij}(\underline{x},t) = c_i^k(\underline{x},t)c_j^k(\underline{x},t)$$

from which it is obvious that $\{b_{ij}\}$ is a non-negative definite matrix.

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[7] We are employing the summation convention in which repeated indices are summed over the appropriate range.

It is well known[8] that the transition probability density function, $p(\underline{x},t|\underline{\xi},\tau)$, for the Markov process with incremental moments specified by equations (3.1) and (3.2) satisfies the pair of partial differential equations

$$\frac{\partial p}{\partial \tau} = - a_i(\underline{\xi},\tau) \frac{\partial}{\partial \xi_i} p - \frac{1}{2} b_{ij}(\underline{\xi},\tau) \frac{\partial^2}{\partial \xi_i \partial \xi_j} p ,$$

and

$$\frac{\partial p}{\partial t} = - \frac{\partial}{\partial x_i} [a_i(\underline{x},t)p] + \frac{1}{2} \frac{\partial^2}{\partial x_i \partial x_j} [b_{ij}(\underline{x},t)p] . \tag{3.3}$$

These equations are termed the backward and forward equations, respectively, since in the backward equation derivatives are taken with respect to the backward variables, $(\underline{\xi},\tau)$, and in the forward equation derivatives are taken with respect to the forward variables, $(\underline{x},t)$. The forward equation is commonly called the Fokker-Planck equation.

If we denote the spatial operator of the forward equation by L_x, the backward and forward equations can be written as

$$\frac{\partial p}{\partial \tau} = - L_\xi^* p ,$$

$$\frac{\partial p}{\partial t} = L_x p ,$$

Where L* is the formal adjoint of L.

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[8] See, for example, Caughey [8] for a derivation of equation (3.3).

If the incremental moments are independent of time, the transition probability density function is stationary, and then the backward and forward equations for $p(\underline{x},t|\underline{\xi})$ are

$$\frac{\partial p}{\partial t} = L^*_{\xi} p \quad , \tag{3.4}$$

$$\frac{\partial p}{\partial t} = L_x p \quad . \tag{3.5}$$

A solution, $p(\underline{x},t|\underline{\xi})$, must satisfy both equations (3.4) and (3.5) and, in order to be a transition probability density function, the following conditions:

$$p \geq 0, \quad \text{all } \underline{x},\underline{\xi} \in R \quad , \tag{3.6}$$

$$\int_R p(\underline{x},t|\underline{\xi})\, d\underline{x} = 1, \qquad t \geq 0 \quad , \tag{3.7}$$

and

$$\lim_{t\to 0} \int_R p(\underline{x},t|\underline{\xi})\, f(\underline{\xi})\, d\xi = f(\underline{x}) \tag{3.8}$$

for any initial probability density, $f(\underline{\xi})$. On occasion, by a solution we shall mean a function, $q(\underline{x},t)$, satisfying (3.5)-(3.7) and

$$\lim_{t\to 0} q(\underline{x},t) = f(\underline{x}) \quad .$$

In terms of the transition probability density function,

$$q(\underline{x},t) = \int_R p(\underline{x},t|\underline{\xi})\, f(\underline{\xi})\, d\underline{\xi} \quad .$$

The distinction should always be clear from the context.

Existence of Solutions

We have demonstrated the existence of a Markov process with specified incremental moments when these moments are essentially linear[9]. It follows that a solution of the Fokker-Planck equation exists for this case. In more general circumstances, no existence theory is available.

As we have taken a Markov process with incremental moments associated with a given set of stochastic differential equations as the representation of the response, it is necessary to demonstrate that a solution of the Fokker-Planck equation can be used to define a Markov process. A solution will certainly have the correct incremental moments. We have seen in Part II that a transition probability density function which satisfies the Chapman-Kolmogorov equation can be used to define a Markov process. All that remains then is to establish that a solution of the Fokker-Planck equation does satisfy the Chapman-Kolmogorov equation. This, in turn, depends on establishing the uniqueness of solutions.

Uniqueness of Solutions

Gray [8, appendix] has provided a uniqueness theorem involving conditions which are essentially those required for the derivation of the Fokker-Planck equation. The result is that we can say, quite generally, that solutions of the Fokker-Planck equation are unique.

[9]"Essentially linear" refers to the conditions specified in Part I.

Actually, Gray has shown that if $q_1(\underline{x},t)$ and $q_2(\underline{x},t)$ both satisfy the same Fokker-Planck equation, both satisfy conditions (3.6) and (3.7) and for some t_o, $q_1(\underline{x},t_o) = q_2(\underline{x},t_o)$, then $q_1(\underline{x},t) \equiv q_2(\underline{x},t)$ for $t \geq t_o$.

It follows that the transition probability density function must also be unique. For suppose there were two transition probability density functions. That is, suppose $p_1(x,t|\underline{\xi})$ and $p_2(\underline{x},t|\underline{\xi})$ are solutions to the Fokker-Planck equation as defined earlier. Then, if $f(\underline{\xi})$ is any initial probability density function

$$q_1(\underline{x},t) \equiv \int_R p_1(\underline{x},t|\underline{\xi})\ f(\underline{\xi})\ d\underline{\xi}$$

$$q_2(\underline{x},t) \equiv \int_R p_2(\underline{x},t|\underline{\xi})\ f(\underline{\xi})\ d\underline{\xi}$$

satisfy the conditions for Gray's uniqueness theorem, so that

$$q_1(\underline{x},t) \equiv q_2(\underline{x},t), \quad t \geq 0 .$$

This means

$$\int_R f(\underline{\xi})\,|p_1(\underline{x},t|\underline{\xi}) - p_2(\underline{x},t|\underline{\xi})\ d\underline{\xi} = 0$$

for any initial probability density function, $f(\underline{\xi})$. It follows that

$$p_1(\underline{x},t|\underline{\xi}) \equiv p_2(\underline{x},t|\underline{\xi}), \quad t>0.$$

Having a rather general uniqueness theorem, we shall now show that a unique transition probability density function satisfies the Chapman-Kolmogorov equation. Introducing the notation

$$E_t[f(\underline{x})] \equiv \int_R p(\underline{x},t|\underline{\xi})\ f(\underline{\xi})\ d\xi\ ,$$

the Chapman-Kolmogorov equation (equation (2.1)) implies that

$$E_{t+\tau}[f(\underline{x})] = E_t[E_\tau[f(\underline{\xi})](\underline{x})] \tag{3.9}$$

for any initial probability density, $f(\underline{\xi})$. Now, each side of equation (3.9) satisfies the Fokker-Planck equation and conditions (3.6) and (3.7). Furthermore, at t=0, both equal $E_\tau[f(\underline{\xi})]$. By the uniqueness theorem, they must be equal for all $t\geq 0$. This establishes equation (3.9) for arbitrary initial probability densities and thus implies the validity of the Chapman-Kolmogorov equation for the transition probability density function by the same argument as was used to establish the uniqueness of the transition probability density function.

Convergence to the Steady-State[10]

Any solution of the steady-state Fokker-Planck equation

$$Lp_s(\underline{x}) = 0$$

is termed a steady-state solution and will be denoted as $p_s(\underline{x})$. We have seen that solutions of the Fokker-Planck equation are unique. The same proof shows that steady-state solutions are unique. One has the inclination to regard such a solution as the large time limit of solutions to the Fokker-Planck equation. We shall now examine this proposition and determine sufficient conditions for its validity.

[10]The material of this section has been accepted for publication [9].

Conservation of Probability

If the system variables can take on values in a phase space R, that $p(\underline{x},t|\underline{\xi},\tau)$ is a probability density in $\underline{x}$ requires

$$\int_R p(\underline{x},t|\underline{\xi},\tau)\, d\underline{x} = 1 \ .$$

The requirement that the total probability remain normalized to unity implies that

$$\frac{d}{dt}\int_R p(\underline{x},t|\underline{\xi},\tau)\, d\underline{x} = 0 \ .$$

By integrating the Fokker-Planck equation, expressed by equation (3.3), over R, this is seen to require

$$0 = -\int_R \frac{\partial}{\partial x_i}\left[a_i p - \frac{1}{2}\frac{b_{ij}p}{\partial x_j}\right] d\underline{x}$$

$$= \int_{\partial R}\left[a_i p - \frac{1}{2}\frac{\partial b_{ij}p}{\partial x_i}\right] n_i\, d\underline{x} \ ,$$

where the n_i are the components of the outward normal vector to the surface ∂R bounding the region R. To conserve probability, it is sufficient to require

$$\left[a_i p - \frac{1}{2}\frac{\partial b_{ij}p}{\partial x_i}\right] n_i = 0, \quad \underline{x}\in \partial R \ , \tag{3.10}$$

which can be interpreted as a requirement that the flux of probability through ∂R vanish at any point on ∂R. When R is an infinite domain, equation (3.10) should be taken in a limiting sense.

Definition: A well-behaved solution of the time-dependent Fokker-Planck equation, (3.3), is one which satisfies the boundary conditions

$$\left\{ \left[a_i p - \frac{1}{2} \frac{\partial b_{ij} p}{\partial x_j} \right]_{p=0} n_i = 0, \quad \underline{x} \in \partial R \right. , \tag{3.11}$$

and the integral conditions

$$\int_R p_s^{-1} p^2 \, d\underline{x} < \infty ,$$

$$\int_R p_s^{-1} \left(\frac{\partial p}{\partial t} \right)^2 d\underline{x} < \infty , \tag{3.12}$$

for all $t \geq 0$, with the convergence being uniform in t if the region R is infinite.

It is easily shown that a linear combination of two well-behaved solutions is also well-behaved. Note that $p_s(\underline{x})$ itself is well-behaved if it satisfies condition (3.11) (which we shall assume).

Let p_1 and p_2 be any two well-behaved solutions. Define u by

$$p_2 = p_1 + p_s u \quad . \tag{3.13}$$

Of central importance in establishing conditions for convergence to the steady-state is the following:

Lemma: If p_1 and p_2 are well-behaved solutions of the Fokker-Planck equation, (3.3), then

$$\frac{d}{dt}\int_R p_s u^2 \, d\underline{x} = -\int_R p_s b_{ij} \frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j} \, d\underline{x} \, . \tag{3.14}$$

Using equation (3.3), u satisfies

$$p_s \frac{\partial u}{\partial t} = L_x(up_s) \equiv p_s A(u) \, . \tag{3.15}$$

Introduce the inner product

$$(u,v) = \int_R p_s uv \, d\underline{x} \, , \tag{3.16}$$

with the corresponding norm

$$\|u\| \equiv (u,u)^{1/2} \, .$$

Then

$$(Au,v) = \int_R v L_x(up_s) \, d\underline{x} \, .$$

Integrating by parts, the surface integrals arising are seen to vanish because of the boundary conditions imposed, so that

$$(Au,v) = (u,A^*v) \, ,$$

where

$$A^* = a_i \frac{\partial}{\partial x_i} + \frac{1}{2} b_{ij} \frac{\partial^2}{\partial x_i \partial x_j} .$$

Then

$$(Au,u) = (u,A^*u)$$

$$= \int_R p_s u A^* u \, d\underline{x} \tag{3.17}$$

$$= -\frac{1}{2} \int_R p_s \, b_{ij} \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j} \, d\underline{x} \quad .$$

Using equation (3.15),

$$\int_R u p_s \frac{\partial u}{\partial t} \, d\underline{x} = \frac{1}{2} \frac{d}{dt} \int_R p_s u^2 \, d\underline{x} = (Au,u), \tag{3.18}$$

so that the desired result follows.

The interchange of order of integration and differentiation involved in equation (3.18) is justified by the integral conditions (3.12) included in the definition of well-behaved solutions. It is interesting to note that these integral conditions imply that

$$\|u\| < \infty, \qquad \|Au\| < \infty,$$

for each $t>0$. Thus we are limiting the discussion to that domain of the linear operator A such that the range of A is in the Hilbert space defined by the inner product (3.16).

Uniqueness

Uniqueness of well-behaved solutions follows easily from this lemma. Suppose p_1 and p_2 are well-behaved solutions of equation (3.3), and u is defined by equation (3.13). If p_1 and p_2 are identical at some time t_o, u is identically zero at t_o. Then the statement of the lemma reduces to

$$\frac{d}{dt} \|u\|^2 = 0$$

for all $t \geq t_o$. As $\|u\| \geq 0$ it follows that $u \equiv 0$, all $t \geq t_o$, i.e., $p_1 \equiv p_2$, $t \geq t_o$.

The result applies equally well if $\partial p/\partial t = 0$, i.e., in the steady-state case, for here equation (3.14) says

$$0 = \int_R p_s b_{ij} \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j} d\underline{x} ,$$

which requires u to be a constant since $\{b_{ij}\}$ is positive definite. (Actually, $\{b_{ij}\}$ is necessarily only non-negative definite. When $\{b_{ij}\}$ is not positive definite, considerations such as those pursued in later discussions of convergence to the steady-state lead to the desired result.) On the other hand, integrating equation (3.13) over R, we find

$$\int_R p_s u \, d\underline{x} = 0$$

so thatwe must have $u \equiv 0$, i.e., $p_1 \equiv p_2 \equiv p_s$.

We can summarize this discussion briefly in the

Theorem: (Uniqueness)

A well-behaved solution of the Fokker-Planck equation is unique.

Convergence to the Steady-State

Suppose in the definition of u, equation (3.13), we set $p_2=p$ and $p_1=p_s$, so that we have

$$p = p_s + p_s u \quad . \tag{3.19}$$

If we can show that $u \to 0$ as $t \to \infty$, we will have proved that $p \to p_s$, the time-dependent solution converges to the steady-state solution.

The statement of the lemma, equation (3.14), can be written as

$$\frac{1}{2}\frac{d}{dt}\,|u|^2 = (Au,u) \quad . \tag{3.20}$$

If we can bound the right hand side of equation (3.20) by

$$(Au,u) \leq - k\,\|u\|^2 \quad , \quad k > 0 \quad , \tag{3.21}$$

we will have

$$\frac{1}{2}\frac{d}{dt}\,\|u\|^2 \leq - k\,|u|^2 \quad ,$$

so that

$$\|u(t)\| \leq \|u(t_o)\|\, \exp[-k(t-t_o)], \quad t \geq t_o.$$

Thus we would have $\|u\| \to 0$ at $t \to \infty$. Note that from equation (3.19)

$$\int_R |p-p_s| d\underline{x} = \int_R |p_s u| d\underline{x} \leq \left[\int_R p_s \, d\underline{x} \int_R p_s u^2 \, d\underline{x} \right]^{1/2} = \|u\|$$

so that $\|u\| \to 0$ implies that $p \to p_s$ in L_1 (exponentially).

To establish conditions for the validity of the inequality (3.21), consider minimizing the Rayleigh quotient

$$\lambda = \frac{-(Au,u)}{(u,u)} \tag{3.22}$$

subject to

$$\|u\| < \infty, \tag{3.23}$$

and

$$\int_R p_s u \, d\underline{x} = 0 \ . \tag{3.24}$$

Condition (3.23) arises naturally from the requirement that p be well-behaved; λ will be bounded for the same reason. Condition (3.24) arises upon integrating equation (3.19).

The following Euler equation is a necessary condition for an extremum:

$$\frac{\partial}{\partial x_j}\left(p_s b_{ij}\frac{\partial u}{\partial x_i}\right) + 2\lambda u p_s = 0 \quad .$$

To proceed, it will be assumed that $\{b_{ij}\}$ is a constant matrix. Let

$$u = v p_s^{-1/2} \tag{3.25}$$

to obtain the equation for v:

$$b_{ij}\frac{\partial v^2}{\partial x_i \partial x_j} + v(2\lambda - q(\underline{x})) = 0 \ ,$$

with

$$q(\underline{x}) = -\frac{1}{4}p_s^{-2}b_{ij}\frac{\partial p_s}{\partial x_i}\frac{\partial p_s}{\partial x_j} + \frac{1}{2}p_s^{-1}b_{ij}\frac{\partial^2 p}{\partial x_i \partial x_j} \quad .$$

Since $\{b_{ij}\}$ is symmetric and non-negative definite, it can be assumed that principal coordinates have been chosen so that

$$\{b_{ij}\} = I_N \ ,$$

that is, the identity matrix of order N, when $\{b_{ij}\}$ is positive definite; or

$$\{b_{ij}\} = \begin{pmatrix} I_m & & 0 \\ & 0 & \\ 0 & & \ddots \\ & & & 0 \end{pmatrix} , \tag{3.26}$$

with $m<N$ when $\{b_{ij}\}$ is non-negative definite. In the following, the vector $\underline{x}$ is to be replaced by $(\underline{\xi},\underline{\eta})$, where ξ_i correspond to 1's in the matrix $\{b_{ij}\}$ according to (3.26), and η_i correspond to 0's. The Euler equation reduces to

$$\frac{\partial^2 v}{\partial \xi_i^2} + v(2\lambda - q(\underline{\xi},\underline{\eta})) = 0 \quad , \tag{3.27}$$

where

$$q(\underline{\xi},\underline{\eta}) = -\frac{1}{4} p_s^{-2} \left(\frac{\partial p_s}{\partial \xi_i}\right)^2 + \frac{1}{2} p_s^{-1} \frac{\partial^2 p_s}{\partial \xi_i^2} \quad .$$

Note that the requirement that $(u,u)<\infty$ corresponds to $v \epsilon L_2$ because of (3.25). A sufficient condition for the eigenvalue problem (3.27) to *have a discrete spectrum is that* $q(\underline{\xi},\underline{\eta}) \to \infty$ *as* $\|\xi\| = \xi_i^2 \to \infty$, *and that* $q(\underline{\xi},\underline{\eta})$ be piecewise continuous[11].

If p_s has the exponential form

$$p_s(\underline{\xi},\underline{\eta}) = c \exp[-q(\underline{\xi},\underline{\eta})] \quad ,$$

then

$$q(\underline{\xi},\underline{\eta}) = \frac{1}{4} \left(\frac{\partial q}{\partial \xi_i}\right)^2 - \frac{1}{2} \frac{\partial^2 q}{\partial \xi_i^2} \quad .$$

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[11] Titchmarsh, E. C. [10, p. 150].

It is clear from (3.22) and (3.17) that $\lambda \geq 0$. Consider $\lambda=0$. Employing the $(\underline{\xi},\underline{\eta})$ variables,

$$(Au,u) = -\frac{1}{2}\int_R p_s \left(\frac{\partial u}{\partial \xi_i}\right)^2 d\underline{\xi}\, d\underline{\eta} \quad .$$

Then $\lambda=0$ implies that $\partial u/\partial \xi_i=0$, $i=1,\ldots,m$. If $\{b_{ij}\}$ were positive definite so that $m=N$, this would require that u be a constant for fixed t. However, this would violate condition (3.24), unless the constant is chosen to be zero. If $\{b_{ij}\}$ is only non-negative definite, it is possible to have $u=u(\underline{\eta},t)$ and still have $\lambda=0$. In this case, the Fokker-Planck equation reduces to

$$\frac{\partial u}{\partial t} = \sum_{i=m+1}^{N} a_i \frac{\partial u}{\partial \eta_{i-m}} \quad . \tag{3.28}$$

If (3.28) has only the solution u = constant, condition (3.24) is again violated if the constant is nonzero.

Having ruled out the possibility that $\lambda=0$, it follows that

$$(Au,u) \leq -\lambda_{min} \|u\|^2 \quad ,$$

where λ_{min} is the first nonzero eigenvalue of (3.27). The discreteness of the spectrum guarantees that λ_{min} is positive. We need only identify k in the inequality (3.21) as λ_{min} to obtain the desired result. In summary, we have the following.

Theorem: (Convergence to the Steady-State)

Suppose there exists a solution p_s to the steady-state Fokker-Planck equation,

$$Lp_s = -\frac{\partial}{\partial x_i}\left[a_i p_s - \frac{1}{2}\frac{\partial b_{ij} p_s}{\partial x_j} \right] = 0, \ \underline{x}\in R, \ \{b_{ij}\} \text{ constant.}$$

Further, suppose there exists a well-behaved solution of the time dependent Fokker-Planck equation,

$$\frac{\partial p}{\partial t} = Lp, \ \underline{x}\in R .$$

If p_s is such that

$$q(x) = -\frac{1}{4} p_s^{-2} b_{ij} \frac{\partial p_s}{\partial x_i}\frac{\partial p_s}{\partial x_j} + \frac{1}{2} p_s^{-1} b_{ij} \frac{\partial^2 p_s}{\partial x_i \partial x_j} \tag{3.29}$$

as $\|x\| = \sum_{i=1}^{N} x_i^2 \to \infty$, and is piecewise continuous, then

$$\lim_{t\to\infty} p(\underline{x},t|\underline{x}_o,t_o) = p_s(\underline{x})$$

in the sense of convergence in L_1.

If the matrix $\{b_{ij}\}$ is only non-negative definite, there is a further requirement that the only solution of

$$\frac{\partial u}{\partial t} = \sum_{i=m+1}^{N} a_i \frac{\partial u}{\partial n_{i-m}}, \qquad \frac{\partial u}{\partial \xi} = 0, \quad i=1,\ldots,m$$

is a constant, where ξ_i are the only variables with respect to which second derivatives are taken in the operator L. In this case p_s must be such that

$$q(\underline{\xi},\underline{\eta}) \equiv -\frac{1}{4}\, p_s^{-2} \left(\frac{\partial p_s}{\partial \xi_i}\right)^2 + \frac{1}{2}\, p_s^{-1} \frac{\partial^2 p_s}{\partial \xi_i^2} \to \infty$$

as $|\xi| = \sum_{i=1}^{m} \xi_i^2 \to \infty$ for any fixed $\underline{\eta}$.

For an important class of problems, the last condition required in the case of $\{b_{ij}\}$ non-negative definite is readily seen to hold. For Fokker-Planck equations corresponding to systems of second order differential equations, the coefficients a_i turn out to be just ξ_i. That u a constant is the only solution of

$$\frac{\partial u}{\partial t} = -\sum_{i=1}^{m} \xi_i \frac{\partial u}{\partial \eta_i}, \quad \frac{\partial u}{\partial \xi_i} = 0, \quad i=1,\ldots,m$$

is obvious.

Transformation of Coordinates

We have seen that the system of stochastic differential equations

$$(3.30) \qquad dx_i(t) = a_i(\underline{x},t)\, dt + c_i^{\ell}(\underline{x},t)\, dw_{\ell}(t), \qquad i=1,\ldots,n,$$

where the w_{ℓ} are independent Wiener processes with

$$E[(w_\ell(t) - w_\ell(s))^2] = |t-s|$$

and

$$c_i^\ell(\underline{x},t) \geq 0, \qquad b_{ij} = c_i^\ell c_j^\ell$$

are to be associated with the backward and forward equations

$$\frac{\partial}{\partial \tau} p(\underline{x},t|\underline{\xi},\tau) = -a_i(\underline{\xi},\tau) \frac{\partial}{\partial \xi_i} p - \frac{1}{2} b_{ij}(\underline{\xi},\tau) \frac{\partial^2}{\partial \xi_i \partial \xi_j} p,$$

$$\frac{\partial}{\partial \tau} p(\underline{x},t|\underline{\xi},\tau) = - \frac{\partial}{\partial x_i} a_i(\underline{x},t)\ p + \frac{1}{2} \frac{\partial^2}{\partial x_i \partial x_j} b_{ij}(\underline{x},t)\ p.$$

One often finds that a transformation to a new set of coordinates is of interest. Consider a one-to-one transformation as defined by

(3.31) $$y_i = f_i(\underline{x},t) \ ,$$

(3.32) $$x_i = f_i^{-1}(\underline{y},t), \qquad i = 1,\ldots,n.$$

It is easy to show that $\{\underline{y}(t)\}$ is also an n-component Markov process

It seems reasonable that if we set

$$q(\underline{y},t|\underline{\eta},\tau) = p(\underline{x},t|\underline{\xi},\tau)$$

with $(\underline{y},t)$ related to $(\underline{x},t)$, and $(\underline{\eta},t)$ related to $(\underline{\xi},\tau)$ by the transformation (3.31), the backward and forward equations for $q(\underline{y},t|\underline{\eta},\tau)$ may be obtained by direct substitution:

$$\frac{\partial q}{\partial \tau}(\underline{y},t|\underline{\eta},\tau) = - A_i(\underline{\eta},\tau)\frac{\partial}{\partial \eta_i} q - \frac{1}{2} B_{ij}(\underline{\eta},\tau)\frac{\partial^2}{\partial \eta_i \partial \eta_j} q ,$$

$$\frac{\partial q}{\partial t}(\underline{y},t|\underline{\eta},\tau) = - \frac{\partial}{\partial y_i} A_i(\underline{y},t)\, q + \frac{1}{2}\frac{\partial^2}{\partial y_i \partial y_j} B_{ij}(\underline{y},t)\, q ,$$

where one easily finds that

$$(3.33)\quad A_i(\underline{y},t) = \frac{\partial f_i}{\partial t} + \frac{\partial f_i}{\partial x_k} a_k + \frac{1}{2}\frac{\partial^2 f_i}{\partial x_j \partial x_k} b_{jk} ,$$

$$(3.34)\quad B_{ij}(\underline{y},t) = \frac{\partial f_i}{\partial x_k}\frac{\partial f_j}{\partial x_\ell} b_{k\ell} .$$

In these, f_i, a_k, b_{kl} are to be expressed, finally, in terms of $\underline{y}$ variables by means of the inverse transformation (3.32).

Ito [11] has provided a formula connecting the stochastic differentials of $\underline{y}$ to those of $\underline{x}$ which verifies the suggestion of the previous paragraph. For our purposes, we can summarize Ito's result to say that if $\underline{x}$ is an n-component Markov process defined by the system of stochastic differentials, (3.30), and $\underline{y}$ is obtained from $\underline{x}$ by the one-to-one transformation, (3.31), then $\underline{y}$ is an n-component Markov process defined by the following system of stochastic differentials:

$$dy_i(t) = A_i(\underline{y},t)\ dt + C_i^{\ell}(\underline{y},t)\ dw_{\ell}(t)$$

where $A_i(\underline{y},t)$ is defined by (3.33) and

$$C_i^{\ell}(\underline{y},t) = \frac{\partial f_i}{\partial x_k}\ C_k^{\ell}\ .$$

It follows easily that $B_{ij}(\underline{y},t) = C_i^{\ell}(\underline{y},t)\ C_j^{\ell}(\underline{y},t)$ is given by (3.34).

As we have noted, a different interpretation of the system of stochastic differentials is possible and may lead to a different set of incremental moments. Even with his interpretation, Stratonovich [3] has shown that the correct backward and forward equations may be obtained by direct transformation of these equations, if one takes note of his version of the formula connecting stochastic differentials and uses a consistent interpretation of the stochastic differential system.

Moments from the Fokker-Planck Equation

Once we have determined the transition probability density function, various statistics can be derived from it, as we saw in Part II. It is important to note that many statistics can be found directly by means of the Fokker-Planck equation without a need for determining the transition probability density function.

Suppose $f(\underline{x})$ is any function of the n variables which are the components of the Markov Process. Then

$$\frac{d}{dt} E_t[f|\underline{\xi}] = \frac{d}{dt} \int_R f(\underline{x})\ p(\underline{x},t|\underline{\xi})\ d\underline{x} = \int_R f(\underline{x})\ \frac{\partial p}{\partial t}\ (\underline{x},t|\underline{\xi})\ d\underline{x}$$

and using the Fokker-Planck equation,

$$\frac{d}{dt} E_t[f|\underline{\xi}] = \int_R f(\underline{x})\ L_x p(\underline{x},t|\underline{\xi})\ d\underline{x} = \int_R p(\underline{x},t|\underline{\xi})\ L_x^* f(\underline{x})\ d\underline{x}$$

$$= E_t[L_x^* f|\underline{\xi}]\ . \tag{3.35}$$

Using the backward equation

$$\frac{d}{dt} E_t[f|\underline{\xi}] = \int_R f(\underline{x})\ L_{\xi}^* p(\underline{x},t|\underline{\xi})\ d\underline{x} = L_{\xi}^* \int_R f(\underline{x})\ p(\underline{x},t|\underline{\xi})\ d\underline{x}$$

$$= L_{\xi}^*\ E_t[f|\underline{\xi}]\ . \tag{3.36}$$

Equation (3.35) is an ordinary differential equation for $E_t[f|\underline{\xi}]$ with the initial condition

$$\lim_{t\to 0} E_t[f|\underline{\xi}] = f(\underline{\xi}).$$

For a linear stochastic differential equation of order n with white noise, if we choose

$$f_i(\underline{x}) = x_i, \qquad i = 1,\ldots,n \tag{3.37}$$

we obtain a closed set of ordinary differential equations for these first moments[12]. In fact, if a set of functions $f_i(\underline{x})$ is chosen to represent all moments of order m, then a closed set of ordinary differential equations for these m^{th} moments, which also involve only lower order moments, will be found.

Unfortunately, when the stochastic differential equation is non-linear, this simple situation does not arise. Typically, one finds that by choosing $f_i(\underline{x})$ as in (3.37) a set of ordinary differential equations arises which also involves higher order moments.

Equation (3.36) may be useful in these latter cases. This is a partial differential equation, clearly of the same form as the backward equation.

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[12] Caughey, T.K., and Dienes, J.K. [12, pp. 304-306], or Bogdanoff, J.L., and Kozin, F. [13].

PART IV

SOLUTIONS OF THE FOKKER-PLANCK EQUATION ASSOCIATED WITH FIRST AND SECOND ORDER STOCHASTIC DIFFERENTIAL EQUATIONS

Exact Solutions

In the first three sections we have developed a theory of stochastic differential equations which has led us to the Fokker-Planck equation satisfied by the transition probability density function. In the remainder of this thesis we shall be concerned with the application of this theory to first and second order stochastic differential equations.

Exact solutions of the Fokker-Planck equation for the transition probability density function have been found for two types of stochastic differential equations: 1) linear equations, and 2) certain first order nonlinear equations. The steady-state density can always be obtained by quadratures for a first order system, and has also been found for a certain class of nonlinear oscillators.

As we have previously remarked, the transition probability density function for a linear system is Gaussian due to the Gaussian nature of the Wiener process, so that the transition probability density function is completely characterized by the means and second moments. Even without using this fact, the Fokker-Planck equation is easily solved by means of the Fourier transform[13].

[13] Wang and Uhlenbeck [14] solved the Fokker-Planck equation for a system of linear oscillators by this method.

The transition probability density function is also known for certain one-dimensional systems. Dienes [15] found it for the stochastic differential equation

$$\frac{dy}{dt} + k \text{ sign } (y) = \frac{dw}{dt} ,$$

where w(t) is a Wiener process. Atkinson [16] has studied piecewise linear, one-dimensional systems. Wong [17] found eigenfunction expansions for the transition probability density functions satisfying Fokker-Planck equations of the form

$$\frac{\partial p}{\partial t} = \frac{\partial}{\partial x} \left[(ax+b)\ p + \frac{\partial}{\partial x} (cx^2 + dx + e)\ p \right] .$$

Exact solutions for the steady-state density for any one-dimensional system can be determined by direct integration. The general steady-state Fokker-Planck equation for a one-dimensional system is

$$0 = - \frac{\partial}{\partial x} \left[a(x)\ p - \frac{1}{2} \frac{\partial}{\partial x} (b(x)\ p) \right] .$$

One integration yields

$$a(x)\ p - \frac{1}{2} \frac{\partial}{\partial x} [b(x)\ p] = k$$

and another gives

$$p(x) = \frac{C}{b(x)} \exp\left[2\int_0^x \frac{a(\xi)}{b(\xi)}\, d\xi \right] - \frac{2k}{b(x)} \exp\left[2\int_0^x \frac{a(\xi)}{b(\xi)}\, d\xi \right]$$

$$\int_0^x \exp\left[-2\int_0^\xi \frac{a(\eta)}{b(\eta)}\, d\eta \right] d\xi$$

where k is arbitrary. Usually, the conditions $p \geq 0$, and $\int_{-\infty}^{\infty} p(x)\, dx = 1$ require that k=0 so that we are left with

$$p(x) = \frac{C}{b(x)} \exp\left[2\int_0^x \frac{a(\xi)}{b(\xi)}\, d\xi \right]$$

with C chosen to normalize p(x).

Exact solutions for nonlinear systems of second order have been found only for the steady-state density for oscillators of the form

$$\ddot{x} + f(H)\,\dot{x} + g(x) = \dot{w}$$

where

$$H = \frac{1}{2} x^2 + \int_0^x g(\eta)\, d\eta \ .$$

Caughey [18] found that the steady-state density, which satisfies the steady-state Fokker-Planck equation

$$(4.1) \qquad 0 = \left[g(x) \frac{\partial p}{\partial \dot{x}} - \dot{x} \frac{\partial p}{\partial x} \right] + \frac{\partial}{\partial \dot{x}} \left[f(H)\, \dot{x} p + D \frac{\partial p}{\partial \dot{x}} \right] ,$$

is given by[14]

$$p(x,\dot{x}) = A \exp \left[- \frac{1}{D} F(H) \right]$$

where

$$F(H) = \int_0^H f(\eta)\, d\eta$$

and A is chosen to normalize $p(x,\dot{x})$.

The solutions for these nonlinear systems are found by noting the fortunate circumstance that separating equation (4.1) as

$$0 = g(x) \frac{\partial p}{\partial \dot{x}} - \dot{x} \frac{\partial p}{\partial x}$$

$$0 = \frac{\partial}{\partial \dot{x}} \left[f(H)\, \dot{x} p + D \frac{\partial p}{\partial \dot{x}} \right]$$

yields the unique solution. This pair of equations can easily be solved by Lagrange's method[15] to yield the solution above.

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[14] In Appendix I, the uniqueness of this solution and the uniqueness and convergence of well-behaved time-dependent solutions, as $t \to \infty$ to this solution are established under appropriate conditions on F(H) and g(x).

[15] Piaggio, H. [19].

The exact steady-state probability density for the set of coupled, nonlinear oscillators

$$\ddot{x}_i + \beta_i \dot{x}_i + \frac{\partial V(\underline{x})}{\partial x_i} = \frac{dw_i(t)}{dt} , \quad i = 1,\ldots,n,$$

$$E[dw_i(t)\ dw_j(t)] = 2D_i\ \delta_{ij}\ dt$$

can be found by the same method, provided $\frac{\beta_i}{D_i}$ = constant, to be

$$p(\underline{x},\dot{\underline{x}}) = A \exp\left[- \frac{\beta_i}{2D_i} \left(\sum_{i=1}^{n} \dot{x}_i^2 + V(\underline{x}) \right) \right] .$$

Unfortunately, this trick does not appear to work for other types of nonlinear systems, e.g., the van der Pol oscillator.

Approximation Methods

The paucity of exact solutions for nonlinear systems, particularly in the case of second order systems, suggests the need for approximate methods. One approach is to replace the original nonlinear system by an "equivalent" linear system[16]. A second approach is to use a perturbation procedure directly on the stochastic differential equation[17]. These techniques depend in an essential way on an ability to reduce

[16] See Caughey, T. K. [20] for a review of this method.

[17] Crandall, S. [21].

the nonlinear system to one or a succession of linear systems, for which, as we have seen, complete solutions are available. In so doing, the Fokker-Planck equation is really ignored.

In the table below, the appropriate method or methods for obtaining approximate solutions are indicated for the various types of first and second order, nonlinear stochastic differential equations. Among the methods indicated are two new ones which are based on the Fokker-Planck equation: perturbation (or approximation) of eigenvalues and asymptotic expansions.

First Order:	Equivalent linearization
	Perturbation method on the stochastic differential equation
	Perturbation of eigenvalues
	Approximation of Eigenvalues
Second Order (oscillators):	
Passive	Equivalent linearization
	Perturbation method on the stochastic differential equation
Self-excited	Approximation of eigenvalues
	Asymptotic expansions

First-Order Stochastic Differential Equations

Method of Equivalent Linearization

First, we shall illustrate the two methods which deal directly with the stochastic differential equation. The idea of the method of equivalent linearization is to replace the nonlinear system by a linear system which is equivalent in some sense. Consider, as an example, the first order stochastic differential equation

$$\frac{dx}{dt} + x + \varepsilon x^3 = \frac{dw}{dt}, \qquad \varepsilon > 0$$

with $w(t)$ a Wiener process and $E[dw(t)^2] = 2D\,dt$. We wish to replace this equation by

$$\frac{dx}{dt} + \beta x = \frac{dw}{dt}$$

with β chosen to minimize the equation deficiency term,

$$e(x) = x + \varepsilon x^3 - \beta x,$$

in some sense. A convenient choice for purposes of computation is to minimize $E[e^2(x)]$ in the steady-state. The steady-state Fokker-Planck equation,

$$\frac{\partial}{\partial x}\left[(x+\varepsilon x^3)\, p_s + D \frac{\partial p_s}{\partial x} \right] = 0,$$

has the unique solution

$$p_s(x) = \frac{\exp\left[-\frac{x^2}{2D} - \varepsilon\frac{x^4}{4D}\right]}{\int_{-\infty}^{\infty} \exp\left[-\frac{x^2}{2D} - \varepsilon\frac{x^4}{4D}\right] dx} .$$

Then

$$E[e^2(x)] = \frac{\int_{-\infty}^{\infty} \left[x + \varepsilon x^3 - \beta x\right]^2 \exp\left[-\frac{x^2}{2D} - \varepsilon\frac{x^4}{4D}\right] dx}{\int_{-\infty}^{\infty} \exp\left[-\frac{x^2}{2D} - \varepsilon\frac{x^4}{4D}\right] dx}$$

To determine the extrema of $E[e^2(x)]$, we set $\frac{d}{d\beta} E[e^2(x)] = 0$:

$$0 = \int_{-\infty}^{\infty} x\left[x + \varepsilon x^3 - \beta x\right] \exp\left[\frac{x^2}{2D} - \varepsilon\frac{x^4}{4D}\right] dx.$$

So

$$\beta = \frac{\int_{-\infty}^{\infty} (x^2 + \varepsilon x^4) \exp\left[- \frac{x^2}{2D} - \varepsilon \frac{x^4}{4D} \right] dx}{\int_{-\infty}^{\infty} x^2 \exp\left[- \frac{x^2}{2D} - \varepsilon \frac{x^4}{4D} \right] dx} .$$

Since

$$\frac{d^2}{d\beta^2} E[e^2(x)] = \frac{\int_{-\infty}^{\infty} x^2 \exp\left[- \frac{x^2}{2D} - \varepsilon \frac{x^4}{4D} \right] dx}{\int_{-\infty}^{\infty} \exp\left[- \frac{x^2}{2D} - \varepsilon \frac{x^4}{4D} \right] dx} \quad 0 \ ,$$

the extremum so determined is actually a minimum, as desired. β can be computed to be

$$\beta = 1 + 3D\varepsilon + O(\varepsilon^2) .$$

The transition probability density function of the equivalent linear system is Gaussian, and therefore characterized by the mean and variance. By operating with the formal properties of the differentials of the Wiener process, these can be determined directly from the convolution form of the solution of the linear stochastic differential equation.

We shall, however, show here how the desired moments are obtained from the appropriate Fokker-Planck equation.

The Fokker-Planck equation associated with the equivalent linear stochastic differential equations is

$$\frac{\partial p}{\partial t} = + \frac{\partial}{\partial x} [\beta x p] + D \frac{\partial^2}{\partial x^2} p \quad .$$

Moment equations are obtained in the manner suggested in Part III. There, we found

$$\frac{d}{dt} E_t [f|\xi] = E_t [L_x^* f|\xi]$$

with the initial condition

$$E_o[f|\xi] = f(\xi) \ .$$

We identify, in this case,

$$L_x^* = - \beta x \frac{\partial}{\partial x} + D \frac{\partial^2}{\partial x^2} \quad .$$

Then the equation for the first moment is

$$\frac{d}{dt} E_t[x|x_o] = E_t[-\beta x|x_o] = - \beta E_t[x|x_o] \ ,$$

which has the solution

$$E_t[x|x_o] = x_o e^{-\beta t} \quad .$$

The equation for the second moment is

$$\frac{d}{dt} E_t[x^2|x_o] = E_t[-2\beta x^2 + 2D|x_o] = -2\beta\, E_t[x^2|x_o] + 2D,$$

which has the solution

$$E_t[x^2|x_o] = \frac{D}{\beta}(1-e^{-2\beta t}) + x_o^2\, e^{-2\beta t} .$$

The variance is then

$$E_t\left\{[x-E_t(x|x_o)]^2 \mid x_o\right\} = \frac{D}{\beta}(1-e^{-2\beta t}) .$$

It follows that the transition probability density function of the equivalent linear system is

$$P_{el}(x) = \sqrt{\frac{\beta}{2\pi D(1-e^{-2\beta t})}}\ \exp\left[-\frac{\beta(x-x_o e^{-\beta t})^2}{2D(1-e^{-2\beta t})}\right] .$$

Note that the variance in the steady-state is

$$\frac{D}{\beta} = \frac{D}{1+3D\varepsilon} + O(\varepsilon^2)$$

$$= D[1-3D\varepsilon] + O(\varepsilon^2)$$

and coincides with the exact steady-state variance to this order.

An approximate autocorrelation function can now be computed.

$$R_x(\tau) = \int_{-\infty}^{\infty}\int x_1 x_2 \; p_{el}(x_2,\tau|x_1) \; p_s(x_1) \; dx_1 \; dx_2$$

$$= D[1-3\varepsilon D] \; e^{-\beta t} + O(\varepsilon^2) \quad .$$

Through the Wiener-Khintchine relation, (2.5), we have the spectral density

$$\phi_x(\omega) = \frac{\pi D(1-3\varepsilon D)}{\beta} \; \frac{1}{\omega^2+\beta^2} + O(\varepsilon^2) \quad .$$

The significant features of this method are: 1) the nonlinear equation is replaced by a linear equation for which the transition probability density function is known, and 2) the method is limited by the fact that one can not iterate to reduce the equation deficiency any further.

Perturbation Method on the Stochastic Differential Equation

The second method which involves operations on the stochastic differential equation overcomes the one significant drawback of the method of equivalent linearization, but does have its own disadvantages. Again we consider the first order stochastic differential equation

$$\frac{dx}{dt} + x + \varepsilon x^3 = \frac{dw}{dt} \quad ,$$

with w(t) as before. We seek a solution in the form

$$x = x_o + \varepsilon x_1 + \varepsilon^2 \; x_2 + \ldots \quad .$$

Substituting into the stochastic differential equation and forming successive equations by equating groups of terms of like powers of ε to zero, we obtain as the first two

$$\frac{dw_o}{dt} + x_o = \frac{dw}{dt} ,$$

$$\frac{dx_1}{dt} + x_1 = - x_o^3 .$$

This method is most useful for determining the autocorrelation function directly. For a zero mean process,

$$R_x(\tau) = \lim_{t\to\infty} E[x(t)\ x(t+\tau)]$$

$$= \lim_{t\to\infty} \{E[x_o(t)x_o(t+\tau)] + \varepsilon E[x_o(t)x_1(t+\tau)] + \varepsilon E[x_o(t+\tau)x_1(t)] + 0(\varepsilon^2)\}$$

The 0(1) term in this expression is just the autocorrelation function of a linear system. We found this in employing the method of equivalent linearization to be

$$\lim_{t\to\infty} E[x_o(t)x_o(t+\tau)] = De^{-\tau} .$$

The succeeding expectations are found by expressing $x_1(t)$, etc., in convolution form, e.g.,

$$x_1(t) = -\int_0^t e^{-t+s}\ x_o^3(s)\ ds$$

and using the Gaussian property of $x_o(s)$. Now

$$E[x_o(t+\tau)x_1(t)] = -\int_0^t e^{-t+s}\, E[x_o(t+\tau)\, x_o^3(s)]\, ds \quad .$$

The interchange of order of integration and expectation is justified by the fact that $x_o(t+\tau)x_o^3(s)$ has finite mean square[18].

Due to the Markovian nature of $x_o(t)$, we have for $s<t+\tau$,

$$E[x_o(t+\tau)x_o^3(s)\,|a] = E[x_o^3(s)\; E[x_o(t+\tau)\,|x_o(s)]\,|a] \ ,$$

where we have taken the initial condition, $x_o(0) = a$, $x_j(0) = 0, j>0$. In employing the method of equivalent linearization, we found the conditional mean and mean square

$$E[x(t)\,|a] = ae^{-t} \ ,$$

$$E[x^2(t)\,|a] = D(1-e^{-2t}) + a^2\, e^{-2t} \quad .$$

Then it follows that

$$E[x_o(t+\tau)\; x_o^3(s)\,|a] = E[x_o^3(s)x_o(s)e^{-t-\tau+s}|a] \ .$$

We have identically that

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[18] Parzen, E. [6, p. 79].

$$E[x_o^4(s)] = E\left\{[x_o(s) - ae^{-s}]^4 + 4ae^{-s}[x_o(s) - ae^{-s}]^3 + 6a^2e^{-2s}[x_o(s) - ae^{-s}]^2 + 4a^3e^{-3s}[x_o(s) - ae^{-s}] + a^4e^{-4s}\right\} .$$

If y_1, y_2, y_3, y_4, are jointly normally distributed with zero means, then[19]

$$E[y_1y_2y_3y_4] = E[y_1y_2]\,E[y_3y_4] + E[y_1y_3]\,E[y_2y_4] + E[y_1y_4]\,E[y_2y_3] .$$

Also,

$$E[y_1y_2y_3] = 0 .$$

Since $x_o(s) - ae^{-s}$ has zero mean, we then have

$$E[x_o^4(s)] = 3[E([x_o(s) - ae^{-s}]^2)]^2 + 6a^2e^{-2s}\,E[x_o(s) - ae^{-s}]^2 + a^4e^{-4s}$$

$$= 3D^2[1-e^{-2s}]^2 + 6a^2e^{-2s}\,D(1-e^{-2s}) + a^4e^{-4s} .$$

Substituting into the integral, and performing the integrations, we obtain

$$E[x_o(t+\tau)x_1(t)] = -e^{-2t-\tau}\left\{3D^2\left[\frac{1}{2}(e^{2t}-1) - 2t - \frac{1}{2}(e^{-2t}-1)\right] + 6a^2D[t + \frac{1}{2}(e^{-2t}-1)] - \frac{1}{2}a^4(e^{-2t}-1)\right\} .$$

[19] Parzen, E. [6, p. 93]

Then

$$\lim_{t\to\infty} E[x_o(t+\tau)\ x_1(t)] = -\frac{3D^2}{2} e^{-\tau} \ .$$

The computation of $E[x_o(t)x_1(t+\tau)]$ proceeds in a similar way.

$$E[x_o(t)\ x_1(t+\tau) = -\int_0^t e^{-t-\tau+s}\ E[x_o(t)x_o^3(s)]\ ds$$

$$-\int_t^{t+\tau} e^{-t-\tau+s} E[x_o(t)\ x_o^3(s)]\ ds.$$

Each of these integrals can be handled as was done above. The net result is that

$$\lim_{t\to\infty} E[x_o(t)\ x_1(t+\tau)] = -\frac{3D^2}{2} e^{-\tau} - 3D^2\tau e^{-\tau} \ .$$

Then we have for the autocorrelation function

$$R_x(\tau) = De^{-\tau}\ [1-3\varepsilon D-3\varepsilon D\tau] + O(\varepsilon^2) \ .$$

It is interesting to note that this is precisely the expression we get, to $O(\varepsilon)$, by expanding the result obtained by the method of equivalent linearization if we use the expansion of β in powers of ε also found there.

The advantage, in principle, of this perturbation method is that higher order corrections may be found. Actually calculating the

the higher order corrections, however, is quite tedious as the amount of work required grows enormously with each succeeding power of ε. The result found here illustrates the significant disadvantage of this perturbation method, specifically, that one obtains a nonuniform expansion of the autocorrelation function with respect to the correlation time τ, that is, the $O(\varepsilon)$ correction is not uniformly small for all τ as compared to the $O(1)$ term.

Eigenfunction Expansions; Approximation of Eigenvalues

The third method which we present here overcomes the disadvantages inherent in both the method of equivalent linearization and the perturbation method on the stochastic differential equation. In order to make the benefits of this method evident, we shall first consider the development and consequences of obtaining an eigenfunction expansion for the transition probability density function.

A system described by the first-order stochastic differential equation

$$\frac{dx}{dt} + f(x) = \frac{dw}{dt} \quad ,$$

where $w(t)$ is a Wiener process with $E[dw(t)^2] = 2Ddt$, has a transition probability density function which satisfies the Fokker-Planck equation

$$\frac{\partial}{\partial t} p(x,t|\xi) = \frac{\partial}{\partial x} [f(x)p] + D \frac{\partial^2 p}{\partial x^2} .$$

The steady-state solution, denoted by $p_s(x)$, is easily found to be

$$p_s(x) = C \exp\left[- \frac{F(x)}{D} \right] ,$$

where C is the normalization constant and

$$F(x) = \int_0^x f(\eta)\, d\eta .$$

The only condition required of $f(x)$ is that $p_s(x)$ be integrable over the infinite real line. For example, when $f(x) = \beta x$, we find

$$p_s(x) = \frac{1}{\sqrt{2\pi D/\beta}} \exp\left[- \frac{\beta x^2}{2D} \right] .$$

If we make the substitution

$$p = p_s + p_s u \tag{4.2}$$

in the Fokker-Planck equation we obtain

$$\frac{\partial u}{\partial t} = - Au$$

with

$$Au = - \frac{1}{p_s} \frac{\partial}{\partial x} [f(x)\ p_s u] - \frac{D}{p_s} \frac{\partial^2}{\partial x^2} [p_s u]$$

$$= f(x) \frac{\partial u}{\partial x} - D \frac{\partial^2 u}{\partial x^2} .$$

Introducing the inner product

$$(u,v) = \int_{-\infty}^{\infty} p_s(x)\, u(x)\, v(x)\, dx, \tag{4.3}$$

we define A* by

$$(Au,v) = (u,A^*v) \ .$$

It is easily seen that $A = A^*$.

Now, if we set

$$u(x,t|\xi) = e^{-\lambda t}\, v(x)\, C(\xi)$$

we obtain the eigenvalue problem

$$Av - \lambda v = 0 \ . \tag{4.4}$$

Suppose that we have solved the eigenvalue problem and have obtained a complete set of eigenfunctions, $\psi_j(x)$, orthonormal in the inner product specified by (4.3). Then we can expand u as

$$u(x,t|\xi) = \sum_{n=1}^{\infty} e^{-\lambda_n t}\, C_n(\xi)\, \psi_n(x) \ ,$$

where

$$C_n(\xi) = (u(x,0|\xi),\, \psi_n(x)) = \int_{-\infty}^{\infty} p(x,0|\xi)\psi_n(x)\, dx - \int_{-\infty}^{\infty} p_s(x)\, \psi_n(x)\, dx.$$

Note that $\lambda=0$ is an eigenvalue corresponding to the eigenfunction $\psi_o(x) = 1$. We have explicitly removed this term in the representation (4.2) so that the second integral vanishes.

Applying the initial condition for the transition probability density function we find

$$C_n(\xi) = \psi_n(\xi)$$

so that

$$u(x,t|\xi) = \sum_{n=1}^{\infty} e^{-\lambda_n t} \psi_n(\xi) \psi_n(x),$$

and then

(4.5)

$$p(x,t|\xi) = p_s(x) [1+u(x,t|\xi)] = p_s(x) \sum_{n=0}^{\infty} e^{-\lambda_n t} \psi_n(\xi) \psi_n(x),$$

where we have taken $\lambda_o = 0$ and $\psi_o(x) = 1$.

Autocorrelation Function

Assuming that $E[x(t)] = 0$, the autocorrelation function as given by equation (2.4) is

$$R_x(\tau) = \lim_{t\to\infty} \int\int_{-\infty}^{\infty} x_t x_{t+\tau} \, p(x_t,t|\xi) \, p(x_{t+\tau},\tau|x_t) \, dx_t \, dx_{t+\tau} .$$

Using the series expansion, equation (4.5), we have

$$R_x(\tau) = \lim_{t\to\infty} \int_{-\infty}^{\infty} x_t x_{t+\tau} \left[p_s(x_t) \sum_0^{\infty} e^{-\lambda_n t} \psi_n(x_t)\, \psi_n(\xi) \right]$$

$$\left[p_s(x_{t+\tau}) \sum_0^{\infty} e^{-\lambda_m \tau} \psi_m(x_{t+\tau})\, \psi_m(x_t) \right] dx_t\, dx_{t+\tau} .$$

Taking the indicated limit inside the integral

$$R_x(\tau) = \int_{-\infty}^{\infty} x_t x_{t+\tau}\, p_s(x_t)\, p_s(x_{t+\tau}) \sum_0^{\infty} e^{-\lambda_m \tau} \psi_m(x_{t+\tau})\, \psi_m(x_t)\, dx_t\, dx_{t+\tau}$$

$$(4.6) = \sum_{m=0}^{\infty} e^{-\lambda_m \tau} \alpha_m^2$$

with

$$(4.7) \qquad \alpha_m = \int_{-\infty}^{\infty} x p_s(x)\, \psi_m(x)\, dx = (x, \psi_m(x)) .$$

From equation (4.6), we easily obtain the spectral density function by means of one of the Wiener-Khintchine relations, equation (2.5),

$$\phi_x(\omega) = \frac{2}{\pi} \int_0^{\infty} R_x(\tau) \cos \omega\tau\, d\tau$$

$$\phi_x(\omega) = \frac{2}{\pi} \sum_{m=0}^{\infty} \frac{\alpha_m^2 \lambda_m}{\lambda_m^2 + \omega^2} \quad .$$

Example 1: $f(x) = x$ (linear case)

The eigenvalue problem, equation (4.4), can be written as

$$- D \frac{d^2v}{dx^2} + f(x) \frac{dv}{dx} - \lambda v = 0$$

and in this particular case, we have

$$- D \frac{d^2v}{dx^2} + \beta x \frac{dv}{dx} - \lambda v = 0.$$

Introduce the change of variable

$$\xi = \sqrt{\frac{\beta}{2D}} \; x$$

to obtain

$$\frac{d^2v}{d\xi^2} - 2\xi \frac{dv}{d\xi} + \frac{2\lambda}{\beta} v = 0,$$

which is Hermite's equation. Thus

$$p_s(\xi) = \frac{1}{\sqrt{\pi}} \exp(-\xi^2) \quad ,$$

and the orthonormal eigenfunctions, $\psi_n(\xi)$, are essentially the Hermite polynomials, $H_n(\xi)$:

$$\psi_n(\xi) = \frac{H_n(\xi)}{N_n}, \quad N_n = 2^{n/2}\,(n!)^{1/2}, \quad H_n(\xi) = (-1)^n\, e^{\xi^2}\, \frac{d^n e^{-\xi^2}}{d\xi^n}$$

with corresponding eigenvalues

$$\lambda_n = \beta n, \qquad n = 0, 1, 2, \ldots \quad .$$

The α_m are easily computed:

$$\alpha_m = (x, \psi_m(x)) = \sqrt{\frac{2D}{\beta}} \left(\xi, \frac{H_m(\xi)}{N_m}\right)$$

$$= \sqrt{\frac{2D}{\beta}} \left(\frac{1}{2} H_1(\xi), \frac{1}{N_m} H_m(\xi)\right) = \sqrt{\frac{D}{\beta}}\ \delta_{1m} \ .$$

The autocorrelation function and spectral density are then, respectively,

$$R_x(\tau) = \frac{D}{\beta}\, e^{-\beta\tau},$$

and

$$\phi_x(\omega) = \frac{2D}{\pi(\beta^2+\omega^2)} \quad .$$

Perturbation Procedure for Nearly Linear Systems

Systems which are nearly linear will have eigenfunctions and eigenvalues which are nearly those of the linear system. To exploit this idea, suppose that

$$f(x) = \beta x + \varepsilon g(x)$$

where ε is supposed to be a small parameter. The eigenvalue problem, equation (4.4), then becomes

$$-D \frac{d^2 v^n}{dx^2} + (\beta x + \varepsilon g(x)) \frac{dv^n}{dx} - \lambda^n v^n = 0$$

where we shall employ superscripts to denote separate eigenfunctions and their corresponding eigenvalues. Again, introduce the transformation

$$\xi = \sqrt{\frac{\beta}{2D}}\ x$$

to obtain

$$\frac{d^2 v^n}{d\xi^2} - \left[2\xi + \varepsilon \sqrt{\frac{2}{\beta D}}\ g\left(\sqrt{\frac{2D}{\beta}}\ \xi \right) \right] \frac{dv^n}{d\xi} + \left(\frac{2\lambda^n}{\beta} \right) v^n = 0 \quad .$$

Defining

$$\mu^n = \frac{\lambda^n}{\beta}$$

$$h(\xi) = \sqrt{\frac{2}{\beta D}}\ g\left(\sqrt{\frac{2D}{\beta}}\ \xi \right)$$

we have

$$\frac{d^2 v^n}{d\xi^2} - 2\xi \frac{dv^n}{d\xi} + 2\mu^n v^n = \varepsilon h(\xi) \frac{dv^n}{d\xi} \quad . \tag{4.8}$$

We will obtain an asymptotic expansion for $v^n(\xi)$ by assuming expansions of the form

$$(4.9)\quad v^n(\xi) = v_o^n + \varepsilon v_1^n + \varepsilon^2 v_2^n + \ldots ,$$

$$\mu^n = \mu_o^n + \varepsilon\mu_1^n + \varepsilon^2 \mu_2^n + \ldots .$$

Substituting into equation (4.8), we obtain a set of equations by equating each coefficient of a power of ε to zero. We show the first three explicitly:

$$\frac{d^2 v_o^n}{d\xi^2} - 2\xi \frac{dv_o^n}{d\xi} + 2\mu_o^n v_o^n = 0 ,$$

$$\frac{d^2 v_1^n}{d\xi^2} - 2\xi \frac{dv_1^n}{d\xi} + 2\mu_o^n v_1^n = h(\xi) \frac{dv_o^n}{d\xi} - 2\mu_1^n v_o^n ,$$

$$\frac{d^2 v_2^n}{d\xi^2} - 2\xi \frac{dv_2^n}{d\xi} + 2\mu_o^n v_2^n = h(\xi) \frac{dv_1^n}{d\xi} - 2\mu_2^n v_o^n - 2\mu_1^n v_1^n .$$

The differential operator appearing on the left hand side in each equation is the Hermite differential operator. The terms arising from the small non-linearity are viewed as forcing terms.

The relationships needed to obtain v_j^n and μ_j^n are developed in Appendix B. The results are stated here. We assume an expansion in terms of the unperturbed eigenfunctions. Specifically

$$v_1^n = \sum_{j=0}^{\infty} a_{nj}\, v_o^j \ ,$$

$$v_2^n = \sum_{j=0}^{\infty} b_{nj}\, v_o^j \ ,$$

where the unperturbed eigenfunctions are chosen to be orthonormal, i.e.,

$$\int_{-\infty}^{\infty} p_s v_o^n\, v_o^\ell \, d\xi = \delta_{n\ell}$$

where $p_s(\xi)$ is the steady-state solution when $\varepsilon=0$. (See Example 1.) The steady-state solution for nonzero ε, designated here as $\overline{p_s}(\xi)$, is expanded as

$$\overline{p_s}(\xi) = p_s(\xi)\ [1 + \varepsilon s_1(\xi) + \varepsilon^2 s_2(\xi) + \ \ldots\ ] \ . \tag{4.10}$$

We then define further expansion coefficients

$$d_{n\ell} = \int_{-\infty}^{\infty} p_s h v_o^n\, v_o^\ell \, d\xi \ ,$$

$$e_{n\ell} = \int_{-\infty}^{\infty} p_s s_1 v_o^n\, v_o^\ell \, d\xi \ ,$$

$$f_{n\ell} = \int_{-\infty}^{\infty} p_s s_2 v_o^n v_o^\ell \, d\xi \ .$$

First order approximation:

$$\mu_1^n = \left(\frac{n}{2}\right)^{1/2} d_{n,n-1} \quad ,$$

$$a_{n\ell} = \frac{\left(\frac{n}{2}\right)^{1/2} d_{\ell,n-1}}{n-\ell} \quad , \qquad n \neq \ell,$$

where we have noted that $\mu_o^n = n$. And

$$a_{nn} = -\frac{1}{2} e_{nn} \quad .$$

Second order approximations:

$$\mu_2^n = \sum_{k=0}^{\infty} \left(\frac{k}{2}\right)^{1/2} a_{nk} d_{n,k-1} \quad ,$$

$$b_{n\ell} = \frac{1}{n-\ell} \left[-\mu_1^n a_{n\ell} + \sum_{k=0}^{\infty} \left(\frac{k}{2}\right)^{1/2} a_{nk} d_{\ell,k-1} \right] \qquad n \neq \ell,$$

$$b_{nn} = -\frac{1}{2} \sum_{j=0}^{\infty} (a_{nj})^2 - \sum_{j=0}^{\infty} e_{nj} a_{nj} - f_{nn} \quad .$$

From equation (4.7), we see that we need to compute

$$\alpha_m^2 = \left(\int_{-\infty}^{\infty} \overline{p_s}(\xi)\, v^m(\xi)\, d\xi \right)^2 \quad ,$$

where $\overline{p_s}(\xi)$ and $v^n(\xi)$ are the perturbed steady-state solution and perturbed eigenfunctions, respectively. Substituting the expansions (4.9) and (4.10), we obtain

$$\alpha_m = \int_{-\infty}^{\infty} \xi p_s v_o^m d\xi + \varepsilon \left[\int_{-\infty}^{\infty} \xi p_s v_o^m s_1 d\xi + \int_{-\infty}^{\infty} \xi p_s \sum_{j=0}^{\infty} a_{mj} v_o^j d\xi \right]$$

$$+ \varepsilon^2 \left[\int_{-\infty}^{\infty} \xi p_s s_2 v_o^m d\xi + \int_{-\infty}^{\infty} \xi p_s s_1 \sum_{j=0}^{\infty} a_{mj} v_o^j d\xi + \int_{-\infty}^{\infty} \xi p_s \sum_{j=0}^{\infty} b_{mj} v_o^j d\xi \right]$$

$$+ 0\ (\varepsilon^3)$$

$$= \frac{1}{\sqrt{2}} \delta_{lm} + \varepsilon \left[\frac{1}{\sqrt{2}} e_{lm} + \frac{1}{\sqrt{2}} a_{ml} \right]$$

$$+ \varepsilon^2 \left[\frac{1}{\sqrt{2}} f_{lm} + \frac{1}{\sqrt{2}} \sum_{j=0}^{\infty} a_{mj} e_{lj} + \frac{1}{2} b_{ml} \right] + 0\ (\varepsilon^3) \quad .$$

Then

$$\alpha_m^2 = \frac{1}{2}\,\delta_m + \varepsilon\delta_{1m}\,(a_{11} + e_{11}) + \varepsilon^2\left[\frac{1}{2}\,(e_{1m} + a_{1m})^2\right.$$

$$\left. + \delta_{1m}\,(f_{11} + \sum_{j=0}^{\infty} a_{1j}\,e_{1j} + b_{11})\right] + 0\,(\varepsilon^3) \quad .$$

Using the expressions for a_{11} and b_{11} above, this reduces to

$$\alpha_m^2 = \frac{1}{2}\,\delta_{1m} + \frac{1}{2}\,\varepsilon\delta_{1m}\,e_{11} + \frac{1}{2}\,\varepsilon^2\left[(e_{1m} + a_{1m})^2 - \sum_{j=0}^{\infty}(a_{1j})^2\right] + 0\,(\varepsilon^3).$$

Example 2:

As an example of the application of this method, we shall again consider the first-order stochastic differential equation treated earlier by the methods of equivalent linearization and direct perturbation on the stochastic differential equation. That is, we take

$$f(x) = \beta x + \varepsilon x^3 \quad .$$

In this case

$$h(\xi) = \sqrt{\frac{2}{D}}\; g\;(\sqrt{2D}\;\xi) = 4D\xi^3 \quad .$$

$\overline{p_s}(\xi)$ is easily found to be

$$\overline{p_s}(\xi) = A\,\exp\,[-\xi^2 - \varepsilon D\xi^4] \quad .$$

Expanding

$$\overline{p_s}(\xi) = Ae^{-\xi^2}\left[1 - \varepsilon D\xi^4 + \frac{1}{2}\varepsilon^2 D^2 \xi^8 + \ldots\right] .$$

Integrating we find

$$1 = \int_{-\infty}^{\infty} \overline{p_s}(\xi)\, d\xi = A\left[\sqrt{\pi} - \frac{3\varepsilon}{4} D\sqrt{\pi} + \frac{105}{32}\varepsilon^2 D^2 \sqrt{\pi} + \ldots\right]$$

so that

$$A = \frac{1}{\sqrt{\pi}}\left[1 + \frac{3}{4}\varepsilon D - \frac{87}{32}\varepsilon^2 D^2 + \ldots\right] .$$

Then

$$\overline{p_s}(\xi) = p_s(\xi)\left[1 - \varepsilon D\left(\xi^4 - \frac{3}{4}\right) + \varepsilon^2 D^2\left(-\frac{87}{32} + \frac{1}{2}\xi^8 - \frac{3}{4}\xi^4\right) + \ldots\right] .$$

We identify

$$s_1(\xi) = \left(\frac{3}{4} - \xi^4\right) D$$

$$s_2(\xi) = \left(-\frac{87}{32} - \frac{3}{4}\xi^4 + \frac{1}{2}\xi^8\right) D^2 .$$

The quantities a_{1j}, a_{j1}, and e_{1j}, which are needed to obtain α_m^2 up to $O(\varepsilon^2)$, are easily computed and are listed below:

$$a_{11} = \frac{3}{2} D \qquad e_{11} = -3D$$

$$a_{12} = 0 \qquad e_{12} = 0$$

$$a_{13} = -\sqrt{\frac{3}{2}} D \qquad e_{13} = -\frac{5}{2}\sqrt{6}\, D$$

$$a_{1j} = 0,\ j>3 \qquad e_{14} = 0$$

$$a_{21} = 0 \qquad e_{15} = -\sqrt{\frac{15}{2}} D$$

$$a_{31} = 3\sqrt{6}\, D \qquad e_{1j} = 0,\ j>0$$

$$a_{41} = 0$$

$$a_{51} = \frac{1}{2}\sqrt{30}\, D$$

$$a_{j1} = 0,\ j>5$$

From these, we find

$$\alpha_j^2 = 0,\ j \text{ even}$$

$$\alpha_1^2 = \frac{1}{2} - \frac{3}{2}\varepsilon D - \frac{3}{2}\varepsilon^2 D^2 + O(\varepsilon^3)$$

$$\alpha_3^2 = \frac{3}{4}\varepsilon^2 D^2 + O(\varepsilon^3)$$

$$\alpha_j^2 = O(\varepsilon^3),\ j\geq 5$$

To obtain the autocorrelation function to $O(\varepsilon^2)$, we need μ^1 and μ^3, which we shall also compute to $O(\varepsilon^2)$. Now

$$\mu^1 = 1 + \varepsilon\mu_1^1 + \varepsilon^2\,\mu_2^1 + 0(\varepsilon^3)$$

$$\mu^3 = 3 + \varepsilon\mu_1^3 + \varepsilon^2\,\mu_2^3 + 0(\varepsilon^3)\ .$$

The expressions for μ_1^1, μ_2^1, and μ_2^3 reduce to

$$\mu_1^1 = \sqrt{\frac{1}{2}}\ d_{10}$$

$$\mu_2^1 = \sqrt{\frac{1}{2}}\ a_{11}\ d_{10} + \frac{3}{2}\ a_{13}\ d_{12}$$

$$\mu_1^3 = \sqrt{\frac{3}{2}}\ d_{32}$$

$$\mu_2^3 = \sqrt{\frac{1}{2}}\ a_{31}\ d_{30} + \sqrt{\frac{3}{2}}\ a_{33}\ d_{32} + \sqrt{\frac{5}{2}}\ a_{35}\ d_{34}\ .$$

The quantities appearing here are computed to be

$$d_{10} = 3\sqrt{2}\ D$$

$$d_{30} = 2\sqrt{3}\ D$$

$$d_{32} = 9\sqrt{6}\ D$$

$$d_{34} = 24\sqrt{2}\ D$$

$$a_{33} = \frac{31}{4}\ D$$

$$a_{35} = -3\sqrt{5}\ D\ .$$

From these, we find

$$\mu^1 = 1 + 3\,\varepsilon D - \frac{15}{2}\,\varepsilon^2 D^2 + 0\,(\varepsilon^3)$$

$$\mu^3 = 3 + 27\,\varepsilon D - \frac{531}{4}\,\varepsilon^2 D^2 + 0\,(\varepsilon^3)\ .$$

The uniformly valid asymptotic expansion of the autocorrelation up to $0(\varepsilon^2)$ is then

$$R_x(\tau) = D\left\{\left[\,1-3\varepsilon D - 3\varepsilon^2 D^2 + 0(\varepsilon^3)\,\right]\right.$$

$$\exp\left[\,-\tau(1+3\varepsilon D - \frac{15}{2}\ \varepsilon^2 D^2 + 0(\varepsilon^3))\,\right] + \left[\,\frac{3}{2}\,\varepsilon^2 D^2 + 0(\varepsilon^3)\,\right]$$

$$\left.\exp\left[\,-\tau(3+27\varepsilon D - \frac{531}{4}\,\varepsilon^2 D^2 + 0(\varepsilon^3))\,\right]\right\}\ .$$

If only $0(\varepsilon)$ terms are retained, this result coincides with that obtained by the method of equivalent linearization, and also, if expanded completely in powers of ε, with that of the method of direct perturbation on the stochastic differential equation. It is clear, however, that the method of equivalent linearization could not produce a result accurate to $0(\varepsilon^2)$.

Example 3: Piecewise linear

Consider a symmetric piecewise linear system described by

$$f(x) = \beta x + \varepsilon(|x| - x_o)\, H(|x| - x_o)\, \text{sign}\,(x)$$

(H(x) is the unit step function). The steady-state solution is then

$$\overline{p_s}(x) = C \exp\left[- \frac{\beta x^2}{2D} - \frac{\varepsilon}{2D} (|x| - x_o)^2 H(|x| - x_o) \right]$$

In terms of the variable $\xi = \sqrt{\frac{\beta}{2D}}\, x$, with $\xi_o = \sqrt{\frac{\beta}{2D}}\, x_o$,

$$\overline{p_s}(\xi) = C \exp\left[- \xi^2 - \frac{\varepsilon}{\beta} (|\xi| - \xi_o)^2 H(|\xi| - \xi_o) \right] .$$

Normalizing $\overline{p_s}(\xi)$, one finds

$$\frac{1}{\sqrt{\pi}\, C} = \text{erf}(\xi_o) + (1 + \frac{\varepsilon}{\beta})^{-1/2} \exp\left[- \frac{\varepsilon}{\beta} (1 + \frac{\varepsilon}{\beta})^{-1} \xi_o^2 \right] \text{erfc}\left[(1 + \frac{\varepsilon}{\beta})^{1/2} \xi_o \right]$$

where we use the definition of erf(x)

$$\text{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2}\, dt$$

and

$$\operatorname{erfc}(x) = \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-t^2}\, dt = 1 - \operatorname{erf}(x) \ .$$

Expanding $\overline{p_s}(\xi)$ in powers of ε, we find

$$\overline{p_s}(\xi) = \frac{1}{\sqrt{\pi}}\ e^{-\xi^2}\ [1 + \varepsilon s_1(\xi) + ,\ldots,]$$

with

$$s_1(\xi) = \frac{1}{\beta}\left[\left(\frac{1}{2} + \xi_o^2\right)\ \operatorname{erfc}(\xi_o) - \frac{\xi_o}{\sqrt{\pi}}\ e^{-\xi_o^2} - (|\xi| - \xi_o)^2\ H(|\xi| - \xi_o) \right] .$$

In this case, we identify

$$h(\xi) = \frac{2}{\beta}\ (|\xi| - \xi_o)\ H(|\xi| - \xi_o)\ \operatorname{sign}\ (\xi) \ .$$

The expansion coefficients required for $0(\varepsilon)$ corrections are easily computed:

$$e_{11} = \int_{-\infty}^{\infty} p_s s_1\ [v_o^1]^2\ d\xi = -\frac{1}{\beta}\ \operatorname{erfc}(\xi_o)$$

$$d_{10} - \int_{-\infty}^{\infty} p_s h v_o^1\ v_o^o\ d\xi = \frac{2}{\beta}\sqrt{\frac{2}{\pi}}\ \xi_o\ e^{-\xi_o^2} + \frac{\sqrt{2}}{\beta}\ \operatorname{erfc}(\xi_o) \ .$$

Then $\alpha_1^2 = \frac{1}{2} - \frac{\varepsilon}{2\beta}\ \operatorname{erfc}(\xi_o) + 0(\varepsilon^2)$,

$$\mu_1^1 = \frac{2}{\beta\sqrt{\pi}}\ \xi_o\ e^{-\xi_o^2} + \frac{1}{\beta}\ \operatorname{erfc}(\xi_o) \ .$$

The autocorrelation function is then found to be

$$R_x(\tau) = \frac{D}{\beta}\left[1 - \frac{\varepsilon}{\beta}\,\mathrm{erfc}(\xi_o)\right]\exp\left[-\beta\tau-\varepsilon\left(\frac{2}{\sqrt{\pi}}\,\xi_o e^{-\xi_o^2} + \mathrm{erfc}(\xi_o)\right)\right] + O(\varepsilon^2)$$

and the spectral density is then

$$\phi_x(\omega) = \frac{2D\left(1 - \frac{\varepsilon}{\beta}\,\mathrm{erfc}(\xi_o)\right)}{\pi\left[\omega^2 + \left(\beta + \frac{2\varepsilon}{\sqrt{\pi}}\,\xi_o\, e^{-\xi_o^2} + \varepsilon\,\mathrm{erfc}(\xi_o)\right)^2\right]} + O(\varepsilon^2)$$

Power Law Nonlinearity

We will now consider the case of a power law nonlinearity, i.e.,

$$f(x) = (2p + 2)\, x^{2p+1}$$

for which the steady-state solution is

$$p_s(x) = \frac{p + 1}{\Gamma\left(\frac{1}{2p + 2}\right)}\, \exp\,(-x^{2p + 2}) \quad .$$

The eigenvalue problem in this case becomes (taking D=1)

$$\frac{d^2u}{dx^2} - (2p + 2)\, x^{2p + 1}\,\frac{du}{dx} + \lambda u = 0 \quad . \tag{4.11}$$

We note that we are assured from a general argument that the eigenvalues of this equation are discrete. As we are no longer in the nearly linear case and a perturbation procedure is not appropriate,

we resort to the Ritz variational method. This is satisfactory if we are primarily interested in an expression for the autocorrelation function, since only the first non-zero eigenvalue and its corresponding eigenfunction make a significant contribution.

It is well known that the first non-zero eigenvalue of equation (4.11) has the property

$$\lambda_1 = \min \frac{(Lu,u)}{(u,u)} \tag{4.12}$$

where we have again chosen the inner product

$$(u,v) = \int_{-\infty}^{\infty} p_s u\, v\, dx$$

and L is the differential operator

$$\frac{d^2u}{dx^2} - (2p + 2)\, x^{2p+1} \frac{du}{dx} .$$

One can readily show that

$$(Lu,u) = \int_{-\infty}^{\infty} p_s \left|\frac{du}{dx}\right|^2 dx .$$

The minimum in (4.12) is then taken over non-constant functions, and the function which achieves the minimum is the corresponding eigenfunction.

Succeeding eigenvalues and eigenfunctions can be found, after λ_1, u_1 are found, by minimizing the Rayleigh quotient, $(Lu,u)/(u,u)$, for all functions orthogonal to the eigenfunctions previously found.

Equation (4.11) is seen to be invariant with respect to a reflection in the x-axis. Thus the eigenfunctions will be even and odd functions - - being even for λ_o, λ_2,..., and odd for λ_1, λ_3,... . In particular, the eigenfunction corresponding to λ_1 is odd. Thus it is reasonable to use only odd functions to approximate the Rayleigh quotient.

One possible approach, then, is to approximate u_1 by a finite odd polynomial with some of the coefficients as parameters. After calculating the Rayleigh quotient, we then choose the parameters to minimize the Rayleigh quotient.

We illustrate this with the simplest case:

Example 4: $f(x) = x^3$

The steady-state solution is given by

$$p_s(x) = \frac{2}{\Gamma\left(\frac{1}{4}\right)} \exp(-x^4) \quad .$$

For the trial function $u = x$, we find $\lambda_1 = 2.959$.

For $u = x + ax^3$; $a = -.209$, $\lambda_1 = 2.757$

For $u = x + ax^3 + bx^5$; $a = -.117$, $b = -.042$, $\lambda_1 = 2.714$.

Normalizing the last approximation, we find the approximate eigenfunction

$$u_1 = \left(\frac{1}{.27258}\right)^{1/2} [x - .117\, x^3 - .042\, x^5] \ .$$

From this, we can calculate α_1^2 appearing in the expansion for the autocorrelation function:

$$\alpha_1^2 = \left(\int_{-\infty}^{\infty} x p_s\, u_1\, dx \right)^2 = .32610,$$

so that the first term appearing into the autocorrelation function is approximately

$$.32610 \exp [-2.714\tau] \ .$$

Contributions from the succeeding eigenvalues can be significant only near the τ origin. Now, we have exactly

$$R_x(0) = \int_{-\infty}^{\infty} x^2 p_s(x)\, dx = \frac{\Gamma\left(\frac{3}{4}\right)}{\Gamma\left(\frac{1}{4}\right)} = .33799.$$

Comparing this with the approximate autocorrelation function determined above, the exact value of $R_x(0)$ is seen to be only 3.65 percent higher than is found from the approximation.

Second Order Stochastic Differential Equations

We have seen that all the approximation methods find application in the study of first order stochastic differential equations, with

the method of approximating eigenvalues providing the most complete description. In the case of second order stochastic differential equations, the methods of equivalent linearization and perturbation directly on the stochastic differential equation appear to be most useful in the study of passive type oscillators, while the methods of approximating eigenvalues and asymptotic expansions are most appropriate in the study of self-excited oscillators. We might point out, however, that it is possible to use, for example, the method of equivalent linearization in modified form in the study of self-excited oscillators[20]. The indications of which methods are most appropriate are intended to point out which methods provide the better approximations by means of tractable computations.

The methods of equivalent linearization and perturbation directly on the stochastic differential equation proceed in the same spirit as explained in some detail for first order stochastic differential equations, though, of course, the details certainly differ. For example, in the method of equivalent linearization for second order stochastic differential equations, the equivalent linear system which is used to represent the original nonlinear system is taken to be in the form

$$\ddot{x} + \beta\dot{x} + \omega x = \frac{dw}{dt}$$

with β and ω taken to minimize the mean square of the corresponding equation deficiency in the steady-state.

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[20] Caughey, T. K. [22].

We shall now try to indicate why one cannot successfully extend the application of the method of equivalent linearization in a straightforward way to the study of self-excited oscillators. For this purpose consider the van der Pol equation with white noise disturbance:

$$\ddot{x} - \varepsilon(1 - x^2)\,\dot{x} + x = \frac{dw}{dt} \quad .$$

Replacing this equation by the equivalent linear equation suggested above leads to the equation deficiency

$$\dot{x}[\,-\beta - \varepsilon + \varepsilon x^2] + x(1 - \omega) \quad .$$

Since the steady-state solution for the van der Pol oscillator is not known, the expectations in the steady-state will be calculated by using the steady-state of the equivalent linear system. Now

$$E[e^2(x,\dot{x})] = (\beta+\varepsilon)^2\, E[\dot{x}^2] + 2\varepsilon E\,[\dot{x}^2x^2] + \varepsilon^2 E\,[\dot{x}^2x^4] - 2(\beta+\varepsilon)(1-\omega)\, E[x\dot{x}]$$

$$+ 2\varepsilon(1-\omega)\, E[\dot{x}x^3] + (1-\omega)^2\, E[x^2] \quad .$$

To determine the extrema, we set

$$\frac{\partial}{\partial\beta}\, E[e^2(x,\dot{x})] = 0 = 2(\beta+\varepsilon)\, E[\dot{x}^2] - 2(1-\omega)\, E[x\dot{x}]$$

$$\frac{\partial}{\partial\omega}\, E[e^2(x,\dot{x})] = 0 = 2(\beta+\varepsilon)\, E[x\dot{x}] - 2\varepsilon E[\dot{x}x^3] - 2(1-\omega)\, E[x^2] \quad .$$

Since

$$\frac{\partial^2}{\partial\beta^2} E[e^2(x,\dot{x})] = 2E[\dot{x}^2]$$

$$\frac{\partial^2}{\partial\omega^2} E[e^2(x,\dot{x})] = 2E[x^2]$$

$$\frac{\partial^2}{\partial\omega\partial\beta} E[e^2(x,\dot{x})] = 2E[x\dot{x}]$$

and since for any second order system in the steady-state, $E[x\,\dot{x}] = 0$, one can see that these conditions actually determine a minimum. We can solve for β and ω to find

$$\beta = -\varepsilon,$$

$$\omega = 1+\varepsilon \frac{E[\dot{x}x^3]}{E[x^2]} .$$

For the equivalent linear system in the steady-state,

$$E[x^2] = \frac{D}{\beta\omega^2} ,$$

$$E[\dot{x}x^3] = 0.$$

Then we take $\omega = 1$.

There are several things wrong with this result, but one need only point out the most obvious feature. For $\varepsilon>0$, as is the case for a van der Pol oscillator, $\beta<0$ and therefore implies an unstable

oscillator. In fact, an "equivalent linear" steady-state does not even exist though the formal calculations above can still be made. A van der Pol oscillator is, however, certainly stable.

Difficulties also arise when attempting to extend the method of perturbation directly on the stochastic differential equation to the study of self-excited oscillators.

Since the methods of equivalent linearization and perturbation directly on the stochastic differential equation appeared to be inadequate for the study of self-excited oscillators, it was necessary to develop new methods which we will now discuss.

Eigenfunction Expansion for Self-Excited Oscillator

The eigenfunction expansion approach in conjunction with a modified method of averaging can also be used to study self-excited oscillators. We shall be concerned with oscillators described by

$$\ddot{x} + \varepsilon f(x,\dot{x})\,\dot{x} + x = \frac{dw}{dt} \tag{4.13}$$

which exhibit limit cycle behavior in the absence of stochastic excitation, where $w(t)$ is a Wiener process with $E[dw(t)^2] = 2D\varepsilon^2\,dt$ and ε is a small parameter.

In order to obtain a Fokker-Planck equation for which approximate eigenvalues and eigenfunctions can be found, we shall employ a modified method of averaging. For ε small, it is clear from equation

(4.13) that the limit cycle will be nearly circular. The addition of stochastic excitation of $0(\varepsilon)$ will introduce $0(\varepsilon)$ pertubations in this nearly circular limit cycle. Thus we seek a solution of (4.13) in the form

$$r(t) = \rho(t) + \varepsilon\rho_1(\rho,\phi,t) + \varepsilon^2\rho_2(\rho,\phi,t) + , \ldots \tag{4.14}$$

$$\theta(t) = t + \phi(t) + \varepsilon\phi_1(\rho,\phi,t) + \varepsilon^2\phi_2(\rho,\phi,t) + ,\ldots \tag{4.15}$$

where we set

$$x = r\cos\theta \ , \tag{4.16}$$

$$\dot{x} = r\sin\theta \ . \tag{4.17}$$

Differentiating (4.16), this is seen to require

$$-r\sin\theta = \dot{r}\cos\theta - r\dot{\theta}\sin\theta \ ,$$

and equation (4.13) gives

$$\ddot{x} = -\dot{r}\sin\theta - r\dot{\theta}\cos\theta$$

$$= \varepsilon f(r\cos\theta, -r\sin\theta)\, r\sin\theta - r\cos\theta + \frac{dw}{dt} \ .$$

Solving for $\dot{r}$, $\dot{\theta}$

$$\dot{r} = -\varepsilon f(r\cos\theta, -r\sin\theta)\, r\sin^2\theta - \frac{dw}{dt}\sin\theta \ , \tag{4.18}$$

$$r\dot{\theta} = -\varepsilon f(r\cos\theta, -r\sin\theta)\, r\sin\theta\cos\theta + r - \frac{dw}{dt}\cos\theta \ . \tag{4.19}$$

It will be convenient in what follows to express the terms appearing in (4.18), (4.19) in Fourier series:

$$rf(r\cos\theta,-r\sin\theta)\sin^2\theta = a_o(r) + \sum a_n(r)\cos n\theta + \sum b_n(r)\sin n\theta$$

$$rf(r\cos\theta,-r\sin\theta)\sin\theta\cos\theta = c_o(r) + \sum c_n(r)\cos n\theta + \sum d_n(r)\sin n\theta$$

From equations (4.14) and (4.15)

$$\frac{dr}{dt} = \frac{d\rho}{dt} + \epsilon\left(\frac{\partial\rho_1}{\partial\rho}\frac{d\rho}{dt} + \frac{\partial\rho_1}{\partial\phi}\frac{d\phi}{dt} + \frac{\partial\rho_1}{\partial t}\right) + O(\epsilon^2)$$

$$\frac{d\theta}{dt} = 1 + \frac{d\phi}{dt} + \epsilon\left(\frac{\partial\phi_1}{\partial\rho}\frac{d\rho}{dt} + \frac{\partial\phi_1}{\partial\phi}\frac{d\phi}{dt} + \frac{\partial\phi_1}{\partial t}\right) + O(\epsilon^2)$$

so (4.18), (4.19) become

(4.20)

$$\frac{d\rho}{dt} + \epsilon\left(\frac{\partial\rho_1}{\partial\rho}\frac{d\rho}{dt} + \frac{\partial\rho_1}{\partial\phi}\frac{d\phi}{dt} + \frac{\partial\rho_1}{\partial t}\right) = -\epsilon a_o(r) - \epsilon\sum a_n(r)\cos n\theta$$

$$- \epsilon\sum b_n(r)\sin n\theta$$

$$- \frac{dw}{dt}\sin\theta + O(\epsilon^2)$$

(4.21)
$$(\rho + \varepsilon\rho_1)\left[1 + \frac{d\phi}{dt} + \varepsilon\left(\frac{\partial\phi_1}{\partial\rho}\frac{\partial\rho}{\partial t} + \frac{\partial\phi_1}{\partial\phi}\frac{\partial\phi}{\partial t} + \frac{\partial\phi_1}{\partial t}\right)\right]$$

$$= (\rho + \varepsilon\rho_1) - \varepsilon c_o(r) - \sum c_n(r)\cos n\theta$$

$$- \varepsilon \sum d_n(r)\sin n\theta - \frac{dw}{dt}\cos\theta + O(\varepsilon^2) .$$

The basic idea of the method of averaging is that ρ, ϕ should include all the persistent changes in r and θ, while the correction terms, ρ_1, ϕ_1, etc., should be periodic in t for fixed ρ, ϕ. With this in mind, the terms $\frac{dw}{dt}\sin\theta$, $\frac{dw}{dt}\cos\theta$, appearing in equations (4.20) and (4.21) produce persistent changes in the variances in r and θ so that we choose

$$\frac{d\rho}{dt} = -\varepsilon a_o(r) - \frac{dw}{dt}\sin\theta + O(\varepsilon^2)$$

$$(\rho + \varepsilon\rho_1)\frac{d\phi}{dt} = -\varepsilon c_o(r) - \frac{dw}{dt}\cos\theta + O(\varepsilon^2) .$$

Furthermore, to $O(\varepsilon^2)$ we can substitute for r, θ

(4.22)
$$\frac{d\rho}{dt} = -\varepsilon a_o(\rho) - \frac{dw}{dt}\sin(t + \phi) + O(\varepsilon^2),$$

$$(4.23) \qquad \rho \frac{d\phi}{dt} = - \varepsilon c_o(\rho) - \frac{dw}{dt} \cos (t + \phi) + O(\varepsilon^2).$$

No attempt will be made to calculate ρ_1 and ϕ_1.

From equations (4.22) and (4.23) we can form a Fokker-Planck equation from terms up to $O(\varepsilon)$

$$\frac{\partial p}{\partial t} = - \frac{\partial}{\partial \rho} a_\rho p - \frac{\partial}{\partial \phi} a_\phi p + \frac{1}{2} \frac{\partial^2 b_{\rho\rho} p}{\partial \rho^2} + \frac{\partial^2 b_{\rho\phi} p}{\partial \rho \partial \phi} + \frac{1}{2} \frac{\partial^2 b_{\phi\phi} p}{\partial \phi^2} \quad .$$

In what follows we shall assume that $c_o(\rho) \equiv 0$. Now

$$a_\rho = \lim_{\Delta t \to 0} E\left[\frac{\Delta \rho}{\Delta t} \right] = - \varepsilon a_o(\rho) - \lim_{\Delta t \to 0} E\left[\frac{1}{\Delta t} \int_t^{t+\Delta t} \sin(\tau + \phi)\, dw(\tau) \right]$$

One should refrain from concluding that $E[\sin (\tau + \phi)\, dw] = 0$ from the fact that $E[dw] = 0$ as ϕ is a random variable and in general is correlated with $w(\tau)$. However, in this case, ϕ is slowly varying so that $\phi(\tau)$ can be replaced by $\phi(t)$ (initial point in the integration) with error $O(\varepsilon \Delta t)$. It follows that $E[\sin (\tau + \phi)\, dw]$ is $O(\varepsilon^2)$ so can be neglected. Thus, to lowest orders in ε

$$a_\rho = -\varepsilon a_o(\rho)$$

$$a_\phi = 0$$

$$b_{\rho\rho} = 2D \sin^2(t + \phi)\, \varepsilon^2$$

$$b_{\rho\phi} = \frac{1}{\rho} 2D \sin(t + \phi) \cos(t + \phi)\, \varepsilon^2$$

$$b_{\phi\phi} = \frac{1}{\rho^2} 2D \cos^2(t + \phi)\, \varepsilon^2$$

and the corresponding Fokker-Planck equation is

$$\frac{\partial p}{\partial t} = \varepsilon \frac{\partial}{\partial \rho} [a_o p] + D\varepsilon^2 \sin^2(t + \phi) \frac{\partial^2 p}{\partial \rho^2} + 2D\varepsilon^2 \frac{\partial^2}{\partial \rho \partial \phi} \left[\frac{1}{\rho} \sin(t + \phi) \cos(t + \phi)\, p \right] + \frac{D\varepsilon^2}{\rho^2} \frac{\partial^2}{\partial \phi^2} [\cos^2(t + \phi)\, p] . \tag{4.24}$$

We seek the positive solution of equation (4.24), normalized by

$$\int_0^\infty \int_{-\infty}^{\infty} p(\rho,\phi,t|\rho_o,\phi_o)\, d\rho\, d\phi = 1. \tag{4.25}$$

Note that ϕ is not restricted to an interval of 2π. We cannot solve equation (4.24) as it stands. However, since ρ, ϕ are slowly varying it seems reasonable to conclude that $p(\rho,\phi,t)$ is slowly varying. Suppose we average equation (4.24) by

$$\overline{(\quad)} \equiv \frac{1}{2\pi} \int_t^{t+2\pi} (\quad)\, d\tau .$$

Now

$$\overline{\cos\, 2(t+\phi)p} = \frac{1}{2\pi} \int_t^{t+2\pi} \cos\, 2(\tau+\phi)\; p(\rho,\phi,\tau)\; d\tau = \frac{1}{2\pi}\, \bar{p} \int_t^{t+2\pi} \cos\, 2(\tau+\phi)\; d\tau$$

$$+ O(\varepsilon) = O(\varepsilon)$$

so this can be neglected in comparison to other terms appearing, as can $\overline{\sin\, 2(t+\phi)p}$. Thus we obtain

$$(4.26) \qquad \frac{\partial \bar{p}}{\partial t} = \varepsilon\, \frac{\partial}{\partial \rho}\, [a_o \bar{p}] + \frac{1}{2}\, D\varepsilon^2\, \frac{\partial^2 \bar{p}}{\partial \rho^2} + D\varepsilon^2\, \frac{1}{\rho^2}\, \frac{\partial^2 \bar{p}}{\partial \phi^2} \quad .$$

The steady-state solution must satisfy

$$(4.27) \qquad 0 = \varepsilon\, \frac{\partial}{\partial \rho}\, [a_o \bar{p}_s] + \frac{1}{2}\, D\varepsilon^2\, \frac{\partial^2 \bar{p}_s}{\partial \rho^2} + \frac{1}{2}\, D\varepsilon^2\, \frac{1}{\rho^2}\, \frac{\partial^2 \bar{p}_s}{\partial \phi^2} \quad .$$

A solution of (4.27) can be got by setting $\frac{\partial \bar{p}_s}{\partial \phi} = 0$ and then

$$0 = \varepsilon\, \frac{\partial}{\partial \rho} \left[a_o p_s + \frac{D\varepsilon}{2}\, \frac{\partial \bar{p}_s}{\partial \rho} \right]$$

so

$$(4.28) \qquad \bar{p}_s = C \exp \left[-\frac{2}{D\varepsilon} \int_{\rho_0}^{\rho} a_o(\xi)\; d\xi \right]$$

where we will take C so that

$$\frac{1}{C} = \int_0^\infty \exp\left[-\frac{2}{D\varepsilon} \int_{\rho_o}^{\rho} a_o(\xi)\, d\xi \right] d\rho. \tag{4.29}$$

Note that we do not satisfy the full normalization condition, equation (4.25); nonetheless, $p_s(\rho)$ as given by (4.28) and (4.29) is a useful quantity. We take ρ_o so that

$$a_o(\rho_o) = 0, \qquad \rho_o \neq 0 .$$

We are guaranteed this possibility by the assumption that the unforced oscillator exhibits limit cycle behavior.

We can now proceed to solve equation (4.26) by obtaining an eigenfunction expansion. Consider solutions of the form

$$\bar{p} = e^{\lambda t} R(\rho)\, e^{i\mu\phi}.$$

Then

$$\frac{1}{2} D\varepsilon^2 \frac{d^2R}{d\rho^2} + \varepsilon \frac{d}{d\rho} a_o R = \left(\lambda + \frac{1}{2} \frac{D\varepsilon^2\mu^2}{\rho^2} \right) R .$$

Further, let $R(\rho) = u(\rho)\, \bar{p}_s$ to obtain

$$\frac{1}{2} D\varepsilon^2 \frac{d^2u}{d\rho^2} - \varepsilon a_o \frac{du}{d\rho} = \left(\lambda + \frac{D\varepsilon^2\mu^2}{2\rho^2} \right) u \quad . \tag{4.30}$$

This equation is self-adjoint, with the inner product

$$(u,v) = \int_0^\infty \bar{p}_s \, uv \, d\rho \quad .$$

If we make the change of variable

$$u = v \exp\left[\frac{1}{D\varepsilon} \int_{\rho_o}^{\rho} a_o(\xi) \, d\xi \right]$$

we obtain the equation

$$v'' + v\left[\frac{\varepsilon a_o'}{2} - \frac{a_o^2}{D} - \frac{2\lambda}{D\varepsilon^2} - \frac{\mu^2}{\rho 2} \right] = 0 \quad . \tag{4.31}$$

If $a_o(\rho) \to \infty$ as $\rho \to \infty$, the eigenvalues, λ, for (4.31) form a discrete spectrum. We shall assume this to be the case.

Consider the consequences of obtaining the transition probability density function in the form

$$p(\rho,\phi,t|\rho_o,\phi_o) = \int_{-\infty}^{\infty} \sum_n u_{n\mu}(\rho) \, \bar{p}_s(\rho) \, e^{i\mu\phi} e^{\lambda_{n\mu} t} c_{n\mu}(\rho_o,\phi_o) \, d\mu \quad .$$

As $t \to 0$, $p(\rho,\phi,t|\rho_o,\phi_o) \to \delta(\rho-\rho_o) \quad \delta(\phi-\phi_o)$ so

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{0}^{\infty} u_{k\omega}(\rho)\, e^{-i\omega\phi}\, \delta(\rho-\rho_o)\, \delta(\phi-\phi_o)\, d\phi\, d\rho$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{0}^{\infty} \sum_{n} u_{n\mu}(\rho)u_{k\omega}(\rho)\, \bar{p}_s(\rho)\, e^{i\phi(\mu-\omega)}\, c_{n\mu}(\rho_o,\phi_o)\, d\mu\, d\phi\, d\rho\ .$$

By integrating first over ϕ, then μ and ρ we find

$$c_{k\omega}(\rho_o,\phi_o) = \frac{1}{2\pi}\, u_{k\omega}(\rho_o)\, e^{-i\omega\phi_o}$$

as $u_{n\mu}$ can be chosen to form an orthonormal set for fixed μ. Thus

(4.32)

$$\bar{p}\,(\rho,\phi,t|\rho_o,\phi_o) = \int_{-\infty}^{\infty} \sum_{n} u_{n\mu}(\rho)\, u_{n\mu}(\rho_o)\, \bar{p}_s(\rho)\, \frac{1}{2\pi}\, e^{i\mu(\phi-\phi_o)}\, e^{\lambda_{n\mu}t}\, d\mu\ .$$

Autocorrelation Function

The autocorrelation function is defined by

$$R_x(\tau) = \lim_{t\to\infty} E[x(t)\, x(t+\tau)]\ .$$

Now as $x(t) = r(t)\cos\theta(t)$, from equations (4.14) and (4.15)

$$x(t) = \rho_t \cos(t+\phi_t) + 0(\varepsilon)$$

where $\rho_t = \rho(t)$, etc. Thus

$$E[x(t)x(t+\tau)] = E[\rho_t \rho_{t+\tau} \cos(t+\phi_t)\ \cos\ (t+\tau+\phi_{t+\tau})] + 0(\varepsilon) \quad .$$

The expectation shown here should be calculated using the solution of equation (4.24). As we do not have this, but rather a time averaged solution, $\overline{p}$, to equation (4.26), to be consistent we must ask for the time averaged correlation function, $E[\overline{x(t)x(t+\tau)}]$, the averaging being with respect to t. It is easily seen that this time averaging has the effect of discarding that part which is oscillatory in t. Note that when calculating $E[\overline{x(t)x(t+\tau)}]$, ρ_t, ϕ_t, etc., are not considered functions of t but rather are independent variables appearing in $p(\rho,\phi,t)$. Now

$$\cos(t+\phi_t)\ \cos(t+\tau+\phi_{t+\tau}) = \frac{1}{2}\cos(\tau+\phi_{t+\tau}-\phi_t) + \frac{1}{2}\cos(2t+\tau+\phi_t+\phi_{t+\tau})$$

so we clearly only retain the first term, i.e., we consider

$$E[x(t)x(t+\tau)] = \int_0^\infty\int_0^\infty\int_{-\infty}^{\infty}\int\int\frac{1}{4}\,\rho_t\rho_{t+\tau}[e^{i\tau+i\phi_{t+\tau}-i\phi_t}+e^{-i\tau-i\phi_{t+\tau}+i\phi_t}]$$

$$\times \sum_n u_{n\mu}(\rho_t)\ u_{n\mu}(\rho_o)\ \overline{p}_s(\rho_t)\ \frac{1}{2\pi}\, e^{i\mu(\phi_t-\phi_o)}\, e^{\lambda_{n\mu}t}$$

$$\times \sum_k u_{k\omega}(\rho_{t+\tau})\ u_{k\omega}(\rho_t)\ \overline{p}_s(\rho_{t+\tau})\ \frac{1}{2\pi}\, e^{i\omega(\phi_{t+\tau}-\phi_t)}\, e^{\lambda_{k\omega}\tau}$$

$$d\mu\ d\omega\ d\rho_t\ d\rho_{t+\tau}\ d\phi_t\ d\phi_{t+\tau} \quad .$$

Integrate first over ϕ_t:

$$\int_{-\infty}^{\infty} d\phi_t \left[A e^{i\phi_t} e^{i\mu\phi_t - i\omega\phi_t} + B e^{i\phi_t} e^{i\mu\phi_t - i\omega\phi_t}\right]$$

$$= 2\pi A\, \delta(\mu-\omega-1) + 2\pi B\, \delta(\mu-\omega+1) \quad .$$

Now integrate over $\phi_{t+\tau}$:

$$\int_{-\infty}^{\infty} d\phi_{t+\tau} \left[2\pi A'\, \delta(\mu-\omega-1)\, e^{i\phi_{t+\tau}} + 2\pi B'\, \delta(\mu-\omega+1)\, e^{-i\phi_{t+\tau}}\right] e^{i\omega\phi_{t+\tau}}$$

$$= (2\pi)^2 A'\, \delta(\mu-\omega-1)\, \delta(\omega+1) + (2\pi)^2 B'\, \delta(\mu-\omega+1)\, \delta(\omega-1).$$

Now we can easily perform the integrations over ω, then μ to obtain

$$E[\overline{x(t)x(t+\tau)}] = \int_0^{\infty}\int_0^{\infty} \frac{1}{4} \rho_t \rho_{t+\tau} \sum_n u_{no}(\rho_t)\, u_{no}(\rho_o)\, \bar{p}_s(\rho_t)\, e^{\lambda_{no} t}$$

$$X \left\{ e^{i\tau} \sum_k u_{k,-1}(\rho_{t+\tau})\, u_{k,-1}(\rho_t)\, \bar{p}_s(\rho_{t+}\,)\, e^{\lambda_{k,-1}\tau} \right.$$

$$\left. + e^{-i\tau} \sum_k u_{k1}(\rho_{t+\tau})\, u_{k1}(\rho_t)\, \bar{p}_s(\rho_{t+\tau})\, e^{\lambda_{k1}\tau} \right\} d\rho_t\, d\rho_{t+\tau}.$$

Noting that equation (4.31) is not affected by the sign of μ, it follows that $u_{k,-1}(\rho) = u_{k1}(\rho)$, $\lambda_{k,-1} = \lambda_{k1}$, so that

$$E[\overline{x(t)x(t+\tau)}] = \int_0^\infty\int_0^\infty \frac{1}{2}\rho_t\rho_{t+\tau}\cos\tau \sum_n u_{no}(\rho_t)\, u_{no}(\rho_o)\, \bar{p}_s(\rho_t)\, e^{\lambda_{no}t}$$

$$\times \sum_k u_{k1}(\rho_{t+\tau})\, u_{k1}(\rho_t)\, \bar{p}_s(\rho_{t+\tau})\, e^{\lambda_{k1}\tau}\, d\rho_t\, d\rho_{t+\tau} .$$

Now take $t\to\infty$. Recalling the steady-state solution, it is seen that $\lambda_{oo} = 0$ but $\lambda_{no}<0$, $n>0$ and also $u_{oo}=1$. Then

$$R_x(\tau) = \lim_{t\to\infty} E[\overline{x(t)x(t+\tau)}]$$

$$= \frac{1}{2}\cos \sum_k e^{\lambda_{k1}\tau}\left[\int_0^\infty \bar{p}_s(\rho)\, u_{k1}(\rho)\, d\rho\right]^2 .$$

Note the similarity of this result to that for one-dimensional systems, equation (4.6). All that is needed to compute $R_x(\tau)$ are the eigenfunctions and eigenvalues of equation (4.31) with $\mu=1$. This we propose to do approximately by means of the Rayleigh-Ritz variational principle. Because of the self-adjointness of the differential operator, L, appearing in equation (4.31)

$$L \equiv -\frac{d^2}{d\rho^2} + \frac{2a_o(\rho)}{D\epsilon}\frac{d}{d\rho} + \frac{1}{\rho^2} ,$$

$$Lu = -\frac{2\lambda}{D\epsilon^2} ,$$

we have

$$-\frac{2\lambda_{01}}{D\varepsilon^2} = \min_{u \perp 1} \frac{(u,Lu)}{(u,u)} \quad ,$$

$$-\frac{2\lambda_{n1}}{D\varepsilon^2} = \min_{u \perp u_0, u_1, \ldots, u_{n-1}} \frac{(u,Lu)}{(u,u)} \quad ,$$

where u_n are the eigenfunctions corresponding to λ_{n1}.

Approximate Eigenvalues

Because of the $\frac{1}{\varepsilon}$ factor in $\bar{p}_s(\rho)$, matters can be significantly simplified. Noting that ρ_o is chosen so that $a_o(\rho_o)=0$, we have

$$a_o(\xi) = (\xi-\rho_o)\, a_o'(\rho_o) + \ldots$$

so

$$\int_{\rho_o}^{\rho} a_o(\xi)\, d\xi = \frac{1}{2}\,(\xi-\rho_o)^2\, a_o'(\rho_o) + \ldots \quad .$$

Because of the $\frac{1}{\varepsilon}$ factor, one term in the expansion should suffice. Thus, we have approximately

$$\bar{p}_s(\rho) = C \exp\left[-\frac{a_o'(\rho)}{D\varepsilon}\,(\xi-\rho_o)^2 \right]$$

and to this order of approximation

$$C = \left[\frac{a_o'(\rho_o)}{\pi D \epsilon} \right]^{1/2} .$$

Now

$$(u,Lu) = \int_0^\infty \bar{p}_s(\rho)\, u \left[-\frac{d^2u}{d\rho^2} + \frac{2a_o}{D\epsilon}\frac{du}{d\rho} + \frac{u}{\rho^2} \right] d\rho$$

$$= \int_0^\infty \bar{p}_s(\rho) \left[\left(\frac{du}{d\rho}\right)^2 + \left(\frac{u}{\rho}\right)^2 \right] d\rho .$$

It is noted that u must vanish at $\rho=0$ in order that (u,Lu) be finite. On the other hand, $\bar{p}_s(\rho)$ is exponentially small at the origin so that a minor change in any trial function will meet this condition without changing the value of (u,Lu) to any order of ϵ in which we will be interested. So we shall ignore this point and consider that contributions to the integral from the region near $\rho=0$ are negligible. In fact, we shall approximate

$$\frac{1}{\rho^2} \approx \frac{1}{\rho_o^2} \left[1 - \frac{2(\rho-\rho_o)}{\rho_o} + \frac{3(\rho-\rho_o)^2}{\rho_o^2} + \cdots \right] .$$

Consider the trial function u=1. Then

$$(u,u)=1$$

$$(u,Lu) = \int_{-\infty}^{\infty} \overline{p}_s(\rho) \frac{1}{\rho_o^2}\left[1 - \frac{2(\rho-\rho_o)}{\rho_o} + O(\rho-\rho_o)^2 \right] d\rho \ ,$$

where we have extended the lower limit on the integration to $-\infty$ in view of the negligible error committed thereby. Then

$$(u,Lu) = \frac{1}{\rho_o^2} + \frac{3}{\rho_o^4} \frac{D\varepsilon}{2a_o'(\rho_o)} + O(\varepsilon^2) \ ,$$

and we have the approximate eigenvalue

$$\lambda_{01} = -\frac{D\varepsilon^2}{2}\left[\frac{1}{\rho_o^2} + \frac{3D\varepsilon}{2\rho_o^4 a_o'(\rho_o)} + O(\varepsilon^2) \right] .$$

It is easily seen that any other choice of a trial function does not improve the $O(\varepsilon^2)$ term in λ_{01} so that we have

$$\lambda_{01} = -\frac{D\varepsilon^2}{2\rho_o^2} + O(\varepsilon^3) \ ,$$

and also that $u_o=1+O(\varepsilon)$. To obtain λ_{11}, which is as far as we will go, we must use trial functions orthogonal to 1 (to $O(\varepsilon)$). Consider $u=\rho-\rho_o$. Then

$$(u,u) = \frac{D\varepsilon}{2a_o'(\rho_o)}$$

$$(u,Lu) = 1 + \frac{1}{\rho_o^2}\left[\frac{D\varepsilon}{2a_o'(\rho_o)} + \frac{3}{\rho_o^2}\frac{D\varepsilon^2}{2a_o'(\rho_o)} + O(\varepsilon^3)\right]$$

so

$$\lambda_{11} = -\varepsilon a_o'(\rho_o)\left[1 + \frac{D\varepsilon}{2\rho_o^2 a_o'(\rho_o)} + O(\varepsilon^2)\right]$$

Again, it is readily seen that any other choice of trial function which is orthogonal to 1 does not improve upon the first term in the *approximation, so that*

$$\lambda_{11} = -\varepsilon a_o'(\rho_o) + O(\varepsilon^2) \quad .$$

In summary, we have

$$u_o = 1 + O(\varepsilon), \qquad \lambda_{01} = \frac{-D\varepsilon^2}{2\rho_o^2} + O(\varepsilon^3) \quad ,$$

$$u_1 = \sqrt{\frac{2a_o'(\rho_o)}{D\varepsilon}}\,(\rho-\rho_o) + O(\sqrt{\varepsilon}), \quad \lambda_{11} = -a_o'(\rho_o) + O(\varepsilon^2) \quad .$$

We are now in a position to obtain an approximate autocorrelation function and from this an approximate spectral density. Now

$$\int_{-\infty}^{\infty} \rho \bar{p}_s(\rho)\, u_{01}\, d\rho = \rho_o,$$

$$\int_{-\infty}^{\infty} \rho \bar{p}_s(\rho)\, u_{11}\, d\rho = \sqrt{\frac{D\varepsilon}{2a_o'(\rho_o)}} \quad .$$

So we have

(4.34)

$$R_x(\tau) = \frac{1}{2} \cos \tau \left[\rho_o^2 \exp\left(- \frac{D\varepsilon^2 \tau}{2\rho_o^2} \right) + \frac{D\varepsilon}{2a_o'(\rho_o)} \exp\left(-\varepsilon a_o'(\rho_o)\, \tau \right) \right] + O(\varepsilon) \quad .$$

The $O(\varepsilon)$ term in the autocorrelation function as explicitly shown is probably in error as we have made numerous approximations which could have produced errors of $O(\varepsilon)$. However we will carry it along in the hope that it indicates, at least qualitatively, the nature of the $O(\varepsilon)$ term.

The spectral density, related to the autocorrelation function through the Wiener-Khintchine relations, is easily computed:

$$\phi_x(\omega) = \frac{\rho_o^2}{\pi} \frac{\frac{D\varepsilon^2}{2\rho_o^2} \left[\left(\frac{D\varepsilon^2}{2\rho_o^2} \right)^2 + 1+\omega^2 \right]}{\left[\left(\frac{D\varepsilon^2}{2\rho_o^2} \right)^2 + (1-\omega)^2 \right] \left[\left(\frac{D\varepsilon^2}{2\rho_o^2} \right)^2 + (1+\omega)^2 \right]} + \frac{D\varepsilon^2}{2\pi} \frac{[\varepsilon a_o'(\rho_o)]^2 + 1+\omega^2}{[(\varepsilon a_o'(\rho_o))^2 + (1-\omega)^2][(\varepsilon a_o'(\rho_o))^2 + (1+\omega)^2]} \quad .$$

This expression can be simplified by neglecting certain small contributions of $0(\varepsilon^2)$ without changing the validity of the expression to $0(\varepsilon)$:

$$\phi_x(\omega) = \frac{D\varepsilon^2}{2\pi} \frac{1+\omega^2}{(1+\omega)^2} \frac{1}{\left[\left(\frac{D\varepsilon^2}{2\rho_o}\right)^2 + (1-\omega)^2\right]}$$

(4.35)

$$+ \frac{D\varepsilon^2}{2\pi} \frac{1+\omega^2}{(1+\omega)^2} \frac{1}{\left[(\varepsilon a_o'(\rho_o))^2 + (1-\omega)^2\right]} .$$

Asymptotic Expansion for the Transition Probability Density Function for Self-Excited Oscillators[21]

We shall again take up a consideration of the time-dependent statistics of self-excited oscillators. The method used here combines the time-averaging technique used previously in obtaining an eigenfunction expansion of the transition probability density function and the idea of a perturbation expansion.

[21]The material of this section has been accepted for publication [9].

Again, we shall be concerned with oscillators described by the equation

$$\ddot{x} + \varepsilon f(x,\dot{x})\,\dot{x} + x = \frac{dw}{dt} \tag{4.36}$$

which exhibit limit cycle behavior in the absence of stochastic excitation, where w(t) is a Wiener process with $E[dw(t)^2] = 2D\varepsilon^2\,dt$ and ε is small parameter. The associated Fokker-Planck equation is

$$\frac{\partial p}{\partial t} = -\dot{x}\frac{\partial p}{\partial x} + x\frac{\partial p}{\partial \dot{x}} + \varepsilon\frac{\partial}{\partial \dot{x}}[f(x,\dot{x})\dot{x}]\,p + \varepsilon^2 D\frac{\partial^2 p}{\partial \dot{x}^2} \quad .$$

We have seen that if $f(x,\dot{x})$ is of the special form

$$f(x,\dot{x}) = f(H),$$

where

$$H = \frac{1}{2}(x^2 + \dot{x}^2) \quad ,$$

the exact steady-state solution is

$$p_s(x,\dot{x}) = C \exp\left[-\frac{F(H)}{D\varepsilon}\right] \quad ,$$

in which

$$F(H) = \int_0^H f(\eta)\,d\eta$$

and C is chosen to normalize $p_s(x,\dot{x})$. This limit cycle oscillator is characterized by a nonlinear damping coefficient, f(H), such that

$$f(H_o) = 0, \quad H_o > 0$$

and

$$f'(H_o) > 0.$$

The steady-state solution clearly has its maximum when $H=H_o$ and is exponentially small except for a small neighborhood about this value. Now

$$F(H) = F(H_o) + \frac{1}{2} f'(H_o) \ (H-H_o)^2 + \ldots \quad .$$

So we have approximately

$$p_s(x,\dot{x}) \approx C' \exp \left[- \frac{1}{2D\epsilon} \ f'(H_o)(H-H_o)^2 \right] ,$$

for H near H_o. Thus, $p_s(x,\dot{x})$ is exponentially small unless $H-H_o = 0(\sqrt{\epsilon})$.

For oscillators the spectral density function, and therefore the autocorrelation function, is always of special interest. The autocorrelation function, being given by

$$R_x(\tau) = \int\int\int\int_{-\infty}^{\infty} x_1 x_2 p_s(x_1,\dot{x}_1) \ p(x_2,\dot{x}_2,\tau|x_1,\dot{x}_1) \ dx_1 d\dot{x}_1 dx_2 d\dot{x}_2$$

is primarily determined by values of the transition probability density function for x and $\dot{x}$ near the limit cycle, since, as we have seen, $p_s(x,\dot{x})$ is exponentially small elsewhere. We should expect that this situation would exist for other types of self-excited oscillators.

Therefore, we shall seek an asymptotic expansion of the transition probability density function in the neighborhood of the limit cycle.

Now, the lowest order terms in the Fokker-Planck equation describe the motion of a linear, undamped oscillator, so we introduce rotating coordinates

$$x = a \cos(\theta + t) ,$$

$$\dot{x} = -a \sin(\theta + t) , \tag{4.37}$$

with $0<\theta<2\pi$, $a>0$, and

$$q(a,\theta,t) = p(x,y,t) .$$

Then these terms reduce to $\frac{\partial q}{\partial t}$, so we now have

$$\begin{aligned}\frac{\partial q}{\partial t} =& \varepsilon \left[\sin(\theta+t) \frac{\partial}{\partial a} + \frac{1}{a} \cos(\theta+t) \frac{\partial}{\partial \theta} \right] f(a\cos(\theta+t), -a\sin(\theta t))a \sin(\theta+t)q \\ &+ \varepsilon^2 D \left\{ \sin^2(\theta+t) \frac{\partial^2 q}{\partial a^2} + \frac{1}{a} \cos^2(\theta+t) \frac{\partial q}{\partial a} + \frac{1}{a} \sin 2(\theta+t) \frac{\partial^2 q}{\partial a \partial \theta} \right. \\ &\left. - \frac{\sin 2(\theta+t)}{a^2} \frac{\partial q}{\partial \theta} + \frac{1}{a^2} \cos^2(\theta+t) \frac{\partial^2 q}{\partial \theta^2} \right\} .\end{aligned} \tag{4.38}$$

Note specifically that $\frac{\partial q}{\partial t} = O(\varepsilon)$. This suggests the introduction of a second, slow time variable

$$\tau = \varepsilon t, \tag{4.39}$$

and we now consider q to be a function of two independent time variables, $q = q\ (a,\theta,t,\tau)$.

We will require that q be periodic in t. There may be some question about the possibility of demanding periodicity in the fast time variable without introducing correction terms. Some comments are in order. For Hamiltonian type oscillators, it can be shown that the expected frequency is exactly the deterministic frequency, independent of the noise level. And for other self-excited oscillators with small damping, on a deterministic basis, frequency corrections are known to be $0(\varepsilon^2)$. So it appears that if corrections to the fast time variable are required in order to obtain periodicity they will be $0(\varepsilon^2)$ - - and we won't be concerned with corrections of this high order.

With the introduction of the second time variable $\frac{\partial}{\partial t}$ is replaced by $\frac{\partial}{\partial t} + \varepsilon \frac{\partial}{\partial \tau}$.

To examine solutions near the limit cycle, we introduce stretched coordinates (ξ,ψ) by

$$a = a_o + \sqrt{\varepsilon}\xi, \tag{4.40}$$
$$\theta = \theta_o + \sqrt{\varepsilon}\psi,$$

taking a_o to be the limit cycle amplitude and θ_o to be the initial phase. The choice of stretching for the amplitude, which has the interpretation that we are looking at the $\sqrt{\varepsilon}$ neighborhood of the limit cycle amplitude, is suggested by the nature of the steady-state

solution, as discussed before. The choice of stretching for the angle variable is then dictated by the requirement that q be a probability density in ξ and ψ. Specifically, when the partial differential equation for q_o is determined, it must involve both ξ and ψ, since otherwise q_o would be independent of one of these variables, a result which would be incompatible with the fact that these variables can take on values over infinite intervals.

Finally, the following form of solution is assumed:

$$(4.41) \qquad q = q_o(\xi,\psi,t,\tau) + \sqrt{\varepsilon}\, q_1(\xi,\psi,t,\tau) + \varepsilon q_2(\xi,\psi,t,\tau) + \ldots \quad .$$

First Term of the Asymptotic Expansion

Using equations (4.39)-(4.41), all terms appearing in equation (4.38) are expanded in ascending powers of $\sqrt{\varepsilon}$; the successive groups of terms forming the coefficients of the successive powers of $\sqrt{\varepsilon}$ yield the equations

$$(4.42) \qquad \frac{\partial q_o}{\partial t} = 0,$$

$$(4.43) \qquad \frac{\partial q_1}{\partial t} = \frac{1}{2} \sin 2\phi \; f(a_o\cos\phi, -a_o\sin\phi) \frac{\partial q_o}{\partial \psi} + a_o \sin^2\phi \; f(a_o\cos\phi, -a_o\sin\phi) \frac{\partial q_o}{\partial \xi} \quad ,$$

$$(4.44)\quad \frac{\partial q_2}{\partial t} + \frac{\partial q_o}{\partial \tau} = \frac{1}{2}\sin 2\phi\, f(a_o\cos\phi, -a_o\sin\phi)\,\frac{\partial q_1}{\partial \psi}$$

$$+ a_o \sin^2\phi\, f(a_o\cos\phi, -a_o\sin\phi)\,\frac{\partial q_1}{\partial \psi}$$

$$+ \sin^2\phi\,\frac{\partial}{\partial \xi}\left[a_o \xi f_1(a_o\cos\phi, -a_o\sin\phi)\, q_o\right.$$

$$\left. + \xi f(a_o\cos\phi, -a_o\sin\phi)\, q_o\right]$$

$$+ \frac{1}{2}\sin 2\phi\, f_1(a_o\cos\phi, -a_o\sin\phi)\,\xi\,\frac{\partial q_o}{\partial \psi}$$

$$+ \frac{1}{2}\sin 2\phi\,\frac{\partial}{\partial \phi}\left[f(a_o\cos\phi, -a_o\sin\phi)\, q_o\right]$$

$$+ \cos^2\phi\, f(a_o\cos\phi, -a_o\sin\phi)\, q_o$$

$$+ \frac{1}{2} D \left[\frac{\partial^2 q_o}{\partial \xi^2} + \frac{1}{a_o^2}\frac{\partial^2 q_o}{\partial \psi^2} - \cos 2\phi\,\frac{\partial^2 q_o}{\partial \xi^2} + \right.$$

$$\left. + \frac{2\sin 2\phi}{a_o}\,\frac{\partial^2 q_o}{\partial \xi \partial \psi} + \frac{\cos 2}{a_o^2}\,\frac{\partial^2 q_o}{\partial \psi^2} \right] .$$

In these, $\phi = \theta + t + \sqrt{\varepsilon}\psi$, and $f(a\cos\phi, -a\sin\phi)$ has been expanded as

$$f(a\cos\phi, -a\sin\phi) = f(a_o\cos\phi, -a_o\sin\phi) + \sqrt{\varepsilon}\xi\, f_1(a_o\cos\phi, -a_o\sin\phi)$$

$$+ \varepsilon\xi^2 f_2(a_o\cos\phi, -a_o\sin\phi) + \ldots \quad .$$

Equation (4.42) says that $q_o = q_o(\xi, \psi, \tau)$, independent of t.

We shall proceed to solve these equations successively by using the requirement of periodicity in t and time-averaging. To facilitate this procedure, we define the following time-averaged quantities:

$$\overline{(\ )} \equiv \frac{1}{2\pi}\int_0^{2\pi} (\ )\ dt$$

$$\begin{aligned}
\lambda_1(a_o) &= \overline{f(a_o \sin\phi, -a_o\cos\phi)}\ , \\
\lambda_2(a_o) &= \overline{\sin 2\phi\ f(a_o \sin\phi, -a_o\cos\phi)}\ , \\
\lambda_3(a_o) &= \overline{\cos 2\phi\ f(a_o \sin\phi, -a_o\cos\phi)}\ , \\
\lambda_4(a_o) &= \overline{f_1(a_o \sin\phi, -a_o\cos\phi)}\ , \\
\lambda_5(a_o) &= \overline{\sin 2\phi\ f_1(a_o \sin\phi, -a_o\cos\phi)}\ , \\
\lambda_6(a_o) &= \overline{\cos 2\phi\ f_1(a_o \sin\phi, -a_o\cos\phi)}\ .
\end{aligned} \tag{4.45}$$

Time-averaging equation (4.43), we obtain

$$0 = \frac{1}{2}\lambda_2(a_o)\ \frac{\partial q_o}{\partial\psi} + \frac{1}{2}a_o\,[\lambda_1(a_o) - \lambda_3(a_o)]\ \frac{\partial q_o}{\partial\xi}\ . \tag{4.46}$$

In typical cases, the nonlinear damping coefficient is such that $\lambda_2(a_o)\equiv 0$. This is the case, for example, whenever $f(x,\dot{x})$ is an even function of x and $\dot{x}$. We can then satisfy equation (4.46) by setting

$$\lambda_1(a_o) = \lambda_3(a_o)\ ,$$

which is, in fact, the equation which determines the amplitude of the limit cycle to lowest order in ε[22].

To continue, define time-varying quantities $\mu_j(a_o,\phi)$ by

$$f(a_o\sin\phi,-a_o\cos\phi) = \lambda_1(a_o) + \mu_1(a_o,\phi)$$

etc., corresponding to the definitions of $\lambda_j(a_o)$, (4.45). Then the time-averaged equation (4.46) reduces the original equation (4.43) to

$$\frac{\partial q_1}{\partial t} = \frac{1}{2}\,\mu_2(a_o,\phi)\,\frac{\partial q_o}{\partial \psi} + \frac{1}{2}\,a_o(\mu_1(a_o,\phi) - \mu_3(a_o,\phi))\,\frac{\partial q_o}{\partial \xi}\ ,$$

which determines the t dependence of q_1. In fact,

$$q_1(\xi,\psi,t,\tau) = g_1(\xi,\psi,\tau) + \frac{1}{2}\int_0^t \mu_2(a_o,\phi)\,dt\,\frac{\partial q_o}{\partial \psi} + \frac{1}{2}\int_0^t [\mu_1(a_o,\phi) - \mu_3(a_o,\phi)]\,dt\,\frac{\partial q_o}{\partial \xi}\ ,$$

where $g_1(\xi,\psi,\tau)$ is yet to be determined.

Now consider equation (4.44). Again, we time-average and use the equation determining a_o to obtain

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[22] See, for example, Minorsky, N. [23, pp. 200-201]. The function f there is to be identified with our quantity xf. Also note there that $\lambda_2(a_o)\equiv 0$ implies that the frequency is not affected by the amplitude of oscillations.

$$(4.47)\quad \frac{\partial q_o}{\partial \tau} = \left[\frac{1}{2} a_o \left(\lambda_4(a_o) - \lambda_6(a_o)\right) \right] \frac{\partial}{\partial \xi} \xi q_o + \frac{1}{2} \lambda_5(a_o)\ \xi \frac{\partial q_o}{\partial \psi}$$

$$+ \frac{1}{2} D \left[\frac{\partial^2 q_o}{\partial \xi^2} + \frac{1}{a_o^2} \frac{\partial^2 q_o}{\partial \psi^2} \right]$$

Noting that $\lambda_5(a) = \dfrac{d\lambda_2(a)}{da}$, it follows that $\lambda_5(a) \equiv 0$ also. For convenience, define

$$(4.48)\qquad \beta = \frac{1}{2} a_o \left[\lambda_4(a_o) - \lambda_6(a_o)\right] .$$

Then equation (4.47) becomes

$$(4.49)\quad \frac{\partial q_o}{\partial \tau} = \frac{\partial}{\partial \xi} \left[\beta \xi q_o\right) + \frac{1}{2} D \left[\frac{\partial^2 q_o}{\partial \xi^2} + \frac{1}{a_o^2} \frac{\partial^2 q_o}{\partial \psi^2} \right] .$$

In terms of the original variables, the solution desired must satisfy the initial condition

$$p(x,\dot{x},0|x_o,\dot{x}_o) = \delta(x-x_o)\ \delta(\dot{x}-\dot{x}_o)$$

and the normalization condition

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} p(x,\dot{x},t|x_o,\dot{x}_o)\ dx\ d\dot{x} = 1.$$

Using (4.37), the initial coordinates x_o and $\dot{x}_o$ define initial coordinates r_o and θ_o, and the initial condition becomes

$$p(a,\theta,0|r_o,\theta_o) = \frac{1}{a}\,\delta(a-r_o)\quad\delta(\theta-\theta_o)\ .$$

The normalization condition becomes

$$\int_{-\infty}^{\infty}\int_{0}^{2\pi} p(a,\theta,t|r_o,\theta_o)\ \text{ad ad}\ \theta = 1\ .$$

Now using the definition of the stretched coordinates, equation (4.39), the initial and normalization conditions in terms of the ξ, ψ variables are obtained:

$$(4.50)\qquad p(\xi,\psi,0|\xi_o,\psi_o) = \frac{1}{\varepsilon}\ \frac{1}{a_o+\sqrt{\varepsilon}\xi}\quad\delta(\xi-\xi_o)\quad\delta(\psi)$$

where

$$\xi_o = \frac{r_o-a_o}{\sqrt{\varepsilon}}\ ;$$

and

$$(4.51)\qquad \int_{-\theta_o/\sqrt{\varepsilon}}^{2\pi/\sqrt{\varepsilon}}\ \int_{-a_o/\sqrt{\varepsilon}}^{\infty} p(\xi,\psi,t,\tau|\xi_o,\psi_o)(a_o+\varepsilon\xi)\quad d\xi\ d\psi = \frac{1}{\varepsilon}\ .$$

Only the lowest order terms in these two conditions will be taken, so that the initial and normalization conditions to be associated with equation (4.49) are

$$q_o(\xi,\psi,0|\xi_o,\psi_o) = \frac{1}{a_o\varepsilon}\ \delta(\xi-\xi_o)\ \delta(\psi) \quad , \tag{4.52}$$

and

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} q_o(\xi,\psi,\tau|\xi_o,\psi_o)\ d\xi\ d\psi = \frac{1}{a_o\varepsilon} \quad . \tag{4.53}$$

The solution of equation (4.49) is most readily obtained by noting that the system of stochastic differential equations

$$\frac{d\xi}{d\tau} = -\beta\xi + \frac{dw_1}{d\tau} \quad ,$$

$$\frac{d\psi}{d\tau} = \frac{dw_2}{d\tau} \quad ,$$

where $w_1(\tau)$ and $w_2(\tau)$ are two independent Wiener processes with zero means and $E[dw_1(\tau)^2] = D d\tau$, $E[dw_2(\tau)^2] = \frac{D}{a_o^2}\, d\tau$, has equation (4.49) as the corresponding Fokker-Planck equation. As this system of stochastic differential equations is linear and Wiener processes are Gaussian, q_o must be Gaussian.

The first and second moments required to define q_o can be obtained from the Fokker-Planck in the manner suggested in Part III.

As an illustration of the use of the formal properties of differentials of Wiener processes, we shall, however, obtain these directly from the stochastic differential equations. Since these equations are linear, the solutions can be represented as

$$\xi(\tau) = \xi_o e^{-\beta\tau} + \int_0^\tau e^{-\beta(\tau-s)}\, dw_1(s) \quad ,$$

$$\psi(\tau) = w_2(\tau) \quad .$$

From these and the fact that $E[dw_1(s)] = E[dw_2(s)] = 0$, we have the means

$$E[\xi(\tau)] = \xi_o e^{-\beta\tau} \quad ,$$

$$E[\psi(\tau)] = 0 \quad .$$

Proceeding to the second moments

$$\text{var}\,[\xi(\tau)] = \left\{ E\left[\xi(\tau) - \xi_o e^{-\beta\tau}\right]^2 \right\}$$

$$= E\left[\int_0^\tau e^{-\beta(\tau-s)}\, dw_1(s) \int_0^\tau e^{-\beta(\tau-\sigma)}\, dw_1(\sigma) \right]$$

$$= \int_0^\tau \int_0^\tau e^{-2\beta\tau+\beta s+\beta\sigma} E[dw_1(s)\, dw_1(\sigma)] \quad .$$

Now we use the formal property

$$E[dw_1(s)\, dw_1(\sigma)] = D\delta(s-\sigma)\, ds\, d\sigma$$

so that

$$\text{var}[\xi(\tau)] = D\int_0^\tau e^{-2\beta(\tau-s)}\, dw = \frac{D}{2\beta}[1-e^{-2\beta\tau}] \quad .$$

The variance of $\psi(\tau)$ can be gotten immediately from the property of a Wiener process:

$$\text{var}\,[\psi(\tau)] = \frac{D\tau}{a_o^2} \, .$$

The solution of equation (4.49) subject to the conditions of equations (4.52) and (4.53) is then

$$q_o(\xi,\psi,\tau|\xi_o,0) = \frac{1}{a_o\epsilon}\sqrt{\frac{\beta}{\pi D(1-e^{-2\beta\tau})}}\sqrt{\frac{a_o^2}{2\pi D\tau}}\exp\left[-\frac{\beta(\xi-\xi_o e^{-\beta\tau})^2}{D(1-e^{-2\beta\tau})}\right]$$

$$\text{X}\quad \exp\left[\frac{-a_o^2\psi^2}{2D\tau}\right] \, .$$

Example 5: Circular limit cycle.

Let $f(x,\dot{x}) = -1 + g(H)$

where $H = \frac{1}{2}(x^2 + \dot{x}^2)$,

and assume that

$g(H_o) = 0, \quad H_o>0$,

$g'(H_o)>0.$

The quantities $\lambda_1(a)$, $\lambda_2(a)$, $\lambda_3(a)$ are easily computed:

$$\lambda_1(a) = \frac{1}{2\pi} \int_0^{2\pi} \left[-1 + g\left(\frac{1}{2} a^2\right) \right] d\phi = -1 + g\left(\frac{1}{2} a^2\right) ,$$

$$\lambda_2(a) = \frac{1}{2\pi} \int_0^{2\pi} \left[-1 + g\left(\frac{1}{2} a^2\right) \right] \sin 2\phi d\phi = 0 ,$$

$$\lambda_3(a) = \frac{1}{2\pi} \int_0^{2\pi} \left[-1 + g\left(\frac{1}{2} a^2\right) \right] \cos 2\phi d\phi = 0,$$

so a_o is defined by

$$g\left(\frac{1}{2} a_o^2\right) = 1.$$

Noting that $\lambda_4(a) = \frac{d\lambda_1(a)}{da}$, $\lambda_6(a) = \frac{d\lambda_3(a)}{da}$, we find

$$\beta = \frac{1}{2} a_o^2 g'\left(\frac{1}{2} a_o^2\right) .$$

The transition probability density function, to lowest order in ε, is then given by equation (4.54) with a_o, β defined above.

Example 6: van der Pol's oscillator.

The van der Pol oscillator is described by

$$f(x, \dot{x}) = -1 + x^2 .$$

In this case

$$\lambda_1(a) = \frac{1}{2\pi}\int_0^{2\pi}\left[-1 + a^2\cos^2\phi\right]d\phi = -1 + \frac{a^2}{2} \quad ,$$

$$\lambda_2(a) = \frac{1}{2\pi}\int_0^{2\pi}\left[-1 + a^2\cos^2\phi\right]\sin 2\phi d\phi = 0 \quad ,$$

$$\lambda_3(a) = \frac{1}{2\pi}\int_0^{2\pi}\left[-1 + a^2\cos^2\phi\right]\cos 2\phi d\phi = \frac{a^2}{4} \quad ,$$

so a_o is determined by

$$-1 + \frac{a_o^2}{2} - \frac{a_o^2}{4} = 0 \quad .$$

Thus $a_o = 2$, and then $\beta = 1$.

Higher Order Approximations

In the following paragraphs we will show how the procedure used above may be iterated to obtain each of the functions q_j in the expansion of p successively.

First, the definitions of the stretched coordinates and the slow time are used in the coefficients and differential operators of equation (4.38), and the resulting operators are collected in groups, each of

which has a common power of $\sqrt{\varepsilon}$. For definiteness, suppose equation (4.38) is represented by

$$Mp = 0.$$

Then the substitutions mentioned result in the expansion

$$M = \sum_{j=0}^{\infty} (\sqrt{\varepsilon})^j M_j \ .$$

Second, equation (4.41) is used as the expansion of p. Substituting equation (4.41) into equation (4.38), we obtain the following set of equations, each corresponding to a different power of $\sqrt{\varepsilon}$:

$$\sum_{j=0}^{k} M_j \, q_{k-j} = 0 \quad ((\sqrt{\varepsilon})^k \text{ equation}) \ .$$

The first few equations are written out explicitly:

$$M_o q_o = 0 \ ,$$

$$M_1 q_o + M_o q_1 = 0 \ ,$$

$$M_2 q_o + M_1 q_1 + M_o q_2 = 0 \ ,$$

$$M_3 q_o + M_2 q_1 + M_1 q_2 + M_o q_3 = 0 \ .$$

In our previous work, M_o was identified as $-\frac{\partial}{\partial t}$. The other operators can be separated into two terms so that M_j is written as

$$M_j = M_j^d + M_j^p \ .$$

The superscript 'd' is intended to indicate that these terms are independent of t, the 'p', that these terms are periodic in t with zero time average.

From the first of these equations, we find

$$q_o = q_o(\xi,\psi,\tau) \quad .$$

We proceed to solve the equations sequentially by separating each equation into two parts. One part is independent of t and is obtained by time-averaging. The second part is the remainder of the equation after the t independent part is removed. For example, the next equation to be considered can be written as

$$\left(M_1^d + M_1^p\right) q_o - \frac{\partial}{\partial t} q_1 = 0 \quad .$$

The t independent part of this equation is

$$M_1^d \, q_o = 0.$$

Setting $M_1^d = 0$ determines the amplitude of the limit cycle to lowest order in ε. The t dependent part of this equation is then

$$\frac{\partial}{\partial t} q_1 = M_1^p \, q_o \quad ,$$

and serves to determine the t dependence of q_1. In fact

$$q_1(\xi,\psi,t,\tau) = g_1(\xi,\psi,\tau) + \int_0^t M_1^p(s)\, ds \; q_o(\xi,\psi,\tau) \quad .$$

We proceed to the next equation and treat it in the same manner. The t independent part is

$$M_2^d \, q_o = 0.$$

Since the time average of $M_1^p \, q_1$ is zero, and $M_1^d = 0$, this equation determines q_o. The t dependent part of the equation is

$$\frac{\partial}{\partial t} q_2 = M_1^p \, q_1 + M_2^p \, q_o$$

and determines the t dependence of q_2:

$$q_2(\xi,\psi,t,\tau) = g_2 \, (\xi,\psi,\tau) + \int_0^t M_1^p \, q_1 \, ds + \int_0^t M_2^p \, ds \, q_o \, .$$

Continuing in this fashion, all q_j can be determined. The t dependence of q_j will be determined by the t dependent part of the $(\sqrt{\epsilon})^j$ equation and $g_j(\xi,\psi,\tau)$ will be determined by the t independent part of the $(\sqrt{\epsilon})^{j+2}$ equation.

There still remains the question of initial conditions and normalization. For initial conditions, equation (4.50) is expanded in ascending powers of $\sqrt{\epsilon}$, with the successive terms providing initial conditions for each q_j in order. The normalization condition, equation (4.51), is used successively with all the q_j so far determined, each time satisfying equation (4.51) to one power of $\sqrt{\epsilon}$ higher.

While we have indicated a procedure which, in principle, allows one to obtain the asymptotic expansion of the transition probability density function to any order of $\sqrt{\varepsilon}$ desired, the calculations required increase rapidly with each succeeding term in the expansion. To indicate what calculations must be made for $q(\xi,\psi,t,\tau)$, for example, we shall consider the circular limit cycle case, since here there are certain simplifications which make the discussion easier.

In example 5, we chose

$$f(x,\dot{x}) = -1 + g\left(\frac{1}{2}\dot{x}^2 + \frac{1}{2}x^2\right) ,$$
$$= -1 + g\left(\frac{1}{2}a^2\right) .$$

Clearly, then, all $\mu_1(a,\phi) \equiv 0$, and the operator M_1 reduces to

$$M_1 = M_1^d = \frac{1}{2} a_o \lambda_1(a_o) \frac{\partial}{\partial \xi} .$$

In particular, $M_1^p \equiv 0$ so that

$$\frac{\partial q_1}{\partial \tau} = M_1^p q_o = 0,$$

i.e., $q_1 = q_1(\xi,\psi,\tau)$. Following the general procedure described above, one finds the equation for q_1:

$$M_2^d q_1 = - M_3^d q_o .$$

The Green's function for the operator M_2^d is essentially q_o, differing only by a constant factor. Noting the initial condition on q_1,

$$q_1(\xi,\psi,0|\xi_o,0) = -\frac{\xi_o}{\varepsilon a_o^2}\ \delta(\xi-\xi_o)\ \delta(\psi)\ ,$$

the solution of the equation for q_1 can be written as

$$q_1(\xi,\psi,\tau|\xi_o,0) = -\frac{\xi_o}{a_o}\, q_o(\xi,\psi,\tau|\xi_o,0) + \int_0^\tau \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \varepsilon a_o\ q_o(\xi,\psi,\tau-\sigma|x,\omega)\ M_3^d(x,\omega)\ q_o(x,\omega,\sigma|\xi_o,0)\ dx\ d\omega\ d\sigma\ .$$

It can be verified that this solution is properly normalized.

Autocorrelation Function and Spectral Density

The autocorrelation function is defined by

$$R_x(s) = \lim_{t\to\infty} \int\int\int\int_{-\infty}^{\infty} x_t x_{t+s} p(x_t,\dot{x}_t,t|x_o,\dot{x}_o)\ p(x_{t+s},\dot{x}_{t+s},s|x_t,\dot{x}_t)\ dx_t\ d\dot{x}_t\ dx_{t+s}\ d\dot{x}_{t+s}\ .$$

Expressing all quantities in terms of the ξ,ψ variables, this becomes

$$R_x(s) = \lim_{t\to\infty} \int_{-a_o/\sqrt{\varepsilon}}^{\infty} \int_{-a_o/\sqrt{\varepsilon}}^{\infty} \int_{-\theta_o/\sqrt{\varepsilon}}^{2\pi/\sqrt{\varepsilon}} \Big\{ (a_o+\sqrt{\varepsilon}\xi_t)\ \cos(\sqrt{\varepsilon}\psi_t+\theta_o+t)$$

$$(a_o+\sqrt{\varepsilon}\xi_{t+s})\ \cos(\sqrt{\varepsilon}\psi_{t+s}+\theta_o+t+s)$$

$$\times \sum_{j=0}^{\infty} q_j(\xi_t,\psi_t,t,\tau|\xi_o,\psi_o) \sum_{k=0}^{\infty} q_k(\xi_{t+s},\psi_{t+s},s,\sigma|\xi_t,\psi_t)$$

$$(a_o+\sqrt{\varepsilon}\xi_{t+s})\ \varepsilon^2\, d\xi_t\, d\xi_{t+s}\, d\psi_t\, d\psi_{t+s} \quad .$$

In these $\tau = \varepsilon t$, $\sigma = \varepsilon s$. Only the lowest order contributions are considered as only q_o was obtained above:

$$R_x(s) = \lim_{t\to\infty} \Bigg\{ \int_{-\infty}^{\infty}\int\int\int\int a_o^4 \cos\,(\sqrt{\varepsilon}\psi_t+\theta_o+t)\ \cos\,(\sqrt{\varepsilon}\psi_{t+s}+\theta_o+t+s)$$

$$\times \frac{1}{a_o\varepsilon} \sqrt{\frac{\beta}{\pi D(1-e^{-2\beta\tau})}}\ \sqrt{\frac{a_o^2}{2\pi D\tau}}\ \exp\left[\frac{-\beta(\xi_t-\xi_o e^{-\beta\tau})^2}{D(1-e^{-\beta\tau})}\right]$$

$$\times \exp\left[\frac{-a_o^2\psi_t^2}{2D\tau}\right]$$

$$\times \frac{1}{a_o\varepsilon} \sqrt{\frac{\beta}{\pi D(1-e^{-2\beta\sigma})}}\ \sqrt{\frac{a_o^2}{2\pi D\sigma}}\ \exp\left[\frac{-\beta(\xi_{t+s}-\xi_t e^{-\beta\sigma})^2}{D(1-e^{-\beta\sigma})}\right]$$

$$\times \exp\left[-\frac{a_o^2(\psi_{t+s}-\psi_t)^2}{2D\sigma}\right]\ \varepsilon^2 d\xi_t d\xi_{t+s} d\psi_t d\psi_{t+s} + O(\sqrt{\varepsilon}) \Bigg\} \quad .$$

The integrations over ξ_t, ξ_{t+s} leave only Gaussian distributions in ψ_t, ψ_{t+s}. These integrals are also readily evaluated and result in

$$R_x(s) = \lim_{t\to\infty} \left\{ \frac{1}{2} a_o^2 \cos(s) \exp\left[-\frac{D\epsilon^2 s}{2a_o^2} \right] + \frac{1}{2}\left[a_o^2 \cos(2\theta_o + 2t + s)\right] \right.$$

$$\left. \times \exp\left[-\frac{D\epsilon^2 s}{2a_o^2} - \frac{2D\epsilon^2 t}{a_o^2} \right] + O(\sqrt{\epsilon}) \right\} .$$

Taking the limit, we have

$$R_x(s) = \frac{1}{2} a_o^2 \cos(s) \exp\left[-\frac{D\epsilon^2 s}{2a_o^2} \right] + O(\sqrt{\epsilon}) .$$

The spectral density, defined by

$$\phi_x(\omega) = \frac{2}{\pi} \int_0^\infty R_x(s) \cos \omega s \, ds$$

is then found to be, to lowest order,

$$\phi_x(\omega) = \frac{D\epsilon^2}{4\pi} \left[\frac{1}{\left(\frac{D\epsilon^2}{2a_o^2}\right)^2 + (\omega-1)^2} + \frac{1}{\left(\frac{D\epsilon^2}{2a_o^2}\right)^2 + (\omega+1)^2} \right] .$$

If we take ω positive, only the first term has a peak (at $\omega=1$) while the second term contributes a quantity which is $O(\varepsilon^4)$ smaller than the first term. Note that the peak level is inversely proportional to the noise parameter, $\varepsilon^2 D$, and that the bandwidth is directly proportional to the noise parameter.

As all second order oscillators behave $\sim \frac{1}{\omega^4}$ for large ω[23], the spectral density given above can be accurate only for moderate values of ω.

Possible Extension of Fokker-Planck Methods

The eigenfunction expansion and asymptotic expansion methods we have developed here provide quite satisfactory results for self-excited oscillators. There are definite limitations on the application of these methods, however. For one thing, we have always limited ourselves to finding the first term in the expansions, inasmuch as the practical difficulties involved in getting more terms were considerable. And we have always considered so-called "good" oscillators, that is, oscillators with linear restoring forces.

In these good oscillators, setting $\varepsilon=0$ leads to the conservative system

$$\ddot{x} + x = 0$$

[23] See Example 8 for "small time expansion of the autocorrelation function."

which has sines and cosines for solution. In the procedure used to obtain the asymptotic expansion of the transition probability density function, expansions in terms of these trigonometric functions, followed by an averaging technique, were used. In oscillators with nonlinear restoring forces, setting $\varepsilon=0$ leads to the conservative system

$$\ddot{x} + g(x) = 0 ,$$

which, for polynomials for $g(x)$, has elliptic functions for solutions. It seems reasonable that the same procedure for obtaining an asymptotic expansion might be successful if expansions in trigonometric functions were replaced by expansions in elliptic functions. So far, however, investigations in this direction have not been successful.

We recall that these Fokker-Planck methods were developed because other methods which were successful in treating passive systems were inadequate for the study of self-excited oscillators. One might ask now if these new methods might themselves be useful in the study of passive systems. These methods can be used, at least for good oscillators, but they provide no practical advantage over the methods of equivalent linearization and perturbation directly on the stochastic differential equation. The first term in such an asymptoic expansion merely describes a linear oscillator and higher order terms again prove difficult to obtain. Since the other methods also describe the response in terms of a linear oscillator and these other methods are easier to employ, there seems to be no advantage in seeking an asymptotic expansion.

Small Time Expansion of the Autocorrelation Function

An expansion of the autocorrelation function for small time for any component of a Markov process can be found by means of the Fokker-Planck equation. The autocorrelation was defined earlier as

$$R_x(\tau) = \lim_{t\to\infty} \iint_R x_1 y_1 \; p(\underline{x},t|\underline{x}_o) \; p(\underline{y},\tau|\underline{x}) \; d\underline{x} \; d\underline{y}$$

or

$$R_x(\tau) = \iint_R x_1 y_1 \; p_s(\underline{x}) \; p(\underline{y},\tau|\underline{x}) \; d\underline{x} \; d\underline{y}$$

if $\lim_{t\to\infty} p(\underline{x},t|\underline{x}_o)$ exists. Note that this latter expression can be written as

$$R_x(\tau) = \int_R x_1 \; p_s(\underline{x}) \; E[y_1,\tau|\underline{x}] \; d\underline{x}$$

with

$$E[y_1,\tau|\underline{x}] = \int_R y_1 \; p(\underline{y},\tau|\underline{x}) \; d\underline{y} \quad .$$

We have seen (equation (3.36)) that $E[y_1,\tau|\underline{x}]$ satisfies the backward equation

$$\frac{d}{d\tau} E[y_1,\tau|\underline{x}] = L^*_{\underline{x}} \, E[y_1,\tau|\underline{x}] \quad .$$

Therefore

$$\frac{d^n}{d\tau^n} R_x(\tau) = \int_R x_1 \, p_s(\underline{x}) \, \frac{d^n}{d\tau^n} E[y_1,\tau|\underline{x}] \, d\underline{x}$$

$$= \int_R x_1 \, p_s(\underline{x}) \, [L_{\underline{x}}^*]^n \, E[y_1,\tau|\underline{x}] \, d\underline{x} \quad .$$

Integrating by parts

$$\frac{d^n}{d\tau^n} R_x(\tau) = \int_R [L_{\underline{x}}]^n \, [x_1 \, p_s(\underline{x})] \, E[y_1,\tau|\underline{x}] \, d\underline{x} \quad .$$

Letting $\tau \to 0$, and noting that $\lim_{\tau\to 0} E[y_1,\tau|\underline{x}] = x_1$,

$$\frac{d^n}{d\tau^n} R_x(\tau) \Big|_{\tau=0} = \int_R x_1 [L_{\underline{x}}]^n \, [x_1 \, p_s(\underline{x})] \, d\underline{x} \quad .$$

Integrating by parts again

$$\frac{d^n}{d\tau^n} R_x(\tau) \Big|_{\tau=0} = \int_R x_1 \, p_s(\underline{x}) \, [L_{\underline{x}}^*]^n \, (x_1) \, d\underline{x} \quad .$$

Then, if $R_x(\tau)$ is analytic at the origin, we have

$$R_x(\tau) = \sum_{n=0}^{\infty} \frac{\tau^n}{n!} \left[\int_R x_1 \, p_s(\underline{x}) \, [L_{\underline{x}}^*]^n \, (x_1) \, d\underline{x} \right] .$$

Example 7:

For a first order system described by

$$\dot{x} + \beta x = \frac{dw}{dt} \quad ,$$

with $E[dw(t)^2] = 2D\,dt$, the backward operator is

$$L^* = -\beta x \frac{\partial}{\partial x} + D \frac{\partial^2}{\partial x^2} \quad .$$

Then $L^*(x) = -\beta x$,

and generally

$$[L^*]^n (x) = (-\beta)^n x.$$

Using the steady-state solution found earlier (Example 1) we easily find

$$\int_{-\infty}^{\infty} x^2 p_s(x)\, dx = \frac{D}{\beta} \quad ,$$

so

$$\frac{d^n}{d\tau^n} R_x(\tau) = D\beta^{n-1} (-1)^n \quad .$$

Here, we can actually sum the series to find

$$R_x(\tau) = \sum_{n=0}^{\infty} \frac{\tau^n}{n!} D\beta^{n-1} (-1)^n = \frac{D}{\beta} e^{-\beta\tau} \quad ,$$

the result we found in Example 1 by means of an eigenfunction expansion.

Example 8: Oscillators

A general oscillator is described by

$$\ddot{x} + f(x,\dot{x})\,\dot{x} + g(x) = \frac{dw}{dt}$$

where $w(t)$ is a Wiener process with $E[dw(t)^2] = 2D\,dt$. The associated Fokker-Planck equation is

$$\frac{\partial p}{\partial t} = L_{\underline{x}}\, p = -\dot{x}\,\frac{\partial p}{\partial x} + \frac{\partial}{\partial \dot{x}}\,[f(x,\dot{x})\dot{x}+g(x)]\,p + D\,\frac{\partial^2 p}{\partial \dot{x}^2}\ ,$$

and the backward operator is

$$L^*_{\underline{x}} = \dot{x}\,\frac{\partial}{\partial x} - [f(x,\dot{x})\dot{x}+g(x)]\,\frac{\partial}{\partial \dot{x}} + D\,\frac{\partial^2}{\partial \dot{x}^2}\ .$$

We can use this small τ expansion of the autocorrelation function to obtain a large frequency expansion of the spectral density function. Using the definition of the spectral density function

$$\phi_x(\omega) = \frac{2}{\pi}\int_0^\infty R_x(\tau)\,\cos\omega\tau\,d\tau\ ,$$

repeated integration by parts yields the following asymptotic expansion for $\phi_x(\omega)$:

$$\phi_x(\omega) \sim \frac{2}{\pi}\sum_{n=1}^{\infty}\left.\frac{d^{2n-1}R_x(\tau)}{d\tau^{2n-1}}\right|_{\tau=0}\frac{(-1)^{n+1}}{\omega^{2n}}\ .$$

Note particularly that $\varphi_x(\omega)$ involves only odd order derivatives of the autocorrelation function. Now

$$\left.\frac{dR_x(\tau)}{d\tau}\right|_{t=0} = \int_{-\infty}^{\infty}\int x\, p_s(x,\dot{x})\, L_x^*(x)\, dx\, d\dot{x}$$

$$= \int_{-\infty}^{\infty}\int x\, p_s(x,\dot{x})\dot{x}\, dx\, d\dot{x} = 0,$$

as one can easily verify directly from the steady-state Fokker-Planck equation.

The implication of this result is that the first possibly non-zero term in the expansion of $\phi_x(\omega)$ is $\frac{2}{\pi\omega^4}\left.\frac{d^3R_x(\tau)}{d\tau^3}\right|_{\tau=0}$, i.e.,

$$\phi_x(\omega) \sim \frac{\text{Constant}}{\omega^4}, \quad \text{as } \omega\to\infty.$$

PART V

FORMULATION OF A CLASS OF FIRST PASSAGE PROBLEMS

The term first passage problem refers to any one of a number of problems whose solution involves the knowledge of the distribution of times that certain events occur. In one dimensional systems, a typical first passage problem is the determination of the distribution of times that the system variable first takes on a value outside a specified interval. For two dimensional systems (oscillators), typical first passage problems are the determination of the distribution of times that the amplitude of oscillation first exceeds a given bound and the determination of the distribution of times between zero crossings.

For one dimensional systems, an elegant solution of the first passage problem has been provided in two papers by Siegert [24] and Darling and Siegert [25]. Their method involves the introduction of Laplace transforms, since the integral equations they derive are all of convolution type (cf. equation (2.7)).

For oscillators, represented as two dimensional Markov processes, there has been no such progress, despite the attention given the problem by many investigators. Indeed, even the formulation of many examples of first passage problems have not even been achieved. In particular, there is yet to be found a formulation of the problem of

determining the distribution of times that the amplitude of an oscillator first exceeds a given bound.

At the end of Part II, the first passage density function was related to the transition probability density function by means of an integral equation. The principal purpose there was to indicate that a connection was possible in order to further emphasize the importance of the transition probability density function in describing a Markov process. In the following paragraphs we shall make use of this integral equation to formulate the problem of determining the distribution of times for zero crossings.

In Part II, we found that the conditional probability density of first passage times through an element of surface at $\underline{\sigma}$ of a connected region, Γ, denoted by $f(\Gamma,\underline{\sigma},t|\underline{\xi})$, satisfies the integral equation

$$p(\underline{x},t|\underline{\xi}) = \int_{\partial\Gamma}\int_{0}^{t} f(\Gamma,\underline{\sigma},s|\underline{\xi})\ p(\underline{x},t-s|\underline{\sigma})\ d\underline{\sigma}\ ds$$

where $\underline{x}\varepsilon\Gamma$, $\underline{\xi}\notin\Gamma$ and $\partial\Gamma$ denotes the boundary of Γ. In Part III, we saw that $p(\underline{x},t|\underline{\xi})$ satisfies the backward equation, (3.4). This implies that

$$\begin{aligned} 0 &= \left(\frac{\partial}{\partial t} - L_{\xi}^{*}\right)\int_{\partial\Gamma}\int_{0}^{t} f(\Gamma,\underline{\sigma},s|\underline{\xi})\ p(\underline{x},t-s|\underline{\sigma})\ d\underline{\sigma}\ ds \\ &= \int_{\partial\Gamma}\int_{0}^{t} f(\Gamma,\underline{\sigma},s|\underline{\xi})\ \frac{\partial p}{\partial t}(\underline{x},t-s|\underline{\sigma}) - p(\underline{x},t-s|\underline{\sigma})\ L_{\xi}^{*} f(\Gamma,\underline{\sigma},s|\underline{\xi})\ d\underline{\sigma}ds \\ &\quad + \lim_{s\to t} - \int_{\partial\Gamma} f(\Gamma,\underline{\sigma},s|\underline{\xi})\ p(\underline{x},t-s|\underline{\sigma})\ d\underline{\sigma}. \end{aligned}$$

Noting that

$$\frac{\partial p}{\partial t}(\underline{x},t-s|\underline{\sigma}) = -\frac{\partial}{\partial s} p(\underline{x},t-s|\underline{\sigma})$$

and integrating by parts in the time integration, we find

$$0 = \int_{\partial\Gamma}\int_{0}^{t} p(\underline{x},t-s|\underline{\sigma})\left(\frac{\partial}{\partial s} - L_{\xi}^{*}\right) f(\Gamma,\underline{\sigma},s|\underline{\xi})\, d\underline{\sigma}\, ds$$

$$+ \lim_{s\to 0+}\int_{\partial\Gamma} p(\underline{x},t-s|\underline{\sigma})\, f(\Gamma,\underline{\sigma},s|\underline{\xi})\, d\underline{\sigma} \quad .$$

The second integral will vanish if we require

$$\lim_{s\to 0} f(\Gamma,\underline{\sigma},s|\underline{\xi}) = 0 \tag{5.1}$$

for any $\underline{\xi}\notin\Gamma$. This condition seems to be a reasonable one to impose, since initially first passage could occur at $\underline{\sigma}$ only if $\underline{\xi} = \underline{\sigma}$, which is ruled out since we have taken $\underline{\xi}\notin\Gamma$. This leaves us with

$$0 = \int_{\partial\Gamma}\int_{0}^{t} p(\underline{x},t-s|\underline{\sigma})\left(\frac{\partial}{\partial s} - L_{\xi}^{*}\right) f(\Gamma,\underline{\sigma},s|\underline{\xi})\, d\underline{\sigma}\, ds$$

from which it seems reasonable to conclude that

$$\frac{\partial}{\partial s} f(\Gamma,\underline{\sigma},s|\underline{\xi}) = L_{\xi}^{*} f(\Gamma,\underline{\sigma},s|\underline{\xi}) \quad , \tag{5.2}$$

i.e., the density of first passage times also satisfies the backward equation, a result which Siegert [24] found to be true for one

dimensional Markov processes. We have not been able to prove that this is necessarily so for higher dimensional Markov processes.

Before discussing boundary conditions for the first passage density function, we shall first consider properties of the boundaries themselves. There are special features of boundaries for Markov processes associated with second (and higher) order stochastic differential equations which do not appear for one dimensional Markov processes. In particular, consider the second order stochastic differential equation

$$\frac{d^2x}{dt^2} + f\left(x, \frac{dx}{dt}\right)\frac{dx}{dt} + g(x) = \frac{dw}{dt} \quad .$$

Introducing $y = \frac{dx}{dt}$, this can be written as a pair of equations,

$$\begin{aligned} \frac{dx}{dt} &= y \ , \\ \frac{dy}{dt} &= -f(x,y)y - g(x) + \frac{dw}{dt} \quad . \end{aligned} \tag{5.3}$$

If $g(x)$ is a restoring force so that $xg(x)>0$ for $|x|>0$, and we consider the autonomous system, $\frac{dw}{dt} \equiv 0$, we can assign a definite direction at each point of the x-y or phase plane which indicates the incremental change in the solution trajectory corresponding to a positive increment of time. Then for any region Γ, we can specify at each point of its boundary, $\partial\Gamma$, whether or not trajectories are directed into Γ, or out of Γ, except for the exceptional points where the trajectories are tangent to $\partial\Gamma$.

For the two dimensional Markov process associated with the pair of stochastic differential equations above, the points of the boundary $\partial\Gamma$, of certain regions Γ can be similarly categorized. Here, of course, we must speak in terms of probability, and it is appropriate to consider

$$\lim_{t\to 0} \int_{\Gamma} p(\underline{x},t|\underline{\xi})\, d\underline{x} \tag{5.4}$$

for various points $\underline{\xi}\varepsilon\partial\Gamma$. If this limit is 1, then all stochastic trajectories which pass through the point $\underline{\xi}$ enter the region Γ in the immediate future; if this limit is 0, then no stochastic trajectory which passes through the point $\underline{\xi}$ enters the region Γ in the immediate future. The latter statement also has the interpretation that no stochastic trajectory enters the region Γ through the boundary point $\underline{\xi}$.

For the linear second order oscillator, the limit can be calculated explicitly for specified regions Γ and boundary points $\underline{\xi}$. Of particular interest are those regions Γ for which the limit always yields either 0 or 1. One finds that the only such regions are half-spaces defined either by $x\geq a$ or by $x\leq a$. We shall consider the calculation in detail for boundary points on the line $x=a$ with Γ defined by $x\geq a$.

To be specific, the linear oscillator being considered is

$$\frac{d^2x}{dt^2} + \beta\frac{dx}{dt} + x = \frac{dw}{dt} \quad ,$$

where w(t) is a Wiener process with $E[(dw(t))^2] = 2D\,dt$. The transition probability density function, being Gaussian, is characterized by the means and covariances[24].

$$m_x \equiv E[x] = \frac{y_o}{\omega_1} e^{-\beta t/2} \sin \omega_1 t + \frac{a}{\omega_1} e^{-\beta t/2} (\omega_1 \cos\omega_1 t + \frac{\beta}{2} \sin\omega_1 t) \quad ,$$

$$m_y \equiv E[y] = \frac{y_o}{\omega_1} e^{-\beta t/2} (\omega_1 \cos\omega_1 t - \frac{\beta}{2} \sin\omega_1 t) - \frac{a}{\omega_1} e^{-\beta t/2} \sin\omega_1 t \quad ,$$

$$\mathrm{var}(x) = \frac{D}{\beta} \left[1 - \frac{e^{-\beta t}}{\omega_1^2} (\omega_1^2 + \frac{1}{2} \beta^2 \sin^2\omega_1 t + \beta\omega_1 \sin\omega_1 t \cos\omega_1 t) \right] \quad ,$$

$$\mathrm{cov}(x,y) = \frac{D}{\omega_1^2} e^{-\beta t} \sin^2\omega_1 t \quad ,$$

$$\mathrm{var}(y) = \frac{D}{\beta} \left[1 - \frac{1}{\omega_1^2} e^{-\beta t} (\omega_1^2 + \frac{1}{2} \beta^2 \sin^2\omega_1 t - \beta\omega_1 \sin\omega_1 t \cos\omega_1 t) \right] \quad ,$$

where we have again introduced $y = \frac{dx}{dt}$ and have taken as initial conditions, $x(0) = a$, $y(0) = y_o$. Also, $\omega_1^2 = 1 - \beta^2/4$.

In performing the integration over the region Γ, we can first integrate over all y, leaving only a Gaussian distribution in x, which is

$$p(x,t\,|a,y_o) = \frac{1}{\sqrt{2\pi\ \mathrm{var}(x)}} \exp \left[- \frac{(x-m_x)^2}{2\ \mathrm{var}(x)} \right] .$$

[24] Wang, M.C. and Uhlenbeck, G.E. [14].

Now, integrating over x,

$$\int_a^\infty p(x,t|a,y_o)\ dx = \frac{1}{2}\ \text{erfc}\left[\frac{a-m_x}{\sqrt{2\ \text{var}(x)}}\right],$$

where the complementary error function is defined by

$$\text{erfc}\ (x) = \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-t^2}\ dt.$$

For small t,

$$a-m_x = -\ y_o t + 0(t^2)$$

$$\text{var}(x) = \frac{2}{3}\ Dt^3 + 0(t^4)\quad .$$

Then,

$$\lim_{t\to 0} \frac{a-m_x}{\sqrt{2\text{var}(x)}} = \lim_{t\to 0} \frac{-y_o t}{\sqrt{\frac{2}{3}\ Dt^3}} = \mp\infty\ ,$$

with the upper sign holding if $y_o>0$, the lower sign if $y_o<0$. It follows that

$$\lim_{t\to 0} \int_{x\geq a} p(x,y,t|a,y_o)\ dx\ dy = \begin{cases} 1, & \text{if } y_o>0, \\ \frac{1}{2}, & \text{if } y_o = 0, \\ 0, & \text{if } y_o<0. \end{cases}$$

It is clear, in fact, that if Γ is any half-space of the sort indicated, the limit above will be 0 or 1 according to whether the directions associated with the autonomous system are out of or into the region.

From these last comments and the fact that the limit ultimately depended only on the incremental mean and variance of x, it appears that we should be able to deduce similar results for any second order oscillator. For small t, the transition probability density function depends only on the incremental means and covariances and therefore is approximately Gaussian. If we once again take Γ to be a half-space defined either by $x \geq a$ or by $x \leq a$, the integration over y again leaves (approximately) a Gaussian distribution in x. We have only to determine the incremental mean and variance.

We shall again consider the region Γ defined by $x \geq a$. It is clear that once again

$$a - m_x = y_o t + O(t^2).$$

Consider now var(x) for small t for equations (5.3). Integrating each equation once,

$$x(t) = a + \int_0^t y(s)\, ds$$

$$y(t) = y_o - \int_0^t f(x,y)\, y\, ds - \int_0^t g(x)\, ds + w(t).$$

For small t, we can approximate the integrals in the second equation to obtain

$$y(t) \approx y_o - tf(a,y_o)\, y_o - tg(a) + w(t).$$

Then

$$x(t) \approx a + y_o t - \frac{t^2}{2} f(a,y_o) y_o - \frac{t^2}{2} g(a) + \int_0^t \omega(s)\, ds.$$

We can then verify that

$$E[x] = a + y_o t + O(t^2).$$

var(x) can now be calculated:

$$\begin{aligned} \text{var}(s) &= E[(x-E(x))^2] \\ &= E\left[\int_0^t \omega(s)\, ds \int_0^t \omega(\tau)\, d\tau\right] + O(t^4) . \end{aligned}$$

Using the properties of the Wiener process presented in Part I, one easily computes

$$\text{var}(x) = \frac{2Dt^3}{3} + O(t^4).$$

Thus we see that the same results hold for any second order oscillator insofar as the lowest order approximations to the mean and variance of x are concerned. And then the same conclusions concerning the limit (5.4) obtain.

We are now prepared to discuss boundary conditions for the first passage distribution. It is clear that if the limit (5.4) is zero, first passage cannot occur through the specified boundary point. Those points for which the limit is zero make up a portion of the boundary which we shall denote by $\partial\Gamma_{out}$. Then we can write

$$f(\Gamma,\underline{\sigma},s|\underline{\xi}) = 0, \qquad \underline{\sigma}\in\partial\Gamma_{out} \quad . \tag{5.5}$$

Those points for which the limit is 1, make up a portion of the boundary which we shall denote by $\partial\Gamma_{in}$. Then (5.5) allows us to reduce the integral equation to

$$p(\underline{x},t|\underline{\xi}) = \int_{\partial\Gamma_{in}} \int_{0}^{t} f(\Gamma,\underline{\sigma},s|\underline{\xi})\; p(\underline{x},t-s|\underline{\sigma})\; ds\; d\underline{\sigma} \quad .$$

To complete the boundary conditions, we note that if the initial point $\underline{\xi}$ were a point on $\partial\Gamma_{in}$, then first passage would occur immediately. That is,

$$f(\Gamma,\underline{\sigma},s|\underline{\xi}) = \delta(\underline{\sigma}-\underline{\xi})\delta(s)$$

if $\underline{\xi}\in\partial\Gamma_{in}$ and $\underline{\sigma}\in\partial\Gamma_{in}$. This boundary condition is compatible with the integral condition, since if we take $\underline{\xi}\in\partial\Gamma_{in}$ and substitute into the integral equation, this boundary condition reduces the integral equation to an identity.

It should be borne in mind that we were able to set up these boundary conditions only for the situations in which Γ is a half-space defined either by $x \geq a$ or by $x \leq a$. In any other case, the limit (5.4) is not either 0 or 1 and it is not clear what boundary conditions apply.

The formulation of the problem of the determination of times that the amplitude of an oscillator first exceeds a specified bound as a boundary value problem seems to be precluded by the fact that the region Γ in this case is not connected so that the integral equation which has served as our starting point is not valid.

The first passage problem which we can formulate by choosing $a=0$ in the discussions above is closely related to the zero crossing problem. In the following paragraphs, we shall formulate this first passage problem and show its relation to the problem of determining the distribution of the number of zero crossings for an oscillator.

We take Γ to be the half-space defined by $x \geq 0$. The density of first passage times into Γ, that is, the density of zero crossing times with positive slope assuming that the displacement is initially negative, denoted by $f(y,t|\xi,\eta)$, satisfies the backward equation

$$\frac{\partial f}{\partial t} = L^*_{\xi,\eta} f,$$

the initial condition

$$f(y,t|\xi,\eta) = 0, \quad \xi<0,$$

and the boundary condition

$$f(y,t|0,\eta) = \delta(y-\eta)\delta(t), \quad y>0.$$

In the particular case of the linear oscillator, the backward equation is

$$\frac{\partial f}{\partial t} = y\,\frac{\partial f}{\partial x} - (\beta y+x)\,\frac{\partial f}{\partial t} + D\,\frac{\partial^2 f}{\partial y^2} \quad .$$

For simplicity, we shall assume that the oscillator under consideration possesses a symmetry property so that a reflection through the origin leaves the backward equation invariant. This is the case, for example, for the linear oscillator. Then the problem of zero crossings with negative slope is the same as the problem of zero crossings with positive slope, formulated above.

Starting with a negative displacement, ξ, and arbitrary slope,

The probability that at least two zero crossings occur in the time interval (0,t) is given by

$$\int_0^\infty \int_0^\infty \int_0^t \int_0^{t-\tau_2} f(y_2,\tau_2|0,-y,)\ f(y_1,\tau_1|\xi,\eta)\ d\tau_1 d\tau_2 dy_1 dy_2 \ .$$

Note that we have taken advantage of the symmetry to express this probability completely in terms of the density of zero crossing times with positive slope. We can easily generalize this to obtain the expression for the probability of at least n zero crossings in the time interval (0,t):

$$\underbrace{\int_0^\infty \cdots \int_0^\infty}_{n\text{-fold}} \int_0^t \int_0^{t-\tau_n} \cdots \int_0^{t-\tau_n-\ldots-\tau_2} \prod_{j=2}^{n} f(y_j,\tau_j|0,-y_{j-1}) f(y_1,\tau|\xi,\eta) \prod_{j=1}^{n} d\tau_j dy_j .$$

Finally, the probability of exactly n zero crossings in the time interval (0,t) is the probability of at least n zero crossings minus the probability of at least n+1 zero crossings.

APPENDIX A

UNIQUENESS AND CONVERGENCE TO THE STEADY-STATE FOR A CLASS OF NONLINEAR OSCILLATORS

In the following paragraphs, we will show that under a proposed set of conditions on the functions g(x) and f(H) which appear in the nonlinear stochastic differential equation

$$\ddot{x} + f(H)\,\dot{x} + g(x) = \frac{dw}{dt} \quad ,$$

in which

$$H = \frac{1}{2}\,\dot{x}^2 + \int_0^x g(\eta)\,d\eta$$

and w(t) is a Wiener process with

$$E[dw(t)^2] = 2D\,dt,$$

a well-behaved solution of the corresponding Fokker-Planck equation,

$$\frac{\partial p}{\partial t} = -\,\dot{x}\,\frac{\partial p}{\partial x} + \frac{\partial}{\partial \dot{x}}\,[f(H)\,\dot{x} + g(x)]\,p + \frac{\partial^2 p}{\partial \dot{x}^2} \quad ,$$

is unique and converges, as $t \to \infty$, to the solution of the steady-state Fokker-Planck equation

$$0 = -\,\dot{x}\,\frac{\partial p_s}{\partial x} + \frac{\partial}{\partial \dot{x}}\,[f(H)\,\dot{x} + g(x)]\,p_s + D\,\frac{\partial^2 p_s}{\partial \dot{x}^2} \quad .$$

We have previously stated that the steady-state solution is

$$p_s(x,\dot{x}) = A \exp\ [-F(H)/D]\ ,$$

where

$$F(H) = \int_0^H f(\eta)\ d\eta$$

and A is chosen to normalize $p_s(x,\dot{x})$.

We proposed the following conditions:

i) df/dH is continuous and f(H) is a positive strictly increasing function.

ii) $G(x) = \int_0^x g(\eta)\ d\eta$ is a positive, strictly increasing function of $|x|$.

iii) $f^{-2}(H)\ df/dH \to 0$ as $H \to \infty$.

Uniqueness

First, we shall establish that the steady-state solution given above is unique. According to the uniqueness theorem proved in Part III, we need only establish that the solution is well-behaved as defined by the conditions (particularized to this problem)

(A-1) $$p_s(x,\dot{x}) \to 0, \quad -\dot{x}p_s\ \alpha_1 + \alpha_2(f(H)x + g(x))\ p_s - D\alpha_2 \frac{\partial p_s}{\partial \dot{x}} \to 0$$

as $|x| + |\dot{x}| \to \infty$, for any α_1, α_2 such that $\alpha_1^2 + \alpha_2^2 = 1$, and that

$$\text{(A-2)} \qquad \int_{-\infty}^{\infty}\int p_s(x,\dot{x})\, dx\, d\dot{x} = 1 \ .$$

By noting condition i), we see that the steady-state solution has finite integral so that (A-2) is satisfied. Furthermore, due to the exponential nature of $p_s(x,\dot{x})$, each term in the condition (A-1) vanishes as $|x| + |\dot{x}| \to \infty$, so that $p_s(x,\dot{x})$ is indeed well-behaved and therefore unique.

Convergence to the Steady-State

The function $q(x,\dot{x})$, defined by (3.29) in the theorem on convergence to the steady-state, is in this case

$$q(x,\dot{x}) = \frac{1}{4D^2}\,\dot{x}^2\, f^2(H) - \frac{1}{2D^2}\,\dot{x}^2\, df/dH - \frac{1}{2D}\, f(H).$$

Multiplying by $f^{-2}(H)$, we have

$$q(x,\dot{x})\ f^{-2}(H) = \frac{\dot{x}^2}{D^2}\left[\ \frac{1}{2} - \frac{1}{2}\, f^{-2}\, df/dH\ \right] - \frac{1}{2D}\, f^{-1}(H) \ .$$

From the definition of H and condition ii) on G(x) it follows that H is positive and strictly increasing with respect to $|x|$ and $|\dot{x}|$. And by i), f(H) is positive and strictly increasing so that $f^{-1}(H)$ is

bounded as $|\dot{x}|\to\infty$. Condition iii) can now be applied to see that $q(x,\dot{x})f^{-2}(H)\to\infty$ as $|\dot{x}|\to\infty$. Since $f^{-1}(H)$ is bounded as $|\dot{x}|\to\infty$, we must also have that $q(x,\dot{x})\to\infty$ as $|\dot{x}|\to\infty$.

In this case the matrix $\{b_{ij}\}$ discussed in the theorem on convergence to the steady-state is

$$\begin{bmatrix} 0 & 0 \\ 0 & 2D \end{bmatrix} ,$$

i.e., not positive definite, so that we have the further requirement that the only solution of

$$\frac{\partial u}{\partial t} = \dot{x}\,\frac{\partial u}{\partial x} , \qquad \frac{\partial u}{\partial \dot{x}} = 0 ,$$

is a constant. That this is the case can be easily seen by differentiating the first of the pair of these equations with respect to $\dot{x}$ and noting the second equation to find $0 = \frac{\partial u}{\partial x}$ and then also $\frac{\partial u}{\partial t} = 0$.

All the requirements of the theorem being met, we conclude that any well-behaved solution of the time-dependent Fokker-Planck converges, as $t\to\infty$, to the steady-state solution.

APPENDIX B

PERTURBATION EXPANSIONS OF EIGENFUNCTIONS AND EIGENVALUES ASSOCIATED WITH THE FOKKER-PLANCK EQUATION FOR FIRST ORDER STOCHASTIC DIFFERENTIAL EQUATIONS

The material presented here is a straightforward application of the perturbation procedure given in Courant-Hilbert [26, p. 343 ff.].

We shall obtain an asymptotic expansion to $0(\varepsilon^2)$ for the eigenfunctions and eigenvalues of the equation

$$\frac{d^2v^n}{d\xi^2} - 2\xi \frac{dv^n}{d\xi} + 2\mu^n v^n = \varepsilon h(\xi) \frac{dv^n}{d\xi} \tag{B-1}$$

by assuming expansions of the form

$$v^n = v_o^n + \varepsilon v_1^n + \varepsilon^2 v_2^n + \ldots$$

$$\mu^n = \mu_o^n + \varepsilon \mu_1^n + \varepsilon^2 \mu_2^n + \ldots$$

Substituting the expansions into the differential equation, collecting together terms with like powers of ε, and equating each group to zero, we obtain a sequence of equations, of which we display the first three:

$$Lv_o^n + 2\mu_o^n v_o^n = 0$$

$$Lv_1^n + 2\mu_o^n v_1^n = h(\xi) \frac{dv_o^n}{d\xi_n} - 2\mu_1^n v_o^n \tag{B-2}$$

$$Lv_2^n + 2\mu_o^n v_2^n = h(\xi) \frac{dv_1}{d\xi} - 2\mu_2^n v_o^n - 2\mu_1^n v_1^n \tag{B-3}$$

where, in each

$$Lv = \frac{d^2v}{d\xi^2} - 2\xi \frac{dv}{d\xi} \quad .$$

Introduce the inner product

$$(u,v) = \int_{-\infty}^{\infty} p_s(\xi)\, u(\xi)\, v(\xi)\, d\xi \ , \tag{B-4}$$

where $p_s(\xi) = \frac{1}{\sqrt{\pi}}\ e^{-\xi^2}$ is the solution of the steady-state Fokker-Planck equation in which $\varepsilon=0$. One easily finds that

$$(Lu,v) = (u,Lv) \ ,$$

i.e., L is self-adjoint with respect to this inner product. As we saw in Example, 1, Part IV, for the linear first order stochastic differential equation, the eigenfunctions, $\{v_o^n\}$, of equation (B-1) with $\varepsilon=0$ are the Hermite polynomials,

$$H_n(\xi) = (-1)^n\ e^{\xi^2}\ \frac{d^n e^{-\xi^2}}{d\xi^n} \ .$$

Taking

$$v_o^n(\xi) = \frac{1}{N_n} H_n(\xi), \qquad N_n = 2^{n/2}(n!)^{1/2}$$

and

$$\mu_o^n = n, \quad n = 0, 1, 2, \ldots \ ,$$

$\{v_o^n(\xi)\}$ form a complete orthonormal set with the inner product (B-4). We can then expand v_1^n and v_2^n as

$$v_1^n = \sum_{j=0}^{\infty} a_{nj} v_o^j ,$$

$$v_2^n = \sum_{j=0}^{\infty} b_{nj} v_o^j .$$

Because of the orthonormality of $\{v_o^j\}$, we have

$$a_{nj} = (v_1^n, v_o^j) ,$$

$$b_{nj} = (v_2^n, v_o^j) .$$

Multiplying equation (B-2) by $p_s v_o^\ell$ and integrating, we find

$$(v_o^\ell, Lv_1^n) + 2\mu_o^n (v_o^\ell, v_1^n) = \left(v_o^\ell, h \frac{dv_o^n}{d\xi}\right) - 2\mu_1^n (v_o^\ell, v_o^n) .$$

Noting the self-adjointness of L, we can write the lefthand side as

$$(Lv_o^\ell, v_1^n) + 2\mu_o^n (v_o^\ell, v_1^n) = 2(\mu_o^n - \mu_o^\ell) a_{n\ell} .$$

Using the fact that the Hermite polynomials satisfy the relation

$$\frac{dH}{d\xi} n = 2nH_{n-1} ,$$

we see that

$$\frac{dv_o^n}{d\xi} = \sqrt{2n}\, v_o^{n-1} .$$

We can then write the right-hand side as

$$(v_o^\ell,\ h\sqrt{2n}\ v_o^{n-1}) - 2\mu_1^n\ \delta_{n\ell} = \sqrt{2n}\ d_{\ell,n-1} - 2\mu_1^n\ \delta_{n\ell}\ ,$$

where we have defined

$$d_{n\ell} = (v_o^n,\ hv_o^\ell)\ .$$

Thus we obtain

$$(\mu_o^n - \mu_o^\ell)\ a_{n\ell} = \sqrt{\frac{n}{2}}\ d_{\ell,n-1} - \mu_1^n\ \delta_{n\ell}.$$

Setting $n=\ell$, we find

$$\mu_1^n = \sqrt{\frac{n}{2}}\ d_{n,n-1}$$

and for $n \neq \ell$,

$$a_{n\ell} = \frac{\sqrt{\frac{n}{2}}\ d_{\ell,n-1}}{\mu_o^n - \mu_o^\ell} = \frac{\sqrt{\frac{n}{2}}\ d_{\ell,n-1}}{n-\ell}\ .$$

To determine a_{nn}, we apply the normalization condition

$$\int_{-\infty}^{\infty} \bar{p}_s(\xi)\ [v^n(\xi)]^2\ d\xi = 1\ , \tag{B-5}$$

where $\bar{p}_s(\xi)$ is the steady-state solution of the perturbed equation. Expanding $\bar{p}_s(\xi)$ as

$$\overline{p}_s(\xi) = p_s(\xi)\ [1+\epsilon s_1(\xi) + \epsilon^2 s_2(\xi) + \ldots]\ ,$$

we can write (B-5) as

$$\int_{-\infty}^{\infty} p_s(\xi)\ [1+\epsilon s_1 + \epsilon^2 s_2 + \ldots]\ [v_o^n + \epsilon v_1^n + \epsilon^2 v_2^n +]^2\ d\xi = 1\ .$$

Noting that $(v_o^n,\ v_o^n) = 1$, we see that each coefficient of each power of ϵ must vanish. For the ϵ terms, this means

(B-6) $$2(v_o^n,\ v_1^n) + (s_1 v_o^n,\ v_o^n) = 0,$$

and for ϵ^2 terms

(B-7) $$(v_o^n,\ v_2^n) + (v_1^n,\ v_1^n) + 2(s_1 v_o^n,\ v_1^n) + (s_2 v_o^n,\ v_o^n) = 0.$$

Defining

$$e_{nj} = (s_1 v_o^n,\ v_o^j)$$

equation (B-6) becomes

$$a_{nn} = -\frac{1}{2}\ e_{nn}\ .$$

Having determined the first approximation, we find the second in a similar way by using equation (B-3):

$$\left(v_o^\ell,\ Lv_2^n\right) + 2\mu_o^n \left(v_o^\ell,\ v_2^n\right) = \left(v_o^\ell, h\,\frac{dv_1^n}{d\xi}\right) - 2\mu_2^n \left(v_o^\ell,\ v_o^n\right) - 2\mu_1^n \left(v_o^\ell,\ v_1^n\right) .$$

The left hand side reduces to

$$2b_{n\ell}\left(\mu_o^n - \mu_o^\ell\right) .$$

The second term on the right hand side is $-2\mu_2^n\delta_n^\ell$ and the third is $-2\mu_1^n a_{n\ell}$. Noting that

$$\frac{dv_1}{d\xi} = \sum_{j=0}^{\infty} a_{nj}\,\frac{dv_o^j}{d\xi} = \sum_{j=1}^{\infty} (2j)^{1/2}\, a_{nj} v_o^{j-1}$$

The first term on the right hand side becomes

$$\sum_{j=1}^{\infty} (2j)^{1/2}\, a_{nj}\left(v_o^\ell,\ hv_o^{j-1}\right) = \sum_{j=1}^{\infty} (2j)^{1/2}\, a_{nj} d_{\ell;j-1}$$

Thus, we have

$$b_{n\ell}\left(\mu_o^n - \mu_o^\ell\right) = \sum_{k=0}^{\infty} \left(\frac{k}{2}\right)^{1/2} a_{nk} d_{\ell,k-1} - \mu_2^n \delta_n - \mu_1^n a_{n\ell} .$$

Setting $n = \ell$, we obtain

$$\mu_2^n = \sum_{k=0}^{\infty} \left(\frac{k}{2}\right)^{1/2} a_{nk} d_{n,k-1} ,$$

and by taking $n \neq \ell$,

$$b_{n\ell} = \frac{1}{\mu_o^n - \mu_o^\ell} \left[-\mu_o^n a_{n\ell} + \sum_{k=0}^{\infty} \left(\frac{k}{2}\right)^{1/2} a_{nk} d_{\ell,k-1} \right] .$$

b_{nn} is found from the normalization condition by using equation (B-7):

$$b_{nn} = -\frac{1}{2} \sum_{j=0}^{\infty} (a_{nj})^2 - \sum_{j=0}^{\infty} e_{nj} a_{nj} - f_{nn} \quad ,$$

where we define

$$f_{nj} = (s_2 v_o^n, v_o^j) \quad .$$

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