ON THE DYNAMIC BEHAVIOR OF THIN ELASTIC PLATES.

Thesis by

Michael John O'Sullivan

In Partial Fulfillment of the Requirements

For the Degree of

Doctor of Philosophy

California Institute of Technology

Pasadena, California

1968

(Submitted March 8, 1968)

Acknowledgement

The author wishes to thank his research advisor, Dr. James K. Knowles, for the invaluable advice and assistance offered during the investigation of the problems considered in this thesis.

ABSTRACT.

Two wave propagation problems are considered: the propagation of acoustic waves in a fluid slab and the propagation of elastic waves in an elastic slab.

When formulated in terms of nondimensional variables these problems depend explicitly on two small parameters ε and δ. The parameter ε provides a measure of the thinness of the slabs considered and the parameter δ measures the impulsiveness of the applied excitation or loading. Approximation solutions of the problems considered are obtained consisting of several parts, each part having the form of a power series expansion in the parameters ε and δ.

The most important result obtained is the development of the approximate theories - the plate wave equation and the Euler-Bernoulli plate equation - directly from the full equations of dynamic elasticity using a rational perturbation expansion technique.

TABLE OF CONTENTS.

INTRODUCTION.

This thesis is concerned with the discussion of two wave propagation problems: the propagation of acoustic waves in a fluid slab and the propagation of elastic waves in an elastic slab. These two problems can be regarded as particular cases of a general class of problems which arise in the investigation of the physical behavior of a material contained in a geometrically thin region. Such problems are often described mathematically by a boundary value problem or a boundary-initial value problem for a partial differential equation. In many cases it is difficult or impossible to obtain an exact, explicit solution for such problems and therefore it is desirable to be able to find approximations to the exact solution.

Usually the most significant behavior of solutions to boundary value problems for such thin regions can be described in terms of a thickness average of some relevant physical quantity. For example, we are usually more interested in the average flow of fluid through a pipe, than the details of the velocity distribution across the pipe. Consideration of such average quantities allows the number of geometrical dimensions of a problem to be reduced. Thus the static displacement of a thin three-dimensional elastic plate can be approximately described by the average displacement of the plate considered in the corresponding two-dimensional region. Similarly the three-dimensional flow of fluid through a thin pipe can be considered approximately as a one-dimensional problem for the average flow.

Various methods have been devised for obtaining "approximate" theories which describe the average behavior of solutions for a thin region. Many of these methods involve assumptions whose validity is difficult to assess. Perhaps the most systematic formal method consists of formulating the problem in terms of a small parameter ε which measures the thinness of the region. This is followed by a formal perturbation expansion procedure in powers of ε which yields many simpler boundary value or boundary-initial value problems. The lowest order term derived by this procedure is found to be some significant thickness average satisfying an equation suitable for use as an "approximate" theory. A difficulty arises here in that the solution derived by using this perturbation scheme is often not uniformly valid; in the case of the static deflection of an elastic plate the corresponding approximation cannot satisfy all of the originally prescribed boundary conditions specified at the edges of the plate. This difficulty is to be expected as the problem formulated in terms of ε can be considered as one of a singular perturbation type.

To improve the solution predicted by the approximate theory further corrections are necessary. The derivation of these corrections also allows the rational development of boundary conditions and initial conditions suitable for use with the approximate theory.

The perturbation expansion approach discussed above has been applied to the problem of the bending and stretching of thin elastic plates by Friedrichs [5], Friedrichs and Dressler [6], Reiss [35] and Laws [21]. The deformation of a cylindrical shell was investigated using this method by Johnson and Reissner [14] and Reiss [34]. Green [7,8], Green and Naghdi [9] and Green and Laws [10] have extended the perturbation procedure to thin shells of arbitrary shape.

Fox [4] and Knowles [17] used the same procedure to investigate the potential problem for a flat plate with Dirichlet boundary conditions and Westbrook [38] extended this work to the potential problem for a cylindrical shell. For the problems considered in these three papers the authors demonstrated rigorously that the solutions obtained by using the perturbation procedure are uniformly valid.

An investigation proving the accuracy of the results obtained from an application of this perturbation procedure to a nonlinear shell problem was carried out by John [12,13].

Some estimation of the worth of the technique is inherent in the work on Saint-Venant's principle by Knowles [18,19], Horvay [11], Johnson and Little [16], Novozhilov [30] and others.

The present work is concerned with the application of this perturbation expansion procedure to dynamic problems for a "fluid plate", dealt with in chapter I, and an elastic plate, considered in chapter II. In both instances a boundary-initial value problem is considered for a thin two-dimensional rectangular region. Initial quiescence is assumed and impulsive excitations or loads are prescribed at one end of the rectangular region. In addition slowly varying excitations or loads are prescribed on the other boundaries.

When these problems are formulated in non-dimensional form two small parameters ε, δ emerge. The parameter ε measures the thinness of the region considered and δ measures the 'suddenness' of the applied impulsive disturbance. Approximate solutions are obtained consisting of several parts, each in turn a perturbation expansion in ε and δ. A more precise discussion of the problems considered is given in the introduction to the separate chapters.

I. AN APPROXIMATE SOLUTION OF THE WAVE EQUATION IN A THIN DOMAIN.

1. Introduction.

In this chapter an approximate solution of a boundary value-initial value problem for the two-dimensional wave equation in a thin rectangular domain is constructed and discussed. The wave equation mathematically describes many physical situations. Two examples, similar in certain respects to the more complex elasticity problem described in chapter II, are the small motion acoustic behavior of a fluid, and the behavior of an elastic medium sheared in such a way that the particle motion occurs only in one direction (the so-called SH wave). The former of these two cases is used as a model to allow some physical insight into the mathematical procedure that follows.

In physical terms the problem we wish to consider may be described as a waveguide problem. We consider an initially quiescent slab-shaped region of fluid suddenly excited on its boundaries, and we try to determine the subsequent motion of the whole region. For simplicity it is assumed that the prescribed excitations are such that they induce a motion which does not vary along the slab. This means that the motion is two-dimensional and we accordingly speak of a " rectangle of fluid " to connote a typical cross section of the corresponding three-dimensional region.

We shall consider a "thin" rectangle of fluid. That is, a section of a plate whose thickness is small compared with its length and width. In this dynamic case another length scale besides the physical dimensions of the plate is introduced by the time dependence of the applied boundary excitations. It is assumed that large changes in these applied excitations only occur soon after the start of the motion. The duration of these sudden effects multiplied by a characteristic velocity of the problem (the velocity of sound in the fluid) provides a quantity with the dimensions of length which is a measure of the suddenness of loading. We call this quantity the "excitation length". The relative magnitudes of the thickness of the rectangle, the length of the rectangle and this excitation length greatly influence the solution. We do not attempt to solve the problem for completely general boundary conditions, but rather specify in advance that the excitation applied to the long sides of the rectangle is slowly varying with a corresponding length comparable to the excitation length of the rectangle; and we specify that part of the excitation of the ends is slowly varying. In addition there may be a brief initial effect at the ends with an excitation length small in comparison with the length of the rectangle. This rather vague description of the problem is made more precise with the detailed formulation in the next section.

We cannot expect an approximate solution for this problem entirely analogous to those obtained for static problems (Knowles [17], Friedrichs and Dressler [6], Reiss [34], Reissner and Johnson [14] for example) since the dynamic problem for a fluid plate is concerned in an essential way with the phenomenon of propagation. The effect of boundary data specified at the ends of the rectangle will not remain localised in boundary layers near these ends. However for the particular type of boundary excitation considered here it might be expected that propagation effects are mainly concentrated in a pulse travelling along the rectangle and that this pulse could be approximated without having to obtain an exact solution of the problem. This idea is found to have considerable merit and an approximate solution is constructed consisting of three parts: an "inner" approximation satisfying boundary data on the sides of the rectangle, "quasi static" boundary layer approximations valid near each end of the rectangle and a wave front approximation describing the pulse propagation.

For this problem some assessment of the quality of the approximate solution obtained is possible by rigorous comparison with the exact solution.

2. Formulation of the Problem.

The two-dimensional equation of motion for the acoustic behavior of a fluid may be written in terms of a velocity potential Φ in the following form:

$$\Phi,_{XX} + \Phi,_{YY} - \frac{1}{c^2}\Phi,_{TT} = 0 \qquad (2.1)$$

The velocities U,V in the X,Y directions respectively, are given by the formulae:

$$U = \Phi,_X \qquad , \qquad V = \Phi,_Y \qquad . \qquad (2.2)$$

The constant c is the speed of sound in the fluid.

We consider the open rectangle R in the X,Y plane consisting of those points for which $0 < X < \ell$, $0 < Y < h$. It is required to find a function $\Phi(X,Y,T)$ satisfying the differential equation (2.1) for $(X,Y) \in R$, $T > 0$ and fulfilling the following boundary and initial conditions:

$$\left.\begin{aligned} V(X,0,T) &= P(X,T), \\ V(X,h,T) &= 0, \end{aligned}\right\} \quad 0 \leq X \leq \ell \quad , \quad 0 \leq T < \infty;$$

$$\left.\begin{aligned} U(0,Y,T) &= F(Y,T) + A(Y,T), \\ U(\ell,Y,T) &= 0, \end{aligned}\right\} \quad 0 \leq Y \leq h \quad , \quad 0 \leq T < \infty;$$

$$\left.\begin{aligned} \Phi(X,Y,0) &= 0, \\ \Phi,_T(X,Y,0) &= 0, \end{aligned}\right\} \quad (X,Y) \in R.$$

In the construction of an approximate solution for this problem it is found that the prescribed excitation at $X = 0$ produces two effects: a quasi-static boundary layer and a propagating pulse (not including the propagation of the average excitation). We have anticipated this result by writing the prescribed excitation in two parts: $F(Y,T)$ is a slowly varying function of T responsible for the quasi-static boundary layer near $X = 0$, and $A(Y,T)$ represents a sudden excitation. It is assumed that $A(Y,T)$ is of limited duration vanishing outside an interval $[0,T_o]$ say. The details of the assumed smoothness of the functions $P(X,T)$, $F(Y,T)$ and $A(Y,T)$ will be discussed when they become important in later calculations. For the present we assume that given data are such that a solution for φ does exist which gives rise to continuous velocities on the closure of R for each $T \geq 0$.

The rectangle R is thin if the parameter $\varepsilon = \frac{h}{\ell}$ is small in comparison with unity. In order to exhibit the role of ε explicitly we introduce new independent variables x,y,t defined by the change of scale:

$$x = \frac{X}{\ell} \quad , \quad y = \frac{Y}{h} \quad , \quad t = \frac{cT}{\ell} \quad . \tag{2.3}$$

Nondimensional dependent variables ϕ, u v are also introduced by the following definitions:

$$\begin{aligned} \phi(x,y,t;\varepsilon,\delta) &= \frac{1}{h^2}\,\varphi(\ell x, hy, \frac{\ell}{c}t) \quad , \\ u(x,y,t;\varepsilon,\delta) &= \frac{1}{h}\,U(\ell x, hy, \frac{\ell}{c}t) \quad , \\ v(x,y,t;\varepsilon,\delta) &= \frac{1}{h}\,V(\ell x, hy, \frac{\ell}{c}t) \quad . \end{aligned} \tag{2.4}$$

The parameter δ, small compared to unity, is defined by the relation $\delta = \frac{cT_o}{\ell}$, where T_o is the length of time for which the given excitation at x = 0 has a large variation (the time derivatives of the prescribed excitation are not O(1) in comparison with ε for $0 \leq T \leq T_o$).

We are interested in the particular case where the applied excitations depend on ε in such a way that the limit problem as $\varepsilon \to 0$ will produce a displacement in the x direction, u, which is O(1). Accordingly we introduce new functions $p^{(1)}$, $f^{(0)}$ and $a^{(0)}$ through the definitions:

$$\begin{aligned} \varepsilon\, p^{(1)}(x,t) &= \frac{1}{h}\,P(\ell x, \frac{\ell}{c}t) \quad , \\ f^{(0)}(y,t) &= \frac{1}{h}\,F(hy, \frac{\ell}{c}t) \quad , \\ a^{(0)}(y, \frac{t}{\delta}) &= \frac{1}{h}\,A(hy, \frac{\ell}{c}t) \quad . \end{aligned} \tag{2.5}$$

More general applied excitations, say

$$\frac{1}{h}P(\ell x, \frac{\ell}{c} t) = p(x,t;\varepsilon) = \varepsilon p^{(1)}(x,t) + \varepsilon^2 p^{(2)}(x,t) + \dots ,$$

require more cumbersome calculations with no fundamental change of method and will not be considered here.

Using this new notation the problem for ϕ may be written in the following final form:

$$\phi,_{yy} + \varepsilon^2(\phi,_{xx} - \phi,_{tt}) = 0 \ , \ 0 < x, \ y < 1, \ 0 < t < \infty. \quad (2.6)$$

$$\left.\begin{aligned} \phi,_y(x,0,t;\varepsilon,\delta) &= \varepsilon p^{(1)}(x,t) \ , \\ \phi,_y(x,1,t;\varepsilon,\delta) &= 0 \ , \end{aligned}\right\} \ 0 \leq x \leq 1, \ 0 \leq t < \infty. \quad (2.7a)$$

$$\left.\begin{aligned} \phi,_x(0,y,t;\varepsilon,\delta) &= \frac{1}{\varepsilon}f^{(0)}(y,t) + \frac{1}{\varepsilon}\, a^{(0)}(y, \frac{t}{\delta}) \ , \\ \phi,_x(1,y,t;\varepsilon,\delta) &= 0 \ , \end{aligned}\right\} 0 \leq y \leq 1, \ 0 \leq t < \infty. \quad (2.7b)$$

$$\left.\begin{aligned} \phi(x,y,0) &= 0 \ , \\ \phi,_t(x,y,0) &= 0 \ , \end{aligned}\right\} \ 0 < x \ , \quad y < 1 \ . \quad (2.7c)$$

Once the potential ϕ is known, the velocity components follow from the relations:

$$u = \varepsilon\phi,_x \quad , \quad v = \phi,_y \quad . \quad (2.8)$$

This problem may be solved exactly by elementary means. However, since we are only interested in the case where ε and δ are small, we shall attempt to obtain an approximate solution for ϕ, as an asymptotic expansion in ε and δ, directly from the differential equation, boundary conditions and initial conditions without first finding the exact solution. While this scheme is of interest for its own sake, it also provides a pilot problem to suggest procedures in more complicated cases where an exact solution may be difficult or impossible to obtain. Such is the case, for example, in the elasticity problem described in chapter II.

3. Formal Inner Approximation.

The most obvious first approach to the problem (2.6), (2.7) is to investigate approximations having the form of power series in ε. We wish to find an approximation for the velocity u which is $O(1)$ in the limiting case $\varepsilon \to 0$. Consequently we assume a formal expansion

$$u_i(x,y,t;\varepsilon,\delta) = u_i^{(0)}(x,y,t;\delta) + \varepsilon u_i^{(1)}(x,y,t;\delta) + \dots \quad (3.1)$$

The subscript "i" has been added as an abbreviation for "inner" as a reminder that we expect this expansion to only partially fulfill the requirements of the problem. We see from equation (2.8) that the expansions for ϕ_i and v_i associated with (3.1) are:

$$\phi_i(x,y,t;\varepsilon,\delta) = \frac{1}{\varepsilon}\,\phi_i^{(-1)}(x,y,t;\delta) + \phi_i^{(0)}(x,y,t;\delta) + \dots \quad (3.2)$$

$$v_i(x,y,t;\varepsilon,\delta) = \frac{1}{\varepsilon}\,v_i^{(-1)}(x,y,t;\delta) + v_i^{(0)}(x,y,t;\delta) + \dots \quad (3.3)$$

We now substitute the expansion (3.2) into the problem (2.6), (2.7) and consider the equations associated with each particular power of ε as individual problems. We first obtain the following equation for $\phi_i^{(-1)}$:

$$\phi_{i,yy}^{(-1)} = 0 \quad , \quad 0 < x \; , \; y < 1 \; , \; 0 < t < \infty. \quad (3.4)$$

Solutions of this equation certainly cannot, in general, fulfill given conditions at $x = 0,1$. However, the boundary conditions arising from

(2.7a) merely require:

$$\left.\begin{aligned}\phi_{i,y}^{(-1)}(x,0,t;\delta) &= 0\\ \phi_{i,y}^{(-1)}(x,1,t;\delta) &= 0\end{aligned}\right\} \quad 0 \le x \le 1 \quad , \quad 0 \le t < \infty; \tag{3.5}$$

These may be fulfilled by a solution of (3.4) which is a function of x,t only. We use the notation:

$$\phi_i^{(-1)}(x,y,t;\delta) = \bar{\phi}_i^{(-1)}(x,t;\delta) \quad . \tag{3.6}$$

The displacements derived from this potential are:

$$u_i^{(0)} = \bar{\phi}_{i,x}^{(-1)} \quad , \quad v_i^{(-1)} = 0 \quad . \tag{3.7}$$

The equations (3.4) and (3.5) alone do not constitute a well set problem for $\phi_i^{(-1)}$. Having determined the y-dependence of $\phi_i^{(-1)}$, it is not yet obvious what further conditions to impose on $\bar{\phi}_i^{(-1)}$. Equations, similar to (3.4) and (3.5), determine the y-dependence of $\phi_i^{(0)}$, $\phi_i^{(1)}$, ... , and also supply more information about $\phi_i^{(-1)}$.

$\phi_i^{(0)}$ satisfies equations identical to those for $\phi_i^{(-1)}$ in (3.4) and (3.5). Therefore we conclude that:

$$\phi_i^{(0)}(x,y,t;\delta) = \bar{\phi}_i^{(0)}(x,t;\delta) \quad ; \tag{3.8}$$

where at present $\bar{\phi}_i^{(0)}$ is an arbitrary function. The associated displacements are:

$$u_i^{(1)} = \bar{\phi}_{i,x}^{(0)} \quad , \quad v_i^{(0)} = 0 \quad . \tag{3.9}$$

In the first order system arising from (2.7) and (2.8a) there is some interaction between $\phi_i^{(1)}$ and $\phi_i^{(-1)}$; in fact $\phi_i^{(1)}$ satisfies the following differential equation and boundary conditions:

$$\phi^{(1)}_{i,yy} + \phi^{(1)}_{i,xx} - \phi^{(-1)}_{i,tt} = 0 \ , \qquad 0 < x,y < 1 \ , \ 0 < t < \infty \ , \qquad (3.10)$$

$$\left.\begin{aligned} \phi^{(1)}_{i,y}(x,0,t;\delta) &= p^{(1)}(x,t) \\ \phi^{(1)}_{i,y}(x,1,t;\delta) &= 0 \end{aligned}\right\} \qquad 0 \leq x \leq 1 \ , \ 0 < t < \infty . \qquad (3.11)$$

Equation (3.10) and (3.11) can be used to determine the y-dependence of $\phi^{(1)}_i$ and to obtain a condition on $\phi^{(-1)}_i$ as follows:

$$\phi^{(1)}_i(x,y,t;\delta) = (-\frac{1}{3} + y - \frac{y^2}{2})p^{(1)}(x,t) + \overline{\phi}^{(1)}_i(x,t;\delta) \ . \qquad (3.12)$$

The term $-\frac{1}{3}p^{(1)}(x,t)$ is not immediately suggested by equations (3.10) and (3.11), but is added to the arbitrary function $\overline{\phi}^{(1)}_i(x,t)$ to arrange that $\int_0^1 \phi^{(1)}(x,y,t;\delta)dy = \overline{\phi}^{(1)}(x,t;\delta)$. In writing equation (3.12) we have used the condition:

$$\overline{\phi}^{(-1)}_{i,xx} - \overline{\phi}^{(-1)}_{i,tt} = p^{(1)} \qquad , \qquad (3.13)$$

which arises from the fulfillment of the boundary conditions for $\phi^{(1)}_i$ at $y = 0$. Condition (3.13) still does not fully determine $\overline{\phi}^{(-1)}_i$. Boundary conditions at $x = 0,1$ and initial conditions at $t = 0$ are required. From the prescribed initial quiescence condition it is natural to require:

$$\overline{\phi}^{(-1)}_i(x,y,0;\delta) = 0 \ , \qquad \overline{\phi}^{(-1)}_{i,t}(x,y,0;\delta) = 0 \qquad . \qquad (3.14)$$

These conditions may be derived more rationally from investigations of an initial approximation where they are requirements to ensure boundedness of the approximation. Since we are not concerned with non zero initial values and the resulting vibratory type of initial approximation we are obliged to assume conditions (3.14). The question of boundary conditions at $x = 0,1$ will be considered later. Displacements $u^{(1)}_i, v^{(1)}_i$

are associated with the expression for $\phi_i^{(1)}$ obtained in (3.12):

$$u_i^{(1)}(x,y,t;\delta) = \left(-\frac{1}{3} + y - \frac{y^2}{2}\right)p^{(1)}_{,x}(x,t) + \bar{\phi}^{(1)}_{i,x}(x,t;\delta) \quad ,$$
$$v_i^{(1)}(x,y,t;\delta) = (1-y)p^{(1)}(x,t) \quad . \tag{3.15}$$

Proceeding through successive orders we can evaluate $\phi_i^{(-1)}, \phi_i^{(0)}, \phi_i^{(1)}$, in terms of the prescribed boundary excitation $p^{(1)}(x,t)$ and a sequence of functions of x,t only, $\bar{\phi}_i^{(-1)}, \bar{\phi}_i^{(0)}, \bar{\phi}_i^{(1)}, \ldots$, satisfying differential equations similar to (3.13).

Rather than carry out this program in detail, we shall employ a different procedure to obtain the higher order terms more quickly. The alternate procedure is a symbolic one similar to that used by Knowles [17] for a second order elliptic boundary value problem. Introducing the operator $L^2 = \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial t^2}$ we formally consider the equation (2.6) as an ordinary differential equation:

$$\phi_{,yy} + \varepsilon^2 L^2 \phi = 0 \quad , \tag{3.16}$$

with the boundary conditions:

$$\phi_{,y} = \varepsilon p^{(1)} \quad , \qquad y = 0 \quad ,$$
$$\phi_{,y} = 0 \quad , \qquad y = 1 \quad . \tag{3.17}$$

This has the 'symbolic' solution:

$$\phi = \frac{\cos(1-y)\varepsilon L}{L \sin \varepsilon L}\, p^{(1)} \quad . \tag{3.18}$$

The formal expansion for ϕ in powers of ε is obtained by using the Taylor series expansion (cf. Abramowitz and Stegun [1], p.804):

$$\frac{\cos(1-y)z}{z \sin z} = \sum_{n=o}^{\infty} B_{2n}\left(\frac{y}{2}\right) \frac{2^{2n}}{2n!} (-1)^n z^{2n-2} \quad ,$$

where $B_k(r)$ is the Bernoulli polynomial of degree k. Substituting

$z^2 = \varepsilon^2(\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial t^2})$ we obtain the expansion for ϕ:

$$\phi = \sum_{n=o}^{\infty} B_{2n}(\frac{y}{2}) \frac{2^{2n}}{2n!} (-1)^n \varepsilon^{2n-1} (\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial t^2})^{n-1} p^{(1)}(x,t) \quad . \tag{3.19}$$

We interpret the term $(\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial t^2})^{-1} p^{(1)}(x,t)$ in the following way:

$$(\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial t^2})^{-1} p^{(1)}(x,t) = \sum_{n=o}^{\infty} \varepsilon^n \bar{\phi}^{(n-1)}(x,t) \quad ,$$

where $\bar{\phi}^{(-1)}$ satisfies the differential equation $\bar{\phi}^{(-1)}_{,xx} - \bar{\phi}^{(-1)}_{,tt} = p^{(1)}$,

and $\bar{\phi}^{(j)}$ satisfies the differential equation $\bar{\phi}^{(j)}_{,xx} - \bar{\phi}^{(j)}_{,tt} = 0$, $j \geq 0$.

Then we can rewrite the expansion (3.19):

$$\phi = \sum_{n=o}^{\infty} B_{2n}(\frac{y}{2})\frac{2^{2n}}{2n!}(-1)^n \varepsilon^{2n-1}(\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial t^2})^{n-1} p^{(1)} + \sum_{n=o}^{\infty} \varepsilon^{n-1}\bar{\phi}^{(n-1)} \quad . \tag{3.20}$$

The validity of expansion (3.20) of course depends on the differentiability properties of $p^{(1)}$. For example, if $p^{(1)}(x,t)$ is a polynomial in x,t, expansion (3.20) becomes a finite sum. It is easily verified that the expansion (3.20) formally satisfies the differential equation (2.6) and the prescribed boundary conditions at y = 0,1 (2.7a). Also the first few terms of expansion (3.20) agree with the terms of the inner approximation obtained in equations (3.6), (3.8) and (3.12) (with the subscript "i" added in (3.20)). With further calculations we could show this agreement to be valid in general.

Often all that is required for the inner approximation is the first few terms of (3.20). We assume that the derivatives $(\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial t^2})^n p^{(1)}$ for $1 \leq n \leq M_1$ all exist for $0 \leq x \leq 1$, $0 \leq t < \infty$ and are $O(1)$ compared to ε. We write:

$$\phi_i \sim \sum_{n=o}^{M_1} B_{2n}(\frac{y}{2}) \frac{2^{2n}}{2n!} (-1)^n \varepsilon^{2n-1}(\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial t^2})^{n-1} p^{(1)}(x,t) + \sum_{n=0}^{2M_1} \varepsilon^{n-1}\phi_i^{(n-1)}. \tag{3.21}$$

The rigorous interpretation of the relationship connoted by (3.21) will be investigated later.

Let us reexamine equation (3.17), which may be written in the form:

$$(L \sin \varepsilon L)\phi = (\cos(1-y)\varepsilon L)p^{(1)}(x,t) \tag{3.22}$$

This may be considered as an inhomogeneous, infinite order, partial differential equation for ϕ. Then expansion (3.20) is a particular integral of this equation to which we must add a complementary function satisfying the equation $(L \sin \varepsilon L)\ \phi = 0$ and fulfilling the additional constraint that $\phi_{,y} = 0$ at $y = 0,1$. Together, the particular integral and complementary function must satisfy initial conditions and boundary conditions at $x = 0,1$. Since $z \sin z$ has the infinite product expansion:

$$z \sin z = \prod_{n=o}^{\infty} (z^2 - n^2\pi^2) \quad ,$$

we may formally assert that the complementary function satisfies the differential equation:

$$\prod_{n=o}^{\infty} [\varepsilon^2(\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial t^2}) - n^2\Pi^2]\phi = 0 \quad .$$

The significance of this equation in relation to the boundary layer and wave front approximations will become clear in subsequent sections. The inner approximation ϕ_i, written in (3.21), is an approximation to a particular integral of (3.22) and it is found that the boundary layer and wave front approximations together form an approximation to the complementary function of (3.22).

To assess ϕ_i as an approximate solution we substitute it into the equations (2.6), (2.7). We find that ϕ_i satisfies the specified conditions at y = 0,1 exactly and satisfies the differential equation with an error which is $O(\varepsilon^{2M_1+1})$. However, in general ϕ_i is a poor approximation to the boundary conditions at x = 0,1.

The smoothness conditions assumed on $p^{(1)}(x,t)$ are that the derivatives $(\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial t^2})$ p for $1 \leq n \leq M_1$ all exist and are O(1) for $0 \leq x \leq 1$, $0 \leq t < \infty$. This means that all the derivatives of order $\leq M_1-1$ are zero at t = 0. Thus the only contribution to the initial values $\phi_i(x,y,0;\varepsilon,\delta)$ and $\phi_{i,t}(x,y,0;\varepsilon,\delta)$ arises from the functions $\bar{\phi}_i^{(n)}$. We impose the initial conditions (similar to conditions (3.14)):

$$\bar{\phi}_i^{(n)}(x,0;\delta) = 0 \quad , \quad \bar{\phi}_{i,t}^{(n)}(x,0;\delta) = 0 \quad . \tag{3.23}$$

We cannot fully determine $\bar{\phi}_i^{(n)}$ yet as we do not know what conditions to prescribe at x = 0,1. This difficulty will be resolved in the next section.

4. Formal Boundary Layer Approximation.

In the previous section we observed that in general the inner approximation ϕ_i does not satisfy the prescribed boundary conditions at x =0,1. To explore a possible boundary layer correction near x = 0 we introduce a boundary layer variable ζ defined by the scaling:

$$\zeta = \frac{x}{\varepsilon} \quad , \tag{4.1}$$

and we write:

$$\phi(x,y,t;\varepsilon,\delta) = \phi_i(x,y,t;\varepsilon,\delta) + \phi_\ell(\zeta,y,t;\varepsilon,\delta) \quad , \tag{4.2}$$

where ϕ_ℓ represents a tentative boundary layer correction near x = 0. Substituting this expression for ϕ into the original differential equation, and using the property of ϕ_i written in equation (3.23), we obtain the following equation for ϕ_ℓ:

$$\phi_{\ell,\zeta\zeta} + \phi_{\ell,yy} - \varepsilon^2\phi_{\ell,tt} = \psi \qquad , \tag{4.3}$$

where from equation (3.23) we can show that ψ is $O(\varepsilon^{2M_1+1})$. Since we are only interested here in the boundary layer near x = 0 and since we shall require these corrections to tend to zero as ζ increases, it is natural to replace the domain $0 < \zeta < \frac{1}{\varepsilon}$, $0 < y < 1$, $0 < t < \infty$ of the differential equation (4.3) by the region $0 < \zeta < \infty$, $0 < y < 1$, $0 < t < \infty$.

We also require that $\phi_i + \phi_\ell$ fulfills the boundary conditions except at x = 1. This places the following boundary conditions on ϕ_ℓ:

$$\left.\begin{aligned} &\phi_{\ell,y}(\zeta,0,t;\varepsilon,\delta) = 0 \quad ,\\ &\phi_{\ell,y}(\zeta,1,t;\varepsilon,\delta) = 0 \quad ,\end{aligned}\right\} \qquad 0 \le \zeta < \infty,\ 0 \le t < \infty , \tag{4.4a}$$

$$\phi_{\ell,\zeta}(0,y,t;\varepsilon,\delta) = f^{(0)}(y,t) + a^{(0)}(y,t/\delta) - \varepsilon\phi_{i,x}(0,y,t;\varepsilon,\delta). \tag{4.4b}$$

In addition we impose a decay condition:

$$\phi_{\ell,\zeta}(\zeta,y,t;\varepsilon,\delta) \to 0 \qquad \text{as} \qquad \zeta \to \infty \tag{4.4c}$$

We assume an expansion for ϕ_ℓ:

$$\phi_\ell(\zeta,y,t;\varepsilon,\delta) = \phi_\ell^{(0)}(\zeta,y,t;\delta) + \varepsilon\phi_\ell^{(1)}(\zeta,y,t;\delta) + \dots \tag{4.5}$$

so that, following standard procedure, $\phi_\ell^{(0)}$ satisfies the problem below:

$$\phi^{(0)}_{\ell,\zeta\zeta} + \phi^{(0)}_{\ell,yy} = 0 \ ,\ 0 < \zeta < \infty,\ 0 < y < 1,\ 0 < t < \infty, \tag{4.6a}$$

$$\left.\begin{aligned} &\phi^{(0)}_{\ell,y}(\zeta,0,t;\delta) = 0 \ ,\\ &\phi^{(0)}_{\ell,y}(\zeta,1,t;\delta) = 0 \ ,\end{aligned}\right\} \qquad 0 \le \zeta < \infty,\ 0 \le t < \infty, \tag{4.6b}$$

$$\phi^{(0)}_{\ell,\zeta}(0,y,t;\delta) = f^{(0)}(y,t) + a^{(0)}(y,t/\delta) - \bar{\phi}^{(-1)}_{i,x}(0,t;\delta) \qquad (4.6c)$$

$$\phi^{(0)}_{\ell,\zeta} \to 0. \qquad \text{as} \quad \zeta \to \infty \qquad (4.6d)$$

Here it should be remembered that $\bar{\phi}^{(-1)}_{i}(x,t;\delta)$ is still not determined. Without actually solving the system (4.6) we can derive a necessary condition for the existence of a solution with the right properties. This condition provides a boundary condition for $\bar{\phi}^{(-1)}_{i}$ at $x = 0$.

We integrate the differential equation (4.6a) over its domain:

$$0 = \int_0^1\int_0^\infty (\phi^{(0)}_{\ell,yy} + \phi^{(0)}_{\ell,\zeta\zeta})d\zeta dy \ ,$$

$$= -\int_0^1 \phi^{(0)}_{\ell,\zeta}(0,y,t;\delta)dy + \int_0^1 \phi^{(0)}_{\ell,\zeta}(\infty,y,t;\delta)dy + \int_0^\infty \phi^{(0)}_{\ell,y}(\zeta,1,t;\delta)d\zeta - \int_0^\infty \phi^{(0)}_{\ell,y}(\zeta,0,t;\delta)d\zeta \ .$$

Substituting from the boundary conditions (4.6b), (4.6c) and the decay condition (4.6d) we obtain the following condition:

$$\int_0^1 (f^{(0)}(y,t) + a^{(0)}(y,t/\delta) - \bar{\phi}^{(-1)}_{i,x}(0,t;\delta))dy = 0 \ .$$

That is:

$$\bar{\phi}^{(-1)}_{i,x}(0,t;\delta) = \int_0^1 (f^{(0)}(y,t) + a^{(0)}(y,t/\delta))dy \ . \qquad (4.7)$$

This equation provides us with a boundary condition for $\bar{\phi}^{(-1)}_{i}(x,t;\delta)$ at $x = 0$. A similar condition at $x = 1$ is derived by consideration of the boundary layer there.

We can obtain a solution for $\phi_\ell^{(0)}$ having the form:

$$\phi_\ell^{(0)} = -\sum_{n=1}^{\infty} (f_n^{(0)} + a_n^{(0)}) \frac{e^{-n\pi\zeta}}{n\pi} \cos n\pi y \quad , \tag{4.8}$$

where

$$a_n^{(0)}(\tfrac{t}{\delta}) = \int_0^1 a^{(0)}(y,t/\delta) \cos n\pi y \, dy \quad ,$$

and

$$f_n^{(0)}(t) = \int_0^1 f^{(0)}(y,t) \cos n\pi y \, dy \quad .$$

Note that the condition (4.7) ensures that there is no term in the series solution (4.8) which is independent of y and hence non decaying as $\zeta \to \infty$. The variable t only occurs in the solution $\phi_\ell^{(0)}$ as a parameter and hence the behavior of $\phi_\ell^{(0)}$ with respect to t will be similar to the behavior of $f^{(0)}$ and $a^{(0)}$ with respect to t.

At the next order we find that $\phi_\ell^{(1)}$ satisfies equations which are the same as (4.6a), (4.6b) and (4.6d). However at $\zeta = 0$ we now have

$$\phi_{\ell,\zeta}^{(1)}(0,y,t;\delta) = -\overline{\phi}_{i,x}^{(1)}(0,t;\delta) \quad .$$

Proceeding as above we obtain the condition:

$$\overline{\phi}_{i,x}^{(1)}(0,t;\delta) = 0 \tag{4.9}$$

and the solution for $\phi_\ell^{(1)}$:

$$\phi_\ell^{(1)} = 0 \quad . \tag{4.10}$$

At the second order some interaction with $\phi_\ell^{(0)}$ appears. $\phi_\ell^{(2)}$ is a solution of the following problem:

$$\phi_{\ell,\zeta\zeta}^{(2)} + \phi_{\ell,yy}^{(2)} - \phi_{\ell,tt}^{(0)} = 0 \quad , \quad 0 < \zeta < \infty, \ 0 < y < 1, \tag{4.11a}$$

$$\phi^{(2)}_{\ell,y}(\zeta,0,t;\delta) = 0$$

$$0 \le \zeta < \infty,\ 0 \le t < \infty \quad , \qquad (4.11b)$$

$$\phi^{(2)}_{\ell,y}(\zeta,1,t;\delta) = 0$$

$$\phi^{(2)}_{\ell,y}(0,y,t;\delta) = -(-\frac{1}{3} + y - \frac{y^2}{2})p^{(1)}_{,x}(0,t) - \phi^{(1)}_{i,x}(0,t;\delta) \qquad (4.11c)$$

$$\phi^{(2)}_{\ell,\zeta} \to 0 \qquad \text{as} \qquad \zeta \to \infty \qquad (4.11d)$$

Writing the differential equation (4.11a) assumes that $\phi^{(0)}_{\ell,tt}$ is $O(1)$. Therefore we cannot include the function $a^{(0)}$ in the boundary layer approximation; $a^{(0)}$ is $O(1)$ whereas $a^{(0)}_{,tt}$ is $O(\frac{1}{\delta^2})$. This is why the prescribed boundary condition at $x = 0$ is written as the sum of $f^{(0)}(y,t)$ and $a^{(0)}(y,t/\delta)$. It is found that a boundary layer approximation near $x = 0$ which includes terms $\phi^{(0)}_{\ell},\ \phi^{(1)}_{\ell}, \ldots ,\ \phi^{(2M_2)}_{\ell}$ can only accomodate a function $f^{(0)}(y,t)$ for which the derivatives $\frac{\partial^m f^{(0)}}{\partial t^m}$, $0 \le m \le 2M_2$, exist and are $O(1)$ in comparison with ε, for $0 \le y \le 1$ and $0 \le t < \infty$.

We rewrite the solution for $\phi^{(0)}_{\ell}$:

$$\phi^{(0)}_{\ell}(\zeta,y,t) = -\sum_{n=1}^{\infty} f^{(0)}_{n}(t)\,\frac{e^{-n\pi\zeta}}{n}\cos n\pi y \qquad (4.12)$$

From the equations (4.11) we obtain the condition:

$$\overline{\phi}^{(1)}_{i,x}(0,t;\delta) = -\int_0^1 (-\frac{1}{3} + y - \frac{y^2}{2})p^{(1)}_{,x}(0,t)dy = 0 \qquad (4.13)$$

and the solution for $\phi^{(2)}_{\ell}$:

$$\phi^{(2)}_{\ell}(\zeta,y,t) = \sum_{n=1}^{\infty} (f^{(0)}_{n,tt}(1+n\pi\zeta) - 4p^{(1)}_{,x}(0,t))\,\frac{e^{-n\pi\zeta}}{2n^3\pi^3}\cos n\Pi y \ . \qquad (4.14)$$

More complicated higher order boundary layer corrections may be found by proceeding systematically in this manner. The following condition is obtained:

$$\overline{\phi}_{i,x}^{(n)}(0,t;\delta) = 0 \quad , \quad n \geq 0 \tag{4.15}$$

providing a boundary condition at x = 0 for successive orders $\overline{\phi}_i^{(n)}$.

$$\phi_\ell = \sum_{n=0}^{2M_2} \varepsilon^n \phi_\ell^{(n)}(\frac{x}{\varepsilon}, y, t) \quad . \tag{4.16}$$

A similar correction say $\phi_{\ell'}$, where

$$\phi_{\ell'} = \sum_{n=0}^{2M_3} \varepsilon^n \phi_{\ell'}^{(n)}(\frac{1-x}{\varepsilon}, y, t) \quad , \tag{4.17}$$

can be developed to approximate the solution near the end x = 1. This process will also determine boundary conditions for $\overline{\phi}_i^{(-1)}$, $\overline{\phi}_i^{(0)}$,..., $\overline{\phi}_i^{(2M_1)}$ at x = 1. These together with boundary conditions (4.14), (4.13), (4.9) and (4.7), plus the initial conditions (3.25) determine solutions of the differential equations for $\overline{\phi}_i^{(-1)}$, $\overline{\phi}_i^{(0)}$,..., $\overline{\phi}_i^{(2M_1)}$. In fact $\overline{\phi}_i^{(0)} \equiv \overline{\phi}_i^{(1)} \equiv \ldots \equiv \overline{\phi}_i^{(2M_1)} \equiv 0$. In the next section the initial values of boundary layer approximations will be significant. With this knowledge we make a further assumption that the prescribed data at x = 0 may be divided into $f^{(0)}(y,t)$ and $a^{(0)}(y,t/\delta)$ so that $a^{(0)}(y,t/\delta)$ includes all the initial effects. Thus $\frac{\partial^m f^{(0)}}{\partial t^m}(y,0) = 0$, $0 \leq m \leq 2M$ and this ensures that the boundary layer approximations contribute nothing to the initial values at the orders considered.

We now consider the convergence of the series representation of the solutions for $\phi_\ell^{(0)}$, $\phi_\ell^{(2)}$ in equations (4.12), (4.14). From these two orders it is obvious that progressively higher and higher powers of $\frac{1}{n\pi}$ are introduced into the Fourier coefficients of the higher order boundary layer corrections, say $\phi_\ell^{(k)}$. This means that we need not prescribe very restrictive smoothness conditions on $f^{(0)}(y,t)$ for $0 \leq y \leq 1$. We assume that $f^{(0)}(y,t)$ has a continuous y-derivative for $0 \leq y \leq 1$ and $0 \leq t < \infty$. We assume that the series

$$\sum_{n=1}^{\infty} \frac{\partial^{2M_2} f_n^{(0)}(t)}{\partial t^{2M_2}} \cos n\pi y$$

converges uniformly to the sum $\dfrac{\partial^{2M_2} f\quad(y,t)}{\partial t^{2M_2}} - \displaystyle\int_0^1 \frac{\partial^{2M_2} f\quad(y,t)}{\partial t^{2M_2}}\, dy$.

It should be emphasized that in this section we have obtained boundary conditions for $\phi_i^{(n)}$ and demonstrated the inability of the boundary layer to handle prescribed data like $a^{(0)}(y,t/\delta)$ without finding the boundary layer approximation explicitly.

5. Wave Front Approximation.

The composite approximation $\phi_i + \phi_\ell + \phi_{\ell'}$ satisfies the differential equation (2.6), the boundary conditions (2.7a), the initial conditions (2.7c) and part of the boundary conditions (2.7b), all with an error which is $O(\varepsilon^{2M_1})$. The remaining part of the prescribed boundary conditions at $x = 0$ is the rapidly varying portion $a^{(0)}(y,t/\delta)$; this is expected to produce a pulse propagating along the rectangle. It is reasonable to investigate scalings of t and x which make the wave operator $\left(\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial t^2}\right)$ the most important part of the equation (2.6). This procedure produces approximations to the wave front behavior which are not uniformly valid. To obtain an approximation having wider application we introduce the following variables suggested by the boundary condition at $x = 0$ and the differential equation:

$$\zeta = \frac{t-x}{\delta} \qquad\qquad \eta = \frac{\delta}{\varepsilon^2}(t+x) \quad . \tag{5.1}$$

We write an approximation for ϕ:

$$\phi(x,y,t;\varepsilon,\delta) = \phi_i(x,y,t;\varepsilon,\delta)+\phi_\ell(\frac{x}{\varepsilon},y,t;\varepsilon)+\phi_{\ell'}(\frac{1-x}{\varepsilon},y,t;\varepsilon)+\phi_f(\eta,y,\xi;\varepsilon,\delta) \;, \tag{5.2}$$

where ϕ_f represents the pulse approximation. Substituting this form for ϕ into the original system (2.7) and (2.8) we find that ϕ_f satisfies the following problem:

$$\phi_{f,yy} - 4\phi_{f,\xi\eta} = O(\varepsilon^{2M_1}) \qquad , \tag{5.3a}$$

for $\frac{\delta^2}{\varepsilon^2}\xi < \eta < \infty$, $0 < y < 1$, $0 < \xi < \infty$;

$$\begin{aligned} \phi_{f,y}(\eta,0,\xi;\varepsilon,\delta) &= 0 \qquad , \\ \phi_{f,y}(\eta,1,\xi;\varepsilon,\delta) &= 0 \qquad , \end{aligned} \tag{5.3b}$$

for $\frac{\delta^2}{\varepsilon^2}\xi \leq \eta < \infty$, $0 \leq \xi < \infty$;

$$\phi_{f,\xi} - \frac{\delta^2}{\varepsilon^2}\phi_{f,\eta} = -\frac{\delta}{\varepsilon}a^{(0)}(y,\xi) + O(\delta\varepsilon^{M_2}) \qquad , \tag{5.3c}$$

$\eta = \frac{\delta^2}{\varepsilon^2}\xi$, $0 \leq y \leq 1$, $0 \leq \xi < \infty$. We know that no disturbance can travel faster than the acoustic velocity; thus as the fluid is intially at rest ϕ_f is zero for $\xi < 0$. Also we require that ϕ_f be continuous at $\xi = 0$. That is:

$$\phi_f(\eta,y,0;\varepsilon,\delta) = 0 \qquad , \qquad 0 \leq \eta < \infty \qquad , 0 \leq y \leq 1 \;. \tag{5.4}$$

We do not attempt to enforce a boundary condition at $x = 1$. The effect of the solution we shall obtain for ϕ_f at the end $x = 1$ will be used as a boundary condition for another wave front approximation.

The most important aspect of the development that follows is our assumption that the ratio $\frac{\delta^2}{\varepsilon^2}$ is small. Then at the lowest order we can approximate the domain described in equation (5.3a) by the region $0 < y < 1$, $0 < \xi < \infty$, $0 < \eta < \infty$. And we apply the boundary condition (5.3c) at $\zeta = 0$.

This places a further restriction on the prescribed end conditions. The problem for ϕ_f as written in (5.3) and (5.4) does not depend on δ and ε separately but only their ratio. We introduce a new parameter $\nu = \frac{\delta}{\varepsilon}$ and we shall consider only the case where ν is small.

Using this new parameter ν, we can write the problem for ϕ_f as follows:

$$\phi_{f,yy} - 4\phi_{f,\zeta,\xi} = 0(\varepsilon^{2M_2}) \quad , \qquad (5.5a)$$

for $\nu^2\xi < \zeta < \infty$, $0 < y < 1$, $0 < \xi < \infty$;

$$\left.\begin{aligned} &\phi_{f,y}(\zeta,0,\xi;\nu) = 0 \quad , \\ &\phi_{f,y}(\eta,1,\xi;\nu) = 0 \quad , \end{aligned}\right. \qquad (5.5b)$$

for $\nu^2\xi \leq \eta < \infty$, $0 \leq \xi < \infty$;

$$\phi_{f,\xi} - \nu^2\phi_{f,\eta} = -\nu\, a^{(0)}(y,\xi) + 0(\delta\varepsilon^{M_2}) \quad , \qquad (5.5c)$$

for $\eta = \nu^2\xi$, $0 \leq y \leq 1$, $0 \leq \xi < \infty$;

$$\phi_f(\eta,y,0;\nu) = 0 \quad , \qquad (5.5d)$$

for $\nu^2\xi \leq \eta < \infty$, $0 \leq y \leq 1$.

We now assume an expansion for ϕ_f of the form:

$$\phi_f(\eta,y,\xi;\nu) = \nu\phi_f^{(1)}(\eta,y,\xi) + \nu^2\phi_f^{(2)}(\eta,y,\xi) + \ldots \qquad (5.6)$$

Following the usual procedure we can establish that $\phi_f^{(1)}(\eta,y,\xi)$ satifies the equations below:

$$\phi^{(1)}_{f,yy} - 4\phi^{(1)}_{f,\xi\eta} = 0 < \eta < \infty \ , \quad 0 < y < 1,\ 0 < \xi < \infty,$$

$$\left.\begin{aligned} &\phi^{(1)}_{f,y}(\eta,0,\xi) = 0 \\ &\phi^{(1)}_{f,y}(\eta,1,\xi) = 0 \end{aligned}\right\} \quad 0 < \eta < \infty,\ 0 < \xi < \infty;$$

$$\phi^{(1)}_{\eta,\xi}(0,y,\xi) = -a^{(0)}(y,\xi) \quad , \quad 0 \leq y \leq 1, \; 0 \leq \xi < \infty;$$

$$\phi_f^{(1)}(\eta,y,0) = 0 \quad , \; 0 \leq \eta \leq \infty, \; 0 \leq y \leq 1.$$

Note that we have replaced the boundary condition at $\eta = \nu^2\xi$ by one at $\eta = 0$. This is a procedure similar to the replacement of the domain $0 < y < 1$, $0 < \xi < \frac{1}{\varepsilon}$ by the semi infinite strip $0 < y < 1$, $0 < \xi < \infty$ in the analysis of the boundary layer in section 4. However in this case errors produced by this approximation of the domain are not negligible at higher orders.

We can now solve the above system for $\phi_f^{(1)}$ obtaining the solution in the form:

$$\phi_f^{(1)} = -\sum_{n=1}^{\infty}\int_0^{\xi} a_n^{(0)}(z)J_o(n\pi\sqrt{\eta(\xi-z)}\,)dz \,.\, \cos n\pi y \tag{5.7}$$

where

$$a_n^{(0)}(\xi) = \int_0^1 a^{(0)}(y,\xi)\, \cos n\pi y \; dy \qquad ,$$

and J_o is the Bessel function of order zero. Displacements corresponding to this potential are:

$$\begin{aligned} u_f^{(0)} &= -a^{(0)}(y,\xi) + \sum_{n=1}^{\infty}\int_0^{\xi} a_n^{(0)}(z)\,\frac{n\pi\sqrt{\eta}}{2\sqrt{\xi-z}}\,J_1(n\pi\sqrt{\eta(\xi-z)}\,)dz.\cos n\pi y. \\ v_f^{(1)} &= \sum_{n=1}\int_0^{\xi} a_n^{(0)}(z)J_o(n\pi\sqrt{\eta(\xi-z)}\,)dz.n\pi\;\sin n\pi y. \end{aligned} \tag{5.8}$$

Before considering the convergence of the above expansions and behavior far from the wave front, of the functions they represent we examine the procedure required to obtain higher order terms $\phi_f^{(2)}$, $\phi_f^{(3)}, \ldots$. As the system (5.5) is even in ν, $\phi_f^{(2)}$ does not occur. $\phi_f^{(3)}$ satisfies the following equations

$$\phi^{(3)}_{f,yy} - 4\phi^{(3)}_{f,\eta\xi} = 0 \qquad , 0 < y < 1,\ 0 < \xi < \infty,\ 0 < \eta < \infty;$$

$$\left.\begin{aligned}\phi^{(3)}_{f,y}(\eta,0,\xi) &= 0 ,\\ \phi^{(3)}_{f,y}(\eta,1,\xi) &= 0 ,\end{aligned}\right\} \quad 0 \le \eta < \infty,\ 0 \le \xi < \infty; \tag{5.9}$$

$$\phi^{(3)}_{f,\xi}(0,y,\xi) = \lim_{\nu\to o} \frac{1}{\nu^2}\{-a^{(0)}(y,\xi)-\phi^{(1)}_{f,\xi}(\nu^2\xi,y,\xi)+\nu^2\phi^{(1)}_{f,\eta}(\nu^2\xi,y,\xi)\},$$

$$0 \le \xi < \infty,\ 0 \le y \le 1;$$

$$\phi_f^{(3)}(\eta,y,0) = 0, \qquad 0 \le \eta < \infty,\ 0 \le y \le 1.$$

The condition on $\phi^{(3)}_{f,\xi}$ at $\eta = 0$ requires further explanation

If the procedure we are following is to be of use, the approximations $\nu\phi_f^{(1)}$, $\nu^3\phi_f^{(3)}, \ldots$ must be successively of lower and lower order uniformly in η ,y, and ξ. Also at x = 0 ($\eta = \nu^2\xi$)

$$E^{(2k-1)} = \{a^{(0)}(y,\xi)-(\phi_{f,}^{(1)}+\nu^2\phi_{f,}^{(3)}+\ldots+\nu^{2k}\phi_{f,}^{(2k-1)})$$

$$+\nu^2(\phi_{f,}^{(1)}+\nu^2\phi^{(3)}_{f,\eta}+\ldots+\nu^{2k}\phi^{(2k-1)}_{f,\eta})$$

must be uniformly of $0(\nu^{2k+2})$.We examine $E^{(1)}=\{-a^{(0)}(y,\xi)-\phi^{(1)}_{f,\xi}(\nu^2\xi,y,\xi) +\nu^2\phi^{(1)}_{f,\eta}(\nu^2,y,\xi)\}$ in detail. We find the result:

$$E^{(1)} = \sum_{n=1}^{\infty}\int_o^{\xi} za_n^{(0)}(z)\frac{n\pi\nu}{2\sqrt{\xi(\xi-z)}}J_1(n\pi\nu\sqrt{\xi(\xi-z)})dz.\cos n\pi y. \tag{5.11}$$

Using the following inequality:

$$|J_1(n\pi\nu\sqrt{(\xi-z)})| \le \frac{n\pi\nu\sqrt{\xi(\xi-z)}}{2} \quad ,$$

we can estimate the magnitude of $E^{(1)}$:

$$|E^{(1)}| \le \frac{\nu^2}{4} \sum_{n=1}^{\infty} n^2\pi^2 \int_0^{\xi} z|a_n^{(0)}(z)|\,dz \quad .$$

We have prescribed that $a^{(0)}(y,\xi) = 0$ for $\xi \ge 1$. Therefore

$$|E^{(1)}| \le \frac{\nu^2}{4} \sum_{n=1}^{\infty} n^2\pi^2 \int_0^{1} z|a_n^{(0)}(z)|\,dz \quad . \tag{5.12}$$

At present we assume that this sum is convergent and then we conclude $|E^{(1)}|$ is uniformly $0(\nu^2)$.

Writing the Bessel function J_1 in equation (5.11) in power series form we find that

$$E^{(1)} = \frac{\nu^2}{4} \sum_{n=1}^{\infty} n^2\pi^2 \int_0^{z} z\, a_n^{(0)}(z)dz \; . \; \cos n\pi y + 0(\nu^4) \quad .$$

These results then provide a boundary condition for $\phi_{f,\xi}^{(3)}$ at $\eta = \nu^2\xi$ which we approximate by the following condition at $\eta = 0$.

$$\phi_{f,\xi}^{(3)}(0,y,\xi) = \frac{1}{4} \sum_{n=1}^{\infty} n^2\pi^2 \int_0^{z} z\, a_n^{(0)}(z)dz . \cos n\pi y \; ,$$

We can then obtain a solution for $\phi_f^{(3)}$:

$$\phi_f^{(3)} = -\sum_{n=1}^{\infty} \frac{n\pi}{2} \int_0^{\xi} z\, a_n^{(0)}(z)\sqrt{\frac{\xi-z}{\eta}}\, J_1(n\pi\sqrt{\eta(\xi-z)}\,)dz . \cos n\pi y. \tag{5.13}$$

Proceeding to higher orders we can obtain a solution

$$\phi_f^{(2k+1)} = -\sum_{n=1}^{\infty} \left(\frac{n\pi}{2}\right) \int_0^{\xi} \frac{z^k}{k!}\, a_n^{(0)}(z) \left(\frac{\xi-z}{\eta}\right)^{\frac{k}{2}} J_k(n\pi\sqrt{\eta(\xi-z)}dz . \cos n\pi y. \tag{5.14}$$

We also obtain an estimate for the error $E^{(2k+1)}$.

$$|E^{(2k+1)}| \le \nu^{2k+2} \sum_{n=1}^{\infty} \left(\frac{n\pi}{2}\right)^{k+1} \int_0^1 \frac{z^k}{k!} \, |a_n^{(0)}(z)| \, dz \qquad , \quad (5.15)$$

which is uniformly $O(\nu^{2k})$ provided the series above converges. We now take up the question of the convergence of the series in the error estimates (5.12) and (5.15) and in the solutions (5.7), (5.13), (5.14). We assume that $a^{(0)}(y,\xi) - \int_0^1 a^{(0)}(y,\xi)dy$ has continuous k+1 th y derivative and piecewise continuous k+2 th y derivative for $0 \le y \le 1$ and $0 \le \xi \le 1$. These conditions ensure the convergence of $\sum_{n=1}^{\infty} (n\pi)^{k+1} |a_n^{(0)}(\xi)|$ Also $\sum_{n=1}^{\infty} (n\pi)^{k+1} a_n^{(0)}(\xi) \begin{smallmatrix} \cos n\pi y \\ \sin n\pi y \end{smallmatrix}$ converges to $\dfrac{\partial^{(k+1)} a^{(0)}(y,\xi)}{\partial y^{k+1}}$ uniformly for $0 \le y \le 1$, for all $0 \le \xi \le 1$. We further assume that the convergence of $\sum_{n=1}^{\infty} (n\pi)^{k+1} |a_n^{(0)}(\xi)|$ is uniform for $0 \le \xi \le 1$. This is true, for example, if $a^{(0)}(\xi,y) = a_I^{(0)}(\xi) a_{II}^{(0)}(y)$. The above conditions are sufficient to ensure that the errors $E^{(1)}$, $E^{(3)}, \dots, E^{(2k+1)}$ are uniformly $O(\nu^2)$, $O(\nu^4), \dots, O(\nu^{2k+2})$ respectively. Also the series solutions for $\phi^{(1)}$, $\phi^{(3)}, \dots, \phi^{(2k+1)}$ are all uniformly convergent.

From equation (5.15) we can see that the condition on $a^{(0)}$

$$a^{(0)}(y,\xi) = 0 \qquad\qquad \xi \ge 1,$$

is unnecessarily strong. The condition,

$$a^{(0)}(y,\xi) = o\left(\frac{1}{\xi^{k+2}}\right) \text{ for large } \xi, \text{ is sufficient.}$$

We are interested in the behavior of $\phi_f^{(2k+1)}$ far behind the wavefront. That is for large ζ (but for $\xi \le \frac{\eta}{\nu^2}$). For large ξ

$$\nu^{2k} \phi_f^{(2k+1)} = -\nu^{2k} \sum_{n=1}^{\infty} \left(\frac{n\pi}{2}\right)^k \int_0^1 \frac{z^k}{k!} a_n^{(0)}(z) \left(\frac{\xi - z}{\eta}\right)^{\frac{k}{2}} J_k(n\pi\sqrt{\eta(\xi - z)} dz . \cos n\pi y.$$

Using the assumed properties of $a^{(0)}(y,z)$ and the fact that $\frac{\xi-z}{\eta} \leq \frac{\xi}{\eta} < \frac{1}{\nu^2}$, we can see that although the approximate solution $\sum_{j=0}^{k} \phi_f^{(2j+1)}$ produces errors at x = 0 uniformly at $O(\nu^{2k})$ the separate orders are not uniformly of $O(\nu^{2j})$ for all ξ,η,y. We can demonstrate this fact by substituting $\xi = \frac{t-x}{\delta}$ and $\eta = \frac{\delta}{\varepsilon^2}(t+x)$ in part of $\phi_f^{(2k+1)}$.

$$\nu^{2k}\phi_f^{2k+1} = -\nu^k \sum_{n=1}^{\infty} \left(\frac{n\pi}{2}\right)^k \int_0^{\xi} \frac{z^k}{k!}\, a_n^{(0)}(z) \left(\frac{t-x-\delta z}{t+x}\right)^{\frac{k}{2}} J_k(n\pi\sqrt{\eta(\xi-z)})\, dz \tag{5.16}$$

$$\times \cos n\pi y.$$

We can readily verify that $\nu^{2k}\phi_f^{(2k+1)}$ is uniformly of $O(\nu^k)$ and decays like $\xi^{-1/4}$ far from the wave front.

The assumption that $\bar{a}^{(0)}(y,\xi) = a^{(0)}(y,\xi) - \int_0^1 a^{(0)}(y,\xi)dy$ possesses k+1 continuous derivatives on the open interval $0 < y < 1$ is not a very restrictive practical condition. However the assumption that this condition also holds at the end points y = 0, y = 1, is rather severe. As a result we can only prescribe functions $\bar{a}^{(0)}(y,\xi)$ for which the odd y derivatives of order less than or equal to k+1 are all zero at y = 0 and y = 1. If $\bar{a}^{(0)}(y,\xi)$ does not satisfy these conditions we have to consider "corner approximations". We have been unable to develop an asymptotic procedure for examining these "corner approximations". The solution we have developed for ϕ_f was obtained without using any boundary condition at the end x = 1. The solution ϕ_f first effects the end x = 1 at time t = 1. We use the values of ϕ_f at x = 1 and $t \geq 1$ as new data for boundary layer approximations and wave front approximations similar to these of section 4 and this section. This proccess must be repeated after time $t \geq 2$ at the end x = 0, time $t \geq \xi$ at the end x = 1 and so on.

In practice we will probably only be interested in the approximate solution of the problem up until the first reflection.

6. Summary of Results.

In this section we summarize the results of earlier sections. We have developed an approximate solution of the problem (2.6), (2.7) of the form:

$$\phi(x,y,t;\varepsilon,\delta) = \phi_i(x,y,t;\varepsilon,\delta) + \phi_\ell(\tfrac{x}{\varepsilon},y,t;\varepsilon) + \phi_{\ell'}(\tfrac{1-x}{\varepsilon},y,t;\varepsilon)+\phi_f(\tfrac{\delta}{\varepsilon^2}(t+x),y,\tfrac{t-x}{\delta};\ \tfrac{\delta}{\varepsilon}). \tag{6.1}$$

The usefulness of this expression depends on the prescribed properties of the excitations $p^{(1)}$, $f^{(0)}$ and $a^{(0)}$. The inner solution ϕ_i satisfies the boundary conditions at $y = 0$ and $y = 1$ (2.7a) exactly. Also ϕ_i describes the propagation of the y-average of the excitation prescribed at the end $x = 0$ (2.7b) exactly. However, the accuracy with which ϕ_i satisfies the differential equation (2.6) and the initial conditions depends respectively on the smoothness and initial growth of the excitation $p^{(1)}(x,t)$ (the smoother $p^{(1)}$, the more accurate ϕ_i). The functions ϕ_ℓ and $\phi_{\ell'}$ describe the quasi-static boundary layer effects at the ends $x = 0$ and $x = 1$, respectively, arising from the slowly varying excitations prescribed at the ends (except for the y-average of the excitations whose propagation is described by the inner solution ϕ_i). The function ϕ_f describes the propagation effects stimulated by the rapidly varying part of the prescribed excitation at the end $x = 0$. The development of the terms ϕ_ℓ and ϕ_f and the order of accuracy with which they satisfy the differential equation (2.6) and the end condition (2.7b) depends on the prescribed excitation at $x = 0$. This excitation is represented as the sum of two functions $f^{(0)}(y,t)+a^{(0)}(y,t/\delta)$. Here $f^{(0)}(y,t)$ represents the slowly varying part of the prescribed excitation and the development and accuracy of ϕ_ℓ depends on the smoothness of $f^{(0)}$ (the smoother $f^{(0)}$, the more accurate ϕ_ℓ [cf. section 4]). The function $a^{(0)}(y,t/\delta)$ represents the rapidly varying part of the prescribed excitation. The development of ϕ_f depends on $a^{(0)}$ vanishing for $t \geq \delta$, where $\frac{\delta}{\varepsilon}$ is small (that is the sudden part of the excitation must be over in a time shorter than that taken for an acoustic wave to travel across the rectangle). The accuracy of ϕ_f also depends on the smoothness of $a^{(0)}$ with respect to y (the smoother $a^{(0)}$ is with respect to y the more accurate is ϕ_f [cf. section 5]).

The division of the prescribed data at $x = 0$ into functions $f^{(0)}(y,t)$ and $a^{(0)}(y,t/\delta)$ is a rather arbitrary process. There is no very precise mathematical way to define the admissible class of excitations at $x = 0$. From practical considerations we know that the impulsive excitations considered here are described by functions of time which vary rapidly for a short time and then ocillate more slowly back to some final static value. Given the graph of such an excitation (say obtained from some measuring device) we must decide on a time scale for the rapid excitation, say T_o. We smoothly join the section of the prescribed excitation for $t > T_o$ to a zero value at the origin $t = 0$. This determines our functions $a^{(0)}(y,t/\delta)$ and $f^{(0)}(y,t)$. The method developed is only applicable if the time scale T_o leads to a parameter δ (rapidity of excitation) which is small compared with ε (thinness). Even if the condition $\frac{\delta}{\varepsilon} < 1$ is satisfied, the other smoothness conditions required of the prescribed excitations, especially initial smoothness of the function $p^{(1)}(x,t)$ and smoothness of $a^{(0)}(y,t/\delta)$ near $y = 0,1$ limit the generality of the procedure. However in practice the leading term in each approximation may be all that is required, and these can be obtained for prescribed data of a reasonably general nature.

We have established that the expression for ϕ written in (6.1) satisfies the conditions of the problem (2.6), (2.7) accurately. If we write $\psi = \phi_{exact} - \phi$, then we have shown that ψ satisfies a system similar to (2.6), (2.7). This is:

$$\psi,_{yy} + \varepsilon^2(\psi,_{xx} - \psi,_{tt}) = \varepsilon^M A(x,y,t;\varepsilon,\delta) \quad , \qquad (6.2)$$

for $0 < x < 1$, $0 < y < 1$, $0 < t < \infty$;

$$\psi,_y(x,0,t;\varepsilon,\delta) = 0, \ \psi,_y(x,1,t;\varepsilon,\delta) = 0 \quad , \qquad (6.3a)$$

for $0 \le x \le 1$, $0 \le t < \infty$;

$$\begin{aligned} \psi,_x(0,y,t;\varepsilon,\delta) &= \varepsilon^M B(x,y,t;\varepsilon,\delta) \ , \\ \psi,_x(1,y,t;\varepsilon,\delta) &= \varepsilon^M C(x,y,t;\varepsilon,\delta) \quad , \end{aligned} \qquad (6.3b)$$

for $0 \le y \le 1$, $0 \le t < \infty$;

$$\psi(x,y,0;\varepsilon,\delta) = 0 \quad , \quad \psi,_t(x,y,0;\varepsilon,\delta) = 0 \quad , \qquad (6.3c)$$

for $0 \le x \le 1$, $0 \le y \le 1$. Here ε^M is the largest of ε^{2M_1}, ε^{2M_2}, ν^{2M_3} and A, B, C are functions of x,y,t determined from the expressions for ϕ_i, ϕ_ℓ, ϕ_ℓ, and ϕ_f. It is found that $\int_0^1 A\,dy = \int_0^1 B\,dy = \int_0^1 C\,dy = 0$. In order that ϕ should be considered a good approximation to the exact solution exact the solution of the system (6.2), (6.3) for ψ should be $O(\varepsilon^M)$.

This result can be obtained using elementary methods to solve (6.2) , (6.3) exactly; or better, by using the results obtained by Shield [36].

II. AN APPROXIMATE SOLUTION OF THE DYNAMIC EQUATIONS OF ELASTICITY IN A THIN REGION.

1. Introduction.

In this chapter an approximate solution is constructed for a boundary-initial value problem in the linear two-dimensional plane strain theory of elasticity. We consider an initially quiescent elastic slab suddenly loaded on its boundaries, and then try to determine the subsequent motion of the whole slab. For simplicity, it is assumed that the prescribed loads lead to a state of dynamic plane strain in the slab. This means the motion is two-dimensional, and we accordingly speak of an "elastic rectangle" to connote a typical cross section of the actual three-dimensional region. Furthermore in this thesis we are interested in thin plates and therefore we consider an elastic rectangle whose thickness is small compared with its length.

In a dynamic problem certain characteristic time scales (for example: a pulse duration, or a period of oscillation) are introduced by the prescribed loads. Suppose T_c is an example of such a characteristic time. This time, T_c, multiplied by a characteristic velocity for the problem (say the shear speed), gives a quantity having the dimension of length. We call this length the "loading length". The relative magnitudes of the length of the rectangle, the width of the rectangle and various loading lengths associated with the prescribed loads, greatly influence the nature of the solution. We do not attempt to solve the problem for completely general prescribed loads. We specify in advance, that the stresses applied to the long sides of the rectangle are slowly varying with a loading length of the same order of magnitude as the length of the rectangle. We further specify that part of the loading at the ends is similarly slowly varying. In addition there is a brief initial impulse at one end which has a small loading length compared with the length of the rectangle. This rather vague description of the problem is made more precise with the detailed formulation in the next section.

It is well known that two types of waves, the so-called shear and dilatation waves, arise from a disturbance in an elastic medium. These waves interact near a surface to produce a surface (Rayleigh) wave. The differential equations and boundary conditions required to describe these phenomena are naturally more complicated than the simple wave equation and boundary conditions of the acoustic problem of chapter I. However some of the features of the approximate solution obtained in this earlier work occur in the more complex and difficult elasticity problem of this chapter.

It is possible to construct an inner approximation which satisifes the specified loading conditions on the long sides of the rectangle and describes the propagation effects arising from a slowly varying average normal stress at the ends in a manner exactly analogous to that used for the inner approximation of chapter I(or the inner approximation for the deformation of an elastic plate c.f. Friedrichs and Dressler [6]). Again it is found that part of the slowly varying end loads, with certain averages having zero values (zero average normal stress, zero average shear stress, zero bending moment), produce effects which are important near the ends and can be described by quasi-static boundary layer approximations (c.f. Novozhilov [30]). To describe the effects generated by the average shear stress and the bending moment applied at the ends, a "diffusion approximation" is introduced. This has no analogy in the acoustic problem or the static elasticity problem. We use the term "diffusive" to describe the spreading of the slowly varying part of the average shear stress and bending moment into the interior of the elastic rectangle. Furthermore, propagation effects produced by the impulsively applied loads at the end of the rectangle require the development of three wave front approximations associated with dilatation waves, shear waves and Rayleigh waves. As will become apparent in this chapter, the structure of these wave front approximations is more complicated than that developed for the acoustic problem of chapter I.

Except for particular problems corresponding to very special end loads, "corner approximations" are required. These have not been obtained in what follows because they are connected with the unsolved problem of a dynamically loaded elastic quarter space. Since these corner approximations and the wave front approximation moving with the Rayleigh wave speed are inter-related, the latter is also not completely developed.

The main difference between the methods developed in this chapter for the elastic rectangle and those of chapter I developed for the fluid rectangle is that in the elasticity case the slowly varying effects and the rapidly varying effects interact through the boundary conditions. The rapidly varying effects in the elastic rectangle cause other effects at the boundaries which may not all be rapidly varying. In the acoustic case considered in chapter I it was possible to completely separate the slowly varying and rapidly varying portions of the solution.

In spite of the deficiencies mentioned above, this work may help to elucidate the various phenomena occuring in an elastic plate after impulsive loading. Also it may help to show more clearly the significance of the approximate theories - the plate wave theory and the Euler-Bernoulli plate theory - in relation to an exact solution of a dynamic elastic plate problem. In this respect the present work can be regarded as a supplement to the work of Miklowitz [23,24,25,26], Mindlin [28,29] and others (summarized in a review article by Miklowitz [27]).

The results obtained in subsequent sections are summarized in section 11. The reader may understand the development of this work more readily by first reading section 11.

2. Formulation of the Problem.

The two-dimensional equations of motion for an elastic medium may be written in terms of the three components of stress σ_{XX}, σ_{XX}, σ_{YY} and the two displacements U,V, in the X and Y directions, respectively, in the following form:

$$\sigma_{XX,X} + \sigma_{XY,Y} = \rho U,_{TT} \quad ,$$
$$\sigma_{XY,X} + \sigma_{YY,Y} = \rho V,_{TT} \quad ; \qquad (2.1)$$

where ρ is the density of the medium. The stresses are given in terms of the displacements and the Lame constants λ,μ by the following constitutive equations:

$$\sigma_{XX} = (\lambda+2\mu)U,_X + \lambda V,_Y \quad ,$$

$$\sigma_{XY} = \mu U,_{Y} + \mu V,_{X} \quad , \qquad (2.2)$$

$$\sigma_{YY} = U,_{X} + (\lambda+2\mu)V,_{Y} \quad .$$

The problem may also be considered in terms of the two potentials Φ, Ψ for which:

$$U = \Phi,_{X} + \Psi,_{Y} \quad ,$$

$$V = \Phi,_{Y} - \Psi,_{X} \quad , \qquad (2.3)$$

(c.f. Love [22], p. 47).

In terms of these potentials the equations of motion are:

$$\Phi,_{XX} + \Phi,_{YY} - \frac{1}{c_1^2}\Psi,_{TT} = 0 \quad ,$$

$$\Psi,_{XX} + \Psi,_{YY} - \frac{1}{c_2^2}\Psi,_{TT} = 0 \quad . \qquad (2.4)$$

where c_1 is the speed of propagation of dilatation waves, and c_2 speed of propagation of shear waves. These velocities can be expressed in terms of λ, μ, ρ:

$$c_1^2 = \frac{\lambda+2\mu}{\rho} \quad , \qquad c_2^2 = \frac{\mu}{\rho} \quad .$$

We consider the open rectangle R in the X,Y plane consisting of points for which $0 < X < \ell$, $-h < Y < h$. It is required to find functions $U(X,Y,T)$ and $V(X,Y,T)$ satisfying the differential equations:

$$(\lambda+2\mu)U,_{XX} + (\lambda+\mu)V,_{XY} + \mu U,_{YY} - \rho U,_{TT} = 0 \quad ,$$

$$\mu V,_{XX} + (\lambda+\mu)U,_{XY} + (\lambda+2\mu)V,_{YY} - \rho V,_{TT} = 0 \quad , \qquad (2.5)$$

for $(X,Y) \in R$, $T > 0$, and fulfilling the following boundary and initial conditions.

$$\left.\begin{aligned}\sigma_{YY}(X,Y,T) &= P_1(X,T) ,\\ \sigma_{YY}(X,-h,T) &= P_2(X,t ,\end{aligned}\right\} \quad 0 \le X \le \ell,\ 0 \le T < \infty, \qquad (2.6a)$$

$$\left.\begin{aligned}\sigma_{XY}(X,h,T) &= Q_1(X,T) ,\\ \sigma_{XY}(X,-h,T) &= Q_2(X,T) ,\end{aligned}\right\} \quad 0 \le X \le \ell,\ 0 \le T < \infty, \qquad (2.6b)$$

$$\left.\begin{aligned}\sigma_{XX}(0,Y,T) &= F(Y,T) + A(Y,T)\\ \sigma_{XX}(\ell,Y,T) &= 0\end{aligned}\right\} -h \le Y \le h,\ 0 \le T < \infty, \qquad (2.6c)$$

$$\left.\begin{aligned}\sigma_{XY}(0,Y,T) &= G(Y,T) + B(Y,T) ,\\ \sigma_{XY}(\ell,Y,T) &= 0\end{aligned}\right\} -h \le Y \le h,\ 0 \le T < \infty, \qquad (2.6d)$$

$$\left.\begin{aligned}U(X,Y,0) &= 0\\ V(X,Y,0) &= 0,\\ U_{,T}(X,Y,0) &= 0,\\ V_{,T}(X,Y,0) &= 0,\end{aligned}\right\} \quad (X,Y) \in R. \qquad (2.6e)$$

Here $P_1, P_2, Q_1, Q_2, F, G, A, B$ are prescribed functions. The division of the boundary conditions at the end $X = 0$ into two parts is a notational convenience which anticipates results developed in later sections. It is found that the slowly varying parts of the applied stresses at $X = 0$, which we denote by F and G, and the rapidly varying parts of the stresses denoted by A and B, produce quite different effects. The properties required of F, G. A and B depend on the order of accuracy needed for the approximate solution and will be discussed subsequently. At this state we assume that the sudden part of the loading has a limited duration, namely A and B are zero for $T \geq T_o$. The smoothness properties of P_1, P_2, Q_1 and Q_2 are also discussed in later sections when they become important in the construction of the approximate solution. For the present we assume that the given data are such that solutions for Φ and Ψ exist which give rise to continuous displacements on the closure of R for each $T \geq 0$.

The rectangle R is thin if $\varepsilon = \frac{h}{L}$ is small compared with unity. In order to exhibit the role of ε more explicitly, we introduce new independent variables x,y and t, defined by the change of scale

$$x = \frac{X}{\ell} \quad , \; y = \frac{Y}{h} \; , \; t = \frac{c_2 T}{\ell} \quad . \tag{2.7}$$

Another length scale $c_2 T_o$ is formed by multiplying the pulse duration, T_o, of the end loading by the speed of propagation of shear waves, c_2. We define a parameter δ measuring the suddenness of loading by the equation $\delta = \frac{c_2 T_o}{\ell}$.

Non-dimensional dependent variables $u, v, \tau_{xx}, \tau_{xy}, \tau_{yy}, \phi$ and ψ are introduced by the following definitions:

$$\begin{aligned}
u(x,y,t;\varepsilon,\delta) &= \frac{1}{h} U(\ell x, hy, \frac{\ell}{c_2} t) \quad , \\
v(x,y,t;\varepsilon,\delta) &= \frac{1}{h} V(\ell x, hy, \frac{\ell}{c_2} t) \; , \\
\tau_{xx}(x,y,t;\varepsilon,\delta) &= \frac{1}{\mu} \sigma_{XX}(\ell x, hy, \frac{\ell}{c_2} t) \; , \\
\tau_{xy}(x,y,t;\varepsilon,\delta) &= \frac{1}{\mu} \sigma_{XY}(\ell x, hy \; \frac{\ell}{c_2} t) \; , \\
\tau_{yy}(x,y,t;\varepsilon,\delta) &= \frac{1}{\mu} \sigma_{YY}(\ell x, hy, \frac{\ell}{c_2} t) \; , \\
\phi(x,y,t;\varepsilon,\delta) &= \frac{1}{h^2} \Phi(\ell x, hy, \frac{\ell}{c_2} t) \; , \\
\psi(x,y,t;\varepsilon,\delta) &= \frac{1}{h^2} \Psi(x, hy, \frac{\ell}{c_2} t) \; .
\end{aligned} \tag{2.8}$$

We are interested in the particular case where the given loads produce displacements u and v which are $O(1)$ in the limiting case $\varepsilon \to 0$. This occurs when the loads, written in terms of scaled variables, have the orders of magnitude displayed in equation (2.9) below. The prescribed loads are given in terms of the new functions $p_1^{(2)}, p_2^{(2)}, q_1^{(2)}, q_2^{(2)}, f^{(1)}, a^{(1)}, g^{(3/2)}$ and $b^{(3/2)}$ when written in scaled variables. Thus:

$$\begin{aligned}
&\varepsilon^2 p_1^{(2)}(x,t) = \frac{1}{\mu} P_1(\ell x, \frac{\ell}{c_2} t), \quad \varepsilon^2 p_2^{(2)}(x,t) = \frac{1}{\mu} P_2(\ell x, \frac{\ell}{c_2} t) \ , \\
&\varepsilon^2 q_1^{(2)}(x,t) = \frac{1}{\mu} Q_1(\ell x, \frac{\ell}{c_2} t), \quad \varepsilon^2 q_2^{(2)}(x,t) = \frac{1}{\mu} Q_2(\ell x, \frac{\ell}{c_2} t) \ , \\
&\varepsilon f^{(1)}(y,t) = \frac{1}{\mu} F(hy, \frac{\ell}{c_2} t) \ , \quad \varepsilon a^{(1)}(y,t/\delta) = \frac{1}{\mu} A(hy, \frac{\ell}{c_2} t) \ , \\
&\varepsilon^{3/2} g^{(3/2)}(y,t) = \frac{1}{\mu} G(hy, \frac{\ell}{c_2} t) \ , \quad \varepsilon^{3/2} b^{(3/2)}(y,t/\delta) = \frac{1}{\mu} B(hy, \frac{\ell}{c_2} t) \ .
\end{aligned} \tag{2.9}$$

Here the superscripts on the above functions refer to the associated power of ε. As in the acoustic problem of chapter I, we have assumed the given data to have a very special dependence on ε, the thinness parameter, when expressed in scaled variables. The problem associated with the prescribed loads of (2.9) can be considered as one of a whole hierarchy of problems associated with more general loads. The essential features of the dynamics of an elastic plate are included in the problem with prescribed loads given by equation (2.9).

Using this new notation, and defining a parameter α by the equation $\alpha^2 = \frac{c_1^2}{c_2^2}$, we are lead to the following final formulation:

$$\begin{aligned}
&u,_{yy} + \varepsilon(\alpha^2-1)v,_{xy} + \varepsilon^2(\alpha^2 u,_{xx} - u,_{tt}) = 0 \ , \\
&v,_{yy} + \varepsilon\frac{\alpha^2-1}{\alpha^2} u,_{xy} + \varepsilon^2(\frac{1}{\alpha^2} v,_{xx} - \frac{1}{\alpha^2} v,_{tt}) = 0 \ ,
\end{aligned} \tag{2.10}$$

for $0 < x < 1$, $-1 < y < 1$, $0 < t < \infty$. The boundary conditions and initial conditions are:

$$\tau_{xy}(x,t,t;\varepsilon,\delta) = \varepsilon^2 q_1^{(2)}(x,t) \quad ,$$
$$\tau_{xy}(x,-1,t;\varepsilon,\delta) = \varepsilon^2 q_2^{(2)}(x,t) \quad , \tag{2.11a}$$

for $0 \le x \le 1$, $0 \le t < \infty$;

$$\tau_{yy}(x,1,t;\varepsilon,\delta) = \varepsilon^2 p_1^{(2)}(x,t) \quad ,$$
$$\tau_{yy}(x,-1,t;\varepsilon,\delta) = \varepsilon^2 p_2^{(2)}(x,t) \quad , \tag{2.11b}$$

for $0 \le x \le 1$, $0 \le t < \infty$;

$$\tau_{xx}(0,y,t;\varepsilon,\delta) = \varepsilon f^{(1)}(y,t) + \varepsilon a^{(1)}(y,t/\delta) \quad ,$$
$$\tau_{xx}(1,y,t;\varepsilon,\delta) = 0 \quad , \tag{2.11c}$$

for $0 \le y \le 1$, $0 \le t < \infty$;

$$\tau_{xy}(0,y,t;\varepsilon,\delta) = \varepsilon^{3/2} g^{(3/2)}(y,t) + \varepsilon^{3/2} b^{(3/2)}(y,t/\delta) \quad ,$$
$$\tau_{xy}(1,y,t;\varepsilon,\delta) = 0 \quad , \tag{2.11d}$$

for $0 \le y \le 1$, $0 \le t < \infty$;

$$u(x,y,0;\varepsilon,\delta) = 0 \ ,$$
$$v(x,y,0;\varepsilon,\delta) = 0 \ ,$$
$$u_{,t}(x,y,0;\varepsilon,\delta) = 0 \ , \tag{2.11e}$$
$$v_{,t}(x,y,0;\varepsilon,\delta) = 0 \ ,$$

for $0 \le x \le 1$, $-1 \le y \le 1$.

The stresses are given in terms of the displacements according to the equations below:

$$\tau_{xx} = (\alpha^2-2)v,_y + \varepsilon\alpha^2 u,_x ,$$

$$\tau_{xy} = u,_y + \varepsilon v,_x \qquad , \qquad (2.12)$$

$$\tau_{yy} = \alpha^2 v,_y + \varepsilon(\alpha^2-2)u,_x \qquad .$$

The potentials ϕ and ψ associated with the displacements u,v are calculated from the equations:

$$u = \psi,_y + \varepsilon\psi,_x \qquad ,$$

$$\qquad (2.13)$$

$$v = \phi,_y - \varepsilon\psi,_x \qquad .$$

The problem (2.10), (2.11), (2.12) can not be solved explicitly and exactly by simple methods. Thus the construction of an approximate solution directly from the differential equations, boundary conditions and initial conditions in the form of an asymptotic series in ε and δ will be of interest. We attempt to carry out this scheme in later sections.

3. Formal Inner Approximation.

In this section we utilise a procedure previously developed for the static elasticity problem [c.f. Friedrichs and Dressler [6]) utilized in Chapter I for the wave equation. First, approximations having the form of power series in ε are examined. Furthermore, as stated in the last section, we are interested in the problem where the displacements u,v are O(1) in the limiting case $\varepsilon \to 0$. We call the approximations investigated in this section u_i and v_i, and we assume expansions for them as follows:

$$u_i(x,y,t;\varepsilon,\delta) = u_i^{(0)}(x,y,t;\delta) + \varepsilon u_i^{(1)}(x,y,t;\delta) + \dots ,$$

$$v_i(x,y,t;\varepsilon,\delta) = v_i^{(0)}(x,y,t;\delta) + \varepsilon v_i^{(1)}(x,y,t;\delta) + \dots . \tag{3.1}$$

Here the subscript "i" is used as an abbreviation for "inner". We also assume expansions for the associated stresses τ_{xxi}, τ_{xyi} and τ_{yyi} of the form:

$$\begin{aligned}\tau_{xxi} &= \tau_{xxi}^{(0)} + \tau_{xxi}^{(1)} + \dots ,\\ \tau_{xyi} &= \tau_{xyi}^{(0)} + \tau_{xyi}^{(1)} + \dots ,\\ \tau_{yyi} &= \tau_{yyi}^{(0)} + \varepsilon\tau_{yyi}^{(1)} + \dots .\end{aligned} \tag{3.2}$$

We now substitute these expansions into the system (2.10), (2.11) and (2.12) and solve the resulting equations, by considering the coefficients of each power of ε separately. This leads to the following differential equations for $u_i^{(0)}$ and $v_i^{(0)}$:

$$\begin{aligned}u_{i,yy}^{(0)} &= 0 ,\\ v_{i,yy}^{(0)} &= 0 ,\end{aligned} \tag{3.3}$$

for $0 < x < 1$, $-1 < y < 1$, $0 < t < \infty$. Solutions of these equations cannot in general satisfy the prescribed boundary conditions at $x = 0,1$. This statement is also true for the solutions of the equations for $u_i^{(1)}, v_i^{(1)}, u_i^{(2)}, v_i^{(2)}, \dots$ which are provided by this procedure. Consequently in this section we are content to fulfill the prescribed boundary conditions at $y = 1$. The approximations so obtained are not fully determined, and have the capability of accomodating only special types of boundary conditions at $x = 0,1$. These conditions are developed in the later investigations of the "boundary layers".

The boundary conditions for $u_i^{(0)}$ and $v_i^{(0)}$ at $y = \pm 1$ which follow from (2.11a), (2.11b) and (2.12) are:

$$\tau_{xyi}^{(0)}(x,\pm 1,t;\delta) = u_{i,y}^{(0)}(x,\pm 1,t;\delta) = 0 \quad ,$$
$$\tau_{yyi}^{(0)}(x,\pm 1,t;\delta) = v_{i,y}^{(0)}(x,\pm 1,t;\delta) = 0 \quad , \tag{3.4}$$

for $0 \leq x \leq 1$, $0 \leq t < \infty$.

We conclude from (3.3) and (3.4) that $u_i^{(0)}$, $v_i^{(0)}$ have the form:

$$u_i^{(0)}(x,y,t;\delta) = \bar{u}_i^{(0)}(x,t;\delta) \ ,$$
$$v_i^{(0)}(x,y,t;\delta) = \bar{v}_i^{(0)}(x,t;\delta) \quad ; \tag{3.5}$$

$\bar{u}_i^{(0)}$ and $\bar{v}_i^{(0)}$ are as yet arbitrary functions of x and t. Corresponding stresses are all zero, that is, $\tau_{xxi}^{(0)} = 0$, $\tau_{xyi}^{(0)} = 0$, $\tau_{yyi}^{(0)} = 0$. To obtain more information about $u_i^{(0)}$ and $v_i^{(0)}$ we must investigate higher order terms.

At the first order there is some interaction with the zeroth order. The differential equations and boundary conditions for $u_i^{(1)}$, $v_i^{(1)}$ resulting from (2.10), (2.11) and (2.12) are:

$$u_{i,yy}^{(1)} = 0 \quad , \qquad v_{i,yy}^{(1)} = 0 \quad ,$$

for $0 < x < 1$, $-1 < y < 1$, $0 < t < \infty$;

$$\tau_{xyi}^{(1)}(x,\pm 1,t;\delta) = u_{i,y}^{(1)}(x,\pm 1,t;\delta) + v_{i,x}^{(0)}(x,\pm 1,t;\delta) = 0 \quad ,$$

$$\tau_{yyi}^{(1)}(x,\pm 1,t;\delta) = (\alpha^2-2)u_{i,x}^{(1)}(x,\pm 1,t;\delta) + \alpha^2 v_{i,y}^{(1)}(x,\pm 1,t;\delta) = 0,$$

for $0 \leq x \leq 1,\ 0 \leq t < \infty$.

The most general solution of this system is:

$$u_i^{(1)}(x,y,t;\delta) = -y\bar{v}_{i,x}^{(0)}(x,t;\delta) + \bar{u}_i^{(1)}(x,t;\delta) \quad ,$$
$$v_i^{(1)}(x,y,t;\delta) = -y\,\frac{\alpha^2-2}{\alpha^2}\,\bar{u}_{i,x}^{(0)}(x,t;\delta) + \bar{v}_i^{(1)}(x,t;\delta) \quad , \qquad (3.6)$$

where $\bar{u}_i^{(1)}$, $\bar{v}_i^{(1)}$ are arbitrary functions of x and t. The corresponding stresses are:

$$\tau_{xxi}^{(1)} = \frac{4(\alpha^2-1)}{\alpha^2}\,\bar{u}_{i,x}^{(0)}\ ,\quad \tau_{xyi}^{(1)} = 0\ ,\quad \tau_{yyi}^{(1)} = 0\ . \qquad (3.7)$$

This procedure will determine the structure of the y-dependence of successive terms $u_i^{(0)}$, $v_i^{(0)}$, $u_i^{(1)}$, $v_i^{(1)}$, ..., but only in terms of the arbitrary functions $\bar{u}_i^{(0)}$, $\bar{v}_i^{(0)}$, $\bar{u}_i^{(1)}$, $\bar{v}_i^{(1)}$,... . However, at the next order, further restrictions on the behavior of $\bar{u}_i^{(0)}$, $\bar{v}_i^{(0)}$ are obtained.

The differential equations and boundary conditions for $u_i^{(2)}$ and $v_i^{(2)}$ are:

$$u_{i,yy}^{(2)} + (\alpha^2-1)v_{i,xy}^{(1)} + \alpha^2 u_{i,xx}^{(0)} - u_{i,tt}^{(0)} = 0 \quad ,$$
$$v_{i,yy}^{(2)} + \frac{\alpha^2-1}{\alpha^2}\,u_{i,xy}^{(1)} + \frac{1}{\alpha^2}v_{i,xx}^{(0)} - \frac{1}{\alpha^2}\,v_{i,tt}^{(0)} = 0 \quad ,$$

for $0 < x < 1,\ -1 < y < 1,\ 0 < t < \infty$;

$$\tau_{xyi}^{(2)} = u_{i,y}^{(2)} + v_{i,x}^{(1)} = \begin{cases} q_1^{(2)}(x,t), & y = 1 \quad , \\ q_2^{(2)}(x,t), & y = -1 \quad , \end{cases}$$

$$\tau_{yyi}^{(2)} = (\alpha^2-2)u_{i,x}^{(1)} + \alpha^2 v_{i,y}^{(2)} = \begin{cases} p_1^{(2)}(x,t)\ , & y = 1, \\ p_2^{(2)}(x,t)\ , & y = -1\ , \end{cases}$$

for $0 \leq x \leq 1,\ 0 \leq t < \infty$. After substituting the earlier results from

from (3.5) and (3.6) we can solve the above system to obtain the solutions:

$$u_i^{(2)} = \frac{y^2}{2}\left[\frac{\alpha^2-2}{\alpha^2}\,\bar{u}^{(0)}_{i,xx} - \frac{1}{2}\left(q_1^{(2)} - q_2^{(2)}\right)\right] - y\left[\bar{v}^{(1)}_{i,x} - \frac{1}{2}\left(q_1^{(2)} + q_2^{(2)}\right)\right] + \bar{u}_i^{(2)},$$

$$v_i^{(2)} = \frac{y}{2}\left[\frac{1}{2\alpha^2}\left(p_1^{(2)}-p_2^{(2)}\right) - \frac{\alpha^2-2}{\alpha^2}\bar{v}^{(0)}_{i,xx}\right] - y\left[\frac{\alpha^2-2}{\alpha^2}\bar{u}^{(1)}_{i,x} - \frac{1}{2\alpha^2}\left(p_1^{(2)}+p_2^{(2)}\right)\right] + \bar{v}_i^{(2)}. \tag{3.8}$$

where $\bar{u}_i^{(2)}$ and $\bar{v}_i^{(2)}$ are arbitrary functions of x and t. The integration of the differential equations governing $u_i^{(2)}$ and $v_i^{(2)}$ leads to the result that they are both quadratic functions of the variable y. Using this information in the boundary conditions we find the solutions (3.8) and also the following restrictions on the behavior of $\bar{u}_i^{(0)}$ and $\bar{v}_i^{(0)}$:

$$\bar{u}^{(0)}_{i,tt} - \frac{4(\alpha^2-1)}{\alpha^2}\,\bar{u}^{(0)}_{i,xx} = \frac{1}{2}\left(q_1^{(2)} - q_2^{(2)}\right) , \tag{3.9}$$

$$\bar{v}^{(0)}_{i,tt} = \frac{1}{2}\left(p_1^{(2)} - p_2^{(2)}\right) . \tag{3.10}$$

When recast into physical variables, equation (3.9) is readily identifiable as the "plate" wave equation (the plate wave speed is $\left[\frac{E}{\rho(1-\sigma^2)}\right]^{\frac{1}{2}} = \left[\frac{\mu}{\rho}\,\frac{4(\alpha^2-1)}{\alpha^2}\right]^{\frac{1}{2}}$; E is Young's modulus and σ is Poisson's ratio (c.f. Kolsky [20], p.81). Equation (3.10) can be interpreted as a statement of Newton's law of motion for the mean behavior of each section of the plate. It is interesting to note that, as with the static elasticity problem, the inner approximation can be identified with one one of the approximate theories; in this case, the plate wave equation. We assume that the inner approximation satisfies the initial quiescence condition (2.11e). Thus $v_i^{(0)}$ satisfies the conditions:

$$v_i^{(0)}(x,y,0;\delta) = 0 , \quad v^{(0)}_{i,t}(x,y,0;\delta) = 0 ,$$

and we can integrate equation (3.10) to obtain:

$$\bar{v}_i^{(0)} = \frac{1}{2}\int_0^t dt_1 \int_0^t [p_1^{(2)}(x,t_2)-p_2^{(2)}(x,t_2)]dt_2 \quad . \qquad (3.11)$$

However, the initial conditions:

$$\bar{u}_i^{(0)}(x,y,0;\delta) = 0, \; \bar{u}_{i,t}^{(0)}(x,y,0;\delta) = 0 \quad ,$$

are not sufficient to determine $\bar{u}_i^{(0)}$ as a solution of equation (3.9); boundary conditions at x = 0,1 are also required. If the prescribed normal stress at x = 0 was just a function of t (that is, constant across the thickness of the plate) then, using equation (3.7), we could write a suitable boundary condition for $\bar{u}_i^{(0)}$. In general this is not the case and it is not immediately obvious what condition to specify for $\bar{u}_i^{(0)}$ at x = 0 and x = 1. Suitable boundary conditions for $\bar{u}_i^{(0)}$ are determined in the course of the boundary layer investigations of section 5.

The stresses associated with $u_i^{(2)}$ and $v_i^{(2)}$ are:

$$\tau_{xxi}^{(2)} = y[\frac{\alpha^2-2}{2\alpha^2}(p_1^{(2)}-p_2^{(2)}) + \frac{4(\alpha^2-1)}{\alpha^2}\bar{v}_{i,xx}^{(0)}] + \frac{\alpha^2-2}{2\alpha^2}(p_1^{(2)}+p_2^{(2)}) + \frac{4(\alpha^2-1)}{\alpha^2}\bar{u}_{i,x}^{(1)}$$

$$\tau_{xyi}^{(2)} = \frac{y}{2}(q_1^{(2)} - q_1^{(2)}) + \frac{1}{2}(q_1^{(2)} + q_2^{(2)}) \quad , \qquad (3.12)$$

$$\tau_{yyi}^{(2)} = \frac{y}{2}(p_1^{(2)} - p_2^{(2)}) + \frac{1}{2}(p_1^{(2)} + p_2^{(2)}) \quad .$$

Proceeding through successive orders it is possible to evaluate $u_i^{(0)}$, $v_i^{(0)}$, $u_i^{(1)}$, $v_i^{(1)}$,..., in terms of the boundary data $p_1^{(2)}$, $p_2^{(2)}$, $q_1^{(2)}$, $q_2^{(2)}$ and a sequence of functions of x and t only, $\bar{u}_i^{(0)}$, $\bar{v}_i^{(0)}$, $\bar{u}_i^{(1)}$, $\bar{v}_i^{(1)}$,..., satisfying differential equations similar to (3.9) and (3.10). It is found that the successive orders of the displacements depend on successively higher derivatives of the boundary data (as was the case for the analogous terms of the inner approximation derived for the acoustic problem of chapter I). Thus the smoothness of the prescribed loads determines the order to which the process may be continued.

To examine the structure of successive orders in more detail we now proceed to the third and fourth order equations. By examining these higher orders we can discover the relation between the functions $\bar{u}_i^{(0)}$ and $\bar{u}_i^{(2)}$, and show the inability of the inner approximation to deal with the rapidly varying part of the applied end loads.

The differential equations and boundary conditions governing the behavior of $u_i^{(3)}$ and $v_i^{(3)}$ are:

$$u^{(3)}_{i,yy} + (\alpha^2-1)v^{(2)}_{i,xy} + \alpha^2 u^{(1)}_{i,xx} - u^{(1)}_{i,tt} = 0 \quad ,$$

$$v^{(3)}_{i,yy} + \frac{\alpha^2-1}{\alpha^2} u^{(2)}_{i,xy} + \frac{1}{\alpha^2} v^{(1)}_{i,xx} - v^{(1)}_{i,tt} = 0 \quad ,$$

for $0 < x < 1$, $-1 < y < 1$, $0 < t < \infty$;

$$\tau^{(3)}_{xyi} = u^{(3)}_{i,y} + v^{(2)}_{i,x} = 0 \quad , \qquad \tau^{(3)}_{yyi} = (\alpha^2-2)u^{(2)}_{i,x} + \alpha^2 v^{(3)}_{i,y} = 0 \ , \quad y = \pm 1,$$

for $0 < x < 1$, $0 < t < \infty$. After substituting the earlier results from (3.5), (3.6) and (3.8), and performing some tedious manipulations we can solve the above system to obtain the solutions:

$$u_i^{(3)} = \frac{y^3}{6}\left[- \frac{2\alpha^2-1}{\alpha^2}\bar{v}^{(0)}_{i,ttx} + \frac{3\alpha^2-2}{\alpha^2}\bar{v}^{(0)}_{i,xxx}\right] +$$
$$+ \frac{y^2}{2}\left[\bar{u}^{(1)}_{i,tt} - \frac{3\alpha^2-2}{\alpha^2}\bar{u}^{(1)}_{i,xx} - \frac{\alpha^2-1}{2\alpha^2}(p_1^{(2)} + p_2^{(2)})\right] +$$
$$+ y\left[- \bar{v}^{(2)}_{i,x} + \frac{\alpha^2-1}{\alpha^2}\bar{v}^{(0)}_{i,ttx} - \frac{2(\alpha^2-1)}{\alpha^2}\bar{v}^{(0)}_{i,xxx}\right] + \bar{u}_i^{(3)} \quad (3.13)$$

$$v_i^{(3)} = \frac{y^3}{6}\left[-\frac{\alpha^4-2}{\alpha^4}\bar{u}^{(0)}_{i,xtt}+\frac{3\alpha^2-4}{\alpha^2}\bar{u}^{(0)}_{i,xxx}\right]+\frac{y^2}{2}\left[\frac{\alpha^2-2}{\alpha^2}\bar{v}^{(1)}_{i,xx}+\frac{1}{\alpha^2}\bar{v}^{(1)}_{i,tt}-\right.$$

$$\left.\frac{\alpha^2-1}{2\alpha^2}(q^{(2)}_{1,x}+q^{(2)}_{2,x})\right] + y\left[-\frac{\alpha^2-2}{\alpha^2}\bar{u}^{(2)}_{i.x}+\frac{\alpha^2-1}{\alpha^4}\bar{u}^{(0)}_{i,ttx}-\frac{2(\alpha^2-1)}{\alpha^4}\bar{u}^{(0)}_{i,xxx}\right]+\bar{v}_i^{(3)}$$

In writing these solutions we have assumed that $p_1^{(2)}$, $p_2^{(2)}$, $q_1^{(2)}$, $q_2^{(2)}$ are at least differentiable, and their first derivatives with respect to x are O(1) in comparison with ε. The development of the solutions (3.13) also produces conditions restricting the behavior of $\bar{u}_i^{(1)}$ and $\bar{v}_i^{(1)}$ These are:

$$\bar{u}^{(1)}_{i,tt} - \frac{4(\alpha^2-1)}{\alpha^2}\bar{u}^{(1)}_{i,xx} = \frac{\alpha^2-2}{2\alpha^2}(p^{(2)}_{1,x}+p^{(2)}_{2,x}) \quad , \qquad (3.14)$$

$$\bar{v}^{(1)}_{i,tt} = \frac{1}{2}(q^{(2)}_{1,x} + q^{(2)}_{2,x}) \quad . \qquad (3.15)$$

The right-hand side of (3.14) represents the squeezing action of the normal stress applied on the long sides. The right-hand side of (3.15) shows that the shear gradient produces a lateral deflection of the plate. It is worth noting that both of these effects are important in the development of some of the "approximate"plate and rod theories. The Rayleigh correction to the plate wave equation based on squeezing motion (or lateral inertia) is described by (3.14) and the Timoshenko shear correction to the approximate beam equation accounts for the effect described by equation (3.15) (c.f. Miklowitz [27]).

The system of equations for $u_i^{(4)}$ and $v_i^{(4)}$ were investigated and solutions obtained. These are lengthy and contain no significant new information and hence we shall not record them here. As part of the investigation of the solutions for $u_i^{(4)}$ and $v_i^{(4)}$ we can derive the following equations governing the behavior of $\bar{u}_i^{(2)}$ and $\bar{v}_i^{(2)}$:

$$\bar{u}^{(2)}_{i,tt} - \frac{4(\alpha^2-1)}{\alpha^2}\bar{u}^{(2)}_{i,xx} + \frac{1}{12}(q^{(2)}_{1,tt} - q^{(2)}_{2,tt}) - \frac{7\alpha^4-14\alpha^2+8}{6\alpha^4}(q^{(2)}_{1,xx} - q^{(2)}_{2,xx})$$

$$- \frac{4}{3}\frac{(\alpha^2-1)(\alpha^2-2)^2}{\alpha^6}\bar{u}^{(0)}_{i,xxxx} = 0 \quad , \tag{3.16}$$

$$\bar{v}^{(2)}_{i,tt} + \frac{1}{12\alpha^2}(p^{(2)}_{1,tt} - p^{(2)}_{2,tt}) - \frac{3\alpha^2-2}{12\alpha^2}(p^{(2)}_{1,xx} - p^{(2)}_{2,xx}) + \frac{4}{3}\frac{(\alpha^2-1)}{\alpha^4}\bar{v}^{(0)}_{i,xxxx} = 0. \tag{3.17}$$

Here we have assumed that the second derivatives, $q^{(2)}_{1,tt}$, $q^{(2)}_{1,xx}$, $p^{(3)}_{1,tt}$, $p^{(2)}_{1,xx}$, ... , exist and are $0(1)$. Also it is clear that $\bar{u}^{(0)}_i$ must be at least four times differentiable with respect to x to allow for a continuous solution for $\bar{u}^{(2)}_i$. A more precise statement of this fact is made later when suitable boundary conditions for (3.9), (3.14) and (3.16) are discussed. This will be done in section 5 which is devoted to the investigation of "boundary layers" near each end of the plate.

In principle it is possible to proceed to higher orders for solutions for $u^{(5)}_i$, $v^{(5)}_i$,... ; at the same time we can develop further equations restricting the behavior of $\bar{u}^{(3)}_i$, $\bar{v}^{(3)}_i$,... . However, the calculations involved would be prohibitively lengthy. In this case a symbolic procedure similar to that used in section 3 of chapter I in dealing with the inner approximation for the wave equation problem is of little use. A "symbolic" solution analogous to equation (I.3.18) can be developed but no convenient expansion similar to (I.3.19) is found.

However for a practical problem the first three terms $u^{(0)}_i$, $u^{(1)}_i$, $u^{(3)}_i$ may be all that are required. We shall adopt a truncation of the original expansion (3.1) as our inner approximation:

$$\begin{aligned} u_i &= u^{(0)}_i + \varepsilon u^{(1)}_i + \varepsilon^2 u^{(2)}_i \quad , \\ v_i &= v^{(0)}_i + \varepsilon v^{(1)}_i + \varepsilon^2 v^{(2)}_i \quad , \end{aligned} \tag{3.18}$$

where $u_i^{(0)}$, $v_i^{(0)}$, $u_i^{(1)}$, $v_i^{(0)}$, $u_i^{(2)}$ and $v_i^{(2)}$ are determined from equations (3.5), (3.6) and (3.8). In full these are:

$$u_i = \bar{u}_i^{(0)} + \varepsilon\{-y\,\bar{v}_{i,x}^{(0)} + u_i^{(1)}\} + \varepsilon^2\{\frac{y}{2}[\frac{\alpha^2-2}{\alpha^2}\bar{u}_{i,xx}^{(0)} - \frac{1}{2}(q_1^{(2)}-q_2^{(2)})]$$

$$+ y[\frac{1}{2}(q_1^{(2)}+q_2^{(2)}) - \bar{v}_{i,x}^{(1)}] + \bar{u}_i^{(2)}\} \quad , \qquad (3.19)$$

$$v_i = \bar{v}_i^{(0)} + \varepsilon\{-y\,\frac{\alpha^2-2}{\alpha^2}\bar{u}_{i,x}^{(0)} + \bar{v}_i^{(1)}\} + \varepsilon^2\{\frac{y}{2}[\frac{1}{2\alpha^2}(p_1^{(2)}-p_2^{(2)}) - \frac{\alpha^2-2}{\alpha^2}\bar{v}_{i,xx}^{(0)}]$$

$$+ y[\frac{1}{2\alpha^2}(p_1^{(2)} + p_2^{(2)}) - \frac{\alpha^2-2}{\alpha^2}\bar{u}_{i,x}^{(1)}] + \bar{v}_i^{(2)}\} \quad .$$

Here $\bar{u}_i^{(0)}$, $\bar{v}_i^{(0)}$, $\bar{u}_i^{(1)}$, $\bar{v}_i^{(1)}$, $\bar{u}_i^{(2)}$ and $\bar{v}_i^{(2)}$ are all functions of x and t only, satisfying equations (3.9), (3.10), (3.14), (3.15), (3.16), (3.17), respectively.

We require that the inner approximations given by equation (3.19) fulfill the initial quiescence condition. That is:

$$\begin{aligned} u_i(x,y,0;\varepsilon,\delta) &= 0 \quad , \quad & u_{i,t}(x,y,0;\varepsilon,\delta) &= 0 \\ v_i(x,y,0;\varepsilon,\delta) &= 0 \quad , \quad & v_{i,t}(x,y,0;\varepsilon,\delta) &= 0 \end{aligned} \qquad (3.20)$$

This condition provides the initial conditions for $\bar{u}_i^{(0)}$, $\bar{v}_i^{(0)}$, ... :

$$\begin{aligned} \bar{u}_i^{(0)} = \bar{u}_{i,t}^{(0)} = \bar{u}_i^{(1)} = \bar{u}_{i,t}^{(1)} = \bar{u}_i^{(2)} = \bar{u}_{i,t}^{(2)} = 0 \quad &, \quad t = 0 \ , \\ \bar{v}_i^{(0)} = \bar{v}_{i,t}^{(0)} = \bar{v}_i^{(1)} = \bar{v}_{i,t}^{(1)} = \bar{v}_i^{(2)} = \bar{v}_{i,t}^{(2)} = 0 \quad &, \quad t = 0 \ , \end{aligned} \qquad (3.21)$$

for $0 \le x \le 1$, $-1 \le y \le 1$. The equation (3.20) also requires that the prescribed functions $p_1^{(2)}$, $p_2^{(2)}$, $q_1^{(2)}$ and $q_2^{(2)}$ all vanish at t = 0 and have vanishing first time derivatives at t = 0.

The stresses corresponding to the inner approximation for the displacements written in equation(3.20) are:

$$\tau_{xxi} = \varepsilon \{ \frac{4(\alpha^2-1)}{\alpha^2} \bar{u}^{(0)}_{i,x} \} + \varepsilon^2 \{ y[\frac{\alpha^2-2}{2\alpha^2} (p_1^{(2)} - p_2^{(2)}) + \frac{4(\alpha^2-1)}{\alpha^2} \bar{v}^{(0)}_{i,xx}] + [\frac{\alpha^2-2}{2\alpha^2} (p_1^{(2)} + p_2^{(2)}) + \frac{4(\alpha^2-1)}{\alpha^2} \bar{u}^{(1)}_{i,x}]\} + O(\varepsilon^3) \quad ,$$

$$\tau_{yyi} = \varepsilon^2 \{ \frac{y}{2}(p_1^{(2)} - p_2^{(2)}) + \frac{1}{2}(p_1^{(2)} + p_2^{(2)})\} + \varepsilon^3 \{(y^2-1)[-\frac{\alpha^2-1}{2\alpha^2} (q^{(2)}_{1,x} - q^{(2)}_{2,x}) - \frac{2(\alpha^2-1)(\alpha^2-2)}{\alpha^4} \bar{u}^{(0)}_{i,xxx}]\} \quad , \tag{3.22}$$

$$\tau_{xyi} = \varepsilon^2 \{ \frac{y}{2}(q_1^{(2)} - q_2^{(2)}) + \frac{1}{2}(q_1^{(2)} + q_2^{(2)})\} + \varepsilon^3 \{(y^2-1)[-\frac{\alpha^2-1}{2\alpha^2} (p^{(2)}_{1,x} - p^{(2)}_{2,x}) + \frac{2(\alpha^2-1)}{\alpha^2} \bar{v}^{(0)}_{i,xxx}]\} \quad .$$

Note that the terms in ε^3 in τ_{xyi}, τ_{yyi} are written explicitly to show that the boundary conditions at $y = \pm 1$ are satisfied by the inner approximation (3.21). The term in ε^3 in τ_{xxi} will not be used and therefore is not written explicitly.

We can now use the results of (3.21) and (3.22) to test u_i, v_i as solutions of the original problem (2.10) and (2.11). We find:

$$u_{i,yy} + \varepsilon(\alpha^2-1)v_{i,xy} + \varepsilon^2(\alpha^2 u_{i,xx} - u_{i,tt}) = O(\varepsilon^3) \quad ,$$

$$\alpha^2 v_{i,yy} + \varepsilon(\alpha^2-1)u_{i,xy} + \varepsilon^2(v_{i,xx} - v_{i,tt}) = O(\varepsilon^3) \quad ,$$

$$\tau_{xyi}(x,1,t;\varepsilon,\delta) = \varepsilon^2 q_1^{(2)} \quad , \qquad \tau_{xyi}(x,-1,t;\varepsilon,\delta) = \varepsilon^2 q_2^{(2)} \quad ,$$

$$\tau_{yyi}(x,1,t;\varepsilon,\delta) = \varepsilon^2 p_1^{(2)} \quad , \qquad \tau_{yyi}(x,-1,t;\varepsilon,\delta) = \varepsilon^2 p_2^{(2)} \quad .$$

Thus the prescribed boundary conditions at $y = \pm 1$ are satisfied exactly by u_i, v_i and the differential equations are satisfied with an error of $O(\varepsilon^3)$. However we can see from equation (3.22) that in general u_i and v_i cannot fulfill the prescribed stress conditions at $x = 0,1$. As discussed following equation (3.20), the inner approximation satisfies prescribed initial conditions. Before proceeding to the later sections of this chapter, where we obtain corrections to u_i and v_i to improve the approximations near $x = 0,1$, we briefly examine some aspects of a formal "symbolic" approach used for finding higher order terms of the inner approximation.

As in chapter I we introduce the notation $L \equiv \frac{\partial}{\partial t}$ and $D \equiv \frac{\partial}{\partial x}$, and formally consider the equations (2.10) as differential equations in the variable y:

$$\begin{aligned} &u,_{yy} + (\alpha^2-1)\varepsilon D\, v,_y + \alpha^2\varepsilon^2 D\, u - \varepsilon^2 L^2 u = 0 \quad , \\ &v,_{yy} + \frac{\alpha^2-1}{\alpha^2}\varepsilon D\, u,y + \frac{1}{\alpha^2}\varepsilon^2 D^2 v - \frac{1}{\alpha^2}\varepsilon^2 L^2 v = 0 \quad , \end{aligned} \tag{3.23}$$

for $-1 < y < 1$, with the boundary conditions,

$$u,_y + \varepsilon Dv = \begin{cases} \varepsilon^2 q_1^{(2)} & , \quad y = 1 \quad , \\ \varepsilon^2 q_2^{(2)} & , \quad y = -1 \quad , \end{cases}$$

$$(\alpha^2-2)\varepsilon Du + \alpha^2 v,_y = \begin{cases} \varepsilon^2 p_1^{(2)} & , \quad y = 1 \quad , \\ \varepsilon^2 p_2^{(2)} & , \quad y = -1 \quad . \end{cases} \tag{3.24}$$

Now "symbolic solutions" for u,v analogous to (I.3.18) can be obtained. These "symbolic solutions" divide naturally into odd and even functions of y. The calculations are lengthy and will not be included here. It is found that the part of v which is even in y, (v_{even}), and the part of u which is odd in y, (u_{odd}), both satisfy equations of the form:

$$[\ (L^2 - 2D^2)^2 \cos\varepsilon\sqrt{D^2 - L^2}\ \frac{\sin\varepsilon\sqrt{D^2 - L^2/\alpha^2}}{\varepsilon\sqrt{D^2 - L^2/\alpha^2}} - \qquad (3.25)$$

$$- \frac{4D^2}{\varepsilon^2}\sqrt{D^2 - L^2}\ \sin\varepsilon\sqrt{D^2 - L^2}\ \cos\varepsilon\sqrt{D^2 - L^2/\alpha^2}\ ]v_{even} = \ldots .$$

When we put $\varepsilon = 0$ in (3.25) we obtain equation (3.10). Assuming an expansion $v_{even} = v^{(0)}_{even} + \varepsilon v^{(1)}_{even} + \ldots$, and equating the coefficient of ε equal to zero in (3.25) leads us to an equation similar to (3.15). It is interesting to examine a truncation of each side of (3.25) at $O(\varepsilon^2)$. The left hand side becomes:

$$[L^2 + \frac{\varepsilon^2}{3}\{(\frac{3}{2} + \frac{1}{2\alpha^2})L^2 - (\frac{4(\alpha^2-1)}{\alpha^2} + 2)L^2D^2 + \frac{4(\alpha^2-1)}{\alpha^2}D^4\}\ ]\ v_{even} = \ldots,$$

This equation has a strong resemblance to the Timoshenko equation (cf. Kolsky [20], p. 53) for the dynamic behavior of plates, which in our present notation may be written:

$$[L^2 + \frac{\varepsilon^2}{3}\{\ \frac{1}{K}L^4 - (\frac{4(\alpha^2-1)}{\alpha^2} + \frac{1}{K})L^2D^2 + \frac{4(\alpha^2-1)}{\alpha^2}D^4\}\ ]\ v_{even} = \ldots ,$$

where K is a "correction factor" usually given a value between 2/3 and 1. The above discussion is not intended to be a rational development of the Timoshenko equation. In fact the arbitary truncation process has no rational basis within the scope of the perturbation techniques used in this thesis. It suggests however that the Timoshenko equation is a combination of the first two orders of the inner approximation.

4. Formal Diffusive Approximation.

In general the inner approximation (3.19) cannot fulfill the prescribed stress condition at $x = 0,1$. To explore the transfer of the prescribed loads into the interior of the plate we must study the solution near the ends in more detail. We expect the rapidly varying prescribed loads at x = 0 to produce propagating pulses which can be described by wave front approximations (see section 6). Using the results of chapter I as a guide we might expect that the slowly varying prescribed loads at x = 0 produce effects which can be described by a "boundary layer" approximation, important only near the end, plus the inner approximation. A preliminary "boundary layer" investigation was conducted near the end x = 0. It was found that the slowly varying part of the prescribed loads produces an effect local to the end only if certain conditions are satisfied. These are: (i) The prescribed average normal stress is zero, (ii) the prescribed average shear stress is zero, (iii) the bending moment resulting from the prescribed normal stress is zero. In general the prescribed loads do not fulfill these three conditions and this means that effects other than a quasi-static boundary layer are produced by the slowly varying prescribed loads at x = 0. The average normal stress at x = 0 can be used to provide a suitable boundary condition for the arbitrary functions $\bar{u}_i^{(0)}$, $\bar{u}_i^{(1)}$, $\bar{u}_i^{(2)}$ in the inner solution. Thus the average normal stress will propagate with a velocity equal to the "plate" speed. So far we have no means of dealing with the average shear stress and the bending moments. Thus before examining in more detail the "boundary layer" approximation near x = 0 and its interaction with the inner approximation, we study the effects produced by the average shear stress and the bending moment prescribed at x = 0.

We consider the idea that the part of the total solution given by the slowly varying prescribed average shear and bending moment at $x = 0$, is important near the end but decays away from the end. However, this decay is more gradual than that of the boundary layer approximation and we allow the possibility that the decay depends on time; in other words perhaps the effect of the average shear stress and bending moment "spreads" or "diffuses" away from the end.

In order to examine a neighborhood of the end $x = 0$ on a length scale longer than that associated with the boundary layer of the next section, but shorter than that of the inner approximation we introduce a new variable z, by the equation:

$$z = \frac{x}{\sqrt{\varepsilon}} \qquad . \tag{4.1}$$

The reasons for the choice of this scaling are not obvious. Trial and error methods using arbitrary scale factors, say ε^a, showed that $a = \frac{1}{2}$ was a significant value allowing the development of an approximation having the required properties.

We assume approximate solutions for the displacements u,v of the form

$$\begin{aligned} u(x,y,t;\varepsilon,\delta) &= u_i(x,y,t;\varepsilon) + u_d(z,y,t;\varepsilon) \quad , \\ v(x,y,t;\varepsilon,\delta) &= v_i(x,y,t;\varepsilon) + v_d(z,y,t;\varepsilon) \quad , \end{aligned} \tag{4.2}$$

where u_d, v_d represent the diffusive approximation describing the effect discussed above. We do not include the parameter δ in both the inner approximation and the diffusive approximation since we shall find later that these approximations can only handle the slowly varying part of the prescribed loads.

We require that $u = u_i + u_d$, $v = v_i + v_d$ satisfy the differential equation (2.10) and the boundary conditions at $y = \pm 1$, (2.11ab). Also we require that the stresses derived from these displacements, which we write as $\tau_{xx} = \tau_{xxi} + \tau_{xxd}$, $\tau_{xy} = \tau_{xyi} + \tau_{xyd}$, $\tau_{yy} = \tau_{yyi} + \tau_{yyd}$, are capable of fulfilling the macroscopic features of the prescribed loads at $x = 0$ (average normal stress, average shear stress, bending moment). However as we have not yet examined the boundary layer near $x = 0$ in detail, it is not yet obvious what boundary conditions to impose on the diffusive approximation and inner approximation at orders higher than that at which the loads are prescribed.

In this section we determine some properties of the diffusive approximation but do not determine it completely. The boundary layer investigation of the next section will demonstrate how to completely determine both the inner approximation and the diffusive approximation.

Using the properties of u_i and v_i derived in the last section (see (3.19), (3.22)) and introducing the new variable z we can derive the following system of equations governing the behavior of u_d and v_d:

$$\begin{aligned} &u_{d,yy} + \sqrt{\varepsilon}(\alpha^2-1)v_{d,yz} + \varepsilon\alpha^2 u_{d,zz} - \varepsilon^2 u_{d,tt} = O(\varepsilon^3) \quad , \\ &v_{d,yy} + \sqrt{\varepsilon}\,\frac{\alpha^2-1}{\alpha^2}\,u_{d,yz} + \varepsilon\frac{1}{\alpha^2}\,v_{d,zz} - \varepsilon^2\frac{1}{\alpha^2}\,v_{d,tt} = O(\varepsilon^3) \quad , \end{aligned} \tag{4.3}$$

for $0 < z < \dfrac{1}{\sqrt{\varepsilon}}$, $-1 < y < 1$, $0 < t < \infty$;

$$\tau_{xyd}(z,\pm 1,t;\varepsilon) = 0 \quad , \quad \tau_{ttd}(z,\pm 1,t;\varepsilon) = 0 \quad , \tag{4.4}$$

for $0 \leq z \leq \dfrac{1}{\sqrt{\varepsilon}}$, $0 \leq t < \infty$; here the stresses are given in terms of the displacements by the following equations:

$$\tau_{xxd} = (\alpha^2-2)v_{d,y} + \sqrt{\varepsilon}\,\alpha^2 u_{d,x},$$

$$\tau_{xyd} = u_{d,y} + \sqrt{\varepsilon}\,v_{d,x},$$

$$\tau_{yyd} = \alpha^2 v_{d,y} + \sqrt{\varepsilon}\,(\alpha^2-2)u_{d,x} \quad . \tag{4.5}$$

Since we expect u_d and v_d to be small for large z, that is, far away from the end, it is natural to replace the domain of equation (4.3) by the domain $0 < z < \infty$, $-1 < y < 1$, $0 < t < \infty$. Then, instead of a boundary condition at the end $z = \frac{1}{\sqrt{\varepsilon}}$, we impose the following decay condition:

$$\lim_{z\to\infty} u_d, v_d = 0 \quad , \quad -1 < y < 1,\ 0 < t < \infty \quad . \tag{4.6}$$

We shall discuss the errors introduced by this approximation of domain at a later stage.

The appropriate formal expansions for u_d and v_d are:

$$u_d(z,y,t;\varepsilon) = u_d^{(0)}(z,y,t) + \sqrt{\varepsilon}\, u_d^{(\frac{1}{2})}(z,y,t) + \dots ,$$

$$v_d(z,y,t;\varepsilon) = v_d^{(0)}(z,y,t) + \sqrt{\varepsilon}\, v_d^{(\frac{1}{2})}(z,y,t) + \dots , \tag{4.7}$$

and proceeding in the standard way the following differential equations and boundary conditions for $u_d^{(0)}$ and $v_d^{(0)}$ are obtained:

$$u_{d,yy}^{(0)} = 0 \quad , \quad v_{d,yy}^{(0)} = 0 \quad ,$$

for $0 < z < \infty$, $-1 < y < 1$, $0 < t < \infty$;

$$\tau_{xyd}^{(0)}(z, \pm 1,t) = 0 \quad , \quad \tau_{yyd}^{(0)}(z,\pm 1,t) = 0 \quad ,$$

for $0 \leq z < \infty$, $0 < t < \infty$;

$$\lim_{z\to\infty} u_d^{(0)}, v_d^{(0)} = 0 \quad ,$$

for $-1 \leq y \leq 1$, $0 \leq t < \infty$. We do not expect this system to determine $u_d^{(0)}$, $v_d^{(0)}$; however it does predict that the solutions have the following form:

$$u_d^{(0)}(z,y,t) = \bar{u}_d^{(0)}(z,t) \quad , \quad v_d^{(0)}(z,y,t) = \bar{v}_d^{(0)}(z,t) , \tag{4.8}$$

where $\bar{u}_d^{(0)}$ and $\bar{v}_d^{(0)}$ are arbitrary apart from the condition:

$$\lim_{z\to\infty} \bar{u}_d^{(0)} \quad , \quad \bar{v}_d^{(0)} = 0 \quad . \tag{4.9}$$

To obtain more information concerning the functions $\bar{u}_d^{(0)}$ and $\bar{v}_d^{(0)}$ we must investigate higher order terms. Unfortunately to develop all the conditions governing the behavior of $\bar{v}_d^{(0)}$ possible in this section, we have to investigate the form of the solutions for $u_d^{(\frac{1}{2})}$, $v_d^{(\frac{1}{2})}$, ... , $u_d^{(2)}$, $v_d^{(2)}$.

Before proceeding to the next order we note that the stresses corresponding to $u_d^{(0)}$, $v_d^{(0)}$ all vanish:

$$\tau_{xxd}^{(0)} = 0 \quad , \quad \tau_{xyd}^{(0)} = 0 \quad , \quad \tau_{yyd}^{(0)} = 0 \quad . \tag{4.10}$$

The differential equations and boundary conditions for $u_d^{(\frac{1}{2})}$ and $v_d^{(\frac{1}{2})}$ are:

$$u_{d,yy}^{(1/2)} + (\alpha^2-1)v_{d,zy}^{(0)} = 0 \quad , \quad v_{d,yy}^{(1/2)} + \frac{\alpha^2-1}{\alpha^2} u_{d,zy}^{(0)} = 0$$

for $0 < z < \infty$, $-1 < y < 1$, $0 < t < \infty$;

$$\tau_{xyd}^{(1/2)}(z,\pm 1,t) = 0 \quad , \quad \tau_{yyd}^{(1/2)}(z,\pm 1,t) = 0 \quad ,$$

for $0 \leq z < \infty$, $0 \leq t < \infty$;

$$\lim_{z\to\infty} u_d^{(1/2)}, v_d^{(1/2)} = 0 \quad ,$$

for $-1 \leq y \leq 1$, $0 \leq t < \infty$. Using the expressions for stresses in terms of displacements written in (4.5) and the results for $u_d^{(0)}$ and $v_d^{(0)}$ of (4.8) we can obtain the form of the solutions for $u_d^{(1/2)}$ and $v_d^{(1/2)}$ suggested by the above system. These are:

$$u_d^{(1/2)}(x,y,t) = -y\,\bar{v}_{d,z}^{(0)}(z,t) + \bar{u}_d^{(1/2)}(z,t) \quad ,$$
$$v_d^{(1/2)}(z,y,t) = -y\,\frac{\alpha^2-2}{\alpha^2}\,\bar{u}_{d,z}^{(0)}(z,t) + \bar{v}_d^{(1/2)}(z,t) \quad , \tag{4.11}$$

Here $\bar{u}^{(1/2)}(z,t)$ and $\bar{v}^{(1/2)}(z,t)$ are arbitrary except for the condition:

$$\lim_{z\to\infty} \bar{u}_d^{(1/2)} \quad , \quad \bar{v}_d^{(1/2)} = 0 \quad , \quad 0 \leq t < \infty \quad . \tag{4.12}$$

The stresses corresponding to $u_d^{(1/2)}$ and $v_d^{(1/2)}$ are:

$$\tau_{xxd}^{(1/2)} = \frac{4(\alpha^2-1)}{\alpha^2}\,\bar{u}_{d,z}^{(0)} \quad , \quad \tau_{xyd}^{(1/2)} = 0, \quad \tau_{yyd}^{(1/2)} = 0 \; , \tag{4.13}$$

We now move on to the next order and examine the following differential equations and boundary conditions for $u_d^{(1)}$ and $v_d^{(1)}$:

$$u_{d,yy}^{(1)} + (\alpha^2-1)v_{d,zy}^{(1/2)} + \alpha^2 u_{d,zz}^{(0)} = 0 \quad ,$$
$$v_{d,yy}^{(1)} + \frac{\alpha^2-1}{\alpha^2}\,u_{d,zy}^{(1/2)} + \frac{1}{\alpha^2}\,v_{d,zz}^{(0)} = 0 \quad ,$$

for $0 < z < \infty$, $-1 < y < 1$, $0 < t < \infty$;

$$\tau_{xyd}^{(1)}(z,\pm1,t) = 0 \quad , \quad \tau_{yyd}^{(1)}(z,\pm1,t) = 0 \quad ,$$

for $0 < z < \infty$, $0 < t < \infty$;

$$\lim_{z\to\infty} u_d^{(1)} \quad , \quad v_d^{(1)} = 0 \quad ,$$

for $-1 < y < 1$, $0 < t < \infty$. After substituting the expressions obtained earlier for $u_d^{(0)}$, $v_d^{(0)}$, (4.8), and $u_d^{(1/2)}$, $v_d^{(1/2)}$, (4.11), we can find the form of the solutions of the above system to be:

$$u_d^{(1)}(z,y,t) = -y\,\bar{v}_{d,z}^{(1/2)}(z,t) + \bar{u}_d^{(1)}(z,t) \quad , \qquad (4.14)$$

$$v_d^{(1)}(z,y,t) = \frac{y^2}{2}\,\frac{\alpha^2-2}{\alpha^2}\,\bar{v}_{d,zz}^{(0)}(z,t) - y\,\frac{\alpha^2-2}{\alpha^2}\,\bar{u}_{d,z}^{(1/2)}(z,t) + \bar{v}_d^{(1)}$$

In deriving these expressions we obtain restrictions on the previously arbitrary function $\bar{u}_d^{(0)}$. The satisfaction of the condition $\tau_{xyd}^{(1)}(z,\pm1,t)=0$ requires that:

$$\bar{u}_{d,zz}^{(0)} = 0 \qquad (4.15)$$

This differential equation, together with the decay condition (4.9), provides the following solution for $\bar{u}_d^{(0)}$:

$$\bar{u}_d^{(0)} \equiv 0 \quad . \qquad (4.16)$$

The stresses corresponding to $\bar{u}_d^{(1)}$ and $\bar{v}_d^{(1)}$ are:

$$\tau_{xxd}^{(1)} = -y\frac{4(\alpha^2-1)}{\alpha^2}\,\bar{v}_{d,zz}^{(0)} + \frac{4(\alpha^2-1)}{\alpha^2}\,\bar{u}_{d,z}^{(1/2)} \quad , \quad \tau_{xyd}^{(1)}=0 \quad , \quad \tau_{yyd}^{(1)} = 0 \ . \qquad (4.17)$$

Since $\tau_{xxd}^{(1)}$ is partly an odd function of y, it may be able to describe the prescribed bending moment at x = 0 (for example, by taking $\bar{v}_{d,zz}^{(0)}(0,t) = -\frac{3\alpha^2}{8(\alpha^2-1)}\int_{-1}^{1} y\, f^{(1)}(y,t)dy$). In (4.14) the functions $\bar{u}_d^{(1)}$ and $\bar{v}_d^{(1)}$ are arbitrary except for the condition:

$$\lim_{z\to\infty} \bar{u}_d^{(1)},\ \bar{v}_d^{(1)} = 0 \quad , \qquad -1 \le y \le 1,\ 0 \le t < \infty. \qquad (4.18)$$

To find out more about $\bar{v}_d^{(0)}$, $\bar{u}_d^{(1/2)}$, $\bar{v}_d^{(1/2)}$,..., we proceed with the above scheme. The system of equations for $u_d^{(3/2)}$, $v_d^{(3/2)}$ is:

$$u_{d,yy}^{(3/2)} + (\alpha^2-1)v_{d,zy}^{(1)} + \alpha^2 u_{d,zz}^{(1/2)} = 0 \quad ,$$

$$v_{d,yy}^{(3/2)} + \frac{\alpha^2-1}{\alpha^2}\,u_{d,zy}^{(1)} + \frac{1}{\alpha^2}\,v_{d,zz}^{(1/2)} = 0 \quad ,$$

for $0 < z < \infty$, $-1 < y < 1$, $0 < t < \infty$;

$$\tau^{(3/2)}_{xyd}(z,\pm 1,t) = 0 \quad , \quad \tau^{(3/2)}_{yyd}(z,\pm 1,t) = 0 \quad ,$$

for $0 \leq z < \infty$, $0 \leq t < \infty$;

$$\lim_{z\to\infty} u^{(3/2)}_d \quad , \quad v^{(3/2)}_d = 0 \quad ,$$

for $-1 \leq y \leq 1$, $0 \leq t < \infty$. Using the results obtained from the lower order systems we find that $u^{(3/2)}_d$, $v^{(3/2)}_d$ have the following form:

$$u^{(3/2)}_d(z,y,t) = \frac{y^3}{6}\ \frac{3\alpha^2-2}{\alpha^2}\,\bar{v}^{(0)}_{d,zzz} - y\left(\bar{v}^{(1)}_{d,z} + \frac{2(\alpha^2-1)}{\alpha^2}\,\bar{v}^{(0)}_{d,zzz}\right) + \bar{u}^{(3/2)}_d$$

$$v^{(3/2)}_d(z,y,t) = \frac{y^2}{2}\ \frac{\alpha^2-2}{}\,\bar{v}^{(1/2)}_{d,zz} - y\,\frac{\alpha^2-2}{\alpha^2}\,\bar{u}^{(1)}_{d,z} + \bar{v}^{(3/2)}_d \quad . \qquad (4.19)$$

Here $u^{(3/2)}_d$ and $\bar{v}^{(3/2)}_d$ are arbitrary functions of z and t satisfying the condition:

$$\lim_{z\to\infty} \bar{u}^{(3/2)}_d \quad , \quad \bar{v}^{(3/2)}_d = 0 \quad . \qquad (4.20)$$

In deriving the results of (4.20) we find a restriction on $\bar{u}^{(1/2)}_d$. The condition $^{(3/2)}_{xyd}(z,\ 1,t) = 0$ requires that:

$$\bar{u}^{(1/2)}_{d,zz} = 0 \quad . \qquad (4.21)$$

This equation with the decay condition (4.12) has the solution:

$$\bar{u}^{(1/2)}_d \equiv 0 \quad ; \qquad (4.22)$$

a result exactly similar to (4.16). The stresses corresponding to $\bar{u}^{(3/2)}_d$, $\bar{v}^{(3/2)}_d$ are:

$$\tau^{(2/3)}_{xxd} = y\,\frac{4(\alpha^2-1)}{\alpha^2}\,\bar{v}^{(1/2)}_{d,zz} + \frac{4(\alpha^2-1)}{\alpha^2}\,\bar{u}^{(1)}_{d,z} \quad ,$$

$$\tau^{(2/3)}_{xyd} = (y^2-1)\,\frac{2(\alpha^2-1)}{\alpha^2}\,\bar{v}^{(0)}_{d,zzz} \quad , \qquad (4.23)$$

$$\tau^{(3/2)}_{yyd} = 0$$

The form of these stresses corresponds to an associated non-zero bending moment and non-zero average shear stress.

The system of equations governing the behavior of $u_d^{(2)}$ and $v_d^{(2)}$ are:

$$u^{(2)}_{d,yy} + (\alpha^2-1)v^{(3/2)}_{d,zy} + \alpha^2 u^{(1)}_{d,zz} - u^{(0)}_{d,tt} = 0 \quad ,$$

$$v^{(2)}_{d,yy} + \frac{\alpha^2-1}{\alpha^2}\, u^{(3/2)}_{d,zy} + \frac{1}{\alpha^2}\, v^{(1)}_{d,zz} - \frac{1}{\alpha^2}\, v^{(1)}_{d,tt} = 0 \quad ,$$

for $0 < z < \infty$, $-1 < y < 1$, $0 < t < \infty$;

$$\tau^{(2)}_{xyd}(z,\pm 1,t) = 0 \quad , \quad \tau^{(2)}_{yyd}(z,\pm 1,t) = 0 \quad ,$$

for $0 \le z < \infty$, $0 \le t < \infty$;

$$\lim_{z\to\infty} u_d^{(2)} \quad , \quad v_d^{(2)} = 0 \quad ,$$

for $-1 \le y \le 1$, $0 \le t < \infty$. Using the results of earlier orders we can obtain the following forms for $u_d^{(2)}$ and $v_d^{(2)}$:

$$u_d^{(2)}(z,y,t) = \frac{y^3}{6}\ \frac{3\alpha^2-2}{\alpha^2}\ \bar{v}^{(1/2)}_{d,zzz} - y\left(\bar{v}^{(3/2)}_{d,z} + \frac{2(\alpha^2-1)}{\alpha^2}\bar{v}^{(1/2)}_{d,zzz}\right) + \bar{u}_d^{(2)} \quad ,$$

$$v_d^{(2)}(z,y,t) = -\frac{y^4}{24}\,\frac{3\alpha^2-4}{\alpha^2}\,\bar{v}^{(0)}_{d,zzz} + \frac{y^2}{2}\left(\frac{\alpha^2-2}{\alpha^2}\,\bar{v}^{(2)}_{d,zz} + \frac{2}{3}(\alpha^2-1)(3\alpha^2-5)\bar{v}^{(0)}_{d,zzzz}\right)$$

$$- y\,\frac{\alpha^2-2}{\alpha^2}\,\bar{u}^{(3/2)}_{d,z} + \bar{v}_d^{(2)} \quad .$$

In deriving these results some interesting restrictions are obtained for $\bar{u}^{(1)}$ and $\bar{v}_d^{(0)}$. The boundary condition $\tau^{(2)}_{xyd}(z,\pm 1,t) = 0$, requires that $\bar{u}_d^{(1)}$ satisfies the differential equation:

$$\bar{u}^{(1)}_{d,zz} = 0 \quad .$$

With the decay condition (4.18) this means that

$$\bar{u}_d^{(1)} \equiv 0 \quad .$$

Satisfying the boundary condition $\tau^{(2)}_{yyd}(z, 1,t) = 0$ requires the following restriction on $\bar{v}^{(0)}_d$:

$$\bar{v}^{(0)}_{d,tt} + \frac{4}{3}\,\frac{\alpha^2-1}{\alpha^2}\,\bar{v}^{(0)}_{d,zzzz} = 0 \qquad . \tag{4.27}$$

We can write this in the original physical variables as follows:

$$\bar{v}^{(0)}_{d,TT} + \frac{h^3}{3}\,\frac{E}{\rho(1-\sigma^2)}\,\bar{v}^{(0)}_{d,XXXX} = 0 \qquad , \tag{4.28}$$

where E is Young's modulus, σ is Poisson's ratio, ρ is the density for the material of the plate, and 2h is the plate thickness. Equation (4.28) is easily recognizable as the Euler-Bernoulli equation for the dynamic bending of a plate (cf: Love [22], p. 496) . To fully determine solutions for $\bar{v}^{(0)}_d$ from (4.27) we require two boundary conditions at z = 0 as well as the decay condition (4.9). We expect the conditions at z = 0 to be connected with the average shear stress and bending moment of the applied loads. These will be discussed in detail in the next section. Initial conditions are also required to determine $\bar{v}^{(0)}_d$ and we assume that u_d and v_d fulfill the initial quiescence condition. Then we find that:

$$\bar{v}^{(0)}_d(z,0) = 0 \qquad , \qquad \bar{v}^{(0)}_{d,t}(z,0) = 0 \qquad . \tag{4.29}$$

We cannot say much more about the diffusive approximation until the investigation of the boundary layer approximation is completed in the next section. Before temporarily leaving the study of the diffusive approximation (we reconsider the inner and diffusive approximations in section 6) we summarize the results obtained so far. We know that u_d, v_d up to $O(\varepsilon^2)$ have solution which produce stresses having the following forms:

$$\tau_{xxd}=\varepsilon[-y\frac{4(\alpha^2-1)}{\alpha^2}\bar{v}^{(0)}_{d,zz}]+\varepsilon^{3/2}[-y\frac{4(\alpha^2-1)}{\alpha^2}\bar{v}^{(1/2)}_{d,zz}]+\varepsilon^2[\frac{y^3}{6}\frac{8(\alpha^2-1)}{\alpha^2}\bar{v}^{(0)}_{d,zzzz}$$

$$+ y(-\frac{4(\alpha^2-1)}{\alpha^2}\bar{v}^{(1)}_{d,zz} + \frac{2}{3}(\alpha^2-1)(3\alpha^4-11\alpha^2+7)\bar{v}^{(0)}_{d,zzzz})] \quad ,$$

$$\tau_{xyd}=\varepsilon^{3/2}[(y^2-1)\frac{2(\alpha^2-1)}{\alpha^2}\bar{v}^{(0)}_{d,zzz}]+\varepsilon^2[(y^2-1)\frac{2(\alpha^2-1)}{\alpha^2}\bar{v}^{(1/2)}_{d,zzz}] \quad , \qquad (4.30)$$

$$\tau_{yyd}=\varepsilon^2[-\frac{2}{3}(y^3-y)\frac{\alpha^2-1}{\alpha^2}\bar{v}^{(0)}_{d,zzzz}] \quad .$$

In a similar manner we can develop a diffusive approximation important near the end $x = 1$ by introducing the scaled variable z' defined by the equation:

$$z' = \frac{1-x}{\sqrt{\varepsilon}} \quad . \qquad (4.31)$$

If we write this diffusive approximation as $u_{d'}$ and $v_{d'}$ it can be shown that the lowest order term, $v^{(0)}_{d'}$, is independent of y, say $\bar{v}^{(0)}_{d'}(z',t)$ and satisfies the differential equation:

$$\bar{v}^{(0)}_{d',tt} + \frac{4(\alpha^2-1)}{3\alpha^2}\bar{v}^{(0)}_{d',z'z'z'z'} = 0 \quad , \qquad (4.32)$$

for $0 < t < \infty$, $0 < z' < \infty$; and the initial conditions

$$\bar{v}^{(0)}_{d'}(z',0) = 0 \quad , \quad \bar{v}^{(0)}_{d',t}(z',0) = 0 \quad ,$$

for 0 z' . (4.33)

Similar results can be found for higher order terms $\bar{v}^{(1/2)}_{d'}$, $\bar{v}^{(1)}_{d'}$.

5. Formal Boundary Layer Approximations.

The approximate solution consisting of the sum of the inner approximation and the diffusive approximation cannot in general satisfy the prescribed boundary conditions at $x = 0,1$. To explore the transfer of the prescribed loads at $x = 0$ into the interior, and to investigate the solution near $x = 0$, we introduce a 'boundary layer' variable defined by the scaling:

$$\zeta = \frac{x}{\varepsilon} \quad . \tag{5.1}$$

The following investigation will provide boundary conditions to fully determine the inner approximation and the diffusive approximation. We assume approximations of the form:

$$\begin{aligned} u(x,y,t;\varepsilon,\delta) &= u_i(x,y,t;\varepsilon) + u_d(\tfrac{x}{\sqrt{\varepsilon}},y,t;\varepsilon) + u_\ell(\zeta,y,t;\varepsilon) \ , \\ v(x,y,t;\varepsilon,\delta) &= v_i(x,y,t;\varepsilon) + v_d(\tfrac{x}{\sqrt{\varepsilon}},y,t;\varepsilon) + v_\ell(\zeta,y,t;\varepsilon) \ , \end{aligned} \tag{5.2}$$

where u_ℓ, v_ℓ represent tentative boundary layer corrections near $x = 0$. We have not included the parameter δ in any of the above approximations. This anticipates the conclusion that the inner approximation, diffusive approximation and boundary layer approximation can only describe slowly varying effects. We require $u = u_i+u_d+u_\ell$, $v = v_i+v_d+v_\ell$ to satisfy the differential equations (2.10), and the corresponding stresses to satisfy the boundary conditions (2.11abcd). Then using the results of sections 3 and 4, we obtain the following differential equations and boundary conditions for u_ℓ and v_ℓ:

$$\begin{aligned} u_{\ell,yy} + (\alpha^2-1)v_{\ell,\zeta y} + \alpha^2 u_{\ell,\zeta\zeta} - \varepsilon^3 u_{\ell,tt} &= O(\varepsilon^3) \ , \\ \alpha^2 v_{\ell,yy} + (\alpha^2-1)u_{\ell,\zeta y} + v_{\ell,\zeta\zeta} - \varepsilon^3 v_{\ell,tt} &= O(\varepsilon^3) \ , \end{aligned} \tag{5.3}$$

for $0 < \zeta < 1/\varepsilon$, $-1 < y < 1$, $0 < t < \infty$;

$$\tau_{xy\ell}(\zeta,\pm 1,t;\varepsilon) = 0 \quad , \quad \tau_{yy\ell}(\zeta,\pm 1,t;\varepsilon) = 0 \quad , \qquad (5.4a)$$

for $0 \le \zeta \le \frac{1}{\varepsilon}$, $0 \le t < \infty$;

$$\tau_{xx\ell}(0,y,t;\varepsilon) = \varepsilon f^{(1)}(y,t) - \tau_{xxi}(0,y,t;\varepsilon) - \tau_{xxd}(0,y,t;\varepsilon) \quad , \qquad (5.4b)$$

$$\tau_{xy\ell}(0,y,t;\varepsilon) = \varepsilon^{3/2} g^{(3/2)}(y,t) - \tau_{xyi}(0,y,t;\varepsilon) - \tau_{xyd}(0,y,t;\varepsilon) \quad . \qquad (5.4c)$$

for $-1 \le y \le 1$, $0 \le t < \infty$. Since we expect such a boundary layer approximation to be important only near the end $x = 0$, it is natural to replace the domain of the differential equation (5.3) by $0 < \zeta < \infty$, $-1 < y < 1$, $0 < t < \infty$. Thus instead of boundary conditions at $\zeta = \frac{1}{\varepsilon}$ we impose the following decay condition:

$$\lim_{\zeta\to\infty} \tau_{xx\ell}, \tau_{xy\ell}, \tau_{yy\ell} = 0 \quad , \quad -1 \le y \le 1, \quad 0 \le t < \infty . \qquad (5.5)$$

(For the analogous static problem, the error introduced in this way turns out to be $O(e^{-x/\varepsilon})$).

We now assume expansions for u_ℓ , v_ℓ of the form:

$$\begin{aligned} u_\ell(\zeta,y,t;\varepsilon) &= \varepsilon u_\ell^{(1)}(\zeta,y,t) + \varepsilon^{3/2} u_\ell^{(3/2)}(\zeta,y,t) + \ldots \quad , \\ v_\ell(\zeta,y,t;\varepsilon) &= \varepsilon v_\ell^{(1)}(\zeta,y,t) + \varepsilon^{3/2} v_\ell^{(3/2)}(\zeta,y,t) + \ldots \quad . \end{aligned} \qquad (5.6)$$

Following the standard procedure we obtain a system of differential equations and boundary conditions, which, written in terms of the stresses $\tau^{(1)}_{xx\ell}, \tau^{(1)}_{xy\ell}, \tau^{(1)}_{yy\ell}$, is:

$$\tau^{(1)}_{xx\ell,\zeta} + \tau^{(1)}_{xy\ell,y} = 0 \quad , \quad \tau^{(1)}_{xy\ell,\zeta} + \tau^{(1)}_{yy\ell,y} = 0 \quad , \qquad (5.7)$$

for $0 < \zeta < \infty$, $-1 < y < 1$, $0 < t < \infty$;

$$\tau^{(1)}_{xy}(\zeta,\pm 1,t) = 0 \quad , \quad \tau^{(1)}_{yy}(\zeta,\pm 1,t) = 0 \quad , \tag{5.8}$$

for $0 \leq \zeta < \infty$, $0 \leq t < \infty$;

$$\tau^{(1)}_{xx\ell}(0,y,t) = f^{(1)}(y,t) - \frac{4(\alpha^2-1)}{\alpha^2}\,[\bar{u}^{(0)}_{i,x}(0,t) - y\bar{v}^{(0)}_{d,zz}(0,t)] \; , \tag{5.9a}$$

$$\tau^{(1)}_{xy\ell}(0,y,t) = 0 \quad , \tag{5.9b}$$

for $-1 \leq y \leq 1$, $0 \leq t < \infty$;

$$\lim_{\zeta\to\infty} \tau^{(1)}_{xx\ell} \, , \tau^{(1)}_{xy\ell}, \tau^{(1)}_{yy\ell} = 0 \quad , \tag{5.10}$$

for $-1 \leq y \leq 1$, $0 \leq t < \infty$. These equations are merely the equations of static elasticity with t appearing as a parameter. No convenient explicit solution for the stresses can be found (cf. Johnson and Little [16]). However by integrating equations (5.7) over their domain, we obtain a condition (restricting the prescribed stresses at $\zeta = 0$) necessary for the existence of a solution for the above system. From (5.7) it can be observed that if a solution exists, then:

$$0 = \int_{-1}^{1} \int^{\infty} [\tau^{(1)}_{xx\ell,\zeta} + \tau^{(1)}_{xy\ell,y}] \, d\zeta dy \quad ,$$

$$0 = \int_{-1}^{1} \int^{\infty} [\tau^{(1)}_{xy\ell,\zeta} + \tau^{(1)}_{yy\ell,y}] \, d\zeta dy \quad .$$

Assuming that we can reverse the order of integration where necessary, using the boundary condition (5.8) and the decay condition (5.10) we find:

$$\int_{-1}^{1} \tau^{(1)}_{xx\ell}(0,y,t)dy = 0 \quad , \quad \int_{-1}^{1} \tau^{(1)}_{xy\ell}(0,y,t)dy = 0 \quad . \tag{5.11}$$

If a solution exists the first equation of (5.7) shows that:

$$0 = \int_{-1}^{1}\int_{0}^{\infty} y(\tau^{(1)}_{xx\ell,\zeta}+\tau^{(1)}_{xy\ell,y})d\zeta dy$$

After integrating by parts we can simplify this equation using the second result of (5.11) and the boundary condition (5.7). We find that:

$$\int_{-1}^{1} y\ \tau^{(1)}_{xx\ell}(0,y,t)dy = 0 \quad . \tag{5.12}$$

Conditions (5.11) and (5.12) are satisfied by the prescribed stresses (5.9a), (5.9b) if:

$$\bar{u}^{(0)}_{i,x}(0,t) = \frac{\alpha^2}{8(\alpha^2-1)} \int_{-1}^{1} f^{(1)}(y,t)dy \quad , \tag{5.13}$$

$$\bar{v}^{(0)}_{d,zz}(0,t) = -\frac{3\alpha^2}{8(\alpha^2-1)} \int_{-1}^{1} y\ f^{(1)}(y,t)dy \quad , \tag{5.14}$$

for $0 \leq t < \infty$. These conditions provide suitable boundary conditions for $\bar{u}^{(0)}_i$ and $\bar{v}^{(0)}_d$ at the end $x = 0$. The equations (5.12) and (5.14) are the main results of this section. We do not attempt to develop explicit solutions for $u^{(1)}_\ell$ and $v^{(1)}_\ell$, but merely note before proceeding that Johnson and Little [16] have obtained useful results in relation to eigenfunction expansion solutions for the system of equations (5.7), (5.8) and (5.10).

To demonstrate the complex interaction bewteen the diffusive approximation, inner approximation and boundary layer approximations at higher orders we now investigate the next two orders.

The stresses $\tau^{(3/2)}_{xx\ell}$, $\tau^{(3/2)}_{xy\ell}$ and $\tau^{(3/2)}_{yy\ell}$ satisfy the following equations:

$$\tau^{(3/2)}_{xx\ell,\zeta} + \tau^{(3/2)}_{xy\ell,y} = 0 \quad , \tag{5.15}$$

$$\tau^{(3/2)}_{xy\ell,\zeta} + \tau^{(3/2)}_{yy\ell,y} = 0 \quad ,$$

for $0 < \zeta < \infty$, $-1 < y < 1$, $0 < t < \infty$;

$$\tau^{(3/2)}_{xy\ell}(\zeta,\pm 1,t) = 0 \quad , \quad \tau^{(3/2)}_{yy\ell}(\zeta,\pm 1,t) = 0 \qquad , \tag{5.16}$$

for $0 \leq \zeta < \infty$, $0 \leq t < \infty$;

$$\tau^{(3/2)}_{xx\ell}(0,y,t) = -y\,\frac{4(\alpha^2-1)}{\alpha^2}\,\bar{v}^{(1/2)}_{d,zz}(0,t) \qquad , \tag{5.17}$$

$$\tau^{(3/2)}_{xy\ell}(0,y,t) = g^{(3/2)}(y,t)-(y^2-1)\frac{2(\alpha^2-1)}{\alpha^2}\,\bar{v}^{(0)}_{d,zzz}(0,t) \qquad , \tag{5.18}$$

for $-1 \leq y \leq 1$, $0 \leq t < \infty$;

$$\lim_{\zeta\to\infty} \tau^{(3/2)}_{xx\ell}, \tau^{(3/2)}_{xy\ell}, \tau^{(3/2)}_{yy\ell} = 0 \qquad , \tag{5.19}$$

for $-1 \leq y \leq 1$, $0 \leq t < \infty$. Using methods similar to those used in investigating the system (5.7)-(5.10) we can find a necessary condition for existence of a solution of the above system. It is:

$$\int_{-1}^{1} \tau^{(3/2)}_{xx\ell}(0,y,t)dy = 0 \qquad , \tag{5.20}$$

$$\int_{-1}^{1} \tau^{(3/2)}_{xy\ell}(0,y,t)dy = 0 \qquad , \tag{5.21}$$

$$\int_{-1}^{1} y\tau^{(3/2)}_{xx\ell}(0,y,t)dy = 0 \qquad , \tag{5.22}$$

for $0 \leq t < \infty$. Substituting the prescribed conditions (5.9ab) into the equations we obtain the results below:

$$\bar{v}^{(0)}_{d,zzz}(0,t) = \frac{3\alpha^2}{8(\alpha^2-1)} \int_{-1}^{1} g^{(3/2)}(y,t)dy \qquad , \tag{5.23}$$

$$\bar{v}^{(1/2)}_{d,zz}(0,t) = 0 \qquad , \tag{5.24}$$

for $0 \le t < \infty$. These results provide suitable boundary conditions for $\bar{v}_d^{(0)}$, $\bar{v}_d^{(1/2)}$.

We consider one additional order before examining $\tau_{xx\ell}^{(1)}, \ldots$ $\tau_{xy\ell}^{(3/2)}, \ldots$ in more detail. The system of equations governing the behavior of $u_\ell^{(2)}$, $v_\ell^{(2)}$ interact with $u_\ell^{(1)}$, $v_\ell^{(1)}$. We can write them in terms of stresses as follows:

$$\tau_{xx\ell,\zeta}^{(2)} + \tau_{xy\ell,y}^{(2)} = u_{\ell,tt}^{(1)} \quad , \quad \tau_{xy\ell,\zeta}^{(2)} + \tau_{yy\ell,y}^{(2)} = v_{\ell,tt}^{(1)} \quad , \tag{5.25}$$

for $0 < \zeta < \infty$, $-1 < y < 1$, $0 < t < \infty$;

$$\tau_{xx\ell}^{(2)}(\zeta,\pm 1,t) = 0 \quad , \quad \tau_{yy\ell}^{(2)}(\zeta,\pm 1,t) = 0 \quad , \tag{5.26}$$

for $0 \le \zeta < \infty$, $0 \le t < \infty$;

$$\begin{aligned}\tau_{xx\ell}^{(2)}(0,y,t) = {} & -y\Big[\frac{(\alpha^2-2)}{2\alpha^2}\,(p_1^{(2)}(0,t) - p_2^{(2)}(0,t)) + \frac{4(\alpha^2-1)}{\alpha^2}\,\bar{v}_{i,xx}^{(0)}(0,t)\Big] - \\ & - \Big[\frac{\alpha^2-2}{2\alpha^2}(p^{(2)}(0,t)+p^{(2)}(0,t)) + \frac{4(\alpha^2-1)}{\alpha^2}\,\bar{u}^{(1)}(0,t)\Big] - \frac{y^3}{3}\frac{4(\alpha^2-1)}{\alpha^2}\bar{v}_{d,zzzz}^{(0)}(0,t) \\ & - \Big[-\frac{4(\alpha^2-1)}{\alpha^2}\,\bar{v}_{d,zz}^{(1)}(0,t) + \frac{2}{3}(\alpha^2-1)(3\alpha^4-11\alpha^2+7)\bar{v}_{d,zzzz}^{(0)}(0,t)\Big] \,,\end{aligned} \tag{5.27}$$

$$\begin{aligned}\tau_{xy\ell}^{(2)}(0,y,t) = {} & -\frac{y}{2}(q_1^{(2)}(0,t) - q_2^{(2)}(0,t)) - \frac{1}{2}(q_1^{(2)}(0,t) + q_2^{(2)}(0,t)) - \\ & - (y^2-1)\,\frac{2(\alpha^2-1)}{\alpha^2}\,\bar{v}_{d,zzz}^{(1/2)}(0,t) \quad ,\end{aligned} \tag{5.28}$$

for $-1 \le y \le 1$, $0 \le t < \infty$;

$$\lim_{\zeta\to\infty} \tau_{xx\ell}^{(2)},\ \tau_{xy\ell}^{(2)},\ \tau_{yy\ell}^{(2)} = 0 \quad , \tag{5.29}$$

for $-1 \le y \le 1$, $0 \le t < \infty$. Even at this order interaction between the boundary layer approximation, the diffusive approximation, and the inner approximation has become quite complex. Proceeding as before we obtain a necessary condition for the existence of a solution of the above system as follows:

$$\int_{-1}^{1} \tau_{xx\ell}^{(2)}(0,y,t)dy = \int_{-1}^{1}\int^{\infty} u_{\ell,tt}^{(1)}(\zeta,y,t)d\zeta dy \quad , \tag{5.30}$$

$$\int_{-1}^{1} \tau_{xy\ell}^{(2)}(0,y,t)dy = \int_{-1}^{1}\int_{o}^{\infty} v_{\ell,tt}^{(1)}(\zeta,y,t)d\zeta dy \quad , \tag{5.31}$$

$$\int_{-1}^{1} y\tau_{xx\ell}^{(2)}(0,y,t)dy = \int_{-1}^{1}\int_{o}^{\infty} yu_{\ell,tt}^{(1)}(\zeta,y,t)d\zeta dy - \int_{-1}^{1}\int_{o}^{\infty}\int_{\zeta}^{\infty} v_{\ell,tt}^{(1)}(z,y,t)dzd\zeta dy, \tag{5.32}$$

for $0 \leq t < \infty$. We assume at present that the infinite integrals on the right hand sides of equations (5.30), and (5.32) all converge (this assumption is discussed subsequently). We can therefore substitute the stresses prescribed at $\zeta = 0$ to obtain the following conditions:

$$\bar{u}_{i,x}^{(1)}(0,t) = \frac{2-\alpha^2}{16(\alpha^2-1)}[p_1^{(2)}(0,t)+p_2^{(2)}(0,t)] - \frac{\alpha^2}{8(\alpha^2-1)}\int_{-1}^{1}\int_{o}^{\infty} u_{\ell,tt}^{(1)}(\zeta,y,t)d\zeta dy, \tag{5.33}$$

$$\bar{v}_{d,zzz}^{(1/2)}(0,t) = \frac{3\alpha^2}{16(\alpha^2-1)}[q_1^{(2)}(0,t)+q_2^{(2)}(0,t)] + \frac{3\alpha^2}{8(\alpha^2-1)}\int_{-1}^{1}\int_{o}^{\infty} v_{\ell,tt}^{(1)}(\zeta,y,t)d\zeta dy, \tag{5.34}$$

$$\bar{v}_{d,zz}^{(1)}(0,t) = \int_{-1}^{1}\int_{o}^{\infty} yu_{\ell,tt}^{(1)}(\zeta,y,t)d\zeta dy - \int_{-1}^{1}\int_{o}^{\infty}\int_{\zeta}^{\infty} v_{\ell,tt}^{(1)}(z,y,t)dzd\zeta dy +$$
$$+\frac{\alpha^2-2}{8(\alpha^2-1)}[p_1^{(2)}(0,t)-p_2^{(2)}(0,t)] + \frac{1}{30}(6+35\alpha^2-55\alpha^4+15\alpha^6)\bar{v}_{d,zzzz}^{(0)}(0,t) +$$
$$+\bar{v}_{i,xx}^{(0)}(0,t) \quad , \tag{5.35}$$

for $0 \leq t < \infty$.

These equations show that at this order, interaction is occurring between all three of the approximations considered (inner, diffusive and boundary layer).

Until now we have assumed that the boundary layer approximation is incapable of dealing with rapidly varying part of the prescribed loads. We are now in a position to show why this assumption is valid. The solutions for $u_\ell^{(1)}$ and $v_\ell^{(1)}$ only depend on t as a parameter. Thus if the prescribed loads depend on t/δ then $u_\ell^{(1)}$ and $v_\ell^{(1)}$ also depend on t/δ and $u_{\ell,tt}^{(1)}$ and $v_{\ell,tt}^{(1)}$ would not be $O(1)$ in comparison with ε. This would make (5.25) invalid, since in writing these differential equations we tacitly assumed that $u_{\ell,tt}^{(1)}$ and $v_{\ell,tt}^{(1)}$ exist and are $O(1)$ for all $0 \leq t < \infty$. From these facts we conclude that the function $f^{(1)}$ occurring in the boundary conditions for $u_\ell^{(1)}$, $v_\ell^{(1)}$ must have a second time derivative which is $O(1)$ for $0 \leq t < \infty$. A similar argument applied to $u_\ell^{(3/2)}$, $v_\ell^{(3/2)}$ shows that the function $g^{(3/2)}(y,t)$ must have a second time derivative which is $O(1)$ for $0 \leq t < \infty$. If greater accuracy was required and the corrections $u_\ell^{(2)}$, $v_\ell^{(2)}$, $u_\ell^{(5/2)}$, $v_\ell^{(5/2)}$ obtained then we would require $f^{(1)}(y,t)$ and $g^{(3/2)}(y,t)$ to possess fourth time derivatives, $O(1)$ $0 \leq t < \infty$, and so on.

We now briefly consider the error introduced by the approximation of domain and the related question of the convergence of the integrals in equations (5.33), (5.34) and (5.35). The results of Knowles [19] show that the stresses $\tau_{xx\ell}^{(1)}$ and $\tau_{xy\ell}^{(1)}$ are $O(e^{-1/\varepsilon})$ at $\zeta = 1/\varepsilon$ and therefore need not be considered in any analysis at the end $x = 1$. A similar result can be deduced from the form of the eigenfunction expansion representation for solutions of (5.7)-(5.10) obtained by

Johnson and Little [16]. These authors found that each term of an eigenfunction expansion, for say $\tau^{(1)}_{xx\ell}$, involved a decaying exponential in the variable ζ. The overall exponential decay of $\tau^{(1)}_{xx\ell}$ and the convergence of the integrals in (5.33), (5.34) and (5.35) can probably be proved as a consequence of the exponential form of the eigenfunction expansions; however this result has not yet been established. It would be even more desirable to use the more direct energy estimate methods used by Knowles[19] to prove the convergence of these integrals.

Results similar to those discussed above can be developed for the end x = 1 by introducing the variable ζ' defined by the equation:

$$\zeta' = \frac{1-x}{\varepsilon} \quad . \tag{5.36}$$

and considering the boundary layer corrections $u_{\ell'}(\zeta',y,t)$ and $v_{\ell'}(\zeta',y,t)$. We will not consider these corrections in detail as the analysis is substantially the as that for u_ℓ and v_ℓ. However, it is of interest to note that we would obtain the following boundary conditions to use with the inner approximation and the diffuse approximation at x = 1:

$$\bar{u}^{(0)}_{i,x}(1,t) = 0 \quad , \tag{5.37}$$

$$\bar{v}^{(0)}_{d',z'z'}(0,t) = 0 \quad , \tag{5.38}$$

$$\bar{v}^{(0)}_{d',z'z'z'}(0,t) = 0 \quad , \tag{5.39}$$

for $0 \leq t < \infty$. These last two conditions, together with the initial condition (4.33), and the differential equation (4.32) show that $\bar{v}^{(0)}_{d'} \equiv 0$. In fact the lowest order boundary layer terms at x = 1, $u^{(1)}_{\ell'}$ and $v^{(1)}_{\ell'}$ are also identically zero.

6. Further Consideration of the Inner Approximation and the Diffusive Approximation.

With the additional information obtained in the last section we are now able to completely determine the functions $\overline{u}_i^{(0)}$, $\overline{u}_i^{(1)}$, which are part of the inner approximation, and the functions $\overline{v}_d^{(0)}, \overline{v}_d^{(1/2)}, \overline{v}_d^{(1)}$ $\overline{v}_{d'}^{(0)}, \overline{v}_{d'}^{(1/2)}, \overline{v}_{d'}^{(1)}$ which occur in the two diffusive approximations.

First we consider $\overline{u}_i^{(0)}$ and $\overline{u}_i^{(1)}$. In section 3 it was shown that $\overline{u}_i^{(0)}$ satisfies the following differential equation (3.9):

$$\overline{u}_{i,tt}^{(0)} - \frac{4(\alpha^2-1)}{\alpha^2}\overline{u}_{i,xx}^{(0)} = \frac{1}{2}\left(q_1^{(2)} - q_2^{(2)}\right) ,$$

for $0 < x < 1$, $0 < t < \infty$; in section 5 we found the following boundary conditions (5.13), (5.37) :

$$\overline{u}_{i,x}^{(0)}(0,t) = \frac{\alpha^2}{8(\alpha^2-1)}\int_{-1}^{1} f^{(1)}(y,t)dy , \qquad \overline{u}_{i,x}^{(0)}(1,t) = 0 ,$$

for $0 \leq t < \infty$; and we have assumed that $\overline{u}_i^{(0)}$ satisfies the initial conditions:

$$\overline{u}_i^{(0)}(x,0) = 0 , \qquad \overline{u}_{i,t}^{(0)}(x,0) = 0 ,$$

for $0 \leq x \leq 1$. The above system of equations completely determines $\overline{u}_i^{(0)}$ which can be found explicitly using standard techniques. We shall not exhibit the solution for $\overline{u}_i^{(0)}$ here. It can easily be verified that the derivatives $\overline{u}_{i,tt}^{(0)}$ and $\overline{u}_{i,xx}^{(0)}$ are $0(1)$ in comparison with ε, and the development of $\overline{u}_i^{(0)}$ valid, only if the rapidly varying prescribed normal stress is excluded from the boundary condition (5.13). We can summarize the results for $\bar{u}_i^{(1)}$ as a solution of the following system of equations:

$$u_{i,tt}^{(1)} - \frac{4(\alpha^2-1)}{\alpha^2} u_{i,xx}^{(1)} = \frac{\alpha^2-2}{2\alpha^2} (p_{1,x}^{(2)} + p_{2,x}^{(2)}) \quad ,$$

for $0 < x < 1$, $0 < t < \infty$;

$$\bar{u}_i^{(1)}(0,t) = \frac{(2-\alpha^2)}{16(\alpha^2-1)} [p_1^{(2)}(0,t)+p_2^{(2)}(0,t)] - \frac{\alpha^2}{8(\alpha^2-1)} \int_{-1}^{1}\int_{0}^{\infty} u_{\ell,tt}^{(1)}(\zeta,y,t)d\zeta\, dy,$$

$$\bar{u}_i^{(1)}(1,t) = \frac{(2-\alpha^2)}{16(\alpha^2-1)} [p^{(2)}(1,t)+p^{(2)}(1,t)] \quad ,$$

for $0 \leq t < \infty$;

$$\bar{u}_i^{(1)}(x,t) = 0 \quad , \qquad \bar{u}_{i,t}^{(1)}(x,t) = 0 \quad ,$$

for $0 \leq x \leq 1$. From these equations it may be observed that the first term of the boundary layer approximation, $u_\ell^{(1)}$, must be obtained before we can solve for $\bar{u}_i^{(1)}$.

The lowest order term in the diffusive approximation near $x = 0$, $\bar{v}_d^{(0)}$, is a solution of the following system (cf. (4.27), (5.14)):

$$\bar{v}_{d,tt}^{(0)} + \frac{4}{3}\frac{\alpha^2-1}{\alpha^2} \bar{v}_{d,zzzz}^{(0)} = 0 \quad ,$$

for $0 < z < \infty$, $0 < t < \infty$;

$$\bar{v}_{d,zz}^{(0)} = - \frac{3\alpha^2}{8(\alpha^2-1)} \int_{-1}^{1} y\, f^{(1)}(y,t)dy \quad ,$$

$$\bar{v}_{d,zzzz}^{(0)}(0,t) = \frac{3\alpha^2}{8(\alpha^2-1)} \int_{-1}^{1} g^{(3/2)}(y,t)dy \quad ,$$

for $0 \leq t < \infty$;

$$\lim_{z\to\infty} \bar{v}_d^{(0)}(z,t) = 0 \quad , \qquad \lim_{z\to\infty} \bar{v}_d^{(0)}(z,t) = 0 \quad ,$$

for $0 \leq t < \infty$;

$$\bar{v}_d^{(0)}(z,0) = 0 \quad , \qquad \bar{v}_{d,t}^{(0)}(z,0) = 0 \quad ,$$

for $0 \leq z < \infty$. This system completely determines $\bar{v}_d^{(0)}$ and a solution can readily be found using standard techniques. In an earlier section we assumed that the effect of the diffusive approximation $\bar{v}_d^{(0)}$ was small at $z = \frac{1}{\sqrt{\varepsilon}}$ (x=1). To verify this assumption we must examine the solution for $\bar{v}_d^{(0)}$ in detail. We can obtain a solution for $\bar{v}_d^{(0)}$ having the following form:

$$\bar{v}_{d,zz}^{(0)} = \frac{k\sqrt{2}}{\sqrt{\pi}}\int_o^t \left\{ \frac{S^{(3/2)}(t-\tau)}{\sqrt{\tau}} - \frac{z}{2k^2\tau^{3/2}} M^{(1)}(t-\tau) \right\} \sin\left(\frac{z^2}{4k^2\tau}\right) d\tau \,, \qquad (6.1)$$

where the notation below has been used:

$$M^{(1)}(t) = \frac{3\alpha^2}{8(\alpha^2-1)} \int_{-1}^{1} y\; f^{(1)}(y,t)\,dy \qquad ,$$

$$S^{(3/2)}(t) = \frac{3\alpha^2}{8(\alpha^2-1)} \int_{-1}^{1} g^{(3/2)}(y,t)\,dy \qquad .$$

The behavior at the end $z = \frac{1}{\sqrt{\varepsilon}}$, is found by substituting this value for z in (6.1). We find that:

$$\bar{v}_{d,zz}^{(0)}\left(\frac{1}{\sqrt{\varepsilon}},t\right) = k\ \frac{2}{\pi}\int_o^t \left\{ \frac{S^{(3/2)}(t-\tau)}{\sqrt{\tau}} - \frac{M^{(1)}(t-\tau)}{2k^2\tau^{3/2}\sqrt{\varepsilon}} \right\} \sin\left(\frac{1}{4k^2\tau\varepsilon}\right) d\tau \,. \qquad (6.2)$$

For finite values of t, using the Riemann-Lebesgue lemma (Apostol [2]) and the properties of $f^{(1)}(y,t)$, $g^{(3/2)}(y,t)$ discussed earlier (existence of two t-derivatives for $0 \leq t < \infty$) we find that the right side of (6.2) is at least as small as $O(\varepsilon^{3/2})$. This result is obtained by using integration by parts in the right-hand side of (6.2). This estimate of $\bar{v}_{d,zz}^{(0)}\left(\frac{1}{\sqrt{\varepsilon}}, t\right)$ is not uniformly valid. In fact for t as large as $O(\frac{1}{\varepsilon})$ it is not valid. But we are only concerned with times before the first wave (the dilatation wave, see section 8) travels the length of the rectangle. Thus the original assumption that the diffusive approximation $\bar{v}_d^{(0)}$ does not produce significant effects at the end x = 1 is valid. We can also show that $\bar{v}^{(0)}_{d,zzz}(\frac{1}{\sqrt{\varepsilon}},t)$ is smaller than $O(\varepsilon)$. Thus $\tau_{xyd}(\frac{1}{\sqrt{\varepsilon}},t;\varepsilon)$ is smaller than $O(\varepsilon^{5/2})$ and $\tau_{xxd}(\frac{1}{\sqrt{\varepsilon}},t;\varepsilon)$ is smaller than $O(\varepsilon^{5/2})$. These stresses contribute to higher order effects at the end x = 1 but only at orders beyond those of interest for the present work. Further discussion of the contribution of $\bar{v}_d^{(0)}$ at the end x = 1 for long times can be found in section 10 of this chapter.

The next term in the expansion for v_d is $\bar{v}_d^{(1/2)}$ which satisfies the following system of equations:

$$\bar{v}^{(1/2)}_{d,tt} + \frac{4}{3}\frac{\alpha^2-1}{\alpha^2}\bar{v}^{(1/2)}_{d,zzzz} = 0 \quad ,$$

for $0 < z < \infty$, $0 < t < \infty$;

$$\bar{v}^{(1/2)}_{d,zz}(0,t) = 0 \quad ,$$

$$\bar{v}^{(1/2)}_{d,zzz}(0,t) = -\frac{3\alpha^2}{16(\alpha^2-1)}[q_1^{(2)}(0,t)+q_2^{(2)}(0,t)] - \frac{3\alpha^2}{8(\alpha^2-1)}\int_{-1}^{1}\int_{0}^{\infty} v^{(1)}_{\ell,tt}\,d\zeta\,dy \quad ,$$

for $0 \le t < \infty$;

$$\lim_{z\to\infty} \bar{v}_d^{(1/2)}(z,t) = 0 \quad ,$$

for $0 \le t < \infty$;

$$\bar{v}_d^{(1/2)}(z,0) = 0 \quad , \quad \bar{v}^{(1/2)}_{d,t}(z,0) = 0 \quad ,$$

for $0 \le z < \infty$; note that the boundary layer approximation term $v_\ell^{(1)}$ must be determined before a solution for $\bar{v}_d^{(1/2)}$ can be found.

A similar system of equations can be written for $\bar{v}_{d'}^{(1/2)}$, the lowest order term in the diffusive approximation at the end x = 1.

This completes the consideration of the inner approximation, the diffusive approximations and boundary layer approximations. These approximations together describe only the effects produced in the elastic rectangle by the slowly varying part of the precsribed loads. We now consider effects resulting from the rapidly varying part of the prescribed loads.

7. Formal Wave Front Approximations.

In constructing the approximations of the previous sections, we have taken into account all the conditions of the original problem except that part of the applied load at the end $x = 0$ which varies rapidly. In an elastic plate we expect such loading to give rise to significant stresses propagating in pulses associated with three different speeds: the dilatation speed, the shear speed and possibly the Rayleigh speed (we use the symbol β to represent the ratio of the Rayleigh speed to the shear speed). The rapidly varying parts of the prescribed loads are functions of t/δ and we expect this scale to be important in describing propagative effects. Consequently we introduce a wave front coordinate ξ defined by the equation:

$$\xi = \frac{ct-x}{\delta} \quad . \tag{7.1}$$

At present the constant c is undetermined. We will deduce the significant possible values for c in the course of the construction of the wave front approximations and at present assume only that it has a value in the range $0 \leq c \leq \alpha$. We introduce another new variable emphasizing the wave character of the problem. This variable, η , is defined by:

$$\eta = \frac{\delta}{\varepsilon^2}(ct+x) \quad . \tag{7.2}$$

The choice of scale in the equation above for η ensures that wave front approximations obtained later are valid everywhere in the plate and not just near the wave fronts. It will turn out that the scheme developed in this section is of use only when the parameter δ is small in comparison with ε; that is, when the rapidly varying part of the load has a shorter duration than the time required for a shear wave to travel across the thickness of the plate.

Additional corrections u_c, v_c are added to the approximate displacements u, v as follows:

$$\begin{aligned} u(x,y,t;\varepsilon,\delta) &= u_i(x,y,t;\varepsilon) + u_\ell(\tfrac{x}{\varepsilon},y,t;\varepsilon) + u_{\ell'}(\tfrac{1-x}{\varepsilon},y,t;\varepsilon) + \\ &\quad + u_d(\tfrac{x}{\sqrt{\varepsilon}},y,t;\varepsilon) + u_{d'}(\tfrac{1-x}{\sqrt{\varepsilon}},y,t;\varepsilon) + u_c(\eta,y,\xi;\varepsilon,\delta), \\ v(x,y,t;\varepsilon,\delta) &= v_i(x,y,t;\varepsilon) + v_\ell(\tfrac{x}{\varepsilon},y,t;\varepsilon) + v_{\ell'}(\tfrac{1-x}{\varepsilon},y,t;\varepsilon) + \\ &\quad + v_d(\tfrac{x}{\sqrt{\varepsilon}},y,t;\varepsilon) + v_{d'}(\tfrac{1-x}{\sqrt{\varepsilon}},y,t;\varepsilon) + v_c(\eta,y,\xi;\varepsilon,\delta) \quad ; \end{aligned}$$

where u_c, v_c represent certain wave front approximations. We require $u_i+u_\ell+u_{\ell'}+u_d+u_{d'}+u_c$ and $v_i+v_\ell+v_{\ell'}+v_d+v_{d'}+v_c$ to satisfy the differential equations (2.10), the boundary conditions at $y = \pm 1$ (2.11a), (2.11b), the boundary conditions at $x = 0$ (2.11c), (2.11d), and initial condition (2.11e). At first solutions are considered only until the first wave has reached the end $x = 1$. Here it is found to be convenient to consider the problem for u_c, v_c in terms of the corresponding potentials ϕ_c, ψ_c defined by:

$$\begin{aligned} u_c &= \psi_{c,y} - \frac{\varepsilon}{\delta}\phi_{c,\xi} + \frac{\delta}{\varepsilon}\phi_{c,\eta} \quad , \\ v_c &= \phi_{c,y} - \frac{\varepsilon}{\delta}\psi_{c,\xi} + \frac{\delta}{\varepsilon}\psi_{c,\eta} \quad . \end{aligned} \tag{7.4}$$

Using the properties of $u_i, v_i, \ldots,$ developed earlier we find that ϕ_c, ψ_c are solutions of the following problem

$$\frac{\varepsilon^2}{\delta^2}\left(1-\frac{c^2}{\alpha^2}\right)\phi_{c,\xi\xi}+\phi_{c,yy}-2\left(1+\frac{c^2}{\alpha^2}\right)\phi_{c,\xi\eta}+\frac{\delta^2}{\varepsilon^2}\left(1-\frac{c^2}{\alpha^2}\right)\phi_{c,\eta\eta}=0(\varepsilon^3) \quad ,$$

(7.5)

$$\frac{\varepsilon^2}{\delta^2}(1-c^2)\psi_{c,\xi\xi}+\psi_{c,yy}-2(1+c^2)\psi_{c,\xi\eta}+\frac{\delta^2}{\varepsilon^2}(1-c^2)\psi_{c,\eta\eta}=0(\varepsilon^3) \quad ,$$

for $\frac{\delta^2}{\varepsilon^2}\xi < \eta < \infty$, $-1 < y < 1$, $0 < \xi < \infty$;

$$\tau_{xyc}(\eta,\pm 1,\xi;\varepsilon,\delta) = 0 \quad , \quad \tau_{yyc}(\eta,\pm 1,\xi;\varepsilon,\delta) = 0 \quad , \qquad (7.6a)$$

for $\frac{\delta^2}{\varepsilon^2}\xi \le \eta < \infty$, $0 \le \xi < \infty$;

$$\tau_{xxc}\left(\frac{\delta^2}{\varepsilon^2}\xi,y,\xi;\varepsilon,\delta\right) = \varepsilon\, a^{(1)}(y,\xi/c) \quad ,$$

$$\tau_{xyc}\left(\frac{\delta^2}{\varepsilon^2}\xi,y,\xi;\varepsilon,\delta\right) = \varepsilon^{3/2} b^{(3/2)}(y,\xi/c) \quad , \qquad (7.6b)$$

for $-1 \le y \le 1$, $0 \le \xi < \infty$. Here the stresses τ_{xxc}, τ_{xyc}, τ_{yyc} are given in terms of the potentials ϕ_c and ψ_c by the following equations:

$$\tau_{xxc} = \frac{\varepsilon^2}{\delta^2}c^2\psi_{c,\xi\xi} - \frac{\varepsilon}{\delta}2\psi_{c,\xi y} + 2c^2\phi_{c,\xi\eta} - 2\phi_{c,yy} + \frac{\delta}{\varepsilon}2\psi_{c,\eta y} + \frac{\delta^2}{\varepsilon^2}\phi_{c,\eta\eta} \quad ,$$

$$\tau_{xyc} = \frac{\varepsilon^2}{\delta^2}(c^2-2)\psi_{c,\xi\xi} - \frac{\varepsilon}{\delta}2\phi_{c,\xi y} + 2(c^2+2)\psi_{c,\xi\eta} + \frac{\delta}{\varepsilon}2\phi_{c,y\eta} + \frac{\delta^2}{\varepsilon^2}(c^2-2)\psi_{c,\eta\eta} \quad ,$$

$$\tau_{yyc} = \frac{\varepsilon^2}{\delta^2}(c^2-2)\phi_{c,\xi\xi} + \frac{\varepsilon}{\delta}2\psi_{c,\xi y} + 2(c^2+2)\phi_{c,\xi\eta} - \frac{\delta}{\varepsilon}2\psi_{c,y\eta} + \frac{\delta^2}{\varepsilon^2}(c^2-2)\phi_{c,\eta\eta} \quad .$$

(7.7)

Throughout the problem posed above, except in the definition of the domain of interest, the parameter δ occurs only in the combination δ/ε. For the approximations developed in this section it is assumed that the parameter ν defined by:

$$\nu = \frac{\delta}{\varepsilon} \quad , \qquad (7.8)$$

is small.

Two obvious special choices for c arising from a study of equation (7.5) are $c = \alpha$ and $c = 1$. We shall find that the dilatation front approximation ($c = \alpha$) describes the propagation of the prescribed normal stress from the end into the interior of the rectangle, and the shear front approximation ($c = 1$) describes the propagation of the given shear stress from the end into the interior of the rectangle. In the study of these wave front approximations we find that they generate high order effects important near the sides $y = \pm 1$. In fact we find that the dilatation front approximation generates higher order shear stresses at $y = \pm 1$ and that the shear wave front approximation generates higher order normal stress at $y = \pm 1$. To find a method suitable for dealing with the 'secondary' stresses we must examine the sides $y = \pm 1$ more closely. Near $y = 1$ we introduce a 'boundary variable':

$$r = \frac{1-y}{\nu} \quad . \tag{7.9}$$

Together with each wave front approximation ϕ_c, ψ_c we introduce wave front boundary corrections $\phi_{c\ell}(\eta, r, \xi; \varepsilon, \nu)$, $\psi_{c\ell}(\eta, r, \xi; \varepsilon, \nu)$. These corrections must satisfy the differential equations (7.5) and certain stress boundary conditions at $r = 0$. That is, $\phi_{c\ell}$ and $\psi_{c\ell}$ satisfy the following differential equations:

$$\frac{1}{\nu^2}\left[\left(1-\frac{c^2}{\alpha^2}\right)\phi_{c\ell,\xi\xi}+\phi_{c\ell,rr}\right]-2\left(1+\frac{c^2}{\alpha^2}\right)\phi_{c\ell,\xi\eta}+\nu^2\left(1-\frac{c^2}{\alpha^2}\right)\phi_{c\ell,\eta\eta}=0 \quad ,$$

$$\frac{1}{\nu^2}\left[(1-c^2)\psi_{c\ell,\xi\xi}+\psi_{c\ell,rr}\right]-2(1+c^2)\psi_{c\ell,\xi\eta}+\nu^2(1-c^2)\psi_{c\ell,\eta\eta}=0 \quad . \tag{7.10}$$

We consider these boundary corrections to the wave front approximations in specific detail in the next two sections.

Before proceeding it is necessary to define the properties of ε and δ. We do not attempt to deal with the problem where ε and δ are independent small parameters. In general it is assumed that $\nu = \varepsilon^a$, $a > 0$, and expansions for ϕ_c, ψ_c in powers of ε^b are assumed where b is the highest common factor of a and 1/2. We shall only consider the special case a = 1/2 in detail. It is easily seen that the methods developed for this special case also apply for other values of a. We now consider the two cases $c = \alpha, 1$ in detail. Since the normal stress at the end x = 0 is given at a lower order than the shear stress, the case $c = \alpha$ is considered first.

8. The Formal Dilatation Wave Front Approximation and Associated Boundary Corrections.

First consider the wave front approximation associated with the choice of $c = \alpha$ in equations (7.5)-(7.7). We denote the corresponding potentials as ϕ_α, ψ_α, and use the independent variables ξ_α, η_α defined by the equations:

$$\xi_\alpha = \frac{\alpha t - x}{\varepsilon^{3/2}} \quad , \qquad \eta_\alpha = \frac{\alpha t + x}{\sqrt{\varepsilon}} \quad , \tag{8.1}$$

(cf. equations (7.1), (7.2)). As was stated at the end of section 7, only the special case $\nu = \sqrt{\varepsilon}$ is considered. It is required that ϕ_α, ψ_α should satisfy as many of the following conditions as possible:

$$\begin{aligned} &\phi_{\alpha,yy} - 4\phi_{\alpha,\xi_\alpha\eta_\alpha} = O(\varepsilon^3) \quad , \\ &\frac{1}{\varepsilon}(1-\alpha^2)\psi_{\alpha,\xi_\alpha\xi_\alpha} + \psi_{\alpha,yy} - 2(1+\alpha^2)\psi_{\alpha,\eta_\alpha\xi_\alpha} + \varepsilon(1-\alpha^2)\psi_{\alpha,\eta_\alpha\eta_\alpha} = O(\varepsilon^3) \quad , \end{aligned} \tag{8.2}$$

for $\varepsilon\xi_\alpha < \eta_\alpha < \infty$, $-1 < y < 1$, $0 < \xi_\alpha < \infty$;

$$\tau_{xy\alpha}(\eta_\alpha,\pm 1,\xi_\alpha;\varepsilon) = 0 \quad , \qquad \tau_{yy\alpha}(\eta_\alpha,\pm 1,\xi_\alpha;\varepsilon) = 0 \quad , \tag{8.3a}$$

for $\varepsilon\xi_\alpha \leq \eta_\alpha < \infty$, $0 \leq \xi < \infty$;

$$\begin{aligned} \tau_{xx\alpha}(\varepsilon\xi_\alpha,y,\xi_\alpha;\varepsilon) &= \varepsilon\, a^{(1)}(y,\frac{\xi_\alpha}{\alpha}) \quad , \\ \tau_{xy\alpha}(\varepsilon\xi_\alpha,y,\xi_\alpha;\varepsilon) &= \varepsilon^{(3/2)} b^{(3/2)}(y,\frac{\xi_\alpha}{\alpha}) \quad , \end{aligned} \tag{8.3b}$$

for $-1 \leq y \leq 1$, $0 \leq \xi_\alpha < \infty$;

$$u_\alpha(\eta_\alpha,y,0;\varepsilon) = 0 \quad , \qquad v_\alpha(\eta_\alpha,y,0;\varepsilon) = 0 \quad , \tag{8.3c}$$

for $\varepsilon\xi_\alpha \leq \eta_\alpha < \infty$, $-1 \leq y \leq 1$; where the stresses $\tau_{xx\alpha}, \tau_{xy\alpha}, \tau_{yy\alpha}$ and the displacements u_α, v_α are given in terms of the potentials ϕ_α and ψ_α by the following equations:

$$\tau_{xx\alpha} = \frac{1}{\varepsilon}\alpha^2\phi_{\alpha,\xi_\alpha\xi_\alpha} - \frac{1}{\sqrt{\varepsilon}}2\psi_{\alpha,\xi_\alpha y} + 2\alpha^2\phi_{\alpha,\xi_\alpha\eta_\alpha} - 2\phi_{\alpha,yy} + \sqrt{\varepsilon}2\psi_{\alpha,\eta_\alpha y} + \varepsilon\alpha^2\phi_{\alpha,\eta_\alpha\eta_\alpha} \quad ,$$

$$\tau_{xy\alpha} = \frac{1}{\varepsilon}(\alpha^2-2)\psi_{\alpha,\xi_\alpha\xi_\alpha} - \frac{1}{\sqrt{\varepsilon}}2\phi_{\alpha,\xi_\alpha y} + 2(\alpha^2+2)\psi_{\alpha,\xi_\alpha\eta_\alpha} + \sqrt{\varepsilon}2\phi_{\alpha,\xi_\alpha y} + \varepsilon(\alpha^2-2)\psi_{\alpha,\eta_\alpha\eta_\alpha} ,$$

$$\tau_{yy\alpha} = \frac{1}{\varepsilon}(\alpha^2-2)\phi_{\alpha,\xi_\alpha\xi_\alpha} + \frac{2}{\sqrt{\varepsilon}}\psi_{\alpha,\xi_\alpha y} + 2(\alpha^2+2)\phi_{\alpha,\xi_\alpha\eta_\alpha} - 2\sqrt{\varepsilon}\psi_{\alpha,y\eta_\alpha} + \varepsilon(\alpha^2-2)\phi_{\alpha,\eta_\alpha\eta_\alpha} ,$$

$$u_\alpha = -\frac{1}{\sqrt{\varepsilon}}\phi_{\alpha,\xi_\alpha} + \psi_{\alpha,y} + \sqrt{\varepsilon}\,\phi_{\alpha,\eta_\alpha} \quad , \tag{8.4}$$

$$v_\alpha = \frac{1}{\sqrt{\varepsilon}}\psi_{\alpha,\xi_\alpha} + \phi_{\alpha,y} - \sqrt{\varepsilon}\,\psi_{\alpha,\eta_\alpha} \quad .$$

Expansions for ϕ_α, ψ_α of the following form are assumed:

$$\begin{aligned} \phi_\alpha(\eta_\alpha,y,\xi_\alpha;\varepsilon) &= \varepsilon^2\phi_\alpha^{(2)}(\eta_\alpha,y,\xi_\alpha) + \varepsilon^{5/2}\phi_\alpha^{(5/2)}(\eta_\alpha,y,\xi_\alpha) + \ldots \quad , \\ \psi_\alpha(\eta_\alpha,y,\xi_\alpha;\varepsilon) &= \varepsilon^2\psi_\alpha^{(2)}(\eta_\alpha,y,\xi_\alpha) + \varepsilon^{5/2}\psi^{(5/2)}(\eta_\alpha,y,\xi_\alpha) + \ldots \quad . \end{aligned} \tag{8.5}$$

Then adopting the standard procedure we find that $\phi_\alpha^{(2)}$ and $\psi_\alpha^{(2)}$ satisfy the system of equations written below:

$$\phi^{(2)}_{\alpha,yy} - 4\,\phi^{(2)}_{\alpha,\eta_\alpha\xi_\alpha} = 0 \quad ,$$

$$\psi^{(2)}_{\alpha,\xi_\alpha\xi_\alpha} = 0 \quad ,$$

for $0 < \eta_\alpha < \infty$, $-1 < y < 1$, $0 < \xi_\alpha < \infty$;

$$\tau^{(1)}_{xy\alpha}(\eta_\alpha, \pm 1, \xi_\alpha) = 0 \quad , \quad \tau^{(1)}_{yy\alpha}(\eta_\alpha, \pm 1, \xi_\alpha) = 0 \quad ,$$

for $0 \leq \eta_\alpha < \infty$, $0 \leq \xi_\alpha < \infty$;

$$\tau^{(1)}_{xx\alpha}(0,y,\xi_\alpha) = a^{(1)}(y,\frac{\xi_\alpha}{\alpha}) \quad ,$$

$$\tau^{(1)}_{xy\alpha}(0,y,\xi_\alpha) = 0 \quad ,$$

for $-1 \leq y \leq 1$, $0 \leq \xi_\alpha < \infty$;

$$u^{(3/2)}_\alpha(\eta_\alpha,y,0) = 0 \quad , \quad v^{(3/2)}_\alpha(\eta_\alpha,y,0) = 0 \quad ,$$

for $0 \leq \eta_\alpha < \infty$, $-1 \leq y \leq 1$. Note that the boundary condition at $\eta_\alpha = \varepsilon\,\xi_\alpha$ has been replaced by a similar one at $\eta_\alpha = 0$ and the domain of the problem modified accordingly. This is a procedure similar to the approximation of domain used in deriving the diffusive approximation and boundary layer approximations of earlier sections. Now the errors produced by the approximation of domain are relatively large. However, provided that the prescribed rapidly varying normal stress $a^{(1)}(y,t/\delta)$ satisfies certain conditions (discussed later) we can show that these errors are uniformly smaller than $O(\varepsilon^{5/2})$ in the calculation of the stresses. We can now solve for $\phi_\alpha^{(2)}$ and $\psi_\alpha^{(2)}$ obtaining solutions:

$$\phi^{(2)}_{\alpha,\xi_\alpha} = \sum_{n=1}^{\infty} \int_0^{\xi_\alpha} \frac{1}{\alpha^2} a^{(1)}_{on}(\tfrac{z}{\alpha}) J_o(n\pi\sqrt{\eta_\alpha(\xi_\alpha - z)})dz.\ \sin n\pi y$$

$$+ \sum_{n=1}^{\infty} \int_0^{\xi_\alpha} \frac{1}{\alpha^2}\, a^{(1)}_{en}(\tfrac{z}{\alpha}) J_o(\tfrac{2n-1}{2}\pi\sqrt{\eta_\alpha(\xi_\alpha - z)})dz.\ \cos\frac{2n-1}{2}\pi y \quad , \tag{8.6}$$

$$\psi^{(2)}_{\alpha,\xi_\alpha} = 0 \tag{8.7}$$

where

$$a^{(1)}_{on}(\tfrac{z}{\alpha}) = \int_{-1}^{1} a^{(1)}(y,\tfrac{z}{\alpha}) \sin n\pi y\ dy \quad ,$$

$$a^{(1)}_{en}(\tfrac{z}{\alpha}) = \int_{-1}^{1} a^{(1)}(y,\tfrac{z}{\alpha}) \cos\frac{2n-1}{2}\pi y\ dy \quad ,$$

and J_o is the Bessel function of order zero. The subscripts "o" and "e" on $a^{(1)}_{on}(\frac{z}{\alpha})$, $a^{(1)}_{en}(\frac{z}{\alpha})$ refer to the fact that the part of $a^{(1)}(y,\xi/\alpha)$ which is an even function of y contributes only to $a^{(1)}_{en}$, and the part of $a^{(1)}$ which is an odd function of y contributes only to $a^{(1)}_{on}$. Here we solved for the derivatives $\phi^{(2)}_{\alpha,\xi}$ and $\psi^{(2)}_{\alpha,\xi}$ rather than the potentials $\phi^{(2)}_\alpha, \psi^{(2)}_\alpha$ themselves since only these derivatives are required for the calculation of stresses and displacements at this order. The corresponding stresses and displacements are:

$$\tau^{(1)}_{xx\alpha} = a^{(1)}(y,\xi_\alpha/\alpha) - \sum_{n=1}^{\infty} \int_0^{\xi_\alpha} a^{(1)}_{on}(\tfrac{z}{\alpha}) \frac{n\pi\sqrt{\xi_\alpha}}{2\sqrt{\xi_\alpha - z}}\, J_1(n\pi\sqrt{\eta_\alpha(\xi_\alpha - z)})dz.\sin n\pi y$$

$$- \sum_{n=1}^{\infty} \int_0^{\xi_\alpha} a^{(1)}_{en}(\tfrac{z}{\alpha}) \frac{(2n-1)\pi\sqrt{\xi_\alpha}}{4\sqrt{\xi_\alpha - z}}\, J_1(\tfrac{2n-1}{2}\pi\sqrt{\eta_\alpha(\xi_\alpha - z)})dz.\cos\frac{2n-1}{2}\pi y$$

$$\tau^{(1)}_{xy\alpha} = 0 \quad ,$$

$$\tau^{(1)}_{yy\alpha} = \frac{\alpha^2 - 2}{\alpha^2}\, \tau^{(1)}_{xx\alpha} \quad ,$$

$$u_\alpha^{(3/2)} = -\sum_{n=1}^{\infty}\int_0^{\xi_\alpha} \frac{1}{\alpha^2} a_{on}^{(1)}(\frac{z}{\alpha}) J_o(n\pi\sqrt{\eta_\alpha(\xi_\alpha - z)}) dz.\sin n\pi y$$

$$-\sum_{n=1}^{\infty}\int_0^{\xi_\alpha} \frac{1}{\alpha^2} a_{en}^{(1)}(\frac{z}{\alpha}) J_o(\frac{2n-1}{2}\pi\sqrt{\eta_\alpha(\xi_\alpha - z)}) dz.\cos\frac{2n-1}{2}\pi y \,, \qquad (8.8)$$

$$v_\alpha^{(3/2)} = 0 \qquad .$$

This solution is useful only if the prescribed load $a^{(1)}(y,\frac{\xi_\alpha}{\alpha})$ has the property that:

$$a^{(1)}(\pm 1,\frac{\xi_\alpha}{\alpha}) = 0 \qquad . \qquad (8.9)$$

Equation (8.9) requires that the rapidly varying part of the stress τ_{xx} prescribed at the end x = 0, vanishes at the corners y = ±1. In general this condition may not be fulfilled. We shall discuss how to deal with loads having non-zero values at y = ±1 in a later section dealing with "corner approximations". At present it is assumed that the prescribed normal stress at the end x = 0 fulfills equation (8.9), or that we are considering the part of the normal stress which fulfills condition (8.9).

We now consider the error introduced by the approximation of domain. At $\eta_\alpha = \varepsilon\xi_\alpha$:

$$\tau_{xx\alpha} - \tau_{xx\alpha}^{(1)} = \varepsilon a^{(1)}(y,\frac{\xi_\alpha}{\alpha}) - \varepsilon\alpha^2\phi^{(2)}_{\alpha,\xi_\alpha\xi_\alpha} - \varepsilon^2(2\alpha^2\phi^{(2)}_{\alpha,\xi_\alpha\eta_\alpha} - 2\phi^{(2)}_{\alpha,yy}) - \varepsilon^3\alpha^2\phi^{(2)}_{\alpha,\eta_\alpha\eta_\alpha} \qquad .$$

If our procedure is to be of use then $\tau_{xx\alpha}(\varepsilon\xi_\zeta,y,\xi_\alpha;\varepsilon) - \varepsilon\tau^{(1)}_{xx\alpha}(\varepsilon\xi_\alpha,y,\xi_\alpha)$ is uniformly of higher order than ε. Substituting the solution for $\phi_\alpha^{(2)}$ obtained above in (8.6) we find the following result:

$$\tau_{xx\alpha}(\varepsilon\xi_\alpha,y,\xi_\alpha;\varepsilon) = \varepsilon\tau^{(1)}_{xx\alpha}(\varepsilon\xi_\alpha,y,\xi_\alpha) =$$

$$\varepsilon^{3/2}\Bigg\{\sum_{n=1}^{\infty} n\pi\int_0^{\xi_\alpha} a^{(1)}_{on}(\tfrac{z}{\alpha})[(\frac{z}{\sqrt{\xi_\alpha(\xi_\alpha-z)}} + (3-\frac{8}{\alpha^2})(\frac{\xi_\alpha-z}{\xi_\alpha})^{1/2})J_1(n\pi\sqrt{\varepsilon}\sqrt{\xi_\alpha(\xi_\alpha-z)})$$

$$-(\frac{\xi_\alpha-z}{\xi_\alpha})^{3/2}J_3(n\pi\sqrt{\varepsilon}\sqrt{\xi_\alpha(\xi_\alpha-z)})]dz.\sin n\pi y +$$

$$+\sum_{n=1}^{\infty}\frac{2n-1}{2}\pi\int_0^{\xi_\alpha} a^{(1)}_{en}(\tfrac{z}{\alpha})[(\frac{z}{\sqrt{\xi_\alpha(\xi_\alpha-z)}} + (3-\frac{8}{\alpha^2})(\frac{\xi_\alpha-z}{\xi_\alpha})^{1/2})J_1(\frac{2n-1}{2}\pi\sqrt{\varepsilon}\sqrt{\xi_\alpha(\xi_\alpha-z)})$$

$$-(\frac{\xi_\alpha-z}{\xi_\alpha})^{3/2}J_3(\frac{2n-1}{2}\pi\sqrt{\varepsilon}\sqrt{\xi_\alpha(\xi_\alpha-z)})]dz.\cos\frac{2n-1}{2}\pi y\Bigg\}.$$

Bessel functions satisfy the following inequalities (cf. Watson [37]):

$$|J_k(x)| \le \frac{1}{\sqrt{2}}\ , \quad x \text{ real},\quad k \ge 1\ ,\quad |J_1(x)| \le \frac{x}{4}\ ,\quad x \text{ real}.$$

Using this information we can obtain the estimate below:

$$|\tau_{xx\alpha}(\varepsilon\xi_\alpha,y,\xi_\alpha;\varepsilon) - \varepsilon\tau^{(1)}_{xx\alpha}(\varepsilon\xi_\alpha,y,\xi_\alpha)| \le$$

$$\le \varepsilon^{3/2}\Bigg\{\sum_{n=1}^{\infty}\frac{n\pi}{2}\int_0^{\alpha}|a^{(1)}_{on}(\tfrac{z}{\alpha})|[z\frac{n\pi}{4}\sqrt{\varepsilon} + 2\sqrt{2}(1+\frac{2}{\alpha^2})]dz$$

$$+\sum_{n=1}^{\infty}\frac{2n-1}{2}\pi\int_0^{\alpha}|a^{(1)}_{en}(\tfrac{z}{\alpha})|[z\frac{2n-1}{2}\pi\sqrt{\varepsilon}+2\sqrt{2}(1+\frac{2}{\alpha^2})]dz\Bigg\} \qquad (8.10)$$

The property that $a^{(1)}(y,\xi_\alpha/\alpha) = 0$ for $\xi_\alpha \ge \alpha$ has been used in the above formula (8.10). This property ensures that the integrals in (8.10) converge. Of course a less restrictive condition on $a^{(1)}$ could have been prescribed; for example, $a^{(1)}(y,\xi_\alpha/\alpha) \sim \frac{1}{\xi_\alpha^{1+k}}$ for large ξ_α. However insisting that $a^{(1)}$ is zero for $\xi \ge \alpha$ is a more convenient practical criterion for the division of the given data into slowly and rapidly varying parts. To ensure that the estimate (8.10) is uniformly valid we must prove that the series on the right hand side of (8.10) converge uniformly for $-1 \leq y \leq 1$. A sufficient condition for this is that the prescribed function $a^{(1)}(y,\xi_\alpha/\alpha)$ has a continuous second derivative with respect to y and $a^{(1)}_{,y}(1,\xi_\alpha/\alpha) = a^{(1)}_{,y}(-1,\xi_\alpha/\alpha)$. This means that we can only admit quite smooth functions of y (where our definition of smoothness is the existence of continuous second derivatives). It becomes apparent that in order to develop higher order terms $\phi_\alpha^{(5/2)},\ldots$, we would have to know if $a^{(1)}(y,\xi_\alpha/\alpha)$ has continuous fourth derivatives with respect to y. We do not investigate this matter in any further detail.

We now briefly consider the next order, that is the functions $\phi_\alpha^{(5/2)},\psi_\alpha^{(5/2)}$. These functions satisfy the following system of equations:

$$\phi^{(5/2)}_{\alpha,yy} - 4\phi^{(5/2)}_{\alpha,\eta_\alpha\xi_\alpha} = 0 \qquad \psi^{(5/2)}_{\alpha,\xi_\alpha\xi_\alpha} = 0 \tag{8.11}$$

for $0 < \eta_\alpha < \infty$, $-1 < y < 1$, $0 < \xi_\alpha < \infty$;

$$\tau^{(3/2)}_{xy\alpha}(\eta_\alpha,\pm 1,\xi_\alpha) = 0 \quad , \quad \tau^{(3/2)}_{yy\alpha}(\eta_\alpha,\pm 1,\xi_\alpha) = 0 \quad ,$$

for $0 \leq \eta_\alpha < \infty$, $0 \leq \xi_\alpha < \infty$; (8.12a)

$$\tau^{(3/2)}_{xx\alpha}(0,y,\xi_\alpha) = \lim_{\varepsilon\to 0} \frac{1}{\sqrt{\varepsilon}} \{a^{(1)}(y,\xi_\alpha/\alpha) - \tau^{(1)}_{xx\alpha}(\varepsilon\xi_\alpha,y,\xi)\} \quad ,$$

for $0 \leq \xi_\alpha < \infty$, $-1 \leq y \leq 1$; (8.12b)

$$u^{(2)}_\alpha(\eta_\alpha,y,0) = 0 \quad , \quad v^{(2)}_\alpha(\eta_\alpha,y,0) = 0 \quad , \tag{8.12c}$$

for $0 \leq \eta_\alpha < \infty$, $-1 \leq y \leq 1$. Here, as in the previous order, the domain $\varepsilon\xi < \eta < \infty$, $-1 < y < 1$, $0 < \xi < \infty$, has been approximated by the domain

$0 < \eta < \infty$, $-1 < y < 1$, $0 < \xi < \infty$. To verify that this approximation of domain is valid at this order the properties of

$$\lim_{\varepsilon \to 0} \frac{1}{\sqrt{\varepsilon}} [a^{(1)}(y, \xi_\alpha / \alpha) - \tau^{(1)}_{xx\alpha}(\varepsilon \xi_\alpha, y, \xi_\alpha)]$$

must be determined in detail. In particular we must know the behavior of this expression for large ξ_α. It may be necessary to divide it into a slowly varying function and a rapidly varying function before proceeding. However our primary interest in considering the equations for $\phi^{(5/2)}_\alpha$ and $\psi^{(5/2)}_\alpha$ is not to find explicit solutions for these functions but to demonstrate that both boundary conditions at $y = \pm 1$ cannot be satisfied. We can write the shear stress $\tau^{(3/2)}_{xy\alpha}$ in terms of potentials $\phi^{(5/2)}_\alpha, \psi^{(5/2)}_\alpha$: $\tau^{(3/2)}_{xy\alpha} = -2\phi^{(2)}_{\alpha,\xi_\alpha y} + (\alpha^2 - 2)\psi^{(5/2)}_{\alpha,\xi_\alpha \xi_\alpha}$.
From equation (8.11) it is clear that $\psi^{(5/2)}_{\alpha,\xi_\alpha\xi_\alpha} = 0$, and we have already determined $\phi^{(2)}_\alpha$ completely in equation (8.6). It is obvious from (8.6) that in general $\tau^{(3/2)}_{xy\alpha}$ does not vanish at $y = \pm 1$. We now develop 'boundary approximations' to deal with the stress $\tau^{(3/2)}_{xy\alpha}$ generated at $y = \pm 1$ by the function $\phi^{(2)}_\alpha$.

We first consider the side $y = 1$ and introduce the boundary variable r defined in equation (7.9). The dilatation wave front boundary corrections $\phi_{\alpha\ell}(\eta_\alpha, r, \xi_\alpha; \varepsilon)$ and $\psi_{\alpha\ell}(\eta_\alpha, r, \xi_\alpha; \varepsilon)$ are introduced. From equation (7.10) it can be observed that these function satisfy:

$$\frac{1}{\varepsilon}\phi_{\alpha\ell,rr} - 4\phi_{\alpha\ell,\xi_\alpha\eta_\alpha} = 0 \quad ,$$
$$\frac{1}{\varepsilon}[\psi_{\alpha\ell,rr} - (\alpha^2-1)\psi_{\alpha\ell,\xi_\alpha\xi_\alpha}] - 2(1+\alpha^2)\psi_{\alpha\ell,\xi_\alpha\eta_\alpha} - \varepsilon(\alpha^2-1)\psi_{\alpha\ell,\eta_\alpha\eta_\alpha} = 0, \tag{8.13}$$

$\varepsilon\xi_\alpha < \eta_\alpha < \infty$, $0 < r < \frac{1}{\sqrt{\varepsilon}}$, $0 < \xi_\alpha < \infty$. We also require that $\phi_{\alpha\ell}$ and $\psi_{\alpha\ell}$ deal with the stress $\tau^{(3/2)}_{xy\alpha}$ generated by the wave front approximation at r = 0 (y = 1). That is:

$$\tau_{xy\alpha\ell}(\eta_\alpha, 0, \xi_\alpha) = \varepsilon^{3/2}\, 2\phi^{(2)}_{\alpha,\xi_\alpha y}(\eta_\alpha, 1, \xi_\alpha) \quad ,$$

for $\varepsilon\xi_\alpha \leq \eta_\alpha \leq \infty$, $0 \leq \xi_\alpha < \infty$. There must be no interference with the normal stress condition at y = 1 and consequently:

$$\tau_{yy\alpha\ell}(\eta_\alpha, 0, \xi_\alpha; \varepsilon) = 0 \quad ,$$

for $\varepsilon\xi_\alpha \leq \eta_\alpha \leq \infty$, $0 \leq \xi_\alpha < \infty$. It is assumed that the displacements $u_{\alpha\ell}$, $v_{\alpha\ell}$ corresponding to the potentials $\phi_{\alpha\ell}$ and $\psi_{\alpha\ell}$ must be continuous. Other conditions are required to determine $\phi_{\alpha\ell}$, $\psi_{\alpha\ell}$; however they are not introduced until after the differential equations (8.13) are considered in more detail. We assume expansions of the form:

$$\begin{aligned} \phi_{\alpha\ell}(\eta_\alpha, r, \xi_\alpha; \varepsilon) &= \varepsilon^{5/2}\phi^{(5/2)}_{\alpha\ell}(\eta_\alpha, r, \xi_\alpha) + \varepsilon^{3}\phi^{(3)}_{\alpha\ell}(\eta_\alpha, r, \xi_\alpha) + \ldots , \\ \psi_{\alpha\ell}(\eta_\alpha, r, \xi_\alpha; \varepsilon) &= \varepsilon^{5/2}\psi^{(5/2)}_{\alpha\ell}(\eta_\alpha, r, \xi_\alpha) + \varepsilon^{3}\psi^{(3)}_{\alpha\ell}(\eta_\alpha, r, \xi_\alpha) + \ldots , \end{aligned} \qquad (8.14)$$

and thence, substituting in equation (8.13), we find that the lowest order terms $\psi^{(5/2)}_{\alpha\ell}$ and $\psi^{(5/2)}_{\alpha\ell}$ satisfy the following equations:

$$\phi^{(5/2)}_{\alpha\ell,rr} = 0 \quad , \qquad \psi^{(5/2)}_{\alpha\ell,rr} - (\alpha^2-1)\psi^{(5/2)}_{\alpha\ell,\xi_\alpha\xi_\alpha} = 0 \quad , \qquad (8.15)$$

for $0 < \xi_\alpha < \infty$, $0 < r < \infty$, $0 < \eta_\alpha < \infty$. We might expect that the effect produced by the secondary stress $\tau^{(3/2)}_{xy\alpha}$ at y = 1 would remain local to the side. However the second equation of (8.15) shows that effects propagate even though they occur on the same scale as r. Thus there is no decay condition for large r. We assume that the solutions $\phi^{(5/2)}_{\alpha\ell}$, $\psi^{(5/2)}_{\alpha\ell}$ remain bounded for large r.

The boundary conditions at $r = 0$ require that:

$$\tau^{(3/2)}_{xy\alpha\ell}(\eta_\alpha, 0, \xi_\alpha) = 2\phi^{(2)}_{\alpha,\xi_\alpha y}(\eta_\alpha, 1, \xi_\alpha) \ ,$$

$$\tau^{(3/2)}_{yy\alpha\ell}(\eta_\alpha, 0, \xi_\alpha) = 0 \ ,$$

for $0 \leq \eta_\alpha < \infty$, $0 \leq \xi_\alpha < \infty$. Here the stresses $\tau^{(3/2)}_{xy\alpha\ell}$ and $\tau^{(3/2)}_{yy\alpha\ell}$ are given in terms of the potentials as follows:

$$\tau^{(3/2)}_{xy\alpha\ell} = 2\phi^{(5/2)}_{\alpha\ell,\xi_\alpha r} + (\alpha^2-2)\psi^{(5/2)}_{\alpha\ell,\xi_\alpha\xi_\alpha} \ ,$$

$$\tau^{(3/2)}_{yy\alpha\ell} = (\alpha^2-2)\phi^{(5/2)}_{\alpha\ell,\xi_\alpha\xi_\alpha} - 2\psi^{(5/2)}_{\alpha\ell,\xi_\alpha r} \ .$$

The only solution for $\phi^{(5/2)}_{\alpha\ell}$ which is bounded for large r and which will allow higher order terms $\phi^{(3)}_{\alpha\ell}, \phi^{(7/2)}_{\alpha\ell}$,... to be bounded for large r is $\phi^{(5/2)}_{\alpha\ell} \equiv 0$. Since no propagation in front of the wave front $\xi_\alpha = 0$ is possible, the solution for $\psi^{(5/2)}_{\alpha\ell}$ must be a function of $\frac{\xi_\alpha}{\sqrt{\alpha^2-1}} - r$, and thus it is possible to specify only one condition at $r = 0$. We require $\psi_{\alpha\ell}$ to deal with the shear stress at $r = 0$ as we have already seen that the wave front approximation $\phi^{(5/2)}_\alpha, \psi^{(5/2)}_\alpha$ cannot. Then the equation:

$$\psi^{(5/2)}_{\alpha\ell,\xi_\alpha} = \begin{cases} 0 \ , & \frac{\xi_\alpha}{\sqrt{\alpha^2-1}} - r < 0 \ , \\ \frac{2}{(\alpha^2-1)}\phi^{(3)}_{\alpha,y}\left(\eta_\alpha, 1, \frac{\xi_\alpha}{\sqrt{\alpha^2-1}} - r\right) \ , & \frac{\xi_\alpha}{\sqrt{\alpha^2-1}} - r \geq 0 \ , \end{cases} \qquad (8.16)$$

provides a function which satisfies the differential equation (8.15) and the shear stress boundary condition at $y = 1$. Similarly, near $y = -1$, if we define a boundary variable:

$$r' = \frac{1+y}{\sqrt{\varepsilon}} \ , \qquad (8.17)$$

and a corresponding boundary correction $\phi_{\alpha\ell'}, \psi_{\alpha\ell'}$, we find that:

$$\psi^{(5/2)}_{\alpha\ell',\xi_\alpha} = \begin{cases} 0 \quad , & \dfrac{\xi_\alpha}{\sqrt{\alpha^2-1}} - r' < 0 , \\ \dfrac{2}{\alpha^2-2}\,\phi^{(2)}_{\alpha,y}(\eta_\alpha, -1, -\dfrac{\xi_\alpha}{\sqrt{\alpha^2-1}} - r') , & \dfrac{\xi_\alpha}{\sqrt{\alpha^2-1}} - r' \geq 0 , \end{cases} \tag{8.18}$$

$$\phi^{(5/2)}_{\alpha\ell'} = 0 \quad .$$

Of course $\psi^{(5/2)}_{\alpha\ell}$ and $\psi^{(5/2)}_{\alpha\ell'}$ do not satisfy the normal stress condition at y = 1 and y = -1 respectively. Before considering this fact in more detail we consider the shear stress produced at y = -1 by $\psi^{(5/2)}_{\alpha\ell}$ and the shear stress produced at y = 1 by $\psi^{(5/2)}_{\alpha\ell'}$. At y = 1:

$$\tau^{(3/2)}_{xy\alpha\ell'}(\eta_\alpha, \frac{2}{\sqrt{\varepsilon}}, \xi_\alpha) = \begin{cases} 0 & \xi_\alpha < \dfrac{2}{\sqrt{\varepsilon}}\sqrt{\alpha^2-1} , \\ 2\phi^{(2)}_{\alpha,y\xi_\alpha}(\eta_\alpha, 1, \dfrac{\xi_\alpha}{\sqrt{\alpha^2-1}} - \dfrac{2}{\sqrt{\varepsilon}}) , & \xi_\alpha \geq \sqrt{\alpha^2-1} , \end{cases}$$

and at y = -1:

$$\tau^{(3/2)}_{xy\alpha\ell}(\eta_\alpha, \frac{2}{\sqrt{\varepsilon}}, \xi_\alpha) = \begin{cases} 0 & \xi_\alpha < \dfrac{2}{\sqrt{\varepsilon}}\sqrt{\alpha^2-1} , \\ 2\phi^{(2)}_{\alpha,y}(\eta_\alpha, -1, \dfrac{\xi_\alpha}{\sqrt{\alpha^2-1}} - \dfrac{2}{\sqrt{\varepsilon}}) , & \xi_\alpha \geq \dfrac{2}{\sqrt{\varepsilon}}\sqrt{\alpha^2-1} . \end{cases}$$

We can see from the above equations that the boundary corrections $\psi^{(5/2)}_{\alpha\ell}$ $\psi^{(5/2)}_{\alpha\ell'}$ are only of use for a limited distance behind the front $\xi_\alpha = 0$. In physical terms what this means is that the dilatation wave effects are generating secondary waves propagating diagonally across the rectangle. These secondary waves reach the other side of the plate a distance behind the wave front given by $\xi_\alpha = \frac{2}{\sqrt{\varepsilon}}$. The wave fronts of these secondary waves are given by the equations:

$$\xi_\alpha - \sqrt{\alpha^2-1}\ r = 0 \quad , \qquad \xi_\alpha - \sqrt{\alpha^2-1}\ r' = 0 \quad . \tag{8.19}$$

In physical variables these equations are equivalent to the following wave fronts: $c_1T - X - \sqrt{c_1^2/c_2^2-1}\ (h-Y) = 0$, $c_1T - X - \sqrt{c_1^2/c_2^2-1}\ (h+Y) = 0$. These fronts appear to be related to the Stoneley waves occuring following a moving disturbance in an elastic half space.

For $\xi_\alpha \geq \frac{2}{\sqrt{\varepsilon}} \sqrt{\alpha^2-1}$ the boundary corrections $\psi_{\alpha\ell}^{(5/2)}$, $\psi_{\alpha\ell'}^{(5/2)}$ are not suitable for dealing with the shear stress at y = 1. We can improve $\psi_{\alpha\ell}^{(5/2)}$ $\psi_{\alpha\ell'}^{(5/2)}$ by taking the following expressions instead of (8.16), (8.18):

$$\psi_{\alpha\ell,\xi_\alpha}^{(5/2)} = \begin{cases} 0 \quad , & \xi_\alpha - \sqrt{\alpha^2-1}\ r < 0 \ , \\ \frac{2}{\alpha^2-2}\ \phi_{\alpha,y}^{(2)}(\eta_\alpha,1,\xi_\alpha-\sqrt{\alpha^2-1}\ r) \quad , & 0 \leq \xi_\alpha - \sqrt{\alpha^2-1}\ r < \frac{2}{\sqrt{\varepsilon}} \ , \\ \frac{2}{\alpha^2-2}\ \phi_{\alpha,y}^{(2)}(\eta_\alpha,1,\xi_\alpha-\sqrt{\alpha^2-1}\ r) - \frac{2}{\alpha^2-2}\ \phi_{\alpha,y}^{(2)}(\eta_\alpha,-1,\xi_\alpha-\sqrt{\alpha^2-1}\ r - \frac{2}{\sqrt{\varepsilon}}) \ , & \\ \qquad \frac{2}{\sqrt{\varepsilon}} \leq \xi_\alpha - \sqrt{\alpha^2-1}\ r < \frac{4}{\sqrt{\varepsilon}} \ ; & \end{cases} \tag{8.20}$$

$$\psi_{\alpha\ell',\xi_\alpha}^{(5/2)} = \begin{cases} 0 \quad , & \xi_\alpha - \sqrt{\alpha^2-1}\ r' < 0 \ , \\ \frac{2}{\alpha^2-2}\ \phi_{\alpha,y}^{(2)}\ (\eta_\alpha,-1,\xi_\alpha-\sqrt{\alpha^2-1}\ r') \quad , & 0 \leq \xi_\alpha - \sqrt{\alpha^2-1}\ r' < \frac{2}{\sqrt{\varepsilon}} \ , \\ \frac{2}{\alpha^2-2}\ \phi_{\alpha,y}^{(2)}\ (\eta_\alpha,-1,\xi_\alpha-\sqrt{\alpha^2-1}\ r') - \frac{2}{\alpha^2-2}\phi_{\alpha,y}^{(2)}(\eta_\alpha,1,\xi_\alpha-\sqrt{\alpha^2-1}\ r' - \frac{2}{\sqrt{\varepsilon}}), & \\ \qquad \frac{2}{\sqrt{\varepsilon}} \leq \xi_\alpha - \sqrt{\alpha^2-1}\ r' < \frac{4}{\sqrt{\varepsilon}} \ . & \end{cases}$$

There are obvious extensions of these expressions for ξ_α in the ranges $\frac{4}{\sqrt{\varepsilon}} \leq \xi_\alpha - \sqrt{\alpha^2-1} < \frac{6}{\sqrt{\varepsilon}}$ etc. Thus the dilatation wave front approximation is accompanied by wavelets bouncing back and forth across the rectangle behind the primary wave front.

The expressions for $\psi^{(5/2)}_{\alpha\ell}$ and $\psi^{(5/2)}_{\alpha\ell'}$ written in equation (8.20) do not produce zero normal stresses at $y = 1$. Also if extended far enough these expressions for $\psi^{(5/2)}_{\alpha\ell}$ and $\psi^{(5/2)}_{\alpha\ell'}$ show that these boundary corrections produce both normal and shear stresses at the end $x = 0$. However these stresses are at most $O(\varepsilon^{3/2})$ since $\phi^{(2)}_{\alpha,\xi y}$ is bounded (and in fact decays for large ξ_α, if the given normal stress $a^{(1)}(y,t/\delta)$ has a continuous second y-derivative). Using the information we have obtained about $\psi^{(5/2)}_{\alpha\ell}$ and $\psi^{(5/2)}_{\alpha\ell'}$ we can now rewrite the equations governing the behavior of $\phi^{(5/2)}_\alpha$ which were previously written as equations (8.11) and (8.12abc). The potential $\phi^{(5/2)}_\alpha$ satisfies the equations:

$$\phi^{(5/2)}_{\alpha,yy} - 4\phi^{(5/2)}_{\alpha,\xi_\alpha\eta_\alpha} = 0 \quad , \tag{8.21}$$

for $0 < \eta_\alpha < \infty$, $-1 < y < 1$, $0 < \xi_\alpha < \infty$;

$$\tau^{(3/2)}_{yy\alpha}(\eta_\alpha,+1,\xi_\alpha) = -\tau^{(3/2)}_{yy\alpha\ell}(\eta_\alpha,0,\xi_\alpha), \quad \tau^{(3/2)}_{yy\alpha}(\eta_\alpha,-1,\xi_\alpha) = -\tau^{(3/2)}_{yy\alpha\ell'}(\eta_\alpha,0,\xi_\alpha), \tag{8.22a}$$

for $0 \leq \eta_\alpha < \infty$, $0 \leq \xi_\alpha < \infty$;

$$\tau^{(3/2)}_{xx\alpha}(0,y,\xi_\alpha) = \lim_{\varepsilon\to 0}\{\frac{1}{\sqrt{\varepsilon}}[a^{(1)}(y,\frac{\xi_\alpha}{\alpha}) - \tau^{(1)}_{xx\alpha}(\varepsilon\xi_\alpha,y,\xi_\alpha)] - \tau^{(3/2)}_{xx\alpha\ell}(\varepsilon\xi_\alpha,y,\xi_\alpha) - \tau^{(3/2)}_{xx\alpha\ell'}(\varepsilon\xi_\alpha,y,\xi_\alpha)\} , \tag{8.22b}$$

for $0 \leq \eta < \infty$, $-1 \leq y \leq 1$;

$$u^{(2)}_\alpha(\eta_\alpha,y,0) = 0 \quad , \tag{8.22c}$$

for $0 \leq \eta < \infty$, $-1 \leq y \leq 1$. We shall not attempt to find an explicit solution for $\phi^{(5/2)}_\alpha$. It should be noted that the end condition (8.22b) may be incompatible with the boundary conditions at $y = \pm 1$ (8.22a) making 'corner' approximations unavoidable. We now examine a method suitable for dealing with the shear stress at the end $x = 0$.

9. The Formal Shear Wave Front Approximation and Associated Boundary Corrections.

The wave front approximation associated with a value of $c = 1$ in equations (7.5)-(7.7) is now considered. Corresponding potentials are written ϕ_1 and ψ_1. We use the independent variables ξ_1, η_1 defined by the equations:

$$\xi_1 = \frac{t-x}{\varepsilon^{3/2}} \quad , \quad \eta_1 = \frac{t+x}{\varepsilon^{1/2}} \quad , \tag{9.1}$$

(cf. equations (7.1), (7.2) and (8.1)). Again the special case $\nu = \sqrt{\varepsilon}$ is considered. Then ϕ_1, ψ_1 satisfy the following equations:

$$\begin{aligned} &\frac{1}{\varepsilon}(1-\frac{1}{\alpha^2})\phi_{1,\xi_1\xi_1} + \phi_{1,yy} - 2(1+\frac{1}{\alpha^2})\phi_{1,\xi_1\eta_1} + \varepsilon(1-\frac{1}{\alpha^2})\phi_{1,\eta_1\eta_1} = 0 \, , \\ &\psi_{1,yy} - 4\psi_{1,\eta_1\xi_1} = 0 \, , \end{aligned} \tag{9.2}$$

for $\varepsilon\xi_1 < \eta_1 < \infty$, $-1 < y < 1$, $0 < \xi_1 < \infty$;

$$\tau_{xy1}(\eta_1, \pm 1, \xi_1; \varepsilon) = 0 \quad , \quad \tau_{yy1}(\eta_1, \pm 1, \xi_1; \varepsilon) = 0 \quad , \tag{9.3}$$

for $\varepsilon\xi_1 \leq \eta_1 < \infty$, $0 \leq \xi_1 < \infty$;

$$\begin{aligned} &\tau_{xx1}(\varepsilon\xi_1, y, \eta_1; \varepsilon) = 0 \quad , \\ &\tau_{xy1}(\varepsilon\xi_1, y, \eta_1; \varepsilon) = \varepsilon^{3/2}\, b^{(3/2)}(y, \xi_1) - \tau_{xy}(\varepsilon\alpha\xi_1, y, \alpha\xi_1) - \\ &\qquad - \tau_{xy\alpha\ell}(\varepsilon\alpha\xi_1, r, \alpha\xi_1) - \tau_{xy\alpha\ell'}(\varepsilon\alpha\xi_1, r', \alpha\xi_1) \, , \end{aligned} \tag{9.4}$$

for $-1 \leq y \leq 1$, $0 \leq \xi < \infty$. Here the stresses $\tau_{xx1}, \tau_{xy1}, \tau_{yy1}$ are given in terms of potentials ϕ_1 and ψ_1 by the following equations:

$$\tau_{xx1} = \frac{1}{\varepsilon}\phi_{1,\xi_1\xi_1} - \frac{1}{\sqrt{\varepsilon}}2\psi_{1,\xi_1 y} + 2\phi_{1,\xi_1\eta_1} - 2\phi_{1,yy} + \sqrt{\varepsilon}2\psi_{1,\eta_1 y} + \varepsilon\phi_{1,\eta_1\eta_1} \quad ,$$

$$\tau_{xy1} = -\frac{1}{\varepsilon}\psi_{1,\xi_1\xi_1} - \frac{1}{\sqrt{\varepsilon}}2\phi_{1,\xi_1 y} + 6\psi_{1,\xi_1\eta_1} + \sqrt{\varepsilon}2\phi_{1,y\eta_1} - \varepsilon\psi_{1,\eta_1\eta_1} \quad , \qquad (9.5)$$

$$\tau_{yy1} = -\frac{1}{\varepsilon}\phi_{1,\xi_1\xi_1} + \frac{2}{\sqrt{\varepsilon}}\psi_{1,\xi_1 y} + 6\phi_{1,\xi_1\eta_1} - \sqrt{\varepsilon}2\psi_{1,y\eta_1} - \varepsilon\phi_{1,\eta_1\eta_1} \quad .$$

We assume that the disturbance arising from the shear stress at the end $x = 0$ (cf. equation 9.4) cannot travel along the rectangle faster than the shear speed. Thus for continuity of displacements at the shear wave front we require that:

$$u_1(\eta_1,y,0;\varepsilon) = 0 \quad , \quad v_1(\eta_1,y,0;\varepsilon) = 0 \quad , \qquad (9.6)$$

for $\varepsilon\xi_1 \leq \eta_1 < \infty$, $-1 \leq y \leq 1$. The displacements u_1 and v_1 are given in terms of the potentials ϕ_1 and ψ_1 by the following equations:

$$u_1 = -\frac{1}{\sqrt{\varepsilon}}\phi_{1,\xi_1} + \psi_{1,y} + \sqrt{\varepsilon}\phi_{1,\eta_1} \quad ,$$

$$v_1 = \frac{1}{\sqrt{\varepsilon}}\psi_{1,\xi_1} + \phi_{1,y} - \sqrt{\varepsilon}\psi_{1,\eta_1} \quad . \qquad (9.7)$$

We assume expansions for ϕ_1 and ψ_1 of the form:

$$\phi_1(\eta_1,y,\xi_1;\varepsilon) = \varepsilon^{5/2}\phi_1^{(5/2)}(\eta_1,y,\xi_1) + \varepsilon^3\phi_1^{(3)}(\eta_1,y,\xi_1) + \dots \quad ,$$

$$\psi_1(\eta_1,y,\xi_1;\varepsilon) = \varepsilon^{5/2}\psi_1^{(5/2)}(\eta_1,y,\xi_1) + \varepsilon^3\psi_1^{(3)}(\eta_1,y,\xi_1) + \dots \quad , \qquad (9.8)$$

and following the standard procedure find that $\phi_1^{(5/2)}$ and $\psi_1^{(5/2)}$ satisfy the system of equations written below:

$$(1-\frac{1}{\alpha^2})\phi^{(5/2)}_{1,\xi\xi} = 0, \qquad \psi^{(5/2)}_{1,yy}-4\psi^{(5/2)}_{1,\xi_1\eta_1} = 0$$

for $0 \leq \eta_1 < \infty$, $-1 < y < 1$, $0 < \xi_1 < \infty$;

$$\tau^{(3/2)}_{xy1}(\eta_1,\pm 1,\xi_1) = 0 \quad , \quad \tau^{(3/2)}_{yy1}(\eta_1,\pm 1,\xi_1) = 0 \quad ,$$

for $0 \leq \eta_1 < \infty$, $0 \leq \xi_1 < \infty$;

$$\tau^{(3/2)}_{xx1}(0,y,\xi_1) = 0 \quad ,$$

$$\tau^{(3/2)}_{xy1}(0,y,\xi_1) = b^{(3/2)}(y,\xi_1)-\lim_{\varepsilon\to 0}\frac{1}{\varepsilon^{3/2}}\{\tau^{(3/2)}_{xy\alpha}(\varepsilon\alpha\xi_1,y,\alpha\xi_1) + \tau^{(3/2)}_{xy\alpha\ell}(\varepsilon\alpha\xi_1,r,\alpha\xi_1) + \tau^{(3/2)}_{xy\alpha\ell'}(\varepsilon\alpha\xi_1,r',\alpha\xi_1)$$

for $-1 \leq y \leq 1$, $0 \leq \xi_1 < \infty$;

$$u^{(2)}_1(\eta_1,y,0) = 0, \qquad v^{(2)}_1(\eta_1,y,0) = 0,$$

for $0 \leq \eta_1 < \infty$, $-1 \leq y \leq 1$. Note that we have replaced the condition at $\eta_1 = \varepsilon\xi_1$ by a similar one at $\eta_1 = 0$ and have modified the domain above accordingly. This is a procedure similar to that used in the previous section and the error introduced by this procedure is discussed later. . We can now solve for $\phi^{(5/2)}_\alpha$ and $\psi^{(5/2)}_\alpha$ obtaining the solutions:

$$\phi^{(5/2)}_{1,\xi_1} = 0$$

$$\psi^{(5/2)}_{1,\xi_1} = -\sum_{n=1}^{\infty}\int_0^{\xi_1}\hat{b}^{(1)}_{on}(z)J_o(n\pi\sqrt{\eta_1(\xi_1-z)})dz.\sin n\pi y - \sum_{n=1}^{\infty}\int_0^{\xi_1}\hat{b}^{(1)}_{en}(z)J_o(\frac{2n-1}{2}\pi\sqrt{\eta_1(\xi_1-z)})dz.\cos\frac{2n-1}{2}\pi y \quad , \tag{9.9}$$

where

$$\hat{b}^{(1)}_{on}(z) = \int_{-1}^{1} \hat{b}^{(1)}(y,z)\sin n\pi y \, dy \quad , \hat{b}^{(1)}_{en}(z) = \int_{-1}^{1} \hat{b}^{(1)}(y,z)\cos\frac{2n-1}{2}\pi y \, dy$$

and J_o is the Bessel function of order zero. The subscripts "o" and "e" on $\hat{b}^{(1)}_{on}(\xi_1)$, $\hat{b}^{(1)}_{en}(\xi_1)$ refer to the fact that the part of $\hat{b}^{(1)}(y,\xi_1)$ which is an even function of y contributes to $\hat{b}^{(1)}_{en}$ only and the part of $\hat{b}^{(1)}$ which is an odd function of y contributes to $\hat{b}^{(1)}_{on}$ only. We solved for derivatives $\phi^{(5/2)}_{1,\xi_1}$ and $\psi^{(5/2)}_{1,\xi_1}$ rather than the potentials $\phi^{(5/2)}_1$, $\psi^{(5/2)}_1$ themselves since we require only these derivatives for the calculation of stresses and displacements at this order. The corresponding stresses and displacements are:

$$\tau^{(3/2)}_{xx1} = 0 \quad ,$$

$$\tau^{(3/2)}_{xy1} = \hat{b}^{(1)}(y,\xi_1) - \sum_{n=1}^{\infty}\int_o^{\xi_1} \hat{b}^{(1)}_{on}(z)\frac{n\pi\sqrt{\xi_1}}{2\sqrt{\xi_1-z}} J_1(n\pi\sqrt{\eta_1(\xi_1-z)})dz.\sin n\pi y$$

$$- \sum_{n=1}^{\infty}\int_o^{\xi_1} \hat{b}^{(1)}_{en}(z)\frac{(2n-1)\pi\sqrt{\xi_1}}{4\sqrt{\xi_1-z}} J_1(\frac{2n-1}{2}\pi\sqrt{\eta_1(\xi_1-z)})dz.\cos\frac{2n-1}{2}\pi y \quad ,$$

$$\tau^{(3/2)}_{yy1} = 0 \quad , \qquad (9.10)$$

$$u^{(2)}_1 = 0 \quad ,$$

$$v^{(2)}_1 = - \sum_{n=1}^{\infty}\int_o^{\xi_1} \hat{b}^{(1)}_{on}(z) J_o(n\pi\sqrt{\eta_1(\xi_1-z)})dz.\sin n\pi y$$

$$- \sum_{n=1}^{\infty}\int_o^{\xi_1} \hat{b}^{(1)}_{en}(z) J_o(\frac{2n-1}{2}\pi\sqrt{\eta_1(\xi_1-z)})dz.\cos\frac{2n-1}{2}\pi y.$$

This solution is only useful if the prescribed load $\hat{b}^{(3/2)}(y,\xi_1)$ has the property that:

$$\hat{b}^{(3/2)}(\pm 1,\xi_1) = 0 \quad . \qquad (9.11)$$

In general $\hat{b}^{(3/2)}(y,\xi_1)$ does not satisfy this condition. We shall discuss how to deal with prescribed stresses $b^{(3/2)}$ having non zero values at y = 1 in a later section dealing with 'corner approximations'. At present we assume that $\hat{b}^{(3/2)}$ fulfills equation (9.11) or that we are considering the part of the shear stress $\hat{b}^{(3/2)}$ which fulfills(9.11).

Using methods similar to those of the previous section dealing with the dilatation wave front approximation we can estimate the error introduced by the approximation of domain. At the end $\eta_1 = \varepsilon\xi_1$ we find that:

$$|\varepsilon^{3/2}\hat{b}^{(3/2)}(y,\xi_1)-\varepsilon^{3/2}\tau_{xy1}^{(3/2)}(\varepsilon\xi_1,y,\xi_1)| = O(\varepsilon^2) \quad ,$$

uniformly in ξ_1 and y if $\hat{b}^{(3/2)}(y,\xi_1)$ has continuous second y-derivatives, and if $\hat{b}^{(3/2)}(y,\xi_1)$ decays rapidly enough for large ξ_1. It is difficult to decide if $\hat{b}^{(3/2)}(y,\xi_1)$ satisfies these conditions. Only those prescribed loads $b^{(3/2)}(y,\xi_1)$ which satisfy these smoothness and decay conditions are admissible; however $\hat{b}^{(3/2)}(y,\xi_1)$ includes the shear stresses resulting from lower order dilatation wave effects at x = 0, as well as the prescribed shear stress $b^{(3/2)}(y,\xi_1)$. These 'generated' shear stresses may not be sufficiently smooth or may not decay fast enough to allow the validity of our procedure involving the approximation of the domain. Once we have worked out the details of the functional form of the stresses $\tau_{xy\alpha}^{(3/2)}$, $\tau_{xy\alpha\ell}^{(3/2)}$, $\tau_{xy\alpha\ell'}^{(3/2)}$ this issue is decided. Unfortunately this seems to be practically impossible. At the moment we just assume that $\hat{b}^{(3/2)}(y,\xi_1)$ has a continuous second y derivative and decays faster than $\frac{1}{\xi_1^{1+k}}$ for large ξ_1. Any shear stress which does not satisfy these conditions must be dealt with in other ways. Shear stresses at the end which are not sufficiently smooth with respect to y give rise to 'end waves' propagating back and forth across the ends. We shall discuss them briefly in a later section dealing with corner approximations. Shear stresses at the end which do not decay sufficiently fast must be broken up into slowly varying portions and rapidly varying portions and then used in the appropriate sections (sections 3,4,5,6 for the slowly varying end stresses, and sections 7,8,9 for the rapidly varying end stresses) as end conditions.

Apart from the difficulty of determining all the properties of $\hat{b}^{(3/2)}$, further trouble arising when we proceed to the next order and attempt to develop solutions for $\phi_1^{(3)}$ and $\psi_1^{(3)}$. This difficulty is exactly similar to that encountered when the systems of equations (8.11) and (8.12) for $\phi_\alpha^{(5/2)}$ and $\psi_\alpha^{(5/2)}$ was examined. It was found there that the condition $\tau_{xy}^{(3/2)}(\eta_\alpha, \pm 1, \xi_\alpha) = 0$ could not be satisfied. In the present case we find that:

$$\tau_{yy1}^{(2)} = 2\psi_{1,\xi_1 y}^{(5/2)}$$

In general, the stress defined by the above equation cannot satisfy the condition $\tau_{yy1}^{(2)}(\eta_1, \pm 1, \xi_1) = 0$ ($\psi_{1,\xi_1 y}^{(5/2)}$ is fully determined by equation (9.9) and is not zero at $y = \pm 1$). Therefore we must develop another method for dealing with this normal stress.

In the previous section we examined the side y = 1 by introducing the variable r defined in equation (7.9). In a similar manner we introduce here shear wave front boundary corrections $\phi_{1\ell}(\eta_1,r,\xi_1;\varepsilon)$ and $\psi_{1\ell}(\eta_1,r,\xi_1;\varepsilon)$. Equation (7.10) shows that these functions must satisfy the following differential equations:

$$\frac{1}{\varepsilon}[(1-\frac{1}{\alpha^2})\phi_{1\ell,\xi_1\xi_1}+\phi_{1\ell,rr}]-2(1+\frac{1}{\alpha^2})\phi_{1\ell,\xi_1\eta_1}+\varepsilon(1-\frac{1}{\alpha^2})\phi_{1\ell,\eta_1\eta_1} = 0 \quad ,$$

$$\frac{1}{\varepsilon}[\psi_{1\ell,rr}]-4\psi_{1\ell,\xi_1\eta_1} = 0 \quad , \qquad (9.12)$$

$\varepsilon\xi_1 < \eta_1 < \infty$, $0 < r < \frac{2}{\sqrt{\varepsilon}}$, $-\frac{\alpha-1}{\alpha+1}\frac{\eta_1}{\varepsilon} < \xi_1 < \infty$. Note that we allow ξ_1 to be negative in this case. Although it is assumed that no shear effects can propagate directly down the plate faster than shear wave velocity, we now allow effects produced by the interaction of the shear wave and the sides to appear in front of this wave. It is also required that $\phi_{1\ell}$ and $\psi_{1\ell}$ are capable of dealing with the stress $\tau_{yy}^{(2)}$ generated by the wave front approximation at $r = 0$ $(y = 1)$. That is:

$$\tau_{yy1\ell}(\eta_1,0,\xi_1;\varepsilon) = -\varepsilon^2\ 2\psi^{(5/2)}_{1,\xi_1 y} \quad . \qquad (9.13)$$

In this case we shall find that the effects produced by this secondary stress are important near the side $y = 1$. At present we require that $\phi_{1\ell}$ and $\psi_{1\ell}$ are bounded for r large. Expansions for $\phi_{1\ell}$ and $\psi_{1\ell}$ of the following form are assumed:

$$\phi_{1\ell}(\eta_1,r,\xi_1;\varepsilon) = \varepsilon^3\phi^{(3)}_{1\ell}(\eta_1,r,\xi_1) + \varepsilon^{7/2}\phi^{(7/2)}_{1\ell}(\eta_1,r,\xi_1) + \dots ,$$

$$\psi_{1\ell}(\eta_1,r,\xi_1;\varepsilon) = \varepsilon^3\psi^{(3)}_{1}(\eta_1,r,\xi_1) + \varepsilon^{7/2}\psi^{(7/2)}_{1\ell}(\eta_1,r,\xi_1) + \dots .$$

After substituting these expressions into equation (9.12), we find that the lowest order terms $\phi^{(3)}_{1\ell}$ and $\psi^{(3)}_{1\ell}$ satisfy the following equations:

$$(1-\frac{1}{\alpha^2})\phi^{(3)}_{1\ell,\xi_1\xi_1} + \phi^{(3)}_{1\ell,rr} = 0 \quad ,$$

$$\psi^{(3)}_{1\ell,rr} = 0 \quad , \tag{9.14}$$

for $0 < \eta_1 < \infty$, $0 < r < \infty$, $-\infty < \xi_1 < \infty$. These equations can be solved in conjunction with the boundary condition (9.13) and the boundedness assumption, obtaining the following solutions:

$$\phi^{(3)}_{1\ell,\xi_1} = \frac{r}{\pi}\int_{-\infty}^{\infty} \frac{\psi^{(5/2)}_{1,y}(\eta_1,1,z)}{(\xi_1-z)^2+r^2}\,dz \quad , \qquad \psi^{(3)}_{1\ell} = 0 \ . \tag{9.15}$$

The corresponding stresses are:

$$\tau^{(2)}_{xx1} = (\frac{2}{\alpha^2}-3)\int_{-\infty}^{\infty} \frac{\psi^{(5/2)}_{1,y}(\eta_1,1,z)(1-z)}{[(\xi_1-z)^2+r^2]^2}\,\frac{2r}{\pi}\,dz = (\frac{2}{\alpha^2}-3)\frac{r}{\pi}\int_{-\infty}^{\infty} \frac{\psi^{(5/2)}_{1,yz}(\eta_1,1,z)}{(\xi_1-z)^2+r^2}\,dz,$$

$$\tau^{(2)}_{xy1} = \frac{2}{\pi}\int_{-\infty}^{\infty} \frac{\psi^{(5/2)}_{1,y}(\eta_1,1,z)}{(\xi_1-z)^2+r^2}\,dz - \frac{4r^2}{\pi}\int_{-\infty}^{\infty} \frac{\psi^{(5/2)}_{1,y}(\eta_1,1,z)}{[(\xi_1-z)^2+r^2]^2}\,dz \tag{9.16}$$

$$\tau^{(2)}_{yy1} = \frac{r}{\pi}\int_{-\infty}^{\infty} \frac{\psi^{(5/2)}_{1,yz}(\eta_1,1,z)}{(\xi_1-z)^2+r^2}\,dz \quad .$$

If this procedure for removing the stress at y = 1 by means of corrections $\phi_{1\ell}, \psi_{1\ell}$ is to be of use, the stresses corresponding to $\phi_{1\ell}^{(3)}$ and $\psi_{1\ell}^{(3)}$ written above in equation (9.16) must be smaller than $O(\varepsilon^2)$ at $y = -1$ $(r = \frac{2}{\sqrt{\varepsilon}})$. Before we can estimate the size of $\tau_{xy1}^{(2)}, \tau_{yy1}^{(2)}$ we must examine the properties of $\psi_{1,yz}^{(5/2)}$ in more detail. Integrating the last equation of (9.16) by parts the following relationship is obtained:

$$\tau_{yy1}^{(2)} = \frac{2r}{\pi} \int_0^\infty \frac{\psi_{1,y}^{(5/2)}(\eta_1, 1, z)(z-\xi_1)}{[(z-\xi_1)^2 + r^2]^2} \, dz \quad .$$

A crude estimate shows that $\tau_{yy1}^{(2)}(\eta_1, \frac{2}{\sqrt{\varepsilon}}, \xi_1)$ is $O(\sqrt{\varepsilon})$ uniformly in η_1 and ξ_1 if $\psi_{1,y}^{(5/2)}(\eta_1, 1, \xi_1)$ has a uniform bound of $O(1)$ compared with ε. From equation (9.9) we can calculate $\psi_{1,y}^{(5/2)}(\eta_1, 1, \xi_1)$ as follows:

$$\psi_{1,y}^{(5/2)} = -\sum_{n=1}^{\infty} \int_0^{\xi_1} \hat{b}_{on}^{(3/2)}(z) \frac{2\sqrt{\xi - z}}{\sqrt{\xi}} J_1(n\pi\sqrt{\eta_1(\xi_1 - z)}) \, dz \, (-1)^n$$
$$- \sum_{n=1}^{\infty} \int_0^{\xi_1} \hat{b}_{en}^{(3/2)}(z) \frac{2\sqrt{\xi_1 - z}}{\sqrt{\xi_1}} J_1\left(\frac{2n-1}{2} \pi\sqrt{\eta_1(\xi_1 - z)}\right) dz \, (-1)^n \; . \tag{9.17}$$

We have not been successful in determining conditions on $\hat{b}^{(3/2)}(y,z)$ such that $\psi_{1,y}^{(5/2)}$ is uniformly bounded. It is certainly possible to find examples of interest where $\psi_{1,y}^{(5/2)}$ is not uniformly bounded; the stress $\tau_{yy1}^{(2)} = 2\psi_{1,\xi_1 y}^{(5/2)}$ at y = 1 in general decays no faster than $\frac{1}{\xi_1^{1/4}}$ which predicts that $\psi_{1,y}^{(5/2)}(\eta_1, 1, \xi_1)$ will grow as fast as $\xi_1^{3/2}$ for large ξ_1. Fortunately we can deal with the generated stress $\tau_{yy1}^{(2)}(\eta_1, 1, \xi_1)$ in another way. From equation (9.9) we can easily deduce that $\tau^{(2)}_{yy1}(\eta_1,1,\xi_1)$ is uniformly O(1) in comparison with ε (remembering the assumed properties of $b^{(3/2)}(y,\xi_1)$). Thus, written as a function of the variables x,t, the stress $\tau^{(2)}_{yy1}(\eta_1,1,\xi_1)$ is also uniformly O(1) in comparison with ε. In this form it may be used as a part of the normal stress at y = 1 associated with the inner approximation developed in section 3. We are in fact proposing (without proof) that the generated stress $\tau^{(2)}_{yy1}(\eta_1,1,\xi_1)$ can be divided into two parts: a pulse near the front $\xi_1 = 0$ which stimulates the boundary correction $\phi_{1\ell},\psi_{1\ell}$; and a more slowly decaying part which can be included in the inner approximation. Both the boundary correction and the inner approximation are required as the boundary correction cannot deal with the part of $\tau^{(2)}_{yy1}(\eta_1,1,\xi_1)$ which decays slowly away from the front $\xi_1 = 0$; and the inner approximation cannot deal with the part of $\tau^{(2)}_{yy1}(\eta_1,1,\xi_1)$ which varies too rapidly spatially or with time. It should also be noted that this approach could not be used to deal with stresses generated by the dilatation wave front approximation since these effects travel along the rectangle at a faster velocity than the inner approximation allows (the inner approximation predicts that no disturbance can travel along the rectangle faster than the plate wave speed $\frac{4(\alpha^2-1)}{\alpha^2}$).

If $\tau^{(2)}_{yy1}$ was explicity known we would now be able to develop the boundary corrections $\phi^{(5/2)}_{1\lambda},\psi^{(5/2)}_{1\ell}$ valid near y = 1 and a similar correction, $\phi^{(5/2)}_{1\ell'},\psi^{(5/2)}_{1\ell'}$ valid near y = -1. These corrections would in turn generate shear stresses at the sides y = 1 providing boundary conditions for the next term in the shear wave front approximation $\psi^{(3)}_1$.

10. Further Approximations.

In the course of the development of the results of the preceding section various restrictions, which limit the permissible type of prescribed loads, were obtained. Some of these restrictions were required to avoid consideration of effects variously described by the terminology: corner approximations, initial approximations, Rayleigh wave front approximations, long time approximation and end wave approximations. These effects are not unimportant but so far we have not been able to develop the approximations describing them; therefore we can only demonstrate why they occur and discuss them from an intuitive point of view.

First we consider the 'corner approximations'. In sections 8 and 9, devoted to the development of the dilatation wave front approximation and the shear wave front approximation, we found that it was necessary to assume that the rapidly varying part of the prescribed loads at the end x = 1 satisfies the following conditions (cf. (8.9) and (9.11)):

$$a^{(1)}(\pm 1, \frac{\xi}{\alpha}\alpha) = 0 \quad , \quad \hat{b}^{(3/2)}(\pm 1, \xi_1) = 0$$

We could require that only those prescribed loads which satisfy the first of the conditions (10.1) are considered. However since $\hat{b}^{(3/2)}$ includes both the prescribed stress $b^{(3/2)}$, and stresses generated by the dilatation wave front approximation, it seems very artificial to assume that the second condition of (10.1) is true. The question then arises: what can we do if condition (10.1) is not true? A suitable choice of variable allows the corner x = 0, y = 1 to be examined in detail and a corner approximation investigated. However the equations governing the behavior of this corner approximation turned out to be the full equations of two-dimensional elasticity even at the lowest order, and the relevant domain was an infinite quarter space. This is a classic unsolved problem in elasticity. A careful examination of the equations (7.10) for the boundary corrections to the wave front approximations shows that another choice for c besides $c = \alpha$ and $c = 1$ is significant. For $c = \beta$, where $\beta < 1$ is the real root of the following equation (Rayleigh's equation):

$$(2-\beta^2)^2 - 4\sqrt{1-\beta^2}\sqrt{1-\beta^2/\alpha^2} = 0 \quad , \tag{10.2}$$

Solutions $\phi_{\beta\ell}$,$\psi_{\beta\ell}$ can be found which leave the side $r = 0$ stress free and which correspond to effects decaying away from the side $r = 0$. The difficulty is that it is not possible to decide what conditions should be specified for $\phi_{\beta\ell}$, $\psi_{\beta\ell}$ at the end $x = 0$. If it was possible to develop a 'corner approximation' this would naturally suggest conditions to allow thecomplete determination of the Rayleigh wave front approximation $\phi_{\beta\ell}$, $\psi_{\beta\ell}$. The corner approximations would also be associated with the phenomenon of 'end waves' bouncing back and forth across the end of the rectangle.

In the development of the inner approximation and the diffusive approximations it is assumed that $p_1^{(2)}(x,t)$, $p_2^{(2)}(x,t)$, $q_1^{(2)}(x,t)$ and $q_2^{(2)}(x,t)$, the prescribed stresses at the sides $y = 1$, are all zero initially and have their first two time derivatives zero initially. Intuitively speaking, this means that at the lowest order there is no overall vibration of the rectangle due to waves bouncing from side to side. It also means that we must assume suitable initial conditions for the inner approximation and diffusive approximations. The development of an 'initial approximation' would allow general initial conditions to be specified for the elastic plate problem and would also allow a more convincing development of initial conditions for the corresponding inner approximation. This 'initial approximation' has so far not been sucessfully obtained. Therefore we must assume that the rectangle is initially at rest and restrict the initial growth of the prescribed stresses at y = 1 as described above.

In the development of the diffusive approximations it was assumed that the diffusive approximation valid near the end x = 0 has small effect at the other end x = 1 (and vice-versa for the diffusive approximation significant near x = 1). In section 6 this assumption was shown to be valid for finite times of smaller magnitude than $0(1/\varepsilon)$. This is a completely satisfactory conclusion as we are not concerned with the solution to the problem after the dilatation wave propagating from the end x = 0 reaches the end x = 1, However it is of interest to note that we can develop a long-time approximation by introducing the variables x,y and $\tau = \varepsilon t$ and the corresponding approximations, say $v_t(x,y,\tau;\varepsilon)$, $u_t(x,y,\tau;\varepsilon)$. We find that the lowest order term in the expansion for v_t can be written $\bar{v}_t^{(0)}(x,\tau)$ (using familiar notation) where $\bar{v}_t^{(0)}$ satisfies the following differential equation:

$$\bar{v}^{(0)}_{t,\tau\tau} + \frac{4}{3}\frac{\alpha^2-1}{\alpha^2}\bar{v}^{(0)}_{t,xxxx} = 0 \quad . \tag{10.3}$$

The two diffusive approximations can then be 'matched' on to a solution of (10.3).

11. Summary.

The most significant and complete result obtained in the preceding work was development of the plate wave equation and the Euler-Bernoulli plate equation from the full equations of dynamic elasticity using a rational perturbation expansion technique. It was found in section 3 that a slowly varying normal stress applied at one end of an elastic rectangle propagates according to the plate wave equation. In section 4 it was found that a slowly varying bending moment and average shear stress applied at one end of an elastic rectangle diffuse according to the Euler-Bernoulli plate equation. In later sections some results concerning effects arising from rapidly varying applied loads were obtained. A summary of the derivation of these results is now presented.

In sections 3 - 9 an approximate solution of the original system of equations (2.10), (2.11) was obtained as the sum of several different approximations, each satisfying part of the problem and all interacting with one another. These parts of the overall approximate solution are: the inner approximation, two diffusive approximations, two quasi-static boundary layer approximations, the dilatation wave front approximation and associated boundary correction, the shear wave front approximation and associated boundary correction, the corner approximations and the Rayleigh wave front approximations (these last two were not completely developed).

Before any of these approximations can be obtained the prescribed stresses at the end $x = 0$ must be divided into slowly varying parts and rapidly varying parts. We write (cf. (2.11c) and (2.11d)):

$$\tau_{xx}(0,y,t\,;\varepsilon,\delta) = \varepsilon f^{(1)}(y,t) + \varepsilon a^{(1)}(y,t/\delta) \quad ,$$
$$\tau_{xy}(0,y,t;\varepsilon,\delta) = \varepsilon^{3/2} g^{(3/2)}(y,t) + \varepsilon^{3/2}\, b^{(3/2)}(y,t/\delta) \ . \qquad (11.1)$$

Here $f^{(1)}(y,t)$ and $g^{(3/2)}(y,t)$ are required to have continuous second t-derivatives of $0(1)$ in comparison with ε for $0 \leq t$. This condition is especially important at $t = 0$ since that $f^{(1)}(y,t,)$ and $g^{(3/2)}(y,t)$ rise smoothly from $t = 0$. We require that $a^{(1)}(y,t/\delta)$ and $b^{(3/2)}(y,t/\delta)$ are zero for $t \geq \delta$, where δ/ε is small. Given certain prescribed stresses we can examine their second time derivatives and if these are large only for a short time a scale δ is determined and a division as in (11.1) can be made. If these conditions are not satisfied the present methods would require modification. (for example, the methods developed here cannot be used for a problem where the impulsive part of the load has a duration as long as the time required for a shear wave to propagate across the rectangle; nor can prolonged rapidly oscillating loads be considered). We also require the prescribed end stresses to possess continuous second y-derivatives. This avoids the propagation of waves across the ends.

A similar condition is required for the stresses $p_1^{(2)}, p_2^{(2)}$, $q_1^{(2)}, q_2^{(2)}$ prescribed at $y = 1$. It is required that these given functions are twice continuously differentiable for $0 \leq t < \infty$, $0 \leq x \leq 1$. Having ascertained that the given data satisfies the above requirements we can now develop an approximate solution for the original system (2.10) (2.11). The first step is to calculate the lowest order term

$\phi_\alpha^{(2)}(\frac{\alpha t+x}{\varepsilon^2/\delta}, y, \frac{\alpha t-x}{\delta})$ of the dilatation wave front approximation. For particular case $\delta=\varepsilon^{3/2}$ this may be explicitly written as follows (cf. (8.6)):

$$\phi^{(2)}_{\alpha,\xi_\alpha} = \begin{cases} 0 & \xi_\alpha < 0 \ , \\ \sum_{n=1}^{\infty}\int_0^{\xi_\alpha} \frac{1}{\alpha^2} a_{on}^{(1)}(\frac{z}{\alpha}) J_o(n\pi\sqrt{\eta_\alpha(\xi_\alpha - z)})dz.\sin n\pi y & \\ \quad + \sum_{n=1}^{\infty}\int_0^{\xi_\alpha} \frac{1}{\alpha^2} a_{en}^{(1)}(\frac{z}{\alpha}) J_o(\frac{2n-1}{2}\pi\sqrt{\eta_\alpha(\xi_\alpha - z)})dz.\cos\frac{2n-1}{2}\pi y, & \xi_\alpha \geq 0, \end{cases}$$

where $\xi_\alpha = \frac{\alpha t-x}{\varepsilon^{3/2}}$, $\eta_\alpha = \frac{\alpha t+x}{\varepsilon^{1/2}}$, $a_{on}^{(1)}(\frac{z}{\alpha})\sin n\pi y\ dy$,

and $a_{en}^{(1)}(\frac{z}{\alpha}) = \int_{-1}^{1} a^{(1)}(y,\frac{z}{\alpha})\cos\frac{2n-1}{2}\pi y dy$

The expression (11.2) describes the propagation of the prescribed normal stress $a^{(1)}(y,t/\delta)$ from the end into the interior of the rectangle (with a velocity α). As the disturbance described by $\phi_\alpha^{(2)}$ propagates along the rectangle it produces higher order shear stresses on the sides $y = \pm 1$.

To deal with these generated stresses the dilatation wave front corrections are developed. The lowest order terms $\psi_{\alpha\ell}^{(5/2)}(\frac{\alpha t+x}{\sqrt{\varepsilon}}, \frac{1-y}{\sqrt{\varepsilon}}, \frac{\alpha t-x}{\varepsilon^{3/2}})$ and $\psi_{\alpha\ell'}^{(5/2)}(\frac{\alpha t+x}{\sqrt{\varepsilon}}, \frac{1+y}{\sqrt{\varepsilon}}, \frac{\alpha t-x}{\varepsilon^{3/2}})$ are described by equation (8.20). They correspond to wavelets propagating across the rectangle and moving with the dilatation wave speed. (These effects are related to the Stoneley waves of seismology).

Next we calculate the lowest order term of the shear wave front approximation, $\psi_1^{(5/2)}(\frac{t+x}{\sqrt{\varepsilon}}, y, \frac{t-x}{\varepsilon^{3/2}})$, according to equation (9.9) (for the special case $\delta = \varepsilon^{3/2}$). This expression describes the propagation of the prescribed shear stress $b^{(3/2)}(y,t/\delta)$ and shear stresses generated by the dilatation wave front approximation (described by $\phi_{\alpha}^{(2)}, \psi_{\alpha\ell}^{(5/2)}, \psi_{\alpha\ell'}^{(5/2)}$) at $x = 0$ into the interior of the rectangle (with a velocity 1). The difficulty at this stage is that it is not possible to decide in advance, for a general prescribed stress $a^{(1)}(y,t/\delta)$, what the properties of the generated shear stresses $\tau_{xy\alpha}^{(3/2)}, \tau_{xy\alpha\ell}^{(3/2)}, \tau_{xy\alpha\ell'}^{(3/2)}$, are at $x = 0$. In a particular case we must decide how much of these generated stresses to include with the rapidly varying effects and then consider the remaining part with the slowly varying effects.

Assuming that we have successfully and fully determined the properties of the shear stresses $\tau_{xy\alpha}^{(3/2)}, \tau_{xy\alpha\ell}^{(3/2)}, \tau_{xy\alpha\ell'}^{(3/2)}$ at $x = 0$, we can calculate the shear wave front approximation $\psi_1^{(5/2)}$. This expression generates higher order normal stresses $\tau_{yy1}^{(2)}$ at the sides $y = 1$. To deal with these generated stresses we introduce shear wave front boundary corrections $\psi_{1\ell}^{(3)}(\frac{t+x}{\sqrt{\varepsilon}},\frac{1-y}{\sqrt{\varepsilon}},\frac{t-x}{\varepsilon^{3/2}})$ and $\psi_{1\ell'}^{(3)}(\frac{t+x}{\sqrt{\varepsilon}},\frac{1+y}{\sqrt{\varepsilon}},\frac{t-x}{\varepsilon^{3/2}})$. These corrections describe effects important near each side moving with the shear wave. It is found that $\phi_{1\ell}^{(3)}$ can only accomodate a normal stress at $y = 1$ which decays rapidly away from the wave front $t = x$. In general the generated stress $\tau_{yy1}^{(2)}(\frac{t+x}{\sqrt{\varepsilon}},1,\frac{t-x}{\varepsilon^{3/2}})$ does not satisfy this condition. We assume that we can consider $\tau_{yy1}^{(2)}$ in two parts: a rapidly decaying part producing the correction term $\phi_{\ell}^{(3)}$ and a more slowly varying part which can be considered with the prescribed normal stress at $y = 1$, $p_1^{(2)}$, in the analysis of the inner approximation. This division of $\tau_{yy1}^{(2)}(\frac{t+x}{\sqrt{\varepsilon}},1,\frac{t-x}{\varepsilon^{3/2}})$ has not been established.

It is now necessary to investigate the lowest order term in the corner approximations. These are required to describe the effects produced by non zero prescribed stresses at the corners $x = 0$, $y = \pm 1$. These corner approximations have not been developed. Also connected with these corner approximations are the Rayleigh wave front approximations. The existence of a wave front important only near the sides $y = \pm 1$ and travelling with the Rayleigh wave speed (velocity β in our notation) was established. However this Rayleigh wave front approximation can only be fully determined after the corner approximations have been developed. It may also prove necessary to develop an end wave approximation to account for the normal stresses at $x = 0$ generated by the dilatation wave front approximation $\phi_\alpha^{(2)}$ and the accompanying boundary corrections $\psi_{\alpha\ell}^{(5/2)}$, $\psi_{\alpha\ell'}^{(5/2)}$. It is an obvious deficiency in the present work that the corner approximations and Rayleigh wave front approximations have not been obtained.

The results described so far account for the lowest order effects produced by the rapidly varying part of the prescribed stresses. We now develop the lowest order terms of the inner approximation, $\overline{u}_i^{(0)}(x,t)$ and $\overline{v}_i^{(0)}(x,t)$. This approximation describes the effects produced by slowly varying prescribed and generated stresses at $y = \pm 1$, as well as the propagation of the average prescribed and generated slowly varying normal stress from $x = 0$ into the interior of the rectangle. The equations for $\overline{u}_i^{(0)}$ and $\overline{v}_i^{(0)}$, written below, do not include the generated slowly varying stresses. However the modification required to do this is minor. The equations obtained determining $\overline{u}_i^{(0)}$, $\overline{v}_i^{(0)}$

are:

$$u^{(0)}_{i,tt} - \frac{4(\alpha^2-1)}{\alpha^2} u_{i,xx} = \frac{1}{2}(q_1^{(2)} - q_2^{(2)}) \quad , \tag{11.3}$$

$$u^{(0)}_{i,x}(0,t) = \frac{\alpha^2}{8(\alpha^2-1)} \int_{-1}^{1} f^{(1)}(y,t)dy \quad , \quad u^{(0)}_{i,x}(1,t) = 0 \ , \tag{11.4}$$

$$u^{(0)}_{i}(x,0) = 0 \quad , \quad u^{(0)}_{i,t}(x,0) = 0 \quad , \tag{11.5}$$

$$v^{(0)}_{i,tt} = \frac{1}{2}(p_1^{(2)} - p_2^{(2)}) \quad , \tag{11.6}$$

$$v^{(0)}_{i}(x,0) = 0 \quad , \quad v^{(0)}_{i,t}(x,0) = 0 \quad . \tag{11.7}$$

It is of significance that the equation (11.3) can be identified as the plate wave equation.

The lowest order terms of the two diffusive approximations, $\overline{v}^{(0)}_d(\frac{x}{\sqrt{\varepsilon}},t)$ and $\overline{v}^{(1/2)}_{d'}(\frac{1-x}{\sqrt{\varepsilon}},t)$, describe the spreading of the bending moment and the average shear effects produced by the slowly varying prescribed and generated stresses at each end. The discussion of $\overline{v}^{(0)}_d$ and $\overline{v}^{(1/2)}_{d'}$ in sections 4 and 6 does not include the generated (however the extension of the results to do this is not difficult). The term $\overline{v}^{(0)}_d$ is a solution of the following problem:

$$\overline{v}^{(0)}_{d,tt} + \frac{4}{3}\frac{\alpha^2-1}{\alpha^2} \overline{v}^{(0)}_{d,zzzz} = 0 \quad , \tag{11.8}$$

$$\overline{v}^{(0)}_{d,zz}(0,t) = -\frac{3\alpha^2}{8(\alpha^2-1)} \int_{-1}^{1} yf^{(1)}(y,t)dy \quad ,$$

$$\overline{v}^{(0)}_{d,zzz}(0,t) = \frac{3\alpha^2}{8(\alpha^2-1)} \int_{-1}^{1} g^{(3/2)}(y,t)dy \quad , \tag{11.9}$$

$$\lim_{z\to\infty} \bar{v}_d^{(0)}(z,t) = 0 \quad , \quad \lim_{z\to\infty} \bar{v}_{d,z}^{(0)}(z,t) = 0 \quad , \tag{11.10}$$

$$\bar{v}_d^{(0)}(z,0) = 0 \quad , \quad \bar{v}_{d,t}^{(0)}(z,0) = 0 \quad , \tag{11.11}$$

where $z = \frac{x}{\sqrt{\varepsilon}}$. Equation (11.8) can be identified as the Euler-Bernoulli beam equation (see section 4).

Lastly we obtain the lowest order boundary layer approximations $u_\ell^{(1)}(\frac{x}{\varepsilon},y,t)$, $v_\ell^{(1)}(\frac{x}{\varepsilon},y,t)$, $u_{\ell'}^{(2)}(\frac{1-x}{\varepsilon},y,t,)$ and $v_{\ell'}^{(2)}(\frac{1-x}{\varepsilon},y,t)$ which describe the quasi-static effects arising from the slowing varying stresses at each end. These effects are important only near each end.

In the above summary we have only considered the various approximations obtained for times before the fastest wave (the dilatation wave) reaches the end $x = 1$. To proceed further we would have to consider the stresses generated at $x = 1$ by the approximate solution as new data and repeat the whole procedure with the role of the ends $x = 0$ and $x = 1$ interchanged.

REFERENCES.

[1] Abramowitz, M., and I. A. Stegun, Handbook of Mathematical Functions, National Bureau of Standards Applied Mathematics Series. 55 (1964).

[2] Apostol, T. M., Mathematical Analysis. Massachusetts: Adison-Wesley (1957).

[3] Bellman, R., Perturbation Techniques in Mathematics, Physics and Engineering. New York: Holt, Rinehart and Winston (1964).

[4] Fox, N., On Asymptotic Expansions in Plate Theory. Proc.Roy.Soc. Ser. A. 278, 228 (1964).

[5] Friedrichs, K. O., Kirchhoff's Boundary Conditions and the Edge Effect for Elastic Plates. Proc. Symp. Math. 3, 117. New York: McGraw-Hill (1950).

[6] Friedrichs, K. O., and R. F. Dressler, A Boundary Layer Theory for Elastic Bending of Plates. Comm. Pure Appl. Math., 14, 1 (1961).

[7] Green, A. E., On the Theory of Thin Elastic Shells. Proc.Roy.Soc. Ser. A. 266, 143 (1962).

[8] Green, A. E., Boundary-Layer Equations in the Linear Theory of Thin Elastic Shells. Proc. Roy. Soc. Ser. A. 269, 481 (1962)

[9] Green, A. E., and P. M. Naghdi, Some Remarks on the Linear Theory of Shells. Quart. J. Mech. Appl. Math. 18, 257 (1965).

[10] Green, A. E., and N. Laws, Further Remarks on the Boundary Conditions for Thin Elastic Shells. Proc. Roy. Soc. Ser. A. 289, 171 (1966).

[11] Horvay, G., Biharmonic Eigenvalue Problem of the Semi-infinite Strip. Q. Appl. Math. 15, 65 (1957).

[12] John, F., A priori Estimates Applied to Nonlinear Shell Theory. Proc. Symp. Appl. Math. 17, 102; Amer. Math Soc., Providence, R.I., (1965).

[13] John, F., Estimates for the Derivatives of the Stresses in a Thin Shell and Interior Shell Equations. Comm. Pure Appl. Math. 18, 235 (1965).

[14] Johnson, M.W., and E. Reissner, On the Foundations of the Theory of Thin Elastic Shells. J. Math. and Phys. 37, 371 (1958).

[15] Johnson, M. W., A Boundary Layer Theory for Unsymmetric Deformations of Circular Cylindrical Elastic Shells. J. Math. and Phys. 42, 167 (1963).

[16] Johnson, M. W., and R. W. Little, The Semi-infinite Elastic Strip Problem. Q. Appl. Math. 22, 335 (1965).

[17] Knowles, J. K., The Dirichlet Problem for a Thin Rectangle. To appear.

[18] Knowles, J. K., A Saint-Venant Principle for a Class of Second Order Elliptic Boundary Value Problems. To appear Zeit. fur Angew. Math. und Phys.

[19] Knowles, J. K., On Saint-Venant's Principle in the Two-dimensional Linear Theory of Elasticity. Arch. Rational Mech. Anal. 21, 1 (1966).

[20] Kolsky, H., Stress Waves in Solids. New York: Dover Publications (1963).

[21] Laws, N., *A Boundary Layer Theory for Plates with Initial Stress.* Proc. Cambridge Philos. Soc. 62, 313 (1966).

[22] Love, A. E. H., The Mathematical Theory of Elasticity. New York: Dover Publications (1926).

[23] Miklowitz, J., The Propagation of Compressional Waves in a Dispersive Elastic Rod, Part I--Results from the Theory. J. Appl. Mech. 24, 231 (1957).

[24] Miklowitz, J., and C. R. Nisewanger, The Propagation of Compressional Waves in a Dispersive Elastic Rod, Part II--Experimental Results and Comparison with Theory. J. Appl. Mech. 24, 240 (1957).

[25] Miklowitz, J., Flexural Wave Solutions of Coupled Equations Representing the More Exact Theory of Bending. J. Appl. Mech. 20, 511 (1953).

[26] Miklowitz, J., and R. L. Rosenfeld, Wave Fronts in Elastic Rods and Plates. Proc. Fourth U. S. Nat. Congr. Appl. Mech. ASME. 1, 293 (1962).

[27] Miklowitz, J., Recent Developments in Elastic Wave Propagation. Appl. Mech. Rev. 13, 865 (1960).

[28] Mindlin, R. D., Influence of Rotatory Inertia and Shear on Flexural Motions of Isotropic, Elastic Plates. J. Appl. Mech. 18, 31 (1951).

[29] Mindlin, R. C., An Introduction to the Mathematical Theory of Vibration of Elastic Plates. Monograph, U. S. Army Signal Corps Engr. Labs., Ft. Monmouth, N.Y. Signal Corps Contract DA-36-039,

SC-56772, (1955).

[30] Novozhilov, V. V., and L. I. Slepian, On Saint-Venant's Principle in the Dynamics of Beams. Appl. Math. Mech. 29, 293 (1965).

[31] Reissner, E., On the Derivation of Boundary Conditions for Plate Theory. Proc. Roy. Soc. Ser. A. 276, 178 (1963).

[32] Reissner, E., On Asymptotic Expansions for Circular Cylindrical Shells. J. Appl. Mech. 31, 245 (1964).

[33] Reiss E. L., and S. Locke, On the Theory of Plane Stress. Q. Appl. Math. 19, 195 (1961).

[34] Reiss, E. L., On thr Theory of Cylindrical Shells. Q. J. Mech. Appl. Math. 15, 325 (1962).

[35] Reiss, E. L., Symmetric Bending of Thick Circular Plates. J. Soc. Indust. Appl. Math. 10, 596 (1962).

[36] Shield, R. T., On the Stability of Linear Continuous Systems. J. Appl. Math. Phys. 16, 649 (1965).

[37] Watson, G. N., A Treatise on the Theory of Bessel Functions. Cambridge (1966).

[38] Westbrook, D. R., Application of Asymptotic Integration to Potential Problems for Shells. J. SIAM Appl. Math. 14, 131 (1966).