FREE AND FORCED OSCILLATIONS OF A CLASS OF SELF-EXCITED OSCILLATORS

Thesis by

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ABSTRACT

Free and forced oscillations in oscillators governed by the equation $\ddot{x} - \epsilon[1 - g(x, \dot{x})]\dot{x} + h(x) = e(t)$ are studied with appropriate constraints on $g(x, \dot{x})$, $h(x)$ and $e(t)$. Theorems are proved on the existence and uniqueness of stable periodic solutions for free oscillations using the Poincaré-Bendixson theory in the phase-plane. There follow several examples to illustrate the theorems and limit cycles are obtained for these examples by the Liénard construction. A result on the existence of periodic solutions in the forced case is obtained by use of Brouwer's fixed point theorem. The part on topological methods is concluded by applying Yoshizawa's results on ultimate boundedness of solutions to the forced case.

Approximate analytical solutions are obtained for specific examples for different regions of validity of the parameter ϵ. For free oscillations, the perturbation solution is obtained for small ϵ. A Fourier series approximation is given for other values of ϵ, and the limit cycle for the case $\epsilon \rightarrow \infty$ is obtained. Finally, the first order solution for forced oscillations is obtained by the method of slowly varying parameters and the stability of this solution is examined.

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I. INTRODUCTION

By the turn of this century, a remarkably simple and powerful theory of harmonic or sinusoidal oscillations had been developed which enabled engineers to predict with accuracy the behavior of many machines and electrical networks, and which enabled physicists to explain phenomena in acoustics and other disciplines. This fundamental theory assumes that all physical phenomena obey "linear" laws, for example, Ohm's law, Hooke's law, and so forth.

There were some phenomena, however, which could not be explained by this theory. In the theory of oscillations some of these phenomena which could not be adequately explained by the linear theory were: the generation of "maintained" oscillations, that is, oscillations that had constant amplitude and period over many cycles, frequency synchronization in forced oscillations, the existence of forced oscillations in an oscillator which were different in frequency from the frequency of the forcing function. The linear theory either could not explain these phenomena at all or explained them inadequately.

As an example, consider the problem of explaining the generation of "maintained" or "sustained" oscillations in a resonant electrical or mechanical circuit. The differential equation governing the behavior of such a circuit, by the linear theory, is

$$\ddot{x} + \beta\dot{x} + \omega^2 x = 0. \tag{1.1}$$

Here x is a real variable representing a physical quantity, for example, displacement or charge, the dots above the variable x represent differentiation with respect to time, β is the "damping factor" which is a constant, and ω is the "natural frequency" of the oscillator. If the damping is zero, equation (1.1) reduces to

$$\ddot{x} + \omega^2 x = 0, \tag{1.2}$$

whose solution is $x(t) = A \sin(\omega t + \varphi)$, where A and φ are constants which depend on the initial conditions. This method is a mathematically adequate representation of "maintained" oscillations but, physically, this explanation is open to the following objections:

First, surely the damping factor β cannot be exactly zero. It must be either positive or negative, no matter how small in magnitude. If it is positive, the oscillations will eventually die out. If it is negative, the oscillations will increase without bound. In either case, sustained oscillations are not possible. Second, according to linear theory, the amplitude A of oscillation is determined solely by the initial condition. This would mean, for example, that the power which an electric generator will put into a load can take on any arbitrary value depending on the way in which it was started. A startling prospect which, happily, does not agree with experimental observations: In practice, the amplitude of oscillation in oscillators is independent of the initial conditions.

This inadequacy of the linear theory to explain "sustained" oscillations was apparent to Lord Rayleigh(1)* in 1883, and, with his usual deep insight, he saw a correct way out of the difficulty and proposed that the nonlinear equation

$$\ddot{x} + (k + k'\dot{x}^2)\dot{x} + \omega^2 x = 0 \tag{1.3}$$

was a better mathematical model to the physical situation than a linear equation. He assumed k and k' to be small and gave the solution of (1.3) to be

$$x(t) = A \sin \omega t + \frac{k'\omega A^3}{32} \cos 3\omega t \tag{1.4}$$

where A is independent of the initial conditions and given by

$$k + \frac{3}{4} k'\omega^2 A^2 = 0. \tag{1.5}$$

From (1.5) it is clear that steady vibrations are possible if k and k' have opposite signs, and that these vibrations are stable if k is negative. The physical interpretation of this model is that for small velocities the oscillator has "negative damping" which adds energy to the system, causing the oscillations to grow in magnitude. For large velocities, however, the system has "positive damping" which dissipates energy from the system causing the oscillations to decrease in magnitude. The "maintained" oscillations are those where the energy input exactly balances the energy output during one complete cycle. It would, of course, be incorrect to conclude from this that no

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*Numbers in parenthesis designate references at the end of this thesis.

source of energy is needed: the dissipated energy cannot be recovered to its original form, and therefore, outside the system represented by equation (1.3), there must exist a source of energy.

We now give examples from several disciplines to show the wide variety of applications the nonlinear theory of oscillations has: In radio engineering there is great interest in the working of triode oscillators. In 1920 B. van der Pol(2) proposed that the oscillations of this oscillator were governed by the nonlinear equation

$$\ddot{x} - \epsilon(1 - x^2)\dot{x} + x = 0. \tag{1.6}$$

This is the well known van der Pol equation, and it is actually equivalent to Rayleigh's equation. One can be obtained from the other by a change of variables. Van der Pol's paper generated considerable interest in this type of oscillation and there was a flurry of research activity on this subject by Appleton, van der Pol, van der Mark, Greaves and others. They considered questions about the stability of oscillations, two degree-of-freedom oscillators, frequency demultiplication, and approximate solutions for ϵ small and large. These researches are summarized in excellent review papers by van der Pol(3) and Le Corbeiller(4) where many additional references may be found.

For $\epsilon >> 1$, the motion governed by equation (1.6) is not a smooth periodic function but is very rapidly changing for part of the cycle. The period of oscillation of the oscillator for $\epsilon >> 1$ was found to be not the "natural period" of the oscillator, but the same as the "relaxation time," RC, of a linear RLC (resistance, inductance, capacitance) circuit. Hence such oscillations were called "relaxation oscillations" by van der Pol[5]. Such oscillations have technical use, for example, in scanning in television. The heartbeat has been considered as a relaxation oscillation[6].

The field of control engineering provides further examples of these nonlinear oscillatory phenomena. Minorsky[7] discusses the self-oscillations observed during experimental work on the roll-stabilization of ships by the activated tanks method. Other examples may be found in the books by Popov[8] and Gibson[9].

Many examples of such nonlinear oscillatory phenomena may also be observed in mechanical systems. There are, for example, a whole class of maintained oscillations caused by the presence of dry or rubbing friction in various devices. The oscillations of a violin string excited by a bow are of this type, as is the phenomenon of "chatter" in machine tools. More homely examples are the binding and creaking of doors on unlubricated hinges when opened, and the screeching of a piece of chalk that is held perpendicular to the blackboard when writing. A torsional vibration of this type has been observed in the propeller shafts of ships when rotating at low speeds. The shaft is usually supported by two water lubricated journal bearings and at low speeds no water film can form and the bearings are "dry."

Examples of maintained oscillations caused by fluid dynamic forces are stall flutter in aircraft wings and stall flutter of turbine blades. Many other examples of such mechanical nonlinear vibration phenomena are given by Den Hartog in his book[10]. These oscillations are often called "self-excited" because the alternating force that sustains motion is created or controlled by the motion itself; if the motion stops, the alternating force disappears. This is in contrast to forced vibration where the sustaining alternating force exists independently of the motion and persists even when the motion has stopped. Thus "relaxation" oscillations are a particular type of "self-excited" oscillation, but the two terms are often used interchangeably in the literature.

The precise form of the equations that govern these oscillations are often not determined, but they are known to be governed by equations of the van der Pol type. So, obviously, generalizations were proposed so that many of these oscillations can be studied together as a class. A. Liénard[11] used the more general equation

$$\ddot{x} + \epsilon f(x)\dot{x} + x = 0 \tag{1.7}$$

when giving a method for the graphical construction of solutions. Shohat[12] used a Fourier series approach in getting an analytical approximation for the solution to this equation, and LaSalle[13] studied the limiting case as $\epsilon \to \infty$.

The next obvious generalization is to also have a nonlinear "spring," and for forced vibrations the equation is

$$\ddot{x} + \epsilon f(x)\dot{x} + h(x) = e(t) \tag{1.8}$$

Many different results on the existence, uniqueness, stability and boundedness of solutions of the equation may be obtained, depending on the initial assumptions regarding the functions $f(x)$, $h(x)$ and $e(t)$. Researchers who have obtained results related to this equation are Lefschetz[14], Levinson[15, 16], Cartwright and Littlewood[17, 18, 19], Cartwright[20], Urabe[21] and Reuter[22]. This is not an exhaustive list of references. Many others may be found in Cesari's book[23] which contains an extensive bibliography on the mathematical aspects of this equation and related topics.

Levinson and Smith[24] supposed that the "damping" was a function of both displacement and velocity and proved the existence and uniqueness of periodic solutions of

$$\ddot{x} + f(x, \dot{x})\,\dot{x} + h(x) = 0 \qquad (1.9)$$

under appropriate restrictions of $f(x, \dot{x})$ and $h(x)$. Levinson also obtained existence results for the forced case

$$\ddot{x} + f(x, \dot{x})\,\dot{x} + h(x) = e(t)\ , \qquad (1.10)$$

and Reuter[27] and Antosiewicz[28] obtained boundedness theorems for solutions of (1.10) under different assumptions on $f(x, \dot{x})$, $h(x)$ and $e(t)$.

This is a sufficiently general equation to include a great many practical applications to self-excited oscillations. Levinson and Smith's assumptions about (1.9) allow for great generality but the proofs are lengthy and the theorems somewhat difficult to apply.

Caughey and Malhotra[29] have considered the equation

$$\ddot{x} - \epsilon[1 - g(x, \dot{x})]\dot{x} + h(x) = 0, \qquad (1.11)$$

and have proved the existence and uniqueness of a periodic solution when $g(x, \dot{x})$ and $h(x)$ satisfy appropriate conditions.

In this thesis we consider the "self-excited" oscillations of systems whose motions are described by (1.11), and also the forced oscillations governed by the equation

$$\ddot{x} - \epsilon[1 - g(x, \dot{x})]\dot{x} + h(x) = e(t) . \qquad (1.12)$$

To understand the global properties of the solutions, the equations are first studied by topological methods. The Poincaré-Bendixson theory is applied to the system with no forcing term. Some specific examples are given, and their periodic solutions (limit cycles) are constructed by Liénard's graphical method[11]. The stability of periodic motion is established by Lyapounov's Second Method[30]. The forced problem is very much more difficult to work out, but some progress can be made by the application of the Brouwer fixed point theorem[31]. Boundedness of solutions can be established by use of several theorems, especially those of T. Yoshizawa[32]. Finally, analytical approximations to the solutions are made for particular cases for special regions of validity using the perturbation method, Fourier series approximation and other methods.

II. TOPOLOGICAL METHODS

To study the motions of a system governed by the differential equation

$$\ddot{x} - \epsilon[1 - g(x, \dot{x})]\dot{x} + h(x) = e(t) \tag{2.1a}$$

with the arbitrary initial conditions

$$x(0) = a \quad \text{and} \quad \dot{x}(0) = b \tag{2.1b}$$

it will be necessary to restrict the class of functions $g(x, \dot{x})$, $h(x)$ and $e(t)$ so that the existence and uniqueness of the solution of the initial value problem is assured. Here, this is accomplished by requiring that the Cauchy-Lipschitz existence theorem, as stated by Minorsky[33], be applicable to the equivalent system of first order differential equations

$$\begin{aligned} \frac{dx}{dt} &= \dot{x} \\ \frac{d\dot{x}}{dt} &= \epsilon[1 - g(x, \dot{x})]\dot{x} - h(x) + e(t) . \end{aligned} \tag{2.2}$$

The system of equations (2.2) will have a unique solution to prescribed initial values if the partial derivatives $\frac{\partial g}{\partial x}$, $\frac{\partial g}{\partial \dot{x}}$ and $\frac{\partial h}{\partial x}$ exist and are continuous. This is a sufficient condition.

However, physically, it is of even greater interest to study the conditions under which the existence and uniqueness of <u>periodic</u> solutions of equation (2.1) is assured. Also, we are interested in finding out what these solutions are if they do exist. The precise or quantitative nature of these solutions is very difficult to obtain in general. However, certain good approximations to the solution can be obtained for restricted regions of validity. These approximate solutions will be obtained in the next part. In this part, however, we will study the global or topological properties of the solutions.

One of the most fruitful techniques for the study of second order differential equations is to study these solutions in the $(x, \frac{dx}{dt})$-plane, commonly called the phase-plane. Extended to higher order equations, this phase-plane is called the state-space. But the geometry of three or more dimensions is very difficult and the advantages of planar representation are lost. However, this technique is generally useful only to autonomous systems, that is, systems that do not explicitly depend on time. So we now consider only the autonomous differential equation

$$\ddot{x} - \epsilon[1 - g(x, \dot{x})]\dot{x} + h(x) = 0 \tag{2.3}$$

or the equivalent system of equations:

$$\begin{aligned} \frac{dx}{dt} &= \dot{x} \\ \frac{d\dot{x}}{dt} &= \epsilon[1 - g(x, \dot{x})]\dot{x} - h(x) \end{aligned} \tag{2.4}$$

The solutions $x(t)$ and $\dot{x}(t)$ of equation (2.4) provide a parametric representation of the solution which is a curve in the phase-plane. This curve is known as a "trajectory." By eliminating dt from equations (2.4), the differential equation of the "integral curves" of the autonomous system is obtained:

$$\frac{d\dot{x}}{dx} = \frac{\epsilon[1 - g(x, \dot{x})]\dot{x} - h(x)}{\dot{x}} \quad . \tag{2.5}$$

It should be noted that equations (2.4) and (2.5) are equivalent, that is, they have the same integral curves, with the difference, however, that (2.5) gives a geometrical curve without any reference to what happens in time.

This time independence results in the so-called translation property of autonomous differential equations: To a given trajectory, there corresponds an infinity of solutions (motions) differing from each other only in phase. This property is very convenient for the geometric study of integral curves.

It is easy to see that this procedure breaks down for nonautonomous systems for, in that case

$$\frac{d\dot{x}}{dx} = f(x, \dot{x}, t)$$

which varies in time and it is meaningless even to speak of integral curves in the sense of "trajectories." In fact, if one assumes that the solutions are to be represented in the phase-plane, two different trajectories may, in general, even intersect each other which is contrary to the Cauchy-Lipschitz theorem, and so on. This difficulty can be removed by introducing time as a third dimension, but then the advantages of two-dimensional representation are lost.

The advantage of introducing the idea of trajectories in the phase-plane is that a trajectory which closes on itself represents a, stable or unstable, periodic motion. The problem reduces to one of finding if any such closed trajectories exist. We will make use of the Poincaré-Bendixson theory for two-dimensional autonomous systems, an exposition of which may be found in the books of Minosky[33], Coddington and Levinson[34], Stoker[35], Lefschetz[36] and others.

Existence of Periodic Solutions of the Autonomous System[29]

The existence of periodic solutions can only be assured by further restricting $g(x, \dot{x})$ and $h(x)$:

Assume $g(x, \dot{x}) = 1$ defines a simple, closed, convex curve and let Ω_1 be the compact set $g(x, \dot{x}) \leqq 1$. Define the function

$$V(x, \dot{x}) = \frac{1}{2} \dot{x}^2 + \int_0^x h(\xi)\, d\xi \qquad (2.6)$$

and assume that $h(x)$ is odd and monotone increasing, and that $xh(x) > 0$ for $x \neq 0$. Then $V(x, \dot{x})$ = constant also defines a simple, closed, convex curve. Select a constant C_2 such that $V(x, \dot{x}) = C_2$ inscribes the set Ω_1, and let $V(x, \dot{x}) \leqq C_2$ be the compact set Ω_2. Similarly, select constant C_3 such that $V(x, \dot{x}) = C_3$ circumscribes Ω_1, and let $V(x, \dot{x}) \leqq C_3$ be the compact set Ω_3. Note that $\Omega_2 \subset \Omega_1 \subset \Omega_3$ (Figure 1).

Theorem 2.1. Suppose $g(x, \dot{x})$ defines a simple, closed, convex curve which is four-point symmetric: $g(x, \dot{x}) = g(-x, \dot{x}) = g(x, -\dot{x}) = g(-x, -\dot{x}) > 0$ if both $x, \dot{x} \neq 0$. Suppose also that $\frac{\partial g}{\partial x}$, $\frac{\partial g}{\partial \dot{x}}$ and $\frac{\partial h}{\partial x}$

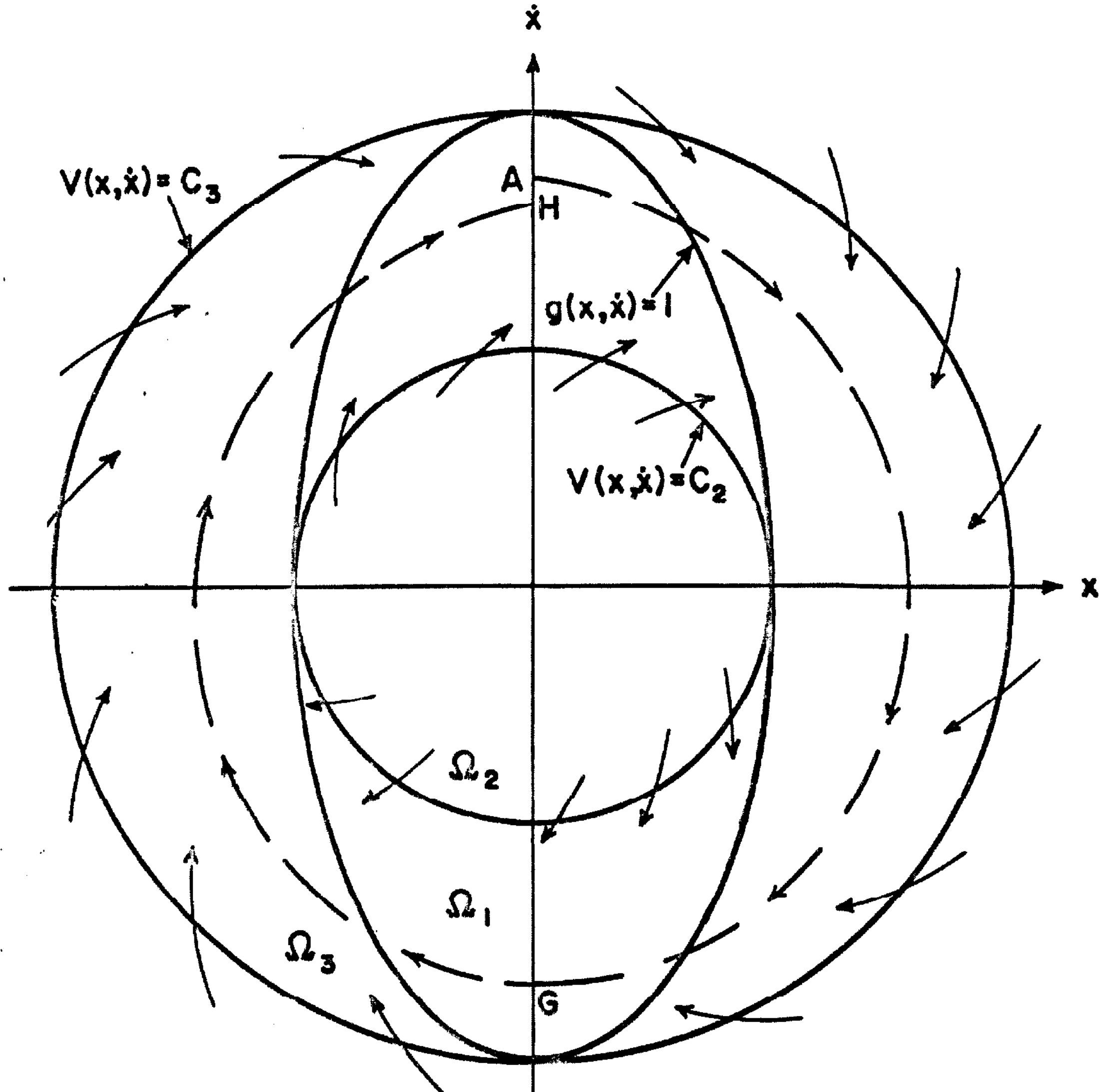


FIGURE 1

exist and are continuous; $h(x)$ is odd and monotone increasing; $xh(x) > 0$ for $x \neq 0$; $h(0) = 0$; and $\epsilon > 0$. Then the autonomous equation (2.3)

$$\ddot{x} - \epsilon[1 - g(x, \dot{x})]\dot{x} + h(x) = 0$$

has at least one periodic solution (limit cycle). Further, any limit cycle that exists will be in the region of the phase-plane bounded by the simple closed curves $V(x, \dot{x}) = C_2$ and $V(x, \dot{x}) = C_3$, that is, in the region $\Omega_3 - \Omega_2$.

Proof: From equation (2.5):

$$\frac{d\dot{x}}{dx} = \frac{\epsilon[1 - g(x, \dot{x})]\dot{x} - h(x)}{\dot{x}}$$

it is readily seen that the only singular or critical point of the equation, where both the numerator and denominator are simultaneously zero, is the origin $x = \dot{x} = 0$. Further, at $\dot{x} = 0$ and $x \neq 0$, $\left|\frac{d\dot{x}}{dx}\right|$ is unbounded so the x-axis does not contain any arc of a trajectory. Since $g(x, \dot{x})$ is assumed four-point symmetric, $g(x, \dot{x}) = 1$ encloses the origin, and the only singular point of the equation is outside the region $\Omega_3 - \Omega_2$.

Differentiating equation (2.6):

$$\frac{dV}{dt} = \dot{V}(x, \dot{x}) = \dot{x}\ddot{x} + \dot{x}h(x)$$

but along a trajectory of (2.3)

$$\ddot{x} = \epsilon[1 - g(x, \dot{x})]\dot{x} - h(x) ,$$

hence

$$\dot{V}(x, \dot{x}) = \epsilon[1 - g(x, \dot{x})]\dot{x}^2 \qquad (2.7)$$

and

$$\dot{V}(x, \dot{x}) < 0 \text{ in } \overline{\Omega}_1$$

$$\dot{V}(x, \dot{x}) = 0 \text{ on the boundary of } \Omega_1, \ g(x, \dot{x}) = 1$$

$$\dot{V}(x, \dot{x}) > 0 \text{ in } \Omega_1 .$$

Hence $\dot{V}(x, \dot{x}) > 0$ in Ω_2 except at the two points of contact with $g(x, \dot{x}) = 1$, where $\dot{V}(x, \dot{x}) = 0$; and $\dot{V}(x, \dot{x}) < 0$ in $\overline{\Omega}_3$ and on the boundary of Ω_3. *Thus each trajectory in* $\overline{\Omega}_3$ *and on the boundary* of Ω_3 is directed inward to $\Omega_3 - \Omega_2$, each trajectory in Ω_2 is directed outward to $\Omega_3 - \Omega_2$, and each trajectory inside $\Omega_3 - \Omega_2$ remains in that finite region. Hence all the conditions of the Poincaré-Bendixson theorem are satisfied, and there exists at least one periodic (limit cycle) in $\Omega_3 - \Omega_2$.

Note that $g(x, \dot{x})$ need not be four-point symmetric for the existence of periodic solutions: all that is really required in the given proof is that $g(x, \dot{x}) = 1$ be a simple closed curve which encloses the origin. The four-point symmetry will, however, be required to show that only one periodic solution exists.

Levinson and Smith[24] have proved a more general form of this theorem: let $xh(x) > 0$ for $x > 0$. Moreover, let $\int_0^{\pm\infty} h(x)\, dx = \infty$. Let $f(0, 0) < 0$ and let there exist some $x_0 > 0$ such that $f(x, \dot{x}) \geqq 0$

for $|x| \geqq x_o$. Further, let there exist an M such that for $|x| < x_o$

$$f(x, \dot{x}) \geqq - M .$$

Finally, let there exist some $x_1 > 0$ such that

$$\int_{x_o}^{x_1} f(x, \dot{x})\, dx \geqq 10\, Mx_o$$

where $\dot{x} > 0$ is an arbitrarily decreasing positive function of x in the above integration. Under these conditions

$$\ddot{x} + f(x, \dot{x})\, \dot{x} + h(x) = 0 \tag{2.8}$$

has at least one periodic solution.

Comparing theorem 2.1 with the above, it is seen that the functions in theorem 2.1 satisfy all the conditions given above. However, theorem 2.1 not only proves the existence of periodic solutions, but also gives upper and lower bounds in the $(x, \dot{x})$-plane for these solutions. Further, its proof is much simpler than that of the more general theorem.

Uniqueness of Periodic Solution of Autonomous System

It has been demonstrated that the autonomous equation (2.3) has at least one periodic solution. It will now be shown that if $g(x, \dot{x})$ satisfies certain additional conditions, this equation has only one periodic solution, that is, the periodic solution is unique, and it is also stable.

Since the curves $V(x, \dot{x}) = C$ and $g(x, \dot{x}) = 1$ are assumed four-point symmetric, simple, closed, and convex, the points of contact between $V(x, \dot{x}) = C_2$ and $g(x, \dot{x}) = 1$ can only be (i) on the x-axis or (ii) on the $\dot{x}$-axis. These two cases will be proved separately.

Theorem 2.2. Suppose $g(x, \dot{x})$ and $h(x)$ satisfy the conditions of theorem 2.1, and that $x \frac{\partial g}{\partial x} > 0$ for $x \neq 0$, and $\dot{x} \frac{\partial g}{\partial \dot{x}} > 0$ for $\dot{x} \neq 0$. Further, assume $[1 - g(x, \dot{x}) - \dot{x} \frac{\partial g}{\partial \dot{x}}] < 0$ in $\Omega_1 - \Omega_2$ if the points of contact are on the x-axis [case (i)]; or, if the points of contact are on the $\dot{x}$-axis [case (ii)], let $[1 - g(x, \dot{x}) - \dot{x} \frac{\partial g}{\partial \dot{x}}] < 0$ in $\Omega_3 - \Omega_1$. Then equation (2.3) has one, and only one, periodic solution.

Proof: Integrate equation (2.7) along the trajectory AGH (Figure 1)

$$\int_A^H \dot{V}(x, \dot{x})\, dt = V(x, \dot{x}) \Big|_A^H = \int_A^H \epsilon[1 - g(x, \dot{x})]\, \dot{x}^2\, dt$$

and, if the trajectory is a limit cycle, $A \equiv H$ and

$$V(x, \dot{x}) \Big|_A^H = \oint \epsilon[1 - g(x, \dot{x})]\, \dot{x}^2\, dt = 0 .$$

Further, since the trajectories are point symmetric, it will be sufficient to demonstrate uniqueness for only a semi-trajectory, that is, to show $V(x, \dot{x}) \Big|_A^G = 0$ for one, and only one, $\dot{x}_A$. There will be no loss in generality if we consider the semi-trajectory to be wholly in $\Omega_3 - \Omega_2$ since $\dot{V}(x, \dot{x}) < 0$ in Ω_3 and $\dot{V}(x, \dot{x}) > 0$ in Ω_2 (except at the points of contact). For convenience, we consider a semi-trajectory starting at $x = 0$, $\dot{x} > 0$ and terminating at $x = 0$, $\dot{x} < 0$.

Case (i): Consider two neighboring semi-trajectories ADG and A'D'G' (Figure 2) which are contained entirely within $\Omega_3 - \Omega_2$.

$$V(x, \dot{x}) \Big|_A^G = \int_A^B \epsilon[1-g]\dot{x}\, dx + \int_B^C + \int_C^D + \int_D^E + \int_E^F + \int_F^G \epsilon[1-g]\dot{x}\, dx$$

$$V(x, \dot{x}) \Big|_{A'}^{G'} = \int_{A'}^{B'} \epsilon[1-g]\dot{x}\, dx + \int_{B'}^{C'} + \int_{C'}^{D'} + \int_{D'}^{E'} + \int_{E'}^{F'} + \int_{F'}^{G'} \epsilon[1-g]\dot{x}\, dx$$

Compare the integrals along ADG and A'D'G' for each of the sub-intervals:

$$\int_{A'}^{B'} = \int_A^B \epsilon[1-g(x, \dot{x}) - \delta(\dot{x})\frac{\partial g}{\partial \dot{x}} - \ldots][\dot{x} + \delta(\dot{x})]\, dx$$

$$\approx \int_A^B \epsilon[1-g]\dot{x}\, dx + \int_A^B \epsilon[1 - g - \dot{x}\frac{\partial g}{\partial \dot{x}}]\, dx$$

$$\therefore \quad \int_{A'}^{B'} < \int_A^B \quad \text{since } [1 - g - \dot{x}\frac{\partial g}{\partial \dot{x}}] < 0 \quad \text{in } \Omega_1 - \Omega_2$$

Now B'C' is in $\overline{\Omega}_1$, hence $\int_{B'}^{C'} < 0$

and BC is in Ω_1, hence $\int_B^C > 0$.

Also, since $\frac{\partial g}{\partial \dot{x}} > 0$ for $\dot{x} > 0$, $\int_{C'}^{D'} < \int_C^D$.

Since D'D" is in $\overline{\Omega}_1$, $\int_{D'}^{D''} < 0$.

Similarly, it can be shown that

Note: BB', CC', D''DD', EE' and FF' are parallel to the ẋ-axis.

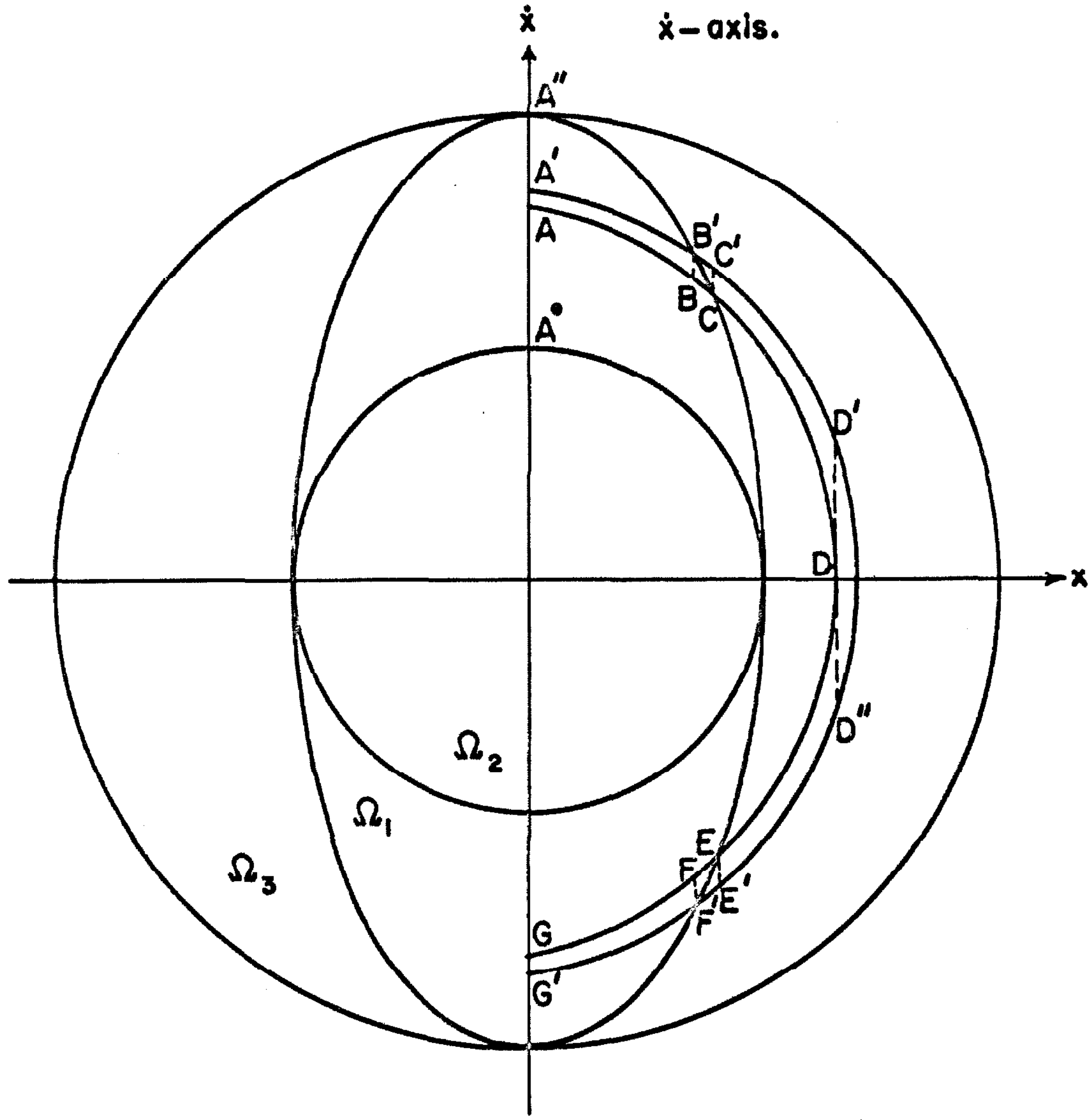


FIGURE 2

Note: The outer points of contact may not lie on either the x-axis or the ẋ-axis but the nature of the proof is exactly the same as before.

$$\int_{D''}^{E'} < \int_{D}^{E}$$

$$\int_{E'}^{F'} < 0 \quad \text{and} \quad \int_{E}^{F} > 0$$

and

$$\int_{F'}^{G'} < \int_{F}^{G} .$$

Hence, over the entire semi-trajectory:

$$V(x, \dot{x}) \Big|_{A'}^{G'} < V(x, \dot{x}) \Big|_{A}^{G} \text{ if } \dot{x}_{A'} > \dot{x}_A \tag{2.9}$$

Now if we choose $A = A^o$, then $V(x, \dot{x}) \Big|_{A}^{G} \geqq 0$ since each trajectory in Ω_2 spirals outwards; and if we choose $A = A''$, then $V(x, \dot{x}) \Big|_{A}^{G} \leqq 0$ since each trajectory on the boundary of Ω_1 spirals inwards. And by (2.9), $V(x, \dot{x}) \Big|_{A}^{G}$ is monotone decreasing for increasing $\dot{x}_A$, hence

$$V(x, \dot{x}) \Big|_{A}^{G} = 0$$

for one, and only one, $\dot{x}_A$ and equation (2.3) has a unique limit cycle which lies in the region $\Omega_3 - \Omega_2$ in the phase-plane.

Case (ii): Again, consider two neighboring trajectories ADG and A'D'G' (Figure 3) which are contained entirely in $\Omega_3 - \Omega_2$.

$$V(x, \dot{x}) \Big|_{A}^{G} = \int_{A}^{B} \epsilon[1-g]\dot{x}\, dx + \int_{B}^{C} + \int_{C}^{E} + \int_{E}^{F} + \int_{F}^{G} \epsilon[1-g]\dot{x}\, dx$$

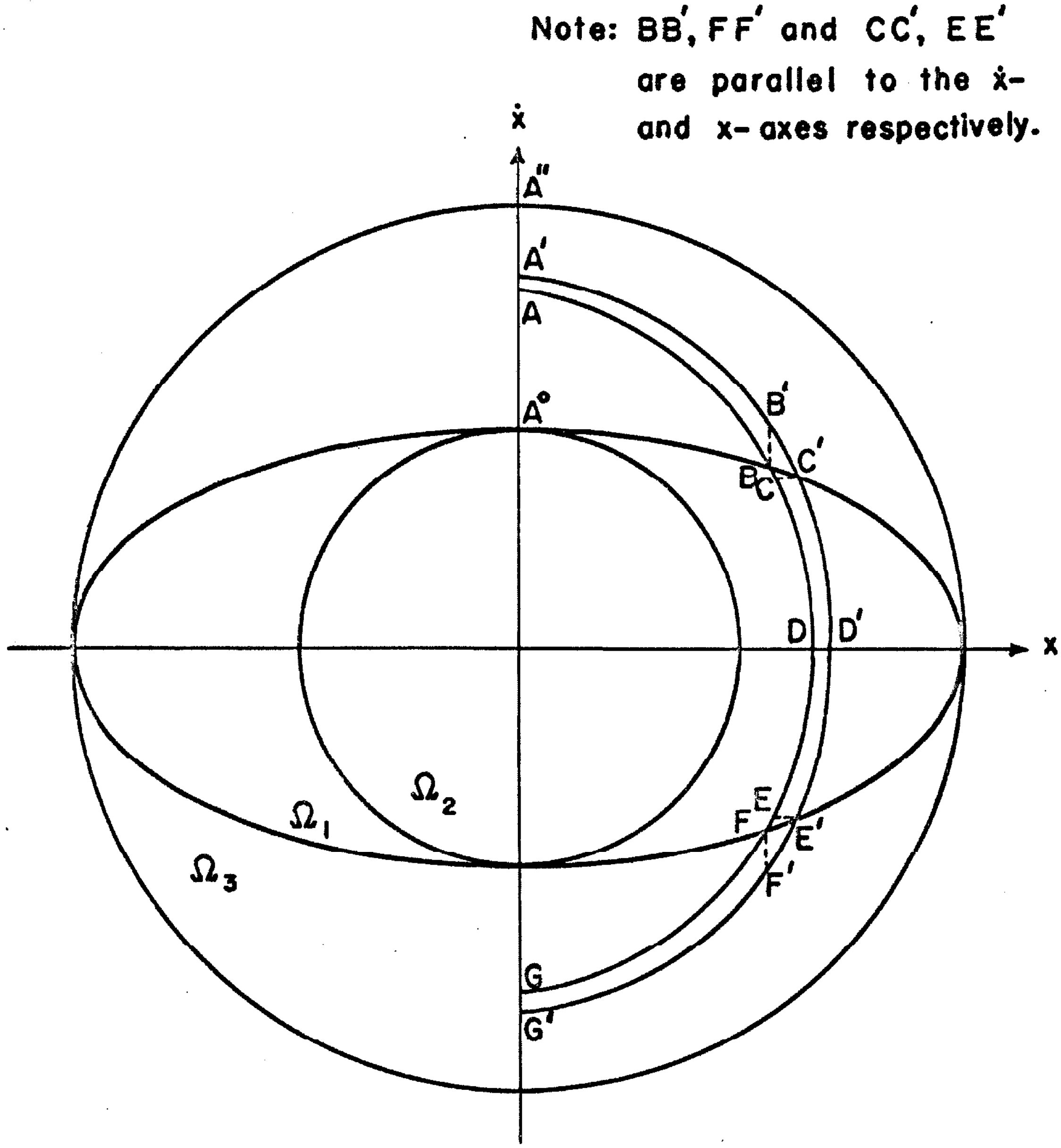


FIGURE 3

Note: The outer points of contact may not lie on either the x-axis or the ẋ-axis but the nature of the proof is exactly the same as before.

$$V(x,\dot{x})\Big|_{A'}^{G'} = \int_{A'}^{B'} \epsilon[1-g]\,\dot{x}\,dx + \int_{B'}^{C'} + \int_{C'}^{E'} + \int_{E'}^{F'} + \int_{F'}^{G'} \epsilon[1-g]\,\dot{x}\,dx$$

Comparing the integrals along each sub-interval:

$$\int_{A'}^{B'} = \int_{A}^{B} \epsilon[1-g(x,\dot{x}) - \delta(\dot{x})\frac{\partial g}{\partial \dot{x}} - \dots][\dot{x} + \delta(\dot{x})]\;dx$$

$$\approx \int_{A}^{B} \epsilon[1-g]\,\dot{x}\,dx + \int_{A}^{B} \epsilon[1-g-x\frac{\partial g}{\partial \dot{x}}]\;dx$$

$$\therefore \qquad \int_{A'}^{B'} < \int_{A}^{B} \quad \text{since} \quad [1-g-\dot{x}\frac{\partial g}{\partial \dot{x}}] < 0 \quad \text{in } \Omega_3 - \Omega_1$$

Now $B'C'$ is in $\overline{\Omega}_1$, hence $\int_{B'}^{C'} \leqq 0$

and BC is in Ω_1, hence $\int_{B}^{C} \geqq 0.$

We can write

$$\int_{C}^{E} \epsilon[1-g]\,\dot{x}\,dx = \int_{C}^{E} \epsilon[1-g]\,\dot{x}\frac{dx}{d\dot{x}}\;d\dot{x}$$

Substituting for $\frac{d\dot{x}}{dx}$ from (2.5)

$$\int_{C}^{E} \epsilon[1-g]\,\dot{x}\,dx = \int_{C}^{E} \frac{\epsilon[1-g]\,\dot{x}^2}{\epsilon[1-g]\,\dot{x} - h(x)}\;d\dot{x}$$

$$\therefore \quad \int_{C'}^{E'} \doteq \int_{C}^{E} \frac{\epsilon[1-g)\dot{x}^2}{\epsilon[1-g]\,\dot{x} - h(x)}\,d\dot{x} + \int_{C}^{E} \frac{\epsilon\dot{x}^2\,\Delta(x)\left\{\frac{\partial g}{\partial x}h(x) + (1-g)\frac{dh}{dx}\right\}}{\left\{\epsilon(1-g)\dot{x} - h(x)\right\}^2}\;d\dot{x}$$

But $(1-g) > 0$ since CE is in Ω_1; $\frac{\partial g}{\partial x}h(x) > 0$ and $\frac{dh}{dx} > 0$ since $h(x)$ is monotone increasing

$$\therefore \int_{C'}^{E'} < \int_{C}^{E} .$$

Similarly, it can be shown that

$$\int_{E'}^{F'} < 0 \quad \text{and} \quad \int_{E}^{F} > 0,$$

and

$$\int_{F'}^{G'} < \int_{F}^{G} .$$

Hence, over the entire semi-trajectory:

$$V(x, \dot{x}) \Big|_{A'}^{G'} < V(x, \dot{x}) \Big|_{A}^{G} \quad \text{provided} \quad \dot{x}_{A'} > \dot{x}_A \tag{2.10}$$

Again, $V(x, \dot{x}) \Big|_{A}^{G} \geqq 0$ if $A = A^o$ and $V(x, \dot{x}) \Big|_{A}^{G} \leqq 0$ if $A = A''$, and by (2.10), $V(x, \dot{x}) \Big|_{A}^{G}$ is monotone decreasing for increasing $\dot{x}_A$.

Hence

$$V(x, \dot{x}) \Big|_{A}^{G} = 0$$

for one, and only one, $\dot{x}_A$ and equation (2.3) has a unique limit cycle which lies in the region $\Omega_3 - \Omega_2$ in the phase-plane.

The limit cycle in both cases is stable for if the motion is perturbed so that the representative point is outside the area enclosed by the limit cycle, then $\dot{V}(x, \dot{x}) < 0$ and the resulting trajectory converges onto the limit cycle. If, on the other hand, the perturbation is such that the resulting representative point is inside the area enclosed by the limit cycle, $\dot{V}(x, \dot{x}) > 0$ and the subsequent trajectory again converges onto the limit cycle. So the unique periodic solution is stable.

Levinson and Smith's theorem on uniqueness is: If $f(x, \dot{x})$ and $h(x)$ satisfy the conditions for the existence of at least one periodic solution of equation (1. 8), and if for every C the minimum of

$$F(x, \dot{x}) = \frac{1}{\dot{x}^2} + \frac{1}{\dot{x}\, f(x, \dot{x})} \frac{\partial f(x, \dot{x})}{\partial \dot{x}}$$

on $R_2(C)$ is positive and exceeds the maximum of $F(x, \dot{x})$ on $R_1(C)$, then equation (1. 8) possesses a unique solution. Here R_1 denotes the region in the $(x, \dot{x})$-plane where $f(x, \dot{x})$ is negative, and R_2 the region where $F(x, \dot{x})$ is positive. That part of the curve $V(x, \dot{x}) = \frac{1}{2}\dot{x}^2 + \int_0^x h(x)\, dx = C$ which lies in R_1 is denoted by $R_1(C)$ and that part of $V(x, \dot{x})$ which lies in R_2 by $R_2(C)$.

The additional conditions presented above are more difficult to verify in applications than those given in theorem 2. 2.

Examples

A few examples are now given to illustrate the type of functions $g(x, \dot{x})$ and $h(x)$ which satisfy the conditions of theorems 2.1 and 2. 2.

(i) As an example of case (i), consider the equation

$$\ddot{x} - \epsilon\left[1 - \frac{x^2}{a^2} - \frac{\dot{x}^2}{b^2}\right]\dot{x} + x = 0, \quad a^2 < b^2 \qquad (2.11)$$

Here $\quad g(x, \dot{x}) = \frac{x^2}{a^2} + \frac{\dot{x}^2}{b^2}, \quad h(x) = x, \quad$ and

$$g(x, \dot{x}) = 1 = \frac{x^2}{a^2} + \frac{\dot{x}^2}{b^2} \quad \text{is an ellipse.}$$

Ω_1 is bounded by the ellipse $g(x, \dot{x}) = 1$, Ω_2 is bounded by the inscribing circle $x^2 + \dot{x}^2 = a^2$, and Ω_3 is bounded by the circumscribing circle $x^2 + \dot{x}^2 = b^2$. It can readily be checked that $g(x, \dot{x})$ and $h(x)$ satisfy all the conditions of theorems 2.1 and 2.2 for the existence and uniqueness of a stable limit cycle (periodic solution), and it would be desirable to know what this limit cycle is. A general solution is not known, but graphical solutions using Liénard's Method[11] have been constructed for the particular case when $a = 1$ and $b = 1.5$ for $\epsilon = 1.0$ and $\epsilon = 10.0$. These graphical results are presented in Figures 4 and 5 respectively. It is seen that the trajectories converge to the limit cycle faster, that is, the transients die out faster, for $\epsilon = 10.0$ than for $\epsilon = 1.0$.

It is observed that

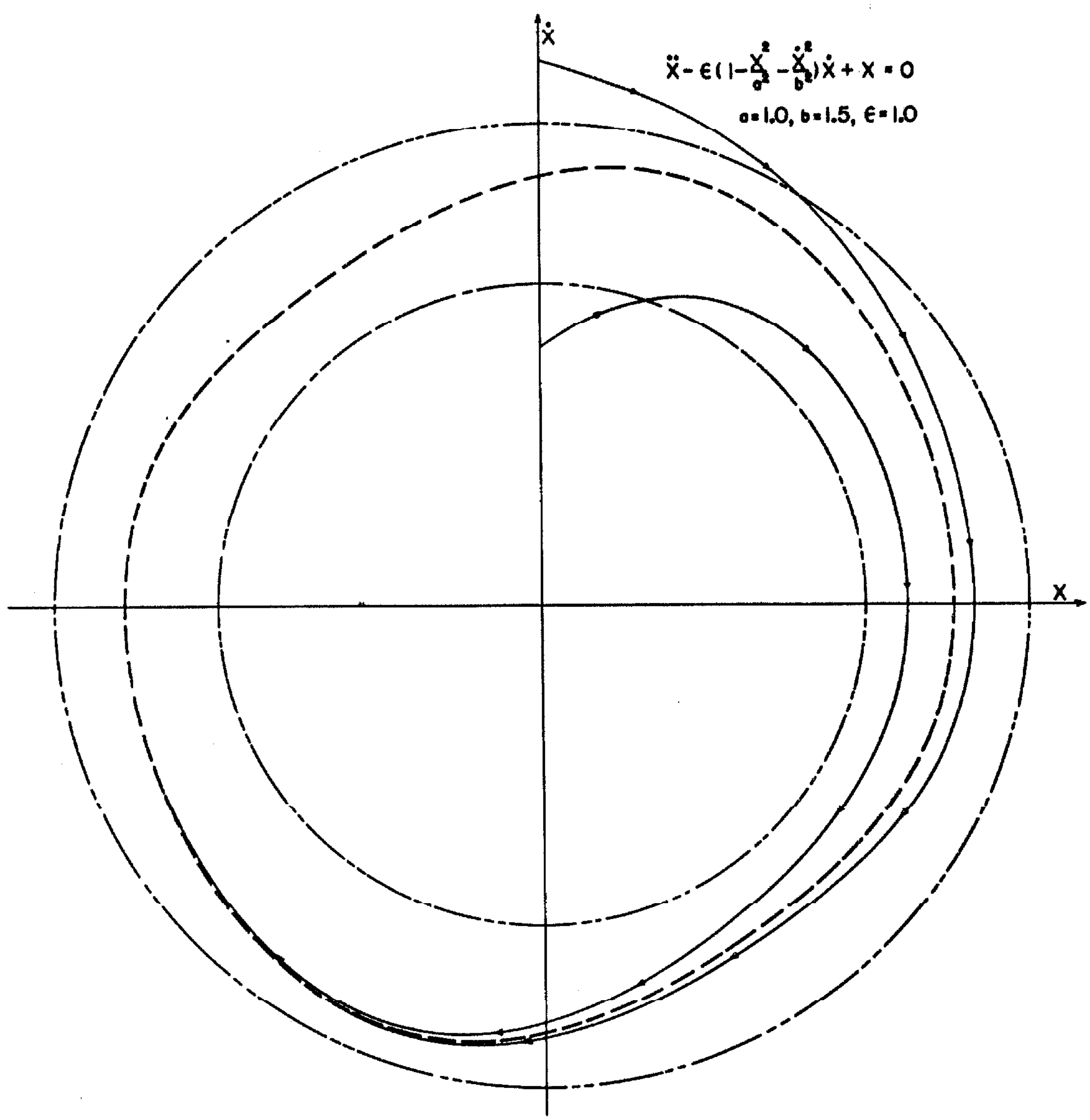
$$g(x, \dot{x}) = (a_1 x^2 + a_2 x^4 + \ldots + a_n x^{2n}) + (b_1 \dot{x}^2 + b_2 \dot{x}^4 + \ldots + b_m \dot{x}^{2m})$$

$$n, m = 1, 2, 3, \ldots$$

and

$$h(x) = C_1 x + C_2 x^3 + \ldots + C_r x^{2r-1} \qquad r = 1, 2, 3, \ldots$$

satisfy all the conditions for $g(x, \dot{x})$ and $h(x)$ of theorems 2.1 and 2.2 for the existence and uniqueness of a stable periodic solution. By the proper choice of the constants $\underline{a}$ and $\underline{b}$, $g(x, \dot{x}) = 1$ can be made to approximate a great number of four-point symmetric, simple,



$\dot{x}$
$\ddot{x} - \epsilon(1 - \frac{x^2}{a^2} - \frac{\dot{x}^2}{b^2})\dot{x} + x = 0$
$a = 1.0,\ b = 1.5,\ \epsilon = 1.0$
x

FIGURE 4

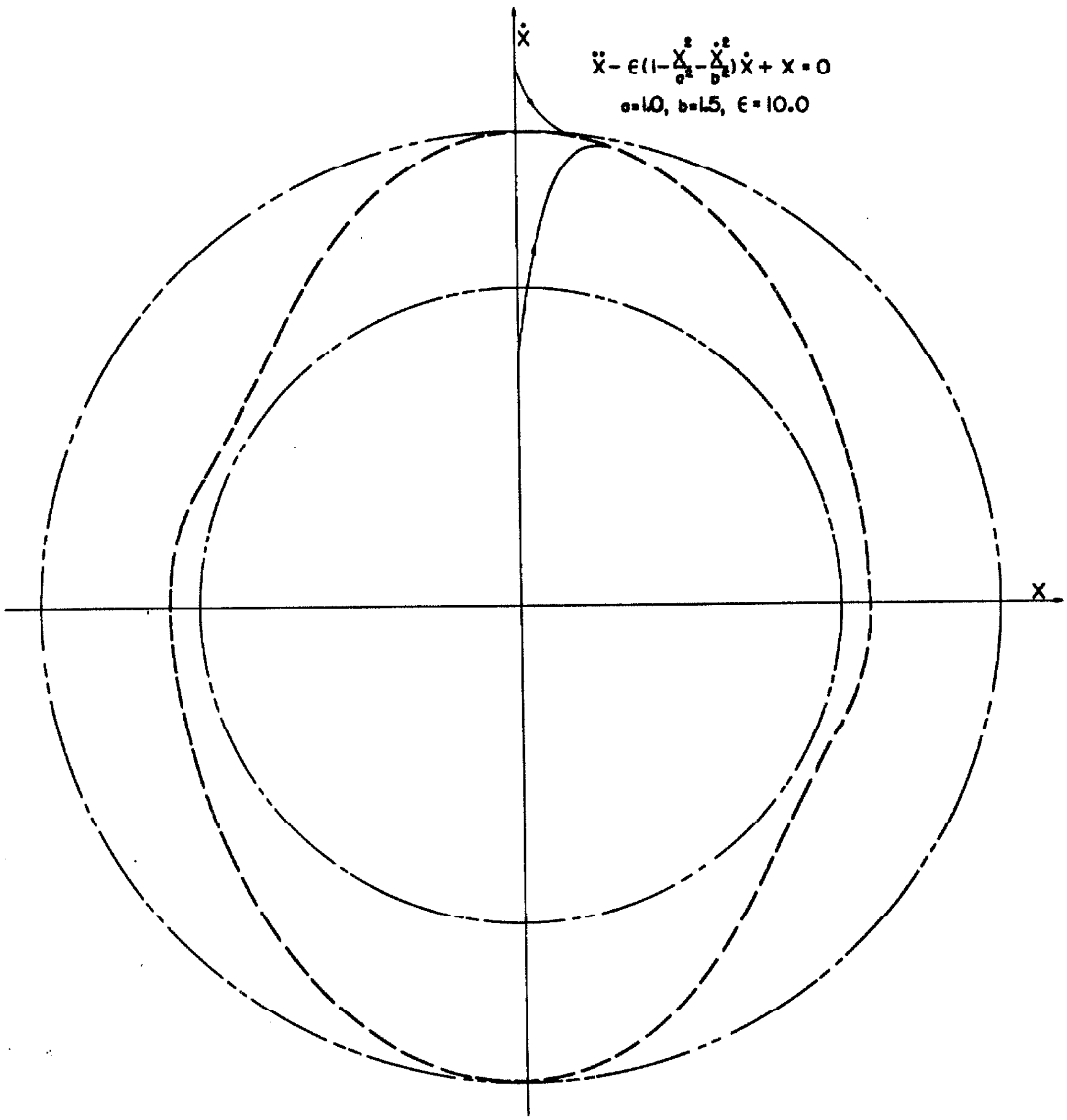

$\ddot{x} - \epsilon(1 - \frac{x^2}{a^2} - \frac{\dot{x}^2}{b^2})\dot{x} + x = 0$
a=1.0, b=1.5, ε=10.0
$\dot{x}$
x

FIGURE 5

closed, convex curves; and by a proper choice of the constants C, h(x) can be made to approximate a great number of odd, monotone increasing functions.

(ii) As an example of case (ii), where the points of contact between $V(x,\dot{x}) = C_2$ and $g(x,\dot{x}) = 1$ are on the $\dot{x}$-axis, consider the equation

$$\ddot{y} - \epsilon\left[1 - \frac{y^2}{a^2} - \frac{\dot{y}^2}{3}\right]\dot{y} + y = 0, \quad a^2 > 3 \tag{2.12}$$

Here Ω_1 is bounded by the ellipse $\frac{y^2}{a^2} + \frac{\dot{y}^2}{3} = 1$, Ω_2 is bounded by the inscribing circle $y^2 + \dot{y}^2 = 3$, and Ω_3 is bounded by the circumscribing circle $y^2 + \dot{y}^2 = a^2$. Again, it can easily be verified that $g(y,\dot{y})$ and $h(y)$ satisfy all the conditions for the existence and uniqueness of a stable periodic solution of equation (2.12).

It is interesting to note that in the limit as $\underline{a} \to \infty$, equation (2.12) becomes the Rayleigh equation

$$\ddot{y} - \epsilon\left(1 - \frac{\dot{y}^2}{3}\right)\dot{y} + y = 0 \tag{2.13}$$

The Rayleigh equation itself does not belong to the class of equations discussed in the theorems because $g(y,\dot{y}) = 1$ is not a closed curve. Thus, on the basis of the theory presented, we cannot strictly say that equation (2.13) has a unique and stable periodic solution. However, a unique and stable periodic solution exists for $\underline{a}$ large but bounded and, physically, if $\underline{a}$ is large enough, it will be impossible to distinguish between the motion for $\underline{a}$ large but bounded compared to the motion for $\underline{a}$ large and unbounded. For this reason, we may say that equation (2.13) has a unique and stable periodic solution as the limiting case of equation (2.12) for $\underline{a} \to \infty$.

Further, if we set $\dot{y} = x$ and $\ddot{y} = \dot{x}$ in equation (2.13) and differentiate with respect to time, we obtain the van der Pol equation

$$\ddot{x} - \epsilon(1 - x^2)\dot{x} + x = 0. \tag{2.14}$$

Since y, $\dot{y} = x$ and $\ddot{y} = \dot{x}$ are uniquely defined on the limit cycle of equation (2.13), $\ddot{x}$ is also uniquely defined by the use of equation (2.14). Hence van der Pol's equation also has a unique and stable periodic solution.

The existence and uniqueness of a stable periodic solution for the Rayleigh and van der Pol equations has been proved directly(23).

A Special Class of Problems

Consider the class of problems where

$$g(x, \dot{x}) = m[V(x, \dot{x})] \tag{2.15}$$

where m is a positive real constant. For convenience, take $m = 2$. Then if $V(x, \dot{x}) = \frac{1}{2}$, $g(x, \dot{x}) = 1$ which is the boundary of Ω_1. Hence for the class of problems where equation (2.15) holds, both the inscribing and circumscribing curves $V(x, \dot{x}) = C_2$ and $V(x, \dot{x}) = C_3$ are simply the boundary of Ω_1, $g(x, \dot{x}) = 2V(x, \dot{x}) = 1$. Hence, the exact solution of

$$\ddot{x} - \epsilon[1 - 2V(x, \dot{x})]\dot{x} + h(x) = 0 \tag{2.16}$$

is given by solving

$$V(x, \dot{x}) = \frac{1}{2}\dot{x}^2 + \int_0^x h(\xi)\, d\xi = \frac{1}{2} \tag{2.17}$$

As an example, suppose $h(x) = x$, then

$$V(x, \dot{x}) = \frac{1}{2}(\dot{x}^2 + x^2) = \frac{1}{2}$$

and it is readily verified that

$$x = \sin(t + \varphi) \tag{2.18}$$

is an exact solution by direct substitution into equation (2.14). The limit cycle is simply the circle

$$x^2 + \dot{x}^2 = 1 .$$

To get an idea of the transient response, trajectories converging to this limit cycle are shown in Figures 6 and 7 for $\epsilon = 0.1$ and $\epsilon = 10$ respectively. The trajectories were drawn using Liénard's graphical construction. It is observed that the trajectories converge slowly to the limit cycle for $\epsilon = 0.1$, but rapidly for $\epsilon = 10$.

It is interesting to note that J. Gibson[9] gives the equation

$$\ddot{x} - \delta(1 - x^2 - \dot{x}^2)\dot{x} + \gamma x = 0$$

as representing a number of control systems where the nonlinearity is frequency dependent, and works it out as an example.

The stability of this exact solution (2.18) of the differential equation (2.16) can be checked by Lyapounov's Second or Direct

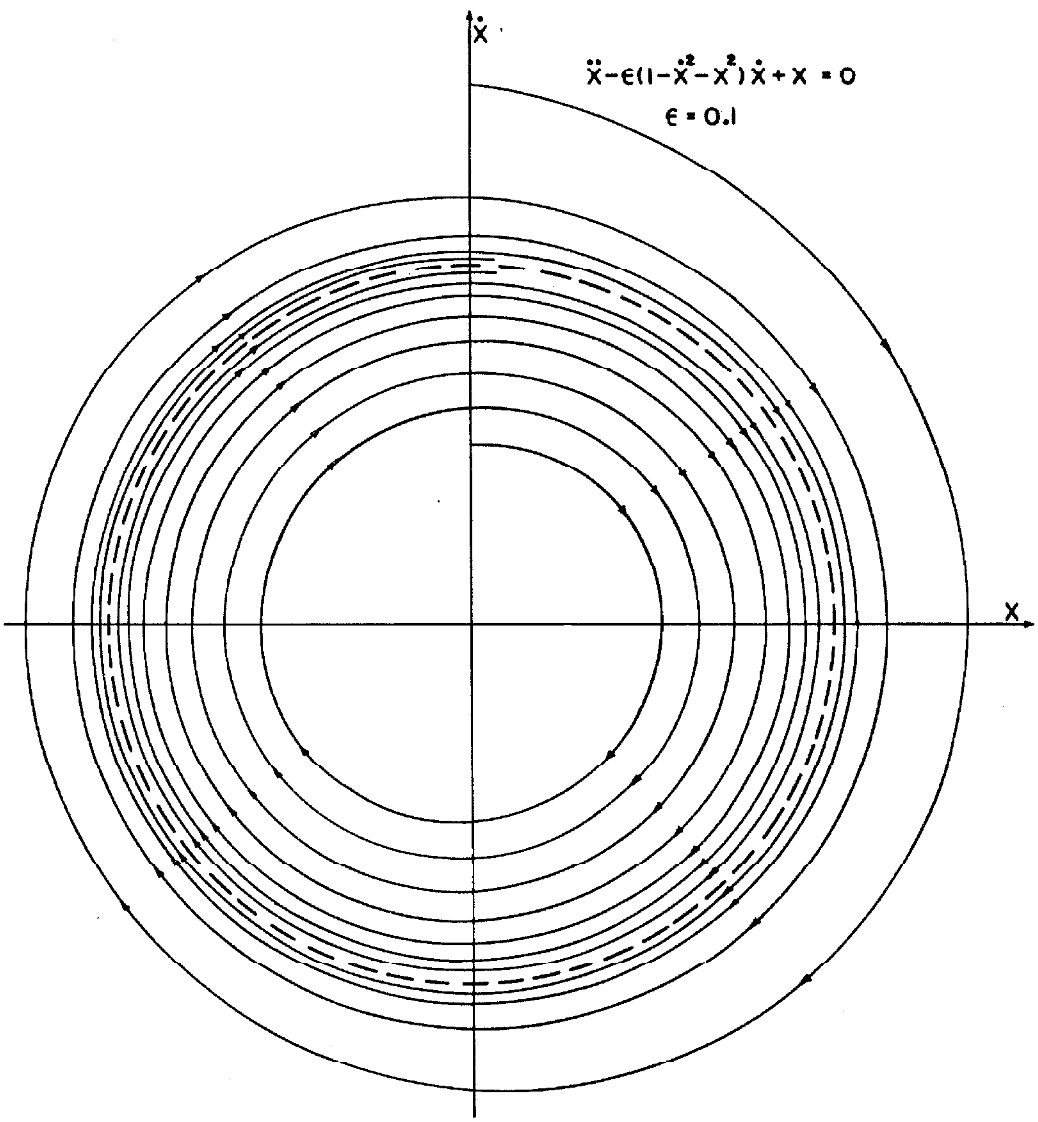


FIGURE 6

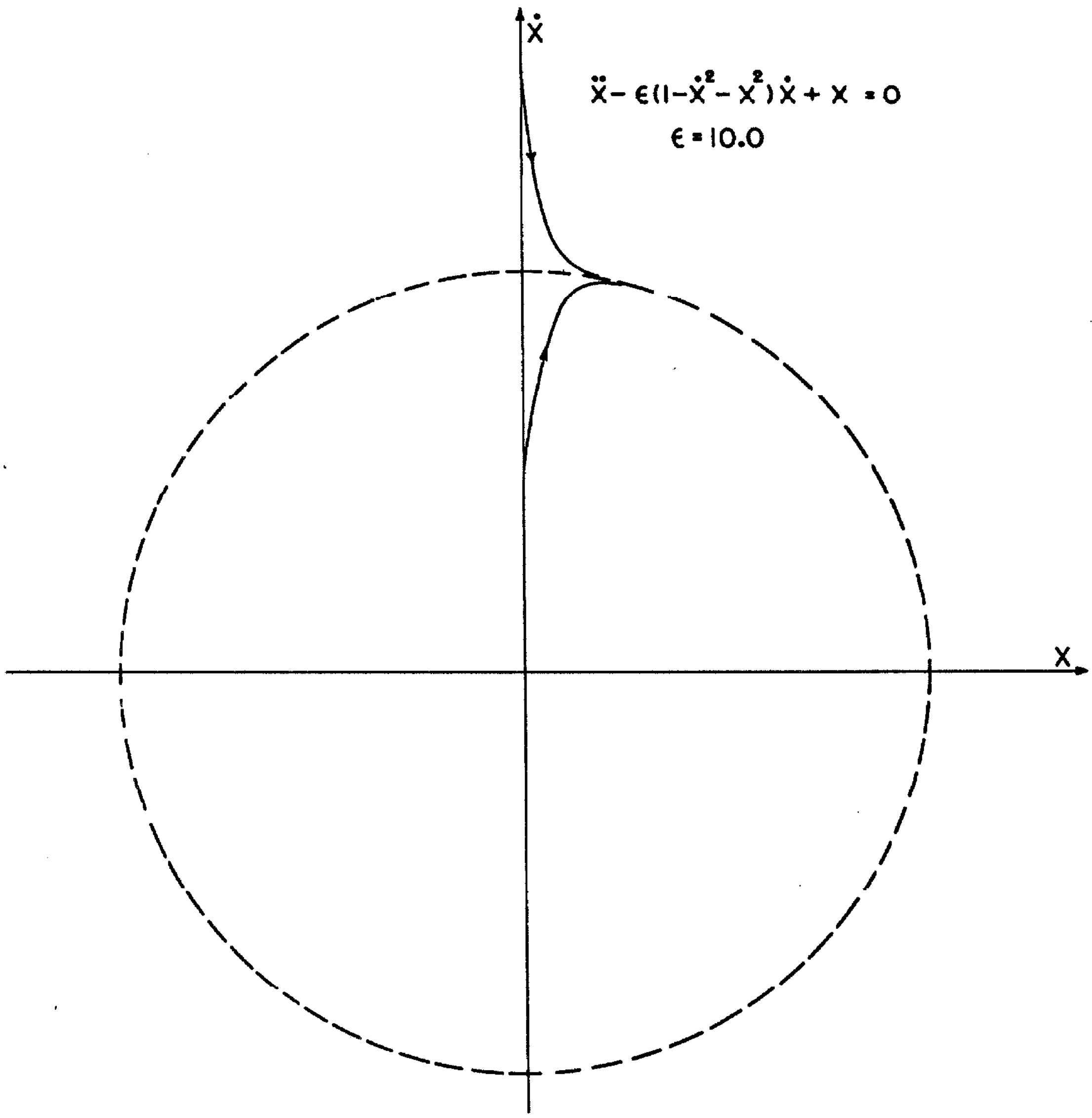


FIGURE 7

Method. A very readable exposition of this method is given by LaSalle and Lefschetz[37].

Since

$$V(x, \dot{x}) = \frac{1}{2}\dot{x}^2 + \int_0^x h(\xi)\, d\xi \ ,$$

therefore

$$h(x) = \frac{\partial V}{\partial x}$$

and equation (2.14) may be rewritten as

$$\ddot{x} - \epsilon[1 - 2V(x, \dot{x})]\,\dot{x} + \frac{\partial V}{\partial x} = 0 \tag{2.19}$$

Suppose $u(t)$ is the solution of (2.17). Perturb the system slightly and let

$$x(t) = u(t) + \eta \tag{2.20}$$

Substituting (2.20) into equation (2.19):

$$\ddot{u} + \ddot{\eta} - \epsilon\left[1 - 2\left\{V(u, \dot{u}) + \frac{\partial V}{\partial x}\eta + \frac{\partial V}{\partial \dot{u}}\dot{\eta} + \ldots\right\}\right](\dot{u} + \dot{\eta})$$
$$+ \left\{\frac{\partial V}{\partial u} + \frac{\partial^2 V}{\partial u^2}\eta + \ldots\right\} = 0$$

Regrouping terms:

$$\ddot{u} - \epsilon[1 - 2V(u, \dot{u})]\dot{u} + \frac{\partial V}{\partial u} + \ddot{\eta} + 2\epsilon\left(\frac{\partial V}{\partial u}\eta + \frac{\partial V}{\partial \dot{u}}\dot{\eta}\right)\dot{u}$$
$$- \epsilon[1 - 2V(u, \dot{u})]\,\dot{\eta} + \frac{\partial^2 V}{\partial u^2}\eta + (\textit{higher order terms of } \eta, \dot{\eta}) = 0$$

Applying (2.19) and noting that $[1 - 2V(u, \dot{u})] = O(\eta)$ since

$1 - 2V(x, \dot{x}) = 0$ is a solution, we get

$$\ddot{\eta} + 2\epsilon\left(\dot{u}\frac{\partial V}{\partial \dot{u}}\right)\dot{\eta} + \left[\frac{\partial^2 V}{\partial u^2} + 2\epsilon\frac{\partial V}{\partial u}\dot{u}\right]\eta = 0 \qquad (2.21)$$

if we neglect higher order terms of η, $\dot{\eta}$. Let

$$V^* = \frac{1}{2}\left[\frac{\partial V}{\partial \dot{u}}\dot{\eta} + \frac{\partial V}{\partial u}\eta\right]^2$$

Then $V^* > 0$ provided either $\dot{\eta} \neq 0$ or $\eta \neq 0$. The possible exceptional case $\frac{\partial V}{\partial \dot{u}}\dot{\eta} + \frac{\partial V}{\partial u}\eta = 0$ is ruled out since this corresponds to a point on the limit cycle.

Differentiating V^*:

$$\frac{dV^*}{dt} = \left[\frac{\partial V}{\partial \dot{u}}\dot{\eta} + \frac{\partial V}{\partial u}\eta\right]\frac{d}{dt}\left[\frac{\partial V}{\partial \dot{u}}\dot{\eta} + \frac{\partial V}{\partial u}\eta\right]$$

But

$$\frac{\partial V}{\partial \dot{u}} = \frac{\partial}{\partial \dot{u}}\left[\frac{1}{2}\dot{u}^2 + \int_0^u h(\xi)\, d\xi\right] = \dot{u}$$

$$\therefore \quad \dot{V}^* = \left[\frac{\partial V}{\partial \dot{u}}\dot{\eta} + \frac{\partial V}{\partial u}\eta\right]\left\{\dot{u}\ddot{\eta} + \ddot{u}\dot{\eta} + \frac{\partial V}{\partial u}\dot{\eta} + \left[\frac{\partial^2 V}{\partial u^2}\dot{u} + \underbrace{\frac{\partial^2 V}{\partial u \partial \dot{u}}}_{=0}\ddot{u}\right]\eta\right\}$$

or

$$\dot{V}^* = \left[\frac{\partial V}{\partial \dot{u}}\dot{\eta} + \frac{\partial V}{\partial u}\eta\right]\left[\dot{u}\ddot{\eta} + \ddot{u}\dot{\eta} + \frac{\partial V}{\partial u}\dot{\eta} + \frac{\partial^2 V}{\partial u^2}\dot{u}\eta\right].$$

Along a trajectory of the differential equation, substitute for $\ddot{u}$ and $\ddot{\eta}$ using equations (2.19) and (2.21) respectively.

$$\therefore \quad \dot{V}^* = -2\epsilon\dot{u}^2\left[\frac{\partial V}{\partial \dot{u}} + \frac{\partial V}{\partial u}\right]\eta$$

$$\dot{V}^* = -4\epsilon\dot{u}^2 V^* \geqq 0$$

V^* satisfies all the conditions for a Lypounov function, and integrating $\dot{V}^*$ we get

$$\dot{V}^* = V_o^* \exp\left[-4\epsilon\int \dot{u}^2\, dt\right]$$

and $V^* \to 0$ as $t \to \infty$. Hence either both η and $\dot{\eta} \to 0$ or $\left[\frac{\partial V}{\partial u}\eta + \frac{\partial V}{\partial \dot{u}}\dot{\eta}\right] \to 0$. In either case, the result is that the perturbed motion tends to the limit cycle as $t \to \infty$. Hence equation (2.16) has the stable limit cycle $V(x, \dot{x}) = \frac{1}{2}$.

Existence of Periodic Solutions in the Non-Autonomous System

We now turn our attention to the non-autonomous differential equation

$$\ddot{x} + \epsilon[g(x, \dot{x}) - 1]\dot{x} + h(x) = e(t) \tag{2.22}$$

or, as is more convenient for our purposes, the equivalent system

$$\left.\begin{aligned} \frac{dx}{dt} &= \dot{x} \\ \frac{d\dot{x}}{dt} &= -\epsilon[g(x, \dot{x}) - 1]\dot{x} - h(x) + e(t) \end{aligned}\right\} \tag{2.23}$$

The difficulties involved in treating this problem in the phase-plane have already been pointed out: e.g., the trajectories may intersect, etc. However, some progress can be made by use of a theorem from topology, the Brouwer Fixed Point Theorem. But as a preliminary step, a closed curve C must be constructed in the $(x, \dot{x})$-plane, enclosing the origin and having the property that every solution $x = x(t)$, $\dot{x} = \dot{x}(t)$ crossing C passes from the exterior of C to its interior. The devices on which such curves C are constructed vary widely from author to author, and here we follow the procedure of N. Levinson[25]. Also see Langenhop[38].

<u>Lemma.</u> Let $g(x, \dot{x}) = 1$ be a simple closed curve surrounding the origin, and $h(x)$ an odd, monotone increasing function for which $xh(x) > 0$ for $x \neq 0$, and $h(0) = 0$. Suppose the partial derivatives $\frac{\partial g}{\partial x}$, $\frac{\partial g}{\partial \dot{x}}$, and $\frac{\partial h}{\partial x}$ exist and are continuous and that $x\frac{\partial g}{\partial x} > 0$ for $x \neq 0$ and $\dot{x}\frac{\partial g}{\partial \dot{x}} > 0$ for $\dot{x} \neq 0$. Suppose $e(t)$ is a bounded continuous function and that $\epsilon > 0$. Then there exists a simple closed curve C in the $(x, \dot{x})$-plane such that a solution of equation (2.22), $[x(t), \dot{x}(t)]$, that crosses this curve passes from the domain exterior to the curve to the domain interior to the curve. Further, through any point in the phase-plane sufficiently remote from the origin, there passes a curve with this property.

Proof of Lemma: It is clear from the hypothesis that there exists a simple closed curve $g(x, \dot{x}) = \text{constant} = 1 + \frac{m}{\epsilon}$, $m > 0$ which encloses $g(x, \dot{x}) = 1$. Curve C will completely enclose the curve $g(x, \dot{x}) = 1 + \frac{m}{\epsilon}$. In the domain outside $g(x, \dot{x}) = 1 + \frac{m}{\epsilon}$:

$$\epsilon[g(x, \dot{x}) - 1] > m > 0 \qquad (2.24)$$

Also, from the assumptions, there exists an M such that

$$\epsilon[g(x, \dot{x}) - 1] > -M \text{ inside } g(x, \dot{x}) = 1 + \frac{m}{\epsilon} \qquad (2.25)$$

Denote the maximum of $|e(t)|$ by E, and select a constant $\underline{a}$ such that

$$ma > 2E \tag{2.26}$$

Again, it will be useful to introduce the "energy" associated with the motion of (2.22):

$$V(x, \dot{x}) = \frac{1}{2}\dot{x}^2 + H(x), \tag{2.27}$$

where

$$H(x) = \int_0^x h(\xi)\, d\xi\ . \tag{2.28}$$

Differentiating $V(x, \dot{x})$ with respect to time

$$\dot{V}(x, \dot{x}) = \dot{x}\ddot{x} + \dot{x}\, h(x)$$

and along a trajectory of (2.22)

$$\dot{V} = -\epsilon[\, g(x, \dot{x}) - 1]\,\dot{x}^2 + e(t)\,\dot{x} \tag{2.29}$$

Now proceed to construct the closed curve C, shown as $P_1P_2P_3 \ldots P_8P_9P_1$ in Figure 8.

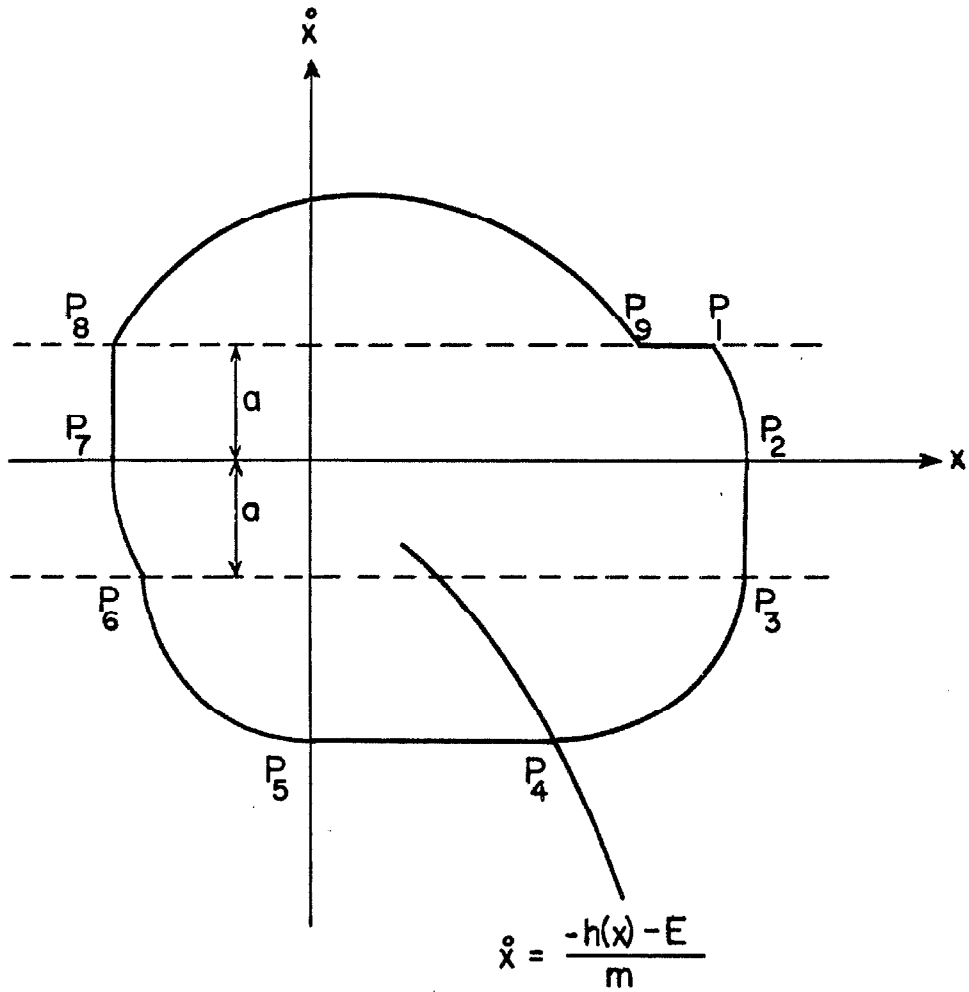


FIGURE 8

Point P_4 is located first. This point lies on the curve

$$\dot{x} = \frac{-h(x) - E}{m} \tag{2.30}$$

sufficiently far out so that if $P_4 = (x_4, \dot{x}_4)$, then for $x \geqq x_4$, $h(x) >> 0$ and $H(x) >> h(x)$. That such a point will exist is clear from the hypothesis. More precisely, choose $x_4 > x_o$ where x_o is sufficiently large so that

$$h(x) = - h(-x) > 2(Ma + E) \tag{2.31}$$

and

$$H(x) = \int_0^x h(\xi)\, d\xi > 2a^2 \tag{2.32}$$

That portion of the curve C consisting of P_3P_4 is determined by the equation

$$V(x, \dot{x}) = \text{constant} = V(x_4, \dot{x}_4) \tag{2.33}$$

Point P_3 is chosen so that $\dot{x}_3 = -a$.

Along any trajectory of (2.22) recall that

$$\dot{V}(x, \dot{x}) = -\epsilon[g(x, \dot{x}) - 1]\dot{x}^2 + e(t)\dot{x}$$

but since P_3P_4 lies outside $g(x, \dot{x}) = 1 + \frac{m}{\epsilon}$, by (2.24)

$$\dot{V}(x, \dot{x}) < -m\dot{x}^2 + E|\dot{x}| = -m\dot{x}^2 \left[1 - \frac{E}{m|\dot{x}|}\right].$$

Since $\dot{x} \geqq a$ on P_3P_4:

$$\dot{V}(x, \dot{x}) < -m\dot{x}^2 \left[1 - \frac{E}{ma}\right]$$

and by (2.26), $E/ma < \frac{1}{2}$, hence

$$\dot{V}(x, \dot{x}) < -\frac{1}{2} m\dot{x}^2 < 0 \tag{2.34}$$

for any solution of (2.22) which intersects the arc P_3P_4. Hence $V(x, \dot{x})$ decreases along a solution that cuts the arc P_3P_4. But $V(x, \dot{x})$ is constant along P_3P_4. Thus a solution of (2.22) which cuts P_3P_4 can only pass from the domain exterior to C into the domain interior to C.

P_2P_3 is the vertical line segment $x = x_3$, $-a \leqq \dot{x} \leqq 0$. Now $\frac{dx}{dt} = \dot{x} < 0$ since P_2P_3 lies below the x-axis. Hence x must decrease along any trajectory intersecting P_2P_3, that is, any solution of (2. 22) which cuts P_2P_3 must go from right to left, from the domain exterior to C into the domain interior to C.

That portion of C made up of P_1P_2 consists of that part of the curve

$$\frac{1}{2}\dot{x}^2 + H(x) - (Ma + E)x = H(x_2) - (Ma + E)x_2 \qquad (2.35)$$

which starts at P_2 and for which $0 \leq \dot{x} \leq a$. On P_1P_2, by differentiating (2. 35) with respect to x, we have

$$\frac{d\dot{x}}{dx} = - \frac{h(x) + Ma + E}{\dot{x}} \qquad (2.36)$$

By eliminating dt from (2. 23), we have for solutions of (2. 22)

$$\frac{d\dot{x}}{dx} = \frac{-\epsilon[g(x, \dot{x})-1]\dot{x}-h(x)+e(t)}{\dot{x}} < \frac{-h(x) + Ma + E}{\dot{x}} \qquad (2.37)$$

And, as an immediate consequence of (2. 31)

$$0 > - h(x) + Ma + E.$$

Thus (2. 36) and (2. 37) indicate that the slope of solutions of the solutions of (2. 22) is more negative on P_1P_2 than the slope of P_1P_2 itself. Thus the solutions of (2. 2) cut P_1P_2 from the exterior of C to its interior.

The determination of P_1P_4 is complete. Next, we turn to P_4P_5. P_4P_5 is the horizontal line segment $\dot{x} = \dot{x}_4$, $0 \leq x \leq x_4$.

Since P_4P_5 is that part of the phase-plane lying below the curve

$$\dot{x} = \frac{-h(x) - E}{m} ,$$

we have

$$0 < - m\dot{x} - h(x) - E . \tag{2.38}$$

Along solutions of (2. 22) outside $g(x, \dot{x}) = 1 + \frac{m}{\epsilon}$

$$\frac{d\dot{x}}{dt} = - h(x) - \epsilon[g(x, \dot{x}) - 1] \dot{x} + e(t) > -h(x) - m\dot{x} - E.$$

And by (2. 38) this means that $\frac{d\dot{x}}{dt} > 0$, that is, $\dot{x}$ increases along solutions of (2. 22) as they cross P_4P_5. So, again, the solutions of (2. 22) can cut P_4P_5 only by passing from the exterior of the curve C into its interior.

P_5P_6 is given by the curve $V(x, \dot{x})$ = constant = $V(x_5, \dot{x}_6)$ with $\dot{x}_6 = - a$. As in the case of P_3P_4, $\dot{V}(x, \dot{x}) < 0$ along solutions of (2. 22) in this region. Thus these solutions cut P_5P_6 from the exterior of curve C to its interior.

P_6P_7, much like P_1P_2, is that portion of the curve

$$\frac{1}{2} \dot{x}^2 + H(x) + (Ma+E)x = \frac{1}{2} a^2 + H(x_6) + (Ma+E)x_6$$

for which $-a \leq \dot{x} \leq 0$ and which starts at P_6. As in the case of P_1P_2 we make use of inequality (2. 31) here. If x_6 is not sufficiently large in magnitude for the inequality

$$-h(-x) > 2(ma + E)$$

to hold, $|x_6|$ can be increased by moving P_4 out so that the inequality does hold and $|x_6| > x_o$.

P_7P_8 is the vertical straight line segment $x = x_7$, $0 \leq \dot{x} \leq a$. Here $\frac{dx}{dt} = \dot{x} > 0$ so the solutions cross P_7P_8 from left to right, from outside C to inside it.

P_8P_9 is the curve $V(x, \dot{x}) = \text{constant} = V(x_8, \dot{x}_8)$ which starts at P_8 and goes to P_9 where $\dot{x}_9 = a$. Here again $\dot{V}(x, \dot{x}) < 0$ on the solutions cutting P_8P_9.

P_9P_1 is the horizontal straight line segment $\dot{x} = a$ between x_9 and x_1. Once it is demonstrated that $x_9 < x_1$ it is easy to show that the solutions of (2.22) cut P_9P_1 from the outside of C to its inside. This must be so because on P_9P_1 for solutions of (2.22):

$$\frac{d\dot{x}}{dt} = -\epsilon[g(x, \dot{x})-1]\dot{x} - h(x) + e(t) < -ma - h(x) + E.$$

And since $ma > 2E$ and $h(x) > 0$ for $x > 0$, we have $\frac{d\dot{x}}{dt} < 0$. Thus P_9P_1 is cut by solutions of (2.22) from top to bottom, from outside C to inside it.

To demonstrate that P_9 lies to the left of P_1, consider the changes in the "energy" $V(x, \dot{x})$ as curve C is traversed. Denote $V(x_n, \dot{x}_n)$ by V_n:

$$V_2 - V_1 = H(x_2) - H(x_1) - \frac{1}{2} a^2$$

and using the defining equation (2.31)

$$V_2 - V_1 = H(x_2) - H(x_1) - \frac{1}{2} a^2 = (Ma + E)(x_2 - x_1) \qquad (2.39)$$

By (2.39)

$$H(x_2) - H(x_1) = \int_{x_1}^{x_2} h(x)\,dx = (Ma+E)(x_2-x_1) + \frac{1}{2}a^2 \qquad (2.40)$$

But $x_1 > x_4 > x_o$ so by (2.31), $h(x) > 2(Ma + E)$ for $x \geqq x_1$. In (2.40) this gives

$$2(Ma + E)(x_2 - x_1) < (Ma + E)(x_2 - x_1) + \frac{1}{2}a^2,$$

or

$$(Ma + E)(x_2 - x_1) < \frac{1}{2}a^2.$$

Thus (2.39) becomes

$$V_2 - V_1 < \frac{1}{2}a^2 \qquad (2.41)$$

Along P_2P_3, $x = x_2 = x_3$, thus

$$V_3 - V_2 = H(x_3) + \frac{1}{2}a^2 - H(x_2) = \frac{1}{2}a^2 \qquad (2.42)$$

By definition

$$V_4 - V_3 = 0. \qquad (2.43)$$

Along P_4P_5, $\dot{x} = \dot{x}_4 = \dot{x}_5$, hencc

$$V_5 - V_4 = H(0) + \frac{1}{2}\dot{x}_5^2 - H(x_4) - \frac{1}{2}\dot{x}_4^2 = -H(x_4) \qquad (2.44)$$

And, much as before,

$$V_6 - V_5 = 0 \qquad (2.45)$$

$$V_7 - V_6 < \frac{1}{2}a^2 \qquad (2.46)$$

$$V_8 - V_7 = \frac{1}{2}a^2 \qquad (2.47)$$

$$V_9 - V_8 = 0 \qquad (2.48)$$

By (2.41) - (2.48), we have

$$V_9 - V_1 < - H(x_4) + 2a^2 \tag{2.49}$$

But in (2.32) it was required that $H(x) > 2a^2$ for $|x| > x_o$. If $x_9 \not> x_o$, take P_4 sufficiently far away to assure that $x_9 > x_o$. Hence

$$V_9 - V_1 < 0,$$

and since $\dot{x}_9 = \dot{x}_1 = a$, therefore $x_9 < x_1$. This completes the proof of the lemma.

We are now in a position to consider the existence of periodic solutions. The general approach is to try to find a solution which closes on itself. However, since the system is non-autonomous, it is clear that unless the forcing function is periodic, the solution may not close on itself again. So we assume $e(t)$ to be periodic with period L.

The Brouwer Fixed Point theorem will be applied. It is formulated, for example, in the books of Cesari(23) and Lefschetz(36). For reference, it is stated here: Let region R be a closed interval in the Euclidean x-space E_n, $x = (x_1, \ldots, x_n)$ with $a_i \leqq x_i \leqq b_i$, $i = 1, 2, \ldots, n$. Suppose that the transformation T, given by $y = f(x)$ where $x \in R$, is a continuous mapping of region R into itself, that is, if $f = (f_1, \ldots, f_n)$, $f_i = f_i(x_1, x_2, \ldots, x_n)$ and $a_i \leqq f_i(x_1, \ldots, x_n) \leqq b_i$, $i = 1, 2, \ldots, n$ for all $x \in R$, then there exists at least one point $x_o \in R$ (fixed point) such that $Tx_o = x_o$, that is, $f(x_o) = x_o$. This will now be applied to the following theorem on periodic solutions:

Theorem 2.3. Let $g(x, \dot{x}) = 1$ be a simple closed curve surrounding the origin, and $h(x)$ an odd monotone increasing function for which $xh(x) > 0$ if $x \neq 0$, and $h(0) = 0$. Suppose the partial derivatives $\frac{\partial g}{\partial x}$, $\frac{\partial g}{\partial \dot{x}}$, and $\frac{\partial h}{\partial x}$ exist and are continuous and that $x \frac{\partial g}{\partial x} > 0$ for $x \neq 0$ and $\dot{x} \frac{\partial g}{\partial \dot{x}} > 0$ for $\dot{x} \neq 0$. Suppose $e(t)$ is a bounded periodic and continuous function with period L, and that $\epsilon > 0$. Then equation (2.22) has at least one periodic solution with the same period as the period of the forcing term $e(t)$.

Proof: Let C be a closed curve around the origin of the $(x, \dot{x})$-plane with the property that every solution of (2.2) which crosses it, goes from the domain outside C into the domain inside C. The existence of such a curve C is assured by the Lemma we proved. Let the region R be the 2-cell of the $(x, \dot{x})$-plane enclosed by C. For every point $P_o = (x_o, \dot{x}_o)$ of R, consider the solution $[x(t), \dot{x}(t)]$ of (2.22) with $x(0) = x_o$, $\dot{x}(0) = \dot{x}_o$, and $P_1 = (x_1, \dot{x}_1)$ in the $(x, \dot{x})$-plane defined by $x_1 = x(L)$ and $\dot{x}_1 = \dot{x}(L)$. P_1 must also be in R as no solutions of (2.22) go from inside R into the domain outside R. Then the transformation T which maps $P_o \in R$ into P_1 is defined in R, is continuous in R, and $TP_o \in R$ for every $P_o \in R$; that is, $T(R) < R$. By the Brouwer theorem we conclude that there exists at least one point $P = (\underline{x}, \underline{\dot{x}})$ in R such that for the corresponding solution $x(t)$, $\dot{x}(t)$ of (2.22) we have

$$x(L) = x(0) = \underline{x}, \quad \dot{x}(L) = \dot{x}(0) = \underline{\dot{x}}$$

and since we also have $e(L) = e(0)$, the solution $[x(t), \dot{x}(t)]$ is periodic with period L. This completes the proof.

It should be observed that in theorem 2.3 the periodic solution need not be stable or unique. Moreover, L need not be the minimum period either of $e(t)$ or of the periodic solutions. Hence, there exists the possibility of sub or super-harmonic solutions of (2.22).

<u>Ultimate Boundedness of Forced Oscillations</u>

Whether a periodic solution exists or not, it is of great engineering interest to know the eventual behavior of the solutions in the forced case. Consider the system of equations of the form

$$\{\dot{x}\} = \{X(x, t)\} , \qquad t \geqq 0 .$$

Suppose $\{x(t)\}$ is a solution such that $\{x(t_o)\} = x_o$. Then, either the solution may be extended for all $t \geqq t_o$ in which case the solution $\{x(t)\}$ is said to be "defined in the future," or there is a time $T \geqq t_o$ such that $||x(t)|| \to +\infty$ as $t \to T$ in which case the solution $\{x(t)\}$ is said to have a "finite escape time." These two possibilities are mutually exclusive. LaSalle and Lefschetz have given some theorems which give conditions for solutions to be defined in the future or to have finite escape times[37]. We state them below without proof.

<u>Theorem</u>: Let Ω be a bounded set containing the origin and let $V(x, t)$ be defined throughout $\overline{\Omega}$ and for all $t \geqq 0$. Moreover, let $V(x, t) \to +\infty$ as $||x|| \to +\infty$ and this uniformly on every finite interval $0 \leqq a \leqq t < b$. Furthermore, let $\dot{V} \leqq G(V, t)$ hold throughout $\overline{\Omega}$ and

for all $t \geqq 0$. If

$$\dot{v} \leqq G(v, t), \quad t \geqq 0 \tag{2.50}$$

has no positive solution with finite escape time then every solution $x(t)$ of

$$\dot{x} = X(x, t), \quad t \geqq 0 \tag{2.51}$$

is defined in the future. Here, x, X, v, V and G are all vector quantities.

The next theorem gives conditions for a finite escape time and may be considered as an instability result.

<u>Theorem</u>: Let Ω be a region such that if a solution $x(t)$ starts in Ω it remains thereafter in Ω. Let $V(x, t)$ be positive for all x in Ω and all $t \geqq 0$. Suppose $\dot{V} \geqq G(v, t)$ holds for all $t \geqq 0$ and all x in Ω. If

$$\dot{v} \geqq G(v, t), \quad t \geqq 0 \tag{2.52}$$

has no positive solution defined in the future, then each solution $x(t)$ of (2.51) with $x(t_o) = x_o$ has a finite escape time. Again, x, X, v, V and G are vector quantities.

These theorems will now be applied to the differential equation (2.22):

$$\ddot{x} - \epsilon[1 - g(x, \dot{x})]\dot{x} + h(x) = e(t)$$

The functions $g(x, \dot{x})$ and $h(x)$ satisfy the conditions of theorem 2.3, and let $e(t)$ be continuous for all $t \geqq 0$. The above equation is equivalent to the system

$$\left.\begin{aligned} \dot{x} &= y \\ \dot{y} &= \epsilon[1 - g(x, y)]\, y - h(x) + e(t) \end{aligned}\right\} \tag{2.53}$$

Let

$$V(x, y) = \frac{1}{2} y^2 + \int_0^x h(u)\, du \tag{2.54}$$

Take the set Ω of the first theorem to be the set $g(x, x) < 1$.

$$\dot{V}(x, y) = \epsilon[1 - g(x, y)]\, y^2 + e(t) y \tag{2.55}$$

and outside this region Ω

$$\dot{V} \leqq |e(t)|\, |y| \leqq \sqrt{2}\, |e(t)|\, V^{1/2} .$$

Thus $\dot{V} \leq G(v, t) = k(t) L(v)$ for $g(x, \dot{x}) \geqq 1$ and for all $t \geqq 0$. Here $k(t) = \sqrt{2}\, |e(t)|$ and $L(v) = v^{1/2}$. The inequality

$$\dot{v} \leqq k(t) L(v)$$

or

$$\frac{\dot{v}}{L(v)} \leqq k(t)$$

or

$$\int_{v(t_o)}^{v(t)} \frac{dv}{L(v)} \leqq \int_{t_o}^{t} k(t)\, dt \qquad t_o \leqq t < T$$

has no positive solution with finite escape time T because

$$\int^{+\infty} \frac{dv}{L(v)} = \int^{+\infty} \frac{dv}{v^{1/2}} = +\infty$$

(the integral does not converge), so the right-hand side is bounded as $t \to T$ while the left-hand side would approach $+\infty$ if there were a positive solution with finite escape time T. Hence every solution of the system (2.53) is defined in the future.

The information that a solution is defined in the future might be sufficient in astronomical problems where the orbits of the planets are only computed for one or two orbits at one time. But in many engineering problems, for example in some resonating circuits or control systems, oscillations occur at high frequencies and "long term" behavior is manifested in a few seconds. Then it is important to know if the solutions are "ultimately bounded."

First the term "ultimately bounded" will be defined and then some results on ultimate boundedness obtained by T. Yoshizawa[32] will be quoted and applied to an example which is a sub-class of the oscillators we are considering in this thesis.

Definition: We say that the system of differential equations (2.51) is ultimately bounded if there is a $b > 0$ such that, corresponding to each solution $x(t)$ of (2.51), there is a $T > 0$ with the property that $||x(t)|| < b$ for all $t > T$.

Theorem: Let $V(x)$ be a scalar function which for all x has continuous first partial derivatives with the property that $V(x) \to \infty$ as $||x|| \to \infty$. If

$$\dot{V}(x) \leqq -\eta < 0$$

for all x outside some closed bounded set M, then (2.51)

$$\dot{x} = X(x, t), \qquad t \geqq 0$$

is ultimately bounded.

We now apply this result to the self-excited oscillators governed by

$$\ddot{x} - \epsilon[1 - \dot{x}^2 - k(x)]\dot{x} + h(x) = e(t), \quad t \geqq 0 \tag{2.56}$$

where $k(x)$ and $h(x)$ have continuous first derivatives; $h(x)$ is odd and monotone increasing, $xh(x) > 0$ for $x \neq 0$, $h(0) = 0$ and $\epsilon > 0$. Suppose $k(x) = k(-x) > 0$, and $e(t)$ is a bounded continuous function with $|e(t)| \leqq E$.

Dr. Caughey[47] suggested the function

$$V = \frac{1}{2}\dot{x}^2 + \frac{1}{2}(x + \dot{x})^2 - \frac{(1+\epsilon)}{2}x^2 + \epsilon\int_0^x uk(u)\,du + 2\int_0^x h(u)\,du + C$$

where constant C is selected to assure that $V > 0$ for all $(x, \dot{x})$. Differentiating V:

$$\dot{V} = \dot{x}\ddot{x} + (\dot{x} + \ddot{x})(x + \dot{x}) - (1+\epsilon)x\dot{x} + \epsilon\dot{x}xk(x) + 2h(x)\dot{x}$$

$$\dot{V} = 2\dot{x}\ddot{x} + \dot{x}^2 + x\ddot{x} - \epsilon x\dot{x} + \epsilon x\dot{x}k(x) + 2h(x)\dot{x}\,. \tag{2.57}$$

But along a trajectory of (2.56)

$$\ddot{x} = \epsilon[1 - \dot{x}^2 - k(x)]\dot{x} - h(x) + e(t),$$

so

$$\dot{V} = 2\epsilon[1 - \dot{x}^2 - k(x)]\,\dot{x}^2 - 2h(x)\,\dot{x} + 2e(t)\,\dot{x} + \dot{x}^2$$
$$+ \epsilon[1 - \dot{x}^2 - k(x)]\,x\dot{x} - xh(x) + xe(t) - \epsilon x\dot{x}$$
$$+ \epsilon x\dot{x}k(x) + 2h(x)\,\dot{x}$$
$$\leqq - 2\epsilon[\dot{x}^2 + k(x) - 1]\,\dot{x}^2 + \frac{\epsilon}{2}\dot{x}^2(x^2 + \dot{x}^2) + \dot{x}^2$$
$$- xh(x) + |x|E + 2|\dot{x}|E$$
$$\leqq -\epsilon[\frac{3}{2}\dot{x}^2 + 2k(x) - \frac{x^2}{e} - (2\epsilon+1)]\,\dot{x}^2 - xh(x) + |x|E + 2|\dot{x}|E.$$

Hence, if $k(x) > \frac{x^2}{4}$ then there exists a closed bounded set M such that in $\overline{M}$, $V \leqq -\eta < 0$. So the system is ultimately bounded.

For completeness, we quote another of Yoshizawa's results on ultimate boundedness of solutions which uses a non-autonomous function $V(x, t)$:

Lemma 1. $V(x, t)$ is a scalar function with continuous first partials for all x and all $t \geqq 0$, and M is a closed set in n-space. For any positive number $\underline{r}$, let M_r denote the set of all points whose distance from M is less than $\underline{r}$. Thus x in M_r means that for some point y in M, $||x - y|| < r$. If $\dot{V}(x, t) \leqq 0$ for all x in $\overline{M}$ and if $V(x_1, t_1) < V(x_2, t_2)$ for all $t_2 \geqq t_1 \geqq 0$, all x_1 in M and all x_2 in $\overline{M}_r$, then each solution of (2.51) which at some time $t_o \geqq 0$ as in M can never thereafter leave M_r.

Lemma 2. If, in addition to the conditions of lemma 1, $V(x, t) \geqq 0$ and $\dot{V}(x, t) \leqq -\eta < 0$ for all $t \geqq 0$ and all x in $\overline{M}$, then each solution of (2.51) that is defined in the future is ultimately inside M_r.

Theorem: If in addition to the conditions of lemma 2, the set M is bounded and $V(x, t) \to \infty$ uniformly for $t > 0$ as $||x|| \to \infty$, then the system (2.51) is ultimately bounded.

The above theorem suffers from the fact that there are a large number of conditions to be verified. But if the simpler theorem does not give the results required, this more general theorem may be useful.

These methods concerning existence of solutions in the future and ultimate boundedness are like Lyapounov's method. Lyapounov's theorem draw conclusions about stability from the inequality $\pm \dot{V} \leqq 0$, where V is taken positive. The methods just described consider the more sophisticated inequality $\pm \dot{v} \leqq G(v, t)$ from which interesting conclusions are drawn. For this reason, these methods are said to be extensions of Lyapounov's Method. Mathematicians have taken great interest in such methods recently and many results and references on this topic are given in the monograph by W. Hahn[48].

III. ANALYTICAL METHODS

In the last part we considered the conditions under which the class of oscillators we are studying have unique and stable periodic solutions for free oscillations and periodic or bounded solutions for forced oscillations. In this part we consider the quantitative properties of these solutions. Unfortunately, no quantitative information can be obtained for the general class of oscillators we are considering. The functions $g(x, \dot{x})$, $h(x)$ and $e(t)$ have to be explicitly specified. In other words, quantitative information can only be obtained for particular examples. These examples will give some idea of the behavior of the solutions, even though a general solution is not available. These methods give approximate solutions for limited regions of validity. Both free and forced oscillations will be studied. We begin by obtaining the perturbation solution for small ϵ in the autonomous case.

Perturbation Solution in Autonomous Case

The perturbation method, which is valid for small ϵ, is due to Poincaré[39] who applied the method to problems in astronomy. A mathematical justification of this method may be found in Stoker's book[35].

The specific equation that will be considered here is

$$\ddot{x} - \epsilon(1 - x^2 - \frac{\dot{x}^2}{b^2})\dot{x} + x = 0 \qquad (b^2 > 1) \qquad (3.1)$$

which is related to equation (2.11). Equation (3.1) can be obtained from (2.11) by a change of variables. Equation (3.1) may be rewritten as:

$$\ddot{x} - \epsilon\left[\dot{x} - \frac{1}{3}\frac{d}{dt}(x^3) - \frac{\dot{x}^3}{b^2}\right] + x = 0 \tag{3.2}$$

Since the frequency of the periodic solution is not known, it is advantageous to replace the independent variable t by $\theta = \omega t$, where ω is the unknown frequency of the periodic nonlinear solution. This change of variables in (3.2) gives

$$\omega^2 \frac{d^2x}{d\theta^2} - \epsilon\left[\omega\frac{dx}{d\theta} - \frac{\omega}{3}\frac{d}{d\theta}(x^3) - \frac{\omega^2}{b^2}\left(\frac{dx}{d\theta}\right)^3\right] + x = 0 \tag{3.3}$$

Assume for x and ω the following power series in ϵ:

$$x = x_o + \epsilon x_1 + \epsilon^2 x_2 + \dots \tag{3.4}$$

$$\omega = \omega_o + \epsilon\omega_1 + \epsilon^2\omega_2 + \dots \tag{3.5}$$

We may now assume that the solution $x(\theta)$ of (3.3) has the period 2π, and that $dx/d\theta = 0$ for $\theta = 0$, that is, that the velocity is zero at time $t = 0$. Hence we require that all the functions $x_i(\theta)$ have the period 2π and that $(dx_i/d\theta)_{\theta=0} = 0$.

Multiplying (3.4) and (3.5) out and inserting into (3.3):

$$\begin{aligned}&\left[\omega_o^2 + \epsilon 2\omega_o\omega_1 + \epsilon^2(2\omega_o\omega_2 + \omega_1^2) + \dots\right]\left[\ddot{x}_o + \epsilon\ddot{x}_1 + \epsilon^2\ddot{x}_2 + \dots\right]\\ &- \epsilon\left[\omega_o + \epsilon\omega_1 + \dots\right]\left[\dot{x}_o + \epsilon\dot{x}_1 + \epsilon^2\dot{x}_2 + \dots\right]\\ &+ \frac{\epsilon}{3}\left[\omega_o + \epsilon\omega_1 + \dots\right]\left[\frac{d}{d\theta}\left\{x_o^3 + \epsilon 3x_o^2x_1 + \dots\right\}\right]\\ &+ \frac{\epsilon}{b^2}\left[\omega_o^3 + \epsilon 3\omega_o^2\omega_1 + \dots\right]\left[\dot{x}_o^3 + \epsilon 3\dot{x}_o^2\dot{x}_1 + \dots\right]\\ &+ \left[x_o + \epsilon x_1 + \epsilon^2 x_2 + \dots\right] = 0\end{aligned} \tag{3.6}$$

Equating like powers in ϵ:

$$\omega_o^2 \ddot{x}_o + x_o = 0 \tag{3.6}$$

$$\omega_o^2 \ddot{x}_1 + x_1 = -2\omega_o\omega_1 \ddot{x}_o + \omega_o \dot{x}_o - \omega_o x_o^2 \dot{x}_o - \frac{\omega_o^3}{b^2} \dot{x}_o^3 \tag{3.7}$$

$$\begin{aligned} \omega_o^2 \ddot{x}_2 + x_2 = & -2\omega_o\omega_1 \ddot{x}_1 - (2\omega_o\omega_2 + \omega_1^2)\ddot{x}_o + (\omega_o \dot{x}_1 + \omega_1 \dot{x}_o) \\ & - [\omega_o(2x_o\dot{x}_o x_1 + x_o^2 \dot{x}_1) + \omega_1 x_o^2 \dot{x}_o] \\ & - \frac{3\omega_o^2 \dot{x}_o^2}{b^2} (\omega_o \dot{x}_1 + \omega_1 \dot{x}_o) \end{aligned} \tag{3.8}$$

$$\omega_o^2 \ddot{x}_3 + x_3 = \dots$$

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Recall that the periodicity and initial conditions are:

$$x_i(\theta + 2\pi) = x_i(\theta)$$

and $\qquad i = 1, 2, 3, \dots \qquad (3.9)$

$$\left(\frac{dx_i}{d\theta}\right)_{\theta=0} = 0$$

The set of differential equations obtained above can be solved sequentially and we begin by solving equation (3.6) for the initial conditions (3.9). The solution is well known and is

$$x_o(\theta) = A_o \cos\theta; \quad \omega_o = 1 \tag{3.10}$$

where A_o is still undetermined and will be obtained in the next step of the sequence.

Substituting (3.10) into equation (3.7):

$$\ddot{x}_1 + x_1 = 2\omega_1 A_o \cos\theta - A_o \sin\theta + A_o^3 \cos^2\theta \sin\theta + \frac{A_o^3}{b^2}\sin^3\theta. \qquad (3.11)$$

By use of trigonometric identities, (3.11) becomes

$$\ddot{x}_1 + x_1 = 2A_o\omega_1 \cos\theta + A_o\left[\frac{A_o^2}{4b^2}(3+b^2) - 1\right]\sin\theta + \frac{A_o^3}{4b^2}(b^2-1)\sin 3\theta .$$

The periodicity condition requires that there be no resonance. Hence there must be no forcing terms of the same frequency as the frequency of the oscillator. That is, the "secular terms" must be zero:

$$A_o\left[\frac{A_o^2}{4b^2}(3+b^2) - 1\right] = 0$$
$$2A_o\omega_1 = 0 \; . \qquad (3.12)$$

The non-trivial solution is:

$$A_o = \frac{2b}{(3+b^2)^{1/2}}$$
$$\omega_1 = 0 \; . \qquad (3.13)$$

Thus equation (3.11) becomes

$$\ddot{x}_1 + x_1 = \frac{2b(b^2-1)}{(3+b^2)^{3/2}}\sin 3\theta . \qquad (3.14)$$

The particular solution is $C_1 \sin 3\theta$, and substitution into (3.14) gives:

$$C_1 = \frac{-b(b^2-1)}{4(3+b^2)^{3/2}} \; . \qquad (3.15)$$

The general solution of (3.14) is:

$$x_1(\theta) = A_1\cos\theta + B_1\sin\theta + C_1\sin 3\theta.$$

Applying the initial condition:

$$(dx_1/d\theta)_{\theta=0} = B_1 + 3C_1 = 0, \quad \therefore B_1 = -3C_1$$

and (3.16)

$$x_1(\theta) = A_1\cos\theta - 3C_1\sin\theta + C_1\sin 3\theta\ ,$$

where A_1 will be determined in the next step.

Substituting ω_o, ω_1, $x_o(\theta)$ and $x_1(\theta)$ obtained above into equation (3.8) gives:

$$\begin{aligned}\ddot{x}_2 + x_2 = {} & 2\omega_2 A_o\cos\theta - A_1\sin\theta - 3C_1\cos\theta + 3C_1\cos 3\theta \\ & + 2A_o^2\sin\theta\cos\theta(A_1\cos\theta - 3C_1\sin\theta + C_1\sin 3\theta) \\ & - A_o^2\cos^2\theta(-A_1\sin\theta - 3C_1\cos\theta + 3C_1\cos 3\theta) \\ & - \frac{3A_o^2}{b^2}\sin^2\theta(-A\sin\theta - 3C_1\cos\theta + 3C_1\cos 3\theta)\ .\end{aligned}$$

Expanding using trigonometric identities and grouping terms:

$$\begin{aligned}\ddot{x}_2 + x_2 = {} & \left[2\omega_2 A_o - 3C_1 + \frac{A_o^2 C_1}{2b^2}(9 + b^2)\right]\cos\theta + A_1\left[-1 + \frac{3A_o^2}{4b^2}(3 + b^2)\right]\sin\theta \\ & + \frac{3A_o^2 A_1}{4b^2}(b^2 - 1)\sin 3\theta + 3C_1\left[1 + \frac{A_o^2}{4b^2}(b^2 - 9)\right]\cos 3\theta \\ & - \frac{A_o^2 C_1}{4b^2}(5b^2 - 9)\cos 5\theta\ .\end{aligned}$$

Again, periodicity requires the secular terms to vanish:

$$2\omega_2 A_o - 3C_1 + \frac{A_o^2 C_1}{2b^2}(9 + b^2) = 0$$

and (3.17)

$$A_1\left[-1 + \frac{3A_o^2}{4b^2}(3 + b^2)\right] = 0 ,$$

whose solution is

$$\omega_2 = \frac{3(b^2 - 1)}{8(3 + b)^2}$$

(3.18)

$$A_1 = 0 .$$

So equation (3.8) becomes

$$\ddot{x}_2 + x_2 = 3C_1\left[1 + \frac{A_o^2}{4b^2}(b^2 - 9)\right]\cos 3\theta - \frac{A_o^2 C_1}{4b^2}(5b^2 - 9)\cos 5\theta$$

or, using equations (3.13) and (3.15):

$$\ddot{x}_2 + x_2 = -\frac{3b(b^2 - 1)(b^2 - 3)}{2(3 + b^2)^{5/2}}\cos 3\theta + \frac{5(b^2 - 1)(5b^2 - 9)}{4(3 + b^2)^{5/2}}\cos 5\theta. \quad (3.19)$$

The particular solution is

$$C_2 \cos 3\theta + D_2 \cos 5\theta$$

and substitution into (3.19) gives

$$C_2 = \frac{3b(b^2 - 1)(b^2 - 3)}{16(3 + b^2)^{5/2}}$$

and (3.20)

$$D_2 = \frac{-b(b^2 - 1)(5b^2 - 9)}{96(3 + b^2)^{5/2}} .$$

The general solution is:

$$x_2(\theta) = A_2\cos\theta + B_2\sin\theta + C_2\cos 3\theta + D_2\sin 5\theta,$$

and since $(dx_2/d\theta)_{\theta=0} = 0$, $B_2 = 0$; and A_2 and ω_3 may be determined in the next step.

It is clear that the computations become longer and more cumbersome with each succeeding equation in the sequence. So we will truncate the series now. Collecting the information gained so far together:

$$x(\theta) = x_o + \epsilon x_1 + \epsilon^2 x_2 + \ldots$$

$$= A_o\cos\theta + \epsilon(-3C_1\sin\theta + C_1\sin 3\theta)$$

$$+ \epsilon^2(A_2\cos\theta + C_2\cos 3\theta + D_2\cos 5\theta) + \ldots$$

or

$$x(\theta) = \frac{2b}{(3+b^2)^{1/2}}\cos\theta + \frac{\epsilon b(b^2-1)}{4(3+b^2)^{3/2}}(3\sin\theta - \sin 3\theta)$$

$$+ \epsilon^2(\quad) + \ldots \tag{3.21}$$

and

$$\omega = \omega_o + \epsilon\omega_1 + \epsilon^2\omega_2 + \ldots$$

or

$$\omega = 1 + \epsilon^2\,\frac{3(b^2-1)}{8(3+b^2)^2} + \ldots \quad . \tag{3.22}$$

These results are what we would expect. If $b = 1$ then we have the special case of equation (2.16) and frequency $\omega = 1$ and amplitude $A_o = 1$, and the coefficients of all ϵ terms are zero in this case. For $b^2 > 1$, according to the geometric theory of the last part, $1 < A_o < b$, and this is certainly correct here. According to equation (3.22) the frequency increases as ϵ^2 from unity, and we expect $\omega > 1$ for $\epsilon > 0$. But it is interesting that the coefficient of ϵ is zero here, a fact which could not be foreseen from geometrical arguments, and which means that frequency changes very slowly with ϵ for small ϵ.

Periodic Oscillations in the Limiting Case $\epsilon \to \infty$

It is of great interest to know the behavior of equation (3.1) for large ϵ. It would appear that in this case, the "damping" term dominates the equation and the limit cycle would be the ellipse $x^2 + \frac{\dot{x}^2}{b^2} = 1$. But this argument is not valid for very small $\dot{x}$ and it is necessary to examine this case more closely. We will now study the limiting case $\epsilon \to \infty$.

From theorems 2.1 and 2.2 we know that for equation (3.1) there exists a stable and unique periodic solution. We will now prove the result that for this equation (3.1):

$$\ddot{x} - \epsilon(1 - x^2 - \frac{\dot{x}^2}{b^2})\dot{x} + x = 0, \quad b^2 > 1$$

in the limiting case $\epsilon \to \infty$, the periodic solution (limit cycle) is the ellipse

$$g(x, \dot{x}) = x^2 + \frac{\dot{x}^2}{b^2} = 1 .$$

To prove this result, define the Lyapounov function

$$V(x, \dot{x}) = \frac{x^2}{2} + \frac{\dot{x}^2}{2b^2} + \frac{1}{2}(1 - \frac{1}{b^2})\dot{x}^2 e^{-\alpha\dot{x}^2} \qquad (3.23)$$

where α is some function of ϵ. Here we take $\alpha = \epsilon$. If $\dot{x}$ is small such that $\alpha\dot{x}^2 << 1$, then $e^{-\alpha\dot{x}^2} \approx 1$ and

$$V(x, \dot{x}) = \text{constant} \approx \frac{1}{2}(x^2 + \dot{x}^2)$$

is approximately a circle. If, on the other hand, $\dot{x}$ is large such that $\alpha\dot{x}^2 >> 1$, then $e^{-\alpha\dot{x}^2} \approx 0$ and

$$V(x, \dot{x}) = \text{constant} \approx \frac{1}{2}(x^2 + \frac{\dot{x}^2}{b^2})$$

approximates an ellipse. Note that we intend to take ϵ to be very large so that $V(x, \dot{x})$ = constant approximates an ellipse everywhere except for very small $\dot{x}$, where it approximates a circle.

Now define constant $C_1 = (\frac{1}{2} - \frac{1}{2\epsilon^{1/2}})$ and note that for large ϵ the near-ellipse

$$V(x, \dot{x}) = C_1 = \frac{1}{2} - \frac{1}{2\epsilon^{1/2}}$$

is contained entirely within the ellipse $g(x, \dot{x}) = 1$. It is further observed that the near-ellipse

$$V(x, \dot{x}) = C_1 = \frac{1}{2} + \frac{1}{2\epsilon^{1/2}}$$

completely surrounds the ellipse $g(x, \dot{x}) = 1$ (Figure 9).

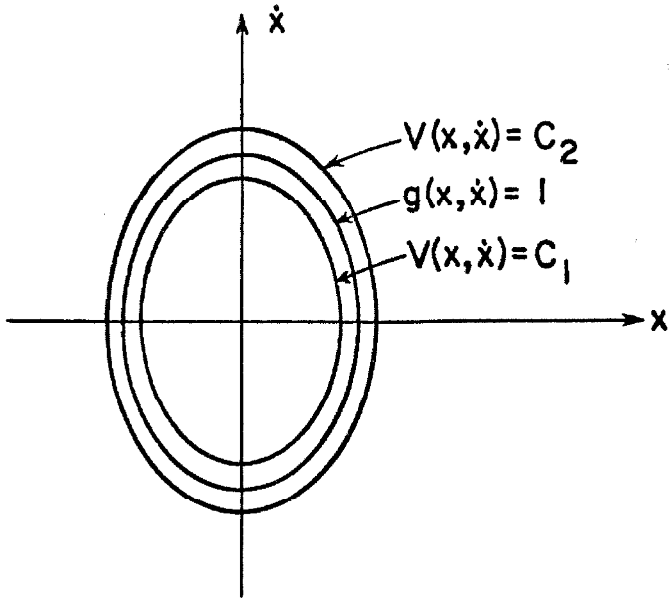

FIGURE 9

The next step is to show that all solutions of (3.1) eventually lie in the region between $V(x, \dot{x}) = C_2$ and $V(x, \dot{x}) = C_1$. This can be done by examining the change of $V(x, \dot{x})$ along solutions of (3.1). So, differentiating $V(x, \dot{x})$ with respect to time:

$$\dot{V}(x, \dot{x}) = x\dot{x} + \frac{\dot{x}\ddot{x}}{b^2} + \left(1 - \frac{1}{b^2}\right)(1 - \alpha\dot{x}^2)\,\dot{x}\ddot{x}\,e^{-\alpha\dot{x}^2} . \qquad (3.23)$$

But along a trajectory of (3.1)

$$\ddot{x} = \epsilon\left(1 - x^2 - \frac{\dot{x}^2}{b^2}\right)\dot{x} - x$$

so that

$$\dot{V}(x, \dot{x}) = x\dot{x}\,(1 - \frac{1}{b^2})\left[1 - (1 - \alpha\dot{x}^2)e^{-\alpha\dot{x}^2}\right]$$

$$+ 2\dot{x}^2\epsilon(1 - x^2 - \frac{\dot{x}^2}{b^2})\left[\frac{1}{b^2} + (1 - \frac{1}{b^2})(1 - \alpha\dot{x}^2)e^{-\alpha\dot{x}^2}\right] \qquad (3.25)$$

Consider the region $V(x, \dot{x}) \leqq C_1$. Here $(1 - x^2 - \frac{\dot{x}^2}{b^2}) > 0$, $x^2 < 1$ and $\dot{x}^2 < b^2$. One can distinguish two cases depending on whether $\alpha\dot{x}^2$ is (i) less than or equal to one or (ii) greater than one.

(i) Suppose $0 < \alpha\dot{x}^2 \leqq 1$, and let

$$\alpha\dot{x}^2 = \theta$$

and

$$(1 - \alpha\dot{x}^2)e^{-\alpha\dot{x}^2} = \beta\,,$$

then

$$\dot{V}(x, \dot{x}) = x\dot{x}(1 - \frac{1}{b^2})(1 - \beta) + \dot{x}^2(1 - x^2 - \frac{\dot{x}^2}{b^2})\left[\frac{1}{b^2} + (1 - \frac{1}{b^2})\beta\right]$$

or

$$\dot{V}(x, \dot{x}) = \frac{x\theta^{1/2}}{\epsilon^{1/2}}(1 - \frac{1}{b^2})(1 - \beta) + \theta(1 - x^2 - \frac{\dot{x}^2}{b^2})\left[\frac{1}{b^2} + (1 - \frac{1}{b^2})\beta\right]. \qquad (3.26)$$

Since $0 < \theta \leq 1$, $0 \leq \beta < 1$ *so that the second term of (3.26) is positive*, while the first term may change signs. But the positive term dominates for sufficiently large ϵ. This is true even for small $\dot{x}$ since $(1 - \beta) \rightarrow 0$ for $\theta \rightarrow 0$. Hence

$$\dot{V}(x, \dot{x}) > 0 \qquad \text{if} \quad \epsilon \text{ is large enough.}$$

(ii) Suppose $\alpha\dot{x}^2 > 1$, then $(1 - \alpha\dot{x}^2)$ is negative and the second term is not necessarily positive. Let us examine this term more closely.

$$\frac{1}{b^2} + (1 - \frac{1}{b^2})(1 - \alpha\dot{x}^2)e^{-\alpha\dot{x}^2} = \frac{1}{b^2} + (1 - \frac{1}{b^2})(1 - \theta)e^{-\theta} \quad , \quad \theta > 1$$

The minimum of $(1 - \theta)e^{-\theta}$ is of interest:

$$\frac{d}{d\theta}\ (1 - \theta)e^{-\theta} = -\ (2 - \theta)e^{-\theta} = 0\ .$$

Therefore, the minimum is at $\theta = 2$. So for the second term to be positive, we must have

$$\frac{1}{b^2} - (1 - \frac{1}{b^2})\frac{1}{e^2} > 0$$

or,

$$b^2 < (1 + e^2) = 8.4 \ldots \quad .$$

Here, $\dot{V}(x, \dot{x})$ again takes the form given in (3.26), and $\dot{V}(x, \dot{x}) > 0$ if ϵ is large enough and $b^2 < (1 + e^2)$. This restriction on b is sufficient but not necessary. The bound could be improved by another choice of the Lyapounov function $V(x, \dot{x})$. However, Lyapounov's theory is not constructive and no definite procedures exist to assure a more suitable choice of $V(x, \dot{x})$. Hence $\dot{V}(x, \dot{x}) > 0$ in the region $V(x, \dot{x}) \leqq C_1$ if ϵ is large enough and if $b^2 < (1 + e^2)$. Hence all trajectories in this region eventually go into the region bounded by $V(x, \dot{x}) = C_2$ and $V(x, \dot{x}) = C_1$.

Now consider <u>the region $V(x, \dot{x}) \geqq C_2$</u>. Here $(1 - x^2 - \frac{\dot{x}^2}{b^2}) < 0$. Again we distinguish two cases depending on whether $\alpha\dot{x}^2$ is less than or equal to unity, or greater than unity.

(i) Suppose $0 < \alpha\dot{x}^2 \leq 1$ then, again $1 > \beta \geq 0$ and

$$\dot{V}(x,\dot{x}) = \frac{x\theta^{1/2}}{\epsilon^{1/2}}(1-\frac{1}{b^2})(1-\beta) + \theta(1-x^2-\frac{\dot{x}^2}{b^2})\left[\frac{1}{b^2}+(1-\frac{1}{b^2})\beta\right]$$

Here the second term is negative while the first one may change signs. For $|x| > M$, the x^2 of the second term dominates the x of the first term so that $\dot{V}(x,\dot{x}) < 0$. For $|x| < M$, the second term dominates if ϵ is large enough and $V(x,\dot{x}) < 0$.

(ii) Suppose $\alpha\dot{x}^2 > 1$. Here again, the second term is negative if $b^2 < (1 + e^2)$, and dominates the first term if ϵ is large enough. Thus $\dot{V}(x,\dot{x}) < 0$ in the region $V(x,\dot{x}) \geqq C_2$. Hence all trajectories in this region eventually go into the region bounded by $V(x,\dot{x}) = C_2$ and $V(x,\dot{x}) = C_1$.

Thus, all trajectories eventually lie in the region bounded by $V(x,\dot{x}) = C_2$ and $V(x,\dot{x}) = C_1$ and the stable and unique limit cycle of (3.1) must lie in this region. Now, in the limiting case $\epsilon \to \infty$

$$V(x,\dot{x}) = \frac{\dot{x}^2}{2} + \frac{x^2}{2b^2} = C_1 = C_2 = \frac{1}{2},$$

so the limit cycle is the ellipse

$$x^2 + \frac{\dot{x}^2}{b^2} = 1 = g(x,\dot{x}),$$

as claimed.

A similar result for the equation

$$\ddot{x} - \epsilon(1 - \frac{x^2}{a^2} - \dot{x}^2)\dot{x} + x = 0, \quad a^2 > 1 \tag{3.27}$$

is that in the limiting case $\epsilon \to \infty$, the periodic solution is represented by the ellipse

$$g(x, \dot{x}) = \frac{x^2}{a^2} + \dot{x}^2 = 1 .$$

It should be noted that equation (3.27) can also be derived from (2.11) by a transformation of variables.

The proof of this result is very similar to the result proved for equation (3.1) so some details will be omitted. In this case we use the Lyapounov function

$$V(x, \dot{x}) = \frac{x^2}{2a^2} + \frac{\dot{x}^2}{2} + \frac{1}{2}\left(\frac{1}{a^2} - 1\right) \dot{x}^2 e^{-\alpha \dot{x}^2} \qquad (3.28)$$

For $\alpha\dot{x}^2 << 1$,

$$V(x, \dot{x}) = \text{constant} \approx \frac{1}{2a^2} (x^2 + \dot{x}^2)$$

approximates a circle, and for $\alpha\dot{x}^2 >> 1$

$$V(x, \dot{x}) = \text{constant} \approx \frac{1}{2}\left(\frac{x^2}{a^2} + \dot{x}^2\right)$$

approximates an ellipse. For ϵ very large, $V(x, \dot{x}) = \text{constant}$ approximates an ellipse everywhere except for very small $\dot{x}$. Again, choose

$$C_1 = \frac{1}{2} - \frac{1}{2\epsilon^{1/2}} \quad \text{and} \quad C_2 = \frac{1}{2} + \frac{1}{2\epsilon^{1/2}} .$$

The curves $V(x, \dot{x}) = C_1$, $C(x, \dot{x}) = C_2$ and $g(x, \dot{x}) = 1$ are shown in Figure 10.

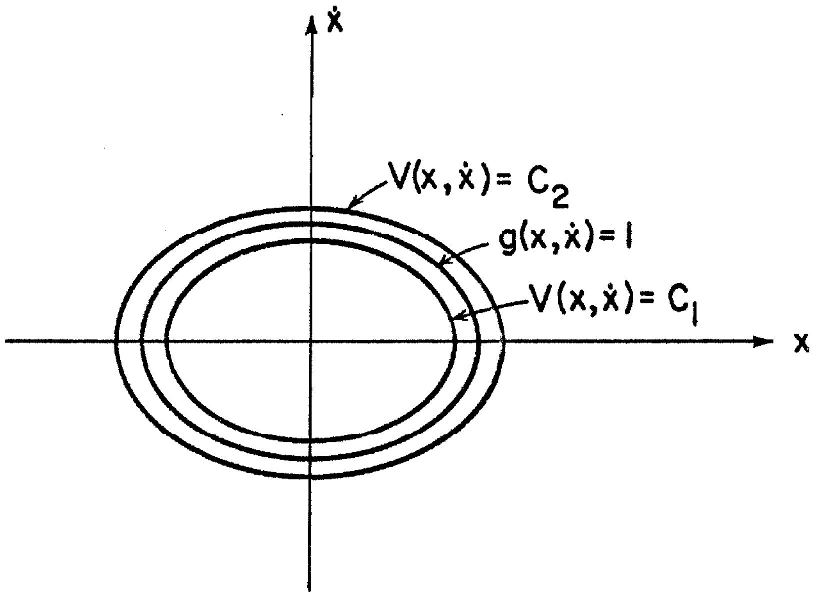

FIGURE 10

Differentiating $V(x, \dot{x})$ in (3. 28) with respect to time:

$$\dot{V}(x, \dot{x}) = \frac{x\dot{x}}{a^2} + \dot{x}\ddot{x} + \left(\frac{1}{a^2} - 1\right)(1 - \alpha\dot{x}^2)\dot{x}\ddot{x}\, e^{-\alpha\dot{x}^2}$$

and along a trajectory of (3. 27):

$$\dot{V}(x, \dot{x}) = \left(\frac{1}{a^2} - 1\right)\left[1 - (1 - \alpha\dot{x}^2)e^{-\alpha\dot{x}^2}\right]x\,\frac{\theta^{1/2}}{\epsilon^{1/2}}$$

$$+ 0\left(1 - \frac{x^2}{a^2} - \dot{x}^2\right)\left[1 + \left(\frac{1}{a^2} - 1\right)(1 - \alpha\dot{x}^2)e^{-\alpha\dot{x}^2}\right] \qquad (3. 29)$$

If $0 < \alpha\dot{x}^2 \leq 1$ then $0 \leq (1 - \alpha\dot{x}^2)e^{-\alpha\dot{x}^2} < 1$, and if $\alpha\dot{x}^2 > 1$ then $(1 - \alpha\dot{x}^2)e^{-\alpha\dot{x}^2} < 0$ so that in the region $V(x, \dot{x}) \leqq C_1$, $V(x, \dot{x}) > 0$

if ϵ is large enough; and in the region $V(x, \dot{x}) \geqq C_2$, $\dot{V}(x, \dot{x}) < 0$ if ϵ is large enough. Thus all trajectories of equation (3.27) eventually are contained in the region bounded by $V(x, \dot{x}) = C_1$ and $V(x, \dot{x}) = C_2$, and the stable and unique limit cycle must lie in this region.

In the limiting case $\epsilon \to \infty$

$$V(x, \dot{x}) = \frac{x^2}{2a^2} + \frac{\dot{x}^2}{2} = C_1 = C_2 = \frac{1}{2},$$

so that the limit cycle is the ellipse

$$\frac{x^2}{a^2} + \dot{x}^2 = 1 = g(x, \dot{x}),$$

as claimed.

These examples show that the crucial point in the proofs is the choice of a suitable Lyapounov function $V(x, \dot{x})$. Unfortunately, there are no standard techniques for generating these functions and the choice is based on experience and trial and error procedures.

Fourier Analysis of Periodic Solutions

We have obtained approximations to the periodic solutions of

$$\ddot{x} - \epsilon(1 - x^2 - \frac{\dot{x}^2}{b^2})\dot{x} + x = 0, \qquad b^2 > 1$$

for small ϵ and for $\epsilon \to \infty$. The behavior for intermediate values of ϵ is also of great practical interest. A stable and unique limit cycle exists, and since $g(x, \dot{x})$ and $h(x)$ have continuous partial derivatives, the limit cycle must be, at least, piecewise-smooth so we can express the solution as a Fourier series:

$$x(t) = \frac{A_o}{2} + \sum_{n=1}^{\infty} (A_n \cos n\omega t + B_n \sin n\omega t) .$$

Geometrically, we know that the limit cycle is point symmetric, that is,

$$x(t + \frac{T}{2}) = - x(t)$$

where T is the period of oscillation. So we must have

$$A_o = A_2 = \ldots = A_{2n} = \ldots = 0$$

and

$$B_2 = \ldots = B_{2n} = \ldots = 0 .$$

Also, the time axis can be shifted to make $B_1 = 0$. Let us truncate the series and assume:

$$x(t) = A \cos \omega t + B \sin 3\omega t + C \cos 3\omega t . \qquad (3.30)$$

Substituting this into (3.1) rewritten as

$$\ddot{x} - \epsilon(\dot{x} - \frac{1}{3} \frac{d}{dt}(x^3) - \frac{\dot{x}^3}{b^2}) + x = 0, \quad (b^2 > 1) \qquad (3.31)$$

we get:

$$-\omega^2(A\cos\omega t + 9B\sin 3\omega t + 9C\cos 3\omega t) + \epsilon\omega(A\sin\omega t - 3B\cos 3\omega t + 3C\sin 3\omega t)$$

$$+ (A\cos\omega t + B\sin 3\omega t + C\cos 3\omega t) + \frac{\epsilon}{3}\frac{d}{dt}\Big\{A^3\cos^3\omega t + B^3\sin^3\omega t + C^3\cos^3 3\omega t$$

$$+ 3A^2B\cos^2\omega t\sin 3\omega t + 3A^2C\cos^2\omega t\cos 3\omega t + 3AB^2\cos\omega t\sin^2 3\omega t$$

$$+ 3AC^2\cos\omega t\cos^2 3\omega t + 3B^2C\sin^2 3\omega t\cos 3\omega t + 3BC^2\sin 3\omega t\cos^2 3\omega t$$

$$+ 6ABC\cos\omega t\sin 3\omega t\cos 3\omega t\Big\} - \frac{\epsilon\omega^3}{b^2}\Big\{A^3\sin^3\omega t - 27B^3\cos^3 3\omega t$$

$$+ 27C^3\sin^3 3\omega t - 9A^2B\sin^2\omega t\cos 3\omega t + 9A^2C\sin^2\omega t\sin 3\omega t$$

$$+ 27AB^2\sin\omega t\cos^2 3\omega t + 27AC^2\sin\omega t\sin^2 3\omega t + 81B^2C\cos^2 3\omega t\sin 3\omega t$$

$$- 81BC^2\cos 3\omega t\sin^2 3\omega t - 54ABC\sin\omega t\cos 3\omega t\sin 3\omega t\Big\} = 0.$$

By use of trigonometric identities the square, cubes and products of the trigonometric functions can be put in terms of the various harmonics. We are only concerned with harmonics up to the third order so higher harmonics are ignored:

$$-\omega^2(A\cos\omega t + 9B\sin 3\omega t + 9C\cos 3\omega t) + \epsilon^{\omega}(A\sin\omega t - 3B\cos 3\omega t + 3C\sin 3\omega t)$$

$$+ A\cos\omega t + B\sin 3\omega t + C\cos 3\omega t) + \frac{\epsilon}{3}\frac{d}{dt}\left\{\frac{A^3}{4}(3\cos\omega t + \cos 3\omega t) + \frac{3B^3}{4}\sin 3\omega t\right.$$

$$+ 3\,\frac{C^3}{4}\cos 3\omega t + \frac{3A^2B}{4}(\sin\omega t + 2\sin 3\omega t) + \frac{3A^2C}{4}(\cos\omega t + 2\cos 3\omega t)$$

$$\left. + \frac{3AB^2}{2}\cos\omega t + \frac{3AC^2}{2}\cos\omega t + \frac{3B^2C}{4}\cos 3\omega t + \frac{3BC^2}{4}\sin 3\omega t\right\}$$

$$-\frac{\epsilon\omega^3}{4b^2}\left\{A^3(3\sin\omega t - \sin 3\omega t) - 81B^3\cos 3\omega t + 81C^3\sin 3\omega t\right.$$

$$+ 9A^2B(\cos\omega t \quad 2\cos 3\omega t) - 9A^2C(\sin\omega t - 2\sin 3\omega t) - 54AB^2\sin\omega t$$

$$\left. + 57AC^2\sin\omega t + 81B^2C\sin 3\omega t - 81BC^2\cos 3\omega t\right\} = 0 \ .$$

Equating the coefficients of $\sin\omega t$, $\cos\omega t$, $\sin 3\omega t$ and $\cos 3\omega t$ to zero, we get the four equations:

$$\frac{\epsilon\omega A}{4}\left(4 - A^2 - AC - 2B^2 - 2C^2 - \frac{3\omega^2A^2}{b^2} + \frac{9\omega^2AC}{b^2} - \frac{54\omega^2B^2}{b^2} - \frac{54\omega^2C^2}{b^2}\right) = 0 \tag{3.32}$$

$$A\left(1 - \omega^2 + \frac{AB}{4}\epsilon\omega - \frac{9AB}{4b^2}\epsilon\omega^3\right) = 0 \tag{3.33}$$

$$B - 9B\omega^2 + \frac{\epsilon\omega}{4}(12C - A^3 - 3C^3 - 6A^2C - 3B^2C)$$
$$+ \frac{\epsilon\omega^3}{4b^2}(A^3 - 81C^3 - 18A^2C - 81B^2C) = 0 \tag{3.34}$$

$$C - 9C\omega^2 - \frac{\epsilon\omega}{4}(12B - 3B^3 - 6A^2B - 3BC^2)$$
$$+ \frac{\epsilon\omega^3}{4b^2}(81B^3 + 18A^2B + 81BC^2) = 0 \tag{3.35}$$

The above are four nonlinear algebraic equations in the four variables ω, A, B, and C. These equations are difficult to solve and the best we can do is to find an approximate solution by numerical methods. A possibility is to try an extension of Newton's method:

Suppose the four equations are written

$$f_j(\bar{x}) = 0, \quad \bar{x} = (x_1, x_2, x_3, x_4) = (\omega, A, B, C) \tag{3.36}$$

$$j = 1, 2, 3, 4 .$$

Newton's method extended to more than one variable gives:

$$\sum_{i=1}^{4} \left.\frac{\partial f_j}{\partial x_i}\right|_{\bar{x}=\bar{x}_{n-1}} \delta x_{i_n} + f_j(\bar{x}_{n-1}) = 0 \tag{3.37}$$

$$n = 1, 2, 3, \ldots \quad ; \quad j = 1, 2, 3, 4$$

which is a vector equation and may be written

$$\Big[F_{ji}\Big]_{\bar{x}=\bar{x}_{n-1}} \{\delta x_n\} = - \{f_j(\bar{x}_{n-1})\} \tag{3.38}$$

$$n = 1, 2, 3, \ldots \; ; \; i, j = 1, 2, 3, 4$$

The solution is then obtained by fixing a value for ϵ and guessing the value of $\bar{x}_{n-1} = \bar{x}_o$ for $n = 1$. Then equation (3.38) gives $\{\delta x_1\}$ and $\bar{x}_1$ can be determined by

$$\{x_n\} = \{\bar{x}_{n-1}\} + \{\delta x_n\} . \tag{3.39}$$

The solution can then be obtained to the desired degree of accuracy by iteration for $n = 2, 3, \ldots$. This iteration can best be done using a digital computing machine. There are two problems associated with this however. It is not certain that the iteration process itself converges and second, even if the iteration process is convergent, the machine computation may not converge.

The computation described above was carried out with limited success using the IBM 7094 digital computer at the Computing Center, California Institute of Technology.

By equations (3.32) - (3.35), the elements of the matrix $[F_{ji}]_{\bar{x}=\bar{x}_n}$ are:

$$F_{11} = A_n - \frac{1}{4}A_n^3 - \frac{1}{4}A_n^2C_n - \frac{1}{2}A_nB_n^2 - \frac{1}{2}A_nC_n^2 - \frac{9\omega_n^2A_n}{4b^2}(A_n^2 - 3A_nC_n + 18B_n^2 + 18C_n^2)$$

$$F_{12} = \omega_n(1 - \frac{3}{4}A_n^2 - \frac{1}{2}A_nC_n - \frac{1}{2}B_n^2 - \frac{1}{2}C_n^2) - \frac{9\omega_n^3}{2b^2}(\frac{1}{2}A_n^2 - A_nC_n + 3B_n^2 + 3C_n^2)$$

$$F_{13} = -\omega_nA_nB_n(1 + \frac{27\omega_n^2}{b^2})$$

$$F_{14} = -\omega_nA_n(\frac{1}{4}A_n + C_n - \frac{9}{4b^2}\omega_n^2A_n + \frac{27}{b^2}\omega_n^2C_n)$$

$$F_{21} = \frac{\epsilon}{4}A_n^2B_n - 2\omega_nA_n - \frac{27\epsilon}{4b^2}\omega_n^2A_n^2B_n$$

$$F_{22} = 1 - \omega_n^2 + \frac{\epsilon}{2}\omega_nA_nB_n - \frac{9\epsilon}{2b^2}\omega_n^3A_nB_n$$

$$F_{23} = \frac{\epsilon}{4}\omega_nA_n^2 - \frac{9\epsilon}{4b^2}\omega_n^3A_n^2$$

$$F_{24} = 0$$

$$F_{31} = 3\epsilon C_n - \frac{\epsilon}{4} A_n^3 - \frac{3\epsilon}{4} C_n(A_n^2 + B_n^2 + C_n^2)$$
$$+ \frac{3\epsilon}{4b^2}\, \omega_n^2 (A_n^3 - 81C_n^3 - 18A_n^2 C_n - 81B_n^2 C_n) - 18\omega_n B_n$$

$$F_{32} = -3\epsilon\omega_n A_n\left(\frac{1}{4} A_n + C_n - \frac{1}{4b^2}\, \omega_n^2 A_n + \frac{3}{b^2}\, \omega_n^2 C_n\right)$$

$$F_{33} = 1 - 9\omega_n^2 - \frac{3\epsilon}{2}\, \omega_n B_n C_n - \frac{81\epsilon}{2b^2}\, \omega_n^3 B_n C_n$$

$$F_{34} = 3\epsilon\omega_n\left(1 - \frac{1}{2} A_n^2 - \frac{1}{2} B_n^2 - \frac{3}{4} C_n^2\right)$$
$$- \frac{9\epsilon}{4b^2}\, \omega_n^3 (2A_n^2 + 9B_n^2 + 27C_n^2)$$

$$F_{41} = -18\omega_n C_n + 3\epsilon B_n\left(-1 + \frac{1}{2} A_n^2 + \frac{1}{4} B_n^2 + \frac{1}{4} C_n^2\right)$$
$$+ \frac{3\epsilon}{4b^2}\, \omega_n^2 (18 A_n^2 B_n + 81B_n^3 + 81B_n C_n^2)$$

$$F_{42} = 3\epsilon\omega_n A_n B_n\left(1 + \frac{3}{b^2}\, \omega_n^2\right)$$

$$F_{43} = 3\epsilon\omega_n\left(-1 + \frac{1}{2} A_n^2 + \frac{3}{4} B_n^2 + \frac{1}{4} C_n^2\right)$$
$$+ \frac{9\epsilon}{4b^2}\, \omega_n^3 (2A_n^2 + 27B_n^2 + 9C_n^2)$$

$$F_{44} = 1 - 9\omega_n^2 + \frac{3\epsilon}{2}\, \omega_n B_n C_n + \frac{81\epsilon}{2b^2}\, \omega_n^3 B_n C_n .$$

And the elements of the vector $-\{f_j(\overline{x}_n)\}$ are:

$$f_1 = \omega_n A_n(-1 + \frac{1}{4} A_n^2 + \frac{1}{4} A_n C_n + \frac{1}{2} B_n^2 + \frac{1}{2} C_n^2)$$
$$+ \frac{3}{4b^2} \omega_n^3 A_n(A_n^2 - 3A_n B_n + 18B_n^2 + 18C_n^2)$$

$$f_2 = A_n(-1 + \omega_n^2 - \frac{\epsilon}{4} \omega_n A_n B_n + \frac{9\epsilon}{4b^2} \omega_n^3 A_n^2 B_n)$$

$$f_3 = -B_n + \epsilon\omega_n(-3C_n + \frac{1}{4} A_n^3 + \frac{3}{2} A_n^2 C_n + \frac{3}{4} B_n^2 C_n + \frac{3}{4} C_n^3)$$
$$+ \frac{\epsilon}{4b^2} \omega_n^3(-A_n^3 + 18A_n^2 C_n + 81B_n^2 C_n + 81C_n^3) + 9\omega_n^2 B_n$$

$$f_4 = -C_n + 3\epsilon\omega_n B_n(1 - \frac{1}{2} A_n^2 - \frac{1}{4} B_n^2 - \frac{1}{4} C_n^2) + 9\omega_n^2 C_n$$
$$- \frac{\epsilon}{4b^2} \omega_n^3(18A_n^2 B_n + 81B_n^3 + 81B_n C_n^2) \quad .$$

Take $b^2 = 2$, then the computer solution for $\epsilon = 10^2$ is:

$\omega = 1.133$

$A = 1.165$

$B = -1.484 \times 10^{-3}$

$C = -4.755 \times 10^{-2}$.

For $\epsilon = 10^4$, the computer solution is:

$\omega = 1.132$

$A = 1.167$

$B = - 1.476 \times 10^{-5}$

$C = -4.760 \times 10^{-2}$.

It is seen that for both solutions, $1 < \omega < b$ and $1 < A < b$ as predicted by the topological methods. The amplitudes B and C of the third harmonic terms are very small, thus justifying the truncation of the Fourier Series at this point. The coefficient B of the sin $3\omega t$ term seems to tend to zero as ϵ increases, but there is no appreciable change in the coefficient C of the cos $3\omega t$ term. The frequency ω is slightly less for $\epsilon = 10^4$ than for $\epsilon = 10^2$. One would expect the reverse to be true as the frequency increases to $b = \sqrt{2}$ as $\epsilon \to \infty$. But the difference is so small that it may oe discounted.

The significant point, however, is that the difference for the two values of ϵ is so small that the convergence of the computations or of the machine calculations is not certain. Due to time limitations this question was not properly resolved, and would be an interesting problem for future investigation.

Forced Oscillations by the Method of Slowly Varying Parameters

We turn now to obtaining an approximate analytical solution for the equation

$$\ddot{x} - \epsilon(1 - x^2 - \frac{\dot{x}^2}{b^2})\dot{x} + x = E \sin \nu t \quad . \tag{3.40}$$

The method of slowly varying parameters has been developed to obtain first-order approximations. For predominantly harmonic solutions, one may start by assuming the solution

$$x(t) = A \sin \nu t + B \cos \nu t \tag{3.41}$$

where A and B vary slowly with time, that is, the variation over the period $2\pi/\nu$ is small. When (3.41) is assumed the method is often known as van der Pol's method[40]. One may equally well choose the equivalent solution

$$x(t) = A \sin(\nu t + \varphi) \tag{3.42}$$

where A and φ vary slowly with time. When (3.42) is assumed the method is often called the Kryloff-Bogoliuboff-Mitropolsky Method[41, 42]. This method has been applied by Dr. Caughey to various nonlinear problems[43, 44]. C. Hayashi has several examples worked out in his book[45]. These predominantly harmonic oscillations are significantly different from linear oscillations as "jump" phenomena may be exhibited in the nonlinear case.

For our purposes it is convenient to suppose the solution has the form of equation (3.42). Assume that the velocity is

$$\dot{x}(t) = A\nu \cos(\nu t + \varphi) \ . \tag{3.43}$$

By differentiating (3.42), it is seen that this assumption about the velocity implies that

$$\frac{dA}{dt} \sin(\nu t + \varphi) + A \frac{d\varphi}{dt} \cos(\nu t + \varphi) = 0 \ . \tag{3.44}$$

From (3.43) we get

$$\ddot{x} = \frac{dA}{dt} \nu \cos(\nu t + \varphi) - A\nu(\nu + \frac{d\varphi}{dt}) \sin(\nu t + \varphi) \ . \tag{3.45}$$

Substituting x, $\dot{x}$ and $\ddot{x}$ into the equation of motion (3.40), using trigonometric identities and grouping terms:

$$A(1 - \nu^2 - \nu \frac{d\varphi}{dt})\sin(\nu t + \varphi) + \left(\frac{dA}{dt}\nu - \epsilon A\nu\left[1 - \frac{A^2}{4}(1 + \frac{3\nu^2}{b^2})\right]\right)\cos(\nu t + \varphi)$$

$$+ \frac{\epsilon A^3 \nu}{4}(1 + \frac{\nu^2}{b^2})\cos 3(\nu t + \varphi) = E\cos\varphi\sin(\nu t + \varphi) - E\sin\varphi\cos(\nu t + \varphi) \; . \tag{3.46}$$

Multiply (3.46) by $\sin(\nu t + \varphi)$, and subtracting from the resulting equation (3.44) times $\nu\cos(\nu t + \varphi)$:

$$A(1 - \nu^2)\sin^2(\nu t + \varphi) - \frac{d\varphi}{dt}A\nu\left[\sin^2(\nu t + \varphi) + \cos^2(\nu t + \varphi)\right]$$

$$+ \epsilon A\nu\left[1 - \frac{A^2}{4}(1 + \frac{3\nu^2}{b^2}) + \frac{A^2}{4}(1 + \frac{1}{b^2})\right]\sin(\nu t + \varphi)\cos(\nu t + \varphi)$$

$$= E\cos\varphi\sin^2(\nu t + \varphi) - E\sin\varphi\sin(\nu t + \varphi)\cos(\nu t + \varphi) . \tag{3.47}$$

Averaging out (3.47)

$$-2\nu A\frac{d\overline{\varphi}}{dt} = (\nu^2 - 1)A + E\cos\varphi \; . \tag{3.48}$$

Similarly, multiply (3.46) by $\cos(\nu t + \varphi)$ and add $\nu\sin(\nu t + \varphi)$ times (3.44) to it. On averaging the resulting equation we get

$$-2\nu\frac{d\overline{A}}{dt} = \epsilon A\nu\left[-1 + \frac{A^2}{4}(1 + \frac{3\nu^2}{b^2})\right] + = E\sin\varphi \; . \tag{3.49}$$

For <u>steady state oscillations</u> we must have

$$\frac{d\overline{\varphi}}{dt} = \frac{d\overline{A}}{dt} = 0 \tag{3.50}$$

so equations (3.48) and (3.49) become

$$(\nu^2 - 1)A_o = E \cos \varphi_o$$

and

$$\epsilon A_o \nu \left[-1 + \frac{A_o^2}{4} (1 + \frac{3\nu^2}{b^2})\right] = E \sin \varphi_o \, .$$

Eliminating φ by squaring and adding

$$E^2 = (\nu^2 - 1)^2 A_o^2 + \epsilon^2 A_o^2 \nu^2 \left[1 - \frac{A_o^2}{4} (1 + \frac{3\nu^2}{b^2})\right]^2 \qquad (3.51)$$

which gives the relationship between the amplitude of oscillation, A_o, and the frequency of the forcing term, ν. The locus of vertical tangency of the amplitude-frequency curves is of great interest, as we shall see, in the stability analytis of these steady state oscillations so now we determine this locus:

$$\frac{d\nu}{dA_o} = 2A_o(\nu^2-1)^2 + 2\epsilon^2 A_o^2 \nu^2 \left[1 - \frac{A_o^2}{4} (1 + \frac{3\nu^2}{b^2})\right]\left[- \frac{A_o}{2} (1 + \frac{3\nu^2}{b^2})\right]$$

$$+ 2\epsilon^2 \nu^2 A_o \left[1 - \frac{A_o^2}{4} (1 + \frac{3\nu^2}{b^2})\right]^2 = 0 \, .$$

Rearranging:

$$\left(\frac{\nu^2-1}{\epsilon\nu}\right)^2 + \left[1 - \frac{A_o^2}{4} (1 + \frac{3\nu^2}{b^2})\right]\left[1 - \frac{3A_o^2}{4} (1 + \frac{3\nu^2}{b^2})\right] = 0 \, . \qquad (3.52)$$

We are particularly interested in small "detuning," that is, $\nu \approx 1$. In such a case (3.52) becomes

$$\left(\frac{\nu^2-1}{\epsilon\nu}\right)^2 + \left[1 - \frac{A_o^2}{4} (1 + \frac{3}{b^2})\right]\left[1 - \frac{3A_o^2}{4} (1 + \frac{3}{b^2})\right] = 0 \, . \qquad (3.53)$$

Recall that the amplitude a_o for free oscillations for small ϵ is given by equation (3.13) as

$$a_o = \frac{2b}{(3 + b)^{1/2}}$$

and so (3.53) may be written

$$\left(\frac{\nu^2 - 1}{\epsilon\nu}\right)^2 + \left(1 - \frac{A_o^2}{a_o^2}\right)\left(1 - \frac{3A_o^2}{a_o^2}\right) = 0 \; . \qquad (3.54)$$

Let

$$X = \frac{\nu^2 - 1}{\epsilon\nu}$$

and

$$Y = \frac{A_o^2}{a_o^2}$$

then (3.54) becomes

$$X^2 + (1 - Y)(1 - 3Y) = 0 \qquad (3.55)$$

which is the equation for the ellipse

$$\frac{X^2}{(1/\sqrt{3})^2} + \frac{(Y - \frac{2}{3})^2}{(1/3)^2} = 1 \; . \qquad (3.56)$$

Thus the approximate result for small detuning is the same as for van der Pol's equation.

The stability of these steady state solutions will now be studied. We use the standard technique of introducing a small disturbance into the solution and determine whether this disturbance dies out or not. So suppose

$$A = A_o + \xi \tag{3.57}$$

and

$$\varphi = \varphi_o + \eta \tag{3.58}$$

where ξ and η are small. Substituting these into (3.48):

$$-2\nu(A_o + \xi)\frac{d}{dt}(\varphi_o + \eta) = (\nu^2 - 1)(A_o + \xi) + E\cos(\varphi_o + \eta). \tag{3.59}$$

Subtract out the steady state solution

$$(\nu^2 - 1)A_o + E_o \cos\varphi_o = 0 \tag{3.60}$$

and neglecting higher order terms, (3.59) becomes

$$-2\nu A_o \frac{d\eta}{dt} = (\nu^2 - 1)\xi - E\eta \sin\varphi_o \,. \tag{3.61}$$

Similarly, putting (3.57) and (3.58) into (3.49) gives

$$-2\nu\frac{d}{dt}(A_o + \xi) = \epsilon(A_o + \xi)\nu\left[-1 + \frac{(A_o + \xi)^2}{4}\left(1 + \frac{3\nu^2}{b^2}\right)\right] + E\sin(\varphi_o + \eta) \tag{3.62}$$

Subtract out the steady state solution

$$\epsilon A_o \nu\left[-1 + \frac{A_o^2}{4}\left(1 + \frac{3\nu^2}{b^2}\right)\right] + E\sin\varphi_o = 0 \tag{3.63}$$

and neglecting higher order terms, (3.62) becomes

$$2\nu\frac{d\xi}{dt} = \epsilon\nu\xi\left[1 - \frac{3A_o^2}{4}\left(1 + \frac{3\nu^2}{b^2}\right)\right] - E\eta\cos\varphi_o \,. \tag{3.64}$$

Assume the distrubances to be

$$\xi = \xi_o e^{\lambda t} \tag{3.65}$$

and

$$\eta = \eta_o e^{\lambda t} \quad . \tag{3.66}$$

Substituting ξ and η into (3.48) and (3.49)

$$-2\nu A_o \eta_o \lambda = (\nu^2 - 1)\xi_o - E\eta_o \sin\varphi_o \tag{3.67}$$

and

$$2\nu\xi_o\lambda = \epsilon\nu\xi_o\left[1 - \frac{3A_o^2}{4}(1 + \frac{3\nu^2}{b^2})\right] - E\eta_o \cos\varphi_o \quad . \tag{3.68}$$

But $E\cos\varphi_o$ and $E\sin\varphi_o$ are known from the steady-state solutions (3.60) and (3.63) so equations (3.67) and (3.68) may be put in the form

$$\begin{bmatrix} (\nu^2-1) & 2\nu A_o\lambda - \epsilon A_o\nu \left[1 - \frac{A_o^2}{4}(1 + \frac{3\nu^2}{b^2})\right] \\ 2\nu\lambda + \epsilon\nu \left[-1 + \frac{3A_o^2}{4}(1 + \frac{3\nu^2}{b^2})\right] & -(\nu^2-1)A_o \end{bmatrix} \begin{Bmatrix} \xi_o \\ \eta_o \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} .$$

For a non-trivial solution the determinant must be zero:

$$(\nu^2-1)A_o + A_o\left\{2\nu\lambda + \epsilon\nu\left[\frac{A_o^2}{4}(1 + \frac{3\nu^2}{4}) - 1\right]\right\}\left\{2\nu\lambda + \epsilon\nu\left[\frac{3A_o^2}{4}(1 + \frac{3\nu^2}{b^2}) - 1\right]\right\} = 0 \quad .$$

Rearranging:

$$\lambda^2 + \frac{\epsilon}{2}\left[A_o^2(1 + \frac{3\nu^2}{b^2}) - 2\right]\lambda + \left(\frac{\nu^2 - 1}{2\nu}\right)^3$$
$$+ \frac{\epsilon^2}{4}\left[\frac{A_o^2}{4}(1 + \frac{3\nu^2}{b^2}) - 1\right]\left[\frac{3A_o^2}{4}(1 + \frac{3\nu^2}{b^2}) - 1\right] = 0 \quad . \tag{3.69}$$

Equation (3.69) is of the form

$$\lambda^2 + m\lambda + n = 0 \tag{3.70}$$

so

$$\lambda = -\frac{m}{2} \pm \sqrt{\left(\frac{m}{2}\right)^2 - n} \quad . \tag{3.71}$$

For the disturbance to die out, that is, for stability, λ must have negative real part so we must have $m > 0$ and $n > 0$.

The requirement $m > 0$ means

$$A_o^2 > \frac{2b^2}{b^2 + 3\nu^2} \tag{3.72}$$

for stability. The requirement $n > 0$ reduces to

$$\left(\frac{\nu^2-1}{\epsilon\nu}\right)^2 + \left[1 - \frac{A_o^2}{4}\left(1 + \frac{3\nu^2}{b^2}\right)\right]\left[1 - \frac{3A_o^2}{4}\left(1 + \frac{3\nu^2}{b^2}\right)\right] > 0 \,. \tag{3.73}$$

So there is marginal stability if

$$\left(\frac{\nu^2-1}{\epsilon\nu}\right)^2 + \left[1 - \frac{A_o^2}{4}\left(1 + \frac{3\nu^2}{b^2}\right)\right]\left[1 - \frac{3A_o^2}{4}\left(1 + \frac{3\nu^2}{b^2}\right)\right] = 0 \tag{3.74}$$

which is exactly the locus of vertical tangents of the amplitude-frequency curves for steady state oscillations (equations 3.52).

For small detuning $\nu \approx 1$ and (3.72) and (3.73) become:

$$A_o^2 > \frac{a_o^2}{2}$$

and

$$\left(\frac{\nu^2-1}{\epsilon\nu}\right)^2 + \left(1 - \frac{A_o^2}{a_o^2}\right)\left(1 - 3\,\frac{A_o^2}{a_o^2}\right) > 0 \,.$$

These are just the conditions for the stability of van der Pol's equation and a good set of response curves may be found on page 82 of McLachlan's book[46].

These analytical results using the method of slowly varying parameters on (3.40), of course, agree with the topological results obtained in Part II. As proved in theorem 2.3, there does exist a periodic solution with the same period as that of the forcing function. This solution is stable only under certain conditions, but may also be unstable. Theorem 2.3 does not predict anything about stability and this had to be investigated separately. It should be pointed out that if a periodic solution is unstable, it does not exist in a physical sense as it will not be observed in practice.

The solutions of equation (3.40)

$$\ddot{x} - \epsilon(1 - x^2 - \frac{\dot{x}^2}{b^2})\dot{x} + x = E \sin \nu t$$

are defined in the future as the equation is a particular case of equation (2.22)

$$\ddot{x} - \epsilon[1 - g(x, \dot{x})] \dot{x} + h(x) = e(t)$$

whose solutions were shown to exist in the future. By the change of variables $x = by$, equation (3.40) becomes

$$\ddot{y} = \epsilon(1 - \dot{y}^2 - b^2y^2)\dot{y} + y = \frac{E}{b} \sin \nu t$$

which is the form of equation (2.56) with $k(y) = b^2y^2$ so that the solutions of (3.40) are also ultimately bounded. The theorems on ultimate boundedness, however, do not actually give bounds but only state that bounds do exist. This is in contrast to the topological results obtained for free oscillations.

IV. SUMMARY AND CONCLUSIONS

Self-excited oscillators are those in which periodic oscillations are possible without the presence of an external periodic forcing function. Such oscillations are physically realizable if a mechanism exists by which the system can absorb energy from its surroundings during part of a cycle to compensate for the energy dissipation during the remainder of the cycle.

In theorem 2.1 the existence of periodic solutions for free vibrations was proved for $g(x, \dot{x}) = 1$ a simple closed curve. This is only slightly more restrictive than the general result obtained by Levinson and Smith, but the proof of the theorem is simpler and it is easier to apply. Not only that, but theorem 2.1 gives bounds in the phase-plane within which such periodic solutions exist and this is of great engineering interest.

To prove that this periodic solution is unique we assumed that $g(x, \dot{x})$ was four-point symmetric, that is, $g(x, \dot{x}) = g(x. -\dot{x}) = g(-x, \dot{x}) = g(-x, -\dot{x}) > 0$. Even with such a restriction, many important problems can still be analyzed. But it is felt that it should be possible to extend the result to curves $g(x, \dot{x}) = 1$ with other axes of symmetry or even to convex curves with no symmetry. This is not very easy to do, however, as many cases will be involved. But extensions would be valuable and merit further investigation. It is true that Levinson and Smith have proved a somewhat more general theorem, but their result is difficult to apply.

The examples which illustrate the topological theory for free oscillations of self-excited oscillators are generalizations of van der Pol and Rayleigh's equations. The limit cycles for these generalizations are obtained graphically for two values of the parameter ϵ. These limit cycles are of interest because, besides illustrating the behavior of the solutions of these self-excited oscillators, worked examples are not abundant in nonlinear vibration theory.

It is shown that for the special class of problems for which $g(x, \dot{x}) = mV(x, \dot{x})$ where $V(x, \dot{x})$ is the Hamiltonian of the system, the upper and lower bounds for the limit cycle are the same and the limit cycle is given exactly by $V(x, \dot{x}) = 1/m$; and for $m = 2$ and $h(x) = x$, the stability of the periodic solution was proved directly by Lyapounov's Second Method.

By use of the Brouwer fixed point theorem it was shown that if the self-excited oscillator is externally excited by a periodic function, then there exists at least one periodic solution with the same period as that of the forcing function. This periodic solution may or may not be stable. Also, it is deduced that super or subharmonics may exist but no conditions that assure the existence of these harmonics are available. These topics are important and need to be studied further.

The concepts of solutions being "defined in the future," having "finite escape times," or being "ultimately bounded" were introduced and some theorems on these were quoted. These results were then applied to self-excited oscillators. These methods tell us if a solution is ultimately bounded or not and if it has a finite escape time but they are not constructive results as no specific bounds are given.

An analytical approximation to the periodic solution for free oscillations of a particular example for small values of the parameter ϵ is satisfactorily obtained by the perturbation method. And the exact limit cycle is obtained for the limiting case $\epsilon \to \infty$. For intermediate values of ϵ, the approximations are not easy to obtain. A Fourier series approach is given and a sample computation was carried out using a digital computer. This calculation had only limited success as convergence of either the method or the machine computation was not certain.

The whole problem of obtaining analytical approximations to the periodic solutions for intermediate values of the parameter ϵ is very difficult. Cartwright and Littlewood have worked this out for van der Pol's equation using real variable theory. But their methods are very complicated and not suitable for direct application to engineering problems. There is also the approach of using different approximating differential equations for different regions of the phase-plane. This is the so-called "boundary layer" approach, and it was applied to van der Pol's equation by Dorodnitsyn[49]. A possibly fruitful approach to solutions of

$$\ddot{x} - \epsilon\left(1 - \frac{x^2}{a^2} - \frac{\dot{x}^2}{b^2}\right)\dot{x} + x = 0$$

is to obtain an approximate solution analytic in ϵ and $\delta = \left(\frac{b}{a} - 1\right)$,

$\delta \approx 0$. Here $a \approx b$ and the solution will be close to harmonic as, by theorem 2.1, it will be bounded within a narrow annular region in the phase-plane. The limiting case $a = b$ has been worked out as the special class of problems where $g(x, \dot{x}) = mV(x, \dot{x})$. The analytic solution may then be extendable to larger δ.

As a last topic, an approximate analytic solution was obtained for the self-excited oscillator with a sinusoidal forcing term for the predominantly harmonic case by the method of slowly varying parameters. The conditions for the stability of these solutions were also obtained. Even the predominantly harmonic case is of interest as jump phenomena may occur which are not possible in the linear case. This method of analysis is available for small ϵ.

It is clear on the basis of the examples that approximate analytic solutions are difficult to obtain even for quasi-linear oscillators, and very often cannot be obtained at all using currently available methods for large nonlinearities. It is for this reason that the qualitative information obtained from topological methods is so important for applications. Applied to the solutions of self-excited oscillators, these topological methods yield elegant and useful results and provide a global understanding of the types of possible behavior such oscillators can exhibit.

V. REFERENCES

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