

A METHOD OF CALCULATING REGGE TRAJECTORIES

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James Lawrence Blue

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ABSTRACT

A method of calculating Regge trajectories is presented and tested. The method is based on dispersion relations and on Cheng's representation for the S-matrix in terms of the Regge trajectories. A non-linear equation must be solved to obtain $\alpha(s)$. Potential theory calculations, using the Cheng representation and a modified representation, are made to test the method. To illustrate the application to relativistic scattering, a crude calculation of the rho trajectory is made. A self-consistent trajectory with the experimental mass and width does not exist, but one may be found if the width is about 200 MeV. Omitted physical effects are discussed, and comparison is made with Chew's strip approximation.

"My bootstraps have taken me the usual distance".

Mignon McLaughlin
The Neurotic's Notebook

"We must continue to grope intelligently".

Urban T. Holmes and Sister Amelia Klenke, O.P.
Chrétien, Troyes, and the Grail

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I. INTRODUCTION

There are several reasons which indicate the desirability of having a method of calculating Regge trajectories; three will be discussed below.

1. Currently it is believed that the high-energy behavior of particle-particle scattering of mesons and hadrons is determined by a combination of terms corresponding to singularities in angular momentum of the partial-wave scattering amplitude $A(s, \ell)$. These singularities are poles in ℓ , called Regge poles after T. Regge who introduced the idea of considering complex angular momentum¹⁾, and branch cuts in ℓ , proved to exist by Mandelstam²⁾. Although at very high energies the contribution of the cuts is probably not negligible compared to the pole contributions, there is some indication that the effects of the cuts are not important for lower energies, below 100 GeV or so^{3,4)}. In addition, the cut terms give no contribution to the scattering amplitude at zero momentum transfer, so for almost-forward scattering, an analysis in terms of only poles is reasonable. These conjectures are supported by the analyses of Phillips and Rarita⁴⁾. They considered πN , KN , and $\bar{K}N$ scattering, both charge-exchange and elastic; the range of the data included was incident momentum from 6 to 20 GeV/c and squared momentum transfer less than 1 (GeV/c)². In a three-pole model (the Pomeranchuk⁵⁾ pole, the Igi pole⁶⁾, and the rho pole), both the total cross-section σ and the differential cross-section $d\sigma/dt$ were fit with excellent results.

Such fits of experimental data typically give $\alpha(0)$ and its slope $\alpha'(0)$; another point on the trajectory is known for trajectories on which lie a resonance or bound particle. These known points on a trajectory provide checks of the calculated trajectory, or, alternatively, may be used as part of the input to a self-consistent calculation of the entire trajectory.

2. Previous bootstrap calculations^{7,8,9)}, which have attempted to determine masses and widths of particles and resonances without introducing Regge trajectories, have been hampered by divergences and by inability to calculate accurately in the high-energy region. Subject to the restrictions discussed in (1), self-consistent calculation of Regge trajectories would determine the high-energy scattering. The divergences will be discussed within the framework of the N/D method¹⁰⁾, which was used in almost all previous calculations.

The N/D method is designed to produce a scattering amplitude which satisfies elastic unitarity and which has the correct analytic behavior. For a given angular momentum ℓ , the partial-wave scattering amplitude $A(s)$ is written as (the ℓ -subscript is suppressed)

$$A(s) = N(s)/D(s). \quad (\text{I.1})$$

The unitarity relation satisfied by $A(s)$ is

$$[A(s) - A^*(s)]/2i = \text{Im } A(s) - A(s)A^*(s). \quad (\text{I.2})$$

This may also be written

$$\text{Im } \frac{1}{A(s)} = -1. \quad (\text{I.3})$$

$D(s)$ is made to have the right-hand cut of $A(s)$, from S_R to ∞ , and no left-hand cut, and $N(s)$ has the left-hand cut of $A(s)$, from S_L to ∞ , and no right-hand cut. The proper analytic structure in s of $A(s)$ and the unitarity relation are automatically satisfied if N and D satisfy the dispersion relations

$$N(s) = \frac{1}{\pi} \int_{S_L}^{-\infty} \frac{ds'}{s' - s} D(s') \operatorname{Im} A(s') \quad (I.4)$$

$$D(s) = 1 - \frac{1}{\pi} \int_{S_R}^{\infty} \frac{ds'}{s' - s} N(s').$$

These comprise coupled integral equations for $N(s)$ and $D(s)$ if the "force term" $\operatorname{Im} A(s')$, for $s < S_L$, is known; the usual procedure is to assume some force term as given by exchange of a few particles.

The difficulty arises because these integrals generally diverge if the force term includes exchange of a particle whose spin is one or higher. This divergence must be removed by some method such as by multiplying the integrands by some arbitrary factor which makes the integral converge⁸⁾, or by setting arbitrary finite upper limits to the integrals⁹⁾. In any case, arbitrary parameters must be introduced, on which the calculated masses and widths must depend.

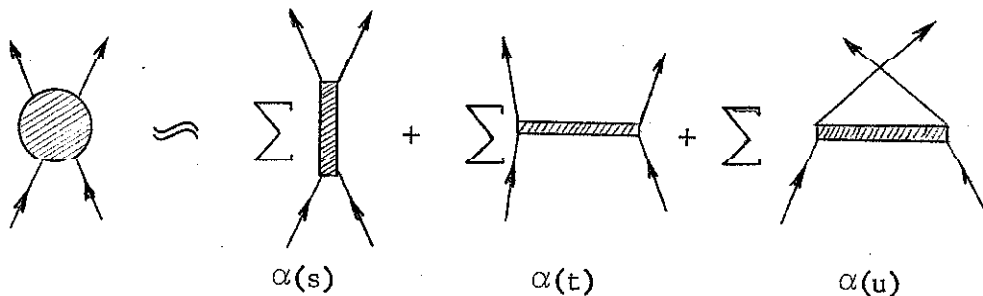
Regge poles provide one method of removing these divergences. In practice, the divergence comes from the $\operatorname{Im} A$ term in the N equation. It is known¹¹⁾ that, if the external particles are spinless, exchange of a particle of spin ℓ produces a term in $\operatorname{Im} A(s)$ proportional to s^ℓ for large $|s|$; however, exchange of a Regge pole $\alpha(t)$ produces a term

in $\text{Im } A(s)$ proportional to $s^{\alpha(\bar{t})}$ for $|s|$ large and for some \bar{t} less than zero. Since $\alpha(t) < 1$ for $t < 0$ ¹²⁾, the exchange of a Regge pole gives a less singular $\text{Im } A(s)$ which would allow convergence of the N integral ($D(s') \rightarrow 1$ as $s' \rightarrow -\infty$, so that the $\text{Im } A(s')$ term determines the convergence of the N integral).

3. A new representation derived by Cheng¹³⁾ suggests that knowledge of all the Regge trajectories suffices to determine the S-matrix, whenever the cuts in angular momentum may be neglected.

An alternative approach to the present method of calculating Regge trajectories is the "strip approximation", first proposed by Chew and Frautschi¹⁴⁾, later revised by Chew¹⁵⁾, and most recently, by Chew and Jones¹⁶⁾. A modified N/D method is used for the calculation; a Regge trajectory $\alpha(s)$ is determined by the value of ℓ for which the denominator function $D_\ell(s)$ is zero: $D_{\alpha(s)}(s) = 0$. The name "strip approximation" comes from the approximation used to determine the force term, $\text{Im } A(s')$ in Eq. (I.4).

Parameters s_1, u_1, t_1 are introduced; s_1 , for instance, is the value of s above which the force term is dominated by the exchange of Regge trajectories in the t and u channels. Schematically, this is



Parameters u_1 and t_1 are defined similarly for the channels in which u and t are the energy, respectively. An empirical form for the contribution of a Regge pole to the total scattering amplitude $T(s,t,u)$ is written, and $T(s,t,u)$ is approximated as a sum of Regge poles in all three channels: s , t , and u . The partial wave projection is then taken of T to give the force term, and the N/D equations are then solved for the Regge trajectories. The input trajectories (into the force term) are adjusted until the output trajectories (from the N/D equation) match them.

This approach has several difficulties which make an alternative method desirable. First, the parameters s_1 , u_1 , t_1 are arbitrary; there is no sharp dividing line between regions in which exchanged Regge trajectories do and do not dominate the scattering amplitude. The hope is expressed that the precise values of s_1 , u_1 , t_1 are unimportant, that the calculated trajectories are insensitive to the choice of these parameters, but there has been no indication that this will be the case.

Secondly, the numerical difficulties involved are enormous. The form chosen for the contribution of a Regge pole to T becomes logarithmically infinite as $s \rightarrow s_1$, so that both N and D become infinite, and a special technique is necessary to solve the N/D equations¹⁷⁾. In a preliminary calculation¹⁸⁾ where all poles except the rho pole were neglected, computing the output trajectory from the input trajectory took 6 minutes of IBM 7094 time; by comparison, the time for the corresponding calculation in part III was 8 seconds.

Thirdly, the strip approximation seems to be inadequate for calculating the trajectories above threshold, where the α 's become complex; Collins and Teplitz conclude that "it may be possible to 'bootstrap' trajectories with some hope of obtaining the physical parameters for $t < 0$, when all the trajectories are included, but there is no sign that we shall be able to obtain the correct particle masses and widths".

In the present calculation, the goal is to calculate Regge trajectories for relativistic scattering, but first non-relativistic potential scattering will be studied. The reason for this is twofold: First, there exist exact methods in potential theory for computing Regge trajectories, and calculations have been performed^{19,20)}. Any relativistic calculation is of necessity approximate; the corresponding approximation in potential theory allows comparison with exact results. This at least provides a negative check; if the potential theory calculation gives good results, this does not prove the corresponding relativistic calculation will give good results, but if the potential theory calculation gives poor results, less reliance may be held in the worth of the corresponding relativistic calculation. Second, in the relativistic scattering problem, various assumptions must be made; comparison with potential theory gives some indication of the proper assumptions to make.

The exact methods in potential theory are explicitly based upon the Schrödinger equation, and so are not suitable bases for the extension to relativistic scattering. The method that will be used is similar to those previously used by Cheng and Sharp²¹⁾, and by

Frautschi, Kaus, and Zachariasen²²⁾; no explicit use is made of the Schrödinger equation. The method, presented in Section II, is based upon proven dispersion relations, proven asymptotic behavior of the scattering amplitude and of Regge trajectories, and unitarity. These are directly extendable to relativistic scattering; all but the asymptotic behavior of the scattering amplitude have been proven for relativistic scattering, at least when the external particles are spinless.

The approach used here differs from that of the earlier computations in that Cheng's new representation is utilized. This representation is unitary, even in its approximations, in contrast to earlier calculations^{21,22)} where unitarity was only approximately satisfied. Dispersion relations for the Regge poles and for the residues of the poles of the scattering amplitude are combined with the Cheng representation, and a set of integral equations for the Regge trajectories is derived. For a practical calculation, all but a few trajectories must be neglected; the complexity of the integral equations that must be solved almost necessitates keeping only one trajectory. For this reason, emphasis is given to finding a method which gives accurate results in a one-trajectory approximation.

Sample potential theory calculations were performed using the Cheng representation and retaining one trajectory; qualitative agreement with the exact trajectory was found. The Cheng representation for the S-matrix contains little information about the potential, and has other defects in the one-trajectory approximation, as discussed in Section II. A modification of this representation, found inde-

pendently by the present author and by Abbe, Kaus, Nath, and Srivastava²³⁾, is then derived. This modified representation has improved properties, and the solution of the derived integral equation for the Regge trajectory is quantitatively correct in a one-trajectory approximation.

In Section III, the method is applied to an illustratory calculation of the rho trajectory. The bootstrap hypothesis²⁴⁾ is, roughly, that all particles are either bound states or resonances of all the other particles, and that the forces between particles are produced by the exchange of these particles. Thus, the rho meson is composed of all sets of particles which may be found in a state with spin one and isotopic spin one, such as 2π , 4π , $K\bar{K}$, and many others. The forces between two pions come from the exchange between the pions of 2π , 4π , and many others.

In this calculation, the simplest possible situation⁷⁾ is considered as an illustration; the rho is computed as a resonance in $\pi\pi$ scattering only. The force between the pions is approximated by rho exchange only; this force is attractive in the $I = 1$, $\ell = 1$ channel, and may be strong enough to produce a resonance or bound state. If the mass and width of the input rho are chosen properly, the force may be such as to produce an output rho of the same mass and width. The possibility of this occurring depends, of course, upon the validity of the approximate method of calculation used. The present calculation fails to generate a self consistent rho of the experimental mass and width, but this is not surprising in view of the drastic approximation used. This approximation will serve to demonstrate the method of

calculating trajectories, but neglect of other channels and other contributions to the force indicates that the numerical results will only be of qualitative significance.

The method and the assumptions and approximations made, are further discussed in Section III. Briefly, the method of calculation is completely analogous to the potential theory computation. Dispersion relations are written for the Regge poles and residues, and a representation for the S-matrix is derived. These are combined, in a one-trajectory approximation to give an integral equation whose solution gives the Regge trajectory on which the rho lies.

Finally, the qualitative changes in the rho trajectory due to several physical effects are investigated, especially inelasticity and short-range forces.

II. NON-RELATIVISTIC (POTENTIAL THEORY) CALCULATIONS

For certain potentials, notably single Yukawa potentials, the Regge trajectories have been calculated analytically²⁰⁾; comparison of these exact results with the approximate calculations done here is the crucial test for the validity of the methods used. In addition, many of the results which may be proved in potential theory may be generalized to the relativistic calculations.

The approach followed is similar to that used by Cheng and Sharp²¹⁾ and by Frautschi, Kaus, and Zachariasen²²⁾. First, dispersion relations are written for the Regge trajectory, $\alpha(s)$, and for the residue, $\beta(s)$, of the Regge pole at $\ell = \alpha(s)$. Then various representations of the S-matrix in terms of its Regge poles are presented and discussed. In the approximation that only one trajectory is retained, an integral equation for $\text{Im } \alpha(s)$ is derived and solved for the Regge trajectory associated with a single Yukawa potential, $V(r) = -g^2 e^{-\mu r}/r$.

The scattering amplitude, $A(s, \ell)$, used is defined by

$$S(s, \ell) = 1 + 2iq A(s, \ell), \quad (\text{II.1})$$

with the S-matrix obeying the unitarity condition

$$S(s, \ell) S^*(s, \ell^*) = 1. \quad (\text{II.2})$$

The residue $\beta_n(s)$ is the residue of $A(s, \ell)$ at $\ell = \alpha_n(s)$

$$\beta_n(s) = \lim_{\ell \rightarrow \alpha_n(s)} \{[\ell - \alpha_n(s)] A(s, \ell)\}. \quad (\text{II.3})$$

A. Dispersion Relations for the Regge Pole Parameters

The method used for the calculation of Regge trajectories arising from a given potential is based on the proven analytic properties of the trajectory $\alpha(s)$ and its associated residue $\beta(s)$. It has been shown^{25,26)} that the leading trajectories (i.e., those for which $\text{Re } \alpha(s)$ is largest) obey a dispersion relation in the energy s :

$$\alpha_n(s) = -n + \frac{1}{\pi} \int_0^{\infty} \frac{ds'}{s' - s - i\epsilon} \text{Im } \alpha_n(s'), \quad n = 1, 2, \dots \quad (\text{II.4})$$

For the non-leading trajectories, α and β are not real-analytic functions with only a right-hand cut^{27,28)}; these trajectories may cross each other, and their α 's and β 's will not obey simple dispersion relations of the type written here. For example, if $V(r) = -g^2 e^{-\mu r}/r$ with $g^2 = 1.8$, $\mu = 1$, $m = 1/2$, the first and second trajectories obey the simple dispersion relations, but the third does not. However, the S-matrix is accurately obtained by using only one or two trajectories in this case^{23,29)}, so apparently these "abnormal" trajectories may be neglected. In addition, the reduced residue $b_n(s)$, defined by

$$b_n(s) = \beta_n(s) / q^{2\alpha_n(s)}, \quad (\text{II.5})$$

is an analytic function of s and has only a right-hand cut, from $s = 0$ to $s = \infty$ along the real axis. Here q is the momentum of the particle being scattered from the potential $V(r)$, $s = q^2/2m$, with m the particle mass.

Further, if $V(r)$ is a superposition of Yukawa potentials

$$V(r) = - \int_{\mu}^{\infty} d\mu' \sigma(\mu') e^{-\mu' r/r} \quad (\text{II.6})$$

with

$$g^2 = \int_{\mu}^{\infty} d\mu' \sigma(\mu'), \quad (\text{II.7})$$

the following asymptotic behavior holds:

$$\alpha_n(s) \rightarrow -n + i g^2 m/q \text{ as } s \rightarrow \infty \quad (\text{II.8})$$

$$b_n(s) \rightarrow g^2 m q^{2n-2} \text{ as } s \rightarrow \infty. \quad (\text{II.9})$$

This may be seen by noting that, as $s \rightarrow \infty$, the S-matrix $S(s, \ell)$ approaches the S-matrix for the Coulomb potential $V_{\text{coul}}(r) = -g^2/r$. This S-matrix has been calculated exactly^{30,31}:

$$S_{\text{coul}}(s, \ell) = \frac{\Gamma(1 + \ell - i g^2 m/q)}{\Gamma(1 + \ell + i g^2 m/q)}. \quad (\text{II.10})$$

The α 's are given by the locations of the poles in ℓ , with the β 's given by the residue of $A_{\text{coul}}(s, \ell)$.

(In most of the literature, the mass of the scattered particle is explicitly or implicitly taken to be 1/2 to remove factors of 2m from various equations, so that $s = q^2$.)

Since $b_n(s)$ is real-analytic, it will also obey a dispersion relation; because of the asymptotic behavior, however, each higher trajectory will require a dispersion relation with one more subtraction. For this reason, FKZ do not write the usual type of dispersion relation for $b_n(s)$. Instead, they consider the function $\Phi(s)$, where

$$\Phi(s) = \ell n \left\{ \frac{b_n(s)}{\prod_{i=1}^{n-1} [2m(s - s_i)]} \right\}. \quad (\text{II.11})$$

The s_i are the $(n - 1)$ zeroes of $b_n(s)$. $\Phi(s)$ is real-analytic with a right-hand cut starting at $s = 0$ and has no other singularities. Its asymptotic value, as $s \rightarrow \infty$, is $\ell n (mg^2)$; therefore, it obeys a dispersion relation

$$\Phi(s) = \ell n (mg^2) + \frac{1}{\pi} \int_0^{\infty} \frac{ds'}{s' - s - i\epsilon} \text{Im } \Phi(s'). \quad (\text{II.12})$$

In addition ($q'^2 = 2ms'$)

$$\text{Im } \Phi(s') = \arg [b_n(s')] = \arg [\beta_n(s')] - \text{Im } \alpha_n(s') \ell n q'^2. \quad (\text{II.13})$$

Then we obtain the exact relation

$$\beta_n(s) = mg^2 q^{2\alpha(s)} \prod_{i=1}^{n-1} [2m(s - s_i)] \exp \left\{ \frac{1}{\pi} \int_0^{\infty} \frac{ds'}{s' - s - i\epsilon} \left[\arg \beta_n(s') - \text{Im } \alpha_n(s') \ell n q'^2 \right] \right\}. \quad (\text{II.14})$$

The dispersion relations for $\alpha_n(s)$ and $\beta_n(s)$ form part of a scheme for calculating α and β ; the necessary further relation comes from considering a specific representation for the scattering amplitude in terms of the α 's and β 's, and from requiring the S-matrix to be unitary at some point or points in the ℓ -plane. The Cheng representation and its successors are automatically unitary at all values of ℓ , but give the scattering amplitude in terms of the α 's only; thus another relation is obtainable from the definition of β , Eq. (II.3).

Although unsubtracted dispersion relations may be used, as written above, subtracted dispersion relations might be of use. These are ($q_0^2 = 2ms_0$)

$$\alpha_n(s) = \alpha_n(s_0) + \frac{s - s_0}{\pi} \int_0^\infty \frac{ds' \operatorname{Im} \alpha(s')}{(s' - s - i\epsilon)(s' - s_0 - i\epsilon)} . \quad (\text{II.15})$$

$$\beta_n(s) = \beta_n(s_0) \frac{q_n(s)}{q_n(s_0)} \prod_{i=1}^{n-1} \left(\frac{s - s_i}{s_0 - s_i} \right) \exp \left\{ \frac{s - s_0}{\pi} \int_0^\infty \frac{ds'}{(s' - s - i\epsilon)(s' - s_0 - i\epsilon)} \left[\arg \beta_n(s') - \operatorname{Im} \alpha(s') \ln q'^2 \right] \right\} . \quad (\text{II.16})$$

B. Representation for the Partial-Wave Scattering Amplitude $A(s, \ell)$.

1. The "Universal" Representation

The first representation was used both by CS and by FKZ, although neither wrote it down explicitly. Regge's ¹⁾ representation expresses the total scattering amplitude in terms of the partial-wave amplitudes and the Regge poles:

$$T(s, \cos \theta) = \frac{1}{2i} \int_{-1/2-i\infty}^{-1/2+i\infty} d\ell (2\ell + 1) \frac{A(s, \ell)}{\sin \pi \ell} P_\ell(-\cos \theta) - \pi \sum_n \frac{(2\alpha_n(s) + 1) \beta_n(s)}{\sin \pi \alpha_n(s)} P_{\alpha_n(s)}(-\cos \theta) , \quad (\text{II.17})$$

where the summation is over those poles for which $\operatorname{Re} \alpha_n(s) > -1/2$.

The partial-wave amplitude is defined, for non-negative integer ℓ , by ($x = \cos \theta$)

$$A(s, \ell) = \frac{1}{2} \int_{-1}^1 dx P_{\ell}(x) T(s, x). \quad (\text{II.18})$$

The relation ³²⁾

$$\frac{1}{2} \int_{-1}^1 dx P_{\ell}(x) P_{\alpha}(-x) = \frac{\sin \pi \alpha}{\pi(\alpha + \ell)(\alpha + \ell + 1)}, \quad (\text{II.19})$$

which is valid for $\ell = 0, 1, 2, \dots$, is necessary. (In the limit $\alpha \rightarrow n = \text{integer}$, the well-known orthogonality relation for the Legendre polynomials results in

$$\frac{1}{2} \int_{-1}^1 dx P_{\ell}(x) P_n(-x) = \frac{(-1)^n}{2\ell + 1} \delta_{n\ell}, \quad (\text{II.20})$$

after we use the identity $P_n(x) = (-1)^n P_n(-x)$.

Taking the partial-wave projection of (II.17) gives

$$A(s, \ell) = \sum_n \frac{2\alpha_n(s) + 1}{\ell + \alpha_n(s) + 1} \frac{\beta_n(s)}{\ell - \alpha_n(s)} + B(s, \ell). \quad (\text{II.21})$$

The sum is again over the Regge poles with $\text{Re } \alpha_n(s) > -1/2$, and $B(s, \ell)$, the "background integral", is the partial-wave projection of the integral term. This background integral, which is assumed to be small compared with the first term when ℓ is near $\alpha_n(s)$, is neglected. In practice, only one term of the sum is used, so the final result is

$$A(s, \ell) = \frac{2\alpha(s) + 1}{\ell + \alpha(s) + 1} \frac{\beta(s)}{\ell - \alpha(s)}, \quad (\text{II.22})$$

which is used even for $\text{Re } \alpha < -1/2$. This may be called the "universal" representation (after FKZ), since it is independent of the potential; calling it the Regge representation would be grossly unfair to Regge.

FKZ and CS do not actually use this representation except at $\ell = \alpha(s)$, where it is inserted into the unitarity relation. $A(s, \ell)$ obeys the condition

$$A(s, \ell) - A^*(s, \ell^*) = 2iq A(s, \ell) A^*(s, \ell^*). \quad (\text{II.22a})$$

The representation (II.22) will not, in general, obey this equation. Both FKZ and CS require (II.22a) to be satisfied for $\ell = \alpha(s)$. This gives

$$\beta(s) = \text{Im } \alpha(s)/q, \quad (\text{II.23})$$

and

$$A(s, \ell) = \frac{2\alpha(s) + 1}{\ell + \pi(s) + 1} \frac{\text{Im } \alpha(s)}{\sqrt{s}} \frac{1}{\ell - \alpha(s)}. \quad (\text{II.24})$$

$A(s, \ell)$ is related to the S-matrix by

$$S(s, \ell) = 1 + 2i q A(s, \ell). \quad (\text{II.25})$$

In order to compare more easily with the Cheng representation later, we calculate as follows

$$S(s, \ell) = 1 + \frac{2i \text{Im } \alpha(s)}{\ell - \alpha(s)} \frac{2\alpha(s) + 1}{\ell + \alpha(s) + 1}. \quad (\text{II.26})$$

Near $\ell = \alpha(s)$,

$$S(s, \ell) \xrightarrow{\ell \rightarrow \alpha(s)} \frac{2i \text{Im } \alpha(s)}{\ell - \alpha(s)}. \quad (\text{II.27})$$

Near $\ell = \alpha(s)$ this may also be written

$$S(s, \ell) \xrightarrow{\ell \rightarrow \alpha(s)} \frac{\ell - \alpha^*(s)}{\ell - \alpha(s)}. \quad (\text{II.28})$$

Since this representation is used only at $\ell = \alpha(s)$, all results of CS and FKZ using the "universal" representation are exactly the same as if they had explicitly used

$$S(s, \ell) = \frac{\ell - \alpha^*(s)}{\ell - \alpha(s)}, \quad (\text{II.29})$$

which is unitary for all s, ℓ .

2. The Khuri Representation

The Khuri representation³³⁾ was the first to incorporate the background integral term of the partial-wave scattering amplitude into a tractable form. In the "universal representation," the neglect of the background integral term removed all dependence of the trajectories on the potential; the Khuri representation attempted to repair this failure. Khuri derived

$$A(s, \ell) = \sum_n \frac{\beta_n(s)}{\ell - \alpha_n(s)} e^{-[\ell - \alpha_n(s)]\xi(s)}, \quad (\text{II.30})$$

where $\cosh \xi = 1 + m^2/2s$, and the potential is given as

$$V(r) = - \int_m^\infty \frac{e^{-\mu r}}{r} \sigma(\mu) d\mu; \quad g^2 = \int_m^\infty \sigma(\mu) d\mu. \quad (\text{II.31})$$

The sum runs over all the Regge trajectories. In deriving this result, Khuri made use of the proven result¹⁾ that as $|\ell| \rightarrow \infty$, $\text{Re } \ell > -1/2$,

$$A(s, \ell) = O(e^{-\xi(s) \text{Re } \ell / \sqrt{D}}). \quad (\text{II.32})$$

He found it necessary also to assume the same behavior for $\text{Re } \ell < -1/2$; since Cheng³⁴⁾ has shown this to be incorrect, the Khuri representation is invalid.

FKZ and Ahmadzadeh²⁹⁾ have also presented a modified Khuri representation; its derivation is similar to that of the modified Cheng representation, to be given below, and so will not be derived here. The modified representation has three major advantages over the unmodified: firstly, in the modified representation the scattering amplitude approaches the known high-energy limit (the Born term) as $s \rightarrow \infty$, even in the one-trajectory approximation. Secondly, the modified representation has the proper cuts in s for a single Yukawa potential, even in the one-trajectory approximation. Thirdly, the convergence of the series for the scattering amplitude is much quicker than for the unmodified representation; one or two poles are an excellent approximation²⁹⁾.

For completeness' sake, we write down the two Khuri representations; the modified representation is given for the potential $V(r) = -g^2 e^{-\mu r}/r$.

Unmodified:

$$A(s, \ell) = \sum_{n=1}^{\infty} \frac{\beta_n(s)}{\ell - \alpha_n(s)} e^{-[\ell - \alpha_n(s)] \xi(s)}. \quad (\text{II.33})$$

Modified:

$$A(s, \ell) = \sum_{n=1}^{\infty} \frac{\beta_n(s) e^{-[\ell - \alpha_n(s)] \xi(s)}}{\ell - \alpha_n(s)} + \frac{g^2}{2q^2} \left[Q_\ell \left(1 + \frac{\mu^2}{2q^2} \right) - \sum_{n=1}^{\infty} \frac{e^{-(\ell+n)\xi(s)}}{\ell+n} P_{n-1} \left(1 + \frac{\mu^2}{2q^2} \right) \right] \quad (\text{II.34})$$

where $\cosh \bar{\xi} = 1 + (2\mu)^2/2q^2$, and Q_ℓ is the usual Legendre function of the second kind. The one-trajectory approximation consists of using only the first term of each sum.

Neither representation is unitary; FKZ require unitarity at $\ell = \alpha(s)$ to get an expression for $\beta(s)$. This gives, in the one-trajectory approximation,

Unmodified:

$$\beta(s) = \frac{\text{Im}\alpha}{q} e^{-2i \bar{\xi}(s)} \text{Im} \alpha(s) \quad (\text{II.35})$$

Modified:

$$\beta(s) = \text{Im} \alpha(s) e^{-2i \bar{\xi}(s)} \text{Im} \alpha(s) \left\{ \frac{1}{q} + \frac{ig^2}{2} \left[Q_{\alpha^*(s)}(1 + \mu^2/2s) - \frac{e^{-[1 + \alpha^*(s)] \bar{\xi}(s)}}{1 + \alpha^*(s)} \right] \right\} \quad (\text{II.36})$$

3. The Cheng Representation

This representation, derived by H. Cheng¹³⁾, allows the S-matrix to be expressed in terms of the Regge poles only. Its construction was made possible by the proof³⁴⁾ by Cheng and Wu that $(\lambda = \ell + 1/2)$,

$$S(s, \ell) \rightarrow e^{2\pi i \lambda} \text{ as } |\lambda| \rightarrow \infty \text{ with } \text{Re } \lambda < 0. \quad (\text{II.37})$$

This was proved for several potentials: a single Yukawa superposition with $\sigma(\mu) = e^{-\mu}$, a square well, and any potential which has a power series expansion $V(r) = \sum_{n=-1}^{\infty} a_n r^n$ if the potential is cut off at a finite r . It presumably holds for a wider class of potentials, but this exact form is unnecessary for the derivations of the Cheng

Representation and those to follow.

Cheng's derivation is as follows: the unitarity condition for S , for $s > 0$,

$$S(s, \ell) S^*(s, \ell^*) = 1, \quad (\text{II.38})$$

requires that if S has a pole at $\ell = \alpha(s)$, S must also have a zero at $\ell = \alpha^*(s)$, for s above threshold. Now consider the contour integral ($\lambda' = \ell' + 1/2$, $\lambda = \ell + 1/2$)

$$I = \frac{1}{2\pi i} \int_{C_R} d\lambda' \frac{e^{\lambda' \xi(s)} \ln S(s, \ell')}{\lambda' - \lambda}. \quad (\text{II.39})$$

Here C_R is a circle of radius R , centered at the origin, and $R \rightarrow \infty$.

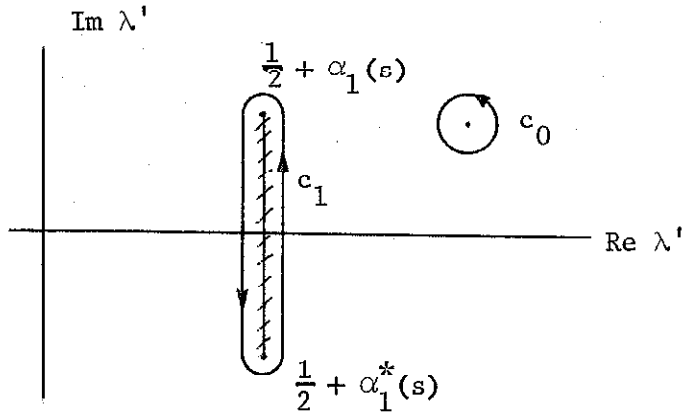
Also $\cosh \xi = 1 + \mu^2/2s$. The function $\ln S(s, \ell)$ is an analytic function of ℓ (or λ) with branch points at the poles $\ell = \alpha_n(s)$ and zeroes $\ell = \alpha_n^*(s)$ of $S(s, \ell)$. [For this potential $S(s, \ell)$ has no cuts in ℓ .]

The branch cuts in λ' are chosen to run from $\alpha_n(s) + 1/2$ to $\alpha_n^*(s) + 1/2$, although they may equally well be chosen to run from the branch points to $-\infty$. In the region $\text{Re } \lambda' > 0$, as $|\lambda'| \rightarrow \infty$ ¹⁾

$$S(s, \ell') \rightarrow 1 + O(e^{-\lambda' \xi} / \sqrt{\lambda'}), \quad (\text{II.40})$$

so that the integrand is $O(\lambda'^{-3/2})$. In the region $\text{Re } \lambda' < 0$, the integrand is $O(e^{\lambda' \xi})$ as $|\lambda'| \rightarrow \infty$, so that the integral I goes to zero as $R \rightarrow \infty$.

The contour C_R may be deformed to the contour C , composed of C_0 around $\lambda' = \lambda$ and the C_n around the branch cuts. For simplicity, only one cut is shown.



The integral may then be evaluated to give

$$I = 0 = e^{\lambda \xi(s)} \ell_n S(s, \ell) + \sum_{n=1}^{\infty} \int_{\alpha_n^* + 1/2}^{\alpha_n + 1/2} \frac{d\lambda' e^{\lambda' \xi}}{\lambda' - \lambda} . \quad (\text{II.41})$$

then

$$S(s, \ell) = \exp \left\{ \sum_n \int_{\alpha_n(s)}^{\alpha_n^*(s)} d\ell' \frac{e^{(\ell' - \ell) \xi(s)}}{\ell' - \ell} \right\} , \quad (\text{II.42})$$

which is the Cheng representation. Note that knowledge of the Regge trajectories is sufficient to give the S-matrix. The Cheng representation may also be written

$$S(s, \ell) = \prod_{n=1}^{\infty} S_n(s, \ell) , \quad (\text{II.43})$$

where

$$S_n(s, \ell) = \exp \left\{ \int_{\alpha_n(s)}^{\alpha_n^*(s)} d\ell' \frac{e^{(\ell' - \ell) \xi(s)}}{\ell' - \ell} \right\} . \quad (\text{II.44})$$

Here, each S_n may be regarded as the contribution of one of the Regge poles to the scattering matrix. In contrast to the earlier ones, this representation has the advantage of being rigorously correct,

but perhaps the most important feature of this representation is that each S_n is unitary:

$$S_n(s, \ell) S_n^*(s, \ell^*) = 1. \quad (\text{II.45})$$

This means that the unitarity of S does not depend on cancellations among the various terms, and that in a practical calculation, where only a finite number of terms are retained of the infinite product, the approximate S -matrix used is unitary.

We may also write each $S_n(s, \ell)$ as

$$S_n(s, \ell) = \frac{\ell - \alpha_n^*(s)}{\ell - \alpha_n(s)} \exp \left\{ \int_{\alpha_n(s)}^{\alpha_n^*(s)} d\ell' \frac{e^{(\ell' - \ell)\xi(s)} - 1}{\ell' - \ell} \right\}. \quad (\text{II.46})$$

Then, using the definition $\beta_n(s)$ it is easy to obtain

$$\beta_n(s) = \frac{\text{Im } \alpha_n(s)}{q} \exp \left\{ \int_{\alpha_n(s)}^{\alpha_n^*(s)} d\ell' \frac{e^{[\ell' - \alpha_n(s)]\xi(s)} - 1}{\ell' - \alpha_n(s)} \right\} \prod_{n' \neq n} S_{n'}(s, \alpha_n(s)). \quad (\text{II.47})$$

In the one-trajectory approximation the product is set equal to one.

Note that, for $\text{Im } \alpha(s)$ small, the integrand may be expanded to give²³⁾

$$\beta(s) = \frac{\text{Im } \alpha(s)}{q} e^{-2i\xi(s)\text{Im } \alpha(s)}, \quad (\text{II.48})$$

just as in the Khuri representation. An approximation one step cruder, for small $\text{Im } \alpha$, is obtained by replacing the exponential by 1; this gives the "universal" result

$$\beta(s) = \frac{\text{Im } \alpha(s)}{q}. \quad (\text{II.49})$$

Equating equations (II.47) and (II.14), in the one-trajectory

approximation, gives the integral equation

$$\text{Im } \alpha(s) = m g^2 q (1 + 2\alpha(s)) \exp \left\{ \int_{\alpha^*}^{\alpha} dl' \frac{e^{(l' - \alpha)\xi} - 1}{l' - \alpha} + \frac{1}{\pi} \int_0^{\infty} \frac{ds'}{s' - s - i\epsilon} [\theta(s') - \ln q'^2 \text{Im } \alpha(s')] \right\}. \quad (\text{II.50})$$

Here, $\theta(s) = \arg \beta(s)$. Letting

$$D(s) = \int_{\alpha(s)}^{\alpha^*(s)} dl' \frac{e^{(l' - \alpha(s))\xi(s)} - 1}{l' - \alpha(s)}, \quad (\text{II.51})$$

and changing the integral over s' to a Cauchy principle value, we obtain the integral equation

$$\text{Im } \alpha(s) = g^2 m q [1 + 2 \text{Re } \alpha(s)] \exp \left\{ - \text{Re } D(s) + \frac{1}{\pi} \mathcal{P} \int_0^{\infty} \frac{ds'}{s' - s} [\text{Im } D(s') - \text{Im } \alpha(s') \ln q'^2] \right\}. \quad (\text{II.52})$$

The Regge trajectory $\alpha(s)$ is determined by the solution of this equation, together with the dispersion relation for $\text{Re } \alpha(s)$, derived from (II.4) by changing the integral to a Cauchy principle value integral:

$$\text{Re } \alpha(s) = -1 + \frac{1}{\pi} \mathcal{P} \int_0^{\infty} \frac{ds'}{s' - s} \text{Im } \alpha(s'). \quad (\text{II.53})$$

Writing the integral equation in the above form allows an easy comparison with the equations which would be obtained for the earlier representations: these would give the identical equation except for the replacement of $D(s)$ by another function $D'(s)$.

Universal : $D'(s) = 0$ (II.54)

Khuri : $D'(s) = -2i \xi(s) \text{Im } \alpha(s)$ (II.55)

Modified Khuri : $\text{Im } D'(s) = -2 \xi(s) \text{Im } \alpha(s) + \text{Im } Z(s);$

$\text{Re } D'(s) = \ln [\text{Re } Z(s)];$

with $Z(s) = \frac{ig^2}{q} \left[Q_{\alpha^*}(s) \left(1 + \frac{\mu^2}{2q^2} \right) - \frac{e^{[1 + \alpha^*(s)] \xi(s)}}{1 + \alpha^*(s)} \right].$ (II.56)

In any of the approximations, the integral equation embodies the known²⁵⁾ asymptotic behavior of $\alpha(s)$:

$\alpha(s) \rightarrow -1 + ig^2_{\mu/q}, \text{ as } s \rightarrow \infty$ (II.57)

$\text{Im } \alpha(s) \propto s^{1/2} + \text{Re } \alpha(0), \text{ as } s \rightarrow 0.$ (II.58)

The integral equation to be solved if once subtracted dispersion relations are used may be obtained if $s_0 > 0$ by dividing the (unsubtracted) integral equation for $\text{Im } \alpha(s)$ by the equation for $\text{Im } \alpha(s_0)$, and combining the two Cauchy principle value integrals:

$$\text{Im } \alpha(s) = \text{Im } \alpha(s_0) \frac{q}{q} \frac{1 + 2 \text{Re } \alpha(s)}{1 + 2 \text{Re } \alpha(s_0)} \exp \left\{ \text{Re } [D(s_0) - D(s)] + \frac{s - s_0}{\pi} \int_0^\infty \frac{ds'}{(s' - s)(s' - s_0)} [\text{Im } D(s') - \text{Im } \alpha(s') \ln q'^2] \right\}. \quad \text{(II.59)}$$

If $s_0 < 0$, the equation to be solved becomes ($b(s_0)$ is real for $s_0 < 0$):

$\text{Im } \alpha(s) = b(s_0) q^{[1 + 2 \text{Re } \alpha(s)]} \exp \{-D(s) + (\text{same integral})\}.$ (II.60)

In either case, $\text{Re } \alpha(s)$ is given by

$$\operatorname{Re} \alpha(s) = \operatorname{Re} \alpha(s_0) + \frac{s - s_0}{\pi} \int_0^{\infty} \frac{ds'}{(s' - s)(s' - s_0)} \operatorname{Im} \alpha(s'). \quad (\text{II.61})$$

In this form, the integral equation does not have the correct asymptotic properties guaranteed. The $s \rightarrow 0$ limit is correct, but $\operatorname{Re} \alpha(s)$ does not necessarily approach -1 as $s \rightarrow \infty$.

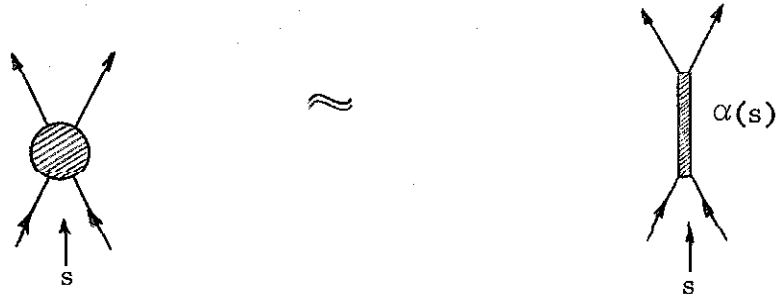
$$\operatorname{Im} \alpha(s) \rightarrow c q [1 + 2 \operatorname{Re} \alpha(\infty)] \text{ as } s \rightarrow \infty, \quad (\text{II.62})$$

but this is not correct unless $\operatorname{Re} \alpha(\infty) = -1$, and the constant c may not be correct either. If enough information about the potential is included, the subtracted equations can be as accurate as the unsubtracted ones.

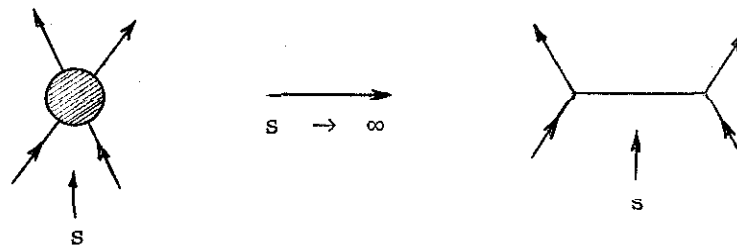
The integral equations (II.52) and (II.59) were solved for the values of the parameters $g^2 = 1.8$, $m = 1/2$, $\mu = 1$, for the potential $V(r) = -g^2 e^{-\mu r}/r$. For (II.59) the subtraction was arbitrarily set at $s_0 = .4$; $\alpha(s_0)$ was taken from the exact solution given by Ahmadzadeh³⁵. The solutions are given in Figures 1 and 2, along with the exact solution of Ahmadzadeh. The solution with the subtraction at $s_0 = -.4$ is very similar to that for $s_0 = +.4$, except slightly shifted toward higher energy.

The deficiencies of this representation for the S-matrix are (at least) twofold. First, very few details of the force are included - the over-all strength g^2 of the potential, and its range μ (in ξ). The second deficiency is easy to see in terms of scattering diagrams; these represent scattering of two particles (of mass $2m$ each) by an interaction potential $V(r)$. We have expressed $S(s, \ell)$ as a sum (or

product) of Regge poles, and approximated by taking only one pole, i.e.,

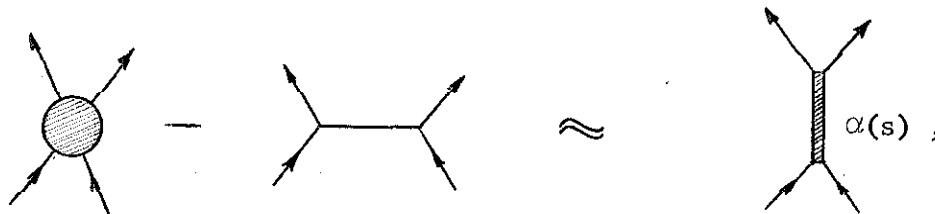


For high energies, however, this is seriously wrong, for it is known that as $s \rightarrow \infty$, $S(s, l)$ approaches the Born term



$$S_{\text{Born}}(s, l) = 1 + 2 \, 1q \left\{ \frac{g^2}{2q^2} Q_l(1 + \mu^2/2q^2) \right\} . \quad (\text{II.63})$$

Therefore it seems reasonable that a better approximation than the one-trajectory approximation would be one trajectory plus the Born term, with some alteration to take care of the fact that part of the Born term will be contained in the one-trajectory term. This may be schematically presented as



where for large s the α term is small compared to the Born term.

For a single Yukawa, the Cheng representation also gives the total scattering amplitude $T(s,t)$ an incorrect cut in t ²³⁾; in reality $S(s,t)$ has a pole at $t = \mu^2$ and a cut from $t = 4\mu^2$ to ∞ , but the Cheng representation in the one-trajectory approximation gives a cut from $t = \mu^2$ to ∞ (from ξ). (In the Regge representation this spurious cut is cancelled between the Regge pole terms and background integral; the infinite product of the Cheng representation is responsible for the cancellation of the spurious cut.) A modified Cheng representation may be constructed, which has improved properties, as follows.

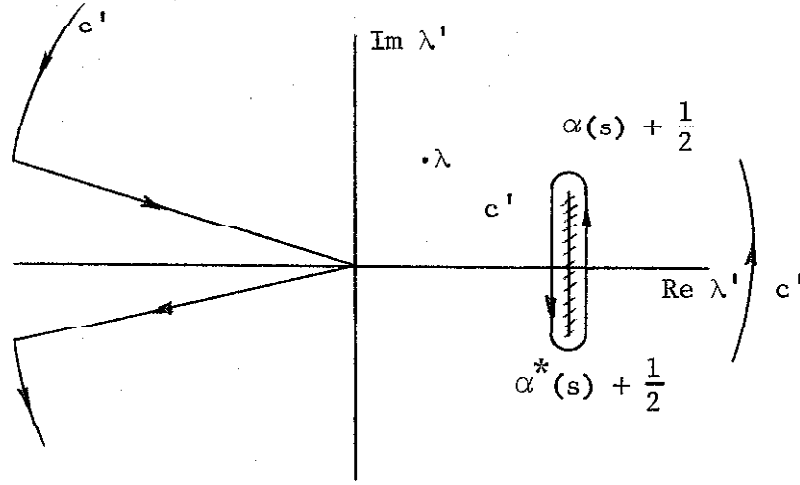
4. The Modified Cheng Representation³⁶⁾

The modified Cheng Representation is, for $V(r)$ a single attractive Yukawa potential, fully equivalent to the unmodified Cheng representation. In practice, however, it is superior, since the modified Cheng Representation approximates $[S(s,l) - S_{\text{Born}}(s,l)]$ by a finite number of Regge poles, while the Cheng representation approximates $S(s,l)$ by a finite number of poles. In addition, the modified Cheng Representation gives the correct analytic properties both for the partial-wave amplitude and for the total amplitudes, even in the one-trajectory approximation.

Instead of the integral I of the last section we consider I' ($\lambda' = l' + 1/2$, $\lambda = l + 1/2$),

$$I' = \frac{1}{2\pi i} \int_{C'} e^{\lambda' \xi(s)} \frac{\ln S(s, l') - (ig^2/q) Q_{l'} \left(1 + \frac{\mu^2}{2q}\right)}{\lambda' - \lambda} d\lambda', \quad (\text{II.64})$$

where C' is shown in the figure. (Only one of the poles is indicated.)



(C' is the limit as the radius of the circle goes to infinity and the wedge removed shrinks to zero angle.) The integrand has no singularities except at $\lambda = \lambda'$, so

$$I' = e^{\lambda \bar{\xi}(s)} \left\{ \ln S(s, l) - \frac{ig^2}{q} Q_l \left(1 + \frac{\mu^2}{2q^2} \right) \right\}. \quad (\text{II.65})$$

The representation³³⁾

$$Q_l(\cosh \xi) = \sum_{n=1}^{\infty} \frac{P_{n-1}(\cosh \xi) e^{-(l+n)\xi}}{l+n}, \quad (\text{II.66})$$

indicates that $Q_l(Z)$ is regular in l except for poles at the negative integers, with residues $P_{n-1}(Z)$. In addition³⁷⁾,

$$Q_l(\cosh \xi) \rightarrow \sqrt{\frac{\pi}{2\lambda \sinh \xi}} e^{-\lambda \xi} \text{ as } |\lambda| \rightarrow \infty \text{ in } |\arg \lambda| < \pi. \quad (\text{II.67})$$

Also,

$$S(s, l) \rightarrow 1 + \frac{ig^2}{q} Q_l \left(1 + \frac{\mu^2}{2q^2} \right) + o \left(\frac{e^{-\lambda \bar{\xi}(s)}}{\sqrt{\lambda}} \right), \quad (\text{II.68})$$

where $\cosh \bar{\xi}(s) = 1 + \frac{(2\mu)^2}{2q^2} 22)$.

It is then easily seen that there is no contribution to I from the infinite semi-circle. Evaluating I' by integrating around the cuts from $\alpha_n(s)$ to $\alpha_n^*(s)$ and including the poles of Q_ℓ , we obtain

$$I' = \frac{ig^2}{q} \sum_{n=0}^{\infty} \frac{e^{(-m-1/2)\xi} P_m(1 + \frac{\mu^2}{2q^2})}{(-m-1/2) - \lambda} + \sum_{n=1}^{\infty} \int_{\alpha_n}^{\alpha_n^*} \frac{e^{(l'+1/2)\xi}}{l' - l} dl' \quad (II.69)$$

Finally, equating the two expressions for I' gives the modified Cheng Representation,

$$S(s, l) = \exp \left\{ \frac{ig^2}{q} Q_\ell \left(1 + \frac{\mu^2}{2q^2} \right) \right\} \prod_{n=1}^{\infty} [S_n(s, l)], \quad (II.70)$$

with

$$S_n(s, l) = \exp \left\{ \int_{\alpha_n}^{\alpha_n^*} dl' \frac{e^{(l'-l)\xi}}{l' - l} - \frac{ig^2}{q} \frac{e^{-(l+n)\xi} P_{n-1}(1 + \mu^2/2q^2)}{l+n} \right\}. \quad (II.71)$$

This expresses $S(s, l)$ as S_{Born} times a product of S_n 's where each S_n may be regarded as the contribution to S from a single Regge trajectory.

This $S(s, l)$ produces the correct analytic structure for the total amplitude $T(s, \cos \theta)$ ²³; T has a pole in $\cos \theta$ at $\cos \theta = 1 + \mu^2/2q^2$, from the Q -function, and a cut from $\cos \theta = 1 + (2\mu)^2/2q^2$ from $\xi(s)$.

This is true even in the one-trajectory approximation.

The residue $\beta_n(s)$ may easily be found from its defining relation.

$$\text{Letting } E_n(s) = \left\{ \int_{\alpha_n}^{\alpha_n^*} dl' \frac{e^{(l'-\alpha_n)\xi} - 1}{l' - \alpha_n} + \frac{ig^2}{q} \left[Q_{\alpha_n} \left(1 + \frac{\mu^2}{2q^2} \right) - \frac{e^{-(n+\alpha_n)\xi} P_{n-1} \left(1 + \frac{\mu^2}{2q^2} \right)}{n + \alpha_n} \right] \right\} \quad (II.72)$$

We may write β_n as

$$\beta_n(s) = \frac{\text{Im } \alpha_n(s)}{q} \exp \{E_n(s)\} \prod_{n' \neq n}^{\infty} S_{n'}(s, \alpha_n). \quad (\text{II.73})$$

This equation combined with the representation (Eq. II.70) allows the elimination of the $\beta_n(s)$, and the writing of a set of coupled integral equations for the $\text{Im } \alpha_n(s)$, as before. In the one-trajectory approximation we obtain (dropping the subscripts $n = 1$)

$$\text{Im } \alpha(s) = mg^2 q^{2(1/2 + \text{Re } \alpha)} \exp \left\{ - \text{Re } E(s) + \frac{1}{\pi} \int_0^{\infty} \frac{ds'}{s' - s} \left[\text{Im } E(s') - \text{Im } \alpha(s') \right] \right\}. \quad (\text{II.74})$$

This then, is exactly equation (II.52) of the Cheng Representation, except that D is replaced by E . The solution of this integral equation produces the same asymptotic properties for $\text{Im } \alpha$ as does the unmodified equation:

$$\alpha(s) \rightarrow -1 + i g^2 m/q \text{ as } s \rightarrow \infty, \quad (\text{II.75})$$

$$\text{Im } \alpha(s) \propto s^{1/2 + \text{Re } \alpha(0)} \text{ as } s \rightarrow 0. \quad (\text{II.76})$$

This equation and the corresponding subtracted equation were solved for the same choice of parameters as were the unmodified equations, $V(r) = -g^2 e^{-\mu r}/r$, $g^2 = 1.8$, $\mu = 1$, $m = 1/2$. The subtraction point s_0 was chosen to be $s_0 = +.1$, because the true trajectory and the one calculated from the unsubtracted modified Cheng representation were quite close at that value of s ; $\alpha(s_0)$ was taken from the true trajectory. The resulting trajectory is in each case quite close to the true

trajectory, and much better than the corresponding trajectory calculated from the unmodified Cheng representation. In particular, $\text{Re } \alpha(\infty)$ for the subtracted case is almost exactly - 1, though this was not forced.

III. RELATIVISTIC CALCULATION OF THE RHO TRAJECTORY³⁸⁾

To demonstrate the application of the method to a relativistic problem, the rho trajectory will be calculated. The approximations involved are such that only the qualitative features of the trajectory are to be believed; little credence should be held in the numerical values found.

The rho meson has isotopic spin $I = 1$, spin $J = 1$, parity $-$, G-parity $+$, mass $m_\rho \approx 765$ MeV, width for decay into two pions $\Gamma \approx 120$ MeV³⁹⁾. In accordance with the bootstrap hypothesis, the rho may be considered as being composed of (or having an amplitude to be found as) all systems of particles with the same quantum numbers. A few of the lowest-mass systems include 2π , 4π , 6π , ..., $\pi\omega$, $K\bar{K}$, $N\bar{N}$. A complete calculation would require including all these systems, and finding the rho as a resonance in each channel: $2\pi \rightarrow \rho \rightarrow 2\pi$, $K\bar{K} \rightarrow \rho \rightarrow 6\pi$, and all the others. The customary argument here⁹⁾ is that the most important channels are those with the lowest thresholds; thus, the most important is $2\pi \rightarrow 2\pi$, with the channels with higher thresholds less important; this is equivalent to saying that the rho is primarily composed of two pions, with a lesser admixture of heavier systems. This is a reasonable argument, but the difficulty is in deciding where to draw the line. In this calculation, the rho will be considered as being wholly a 2π resonance. This is correct in reactions with center-of-mass energy less than (m_π^*) , but less accurate for

*) The units are such that $\hbar = c = 1$.

higher energies; the approximation is certainly a crude one, but suffices to demonstrate the method which may be used in more accurate calculations.

The usual Mandelstam variables are defined:

$$\begin{aligned} s &= 4(q^2 + m_\pi^2), \\ t &= -2q^2 (1 - z_s), \\ u &= -2q^2 (1 + z_s), \end{aligned} \quad (\text{III.1})$$

with q the s -channel center-of-mass momentum of one of the pions, m_π the pion mass (≈ 139.6 MeV), and z_s the cosine of the s -channel scattering angle in the center-of-mass system. The total s -channel amplitude $T^I(s, t, u)$ is defined in terms of the partial-wave amplitude by

$$T^I(s, t, u) = \sum_{\ell=0}^{\infty} (2\ell + 1) A^I(s, \ell) [P_\ell(z) + (-1)^\ell P_\ell(-z)]. \quad (\text{III.2})$$

Since

$$z = 1 + \frac{2t}{s - 4m_\pi^2} \quad \text{and} \quad -z = 1 + \frac{2u}{s - 4m_\pi^2}, \quad (\text{III.3})$$

Bose symmetry is satisfied:

$$T^I(s, t, u) = (-1)^\ell T^I(s, u, t). \quad (\text{III.4})$$

(The Bose symmetry allows some simplification from the + and - amplitudes of Frautschi, Gell-Mann and Zachariasen¹¹⁾. If. Eq. (III.2) is rewritten as

$$T^I(s, t, u) = \bar{T}^I(s, t) + (-1)^\ell \bar{T}^I(s, u), \quad (\text{III.5})$$

the Sommerfeld-Watson transformation may be made on the \bar{T} amplitudes.)

The S-matrix is given in terms of A^I by

$$S^I(s, \ell) = 1 + 2i \frac{q}{w} A^I(s, \ell), \quad (\text{III.6})$$

where $w = \sqrt{s/4}$ is the energy of one of the pions in the center-of-mass system.

A. The Relativistic Cheng Representation

The derivation of the relativistic analog of the Cheng representation is quite similar to the non-relativistic derivation. For $\text{Re } \ell > -1/2$, it is reasonable to assume that

$$S^I(s, \ell) \rightarrow 1 + O(e^{-\ell \xi} / \sqrt{\ell}) \text{ as } |\ell| \rightarrow \infty, \quad (\text{III.7})$$

where $\cosh \xi = 1 + (2m_\pi)^2 / 2q^2$, since the least massive system which may be exchanged between two pions consists of two pions. An heuristic derivation of this limit may be obtained by inspection of the Froissart-Gribov extension to complex ℓ of the partial-wave projection formula^{40,41)}

$$A(s, \ell) = \frac{1}{2\pi q} \int_{t_0}^{\infty} dt' Q_\ell \left(1 + \frac{t'}{2s} \right) T_t(s, t'). \quad (\text{III.8})$$

Here, $2\pi i T_t(s, t')$ is the discontinuity across the t -cut of $T(s, t, u)$; t_0 is the beginning of the cut and is equal to the square of the lightest mass system which can be an intermediate state in the t -channel. Because of the asymptotic behavior of the Q -function, interchange of the integral and the limit $|\ell| \rightarrow \infty$ gives the desired result (7). For $\text{Re } \ell < -1/2$, it is not necessary to assume that $S^I(s, \ell) \rightarrow e^{2\pi i(\ell + 1/2)}$, as does the non-relativistic S , but only that

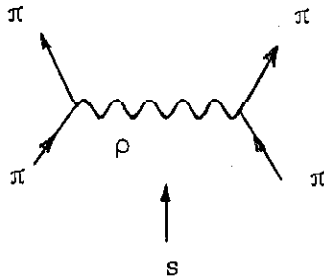
$\ln S^I(s, l) e^{\ell \xi} \rightarrow 0$ as $|\ell| \rightarrow \infty$, with $\text{Re } l < -1/2$. In addition, it has been assumed that the S-matrix has no cuts in angular momentum; if there were angular momentum cuts, then the resultant expression would include a term for the integral around the cut. Then exactly the same derivation holds as for the non-relativistic case, and we obtain

$$S^I(s, l) = \prod_{n=1}^{\infty} \exp \left\{ \int_{\alpha_n(s)}^{\alpha_n^*(s)} \frac{e^{(l' - l)\xi(s)} d\ell'}{l' - l} \right\}. \quad (\text{III.9})$$

This expression is, except for the difference in definition of s , the same as the Cheng Representation.

The derivation of a relativistic analog to the Modified Cheng Representation requires construction of the force term, $S_{\text{Born}}^I(s, l)$. The force between two pions comes from the exchange of particles or trajectories between the pions. The particle exchangeable between two pions include $2\pi, 4\pi, 6\pi, \dots, \rho, f^0, K\bar{K}, N\bar{N}, \dots$; alternatively, Regge trajectories with isotopic spins $I = 0, 1, \text{ or } 2$ may be exchanged between two pions. In particular the exchange of a rho, or of the trajectory on which the rho lies, produces an attractive force in the $I = 1$ channel. The customary approximation^{7, 9)} used in calculating the force is first to neglect exchange of all but the lightest set of particles, which is 2π ; and second to approximate this by a single-particle exchange (of the ρ) which, it is hoped, dominates the 2π exchange. In this calculation, too, the force due to the exchange of an elementary rho meson will be used to find S_{Born}^I , although in any quantitative calculation, this approximation would be inadequate. Since the purpose here is to demonstrate the method, using

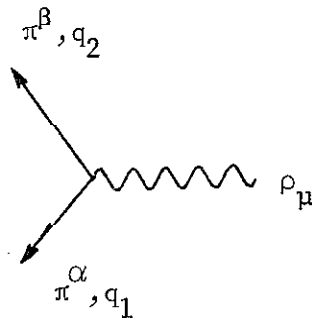
rho-exchange as the force will suffice, because the force thus found is attractive and of approximately the correct size to produce the rho resonance itself, the rho-exchange force term. This is given by the $I = 1$ projection of the diagram



Use of the effective interaction Lagrangian

$$\mathcal{L}_{\text{int}} = 2i \gamma (q_1 + q_2)_\mu \epsilon_{\alpha\beta\gamma}$$

for the vertex



gives

$$S_{\text{Born}}^{I=1}(s, \ell) = 1 + 2i \frac{q}{w} \left(2 + \frac{s + m^2}{2q^2} \frac{\rho}{\rho} \right) Q_\ell \left(1 + \frac{m^2}{2q^2} \right). \quad (\text{III.10})$$

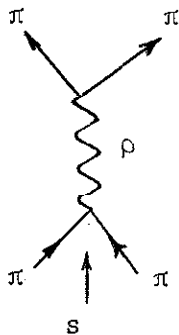
(In what follows, the $I = 1$ superscript will be understood.)

The $S(s, \ell)$ we calculate will have a pole at $\ell = 1 + i\epsilon$ for some energy μ^2 and some small ϵ ; one condition for a self-consistent S-matrix is that $\mu^2 = m_\rho^2$. A second condition comes from requiring the width of the output rho (i.e., the resonance at $\ell = 1, s = \mu^2 = m_\rho^2$) to be the same as that of the input rho. The condition on the coupling constant is

$$\frac{1}{\gamma^2} = -\frac{1}{3\pi} \operatorname{Re} \frac{d}{ds} \left[\frac{g^2}{A(s, 1)} \right]_{s = m_\rho^2} \quad (\text{III.11})$$

Actually, this calculation is only approximately self-consistent; true self-consistency would mean using the rho trajectory as the Born term, or as part of the Born term, and then requiring the calculated trajectory to match this. The method used here is equivalent to matching the trajectories and slopes at one point, $s = m_\rho^2$ (2).

To derive this expression, we first note that the $I = 1$ amplitude for $A(s, 1)$, from the diagram



is

$$A(s, 1) = \frac{-\gamma^2}{3\pi} \frac{g^2}{s - m_\rho^2}, \quad (\text{III.12})$$

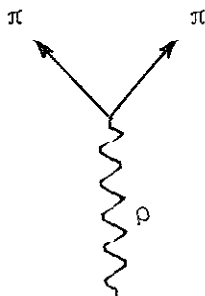
if the intermediate state is an elementary rho. Near $s = m_\rho^2$, the amplitude for resonance as intermediate state in the same process is

the same except $(s - m_\rho^2)$ in the denominator has an added imaginary term. The condition (III.11) results from requiring the input and output coupling constant γ^2 , and thus the widths, to be the same.

The experimental width $\Gamma(\rho \rightarrow 2\pi)$ is related to the coupling constant γ^2 by

$$\Gamma = \frac{\gamma^2}{12\pi} \frac{(m_\rho^2 - 4m_\pi^2)^{3/2}}{m_\rho} ; \quad (\text{III.13})$$

this is obtained from the definition of Γ as the rate for the process $\rho \rightarrow 2\pi$, i.e., the diagram



with the final pions in an $I = 1$ state.

Cheng and Sharp use the approximate expression for the output width as part of their self-consistency requirement:

$$\Gamma = \frac{\text{Im } \alpha(m_\rho^2)}{m_\rho \frac{d}{ds} [\text{Re } \alpha(s)]_{s = m_\rho^2}} . \quad (\text{III.14})$$

This comes from the following agreement⁴³⁾. For ℓ near $\alpha(s)$,

$$A(s, \ell) \propto \frac{1}{\ell - \alpha(s)} . \quad (\text{III.15})$$

With $\alpha(s)$ near ℓ , s is also near m_ρ^2 , and

$$\alpha(s) \approx \ell + (s - m_\rho^2) \frac{d}{ds} \operatorname{Re} \alpha(m_\rho^2) + i \operatorname{Im} \alpha(m_\rho^2), \quad (\text{III.16})$$

so that

$$A(s, \ell) \propto \left\{ \frac{d}{ds} \operatorname{Re} \alpha(m_\rho^2) \left[s - m_\rho^2 + i \operatorname{Im} \alpha(m_\rho^2) / \frac{d}{ds} \operatorname{Re} \alpha(m_\rho^2) \right] \right\}^{-1}. \quad (\text{III.17})$$

The width Γ may be identified by writing $A(s, \ell)$ in the form

$$A(s, \ell) \propto \frac{1}{E - m_\rho + i \frac{\Gamma}{2}}, \quad (\text{III.18})$$

with $E = \sqrt{s}$ the total center of mass energy. Using

$s - m_\rho^2 \approx 2m_\rho(E - m_\rho)$, we obtain

$$A(s, \ell) \propto \left\{ \frac{d}{ds} \left[\operatorname{Re} \alpha(m_\rho^2) \right] 2m_\rho \left[E - m_\rho + \frac{i \operatorname{Im} \alpha(m_\rho^2)}{2m_\rho \frac{d}{ds} \operatorname{Re} \alpha(m_\rho^2)} \right] \right\}^{-1}, \quad (\text{III.19})$$

which gives Γ as Cheng and Sharp have it.

To see that this Γ is approximately the same as the one we obtain, note that we have

$$S(s, \ell) = \frac{\ell - \alpha^*(s)}{\ell - \alpha(s)} e^{T(s)}, \quad (\text{III.20})$$

where presumably the second factor varies much slower than the first factor near the resonance at $\ell = \operatorname{Re} \alpha$. (Cheng and Sharp had $T(s) = 0$.)

Since

$$\begin{aligned} \alpha(m_\rho^2) &= 1 + i \operatorname{Im} \alpha(m_\rho^2) \\ S(m_\rho^2, 1) &= - \exp \left[T(m_\rho^2) \right]. \end{aligned} \quad (\text{III.21})$$

Using the requirement (III.11), assuming the numerator varies slowly compared with the denominator,

$$\frac{1}{\gamma^2} \approx \frac{-(m_\rho^2 - 4m_\pi^2)^{3/2}}{6\pi m_\rho} \operatorname{Re} \left[\frac{d}{ds} \frac{i}{S(s,1) - 1} \right]_{s = m_\rho^2} \quad (\text{III.22})$$

$$\left. \frac{dS(s,1)}{ds} \right|_{s = m_\rho^2} = \left[-2i \frac{\frac{d}{ds} \operatorname{Re} \alpha(s)}{\operatorname{Im} \alpha(s)} - \frac{dT(s)}{ds} \right]_{s = m_\rho^2} \exp \left[T(m_\rho^2) \right]. \quad (\text{III.23})$$

$$\frac{1}{\gamma^2} \approx \frac{(m_\rho^2 - 4m_\pi^2)^{3/2}}{6\pi m_\rho} \operatorname{Re} \left\{ \left[\frac{2 \operatorname{Re} \alpha'(m_\rho^2)}{\operatorname{Im} \alpha(m_\rho^2)} + T'(m_\rho^2) \right] \frac{e^{T(m_\rho^2)}}{(1 + e^{T(m_\rho^2)})^2} \right\}. \quad (\text{III.24})$$

If we neglect $T'(m_\rho^2)$ and set $T(m_\rho^2) = 0$, we obtain

$$\frac{1}{\gamma^2} \approx \frac{(m_\rho^2 - 4m_\pi^2)^{3/2}}{12\pi m_\rho} \frac{\operatorname{Re} \alpha'(m_\rho^2)}{\operatorname{Im} \alpha(m_\rho^2)}, \quad (\text{III.25})$$

then using Eq. (III.13)

$$\Gamma = \frac{\operatorname{Im} \alpha(m_\rho^2)}{m_\rho \operatorname{Re} \alpha'(m_\rho^2)}, \quad (\text{III.26})$$

so the two methods are roughly equivalent. (In the actual calculation, the two methods differed by about 5 to 10 percent.)

With Eq. (III.10) as the Born term, exactly the same procedure is followed as in deriving the modified Cheng Representation, to obtain

$$S(s, \ell) = \exp \left\{ i \frac{\gamma^2}{2\pi} \frac{q}{w} \left(2 + \frac{s + m_\rho^2}{2q^2} \right) Q_\ell \left(1 + \frac{m_\rho^2}{2q^2} \right) \prod_{n=1}^{\infty} S_n(s, \ell) \right\}, \quad (\text{III.27})$$

where

$$S_n(s, \ell) = \exp \left\{ \int_{\alpha_n}^{\alpha_n^*} \frac{e^{(\ell' - \ell)\xi}}{\ell' - \ell} d\ell' - i \frac{q}{w} \frac{Y^2}{2\pi} \left(2 + \frac{s + m^2}{2q^2} \right) P_{n-1} \left(1 + \frac{m^2}{2q^2} \right) \cdot e^{-(\ell + n)\xi / (\ell + n)} \right\}. \quad (\text{III.28})$$

Here $\xi(s) = \cosh^{-1} \left(1 + \frac{(2m_\pi)^2}{2q^2} \right)$, since $2m_\pi$ is the smallest mass that can be exchanged between two pions. The residue $\beta_n(s)$ of $A(s, \ell)$ at $\ell = \alpha_n(s)$ is then

$$\beta_n(s) = \frac{w}{q} \text{Im } \alpha_n \exp \left\{ D(s) + i \frac{q}{w} \frac{Y^2}{2\pi} \left(2 + \frac{s + m^2}{2q^2} \right) \left[Q_{\alpha_n} - \frac{e^{-(n+\alpha)\xi}}{n + \alpha} P_{n-1} \right] \right\} \cdot \prod_{\substack{n'=1 \\ n' \neq n}}^{\infty} S_{n'}(s, \alpha_n) \quad (\text{III.29})$$

$$= \frac{w}{q} \text{Im } \alpha_n \exp \left\{ F_n(s) \right\} \prod_{\substack{n'=1 \\ n' \neq n}}^{\infty} S_{n'}(s, \alpha_n) \quad (\text{III.30})$$

where

$$D_n(s) = \int_{\alpha_n(s)}^{\alpha_n^*(s)} d\ell' \frac{e^{(\ell' - \alpha_n)\xi} - 1}{\ell' - \alpha_n(s)}, \quad (\text{III.31})$$

and $F_n(s)$ is defined by these equations. In the one-trajectory approximation, the product is, as before, set equal to one; then, the n subscripts may be suppressed:

$$\beta(s) = \frac{w}{q} \text{Im } \alpha(s) e^{F(s)}, \quad (\text{III.32})$$

$$F(s) = \int_{\alpha(s)}^{\alpha^*(s)} d\ell' \frac{e^{(\ell' - \alpha)\xi} - 1}{\ell' - \alpha} + i \frac{q}{w} \frac{Y^2}{2\pi} \left(2 + \frac{s + m^2}{2q^2} \right)$$

$$\cdot \left[Q_{\alpha} \left(1 + \frac{m_{\rho}^2}{2q^2} \right) - \frac{e^{-(1-\alpha)\xi}}{1+\alpha} \right]. \quad (\text{III.33})$$

The second term of F is due to the Born term; inclusion of this term is the only difference, since the ξ is the same in the modified and unmodified cases.

One defect of this representation in the one-trajectory approximation is directly attributable to the use of an elementary rho for the force. The S-matrix does not approach 1 as $s \rightarrow \infty$ (for physical l) as it should, but instead

$$S(s, l) \rightarrow \exp \left\{ 2i \frac{Y^2}{\pi} \left[Q_{\alpha(\infty)} \left(1 + \frac{m_{\rho}^2}{2q^2} \right) - \frac{1}{l+1} \right] \right\} \quad (\text{III.34})$$

Using Eqs. AI.14 and AI.20 of the Appendix

$$S(s, l) \rightarrow \exp \left\{ 2i \frac{Y^2}{\pi} \left[\ln \frac{s}{m_{\rho}^2} - \gamma_E - \psi(2 + \alpha(\infty)) \right] \right\} \quad (\text{III.35})$$

or

$$S(s, l) \rightarrow \exp \left\{ i (a_1 + a_2 \ln s) \right\}, \quad (\text{III.36})$$

with a_1 and a_2 constants defined by these equations. Thus the magnitude of S is correct asymptotically, but not the phase. This incorrect phase comes solely from the Born term; in reality, some mechanism, perhaps the Regge mechanism, damps the Born term and the phase goes to zero.

It may be noted, too, that if all the trajectories are kept, the asymptotic phase of S is correct, so that this spurious infinite phase is directly due to the one-trajectory approximation. [The Q term cancels the sum of the P terms.]

B. Dispersion Relations for α and β

For those trajectories which do not intersect with other trajectories, it has been shown that $\alpha(s)$ and $b(s)$ are real analytic functions, with only a right-hand cut from threshold to infinity along the real s -axis²⁵⁾. Gribov and Pomeranchuk⁴⁴⁾ have shown that $\alpha(s)$ then will obey a once subtracted dispersion relation of the form

$$\alpha(s) = \alpha(s_0) + \frac{s - s_0}{\pi} \int_{s_T}^{\infty} \frac{ds'}{(s' - s - i\epsilon)(s' - s_0 - i\epsilon)} \text{Im } \alpha(s'). \quad (\text{III.37})$$

s_T is the threshold for the reaction, here equal to $4m_\pi^2$. If $\text{Im } \alpha(s) > 0$ as $s \rightarrow \infty$, as is true for non-relativistic trajectories, an unsubtracted dispersion relation will hold:

$$\alpha(s) = \alpha(\infty) + \frac{1}{\pi} \int_{s_T}^{\infty} \frac{ds'}{s' - s - i\epsilon} \text{Im } \alpha(s'). \quad (\text{III.38})$$

In view of the asymptotic behavior $S(s, \ell) \rightarrow 1$ as $s \rightarrow \infty$ for physical ℓ , and the representation III.9, $\text{Im } \alpha(s)$ should go to zero as $s \rightarrow \infty$ unless some complicated cancellation among trajectories occurs.

The high s -limit of the Regge trajectory, $\alpha(\infty)$, is not known. For this reason, it may be preferable to use a subtracted dispersion relation, especially if the subtraction is made at the rho mass. This was done in the calculations performed.

To derive an expression for $\beta(s)$, we use the same technique as in the non-relativistic case. We assume that the residue of the rho trajectory has no zeroes, just as the leading trajectory in potential theory has no zeroes. From Eq. (III.29), after a little algebra,

we may see that

$$b(s) \rightarrow \frac{\text{Im } \alpha(s)}{q^{2\alpha(\infty)}} \exp \left\{ i(a_1 + a_2 \ln s) \right\} \text{ as } s \rightarrow \infty \quad (\text{III.39})$$

(The infinite phase is spurious, and is the same one discussed earlier.)

If we are to write an unsubtracted dispersion relation of the same type as before, we need to know the limit of $\text{Im } \alpha(s)/q^{2\alpha(\infty)}$. Recall that in potential theory, the analogous limit is known:

$b(s) \rightarrow mg^2$ for the first trajectory, where m is the external particle mass. To avoid an assumption about the asymptotic form of $b(s)$, a subtracted dispersion relation will be used

$$\beta(s) = \beta(s_0) \frac{q^{2\alpha(s)}}{q^{2\alpha(s_0)}} \exp \left\{ \frac{s - s_0}{\pi} \int_{4m_\pi}^{\infty} \frac{ds'}{(s' - s - i\epsilon)(s' - s_0 - i\epsilon)} \cdot \left[\arg \beta(s') - \text{Im } \alpha(s') \ln q^2(s') \right] \right\}. \quad (\text{III.40})$$

C. Integral Equation

In the one-trajectory approximation, using this and the previous equation for $\beta(s)$, we obtain the integral equation

$$\text{Im } \alpha(s) = \text{Im } \alpha_0 \frac{w_0}{w} \frac{q^{2\alpha(s)}}{q^{2\alpha(s_0)}} \exp \left\{ \text{Re } [F(s_0) - F(s)] + \frac{s - s_0}{\pi} \int_{4m_\pi}^{\infty} \frac{ds'}{(s' - s)(s' - s_0)} \left[\text{Im } F(s') - \text{Im } \alpha(s') \ln q'^2 \right] \right\}. \quad (\text{III.41})$$

(No use has been made of the dispersion relation for $\alpha(s)$; the same

integral equation obtains for subtracted or unsubtracted dispersion relations for $\alpha(s)$, if a subtracted dispersion relation is used for $b(s)$.)

Since $\text{Im } \alpha \rightarrow 0$ as $s \rightarrow \infty$, $D(s) \rightarrow 0$. $F(s)$ approaches $i(a_1 + a_2 \ln s)$, a_1 and a_2 the constants previously defined, but again this is because of the unphysical nature of the Born term at high s .

The "true" $F(s)$ would go to zero as $s \rightarrow \infty$. For all cases investigated, $\text{Im } \alpha(s) \ln q^2 \rightarrow 0$ as $s \rightarrow \infty$; since for a better choice of $F(s)$, $F(s) \rightarrow 0$ as $s \rightarrow \infty$, then the Cauchy principle value integral would $\rightarrow 0$ as $s \rightarrow \infty$. Then as $s \rightarrow \infty$, the equation reduces to

$$\text{Im } \alpha(s) = (\text{const.}) q^2 \text{Re } \alpha(s). \quad (\text{III.42})$$

Since $\text{Im } \alpha(s) \rightarrow 0$, then $\text{Re } \alpha(\infty) < 0$. If $F(s)$ does not $\rightarrow 0$ as $s \rightarrow \infty$, however, $\text{Re } \alpha(\infty)$ may be positive.

Since $F(s) \rightarrow 0$ as $s \rightarrow 4m_\pi^2$, at threshold we have

$$\text{Im } \alpha(s) \propto q^{2(\frac{1}{2} + \text{Re } \alpha)}, \quad (\text{III.43})$$

just as in potential theory.

D. Calculation

The procedure for finding the rho trajectory was as follows: Choose $s_0 = m_\rho^2$, the experimental value, $\alpha(m_\rho^2) = \alpha_0 = 1 + \text{Im } \alpha_0$. Pick a value for α_0 and for γ^2 . Solve the integral equation III.41 using the definition of $F(s)$ as in Eq. III.33 (or a modified F as described later). Then calculate the output γ^2 by Eq. III.11 and check that it agrees with the input γ^2 . Vary $\text{Im } \alpha_0$ until the input and output values for γ^2 agree (if that is possible, for the given γ^2).

Using an elementary rho as the Born term as previously described, and using the experimental rho mass, it was found that no self-consistent solution exists with the experimental width ($\Gamma \approx 120$ MeV). However, a solution does exist for a larger width; in particular, a self-consistent solution for the rho trajectory was found with $\Gamma = 200$ MeV. This is shown in Figs. 7, 8, and 9. The important features to be noticed are that $\text{Re } \alpha(\infty) \approx .03$, positive, but barely so. $\text{Re } \alpha(0) \approx .9$, and decreases very slowly as s becomes more negative. Analyses of recent experiments⁴⁾ have indicated that $\text{Re } \alpha_{\rho}(0) \approx .54$, indicating a steeper slope than that found here. None of the investigated modifications steepened the slope of $\text{Re } \alpha$ below the threshold; for the three modifications described hereafter, the below-threshold character of $\text{Re } \alpha$ is virtually unchanged. (It may be noted that, for $\text{Re } \alpha$ to be steeper, below threshold, the peak in $\text{Im } \alpha$ must move to lower energies). Similar below-threshold behavior was found in a preliminary calculation by Collins and Teplitz¹⁸⁾; they found $\alpha_{\rho}(0) \approx .75$. The integral equation which is obtained from the unmodified representation was also solved; this gives results close to those obtained by Cheng and Sharp, and radically different from the solution to the modified equation. No solution could be found with the experimental mass and width. A sample solution is shown with $\alpha(m_{\rho}^2) = 1 + .025 i$; this has an output Γ of about 500 MeV (Figs. 5,6).

In none of the calculations, including the modified ones to be described below, did $\text{Re } \alpha(s)$ ever become large enough to produce a second resonance at $\ell = 3$; the maximum $\text{Re } \alpha$ was always less than 1.5.

E. Phenomenological Modifications

Several modifications of the basic integral equation were made to investigate, qualitatively, the effects of physical processes which were left out. In particular, it was desired to know if inclusion of these neglected effects would make it possible to find a rho trajectory with the rho meson at its experimental mass and width. This was found to be the case.

1. Inelastic effects

This was, as expected⁴⁵⁾, found to be important in narrowing the width of the resonance. In the previous calculations, inelastic unitarity, $S^*(s, \ell^*) S(s, \ell) = 1$, was used, ignoring inelastic channels. In reality, this should be a matrix equation,

$$S_{11}^*(s, \ell^*) S_{11}(s, \ell) + \sum_{k>1} S_{k1}^*(s, \ell^*) S_{k1}(s, \ell) = 1 \quad (\text{III.44})$$

plus another equation for each possible reaction, and the sum runs over all states k for which is nearer than the threshold for that state.

[Notation: S_{ij} is the amplitude to go from state j to state i , and state 1 is the $\pi\pi$ state of before]. Thus our equation $S^* S = 1$ should have been

$$S^*(s, \ell^*) S(s, \ell) = f(s, \ell), \quad (\text{III.45})$$

where $f(s, \ell)$ is a real positive function of both s and ℓ . For s below the first inelastic threshold, i.e., $s < (4m_\pi)^2$, $f(s, \ell)$ is identically one, and f is less than (or equal to) one for higher energies;
 $0 \leq f(s, \ell) \leq 1$.

If we regard f as a known function we can use the same

arguments on $S(s, \ell) / f(s, \ell)$ as we did on $S(s, \ell)$ before to derive a new representation which phenomenologically includes inelastic effects. The \sqrt{f} is merely carried through as a factor in all the derivations, if it is assumed that $1/\sqrt{f(s, \ell)}$ has no cuts or poles that would necessitate changes when the contour integral corresponding to II.64 is done. For the simple choice of f chosen below, no problem arises. The end result is that the integral equation, Eq. III.41, must be multiplied on the right side by $1/\sqrt{f(s, \alpha(s))}$, and Eq. III.27 for the S -matrix must be multiplied on the right side by $\sqrt{f(s, \ell)}$.

In order to get an indication of the effect that this would have on the calculated rho trajectory, an arbitrary choice of $\sqrt{f(s, \alpha(s))}$ was chosen:

$$\begin{aligned} \sqrt{f(s, \alpha(s))} &= 1, \quad s < 16m_{\pi}^2 \\ &= \frac{s + 16m_{\pi}^2}{2s}, \quad s > 16m_{\pi}^2. \end{aligned} \quad (\text{III.46})$$

This narrowed the resonance significantly. For an input $\Gamma = 120$ MeV and $\text{Im } \alpha(m_{\rho}^2) = .025$, the output width was $\Gamma = 300$ MeV if inelasticity was ignored, but the output width was $\Gamma = 85$ MeV with the above inelastic modifications. Thus it is evident that inclusion of inelastic effects might well result in a self consistent trajectory with the proper width. The slope of $\text{Re } \alpha$ was not appreciably steepened, so that $\text{Re } \alpha(0)$ is still too high.

2. Effects of Higher-Mass Exchange

The exchange of higher-mass particles than the rho between the pions is also expected to narrow the output rho resonance⁴⁶⁾. This

was tested by adding to the Born term a term for the exchange of another $I = 1$ meson, exactly like the rho except for having twice the mass and half the coupling constant. (No attempt was made to calculate this additional meson self-consistently.)

This too narrowed the rho resonance significantly. For an input $\Gamma = 120$ MeV and $\text{Im } \alpha(m_\rho^2) = .025$, the output width was 300 MeV if higher-mass terms were negligible, but the output width was 185 MeV with the above modifications. Again this had little effect on the slope of $\text{Re } \alpha(s)$.

3. Regge Cut-off on Born Term

The use of an elementary rho as Born term is significantly wrong at high energies, since the Born term should tend to zero, but the rho term becomes proportional to the logarithm of the energy. In earlier calculations by the N/D method, otherwise-divergent integrals were given a finite upper limit to avoid divergence. (This upper limit provided a parameter whose adjustment provides a "crude representation of the natural cut-off provided by the Regge behavior of composite states"⁹⁾.)

In order to obtain this "crude representation of the natural cut-off", the Born term was multiplied by a damping factor at high energies, $f_R(s)$

$$\begin{aligned} f_R(s) &= 1, & s < s_R \\ &= (s_R/s)^\epsilon, & s > s_R \end{aligned} \tag{III.47}$$

This damping factor was chosen more-or-less arbitrarily; $\epsilon = 1/2$

was completely arbitrary, and $s_R = 7000 m_\pi^2$ was chosen because $\text{Im } \alpha$ $\text{Im } \alpha(s_R)$ was small there, but $\text{Re } \alpha(s_R)$ was not yet at its asymptotic value ($\text{Re } \alpha \approx -.1$ instead of $\approx +.03$). Very little difference was seen between the damped and the undamped solutions.

The general effect of such a term as $f_R(s)$ above, for any choice of s_R and γ , was identical qualitatively. For a given choice of $\text{Im } \alpha(m_\rho^2)$ and the input γ^2 , the inclusion of a Regge damping factor $f_R(s)$ had the same qualitative effect as lowering γ^2 , i.e., providing a weaker force. The peak in $\text{Im } \alpha$ was lowered and moved toward higher s -values, thus making $\text{Re } \alpha$ closer to 1 (as may be seen from the dispersion relation for $\text{Re } \alpha(s)$), and reducing the slope of $\text{Re } \alpha$. This latter effect than increased the output width, since

$$\Gamma_{\text{out}} \propto \left[\frac{d}{ds} \text{Re } \alpha(s) \Big|_{s = m_\rho^2} \right]^{-1}.$$

IV. CONCLUSION

A method for calculating Regge trajectories has been presented, and has been tested in potential theory. Good agreement was found with the exact trajectory, in a one-trajectory approximation.

An illustratory calculation of the rho trajectory was performed: the approximations used were sufficiently drastic that the computed trajectory is not to be believed quantitatively, but several qualitative features of the trajectory (Figs. 7, 8, 9) may be noted:

- 1) With the mass of the rho fixed at the experimental value, no trajectory with the experimental width (120 MeV) exists. Trajectories with larger widths (such as 200 MeV) may be calculated. This is in accordance with expectations, based on comparison with early non-Regge calculations^{7,9)}.
- 2) The slope of $\text{Re } \alpha(s)$ is smaller than analyses of experiments⁴⁾ indicates; in particular, $\text{Re } \alpha(0)$ is higher ($\approx .9$) than the indicated experimental value ($\approx .5$).
- 3) As $s \rightarrow \infty$, $\text{Re } \alpha(s) \rightarrow .03$, approximately. As discussed in the text, an improved calculation of the force term would have given $\text{Re } \alpha(\infty) < 0$.
- 4) $\text{Re } \alpha(s)$ never became nearly large enough to produce a second resonance at $\ell = 3$.

Properties (1) and (2) agree with the results of a strip approximation calculation by Collins and Teplitz¹⁸⁾. They used only the rho trajectory as the input term [Eq. (I.4)], and set $\text{Re } \alpha = 1$ at the experimental rho mass. (This corresponds closely to the physical approximations made in Section III, but the mathematical

approximations are very different.) They used a two-parameter form for $\text{Re } \alpha$, and approximately matched input and output trajectories over the range $-200 m_\pi^2 < s < 0$ where α is real. They obtained "good self-consistency" in this region with $\alpha(s) = .33 + .42/(1 - s/75m_\pi^2)$. Their solution breaks down for $s > 0$, however; the output trajectory rises only to (about) .85, so that the rho meson is not generated. Since their input term depends on α only for $s < 0$, they conclude "we regard this as a self-consistent solution, but it is clear that our results cannot be continued into the physical region".

Qualitative consideration was given to several effects. The effect of including inelastic channels was to allow a narrower width for the output rho trajectory, but not to change appreciably either the slope of $\text{Re } \alpha(s)$ or $\text{Re } \alpha(0)$. The inclusion of shorter-range forces had much the same effect.

Since none of the qualitative changes lowered $\text{Re } \alpha(0)$, some consideration should be given to the possibility that the $\text{Re } \alpha(0)$ calculated by Phillips and Rarita is incorrect, and should be higher. No firm conclusion may be drawn, of course, until a much more accurate calculation is made. No test was made of 2) below; this is the most important effect which remains untested, and it is possible that this might steepen $\text{Re } \alpha(s)$ below threshold.

Improved calculations of the rho trajectory should include some (or all) of the changes listed below. The comments with each are partially based on experience gained in calculating with this method, partially on results in the literature, and are partially pure opinion.

1. Exchange of a rho trajectory instead of an elementary rho would permit a truly self-consistent calculation, which is essential theoretically. If the rho trajectory in an improved calculation proves to be as flat below threshold as the ones found here and by Collins and Teplitz, this probably would not make a large difference in the output trajectory. (Collins and Teplitz noted that the difference between their rho trajectory input term and an elementary rho input term was "not great".)
2. Inclusion of an inelastic channel, such as the $\pi\omega$ channel, would reduce the width of the output rho, probably enough to permit a self-consistent rho with the experimental parameters.
3. Inclusion of a force due to $I = 0$ exchange, either the f^0 meson or the Pomeranchuk - f^0 trajectory, might possibly steepen the slope of $\text{Re } \alpha(s)$ below threshold and produce a $\text{Re } \alpha(0)$ nearer the experimental value, but this is only a conjecture. Theoretically, though, in any reasonable approximation this force should be included, and the Pomeranchuk trajectory computed simultaneously with the rho trajectory.
4. Inclusion of a second $I = 1$ trajectory (a two- instead of a one-trajectory approximation) is likely to be less important than 2) and 3). This is based upon experience with the potential theory case, where a one-trajectory model gives a fairly good approximation to the scattering amplitude if the modified Cheng representation is used for the S-matrix.

APPENDICES

All computations were carried out on the IBM 7094 computer at Caltech's Booth Computing Center. The difficulties involved were of two varieties: first, the setting up of the programs for doing the necessary integrals and calculating the necessary functions; and second, the solving of the integral equations for $\text{Im } \alpha(s)$. These will be discussed in Appendices I and II, respectively.

APPENDIX I NUMERICAL TECHNIQUES

The two major computational problems were the calculations of the Hilbert transform integral and the Legendre function $Q_\nu(z)$. Of much less difficulty was the calculation of $D(s, \alpha)$.

A. Hilbert Transform Integrals - Unsubtracted

The Cauchy Principle Value integral

$$I(y) = \frac{1}{\pi} \int_0^{\infty} \frac{dx}{x-y} f(x) \quad (\text{AI.1})$$

was required, where the f 's involved were smoothly varying; which were zero at $x = 0$; and which went to zero as $x \rightarrow \infty$, but might go as slowly as $(\ln x/\sqrt{x})$.

After much experimentation, the following method was derived.

The integral is broken up into four integrals: I_1 is the integral from $x = 0$ to $x = x_1$, I_2 from x_1 to x_2 , I_3 from x_2 to x_3 , and I_4 from x_3 to ∞ . The choosing of these three values x_i is not particularly critical, but best accuracy was obtained when the x_3 to ∞ interval contained none of the "structure" of $f(x)$, but only its smooth decay to zero. (This segment proved most difficult to do accurately.)

For accurate calculation, the singularity must be removed.

For example, by definition

$$I_2 \equiv \frac{1}{\pi} \int_{x_1}^{x_2} \frac{dx}{x-y} f(x). \quad (\text{AI.2})$$

Adding and subtracting $f(y)/x - y$ inside the integral, we obtain

$$I_2 = \frac{1}{\pi} \int_{x_1}^{x_2} dx \frac{f(x) - f(y)}{x - y} + \frac{f(y)}{\pi} \int_{x_1}^{x_2} \frac{dx}{x - y} \quad (A1.3)$$

The second integral is just $\ln \left| \frac{x_2 - y}{x_1 - y} \right|$. Exactly the same technique is used for I_1 and I_3 , but a minor modification is used in I_4 so that the second integral will converge:

$$I_4 = \frac{1}{\pi} \int_{x_3}^{\infty} dx \frac{f(x) - \sqrt{\frac{y}{x}} f(y)}{x - y} + \frac{f(y)}{\pi} \ln \left| \frac{\sqrt{x_3} + \sqrt{y}}{\sqrt{x_3} - \sqrt{y}} \right| \quad (A1.4)$$

We are now left with three finite integrals and one infinite integral, each with a smooth integrand. It was decided to do the integrals by the method of Lagrangian quadratures⁴⁷⁾; 16-point quadratures sufficed to give the desired accuracy. The more usual Simpson's Rule integration was rejected because it would require many more points for the same accuracy, and the calculation of $f(x)$ was relatively time-consuming.

The integrals I_1 , I_2 , and I_3 were reduced to the interval $[-1, 1]$ by linear transformations, but I_4 was more difficult. A simple change of variables from x to u like $u = 1 - 2 x_3/x$ makes I_4 into an integral over u from -1 to $+1$, but has a Jacobian which diverges as $1/v^2$ as $x \rightarrow \infty$. Since the integrand contains terms which may go to zero as slowly as $\ln x/x^{3/2}$, the resultant u -integrand is infinite at the upper end, and the u -integral is only very inaccurately done by Gaussian quadratures. After much searching, the transformation

$$x = x_3 \{1 + a(1 + u) \exp[u/(1 - u)]\} \quad (A1.5)$$

was found to give good accuracy. The parameter "a" was usually chosen to be around 1 or 2. [$u = 0$ corresponds to $x = x_3(1 + a)$].

The resultant accuracy, for a test function like $f(x) = \sqrt{x}/(x_0 + x)$, for which $I(y) = \sqrt{x_0}/(x_0 + y)$, was approximately .05 percent to 0.1 percent except for the first 2 or 3 ($y \rightarrow 0$) and the last 2 or 3 ($y \rightarrow \infty$) points. For the first two or three points the error was around 1 percent; for the last 2 or 3 points the error was small in absolute value, but because of the smallness of $|I(y)|$ the percent error might be as large as 10 or 15 percent.

B. Hilbert Transform Integrals - Subtracted

$$I(y) = \frac{y - x_0}{\pi} \int_0^{\infty} \frac{dx}{(x - y)(x - x_0)} f(x). \quad (\text{AI.6})$$

These were done in the same manner as the unsubtracted integrals except that it was necessary to remove the singularity at $x = x_0$ (if $x_0 > 0$). This was done exactly as was the removal of the $x = y$ singularity, and then the integral was done by the same method.

The accuracy was the same as the unsubtracted, except for the last two or three points. Here, the accuracy was around 1 percent; the absolute error was about the same size as for the unsubtracted integral, but the percentage error was less, since $|I(y)|$ will not in general be small as $y \rightarrow \infty$.

The relativistic Hilbert transforms require a subtracted dispersion integral from $s = 4m^2/\pi$ to $s = \infty$; a linear transformation from s to x was used.

C. D(s, α)

The integral

$$D = \int_{\alpha}^{\alpha^*} d\ell \frac{e^{(\ell - \alpha)\xi} - 1}{\ell - \alpha} \quad (\text{AI.7})$$

is required. A change of variable reduces this to

$$D = \int_{-1}^1 du \{ \exp[- i(1 + u)\xi \text{Im } \alpha] - 1 \} / (1 + u). \quad (\text{AI.8})$$

The integrand is smoothly-varying, and, for the values of $\xi \text{Im } \alpha$ necessary, sufficiently slowly-varying so that 16-point Gaussian quadrature gives 6 or 7 figure accuracy.

D. The Legendre Q-Function

The function $Q_\nu(z)$ is needed for ν complex, and for z real and greater than one. The basic formula is derived from the integral representation⁴⁸⁾

$$Q_\nu(z) = \frac{1}{2^{1+\nu}} \int_0^\pi \frac{(\sin t)^{1+2\nu}}{(z + \cos t)^{1+\nu}} dt, \quad (\text{AI.9})$$

which is valid for $\text{Re } \nu > -1$, $z > 1$. The substitution $y = \cos t$ yields

$$Q_\nu(z) = \int_{-1}^1 \left[\frac{1 - y^2}{2(y + z)} \right]^\nu \frac{dy}{2(y + z)}. \quad (\text{AI.10})$$

It is desired to do the integral by Gaussian quadratures. Q_ν has a pole at $\nu = -1$, from the $(1 - y^2)^\nu$ term; for $\text{Re } \nu < 0$, the integrand near $y = \pm 1$ becomes large, and so the evaluation of the integral by

Gaussian quadratures is inaccurate. The following transformation, however, improves the accuracy considerably: Eq. (AI.9) is identically equal to

$$Q_\nu(z) = \int_{-1}^1 \left\{ \left[\frac{1-y^2}{2(y+z)} \right]^{1+\nu} + \left[\frac{1-y^2}{2(y+z)} \right]^\nu \frac{y^2}{2(y+z)} \right\} dy. \quad (\text{AI.11})$$

Integrate the second term by parts; for $\text{Re } \nu > -1$, the integrated term vanishes and we obtain the formula which was used for calculations:

$$Q_\nu(z) = \int_{-1}^1 \left[\frac{1-y^2}{2(y+z)} \right]^{1+\nu} \left[1 + \frac{1}{2(1+\nu)} - \frac{y}{2(x+y)} \right] dy \quad (\text{AI.12})$$

This is valid for $\text{Re } \nu > -1$, $z > 1$. Note that it contains the pole at $\nu = -1$ explicitly; this formula is better approximated by Gaussian quadrature near $\nu = -1$ than the earlier formula, for this reason as well as for the reason that the $(1-y^2)^\nu$ is replaced by $(1-y^2)^{1+\nu}$.

This formula was used for $z > 1.05$, $\text{Re } \nu > -.9$; for $z > 1.05$, $\text{Re } \nu < -.9$, the recursion relation⁴⁹⁾ for Q_ν was used, and formula (AI.12) was used to calculate $Q_{\nu+1}$ and $Q_{\nu+2}$:

$$(1+\nu)Q_\nu(z) = (2\nu+3)z Q_{\nu+1}(z) - (\nu+2) Q_{\nu+2}(z). \quad (\text{AI.13})$$

For $1 < z < 1.05$, an asymptotic expression was used for $Q_\nu(z)$. The basic formula is⁵⁰⁾

$$Q_\nu(z) = P_\nu(z) \left[\frac{1}{2} \ln \left(\frac{z+1}{z-1} \right) - \gamma_E - \psi(1+\nu) \right] - \frac{\sin \pi \nu}{\pi} \sum_{\ell=1}^{\infty} \frac{\Gamma(\ell-\nu)\Gamma(\ell+\nu+1)[\gamma_E + \psi(\ell+1)]}{(\ell!)^2} \left(\frac{1-z}{2} \right)^\ell \quad (\text{AI.14})$$

Here $\Gamma(t)$ is the usual gamma function; $\psi(t)$ is the usual psi function,

$$\psi(t) = \frac{d}{dt} [\ln \Gamma(t)] \quad (\text{AI.15})$$

and $\gamma_E = 0.5772156649\dots$ $P_\nu(z)$ is given by the hypergeometric series⁵¹⁾

$$P_\nu(z) = F(1 + \nu, -\nu; 1; \frac{1-z}{2}). \quad (\text{AI.16})$$

The gamma-function obeys the relations⁵²⁾

$$\begin{aligned} \Gamma(1+t) &= t \Gamma(t), \\ \Gamma(t) \Gamma(1-t) &= \pi \csc \pi t. \end{aligned} \quad (\text{AI.17})$$

Thus

$$\begin{aligned} \Gamma(\ell - \nu) \Gamma(\ell + 1 + \nu) &= [(\ell - 1 - \nu)(\ell - 2 - \nu)\dots(-\nu)(\ell + \nu)(\ell - 1 + \nu)\dots \\ &\quad (1 + \nu)] \pi \csc \pi \nu \equiv \eta(\ell) \pi \csc \pi \nu, \end{aligned} \quad (\text{AI.18})$$

which defines $\eta(\ell)$, except that $\eta(0) = 1$. Using these relations, the formula for Q_ν can be written

$$Q_\nu(z) = \sum_{\ell=0}^{\infty} \frac{\eta(\ell)}{(\ell!)^2} \left(\frac{1-z}{2}\right)^\ell \left[\frac{1}{2} \ln \left(\frac{z+1}{z-1}\right) + \psi(1+\ell) - \psi(1+\nu) \right]. \quad (\text{AI.19})$$

Since $(1-z)/2$ is no larger than .025 in the range where this formula is to be used, only a few terms of the infinite sum are necessary; five terms were used.

$\psi(1+\ell)$ is easily calculated from its initial value and its recurrence relation⁵³⁾:

$$\begin{aligned} \psi(1) + \gamma_E &= 0 \\ \psi(1+t) &= \psi(t) + \frac{1}{t}. \end{aligned} \quad (\text{AI.20})$$

The basic formula for calculating $\psi(1 + \nu)$ is⁵⁴⁾

$$\psi(1 + \nu) + \gamma_E = \nu \sum_{n=1}^{\infty} \frac{1}{n(n + \nu)} , \quad (\text{AI.21})$$

which is valid except for ν a negative integer. The infinite series was altered for quicker convergence by repeated use of the following technique:

$$\sum \frac{1}{n(n + \nu)} = \sum \frac{1}{n^2} \left[1 - \frac{\nu}{n + \nu} \right] = \zeta(2) - \nu \sum \frac{1}{n^2(n + \nu)} . \quad (\text{AI.22})$$

Here, ζ is the Riemann zeta function. The resultant formula, for N repetitions is

$$\psi(1 + \nu) + \gamma_E = \sum_{m=1}^N (-1)^{1+m} \nu^m \zeta(m+1) - (-\nu)^{N+1} \sum_{n=1}^{\infty} \frac{1}{n^{N+1}(n + \nu)} . \quad (\text{AI.23})$$

This formula was used, with $N = 9$; ten terms of the infinite series were kept.

The resultant accuracy for $Q_\nu(z)$ was around six figures, for tabulated values of ν , $\nu = -1/2, 0, 1/2, 1, \dots$. The program was also checked by testing to see if the recursion relation was satisfied.

APPENDIX 2 SOLVING THE INTEGRAL EQUATIONS

The integral equations derived are distinguished by their non-linearity and complexity. Very few techniques for solving such formidable equations have been discovered; foremost among these techniques are iteration and guessing the answer. Both were used in finding solutions.

The iterative method is simply stated: if the integral equation to be solved is written

$$\text{Im } \alpha(s) = F(s; \text{Im } \alpha(s)) \quad (\text{AII.24})$$

then "straight iteration" gives for the n^{th} guess at $\text{Im } \alpha$

$$\text{Im } \alpha_n(s) = F(s; \text{Im } \alpha_{n-1}(s)). \quad (\text{AII.25})$$

The procedure indicated is the following: calculate $\text{Re } \alpha_{n-1}(s)$ from $\text{Im } \alpha_{n-1}(s)$ by using the dispersion relation, and then insert the resulting $\alpha_{n-1}(s)$ into the right-hand side of the integral equation to be solved. Evaluating this right-hand side gives the next guess for $\text{Im } \alpha(s)$, $\text{Im } \alpha_n(s)$. The solution to the integral equation is the limiting function of the sequence $\{\text{Im } \alpha_n(s)\}$, and $\text{Re } \alpha(s)$ is given by a dispersion integral over this limiting function.

The advantage of the iterative method is the simplicity of the solving algorithm; the disadvantage is that iteration frequently fails to converge, especially because of instability. An example is the equation $x = e^{-x}$. The iterative procedure

$$x_n = e^{-x_{n-1}}, \quad (\text{AII.26})$$

converges to the correct answer $x = .567$; with an initial guess $x_0 = 0$, x_{12} and the succeeding x_n 's were each equal to $.567$; starting with the much better guess $x_0 = .5$, $x_8 = .567$. However, the alternative expression $x = -\ln x$, with the same solution, fails to converge with the obvious iterative procedure

$$x_n = -\ln x_{n-1}. \quad (\text{AII.27})$$

There are, of course, an infinite number of possible iterative procedures for any equation of the form $x = f(x)$; a valid procedure is, for example,

$$x_n = f(x) + G(f(x) - x) \quad (\text{AII.28})$$

for any function G with $G(0) = 0$. Choice of a proper iterative procedure can increase the rapidity of an already-convergent procedure or cause a divergent one to converge. For example,

$$x_n = \frac{1}{2} (x_{n-1} + e^{-x_{n-1}}) \quad (\text{AII.29})$$

converges from $x_0 = 0$ to $x_5 = .567$, in 5 instead of 12 steps; the procedure

$$x_n = \frac{1}{2} (x_{n-1} - \ln x_{n-1}) \quad (\text{AII.30})$$

converges instead of diverging, from $x_0 = .5$ to $x_6 = .567$.

One more complexity is that of the "radius of convergence" of any given procedure; equation (AII.29) will converge for any initial guess x_0 , but equation (AII.30) converges only for $x_0 > 0$.

The subtracted integral equations that were solved for this

paper were relatively easy to solve. The iteration scheme

$$\text{Im } \alpha_n(s) = \frac{1}{2} \{ \text{Im } \alpha_{n-1}(s) + F(s; \text{Im } \alpha_{n-1}(s)) \} \quad (\text{AII.31})$$

was found to converge for both the relativistic and non-relativistic calculations, if the initial guess was close enough. The potential theory iteration converged much more slowly, but convergence could be speeded by adjusting the iteration procedure: the difference $\Delta \alpha_n(s)$ between the n^{th} iteration and the solution was found to oscillate with a period of about 15 iterations, while decaying very slowly. Using the average over 15 iterations as the next input produced rapid convergence thereafter. The rho-trajectory calculations converged well, with or without the modifications, after trial-and-error produced a reasonable guess for $\text{Im } \alpha_0(s)$.

The solving of the unsubtracted potential theory equation was quite a contrast, for no iteration procedure was found to work, although many, of various degrees of complexity, were tried. The procedure (AII.31) was found to slow down the divergence enough to get two or three iterations done before $\text{Im } \alpha_n(s)$ became clearly wrong. The solution was finally obtained by trial-and-error methods, by inspecting a diverging set of iterations graphically and then adjusting the guess for the next input $\text{Im } \alpha_0(s)$ accordingly.

FIGURE CAPTIONS

Graphs - Potential Theory

All are for $V(r) = - 1.8 e^{-r}/r$, $m = 1/2$.

Figure 1: $\text{Im } \alpha(s)$: - true; --- calculated from Cheng representation (unsubtracted); calculated from Cheng representation (subtracted at $s_0 = .4$).

Figure 2: $\text{Re } \alpha(s)$: same key at Fig. 1.

Figure 3: $\text{Im } \alpha(s)$: - true; --- calculated from modified Cheng representation (unsubtracted); calculated from modified Cheng representation (subtracted at $s_0 = .1$).

Figure 4: $\text{Re } \alpha(s)$: Same key as Fig. 2.

Graphs - Rho Trajectory

Figures 5 through 9 are subtracted at $s_0 = m_\rho^2$; Figures 10 through 12 are subtracted at $s_0 = \infty$; s is measured in units of m_π^2 .

Figure 5: $\text{Im } \alpha_\rho(s)$; $\text{Im } \alpha_\rho(m_\rho^2) = .025$.

Figure 6: $\text{Re } \alpha_\rho(s)$; as in Fig. 5.

Figure 7: $\text{Im } \alpha_\rho(s)$; $\Gamma = 200$ MeV.

Figure 8: $\text{Re } \alpha_\rho(s)$; $\Gamma = 200$ MeV, s positive.

Figure 9: $\text{Re } \alpha_\rho(s)$; $\Gamma = 200$ MeV, s negative.

Figure 10: $\text{Im } \alpha_\rho(s)$; $\text{Re } \alpha_\rho(\infty) = -1$.

Figure 11: $\text{Re } \alpha_\rho(s)$; $\text{Re } \alpha_\rho(\infty) = -1$, s greater than 4.

Figure 12: $\text{Re } \alpha_\rho(s)$; $\text{Re } \alpha_\rho(\infty) = -1$, s less than 4.

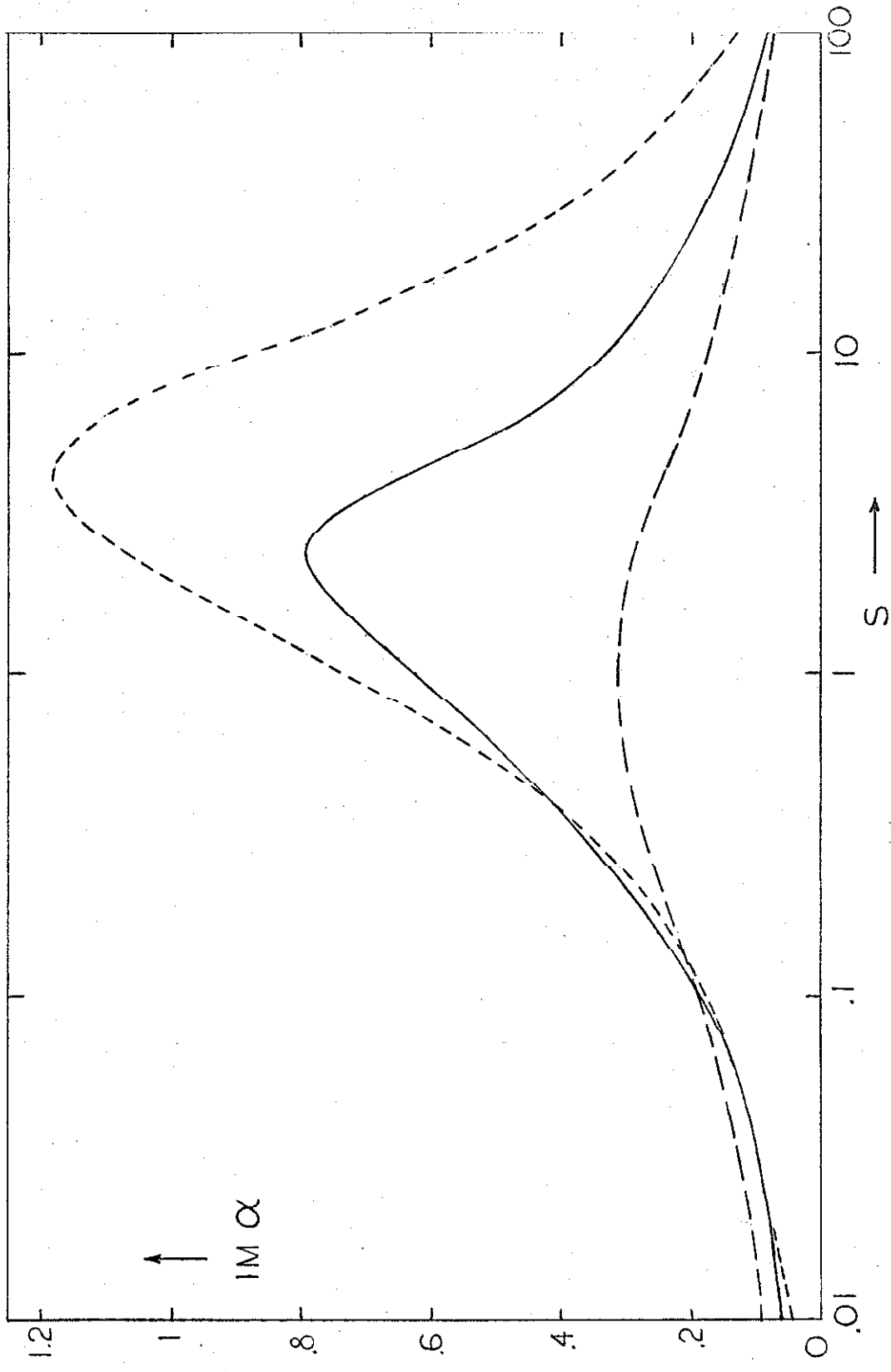


Figure 1

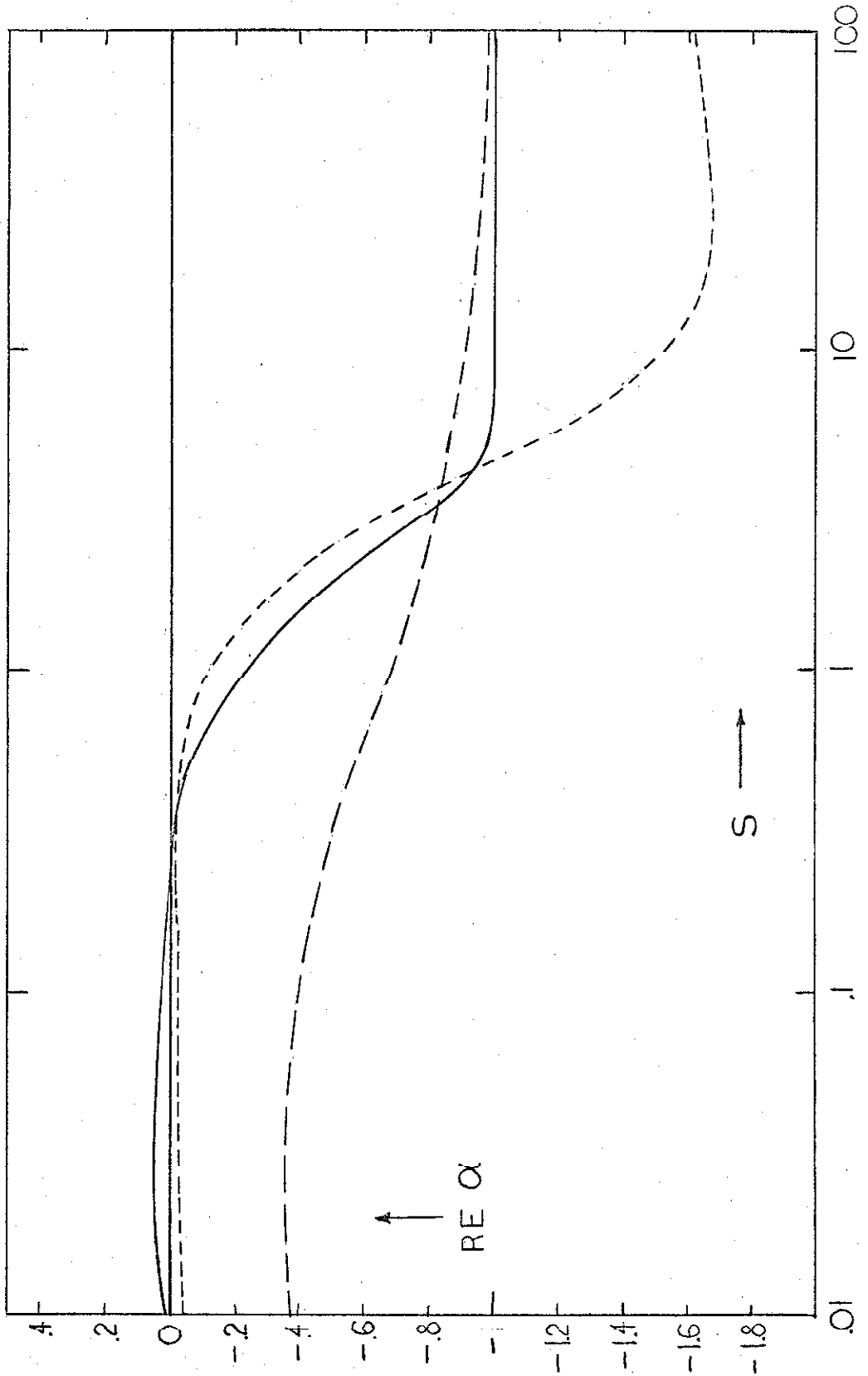


Figure 2

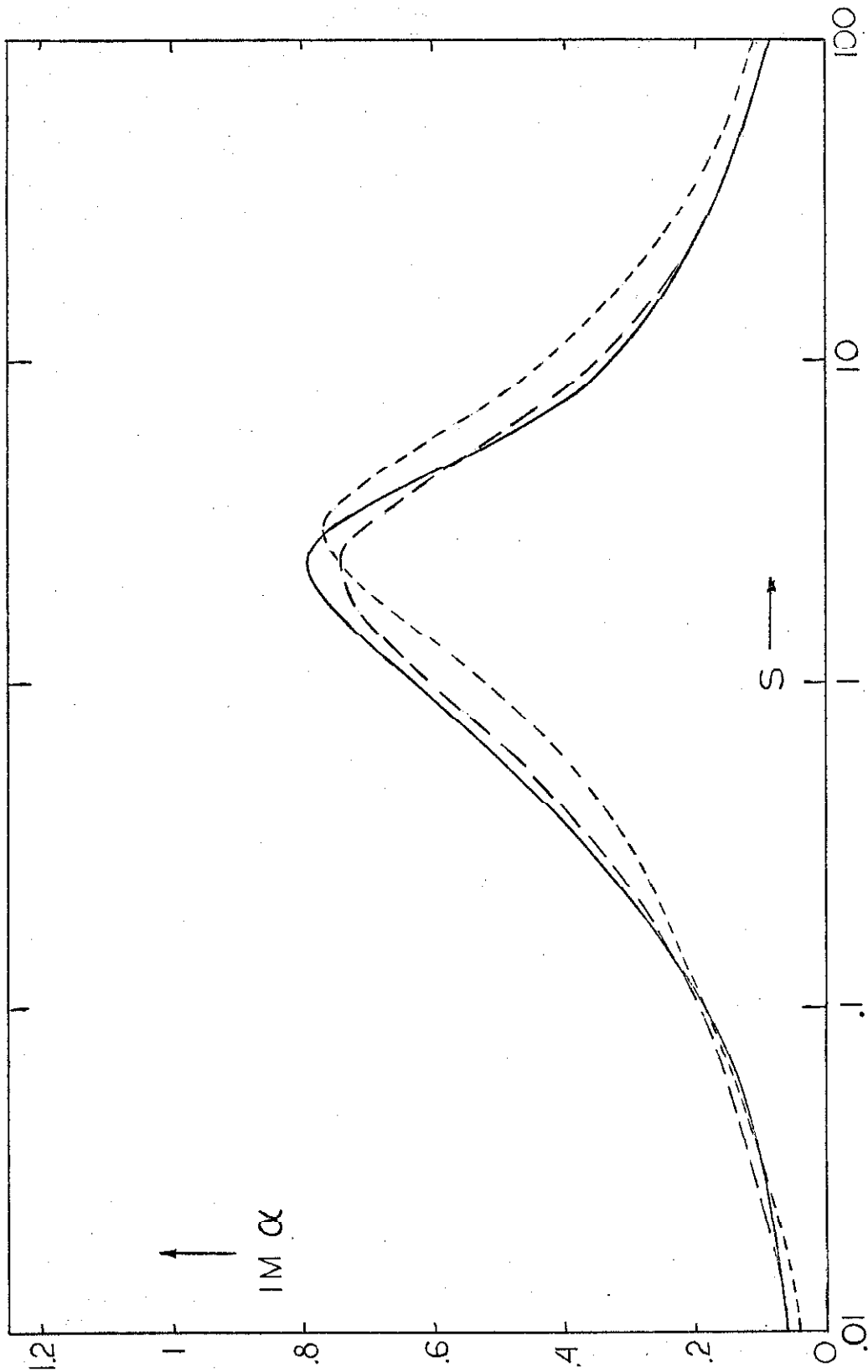


Figure 3

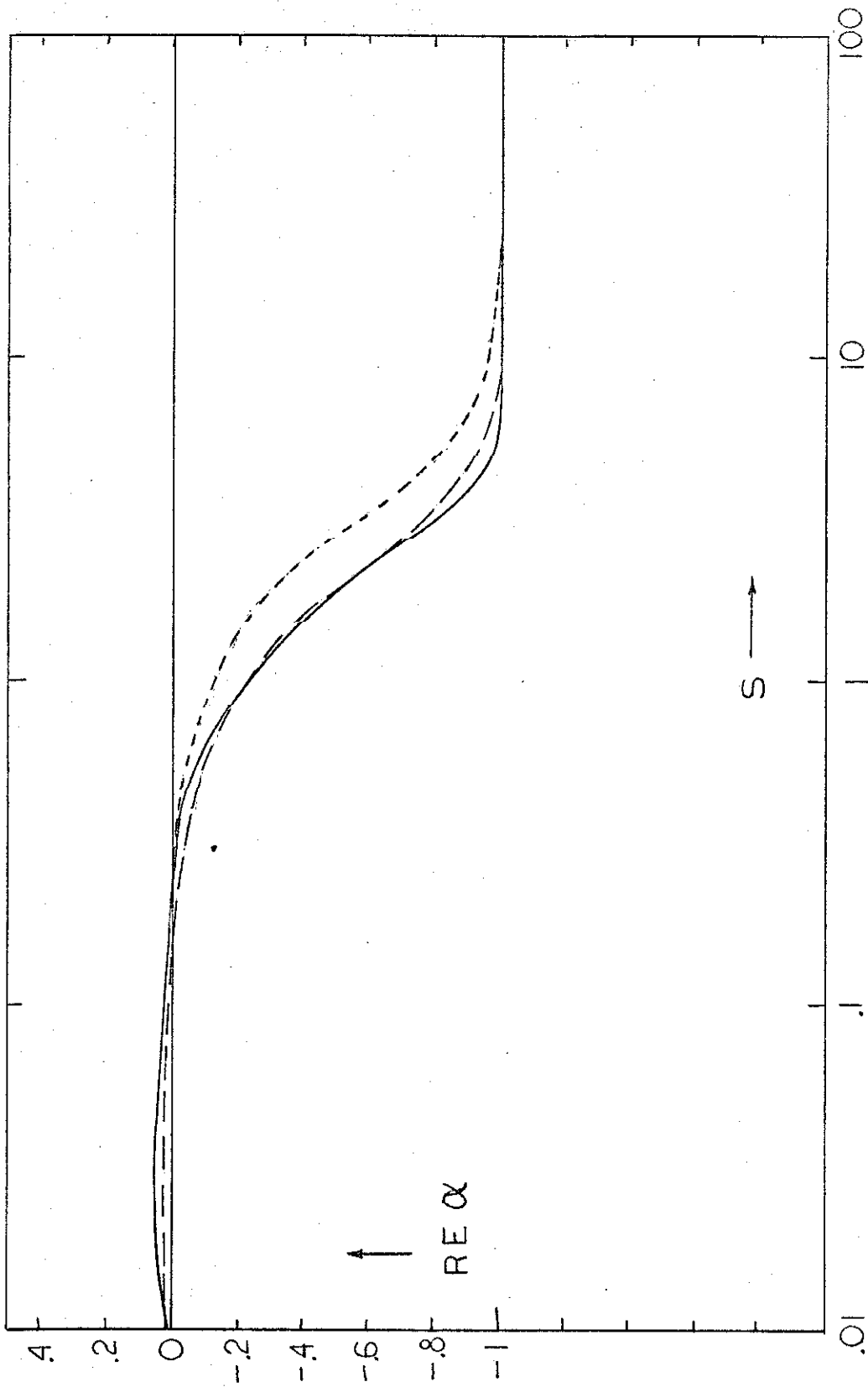


Figure 4

Figure 5

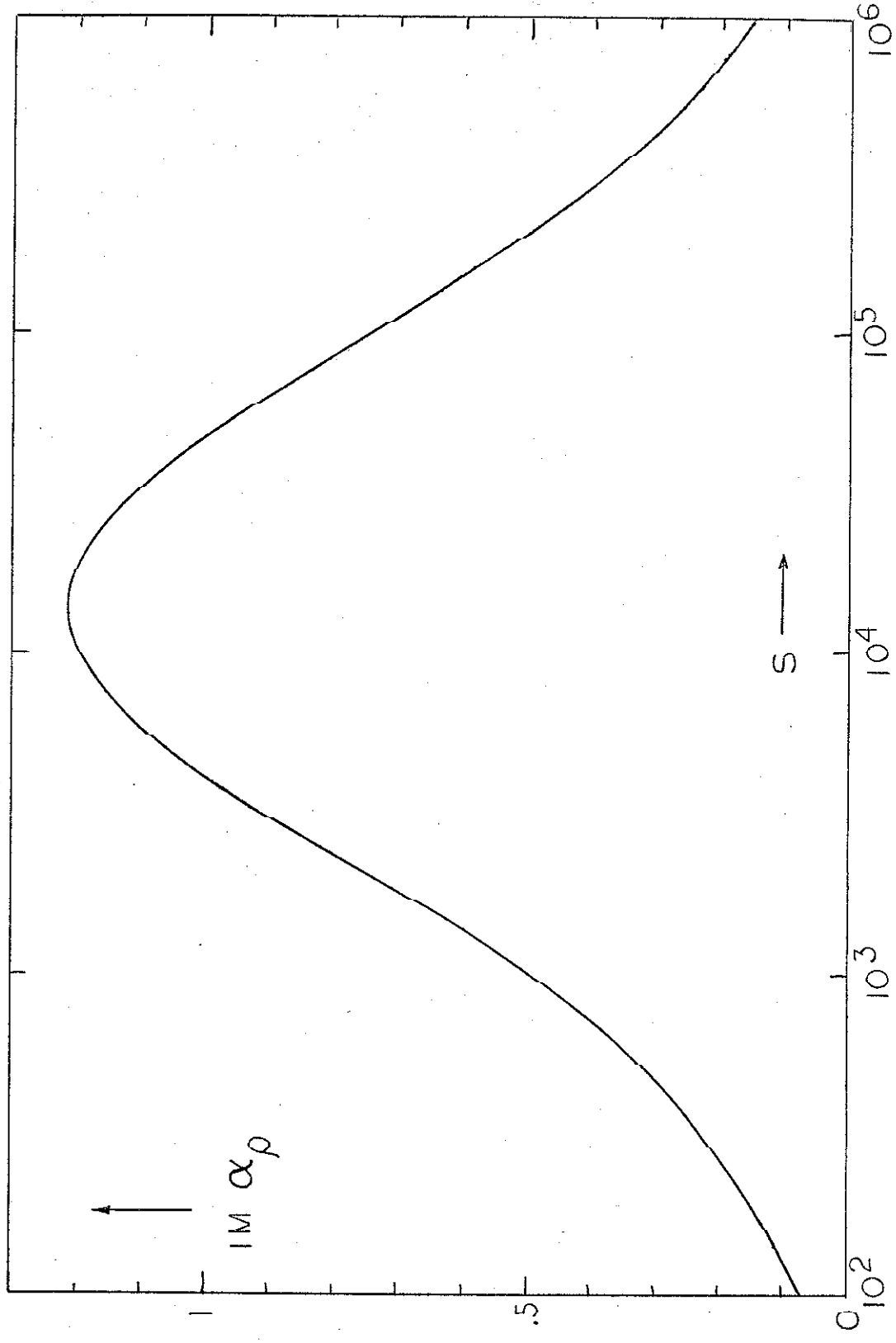
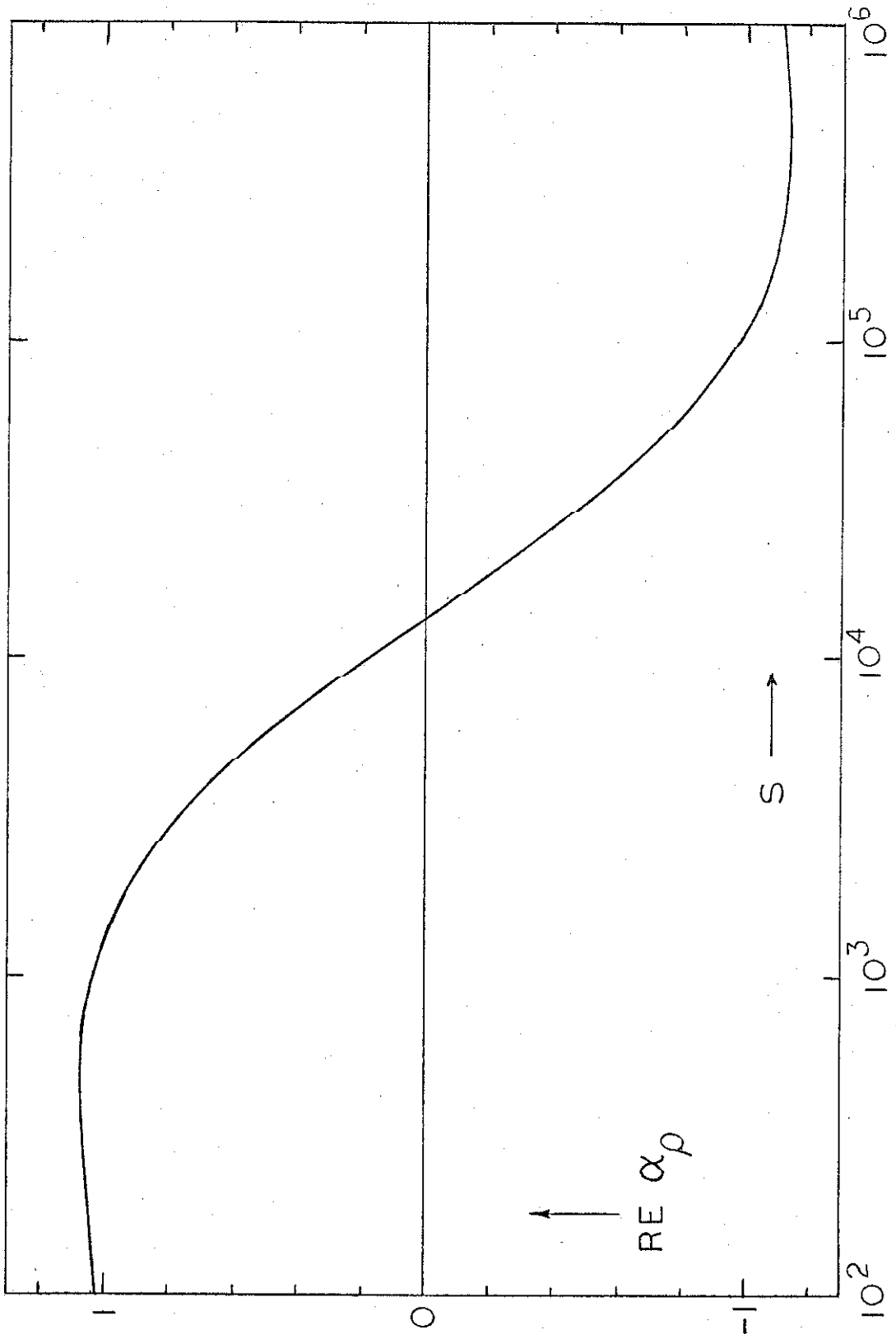
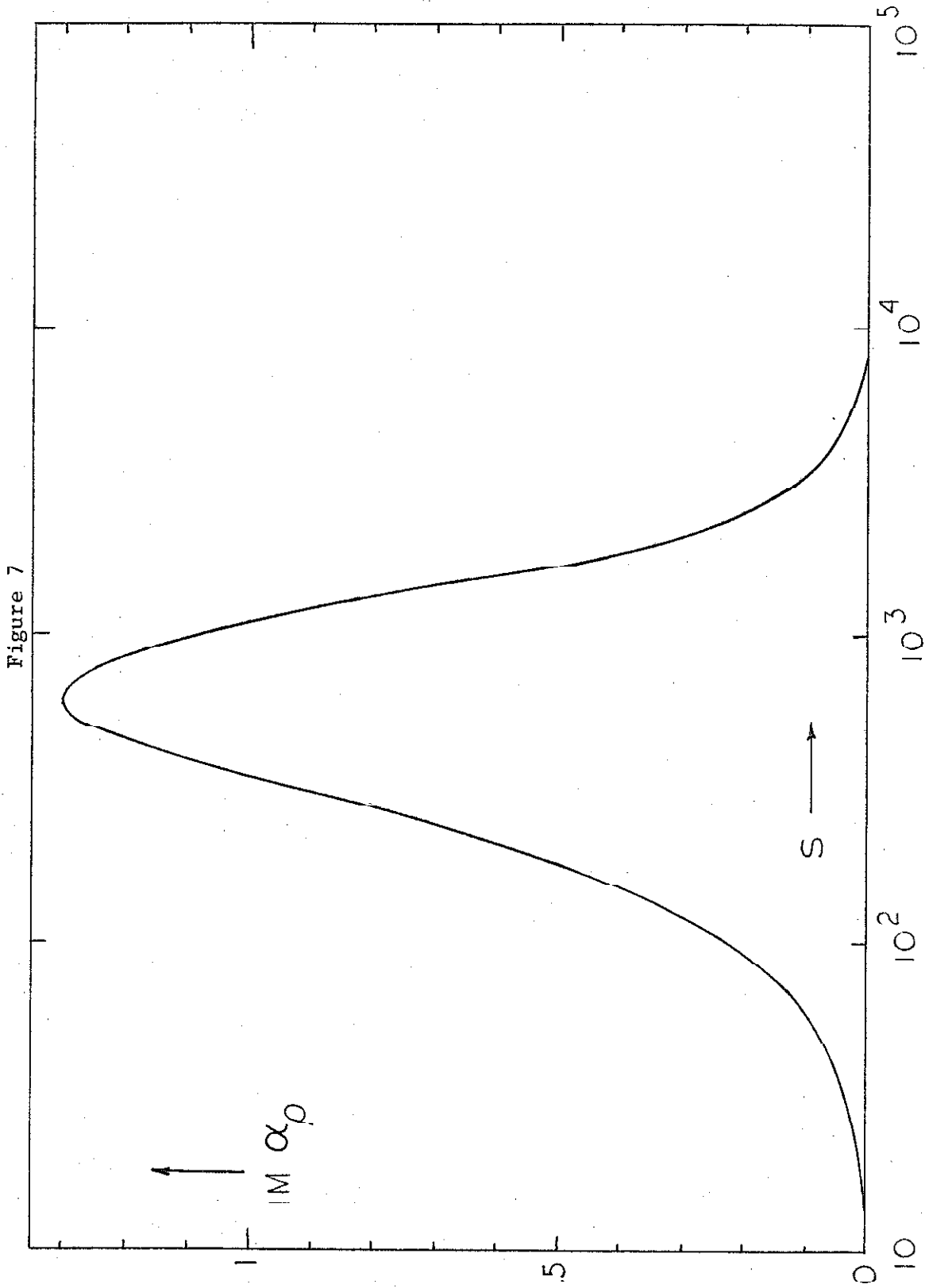


Figure 6





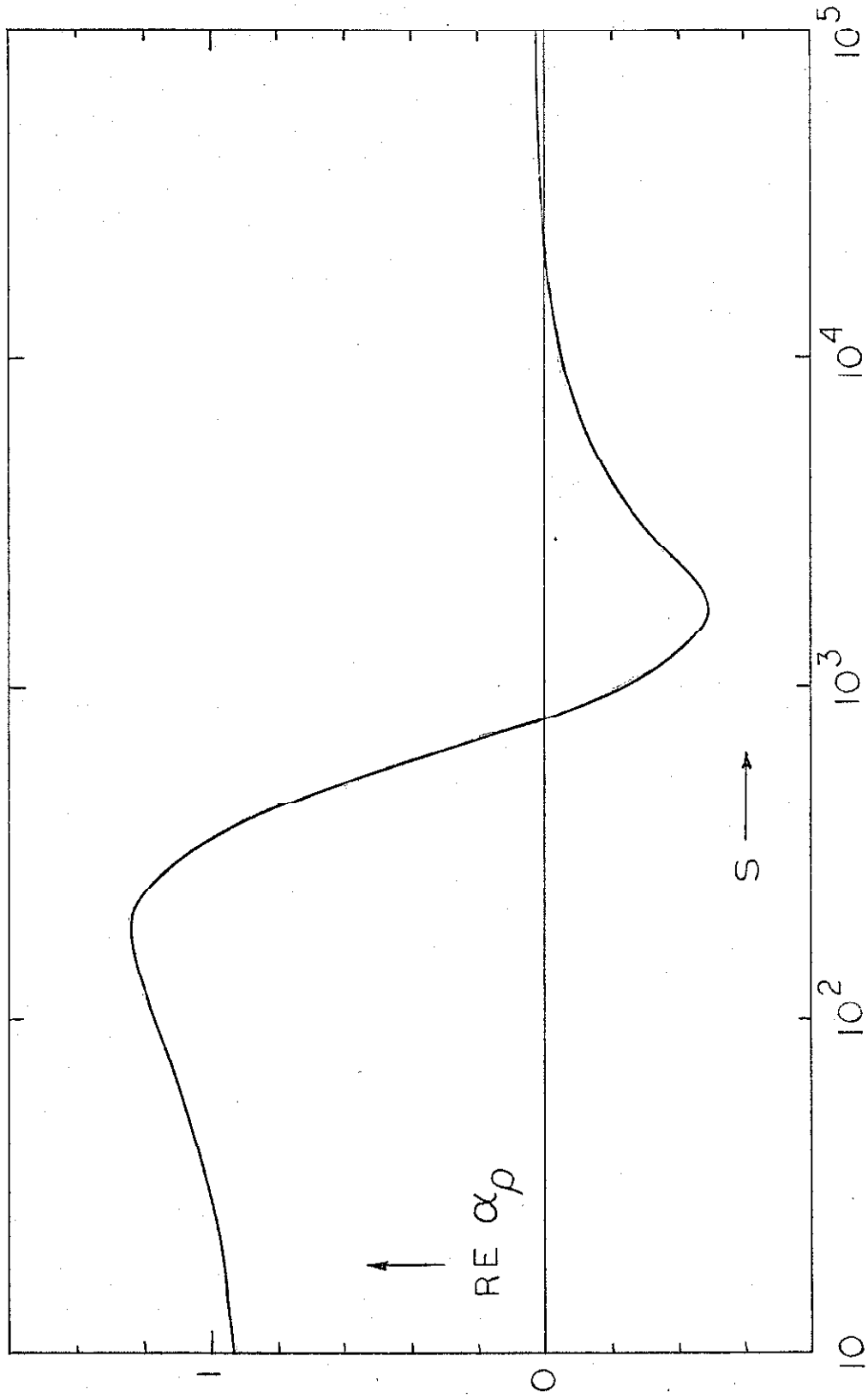


Figure 8

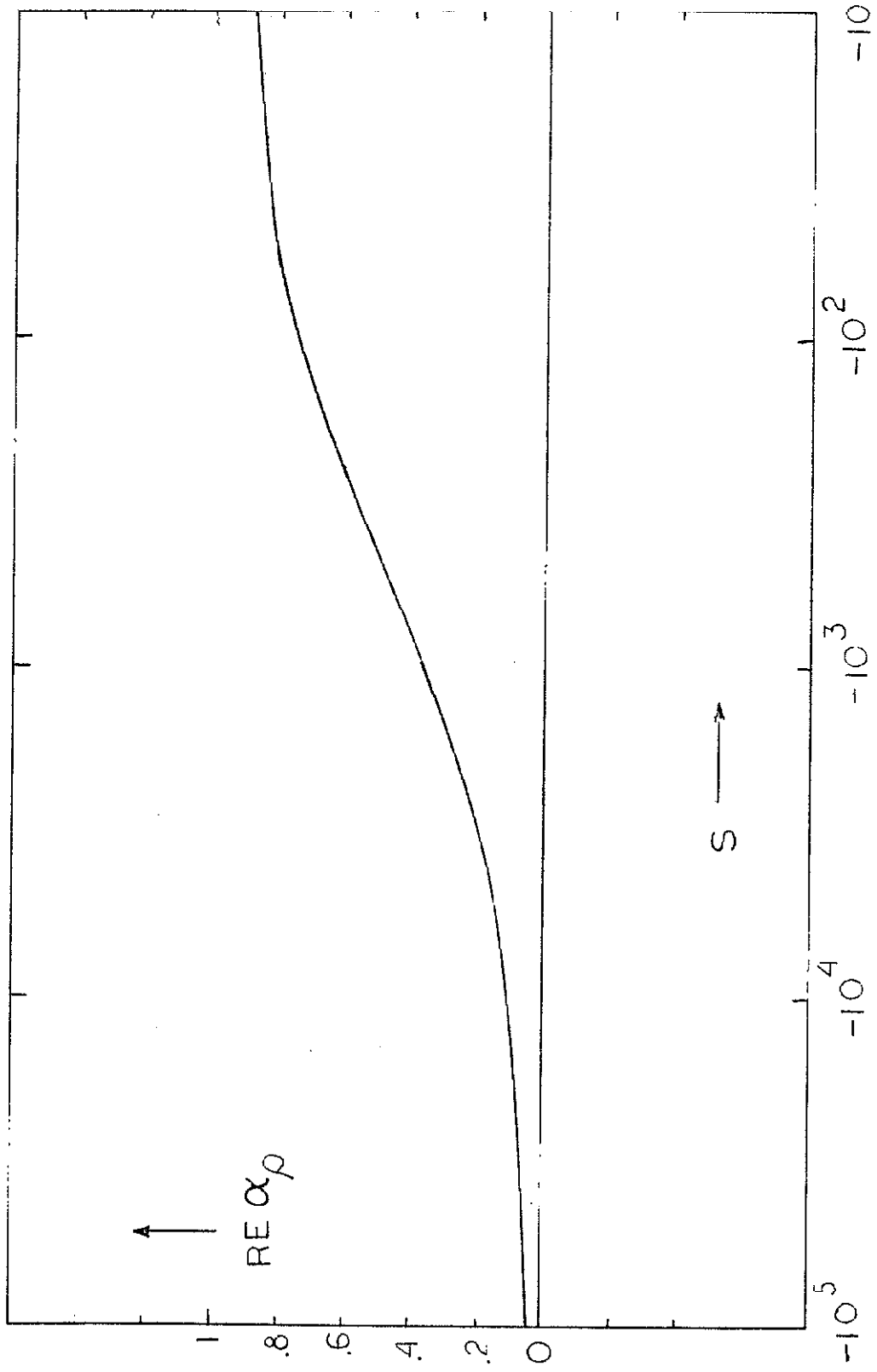


Figure 9

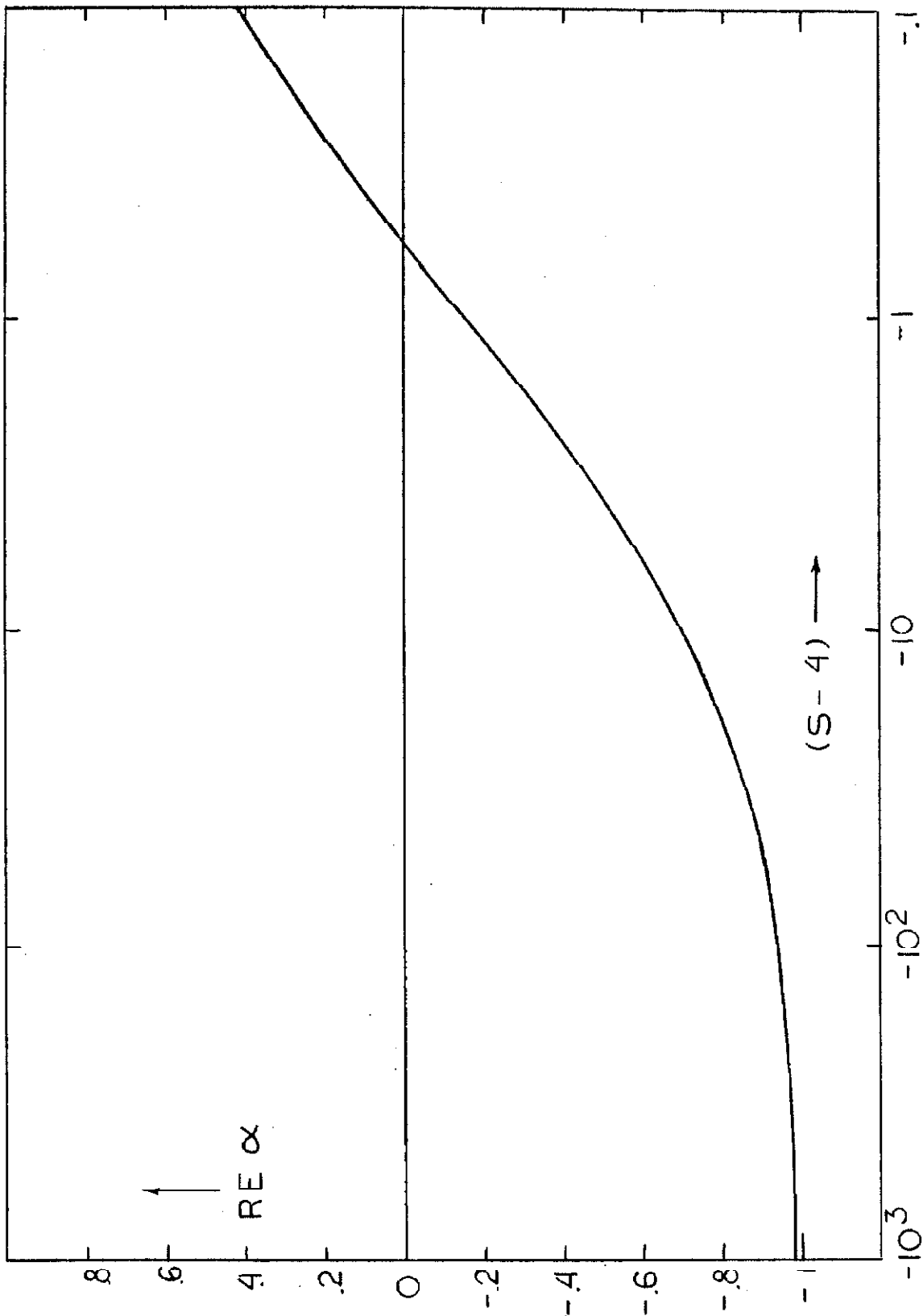


Figure 12

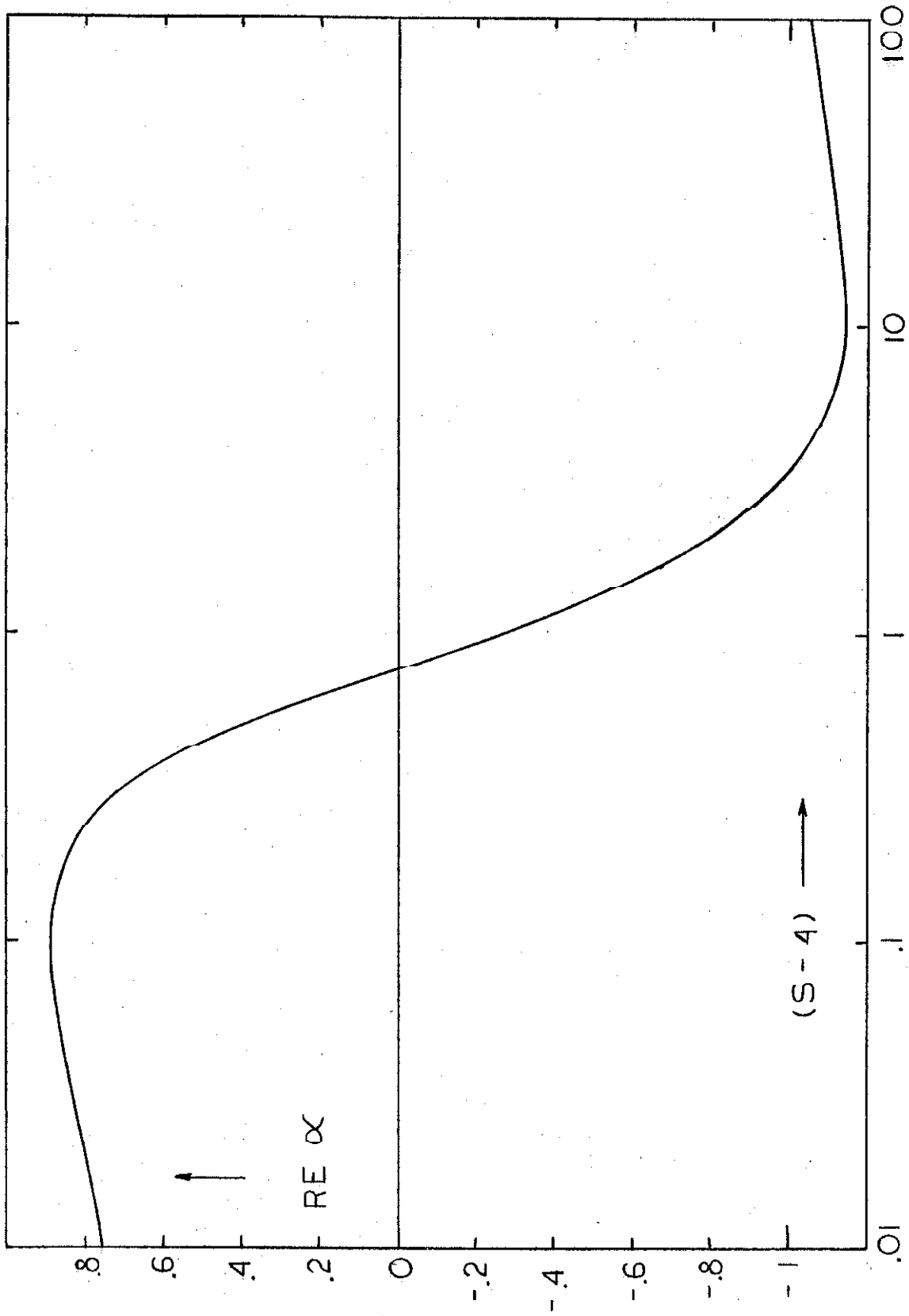


Figure 11

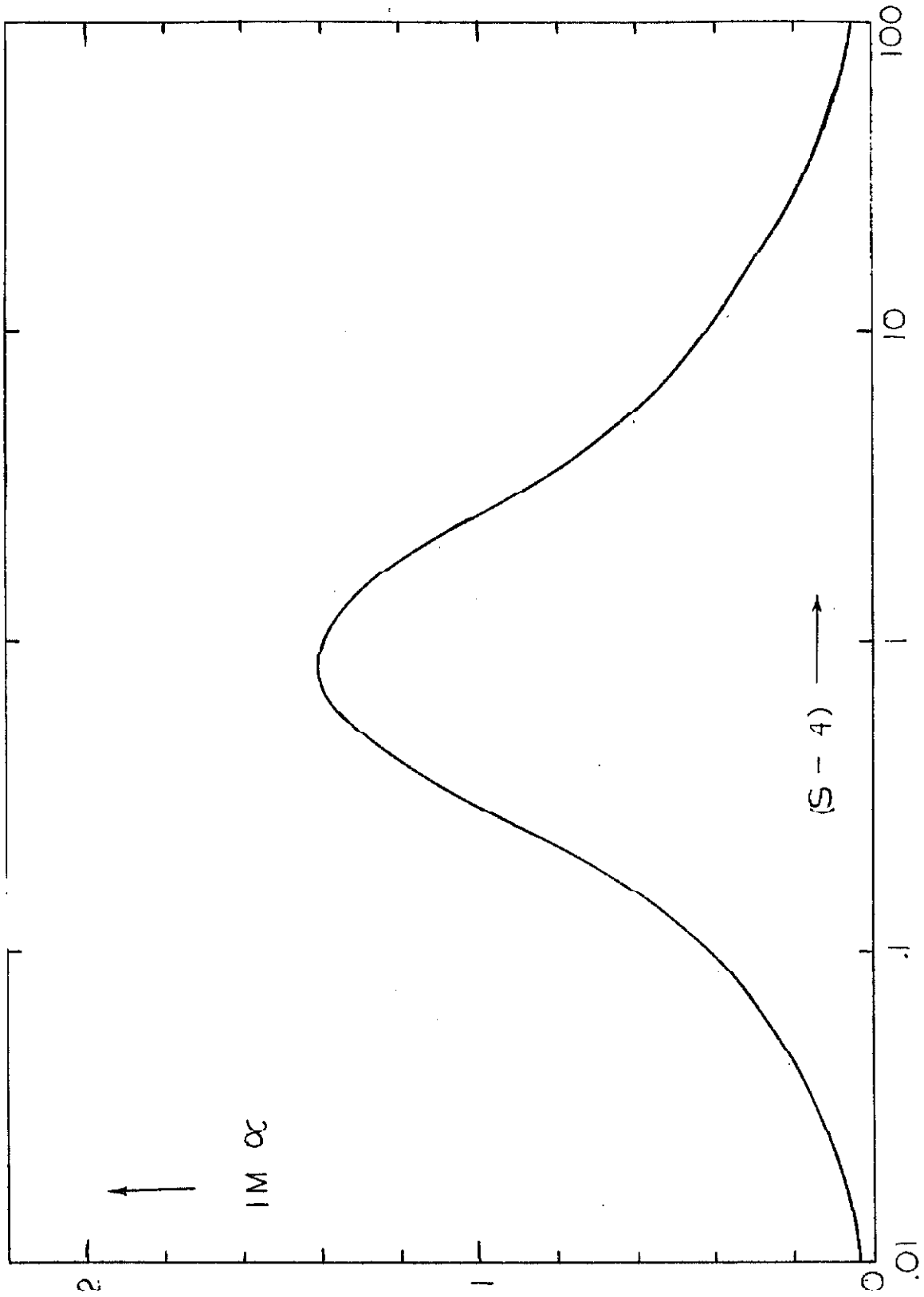


Figure 10

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50. Reference 48, p. 149.
51. Reference 48, p. 148.
52. Reference 32, p. 256.
53. Reference 32, p. 258.
54. Reference 32, p. 259.