MULTIPLE SCATTERING

OF

ACOUSTICAL WAVES

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ABSTRACT

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The general theory of the multiple scattering of acoustical waves by a random distribution of isotropic point scatterers is considered. Configurational averages are taken of the equations of multiple scattering and integral equations governing these configurational averages are obtained; the physical consequences of these equations are examined in detail. A complete theoretical picture is obtained of the propagation of the coherent and incoherent radiation and of the connections between the coherent and incoherent contributions to the average sound intensity and current.

The problem of the transmission of sound from a plane sound source into a scattering half-space is studied; numerical results are presented for the average sound intensity and current. The reflection of an incident plane wave, inclined at an arbitrary angle to a scattering half-space is considered; an expression for the reflection coefficient, including both the coherent and incoherent reflection of sound, is obtained. The foregoing results are then applied to sound propagation in a liquid containing a large number of small gas bubbles.

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I. INTRODUCTION

The general theory of the multiple scattering of particles by a random distribution of scatterers has been extensively considered in recent years with particular application to molecular transport in gases, neutron and gamma-ray transport in matter, radiation transfer in stars, and to a number of other physical phenomena. These problems are usually studied with some variation of the Boltzmann integro-differential equation describing transport processes. This formulation is merely the expression of conservation of particles in phase space; hence the treatment is classical, with no account taken of the wave nature of the particles or photons. Such a theory would be expected to be valid only if the wave length of the particles is much smaller than the average distance of separation between the scatterers.

There is also a large number of problems of multiple scattering in which the wave length is comparable with the average scatterer separation; some examples of this latter type of problem are acoustic wave propagation in bubbly water, elastic wave propagation in polycrystalline materials, and the scattering of electrons or x-rays by the nuclei of liquids or amorphous solids. Any treatment of these problems must include the reflection, refraction, and interference phenomena that are characteristic of wave problems. Hence it must be based on the wave equation, rather than on the simple conservation statement leading to the Boltzmann equation.

The first systematic treatment based on the wave equation was made by Foldy[1], who considered the multiple scattering of scalar waves by a random collection of isotropic point scatterers. This work was a generalization of his previous study[2] of the propagation of sound in water containing a large number of small air bubbles. Foldy's unique contribution was the introduction of the concept of "configurational" averaging of relevant physical quantities by defining a joint probability distribution for the occurrence of a particular scatterer configuration. By averaging the equations of multiple scattering over the statistical ensemble of scatterer configurations, Foldy was able to derive integral equations governing these configurational averages. This procedure was later generalized by Lax[3] to treat the multiple scattering of quantum-mechanical waves by point scatterers having quite general scattering characteristics. Application of the configurational averaging technique has also been used by Twersky[4] to study the scattering and reflection of acoustic waves by a rough surface and by Waterman and Truell[5] to treat scattering regions having non-vanishing dimensions.

Through the use of configurational averaging, the multiple scattering problem admits the natural decomposition into the separate consideration of the "coherent" and the "incoherent" radiation. The sound intensity (or probability density in the quantum-mechanical case) is proportional to the square of the absolute value of the complex velocity potential (or wave function) $\psi\psi^*$. We denote the configurational average of this quantity by $\langle \psi\psi^* \rangle$; this is not, in general, equal to the square of the absolute value of the configurational average

$\langle\psi\rangle\langle\psi\rangle^*$. This latter quantity is called the "coherent" component, and the difference $\langle\psi\psi^*\rangle - \langle\psi\rangle\langle\psi\rangle^*$ the incoherent component, of the mean square wave. The solution of the coherent problem bridges the gap between molecular and continuum physics since it may be shown that the coherent wave $\langle\psi\rangle$ satisfies the wave equation with a complex propagation constant; thus a collection of discrete scatterers imbedded in a matrix medium may be replaced by a continuous absorbing medium having a propagation constant that depends, in general, on position. The coherent wave, being governed by a wave equation, displays the phenomena of refraction and specular reflection. The incoherent radiation, on the other hand, arises as a result of the statistical superposition of the scattered waves due to the random nature of the scatterer distribution; it is governed by an equation that is similar to the Boltzmann integral equation describing the transport of particles.

Although there has been a considerable amount of work[6] since Foldy's paper, the emphasis has been on the coherent radiation. Foldy's treatment of the coherent wave for monopole scatterers has served as a model for those attempting to treat scatterers having more general scattering characteristics. Little attention, however, has been given to the incoherent radiation. Although Waterman and Truell[5] have pointed out that the equations describing the intensity $\langle|\psi|^2\rangle$ and the current $\langle\psi^*\nabla\psi - \psi\nabla\psi^*\rangle$ merit further investigation, apparently they have not been further considered on a fundamental basis.

In this work we again consider the physical situation studied by Foldy - - the multiple scattering of scalar waves by a random distribution of isotropic point scatterers; emphasis is given to the special case of acoustic waves. Our objective is to establish the general theory of multiple scattering of waves and to apply this theory to some acoustical problems of interest.

In Chapter II we review the necessary acoustical and statistical definitions on which our theory is based. The definitions of the configurational and time averages of a physical quantity are compared. In the next chapter, configurational averages are taken of the fundamental equations of multiple scattering in order to obtain integral equations governing the configurational averages of relevant physical quantities. The first three sections of Chapter III follow the work of Foldy[1]; the latter section, which provides the basis for our consistent treatment of the incoherent radiation, contains new results. These integral equations are then reduced to more workable forms in Chapter IV, and their physical consequences are discussed. The treatment of the coherent wave is that given by Foldy. However, as shown in Appendix A, his analysis of the incoherent radiation is incorrect. We present a consistent formulation based on fundamental conservation relations for the coherent and incoherent radiation. This provides a very unified picture of the propagation of the coherent and incoherent radiation and the connection between the coherent and incoherent contributions to the average sound intensity and current. At the end of Chapter IV we consider the special case in which the scatterer density is uniform throughout all space. We find that the incoherent contribution to the average sound intensity satisfies the Boltzmann integral equation describing the transport of particles.

As a specific example of the general theory, we consider in Chapter V the problem of the transmission of sound from a plane sound source into a half-space that is filled with a uniform random distribution of scatterers. Numerical results are presented for the average sound intensity and current at an arbitrary distance from the source. As a further acoustical application, the reflection of an incident plane wave, inclined at an arbitrary angle to a scattering half-space, is studied in Chapter VI. An expression for the reflection coefficient, including both the coherent and incoherent radiation, is obtained. Finally, in the last chapter the foregoing results are applied to sound propagation in a liquid containing a large number of small gas bubbles. Numerical results are given for the variation with frequency of the cross section ratio, phase velocity, attenuation coefficient, and reflection coefficient for two different bubble radii.

II. PRELIMINARIES

A. Acoustical Preliminaries

Let us first consider a homogeneous, ideal fluid having density ρ_o and sound speed c_o; the linearized equations describing the propagation of sound waves in such a medium are[7]

$$\frac{\partial p}{\partial t} + \rho_o c_o^2 \nabla \cdot \vec{v} = 0 \quad , \tag{2.1}$$

$$\rho_o \frac{\partial \vec{v}}{\partial t} + \nabla p = 0 \quad , \tag{2.2}$$

where $\vec{v}(\vec{r}, t)$ is the velocity of the fluid and $p(\vec{r}, t)$ is the perturbation in pressure about the steady-state value p_o. Upon introduction of a velocity potential $\phi(\vec{r}, t)$, defined such that

$$\vec{v} = \nabla \phi \quad , \tag{2.3}$$

Eqs. (2.1) and (2.2) yield the fundamental acoustical equations

$$p = -\rho_o \frac{\partial \phi}{\partial t} \quad , \tag{2.4}$$

$$\frac{\partial^2 \phi}{\partial t^2} - c_o^2 \nabla^2 \phi = 0 \quad . \tag{2.5}$$

Upon multiplication of Eq. (2.1) by p and Eq. (2.2) by $\vec{v}$ and adding, we obtain

$$\frac{\partial}{\partial t}\left(\frac{p^2}{2\rho_o c_o^2} + \frac{\rho_o v^2}{2} \right) + \nabla \cdot (p\vec{v}) = 0 \quad . \tag{2.6}$$

This equation is simply the statement of conservation of sound energy,

$$\frac{\partial E}{\partial t} + \mathrm{div}\, \vec{q} = 0 \quad , \tag{2.7}$$

where the sound energy density $E(\vec{r},t)$ and the sound energy flux density $\vec{q}(\vec{r},t)$ are given by

$$E = \frac{p^2}{2\rho_o c_o^2} + \frac{\rho_o v^2}{2} \quad , \tag{2.8}$$

$$\vec{q} = p\vec{v} \quad . \tag{2.9}$$

For a travelling plane wave, having a velocity potential of the form

$$\phi(x,t) = f(x-c_o t) \quad , \tag{2.10}$$

we have

$$v = \frac{\partial\phi}{\partial x} = f'(x-c_o t) = -\frac{1}{c_o}\frac{\partial\phi}{\partial t} = \frac{p}{\rho_o c} \quad , \tag{2.11}$$

and hence, for this special case, the sound energy density assumes the simpler form:

$$E = \frac{p^2}{\rho_o c_o^2} \quad . \tag{2.12}$$

In our study of the multiple scattering of waves, we shall assume that the sound field is harmonic with a fixed frequency ω. Such an acoustic field may be expressed in terms of a complex velocity potential $\psi(\vec{r})$, as follows:

$$\phi(\vec{r},t) = \mathrm{Re}\{\psi(\vec{r})e^{-i\omega t}\} \quad , \tag{2.13}$$

where Re{ } indicates that the real part of the complex expression is to be taken. Upon substitution of this representation into Eqs. (2.3) and (2.4), the time-dependent velocity and pressure fields are given by

$$\vec{v}(\vec{r},t) = \mathrm{Re}\{\nabla\psi(\vec{r})e^{-i\omega t}\} \quad , \tag{2.14}$$

$$p(\vec{r}, t) = \mathrm{Re}\{i\omega\rho_o \psi(\vec{r}) e^{-i\omega t}\} \quad , \tag{2.15}$$

while the wave equation (2.5) yields

$$(\nabla^2 + k_o^2)\psi(\vec{r}) = 0 \quad , \tag{2.16}$$

where the propagation constant k_o is given by

$$k_o = \frac{\omega}{c_o} \quad . \tag{2.17}$$

Of particular interest to our study will be quantities related to energy. Let us define the "sound intensity" $e(\vec{r})$ as the mean-square pressure divided by $\rho_o c_o^2$, i.e.

$$e(\vec{r}) = \frac{1}{\rho_o c_o^2} \frac{1}{T} \int_o^T p(\vec{r}, t)^2 \, dt \quad , \tag{2.18}$$

where $T = \frac{2\pi}{\omega}$ is the period of oscillation of the sound field. It is noted that for the special case of a harmonic plane wave, the sound intensity equals the mean sound energy density. We shall call the mean sound energy flux density the "sound energy current" $\vec{j}(\vec{r})$; by Eq. (2.9), this quantity is given by

$$\vec{j}(\vec{r}) = \frac{1}{T} \int_o^T p(\vec{r}, t) \vec{v}(\vec{r}, t) dt \quad . \tag{2.19}$$

If we denote the real and imaginary parts of $\psi(\vec{r})$ by $\hat{\psi}(\vec{r})$ and $\check{\psi}(\vec{r})$ respectively, Eqs. (2.14) and (2.15) may then be written as

$$\vec{v}(\vec{r}, t) = \nabla\hat{\psi}(\vec{r}) \cos\omega t + \nabla\check{\psi}(\vec{r}) \sin\omega t \quad , \tag{2.20}$$

$$p(\vec{r}, t) = \omega\rho_o [\hat{\psi}(\vec{r}) \sin\omega t - \check{\psi}(\vec{r}) \cos\omega t] \quad , \tag{2.21}$$

When these representations are substituted into the above definitions, and the integrations are carried out, it is found that the intensity and current may be simply expressed in terms of the complex velocity potential $\psi(\vec{r})$, as follows:

$$e(\vec{r}) = \frac{\rho_o \omega^2}{2c_o^2} [\hat{\psi}(\vec{r})^2 + \check{\psi}(\vec{r})^2] = \frac{\rho_o \omega^2}{2c_o^2} |\psi(\vec{r})|^2 \quad , \tag{2.22}$$

$$\vec{j}(\vec{r}) = \frac{\rho_o \omega}{2} [\hat{\psi}(\vec{r})\nabla\check{\psi}(\vec{r}) - \check{\psi}(\vec{r})\nabla\hat{\psi}(\vec{r})] = \frac{\rho_o \omega}{4i} [\psi^*(\vec{r})\nabla\psi(\vec{r}) - \psi(\vec{r})\nabla\psi^*(\vec{r})] \quad , \tag{2.23}$$

where the star indicates the complex conjugate of the quantity. If the multiplicative physical constants are ignored, these expressions are identical to those for the probability density and probability density current of quantum mechanics[8].

The effect of small viscosity in the fluid may be easily included. For irrotational flow, the momentum equation (2.2) is modified to[7]

$$\rho_o \frac{\partial \vec{v}}{\partial t} + \nabla p = \frac{4}{3} \mu \nabla(\nabla \cdot \vec{v}) \quad , \tag{2.24}$$

where μ is the viscosity of the fluid; the continuity equation (2.1) remains unchanged. Equations (2.1) and (2.24) may be combined to yield a single equation for the velocity potential,

$$\frac{\partial^2 \phi}{\partial t^2} - c_o^2 \nabla^2 \phi = \frac{4\mu}{3\rho_o} \nabla^2 \left(\frac{\partial \phi}{\partial t} \right) \quad . \tag{2.25}$$

For a harmonic sound field, the representation (2.13) then yields

$$(\nabla^2 + \kappa_o^2)\psi(\vec{r}) = 0 \quad , \tag{2.26}$$

where the propagation constant κ_o is now complex,

$$\kappa_o = \frac{\omega}{c_o}\left(1 - i\,\frac{4\mu\omega}{3\rho_o c_o^2}\right)^{-\frac{1}{2}} \quad . \tag{2.27}$$

The real part of κ_o will be denoted by k_o, and the imaginary part by $\frac{\alpha_o}{2}$; since $\frac{4\mu\omega}{3\rho_o c_o^2} \ll 1$ for frequencies of interest, the above relation may be expanded to yield

$$\kappa_o = k_o + i\,\frac{\alpha_o}{2} \simeq \frac{\omega}{c_o} + i\,\frac{2\mu\omega^2}{3\rho_o c_o^3} \quad . \tag{2.28}$$

The intensity of a plane harmonic wave is proportional to $e^{-\alpha_o x}$; therefore α_o is the attenuation constant for a plane wave in the medium.

We shall assume that the scatterers are isotropic point scatterers such that the scattered wave from each scatterer is a spherically symmetric (s-) wave. The scattering characteristics of each scatterer are assumed to be governed by a known complex scattering coefficient function $g(\vec{r}, R)$, defined such that if the velocity potential incident on a scatterer located at $\vec{r}_j$ is $\psi^j(\vec{r}_j)$, then the velocity potential of the scattered wave is given by

$$\psi_s(\vec{r}) = g(\vec{r}_j, R_j)\psi^j(\vec{r}_j)E(\vec{r}, \vec{r}_j) \quad , \tag{2.29}$$

where

$$E(\vec{r}, \vec{r}_j) = \frac{e^{i\kappa_o|\vec{r} - \vec{r}_j|}}{|\vec{r} - \vec{r}_j|} \quad . \tag{2.30}$$

As indicated, the scattering coefficient of each scatterer may depend on its position $\vec{r}_j$ as well as on a parameter R_j; the dependence on frequency ω is not indicated since a harmonic sound field of a fixed frequency is being considered. Although each scatterer is idealized as having vanishing dimensions, we shall eventually apply our theory to configurations of small scatterers. The theory will be applicable provided that the radii of the scatterers are small compared with the wave length in the fluid, and provided that the average distance between scatterers is large compared with their radii. Since the scattering characteristics of a scatterer is primarily determined by its radius, the parameter R_j appearing in the above expression for the scattering coefficient will be eventually identified with the radius of the j'th scatterer. Therefore, throughout our subsequent discussion, we shall refer to this parameter as the "radius" of the scatterer, although a non-zero radius is contradictory to the notion of a "point" scatterer.

Let us consider a single scatterer of radius R excited at the origin by an incident plane wave having the potential

$$\psi_i(x) = A e^{i\kappa_o x} \quad ; \tag{2.31}$$

the total field is then the sum of the incident wave and the scattered wave,

$$\psi(\vec{r}) = \psi_i(x) + \psi_s(r) = Ae^{i\kappa_o x} + A g(0,R) \frac{e^{i\kappa_o r}}{r} \quad . \tag{2.32}$$

The energy flux per unit area in the incident wave at $x = 0$ is proportional to

$$\psi^* \frac{\partial\psi}{\partial x} - \psi \frac{\partial\psi^*}{\partial x} = 2\, ik_o |A|^2 \quad , \tag{2.33}$$

while the total energy flux in the scattered wave crossing a sphere S_r of radius r centered at the origin is, in the limit $r \to 0$, proportional to

$$\lim_{r\to 0} \int_{S_r} (\psi_s^* \nabla\psi_s - \psi_s \nabla\psi_s^*) \cdot d\vec{S}_r = 8\pi i k_o |A|^2 |g(0,R)|^2 \ . \quad (2.34)$$

The scattering cross section is defined as the total energy flux in the scattered wave leaving the scatterer divided by the incident energy flux per unit area at the scatterer. Thus, at an arbitrary field point $\vec{r}$, the scattering cross section is given by

$$\sigma_s(\vec{r}, R) = 4\pi |g(\vec{r}, R)|^2 \quad . \qquad (2.35)$$

The total energy flux passing inward through the sphere S_r, in the limit $r \to 0$, is proportional to

$$-\lim_{r\to 0} \int_{S_r} (\psi^* \nabla\psi - \psi \nabla\psi^*) \cdot d\vec{S}_r = -8\pi i k_o |A|^2 \, |g(0,R)|^2$$

$$+ 4\pi |A|^2 [g(0,R) - g^*(0,R)] \quad . \qquad (2.36)$$

The absorption cross section is defined as the total energy flux absorbed by the scatterer divided by the incident energy flux per unit area at the scatterer; hence it is given by

$$\sigma_a(\vec{r}, R) = -\sigma_s(\vec{r}, R) + \frac{4\pi}{k_o} \, \mathrm{Im}\{g(\vec{r}, R)\} \quad . \quad (2.37)$$

The sum of the scattering and absorption cross sections is called the extinction cross section,

$$\sigma_e(\vec{r}, R) = \frac{4\pi}{k_o} \, \mathrm{Im}\{g(\vec{r}, R)\} \quad . \qquad (2.38)$$

In the analysis of the multiple scattering of waves by a large number of scatterers, it is convenient to employ cross section densities. The *scatterer number density distribution* $n(\vec{r}, R)$ is defined such that $n(\vec{r}, R)dR$ is the number of scatterers per unit volume at location $\vec{r}$ having radii in the interval dR about the radius R. The scattering and absorption cross section densities may be defined by

$$\Sigma_s(\vec{r}) = \int_o^\infty \sigma_s(\vec{r}, R) n(\vec{r}, R) dR \quad , \tag{2.39}$$

$$\Sigma_a(\vec{r}) = \int_o^\infty \sigma_a(\vec{r}, R) n(\vec{r}, R) dR + \alpha_o \quad . \tag{2.40}$$

Absorption of sound due to the viscosity of the fluid medium is included by addition of the attenuation constant α_o. The extinction cross section density is defined as the sum of the above,

$$\Sigma_e(\vec{r}) = \int_o^\infty \sigma_e(\vec{r}, R) n(\vec{r}, R) dR + \alpha_o \quad . \tag{2.41}$$

B. Statistical Considerations

If the exact position $\vec{r}_j$ and the exact radius R_j of each member of a collection of scatterers is specified, we shall say that we have a configuration of the scatterers. In problems dealing with the multiple scattering of waves by a collection of randomly distributed point scatterers, the exact configuration of a given collection of scatterers will not be specified; rather, we will only have information concerning the average distribution of the scatterers, or the probability of occurrence of a particular configuration. For example, if the scatterers are statistically independent of one another, i.e. the position and radius of any given scatterer are completely independent of the positions or radii of all the other scatterers in the collection, then the only information that is available concerning the scatterer distribution is the scatterer number density distribution $n(\vec{r}, R)$. For a random distribution of scatterers, the quantities of interest are the average values of relevant physical quantities, taken over all possible scatterer configurations, consistent with the known statistical data concerning the average or probable distribution of the scatterers. We shall call such an average a configurational average.

In order to establish the probabilistic concepts defining the configurational averages, we shall consider a statistical ensemble consisting of a collection of an infinite number of scatterer configurations. The nature of this ensemble is made precise by specification of its joint probability distribution function $p(\vec{r}_1, R_1; \ldots; \vec{r}_N, R_N)$, defined such that $p(\vec{r}_1, R_1; \ldots; \vec{r}_N, R_N) d\vec{r}_1 \, dR_1 \ldots d\vec{r}_N dR_N$ is the probability of a configuration of N scatterers having the j'th scatterer located in the volume element $d\vec{r}_j$ about the position $\vec{r}_j$ and in the interval dR_j about the radius R_j, for $j = 1, \ldots, N$. This probability is normalized such that the integral over all its coordinates is unity; that is

$$\iint \ldots \iint p(\vec{r}_1, R_1; \ldots; \vec{r}_N, R_N) d\vec{r}_1 dR_1 \ldots d\vec{r}_N dR_N = 1 \quad , \tag{2.42}$$

where the spatial integrations are taken over all space and the integrations over the radii are taken from 0 to ∞. The probability of a configuration having the j'th scatterer occupying $d\vec{r}_j dR_j$, regardless of the locations or radii of all the other scatterers, may be obtained by integrating over all but the j'th coordinates, as follows:

$$p(\vec{r}_j, R_j)d\vec{r}_j dR_j = \iint \overset{j}{\cdots} \iint p(\vec{r}_1, R_1; \ldots; \vec{r}_N, R_N) d\vec{r}_1 dR_1 \ldots d\vec{r}_N dR_N. \tag{2.43}$$

The superscript indicates the omission of the integrations over the j'th coordinates. In like manner, the probability of a configuration having the j'th scatterer occupying $d\vec{r}_j dR_j$ and the k'th scatterer occupying $d\vec{r}_k dR_k$, regardless of the locations or radii of all the other scatterers, is given by

$$p(\vec{r}_j, R_j; \vec{r}_k, R_k)d\vec{r}_j dR_j d\vec{r}_k dR_k = \iint \overset{jk}{\cdots} \iint p(\vec{r}_1 R_1; \ldots; \vec{r}_N, R_N) d\vec{r}_1 dR_1 \ldots d\vec{r}_N dR_N \quad , \tag{2.44}$$

the integrations over both the j'th and k'th coordinates having been omitted.

It is also convenient to introduce the notion of "conditional" probability. The conditional probability for a configuration having the j'th scatterer fixed at location $\vec{r}_j$ and radius R_j is defined by

$$p_j(\vec{r}_1, R_1; \overset{j}{\ldots}; \vec{r}_N, R_N) = \frac{p(\vec{r}_1, R_1; \ldots; \vec{r}_N, R_N)}{p(\vec{r}_j, R_j)} \quad , \tag{2.45}$$

the superscript here indicating the omission of the j'th coordinates within the parenthesis. Similarly, the conditional probability for a configuration having both the j'th and k'th scatterers fixed is defined by

$$p_{jk}^{jk}(\vec{r}_1, R_1; \ldots; \vec{r}_N, R_N) = \frac{p(\vec{r}_1, R_1; \ldots; \vec{r}_N, R_N)}{p(\vec{r}_j, R_j; \vec{r}_k, R_k)} \quad , \tag{2.46}$$

both the j'th and k'th coordinates having been omitted within the parenthesis. The conditional probability thus effects a pseudo-factorization of the complete probability distribution.

It will be assumed that the locations and radii of the scatterers are statistically independent; i.e. correlations between the positions and radii of separate scatterers are neglected. This requires that the joint probability distribution be expressible as the product of the individual probabilities for each scatterer; that is

$$p(\vec{r}_1, R_1; \ldots; \vec{r}_N, R_N) = \prod_{j=1}^{N} p(\vec{r}_j, R_j) \quad . \tag{2.47}$$

As a special case of this, we have

$$p(\vec{r}_j, R_j; \vec{r}_k, R_k) = p(\vec{r}_j, R_j)\, p(\vec{r}_k, R_k) \quad . \tag{2.48}$$

Finally, we note that the probability $p(\vec{r}_j, R_j)$ is simply equal to the average number density $n(\vec{r}_j, R_j)$ of the scatterers at location $\vec{r}_j$ per unit radius interval about R_j, divided by the total number of scatterers present,

$$p(\vec{r}_j, R_j) = \frac{n(\vec{r}_j, R_j)}{N} \quad . \tag{2.49}$$

Let us now consider a complex velocity potential field $\psi(\vec{r})$ produced by the multiple scattering of an incident wave by a configuration of scatterers. Let us indicate the dependence of this quantity on the location $\vec{r}_j$ and the radius R_j of each scatterer, in addition to the field point $\vec{r}$ where it is observed; the configurational average is then defined as

$$\langle\psi(\vec{r})\rangle = \iint \ldots \iint \psi(\vec{r}\,|\,\vec{r}_1, R_1; \ldots; \vec{r}_N, R_N)\, p(\vec{r}_1, R_1; \ldots; \vec{r}_N, R_N)\, d\vec{r}_1\, dR_1 \ldots d\vec{r}_N dR_N \quad , \qquad (2.50)$$

the average being taken over the statistical ensemble of scatterer configurations. The configurational averages of other quantities are defined in a similar fashion. The exciting field of the j'th scatterer $\psi^j(\vec{r}_j)$ depends on the locations and radii of all the other scatterers, in addition to $\vec{r}_j$. Therefore we shall define the conditional configurational average of this quantity by averaging it over a statistical ensemble of scatterer configurations having the j'th scatterer fixed; such an average may be defined in terms of the conditional probability distribution, as follows:

$$\langle\psi^j(\vec{r}_j)\rangle_j = \iint \overset{j}{\ldots} \iint \psi^j(\vec{r}_j\,|\,\vec{r}_1, R_1; \overset{j}{\ldots}; \vec{r}_N, R_N)\, p_j(\vec{r}_1, R_1; \overset{j}{\ldots}; \vec{r}_N, R_N)\, d\vec{r}_1\, dR_1 \overset{j}{\ldots} d\vec{r}_N dR_N \quad . \qquad (2.51)$$

Similarly, the conditional configurational average of $\psi^j(\vec{r}_j)\psi^{k^*}(\vec{r}_k)$, taken over an ensemble of configurations having both the j'th and k'th scatterers fixed, may be expressed as

$$\langle \psi^j(\vec{r}_j)\psi^{k^*}(\vec{r}_k)\rangle = \iint \overset{jk}{\cdots} \iint \psi^j(\vec{r}_j|\vec{r}_1, R_1; \overset{j}{\cdots}; \vec{r}_N, R_N)$$

$$\psi^{k^*}(\vec{r}_k|\vec{r}_1, R_1 \overset{k}{;} \cdots ; \vec{r}_N, R_N)$$

$$\times\, p_{jk}(\vec{r}_1, R_1; \overset{jk}{\cdots}; \vec{r}_N, R_N) d\vec{r}_1\, dR_1 \overset{jk}{\cdots} d\vec{r}_N dR_N \quad . \tag{2.52}$$

We have defined the "configurational" average of a physical quantity as an average taken over a statistical ensemble of scatterer configurations. Hence, the average complex velocity potential $\langle\psi(\vec{r})\rangle$ represents the average value of a set of simultaneous measurements of $\psi(\vec{r})$ taken on the distinct members of a collection of similar scatterer systems. If the average deviation of $\psi(\vec{r})$ from its average value is small, i.e.

$$\langle |\psi(\vec{r}) - \langle\psi(\vec{r})\rangle|^2\rangle = \langle |\psi(\vec{r})|^2\rangle - |\langle\psi(\vec{r})\rangle|^2 \ll |\langle\psi(\vec{r})\rangle|^2 \quad , \tag{2.53}$$

then a single measurement of $\psi(\vec{r})$ on a given system would be expected, with high probability, to be very close to the average value $\langle\psi(\vec{r})\rangle$.

We may also envision a physical situation in which the locations $\vec{r}_j(t)$ and the radii $R_j(t)$ of the scatterers in a single configuration are slowly changing with time such that it continuously passes through the various states of the statistical ensemble (we assume that the time required for the configuration to undergo a significant change is much greater than the period of oscillation of the sound field). The complex velocity potential, that is produced by the multiple scattering of an incident wave by this system, depends on time in addition to the point at which it is observed,

$$\psi(\vec{r}, t) = \psi(\vec{r} \mid \vec{r}_1(t), R_1(t); \; . \; . \; . \; ; \vec{r}_N(t), R_N(t)) \quad . \tag{2.54}$$

Hence a "time" average of this quantity may be defined by

$$\overline{\psi(\vec{r}, t)} = \frac{1}{T} \int_0^T \psi(\vec{r}, t) dt \quad , \tag{2.55}$$

where T is the length of time over which the average is taken. This average value has little practical significance and would be difficult to measure unless the mean square deviation is small, i.e.

$$\frac{1}{T} \int_0^T \left| \psi(\vec{r}, t) - \overline{\psi(\vec{r}, t)} \right|^2 dt = \overline{|\psi(\vec{r}, t)|^2} - \left| \overline{\psi(\vec{r}, t)} \right|^2 \ll \left| \overline{\psi(\vec{r}, t)} \right|^2 \; . \tag{2.56}$$

If this is not the case, the mean square value of the velocity potential (which is proportional to the mean square value of the pressure) has greater physical significance and may be more easily measured. This quantity may be decomposed into a "coherent" and an "incoherent" part, as follows:

$$\overline{|\psi(\vec{r}, t)|^2} = \left| \overline{\psi(\vec{r}, t)} \right|^2 + \left\{ \overline{|\psi(\vec{r}, t)|^2} - \left| \overline{\psi(\vec{r}, t)} \right|^2 \right\} . \tag{2.57}$$

We shall assume that, provided the time average is taken over a sufficiently long period of time for the scatterer configuration to pass through the majority of states in the statistical ensemble, the time average of a physical quantity is essentially equal to the corresponding configurational average; for example:

$$\overline{\psi(\vec{r}, t)} \simeq \langle \psi(\vec{r}) \rangle \quad , \tag{2.58}$$

$$\overline{|\psi(\vec{r},t)|^2} \cong \langle |\psi(\vec{r})|^2 \rangle \quad . \tag{2.59}$$

The latter quantity may also be decomposed into a coherent and an incoherent part,

$$\langle |\psi(\vec{r})|^2 \rangle = |\langle \psi(\vec{r}) \rangle|^2 + \{\langle |\psi(\vec{r})|^2 \rangle - |\langle \psi(\vec{r}) \rangle|^2\} \quad . \tag{2.60}$$

Therefore with this assumption, which is analogous to the ergodic hypothesis of statistical mechanics, we may provide an alternative description of the configurational averages as being equal to the corresponding time averages for a single configuration.

III. CONFIGURATIONAL AVERAGES OF THE EQUATIONS OF MULTIPLE SCATTERING

A. Equations of Multiple Scattering, Configurational Averages

Let us consider a given scatterer configuration and write the self-consistent field equations of multiple scattering, completely accounting for the effect on each scatterer due to the combined presence of all the other scatterers in the configuration. We shall assume that the scatterers act as points which affect an incident wave only through the additive constructive or destructive interference by the scattered waves produced at these points.

The total velocity potential may be expressed as the sum of the potentials of the incident wave and the spherically symmetric waves from each of the scatterers in the configuration,

$$\psi(\vec{r}) = \psi_i(\vec{r}) + \sum_j g_j \psi^j(\vec{r}_j) E(\vec{r}, \vec{r}_j) \quad , \tag{3.1}$$

where we denote

$$g_j = g(\vec{r}_j, R_j) \quad , \tag{3.2}$$

$$E(\vec{r}, \vec{r}_j) = \frac{e^{i\kappa_o |\vec{r} - \vec{r}_j|}}{|\vec{r} - \vec{r}_j|} \quad . \tag{3.3}$$

The wave incident on the j'th scatterer consists of the sum of the incident wave plus the waves from all the other scatterers,

$$\psi^j(\vec{r}) = \psi_i(\vec{r}) + \sum_{k \neq j} g_k \psi^k(\vec{r}_k) E(\vec{r}, \vec{r}_k) \quad . \tag{3.4}$$

These equations are rigorous as they stand and include all orders of multiple scattering.

The quantities of physical interest are the average values of the relevant physical quantities: the pressure, velocity, sound intensity (or mean square pressure), and the sound energy current. By Eqs. (2.20), (2.21), (2.22), and (2.23) these quantities may be related to the averages of $\psi(\vec{r})$, $\nabla\psi(\vec{r})$, $|\psi(\vec{r})|^2$, and $\psi^*(\vec{r})\nabla\psi(\vec{r}) - \psi(\vec{r})\nabla\psi^*(\vec{r})$. Our present objective is to obtain governing integral equations for $\langle\psi(\vec{r})\rangle$ and $\langle\psi(\vec{r})\psi^*(\vec{r}_0)\rangle$. Since the operation of taking a derivative with respect to $\vec{r}$ or $\vec{r}_0$ commutes with the integrations over the scatterer coordinates in the configurational averages, the quantities of interest may be calculated in the following manner:

$$\langle \nabla\psi(\vec{r}) \rangle = \nabla\langle\psi(\vec{r})\rangle \quad , \tag{3.5}$$

$$\langle |\psi(\vec{r})|^2 \rangle = \langle\psi(\vec{r})\psi^*(\vec{r}_0)\rangle\Big|_{\vec{r}=\vec{r}_0} \quad , \tag{3.6}$$

$$\langle\psi^*(\vec{r})\nabla\psi(\vec{r})-\psi(\vec{r})\nabla\psi^*(\vec{r})\rangle = (\nabla - \nabla_0)\langle\psi(\vec{r})\psi^*(\vec{r}_0)\rangle\Big|_{\vec{r}=\vec{r}_0} \quad . \tag{3.7}$$

B. <u>Governing Integral Equation for $\langle\psi(\vec{r})\rangle$</u>

In order to determine an integral equation satisfied by the average field $\langle\psi(\vec{r})\rangle$, let us multiply each term of Eq. (3.1) by the joint probability distribution $p(\vec{r}_1, R_1; \ldots; \vec{r}_N, R_N)$ and integrate over all its coordinates. By Eq. (2.50), the left hand side of the resulting equation is then just the configurational average $\langle\psi(\vec{r})\rangle$. Since the incident field $\psi_i(\vec{r})$ is independent of the locations or the radii of the scatterers, the second term is left unaltered. In order to evaluate the third term, use may be made of the conditional probability decomposition (2.45) and the definition (2.51) of the conditional configurational average of $\psi^j(\vec{r}_j)$, as follows:

$$\iint \cdots \iint \sum_j g_j \psi^j(\vec{r}_j) E(\vec{r}, \vec{r}_j) p(\vec{r}_1, R_1; \ldots; \vec{r}_N, R_N) \, d\vec{r}_1 dR_1 \ldots d\vec{r}_N dR_N$$

$$= \sum_j \iint g(\vec{r}_j, R_j) p(\vec{r}_j, R_j) dR_j \iint \overset{j}{\cdots} \iint \psi^j(\vec{r}_j \,|\, \vec{r}_1, R_1; \overset{j}{\ldots}; \vec{r}_N, R_N)$$

$$p_j(\vec{r}_1, R_1; \overset{j}{\ldots}; \vec{r}_N, R_N) d\vec{r}_1 dR_1 \overset{j}{\ldots} d\vec{r}_N dR_N E(\vec{r}, \vec{r}_j) d\vec{r}_j$$

$$= \frac{1}{N} \sum_j \iint g(\vec{r}_j, R_j) n(\vec{r}_j, R_j) dR_j \langle \psi^j(\vec{r}_j) \rangle_j E(\vec{r}, \vec{r}_j) d\vec{r}_j$$

$$= \frac{1}{N} \sum_j \int G(\vec{r}_j) \langle \psi^j(\vec{r}_j) \rangle_j E(\vec{r}, \vec{r}_j) d\vec{r}_j \quad . \tag{3.8}$$

Here the scatterer coefficient density $G(\vec{r})$ is defined by

$$G(\vec{r}) = \int_0^\infty g(\vec{r}, R) n(\vec{r}, R) dR \quad . \tag{3.9}$$

At this point the following approximation will be introduced

$$\langle \psi^j(\vec{r}_j) \rangle_j \cong \langle \psi(\vec{r}_j) \rangle \quad ; \tag{3.10}$$

that is, the configurational average of the exciting field of the j'th scatterer, averaged over a statistical ensemble of configurations having the j'th scatterer fixed, is approximately equal to the configurational average of the total field at the same point. The validity of this assumption has been discussed by Lax[9], and more recently by Waterman and Truell[5]. The error introduced by such an assumption would be expected to be $O\left(\frac{1}{N}\right)$; therefore the assumption may be considered to be valid when the number N of scatterers is large. With this assumption, the configurational average of Eq. (3.1) then yields

$$\langle \psi(\vec{r}) \rangle = \psi_i(\vec{r}) + \frac{1}{N} \sum_j \int G(\vec{r}_j) \langle \psi(\vec{r}_j) \rangle E(\vec{r}, \vec{r}_j) d\vec{r}_j \quad . \qquad (3.11)$$

Since each of the N terms in the sum is identical, the governing integral equation for the configurational average of the complex velocity potential becomes

$$\langle \psi(\vec{r}) \rangle = \psi_i(\vec{r}) + \int G(\vec{r}\,') \langle \psi(\vec{r}\,') \rangle E(\vec{r}, \vec{r}\,') d\vec{r}\,' \quad . \qquad (3.12)$$

C. Governing Integral Equation for $\langle \psi(\vec{r}) \psi^*(\vec{r}_o) \rangle$

If we replace $\vec{r}$ by $\vec{r}_o$, j by k in the complex conjugate of Eq. (3.1),

$$\psi^*(\vec{r}_o) = \psi_i^*(\vec{r}_o) + \sum_k g_k^* \psi^{k^*}(\vec{r}_o) E^*(\vec{r}_o, \vec{r}_k) \quad , \qquad (3.13)$$

and form the product $\psi(\vec{r})\psi^*(\vec{r}_o)$ with Eq. (3.1), we obtain

$$\psi(\vec{r})\psi^*(\vec{r}_o) = \psi_i(\vec{r})\psi_i^*(\vec{r}_o) + \psi_i(\vec{r})[\psi^*(\vec{r}_o) - \psi_i^*(\vec{r}_o)] + [\psi(\vec{r}) - \psi_i(\vec{r})]\psi_i^*(\vec{r}_o)$$
$$+ \sum_j g_j g_j^* \psi^j(\vec{r}_j) \psi^{j^*}(\vec{r}_j) E(\vec{r}, \vec{r}_j) E^*(\vec{r}_o, \vec{r}_j) + \sum_{j \neq k} \sum g_j g_k^* \psi^j(\vec{r}_j) \psi^{k^*}(\vec{r}_k) E(\vec{r}, \vec{r}_j) E^*(\vec{r}_o, \vec{r}_k) \quad . \qquad (3.14)$$

Let us again multiply each term by the joint probability distribution $p(\vec{r}_1, R_1; \ldots; \vec{r}_N, R_N)$ and integrate over all its coordinates. The last term may be evaluated by use of the conditional probability decomposition (2.46), the condition (2.48) of statistical independence, and the definition (2.52) of the conditional configurational average with two scatterers held fixed, as follows:

$$\iint \ldots \iint \sum_{j \neq k}\sum g_j g_k^* \psi^j(\vec{r}_j)\psi^{k^*}(\vec{r}_k)E(\vec{r}, \vec{r}_j)E^*(\vec{r}_o, \vec{r}_k)$$

$$p(\vec{r}_1, R_1; \ldots; \vec{r}_N, R_N)d\vec{r}_1 dR_1 \ldots d\vec{r}_N dR_N$$

$$= \sum_{j \neq k}\sum \iiiint g(\vec{r}_j, R_j)g^*(\vec{r}_k, R_k)p(\vec{r}_j, R_j; \vec{r}_k R_k)dR_j dR_k$$

$$\overset{jk}{\iint} \ldots \iint \psi^j(\vec{r}_j|\vec{r}_1, R_1; \overset{j}{\ldots}; \vec{r}_N, R_N)\psi^{k^*}(\vec{r}_k|\vec{r}_1, R_1; \overset{k}{\ldots}; \vec{r}_N, R_N)$$

$$\times\, p_{jk}(\vec{r}_1, R_1; \overset{jk}{\ldots}; \vec{r}_N, R_N)d\vec{r}_1 dR_1 \overset{jk}{\ldots} d\vec{r}_N dR_N d\vec{r}_j d\vec{r}_k$$

$$= \frac{1}{N^2}\sum_{j \neq k}\sum \iiint g(\vec{r}_j, R_j)n(\vec{r}_j, R_j)dR_j \int g^*(\vec{r}_k, R_k)n(\vec{r}_k, R_k)dR_k$$

$$\langle \psi^j(\vec{r}_j)\psi^{k^*}(\vec{r}_k)\rangle_{jk}E(\vec{r}, \vec{r}_j)E^*(\vec{r}_o, \vec{r}_k)d\vec{r}_j d\vec{r}_k$$

$$= \frac{1}{N^2}\sum_{j \neq k}\sum \iint G(\vec{r}_j)G^*(\vec{r}_k)\langle \psi^j(\vec{r}_j)\psi^{k^*}(\vec{r}_k)\rangle_{jk}E(\vec{r}, \vec{r}_j)E^*(\vec{r}_o, \vec{r}_k)d\vec{r}_j d\vec{r}_k \quad . \tag{3.15}$$

After averaging the remaining terms of Eq. (3.14), we obtain

$$\langle \psi(\vec{r})\psi^*(\vec{r}_o) \rangle = \psi_i(\vec{r})\psi_i^*(\vec{r}_o) + \psi_i(\vec{r})[\langle \psi(\vec{r}_o)\rangle^* - \psi_i^*(\vec{r}_o)]$$

$$+ [\langle \psi(\vec{r})\rangle - \psi_i(\vec{r})]\, \psi_i^*(\vec{r}_o)$$

$$+ \frac{1}{N} \sum_j \frac{1}{4\pi} \int \Sigma_s(\vec{r}_j)\langle |\psi(\vec{r}_j)|^2\rangle_j\, E(\vec{r}, \vec{r}_j)E^*(\vec{r}_o, \vec{r}_j)d\vec{r}_j$$

$$+ \frac{1}{N^2} \sum_{j \neq k}\sum \iint G(\vec{r}_j)G^*(\vec{r}_k)\langle \psi^j(\vec{r}_j)\psi^{k^*}(\vec{r}_k)\rangle_{jk} E(\vec{r}, \vec{r}_j)E^*(\vec{r}_o, \vec{r}_k)d\vec{r}_j d\vec{r}_k \quad , \tag{3.16}$$

where the scattering cross section density $\Sigma_s(\vec{r})$ has been previously defined by Eq. (2.39). It is again necessary to introduce approximations in order to obtain a governing integral equation; we shall assume that

$$\langle |\psi^j(\vec{r}_j)|^2\rangle_j \cong \langle |\psi(\vec{r}_j)|^2\rangle \tag{3.17}$$

and

$$\langle \psi^j(\vec{r}_j)\psi^{k^*}(\vec{r}_k)\rangle_{jk} \cong \langle \psi(\vec{r}_j)\psi^*(\vec{r}_k)\rangle \quad . \tag{3.18}$$

When these replacements are made in Eq. (3.16) and $\frac{(N-1)}{N}$ is replaced by unity in the last term, the following equation results:

$$\langle \psi(\vec{r})\psi^*(\vec{r}_o)\rangle = \psi_i(\vec{r})\psi_i^*(\vec{r}_o) + \psi_i(\vec{r})[\langle\psi(\vec{r}_o)\rangle^* - \psi_i^*(\vec{r}_o)] + [\langle\psi(\vec{r})\rangle - \psi_i(\vec{r})]\, \psi_i^*(\vec{r}_o)$$

$$+ \frac{1}{4\pi} \int \Sigma_s(\vec{r}\,')\langle |\psi(\vec{r}\,')|^2\rangle E(\vec{r}, \vec{r}\,')E^*(\vec{r}_o, \vec{r}\,')d\vec{r}\,'$$

$$+ \iint G(\vec{r}\,'')G^*(\vec{r}\,''')\langle \psi(\vec{r}\,'')\psi^*(\vec{r}\,''')\rangle E(\vec{r}, \vec{r}\,'')E^*(\vec{r}_o, \vec{r}\,''')d\vec{r}\,''d\vec{r}\,''' \quad . \tag{3.19}$$

This equation may be simplified by using the integral equation (3.12) for the coherent wave to form the product $\langle \psi(\vec{r})\rangle\langle\psi(\vec{r}_o)\rangle^*$, yielding

$$\langle \psi(\vec{r})\rangle \langle\psi(\vec{r}_o)\rangle^* = \psi_i(\vec{r})\psi_i^*(\vec{r}_o) + \psi_i(\vec{r})[\langle\psi(\vec{r}_o)\rangle^* - \psi_i^*(\vec{r}_o)]$$

$$+ [\langle\psi(\vec{r})\rangle - \psi_i(\vec{r})]\,\psi_i^*(\vec{r}_o)$$

$$+ \iint G(\vec{r}\,'')G^*(\vec{r}\,''')\langle\psi(\vec{r}\,'')\rangle\langle\psi(\vec{r}\,''')\rangle^* E(\vec{r}, \vec{r}\,'')E^*(\vec{r}_o, \vec{r}\,''')d\vec{r}\,''d\vec{r}\,''' \quad . \quad (3.20)$$

Upon subtraction of the latter from the former equation, a governing integral equation for the incoherent quantity $\langle \psi(\vec{r})\psi^*(\vec{r}_o)\rangle - \langle\psi(\vec{r})\rangle\langle\psi(\vec{r}_o)\rangle^*$ is obtained:

$$\langle \psi(\vec{r})\psi^*(\vec{r}_o)\rangle - \langle\psi(\vec{r})\rangle\langle\psi(\vec{r}_o)\rangle^* = \frac{1}{4\pi}\int \Sigma_s(\vec{r}\,')\langle|\psi(\vec{r}\,')|^2\rangle E(\vec{r}, \vec{r}\,')E^*(\vec{r}_o, \vec{r}\,')d\vec{r}\,'$$

$$+ \iint G(\vec{r}\,'')G^*(\vec{r}\,''')[\langle\psi(\vec{r}\,'')\psi^*(\vec{r}\,''')\rangle - \langle\psi(\vec{r}\,'')\rangle\langle\psi(\vec{r}\,''')\rangle^*]$$

$$E(\vec{r}, \vec{r}\,'')E^*(\vec{r}_o, \vec{r}\,''')d\vec{r}\,''d\vec{r}\,''' \quad . \quad (3.21)$$

D. Integral Relation Connecting $\langle \psi^*(\vec{r})\nabla\psi(\vec{r}) - \psi(\vec{r})\nabla\psi^*(\vec{r})\rangle$ and $\langle|\psi(\vec{r})|^2\rangle$

Let us now consider a given configuration of scatterers and construct an arbitrary surface S containing an arbitrary number of the scatterers as well as a sphere S_j of radius ρ_j about each scatterer j. Take each radius ρ_j sufficiently small such that none of these surfaces intersects and let V denote the multiply-connected volume enclosed between the surface S and the spheres S_j lying within S. Define:

$$\gamma_j = \gamma(\vec{r}_j) = \begin{cases} 1 & \text{if the j'th scatterer lies within S} \\ 0 & \text{if the j'th scatterer lies outside S.} \end{cases} \tag{3.22}$$

The total field $\psi(\vec{r})$ is regular throughout the volume V, including its boundaries; by applying the divergence theorem, we may write

$$\int_S [\psi^*(\vec{r})\nabla\psi(\vec{r}) - \psi(\vec{r})\nabla\psi^*(\vec{r})] \cdot d\vec{S}$$
$$- \sum_j \gamma_i \int_{S_j} [\psi^*(\vec{r})\nabla\psi(\vec{r}) - \psi(\vec{r})\nabla\psi^*(\vec{r})] \cdot d\vec{S}_j$$
$$= \int_V \nabla \cdot [\psi^*(\vec{r})\nabla\psi(\vec{r}) - \psi(\vec{r})\nabla\psi^*(\vec{r})] \, d\vec{r} \quad . \tag{3.23}$$

Throughout the volume V, the field $\psi(\vec{r})$ satisfies the wave equation

$$(\nabla^2 + \kappa_o^2)\,\psi(\vec{r}) = 0 \quad , \tag{3.24}$$

so that the foregoing equation may be rewritten as

$$\int_S [\psi^*(\vec{r})\nabla\psi(\vec{r}) - \psi(\vec{r})\nabla\psi^*(\vec{r})] \cdot d\vec{S}$$
$$= \sum_j \gamma_j \int_{S_j} [\psi^*(\vec{r})\nabla\psi(\vec{r}) - \psi(\vec{r})\nabla\psi^*(\vec{r})] \cdot d\vec{S}_j$$
$$- \int_V (\kappa_o^2 - \kappa_o^{*2}) \, |\psi(\vec{r})|^2 d\vec{r} \quad . \tag{3.25}$$

The total field $\psi(\vec{r})$ is described by the fundamental equations (3.1) and (3.4) of multiple scattering; in the neighborhood of the j'th scatterer, it may be expressed as the sum of the exciting field $\psi^j(\vec{r})$ and the scattered wave,

$$\psi(\vec{r}) = \psi^j(\vec{r}) + g_j \psi^j(\vec{r}_j) E(\vec{r}, \vec{r}_j) \quad . \tag{3.26}$$

Using this decomposition, the energy flux passing through the spherical surface S_j may be written as the sum of four terms:

$$\int_{S_j} [\psi^*(\vec{r})\nabla\psi(\vec{r}) - \psi(\vec{r})\nabla\psi^*(\vec{r})] \cdot d\vec{S}_j$$

$$= \int_{S_j} [\psi^{j^*}(\vec{r})\nabla\psi^j(\vec{r}) - \psi^j(\vec{r})\nabla\psi^{j^*}(\vec{r})] \cdot d\vec{S}_j$$

$$+ \int_{S_j} [g_j^*\psi^{j^*}(\vec{r}_j)E^*(\vec{r},\vec{r}_j)\nabla\psi^j(\vec{r})$$

$$- g_j\psi^j(\vec{r}_j)E(\vec{r},\vec{r}_j)\nabla\psi^{j^*}(\vec{r})] \cdot d\vec{S}_j$$

$$+ \int_{S_j} [g_j\psi^j(\vec{r}_j)\nabla E(\vec{r},\vec{r}_j)\psi^{j^*}(\vec{r})$$

$$- g_j^*\psi^{j^*}(\vec{r}_j)\nabla E^*(\vec{r},\vec{r}_j)\psi^j(\vec{r})] \cdot d\vec{S}_j$$

$$+ \int_{S_j} |g_j|^2 |\psi^j(\vec{r}_j)|^2 [E^*(\vec{r},\vec{r}_j)\nabla E(\vec{r},\vec{r}_j)$$

$$-E(\vec{r},\vec{r}_j)\nabla E^*(\vec{r},\vec{r}_j)] \cdot d\vec{S}_j \quad . \tag{3.27}$$

We shall now investigate the limit of each of the four terms on the right of this equation, as the radius ρ_j of the spherical surface S_j tends to zero. For the first term, we note that the exciting field and its gradient are regular at $\vec{r}_j$ so that $\psi^{j^*}(\vec{r})\nabla\psi^j(\vec{r}) = O(1)$ as $\rho_j \to 0$, and since the surface area of S_j is $O(\rho_j^2)$ as $\rho_j \to 0$, then the term vanishes in this limit. Similarly, we note that in the second integral $g_j^*\psi^{j^*}(\vec{r}_j)\nabla\psi^j(\vec{r}) = O(1)$ and $E^*(\vec{r},\vec{r}_j) = O(\rho_j^{-1})$ as $\rho_j \to 0$, so that this term also vanishes. For the third term, we have

$$\nabla E(\vec{r},\vec{r}_j)\cdot d\vec{S}_j = (i\kappa_o - \frac{1}{\rho_j}) \frac{e^{i\kappa_o\rho_j}}{\rho_j} \rho_j^2 \, d\Omega \quad , \tag{3.28}$$

where $d\Omega$ is an element of solid angle. Since for $\vec{r}$ lying on S_j,

$$\psi^j(\vec{r}) = \psi^j(\vec{r}_j) + O(\rho_j) \quad \text{as} \quad \rho_j \to 0 \quad , \tag{3.29}$$

then

$$g_j\psi^j(\vec{r}_j)\nabla E(\vec{r},\vec{r}_j)\psi^{j^*}(\vec{r})\cdot d\vec{S} = -g_j|\psi^j(\vec{r}_j)|^2\,d\Omega + O(\rho_j)d\Omega \text{ as } \rho_j \to 0 \quad . \tag{3.30}$$

Therefore, in the limit $\rho_j \to 0$ the third surface integral on the right of Eq. (3.27) becomes $-4\pi(g_j - g_j^*)|\psi^j(\vec{r}_j)|^2$. Finally, we have that

$$[E^*(\vec{r},\vec{r}_j)\nabla E(\vec{r},\vec{r}_j) - E(\vec{r},\vec{r}_j)\nabla E^*(\vec{r},\vec{r}_j)] = i(\kappa_o + \kappa_o^*)d\Omega + O(\rho_j)d\Omega \quad \text{as} \quad \rho_j \to 0 \quad , \tag{3.31}$$

so that in the limit $\rho_j \to 0$ the last surface integral reduces to $8\pi i k_o|g_j|^2|\psi^j(\vec{r}_j)|^2$, where k_o denotes the real part of κ_o. Therefore, in the limit $\rho_j \to 0$ Eq. (3.27) reduces to

$$\lim_{\rho_j \to 0}\int_{S_j}[\psi^*(\vec{r})\nabla\psi(\vec{r}) - \psi(\vec{r})\nabla\psi^*(\vec{r})]\cdot d\vec{S}_j = -2ik_o\left(\frac{4\pi}{k_o}\,\mathrm{Im}\{g_j\} - 4\pi|g_j|^2\right)|\psi^j(\vec{r}_j)|^2 \quad . \tag{3.32}$$

By definition (2.37), the quantity within the parenthesis is the absorption cross section of the j'th scatterer,

$$\sigma_a^j = \sigma_a(\vec{r}_j, R_j) = \frac{4\pi}{k_o}\,\mathrm{Im}\{g_j\} - 4\pi|g_j|^2 \quad . \tag{3.33}$$

Upon substitution of these results back in Eq. (3.25), we obtain

$$\int_S [\psi^*(\vec{r})\nabla\psi(\vec{r}) - \psi(\vec{r})\nabla\psi^*(\vec{r})] \cdot d\vec{S}$$
$$= -2ik_o\left\{\sum_j \gamma_j\sigma_a^j|\psi^j(\vec{r}_j)|^2 + \int_V \alpha_o|\psi(\vec{r})|^2 d\vec{r}\right\} \quad , \qquad (3.34)$$

where the attenuation constant α_o denotes half the imaginary part of κ_o. This equation represents a simple statement of conservation of energy; it states that the mean energy flux through the surface S is equal to the rate of energy absorption by the scatterers plus the rate of energy dissipation in the matrix medium.

Since we are interested in a random distribution of scatterers, rather than a given configuration, we shall take the configurational average of the above equation by multiplying each term by the joint probability distribution $p(\vec{r}_1, R_1; \ldots; \vec{r}_N, R_N)$ and integrating over the coordinates of all the scatterers. The configurational average of the first term of Eq. (3.34) is given by (omitting the dependence of $\psi(\vec{r})$ on the coordinates $\vec{r}_1, R_1; \ldots; \vec{r}_N, R_N$ of the scatterers)

$$\langle \int_S [\psi^*(\vec{r})\nabla\psi(\vec{r}) - \psi(\vec{r})\nabla\psi^*(\vec{r})] \cdot d\vec{S}\rangle$$
$$= \iint \ldots \iiint_S [\psi^*(\vec{r})\nabla\psi(\vec{r}) - \psi(\vec{r})\nabla\psi^*(\vec{r})] \cdot d\vec{S}\, p(\vec{r}_1, R_1; \ldots; \vec{r}_N, R_N)\, d\vec{r}_1 dR_1 \ldots d\vec{r}_N dR_N$$
$$= \int_S \left\{\iint \ldots \iint [\psi^*(\vec{r})\nabla\psi(\vec{r}) - \psi(\vec{r})\nabla\psi^*(\vec{r})]\, p(\vec{r}_1, R_1; \ldots; \vec{r}_N, R_N)\, d\vec{r}_1\, dR_1 \ldots d\vec{r}_N dR_N\right\} \cdot d\vec{S}$$
$$= \int_S \langle\psi^*(\vec{r})\nabla\psi(\vec{r}) - \psi(\vec{r})\nabla\psi^*(\vec{r})\rangle \cdot d\vec{S} \quad . \qquad (3.35)$$

Thus the configurational average of the surface integral of the current is equal to the surface integral of the configurational average of the current. The third term of Eq. (3.34) may be similarly treated with the result:

$$\left\langle \int_V \alpha_o |\psi(\vec{r})|^2 d\vec{r} \right\rangle = \int_V \alpha_o \langle |\psi(\vec{r})|^2 \rangle \, d\vec{r} \quad . \tag{3.36}$$

The configurational average of the second term of Eq. (3.34) may be evaluated by means of the decomposition (2.45) and the definition (2.51) of the conditional configurational average, as follows:

$$\iint \cdots \iint \sum_j \gamma(\vec{r}_j)\sigma_a(\vec{r}_j, R_j)|\psi^j(\vec{r}_j|\vec{r}_1, R_1; \overset{j}{\cdots}; \vec{r}_N, R_N)|^2$$

$$p(\vec{r}_1, R_1; \cdots; \vec{r}_N, R_N) d\vec{r}_1 dR_1 \cdots d\vec{r}_N dR_N$$

$$= \sum_j \int \gamma(\vec{r}_j) \int \sigma_a(\vec{r}_j, R_j) p(\vec{r}_j, R_j) dR_j \iint \overset{j}{\cdots} \iint |\psi^j(\vec{r}_j|\vec{r}_1, R_1; \overset{j}{\cdots}; \vec{r}_N, R_N)|^2$$

$$p_j(\vec{r}_1, R_1; \overset{j}{\cdots}; \vec{r}_N, R_N) d\vec{r}_1 dR_1 \overset{j}{\cdots} d\vec{r}_N dR_N d\vec{r}_j$$

$$= \frac{1}{N} \sum_j \int_V \int \sigma_a(\vec{r}_j, R_j) n(\vec{r}_j, R_j) dR_j \langle |\psi^j(\vec{r}_j)|^2 \rangle_j d\vec{r}_j \quad . \tag{3.37}$$

It is noted that the effect of the function $\gamma(\vec{r}_j)$, defined by Eq. (3.22), is to truncate the volume integration over $\vec{r}_j$ to the volume V lying within the surface S. Let us again employ the approximation (3.17)

$$\langle |\psi^j(\vec{r}_j)|^2 \rangle_j \simeq \langle |\psi(\vec{r}_j)|^2 \rangle \quad , \tag{3.38}$$

which states that the configurational average of the mean-square exciting field of the j'th scatterer, averaged over the statistical ensemble of configurations having this scatterer fixed, is approximately equal to the configurational average of the total mean-square field at the same point. When this approximation is substituted into Eq. (3.37), the left hand side may be simply expressed as

$$\int_V \int_o^\infty \sigma_a(\vec{r}, R) n(\vec{r}, R) dR \langle |\psi(\vec{r})|^2 \rangle \, d\vec{r} \quad .$$

Therefore, the configurational average of Eq. (3.34) results in

$$\int_S \langle \psi^*(\vec{r}) \nabla \psi(\vec{r}) - \psi(\vec{r}) \nabla \psi^*(\vec{r}) \rangle \cdot d\vec{S}$$

$$= -2ik_o \int_V \left\{ \int_o^\infty \sigma_a(\vec{r}, R) n(\vec{r}, R) dR + \alpha_o \right\} \langle |\psi(\vec{r})|^2 \rangle \, d\vec{r} \; ; \quad (3.39)$$

by use of the definition (2.40) of the absorption cross section density, this may be written as

$$\int_S \langle \psi^*(\vec{r}) \nabla \psi(\vec{r}) - \psi(\vec{r}) \nabla \psi^*(\vec{r}) \rangle \cdot d\vec{S}$$

$$= -2ik_o \int_V \Sigma_a(\vec{r}) \langle |\psi(\vec{r})|^2 \rangle \, d\vec{r} \quad . \qquad (3.40)$$

IV. COHERENT AND INCOHERENT SCATTERING

In the previous section, we have obtained governing integral equations (3.12) and (3.21) for $\langle \psi(\vec{r}) \rangle$ and $\langle \psi(\vec{r})\psi^*(\vec{r}_o) \rangle$, as well as an integral relation (3.40) connecting $\langle \psi^*(\vec{r})\nabla\psi(\vec{r}) - \psi(\vec{r})\nabla\psi^*(\vec{r}) \rangle$ and $\langle |\psi(\vec{r})|^2 \rangle$. We shall now examine the physical consequences of these equations and reduce them to more workable forms.

A. The Coherent Wave

Let us first recall Eq. (3.12) governing the configurational average of the complex velocity potential,

$$\langle \psi(\vec{r}) \rangle = \psi_i(\vec{r}) + \int G(\vec{r}\,')\langle \psi(\vec{r}\,') \rangle E(\vec{r}, \vec{r}\,') d\vec{r}\,' \quad . \tag{4.1}$$

The kernel $E(\vec{r}, \vec{r}\,')$ of this equation is proportional to the Green's function of the operator $(\nabla^2 + \kappa_o^2)$,

$$(\nabla^2 + \kappa_o^2) E(\vec{r}, \vec{r}\,') = -4\pi\delta(\vec{r} - \vec{r}\,') \quad , \tag{4.2}$$

where $\delta(\vec{r} - \vec{r}\,')$ is the three-dimensional delta function having the property: that for any continuous function $f(\vec{r})$

$$\int f(\vec{r}\,')\delta(\vec{r} - \vec{r}\,') d\vec{r}\,' = f(\vec{r}) \quad . \tag{4.3}$$

The incident wave $\psi_i(\vec{r})$ is a regular solution of the wave equation, based on the propagation constant κ_o of the homogeneous matrix medium,

$$(\nabla^2 + \kappa_o^2)\psi_i(\vec{r}) = 0 \quad . \tag{4.4}$$

Therefore, if we operate on Eq. (4.1) with $(\nabla^2 + \kappa_o^2)$ and use the property (4.3) of the delta function, we obtain

$$(\nabla^2+\kappa_o^2)\langle\psi(\vec{r})\rangle = -4\pi G(\vec{r})\langle\psi(\vec{r})\rangle \quad (4.5)$$

Hence, the configurational average of the complex velocity potential satisfies the wave equation

$$[\nabla^2+\kappa^2(\vec{r})]\langle\psi(\vec{r})\rangle = 0 \quad , \quad (4.6)$$

where the complex propagation coefficient $\kappa(\vec{r})$ of the scattering medium is given by

$$\kappa(\vec{r}) = k(\vec{r}) + i\,\frac{\alpha(\vec{r})}{2} = [\kappa_o^2 + 4\pi G(\vec{r})]^{\frac{1}{2}} \quad . \quad (4.7)$$

The local phase velocity and attenuation coefficient of the coherent wave in the scattering medium are given by the following expressions:

$$c(\vec{r}) = \frac{\omega}{k(\vec{r})} = \frac{\omega}{\mathrm{Re}\{[\kappa_o^2+4\pi G(\vec{r})]^{\frac{1}{2}}\}} \quad , \quad (4.8)$$

$$\alpha(\vec{r}) = 2\,\mathrm{Im}\{[\kappa_o^2+4\pi G(\vec{r})]^{\frac{1}{2}}\} \quad . \quad (4.9)$$

The problem of determining the coherent wave $\langle\psi(\vec{r})\rangle$ has been reduced to solving a boundary value problem for the wave equation. If the function $G(\vec{r})$ is sectionally continuous and approaches a constant value at infinity, then the integral equation (4.1) implies that $\langle\psi(\vec{r})\rangle$ be everywhere continuous, have a continuous normal derivative across surfaces of discontinuity of $G(\vec{r})$, and represent outward travelling waves at infinity.

The physical behavior of the coherent wave is quite simple.

According to Eq. (4.6), the coherent wave satisfies a wave equation having a complex propagation coefficient that depends, in general, on position. Thus the incident wave and the scattered waves from all the scatterers interfere, on the average, to form a new wave travelling at a different phase velocity and undergoing attenuation. This wave will display the reflection and refraction aspects of coherent scattering at surfaces of discontinuity.

B. Conservation of Coherent and Incoherent Sound Energy

Let us now return to the integral relation (3.40) connecting $\langle \psi^*(\vec{r})\nabla\psi(\vec{r}) - \psi(\vec{r})\nabla\psi^*(\vec{r})\rangle$ and $\langle |\psi(\vec{r})|^2\rangle$. Since the configurational average of the current is regular on and within the surface S, the divergence theorem may be employed in order to convert the surface integral into a volume integral, and since the volume V is arbitrary, the relation may then be written in differential form as

$$\mathrm{div}\langle \psi^*(\vec{r})\nabla\psi(\vec{r}) - \psi(\vec{r})\nabla\psi^*(\vec{r})\rangle = -2ik_o\Sigma_a(\vec{r})\langle |\psi(\vec{r})|^2\rangle \quad . \tag{4.10}$$

Now let us compute the corresponding expression for the coherent wave, which satisfies the wave equation (4.6). We have

$$\begin{aligned} \mathrm{div}[\langle\psi(\vec{r})\rangle^*\nabla\langle\psi(\vec{r})\rangle - \langle\psi(\vec{r})\rangle\nabla\langle\psi(\vec{r})\rangle^*] &= \langle\psi(\vec{r})\rangle^*\nabla^2\langle\psi(\vec{r})\rangle \\ &\quad - \langle\psi(\vec{r})\rangle\nabla^2\langle\psi(\vec{r})\rangle^* \\ &= -[\kappa^2(\vec{r}) - \kappa^{*2}(\vec{r})]\,|\langle\psi(\vec{r})\rangle|^2 \quad . \end{aligned} \tag{4.11}$$

Using the definition (4.7) of the propagation constant of the scattering medium, the notation (2.28) for the real and imaginary part of κ_o, and the definition (2.41) of the extinction cross section density, we may write

$$\kappa^2(\vec{r})-\kappa^{*2}(\vec{r}) = 2i\,\mathrm{Im}\left\{\kappa_o^2 + 4\pi\int_o^\infty g(\vec{r},R)n(\vec{r},R)dR\right\}$$

$$= 2ik_o\left\{\alpha_o + \int_o^\infty \sigma_e(\vec{r},R)n(\vec{r},R)dR\right\} = 2ik_o\Sigma_e(\vec{r}) \quad . \qquad (4.12)$$

Therefore, the divergence of the coherent current satisfies a relation quite analogous to Eq. (4.10) for the total current:

$$\mathrm{div}[\langle\psi(\vec{r})\rangle^*\nabla\langle\psi(\vec{r})\rangle - \langle\psi(\vec{r})\rangle\nabla\langle\psi(\vec{r})\rangle^*] = -2ik_o\Sigma_e(\vec{r})|\langle\psi(\vec{r})\rangle|^2 \quad . \qquad (4.13)$$

In order to understand the physical significance of Eqs. (4.10) and (4.13), let us denote the configurational averages of the total sound intensity and sound energy current by

$$e(\vec{r}) = \frac{\rho_o\omega^2}{2c_o^2}\langle|\psi(\vec{r})|^2\rangle \quad , \qquad (4.14)$$

$$\vec{j}(\vec{r}) = \frac{\rho_o\omega}{4i}\langle\psi^*(\vec{r})\nabla\psi(\vec{r}) - \psi(\vec{r})\nabla\psi^*(\vec{r})\rangle \quad . \qquad (4.15)$$

The "coherent" contributions to the average sound intensity and average sound energy current may be defined as

$$e_c(\vec{r}) = \frac{\rho_o\omega^2}{2c_o^2}|\langle\psi(\vec{r})\rangle|^2 \quad , \qquad (4.16)$$

$$\vec{j}_c(\vec{r}) = \frac{\rho_o\omega}{4i}[\langle\psi(\vec{r})\rangle^*\nabla\langle\psi(\vec{r})\rangle - \langle\psi(\vec{r})\rangle\nabla\langle\psi(\vec{r})\rangle^*] \quad , \qquad (4.17)$$

with the "incoherent" contributions to these quantities being defined as the differences:

$$e_i(\vec{r}) = e(\vec{r}) - e_c(\vec{r}) \quad , \tag{4.18}$$

$$\vec{j}_i(\vec{r}) = \vec{j}(\vec{r}) - \vec{j}_c(\vec{r}) \quad . \tag{4.19}$$

Equations (4.10) and (4.13) may be rewritten, in terms of these definitions, as follows:

$$\text{div}\,\vec{j}(\vec{r}) = -\,c_o\Sigma_a(\vec{r})e(\vec{r}) \quad , \tag{4.20}$$

$$\text{div}\,\vec{j}_c(\vec{r}) = -\,c_o\Sigma_e(\vec{r})e_c(\vec{r}) \quad . \tag{4.21}$$

The first of these is a statement of conservation of the total sound energy; the latter describes the conservation of the coherent portion of the sound energy. By employing the additivity property of the cross section densities, the former equation may be rewritten as

$$\text{div}[\vec{j}_c(\vec{r})+\vec{j}_i(\vec{r})] = -\,c_o[\Sigma_e(\vec{r})-\Sigma_s(\vec{r})]\,e_c(\vec{r}) - c_o\Sigma_a(\vec{r})e_i(\vec{r}) \quad , \tag{4.22}$$

Upon subtraction of the second, we obtain a statement of conservation of the incoherent portion of the sound energy:

$$\text{div}\,\vec{j}_i(\vec{r}) = c_o\Sigma_s(\vec{r})e_c(\vec{r}) - c_o\Sigma_a(\vec{r})e_i(\vec{r}) \quad . \tag{4.23}$$

This last equation now shows the essential connection between the coherent and incoherent contributions to the sound intensity and current.

Let us use the divergence theorem to express Eqs. (4.21) and (4.23) as integral relations relating the energy intensities and currents for an arbitrary volume V bounded by the surface S; we obtain

$$\int_S \vec{j}_c(\vec{r})\cdot d\vec{S} = -\int_V c_o[\Sigma_a(\vec{r}) + \Sigma_s(\vec{r})]\,e_c(\vec{r})d\vec{r} \quad , \tag{4.24}$$

$$\int_S \vec{j}_i(\vec{r})\cdot d\vec{S} = \int_V c_o \Sigma_s(\vec{r}) e_c(\vec{r}) d\vec{r} - \int_V c_o \Sigma_a(\vec{r}) e_i(\vec{r}) d\vec{r} \quad . \qquad (4.25)$$

It is seen that the flux of coherent sound energy across the surface S is equal to the combined rates of absorption and scattering within S. The coherent energy lost as a result of scattering appears as a source of incoherent energy, as evidenced by its presence on the right hand side of Eq. (4.25). This equation states that the flux of incoherent sound energy across the surface S is equal to the rate of production of incoherent sound energy within the volume V by the scattering of the coherent energy minus the rate of absorption of incoherent sound energy throughout V. The integral form of Eq. (4.20),

$$\int \vec{j}(\vec{r})\cdot d\vec{S} = -\int_V c_o \Sigma_a(\vec{r}) e(\vec{r}) d\vec{r} \quad , \qquad (4.26)$$

expresses the over-all statement of conservation of sound energy.

C. Incoherent Scattering

We now return to the governing integral equation (3.21) for $\langle \psi(\vec{r})\psi^*(\vec{r}_o)\rangle - \langle \psi(\vec{r})\rangle\langle\psi(\vec{r}_o)\rangle^*$, from which the incoherent contributions to the intensity and current may be determined, by use of Eqs. (3.6) and (3.7) respectively. Let us assume the representation

$$\langle\psi(\vec{r})\psi^*(\vec{r}_o)\rangle - \langle\psi(\vec{r})\rangle\langle\psi(\vec{r}_o)\rangle^* = \frac{1}{4\pi}\int \Sigma_s(\vec{r}\,')\langle|\psi(\vec{r}\,')|^2\rangle L(\vec{r},\vec{r}_o;\vec{r}\,')d\vec{r}\,' \quad , \qquad (4.27)$$

and substitute it into Eq. (3.21); after the order of integration has been interchanged, we obtain

$$\frac{1}{4\pi}\int \Sigma_s(\vec{r}\,')\langle|\psi(\vec{r}\,')|^2\rangle\Big\{ L(\vec{r},\vec{r}_o;\vec{r}\,')-E(\vec{r},\vec{r}\,')E^*(\vec{r}_o,\vec{r}\,')$$
$$-\iint G(\vec{r}\,'')G^*(\vec{r}\,''')L(\vec{r}\,'',\vec{r}\,''';\vec{r}\,')E(\vec{r},\vec{r}\,'')E^*(\vec{r}_o,\vec{r}\,''')d\vec{r}\,''d\vec{r}\,'''\Big\}d\vec{r}\,'=0 \quad . \tag{4.28}$$

Therefore the representation (4.27) is consistent with the integral equation (3.21), provided that the kernel $L(\vec{r},\vec{r}_o;\vec{r}\,')$ satisfies the following integral equation:

$$L(\vec{r},\vec{r}_o;\vec{r}\,') = E(\vec{r},\vec{r}\,')E^*(\vec{r}_o,\vec{r}\,') + \iint G(\vec{r}\,'')G^*(\vec{r}\,''')L(\vec{r}\,'',\vec{r}\,''';\vec{r}\,')$$
$$E(\vec{r},\vec{r}\,'')E^*(\vec{r}_o,\vec{r}\,''')d\vec{r}\,''d\vec{r}\,''', \tag{4.29}$$

or, equivalently, the differential equation:

$$\{(\nabla^2+\kappa_o^2)(\nabla_o^2+\kappa_o^{*2})-16\pi^2 G(\vec{r})G^*(\vec{r}_o)\}\, L(\vec{r},\vec{r}_o;\vec{r}\,')$$
$$= 16\pi^2\, \delta(\vec{r}-\vec{r}\,')\delta(\vec{r}_o-\vec{r}\,') \quad . \tag{4.30}$$

The latter equation may be obtained by operating on the former equation with $(\nabla^2+\kappa_o^2)(\nabla_o^2+\kappa_o^{*2})$. Equation (4.30) was obtained by Foldy and formed the basis of his subsequent analysis. However, his mathematical development for the kernel $L(\vec{r},\vec{r}_o;\vec{r}\,')$ appears to be invalid, although his result is essentially correct in the limit of very low scatterer density. This point is further discussed in Appendix A.

Instead of considering Eq. (4.29) further, we shall determine the expression for $L(\vec{r},\vec{r}_o;\vec{r}\,')$ by requiring the representation (4.27) to be consistent with the conservation relations (4.10) and (4.13) of the previous section. If we differentiate Eq. (4.27) first with respect to $\vec{r}$, then with respect to $\vec{r}_o$, subtract, set $\vec{r}_o = \vec{r}$, and then take the divergence of each term of the resulting equation, we obtain

$$\text{div}\langle \psi^*(\vec{r})\nabla\psi(\vec{r}) - \psi(\vec{r})\nabla\psi^*(\vec{r})\rangle - \text{div}[\langle \psi(\vec{r})\rangle^* \nabla \langle \psi(\vec{r})\rangle - \langle \psi(\vec{r})\rangle \nabla \langle \psi(\vec{r})\rangle^*]$$
$$= \frac{1}{4\pi} \int \Sigma_s(\vec{r}')\langle |\psi(\vec{r}')|^2\rangle \nabla\cdot[(\nabla L - \nabla_o L)(\vec{r},\vec{r};\vec{r}')]\, d\vec{r}' \quad . \qquad (4.31)$$

Now let us employ Eq. (4.10) and (4.13) in order to eliminate these divergences, yielding

$$-2ik_o\Sigma_a(\vec{r})\langle |\psi(\vec{r})|^2\rangle + 2ik_o\Sigma_e(\vec{r})|\langle \psi(\vec{r})\rangle|^2$$
$$= \frac{1}{4\pi} \int \Sigma_s(\vec{r}')\langle |\psi(\vec{r}')|^2\rangle \nabla\cdot[(\nabla L - \nabla_o L)(\vec{r},\vec{r};\vec{r}')]\, d\vec{r}' \quad . \qquad (4.32)$$

By setting $\vec{r}_o = \vec{r}$ in Eq. (4.27), we have

$$|\langle \psi(\vec{r})\rangle|^2 = \langle |\psi(\vec{r})|^2\rangle - \frac{1}{4\pi} \int \Sigma_s(\vec{r}')\langle |\psi(\vec{r}')|^2\rangle L(\vec{r},\vec{r};\vec{r}')\, d\vec{r}' , \qquad (4.33)$$

and using this to eliminate $|\langle \psi(\vec{r})\rangle|^2$ in Eq. (4.32), we obtain

$$2ik_o\Sigma_s(\vec{r})\langle |\psi(\vec{r})|^2\rangle - 2ik_o\Sigma_e(\vec{r}) \frac{1}{4\pi} \int \Sigma_s(\vec{r}')\langle |\psi(\vec{r}')|^2\rangle L(\vec{r},\vec{r};\vec{r}')\, d\vec{r}'$$
$$= \frac{1}{4\pi} \int \Sigma_s(\vec{r}')\langle |\psi(\vec{r}')|^2\rangle \nabla\cdot[(\nabla L - \nabla_o L)(\vec{r},\vec{r};\vec{r}')]\, d\vec{r}' \quad , \qquad (4.34)$$

which may be expressed in the following manner

$$\frac{1}{4\pi} \int \Sigma_s(\vec{r}')\langle |\psi(\vec{r}')|^2\rangle \{\nabla\cdot[(\nabla L - \nabla_o L)(\vec{r},\vec{r};\vec{r}')] - 8\pi i k_o \delta(\vec{r}-\vec{r}')$$
$$+ 2ik_o\Sigma_e(\vec{r})L(\vec{r},\vec{r};\vec{r}')\}\, d\vec{r}' = 0 \quad . \qquad (4.35)$$

Therefore, in order for the kernel $L(\vec{r},\vec{r}_o;\vec{r}')$ of the representation (4.27) to be consistent with the conservation relations developed in the preceding section, it must satisfy the following relation:

$$\nabla\cdot[(\nabla L-\nabla_o L)(\vec{r},\vec{r};\vec{r}\,')] = 8\pi i k_o \delta(\vec{r}-\vec{r}\,') - 2ik_o \Sigma_e(\vec{r})L(\vec{r},\vec{r};\vec{r}\,') \quad . \qquad (4.36)$$

If we take $L(\vec{r},\vec{r}_o;\vec{r}\,')$ to be a product of a function of $\vec{r}$ times a function of $\vec{r}_o$, this relation may be written, with the aid of Eq. (4.12), as

$$\{[\nabla^2+\kappa^2(\vec{r})]\,L(\vec{r},\vec{r}_o;\vec{r}\,') - [\nabla_o^2+\kappa^{*2}(\vec{r}_o)]\,L(\vec{r},\vec{r}_o;\vec{r}\,')\}_{\vec{r}=\vec{r}_o} = 8\pi i k_o \delta(\vec{r}-\vec{r}_o) \quad . \qquad (4.37)$$

One may readily verify that a solution to this equation is

$$L(\vec{r},\vec{r}_o;\vec{r}\,') = \gamma(\vec{r}\,')K(\vec{r},\vec{r}\,')K^*(\vec{r}_o,\vec{r}\,') \quad , \qquad (4.38)$$

where $K(\vec{r},\vec{r}\,')$ is the outgoing solution of

$$[\nabla^2+\kappa^2(\vec{r})]\,K(\vec{r},\vec{r}\,') = -4\pi\delta(\vec{r}-\vec{r}\,') \qquad (4.39)$$

and

$$\gamma(\vec{r}\,') = \frac{k_o}{\mathrm{Im}\{K(\vec{r},\vec{r}\,')\}\big|_{\vec{r}=\vec{r}\,'}} \quad . \qquad (4.40)$$

Upon substitution of this expression for the kernel into Eq. (4.27), we obtain

$$\langle\psi(\vec{r})\psi^*(\vec{r}_o)\rangle - \langle\psi(\vec{r})\rangle\langle\psi(\vec{r}_o)\rangle^*$$

$$= \frac{1}{4\pi}\int \gamma(\vec{r}\,')\,\Sigma_s(\vec{r}\,')\langle|\psi(\vec{r}\,')|^2\rangle\,K(\vec{r},\vec{r}\,')K^*(\vec{r}_o,\vec{r}\,')d\vec{r}\,' \quad . \qquad (4.41)$$

In particular, the incoherent contribution to the mean square wave is obtained by setting $\vec{r}_o=\vec{r}$, yielding

$$\langle|\psi(\vec{r})|^2\rangle - |\langle\psi(\vec{r})\rangle|^2 = \frac{1}{4\pi}\int \gamma(\vec{r}\,')\Sigma_s\langle|\psi(\vec{r}\,')|^2\rangle\;|K(\vec{r},\vec{r}\,')|^2\,d\vec{r}\,' \quad ; \qquad (4.42)$$

while the following expression may be obtained for the incoherent contribution to the average current

$$\langle\psi^*(\vec{r})\nabla\psi(\vec{r})-\psi(\vec{r})\nabla\psi^*(\vec{r})\rangle - [\langle\psi(\vec{r})\rangle^*\nabla\langle\psi(\vec{r})\rangle-\langle\psi(\vec{r})\rangle\nabla\langle\psi(\vec{r})\rangle^*]$$

$$= \frac{1}{4\pi}\int\gamma(\vec{r}\,')\Sigma_s(\vec{r}\,')\langle|\psi(\vec{r}\,')|^2\rangle[K^*(\vec{r},\vec{r}\,')\nabla K(\vec{r},\vec{r}\,')-K(\vec{r},\vec{r}\,')\nabla K^*(\vec{r},\vec{r}\,')]d\vec{r}\,' . \tag{4.43}$$

D. Wave Propagation in an Infinite Scattering Medium

In order to illustrate the rather general theory presented heretofore, let us now consider the special case in which the entire space is filled by a random distribution of scatterers; assume that the scatterer number density $n(R)$ and the scatterer coefficient $g(R)$ of a single scatterer are independent of position so that the scatterer coefficient density

$$G = \int_0^\infty g(R)n(R)dR \quad , \tag{4.44}$$

propagation coefficient of the scattering medium

$$\kappa = k + i\,\frac{\alpha}{2} = (\kappa_o^2 + 4\pi G)^{\frac{1}{2}} \quad , \tag{4.45}$$

as well as the scattering and extinction cross section densities

$$\Sigma_s = 4\pi\int_0^\infty |g(R)|^2 n(R)dR \quad , \tag{4.46}$$

$$\Sigma_e = \frac{4\pi}{k_o}\int_0^\infty \mathrm{Im}\{g(R)\}n(R)dR + \alpha_o \quad , \tag{4.47}$$

are all constant. We note that the scattering coefficient of a single scatterer, along with the scatterer number density distribution, completely determines the macroscopic properties of the scattering medium.

By Eq. (4.6), the coherent wave $\langle\psi(\vec{r})\rangle$ satisfies the wave equation

$$(\nabla^2+\kappa^2)\langle\psi(\vec{r})\rangle = 0 \quad ; \tag{4.48}$$

the average pressure, coherent contribution to the average sound intensity, and coherent contribution to the average sound energy current are related to this quantity by the following formulas:

$$p(\vec{r},t) = \mathrm{Re}\{i\omega\rho_o\langle\psi(\vec{r})\rangle e^{-i\omega t}\} \quad , \tag{4.49}$$

$$e_c(\vec{r}) = \frac{\rho_o\omega^2}{2c_o^{\,2}} \; |\langle\psi(\vec{r})\rangle|^2 \quad , \tag{4.50}$$

$$\vec{j}_c(\vec{r}) = \frac{\rho_o\omega}{4i} \; [\langle\psi(\vec{r})\rangle^*\nabla\langle\psi(\vec{r})\rangle - \langle\psi(\vec{r})\rangle\nabla\langle\psi(\vec{r})\rangle^*] \quad . \tag{4.51}$$

Therefore the properties of the coherent radiation are completely determined by the solution of the wave equation (4.48).

For the case of constant propagation coefficient, Eq. (4.39) has the solution

$$K(\vec{r},\vec{r}\,') = \frac{e^{i\kappa|\vec{r}-\vec{r}\,'|}}{|\vec{r}-\vec{r}\,'|} \quad , \tag{4.52}$$

so that $\gamma(\vec{r}\,')$, defined by Eq. (4.40), is given by

$$\gamma(\vec{r}\,') = k_o/\mathrm{Im}\left\{\frac{e^{i\kappa|\vec{r}-\vec{r}\,'|}}{|\vec{r}-\vec{r}\,'|}\right\}\Bigg|_{\vec{r}=\vec{r}\,'} = \frac{k_o}{k} = \frac{c}{c_o} \quad , \tag{4.53}$$

i.e. the ratio of the phase velocity of the scattering medium to the phase velocity of the medium when no scatterers are present. By multiplying Eq. (4.42) by $\frac{\rho_o \omega^2}{2c_o^2}$ and substituting these expressions for $K(\vec{r},\vec{r}\,')$ and $\gamma(\vec{r}\,')$, we obtain the following governing equation for the incoherent contribution to the average sound intensity

$$e_i(\vec{r}) = \int \frac{c\Sigma_s}{c_o} \left[e_c(\vec{r}\,') + e_i(\vec{r}\,') \right] \frac{e^{-\alpha|\vec{r}-\vec{r}\,'|}}{4\pi|\vec{r}-\vec{r}\,'|^2} \, d\vec{r}\,' . \tag{4.54}$$

By Eqs. (4.7) and (4.12), we have

$$\frac{c}{c_o} = \frac{2ik_o}{\kappa^2 - \kappa^{*2}} (-i)(\kappa - \kappa^*) = \frac{\alpha}{\Sigma_e} . \tag{4.55}$$

Let us denote the ratio of scattering to extinction cross section densities by

$$\beta = \frac{\Sigma_s}{\Sigma_e} , \tag{4.56}$$

and base distances on the attenuation length in the scattering medium by defining the coordinate $\vec{\tau} = \alpha\vec{r}$; Eq. (4.54) may then be written as

$$e_i(\vec{\tau}) = \beta \int \left[e_c(\vec{\tau}\,') + e_i(\vec{\tau}\,') \right] \frac{e^{-|\vec{\tau}-\vec{\tau}\,'|}}{4\pi|\vec{\tau}-\vec{\tau}\,'|^2} \, d\vec{\tau}\,' . \tag{4.57}$$

Here we have the very interesting result, that for this special case in which the scatterer density is uniform throughout all space, the governing equation for the incoherent contribution to the average sound intensity is identical in form and physical interpretation to the Boltzmann integral equation describing the transport of monoenergetic neutrons in an infinite homogeneous medium[10]. That is, we need only to replace $e_i(\vec{\tau})$ by the neutron density $n(\vec{\tau})$ and $\beta e_c(\vec{\tau})$ by the neutron source density $S(\vec{\tau})$ in order to obtain the fundamental neutron transport equation; β retains the same significance as the ratio of the scattering to total macroscopic cross section. Thus it is seen that the incoherent radiation is governed by an equation displaying "particle" aspects, as opposed to the coherent radiation which is governed by a "wave" equation. It should be noted that in the preceding general theory no assumption has been made that the wavelength is small compared with the average distance between scatterers. Therefore, there is no reason to assume a priori that the incoherent radiation satisfies the transport equation for particles, since there are no wave "packets" which may be treated as independent discrete entities. Indeed, as will be illustrated later by specific examples, this strict particle analogy results only for this one special case in which there are no discontinuities in the average scattering characteristics of the medium, with a resulting absence of any specular reflection or refraction.

By substituting the expression (4.52) for $K(\vec{r},\vec{r}\,')$ into Eq. (4.43) and using the definitions (4.14) and (4.15), we obtain the following expression for the incoherent contribution to the average sound energy current:

$$\vec{j}_i(\vec{r}) = \int c_o\Sigma_s[\,e_c(\vec{r}\,')+e_i(\vec{r}\,')\,]\;\frac{(\vec{r}-\vec{r}\,')}{4\pi|\vec{r}-\vec{r}\,'|^3}\;e^{-\alpha|\vec{r}-\vec{r}\,'|}\,d\vec{r}\,'. \quad (4.58)$$

The physical consequence of this equation is easily understood since it is an exact dual of the corresponding equation for particles. Here $c_o\Sigma_s[\,e_c(\vec{r}\,')+e_i(\vec{r}\,')]$ is the rate that incoherent radiation is scattered away from $d\vec{r}\,'$ (recall from Eq. (4.23) that $c_o\Sigma_s e_c(\vec{r}\,')$ is the rate of production of incoherent sound energy per unit volume due to the scattering of the coherent wave). This, multiplied by the kernel, gives the average current at $\vec{r}$ due to radiation scattered from $d\vec{r}\,'$; integration over all space then yields the total incoherent contribution to the average current. Again, it is only for this special case that such a "particle" interpretation is possible.

The average sound intensity, which is proportional to the average mean square pressure, is obtained by summing the coherent and incoherent contributions,

$$e(\vec{r}) = \frac{1}{\rho_o c_o^2} \langle \overline{p(\vec{r}, t)^2} \rangle = e_c(\vec{r}) + e_i(\vec{r}) \quad , \tag{4.59}$$

and the average sound energy current is computed in a similar manner,

$$\vec{j}(\vec{r}) = \langle \overline{p(\vec{r}, t)\vec{V}(\vec{r}, t)} \rangle = \vec{j}_c(\vec{r}) + \vec{j}_i(\vec{r}) \quad . \tag{4.60}$$

If we assume that our previous "ergodic" hypothesis applies, the above configurational averages of mean quantities are equal to the time averages of the quantities, taken over a sufficiently long period of time.

V. TRANSMISSION OF SOUND FROM A PLANE SOURCE INTO A SEMI-INFINITE SCATTERING MEDIUM

As a specific example of the foregoing theory of the multiple scattering of waves, let us consider the following problem. An infinite plane sound source is located at the plane $x = 0$ and emits a plane sound wave into the half-space $x > 0$, which is occupied by a medium containing a random distribution of scatterers; the scatterer number density and the scatterer coefficient are independent of position throughout the half-space. The motion of the source is harmonic and its velocity is given by

$$V(t) = A \cos \omega t \; . \tag{5.1}$$

Sound will be both absorbed and scattered by the scatterers; sound scattered back to the source at $x = 0$ is reflected back into the medium. The problem is to calculate the average sound intensity and the average sound energy current at an arbitrary distance x from the source.

The macroscopic properties of the scattering medium will be denoted by Eqs. (4.44), (4.45), (4.46), and (4.47) of the preceding section. The coherent wave satisfies the wave equation (4.48); the outgoing solution of this equation satisfying the boundary condition (5.1) at $x = 0$ is

$$\langle \psi(x) \rangle = \frac{A}{i\kappa} e^{i\kappa x} \; , \tag{5.2}$$

and hence the coherent contributions to the average sound intensity and the average sound energy current are given by

$$e_c(x) = \frac{\rho_o \omega^2}{2c_o^2} \left|\langle \psi(x) \rangle\right|^2 = \frac{\rho_o \omega^2 A^2}{2c_o^2 |\kappa|^2} e^{-\alpha x} , \qquad (5.3)$$

$$j_c(x) = \frac{\rho_o \omega}{4i} \left[\langle \psi(x) \rangle^* \frac{\partial}{\partial x} \langle \psi(x) \rangle - \langle \psi(x) \rangle \frac{\partial}{\partial x} \langle \psi(x) \rangle^* \right] = \frac{\rho_o \omega k A^2}{2|\kappa|^2} e^{-\alpha x} . \qquad (5.4)$$

Let us choose the amplitude A such that the average current leaving the source is unity; this requires that

$$\frac{\rho_o \omega k A^2}{2|\kappa|^2} = 1 . \qquad (5.5)$$

The solution for other amplitudes may be determined by a simple scaling operation. With this normalization, the coherent contributions to the intensity and current may be expressed in terms of the dimensionless distance $\tau = \alpha x$, as follows:

$$e_c(\tau) = \frac{c}{c_o^2} e^{-\tau} , \qquad (5.6)$$

$$j_c(\tau) = e^{-\tau} . \qquad (5.7)$$

From Eq. (4.42) we obtain the following integral equation for the average sound intensity:

$$e(x) = e_c(x) + \frac{\Sigma_s}{4\pi} \int \gamma(x') e(x') \left|K(\vec{i}x, \vec{r}\,')\right|^2 d\vec{r}\,' , \qquad (5.8)$$

where $K(\vec{r},\vec{r}\,')$ is the outgoing solution of

$$(\nabla^2 + \kappa^2)K(\vec{r},\vec{r}\,') = -4\pi\,\delta(\vec{r}-\vec{r}\,') \quad . \tag{5.9}$$

We shall impose the boundary condition that at $x = 0$ the normal gradient of the sum of $K(\vec{r},\vec{r}\,')$ and the coherent wave $\langle \psi(x) \rangle$ is equal to the velocity of the source; that is

$$\frac{\partial}{\partial x}\Big[K(\vec{r},\vec{r}\,') + \langle \psi(x) \rangle\Big]\Big|_{x=0} = V(t) \ , \tag{5.10}$$

which requires that the normal gradient of $K(\vec{r},\vec{r}\,')$ vanish at $x = 0$,

$$\frac{\partial}{\partial x} K(\vec{r},\vec{r}\,')\Big|_{x=0} = 0 \ . \tag{5.11}$$

Hence, the movement of the source does not influence the reflection of the incoherent sound.

The solution of the differential equation (5.9) satisfying the boundary condition (5.11) may be easily determined by use of the method of images.[11] Let us represent the position $\vec{r}\,'$ in cylindrical coordinates about the x-axis as $\vec{r}\,' = \vec{i}x' + \vec{e}_r R$; the image point of $\vec{r}\,'$ is $\vec{r}\,'' = -\vec{i}x' + \vec{e}_r R$. The desired solution consists of the sum of the outgoing spherical waves from $\vec{r}\,'$ and from the image point $\vec{r}\,''$,

$$K(\vec{r},\vec{r}\,') = \frac{e^{i\kappa|\vec{r}-\vec{r}\,'|}}{|\vec{r}-\vec{r}\,'|} + \frac{e^{i\kappa|\vec{r}-\vec{r}\,''|}}{|\vec{r}-\vec{r}\,''|} \quad . \tag{5.12}$$

Let us define the one-dimensional kernel

$$L(x,x') = \frac{1}{2\pi}\int_0^\infty |K(\vec{i}x,\vec{r}\,')|^2\, 2\pi R\, dR\ , \tag{5.13}$$

so that the integral equation (5.8) may be written as

$$e(x) = e_c(x) + \frac{\Sigma_s}{2}\int_0^\infty \gamma(x')e(x')L(x,x')\, dx'\ . \tag{5.14}$$

The factor $\gamma(x')$ is easily determined by substitution of the solution (5.12) into the definition (4.40),

$$\gamma(x') = \frac{k_o}{\mathrm{Im}\left\{K(\vec{r},\vec{r}\,')\right\}\Big|_{\vec{r}=\vec{r}\,'}} = \frac{k_o}{k}\left[1 + \frac{\sin 2kx'}{2kx'}e^{-\alpha x'}\right]^{-1}\ . \tag{5.15}$$

At a point many wavelengths removed from the origin, $2kx' \gg 1$, this result becomes essentially equal to the expression (4.53) for the infinite medium. Upon substitution of the solution (5.12) into the expression (5.13) for the one-dimensional kernel, we find that it may be expressed as the sum of three terms,

$$L(x,x') = \sum_{n=1}^{3} L^{(n)}(x,x') = \int_0^\infty \frac{e^{-\alpha|\vec{i}x-\vec{r}\,'|}}{|\vec{i}x-\vec{r}\,'|^2} R\, dR + \int_0^\infty \frac{e^{-\alpha|\vec{i}x-\vec{r}\,''|}}{|\vec{i}x-\vec{r}\,''|^2} R\, dR$$

$$+ 2\,\mathrm{Re}\int_0^\infty \frac{e^{i\kappa|\vec{i}x-\vec{r}\,'|-i\kappa^*|\vec{i}x-\vec{r}\,''|}}{|\vec{i}x-\vec{r}\,'|\,|\vec{i}x-\vec{r}\,''|} R\, dR\ , \tag{5.16}$$

since by Eq. (4.45), $i(\kappa-\kappa^*) = -\alpha$. In the first integral, let us make the change of variables from R to u, by defining $u = |\vec{i}x-\vec{r}\,'|/|x-x'| =$

$[|x-x'|^2 + R^2]^{\frac{1}{2}}/|x-x'|$; this yields

$$L^{(1)}(x,x') = \int_1^{\infty} e^{-\alpha|x-x'|u} \frac{du}{u} = E_1(\alpha|x-x'|) \quad , \tag{5.17}$$

where the exponential integral functions $E_n(t)$ are defined, for $t > 0$ by[12]

$$E_n(t) = \int_1^{\infty} e^{-tu} u^{-n} \, du = \int_0^1 e^{-t/v} v^{n-2} \, dv \quad . \tag{5.18}$$

Similarly, by setting $u = |\vec{i}x - \vec{r}''|/(x+x') = [(x+x')^2 + R^2]^{\frac{1}{2}}/(x+x')$ in the second integral of Eq. (5.16), we obtain

$$L^{(2)}(x,x') = \int_1^{\infty} e^{-\alpha(x+x')u} \frac{du}{u} = E_1(\alpha(x+x')) \quad . \tag{5.19}$$

In the third integral let us set $u = [|\vec{i}x - \vec{r}'| + |\vec{i}x - \vec{r}''|]/[|x-x'| + (x+x')]$; this term may then be transformed to

$$L^{(3)}(x,x') = 2\,\mathrm{Re}\int_1^{\infty} e^{ik[|x-x'|-(x+x')]\frac{1}{u} - \frac{\alpha}{2}[|x-x'|+(x+x')]u} \frac{du}{u} \quad . \tag{5.20}$$

Therefore, the one-dimensional kernel, expressed in terms of the dimensionless coordinate $\tau = \alpha x$, is given by

$$L(\tau,\tau') = E_1(|\tau - \tau'|) + E_2(\tau+\tau') + L^{(3)}(\tau,\tau') \quad , \tag{5.21}$$

where

$$L^{(3)}(\tau,\tau') = \begin{cases} 2\int_0^1 \cos \lambda\tau' v e^{-\tau/v} \frac{dv}{v} & \text{for } \tau' < \tau \\ 2\int_0^1 \cos \lambda\tau v e^{-\tau'/v} \frac{dv}{v} & \text{for } \tau' > \tau , \end{cases} \tag{5.22}$$

with $\lambda = 2k/\alpha$.

Using the relation (4.55) and the definition (4.56) of β, the integral equation (5.14) for the average sound intensity may be written as

$$e(\tau) = e_c(\tau) + \frac{\beta}{2}\int_0^\infty \left[1 + \frac{\sin \lambda\tau'}{\lambda\tau'} e^{-\tau'}\right]^{-1} e(\tau')L(\tau,\tau')\, d\tau' . \tag{5.23}$$

Let us now assume that the attenuation length $1/\alpha$ of the scattering medium is much larger than the wavelength $2\pi/k$ so that $\lambda = 2k/\alpha \gg 1$; this is normally the case for multiple scattering problems of practical interest. Then except for small values of τ and τ', the integrals defining $L^{(3)}(\tau,\tau')$ will have a rapidly oscillating integrand and hence would be expected to be negligible compared with the first two terms of Eq. (5.21); this is substantiated by the asymptotic representation of $L^{(3)}(\tau,\tau')$, which may be determined by integration by parts[13]

$$L^{(3)}(\tau,\tau') \sim \begin{cases} \frac{2}{\lambda\tau'} \sin \lambda\tau' e^{-\tau} & \text{as } \lambda\tau' \to \infty \quad \text{for } \tau' < \tau \\ \frac{2}{\lambda\tau} \sin \lambda\tau e^{-\tau'} & \text{as } \lambda\tau \to \infty \quad \text{for } \tau' > \tau . \end{cases} \tag{5.24}$$

Therefore at all points $\tau \gg 1/\lambda$ (i.e. x several wavelengths removed from the origin), the third term of the kernel (5.21) is negligible over most of the range of integration of τ'. Although this term is of the same order as the first two terms over a limited range about the origin, it oscillates rapidly in τ' and hence contributes negligibly to the integral in Eq. (5.23). Therefore in our subsequent calculations, we shall neglect $L^{(3)}(\tau,\tau')$ entirely; consistent with this approximation, the factor within the bracket will be replaced by unity. The integral equation for the average sound intensity then becomes

$$e(\tau) = e_c(\tau) + \frac{\beta}{2}\int_0^{\infty} e(\tau')[E_1(|\tau-\tau'|)+E_1(\tau+\tau')]\, d\tau' . \qquad (5.25)$$

In the above equation, let us substitute the expression (5.6) for $e_c(\tau)$ and replace τ' by $-\tau'$ in the second term of the integral giving

$$e(\tau) = \frac{c}{c_o^2} e^{-\tau} + \frac{\beta}{2}\int_0^{\infty} e(\tau')E_1(|\tau-\tau'|)\, d\tau' + \frac{\beta}{2}\int_{-\infty}^{0} e(-\tau')E_1(\tau-\tau')\, d\tau' . \qquad (5.26)$$

The following equation, with $e(\tau)$ defined as an even function on $(-\infty,\infty)$, is equivalent to the above:

$$e(\tau) = \frac{c}{c_o^2} e^{-|\tau|} + \frac{\beta}{2}\int_{-\infty}^{\infty} e(\tau')E_1(|\tau-\tau'|)\, d\tau' . \qquad (5.27)$$

This equation may be interpreted as describing the problem of an infinite plane sound source located at $x = 0$ in an infinite scattering medium and emitting plane waves in both the positive and negative directions. The construction of the latter source is such that the scattered sound may pass through it unimpeded. Since, by symmetry, the net flux of incoherent sound energy across the plane $\tau = 0$ is zero in this latter problem, it is essentially equivalent to the former problem (in the limit $\lambda \to \infty$).

The integral equation (5.27) may be solved exactly by use of the Fourier transform,

$$\hat{f}(s) = \mathfrak{F}\{f(\tau)\} = \int_{-\infty}^{\infty} f(\tau)e^{-is\tau}\, d\tau \quad , \tag{5.28}$$

its inversion formula,

$$f(\tau) = \mathfrak{F}^{-1}\{\hat{f}(s)\} = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(s)e^{is\tau}\, ds \quad , \tag{5.29}$$

and the convolution theorem for Fourier transforms,

$$\mathfrak{F}\left\{\int_{-\infty}^{\infty} f(t)g(\tau-t)\, dt\right\} = \mathfrak{F}\{f(\tau)\}\,\mathfrak{F}\{g(\tau)\} = \hat{f}(s)\,\hat{g}(s) \quad . \tag{5.30}$$

Transformation of each term of the integral equation (5.27) reduces it to an algebraic equation in the transforms

$$\hat{e}(s) = \frac{c}{c_o^2}\,\mathfrak{F}\{e^{-|\tau|}\} + \frac{\beta}{2}\,\hat{e}(s)\,\mathfrak{F}\{E_1(|\tau|)\} \quad . \tag{5.31}$$

The above Fourier transforms are easily evaluated:

$$\mathfrak{F}\{E_1(|\tau|)\} = \int_{-\infty}^{\infty}\left\{\int_0^1 e^{-|\tau| \, v} \frac{dv}{v}\right\} e^{-is\tau}\, d\tau$$

$$= \int_0^1\left\{\int_{-\infty}^{\infty} e^{-|\tau|/v - is\tau}\, d\tau\right\} \frac{dv}{v}$$

$$= \int_0^1\left\{\frac{1}{\frac{1}{v} + is} + \frac{1}{\frac{1}{v} - is}\right\}\frac{dv}{v} = \frac{1}{is}\log\frac{1+is}{1-is} = 2\tan^{-1} s/s\,, \tag{5.32}$$

$$\mathfrak{F}\{e^{-|\tau|}\} = \int_{-\infty}^{\infty} e^{-|\tau| - is\tau}\, d\tau = \frac{2}{1+s^2}\quad . \tag{5.33}$$

After substituting these results in Eq. (5.31) and solving for $\hat{e}(s)$, we obtain the following expression for the transform of $e(\tau)$:

$$\hat{e}(s) = \frac{c}{c_o^2}\;\frac{2}{1+s^2}\;\frac{1}{1 - \beta\tan^{-1} s/s}\quad . \tag{5.34}$$

The inversion integral (5.29) may now be applied to yield a representation for the average sound intensity,

$$e(\tau) = \frac{c}{c_o^2}\;\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{2}{1+s^2}\;\frac{e^{is\tau}}{1-\beta\tan^{-1} s/s}\, ds\ . \tag{5.35}$$

This integral may be transformed by contour integration to a form amenable to numerical evaluation; the details of this process are given in Appendix B. The result for $\tau > 0$ is

$$e(\tau) = \frac{c}{c_o^2}\left\{\frac{2\sigma_o}{\beta + \sigma_o^2 - 1} e^{-\sigma_o\tau} - \int_o^\infty g(w;\beta,1)e^{-\tau/\tanh w}\, dw\right\}, \quad (5.36)$$

where

$$g(w;\beta,1) = \frac{\beta(\tanh w)^n}{(1 - \beta w \tanh w)^2 + (\frac{\pi\beta}{2} \tanh w)^2}, \quad (5.37)$$

and σ_o is the positive root of

$$\sigma_o = \tanh \frac{\sigma_o}{\beta} \quad . \quad (5.38)$$

In order to compute the average sound energy current $j(x)$ at a distance x from the source, use may be made of the integral relation (4.26) describing conservation of sound energy. Let us consider a cylindrical volume having unit cross sectional area and bounded by planes at distances x and $a > x$ from the origin. Application of Eq. (4.26) to this volume yields

$$j(a) - j(x) = - c_o\Sigma_a \int_x^a e(x')\, dx' \; ; \quad (5.39)$$

since the current must vanish at a in the limit $a \rightarrow \infty$, we have

$$j(x) = c_o\Sigma_a\int_x^\infty e(x')\, dx' \quad . \quad (5.40)$$

This may be written in terms of the dimensionless length $\tau = \alpha x$ as

$$j(\tau) = \frac{c_o\Sigma_a}{\alpha}\int_\tau^\infty e(\tau')\, d\tau' \quad . \quad (5.41)$$

Upon substitution of the expression (5.36) for $e(\tau')$, we obtain

$$j(\tau) = \frac{c\Sigma_a}{c_o\alpha} \int_\tau^\infty \left\{ \frac{2\sigma_o}{\beta + \sigma_o^2 - 1} e^{-\sigma_o \tau'} - \int_o^\infty g(w;\beta,1) e^{-\tau'/\tanh w} \, dw \;\; d\tau' \right\} ; \tag{5.42}$$

by Eqs. (4.55) and (4.56) this becomes, after interchanging the order of integration, equal to

$$j(\tau) = (1-\beta) \left\{ \frac{2\sigma_o}{\beta + \sigma_o^2 - 1} \int_\tau^\infty e^{-\sigma_o \tau'} \, d\tau' - \int_o^\infty g(w;\beta,1) \int_\tau^\infty e^{-\tau'/\tanh w} \, d\tau' \, dw \right\}. \tag{5.43}$$

Therefore the average sound energy current is given by

$$j(\tau) = (1-\beta) \left\{ \frac{2}{\beta + \sigma_o^2 - 1} e^{-\sigma_o \tau} - \int_o^\infty g(w;\beta,2) e^{-\tau/\tanh w} \, dw \right\}. \tag{5.44}$$

The solutions (5.36) and (5.44) require numerical evaluation. The root σ_o may be determined by iteration of Eq. (5.38), using Newton's method. In order to evaluate the integrals, a value $w_o >> 1$ may be chosen such that $(\beta w - 1) > 0$ and $\tanh w \cong 1$ for $w \geq w_o$. After replacing $\tanh w$ by unity, the portion of the integrals from w_o to ∞ may be carried out analytically, with the result:

$$\int_o^\infty g(w;\beta,n) e^{-\tau/\tanh w} \, dw \cong \int_o^{w_o} g(w;\beta,n) e^{-\tau/\tanh w} \, dw$$

$$+ \frac{1}{\beta} \left[1 - \frac{2}{\pi} \tan^{-1} \frac{2}{\pi\beta} (\beta w_o - 1) \right] e^{-\tau} \; . \tag{5.45}$$

The integrals over the finite range may be evaluated numerically, using Simpson's rule.

The results of the numerical evaluation of the solutions (5.36) and (5.44) for $\frac{c_o^2}{c}\, e(\tau)$ and $j(\tau)$ are displayed in Figures 1 and 2. For small values of $\beta = \Sigma_s/\Sigma_e$, the solutions approach closely the solutions (5.6) and (5.7) for the coherent wave. As the cross section density ratio β approaches unity, the effects of the incoherent radiation on the transmitted sound become important. It is noted that the dependence of $e(\tau)$ and $j(\tau)$ on τ is almost exponential, although this is not apparent from the solution forms.

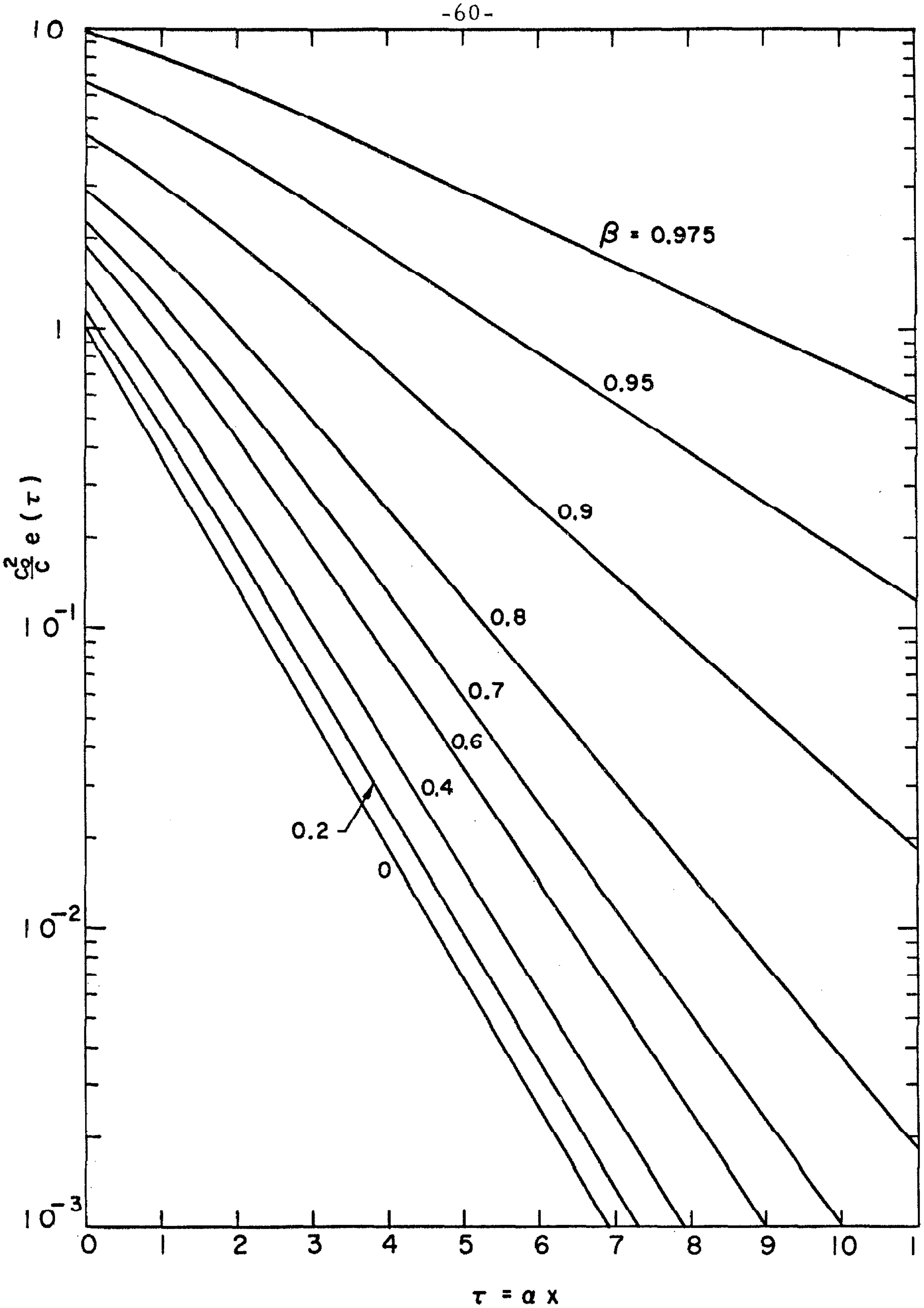

Figure 1 - Sound Intensity vs. Distance

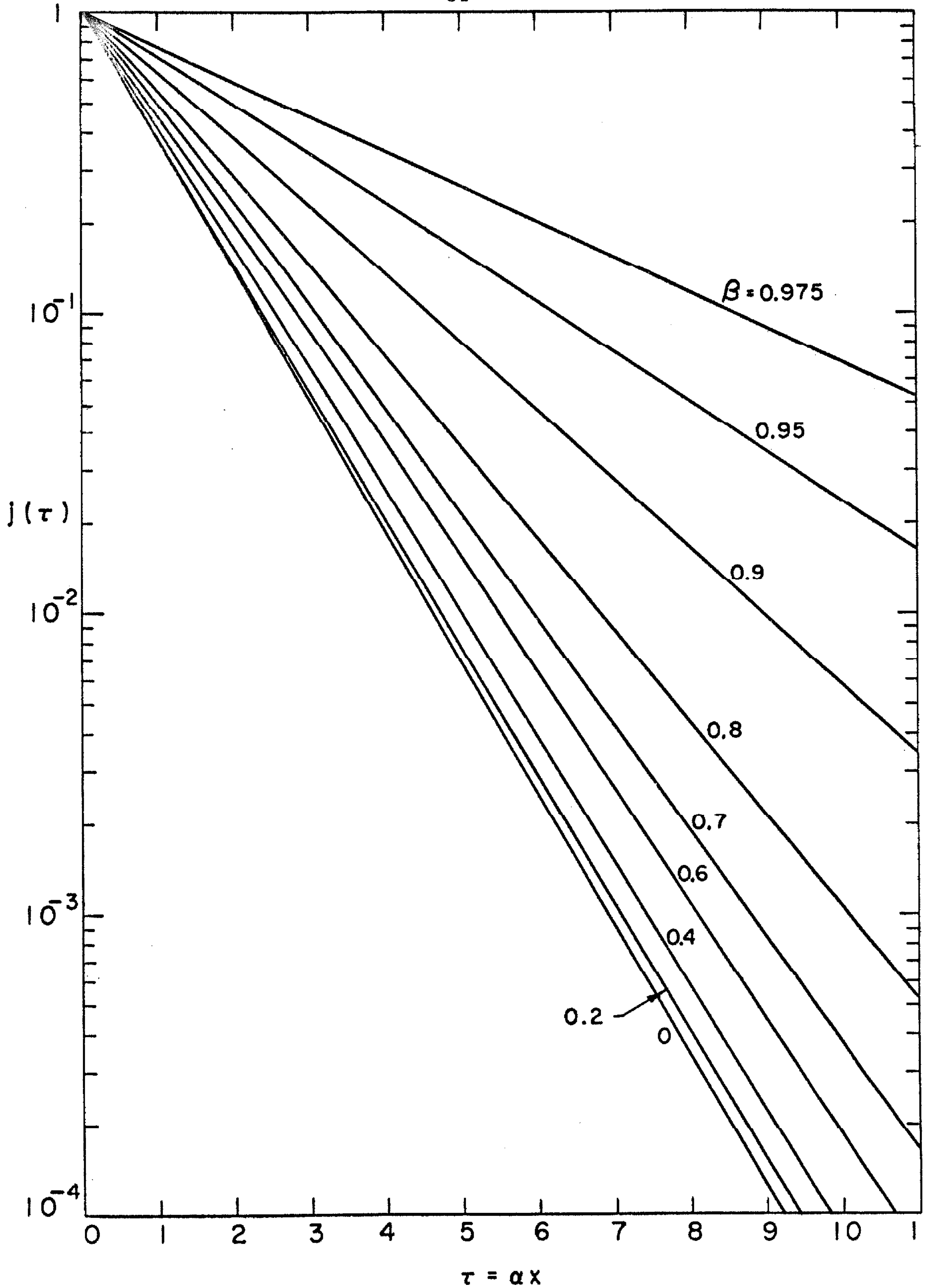

Figure 2 - Sound Energy Current vs. Distance

VI. REFLECTION OF AN INCIDENT WAVE BY A SCATTERING HALF-SPACE

As a further application of the general theory of multiple scattering of waves, we shall now examine the effect of a discontinuity in the scatterer density distribution by considering the specific problem of the reflection of a plane wave by a half-space filled by a scattering medium. The scatterers are assumed to all lie to the right of a given plane; the scatterer density distribution is assumed to be known and independent of position in this region. Hence, if we orient our coordinate axes such that the scatterers are located in the half space $x > 0$, the scatterer density distribution may be expressed by

$$n(\vec{r}, R) = n(R)H(x) \quad , \tag{6.1}$$

where $n(R)$ is the scatterer density per unit radius interval and $H(x)$ is the Heaviside step function, zero for negative argument and unity for positive argument. A plane sound wave is assumed to be incident to the boundary plane of the half-space, making an angle θ_o with the normal. Our objective is to determine an expression for the reflection coefficient, defined as the fraction of the sound energy flux transmitted through a unit area normal to the boundary plane that is reflected back through it. The calculation of this quantity requires separate consideration of the coherent and incoherent contribution to the reflected current.

We shall take the matrix medium to be non-dissipative, with a real propagation constant k_o. The propagation coefficient of the coherent wave is given by

$$\kappa^2(x) = k_o^2 + 4\pi\int_0^\infty g(R)n(\vec{r},R)\,dR = \begin{cases} k_o^2 & \text{for } x < 0 \\ \kappa^2 = k_o^2 + 4\pi G & \text{for } x > 0\ , \end{cases} \qquad (6.2)$$

where

$$G = \int_0^\infty g(R)n(R)\,dR \quad . \qquad (6.3)$$

As before, the real and imaginary parts of the complex propagation coefficient κ define the real propagation constant k and attenuation constant α,

$$\kappa = (k_o^2 + 4\pi G)^{\frac{1}{2}} = k + i\,\frac{\alpha}{2}\,. \qquad (6.4)$$

The coherent wave $\langle\psi(\vec{r})\rangle$ satisfies the differential equation

$$[\nabla^2 + \kappa^2(x)]\,\langle\psi(\vec{r})\rangle = 0 \quad . \qquad (6.5)$$

In addition to this, boundary conditions are established at $x = 0$ by the required continuity of pressure and normal velocity at the interface.

Let us orient the coordinate axes such that the incident plane wave, making an angle θ_o with the x-axis, may be expressed in terms of the spatial variables x and y as

$$\psi_i(x,y) = A e^{ik_o(x\cos\theta_o + y\sin\theta_o)} \quad . \qquad (6.6)$$

The x-component of the sound energy current of the incident wave is proportional to

$$\psi_i^*(x,y)\,\frac{\partial}{\partial x}\,\psi_i(x,y) - \psi_i(x,y)\,\frac{\partial}{\partial x}\,\psi_i^*(x,y) = 2ik_o\mu_o|A|^2 \quad , \qquad (6.7)$$

where we denote $\mu_o = \cos\theta_o$. In the region $x < 0$, the coherent wave is composed of the incident wave and the reflected wave,

$$\langle\psi(x,y)\rangle = Ae^{ik_o(x\cos\theta_o + y\sin\theta_o)} + Be^{ik_o(-x\cos\theta_o + y\sin\theta_o)} \qquad (x < 0) \; , \qquad (6.8)$$

while the transmitted wave in the region $x > 0$ may be written as

$$\langle\psi(x,y)\rangle = Ce^{i\kappa(x\cos\theta + y\sin\theta)} \qquad (x > 0) \quad , \qquad (6.9)$$

where θ is a complex constant. The boundary condition of continuity of pressure at the plane $x = 0$ yields

$$(A + B)e^{ik_o y\sin\theta_o} = Ce^{i\kappa y\sin\theta} \; , \qquad (6.10)$$

while that of continuity of the normal velocity component at $x = 0$ gives

$$k_o\mu_o(A - B)e^{ik_o y\sin\theta_o} = \kappa\cos\theta\, C\, e^{i\kappa y\sin\theta} \; . \qquad (6.11)$$

These are satisfied provided that

$$k_o\sin\theta_o = \kappa\sin\theta \quad , \qquad (6.12)$$

$$A + B = C \quad , \qquad (6.13)$$

$$k_o\mu_o(A - B) = \kappa \cos \theta C \quad . \tag{6.14}$$

The latter two equations may be used to express B and C in terms of A,

$$B = - \frac{\kappa \cos \theta - k_o\mu_o}{\kappa \cos \theta + k_o\mu_o} A \quad , \tag{6.15}$$

$$C = \frac{2k_o\mu_o}{\kappa \cos \theta + k_o\mu_o} A \quad , \tag{6.16}$$

while from the first we obtain

$$\kappa \cos \theta = \kappa(1 - \sin^2 \theta)^{\frac{1}{2}} = (\kappa^2 - k_o^2 \sin^2 \theta_o)^{\frac{1}{2}} = (k_o^2 \cos^2 \theta_o + 4\pi G)^{\frac{1}{2}} \quad . \tag{6.17}$$

Let the real variable μ be defined to denote the real and imaginary parts of $\kappa \cos \theta$ as follows:

$$\kappa \cos \theta = (k_o^2 \mu_o^2 + 4\pi G)^{\frac{1}{2}} = k\mu + i \frac{\alpha}{2\mu} \quad ; \tag{6.18}$$

this is consistent since

$$\mathrm{Im}\,\{k_o^2 \mu_o^2 + 4\pi G\} = \mathrm{Im}\,\{k_o^2 + 4\pi G\} = \mathrm{Im}\,\{(k + i\frac{\alpha}{2})^2\} = k\alpha \quad . \tag{6.19}$$

Substitution of Eqs. (6.12), (6.15), and (6.16) into Eqs. (6.8) and (6.9) yields the following expressions for the coherent wave:

$$\langle \psi(x,y)\rangle = A\left\{ e^{ik_o\mu_o x} - \frac{k\mu - k_o\mu_o + i\frac{\alpha}{2\mu}}{k\mu + k_o\mu_o + i\frac{\alpha}{2\mu}} e^{-ik_o\mu_o x}\right\} e^{ik_o \sin\theta_o y} \quad \text{for } x < 0 , \tag{6.20}$$

$$\langle \psi(x,y)\rangle = \frac{2k_o\mu_o A}{k\mu + k_o\mu_o + i\frac{\alpha}{2\mu}} e^{i(k\mu x + k_o \sin\theta_o y) - \frac{\alpha x}{2\mu}} \quad \text{for } x > 0 . \tag{6.21}$$

The coherent contribution to the x-component of the average sound energy current for $x < 0$ is proportional to

$$\langle \psi(x,y)\rangle^* \frac{\partial}{\partial x} \langle \psi(x,y)\rangle - \langle \psi(x,y)\rangle \frac{\partial}{\partial x} \langle \psi(x,y)\rangle^*$$

$$= 2ik_o\mu_o |A|^2 \left\{ 1 - \frac{(k\mu - k_o\mu_o)^2 + (\frac{\alpha}{2\mu})^2}{(k\mu + k_o\mu_o)^2 + (\frac{\alpha}{2\mu})^2} \right\} . \tag{6.22}$$

Comparison of this result with the expression (6.7) for the x-component of the current of the incident wave yields the following expression for the reflection coefficient of the coherent wave:

$$R_c(\mu_o) = \frac{(k\mu - k_o\mu_o)^2 + (\frac{\alpha}{2\mu})^2}{(k\mu + k_o\mu_o)^2 + (\frac{\alpha}{2\mu})^2} . \tag{6.23}$$

Equation (6.21) describing the coherent wave for $x > 0$ may be written as

$$\langle \psi(x,y)\rangle = \langle \psi(x)\rangle e^{ik_o \sin\theta_o y} , \tag{6.24}$$

where we define

$$\langle \psi(x) \rangle = \frac{2k_o \mu_o A}{k\mu + k_o \mu_o + i\frac{\alpha}{2\mu}} e^{ik\mu x - \frac{\alpha x}{2\mu}} \quad . \tag{6.25}$$

Hence for any two points x and x_o, we have

$$\langle \psi(x,y) \rangle \langle \psi(x_o,y) \rangle^* = \langle \psi(x) \rangle \langle \psi(x_o) \rangle^*$$

$$= \frac{4k_o^2 \mu_o^2 |A|^2}{(k\mu + k_o\mu_o)^2 + (\frac{\alpha}{2\mu})^2} e^{ik\mu(x-x_o) - \alpha(x+x_o)/2\mu} , \tag{6.26}$$

or for the special case $x = x_o$,

$$|\langle \psi(x,y) \rangle|^2 = |\langle \psi(x) \rangle|^2 = \frac{4k_o^2 \mu_o^2 |A|^2}{(k\mu + k_o\mu_o)^2 + (\frac{\alpha}{2\mu})^2} e^{-\alpha x/\mu} \quad . \tag{6.27}$$

Since the corresponding incoherent quantities are also independent of y, the total quantities $\langle \psi(x,y)\psi^*(x_o,y) \rangle$ and $\langle |\psi(x,y)|^2 \rangle$ are also independent of y; in order to emphasize this, as well as to simplify the notation, let us suppress the y-dependence in such quadratic quantities and write

$$\langle \psi(x)\psi^*(x_o) \rangle = \langle \psi(x,y)\psi^*(x_o,y) \rangle \quad , \tag{6.28}$$

$$\langle |\psi(x)|^2 \rangle = \langle |\psi(x,y)|^2 \rangle \quad , \tag{6.29}$$

throughout our subsequent calculations.

Application of the basic integral equation (4.41) for the incoherent quantities to the problem at hand yields

$$\langle \psi(x)\psi^*(x_o)\rangle - \langle \psi(x)\rangle \langle \psi(x_o)\rangle^*$$

$$= \frac{\Sigma_s}{4\pi} \int_{(x'>0)} \gamma(x')\langle |\psi(x')|^2 \rangle K(\vec{i}x, \vec{r}\,')K^*(\vec{i}x_o, \vec{r}\,')\, d\vec{r}\,' \quad , \qquad (6.30)$$

where $K(\vec{r}, \vec{r}\,')$ is the outgoing solution of the wave equation

$$[\nabla^2 + \kappa^2(x)]\, K(\vec{r}, \vec{r}\,') = -4\pi\delta(\vec{r} - \vec{r}\,') \quad , \qquad (6.31)$$

that is continuous and has a continuous normal derivative at the interface $x = 0$. Let us define the following one-dimensional kernel

$$L(x, x_o; x') = \frac{1}{2\pi} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} K(\vec{i}x, \vec{r}\,')K^*(\vec{i}x_o, \vec{r}\,')\, dy'\, dz' \quad , \qquad (6.32)$$

so that the integral equation (6.30) may be written as

$$\langle \psi(x)\psi^*(x_o)\rangle - \langle \psi(x)\rangle \langle \psi(x_o)\rangle^* = \frac{\Sigma_s}{2} \int_0^{\infty} \gamma(x')\langle |\psi(x')|^2 \rangle L(x, x_o; x')\, dx' \quad . \qquad (6.33)$$

Once the expression for the kernel has been obtained, the above integral equation may then be solved for $\langle |\psi(x)|^2 \rangle$ and $\langle \psi^*(x) \frac{\partial}{\partial x} \psi(x) - \psi(x) \frac{\partial}{\partial x} \psi^*(x) \rangle$, which are respectively proportional to the average sound intensity and average sound energy current.

Since the solution $K(\vec{r},\vec{r}\,')$ of Eq. (6.31) depends only on x, x' and $|\vec{r} - \vec{r}\,'|$, we have that

$$K(\vec{i}x, \vec{i}x' + \vec{j}y' + \vec{k}z') = K(\vec{i}x - \vec{j}y' - \vec{k}z', \vec{i}x') = K(\vec{i}x + \vec{j}y' + \vec{k}z', \vec{i}x') \quad , \tag{6.34}$$

and hence the definition (6.32) may be rewritten as

$$L(x, x_o; x') = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} K(\vec{r}, \vec{i}x')K^*(\vec{r}_o, \vec{i}x')\, dy\, dz \quad , \tag{6.35}$$

with $\vec{r} = \vec{i}x + \vec{j}y + \vec{k}z$ and $\vec{r}_o = \vec{i}x_o + \vec{j}y + \vec{k}z$. Because $K(\vec{r}, \vec{i}x')$ has cylindrical symmetry about the x-axis, we may take $\vec{r} = \vec{i}x + \vec{e}_r R$ and represent it by a Bessel integral representation of the form:

$$K(\vec{r}, \vec{i}x') = \int_o^{\infty} \phi(x, x'; \lambda) J_o(\lambda R)\lambda \, d\lambda \quad , \tag{6.36}$$

where $J_o(z)$ is the Bessel function of the first kind of order zero. The operator ∇^2 in (x, R) coordinates becomes

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial R^2} + \frac{1}{R}\frac{\partial}{\partial R} \quad , \tag{6.37}$$

so that

$$[\nabla^2 + \kappa^2(x)]\, K(\vec{r}, \vec{i}x') = \int_o^{\infty} [\frac{\partial^2}{\partial x^2} - \lambda^2 + \kappa^2(x)]\, \phi(x, x'; \lambda) J_o(\lambda R)\lambda \, d\lambda \; . \tag{6.38}$$

The three-dimensional delta function $\delta(\vec{r} - \vec{i}x')$ may be expressed as

$$\delta(\vec{r}-\vec{i}x') = \delta(x-x')\frac{1}{2\pi R}\delta(R) = \delta(x-x')\frac{1}{2\pi}\int_0^\infty J_o(\lambda R)\lambda\, d\lambda \ , \qquad (6.39)$$

where $\delta(x-x')$ and $\delta(R)$ are one-dimensional delta functions. Therefore substitution of the Bessel integral representation (6.36) into the wave equation (6.31) yields

$$\int_0^\infty \left\{ [\frac{\partial^2}{\partial x^2} - \lambda^2 + \kappa^2(x)]\phi(x,x';\lambda) + 2\delta(x-x')\right\} J_o(\lambda R)\lambda\, d\lambda = 0 \ , \qquad (6.40)$$

giving the differential equation

$$[\frac{\partial^2}{\partial x^2} + \kappa^2(x) - \lambda^2]\phi(x,x';\lambda) = -2\delta(x-x') \ , \qquad (6.41)$$

This equation may be replaced by the pair of equations

$$(\frac{\partial^2}{\partial x^2} + \nu_o^2)\phi(x,x';\lambda) = 0 \qquad \text{for } x < 0 \ ,$$

$$(6.42)$$

$$(\frac{\partial^2}{\partial x^2} + \nu^2)\phi(x,x';\lambda) = 0 \qquad \text{for } x > 0,\ x \neq x' \ ,$$

where

$$\nu_o = (k_o^2 - \lambda^2)^{\frac{1}{2}} \ ,$$

$$(6.43)$$

$$\nu = (\kappa^2 - \lambda^2)^{\frac{1}{2}} \ ,$$

are complex functions of the real variable λ; we have the jump boundary conditions at $x = x'$

$$\phi(x'+,x';\lambda) = \phi(x'-,x';\lambda) \quad ,$$

$$\frac{\partial\phi}{\partial x}(x'+,x';\lambda) = \frac{\partial\phi}{\partial x}(x'-,x';\lambda) - 2 \quad . \tag{6.44}$$

Since $K(\vec{r},\vec{i}x')$ must be continuous and have a continuous normal derivative at the interface $x = 0$, we have the following boundary conditions on $\phi(x,x';\lambda)$ at $x = 0$:

$$\phi(0-,x';\lambda) = \phi(0+,x';\lambda) \quad ,$$

$$\frac{\partial\phi}{\partial x}(0-,x';\lambda) = \frac{\partial\phi}{\partial x}(0+,x';\lambda) \quad ; \tag{6.45}$$

we also require the solution to remain bounded for large magnitude of x. The solutions of the differential equations (6.42) that satisfy these boundary conditions are

$$\phi(x,x';\lambda) = \frac{2i}{\nu + \nu_o} e^{i(\nu x' - \nu_o x)} \qquad \text{for } x < 0 \ , \tag{6.46}$$

$$\phi(x,x';\lambda) = \frac{i}{\nu}\left[e^{i\nu|x-x_o|} + \frac{\nu - \nu_o}{\nu + \nu_o} e^{i\nu(x+x')} \right] \quad \text{for } x > 0 \ . \tag{6.47}$$

Now, using the Bessel integral representation (6.36), the one-dimensional kernel (6.35) may be expressed as

$$L(x,x_o;x')$$

$$= \int_o^\infty \left\{ \int_o^\infty \phi(x,x';\lambda) J_o(\lambda R) \lambda \, d\lambda \right\} \left\{ \int_o^\infty \phi^*(x_o,x';\lambda') J_o(\lambda' R) \lambda' \, d\lambda' \right\} R \, dR$$

$$= \int_o^\infty \int_o^\infty \phi(x,x';\lambda) \phi^*(x_o,x';\lambda') \left\{ \int_o^\infty J_o(\lambda R) J_o(\lambda' R) R \, dR \right\} \lambda' \, d\lambda' \, \lambda \, d\lambda \ , \qquad (6.48)$$

where the order of integration has been interchanged. But the Bessel integral representation of an arbitrary function $f(R)$,

$$f(R) = \int_o^\infty \left\{ \int_o^\infty f(R') J_o(\lambda R') R' dR' \right\} J_o(\lambda R) \lambda \, d\lambda$$

$$= \int_o^\infty f(R') \left\{ \int_o^\infty J_o(\lambda R') J_o(\lambda R) \lambda \, d\lambda \right\} R' \, dR' \quad , \qquad (6.49)$$

gives the closure relation

$$\int_o^\infty J_o(\lambda R') J_o(\lambda R) \, \lambda \, d\lambda = \frac{1}{R'} \, \delta(R' - R) \ , \qquad (6.50)$$

and hence Eq. (6.48) becomes

$$L(x,x_o;x') = \int_o^\infty \int_o^\infty \phi(x,x';\lambda) \phi^*(x_o,x';\lambda') \frac{1}{\lambda'} \, \delta(\lambda' - \lambda) \lambda' d\lambda' \, \lambda \, d\lambda \ . \qquad (6.51)$$

Thus we obtain a simple explicit representation of the one-dimensional kernel in terms of the solutions (6.46) and (6.47),

$$L(x,x_o;x') = \int_o^\infty \phi(x,x';\lambda) \phi^*(x_o,x';\lambda) \lambda \, d\lambda \ . \qquad (6.52)$$

By setting $x = x_o$, we obtain as a special case of Eq. (6.33)

$$\langle |\psi(x)|^2 \rangle - |\langle \psi(x) \rangle|^2 = \frac{\Sigma_s}{2} \int_0^\infty \gamma(x') \langle |\psi(x')|^2 \rangle L(x,x')\, dx' \qquad (6.53)$$

where

$$L(x,x') = L(x,x;x') \quad , \qquad (6.54)$$

and by differentiating Eq. (6.33) first with respect to x, then with respect to x_o, subtracting, and setting $x = x_o$, we obtain

$$\langle \psi^*(x) \frac{\partial}{\partial x} \psi(x) - \psi(x) \frac{\partial}{\partial x} \psi^*(x) \rangle - [\langle \psi(x) \rangle^* \frac{\partial}{\partial x} \langle \psi(x) \rangle - \langle \psi(x) \rangle \frac{\partial}{\partial x} \langle \psi(x) \rangle^*]$$

$$= \frac{\Sigma_s}{2} \int_0^\infty \gamma(x') \langle |\psi(x')|^2 \rangle M(x,x')\, dx' \quad , \qquad (6.55)$$

where

$$M(x,x') = \left(\frac{\partial}{\partial x} - \frac{\partial}{\partial x_o} \right) L(x,x_o;x') \Big|_{x=x_o} \quad . \qquad (6.56)$$

Upon substitution of the solution (6.47) for $\phi(x,x';\lambda)$ for positive values of x into the representation (6.52) for $L(x,x_o;x')$, we find that the latter may be expressed as the sum of three terms,

$$L(x,x_o;x') = \sum_{n=1}^{3} L^{(n)}(x,x_o;x') \quad , \qquad (6.57)$$

where

$$L^{(1)}(x,x_o;x') = \int_0^\infty e^{i\nu|x-x'| - i\nu^*|x_o-x'|} \frac{\lambda\, d\lambda}{\nu\nu^*} \quad , \qquad (6.58)$$

$$L^{(2)}(x,x_o;x') = \int_o^{\infty} \left| \frac{\nu-\nu_o}{\nu+\nu_o} \right|^2 e^{i\nu(x+x')-i\nu^*(x_o+x')} \frac{\lambda\, d\lambda}{\nu\nu^*} , \qquad (6.59)$$

$$L^{(3)}(x,x_o;x') = \int_o^{\infty} \left\{ \left(\frac{\nu-\nu_o}{\nu+\nu_o} \right) e^{i\nu(x+x')-i\nu^* |x_o-x'|} + \left(\frac{\nu-\nu_o}{\nu+\nu_o} \right)^* e^{i\nu|x-x'|-i\nu^*(x_o+x')} \right\} \frac{\lambda\, d\lambda}{\nu\nu^*} . \qquad (6.60)$$

Therefore the kernels $L(x,x')$ and $M(x,x')$ may be written as the sums

$$L(x,x') = \sum_{n=1}^{3} L^{(n)}(x,x') \quad , \qquad (6.61)$$

$$M(x,x') = \sum_{n=1}^{3} M^{(n)}(x,x') \quad , \qquad (6.62)$$

where by Eqs. (6.54) and (6.56) the terms are given by

$$L^{(n)}(x,x') = L^{(n)}(x,x;x') \quad , \qquad (6.63)$$

$$M^{(n)}(x,x') = \left(\frac{\partial}{\partial x} - \frac{\partial}{\partial x_o} \right) L^{(n)}(x,x_o;x') \Big|_{x=x_o} . \qquad (6.64)$$

These terms have a simple physical interpretation. The kernel $L(x,x')$ represents the incoherent intensity at x due to a plane of incoherent point scatterers at x'. The first term $L^{(1)}(x,x')$ represents the sum of the intensities of the spherical waves emitted directly by the sources, neglecting the effect of the interface at

$x = 0$. The second term $L^{(2)}(x,x')$ represents the sum of the intensities of the reflected waves from the interface. Finally, the third term $L^{(3)}(x,x')$ describes the interference that takes place between the direct wave from each scatterer and the reflected wave produced by it. Similar statements may be made about $M(x,x')$, which represents the current.

Upon substitution of the expression (6.58) into Eqs. (6.63) and (6.64), we obtain

$$L^{(1)}(x,x') = \int_0^\infty e^{i(\nu-\nu^*)|x-x'|} \frac{\lambda\, d\lambda}{\nu\nu^*} \quad , \tag{6.65}$$

$$M^{(1)}(x,x') = \frac{(x-x')}{|x-x'|} \int_0^\infty i(\nu+\nu^*) e^{i(\nu-\nu^*)|x-x'|} \frac{\lambda\, d\lambda}{\nu\nu^*} \; . \tag{6.66}$$

Let us introduce the change in variables

$$\lambda = \left[k^2 (1 - v^2) + \frac{\alpha^2}{4} \left(\frac{1}{v^2} - 1 \right) \right]^{\frac{1}{2}} \quad , \tag{6.67}$$

where k and α are the propagation and attenuation constants of the scattering medium and v is a real variable ranging from 0 to 1; the following relations are easily verified:

$$\nu = kv + i \frac{\alpha}{2v} \quad , \tag{6.68}$$

$$\frac{\lambda\, d\lambda}{\nu\nu^*} = - \frac{dv}{v} \; . \tag{6.69}$$

After making these substitutions, the expressions (6.65) and (6.66) may be written in terms of the exponential integral functions $E_n(t)$, defined previously by Eq. (5.18), as follows:

$$L^{(1)}(x,x') = \int_o^1 e^{-\alpha|x-x'|/v} \frac{dv}{v} = E_1(\alpha|x-x'|) \quad , \qquad (6.70)$$

$$M^{(1)}(x,x') = 2ik \frac{(x-x')}{|x-x'|} \int_o^1 e^{-\alpha|x-x'|/v} dv = 2ik \frac{(x-x')}{|x-x'|} E_2(\alpha|x-x'|) \ . \qquad (6.71)$$

Although the remaining terms of $L(x,x')$ and $M(x,x')$ may also be transformed to real integrals, the forms of these expressions are rather complicated. Therefore we shall take

$$L(x,x') \cong L^{(1)}(x,x') \quad , \qquad (6.72)$$

$$M(x,x') \cong M^{(1)}(x,x') \quad , \qquad (6.73)$$

and neglect the effect of the interface on the incoherently scattered radiation. It will be seen that the effect of this approximation on the expression for the reflection coefficient will be quite minor. Consistent with this approximation, we shall take the infinite medium expression (4.53) for $\gamma(x')$,

$$\gamma(x') = \frac{k_o}{k} \ . \qquad (6.74)$$

The integral equation (6.53) may now be written as

$$\langle |\psi(\tau)|^2 \rangle - |\langle\psi(\tau)\rangle|^2 = \frac{\beta}{2}\int_0^\infty \langle |\psi(\tau')|^2 \rangle E_1(|\tau-\tau'|)\, d\tau' , \qquad (6.75)$$

where distances are expressed in terms of the dimensionless length $\tau = \alpha x$ and by Eqs. (4.55) and (4.56) β *is the ratio of the scattering* to extinction cross section densities. We may employ Eq. (6.55) in order to compute the incoherent contribution to the current at $x = 0$; the contribution to the reflection coefficient by the incoherent radiation may be obtained by dividing by $-2ik_o\mu_o |A|^2$, which by Eq. (6.7) is proportional to the x-component of the sound energy current in the incident wave; this gives

$$R_i(\mu_o) = \frac{-1}{2ik_o\mu_o|A|^2}\,\frac{\beta}{2}\int_0^\infty \langle |\psi(\tau)|^2 \rangle M(0,\tau)\, d\tau$$

$$= \frac{k}{k_o\mu_o|A|^2}\,\frac{\beta}{2}\int_0^\infty \langle |\psi(\tau)|^2 \rangle E_2(\tau)\, d\tau \quad . \qquad (6.76)$$

The inhomogeneous term of the integral equation (6.75) is given by Eq. (6.27) as

$$|\langle\psi(\tau)\rangle|^2 = \frac{4k_o^2\mu_o^2|A|^2}{(k\mu+k_o\mu_o)^2+(\frac{\alpha}{2\mu})^2}\, e^{-\tau/\mu} = \frac{k_o\mu_o|A|^2}{k\mu}[1 - R_c(\mu_o)]\, e^{-\sigma\tau} , \qquad (6.77)$$

where we denote $\sigma = 1/\mu$. Let us define $B(\tau,\sigma)$ such that

$$\langle |\psi(\tau)|^2\rangle = \frac{k_o \mu_o |A|^2}{k\mu} [1 - R_c(\mu_o)] B(\tau,\sigma) \quad . \tag{6.78}$$

The integral equation (6.75) may then be written, in terms of $B(\tau,\sigma)$ as

$$B(\tau,\sigma) = e^{-\sigma\tau} + \frac{\beta}{2}\int_0^\infty B(\tau',\sigma)E_1(|\tau-\tau'|)\, d\tau' \ , \tag{6.79}$$

while the expression (6.76) becomes

$$R_i(\mu_o) = [1 - R_c(\mu_o)] \frac{\beta\sigma}{2}\int_0^\infty B(\tau,\sigma)E_2(\tau)\, d\tau \quad . \tag{6.80}$$

The integral equation (6.79) turns out to be identical to the "auxilary equation" of radiative transfer.[14] The following result is known for this equation. Let $\hat{B}(s,\sigma)$ denote the Laplace transform of the solution of Eq. (6.79),

$$\hat{B}(s,\sigma) = \int_0^\infty B(\tau,\sigma)e^{-s\tau}\, d\tau \quad ; \tag{6.81}$$

then $\hat{B}(s,\sigma)$ is given by

$$\hat{B}(s,\sigma) = \frac{B(0,s)B(0,\sigma)}{(s+\sigma)} \ , \tag{6.82}$$

and $B(0,\sigma)$ satisfies the non-linear integral equation

$$B(0,\sigma) = 1 + B(0,\sigma)\frac{\beta}{2}\int_1^\infty \frac{B(0,s)}{s(s+\sigma)}\, ds \quad . \tag{6.83}$$

A proof of this assertion is given in Appendix C.

In order to apply this result to the present problem, we note that by interchanging the order of integration,

$$\int_0^\infty B(\tau,\sigma)E_2(\tau)\, d\tau = \int_0^\infty B(\tau,\sigma)\left\{\int_1^\infty e^{-\tau s}\frac{ds}{s^2}\right\} d\tau =$$

$$= \int_1^\infty \left\{\int_0^\infty B(\tau,\sigma)e^{-\tau s}\, d\tau\right\}\frac{ds}{s^2} = \int_1^\infty \hat{B}(s,\sigma)\,\frac{ds}{s^2} \quad , \tag{6.84}$$

and hence Eq. (6.80) may be expressed in terms of the solution $B(0,\sigma)$ of Eq. (6.83) as follows:

$$R_i(\mu_o) = [1 - R_c(\mu_o)]\, B(0,\sigma)\,\frac{\beta\sigma}{2}\int_1^\infty \frac{B(0,s)}{(s+\sigma)s^2}\, ds \quad . \tag{6.85}$$

In order to achieve a more conventional notation, let us set

$$\sigma = \frac{1}{\mu} \quad , \quad s = \frac{1}{\mu'} \tag{6.86}$$

and define

$$H(\mu) = B(0,\mu^{-1}) \quad . \tag{6.87}$$

Then the expression (6.85) for the incoherent contribution to the reflection coefficient becomes

$$R_i(\mu_o) = [1 - R_c(\mu_o)]\, H(\mu)\,\frac{\beta}{2}\int_0^1 \frac{\mu' H(\mu')}{\mu + \mu'}\, d\mu' \quad , \tag{6.88}$$

where by Eq. (6.83) $H(\mu)$ is the solution of the non-linear integral equation

$$H(\mu) = 1 + \mu H(\mu) \frac{\beta}{2} \int_0^1 \frac{H(\mu')}{\mu + \mu'} \, d\mu' \quad . \tag{6.89}$$

The function $H(\mu)$, defined by the above equation, is the well-known H-function introduced by Chandrasekhar to study radiation (particle) transport in semi-infinite atmospheres.[15] It has been shown by Crum[16] that the solution to this equation is given by the following integral in the complex plane:

$$\log H(\mu) = \frac{\mu}{\pi i} \int_0^{i\infty} \log T(w) \frac{dw}{w^2 - \mu^2} \qquad \mathrm{Re}(\mu) > 0 \quad , \tag{6.90}$$

where

$$T(w) = 1 - \frac{\beta}{2} w \log \frac{w+1}{w-1} \quad . \tag{6.91}$$

By the substitution $w = i \cot\theta$, this may be transformed to a real integral expression for $H(\mu)$,

$$H(\mu) = \exp\left\{ - \frac{\mu}{\pi} \int_0^{\pi} \frac{\log(1 - \beta\theta \cot\theta)}{1 - (1-\mu^2)\sin^2\theta} \, d\theta \right\} \quad ; \tag{6.92}$$

values of $H(\mu)$ may be readily determined from this expression by numerical integration. Chandrasekhar has shown that the function $H(\mu)$ also satisfies the alternative integral equation:[15]

$$\frac{1}{H(\mu)} = (1 - \beta)^{\frac{1}{2}} + \frac{\beta}{2}\int_0^1 \frac{\mu' H(\mu')}{\mu + \mu'}\, d\mu' \quad . \tag{6.93}$$

The last result may be used to rewrite the expression (6.88) in closed form,

$$R_i(\mu_o) = [\, 1 - R_c(\mu_o)][\, 1 - (1-\beta)^{\frac{1}{2}} H(\mu)] \quad . \tag{6.94}$$

An approximate expression, valid for $\beta << 1$, may also be determined by taking $H(\mu) = 1$ in Eq. (6.88),

$$R_i(\mu_o) \cong [\, 1 - R_c(\mu_o)]\, \frac{\beta}{2}\int_0^1 \frac{\mu'}{\mu + \mu'}\, d\mu' = [\, 1 - R_c(\mu_o)]\frac{\beta}{2}[\, 1 - \mu \log \frac{\mu + 1}{\mu}] \quad . \tag{6.95}$$

This is the same result that would be obtained if the inhomogeneous term $e^{-\sigma\tau}$ of Eq. (6.79) is substituted for $B(\sigma,\tau)$ in Eq. (6.80); hence it represents the contribution to the reflection coefficient due to the incoherent scattering of the coherent wave, neglecting higher orders of multiple scattering. For values of β less than 0.1, the approximate formula (6.95) differs from the exact result (6.94) by less than a few percent over the entire range of μ. A comparison of the approximate and the exact formulas is given in Appendix D.

Let us summarize our results. The constants k, α, μ, and β are defined in terms of the scatterer coefficient $g(R)$ and scatterer number density $n(R)$ by the following expressions:

$$k = \text{Re}\ \{(k_o^2 + 4\pi G)^{\frac{1}{2}}\} \quad , \tag{6.96}$$

$$\alpha = 2\,\text{Im}\ \{(k_o^2 + 4\pi G)^{\frac{1}{2}}\} \quad , \tag{6.97}$$

$$\mu = \frac{1}{k}\ \text{Re}\ \{(k_o^2 \mu_o^2 + 4\pi G)^{\frac{1}{2}}\} \quad , \tag{6.98}$$

$$\beta = \frac{\Sigma_s}{\Sigma_e} = \frac{k_o \int_o^\infty n(R)\, |g(R)|^2\ dR}{\text{Im}\ \{G\}} \quad , \tag{6.99}$$

where

$$G = \int_o^\infty n(R)\ g(R)\ dR \quad . \tag{6.100}$$

The total reflection coefficient for the incident wave inclined at an angle $\cos^{-1} \mu_o$ with the normal is given by

$$R(\mu_o) = R_c(\mu_o) + [1 - R_c(\mu_o)][1 - (1-\beta)^{\frac{1}{2}} H(\mu)] \quad , \tag{6.101}$$

where the reflection coefficient of the coherent wave is

$$R_c(\mu_o) = \frac{(k\mu - k_o\mu_o)^2 + (\frac{\alpha}{2\mu})^2}{(k\mu + k_o\mu_o)^2 + (\frac{\alpha}{2\mu})^2} \quad , \tag{6.102}$$

and $H(\mu)$ is given by the integral (6.92). A table of values of the factor $[1 - (1 - \beta)^{\frac{1}{2}} H(\mu)]$ is given in Appendix D; for values of $\beta << 1$, the approximation (6.95) may be used.

We recall the single approximation that has been made in the development of Eq. (6.101) -- the neglect of the reflection at the interface of the incoherent radiation. Hence, if the scatterer density in the region $x > 0$ is too large, the expression (6.94) for the incoherent contribution to the reflection coefficient will be in error. In such a case, however, the coherent contribution to the reflection coefficient is much larger than the incoherent contribution; therefore the resulting error in the total reflection coefficient will be comparatively small. Thus the expression (6.101) may be considered to be adequate over the entire range of scatterer densities.

VII. APPLICATION TO SOUND PROPAGATION IN A LIQUID CONTAINING BUBBLES

In the preceding sections of this work we have developed a consistent treatment of the multiple scattering of scalar waves by a random distribution of isotropic point scatterers. Such a mathematical idealization provides an applicable model for the description of the propagation of sound in a liquid containing a large number of small gas bubbles. The presence of bubbles in a liquid may have a dramatic effect on its acoustical characteristics; even a very few bubbles, so widely spaced as to be nearly invisible, may cause a marked change in the phase velocity of the medium and produce a significant attenuation constant. This is due to the possibility of resonance between the sound field and the natural oscillations of the bubbles. Such resonance may cause gas bubbles in a liquid to scatter and absorb underwater sound to a much greater extent than their geometrical cross sections would indicate.

The study of the acoustic properties of a bubbly mixture has considerable practical importance and has been the subject of a number of theoretical[2,17,18,19] and experimental[20,21,22] investigations. It has long been recognized that the presence of small gas bubbles in the wakes of ships is the dominant feature affecting their acoustic properties[23]. The effectiveness of bubbles in the absorption of sound suggests their use as acoustical "screens" in order to attenuate sonar or shock waves in water[24]. Therefore the acoustic behavior of bubbles in a liquid is an essential feature of the physics of underwater sound.

In order for one to apply the general theory of the multiple scattering of waves to sound propagation in a bubbly mixture, the acoustical behavior of a single gas bubble must be determined. The scattering and absorption of sound by a single bubble is well understood and has been discussed by several authors[18, 25,26]. A rigorous treatment of this problem requires that one consider the bubble excited by an incident plane wave and solve the linearized equations of conservation of mass, momentum, and energy both for the gas within the bubble as well as for the liquid outside the bubble; the effect of surface tension and viscosity must be included in the formulation of the boundary condition at the gas-liquid interface. In order to solve for the scattered wave from the bubble, the same partial wave decomposition of the incident plane wave into spherical waves may be employed as is commonly used in atomic and nuclear scattering problems[8]. Although such an ambitious program has been actually carried out[27], the results are much too complicated to be incorporated in the analysis of multiple scattering. Fortunately, such a detailed analysis is not required in order to obtain a satisfactory description of the acoustical behavior of a gas bubble in a liquid. As long as the sound frequency is less than several megacycles, several assumptions may be made which greatly simplify the analysis and lead to a tractable expression for the scattering coefficient of a bubble; we shall now review these assumptions.

1. It is assumed that the radius of the bubble is much smaller than the wave length of sound in the liquid or in the gas. This condition is well satisfied for resonant air bubbles in water, as indicated below.

Radius(cm)	Resonant Freq. (kc)	Wave Length(cm)- Water	Wave Length(cm)- Air
0.1	3.26	46.0	10.4
0.001	326	0.46	0.104

If this condition is satisfied, the bubble may be analyzed as if it was in a pressure field that is uniform in space, but alternating in time. This results in spherically symmetric bubble oscillations, corresponding to the dominant first term (s-wave) of the rigorous solution. The pressure may be considered to be uniform throughout the interior of the bubble.

2. The oscillation of the bubble is considered to be sufficiently small such that the over-all change in radius during oscillation is small compared with the radius. This allows the equations of motion for the liquid and for the gas, as well as the boundary condition at the interface, to be linearized.

3. For the irrotational flow of spherically symmetric bubble oscillations, the viscous terms remaining in the momentum and energy equations arise as a result of the compressibility of the gas or liquid; since the viscosity of the gas and the compressibility of the liquid are both quite small, these terms may be neglected[25]. It is noted, however, that dissipation in the liquid due to viscosity may be

included by the addition of the attenuation coefficient α_o to the absorption and extinction cross section densities. Although the net force produced by the viscous stress is negligible throughout the bulk of the liquid, it does act on the surface of the bubble, and must be included in the boundary condition at the gas-liquid interface. For very small bubbles $(R < 10^{-3}\ \mathrm{cm})$ the effect of surface tension must also be included in the pressure boundary condition at the bubble surface.

4. Because of the large thermal conductivity and heat capacity of a liquid, the temperature variation at the bubble wall may be neglected[25]. However, as the bubble is compressed, the temperature within the bubble will rise and heat will flow from the gas to the liquid; as the bubble expands, heat will flow back into the bubble. This irreversible flow of heat represents an important mechanism by which mechanical sound energy is degraded into thermal energy. Hence, although the pressure may be assumed to be uniform within the bubble, the temperature profile across the bubble must be considered.

5. In order for the analysis of a single bubble in an incident sound field to be applicable to the bubbles in a configuration, the bubbles must be widely spaced compared with their radii. When two bubbles are closely spaced, the scattered wave caused by their mutual oscillation may no longer be considered to be the superposition of two spherical waves originating from their centers; rather, the scattered wave from each will act on the surface of the other to excite higher modes of oscillation. Fundamental to the preceding analysis of the

multiple scattering of waves by a random distribution of isotropic point scatterers was the assumption that the positions and radii of distinct scatterers are statistically independent. Since the scatterers were assumed to be located at points (the "radius" being but a parameter of the scatterer), the configurational averages were taken over all scatterer configurations, including those in which two or more of the scatterers lie arbitrarily close to each other. It is clear that such a theory cannot be rigorously applied to a random distribution of bubbles; the condition of statistical independence cannot apply, since overlapping of bubbles in a configuration is not allowed. Even if a configuration of bubbles does not involve an overlap of two or more bubbles, the spacing of the bubbles may be such as to cause an appreciable amount of non-spherical oscillation. However, if the bubble density is small, such that the average distance between the bubbles is large compared to their radii, the great majority of configurations in the statistically independent ensemble will consist of widely spaced bubbles. Under these conditions, the theory may be expected to be a very good approximation.

Based on these assumptions, the scatterer coefficient of a single gas bubble of radius R, excited by an incident sound wave having frequency ω is given by[18]

$$g(R) = \frac{R}{f(R)-id(R)} \quad , \tag{7.1}$$

where

$$f(R) = \left(\frac{\omega_o}{\omega}\right)^2 \frac{\zeta}{\xi} - 1 \quad , \tag{7.2}$$

$$d(R) = \frac{4\mu}{\rho_o \omega R^2} + \frac{\eta}{\xi^2}\left(\frac{\omega_o}{\omega}\right)^2\left(1+\frac{2\sigma}{Rp_o}\right) + a \quad , \tag{7.3}$$

and ω_o, a, ξ, η and ζ are defined by

$$\omega_o = \left(\frac{3\gamma p_o}{\rho_o R^2}\right)^{\frac{1}{2}} \quad , \tag{7.4}$$

$$a = k_o R = \frac{\omega R}{c_o} \quad , \tag{7.5}$$

$$\xi = 1 + \frac{3(\gamma-1)}{\kappa}\,\frac{\sinh\kappa - \sin\kappa}{\cosh\kappa - \cos\kappa} \quad , \tag{7.6}$$

$$\eta = \frac{3(\gamma-1)}{\kappa}\left[\frac{\sinh\kappa + \sin\kappa}{\cosh\kappa - \cos\kappa} - \frac{2}{\kappa}\right] \quad , \tag{7.7}$$

$$\zeta = 1 + \frac{2\sigma}{Rp_o}\left(1 - \frac{\xi}{3\gamma}\right) \quad , \tag{7.8}$$

where

$$\kappa = \left[\frac{2c_p \omega R^2}{KR_g T_o}\left(1 + \frac{2\sigma}{R}\right)\right]^{\frac{1}{2}} \quad . \tag{7.9}$$

In the above formulas, various physical constants appear; the following numerical values are appropriate for air bubbles in water at 1 atm. and 60°F.

p_o = pressure of liquid	$= 1 \times 10^6$ dynes/cm^2
T_o = temperature of liquid	= 288°K
μ = viscosity of liquid	= 0.01 poise
ρ_o = density of liquid	= 1.0 gm/cm^3
σ = surface tension of liquid	= 75 dynes/cm
c_o = sound speed in liquid	$= 1.5 \times 10^5$ cm/sec

γ	= ratio of specific heats of gas	= 1.4
c_p	= specific heat at constant pressure of gas	= 0.24 cal/gm°C
K	= thermal conductivity of gas	$= 5.6\times 10^{-5}$ cal/cm sec°C
R_g	= gas constant	$= 2.87\times 10^{6}$ dyne cm/gm°C

Let us assume that the number density of bubbles per unit radius interval throughout some region of space is given as $n(R)$. The quantities of physical interest are the phase velocity ratio c/c_o, the attenuation constant α of the bubbly medium, and the scattering to extinction cross section ratio $\beta = \Sigma_s/\Sigma_e$. Upon substitution of the expression (7.1) for the scatterer coefficient of a bubble into the definitions(2.39) and (2.41) we obtain the following expressions for the scattering and extinction cross section densities:

$$\Sigma_s = 4\pi \int_o^\infty \frac{R^2 n(R)}{f(R)^2+d(R)^2}\, dR \quad , \tag{7.10}$$

$$\Sigma_e = \frac{4\pi}{k_o} \int_o^\infty \frac{R d(R) n(R)}{f(R)^2+d(R)^2}\, dR + \alpha_o \quad , \tag{7.11}$$

where by Eq. (2.28) the attenuation constant of the water (without bubbles) is given by

$$\alpha_o = \frac{4\mu\omega^2}{3\rho_o c_o^3} \quad . \tag{7.12}$$

The square of the complex propagation constant of the coherent wave in the bubbly water, given by Eq. (4.7), may be written as

$$\kappa^2 = \left(k + i\,\frac{\alpha}{2}\right)^2 = \left(k_o + i\,\frac{\alpha_o}{2}\right)^2 + 4\pi \int_o^\infty g(R) n(R) dR \;; \tag{7.13}$$

which gives

$$\kappa^2 = k_o^2 X + i k_o \Sigma_e \tag{7.14}$$

where

$$X = 1 + \frac{4\pi}{k_o^2} \int_o^\infty \frac{R f(R) n(R)}{f(R)^2 + d(R)^2} \, dR \; ; \tag{7.15}$$

and $-\frac{\alpha_o^2}{4}$ has been neglected. By equating the real and imaginary parts of Eq. (7.14) we obtain the simultaneous equations

$$k^2 - \frac{\alpha^2}{4} = k_o^2 X \quad , \tag{7.16}$$

$$k\alpha = k_o \Sigma_e \quad . \tag{7.17}$$

These may be first solved for the ratio of the real propagation constants, which is inversely proportional to the phase velocity ratio

$$\frac{k}{k_o} = \frac{c_o}{c} = \left\{ \frac{X + [X^2 + (\Sigma_e / k_o)^2]^{\frac{1}{2}}}{2} \right\}^{\frac{1}{2}} \quad , \tag{7.18}$$

and then the attenuation constant may be computed

$$\alpha = \frac{k_o}{k} \Sigma_e \quad . \tag{7.19}$$

Having determined the phase velocity ratio c/c_o, the attenuation constant α, and the scattering-extinction cross section ratio $\beta = \Sigma_s / \Sigma_e$, we may now estimate the transmission and reflection characteristics of bubbly water by reference to our previous results. The sound intensity (proportional to mean square pressure) and the sound energy current at an arbitrary distance x from an infinite plane sound source may be determined from Figures 1 and

2 respectively, we recall that these plots are normalized such that the sound energy current leaving the source is unity. The coherent reflection coefficient for an incident plane wave normal to a half-space of bubbly water is given by Eq. (6.102) as

$$R_c(1) = \frac{\left(\frac{k}{k_o} - 1\right)^2 + \left(\frac{\alpha}{2k_o}\right)^2}{\left(\frac{k}{k_o} + 1\right)^2 + \left(\frac{\alpha}{2k_o}\right)^2} \quad , \tag{7.20}$$

and the total reflection coefficient is

$$\begin{aligned} R(1) &= R_c(1) + [1 - R_c(1)]\,[1 - (1-\beta)^{\frac{1}{2}} H(1)] \\ &\cong R_c(1) + [1 - R_c(1)]\,\frac{\beta}{2}\,(1 - \log 2) \quad \text{for } \beta \ll 1 \quad . \end{aligned} \tag{7.21}$$

Values of the factor $[1-(1-\beta)^{\frac{1}{2}}H(1)]$ are given in Appendix D.

As a specific example of the use of these formulas, we have carried out calculations for the special case of air bubbles of fixed radius R_o in water at 1 atmosphere and 60° F. If n is the number of bubbles per unit volume, the bubble number density distribution is given by

$$n(R) = n\delta(R - R_o) \quad . \tag{7.22}$$

It is convenient to employ the air-mixture volume ratio u; this is related to n through

$$n = \frac{3u}{4\pi R_o^3} \quad . \tag{7.23}$$

Two different bubble sizes were considered, $R_o = 0.1$ cm and $R_o = 0.001$ cm; these correspond to undamped natural frequencies ω_o

of 3.26 kc and 326 kc respectively. Figure 3 gives the scattering-extinction cross section ratio $\beta = \sigma_s/\sigma_e$; it is noted that this ratio is quite small at frequencies much less than resonance and approaches unity at high frequency. Figures 4 and 5 display the variation of phase velocity with frequency. It is interesting to note that the phase velocity is very markedly decreased by the presence of the bubbles at frequencies below resonance, while at frequencies above resonance it is increased. This theoretical result is substantiated by the experimental results of Silberman[20]. The variation of the attenuation coefficient with frequency is indicated in Figures 6 and 7; the ordinate is the attenuation coefficient α divided by the air-mixture volume ratio u. The curves for $u < 10^{-5}$ essentially coincide with that for $u = 10^{-5}$; at such a low bubble density, the attenuation coefficient is very nearly equal to the extinction cross section density. These results allow the sound transmission characteristics to be directly estimated from Figures 1 and 2. The total reflection coefficient, for a wave incident in the normal direction to a bubbly half-space, is displayed by Figures 8 and 9. It is interesting to compare the shape of these curves with those given by Spitzer[18], who does not include the effect of incoherent scattering. A very significant departure from Spitzer's results is noted at low bubble densities and at high frequencies.

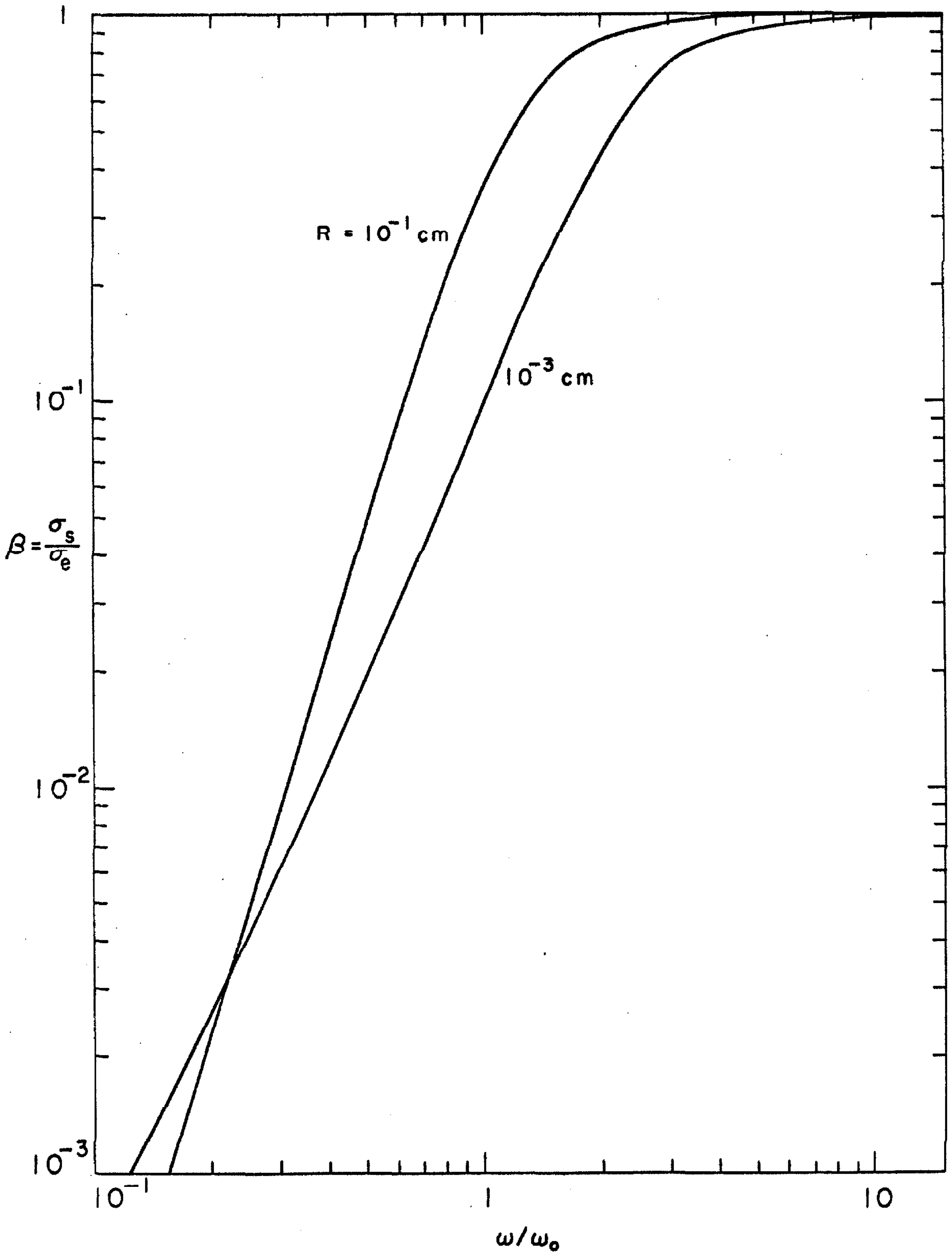

Figure 3 - Variation of Scattering-Extinction Cross Section Ratio with Frequency

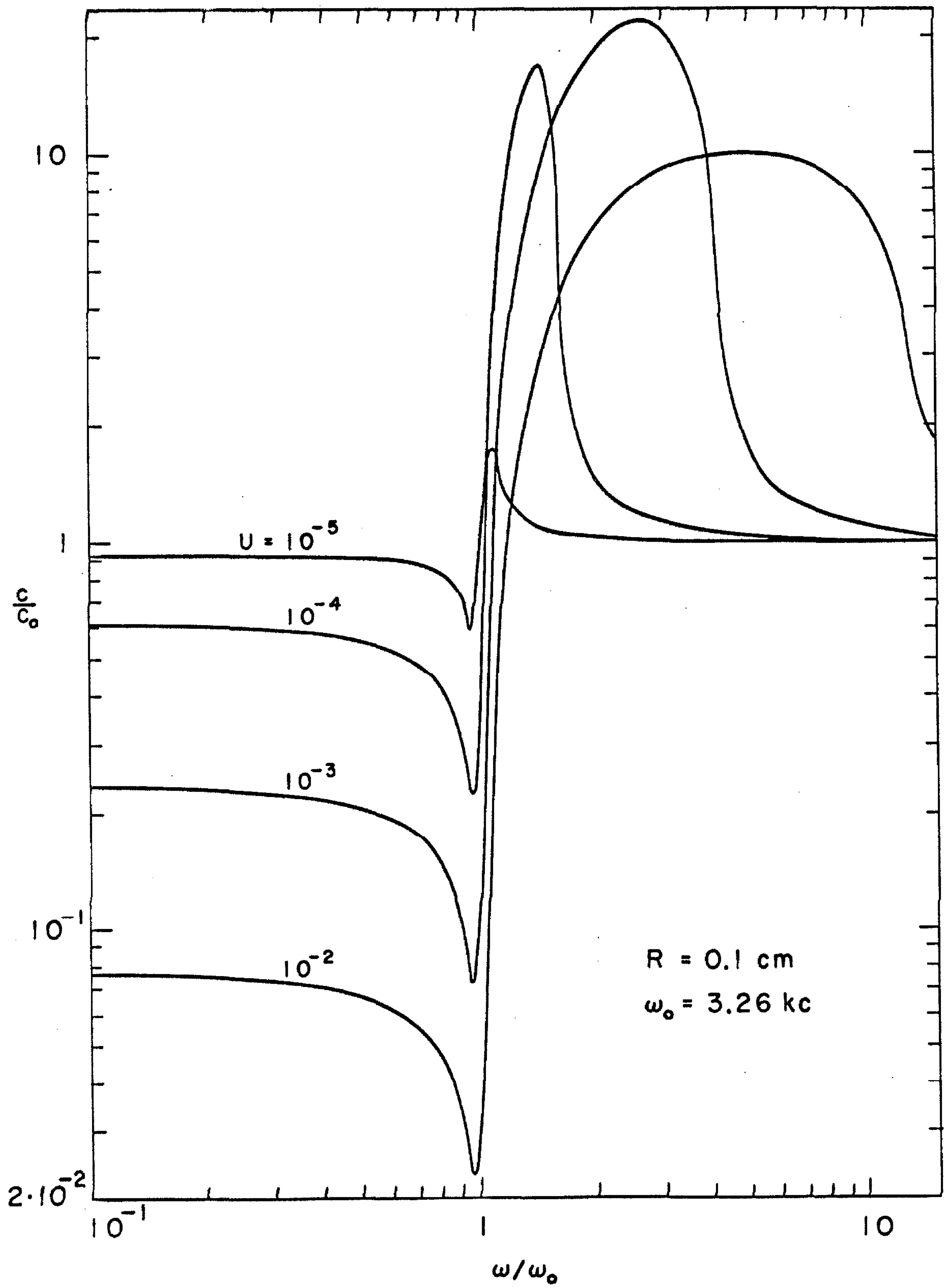

Figure 4 - Variation of Phase Velocity with Frequency

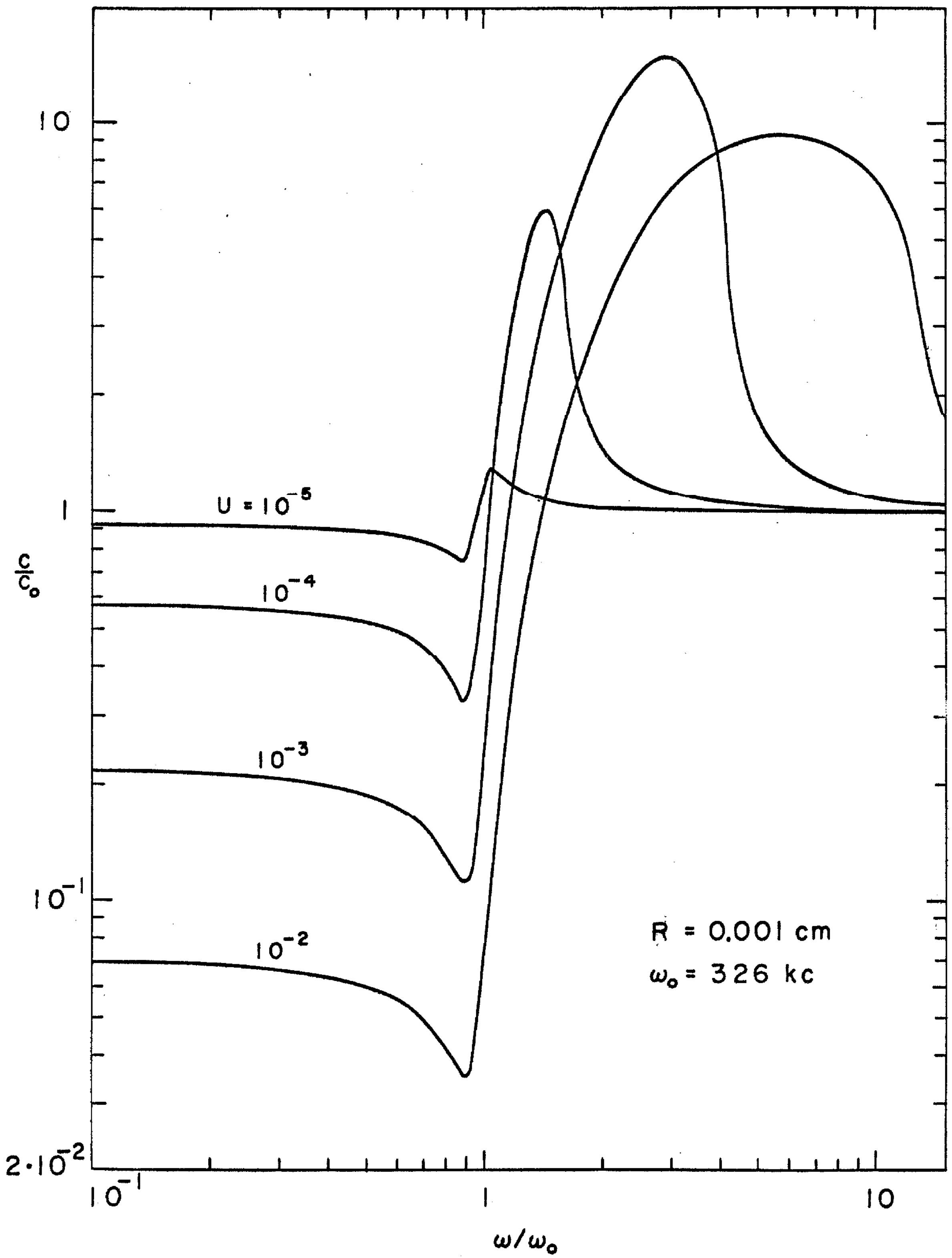

Figure 5 - Variation of Phase Velocity with Frequency

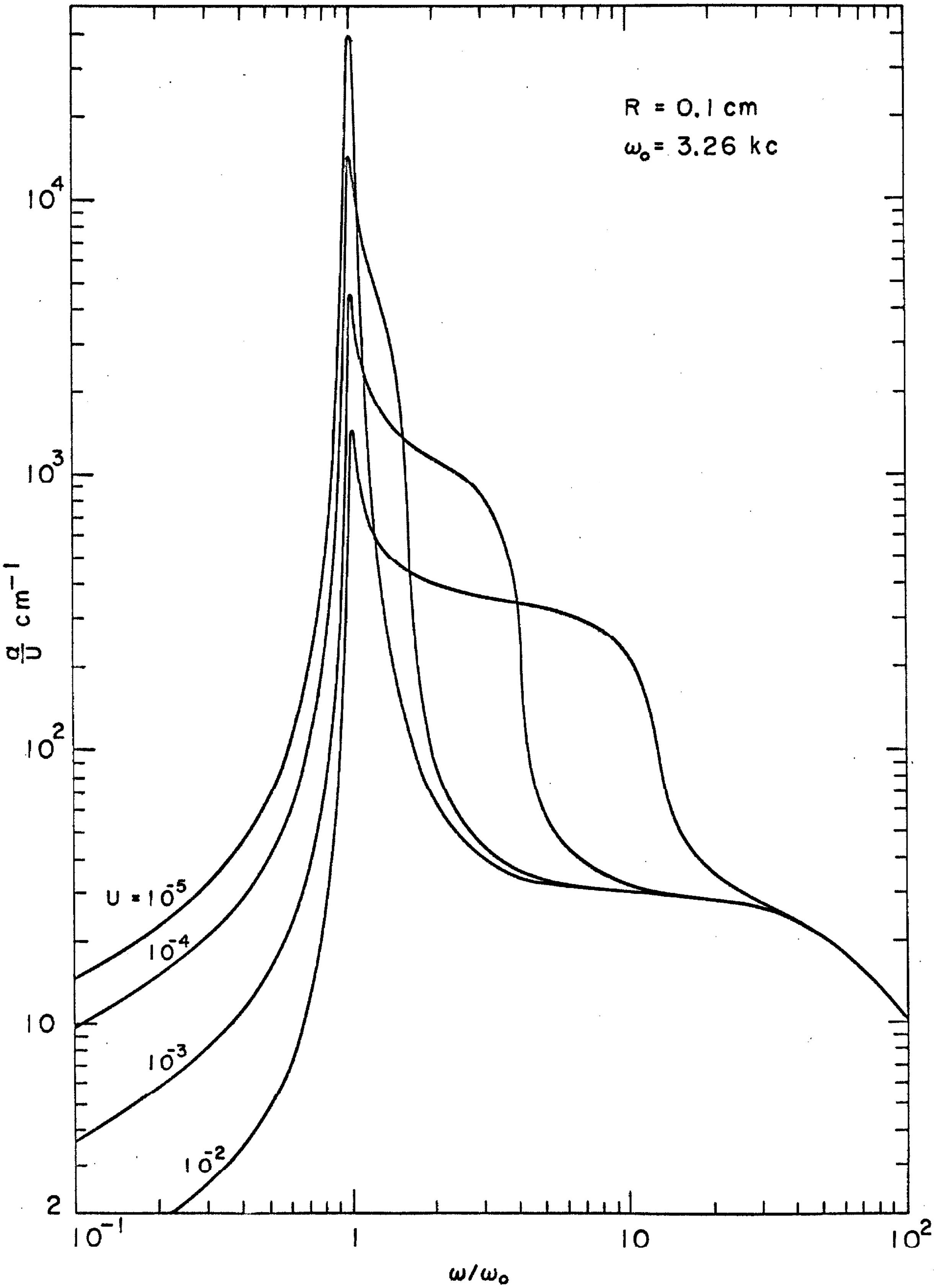

Figure 6 - Variation of Attenuation Coefficient with Frequency

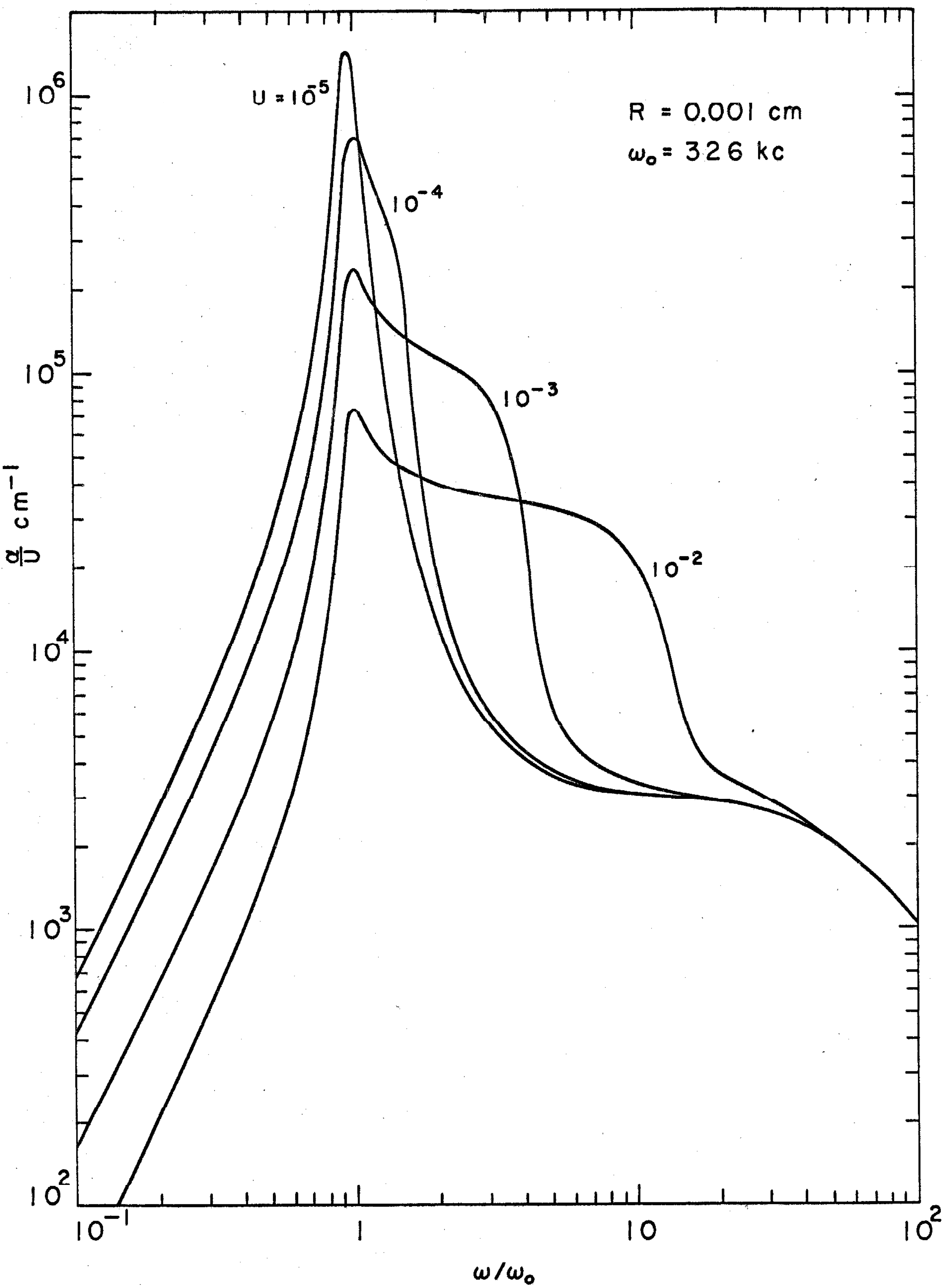

Figure 7 - Variation of Attenuation Coefficient with Frequency

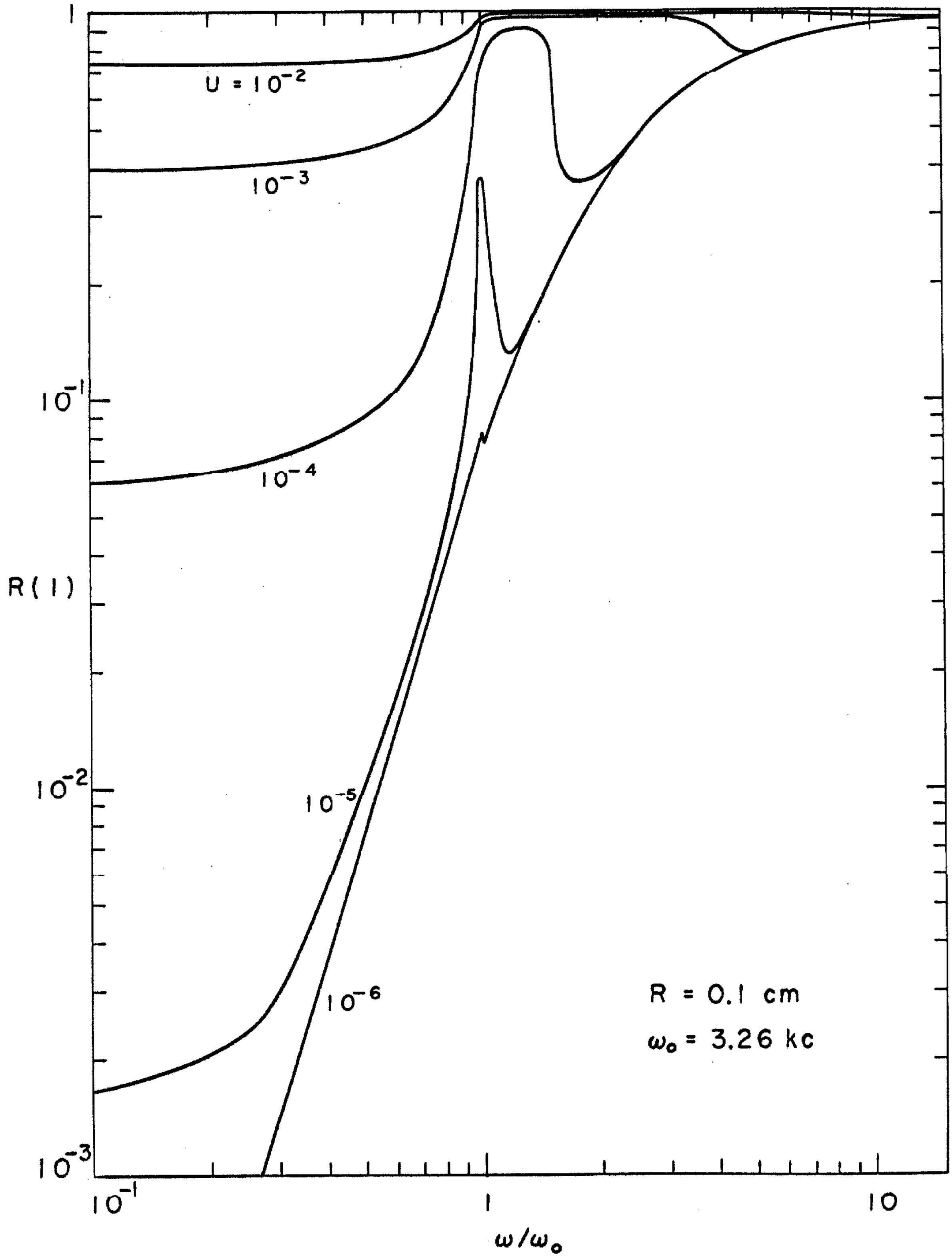

Figure 8 - Reflection Coefficient for a Normal Incident Wave vs. Frequency

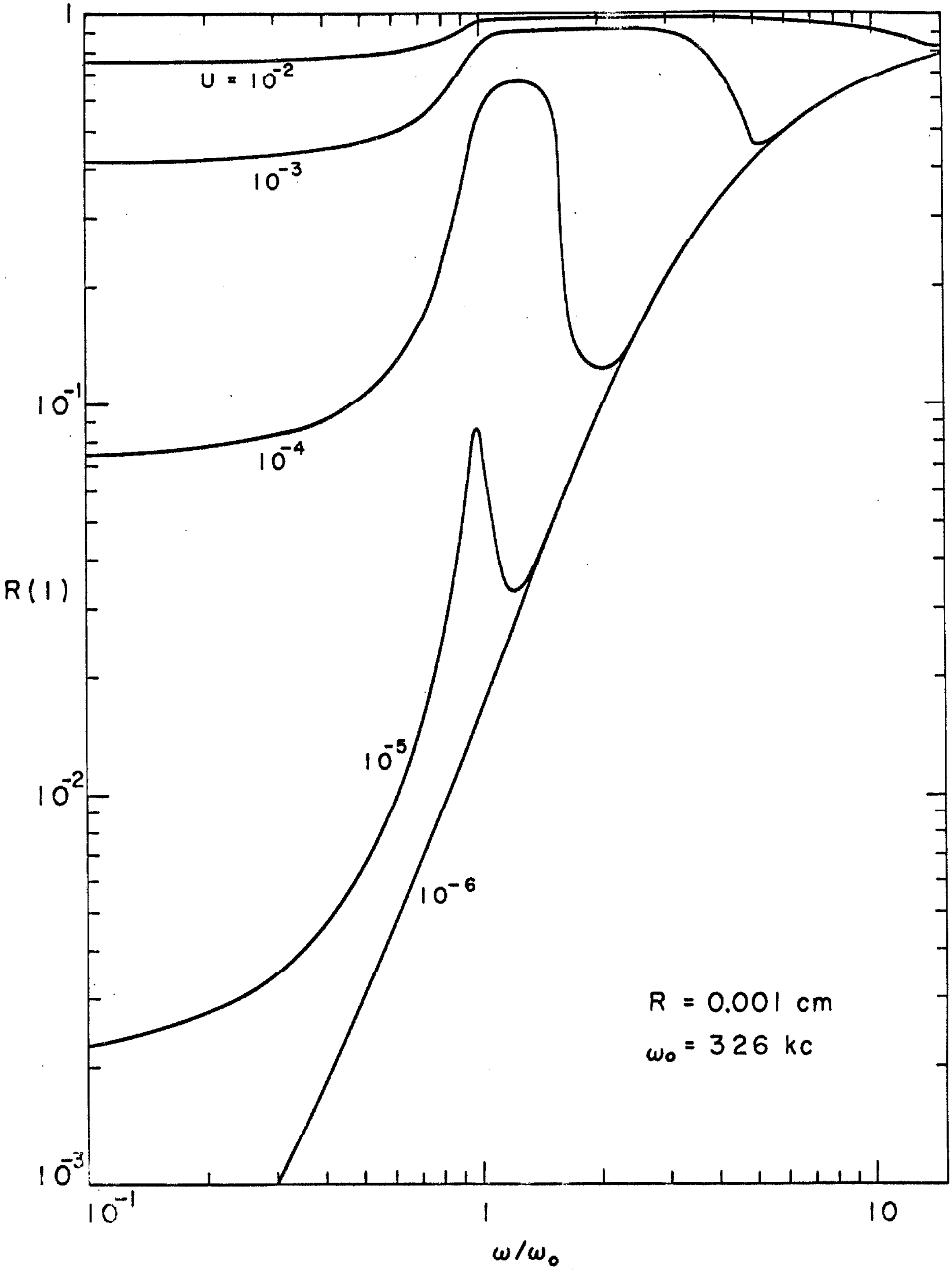

Figure 9 - Reflection Coefficient for a Normal Incident Wave vs. Frequency

APPENDIX A

Criticism of Foldy's Analysis of Equation (4.30) for the Kernel $L(\vec{r}, \vec{r}_o; \vec{r}\,')$

The central problem of Section IV-C was the determination of the kernel $L(\vec{r}, \vec{r}_o; \vec{r}\,')$ of the representation (4.27). Our analysis was based on the conservation relations (4.10) and (4.13) that were derived from the basic equations of multiple scattering; this approach led to the expression (4.38) for the kernel. Since this result differs from that obtained by Foldy[1], we shall now review his treatment and point out what appears to be an essential difficulty with his method.

Foldy based his analysis on the differential equation (4.30) which he first rewrote as the differential equation

$$[\nabla^2 + \kappa^2(\vec{r})][\nabla_o^2 + \kappa^{*2}(\vec{r}_o)]\, L(\vec{r}, \vec{r}_o; \vec{r}\,') = 16\pi^2 \delta(\vec{r} - \vec{r}\,')\delta(\vec{r}_o - \vec{r}\,')$$

$$+ 4\pi\{G(\vec{r})[\nabla_o^2 + \kappa^{*2}(\vec{r}_o)] + G^*(\vec{r}_o)[\nabla^2 + \kappa^2(\vec{r})]\}\, L(\vec{r}, \vec{r}_o; \vec{r}\,') \quad , \qquad (A.1)$$

and then as the integral equation

$$L(\vec{r}, \vec{r}_o; \vec{r}\,') = K(\vec{r}, \vec{r}\,')K^*(\vec{r}_o, \vec{r}\,')$$

$$+ \frac{1}{4\pi} \iint \{G(\vec{r}\,'')[\nabla'''^2 + \kappa^{*2}(\vec{r}\,''')] + G^*(\vec{r}\,''')[\nabla''^2 + \kappa^2(\vec{r}\,'')]\}\, L(\vec{r}\,'', \vec{r}\,'''; \vec{r}\,')$$

$$K(\vec{r}, \vec{r}\,'')K^*(\vec{r}_o, \vec{r}\,''')d\vec{r}\,''d\vec{r}\,''' \, , \qquad (A.2)$$

where $K(\vec{r}, \vec{r}\,')$ is the outgoing solution of the wave equation (4.39). He then wrote the kernel as the Liouville-Neumann iteration series solution of the above integral equation,

$$L(\vec{r},\vec{r}_o;\vec{r}\,') = \sum_{n=o}^{\infty} L_n(\vec{r},\vec{r}_o;\vec{r}\,') \quad , \qquad (A.3)$$

where

$$L_o(\vec{r},\vec{r}_o;\vec{r}\,') = K(\vec{r},\vec{r}\,')K^*(\vec{r}_o,\vec{r}\,') \quad , \qquad (A.4)$$

$$L_n(\vec{r},\vec{r}_o;\vec{r}\,') = \frac{1}{4\pi} \iint \{G(\vec{r}\,'')[\nabla'''^2 + \kappa^{*2}(\vec{r}\,''')]$$

$$+ G^*(\vec{r}\,''')[\nabla''^2 + \kappa^2(\vec{r}\,'')]\} \, L_{n-1}(\vec{r}\,'',\vec{r}\,''';\vec{r}\,')$$

$$\times K(\vec{r},\vec{r}\,'')K^*(\vec{r}_o,\vec{r}\,''')d\vec{r}\,''d\vec{r}\,''' \quad . \qquad (A.5)$$

Such a solution is valid only if the series converges. Foldy asserted that since $K(\vec{r},\vec{r}\,')$ contains a real negative exponential, the successive series terms will die out rapidly because of the increasingly larger number of integrations involving the negative exponential.

We may establish, however, that this is not the case. Let us consider the special case in which $G(\vec{r})$ and $\kappa(\vec{r})$ are constant throughout all space. Then inspection of the differential equation (A.1) for $L(\vec{r},\vec{r}_o;\vec{r}\,')$ indicates that the kernel depends only on the distances $|\vec{r}-\vec{r}\,'|$ and $|\vec{r}_o-\vec{r}\,'|$; therefore we may take the origin of our coordinate system at $\vec{r}\,'$ and write $L(r, r_o)$ for $L(\vec{r},\vec{r}_o;0)$, where r and r_o denote the absolute values of $\vec{r}$ and $\vec{r}_o$. Let us define the repeated three-dimensional Fourier transform of $L(r, r_o)$,

$$\hat{L}(\rho,\rho_o) - \iint L(r, r_o)e^{-i(\vec{\rho}\cdot\vec{r} + \vec{\rho}_o\cdot\vec{r}_o)} d\vec{r}\, d\vec{r}_o \quad , \qquad (A.6)$$

which depends only on the magnitudes ρ and ρ_o. Transformation of each term of the series representation (A.3) yields

$$\hat{L}(\rho,\rho_o) = \sum_{n=o}^{\infty} \hat{L}_n(\rho,\rho_o) \quad . \tag{A.7}$$

The first term of the series (A.3) is

$$L_o(r,r_o) = K(\vec{r},0)K^*(\vec{r}_o,0) = \frac{e^{i\kappa r}}{r}\,\frac{e^{-i\kappa^* r_o}}{r_o} \quad ; \tag{A.8}$$

the repeated Fourier transform of this term may be carried out by employing spherical polar coordinates in $\vec{r}$ and $\vec{r}_o$ space, with the result:

$$\hat{L}_o(\rho,\rho_o) = \iint \frac{e^{i\kappa r}}{r}\,\frac{e^{-i\kappa^* r_o}}{r_o}\, e^{-i(\vec{\rho}\cdot\vec{r}+\vec{\rho}_o\cdot\vec{r}_o)}\, d\vec{r}\, d\vec{r}_o = \frac{16\pi^2}{(\rho^2-\kappa^2)(\rho_o^2-\kappa^{*2})} \quad . \tag{A.9}$$

Equation (A.5) may be written as

$$L_n(r,r_o) = \iint F(r'',r''')\,\frac{e^{i\kappa|\vec{r}-\vec{r}''|}}{|\vec{r}-\vec{r}''|}\,\frac{e^{-i\kappa|\vec{r}_o-\vec{r}'''|}}{|\vec{r}_o-\vec{r}_o'''|}\, d\vec{r}''\, d\vec{r}''' \quad . \tag{A.10}$$

where

$$F(r,r_o) = \frac{1}{4\pi}\,\{G(\nabla_o^2+\kappa^{*2})+G^*(\nabla^2+\kappa^2)\}\, L_{n-1}(r,r_o) \quad ; \tag{A.11}$$

thus it involves a repeated three-dimensional integral convolution. Application of the repeated Fourier transform and use of the convolution theorem for the three-dimensional Fourier transform results in

$$\hat{L}_n(\rho, \rho_o) = \frac{1}{4\pi} \{G(-\rho_o^2+\kappa^{*2})+G^*(-\rho^2+\kappa^2)\} \hat{L}_{n-1}(\rho, \rho_o) \frac{16\pi^2}{(\rho^2-\kappa^2)(\rho_o^2-\kappa^{*2})} \quad . \tag{A.12}$$

From Eqs. (A. 9) and (A. 12) we obtain the following series for the transformed kernel:

$$\hat{L}(\rho, \rho_o) = \frac{16\pi^2}{(\rho^2-\kappa^2)(\rho_o^2-\kappa^{*2})} \sum_{n=o}^{\infty} (-1)^n \left\{ \frac{4\pi G}{(\rho^2-\kappa^2)} + \frac{4\pi G^*}{(\rho_o^2-\kappa^{*2})} \right\}^n \quad . \tag{A.13}$$

The kernel $L(r, r_o)$ may now be determined from the inversion formula

$$L(r, r_o) = \frac{1}{(2\pi)^6} \iint \hat{L}(\rho, \rho_o) e^{i(\vec{\rho} \cdot \vec{r} + \vec{\rho}_o \cdot \vec{r}_o)} d\vec{\rho}\, d\vec{\rho}_o \quad ; \tag{A.14}$$

the binomial may now be expanded and the inverse Fourier transforms computed by residue theory to give an explicit representation of the kernel.

Before doing this, however, it is instructive to consider the direct application of the repeated Fourier transform to the differential equation (A. 1), for $G(\vec{r})$ and $\kappa(\vec{r})$ constant; this yields

$$[(-\rho^2+\kappa^2)(-\rho_o^2+\kappa^{*2})-4\pi G(-\rho_o^2+\kappa^{*2})-4\pi G^*(-\rho^2+\kappa^2)]\, \hat{L}(\rho, \rho_o) = 16\pi^2 \quad , \tag{A.15}$$

or

$$\hat{L}(\rho, \rho_o) = \frac{16\pi^2}{(\rho^2-\kappa^2)(\rho_o^2-\kappa^{*2})} \left\{ 1 + \frac{4\pi G}{\rho^2-\kappa^2} + \frac{4\pi G^*}{\rho_o^2-\kappa^{*2}} \right\}^{-1} \quad . \tag{A.16}$$

If we expand the term within the parenthesis into a geometric series, we obtain the result (A. 13); this geometric series converges if and only if

$$\left| \frac{4\pi G}{\rho^2 - \kappa^2} + \frac{4\pi G^*}{\rho_o^2 - \kappa^{*2}} \right| < 1 \quad . \tag{A. 17}$$

This is definitely not the case for all real positive values of ρ and ρ_o; for example, if we take $\rho = \rho_o = \kappa_o$ (for κ_o real), we obtain

$$\frac{4\pi G}{\rho^2 - \kappa^2} + \frac{4\pi G^*}{\rho_o^2 - \kappa^{*2}} = -2 \quad . \tag{A. 18}$$

It is therefore apparent that the Liouville-Neumann iteration series (A. 3) obtained by Foldy fails to converge and hence cannot be a valid representation of the kernel $L(\vec{r}, \vec{r}_o; \vec{r}\,')$. This becomes even more evident if one substitutes Eq. (A. 13) into Eq. (A. 14) and computes the first two terms of the resulting series for $L(r, r_o)$; the inverse Fourier transforms may be carried out in spherical coordinates in $\vec{\rho}$ and $\vec{\rho}_o$ space, with residue theory being used to evaluate the resulting integrals. The first two terms are

$$L(r, r_o) = \frac{e^{i\kappa r}}{r} \frac{e^{-i\kappa^* r_o}}{r_o} \left[1 - 2i \left(\frac{\pi G r}{\kappa} - \frac{\pi G^* r_o}{\kappa^*} \right) + \ldots \right] . \tag{A. 19}$$

For κ_o real and a sufficiently low scatterer density such that $|\kappa| \approx \kappa_o$, this becomes

$$L(r, r_o) \cong \frac{e^{-\alpha r}}{r^2} \left(1 + \alpha r + \ldots \right) \tag{A. 20}$$

where $\alpha = 2\,\mathrm{Im}(\kappa)$ is the attenuation constant of the scattering medium. We observe that the second term of Foldy's expansion (A. 3) is by no means small compared with the first, thus disproving his assertion; computation of higher terms only provides further evidence of the failure of his series to converge to a physically meaningful result.

The root of this difficulty appears to lie in the differential equation (4.30) or alternatively in the integral equation (4.29) for $L(\vec{r},\vec{r}_o;\vec{r}\,')$. Since the heuristic approximations (3.17) and (3.18) are implicit in Eq. (3.21) leading to Eq. (4.29), it is natural to question the validity of the latter. Although the same type of approximation, involving the replacement of a conditional configurational average by the corresponding configurational average, was employed in deriving the conservation relations (4.10) and (4.13), one has no reason to question the very physical results that were obtained. Therefore we may require that the representation (4.27) be consistent with these relations; this imposes an additional condition of constraint on the kernel $L(\vec{r},\vec{r}_o;\vec{r}\,')$. One may show, after a rather involved calculation that the solution $L(\vec{r},\vec{r}_o;\vec{r}\,')$ of Eq. (4.29) is not quite consistent with Eqs. (4.10) and (4.13). This is the reason that we used the latter equations directly to compute the kernel $L(\vec{r},\vec{r}_o;\vec{r}\,')$, rather than using Eq. (4.30) as did Foldy.

APPENDIX B

Transformation of the Integral in Equation(5.35)

The integral appearing in the solution (5.35), i.e.

$$I(\tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \psi(s,\tau) ds \qquad \text{(B.1)}$$

where

$$\psi(s,\tau) - \frac{2}{1+s^2} \; \frac{e^{is\tau}}{1-\beta \tan^{-1} s/s} \quad , \qquad \text{(B.2)}$$

is similar to those arising from the solution of neutron transport problems[10]. By means of contour integration, the integral may be transformed to a form for which numerical evaluation is possible; the details of this process are given below.

Since

$$\tan^{-1} s = \frac{1}{i} \tanh^{-1} is = \frac{1}{2i} \log \frac{1+is}{1-is} \quad , \qquad \text{(B.3)}$$

the integrand $\psi(s,\tau)$ has branch points at $s = \pm i$; we construct branch cuts in the s-plane from $+i$ to $+i\infty$ and from $-i$ to $-i\infty$, thus making the integrand single-valued. The only other singularities of $\psi(s,\tau)$ arise from the zeros of $1-\beta \tan^{-1} s/s$; these are at $s = \pm s_o = \pm i\sigma_o$, where σ_o is the positive root of

$$1 - \frac{\beta}{\sigma_o} \tanh^{-1} \sigma_o = 0 \quad , \qquad \text{(B.4)}$$

(we assume that $0 < \beta < 1$).

We now consider the contour C depicted in Figure B-1. Since $\psi(s,\tau)$ is analytic on and within C, except for a simple pole at s_o, we have by the residue theorem that

$$\frac{1}{2\pi i}\int_C \psi(s,\tau)ds = \text{Residue of } \psi(s,\tau) \text{ at } s_o = \lim_{s\to s_o}(s-s_o)\psi(s,\tau) \ , \quad \text{(B.5)}$$

where the contour $C = C_1+C_2+C_3+C_4+C_5+C_6$. The residue is given by *the following expression*

$$\text{Residue} = \frac{2}{1+s_o^2}\,\frac{e^{is_o\tau}}{\left[\frac{\partial}{\partial s}(1-\beta\tan^{-1}s/s)\right]_{s=s_o}}$$

$$= \frac{2}{1+s_o^2}\;\frac{e^{is_o\tau}}{-\beta\;\frac{\partial}{\partial s_o}(\tan^{-1}s_o/s_o)} \quad ; \qquad \text{(B.6)}$$

but since s_o is a root of $1-\beta\tan^{-1}s_o/s_o = 0$,

$$\frac{\partial}{\partial s_o}(\tan^{-1}s_o/s_o) = \frac{1}{s_o(1+s_o^2)} - \frac{\tan^{-1}s_o}{s_o^2} - \frac{\beta-(1+s_o^2)}{\beta\, s_o(1+s_o^2)} \ , \qquad \text{(B.7)}$$

and hence

$$\text{Residue} = -\frac{2s_o}{\beta-s_o^2-1}\,e^{is_o\tau} = -i\,\frac{2\sigma_o}{\beta+\sigma_o^2-1}\,e^{-\sigma_o\tau} \ . \qquad \text{(B.8)}$$

Equation (B.5) then becomes

$$\frac{1}{2\pi}\int_C \psi(s,\tau)ds = \frac{2\sigma_o}{\beta+\sigma_o^2-1}\,e^{-\sigma_o\tau} \ . \qquad \text{(B.9)}$$

Now we note that for s lying on the quarter-circles C_2 or C_6, for $\tau \geqslant 0$

$$|\psi(s,\tau)| = O(R^{-2}) \quad \text{as} \quad R\to\infty \ , \qquad \text{(B.10)}$$

so that by a simple integral estimate, we have

$$\lim_{R\to\infty} \int_{C_2+C_6} \psi(s,\tau)ds = 0 \quad . \tag{B.11}$$

For s lying on the circle C_4 ,

$$|\psi(s,\tau)| = O(1/\rho \log \rho) \quad \text{as} \quad \rho \to 0 \quad , \tag{B.12}$$

and hence,

$$\lim_{\rho\to 0} \int_{C_4} \psi(s,\tau)ds = 0 \quad . \tag{B.13}$$

Therefore, from (B.9) we obtain

$$I(\tau) = \lim_{R\to\infty} \frac{1}{2\pi} \int_{C_1} \psi(s,\tau)d\tau = \frac{2\sigma_o}{\beta+\sigma_o^2-1} e^{-\sigma_o\tau}$$

$$- \lim_{\substack{\rho\to 0 \\ R\to\infty}} \frac{1}{2\pi} \int_{C_3+C_5} \psi(s,\tau)ds \quad . \tag{B.14}$$

In order to properly define $\tan^{-1} s$ in the cut plane, let us introduce the auxiliary coordinates indicated in Figure B-2. We have

$$\begin{aligned} 1+is &= i(s-i) = r_1 e^{i(\theta_1 + \frac{\pi}{2})} \qquad \left(0 < r_1 < \infty, \ -\frac{3\pi}{2} < \theta_1 < \frac{\pi}{2}\right) \ , \\ 1-is &= -i(s+i) = r_2 e^{i(\theta_2 - \frac{\pi}{2})} \qquad \left(0 < r_2 < \infty, \ -\frac{\pi}{2} < \theta_2 < \frac{3\pi}{2}\right) \ , \end{aligned} \tag{B.15}$$

so that we may define

$$\tan^{-1} s = \frac{1}{2i} \log \frac{1+is}{1-is} = \frac{1}{2i} \log \frac{r_1}{r_2} + \frac{1}{2}(\theta_1 - \theta_2 + \pi) \quad . \tag{B.16}$$

On the path C_3: $s = iu$, $\infty > u > 1$; $r_1 = u-1$, $r_2 = u+1$; $\theta_1 \to \frac{\pi}{2}$, $\theta_2 = \frac{\pi}{2}$;

$$\tan^{-1} s = \frac{1}{2i} \log \frac{u-1}{u+1} + \frac{\pi}{2} = i \tanh^{-1} \frac{1}{u} + \frac{\pi}{2} \quad . \tag{B.17}$$

On the path C_5: $s = iu$, $1 < u < \infty$; $r_1 = u-1$, $r_2 = u+1$; $\theta_1 \to -\frac{3\pi}{2}$, $\theta_2 = \frac{\pi}{2}$;

$$\tan^{-1} s = \frac{1}{2i} \log \frac{u-1}{u+1} - \frac{\pi}{2} = i \tanh^{-1} \frac{1}{u} - \frac{\pi}{2} \quad . \tag{B.18}$$

Therefore the integrals appearing in Eq. (B.14) are given by

$$\lim_{\substack{\rho \to 0 \\ R \to \infty}} \frac{1}{2\pi} \int_{C_3 + C_5} \psi(s, \tau) ds$$

$$= \frac{1}{2\pi} \int_{\infty}^{1} \frac{2}{1+(iu)^2} \frac{e^{-u\tau}}{1 - \frac{\beta}{u} \tanh^{-1} \frac{1}{u} + i \frac{\pi\beta}{2u}} \, i\,du$$

$$+ \frac{1}{2\pi} \int_{1}^{\infty} \frac{2}{1+(iu)^2} \frac{e^{-u\tau}}{1 - \frac{\beta}{u} \tanh^{-1} \frac{1}{u} - i \frac{\pi\beta}{2u}} \, i\,du$$

$$= - \int_{1}^{\infty} \frac{1/u}{1-u^2} \frac{\beta e^{-u\tau}}{\left(1 - \frac{\beta}{u} \tanh^{-1} \frac{1}{u}\right)^2 + \left(\frac{\pi\beta}{2u}\right)^2} du$$

$$= \int_{0}^{1} \frac{v}{1-v^2} \frac{\beta e^{-\tau/v}}{(1 - \beta v \tanh^{-1} v)^2 + \left(\frac{\pi\beta}{2} v\right)^2} dv \quad , \tag{B.19}$$

where the substitution $u = \frac{1}{v}$ has been made. Let us make the further substitution $v = \tanh w$; the integral then becomes

$$\int_{0}^{\infty} \frac{\beta \tanh w \, e^{-\tau/\tanh w}}{(1 - \beta w \tanh w)^2 + \left(\frac{\pi\beta}{2} \tanh w\right)^2} dw \quad ,$$

We have thus transformed the integral (B.1) to

$$I(\tau) = \frac{2\sigma_o}{\beta + \sigma_o^2 - 1} e^{-\sigma_o \tau} - \int_o^\infty g(w;\beta, 1) e^{-\tau/\tanh w} dw \quad , \qquad (B.20)$$

where

$$g(w;\beta, n) = \frac{\beta (\tanh w)^n}{(1 - \beta w \tanh w)^2 + \left(\frac{\pi\beta}{2} \tanh w\right)^2} \quad . \qquad (B.21)$$

and σ_o is the positive root of

$$\sigma_o = \tanh \frac{\sigma_o}{\beta} \quad . \qquad (B.22)$$

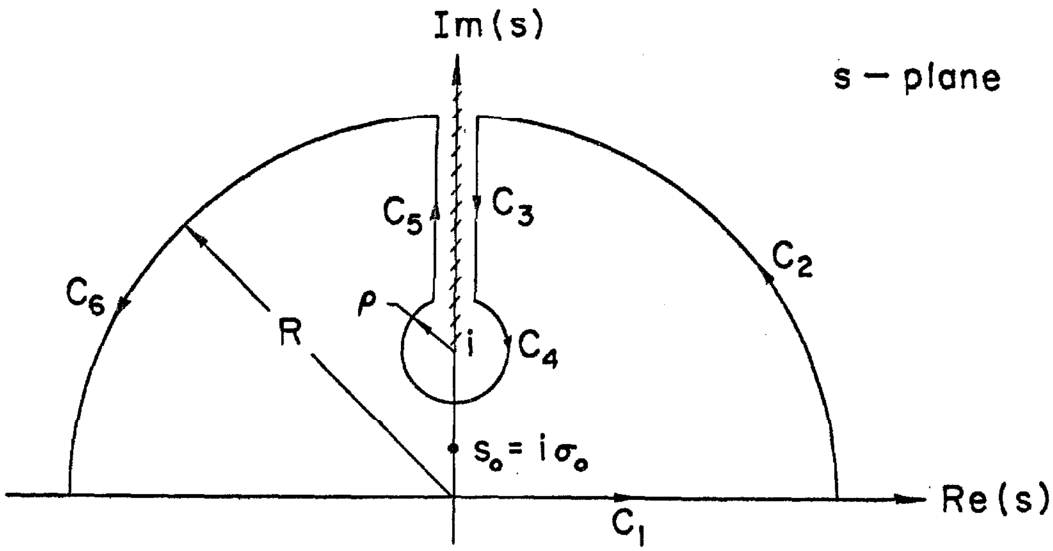

Figure B-1 - Path of Integration in the Complex Plane

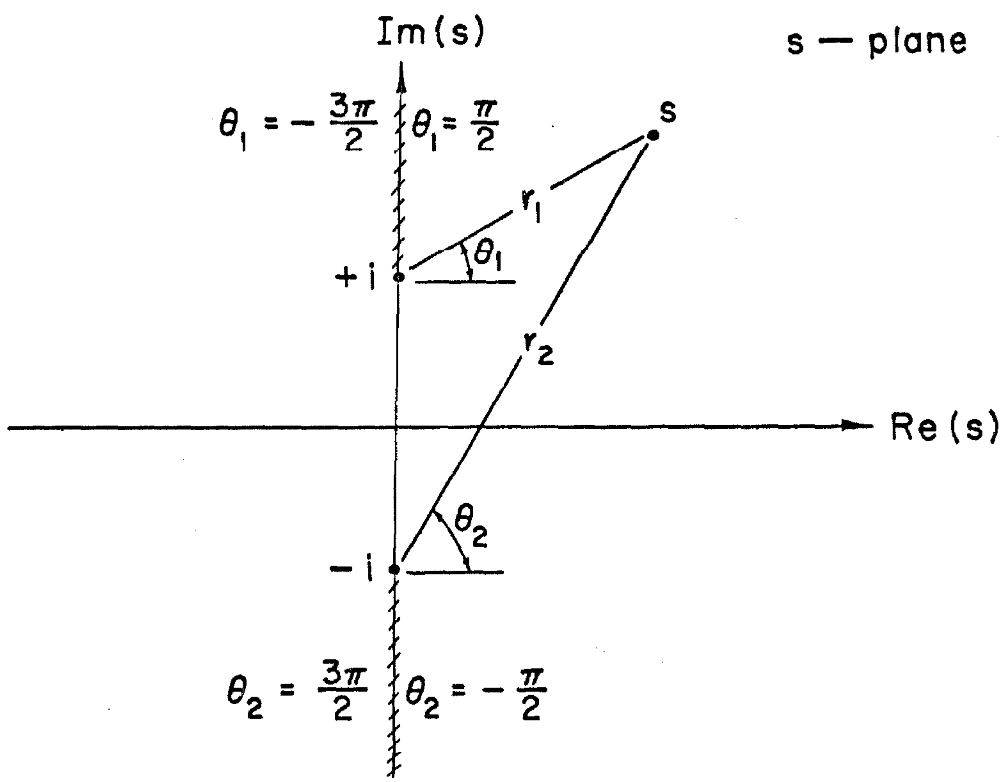

Figure B-2 - Definition of Coordinates

APPENDIX C

Proof of the Results (6.82) and (6.83)

In the analysis of Section 6, we used the following result. Let $B(\tau,\sigma)$ be the solution of the integral equation

$$B(\tau,\sigma) - \Lambda_\tau\{B(t,\sigma)\} = e^{-\sigma\tau} \quad , \tag{C.1}$$

where the operator Λ_τ is defined as

$$\Lambda_\tau\{f(t)\} = \frac{\beta}{2}\int_0^\infty f(t)E_1(|\tau-t|)dt \quad , \tag{C.2}$$

and let $\hat{B}(s,\sigma)$ denote the Laplace transform of $B(\tau,\sigma)$

$$\hat{B}(s,\sigma) = \int_0^\infty B(\tau,\sigma)e^{-s\tau}d\tau \quad ; \tag{C.3}$$

then this quantity is given by

$$\hat{B}(s,\sigma) = \frac{B(0,s)B(0,\sigma)}{(s+\sigma)} \quad , \tag{C.4}$$

and $B(0,\sigma)$ satisfies the non-linear integral equation

$$B(0,\sigma) = 1 + B(0,\sigma)\,\frac{\beta}{2}\int_1^\infty \frac{B(0,s)}{s(s+\sigma)}\,ds \quad . \tag{C.5}$$

A proof of this assertion, following the discussion given by Kourganoff[14], follows.

We shall first show that $\hat{B}(s,\sigma)$ is symmetric in the variables s and σ. We have by (C.1) that

$$B(\tau,\sigma) - \Lambda_\tau\{B(t,\sigma)\} = e^{-\sigma\tau} \quad , \tag{C.6}$$

$$B(\tau, s) - \Lambda_\tau \{B(t, s)\} = e^{-s\tau} \quad . \qquad (C.7)$$

Multiply the first by $B(\tau, s)$, the second by $B(\tau,\sigma)$, integrate each with respect to τ from 0 to ∞, and subtract; we obtain the following relation:

$$- \int_0^\infty B(\tau, s)\Lambda_\tau\{B(t,\sigma)\}\, d\tau + \int_0^\infty B(\tau,\sigma)\Lambda_\tau \{B(t, s)\}\, d\tau$$

$$= \hat{B}(\sigma, s) - \hat{B}(s,\sigma) \quad . \qquad (C.8)$$

But by interchanging the order of integration we may show that

$$\int_0^\infty B(\tau, s)\Lambda_\tau \{B(t,\sigma)\}\, d\tau = \frac{\beta}{2}\int_0^\infty B(\tau, s) \int_0^\infty B(t,\sigma)E_1 (|\tau - t|)dt d\tau$$

$$= \frac{\beta}{2}\int_0^\infty B(t,\sigma) \int_0^\infty B(\tau, s)E_1(|t-\tau|)d\tau dt - \int_0^\infty B(t,\sigma)\Lambda_t\{B(\tau, s)\}\, dt$$

$$= \int_0^\infty B(\tau,\sigma)\Lambda_\tau\{B(t, s)\}\, d\tau \quad , \qquad (C.9)$$

and hence, Eq. (C.8) yields

$$\hat{B}(\sigma, s) = \hat{B}(s,\sigma) \quad . \qquad (C.10)$$

The definition (C.2) may be written, after a linear change in variables, as

$$\Lambda_\tau\{f(t)\} = \frac{\beta}{2} \int_0^\tau f(\tau-u)E_1(u)du + \frac{\beta}{2} \int_0^\infty f(\tau +u)E_1(u)du \quad . \qquad (C.11)$$

By differentiating with respect to τ,

$$\frac{d}{d\tau} \Lambda_\tau\{f(t)\} = \frac{\beta}{2} f(0)E_1(\tau) + \frac{\beta}{2}\int_0^\tau f'(\tau-u)E_1(u)du + \frac{\beta}{2} \int_0^\infty f'(\tau+u)E_1(u)\, du , \qquad (C.12)$$

we obtain the following commutation relation between $\frac{d}{d\tau}$ and Λ_τ

$$\frac{d}{d\tau} \Lambda_\tau \{f(t)\} - \Lambda_\tau\{f'(t)\} = \frac{\beta}{2} f(0)E_1(\tau) \quad . \tag{C.13}$$

Let us denote $B'(\tau,\sigma) = \frac{\partial}{\partial\tau} B(\tau,\sigma)$ and differentiate Eq. (C.1) with respect to τ,

$$B'(\tau,\sigma) - \Lambda_\tau\{B'(t,\sigma)\} = \frac{\beta}{2} B(0,\sigma)E_1(\tau) - \sigma e^{-\sigma\tau} \quad ; \tag{C.14}$$

by setting

$$A(\sigma,u) = \frac{\beta}{2} B(0,\sigma) \frac{1}{u} \tag{C.15}$$

and using the integral definition of $E_1(\tau)$, this may be written as

$$B'(\tau,\sigma) - \Lambda_\tau\{B'(t,\sigma)\} = \int_1^\infty A(\sigma,u)e^{-\tau u}du - \sigma e^{-\sigma\tau} . \tag{C.16}$$

Let us replace σ by u in Eq. (C.1), multiply by $A(\sigma,u)$, and integrate with respect to u from 1 to ∞ to obtain

$$\int_1^\infty B(\tau,u)A(\sigma,u)du - \Lambda_\tau\left\{\int_1^\infty B(t,u)A(\sigma,u)du\right\} = \int_1^\infty A(\sigma,u)e^{-\tau u}du \quad ; \tag{C.17}$$

Let us now subtract this from Eq. (C.16), yielding

$$B'(\tau,\sigma) - \int_1^\infty B(\tau,u)A(\sigma,u)du - \Lambda_\tau\left\{B'(t,\sigma) - \int_1^\infty B(t,u)A(\sigma,u)du\right\} = -\sigma e^{-\sigma\tau}. \tag{C.18}$$

The term on the right may be eliminated by multiplying (C.1) by σ and adding,

$$B'(\tau,\sigma) - \int_1^\infty B(\tau,u)A(\sigma,u)du + \sigma B(\tau,\sigma) - \Lambda_\tau \left\{ B'(\tau,\sigma) - \int_1^\infty B(\tau,u)A(\sigma,u)du + \sigma B(\tau,\sigma) \right\} = 0 \quad . \qquad (C.19)$$

Therefore,

$$B'(\tau,\sigma) - \int_1^\infty B(\tau,u)A(\sigma,u)du + \sigma B(\tau,\sigma) = g(\tau) \quad , \qquad (C.20)$$

where $g(\tau)$ is an arbitrary function satisfying the homogeneous equation

$$g(\tau) - \Lambda_\tau \{g(t)\} = 0 \quad . \qquad (C.21)$$

By taking the Laplace transform of Eq. (C.20),

$$(s+\sigma)\hat{B}(s,\sigma) - B(0,\sigma) - \int_1^\infty \hat{B}(s,u)A(\sigma,u)du = \hat{g}(s) \quad , \qquad (C.22)$$

and replacing $A(\sigma,u)$ by its definition (C.15), we obtain

$$(s+\sigma)\hat{B}(s,\sigma) = B(0,\sigma)\left[1 + \frac{\beta}{2}\int_1^\infty \hat{B}(s,u)\,\frac{du}{u}\right] + \hat{g}(s) \quad . \qquad (C.23)$$

Let us now set $\tau = 0$ in (C.1) and interchange the order of integration, as follows:

$$B(0,s) = 1+\Lambda_o\{B(t,s)\} = 1 + \frac{\beta}{2}\int_o^\infty B(t,s)E_1(t)dt$$

$$= 1 + \frac{\beta}{2}\int_o^\infty B(t,s)\int_1^\infty e^{-tu}\,\frac{du}{u}\,dt = 1 + \frac{\beta}{2}\int_1^\infty\int_o^\infty B(t,s)e^{-tu}dt\,\frac{du}{u}$$

$$= 1 + \frac{\beta}{2}\int_1^\infty \hat{B}(u,s)\,\frac{du}{u} \quad ; \qquad (C.24)$$

this may be written, with the aid of the symmetry property (C.10) of

$\hat{B}(u, s)$, as

$$B(0, s) = 1 + \frac{\beta}{2} \int_1^\infty \hat{B}(s, u) \frac{du}{u} \quad . \tag{C.25}$$

By substituting this equation into Eq. (C.23) and requiring $\hat{B}(s,\sigma)$ to be symmetrical in its coordinates, we obtain the result (C.4),

$$(s+\sigma)\hat{B}(s, \sigma) = B(0, \sigma)B(0, s) \quad , \tag{C.26}$$

with $\hat{g}(s) = 0$. The second assertion (C.5) is obtained by replacing s by σ and u by s in Eq. (C.25) and eliminating $\hat{B}(\sigma, s) = \hat{B}(s, \sigma)$ with Eq. (C.26).

APPENDIX D

Table of Values of the Factor $[1-(1-\beta)^{\frac{1}{2}}H(\mu)]$ Appearing in Equation(6.101)

μ $\beta \rightarrow$	0.0	0.1	0.2	0.3	0.4	0.5	0.6
0.0	0.000	0.051	0.106	0.163	0.225	0.293	0.368
0.2	0.000	0.034	0.071	0.112	0.159	0.213	0.276
0.4	0.000	0.026	0.056	0.090	0.128	0.174	0.230
0.6	0.000	0.022	0.047	0.075	0.109	0.148	0.197
0.8	0.000	0.019	0.040	0.065	0.094	0.130	0.174
1.0	0.000	0.016	0.035	0.057	0.083	0.115	0.155

μ $\beta \rightarrow$	0.7	0.8	0.9	0.925	0.950	0.975	1.000
0.0	0.452	0.553	0.684	0.726	0.776	0.842	1.000
0.2	0.352	0.451	0.592	0.641	0.701	0.783	1.000
0.4	0.299	0.391	0.532	0.584	0.649	0.740	1.000
0.6	0.261	0.348	0.486	0.538	0.605	0.703	1.000
0.8	0.232	0.313	0.447	0.500	0.568	0.671	1.000
1.0	0.209	0.285	0.415	0.467	0.536	0.641	1.000

Validity of the Approximate Expression $\frac{\beta}{2} \cdot 1-\mu \log \frac{\mu+1}{\mu}$ for $\beta = 0.1$

μ	Approximate Value	Exact Value	% Error
0.0	0.0500	0.0513	2.7
0.2	0.0321	0.0336	4.5
0.4	0.0249	0.0264	5.7
0.6	0.0206	0.0219	5.9
0.8	0.0176	0.0187	5.9
1.0	0.0153	0.0164	6.7

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