OSCILLATING AIRFOIL IN PARALLEL STREAMS SEPARATED BY AN INTERFACE

Thesis by

Shih-Chun Lo

In Partial Fulfillment of the Requirements

For the Degree of

Doctor of Philosophy

California Institute of Technology

Pasadena, California

1951

ACKNOWLEDGMENT

First of all, the author wishes to acknowledge the constant help and encouragement which he received from the faculty and the staff of the Guggenheim Aeronautical Laboratory, in particular Dr. Clark B. Millikan and Dr. Ernest E. Sechler.

The inspiration for this attempt to investigate the tail buffeting phenomenon by studying the effect of interface on oscillating airfoil was provided by Dr. Hsue-Shen Tsien, Robert H. Goddard Professor of Jet Propulsion. Not only is the original conception of the paper attributable to him, but the actual execution of the calculation has been possible only with the aid of his constant advice and encouragement. The author is deeply grateful to Professor Tsien for his brilliant and inspiring guidance.

TABLE OF CONTENTS (Cont'd)

		Page
Appendix 10:	Derivation of Eq. (84)	84
Table 1:	Lift and Moment when $h=0$, $0<\ell<\infty$ and $k=0$	86
Table 2:	Lift and Moment when $h=0$, $\ell=1$ and $k=\frac{1}{2}$	87
Table 3:	Aerodynamic Coefficients when $h=0$, $0<\ell<\infty$, $k=0$ and $\frac{a}{c}=.35$	88
Table 4:	Function $M_s(\Omega)$ when $s=0-5$ and $\Omega=0-3$	89
Figure 1:	Wing Wake Acting upon the Oscillating Tail	90
Figure 2:	Interface with Thin Airfoil	91
Figure 3:	Vector Diagram of the Lift and Moment on the Airfoil in Bending Oscillation, as Functions of h, ℓ, k and Ω	92
Figure 4:	Vector Diagram of the Lift and Moment on the Airfoil in "Torsional" Oscillation, as Functions of h, ℓ, k and Ω	93
Figure 5:	Schematic Diagram of a Fluttering Airfoil	94
References		95

ABSTRACT

A new approach to tail buffeting is made by studying the problem of a thin airfoil performing a periodic oscillation of small amplitude in the presence of an interface across which the flow undergoes a constant change in density and velocity. A general solution to the problem is found. Lift and moment for some special cases are obtained in simple forms and are plotted in Figs. 3 and 4 for the two basic modes of oscillation: bending and torsion. A typical application to flutter analysis is made and it is found that tail flutter at low speeds is possible for the tail lying in the wake of the wing.

I. INTRODUCTION AND SUMMARY

Tail buffeting, i.e. tail vibration under the aerodynamic action of the wake shed by the wing at large angle of attack, is a well-known phenomenon. Following the unusual accident of the Junkers airplane at Meopham, the problem took on a serious aspect and led scientific organizations in various countries to undertake detailed investigations. (Refs. 1-4). Based on these and other investigations, a large number of simple devices are now available for the elimination of the tail buffeting. However, theories for predicting the phenomenon are still lacking.

The main difficulty in treating the tail buffeting lies in the fact that the actual nature of the aerodynamic wake behind the bodies has not yet been established. In Ref. 4, Abdrashitov approximated the effect of the wake on the tail by a harmonic disturbance force and found that the characters of the tail vibration are fundamentally determined by

(a) the ratio of the frequency of the flow in the wake of the wing to the natural frequency of the tail surface,

(b) the amplitude of the disturbance forces produced by the wake of the wing,

(c) the magnitude of the speed of flight, and

(d) the vertical position of the tail surface relative to the wing* .

In this paper, an entirely different approach is made. The aerodynamic wake shed by the wing is here approximated by an interface across which the flow undergoes a constant change in density and velocity (Figs. 1 and 2). The problem is then set to determine the possibilities of the tail flutter in the presence of the interface at various speeds of flight. The main aerodynamic effect of the wake is believed to be approximated by the effect of mutual influence between the vorticity on the interface and the vorticity generated by the oscillation of the airfoil.

It is found convenient to discuss the results in terms of the two parameters: $\frac{2h}{c}$, the vertical distance between the airfoil and the interface divided by the half chord, and Ω , the reduced frequency (the product of the half chord and the vibration frequency of the tail divided by the flying speed).

(a) When $\frac{2h}{c} > 1$, the influence of the interface is extremely small for all values of Ω . Therefore the tail flutter has essentially the nature of the wing flutter (Ref. 5), when the vertical distance between the airfoil and the interface is greater than the half chord.

(b) When $\Omega < 1$, the influence of the interface is small for all values of $\frac{2h}{c}$. Hence in high speed range of flight (since $\Omega < 1$ means

-----

* It is noted that he also made an important approximation in taking only the quasi-steady values for the aerodynamic force and moment, i.e. neglecting the effect of the tail wake produced by the non-steady motion of the tail.

$U > \frac{c\omega}{2}$), the tail flutter again has the nature of the wing flutter.

(c) When $\frac{2h}{c} = 0$ and $\Omega > 1$, the influence of the interface is so large as to render the tail flutter possible. Unlike the wing, the tail has the possibility of entering into flutter at low speeds of flight, when it lies in the wake of the wing.

II. FORMULATION OF THE PROBLEM AND THE GENERAL PROCEDURES

To formulate the general problem the following assumptions are made:

(a) the wake given off by the wing may be approximated by an interface across which the flow undergoes a constant change in velocity and density; the interface is flat, of zero thickness and extends to infinity in all directions;

(b) the tail surface is of infinite aspect ratio;

(c) the oscillating motion is two dimensional, i.e. every cross-section taken perpendicular to the span has identical motion and remains in its own plane during the motion;

(d) the flow is incompressible and non-viscous;

(e) the thickness of the tail surface and the amplitude are small in comparison with the chord;

(f) the oscillation is periodic;

(g) the tail has a mean position parallel to the interface.

Using the assumptions (a)-(c), the problem becomes two dimensional. As shown in Fig. 2, the interface is located at $y=h$. The airfoil is put on the x-axis with its leading edge at $x=-1$ and its trailing edge at $x=+1$. The chord is taken as 2 so that all distances in this analysis are measured relative to the half chord. The interface divides the whole space into two regions. The region in which the tail is situated, $y \leq h$ is designated by "1" and the other region, $y \geq h$ is designated by "2". The velocity and density of the undisturbed flow in region 1 are denoted by U_1 and ρ_1 and the corresponding quantities in region 2 are denoted by U_2 and ρ_2.

The assumption (d) makes the flows potential in both regions, i.e. there exist the potential functions, Φ_1 and Φ_2 satisfying the Laplace differential equation for the flows in regions 1 and 2. For determining Φ_1 and Φ_2, complete boundary conditions should be specified. These conditions are

(a) on the surface of the airfoil, the normal component of velocity should be equal to that of the prescribed motion of the airfoil;

(b) at infinity, i.e. points far from the airfoil, the disturbance should vanish, and

(c) at the interface, the velocity vectors on the two sides of the interface are parallel so that the interface remains a streamline and the static pressures on the two sides are equal. For calculating lift and moment, only Φ_1 will be required. However, since both Φ_1 and Φ_2 enter into the conditions at the interface, they must be investigated simultaneously.

Using the assumption (e), the boundary conditions (a) and (c) can be applied at the undisturbed or mean positions. The assumptions (e)-(g) enable one to write for the boundary condition (a):

$$v\,(\theta,t) = e^{i\omega t}\left[\frac{B_o}{2} + \sum_{n=1}^{\infty} B_n \cos n\theta\right] \qquad (1)$$

where v is the velocity component in the y-direction and $\theta = \cos^{-1}x$ as given by Eq. (7) below.

Because the Laplace differential equation is linear, the general solution can be obtained by superposing elementary solutions. The elementary solution satisfies the Laplace differential equation and part of the boundary conditions. The superposition is then made in such a manner that the rest of the boundary conditions are satisfied. The general procedures of the analysis are as follows:

(a) Taking only one term of Eq. (1), $e^{i\omega t}\cos m\theta$, as the boundary condition at the airfoil; ϕ_{om}, the velocity potential for a uniform flow without interface is determined;

(b) introducing the interface, the velocity potentials ϕ_{1m} and ϕ_{2m} are so determined that $\phi_{om}+\phi_{1m}$ for the flow in region 1 and ϕ_{2m} for the flow in region 2 satisfy the conditions at the interface with vanishing disturbance at infinity;

(c) considering $\phi_{om}+\phi_{1m}$ as the elementary solution, the general solution is obtained by superposition

$$\Phi_1 = \sum_{m=0}^{\infty} A_m(\phi_{om}+\phi_{1m}) \qquad (2)$$

where A_m is determined in terms of B_n by the boundary condition at the airfoil, Eq. (1).

(d) using Φ_1, lift and moment acting on the oscillating airfoil are calculated, in particular for the two basic modes of oscillation: bending and torsion;

(e) calculations for some special cases are carried out in detail and others are discussed; and

(f) a typical application to the flutter analysis is made through a numerical example.

It is noted that the airfoil has been set below the interface (i.e. $h > 0$). Since the aerodynamic force and moment bear a definite relation with the motion of the airfoil normal to the interface, the results obtained under the above condition can be easily interpreted for the case of the airfoil lying above the interface.

III. ELEMENTARY SOLUTION

(a) DETERMINATION OF ϕ_{om}

ϕ_{om} is defined as the velocity potential for the oscillating motion of an airfoil described by

$$V(\theta,t) = e^{i\omega t} \cos m\theta \tag{3}$$

in a uniform flow without interface. In accordance with the theory of thin airfoils in oscillating motion (Ref. 6) the whole system is represented by the two vortex sheets:

(a) the bound vortex sheet lying along the chord of the airfoil, i.e. on the x-axis between $x=-1$ and $x=+1$; and

(b) the wake vortex sheet lying on the x-axis from $x=+1$ to $x=+\infty$, assuming that the motion has been occurring so long that transient phenomena have disappeared. The bound vortex sheet is considered to be made up of the two parts: (a) the quasi-steady part which would be produced if the motion were steady, or if the wake had no effect on the motion; and (b) the induced part induced by the wake vortex. Denoting the distribution of the vortex strength (or circulation) along the x-axis by γ , the definition of circulation yields

$$\gamma(x,t) = \lim_{\Delta x \to 0} \frac{1}{\Delta x} \left[\int_{x}^{x+\Delta x} u(x,0+,t)\, dx + \int_{x+\Delta x}^{x} u(x,0-,t)\, dx \right]$$

or, by carrying out the limit procedure,

$$\gamma(x,t) = u(x,0+,t) - u(x,0-,t) \qquad \text{along} \quad y=0 \tag{4}$$

where $u(x,0+,t)$ denotes the velocity component in the x-direction on the upper surface of the sheet $y=0$ and $u(x,0-,t)$ denotes the corresponding quantity on the lower surface of the sheet $y=0$. Writing γ_{om}, γ_{im} and γ_{am} for the strength distributions of the quasi-steady vortex, the induced vortex and the wake vortex respectively, the velocity potential, ϕ_{om} takes the form

$$\phi_{om}(x,y,t) = -\frac{1}{2\pi}\int_{-1}^{1}\left[\gamma_{om}(x',t)+\gamma_{im}(x',t)\right]\tan^{-1}\frac{y}{x-x'}\,dx' +$$

$$-\frac{1}{2\pi}\int_{1}^{\infty}\gamma_{am}(\xi,t)\tan^{-1}\frac{y}{x-\xi}\,d\xi \tag{5}$$

according to the two dimensional potential theory.

<u>Calculation of γ_{om}</u>. The quasi-steady part of the bound vortex is determined by the use of the conformal transformation

$$z = \frac{1}{2}\left(\zeta + \frac{1}{\zeta}\right) \tag{6}$$

where $z = x+iy$ and $\zeta = re^{i\theta}$.

By this transformation, the airfoil in the z-plane is transformed into a unit circle in the ζ-plane with the relation

$$x = \cos\theta \tag{7}$$

The radial and tangential components of the velocity in the ζ-plane, q_r and q_θ, are easily found to be related to the u and v components of the velocity in the z-plane by

$$q_r = v \sin\theta \tag{8}$$

and

$$q_\theta = -u \sin\theta \tag{9}$$

By Eq. (8), Eq. (3) is transformed into the boundary condition on the unit circle,

$$q_r = \frac{e^{i\omega t}}{2}\left[\sin(m+1)\theta - \sin(m-1)\theta\right] \tag{10}$$

The velocity potential in the ζ-plane evidently has the general form

$$\phi(r,\theta,t) = e^{i\omega t}\left[\beta_0\theta + \sum_{n=1}^{\infty} \frac{1}{r^n}\beta_n \sin n\theta\right] \tag{11}$$

where the β's are undetermined constants.

Using Eq. (10), all constants except β_0 in Eq. (11) are determined.

$$
\begin{aligned}
&\text{For } m=0, \quad && \beta_1 = -1 \quad && \beta_2 = \beta_3 = \cdots = 0 \\
&\text{For } m=1, \quad && \beta_2 = -\tfrac{1}{4} \quad && \beta_1 = \beta_3 = \cdots = 0 \\
&\text{For } m \geq 2, \quad && \beta_{m-1} = \tfrac{1}{2(m-1)} \quad \beta_{m+1} = -\tfrac{1}{2(m+1)} \quad && \text{other } \beta\text{'s} = 0
\end{aligned}
\tag{12}
$$

To determine β_0, the Kutta-Joukowsky condition is used. The condition states that the velocity at the trailing edge of the airfoil should be finite. By Eqs. (8) and (9), the condition becomes that

$q_r = q_\theta = 0$ at $\theta = 0$. q_r satisfies the condition as seen from Eq. (10). The condition $q_\theta = 0$ at $\theta = 0$ yields

For $m = 0$ $\beta_o = 1$

(13)

For $m = 1$ $\beta_o = \frac{1}{2}$

For $m \geq 2$ $\beta_o = 0$

γ_{om} can now be calculated directly by the formula

$$\gamma = -\frac{2\, q_\theta}{\sin\theta} \tag{14}$$

which is obtained by the use of Eqs. (4) and (9) and the fact that $q_\theta(-\theta) = q_\theta(+\theta)$ as seen through Eq. (11). Thus

$$\gamma_{oo}(x,t) = -2\, e^{i\omega t}\,\frac{1-\cos\theta}{\sin\theta} = -2\, e^{i\omega t}\sqrt{\frac{1-x}{1+x}}$$

$$\gamma_{o1}(x,t) = -2\, e^{i\omega t}\sin\theta = -2\, e^{i\omega t}\sqrt{1-x^2}$$

(15)

and $\gamma_{om}(x,t) = -2\, e^{i\omega t}\sin m\theta$ for $m \geq 2$

The total circulation of the quasi-steady vortex, Γ_{om}, is obtained by integrating γ_{om} from $x = -1$ to $x = +1$, and is

$$\Gamma_{oo}(t) = -2\pi e^{i\omega t}, \quad \Gamma_{o1}(t) = -\pi e^{i\omega t}, \quad \text{and} \quad \Gamma_{om}(t) = 0 \ \text{for } m \geq 2 \tag{16}$$

Calculation of γ_{1m}. The induced part of the bound vortex may also be evaluated by the conformal transformation given by Eq. (6). The induced vortex distribution on the airfoil due to a single free vortex with circulation Γ' located at $x=\xi$ and $y=0$ has been evaluated in Ref. 6 as

$$\gamma(x) = \frac{1}{\pi}\frac{\Gamma'}{\xi - x}\sqrt{\frac{1-x}{1+x}}\sqrt{\frac{\xi+1}{\xi-1}} \tag{17}$$

Putting $\Gamma' = \gamma_{2m} d\xi$ and integrating the result from $x=1$ to $x=\infty$ yield

$$\gamma_{1m}(x,t) = \frac{1}{\pi}\sqrt{\frac{1-x}{1+x}} \int_1^{\infty} \frac{\gamma_{2m}(\xi,t)}{\xi - x}\sqrt{\frac{\xi+1}{\xi-1}}\, d\xi \tag{18}$$

The total circulation of the induced vortex, Γ_{1m} is obtained by integration of Eq. (18) from $x=-1$ to $x=+1$, and is

$$\Gamma_{1m}(t) = \int_1^{\infty} \gamma_{2m}(\xi,t)\left(\sqrt{\frac{\xi+1}{\xi-1}} - 1\right) d\xi \tag{19}$$

Calculation of γ_{2m}. Assuming that the wake vortex is left in the fluid with invariable strength and position (i.e. the wake vortex is moving away from the airfoil at the velocity U_1 along the x-axis and with fixed strength) its strength distribution, γ_{2m}, can be expressed as

$$\gamma_{2m}(\xi,t) = g_m\, e^{i\omega\left(t - \frac{\xi}{U_1}\right)} \tag{20}$$

where g_m is undetermined constant.

To evaluate g_m, the law of conservation of circulation is used. The law states that the total circulation of the whole system is invariably zero. Denoting the circulation about the airfoil by Γ_m $(=\Gamma_{om}+\Gamma_{1m})$, the increment of Γ_m, $\frac{d\Gamma_m}{dt}dt$ must be equal and opposite to the circulation in the wake between $\xi = 1$ and $\xi = 1+U_1 dt$, or $\gamma_{2m}(1,t)U_1 dt$. Thus, there is obtained

$$\frac{d\Gamma_m}{dt} + \gamma_{2m}(1,t)\,U_1 = 0 \tag{21}$$

Putting Eqs. (16), (19) and (20) into Eq. (21) yields

$$g_0 = \frac{2\pi}{K_0(i\Omega)+K_1(i\Omega)}$$

$$g_1 = \frac{\pi}{K_0(i\Omega)+K_1(i\Omega)} \tag{22*}$$

$$g_m = 0 \qquad \text{for} \quad m \geq 2$$

where $\Omega = \frac{\omega}{U_1}$ and K_0 and K_1 are the modified Bessel functions of the second kind and can be expressed by the Bessel functions of the first kind as

$$K_0(i\Omega) = -\frac{\pi}{2}\left[Y_0(\Omega) + i\,J_0(\Omega)\right]$$
$$K_1(i\Omega) = -\frac{\pi}{2}\left[J_1(\Omega) - i\,Y_1(\Omega)\right] \tag{23}$$

-----

*In this deduction the following integration formula given in Ref. 6 is used

$$\int_1^{\infty} e^{-i\Omega\xi}\left(\sqrt{\frac{\xi+1}{\xi-1}}-1\right)d\xi = K_0(i\Omega) + K_1(i\Omega) - \frac{1}{i\Omega}e^{-i\Omega} \tag{22a}$$

Putting Eq. (22) into Eq.(20) gives

$$\gamma_{20}(\xi,t) = \frac{2\pi}{K_0+K_1}\, e^{i\omega\left(t-\frac{\xi}{U_1}\right)}$$

$$\gamma_{21}(\xi,t) = \frac{\pi}{K_0+K_1}\, e^{i\omega\left(t-\frac{\xi}{U_1}\right)} \tag{24}$$

and $\gamma_{2m} = 0$ for $m \geq 2$

where K_0 and K_1 are used as abbreviations for $K_0(i\lambda)$ and $K_1(i\lambda)$. Eq. (5) together with Eqs. (15), (18) and (24) gives the complete determination of ϕ_{om}.

IV. ELEMENTARY SOLUTION

(b) DETERMINATION OF ϕ_{1m}

ϕ_{1m} is defined as the velocity potential to be added to ϕ_{om} to form an elementary velocity potential satisfying the boundary conditions at the interface and at infinity for the flow in region 1, while another potential ϕ_{2m} is defined as the corresponding elementary velocity potential for the flow in region 2. The boundary conditions at the interface, $y=h$ are given by

$$\frac{v_1}{U_1+u_1} = \frac{v_2}{U_2+u_2} \qquad \text{at} \quad y=h \tag{25}$$

$$p_1 = p_2 \qquad \text{at} \quad y=h \tag{26}$$

Neglecting u_1 and u_2 in comparison with U_1 and U_2 and introducing the velocity potentials, Eq. (25) becomes

$$\frac{\partial \phi_{om}}{\partial y} + \frac{\partial \phi_{1m}}{\partial y} - \frac{U_1}{U_2}\frac{\partial \phi_{2m}}{\partial y} = 0 \qquad \text{at} \quad y=h \tag{27}$$

Eq. (26) can be replaced by

$$\frac{\partial p_1}{\partial x} = \frac{\partial p_2}{\partial x} \qquad \text{at} \quad y=h \tag{28}$$

because $p_1=p_2$ at $y=h$ and $x=-\infty$ for all values of t. Using the linearized Eulerian equations of motion in the x-direction

$$-\frac{1}{\rho_1}\frac{\partial p_1}{\partial x} = \frac{\partial u_1}{\partial t} + U_1 \frac{\partial u_1}{\partial x}$$

and
$$-\frac{1}{\rho_2}\frac{\partial p_2}{\partial x} = \frac{\partial u_2}{\partial t} + U_2 \frac{\partial u_2}{\partial x}$$

Eq. (28) becomes

$$\frac{\partial^2}{\partial t \partial x}\left[\phi_{om} + \phi_{1m} - \frac{\rho_2}{\rho_1}\phi_{2m}\right] + \frac{\partial^2}{\partial x^2}\left[U_1(\phi_{om}+\phi_{1m}) - \frac{\rho_2}{\rho_1}U_2\phi_{2m}\right] = 0 \quad \text{at } y=h \tag{29}$$

where the velocity potentials are introduced.

To determine ϕ_{1m} and ϕ_{2m} by Eqs. (27) and (29), it is convenient to write them in the Fourier integral forms. The appropriate expressions for ϕ_{1m} and ϕ_{2m} satisfying the Laplace differential equation and vanishing at infinity are

$$\phi_{1m}(x,y,t) = e^{i\omega t}\int_0^\infty e^{\lambda y}\left[a_m(\lambda)\cos\lambda x + b_m(\lambda)\sin\lambda x\right]d\lambda \qquad \text{for } y \leq h \tag{30}$$

$$\phi_{2m}(x,y,t) = e^{i\omega t}\int_0^\infty e^{-\lambda y}\left[\alpha_m(\lambda)\cos\lambda x + \beta_m(\lambda)\sin\lambda x\right]d\lambda \qquad \text{for } y \geq h \tag{31}$$

where a_m, b_m, α_m and β_m are undetermined functions of λ.

ϕ_{om} given by Eq. (5) can also be written in the form

$$\phi_{om}(x,y,t) = e^{i\omega t}\int_0^\infty e^{-\lambda y}\left[A_m(\lambda)\cos\lambda x + B_m(\lambda)\sin\lambda x\right]d\lambda \tag{32}$$

where, by the Fourier integral formula

$$e^{i\omega t}e^{-\lambda y}A_m(\lambda) = \frac{1}{\pi}\int_{-\infty}^{\infty}\phi_{om}(s,y,t)\cos\lambda s\, ds \tag{33}$$

$$e^{i\omega t}e^{-\lambda y}B_m(\lambda) = \frac{1}{\pi}\int_{-\infty}^{\infty}\phi_{om}(s,y,t)\sin\lambda s\, ds \tag{34}$$

Putting Eq. (5) into Eq. (33) yields

$$e^{i\omega t} A_m(\lambda) = -\frac{1}{2\pi\lambda}\left\{\int_{-1}^{1}\left[\gamma_{0m}(x',t) + \gamma_{1m}(x',t)\right]\sin\lambda x' dx' + \right.$$

$$\left. + \int_{1}^{\infty} \gamma_{2m}(\xi)\sin\lambda\xi \, d\xi \right\} \tag{35}$$

by the use of the integration formula derived in Appendix 1.

Putting into Eq. (35), Eq. (15) for γ_{0m} , Eq. (18) for γ_{1m} and Eq. (24) for γ_{2m} yields, by the use of the integration formulas derived in Appendix 2,

$$A_0(\lambda) = -\frac{1}{2\pi\lambda}\left\{2\pi J_1(\lambda) - \frac{4\pi}{K_0+K_1}\sum_{s=0}^{\infty}(-)^s J_{2s+1}(\lambda)\, M_{2s+1}(\Omega) + \right.$$

$$\left. + \frac{2\pi e^{-i\Omega}}{K_0+K_1}\;\frac{\lambda\cos\lambda + i\Omega\sin\lambda}{\lambda^2 - \Omega^2}\right\}$$

$$A_1(\lambda) = -\frac{1}{2\pi\lambda}\left\{-\frac{2\pi}{K_0+K_1}\sum_{s=0}^{\infty}(-)^s J_{2s+1}(\lambda)\, M_{2s+1}(\Omega) + \right. \tag{36}$$

$$\left. + \frac{\pi e^{-i\Omega}}{K_0+K_1}\;\frac{\lambda\cos\lambda + i\Omega\sin\lambda}{\lambda^2 - \Omega^2}\right\}$$

$$A_m(\lambda) = -\frac{1}{2\pi\lambda}\left\{(-)^{\frac{m}{2}} 2m\pi\;\frac{J_m(\lambda)}{\lambda}\right\} \qquad \text{for} \qquad m = 2, 4, 6, \cdots$$

$$A_m(\lambda) = 0 \qquad \text{for} \qquad m = 3, 5, 7, \cdots$$

where M_n is given in Appendix 2 and calculated in Appendix 3 and are

$$M_0(\Omega) = -\frac{1}{2}\left[K_0 + K_1 - \frac{1}{i\Omega} e^{-i\Omega}\right]$$

$$M_s(\Omega) = \int_1^\infty e^{-i\Omega\zeta} (\zeta - \sqrt{\zeta^2 - 1})^s \, d\zeta \qquad \text{for} \quad s \geq 1 \tag{37}$$

Similarly, the following expressions are obtained for B_m from Eq. (34),

$$B_0(\lambda) = \frac{1}{2\pi\lambda}\left\{-2\pi J_0(\lambda) - \frac{4\pi}{K_0+K_1}\sum_{s=0}^{\infty}(-)^s J_{2s}(\lambda) M_{2s}(\Omega) + \right.$$

$$\left. + \frac{2\pi e^{-i\Omega}}{K_0+K_1}\,\frac{i\Omega\cos\lambda - \lambda\sin\lambda}{\lambda^2 - \Omega^2}\right\}$$

$$B_{01}(\lambda) = \frac{1}{2\pi\lambda}\left\{-2\pi\,\frac{J_1(\lambda)}{\lambda} - \frac{2\pi}{K_0+K_1}\sum_{s=0}^{\infty}(-)^s J_{2s}(\lambda) M_{2s}(\Omega) + \right.$$

$$\left. + \frac{\pi e^{-i\Omega}}{K_0+K_1}\,\frac{i\Omega\cos\lambda - \lambda\sin\lambda}{\lambda^2 - \Omega^2}\right\} \tag{38}$$

$$B_m(\lambda) = 0 \qquad \text{for} \quad m = 2, 4, 6, \cdots$$

$$B_m(\lambda) = \frac{1}{2\pi\lambda}\left\{(-)^{\frac{m+1}{2}} 2m\pi\,\frac{J_m(\lambda)}{\lambda}\right\} \quad \text{for} \quad m = 3, 5, 7, \cdots$$

Putting Eqs. (30)-(32) into Eqs. (27) and (29) yields four linear algebraic equations for the four unknowns a_m, b_m, α_m and β_m. Solving them, the results for a_m and b_m are

$$a_m(\lambda) = -\frac{e^{-2h\lambda}}{\alpha^2\lambda^2 - \beta^2}\left[(\mu\lambda^2 - \nu)A_m(\lambda) + iq\lambda\, B_m(\lambda)\right]$$

$$b_m(\lambda) = \frac{e^{-2h\lambda}}{\alpha^2\lambda^2 - \beta^2}\left[iq\lambda A_m(\lambda) - (\mu\lambda^2 - \nu)\, B_m(\lambda)\right] \tag{39}$$

where

$$
\begin{aligned}
\alpha &= 1 + l k^2 \\
\beta &= \Omega\,(1 + l k) \\
\mu &= 1 - l^2 k^4 \\
\nu &= \Omega^2 (1 - l^2 k^2) \\
q &= 2\,\Omega\, l k\,(1 - k) \\
\Omega &= \frac{\omega}{U_1} \\
k &= \frac{U_2}{U_1}
\end{aligned}
\tag{40}
$$

and $$l = \frac{\rho_2}{\rho_1}$$

Eq. (30) together with Eqs. (39) (36) and (38) completes the determination of ϕ_{lm}.

V. GENERAL SOLUTION

(a) DETERMINATION OF Φ_1

Φ_1 is defined as the total velocity potential for the flow in region 1 and is obtained by superposing the elementary velocity potentials, $\phi_{om} + \phi_{1m}$ with the unknown coefficients A_m as given by Eq. (2). Φ_1 in this general form satisfies the Laplace differential equation and the boundary conditions at the interface and at infinity. The only condition left is that at the airfoil given by Eq. (1). To satisfy this condition it is convenient to expand $\frac{\partial \Phi_1}{\partial y}$ at $y=0$ and $-1<x<1$ into the Fourier series. For ϕ_{om}, it is evident that

$$\left.\frac{\partial \phi_{om}}{\partial y}\right|_{y=0} = e^{i\omega t} \cos m\theta \tag{41}$$

For ϕ_{1m}, it may be written that

$$\left.\frac{\partial \phi_{1m}}{\partial y}\right|_{y=0} = e^{i\omega t} \left[\frac{c_{om}}{2} + \sum_{n=1}^{\infty} c_{nm} \cos n\theta \right] \tag{42}$$

where by the Fourier series formula

$$e^{i\omega t} c_{nm} = \frac{2}{\pi} \int_0^{\pi} \left[\frac{\partial}{\partial y} \phi_{1m}(\cos\theta, y, t) \right]_{y=0} \cos n\theta \, d\theta \tag{43}$$

Putting Eq. (30) into Eq. (43) yields

$$\begin{aligned} c_{nm} &= 2(-)^{\frac{n}{2}} \int_0^{\infty} a_m(\lambda) J_n(\lambda) \lambda \, d\lambda \qquad &\text{for } n = 0, 2, 4, \cdots \\ c_{nm} &= 2(-)^{\frac{n-1}{2}} \int_0^{\infty} b_m(\lambda) J_n(\lambda) \lambda \, d\lambda \qquad &\text{for } n = 1, 3, 5, \cdots \end{aligned} \tag{44}$$

by using the integration formula derived in Appendix 4.

Putting Eq. (39) into Eq. (44) and using Eqs. (36) and (38), the final results obtained after some simple algebra are

$$(-)^n c_{2n,0} = \frac{2}{\alpha^2}\left[\mu N_{2n,1}(h) + q\delta\, Q_{2n,1}(h,\delta) + iq\, P_{2n,0}(h,\delta)\right] +$$

$$- \frac{4}{\alpha^2}\frac{1}{K_0+K_1}\left[\mu R_{2n}(\Omega,h) + q\delta\, S_{2n}(\Omega,h,\delta) - iq\, T_{2n}(\Omega,h,\delta)\right] +$$

$$- \frac{2e^{-i\Omega}}{K_0+K_1}\left[V_{2n}(h,\Omega) - \frac{2}{\alpha}\, V_{2n}(h,\delta)\right]$$

$$(-)^{n+1} c_{2n+1,0} = \frac{2}{\alpha^2}\left[-\mu N_{2n+1,0}(h) - q\delta\, Q_{2n+1,0}(h,\delta) + iq\, P_{2n+1,1}(h,\delta)\right] +$$

$$- \frac{4}{\alpha^2}\frac{1}{K_0+K_1}\left[\mu R_{2n+1}(\Omega,h) + q\delta\, S_{2n+1}(\Omega,h,\delta) + iq\, T_{2n+1}(\Omega,h,\delta)\right] +$$

$$+ \frac{2e^{-i\Omega}}{K_0+K_1}\left[V_{2n+1}(h,\Omega) - \frac{2}{\alpha}\, V_{2n+1}(h,\delta)\right]$$

$$(-)^n c_{2n,1} = \frac{2iq}{\alpha^2}\, Q_{2n,1}(h,\delta) +$$

$$- \frac{2}{\alpha^2}\frac{1}{K_0+K_1}\left[\mu R_{2n}(\Omega,h) + q\delta\, S_{2n}(\Omega,h,\delta) - iq\, T_{2n}(\Omega,h,\delta)\right] +$$

$$- \frac{e^{-i\Omega}}{K_0+K_1}\left[V_{2n}(h,\Omega) - \frac{2}{\alpha}\, V_{2n}(h,\delta)\right]$$

(45)

$$(-)^{n+1} c_{2n+1,1} = -\frac{2}{\beta^2}\left[\nu\, O_{2n+1,1}(h) + q\delta\; P_{2n+1,1}(h,\delta)\right] +$$

$$-\frac{2}{\alpha^2}\frac{1}{K_0+K_1}\left[\mu\, R_{2n+1}(\Omega,h) + q\delta\, S_{2n+1}(\Omega,h,\delta) + iq\; T_{2n+1}(\Omega,h,\delta)\right] +$$

$$+\frac{e^{-i\Omega}}{K_0+K_1}\left[V_{2n+1}(h,\Omega) - \frac{2}{\alpha}\; V_{2n+1}(h,\delta)\right]$$

$$(-)^{n+m} c_{2n,2m} = \frac{4m}{\beta^2}\left[\nu\, O_{2n,2m}(h) + q\delta\; P_{2n,2m}(h,\delta)\right] \qquad \text{for} \quad m \geq 1$$

$$(-)^{n+m} c_{2n+1,2m+1} = \frac{2(2m+1)}{\beta^2}\left[\nu\, O_{2n+1,2m+1}(h) + q\delta\; P_{2n+1,2m+1}(h,\delta)\right] \qquad \text{for} \quad m \geq 1$$

$$(-)^{n+m} c_{2n,2m+1} = \frac{2iq(2m+1)}{\alpha^2}\; Q_{2n,2m+1}(h,\delta) \qquad \text{for} \quad m \geq 1$$

$$(-)^{n+m} c_{2n+1,2m} = -\frac{4iqm}{\alpha^2}\cdot Q_{2n+1,2m}(h,\delta) \qquad \text{for} \quad m \geq 1$$

where the following definite integrals (with their principal values, if improper) are defined.

$$N_{nm}(h) = \int_0^\infty e^{-2h\lambda} J_n(\lambda)\, J_m(\lambda)\, d\lambda \qquad \text{for} \quad |n-m| = 1, 3, 5, \cdots$$

$$O_{nm}(h) = \int_0^\infty e^{-2h\lambda} J_n(\lambda)\, J_m(\lambda)\, \frac{d\lambda}{\lambda} \qquad \text{for } |n-m| = 0, 2, 4, \cdots \text{ and } m \geq 1$$

$$P_{nm}(h,\delta) = \int_0^\infty e^{-2h\lambda} J_n(\lambda)\, J_m(\lambda)\, \frac{\lambda\, d\lambda}{\lambda^2-\delta^2} \qquad \text{for} \quad n-m = 0, 2, 4, \cdots$$

$$Q_{nm}(h,\delta) = \int_0^\infty e^{-2h\lambda} J_n(\lambda)\, J_m(\lambda)\, \frac{d\lambda}{\lambda^2-\delta^2} \qquad \text{for} \quad n-m = 1, 3, 5, \cdots$$

$$E_n(h,\delta) = \int_0^\infty e^{-2h\lambda} J_n(\lambda) \cos\lambda\, \frac{\lambda\, d\lambda}{\lambda^2-\delta^2} \qquad \text{for} \quad n = 0, 2, 4, \cdots$$

$$F_n(h,\delta) = \int_0^\infty e^{-2h\lambda} J_n(\lambda) \cos\lambda\, \frac{d\lambda}{\lambda^2-\delta^2} \qquad \text{for} \quad n = 1, 3, 5, \cdots$$

$$G_n(h,\delta) = \int_0^\infty e^{-2h\lambda} J_n(\lambda) \sin\lambda \frac{\lambda\, d\lambda}{\lambda^2-\delta^2} \qquad \text{for} \quad n = 1, 3, 5, \cdots$$

$$H_n(h,\delta) = \int_0^\infty e^{-2h\lambda} J_n(\lambda) \sin \frac{d\lambda}{\lambda^2-\delta^2} \qquad \text{for} \quad n = 0, 2, 4, \cdots$$

the following functions associated with the definite integrals are defined

$$R_{2n}(\Omega,h) = \sum_{s=0}^{\infty} (-)^s M_{2s+1}(\Omega) N_{2n,2s+1}(h) \tag{46}$$

$$R_{2n+1}(\Omega,h) = \sum_{s=0}^{\infty} (-)^s M_{2s}(\Omega) N_{2n+1,2s}(h)$$

$$S_{2n}(\Omega,h,\delta) = \sum_{s=0}^{\infty} (-)^s M_{2s+1}(\Omega) Q_{2n,2s+1}(h,\delta)$$

$$S_{2n+1}(\Omega,h,\delta) = \sum_{s=0}^{\infty} (-)^s M_{2s}(\Omega) Q_{2n+1,2s}(h,\delta)$$

$$T_{2n}(\Omega,h,\delta) = \sum_{s=0}^{\infty} (-)^s M_{2s}(\Omega) P_{2n,2s}(h,\delta)$$

$$T_{2n+1}(\Omega,h,\delta) = \sum_{s=0}^{\infty} (-)^s M_{2s+1}(\Omega) P_{2n+1,2s+1}(h,\delta)$$

$$V_{2n}(h,\delta) = E_{2n}(h,\delta) + i\delta H_{2n}(h,\delta)$$

$$V_{2n+1}(h,\delta) = G_{2n+1}(h,\delta) - i\delta F_{2n+1}(h,\delta)$$

$$\delta = \frac{1+\ell k}{1+\ell k^2}\Omega$$

and the other notations are given by Eq. (40).

Adding Eqs. (41) and (42) and superposing the results give

$$\left.\frac{\partial \Phi_1}{\partial y}\right|_{y=0} = e^{i\omega t} \sum_{m=0}^{\infty} A_m \left[\cos m\theta + \frac{c_{om}}{2} + \sum_{n=1}^{\infty} c_{nm} \cos n\theta \right]$$
$$= e^{i\omega t} \left[\left(A_o + \frac{1}{2} \sum_{m=0}^{\infty} c_{om} A_m\right) + \sum_{n=1}^{\infty} \left(A_n + \sum_{m=0}^{\infty} c_{nm} A_m\right) \cos n\theta \right] \tag{47}$$

by interchanging the repeated summations, provided that the resulting series converge.

Equating the coefficients of $\cos n\theta$ in Eqs. (1) and (47), an infinite set of linear algebraic equations for the unknowns, A's are obtained and are as follows

$$A_o + \frac{1}{2} \sum_{m=0}^{\infty} c_{om} A_m = \frac{B_o}{2} \tag{48}$$

$$A_n + \sum_{m=0}^{\infty} c_{nm} A_m = B_n \qquad \text{for} \quad n = 1, 2, 3, \cdots$$

The general solution is then reduced to the determination of the A's from Eq. (48). In order that such an analysis has practical value, it is required that c_{nm} tends to zero so rapidly when m increases for fixed n that approximate solutions can be found without too much labor and with enough accuracy. c_{nm} is expressed in terms of a number of definite integrals as given by Eq. (45). The evaluation of the definite integrals is then the next step of the analysis.

VI. GENERAL SOLUTION

(b) FORMULAS FOR THE DEFINITE INTEGRALS

A number of formulas are obtained for the definite integrals defined in Eq. (46). Some integrals are evaluated under the two different conditions: $h>0$ and $h=0$ due to the difficulties in evaluating them in general forms.

Evaluation of $N_{nm}(h)$. For N_{nm}, the following formula is established in Appendix 5.

$$N_{nm}(h) = \int_0^\infty e^{-2h\lambda} J_n(\lambda) J_m(\lambda)\, d\lambda$$

$$= \frac{\kappa^{1-n-m}}{\pi} \int_0^{\frac{\pi}{2}} \frac{\cos(n-m)\theta \left(\sqrt{1-\kappa^2\sin^2\theta} - h\kappa\right)^{n+m}}{(\cos\theta)^{n+m}\sqrt{1-\kappa^2\sin^2\theta}}\, d\theta \qquad (49)$$

where $\kappa = \frac{1}{\sqrt{1+h^2}}$, $h>0$ and $n+m>-1$.

In addition, a recurrence formula is obtained for N_{nm} in Appendix 6 and is

$$N_{n,n+1} = N_{n,n-1} - 2h\, N_{n,n} + \begin{cases} 1 & \text{for } n=0 \\ 0 & \text{for } n \geq 1 \end{cases} \qquad (50)$$

By the relation that $J_n(\lambda) = (-)^n J_{-n}(\lambda)$, it is evident that

$$N_{n,m} = (-)^n N_{-n,m} = (-)^m N_{n,-m} = (-)^{n+m} N_{-n,-m} \qquad (51)$$

Using Eqs. (50), (51) and $N_{nm} = N_{mn}$, the evaluation of N_{nm} is reduced to that of N_{nn}. The first two integrals of N_{nn} are easily obtained by the use of Eq. (49) and are

$$N_{00}(h) = \frac{\kappa}{\pi} K(\kappa)$$

$$N_{11}(h) = \frac{1}{\pi\kappa}\left[(2-\kappa^2) K(\kappa) - 2E(\kappa)\right]$$

where K and E are the complete elliptic integrals of the first and second kinds respectively.

It is seen that Eq. (49) also holds for $h=0$ when $|n-m|=2s+1$. The following formula is easily obtained

$$N_{nm}(0) = \int_0^\infty J_n(\lambda) J_m(\lambda)\, d\lambda$$

$$= \frac{1}{\pi}\int_0^{\frac{\pi}{2}} \frac{\cos(n-m)\theta}{\cos\theta}\, d\theta = (-)^s \frac{1}{2} \tag{52}$$

where $|n-m| = 2s+1$.

Evaluation of $O_{nm}(h)$. For O_{nm}, the following formula is established in Appendix 5,

$$O_{nm}(h) = \int_0^\infty e^{-2h\lambda} J_n(\lambda) J_m(\lambda) \frac{d\lambda}{\lambda}$$

$$= \frac{2\kappa^{-n-m}}{\pi(n+m)} \int_0^{\frac{\pi}{2}} \frac{\cos(n-m)\theta\, \left(\sqrt{1-\kappa^2\sin^2\theta} - h\kappa\right)^{n+m}}{(\cos\theta)^{n+m}}\, d\theta \tag{53}$$

where $\kappa = \frac{1}{\sqrt{1+h^2}}$, $h > 0$ and $n+m > 0$

O_{nm} can also be evaluated in terms of N_{nm} , i.e.

$$2m\, O_{nm} = N_{n,m-1} + N_{n,m+1} \tag{54}$$

which is obtained by the recurrence formula

$$J_{m-1}(\lambda) + J_{m+1}(\lambda) = 2m \frac{J_m(\lambda)}{\lambda} \tag{55}$$

Also it satisfies the relation

$$O_{nm} = O_{mn} = (-)^n O_{-n,m} = (-)^m O_{n,-m} = (-)^{n+m} O_{-n,-m} \tag{56}$$

Eq. (53) also holds for $h=0$ and $n+m>0$ and becomes

$$O_{nm}(0) = \int_0^\infty J_n(\lambda) J_m(\lambda) \frac{d\lambda}{\lambda}$$

$$= \frac{2}{\pi(n+m)} \int_0^{\frac{\pi}{2}} \cos(n-m)\theta \, d\theta = \frac{2}{\pi} \frac{\sin\frac{1}{2}(n-m)\pi}{n^2-m^2}$$

In particular,

$$O_{nm}(0) = \begin{cases} \frac{1}{2n} & \text{for } n=m \\ 0 & \text{for } n \neq m \end{cases} \quad \text{and } |n-m| = 0, 2, 4, \ldots \tag{57}$$

as required in this analysis.

Evaluation of P_{nm}, Q_{nm}, E_n, F_n, G_n and H_n when $h>0$. The integral P_{nm} can be written as

$$P_{nm}(h,\delta)=\int_0^\infty e^{-2h\lambda}J_n(\lambda)J_m(\lambda)\frac{\lambda\,d\lambda}{\lambda^2-\delta^2}$$

$$=\frac{1}{2}\left\{\int_0^\infty e^{-2h\lambda}J_n(\lambda)J_m(\lambda)\frac{d\lambda}{\lambda+\delta}+\int_0^\infty e^{-2h\lambda}J_n(\lambda)J_m(\lambda)\frac{d\lambda}{\lambda-\delta}\right\} \tag{58}$$

The first definite integral of Eq. (58) can be expressed in terms of the Whittaker function, W_{nm}

$$\int_0^\infty e^{-2h\lambda}J_n(\lambda)J_m(\lambda)\frac{d\lambda}{\lambda+\delta}$$

$$=e^{h\delta}\sum_{s=0}^\infty\frac{(-)^s\left[\Gamma(n+m+2s+1)\right]^2\frac{1}{\sqrt{2h\delta}}\left(\frac{\delta}{8h}\right)^{\frac{n+m+2s}{2}}}{s!\,\Gamma(n+s+1)\,\Gamma(m+s+1)\,\Gamma(n+m+s+1)}\;W_{-\frac{n+m+2s+1}{2},\,\frac{n+m+2s}{2}}(2h\delta) \tag{59}$$

as deduced in Appendix 7.

By the formula for $W_{-\frac{t+1}{2},\frac{t}{2}}$ derived in Appendix 8, Eq. (59) becomes

$$\int_0^\infty e^{-2h\lambda}J_n(\lambda)J_m(\lambda)\frac{d\lambda}{\lambda+\delta}=(-)^{n+m+1}e^{2h\delta}J_n(\delta)J_m(\delta)\;Ei(-2h\delta)+$$

$$+\sum_{s=0}^\infty\frac{(-)^s\;\Gamma(n+m+2s+1)\,(4h)^{-n-m-2s}}{s!\,\Gamma(n+s+1)\,\Gamma(m+s+1)\,\Gamma(n+m+s+1)}\left[\sum_{\mu=0}^{n+m+2s-1}\Gamma(n+m+2s-\mu)(-2h\delta)^\mu\right] \tag{60}$$

where $Ei(-x)$ is the exponential integral defined as

$$Ei(-x)=-\int_x^\infty\frac{e^{-t}}{t}\,dt \qquad \text{for} \quad x>0 \tag{60a}$$

By the use of Eq. (60), the second definite integral of Eq. (58) is obtained

$$\int_0^\infty e^{-2h\lambda} J_n(\lambda) J_m(\lambda) \frac{d\lambda}{\lambda-\delta} = -e^{-2h\delta} J_n(\delta) J_m(\delta) \overline{Ei}(2h\delta) +$$

$$+\sum_{s=0}^{\infty} \frac{(-)^s \Gamma(n+m+2s+1)(4h)^{-n-m-2s}}{s!\,\Gamma(n+s+1)\Gamma(m+s+1)\Gamma(n+m+s+1)} \left[\sum_{\mu=0}^{n+m+2s-1} \Gamma(n+m+2s-\mu)(2h\delta)^{\mu}\right] \tag{61}$$

where

$$\overline{Ei}(x) = \frac{1}{2}\left[Ei(-x e^{i\pi}) + Ei(-x e^{-i\pi}) \right]$$

as given in Ref. 7, pp. 1-2.

Putting Eqs. (60) and (61) into Eq. (58) yields, for $n-m=0,2,4,\cdots$

$$P_{nm}(h,\delta) = -\frac{1}{2} J_n(\delta) J_m(\delta)\left[e^{2h\delta} Ei(-2h\delta) + e^{-2h\delta}\overline{Ei}(2h\delta)\right] +$$

$$+\sum_{s=0}^{\infty} \frac{(-)^s \Gamma(n+m+2s+1)(4h)^{-n-m-2s}}{s!\,\Gamma(n+s+1)\Gamma(m+s+1)\Gamma(n+m+s+1)} \left[\sum_{\mu=0}^{\frac{n+m+2s-2}{2}} \Gamma(n+m+2s-\mu)(2h\delta)^{2\mu}\right] \tag{62}$$

By similar procedures, the following formulas are obtained

$$Q_{nm}(h,\delta) = \int_0^\infty e^{-2h\lambda} J_n(\lambda) J_m(\lambda) \frac{d\lambda}{\lambda^2-\delta^2}$$

$$= -\frac{1}{2\delta} J_n(\delta) J_m(\delta)\left[e^{2h\delta} Ei(-2h\delta) + e^{-2h\delta}\overline{Ei}(2h\delta)\right] +$$

$$+\frac{1}{\delta}\sum_{s=0}^{\infty}\frac{(-)^s\,\Gamma(n+m+2s+1)\,(4h)^{-n-m-2s}}{s!\,\Gamma(n+s+1)\,\Gamma(m+s+1)\,\Gamma(n+m+s+1)}\left[\sum_{\mu=0}^{\frac{n+m+2s-3}{2}}\Gamma(n+m+2s-2\mu-1)(2h\delta)^{2\mu+1}\right]$$

for $n-m=1, 3, 5, \cdots$ (63)

$$E_n(h,\delta)=\int_0^{\infty}e^{-2h\lambda}J_n(\lambda)\cos\lambda\,\frac{\lambda\,d\lambda}{\lambda^2-\delta^2}$$

$$=-\frac{1}{2}J_n(\delta)\cos\delta\left[e^{2h\delta}Ei(-2h\delta)+e^{-2h\delta}\overline{Ei}(2h\delta)\right]+$$

$$+\frac{1}{\sqrt{\pi}}\sum_{s=0}^{\infty}\frac{(-)^s\,\Gamma(n+2s+\frac{1}{2})\,h^{-n-2s}}{\Gamma(2s+1)\,\Gamma(2n+2s+1)}\left[\sum_{\mu=0}^{\frac{n+2s-2}{2}}\Gamma(n+2s-2\mu)(2h\delta)^{2\mu}\right]$$

for $n=0, 2, 4, \cdots$ (64)

$$F_n(h,\delta)=\int_0^{\infty}e^{-2h\lambda}J_n(\lambda)\cos\lambda\,\frac{d\lambda}{\lambda^2-\delta^2}$$

$$=-\frac{1}{2\delta}J_n(\delta)\cos\delta\left[e^{2h\delta}Ei(-2h\delta)+e^{-2h\delta}\overline{Ei}(2h\delta)\right]+$$

$$+\frac{1}{\sqrt{\pi}}\frac{1}{\delta}\sum_{s=0}^{\infty}\frac{(-)^s\,\Gamma(n+2s+\frac{1}{2})\,h^{-n-2s}}{\Gamma(2s+1)\,\Gamma(2n+2s+2)}\left[\sum_{\mu=0}^{\frac{n+2s-3}{2}}\Gamma(n+2s-2\mu-1)(2h\delta)^{2\mu+1}\right]$$

for $n=1, 3, 5, \cdots$ (65)

$$G_n(h,\delta)=\int_0^{\infty}e^{-2h\lambda}J_n(\lambda)\sin\lambda\,\frac{\lambda\,d\lambda}{\lambda^2-\delta^2}$$

$$=-\frac{1}{2}J_n(\delta)\sin\delta\left[e^{2h\delta}Ei(-2h\delta)+e^{-2h\delta}\overline{Ei}(2h\delta)\right]+$$

$$+\frac{1}{\sqrt{\pi}}\sum_{s=0}^{\infty}\frac{(-)^s\,\Gamma(n+2s+\frac{3}{2})\,h^{-n-2s-1}}{\Gamma(2s+2)\,\Gamma(2n+2s+2)}\left[\sum_{\mu=0}^{\frac{n+2s-1}{2}}\Gamma(n+2s-2\mu+1)(2h\delta)^{2\mu}\right]$$

for $n = 1, 3, 5, \ldots$ **(66)**

$$H_n(h,\delta) = \int_0^{\infty} e^{-2h\lambda} J_n(\lambda)\sin\lambda\,\frac{d\lambda}{\lambda^2-\delta^2}$$

$$= -\frac{1}{2\delta} J_n(\delta)\sin\delta\left[e^{2h\delta}\,Ei(-2h\delta) + e^{-2h\delta}\,\overline{Ei}(2h\delta)\right] +$$

$$+\frac{1}{\sqrt{\pi}}\frac{1}{\delta}\sum_{s=0}^{\infty}\frac{(-)^s\,\Gamma(n+2s+\frac{3}{2})\,h^{-n-2s-1}}{\Gamma(2s+2)\,\Gamma(2n+2s+2)}\left[\sum_{\mu=0}^{\frac{n+2s-2}{2}}\Gamma(n+2s-2\mu)\,(2h\delta)^{2\mu+1}\right]$$

for $n = 0, 2, 4, \cdots$ **(67)**

In Eq. (59), using the asymptotic expansion for the Whittaker function for $2h\delta \gg 1$ **gives**

$$\int_0^{\infty} e^{-2h\lambda} J_n(\lambda) J_m(\lambda)\frac{d\lambda}{\lambda+\delta} \sim \frac{2}{\delta}\sum_{s=0}^{\infty}\frac{(-)^s\,\Gamma(n+m+2s+1)(4h)^{-n-m-2s-1}}{s!\,\Gamma(n+s+1)\,\Gamma(m+s+1)\,\Gamma(n+m+s+1)}\left[\sum_{\nu=0}^{\infty}\frac{\Gamma(n+m+2s+\nu+1)}{(-2h\delta)^{\nu}}\right] \quad (68)$$

Also, for $2h\delta \gg 1$

$$\int_0^{\infty} e^{-2h\lambda} J_n(\lambda) J_m(\lambda)\frac{d\lambda}{\lambda-\delta} \sim \frac{2}{\delta}\sum_{s=0}^{\infty}\frac{(-)^{s+1}\,\Gamma(n+m+2s+1)(4h)^{-n-m-2s-1}}{s!\,\Gamma(n+s+1)\,\Gamma(m+s+1)\,\Gamma(n+m+s+1)}\left[\sum_{\nu=0}^{\infty}\frac{\Gamma(n+m+2s+\nu+1)}{(2h\delta)^{\nu}}\right] \quad (69)$$

Putting Eqs. (68) and (69) into Eq. (58) yields, for $2h\delta >> 1$

$$P_{nm}(h,\delta) \sim \frac{2}{\delta} \sum_{s=0}^{\infty} \frac{(-)^{s+1}\Gamma(n+m+2s+1)(4h)^{-n-m-2s-1}}{s!\,\Gamma(n+s+1)\Gamma(m+s+1)\Gamma(n+m+s+1)} \left[\sum_{\nu=0}^{\infty} \frac{\Gamma(n+m+2s+2\nu+2)}{(2h\delta)^{2\nu+1}} \right] \quad (70)$$

Similarly, the following asymptotic expressions are obtained for $2h\delta >> 1$,

$$Q_{nm}(h,\delta) \sim \frac{2}{\delta^2} \sum_{s=0}^{\infty} \frac{(-)^{s+1}\Gamma(n+m+2s+1)(4h)^{-n-m-2s-1}}{s!\,\Gamma(n+s+1)\Gamma(m+s+1)\Gamma(n+m+s+1)} \left[\sum_{\nu=0}^{\infty} \frac{\Gamma(n+m+2s+2\nu+1)}{(2h\delta)^{2\nu}} \right] \quad (71)$$

$$E_n(h,\delta) \sim \frac{1}{2\sqrt{\pi}\,\delta} \sum_{s=0}^{\infty} \frac{(-)^{s+1}\Gamma(n+2s+\frac{1}{2})\,h^{-n-2s-1}}{\Gamma(2s+1)\,\Gamma(2n+2s+1)} \left[\sum_{\nu=0}^{\infty} \frac{\Gamma(n+2s+2\nu+2)}{(2h\delta)^{2\nu+1}} \right] \quad (72)$$

$$F_n(h,\delta) \sim \frac{1}{2\sqrt{\pi}\,\delta^2} \sum_{s=0}^{\infty} \frac{(-)^{s+1}\Gamma(n+2s+\frac{1}{2})\,h^{-n-2s-1}}{\Gamma(2s+1)\,\Gamma(2n+2s+1)} \left[\sum_{\nu=0}^{\infty} \frac{\Gamma(n+2s+2\nu+1)}{(2h\delta)^{2\nu}} \right] \quad (73)$$

$$G_n(h,\delta) \sim \frac{1}{2\sqrt{\pi}\,\delta} \sum_{s=0}^{\infty} \frac{(-)^{s+1}\Gamma(n+2s+\frac{3}{2})\,h^{-n-2s-2}}{\Gamma(2s+2)\,\Gamma(2n+2s+2)} \left[\sum_{\nu=0}^{\infty} \frac{\Gamma(n+2s+2\nu+3)}{(2h\delta)^{2\nu+1}} \right] \quad (74)$$

$$H_n(h,\delta) \sim \frac{1}{2\sqrt{\pi}\,\delta^2} \sum_{s=0}^{\infty} \frac{(-)^{s+1}\Gamma(n+2s+\frac{3}{2})\,h^{-n-2s-2}}{\Gamma(2s+2)\,\Gamma(2n+2s+2)} \left[\sum_{\nu=0}^{\infty} \frac{\Gamma(n+2s+2\nu+2)}{(2h\delta)^{2\nu}} \right] \quad (75)$$

<u>Evaluation of P_{nm} , Q_{nm}, E_n, F_n, G_n and H_n when $h=0$.</u> The following formula is established for P_{nm} in Appendix 9,

$$P_{nm}(0,\delta) = \int_0^{\infty} J_n(\lambda)\, J_m(\lambda)\, \frac{\lambda\, d\lambda}{\lambda^2-\delta^2}$$

$$= -\frac{\pi}{2}\; J_n(\delta)\, Y_m(\delta) \qquad \text{for } n-m=0,2,4,\ldots \quad (76)$$

By the use of Eq. (55), it is easily obtained from Eq. (76) that

$$Q_{nm}(0,\delta)= \int_0^\infty J_n(\lambda)J_m(\lambda)\frac{d\lambda}{\lambda^2-\delta^2}$$

$$= -\frac{\pi}{2\delta}J_n(\delta)\,Y_m(\delta) \qquad \text{for} \qquad n-m=1,3,5,\ldots \tag{77}$$

It should be noticed that the subscripts n and m on the right side of Eqs. (76) and (77) are not interchangeable. Using the expansions (Ref. 8, p. 22),

$$\cos\lambda = J_0(\lambda) + 2\sum_{m=1}^{\infty}(-)^m J_{2m}(\lambda)$$

$$\sin\lambda = 2\sum_{m=0}^{\infty}(-)^m J_{2m+1}(\lambda)$$

the following formulas are obtained

$$E_{2n}(0,\delta)=\int_0^\infty J_{2n}(\lambda)\cos\lambda\,\frac{\lambda d\lambda}{\lambda^2-\delta^2}$$

$$= -\pi\left\{\frac{J_{2n}Y_0-J_0Y_{2n}}{2}+\sum_{m=1}^{n-1}(-1)^m(J_{2n}Y_{2m}-J_{2m}Y_{2n})+\frac{\cos\delta}{2}Y_{2n}\right\} \tag{78}$$

$$F_{2n+1}(0,\delta)=\int_0^\infty J_{2n+1}(\lambda)\cos\lambda\,\frac{d\lambda}{\lambda^2-\delta^2}$$

$$= -\frac{\pi}{\delta}\left\{\frac{J_{2n+1}Y_0-J_0Y_{2n+1}}{2}+\sum_{m=1}^{n}(-)^m(J_{2n+1}Y_{2m}-J_{2m}Y_{2n+1})+\frac{\cos\delta}{2}Y_{2n+1}\right\} \tag{79}$$

$$G_{2n+1}(0,\delta) = \int_0^\infty J_{2n+1}(\lambda)\,\sin\lambda\,\frac{\lambda\,d\lambda}{\lambda^2-\delta^2}$$

$$= -\pi\left\{\sum_{m=0}^{n-1}(-)^m\left(J_{2n+1}Y_{2m+1}-J_{2m+1}Y_{2n+1}\right)+\frac{\sin\delta}{2}\,Y_{2n+1}\right\} \tag{80}$$

$$H_{2n}(0,\delta) = \int_0^\infty J_{2n}(\lambda)\,\sin\lambda\,\frac{d\lambda}{\lambda^2-\delta^2}$$

$$= -\frac{\pi}{\delta}\left\{\sum_{m=0}^{n-1}(-)^m\left(J_{2n}Y_{2m+1}-J_{2m+1}Y_{2n}\right)+\frac{\sin\delta}{2}\,Y_{2n}\right\} \tag{81}$$

where J_n and Y_n are used as abbreviations for $J_n(\delta)$ and $Y_n(\delta)$. The calculation of E_n, F_n, G_n and H_n can be simplified by the use of Eq. (55) and the relation (Ref. 8, p. 77)

$$J_n\,Y_{n+1} - J_{n+1}\,Y_n = -\frac{2}{\pi\delta}$$

The first few V_n defined in Eq. (46) are obtained by the use of the above results and are

$$V_0(0,\delta) = -\frac{\pi}{2}\,e^{i\delta}\,Y_0$$

$$V_1(0,\delta) = \frac{i}{\delta} + \frac{\pi i}{2}\,e^{i\delta}\,Y_1$$

$$V_2(0,\delta) = -\frac{2i}{\delta} - \frac{2}{\delta^2} - \frac{\pi}{2}\,e^{i\delta}\,Y_2 \tag{82}$$

$$V_3(0,\delta) = -\frac{3i}{\delta} - \frac{8}{\delta^2} + \frac{8i}{\delta^3} + \frac{\pi i}{2}\,e^{i\delta}\,Y_3$$

Evaluation of R_n , S_n and T_n when $h=0$. The infinite series R_n defined in Eq. (46) becomes, when $h=0$

$$R_{2n}(\Omega,0)=\frac{(-)^{n-1}}{2}\left[\sum_{s=0}^{n} M_{2s+1}(\Omega)-\sum_{s=n+1}^{\infty} M_{2s+1}(\Omega)\right]$$

$$R_{2n+1}(\Omega,0)=\frac{(-)^{n}}{2}\left[\sum_{s=0}^{n} M_{2s}(\Omega)-\sum_{s=n+1}^{\infty} M_{2s}(\Omega)\right] \tag{83}$$

by the use of Eq. (52). Using the summation formulas derived in Appendix 10, Eq. (83) becomes

$$R_{2n}(\Omega,0)=\frac{(-)^{n-1}}{2}\left\{2\sum_{s=0}^{n} M_{2s+1}(\Omega)+\frac{\pi}{4}\left[Y_0(\Omega)+iJ_0(\Omega)\right]\right\}$$

$$R_{2n+1}(\Omega,0)=\frac{(-)^{n}}{2}\left\{M_0(\Omega)+2\sum_{s=1}^{n} M_{2s}(\Omega)+\frac{\pi}{4}\left[J_1(\Omega)-iY_1(\Omega)\right]-\frac{i}{2}\frac{e^{-i\Omega}}{\Omega}\right\} \tag{84}$$

Using Eqs. (76) and (77), the infinite series S_n and T_n defined in Eq. (46) take the following forms

$$S_{2n}(\Omega,0,\delta)=\frac{\pi}{2\delta}J_{2n}(\delta)\sum_{s=0}^{n-1}(-)^{s+1}M_{2s+1}(\Omega)Y_{2s+1}(\delta)+\frac{\pi}{2\delta}Y_{2n}(\delta)\sum_{s=n}^{\infty}(-)^{s+1}M_{2s+1}(\Omega)J_{2s+1}(\delta)$$

$$S_{2n+1}(\Omega,0,\delta)=\frac{\pi}{2\delta}J_{2n+1}(\delta)\sum_{s=0}^{n}(-)^{s+1}M_{2s}(\Omega)Y_{2s}(\delta)+\frac{\pi}{2\delta}Y_{2n+1}(\delta)\sum_{s=n+1}^{\infty}(-)^{s+1}M_{2s}(\Omega)J_{2s}(\delta) \tag{85}$$

$$T_{2n}(\Omega,0,\delta)=\frac{\pi}{2}J_{2n}(\delta)\sum_{s=0}^{n-1}(-)^{s+1}M_{2s}(\Omega)Y_{2s}(\delta)+\frac{\pi}{2}Y_{2n}(\delta)\sum_{s=n}^{\infty}(-)^{s+1}M_{2s}(\Omega)J_{2s}(\delta) \tag{86}$$

$$T_{2n+1}(\Omega,0,\delta)=\frac{\pi}{2}J_{2n+1}(\delta)\sum_{s=0}^{n-1}(-)^{s+1}M_{2s+1}(\Omega)Y_{2s+1}(\delta)+\frac{\pi}{2}Y_{2n+1}(\delta)\sum_{s=n}^{\infty}(-)^{s+1}M_{2s+1}(\Omega)J_{2s+1}(\delta)$$

VII. GENERAL EXPRESSIONS FOR LIFT AND MOMENT

The lift, L, and the moment, M, are defined as the aerodynamic force and moment acting on the oscillating airfoil. In the following analysis, L and M are taken at the mid-chord point, L is positive when it is in the positive y-direction and M is positive when it causes diving motion. Like Φ_1 in Eq. (2), L and M can be expressed as

$$L = \sum_{m=0}^{\infty} A_m L_m$$
$$M = \sum_{m=0}^{\infty} A_m M_m \tag{87}$$

where L_m and M_m are the lift and moment produced in the flow represented by the elementary potential, $\phi_{om} + \phi_{1m}$, and the constants $A's$ are those of Eq. (2) and are determined by Eq. (48).

In the flow represented by ϕ_{1m}, the static pressure varies continuously, because the Bernoulli's equation gives

$$p_{1m}(x,y,t) = -\rho_1 \left[\frac{\partial\phi_{1m}}{\partial t} + U_1 \frac{\partial\phi_{1m}}{\partial x}\right] + \text{constant}, \quad \text{for} \quad y \leq h$$

where both $\frac{\partial\phi_{1m}}{\partial t}$ and $\frac{\partial\phi_{1m}}{\partial x}$ are continuous as seen from Eq. (30). The forces acting on the upper and lower surface of the thin airfoil are equal and opposite and therefore no lift and moment is produced.

As defined in Section III, the velocity potential ϕ_{om} represents the oscillating motion of a thin airfoil in a uniform flow without interface. The lift and moment produced in such a flow have been calculated in Ref. 6, III, by the momentum consideration of the vortex system representing the motion. Using the notations of the present paper, they are

$$L_m = \rho_1 U_1 \Gamma_{om} - \rho_1 \frac{d}{dt}\int_{-1}^{1} \gamma_{om}(x,t)\, x\, dx + \rho_1 U_1 \int_1^{\infty} \frac{\gamma_{am}(\xi,t)}{\sqrt{\xi^2-1}}\, d\xi$$

$$M_m = \rho_1 U_1 \int_{-1}^{1} \gamma_{om}(x,t)\, x\, dx - \frac{1}{2}\rho_1 \frac{d}{dt}\int_{-1}^{1} \gamma_{om}(x,t)\left(x^2-\frac{1}{2}\right) dx + \qquad (88)$$

$$+ \frac{1}{2}\rho_1 U_1 \int_1^{\infty} \frac{\gamma_{am}(\xi,t)}{\sqrt{\xi^2-1}}\, d\xi$$

where the first terms are the quasi-steady values, the second terms are the contributions of the apparent mass and the last terms are the direct contributions of the wake vorticity.

Putting into Eq. (88), Eq. (15) for γ_{om}, Eq. (24) for γ_{am} and Eq. (16) for Γ_{om}, carrying out the simple integrations and summing the results according to Eq. (87), the final results are

$$L = -\pi\rho_1 U_1 e^{i\omega t}\left[\left(i\Omega + \frac{2K_1}{K_0+K_1}\right) A_0 + \frac{K_1}{K_0+K_1} A_1 - \frac{i\Omega}{2} A_2\right]$$

$$M = \frac{\pi\rho_1 U_1}{2} e^{i\omega t}\left[\frac{2K_1}{K_0+K_1} A_0 - \left(\frac{i\Omega}{4} + \frac{K_0}{K_0+K_1}\right) A_1 - A_2 + \frac{i\Omega}{4} A_3\right] \qquad (89)$$

where K_0 and K_1 are the abbreviations for $K_0(i\Omega)$ and $K_1(i\Omega)$. It is seen from Eq. (89) that only the first four constants, A_0, A_1,

A_2 and A_3 are required to be solved from Eq. (48) for the determination of the lift and the moment.

In the following analysis, the two basic modes of oscillation: bending and torsion, are considered in detail. The bending oscillation is defined as the translatory motion normal to the flight direction and the torsional oscillation is defined as the rotational motion about the mid-chord point. The boundary condition for the former may be expressed as

$$V(t) = \frac{B_o}{2} e^{i\omega t} \tag{90}$$

where B_o has the dimension of velocity.

The boundary condition for the latter may be expressed as

$$V(\theta,t) = e^{i\omega t}\left[\frac{1}{i\Omega} B_1 + B_1 \cos\theta\right] \tag{91}$$

where B_1 has the dimension of angular velocity, the second term represents the upwash due to the angular velocity, and the first term represents the upwash due to the angle of attack at rotated positions. Since the first term of Eq. (91) can be included in Eq. (90), it is convenient for presenting the results on lift and moment to use only the second term of Eq. (91) as the boundary condition for the torsional oscillation. The oscillation represented by the second term of Eq. (91) will be denoted as the "torsional" oscillation in order to differentiate it from the real torsional oscillation which is represented by the complete expression of Eq. (91). The lift and moment for the real torsional oscillation are then the sums of those for the "torsional" oscillation and those for the bending oscillation with $B_o = \frac{a}{i\Omega} B_1$. In accordance with Ref. 6, the lift and moment will be presented in the non-dimensional forms,

$\frac{L}{L_o}$ and $\frac{M}{M_o}$ where, for the bending oscillation

$$L_{oB} = -\pi \rho_1 U_1 B_o e^{i\omega t}$$
$$M_{oB} = \frac{\pi \rho_1 U_1}{a} B_o e^{i\omega t} \tag{92}$$

and for the "torsional" oscillation

$$L_{oT} = -\pi \rho_1 U_1 B_1 e^{i\omega t} \tag{93}$$
$$M_{oT} = -\frac{\pi \rho_1 U_1}{a} B_1 e^{i\omega t}$$

L_{oB}, M_{oB} and L_{oT} in Eqs. (92) and (93) are the respective quasi-steady values in a uniform flow without interface. The corresponding value for M_{oT} is zero. The value of M_{oT} given in Eq. (93) is arbitrary but with the dimension of moment.

In the preceding analysis only the case, $h \geqslant 0$ i.e. the tail lying below the interface is considered. It can now be seen that the lift and moment for the case of the tail lying above the interface are the same as those for the case of the tail lying below the interface with the same vertical distance, h, provided that the density and velocity of the undisturbed flow in which the tail lies are ρ_1 and U_1 and those of the other flow are ρ_2 and U_2 for both cases. Based on the physical fact that the force and moment depend only on the motion of the airfoil normal to the interface, Eq. (89) should give the force and moment acting on the airfoil lying above the interface if force, moment and motion are now positive in the reversed directions. However, a change in the sign of motion means a change in the signs of lift and moment, because $L, M \sim e^{i\omega t}$ in Eq. (89). Therefore Eq. (89) can be used without change for the case of the airfoil lying above the interface.

Before calculating the lift and moment in detail, the ranges for the various parameters under consideration are listed.

$h \ (=\frac{2h}{c},\ c=2)$	$k \ (=\frac{U_2}{U_1})$	$l \ (=\frac{\rho_2}{\rho_1})$	$\Omega \ (=\frac{c\omega}{2U_1})$
$\infty > h > 0$	$\infty \geqq k \geqq 0$	$\infty \geqq l \geqq 0$	$\infty \geqq \Omega \geqq 0$
$h = 0$	$1 \geqq k \geqq 0$		

where the case, $h=0$ and $\infty \geqq k > 1$ is excluded, because its lift and moment can be similarly interpreted from those obtained for the case, $h=0$ and $1 > k \geqq 0$. l is sometimes taken as unity for convenience in the following analysis, which is consistent with the assumption of incompressibility.

VIII. LIFT AND MOMENT WHEN $\frac{2h}{c} > 1$

Due to the difficulties of evaluating the definite integrals in general forms, only special cases are carried out in detail. These special cases, besides having advantages in mathematical manipulation, are also important from the physical point of view. The first case studied is $\frac{2h}{c} >> 1$ or $h >> 1$ in the notations of the present paper. (Since c, the chord, has been taken as 2). Using the formulas of Section VI, the asymptotic results for the case, $h >> 1$, are easily obtained. As seen from the equations listed below, they actually yield good approximations even when $h = 2$, which is very close to the value used in conventional airplane designs. The investigation of this case therefore has practical value. It is found convenient to discuss the case, $h >> 1$, by considering the different values of Ω separately.

(a) When Ω is so small that $h\Omega << 1$. The asymptotic expansions of $N_{nm}(h)$ and $O_{nm}(h)$ for $h >> 1$ are obtained by the use of Eqs. (49) and (53). The approximate expressions of $P_{nm}(h,\delta)$, $Q_{nm}(h,\delta)$, $E_n(h,\delta)$ $F_n(h,\delta)$, $G_n(h,\delta)$ and $H_n(h,\delta)$ for $h >> 1$ and $h\Omega << 1$ are given by Eqs. (62)-(67). Using the above results, the associated functions of Eq. (46) and then the coefficients, C_{nm} of Eq. (45) are calculated. For the bending oscillation, putting $B_1 = B_2 = \cdots = 0$ in Eq. (48) and then solving for A_0, A_1, A_2 and A_3, Eq. (89) gives L and M.

Dividing L and M by L_{oB} and M_{oB} of Eq. (92), the final results are

$$\left(\frac{L}{L_o}\right)_B = 1 + i\Omega\log\Omega + \frac{1+2lk(1-k)-l^2k^4}{(1+lk^2)^2}\, i\Omega\log h\Omega - \frac{1}{4}\,\frac{1-lk^2}{1+lk^2}\,\frac{1}{h^2} + O\left(\Omega, h\Omega^2, \frac{\Omega\log\Omega}{h^2}, \frac{1}{h^4}\right)$$

(94)

$$\left(\frac{M}{M_o}\right)_B = 1 + i\Omega\log\Omega + \frac{1+2lk(1-k)-l^2k^4}{(1+lk^2)^2}\, i\Omega\log h\Omega - \frac{1}{8}\,\frac{1-lk^2}{1+lk^2}\,\frac{1}{h^2} + O\left(\Omega, h\Omega^2, \frac{\Omega\log\Omega}{h^2}, \frac{1}{h^4}\right)$$

For the "torsional" oscillation, putting $B_0 = B_2 = B_3 = \cdots = 0$ in Eq. (48), similar procedures yield L and M. Dividing them by L_{oT} and M_{oT} of Eq. (93) gives

$$\left(\frac{L}{L_o}\right)_T = 1 + i\Omega\log\Omega + \frac{1+2lk(1-k)-l^2k^4}{(1+lk^2)^2}\, i\Omega\log h\Omega - \frac{1}{8}\,\frac{1-lk^2}{1+lk^2}\,\frac{1}{h^2} + O\left(\Omega, h\Omega^2, \frac{\Omega\log\Omega}{h^2}, \frac{1}{h^4}\right)$$

(95)

$$\left(\frac{M}{M_o}\right)_T = -i\Omega\log\Omega - \frac{1+2lk(1-k)-lk^4}{(1+lk^2)^2}\, i\Omega\log h\Omega + O\left(\Omega, h\Omega^2, \frac{\Omega\log\Omega}{h^2}, \frac{1}{h^4}\right)$$

(b) <u>When Ω is of the order of unity.</u> For this case, the asymptotic expansions of $P_{nm}(h,\delta)$, $Q_{nm}(h,\delta)$, $E_n(h,\delta)$, $F_n(h,\delta)$, $G_n(h,\delta)$ and $H_n(h,\delta)$ are given by Eqs. (70)-(75). Using the same expansions for $N_{nm}(h)$ and $O_{nm}(h)$ and following similar procedures, the final results are

$$\left(\frac{L}{L_o}\right)_B = \frac{i\Omega}{2} + \frac{K_1}{K_0+K_1} + \Big\{\left(-\frac{i\Omega}{4} - \frac{1}{2}\,\frac{K_1}{K_0+K_1}\right)\Big[\frac{1}{2\alpha^2}\left(\frac{\mu}{2} - \frac{q}{2\delta} - \frac{iq}{\delta^2}\right) +$$

$$+ \frac{1}{\alpha^2(K_0+K_1)}\left(-\frac{iqM_0}{\delta^2} - \frac{\mu M_1}{2} + \frac{qM_1}{2\delta}\right) + \frac{e^{-i\Omega}}{K_0+K_1}\left(\frac{1}{2\Omega^2} + \frac{i}{2\Omega} - \frac{1}{\alpha\delta^2} - \frac{i}{\alpha\delta}\right)\Big] + \frac{1}{2}\,\frac{K_1}{K_0+K_1}\Big[\frac{1}{4\alpha^2}\left(-\mu + \frac{q}{\delta}\right) +$$

$$+ \frac{M_0}{2\alpha^2(K_0+K_1)}\left(-\mu + \frac{q}{\delta}\right) + \frac{ie^{-i\Omega}}{2(K_0+K_1)}\left(\frac{1}{2\Omega} - \frac{1}{\alpha\delta}\right)\Big]\Big\}\,\frac{1}{h^2} + O\left(\frac{1}{h^4}, \frac{1}{h^4\Omega^2}\right)$$

$$\left(\frac{M}{M_0}\right)_\theta = \frac{K_1}{K_0+K_1} + \Big\{ -\frac{1}{2}\frac{K_1}{K_0+K_1}\Big[\frac{1}{2\alpha^2}\Big(\frac{\mu}{2} - \frac{q}{2\delta} - \frac{iq}{\delta^2}\Big) + \frac{1}{\alpha^2(K_0+K_1)}\Big(-\frac{iqM_0}{\delta^2} - \frac{\mu M_1}{2} + \frac{qM_1}{2\delta}\Big) +$$

$$+ \frac{e^{-i\Omega}}{K_0+K_1}\Big(\frac{1}{2\Omega^2} + \frac{i}{2\Omega} - \frac{1}{\alpha\delta^2} - \frac{i}{\alpha\delta}\Big)\Big] - \Big(\frac{i\Omega}{8} + \frac{1}{2}\frac{K_0}{K_0+K_1}\Big)\Big[\frac{1}{4\alpha^2}\Big(-\mu + \frac{q}{\delta}\Big) +$$

$$+ \frac{M_0}{2\alpha^2(K_0+K_1)}\Big(-\mu + \frac{q}{\delta}\Big) + \frac{ie^{-i\Omega}}{2(K_0+K_1)}\Big(\frac{1}{2\Omega} - \frac{1}{\alpha\delta}\Big)\Big]\Big\}\frac{1}{h^2} + O\Big(\frac{1}{h^4}, \frac{1}{h^4\Omega^2}\Big)$$

(96)

$$\left(\frac{L}{L_0}\right)_T = \frac{K_1}{K_0+K_1} + \Big\{\frac{K_1}{K_0+K_1}\Big[-\frac{\nu}{8\beta^2} + \frac{M_0}{4\alpha^2(K_0+K_1)}\Big(-\mu + \frac{q}{\delta}\Big) + \frac{ie^{-i\Omega}}{4(K_0+K_1)}\Big(\frac{1}{2\Omega} - \frac{1}{\alpha\delta}\Big)\Big] +$$

$$-\Big(\frac{i\Omega}{2} + \frac{K_1}{K_0+K_1}\Big)\Big[-\frac{iq}{4\alpha^2\delta^2} + \frac{1}{2\alpha^2(K_0+K_1)}\Big(-\frac{iqM_0}{\delta^2} - \frac{\mu M_1}{2} + \frac{qM_1}{2\delta}\Big) +$$

$$+ \frac{e^{-i\Omega}}{2(K_0+K_1)}\Big(\frac{1}{2\Omega^2} + \frac{i}{2\Omega} - \frac{1}{\alpha\delta^2} - \frac{i}{\alpha\delta}\Big)\Big]\Big\}\frac{1}{h^2} + O\Big(\frac{1}{h^4}, \frac{1}{h^4\Omega^2}\Big)$$

$$\left(\frac{M}{M_0}\right)_T = \frac{i\Omega}{4} + \frac{K_0}{K_0+K_1} + \Big\{\Big(\frac{i\Omega}{4} + \frac{K_0}{K_0+K_1}\Big)\Big[-\frac{\nu}{8\beta^2} + \frac{M_0}{4\alpha^2(K_0+K_1)}\Big(-\mu + \frac{q}{\delta}\Big) +$$

$$+ \frac{ie^{-i\Omega}}{4(K_0+K_1)}\Big(\frac{1}{2\Omega} - \frac{1}{\alpha\delta}\Big)\Big] + \frac{K_1}{K_0+K_1}\Big[\frac{iq}{4\alpha^2\delta^2} + \frac{1}{2\alpha^2(K_0+K_1)}\Big(-\frac{iqM_0}{\delta} - \frac{\mu M_1}{2} + \frac{qM_1}{2\delta}\Big) +$$

$$+ \frac{e^{-i\Omega}}{2(K_0+K_1)}\Big(\frac{1}{2\Omega^2} + \frac{i}{2\Omega} - \frac{1}{\alpha\delta^2} - \frac{i}{\alpha\delta}\Big)\Big]\Big\}\frac{1}{h^2} + O\Big(\frac{1}{h^4}, \frac{1}{h^4\Omega^2}\Big)$$

Eq. (96) are also valid when $\Omega \gg 1$. However, for this case simpler expressions can be obtained as below.

(c) When $\Omega >> 1$. Introducing the asymptotic expansions of $K_0(i\Omega)$, $K_1(i\Omega)$, $M_0(i\Omega)$ and $M_1(i\Omega)$ for $\Omega >> 1$ into Eq. (96) and using Eq. (40) for various notations, the final results are

$$\left(\frac{L}{L_0}\right)_B = \frac{i\Omega}{2} + \frac{1}{2} - \frac{i}{8\Omega} - \frac{1}{16}\frac{1-lk}{1+lk}\frac{i\Omega+1}{h^2} + O\left(\frac{1}{\Omega^2}, \frac{1}{h^2\Omega}, \frac{1}{h^4}\right)$$

$$\left(\frac{M}{M_0}\right)_B = \frac{1}{2} - \frac{i}{8\Omega} - \frac{1}{16}\frac{1-lk}{1+lk}\left(1+\frac{7}{32\sqrt{\pi\Omega}} - \frac{7i}{32\sqrt{\pi\Omega}}\right)\frac{1}{h^2} + O\left(\frac{1}{\Omega^2}, \frac{1}{h^2\Omega}, \frac{1}{h^4}\right)$$

$$\left(\frac{L}{L_0}\right)_T = \frac{1}{2} - \frac{i}{8\Omega} - \frac{1}{16}\frac{1-lk}{1+lk}\frac{1}{h^2} + O\left(\frac{1}{\Omega^2}, \frac{1}{h^2\Omega}, \frac{1}{h^4}\right) \tag{97}$$

$$\left(\frac{M}{M_0}\right)_T = \frac{i\Omega}{4} + \frac{1}{2} + \frac{i}{8\Omega} + \frac{7(1-i)}{512\sqrt{\pi\Omega}}\frac{1-lk}{1+lk}\frac{1}{h^2} + O\left(\frac{1}{\Omega^2}, \frac{1}{h^2\Omega}, \frac{1}{h^4}\right)$$

It is seen that the thin airfoil oscillating in a uniform flow without interface is a special case corresponding to either $h = \infty$ or $l = k = 1$. The expression of the lift and moment have been obtained in Ref. 6 p. 385 as follows

$$\left(\frac{L}{L_0}\right)_B = \frac{i\Omega}{2} + \frac{K_1}{K_1+K_1}$$

$$\left(\frac{M}{M_0}\right)_B = \frac{K_1}{K_0+K_1} \tag{98}$$

$$\left(\frac{L}{L_0}\right)_T = \frac{K_1}{K_0+K_1}$$

$$\left(\frac{M}{M_0}\right)_T = \frac{i\Omega}{4} + \frac{K_0}{K_0+K_1}$$

which have been evaluated and plotted in Ref. 6, p. 386. Setting $h \to \infty$, Eq. (96) is immediately reduced to Eq. (98) and Eq. (97) is reduced to the asymptotic expansion of Eq. (98). Setting $h \to \infty$

and $h\Omega \to 0$, Eqs. (94) and (95) are reduced to expansion of Eq. (98) for $\Omega \to 0$. Noting this fact, the lift and moment for $h \gg 1$ can be evaluated by the perturbation method using the relations of Eq. (98) as bases and the terms containing h in Eqs. (94)-(97) as perturbations, and superposing the latter to the former. Such a method is good if the perturbations are small relative to the basic values. For $h = 2$, $0 < \ell < \infty$ and $k = 0$ and ∞, the perturbations are found less than 10% of the corresponding values to which they are superposed when $0 \leq \Omega < .04$ by the use of Eqs. (94) and (95) and when $\Omega \geq .5$ by the use of Eq. (97). Believing that the approximations are good for practical usage, the results for $h = 2$, $0 < \ell < \infty$ and $k = 0$ and ∞ are plotted in Figs. 3 and 4 together with those for $h = \infty$ cited above.

The significance of the results is that the wing wake has very little effect upon the tail oscillation when the tail is located away from the wake with a vertical distance equal to or greater than its chord. In other words, the tail flutter under this condition possesses the nature of the wing flutter. Since the flutter speeds for wing are normally high relative to the flight speeds, the tail buffeting which usually appears at low speeds can be avoided by putting the tail sufficiently away from the path of the wing wake.

It may be mentioned that putting $\Omega = 0$ in Eqs. (94) and

(95) yields the ratios of the quasi-steady values of the lift and moment in a flow with interface at $y=h$ to those in a flow without interface, except $\left(\frac{M}{M_0}\right)_T$ of Eq. (95) which states that the quasi-steady moment remains zero in a flow with interface. Moreover, the above ratios are less than one when $\ell k^2 < 1$ (or $\frac{\rho_2 \bar{u}_2^2}{\rho_1 \bar{u}_1^2} < 1$) and greater than one when $\ell k^2 > 1$.

IX. LIFT AND MOMENT WHEN $\frac{2h}{c} = 0$ AND (a) $\Omega << 1$ (b) $\frac{U_2}{U_1} = 0$

In this section, investigations are made for the case of the oscillating tail lying in the wake of the wing, or $h = 0$ in the notations of the present paper. The mathematical calculations are facilitated by the closed forms obtained for the definite integrals at $h = 0$. However, the difficulties in summing the infinite series $S_n(\Omega, 0, \delta)$ and $T_n(\Omega, 0, \delta)$ given by Eqs. (85) and (86) confine the investigation to the two special cases: (a) $\Omega << 1$ and (b) $\frac{U_2}{U_1} = 0$ (or $k = 0$). In the former the series are summed approximately. In the latter the summations are avoided and exact expressions are obtained for lift and moment. This investigation is of interest, because it would reveal the essential features of the interaction between the interface and the tail oscillation if the interaction exists at all, and therefore should lead to an improvement in understanding the real mechanism of the tail buffeting. In particular, case (a) concentrates attention on the flutter at high speeds for $\Omega << 1$ means $U_1 >> \omega$. Case (b) represents the condition under which the greatest possible interaction may be expected. In the latter case, the lift and moment are obtained for the whole range of Ω and thus a typical investigation of the tail flutter at all speeds is made possible.

(a) $\Omega << 1$. $N_{nm}(0)$ and $O_{nm}(0)$ are given by Eqs. (52) and (57). The expansions of $P_{nm}(0,\delta)$, $Q_{nm}(0,\delta)$, $E_n(0,\delta)$, $F_n(0,\delta)$, $G_n(0,\delta)$, $H_n(0,\delta)$ and $R_n(\Omega,0)$ for $\Omega << 1$, are obtained by the use of Eqs. (76)-(81) and (84). Following the procedures stated in the last section, the final results for lift and moment are

$$\left(\frac{L}{L_0}\right)_B = 2\,\frac{1+\ell k^2}{3+\ell k^2} - \frac{1}{2}\,\frac{1-\ell k^2}{3+\ell k^2}\,\frac{1+\ell k}{1+\ell k(1-k)} + i\Omega \log\Omega \left\{ 2\,\frac{1+\ell k^2}{3+\ell k^2} - \frac{1}{2}\,\frac{1-\ell k^2}{3+\ell k^2}\,\frac{1+\ell k}{1+\ell k(1-k)} + \right.$$

$$+ \frac{1}{8}\,\frac{1+\ell k}{(3+\ell k^2)^2}\,\frac{1}{1+\ell k(1-k)} \left[\frac{(1+\ell k)(1-\ell k^2)}{1+\ell k(1-k)} - 4(1+\ell k^2) \right] \left[\frac{(3+\ell k^2)(1-\ell k^2)}{(1+\ell k^2)} + \left(\frac{1-\ell k^2}{1+\ell k^2} - \frac{4}{1+\ell k} \right) \right.$$

$$\left. \left. \left(\frac{4\ell k(1-k)}{1+\ell k^2} + 3(1-\ell k^2) \right) \right] \right\} + O(\Omega)$$

$$\left(\frac{M}{M_0}\right)_B = \left(\frac{L}{L_0}\right)_B + O(\Omega)$$

$$\tag{99}$$

$$\left(\frac{L}{L_0}\right)_T = \frac{1}{2}\,\frac{1+\ell k}{1+\ell k(1-k)} + \frac{i\Omega}{2}\log\Omega\,\frac{1+\ell k}{1+\ell k(1-k)} \left\{ 1 - \frac{1}{4}\,\frac{1-\ell k^2}{1+\ell k^2}(1+\ell k) + \right.$$

$$\left. - \frac{1}{4}\,\frac{1+\ell k}{3+\ell k^2}\,\frac{1}{1+\ell k(1-k)} \left[\frac{1-\ell k^2}{1+\ell k^2} - \frac{4}{1+\ell k} \right] \left[\frac{4\ell k(1-k)}{1+\ell k^2} + 3(1-\ell k^2) \right] \right\} + O(\Omega)$$

$$\left(\frac{M}{M_0}\right)_T = -\frac{i\Omega}{2}\log\Omega\,\frac{1+\ell k}{1+\ell k(1-k)} \left\{ 1 - \frac{1}{4}\,\frac{1-\ell k^2}{1+\ell k^2}\,\frac{1+\ell k}{1+\ell k(1-k)} + \right.$$

$$\left. - \frac{1}{4}\,\frac{1+\ell k}{3+\ell k^2}\,\frac{1}{1+\ell k(1-k)} \left[\frac{1-\ell k^2}{1+\ell k^2} - 4\,\frac{1+\ell k(1-k)}{1+\ell k} \right] \left[\frac{4\ell k(1-k)}{1+\ell k^2} + 3(1-\ell k^2) \right] \right\} + O(\Omega)$$

It is seen that Eq. (99) is very close to the corresponding expansion of Eq. (98) and is reduced to it by putting $\ell = k = 1$. It may then be said that the functional dependence of the lift and moment acting upon the oscillating tail upon the reduced frequency is not much influenced by the wake when the flying speed is so high that $\Omega < 1$. In this high speed range, the tail flutter still has the nature of the wing flutter even though the tail lies in the wake. Setting $\Omega = 0$, Eq. (99) yields the ratios of the quasi-steady lift and moment at $h=0$ to those at $h=\infty$.

(b) $k=0$ *. Putting $k=0$ in Eq. (45), the terms containing $S_n(\Omega,0,\delta)$ and $T_n(\Omega,0,\delta)$ disappear and $c_{nm}=0$ for $m \geq 2$ and $n \neq m$. Due to this fact, the A's can be solved from Eq. (48) in the exact forms. Using Eqs. (76)-(81) and (84) for the various integrals and functions, the final results for lift and moment are

$$\left(\frac{L}{L_0}\right)_B = \frac{1}{c}\left\{\left(\frac{1}{2} - \frac{9i}{2\Omega} - \frac{12}{\Omega^2} + \frac{12i}{\Omega^3}\right)e^{-i\Omega} + \frac{\pi\Omega}{4}\left(\frac{J_0}{2} - i\,Y_0\right) + \right.$$

$$\left. + \frac{\pi J_1}{2}\left(1 - \frac{i\Omega}{4}\right) - \frac{\pi Y_1}{4}(\Omega + i) - \frac{3\pi}{\Omega}(J_2 - i\,Y_2)\right\}$$

$$\left(\frac{M}{M_0}\right)_B = \frac{1}{c}\left\{\left(-\frac{i}{\Omega} - \frac{9}{\Omega^2} + \frac{24i}{\Omega^3} + \frac{24}{\Omega^4}\right)e^{-i\Omega} - \frac{\pi}{2}\left(\frac{J_1}{2} - i\,Y_1\right) + \frac{3\pi}{2\Omega}\left(Y_3 + iJ_3\right)\right\}$$

-----

*On account of the linearization used in deriving Eq. (27), it is necessary to interpret this as $U_2 << U_1$ while the condition, u_2, v_2 (the perturbation velocity components in region 2) << U_2 is still satisfied.

$$\left(\frac{L}{L_0}\right)_T = \frac{1}{c}\left\{\left(-\frac{4i}{\Omega}-\frac{12}{\Omega^2}+\frac{12}{\Omega^3}\right)e^{-i\Omega}+\frac{\pi}{4}J_1-\frac{3\pi}{\Omega}\left(J_2-iY_2\right)\right\}$$

$$\left(\frac{M}{M_0}\right)_T = \frac{1}{c}\left\{\left(\frac{1}{4}-\frac{i}{4\Omega}+\frac{8}{\Omega^2}-\frac{24i}{\Omega^3}-\frac{24}{\Omega^4}\right)e^{-i\Omega}-\frac{\pi Y_0}{2}\left(1+\frac{i\Omega}{4}\right)+\frac{\pi J_0}{2}\left(\frac{\Omega}{8}-i\right)+\right. \tag{100}$$

$$\left.+\frac{\pi J_1}{8}\left(1-\frac{i\Omega}{2}\right)-\frac{\pi Y_1}{2}\left(\frac{\Omega}{8}+\frac{1}{\Omega}+\frac{i}{4}\right)-\frac{\pi}{4}\left(Y_2+iJ_2\right)+\frac{\pi\Omega}{16}Y_3-\frac{3\pi}{2\Omega}\left(Y_3+iJ_3\right)\right\}$$

where

$$c=\left(-\frac{2i}{\Omega}-\frac{2}{\Omega^2}\right)e^{-i\Omega}-\pi\left(Y_0+\frac{iJ_0}{2}\right)-\pi J_1\left(\frac{1}{2}+\frac{i}{\Omega}\right)-\pi Y_1\left(\frac{1}{\Omega}-i\right)$$

and J_n and Y_n are used as abbreviations for $J_n(\Omega)$ and $Y_n(\Omega)$. By the use of Eq. (100), the lift and moment are calculated for $0 \le \Omega \le 24$. The results are tabulated in Table 1 and are plotted out in Figs. 3 and 4. It is seen that when $\Omega < 1$ the variations of lift and moment with Ω are very similar to those of Eq. (98) as shown in case (a) above. When $\Omega > 1$, the variations become entirely different. It is therefore interesting to investigate the asymptotic expansions of Eq. (100) which are obtained as follows:

$$\left(\frac{L}{L_0}\right)_B = \frac{i\Omega}{4}-\frac{1}{6}\left(1-e^{\theta i}\cos\theta\right)+\frac{1-i}{9\sqrt{\pi\Omega}}\left(13-e^{\theta i}\cos\theta\right)+$$

$$+\frac{i}{9\Omega}\left[\frac{299}{16}+\frac{52}{3\pi}+\left(\frac{1}{8}-\frac{4}{3\pi}\right)e^{\theta i}\cos\theta-\frac{1}{4}e^{2\theta i}\cos^2\theta\right]+O\left(\frac{1}{\Omega^{3/2}}\right)$$

$$\left(\frac{M}{M_o}\right)_B = \frac{1}{6}(1+e^{\theta i}\cos\theta) + \frac{1-i}{9\sqrt{\pi\Omega}}(2-e^{\theta i}\cos\theta) +$$

$$+\frac{i}{9\Omega}\left[\frac{115}{16}+\frac{8}{3\pi}-\left(\frac{3}{8}+\frac{4}{3\pi}\right)e^{\theta i}\cos\theta-\frac{1}{4}e^{2\theta i}\cos^2\theta\right]+O\left(\frac{1}{\Omega^{3/2}}\right)$$

$$\left(\frac{L}{L_o}\right)_T = -\frac{1}{6}(1-e^{\theta i}\cos\theta)+\frac{1-i}{9\sqrt{\pi\Omega}}(13-e^{\theta i}\cos\theta) + \tag{101}$$

$$+\frac{i}{9\Omega}\left[\frac{299}{16}+\frac{52}{3\pi}+\left(\frac{1}{8}-\frac{4}{3\pi}\right)e^{\theta i}\cos\theta-\frac{1}{4}e^{2\theta i}\cos^2\theta\right]+O\left(\frac{1}{\Omega^{3/2}}\right)$$

$$\left(\frac{M}{M_o}\right)_T = \frac{i\Omega}{8}+\frac{1}{6}(2-e^{\theta i}\cos\theta)-\frac{1-i}{9\sqrt{\pi\Omega}}(2-e^{\theta i}\cos\theta) +$$

$$-\frac{i}{9\Omega}\left[\frac{115}{16}+\frac{8}{3\pi}-\left(\frac{3}{8}+\frac{4}{3\pi}\right)e^{\theta i}\cos\theta-\frac{1}{4}e^{2\theta i}\cos^2\theta\right]+O\left(\frac{1}{\Omega^{3/2}}\right)$$

where $\theta = \Omega - \frac{\pi}{4}$

By comparing the values calculated by Eq. (101) with those calculated by Eq. (100), it is found that, neglecting the terms of the order $\frac{1}{\Omega}$ and the higher orders the expansions for $\left(\frac{M}{M_o}\right)_B$ and $\left(\frac{M}{M_o}\right)_T$ of Eq. (101) yield very good approximations (error less than 5°/o) when $\Omega > 3$. By comparing the coefficients of the various expansions of Eq. (101), it is expected that the expansions for $\left(\frac{L}{L_o}\right)_B$ and $\left(\frac{L}{L_o}\right)_T$ are accurate when $\Omega > 100$. As $\Omega \to \infty$, the limiting values of $\left(\frac{M}{M_o}\right)_B$, $\left(\frac{L}{L_o}\right)_T$ and the real parts of $\left(\frac{L}{L_o}\right)_B$ and $\left(\frac{M}{M_o}\right)_T$ are periodic in Ω with the period, π.

From Figs. 3 and 4, it is seen that the variations of the lift and moment against Ω are peculiar when $\Omega > 1$. Hence, when the flying speed is so low that $\Omega > 1$, the tail lying in the wake of the wing may be subject to self-excited vibration, i.e. flutter. Such a possibility will later be verified in the numerical example. Before doing this, the cases excluded from this and the last section are investigated.

X. DISCUSSIONS ON OTHER CASES

The cases remaining to be studied are (a) $1 \geq h > 0$ and $\infty \geq k \geq 0$ and (b) $h = 0$, $1 > k > 0$ and $\Omega > 1$. The expressions for lift and moment have not been obtained due to mathematical difficulties. However by considering the continuity character of the various functions with respect to h and k, and using the results for the boundary cases obtained in the last two sections, it is possible to make some qualitative remarks about the lift and moment variations with Ω.

(a) $1 \geq h > 0$ and $\infty \geq k \geq 0$. The difficulty lies in finding the ascending power series in h for the definite integrals of the following type (Eq. 46)

$$P_{nm}(h,\delta) = \int_0^\infty e^{-2h\lambda} J_n(\lambda) J_m(\lambda) \frac{\lambda\, d\lambda}{\lambda^2 - \delta^2}$$

$P_{nm}(h,\delta)$ is evidently continuous in h at $h > 0$, because its first derivative exists there. It is also continuous in h for $h \geq 0$. This is seen by considering the difference

$$P_{nm}(0,\delta) - P_{nm}(\varepsilon,\delta) = \int_0^\infty (1 - e^{-2\varepsilon\lambda}) J_n(\lambda) J_m(\lambda) \frac{\lambda\, d\lambda}{\lambda^2 - \delta^2} \tag{102}$$

where ε is a small positive quantity.

Eq. (102) can be written as

$$P_{nm}(0,\delta) - P_{nm}(\varepsilon,\delta) = \int_0^T (1-e^{-2\varepsilon\lambda}) J_n(\lambda) J_m(\lambda) \frac{\lambda\, d\lambda}{\lambda^2-\delta^2} +$$

$$+ \int_T^\infty (1-e^{-2\varepsilon\lambda}) J_n(\lambda) J_m(\lambda) \frac{\lambda\, d\lambda}{\lambda^2-\delta^2} \tag{103}$$

where T is taken equal to $\varepsilon^{-\frac{1}{2}}$ and is greater than δ. As $\varepsilon \to 0$, the first term of Eq. (103) approaches zero, for $(1-e^{-2\varepsilon\lambda}) \to 0$; and the second term approaches zero for

$$\left| \int_T^\infty (1-e^{-2\varepsilon\lambda}) J_n(\lambda) J_m(\lambda) \frac{\lambda\, d\lambda}{\lambda^2-\delta^2} \right| \leq \int_T^\infty \left| J_n(\lambda) J_m(\lambda) \frac{\lambda}{\lambda^2-\delta^2} \right| d\lambda$$

which approaches zero when $T \to \infty$ i.e. $\varepsilon \to 0$

Hence $P_{nm}(h,\delta)$ is continuous in h at $h=0$ and therefore

$$\lim_{h\to 0} P_{nm}(h,\delta) = P_{nm}(0,\delta) = -\frac{\pi}{2} J_n(\delta)\, Y_m(\delta)$$

as given by Eq. (76). Other definite integrals of Eq. (46) have the same property. Therefore the lift and moment vary continuously from those at $h=2$ to those at $h=0$, when h decreases from 2 to 0. By the use of the findings of the last two sections, the following may be said about the variations of the lift and moment with Ω :

(i) When $\Omega < 1$, they are similar to those for $h = \infty$.

(ii) When $\Omega > 1$ and h is close to one, they are again similar to those for $h = \infty$.

(iii) When $\Omega > 1$ and h is close to zero, they are different from those for $h = \infty$ and become similar to those for $h = 0$.

(b) $h=0,\ 1 > k > 0$ and $\Omega > 1$. The calculation requires the summation of the series of the following type (Eq. 46)

$$\sum_{s=0}^{\infty} (-)^s M_{2s+1}(\Omega)\, Q_{2s+1,2n}(0,\delta)$$

or, by the use of Eq. (77)

$$-\frac{\pi}{2\delta}\, Y_{2n}(\delta) \sum_{s=0}^{\infty} (-)^s M_{2s+1}(\Omega)\, J_{2s+1}(\delta)$$

where $M_s(\Omega)$ is defined in Eq. (37) and is expressible in terms of $J_n(\Omega)$ and $Y_n(\Omega)$ as shown in Appendix 3, and

$$\delta = \frac{1+\lambda k}{1+\lambda k^2}\,\Omega$$

The exact summation has not been obtained. The asymptotic summation for $\Omega >> 1$ is complicated by the fact that there exist three different asymptotic expansions for $J_n(\Omega)$ and $Y_n(\Omega)$ when both n and Ω are large according as $\frac{\Omega}{n}$ is less than, nearly equal to or greater than one. Besides, the usefulness of the asymptotic expansions is in doubt, because at $k=0$ some of them yield good approximation only when $\Omega > 100$ as given by

Eq. (101). The numerical summation for specific values of l, k and Ω can however be easily done. For $l=1$ and $k=\frac{1}{2}$, calculations are made for $\Omega =$ .1, .4, 1.5 and 3 with an error less than 10%. The results are tabulated in Table 2 and shown in Figs. 3 and 4 by broken lines. Though the points are too few to determine the curves precisely, they definitely show that the variations of the lift and moment with Ω for $k=\frac{1}{2}$ are more similar to those for $k=0$ than to those for $k=1$ when $\Omega>1$. Since the lift and moment are continuous in k as seen from Eq. (45), it may be said that the variations of the lift and moment with Ω for $\frac{1}{2}\geq k>0$ have the character pertaining to those for $k=0$. It is interesting to continue this investigation for $k=\frac{3}{4}$ or for $k=1-\varepsilon$ where ε is a positive small quantity to determine the value of k corresponding to the transition in the variations of the lift and moment. However, the more important question is that: Do the peculiar variations of the lift and moment at $k=0$ really cause the tail flutter at low speeds of flight?

XI. TYPICAL APPLICATION TO FLUTTER AND DISCUSSION OF THE RESULTS

To illustrate the essential features of the tail flutter under the influence of the interface, a flutter analysis is here carried out using the lift and moment obtained for the typical case $h=k=0$. The flutter analysis is a two-dimensional one. A tail of unit span with locked elevator is considered. Such a tail has two degrees of freedom, namely bending and torsion. The procedures of the analysis used below follow mainly those given in Ref. 9.

Calculation of the Aerodynamic Coefficients. A representative section of the tail with all related notations is shown in Figure 5. E denotes the elastic center and G the center of gravity. The translatory motion of E and the rotationary motion about E are expressed as

$$Y = \bar{Y}\, e^{i\omega t} \tag{104}$$

and $$\theta = \bar{\theta}\, e^{i\omega t}$$

respectively. The aerodynamic force F and moment T acting at E are expressible in terms of the lift and moment defined in Section VII and $\bar{Y}$ and $\bar{\theta}$ of Eq. (104). According to Fig. 5, it is obtained that

$$F = -m'\omega^2(\bar{A}\,\bar{Y} + c\,\bar{B}\,\bar{\theta})\,e^{i\omega t}$$
$$T = m'c\,\omega^2(\bar{C}\,\bar{Y} + c\,\bar{D}\,\bar{\theta})\,e^{i\omega t} \qquad (105)$$

where $m' = \frac{\pi \rho_1 c^2}{4}$

and $\bar{A}$, $\bar{B}$, $\bar{C}$ and $\bar{D}$ are known as the aerodynamic coefficients and are given by

$$\bar{A} = \frac{2i}{\Omega}\left(\frac{L}{L_o}\right)_B$$

$$\bar{B} = \left[\frac{2i}{\Omega}\left(\frac{1}{2} - \frac{a}{c}\right) + \frac{1}{\Omega^2}\right]\left(\frac{L}{L_o}\right)_B + \frac{i}{2\Omega}\left(\frac{L}{L_o}\right)_T$$

$$(106)$$

$$\bar{C} = \frac{i}{2\Omega}\left(\frac{M}{M_o}\right)_B - \frac{2i}{\Omega}\left(\frac{1}{2} - \frac{a}{c}\right)\left(\frac{L}{L_o}\right)_B$$

$$\bar{D} = \left[\frac{i}{2\Omega}\left(\frac{1}{2} - \frac{a}{c}\right) + \frac{1}{4\Omega^2}\right]\left(\frac{M}{M_o}\right)_B - \frac{i}{8\Omega}\left(\frac{M}{M_o}\right)_T - \left[\frac{2i}{\Omega}\left(\frac{1}{2} - \frac{a}{c}\right)^2 + \right.$$
$$\left. + \frac{1}{\Omega^2}\left(\frac{1}{2} - \frac{a}{c}\right)\right]\left(\frac{L}{L_o}\right)_B + \frac{i}{2\Omega}\left(\frac{a}{c} - \frac{1}{2}\right)\left(\frac{L}{L_o}\right)_T$$

Using Table 1 and taking $\frac{a}{c}$ = .35 in Eq. (106), the aerodynamic coefficients are evaluated for $0 \leq \Omega < 24$ and are tabulated in Table 3.

<u>Calculation of the Flutter Coefficients.</u> Denoting the mass of the tail by m, the mass moment of inertia by J, the elastic constant for translation at E by c_B and the elastic constant for rotation about E by c_T and assuming vanishing damping, the

differential equations of motion are

$$m(\ddot{y} + s\,\ddot{\theta}) + c_B\, y = F$$
$$J\,\ddot{\theta} + m s\,\ddot{y} + c_T\,\theta = T \tag{107}$$

Putting Eqs. (104) and (105) into Eq. (107) yields

$$\bar{A}'\,\bar{Y} + \bar{B}'\,\bar{\theta} = 0$$
$$\bar{C}'\,\bar{Y} + \bar{D}'\,\bar{\theta} = 0 \tag{108}$$

where $\bar{A}'$, $\bar{B}'$, $\bar{C}'$ and $\bar{D}'$ are known as the flutter coefficients and are given by

$$\bar{A}' = m\,\frac{\omega_B^2}{\omega^2} - m + m'\bar{A}$$
$$\bar{B}' = -ms + m'c\,\bar{B}$$
$$\bar{C}' = ms + m'c\,\bar{C} \tag{109}$$
$$\bar{D}' = -J\,\frac{\omega_T^2}{\omega^2} + J + m'c^2\,\bar{D}$$

where ω_B and ω_T are the natural frequencies of the bending vibration and the torsional vibration of the tail in vacuum and are given by $\sqrt{\frac{c_B}{m}}$ and $\sqrt{\frac{c_T}{J}}$ respectively.

Calculation of the Flutter Speeds. From Eq. (108), the criteria for flutter in various modes are immediately obtained.

(a) In pure bending, $\bar{A}' = \bar{C}' = 0$

(b) In pure torsion, $\bar{B}' = \bar{D}' = 0$ (110)

(c) in coupled bending-torsion, $\bar{A}'\bar{D}' - \bar{B}'\bar{C}' = 0$

For a specific structure, $\bar{A}', \bar{B}', \bar{C}'$ and $\bar{D}'$ are functions of Ω and ω or U_1 and ω (for $\Omega = \frac{c\omega}{2U_1}$) as given by Eqs. (109) and (106). The task is to determine the values of U_1 and ω which satisfy the criterion of Eq. (110). There may exist a number of solutions for one criterion. Those with U_1 lying in the speed range of flight or nearest to the range are of interest and these values of U_1 are called as the flutter speeds. In the general wing flutter analysis, only (c) of Eq. (110) has solutions and the flutter speeds are higher than the flying speed.

<u>Numerical Example.</u> The numerical example under consideration is described as follows.

The tail: Span = 1 in., c = 100 in., a = 35 in., s = 5 in.

$m = .009$ lb-sec^2/in., $J = 5.62$ lb-in.-sec.^{2}

$\omega_B = 40$ rad/sec., $\omega_T = 50$ rad/sec.

and Damping constants = 0

The flow: $h = 0$, $\frac{U_2}{U_1} = 0$ and $\rho_1 = \rho_2 = 1.147 \times 10^{-7}$ lb-sec.^{2}/in.^{4}

According to the two dimensional wing flutter analysis, only the coupled bending-torsion is possible in a uniform flow with no interface and the flutter speed for the tail under this condition is equal to 180 m.p.h. by the use of Graph I-A(n) in Ref. 5. The value is rather low relative to the flying speed of modern airplanes.

This is mainly due to the vanishing damping assumed above. For the discussion of the present example, it will be assumed that the flying speed is about 100 m.p.h. so that no flutter occurs for the tail in the absence of interface. To calculate the flutter speeds of the tail in the presence of the interface, Eq. (110) with Eq. (109) and Table 3 are used. The results are as follows:

(a) No flutter occurs in pure bending.

(b) N_o flutter occurs in pure torsion.

(c) Flutter occurs in coupled bending-torsion. And it occurs at speeds below as well as above the flying speed. The lowest speed above the flying speed which may be called the upper flutter speed is equal to 160 m.p.h. and the highest speed below the flying speed which may be called the lower flutter speed lies between zero and 6 m.p.h. No flutter speed lies between 6 m.p.h. and 160 m.p.h. The lower flutter speed has not been determined exactly because it requires the lift and moment of Eq. (100) to be calculated for $\Omega > 24$ (Table 1). If flutter occurred at $\Omega = 24$, the flutter speed would be about 6 m.p.h. The existence of the flutter speeds greater than zero is shown by the use of the asymptotic expansions for the lift and moment. The criterion (c) of Eq. (110) becomes

$$-.05108 + .2130 \sin 2\Omega - .06135 \sin^2 2\Omega + \frac{1}{\sqrt{\pi\Omega}}\Big[4.363 - 1.525 \sin 2\Omega + $$

$$+ .1420 \cos 2\Omega + .08179 (\sin^2 2\Omega - \sin 2\Omega \cos 2\Omega)\Big] + O\left(\frac{1}{\Omega}\right) = 0 \qquad (111)$$

by the use of Eqs. (101), (106) and (109) and assuming that $\Omega < \infty$ .. Eq. (111) has real solutions. Neglecting the terms of the order $\frac{1}{\sqrt{\Omega}}$ and the higher orders in Eq. (111) yields the solutions

$$\Omega = n\pi + .131 \qquad \text{and} \qquad n\pi + 1.440 \tag{112}$$

where n is an integer.

Eq. (112) gives good approximation when $n > 10^4$. It is interesting to note that corresponding to Eq. (112), the angular frequency is equal to 37.7 rad./sec. which lies very close to ω_β . From Eq. (108), is obtained

$$(c-a)\left|\frac{\bar{\theta}}{\bar{Y}}\right| = (c-a)\left|\frac{\bar{A}'}{\bar{B}'}\right| = (c-a)\left|\frac{\bar{C}'}{\bar{D}'}\right| = \lambda \tag{113}$$

which represents the ratio of the displacement of the trailing edge due to torsion to that due to bending. Using Eq. (112), λ is equal to .845, which indicates that bending is predominant during the flutter at low speeds. For the flutter at 160 m.p.h., the angular frequency is 50 rad/sec. and λ is 3.44 which indicates that torsion is predominant at the upper flutter speed. This will later be discussed. Taking $n=10^4$ in Eq. (112) yields a flutter speed equal to .00342 m.p.h. Taking the term of $\frac{1}{\sqrt{\Omega}}$ into consideration in Eq. (111) gives flutter speeds of the order .3 m.p.h. and λ remaining less than one. Finally setting $\Omega=\infty$ the criterion (c) of Eq. (110) yields

$$\omega = 37.9 \text{ and } 52.0 \text{ rad./sec.}$$

which are the natural frequencies of the bending vibration and the torsional vibration of the tail in the still air with the density given above.

Discussion of the Results. The most important results of the example are summarized and discussed as follows.

(a) There exist a lower flutter speed as well as an upper flutter speed. The upper flutter speed is very close to the flutter speed obtained by the wing flutter analysis, i.e. neglecting the effect of the interface. The existence of the lower flutter speed is entirely new to the usual flutter analysis and is caused by the presence of the interface. The occurrence of the tail buffeting near stalling speeds is qualitatively confirmed.

(b) At the upper flutter speed, the predominant mode of vibration is torsion and at the low speeds of flutter, the predominant mode is bending. The former is found in the wing flutter analysis. The latter is again new to the usual flutter analysis and confirms the finding of the English investigators that the failure of the JU 13 arose from the flexural stresses on the tail (Ref. 1). Based on the discussions of the lift and moment in Section X, it is believed that these results are also valid when the flow is described by $h << 1$ and $0 \leq k < 1$ while the other data remain unchanged. In this example, it is seen that the theory and the observed facts agree in the essential features and therefore a continuation of the investigation along the line of approach of the present thesis may be profitable.

APPENDIX I

INTEGRATION FORMULAS USED IN EQ. (35)

The integration formulas used in Eq. (35) are

$$\int_{-\infty}^{\infty} \cos\lambda\eta \, \tan^{-1}\frac{y}{\eta-x}\, d\eta = \frac{\pi}{\lambda} e^{-\lambda y} \sin\lambda x \qquad \text{for } \lambda y > 0 \qquad (114)$$

and

$$\int_{-\infty}^{\infty} \sin\lambda\eta \, \tan^{-1}\frac{y}{\eta-x}\, d\eta = -\frac{\pi}{\lambda} e^{-\lambda y} \cos\lambda x \qquad \text{for } \lambda y > 0 \qquad (115)$$

<u>Derivation</u>. The integral of Eq. (114) is

$$\int_{-\infty}^{\infty} \cos\lambda\eta \, \tan^{-1}\frac{y}{\eta-x}\, d\eta$$

$$= \frac{y}{\lambda}\int_{-\infty}^{\infty} \frac{\sin\lambda\eta}{y^2+(\eta-x)^2}\, d\eta \quad , \qquad \text{by partial integration}$$

$$= \frac{2\sin\lambda x}{\lambda}\int_{-\infty}^{\infty} \frac{\cos\lambda y\xi}{1+\xi^2}\, d\xi \quad , \qquad \text{by setting} \quad \eta - x = \xi y$$

$$= \frac{\pi}{\lambda} e^{-\lambda y} \sin\lambda x \; , \qquad \text{for } \lambda y > 0 \text{ by Formula 490 in Ref. 10}$$

Similarly, Eq. (115) is derived.

APPENDIX 2

INTEGRATION FORMULAS USED IN EQ. (36)

The integration formulas used in Eq. (36) are

$$\int_{-1}^{1}\sqrt{\frac{1-x}{1+x}}\sin\lambda x\,dx = -\pi J_1(\lambda) \tag{116}$$

$$\int_{-1}^{1}\sqrt{\frac{1-x}{1+x}}\cos\lambda x\,dx = \pi J_0(\lambda) \tag{117}$$

$$\int_{-1}^{1}\sin m\theta\,\sin\lambda x\,dx = \begin{cases} (-)^{\frac{m-2}{2}} m\pi J_m(\lambda)/\lambda, & \text{for } m = 2,4,6,\ldots \\ 0, & \text{for } m = 1,3,5,\ldots \end{cases} \tag{118}$$

$$\int_{-1}^{1}\sin m\theta\,\cos\lambda x\,dx = \begin{cases} 0, & \text{for } m = 2,4,6\ldots \\ (-)^{\frac{m-1}{2}} m\pi J_m(\lambda)/\lambda & \text{for } m = 1,3,5,\ldots \end{cases} \tag{119}$$

where $x = \cos\theta$ or Eq. (7).

$$\int_{1}^{\infty} e^{-i\Omega\xi}\sin\lambda\xi\,d\xi = e^{-i\Omega}\frac{\lambda\cos\lambda + i\Omega\sin\lambda}{\lambda^2-\Omega^2} \tag{120}$$

$$\int_{1}^{\infty} e^{-i\Omega\xi}\cos\lambda\xi\,d\xi = e^{-i\Omega}\frac{i\Omega\cos\lambda - \lambda\sin\lambda}{\lambda^2-\Omega^2} \tag{121}$$

$$\int_{-1}^{1}\sqrt{\frac{1-x}{1+x}}\sin\lambda x\,dx\int_{1}^{\infty}\frac{e^{-i\Omega\xi}}{\xi-x}\sqrt{\frac{\xi+1}{\xi-1}}\,d\xi = -2\pi\sum_{s=0}^{\infty}(-)^{s}J_{2s+1}(\lambda)\,M_{2s+1}(\Omega) \tag{122}$$

$$\int_{-1}^{1}\sqrt{\frac{1-x}{1+x}}\cos\lambda x\,dx\int_{1}^{\infty}\frac{e^{-i\Omega\xi}}{\xi-x}\sqrt{\frac{\xi+1}{\xi-1}}\,d\xi = -2\pi\sum_{s=0}^{\infty}(-)^{s}J_{2s}(\lambda)\,M_{2s}(\Omega) \tag{123}$$

where M_n is given by Eq. (37) and calculated in the next appendix.

Derivations. Eqs. (116)-(119) are immediately obtained by putting

$x = \cos\theta$ and then using the expansion formulas (Ref. 8, p. 22).

$$\cos(\lambda \cos\theta) = J_0(\lambda) + 2\sum_{n=1}^{\infty}(-)^n J_{2n}(\lambda)\cos 2n\theta \tag{124}$$

$$\sin(\lambda \cos\theta) = \qquad 2\sum_{n=0}^{\infty}(-)^n J_{2n+1}(\lambda)\cos(2n+1)\theta \tag{125}$$

The integral of Eq. (120) is

$$\int_1^{\infty} e^{-i\Omega\zeta}\sin\lambda\zeta\, d\zeta$$

$$= e^{-i\Omega}\left[\cos\lambda\int_0^{\infty} e^{-i\Omega x}\sin\lambda x + \sin\lambda\int_0^{\infty} e^{-i\Omega x}\cos\lambda x\right]dx, \quad \text{by setting } \zeta = x+1$$

$$= e^{-i\Omega}\,\frac{\lambda\cos\lambda + i\Omega\sin\lambda}{\lambda^2 - \Omega^2}, \qquad \text{by Formulas 414 and 415 of Ref. 10.}$$

Similarly, Eq. (121) is derived.

The integral of Eq. (122) is

$$\int_{-1}^{1}\sqrt{\frac{1-x}{1+x}}\sin\lambda x\, dx\int_1^{\infty}\frac{e^{-i\Omega\zeta}}{\zeta - x}\sqrt{\frac{\zeta+1}{\zeta-1}}\, d\zeta$$

$$= \int_1^{\infty} e^{-i\Omega\zeta}\sqrt{\frac{\zeta+1}{\zeta-1}}\, d\zeta\int_{-1}^{1}\sqrt{\frac{1-x}{1+x}}\,\frac{\sin\lambda x}{\zeta - x}\, dx, \qquad \text{provided that the resulting integral exists}$$

$$= -\frac{1}{2}\int_1^{\infty} e^{-i\Omega\zeta}\sqrt{\zeta^2-1}\, d\zeta\int_{-\pi}^{\pi}\frac{\sin(\lambda\cos\theta)}{\zeta - \cos\theta}\, d\theta, \qquad \text{by putting } x = \cos\theta$$

$$= -\sum_{s=0}^{\infty}(-)^s J_{2s+1}(\lambda)\int_1^{\infty} e^{-i\Omega\zeta}\sqrt{\zeta^2-1}\, d\zeta\int_{-\pi}^{\pi}\frac{\cos(2s+1)\theta}{\zeta-\cos\theta}\, d\theta, \quad \text{by using Eq. (125)}$$

provided the resulting series exist.

It is then required to evaluate the integral $\int_{-\pi}^{\pi}\frac{\cos n\theta}{\zeta - \cos\theta}\, d\theta$ where n is an integer. Evidently,

$$\int_{-\pi}^{\pi} \frac{\cos n\theta}{\xi - \cos\theta}\, d\theta = \int_{-\pi}^{\pi} \frac{e^{in\theta}}{\xi - \cos\theta}\, d\theta$$

$$= 2i \oint_{|z|=1} \frac{z^n}{z^2 - 2\xi z + 1}\, dz \qquad \text{by setting} \quad e^{i\theta} = z$$

$$= 2\pi \frac{(\xi - \sqrt{\xi^2 - 1})^n}{\sqrt{\xi^2 - 1}} \qquad \text{for } \xi > 1 \text{ by the residue theorem.}$$

This completes the derivation of Eq. (122). Similar procedures are used to derive Eq. (123).

$$\int_{-1}^{1} \sqrt{\frac{1-x}{1+x}} \cos\lambda x\, dx \int_{1}^{\infty} \frac{e^{-i\Omega\xi}}{\xi - x} \sqrt{\frac{\xi+1}{\xi-1}}\, d\xi$$

$$= \int_{1}^{\infty} e^{-i\Omega\xi} \sqrt{\frac{\xi+1}{\xi-1}}\, d\xi \int_{-1}^{1} \sqrt{\frac{1-x}{1+x}}\, \frac{\cos\lambda x}{\xi - x}\, dx$$

$$= \int_{1}^{\infty} e^{-i\Omega\xi} \sqrt{\frac{\xi+1}{\xi-1}}\, d\xi \left[\int_{-1}^{1} \frac{\cos\lambda x}{\sqrt{1-x^2}}\, dx - (\xi - 1) \int_{-1}^{1} \frac{\cos\lambda x}{\sqrt{1-x^2}}\, \frac{dx}{\xi - x} \right]$$

$$= \int_{1}^{\infty} e^{-i\Omega\xi} \sqrt{\frac{\xi+1}{\xi-1}}\, d\xi \left[\int_{0}^{\pi} \cos(\lambda\cos\theta)\, d\theta - (\xi - 1) \int_{0}^{\pi} \frac{\cos(\lambda\cos\theta)}{\xi - \cos\theta}\, d\theta \right]$$

$$= \pi \int_{1}^{\infty} e^{-i\Omega\xi} \sqrt{\frac{\xi+1}{\xi-1}}\, d\xi \left[J_0(\lambda) - \sqrt{\frac{\xi-1}{\xi+1}} \left\{ J_0(\lambda) + 2 \sum_{s=0}^{\infty} (-)^s J_{2s}(\lambda) (\xi - \sqrt{\xi^2-1})^{2s} \right\} \right]$$

$$= -2\pi \sum_{s=0}^{\infty} (-)^s J_{2s}(\lambda)\, M_{2s}(\Omega)$$

where

$$M_0(\Omega) = \frac{-1}{2}\int_1^{\infty} e^{-i\Omega\xi}\left(\sqrt{\frac{\xi+1}{\xi-1}} - 1\right) d\xi \quad ,$$ by Eq. (22a)

$$= -\frac{1}{2}\left[K_0 + K_1 - \frac{e^{-i\Omega}}{i\Omega} \right] \tag{37}$$

and $M_s(\Omega) = \int_1^{\infty} e^{-i\Omega\xi}(\xi - \sqrt{\xi^2-1})^s d\xi$ for $s = 1, 2, 3, \ldots$

APPENDIX 3

CALCULATION OF $M_s(\Omega)$

General Formula for $M_s(\Omega)$, $s = 1,2,3\ldots$. The integral for $M_s(\Omega)$ given by Eq. (37) is

$$M_s(\Omega) = \int_1^\infty e^{-z\xi}(\xi - \sqrt{\xi^2-1})^s d\xi \qquad \text{where} \quad z = i\Omega$$

$$= \sum_{n=0}^{s} (-)^n \frac{\Gamma(s+1)}{\Gamma(s-n+1)\Gamma(n+1)} \int_1^\infty e^{-z\xi}\xi^{s-n}(\xi^2-1)^{\frac{n}{2}} d\xi$$, by the binomial theorem

Using the integration formula obtained from Eq. (29) on p. 50 of Ref. 11,

$$\int_1^\infty e^{-z\xi}(\xi^2-1)^{\frac{n}{2}} d\xi = \frac{\Gamma(\frac{n}{2}+1)}{\sqrt{\pi}}\left(\frac{2}{z}\right)^{\frac{n+1}{2}} K_{\frac{n+1}{2}}(z) \tag{126a}$$

where $|arg.\, z| < \pi$, $z \neq 0$ and $R/(\frac{n}{2}+1) > 0$, yields immediately

$$M_s(\Omega) = (-)^s \frac{\Gamma(s+1)}{\sqrt{\pi}} \sum_{n=0}^{s} \frac{\Gamma(\frac{n}{2}+1)\, 2^{\frac{n+1}{2}}}{\Gamma(s-n+1)\Gamma(n+1)} \quad \frac{d^{(s-n)}}{dz^{(s-n)}}\left[z^{-\frac{n+1}{2}} K_{\frac{n+1}{2}}(z)\right] \quad \text{for } s \geq 1 \tag{126}$$

The right hand side of Eq. (126) can be put in terms of $K_\nu(z)$ by the use of the formula (Ref. 8, p. 79)

$$\frac{d}{dz}\left[\frac{K_\nu(z)}{z^\nu}\right] = -\frac{K_{\nu+1}(z)}{z^\nu} \tag{127}$$

Furthermore, $K_n(i\Omega)$ where $n = 0,1,2,\ldots$ can be put in terms of $J_n(\Omega)$ and $Y_n(\Omega)$ by the formula (Ref. 9, pp. 73-78)

$$K_n(i\Omega) = -\frac{\pi}{2}\, e^{-\frac{n\pi i}{2}}\left[Y_n(\Omega) + i J_n(\Omega)\right] \tag{128}$$

and $K_{n+\frac{1}{2}}(i\Omega)$ where $n = 0,1,2,\ldots$ can be put in terms of simple transcendental functions by the use of the asymptotic expansion of $K_\nu(z)$,

$$K_\nu(z) \sim \left(\frac{\pi}{2z}\right)^{\frac{1}{2}} e^{-z}\left[1 + \sum_{s=1}^{\infty} \frac{\{4\nu^2-1^2\}\{4\nu^2-3^2\}\cdots\{4\nu^2-(2s-1)^2\}}{s!\; 2^{3s}\; z^s}\right] \tag{129}$$

which obviously terminates for $\nu = n+\frac{1}{2}$ and thus yields exact values.

Evaluation of $M_s(\Omega)$, $s = 0-5$. Using Eqs. (126)-(129), M_s $(s=1-5)$ are obtained as follows together with M_0.

$$M_0(\Omega) = -\frac{i}{2\Omega}e^{-i\Omega} + \frac{\pi}{4}(Y_0 + iJ_0) + \frac{\pi}{4}(J_1 - iY_1)$$

$$M_1(\Omega) = \left(-\frac{i}{\Omega} - \frac{1}{\Omega^2}\right)e^{-i\Omega} - \frac{\pi}{2\Omega}(Y_1 + iJ_1)$$

$$M_2(\Omega) = \left(-\frac{i}{\Omega} - \frac{4}{\Omega^2} + \frac{4i}{\Omega^3}\right)e^{-i\Omega} - \frac{\pi}{\Omega}(J_2 - iY_2)$$

$$M_3(\Omega) = \left(-\frac{i}{\Omega} - \frac{9}{\Omega^2} + \frac{24i}{\Omega^3} + \frac{24}{\Omega^4}\right)e^{-i\Omega} + \frac{3\pi}{2\Omega}(Y_3 + iJ_3)$$

$$M_4(\Omega) = \left(-\frac{i}{\Omega} + \frac{12}{\Omega^2} - \frac{88i}{\Omega^3} - \frac{228}{\Omega^4} + \frac{228i}{\Omega^5}\right)e^{-i\Omega} + \frac{2\pi}{\Omega}(J_4 - iY_4)$$

$$M_5(\Omega) = \left(-\frac{i}{\Omega} - \frac{25}{\Omega^2} + \frac{200i}{\Omega^3} + \frac{840}{\Omega^4} - \frac{1920i}{\Omega^5} - \frac{1920}{\Omega^6}\right)e^{-i\Omega} + \frac{5\pi}{2\Omega}(Y_3 + iJ_3) + \tag{130}$$

$$+ \frac{10\pi}{\Omega^2}(Y_4 + iJ_4)$$

where J_n and Y_n are abbreviations of $J_n(\Omega)$ and $Y_n(\Omega)$.

Eq. (130) are evaluated for $\Omega = 0, .1, .2, .4, 1, 2$ and 3 and the results are given in Table 4.

Asymptotic Expansion of $M_s(\Omega)$ for $\Omega >> 1$. By the use of Eq. (126), it is obtained that, putting $z = i\Omega$

$$M_s(-iz) \sim \left(\frac{2}{\pi z}\right)^{\frac{1}{2}} K_{s+\frac{1}{2}}(z) - \frac{s}{z} K_s(z) + \frac{s(s-1)}{2z}\left(\frac{2}{\pi z}\right)^{\frac{1}{2}} K_{s-\frac{1}{2}}(z) + \cdots \qquad (131)$$

Introducing Eq. (129) into Eq. (131) yields, for $\Omega >> 1$

$$M_s(\Omega) \sim \frac{e^{-i\Omega}}{i\Omega}\left[1 - s\sqrt{\frac{\pi}{2i\Omega}} + \frac{(s+\frac{1}{2})^2 - \frac{1}{4} + s(s-1)}{2i\Omega} + O\left(\frac{1}{\Omega^{3/2}}\right)\right] \qquad (132)$$

Expansion of $M_s(\Omega)$ for $s >> 1$. The integral of $M_s(\Omega)$ is

$$M_s(\Omega) = \int_1^\infty e^{-i\Omega\xi}(\xi - \sqrt{\xi^2 - 1})^s d\xi \qquad \text{by setting} \quad \xi = \cosh x$$

$$= \frac{1}{2}\left[\int_0^\infty e^{-(s-1)x} e^{-i\Omega \cosh x} dx - \int_0^\infty e^{-(s+1)x} e^{-i\Omega\cosh x} dx\right] \qquad (133)$$

It is then required to consider $\int_0^\infty e^{-\nu x} e^{-i\Omega\cosh x} dx$. Noting that $e^{-i\Omega\cosh x}$ is analytic in x, when $|x| \leq M$ where M is a positive large value and has the expansion, when $|x| \leq M$

$$e^{-i\Omega\cosh x} = e^{-i\Omega}\left[1 - \frac{i\Omega}{2}x^2 - \left(\frac{\Omega^2}{8} + \frac{i\Omega}{24}\right)x^4 + \cdots\right]$$

and when x is positive and $x \geq M$

$$\left| e^{-i\Omega\cosh x}\right| < e^x$$

the Watson's Lemma (Ref. 12, p. 218) gives

$$\int_0^\infty e^{-\nu x} e^{-i\Omega \cosh x}\, dx \sim \frac{e^{-i\Omega}}{\nu}\left[1 - \frac{i\Omega}{\nu^2} - \frac{3\Omega^2 + i\Omega}{\nu^4} + \cdots\right] \tag{134}$$

Putting Eq. (134) into Eq. (133) yields

$$M_s(\Omega) \sim \frac{e^{-i\Omega}}{s^2-1}\left[1 + \frac{(s-1)^3-(s+1)^3}{2(s^2-1)^2}\, i\Omega + \frac{(s-1)^5-(s+1)^5}{2(s^2-1)^4}(3\Omega^2 + i\Omega) + \cdots\right] \tag{135}$$

where $s >> 1$.

APPENDIX 4

INTEGRATION FORMULAS USED IN EQ. (44)

The integration formulas used in Eq. (44) are

$$\int_0^{\pi} \cos n\theta \, \cos(\lambda \cos\theta) \, d\theta = \begin{cases} (-)^{\frac{n}{2}} \pi J_n(\lambda) & \text{for } n = 0, 2, 4, \ldots \\ 0 & \text{for } n = 1, 3, 5, \ldots \end{cases} \tag{136}$$

$$\int_0^{\pi} \cos n\theta \, \sin(\lambda \cos\theta) \, d\theta = \begin{cases} 0 & \text{for } n = 0, 2, 4 \ldots \\ (-)^{\frac{n-1}{2}} \pi J_n(\lambda) & \text{for } n = 1, 3, 5 \ldots \end{cases} \tag{137}$$

which are easily derived by the use of Eqs. (124) and (125).

APPENDIX 5

DERIVATIONS OF EQS. (49) AND (53)

The integral of Eq. (49) is

$$\int_0^\infty e^{-2h\lambda} J_n(\lambda)\, J_m(\lambda)\, d\lambda \quad , \qquad \text{by Eq. (1) on p. 150 of Ref. 8,}$$

$$= \frac{2}{\pi}\int_0^\infty e^{-2h\lambda}\, d\lambda \int_0^{\frac{\pi}{2}} J_{n+m}(2\lambda\cos\theta)\cos(n-m)\theta\, d\theta$$

$$= \frac{2}{\pi}\int_0^{\frac{\pi}{2}} \cos(n-m)\theta\, d\theta \int_0^\infty e^{-2h\lambda} J_{n+m}(2\lambda\cos\theta)\, d\lambda \, , \qquad \text{by Eq. (3) on p. 386 of Ref. 8}$$

$$= \frac{\kappa^{1-n-m}}{\pi}\int_0^{\frac{\pi}{2}} \frac{\cos(n-m)\theta\,\left(\sqrt{1-\kappa^2\sin^2\theta}-h\kappa\right)^{n+m}}{(\cos\theta)^{n+m}\sqrt{1-\kappa^2\sin^2\theta}}\, d\theta$$

where $\kappa = \frac{1}{\sqrt{1+h^2}}$, $Rl(n+m) > -1$ and $Rl(h) > 0$.

The integral of Eq. (53) is

$$\int_0^\infty e^{-2h\lambda} J_n(\lambda) J_m(\lambda)\, \frac{d\lambda}{\lambda}$$

$$= \frac{2}{\pi}\int_0^{\frac{\pi}{2}} \cos(n-m)\theta\, d\theta \int_0^\infty e^{-2h\lambda} J_{n+m}(2\lambda\cos\theta)\, \frac{d\lambda}{\lambda} \, , \qquad \text{by Eq. (7) on p. 386 of Ref. 8}$$

$$= \frac{2\kappa^{-n-m}}{\pi(n+m)}\int_0^{\frac{\pi}{2}} \frac{\cos(n-m)\theta}{(\cos\theta)^{n+m}}\left(\sqrt{1-\kappa^2\sin^2\theta}-h\kappa\right)^{n+m} d\theta$$

where $\kappa = \frac{1}{\sqrt{1+h^2}}$, $Rl(n+m) > 0$ and $Rl(h) > 0$.

APPENDIX 6

DERIVATION OF EQ. (50)

The function of Eq. (50) is defined by

$$\mathcal{N}_{nm}(h) = \int_0^\infty e^{-2h\lambda} J_n(\lambda) J_m(\lambda)\, d\lambda$$

Considering $\quad I = \int_0^\infty \frac{d}{d\lambda} \left[e^{-2h\lambda} J_n(\lambda) J_m(\lambda) \right] d\lambda$

$$I = e^{-2h\lambda} J_n(\lambda) J_m(\lambda) \Big|_0^\infty = \begin{cases} -1 & \text{when } n = m = 0 \\ 0 & \text{when } n \text{ and } m \text{ are otherwise.} \end{cases}$$

and, also

$$I = \int_0^\infty e^{-2h\lambda} \left[-2h\, J_n(\lambda) J_m(\lambda) + J_n'(\lambda) J_m(\lambda) + J_n(\lambda) J_m'(\lambda) \right] d\lambda$$

Using the recurrence relation of $J_n(\lambda)$

$$J_n'(\lambda) = \frac{1}{2} \left[J_{n-1}(\lambda) - J_{n+1}(\lambda) \right]$$

and equating the two expressions for I yield the recurrence formula for $\mathcal{N}_{nm}(h)$

$$\mathcal{N}_{n,\ +1} + \mathcal{N}_{n+1,m} = \mathcal{N}_{n-1,m} - 4h\, \mathcal{N}_{n,m} + \mathcal{N}_{n,m-1} + \begin{cases} 2 & \text{when } n = m = 0 \\ 0 & \text{otherwise} \end{cases} \qquad (138)$$

where N_{nm} are used as abbreviation of $N_{nm}(h)$

Putting $n=m$ in Eq. (138) yields immediately Eq. (50).

APPENDIX 7

DERIVATION OF EQ. (59)

The integral of Eq. (59) is

$$\int_0^\infty e^{-2h\lambda} J_n(\lambda) J_m(\lambda) \frac{d\lambda}{\lambda+\delta} \qquad \text{by Eq. (1) on p. 147 of Ref. 8,}$$

$$= \sum_{s=0}^{\infty} \frac{(-)^s \Gamma(n+m+2s+1)\, 2^{-n-m-2s}}{s!\,\Gamma(n+s+1)\Gamma(m+s+1)\Gamma(n+m+s+1)} \int_0^\infty \frac{e^{-2h\lambda} \lambda^{n+m+2s}}{\lambda+\delta} d\lambda \tag{139}$$

provided the resulting series converge.

It is now required to evaluate the integral

$$\int_0^\infty \frac{e^{-2h\lambda}\lambda^t}{\lambda+\delta} d\lambda$$

Using the integral representation of the Whittaker function, W_{nm} given on p. 340 of Ref. 13 yields

$$\int_0^\infty \frac{e^{-2h\lambda}\lambda^t}{\lambda+\delta} d\lambda = e^{h\delta} (2h)^{-\frac{t+1}{2}} \delta^{\frac{t-1}{2}} \Gamma(t+1)\, W_{-\frac{t+1}{2}, \frac{t}{2}}(2h\delta) \tag{140}$$

where $|\arg h\delta| < \pi$ and $Rl(t) > -1$

Putting Eq. (140) into Eq. (139) gives immediately Eq. (59).

APPENDIX 8

EXPANSION FORMULA FOR $W_{-\frac{t+1}{2}, \frac{t}{2}}$

Putting $\chi = -\frac{t+1}{2}$ and $\mu = \frac{t}{2}$ in the expansion formula of $W_{\chi,\mu}(x)$ given on p. 116 of Ref. 14 yields

$$W_{-\frac{t+1}{2}, \frac{t}{2}}(x) = \frac{(-)^t e^{-\frac{x}{2}} x^{\frac{t+1}{2}}}{\Gamma(t+1)} \left[\sum_{\nu=0}^{\infty} \frac{x^\nu}{\nu!} \left\{ \psi(\nu+1) - \log x \right\} + \right.$$

$$\left. + (-x)^{-t} \sum_{\mu=0}^{t-1} \Gamma(t-\mu)(-x)^{\mu} \right] \tag{141}$$

where $\psi(\nu+1) = \frac{d}{d\nu}\left[\log \Gamma(\nu+1)\right] = -\gamma + \sum_{s=1}^{\nu} \frac{1}{s}$

γ = Euler's constant = .5772

$|\arg x| < \frac{3\pi}{2}$

and $t = 0, 1, 2 \ldots$

Eq. (60a) gives

$$Ei(-x) = -\int_x^{\infty} \frac{e^{-t}}{t} dt \qquad \text{where} \quad x > 0$$

$$= -e^{-x} \int_0^{\infty} \frac{e^{-\mu}}{\mu + x} d\mu \qquad \text{by putting} \quad t = \mu + x$$

$$= -e^{-x} \int_0^{\infty} \frac{e^{-\frac{x}{\delta}\lambda}}{\lambda + \delta} d\lambda \qquad \text{by putting} \quad \mu = \frac{x}{\delta}\lambda$$

$$= -e^{-x} e^{\frac{x}{2}} x^{-\frac{1}{2}} W_{-\frac{1}{2}, 0}(x) \qquad \text{by Eq. (140),}$$

$$= -e^{-x} \sum_{\nu=0}^{\infty} \frac{x^{\nu}}{\nu!} \left[\psi(\nu+1) - \log x \right] \qquad \text{by Eq. (141)}$$

Thus, for $x > 0$

$$\sum_{\nu=0}^{\infty} \frac{x^{\nu}}{\nu!} \left[\psi(\nu+1) - \log x \right] = -e^{x}\, Ei(-x) \tag{142}$$

Putting Eq. (142) into Eq. (141) gives the expansion formula for $\overline{W}_{-\frac{t+1}{2}, \frac{t}{2}}$

$$\overline{W}_{-\frac{t+1}{2}, \frac{t}{2}}(x) = \frac{(-)^{t} x^{\frac{t+1}{2}}}{\Gamma(t+1)} \left[-e^{\frac{x}{2}}\, Ei(-x) + e^{-\frac{x}{2}} (-x)^{-t} \sum_{\mu=0}^{t-1} \Gamma(t-\mu)\, (-x)^{\mu} \right] \tag{143}$$

where $t = 0, 1, 2, \ldots$ and $x > 0$.

APPENDIX 9

DERIVATION OF EQ. (76)

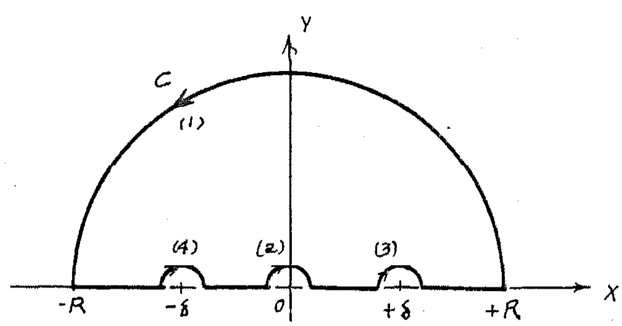
The integral of Eq. (76) is

$$\int_0^\infty J_n(\lambda)\, J_m(\lambda)\, \frac{\lambda\, d\lambda}{\lambda^2 - \delta^2}$$

Using the contour integration of the Hankel's type (§13.53, p. 428 Ref. 8), the following integral is considered

$$I = \oint_C J_n(z)\, H_m^{(1)}(z)\, \frac{z\, dz}{z^2 - \delta^2}$$

where δ is real and positive and the contour C is shown in



the figure where $z = x + iy$.

By the residue theorem, $I = 0$.

Denoting the integrand of I by $Q(z)$, I can be written as

$$I = \int_{(1)} Q(z)\, dz + \int_{(2)} Q(z)\, dz + \int_{(3)} Q(z)\, dz + \int_{(4)} Q(z)\, dz + P\int_{-R}^{R} Q(x)\, dx$$

where the paths (1), (2), (3) and (4) are shown in the figure and P denotes the principal value of the integral.

$$\int_{(1)} Q(z)\,dz = \frac{1}{\pi}\int_{(1)} \frac{e^{i(2z-\frac{m+n+1}{2}\pi)}}{z^2-s^2}\,dz + \frac{1}{\pi}\int_{(1)} \frac{e^{i\frac{n-m}{2}\pi}}{z^2-s^2}\,dz \qquad \text{when } |z|\to\infty$$

where the first term approaches zero by the Jordan's Lemma (§6.222, p. 115, Ref. 13) and the second term approaches zero by the results of § 6.22 on p. 113 of Ref. 13.

$$\left|\int_{(2)} Q(z)\,dz\right| \le M\left|\int_{(2)} z^{n-m+1}\,dz\right| \qquad \text{where M is a constant, when } |z|\to 0$$

$$\to 0 \qquad \text{provided that} \quad n+2>m$$

$$\int_{(3)} Q(z)\,dz = -\pi i\left\{\text{Residue of } Q(z) \text{ at } z=s\right\}$$

$$= -\frac{\pi i}{2} J_n(s)\left[J_m(s)+iY_m(s)\right]$$

$$\int_{(4)} Q(z)\,dz = -\pi i\left\{\text{Residue of } Q(z) \text{ at } z=-s\right\}$$

$$= -\frac{\pi i}{2} J_n(s)\left[-J_m(s)+iY_m(s)\right]$$

$$P\int_{-R}^{R} Q(x)\,dx = P\int_{-\infty}^{\infty} J_n(x)\,H_m^{(1)}(x)\,\frac{x\,dx}{x^2-s^2} \qquad \text{when} \quad R\to\infty$$

$$= P\int_0^{\infty}\left[J_n(x)\,H_m^{(1)}(x) - J_n(-x)\,H_m^{(1)}(-x)\right]\frac{x\,dx}{x^2-s^2}$$

$$= 2P\int_0^{\infty} J_n(x)J_m(x)\,\frac{x\,dx}{x^2-s^2} \qquad \text{provided } n-m = 0, 2, 4, \ldots$$

by the use of the relations that $J_n(-x) = (-)^n J_n(x)$ and $H_m^{(1)}(-x) = (-)^m \left[H_m^{(1)}(x) - 2J_m(x) \right]$.

Using the above results yields immediately Eq. (76).

APPENDIX 10

DERIVATION OF EQ. (84)

Eq. (83) can be written as

$$R_{2n}(\Omega,0) = \frac{(-)^{n-1}}{2}\left[2\sum_{s=0}^{n} M_{2s+1}(\Omega) - \sum_{s=0}^{\infty} M_{2s+1}(\Omega)\right]$$

$$R_{2n+1}(\Omega,0) = \frac{(-)^{n}}{2}\left[M_0(\Omega) + 2\sum_{s=1}^{n} M_{2s}(\Omega) - \sum_{s=1}^{\infty} M_{2s}(\Omega)\right] \tag{144}$$

where $\quad M_s(\Omega) = \int_1^\infty e^{-i\Omega\xi}(\xi - \sqrt{\xi^2-1})^s d\xi \qquad$ for $\quad s \geq 1$.

It is required to sum $\sum_0^\infty M_{2s+1}$ and $\sum_1^\infty M_{2s}$.

$$\sum_{s=0}^{\infty} M_{2s+1}(\Omega) = \sum_{s=0}^{\infty}\int_1^\infty e^{-i\Omega\xi}(\xi - \sqrt{\xi^2-1})^{2s+1} d\xi$$

$$= \lim_{\varepsilon\to 0}\int_{1+\varepsilon}^\infty e^{-i\Omega\xi}\sum_{s=0}^{\infty}(\xi - \sqrt{\xi^2-1})^{2s+1} d\xi$$

$$= \lim_{\varepsilon\to 0}\int_{1+\varepsilon}^\infty e^{-i\Omega\xi}\frac{\xi - \sqrt{\xi^2-1}}{1-(\xi - \sqrt{\xi^2-1})^2} d\xi$$

$$= \frac{1}{2}\int_1^\infty e^{-i\Omega\xi}(\xi^2-1)^{-\frac{1}{2}} d\xi$$

$$= \frac{1}{2} K_0(i\Omega) \qquad \text{by Eq. (126a)} \tag{145}$$

$$\sum_{s=1}^{\infty} M_{2s}(\Omega) = \sum_{s=1}^{\infty} \int_1^{\infty} e^{-i\Omega\xi}(\xi-\sqrt{\xi^2-1})^{2s}\,d\xi$$

$$= \frac{1}{2}\int_1^{\infty} e^{-i\Omega\xi}\left(\frac{\xi}{\sqrt{\xi^2-1}}-1\right)d\xi \qquad \text{by similar steps above}$$

$$\int_1^{\infty} e^{-i\Omega\xi}\frac{\xi}{\sqrt{\xi^2-1}}\,d\xi = -\frac{d}{d(i\Omega)}\left[\int_1^{\infty} e^{-i\Omega\xi}\frac{d\xi}{\sqrt{\xi^2-1}}\right]$$

$$= -\frac{d}{d(i\Omega)}\left[K_0(i\Omega)\right] \qquad \text{by Eq. (126a)}$$

$$= K_1(i\Omega) \qquad \text{by Eq. (127)}$$

$$\int_1^{\infty} e^{-i\Omega\xi}\,d\xi = \frac{1}{\sqrt{\pi}}\sqrt{\frac{2}{i\Omega}}\,K_{\frac{1}{2}}(i\Omega) \qquad \text{by Eq. (126a)}$$

Thus,

$$\sum_{s=1}^{\infty} M_{2s}(\Omega) = \frac{1}{2}\left[K_1(i\Omega)-\sqrt{\frac{2}{i\Omega}}\,K_{\frac{1}{2}}(i\Omega)\right] \tag{146}$$

Putting Eqs. (145) and (146) into Eq. (144) and using Eqs. (128) and (129) give immediately Eq. (84).

TABLE 1: LIFT AND MOMENT WHEN $h=0, 0<\ell<\infty$ AND $k=0$ *[1]

Ω	$R(\frac{L}{L_o})_B$	$I(\frac{L}{L_o})_B$	$R(\frac{M}{M_o})_B$	$I(\frac{M}{M_o})_B$	$R(\frac{L}{L_o})_T$	$I(\frac{L}{L_o})_T$	$R(\frac{M}{M_o})_T$	$I(\frac{M}{M_o})_T$ *[2]
0	.50000	0	.50000	0	.50000	0	0	0
.1	.403	-.086	.404	-.106	.400	-.111	.095	.118
.2	.345	-.076	.341	-.137	.346	-.125	.164	.172
.4	.305	-.0239	.274	-.0883	.302	-.113	.169	.127
.5	.3016	.0214	.315	-.0693	.3016	-.0988	.185	.132
1.0	.2661	.2165	.2961	.01290	.2661	-.0336	.204	.1119
1.5	.19386	.3730	.2368	.0589	.19386	-.00212	.2633	.1287
2.0	.12747	.46103	.17973	.03049	.12747	-.03900	.3203	.21949
2.5	.11954	.51984	.18093	-.03172	.11957	-.1052	.31907	.34420
3.0	.16426	.60785	.23566	-.06675	.16426	-.1422	.26434	.44177
3.5	.21720	.75359	.29932	-.04392	.21720	-.12146	.20062	.48149
4.0	.22704	.94158	.31911	.02221	.22704	-.058439	.18090	.47780
4.5	.17403	1.1156	.27372	.07509	.17403	-.009381	.22628	.48739
5.0	.10005	1.2255	.20503	.06267	.10005	-.02451	.29500	.56233
6.0	.10667	1.3538	.22141	-.05874	.10667	-.14562	.27860	.80878
7.0	.20177	1.6653	.32890	.00203	.20177	-.08414	.17111	.87296
8.0	.09225	1.9857	.22850	.07537	.09225	-.01433	.27152	.92464
9.0	.06450	2.1128	.20623	-.04826	.06450	-.1372	.29374	1.1733
10.	.17816	2.6281	.32839	-.020251	.17816	-.10734	.17161	1.2703
11	.09474	2.7416	.25207	.08027	.09474	-.00839	.24795	1.2947
12	.032720	2.8787	.19371	-.032977	.032720	-.12113	.33848	1.4254
13	.15341	3.1234	.32025	-.040781	.15341	-.12665	.17975	1.6658
14	.10251	3.4921	.30785	.06343	.10251	-.008082	.3826	1.8120
15	.010408	3.6495	.18596	-.013980	.010408	-.10042	.31417	1.8890
16	.12661	3.8585	.30620	-.05763	.12661	-.14177	.19384	2.0578
17	.11211	4.2362	.29674	.07038	.11211	-.013792	.2032	2.0546
18	-.003047	4.4239	.18383	.006937	-.003047	-.077413	.31619	2.2430
19	.09798	4.5984	.28773	-.069458	.09798	-.15172	.2122	2.4446
20	.12085	4.9748	.31498	.05670	.12085	-.025092	.1850	2.5971
21	-.00815	5.1956	.18802	.027886	-.00815	-.054290	.31199	2.5971
22	.06849	5.3442	.26667	-.07535	.06849	-.15575	.2333	2.8253
23	.12670	5.7090	.32862	.038560	.12670	-.040996	.17136	2.8364
24	-.00575	5.9669	.19819	.046991	-.00575	-.03306	.30182	2.9530

*[1] cf. Eq. (100)

*[2] R = Real and I = Imaginary

TABLE 2: LIFT AND MOMENT WHEN $h=o$, $\ell=1$ AND $k=\frac{1}{2}$

Ω	$R(\frac{L}{L_o})_B$	$I(\frac{L}{L_o})_B$	$R(\frac{M}{M_o})_B$	$I(\frac{M}{M_o})_B$	$R(\frac{L}{L_o})_T$	$I(\frac{L}{L_o})_T$	$R(\frac{M}{M_o})_T$	$I(\frac{M}{M_o})_T$
0	.63077	0	.63077	0	.60000	0	0	0
.1	.503	-.108	.505	-.137	.477	-.141	.139	.153
.4	.384	.0149	.399	-.101	.386	-.132	.223	.170
1.5	.260	.454	.265	.033	.210	-.067	.376	.236
3.0	.425	1.25	.603	-.03	.386	-.091	.408	.349

TABLE 3: AERODYNAMIC COEFFICIENTS WHEN $h=0, 0<f<\infty$, $k=0$ AND $\frac{a}{c}=.35$ *1

Ω	A_R	A_I	B_R	B_I	C_R	C_I	D_R	D_I *2
0	∞	∞	∞	$-\infty$	∞	∞	∞	$-\infty$
.5	-.106	1.206	1.289	.5882	.0851	.134	.165	-.1565
1.0	-.4330	.5322	.2179	.4294	.05850	.0683	.05436	-.0645
1.5	-.4973	.2587	.01233	.2692	.05500	.04013	.03223	-.04394
2.0	-.46103	.12747	-.027537	.16626	.061532	.025812	.02794	-.036311
2.5	-.41587	.095632	-.02222	.12143	.068725	.021841	.028771	-.030099
3.0	-.40524	.10951	-.018834	.11134	.07192	.022851	.029447	-.023679
3.5	-.43062	.12411	-.029511	.11117	.070868	.024144	.028672	-.018322
4.0	-.47079	.11352	-.049124	.10426	.067843	.022861	.026870	-.014962
4.5	-.49582	.077346	-.064738	.086030	.066031	.018811	.025377	-.013701
5.0	-.49020	.040024	-.067077	.065029	.067263	.014499	.025230	-.013427
6.0	-.45126	.035482	-.052598	.051837	.072585	.013125	.027011	-.011220
7.0	-.47579	.057648	-.061206	.057044	.071225	.014846	.026426	$-.0^{2}80779$
8.0	-.49643	.02306	-.072127	.040252	.069753	.010822	.025454	$-.0^{2}78438$
9.0	-.46951	.01434	-.062008	.031817	.073107	.009307	.026636	$-.0^{2}72827$
10	-.52562	.03564	-.071694	.040535	.079856	.011074	.027606	$-.0^{2}58121$
11	-.49848	.0172	-.073608	.02954	.071123	$.0^{2}888$	.025214	$-.0^{2}53650$
12	-.47979	.005457	-.066694	.022173	.073342	$.0^{2}72528$	.025087	$-.0^{2}56985$
13	-.48052	.023602	-.066300	.027923	.073648	$.0^{2}8777$	.026671	$-.0^{2}41295$
14	-.49887	.01464	-.074020	.023675	.072567	$.0^{2}8798$	.027335	$-.0^{2}52372$
15	-.48660	$.0^{2}1389$	-.069596	.016775	.07346	$.0^{2}59895$	.026459	$-.0^{2}42202$
16	-.48231	.01583	-.067422	.021403	.074148	$.0^{2}71949$	.026759	$-.0^{2}33458$
17	-.49838	.01320	-.073962	.019934	.072686	$.0^{2}67492$	.026147	$-.0^{2}3114$
18	-.49154	$-.0^{3}3386$	-.071592	.013519	.073540	$.0^{2}51572$	.026428	$-.0^{2}34523$
19	-.48404	.0103	-.068340	.016863	.074432	$.0^{2}6025$	.026807	$-.0^{2}2838$
20	-.49748	.01206	-.073693	.017271	.073204	$.0^{2}60617$	.026292	$-.0^{2}2530$
21	-.49482	$-.0^{3}775$	-.072950	.011470	.073560	$.0^{2}4594$	.026409	$-.0^{2}28903$
22	-.48584	$.0^{2}622$	-.069193	.013533	.074587	$.0^{2}51268$	.026826	$-.0^{2}2486$
23	-.49643	.01101	-.073331	.015199	.073624	$.0^{2}54914$	.026444	$-.0^{2}21213$
24	-.49724	$-.0^{3}48$	-.073907	.010167	.073607	$.0^{2}42009$	.026405	$-.0^{2}24574$

*1 cf. Eq. (106)

*2 $.0^{2} = .00$ and $.0^{3} = .000$

TABLE 4. FUNCTION $M_s(\Omega)$ WHEN $s=0\text{-}5$ AND $\Omega=0\text{-}3$ *

(a) Real Part

Ω	$R(M_0)$	$R(M_1)$	$R(M_2)$	$R(M_3)$	$R(M_4)$	$R(M_5)$
0	$-\infty$	∞	$-\infty$	$-\infty$	$-\infty$	$-\infty$
.1	-1.66494	.95834	.29403	.1224	.07	0
.2	-1.26762	.61028	.25644	.099	.065	.015
.4	-.80878	.26329	.18359	.09910	.05786	.02
1.0	$-.0^{2}58036$	-.15465	$.0^{2}2224$	.02509	.0223	.017
2.0	.62648	-.26655	-.13808	-.071927	-.039834	-.02357
3.0	.53875	-.10704	-.095171	-.070878	-.050539	-.036195

(b) Imaginary Part

Ω	$I(M_0)$	$I(M_1)$	$I(M_2)$	$I(M_3)$	$I(M_4)$	$I(M_5)$
0	$-\infty$	$-\infty$	$-\infty$	∞	∞	$-\infty$
.1	.88126	-.75113	.083965	-.0190	.00	0
.2	.93792	-.71507	-.12391	-.042	-.01	-.01
.4	1.00166	-.63858	-.17586	-.064142	-.03105	-.0181
1.0	.94440	-.39006	-.19898	-.10292	-.0584	-.036
2.0	.36394	-.017561	-.060526	-.054579	-.041417	-.03051
3.0	-.29424	.16815	.078081	.034786	.015171	$.0^{2}6440$

*Cf. Eq. (130)

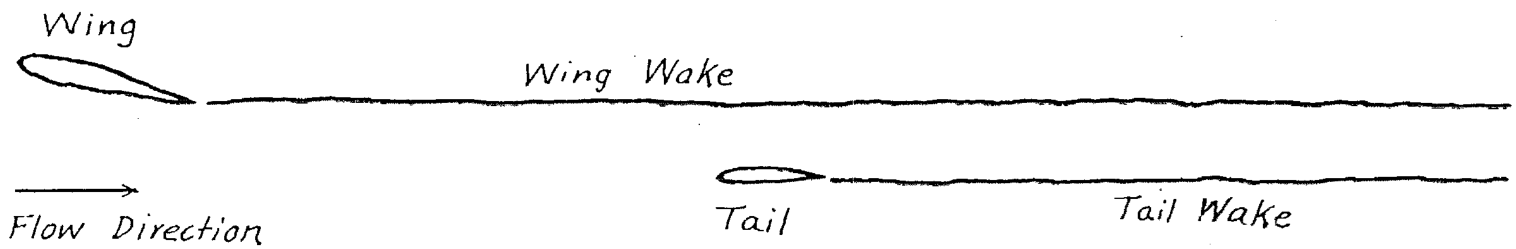


FIGURE 1: WING WAKE ACTING UPON THE OSCILLATING TAIL

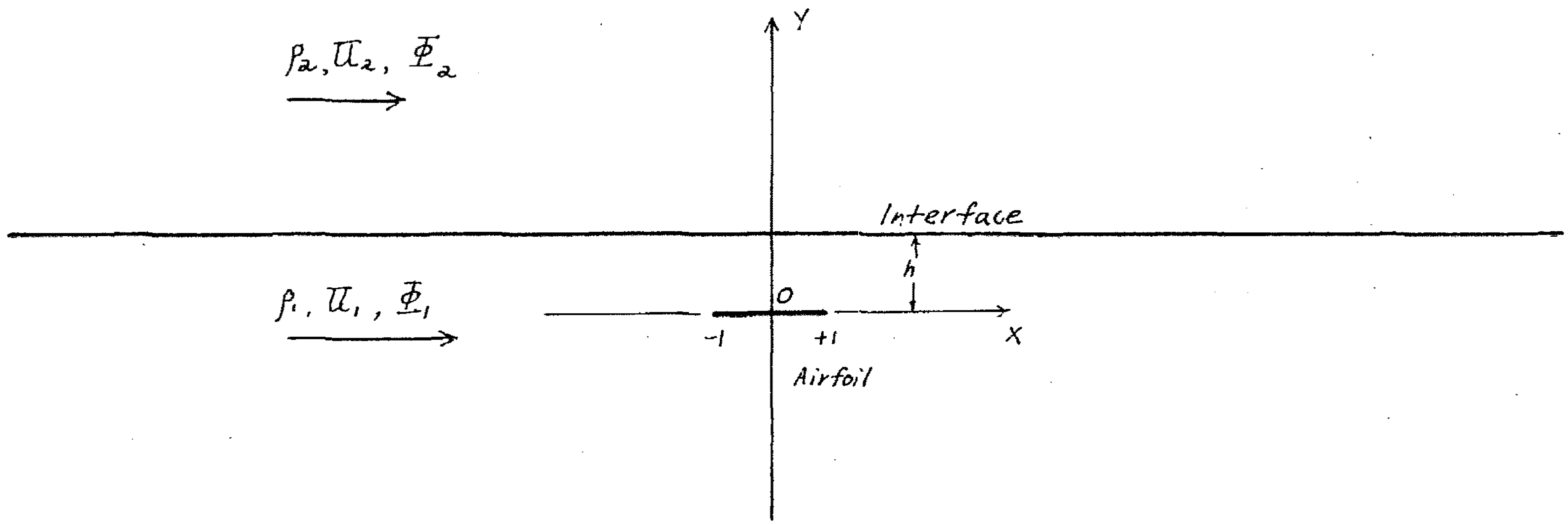


FIGURE 2: INTERFACE WITH THIN AIRFOIL

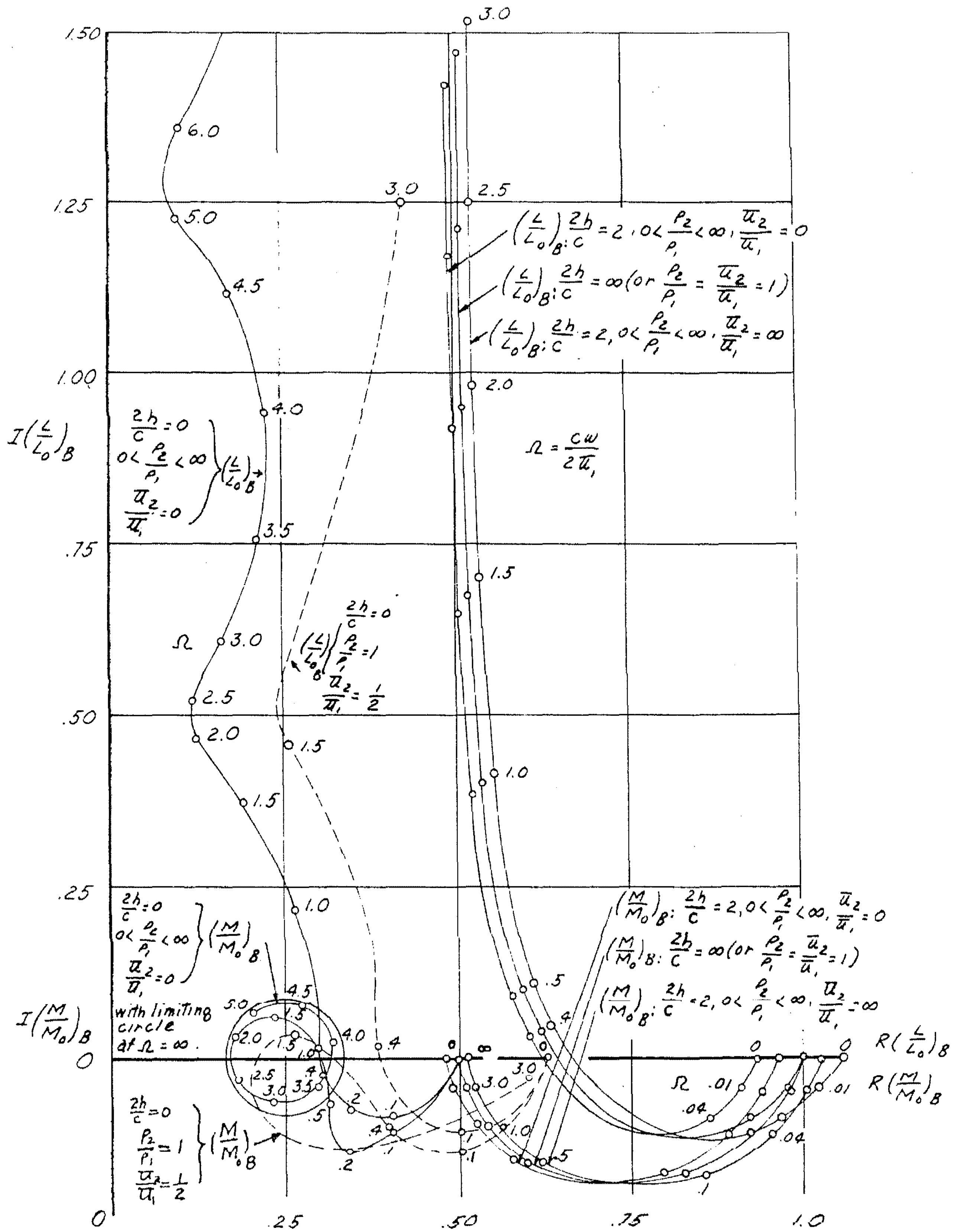


FIGURE 3:

VECTOR DIAGRAM OF THE LIFT AND MOMENT ON THE AIRFOIL IN BENDING OSCILLATION, AS FUNCTIONS OF $h, \ell, k, + \Omega$.

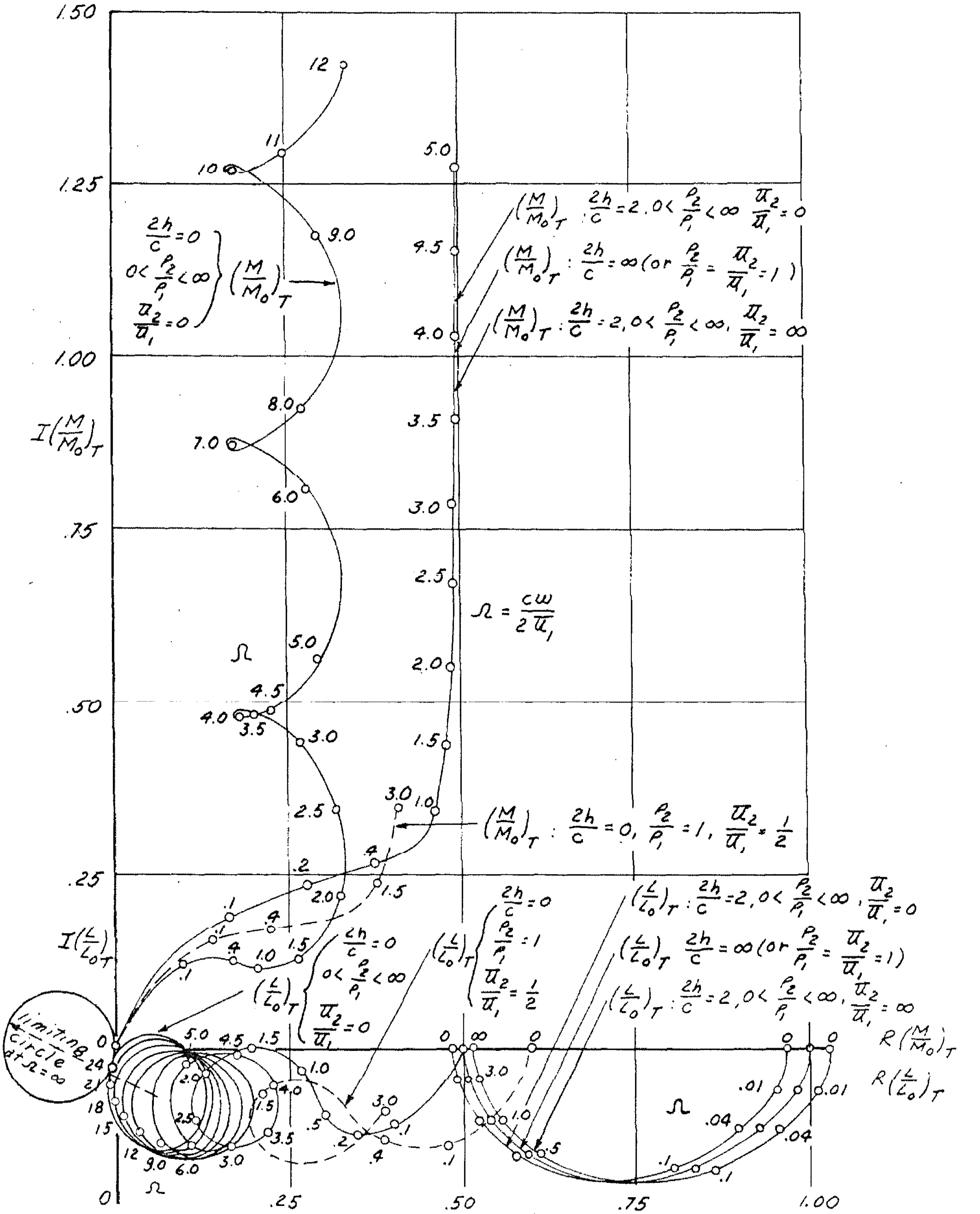


FIGURE 4:

VECTOR DIAGRAM OF THE LIFT AND MOMENT ON THE AIRFOIL IN "TORSIONAL" OSCILLATION, AS FUNCTIONS OF $h, \ell, k, \& \Omega$.

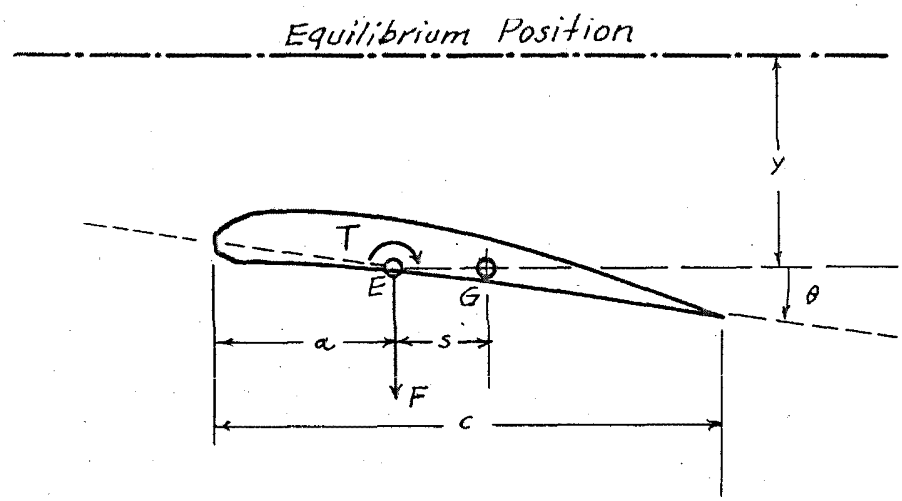


FIGURE 5: SCHEMATIC DIAGRAM OF A FLUTTERING AIRFOIL

REFERENCES

1. Technical Report by the Accidents Investigation Subcommittee on the Accident to the Aeroplane G-AAZK at Meopham, Kent, on 21st July, 1930. R. & M. No. 1360, British A.R.C., (1931).

   Duncan, W. J., Ellis, D. L., and Scruton, C.: First Report on the General Investigation of Tail Buffeting. R. & M. No. 1457, British A.R.C., (1932).

   Duncan, W. J., Ellis, D. L., and Smyth, E.: Second Report on the General Investigation of Tail Buffeting. R. & M. No. 1541, British A.R.C., (1933).

2. Blenk, H., Hertel, H., and Thalan, K.: The German Investigation of the Accident at Meopham (England). T.M. No. 669, N.A.C.A., (1932).

   Biechfeler, C.: Tests for the Elimination of Tail Flutter. T.M. No. 710, N.A.C.A., (1933).

3. White, J. A., and Hood, M. J.: Wing-Fuselage Interference, Tail Buffeting, and Air Flow about the Tail of a Low-Wing Monoplane. Rep. No. 482, N.A.C.A., (1934).

4. Abdrashitoz, G.,: Tail Buffeting. T.M. No. 1041, N.A.C.A. 1943.

5. Theodorsen, T., and Garrick, I.E.: Mechanism of Flutter, A Theoretical and Experimental Investigation of the Flutter Problem. Rep. No. 685, N.A.C.A. (1940).

6. v. Kármán, Th. and Sears, W. R.: Airfoil Theory for Non-Uniform Motion. Journal of the Aeronautical Sciences, Vol. 5, pp. 379-390. (1938).

7. Jahnke, E. and Emde, F.: Tables of Functions with Formulae and Curves. Fourth edition, Dover Publications, New York (1945).

8. Watson, G. N.: A Treatise on the Theory of Bessel Functions. Second Edition, Cambridge University Press, London (1944).

9. Myklestad, N. O.: Vibration Analysis, Section VI-6, pp. 246-259. First edition, McGraw-Hill Book Company, New York (1944).

10. Pierce, B. O.: A Short Table of Integrals, Third revised edition Ginn and Company, Boston (1929).

11. Gray, A., Mathews, G. B. and MacRobert, T. M.: A Treatise on Bessel Functions and Their Applications to Physics. Second edition, MacMillan and Company, Ltd., London (1931).

12. Copson, E. T.: An Introduction to the Theory of Functions of a Complex Variable. First edition, Oxford University Press, London (1935).

13. Whittaker, E. T. and Watson, G. N.: A Course of Modern Analysis, Fourth edition reprinted, Cambridge University Press, London (1950).

14. Magnus, W. and Oberhettinger, F.: Formeln und Sätze für die Speziellen Funktionen der Mathematischen Physik. Zweite Auflage, Springer-Verlag Ohg., Berlin (1948).