# TIME DOMAIN ANALYSIS OF SWITCHING REGULATORS

Thesis by

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In Partial Fulfillment of the Requirements

for the Degree of

Doctor of Philosophy

California Institute of Technology

Pasadena, California

1977

(Submitted December 20, 1976)

To my parents,

whose support and understanding have helped me through many difficulties.

# ACKNOWLEDGMENTS

I would like to express my sincere thanks to my advisor, Dr. T. K. Caughey, whose patient encouragement and valuable insights made this thesis possible. I very much appreciate the support provided by Caltech and the A.R.C.S. Foundation with their Fellowships. Special thanks are due to Sharon Vedrode for her competent typing of the thesis and to Cecilia Lin for her expert drawings of the figures.

## ABSTRACT

The exact expressions for the local stability of a buck regulator are found, and these expressions are simplified when the ratio of the natural frequency to the switching frequency is small. Simplified expressions are also found for the local stability of the boost and buck-boost regulators when the frequency ratio is small and the damping factor is less than the nondimensional switching period, $\xi \leqslant \tau_s$. The feedback constants of the linear control law determine both the local stability and regulation of the linear discrete regulator.

Liapunov's direct stability method as applied to discrete systems and the method of paired systems due to Kalman are used to obtain sufficient conditions for global stability. The paired system technique is also used to analyze the switching regulators for their global convergence properties.

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# CHAPTER I - INTRODUCTION

The switching regulator is a product of the space age. In space, energy is expensive, and an efficient method for regulating voltage is required. A resistive regulator dissipates energy in the resistance used to regulate the voltage. A switching regulator turns a switch on and off in such a way as to maintain the proper output voltage. Since there is little resistance associated with the switch, the efficiency of the switching regulator is very high. These devices are being used increasingly on earth due to the increased cost of energy. In fact, the energy crisis of the last few years has generated a lot of interest in switching regulators.

A partial answer to solving the energy problem is to engineer better control systems to make all processes using energy more efficient. Dr. R. H. Cannon in a talk given at Caltech advocated this approach to the energy problem and gave some examples to illustrate how better control systems could result in significant energy savings. The revolutionary developments in solid state technology have occurred soon enough to be of use in helping to solve our energy problems. Since solid state devices make up an essential part of all switching regulators, further advances in solid state technology should help to improve the regulators.

It is indeed fortunate that great advances in solid state technology and therefore our computing capabilities have preceded the energy crisis. In refs. [15] and [16] Norbert Weiner talks about the relationship between life, energy, entropy, and information.

A thermodynamic point of view of life is very interesting because the life process apparently contradicts the second law of thermodynamics. The idea that matter could spontaneously organize itself to produce a living organism is contrary to our intuition and to the second law which states that a system tends toward maximum disorder. This apparent conflict is solved by noting that a living organism is not a closed system and requires energy to survive. The increase in entropy caused by the degradation of energy is more than the decrease in entropy resulting from the increase in order of the organism. The fundamental issue of life is concerned with entropy.

If society is viewed as a large organism, then it too stays organized by degrading energy, and as civilization has advanced its energy requirements have increased. Like a biological organism, the fundamental issue confronting society's survival is entropy. The use of better controls to save energy may be more than just a stopgap solution to our energy problems until other energy sources are found. In ref. [15] Weiner states:

> "Just as the amount of information in a system is a measure of its degree of organization, so the entropy of a system is a measure of its degree of disorganization; and the one is simply the negative of the other."

The control system decreases entropy by making the proper decisions based on the information obtained from the system. This approach, besides appearing to me to be more fundamental than trying to obtain new energy sources, is based on a technology which has already been developed to a high level of sophistication and is still undergoing rapid development.

In an age of specialization the applicability of control theory to many different types of systems is unique. It is not unusual for a mechanical engineer, such as myself, to work on an electrical problem. In fact, because the control devices are usually electrical, an engineer interested in controls should have some kind of a background in electronics. The techniques used in this thesis are applicable to nonlinear, discrete systems whether the systems are electrical, mechanical, or biological. Many times a system of one discipline will have a direct analog in another. One analog for the electrical switching regulator of this thesis is the pressure switching regulator. The equations of the two systems are identical when the pressure and flow rate is substituted for the voltage and current of the electrical system. The major problem with implementing the pressure regulator is that of noise. The noise results from the rapid switching which is necessary to operate the pressure regulator.

The analysis of switching regulators is done in two parts. In the first part the equations governing the switching regulators are linearized, and the linear regulator is analyzed. In the second part, techniques which can be used to determine the global properties of the nonlinear regulator are demonstrated. In the past switching regulators have been linearized in the time domain, refs. [3, 4, 7], and by using describing functions, refs. [12,13,14,17].

The most popular method at present is to linearize the equations and then carry out the analysis in the frequency domain. With linear continuous systems many of the techniques used in analysis and design are done in the frequency domain. The advantages gained by doing the analysis in the frequency domain usually outweigh the trouble of transforming the continuous system from the time to the frequency domain. With discrete systems it is not clear what advantages, if any, are gained by transforming from the time to the frequency domain. The debate over what domain the analysis should be performed in has been going on for twenty years. R. E. Kalman and J. E. Bertram in ref. [9] make a strong case for performing the analysis in the time domain. There are a number of reasons, however, why engineers still like to do analysis for discrete systems in the frequency domain. The most important reason for continuing to use the frequency domain is that the engineer is used to thinking in those terms from analyzing continuous systems. Another important reason for using the frequency domain is that discrete systems many times interface with continuous systems where specifications are given in terms of the frequency domain. In this thesis the analysis is performed in the time domain.

The recursion formula relating the state of the $(n+1)^{st}$ iteration to the state of the $n^{th}$ iteration is derived for the buck, the boost, and the buck-boost regulator in chapters 2 through 4 respectively. These chapters are all organized in the same manner. After the recursion formula is derived, the steady-state without feedback is found, and it is used in the control law as the reference vector for feedback. The control laws examined in this thesis are either linear or can be approximated adequately by their linear part. In analyzing the linear regulator there is no loss of generality in assuming a linear control law. In fact, the entire character of the linear regulator can be determined from the feedback constants of the control law. Once the general form of the control law is specified, the regulation properties are determined, and then the local stability is derived. The regulation and local stability of the linear regulator are related through a quantity defined as the closed loop gain. The closed loop gain, which was originally defined by Dr. Yuh in ref. [17], gives an indication of how well a regulator regulates, and it is therefore used as a figure of merit in the stability analysis.

The regulation properties are obtained by first assuming that the ratio of the natural frequency to the switching frequency is very small (i.e. $\omega_k/\omega_s \ll 1$ or $\tau_s \ll 1$) and then linearizing the recursion formula about the design point. The linearization is valid when the change in on-time, $\Delta\tau_0$, and/or switching period, $\Delta\tau_s$, are small so that the new equilibrium point is close to the design point. In the case of the boost and buck-boost regulators the additional assumption that the damping factor is less than or equal to the switching period is made to simplify the expressions (i.e. $\xi \le \tau_s$). The analysis shows that the closed loop gain gives a good indication of the regulation properties of the regulator and is therefore used in the local stability analysis as a figure of merit.

After the closed loop gain has been defined for the various regulators, the conditions for local stability are obtained. In the case of the buck regulator the local stability criteria are exact. These criteria can be simplified by making the assumption that $\tau_s \ll 1$. In the case of the boost and buck-boost regulators the assumptions that $\tau_s \ll 1$ and $\xi \leq \tau_s$ are used to obtain simplified stability criteria. The assumption that $\tau_s \ll 1$ is usually valid since the ripple voltage, which is usually small, is directly proportional to the switching period. The local stability criteria are expressed in relation to the closed loop gain so that the various control laws can be compared for regulation. The closed loop gain can be increased by increasing the feedback constants, but if the feedback constants are made too large, the regulator becomes unstable. The properties of the linear regulator are described when the feedback constants are specified.

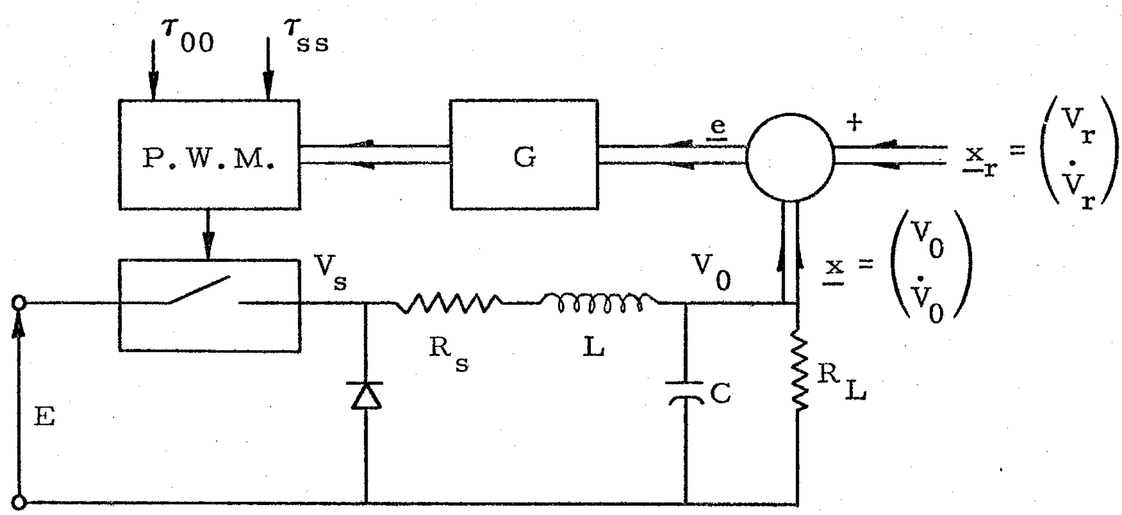
In Chapter 5 techniques for analyzing the global properties of the nonlinear regulator are discussed, and an example is given to illustrate how these techniques can be applied. The first technique discussed is Liapunov's direct method for determining stability as applied to discrete systems. The other technique used is that of pairing a continuous system to the discrete system, and this technique which is due to R. E. Kalman, ref. [8], is known as the paired system method. The analysis is performed in the discrete phase plane. The discrete phase plane for these regulators has regions where the system saturates so that in these regions the system is linear relative to some center. It is in these regions that a continuous system can be paired to the discrete system. Sufficient conditions for global stability are obtained by showing that the Liapunov function always decreases even though it might not decrease every step. If for some step the Liapunov function does not decrease, then it is possible, at least for the example given, to follow the discrete system by means of the method of paired systems until it does decrease.

The true trajectories of a buck regulator in the saturated regions are the same as the trajectories of the paired continuous system. The reason the trajectories are identical for the buck regulator is that the voltage is either on or off the entire switching period. If the discrete system is continuously monitored, and the system made so that trajectories can be changed during the switching period, then the discrete system becomes a continuous system, and a switching line can be defined. In fact, the optimal switching curve which minimizes the time the system takes to reach the origin can be found for such a regulator. For the other regulators, it is possible to find the optimal switching curve associated with the trajectories of the paired systems, and this curve is believed to approximate the optimal switching curve of the discrete system.

# CHAPTER II - BUCK REGULATOR

## 2.1 Recursion Formula

The buck regulator will be the first type of switching regulator to be analyzed. This regulator provides an efficient way of stepping down the supply voltage in a D.C. system to the desired output level. A block diagram of the buck regulator is shown in fig. 2.1.




Buck Regulator

Fig. 2.1

The P.W.M., pulse width modulator, controls the switch such that when no error exists (i.e. $\underline{e} \equiv \underline{0}$), the on-time and switching period will be $\tau_{00}$ and $\tau_{ss}$ respectively. When the error vector is not zero, the input to the P.W.M. will be the gain matrix, G, multiplied by the error, $\underline{e}$. The on-time and/or switching period will be modified to decrease this error.

The differential equation for the filter is

$$\frac{d^2V_0}{d\tau^2} + 2\xi \frac{dV_0}{d\tau} + V_0 = \kappa V_s(\tau)$$

where

$$\omega_\kappa^2 = \frac{1}{LC}\left(1 + \frac{R_s}{R_L}\right) \qquad \tau = \omega_\kappa t$$

$$\xi = \frac{1}{2R_L}\sqrt{\frac{L}{C}}\left[\frac{1 + \mu_s \frac{R_L}{L}}{\sqrt{1 + R_s/R_L}}\right]$$

with

$$\mu_s = R_s C \ , \qquad \kappa = R_L/(R_L + R_s)$$

and

$$V_s(\tau) = \begin{cases} E & \text{switch-on} \\ 0 & \text{switch-off} \end{cases}$$

The differential equation is dimensionless with respect to time. The solution to the equation in state vector notation is

$$\underline{x}(\tau) = Y(\tau)\,\underline{x}(0) + \int_0^\tau Y(\tau - s)\,\underline{b}(s)\,ds$$

where

$$\underline{b}(s) = \begin{pmatrix} 0 \\ \kappa V_s(s) \end{pmatrix} \qquad \underline{x}(\tau) = \begin{pmatrix} x \\ \dot{x} \end{pmatrix} = \begin{pmatrix} V_0 \\ \frac{dV_0}{d\tau} \end{pmatrix}$$

and

$$Y(\tau) = \begin{pmatrix} y_{11}(\tau) & y_{12}(\tau) \\ y_{21}(\tau) & y_{22}(\tau) \end{pmatrix} \text{ - principal matrix solution}$$

The principal matrix solution can assume three different forms depending on the damping factor $\xi$.

$\underline{\xi < 1}$:

$$y_{11}(\tau) = e^{-\xi\tau}\left(\cos\omega_d\tau + \frac{\xi}{\sqrt{1-\xi^2}}\sin\omega_d\tau\right)$$

$$y_{12}(\tau) = e^{-\xi\tau}\frac{\sin\omega_d\tau}{\sqrt{1-\xi^2}}$$

$$y_{21}(\tau) = -y_{12}(\tau)$$

$$y_{22}(\tau) = e^{-\xi\tau}\left(\cos\omega_d\tau - \frac{\xi}{\sqrt{1-\xi^2}}\sin\omega_d\tau\right)$$

$\underline{\xi = 1}$:

$$y_{11}(\tau) = e^{-\tau}(1+\tau)$$

$$y_{12}(\tau) = \tau e^{-\tau}$$

$$y_{21}(\tau) = -y_{12}(\tau)$$

$$y_{22}(\tau) = e^{-\tau}(1-\tau)$$

$\underline{\xi > 1}$:

$$y_{11}(\tau) = e^{-\xi\tau}\left(\cosh\omega_c\tau + \frac{\xi}{\sqrt{\xi^2-1}}\sinh\omega_c\tau\right)$$

$$y_{12}(\tau) = e^{-\xi\tau} \frac{\sinh \omega_c \tau}{\sqrt{\xi^2 - 1}}$$

$$y_{21}(\tau) = -y_{12}(\tau)$$

$$y_{22}(\tau) = e^{-\xi\tau}(\cosh \omega_c \tau - \frac{\xi}{\sqrt{\xi^2 - 1}} \sinh \omega_c \tau)$$

where

$$\omega_d = \sqrt{1 - \xi^2}$$

$$\omega_c = \sqrt{\xi^2 - 1}$$

The switch's voltage, $V_s(\tau)$, is only on during a portion of the switching period so that the value of the state vector at the end of the switching period is

$$\underline{x}(\tau_s) = Y(\tau_s)\,\underline{x}(0) + \int_0^{\tau_0} Y(\tau_s - s)\,\underline{b}'\,ds$$

where

$$\underline{b}' = \begin{pmatrix} 0 \\ \kappa E \end{pmatrix} \quad \text{- constant vector}$$

Since this relation will hold for any switching cycle, a recursion equation can be derived for the state vector.

$$\underline{x}_{n+1} = Y(\tau_s)\,\underline{x}_n + \int_0^{\tau_0} Y(\tau_s - s)\,\underline{b}'\,ds$$

This equation is easily integrated to give

$$\left.\begin{aligned} \underline{x}_{n+1} &= Y(\tau_s)\underline{x}_n + \underline{f}(\tau_0,\tau_s) \\ \text{where} \\ \underline{f}(\tau_0,\tau_s) &= \kappa E\begin{pmatrix} y_{11}(\tau_s-\tau_0) - y_{11}(\tau_s) \\ \\ y_{12}(\tau_s) - y_{12}(\tau_s-\tau_0) \end{pmatrix} \end{aligned}\right\} \quad (2.1)$$

Eqn. (2.1) is the recursion formula for a buck regulator. This equation along with the control law is all that is needed to completely describe the system.

The P.W.M. controls the regulator by varying the on-time and/or switching period. The control laws examined in this thesis are either linear or can be approximated adequately by their linear part so that

$$\tau_0(\underline{x}_n) = \tau_{00} + a_1(x_r-x_n) + b_1(\dot{x}_r-\dot{x}_n) \quad (2.2a)$$

and

$$\tau_s(\underline{x}_n) = \tau_{ss} + a_2(x_n-x_r) + b_2(\dot{x}_n-\dot{x}_r) \quad (2.2b)$$

Eqn. (2.2a) is the control law for the on-time whereas eqn. (2.2b) is the control law for the switching period. The coefficients in these general control laws will be different for the various P.W.M.s analyzed.

## 2.2 Regulation and Local Stability

After the control laws are given, the recursion formula, eqn. (2.1), is completely determined, and the steady-state can be found. If no control is used the on-time and switching period are constant, and the state vector is

$$\underline{x}_{n+1} = \underline{x}_n = \underline{x}_{ss} \qquad \text{steady-state value without feedback}$$

so

$$\underline{x}_{ss} = [I - Y(\tau_{ss})]^{-1}\underline{f}(\tau_{00}, \tau_{ss})$$

where

$$\tau_0(\underline{x}_{ss}) = \tau_{00}$$

$$\tau_{ss}(\underline{x}_{ss}) = \tau_{ss}$$

The matrix inversion can be carried out, and the value of the state vector is

$$x_{ss} = \kappa E \left\{ \frac{y_{11}(D_0'\tau_{ss}) - y_{11}(\tau_{ss}) + e^{-2\xi\tau_{ss}}[1 - e^{2\xi D_0 \tau_{ss}} y_{22}(D_0\tau_{ss})]}{1 + e^{-\xi\tau_{ss}}[e^{-\xi\tau_{ss}} - 2\cos(\omega_d \tau_{ss})]} \right\} \tag{2.3a}$$

$$\dot{x}_{ss} = \kappa E \left\{ \frac{y_{21}(D_0'\tau_{ss}) - y_{21}(\tau_{ss}) \quad e^{-2\xi D_0'\tau_{ss}} y_{21}(D_0\tau_{ss})}{1 + e^{-\xi\tau_{ss}}[e^{-\xi\tau_{ss}} - 2\cos(\omega_d \tau_{ss})]} \right\}$$

where

$$D_0' = 1 - D_0$$

If the assumption is made that $\tau_{ss} \ll 1$, then the above equations can be expanded in a Taylor series to give a first order approximation

$$x_{ss} = D_0 E$$
$$\dot{x}_{ss} = -\tfrac{1}{2}\tau_{ss} D_0 D_0' E \qquad (2.3b)$$

Eqn. (2.3) was derived for an uncontrolled regulator so if any of the parameters change, a new steady-state will result. The purpose of feedback is to minimize the change of state due to a change in parameters. For simplicity the reference vector is taken to be the steady-state value under design conditions so that when the regulator is operating at the design point, $\underline{x}_{ss}$, there is no error. Eqn. (2.2) can now be rewritten with $\underline{x}_{ss}$ substituted for $\underline{x}_r$.

The equilibrium point of the regulator with feedback cannot be solved analytically. It can be obtained with a computer by solving the following transcendental equation

$$\underline{x}_{sf} = Y[\tau_s(\underline{x}_{sf})]\underline{x}_{sf} + \underline{f}[\tau_0(\underline{x}_{sf}), \tau_s(\underline{x}_{sf})] \qquad (2.4a)$$

where

$$\underline{x}_{sf} = \text{steady-state vector with feedback}$$

If the assumption that $\tau_{ss} \ll 1$ is made, then the equations can be simplified to

$$x_{sf} \cong \kappa E D$$
$$\dot{x}_{sf} \cong -\tfrac{1}{2}\kappa E \tau_s D D' \qquad (2.4b)$$

where

$$D = \frac{\tau_0(\underline{x}_{sf})}{\tau_s(\underline{x}_{sf})} \quad \text{and} \quad D' = 1 - D$$

Eqn. (2.4b) is still a nonlinear equation in the variable $\underline{x}_{sf}$. In Appendix (I.A) eqn. (2.4b) is linearized about the design point to give

$$x_{sf} = \frac{x_{ss} + \frac{\kappa D_0 E}{S}\left[1 + \frac{\kappa D_0 E}{2}(b_1 + b_2) + \frac{\dot{x}_{ss}}{D_0 \tau_{ss}}(b_1 + D_0 b_2)\right]}{1 + \frac{1}{S}\left[1 + \frac{\kappa E}{2}\left[(D_0 - D_0')b_1 + D_0^2 b_2\right)\right]}$$

$$\dot{x}_{sf} = \frac{S1\dot{x}_{ss} - \frac{1}{2}\kappa\tau_{ss} D_0 D_0' E\left[1 + \frac{\kappa D_0 E}{\tau_{ss}}\left(\frac{a_1 + a_2}{D_0'}\right) - \frac{x_{ss}}{\tau_{ss}}\left[\left(\frac{D_0 - D_0'}{D_0 D_0'}\right)a_1 + \frac{D_0}{D_0'}a_2\right]\right]}{S1 + \left[1 + \frac{\kappa E}{\tau_{ss}}(a_1 + D_0 a_2)\right]} \qquad (2.5)$$

where

$$S = \frac{\kappa E}{\tau_{ss}}(a_1 + D_0 a_2) + \frac{\kappa^2 E^2 D_0 D_0'}{2\tau_{ss}}(a_1 b_2 - b_1 a_2)$$

$$S1 = \frac{\kappa E}{\tau_{ss}}\left[(D_0 - D_0')b_1 + D_0^2 b_2\right) + \frac{\kappa^2 D^2 D_0 D_0'}{2\tau_{ss}}(a_1 b_2 - b_1 a_2)$$

The above equation, eqn. (2.5), was derived mainly to define the loop gain, S. A large value for the loop gain implies that the voltage is close to the desired voltage and thereby indicates the regulator's ability to regulate. The loop gain cannot be increased to any desired value because at some point the regulator will become unstable. The maximum value the loop gain can achieve before it becomes unstable, $S_{max}$, is used as a figure of merit in the stability analysis. J.M. Yuh defines the loop gain in ref. [17] on page 118.

The validity of eqn. (2.5) is restricted to small changes in the on-time or switching period, see App. (I.A). However, this restriction does tolerate a fairly large variation in the load and input voltage parameters. In fact, according to the linearized equations, variations in the load have little effect on the equilibrium point, $\underline{x}_{sf}$, the only dependence being due to the changes in the parameter

$$\kappa = \frac{R_L}{R_L + R_s} = \frac{1}{1 + R_s/R_L}$$

Since the regulator is designed to be efficient, the ratio of the resistances, $R_s/R_L$, must be small for all operating conditions. Switching regulators do not have the same load regulation problems as the conventional, resistive type regulators.

The exact solution to eqn. (2.4) for the equilibrium point, $\underline{x}_{sf}$, also shows very little dependence on load changes. The exact and approximate equilibrium points will be compared for some of the control laws analyzed. It is necessary to know the equilibrium point because the stability of the system is defined relative to it. The system is said to be asymptotically stable if it returns to the equilibrium point after being disturbed. For small disturbances from equilibrium, the recursion formula, eqn. (2.1), can be approximated by the first two terms in its Taylor series expansion let $$g(\underline{x}_n) = Y(\tau_s)\,\underline{x}_n + \underline{f}[\tau_0(\underline{x}_n), \tau_s(\underline{x}_n)]$$

then $$\underline{x}_{n+1} = \underline{g}(\underline{x}_n) \tag{2.1}$$

and $$\underline{x}_{n+1} \simeq \underline{g}(\underline{x}_{sf}) + \frac{\partial \underline{g}}{\partial \underline{x}_n}\bigg|_{\underline{x}_{sf}} \delta \underline{x}_n + \cdots$$

Since the expansion is carried out about the equilibrium point,

$$\underline{x}_{sf} = \underline{g}(\underline{x}_{sf})$$

so that

$$\delta \underline{x}_{n+1} = \underline{x}_{n+1} - \underline{x}_{sf} \cong \frac{\partial \underline{g}}{\partial \underline{x}_n}\bigg|_{\underline{x}_{sf}} \delta \underline{x}_n$$

The variation in the state of the $(n+1)^{st}$ iteration is related to the state at the $n^{th}$ iteration by

$$\delta \underline{x}_{n+1} = P\, \delta \underline{x}_n \tag{2.2}$$

where

$$P = \frac{\partial \underline{g}}{\partial \underline{x}_n}\bigg|_{\underline{x}_{sf}} \quad \text{- constant matrix}$$

Eqn. (2.2) is called the variational equation and it completely determines the local stability of the regulator. The perturbations, $\delta \underline{x}_n$, will decrease if and only if the modulus of the eigenvalues of the P matrix are less than one. An instructive proof of this statement is given by T. K. Caughey and S. F. Masri in ref. [5].

The elements of the P matrix are evaluated in Appendix (I. B). They are

$$P = \begin{pmatrix} [y_{11}(\tau_s) - a_1\kappa E y_{12}(\tau_s-\tau_0) - a_2h_1][y_{12}(\tau_s) - b_1\kappa E y_{12}(\tau_s-\tau_0) - b_2h_1] \\ [y_{21}(\tau_s) - a_1\kappa E y_{22}(\tau_s-\tau_0) - a_2h_2][y_{22}(\tau_s) - b_1\kappa E y_{22}(\tau_s-\tau_0) - b_2h_2] \end{pmatrix} \quad (2.3)$$

where

$$h_1 = y_{12}(\tau_s)x_{sf} - y_{22}(\tau_s)\dot{x}_{sf} + \kappa E[y_{12}(\tau_s-\tau_0) - y_{12}(\tau_s)]$$

$$h_2 = y_{22}(\tau_s)x_{sf} + [y_{12}(\tau_s) + 2\xi y_{22}(\tau_s)]\dot{x}_{sf} + \kappa E[y_{22}(\tau_s-\tau_0) - y_{22}(\tau_s)]$$

The stability criteria when the eigenvalues of the P-matrix are complex is

$$\text{Det. } (P) < 1$$

and is

$$1 + \text{TR}(P) + \text{Det. } (P) > 0$$

when the eigenvalues are real, see App. (I.B). These stability criterions reduce to

$$\begin{aligned} &(a_2 - 2\xi b_2)\dot{x}_{sf} + b_2(\kappa E - x_{sf}) \\ &\quad + (a_1 + a_2)\kappa E e^{2\xi\tau_0} y_{12}(\tau_0) - (b_1 + b_2)\kappa E e^{2\xi\tau_0} y_{11}(\tau_0) \\ &\quad + (a_1b_2 - a_2b_1)\kappa E e^{2\xi\tau_0}[(\kappa E - x_{sf})y_{12}(\tau_0) + \dot{x}_{sf}\, y_{22}(\tau_0)] \\ &\quad < e^{2\xi\tau_s} - 1 \end{aligned} \quad (2.4a)$$

for complex eigenvalues, and to

$$1 + e^{-\xi\tau_s}(2\cos\omega_d\tau_s + e^{-\xi\tau_s}) + (a_1+a_2)\kappa E[e^{-2\xi(\tau_s-\tau_0)}y_{12}(\tau_0) - y_{12}(\tau_s-\tau_0)]$$

$$- (b_1+b_2)\kappa E[e^{-2\xi(\tau_s-\tau_0)}y_{11}(\tau_0)+y_{22}(\tau_s-\tau_0)] - a_2[(x_{sf} - \kappa E)y_{12}(\tau_s) - \mu\dot{x}_{sf}]$$

$$- b_2[\mu(x_{sf} - \kappa E + 2\xi\dot{x}_{sf}) + y_{12}(\tau_s)\dot{x}_{sf}]$$

$$+ (a_1b_2 - a_2b_1)\kappa E\, e^{-2\xi(\tau_s-\tau_0)}[(\kappa E - x_{sf})y_{12}(\tau_0) + \dot{x}_{sf}\, y_{22}(\tau_0)]$$

$$> 0 \qquad (2.4b)$$

where

$$\mu = e^{-2\xi\tau_s} + y_{22}(\tau_s)$$

for real eigenvalues. The dependence of the stability on the parameters and the equilibrium point is clearly shown. It is interesting to note that there is no explicit dependence on the equilibrium point when only the on-time is varied. The equations derived in this section will be used in the next section to evaluate the stability of various P.W.M.'s.

## 2.3 Comparison of P.W.M.s

The regulation and stability characteristics of the P.W.M.s are related by the loop gain. When the loop gain is made large in order to decrease the change in voltage due to changes in parameters, then the system becomes unstable. The largest loop gain, $S_{max}$, for which the regulator remains stable is determined, and it is used to compare the different P.W.M. s. The stability is defined relative to the equilibrium point, $\underline{x}_{sf}$, and will not be valid for other equilibrium points. When the parameters change, the equilibrium changes and even the loop gain, which is also a function of the parameters, will change.

2.3.1 Uniformly Sampled Voltage P.W.M.

This P.W.M. samples the voltage at the beginning of each switching cycle. The control law for the V.O.T., variable on-time, regulator is

$$\tau_0(x_n) = \tau_{00} + a_1(x_{ss} - x_n)$$

so that

$$b_1 = a_2 = b_2 = 0$$

The loop gain of this P.W.M., from eqn. (2.5), is

$$S = \frac{\kappa a_1 E}{\tau_{ss}}$$

and because the eigenvalues of the P matrix are complex, the stability criterion is given by eqn. (2.4a)

$$a_1 \kappa E\, e^{2\xi\tau_0} y_{12}(\tau_0) < e^{2\xi\tau_{ss}} - 1$$

The stability criterion can be rewritten as

$$\frac{a_1 \kappa E}{\tau_{ss}} < \frac{e^{2\xi\tau_{ss}} - 1}{\tau_{ss} e^{2\xi\tau_0}\, y_{12}(\tau_0)}$$

or

$$S < \frac{e^{2\xi\tau_{ss}} - 1}{\tau_{ss} e^{2\xi\tau_0}\, y_{12}(\tau_0)} = S_{max} \qquad (2.5a)$$

where

$$\tau_0 = \tau_{00} + a_1(x_{ss} - x_{sf})$$

Eqn. (2.5a) can be further simplified if the assumption that $\tau_s = 2\pi(\omega_\kappa/\omega_s) \ll 1$ is made, then

$$S < \frac{2\xi}{\tau_0} \approx S_{max} \qquad (2.5b)$$

The local asymptotic stability about the equilibrium point, $\underline{x}_{sf}$, is guaranteed for closed loop gains, S, less than the critical loop gain, $S_{max}$.

The equilibrium point, $\underline{x}_{sf}$, can be approximated by eqn. (2.5), and it reduces, for this P.W.M., to

$$x_{sf} = \frac{x_{ss} + \kappa D_0 E/S}{1 + 1/S}$$

$$\dot{x}_{sf} = \frac{\dot{x}_{ss} + \frac{1}{2}\tau_{ss}D_0(x_{ss} - \kappa D_0 E) - \frac{1}{2}\kappa\tau_{ss}D_0 D_0' E/S}{1 + 1/S} \qquad (2.6)$$

When the loop gain is zero, eqn. (2.6) gives, as it should, the same equilibrium values as the uncontrolled regulator, eqn. (2.3b). If the loop gain were made very large (i.e. $S \to \infty$), the equilibrium voltage would approach the reference value, but the derivative of the voltage would not.

If the switching period is varied instead of the on-time, then the control law becomes

$$\tau_s(x_n) = \tau_{ss} + a_2(x_n - x_{ss})$$

so that

$$a_1 = b_1 = b_2 = 0$$

The loop gain of this P.W.M. is

$$S = \frac{\kappa a_2 D_0 E}{\tau_s}$$

and the stability criterion is

$$a_2 \dot{x}_{sf} + a_2 \kappa E e^{2\xi\tau_{00}} y_{12}(\tau_{00}) < e^{2\xi\tau_s} - 1$$

or

$$S < \frac{D_0(e^{2\xi\tau_s} - 1)}{\tau_s\left[\frac{\dot{x}_{sf}}{\kappa E} + e^{2\xi\tau_{00}} y_{12}(\tau_{00})\right]} \tag{2.7a}$$

If the assumption is made that $\tau_s \ll 1$, and the equilibrium point is taken to be the design point, $\underline{x}_{ss}$, then eqn. (2.7a) can be approximated as

$$S < \frac{4\xi}{(\tau_{ss} + \tau_{00})} \approx S_{max} \tag{2.7b}$$

The local stability of this regulator, unlike the V.O.T. regulator, depends explicitly on the equilibrium point.

The approximate value for the equilibrium point, $\underline{x}_{sf}$, of the V.S.P., variable switching period, controlled regulator is

$$x_{sf} = \frac{x_{ss} + \kappa D_0 E/S}{1 + 1/S}$$

$$\dot{x}_{sf} = \frac{\dot{x}_{ss} + \frac{1}{2}\tau_{ss}(x_{ss} - \kappa D_0 E) - \frac{1}{2}\kappa\tau_{ss} D_0 D_0' E/S}{1 + 1/S} \qquad (2.8)$$

Except for a slight modification in $\dot{x}_{sf}$, the equilibrium for the V.S.P. regulator, eqn. (2.8), is identical to the equilibrium for the V.O.T. regulator, eqn. (2.6). The closed loop gain, S, as defined, will affect the regulation of both regulators in the same manner.

In figs. 2.2 and 2.3 the maximum closed loop gain, $S_{max}$, is plotted against the damping factor, $\xi$. The recursion formula, eqn. (2.1), along with the proper control law was used on a digital computer to simulate the regulator. The curve for the local stability was verified by slightly decreasing and then increasing the closed loop gain from the predicted value. If the system was stable at the lower value and unstable at the higher for a small perturbation from equilibrium, the predicted value was judged correct. When a comparison is made between the two control laws, the variable on-time control is found to be superior to the variable switching period control, and the two approach each other as the on-time approaches the switching period (i.e. $D_0 \to 1.0$). The exact stability expressions can be approximated by simple equations such as eqn. (2.5b) when $\tau_s \ll 1$. These simpler relationships are reasonably accurate, and they could certainly be used in the first stages of a design to compare different control laws.

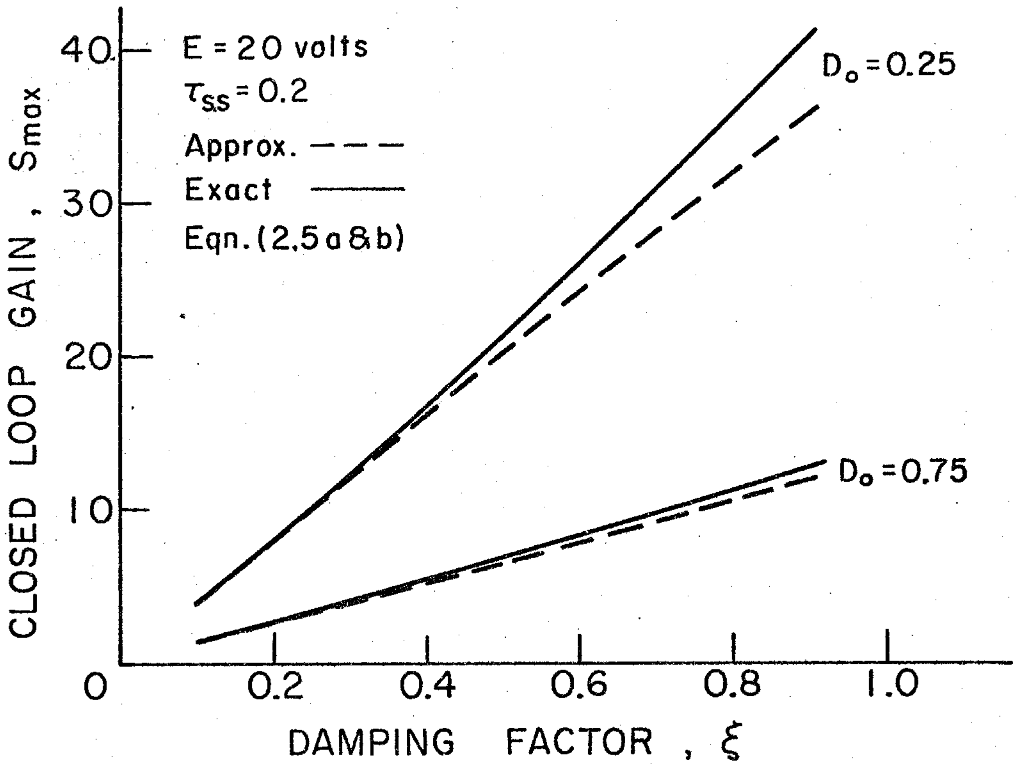



Fig. 2.2. Stability of V.O.T. Controlled Buck Regulator

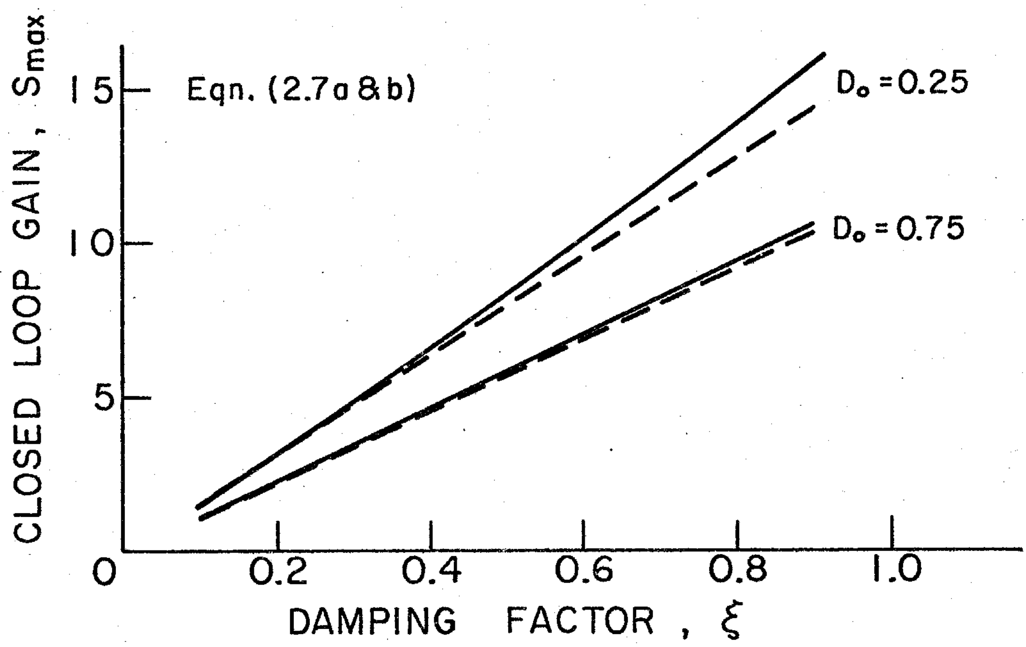



Fig. 2.3. Stability of V.S.P. Controlled Buck Regulator

The purpose of these switching regulators is to regulate voltage so that when the input parameters change the output voltage will remain constant. In figs. 2.4 and 2.5 the output voltage is found for a wide range of input voltages and loads by solving eqn. (2.4a) numerically. This solution is then compared with the approximate solution of eqn. (2.5). The domain of the parameters are restricted to those values which produce an on-time greater than zero but less than the switching period, $\tau_s$. The domain of the load ratio, $R_L^*/R_L$, where the * quantities refer to the input variables at the design point, is also limited to those values which result in a damping factor less than one. The maximum loop gain was used to determine the equilibrium output voltage. This value for the loop gain could not be used in practice since the system would converge very slowly, if at all, to the equilibrium point. In fact the regulator is locally unstable for voltage ratios, $E/E^*$, greater than one and will never converge in this range of parameters. The graphs do indicate the best possible regulation attainable using these P.W.M.s. The curves for the approximate and exact solution of the voltage agree well over a wide range of parameters. The dependence of the voltage on the load ratio is very small in comparison with the dependence on the input voltage.

The main function of these regulators is to regulate against input voltage variations and not load variations. An open loop controller which would change the design duty cycle, $D_0 = \tau_{00}/\tau_{ss}$, by determining the ratio of the reference to the input voltage would enhance the regulation properties of these devices. Such a controller would be required to sense and average the input voltage to obtain the D.C. value, and then change either the on-time, $_{00}$, or the switching period, $_{ss}$, accordingly.

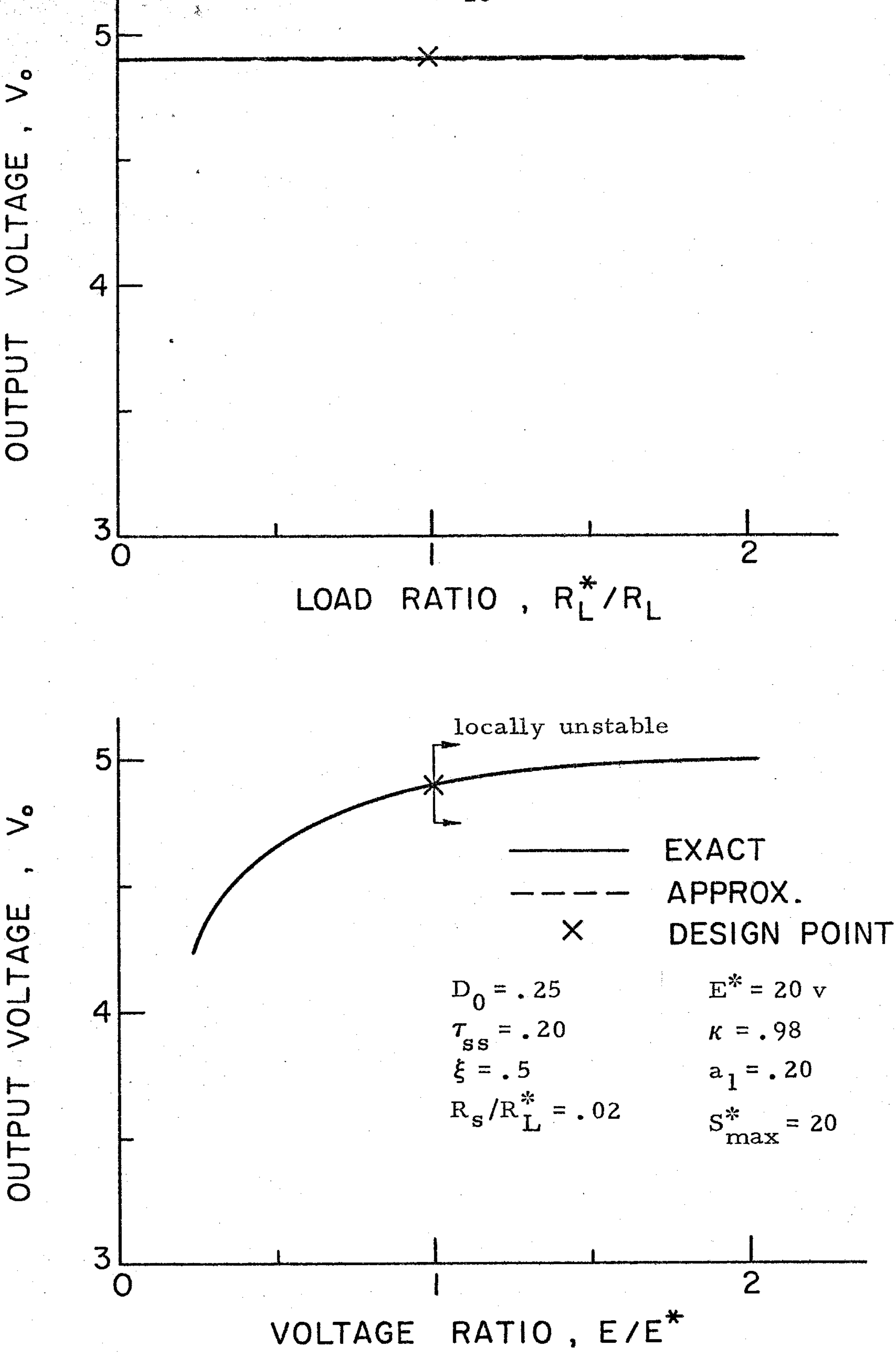



Fig. 2.4. Voltage Regulation for V.O.T. Controlled Buck Regulator

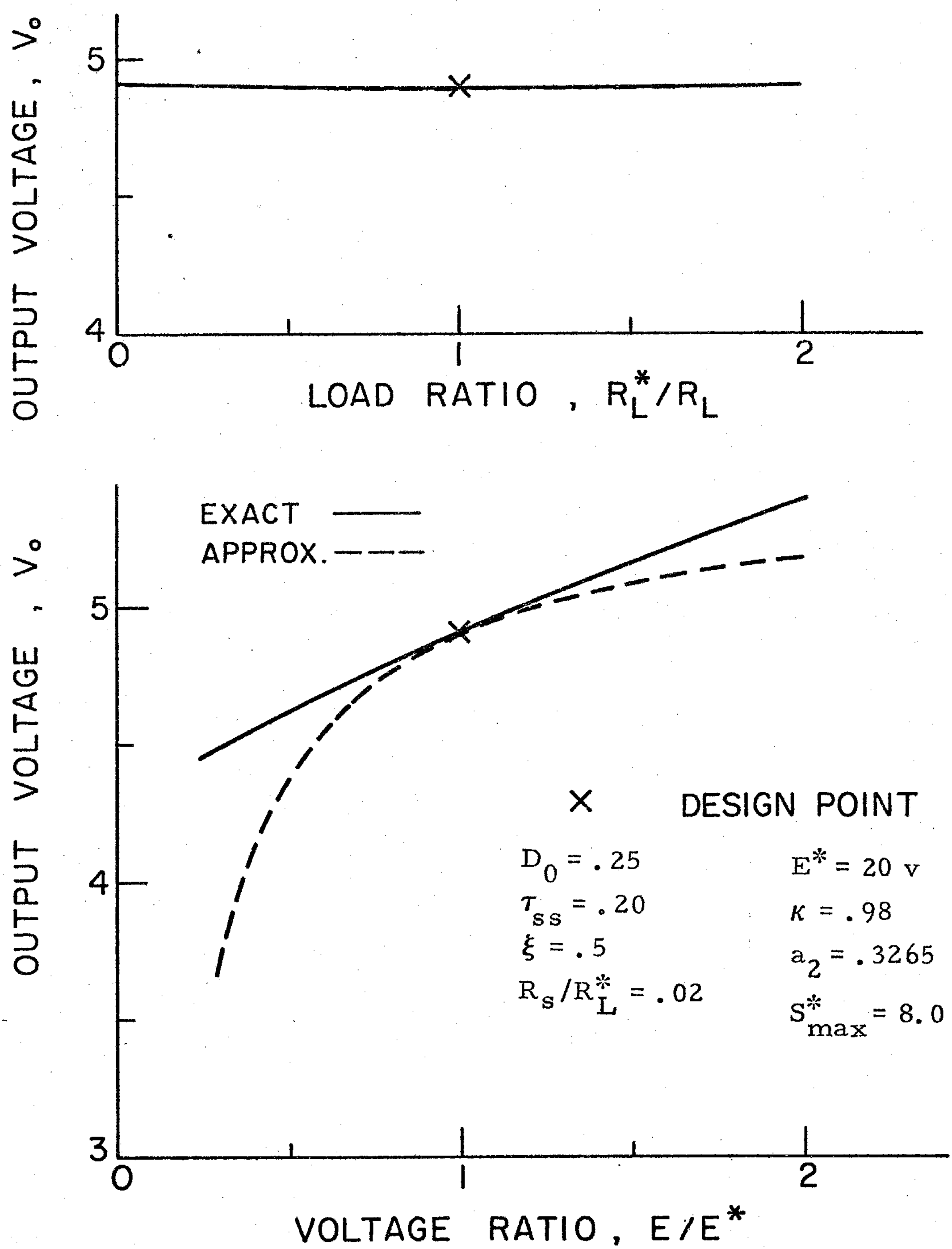



Fig. 2.5. Voltage Regulation for V.S.P. Controlled Regulator

2.3.2 Error Integrating P.W.M.

The error integrating P.W.M. integrates the output voltage of the regulator to a prescribed value. The time it takes for this integration process to occur is the duration of the pulse, $\tau_0$. The prescribed value is a constant, $E_R \tau_{ss}$, and the control law is given implicitly by

$$E_R \tau_{ss} = \int_{n\tau_{ss}}^{n\tau_{ss}+\tau_0} V_0(t)\,dt = \int_0^{\tau_0} \{y_{11}(t)x_n + y_{12}(t)\dot{x}_n + \kappa E[1 - y_{11}(t)]\}\,dt \tag{2.9}$$

where $\underline{x}_n$ is the state at the beginning of the $n^{th}$ switching period. A small variation in the state at the beginning of the $n^{th}$ switching period will result in a small variation of the on-time, $\tau_0$,

$$0 = \int_0^{\tau_0} \{y_{11}(t)\delta x_n + y_{12}(t)\delta \dot{x}_n\}\,dt$$

$$+ \{y_{11}(\tau_0)x_{sf} + y_{12}(\tau_0)\dot{x}_{sf} + \kappa E[1 - y_{11}(\tau_0)]\}\,\Delta\tau_0$$

where $\underline{x}_{sf}$ is the equilibrium point of the regulator. The coefficient of the $\Delta\tau_0$ term is just the voltage evaluated at $\tau_0$, $x(\tau_0)$. After the integral has been evaluated, the variation in on-time is found to be

$$\Delta\tau_0 = -\frac{1}{x(\tau_0)}\{y_{12}(\tau_0) + 2\xi[1 - y_{11}(\tau_0)]\}\,\delta x_n - \frac{1}{x(\tau_0)}[1 - y_{11}(\tau_0)]\,\delta\dot{x}_n \tag{2.10}$$

The coefficients of $\delta x_n$ and $\delta\dot{x}_n$ will take the place of $a_1$ and $b_1$, respectively, in the stability analysis, see App. (I.B). The closed loop gain, S, could be defined as

$$S \doteq \frac{a_1 \kappa E}{\tau_{ss}} = + \frac{\kappa E}{x(\tau_0)} \left( \frac{y_{12}(\tau_0) + 2\xi[1 - y_{11}(\tau_0)]}{\tau_{ss}} \right) \tag{2.11a}$$

It is not necessary, and in fact it would be inconsistent, to use the above form for the closed loop gain since in arriving at the figure of merit it was assumed that $\tau_s \ll 1$. A more consistent definition for the closed loop gain is

$$S = \frac{\kappa E \tau_0}{x(\tau_0)\tau_{ss}} \tag{2.11b}$$

where the assumption that $\tau_{ss} \ll 1$ has been used to reduce eqn. (2.11a). The closed loop gain given in eqn. (2.11b) is the figure of merit for the error integrating P.W.M. The determination of $x(\tau_0)$ involves solving eqn. (2.9) along with eqn. (2.1) for the on-time, $\tau_0$, and the state, $\underline{x}_{sf}$. If it can be assumed that the time average of the voltage over the interval $\tau_0$ is approximately $x(\tau_0)$, then from eqn. (2.9)

$$E_R \tau_{ss} \approx x(\tau_0)\tau_0$$

and

$$S \approx \frac{\kappa E}{E_R} \frac{\tau_0^2}{\tau_{ss}^2} \tag{2.11c}$$

The closed loop gain defined by eqn. (2.11c) is the same one used by Dr. Yuh in ref. [17].

The local stability of the error integrating P.W.M. is given by eqn. (2.4a)

$$\frac{\kappa E}{x(\tau_0)} \frac{\tau_0}{\tau_{ss}} \{y_{12}(\tau_0) + 2\xi[1 - y_{11}(\tau_0)]\} e^{2\xi\tau_0} y_{12}(\tau_0)$$

$$- \frac{\kappa E}{x(\tau_0)} \frac{\tau_0}{\tau_{ss}} [1 - y_{11}(\tau_0)] e^{2\xi\tau_0} y_{11}(\tau_0) < \frac{\tau_0}{\tau_{ss}} (e^{2\xi\tau_{ss}} - 1)$$

or

$$S < \frac{\tau_0}{\tau_{ss}} \frac{(e^{2\xi\tau_{ss}} - 1)}{e^{2\xi\tau_0}[y_{12}(\tau_0)\{y_{12}(\tau_0) + 2\xi[1 - y_{11}(\tau_0)]\} - y_{11}(\tau_0)\{1 - y_{11}(\tau_0)\}]}$$

and after simplifying

$$S < \frac{\tau_0}{\tau_{ss}} \frac{(e^{2\xi\tau_{ss}} - 1)}{[1 - e^{2\xi\tau_0} y_{22}(\tau_0)]} = S_{max} \qquad (2.12a)$$

When $\tau_{ss} \ll 1$, eqn. (2.12a) reduces to

$$S < \frac{4\xi}{\tau_0} \approx S_{max} \qquad (2.12b)$$

The local stability of the error integrating P.W.M. is given by eqn. (2.12). The maximum loop gain, $S_{max}$, attainable by this P.W.M., eqn. (2.12b), is double that of the uniformly sampled voltage P.W.M., eqn. (2.5b).

### 2.3.3 Dither Stabilized P.W.M.

The on-time, $\tau_0$, of the dither stabilized P.W.M. is determined when the sum of the output and ramp voltage reach a specified value, $C_1$.

$$C_1 = \frac{E_b}{\tau_{ss}} \tau_0 + V_0(\tau_0)$$

or

$$C_1 = \frac{E_b}{\tau_{ss}} \tau_0 + y_{11}(\tau_0)x_n + y_{12}(\tau_0)\dot{x}_n + \kappa E[1 - y_{11}(\tau_0)]$$

The variation of the on-time due to a variation of the state at the beginning of the $n^{th}$ switching period is given by:

$$0 = \left[\frac{E_b}{\tau_{ss}} + y_{21}(\tau_0)x_{sF} + y_{22}(\tau_0)\dot{x}_{sF} - \kappa E y_{21}(\tau_0)\right]\Delta\tau_0$$

$$+ y_{11}(\tau_0)\delta x_n + y_{12}(\tau_0)\delta\dot{x}_n$$

now

$$\dot{x}(\tau_0) = y_{21}(\tau_0)x_{sF} + y_{22}(\tau_0)\dot{x}_{sF} - \kappa E y_{21}(\tau_0) = \dot{V}_0(\tau_0)$$

so

$$\Delta\tau_0 = -\frac{\tau_{ss} y_{11}(\tau_0)}{E_b + \tau_{ss}\dot{x}(\tau_0)}\delta x_n - \frac{\tau_{ss} y_{12}(\tau_0)}{E_b + \tau_{ss}\dot{x}(\tau_0)}\delta\dot{x}_n \qquad (2.13)$$

The coefficients of $\delta x_n$ and $\delta\dot{x}_n$ play the role of $a_1$ and $b_1$ respectively in the stability analysis, so that the loop gain is

$$S = \frac{a_1 \kappa E}{\tau_{ss}} = \frac{y_{11}(\tau_0)\kappa E}{E_b + \tau_{ss}\dot{x}(\tau_0)} \qquad (2.14a)$$

As in the case of the error integrating P.W.M., a more consistent form of the loop gain is

$$S = \frac{\kappa E}{E_b + \tau_{ss}\dot{x}(\tau_0)} \qquad (2.14b)$$

where the assumption that $\tau_{ss} \ll 1$ has been made. If the added assumption that $\dot{x}(\tau_0) \ll E_b/\tau_{ss}$, then eqn. (2.14b) can be approximated as

$$S = \frac{\kappa E}{E_b} \qquad (2.14c)$$

This last assumption is generally valid, and the closed loop gain defined by eqn. (2.14c) was used by J.M. Yuh in ref. [17].

Since the eigenvalues of the P-matrix are real, the local stability of the dither stabilized P.W.M. is given by eqn. (2.4b)

$$\begin{aligned} 1 &+ e^{-\xi\tau_s}(2\cos\omega_d\tau_s + e^{-\xi\tau_s}) \\ &+ \left[\frac{\kappa E}{E_b + \tau_{ss}\dot{x}(\tau_0)}\right]\tau_{ss}y_{11}(\tau_0)\left[e^{-2\xi(\tau_s-\tau_0)}y_{12}(\tau_0) - y_{12}(\tau_s-\tau_0)\right] \\ &- \left[\frac{\kappa E}{E_b + \tau_{ss}\dot{x}(\tau_0)}\right]\tau_{ss}y_{12}(\tau_0)\left[e^{-2\xi(\tau_s-\tau_0)}y_{11}(\tau_0) + y_{22}(\tau_0)\right] > 0 \end{aligned}$$

substituting in S and simplifying gives

$$S < \frac{(1 + e^{-2\xi\tau_s} + 2e^{-\xi\tau_s}\cos\omega_d\tau_s)}{\tau_{ss}[y_{11}(\tau_0)y_{12}(\tau_s-\tau_0) + y_{12}(\tau_0)y_{22}(\tau_s-\tau_0)]}$$

which reduces to

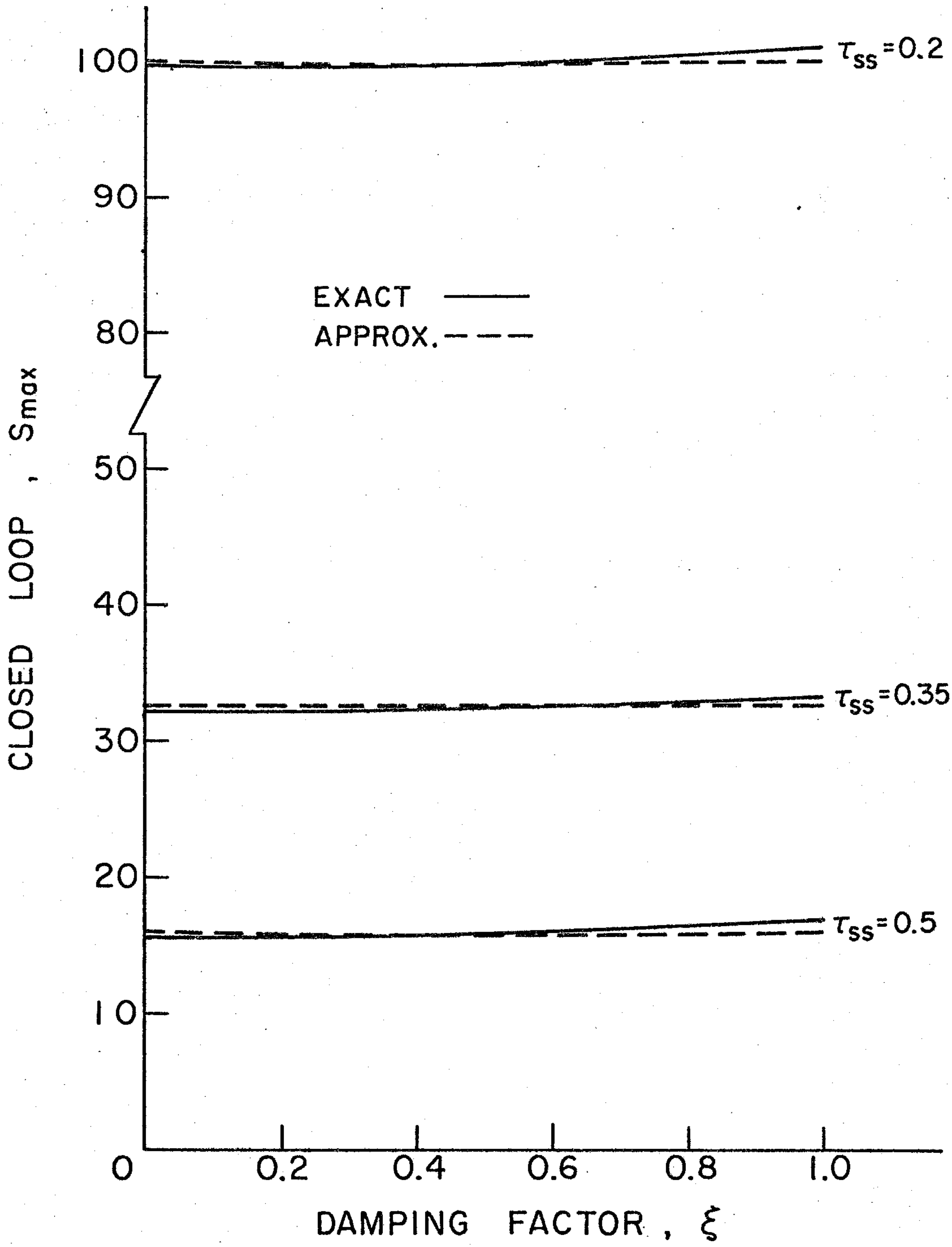



Fig. 2.6. Stability of Dither Stabilized Buck Regulator

$$S < \frac{(1 + e^{-2\xi\tau_{ss}} + 2e^{-\xi\tau_{ss}} \cos \omega_d \tau_{ss})}{\tau_{ss} y_{12}(\tau_{ss})} = S_{max} \qquad (2.15a)$$

If the assumption is made that $\tau_{ss} \ll 1$, then eqn. (2.15a) reduces to

$$S < \frac{4}{\tau_{ss}^2} \approx S_{max} \qquad (2.15b)$$

The maximum closed loop gain for the dither stabilized P.W.M. is independent of the on-time and almost independent of the damping factor. In fig. 2.6 the exact and approximate maximum closed loop gain are compared for various damping factors.

The dither stabilized P.W.M., like the error integrating P.W.M., has a control law which is implicitly dependent on the state over an interval of time. Since the local stability analysis requires an explicit dependence on the state only at the beginning of the switching period, it is necessary to find the variation of the on-time for a variation of the initial state. In this manner the control law is effectively linearized about the equilibrium point, and the approximate feedback constants can be obtained for use in the general stability equations. The loop gain of the dither stabilized P.W.M. is the largest of the P.W.M. s analyzed.

### 2.3.4 Zero Eigenvalue P.W.M.

The previous analysis has been concerned with examining the local stability and its connection with regulation. Increasing the loop gain improves the regulation but decreases the stability. A large loop gain also has the added disadvantage of increasing the time it takes the system to return to equilibrium after being disturbed. The zero eigenvalue P.W.M. is important because it converges rapidly, in two steps for the linearized model. In ref. [4] the authors define the optimal regulator to be the one which has zero eigenvalues. If the eigenvalues of the P matrix are to be zero, the determinate and trace must also be zero. When only the on-time is varied for control, the determinate and trace of the P matrix are

$$\text{Det.}(P) = e^{-2\xi\tau_{ss}} + a_1\kappa E e^{-2\xi(\tau_{ss}-\tau_0)} y_{12}(\tau_0) - b_1 \kappa E e^{-2\xi(\tau_{ss}-\tau_0)} y_{11}(\tau_0) = 0$$

$$\text{TR.}(P) = y_{11}(\tau_{ss}) + y_{22}(\tau_{ss}) - a_1\kappa E y_{12}(\tau_{ss}-\tau_0) - b_1\kappa E y_{22}(\tau_{ss}-\tau_0) = 0$$

The two feedback constants, $a_1$ and $b_1$, can be chosen so that the above equations are satisfied. In matrix form the solution for these constants is

$$\begin{pmatrix} a_1 \\ b_1 \end{pmatrix} = \frac{1}{\kappa E y_{12}(\tau_{ss})} \begin{pmatrix} -y_{22}(\tau_{ss}-\tau_0) & y_{11}(\tau_0) \\ y_{12}(\tau_{ss}-\tau_0) & y_{12}(\tau_0) \end{pmatrix} \begin{pmatrix} e^{-2\xi\tau_0} \\ y_{11}(\tau_{ss}) + y_{22}(\tau_{ss}) \end{pmatrix}$$

and after further simplifications

$$a_1 = \frac{y_{11}(\tau_{ss} + \tau_0)}{\kappa E y_{12}(\tau_{ss})} \tag{2.16a}$$

$$b_1 = \frac{y_{12}(\tau_{ss} + \tau_0)}{\kappa E y_{12}(\tau_{ss})}$$

The feedback constants, as given by eqn. (2.16), are actually functions of the input voltage, the damping factor, and the on-time. It would be necessary to change these constants as the parameters change in order to maintain the convergence properties of the regulator. Alternatively, a set of parameters can be chosen, usually the design parameters, which will be the only set for which the regulator will be optimal.

The closed loop gain for this P.W.M. is then

$$S = \frac{a_1 \kappa E}{\tau_{ss}} = \frac{y_{11}(\tau_{ss}+\tau_0)}{\tau_{ss} y_{12}(\tau_{ss})} \qquad (2.17a)$$

If the constants are fixed so that an increase in the input voltage will increase the loop gain, then the maximum loop gain possible is given by eqn. (2.4b)

$$\frac{1}{\tau_{ss}}[1 + e^{-2\xi\tau_{ss}} + 2e^{-\xi\tau_{ss}} \cos(\omega_d \tau_{ss})]$$

$$+ \frac{a_1 \kappa E}{\tau_{ss}}[e^{-2\xi(\tau_{ss}-\tau_0)} y_{12}(\tau_0) - y_{12}(\tau_s-\tau_0)]$$

$$- \frac{a_1 \kappa E}{\tau_{ss}} \frac{b_1}{a_1}[e^{-2\xi(\tau_{ss}-\tau_{00})} y_{11}(\tau_0) + y_{22}(\tau_s-\tau_0)] > 0$$

but

$$\frac{b_1}{a_1} = \frac{y_{12}(\tau_{ss}+\tau_0)}{y_{11}(\tau_{ss}+\tau_0)}$$

so

$$S[\,y_{12}(\tau_{ss}-\tau_0) - e^{-2\xi(\tau_{ss}-\tau_0)}\,y_{12}(\tau_0)]$$

$$+\ S\,\frac{y_{12}(\tau_{ss}+\tau_0)}{y_{11}(\tau_{ss}+\tau_0)}\,[e^{-2\xi(\tau_{ss}-\tau_{00})}y_{11}(\tau_0)+y_{22}(\tau_s-\tau_0)]$$

$$<\ \frac{1}{\tau_{ss}}\,[\,1+e^{-2\xi\tau_{ss}}+2e^{-\xi\tau_{ss}}\cos(\omega_d\tau_{ss})]$$

which finally simplifies to

$$S\ <\ \frac{y_{11}(\tau_{ss}+\tau_0)[\,1+e^{-2\xi\tau_{ss}}+2e^{-\xi\tau_{ss}}\cos(\omega_d\tau_{ss})]}{\tau_{ss}[\,e^{-2\xi\tau_{ss}}\,y_{12}(\tau_{ss})+y_{12}(2\tau_{ss})]} = S_{max} \qquad (2.18a)$$

If the assumption that $\tau_{ss} \ll 1$ is made, eqn. (2. 17a) and eqn. (2. 18a) reduce to

$$S\ \approx\ \frac{1}{\tau_{ss}^2} \qquad (2.17b)$$

and

$$S\ <\ \frac{4/3}{\tau_{ss}^2}\ \approx\ S_{max} \qquad (2.18b)$$

The zero eigenvalue P. W. M. is a good regulator, eqn. (2. 17), and it converges rapidly, but a thirty-three percent increase in the input voltage will make it locally unstable.

The difficulty with this regulator is that it must operate with a high loop gain to achieve rapid convergence. If the loop gain could be made independent of the input voltage, as it actually should be to have rapid convergence for any set of parameters, then the regulator would be guaranteed to be at least locally stable. If the input voltage was sensed, an analog multiplier could be used to make the feedback constants, when $\tau_s \ll 1$, equal to

$$a_1 \approx \frac{1}{\kappa E \tau_{ss}}$$

$$b_1 \approx \frac{\tau_{ss} + \tau_0}{\kappa E \tau_{ss}}$$

A method of making the loop gain independent of the input voltage would be beneficial to all the P.W.M. s.

In fig. 2.7 the equilibrium voltage is given as a function of the input parameters. The range of the input parameters is limited for a number of reasons. When the load ratio is greater than two, the damping factor, $\xi$, is greater than one, and the form of the recursion formula changes. The bound on the lower voltage ratio occurs when the input voltage is less than the desired output voltage so that the on-time is equal to the switching period. The maximum voltage ratio is limited by local stability considerations, eqn. (2.18b), such that a thirty-three percent increase of the input voltage makes the system locally unstable. An important design consideration is the sensitivity of the local stability to changes in the input parameters. A more important consideration, but a harder one to obtain, is the sensitivity of the global stability to changes in the input parameters. In this example, all the points shown in fig. 2.7 are at least locally stable, and the system does converge to these equilibrium points with zero initial conditions.

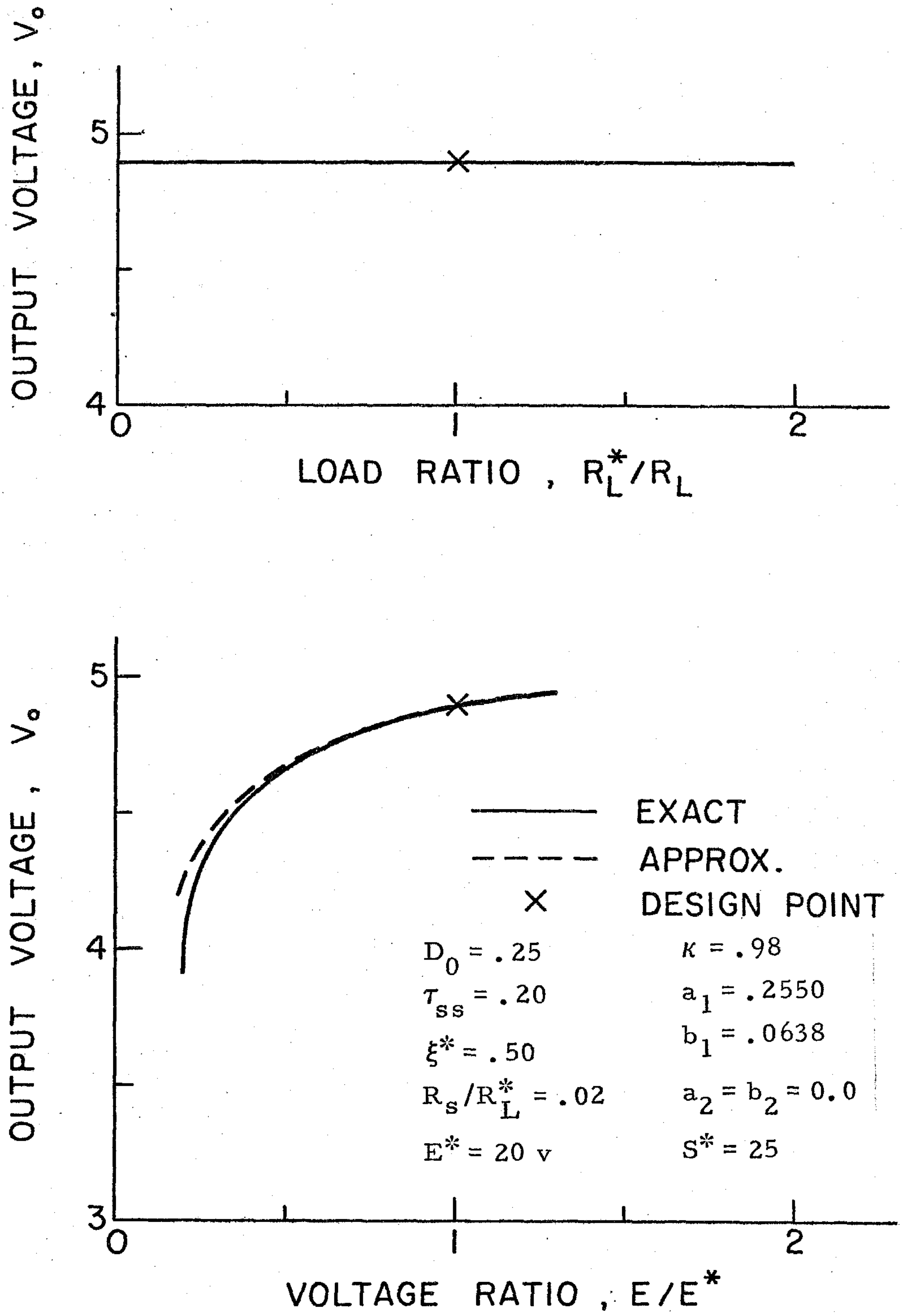



Fig. 2.7. Voltage Regulation for Zero Eigenvalue P.W.M.

2.3.5 Zero Matrix P.W.M.

The zero eigenvalue P.W.M. converges to the equilibrium point in two steps. The zero matrix P.W.M. will converge in one step. The zero matrix P.W.M. requires four constants to be determined so that every element in the P-matrix, eqn. (2.3), is zero.

$$P = \begin{pmatrix} [y_{11}(\tau_s) - a_1\kappa E y_{12}(\tau_s - \tau_0) - a_2 h_1] & [y_{12}(\tau_s) - b_1\kappa E y_{12}(\tau_s - \tau_0) - b_2 h_1] \\ [y_{21}(\tau_s) - a_1\kappa E y_{22}(\tau_s - \tau_0) - a_2 h_2] & [y_{22}(\tau_s) - b_1\kappa E y_{22}(\tau_s - \tau_0) - b_2 h_2] \end{pmatrix}$$

where

$$h_1 = y_{12}(\tau_s)x_{sf} - y_{22}(\tau_s)\dot{x}_{sf} + \kappa E[y_{12}(\tau_s - \tau_0) - y_{12}(\tau_s)]$$

$$h_2 = y_{22}(\tau_s)x_{sf} + [y_{12}(\tau_s) + 2\xi y_{22}(\tau_s)]\dot{x}_{sf} + \kappa E[y_{22}(\tau_s - \tau_0) - y_{22}(\tau_s)]$$

If both V.O.T. and V.S.P. control is used, then the four constants $a_1$, $b_1$, $a_2$, and $b_2$ can be chosen to make the P-matrix identically zero. The solution is

$$a_1 = \frac{h_2 y_{11}(\tau_s) - h_1 y_{21}(\tau_s)}{\kappa E[h_2 y_{12}(\tau_s - \tau_0) - h_1 y_{22}(\tau_s - \tau_0)]}$$

$$a_2 = \frac{y_{21}(\tau_s)y_{12}(\tau_s - \tau_0) - y_{11}(\tau_s)y_{22}(\tau_s - \tau_0)}{h_2 y_{12}(\tau_s - \tau_0) - h_1 y_{22}(\tau_s - \tau_0)} \qquad (2.19a)$$

$$b_1 = \frac{h_2 y_{12}(\tau_s) - h_2 y_{22}(\tau_s)}{\kappa E[h_2 y_{12}(\tau_s - \tau_0) - h_1 y_{22}(\tau_s - \tau_0)]}$$

$$b_2 = \frac{y_{12}(\tau_s - \tau_0)\, y_{22}(\tau_s) - y_{22}(\tau_s - \tau_0)\, y_{12}(\tau_s)}{h_2 y_{12}(\tau_s - \tau_0) - h_2 y_{22}(\tau_s - \tau_0)} \qquad (2.19a) \text{ cont.}$$

These expressions can be greatly simplified by assuming that $\tau_{ss} \ll 1$, then

$$h_1 \approx -\dot{x}_{sf} \approx \tfrac{1}{2}\kappa E D_0 D_0' \tau_s$$

$$h_2 \approx x_{sf} \approx \kappa E D_0$$

and

$$a_1 \approx \frac{2}{\kappa E D_0' \tau_s}$$

$$a_2 \approx -\frac{2}{\kappa E D_0 D_0' \tau_s}$$

$$b_1 \approx \frac{1 + D_0}{\kappa E D_0'} \qquad (2.19b)$$

$$b_2 \approx -\frac{2}{\kappa E D_0'}$$

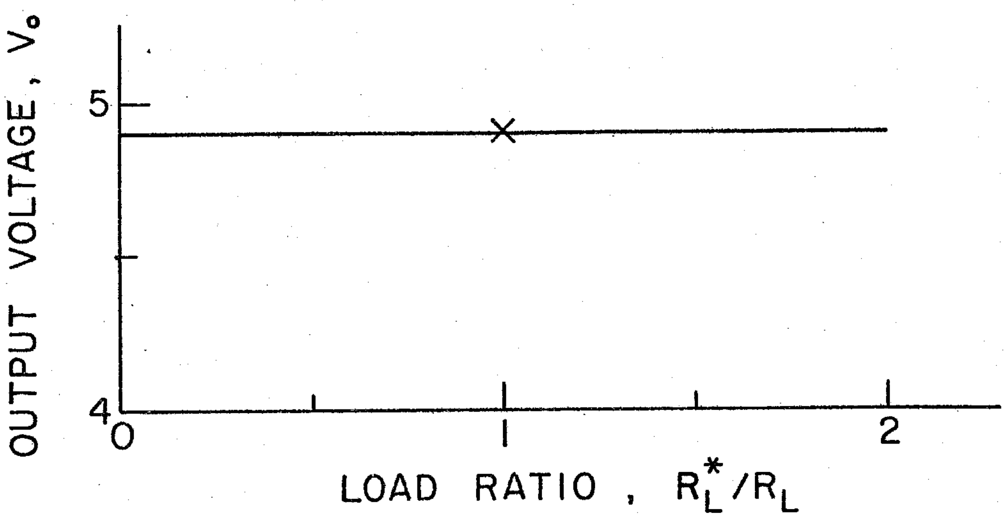
The loop gain is given by eqn. (2. 5) which simplifies, for this regulator, to

$$S = (a_1 b_2 - b_1 a_2)\, \frac{\kappa^2 E^2 D_0 D_0'}{2\tau_{ss}} \qquad (2.20a)$$

since

$$a_1 + D_0 a_2 \approx 0$$

After substituting the values of the constants, eqn. (2. 19b), into the equation for the loop gain, it becomes



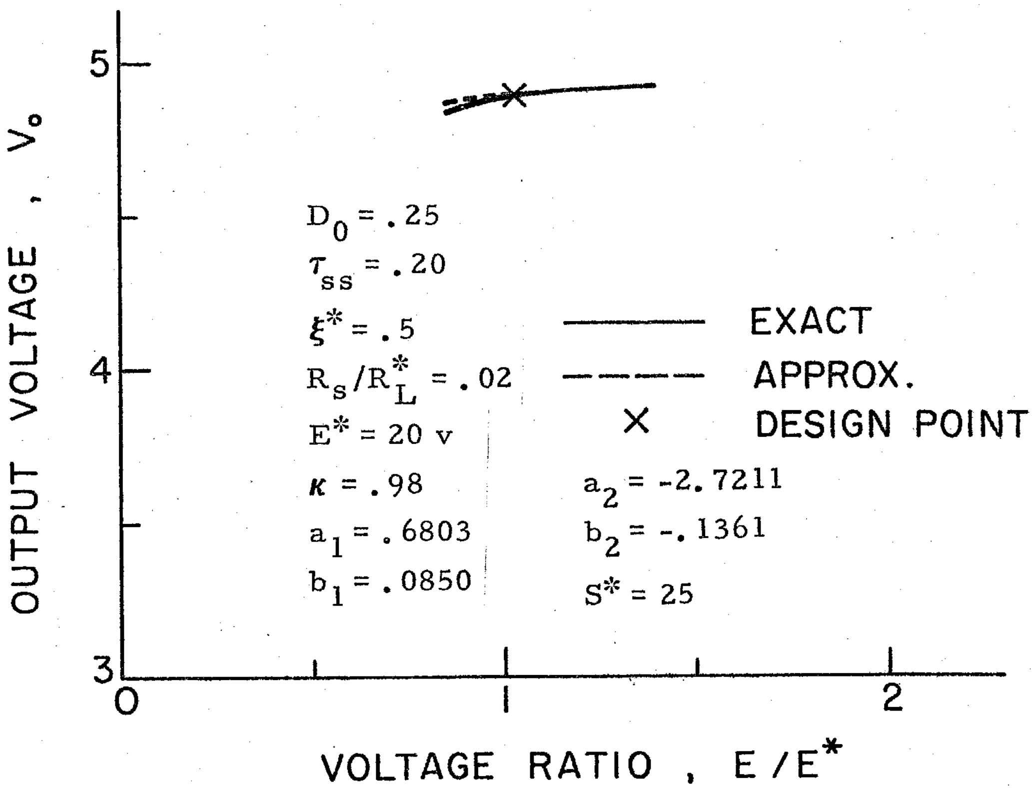




Fig. 2.8. Voltage Regulation for Zero Matrix P.W.M.

$$S = \frac{1}{\tau_{ss}^2} \qquad (2.20b)$$

This is the same value of the loop gain found for the zero eigenvalue P.W.M., compare eqn. (2.20b) with eqn. (2.17b). The loop gains of the two P.W.M. s are defined differently, with the loop gain of the zero matrix P.W.M. dependent upon a product of the various feedback constants, eqn. (2.20a). Like the zero eigenvalue P.W.M., the loop gain of the zero matrix P.W.M. is a function of the parameters, and the system becomes unstable when the input voltage becomes large.

In fig. 2.8 the exact and approximate equilibrium point is plotted against a range of parameters. The regulation is very good and so is the comparison between the exact and approximate solutions. The equilibrium points shown are at least locally stable since the system converges to them from the approximate values. Stability considerations severely limit the range of the voltage ratio. In fact this system is not even globally stable at the design point. The switching period and on-time for this P.W.M. are also very sensitive to changes in the input voltage. For the example shown, a fifteen percent decrease in the input voltage causes the switching period and on-time to more than double.

2.3.6 Discontinuous P.W.M.

At very low damping factors, $\xi < .1$, which usually means a high load resistance, the current in the inductor can become zero.

When this occurs, the P.W.M. is said to be operating in a discontinuous conducting mode, see ref. [3]. In fig. 2.9 the inductor current is plotted against the time. The inductor current goes to zero at time $\tau_c$, and the previous analysis, eqn. (2.1), is only valid during the switching period until $\tau = \tau_c$.

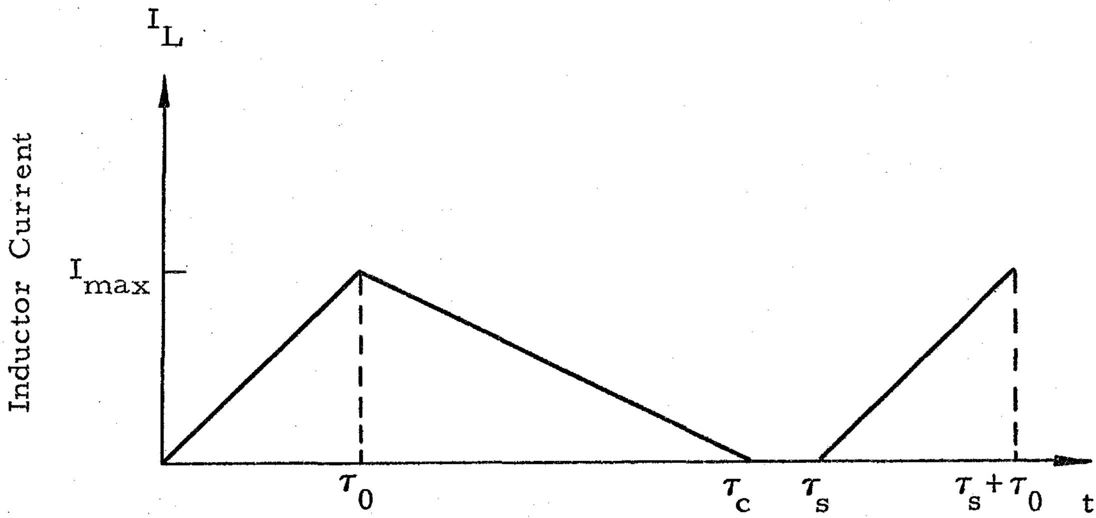



Fig. 2.9

Inductor Current Waveform

The state at which the inductor current goes to zero, $\underline{x}(\tau_c)$, is

$$\underline{x}(\tau_c) = Y(\tau_c)\underline{x}(0) + \kappa E \begin{pmatrix} y_{11}(\tau_c - \tau_0) - y_{11}(\tau_c) \\ y_{12}(\tau_c) - y_{12}(\tau_c - \tau_0) \end{pmatrix} \qquad (2.21)$$

After the inductor current goes to zero, but before the next switching period, $\tau_c \leqslant \tau \leqslant \tau_{ss}$, the capacitor discharges into the load, and the differential equation describing the system is

$$\dot{x}(\tau) + 2\xi_0 x(\tau) = 0 \qquad (2.22)$$

where

$$2\xi_0 = \frac{1}{\omega_\kappa R_L C}$$

The relation between the two damping factors, $\xi$ and $\xi_0$, is

$$\xi = \xi_0 + \kappa\omega_\kappa \mu_s / 2$$

and when $\omega_\kappa \mu_s \ll 1$ (i.e. $R_s \approx 0$), it is reasonable to assume them equal. The differential equation, eqn. (2.22), is easily solved, and the voltage at the end of the switching period is

$$x(\tau_{ss}) = e^{-2\xi_0(\tau_{ss} - \tau_c)} x(\tau_c)$$

When the voltage, $x(\tau_c)$, from eqn. (2.21) is substituted into the above equation, the result is

$$x(\tau_{ss}) = e^{-2\xi_0(\tau_{ss} - \tau_c)}\{y_{11}(\tau_c)x(0) + y_{12}(\tau_c)\dot{x}(0) + \kappa E[y_{11}(\tau_c - \tau_0) - y_{11}(\tau_c)]\}$$

The voltage and its derivative at the beginning of the switching period are related by eqn. (2.22) so that

$$x_{n+1} = e^{-2\xi_0(\tau_{ss} - \tau_c)}[y_{11}(\tau_c) - 2\xi_0 y_{12}(\tau_c)]x_n$$
$$+ \kappa E e^{-2\xi_0(\tau_{ss} - \tau_c)}[y_{11}(\tau_c - \tau_0) - y_{11}(\tau_c)] \qquad (2.23)$$

The matrix recursion formula of eqn. (2.1) is reduced to a scalar equation, eqn. (2.23), for the discontinuous P.W.M. It would appear that the stability analysis should simplify for the scalar case, but it is necessary to solve for the time at which the inductor current becomes zero, $\tau_c$. The added equation needed to solve for the time $\tau_c$ is obtained by substituting the voltage and its derivative from eqn. (2.21) into eqn. (2.22) to give

$$y_{21}(\tau_c)x_n + \kappa E\{y_{12}(\tau_c) - y_{12}(\tau_c - \tau_0) + 2\xi_0[y_{11}(\tau_c - \tau_0) - y_{11}(\tau_c)]\} = 0 \qquad (2.24)$$

where the assumption that $\xi \approx \xi_0$ was used to help simplify the equation. The on-time, $\tau_0$, is defined by the equation

$$\tau_0(x_n) = \tau_{00} + a_1(x_{ss} - x_n) \qquad (2.25)$$

Since the voltage and its derivative are not independent, it is only necessary to have feedback on one.

The above equations, eqn. 2.23-2.25, completely describe the system, and they are all that is needed to solve for the equilibrium voltage, $x_{sf}$, the time, $\tau_c$, and the on-time, $\tau_0$.

$$x_{sf} = \frac{\kappa E[y_{11}(\tau_c - \tau_c) - y_{11}(\tau_c)]}{e^{2\xi_0(\tau_{ss} - \tau_c)} - [y_{11}(\tau_c) - 2\xi_0 y_{12}(\tau_c)]} \qquad (2.23)$$

$$x_{sf} = \frac{\kappa E}{y_{12}(\tau_c)}\{y_{12}(\tau_c) - y_{12}(\tau_c - \tau_0) + 2\xi_0[y_{11}(\tau_c - \tau_0) - y_{11}(\tau_c)]\} \qquad (2.24)$$

and

$$\tau_0 = \tau_{00} + a_1(x_{ss} - x_{sf}) \qquad (2.25)$$

A computer is needed to solve these equations exactly. If the assumption is made that $\tau_{ss} \ll 1$, then the above equations simplify to

$$x_{sf} \approx \frac{\frac{1}{2}\kappa E(2\tau_c\tau_0 - \tau_0^2)}{2\xi_0\tau_{ss}(1 + \xi_0\tau_{ss}) + \frac{1}{2}\tau_c^2 - 4\xi_0^2\tau_{ss}\tau_c} \tag{2.23}$$

now

$$\xi_0 \ll 1 \qquad \xi_0^2 \approx 0$$

so

$$x_{sf} \approx \frac{\kappa E\tau_0(2\tau_c - \tau_0)}{\tau_c^2 + 4\xi_0\tau_{ss}} \tag{2.23}$$

and

$$x_{sf} = \frac{\kappa E\tau_0}{\tau_c} \tag{2.24}$$

The approximate solution for $\tau_c$ from these two equations is

$$\tau_c^2 - \tau_0\tau_c - 4\xi_0\tau_{ss} = 0 \tag{2.26}$$

or

$$\tau_c = \frac{\tau_0}{2} + \sqrt{\left(\frac{\tau_0}{2}\right)^2 + 4\xi_0\tau_{ss}}$$

The relation between $\tau_c$ and $\tau_0$ is nonlinear. The magnitude of $\tau_c$ is constrained to be less than the switching period, $\tau_{ss}$, or else the current in the inductor will never be zero.

$$\tau_{ss} \geq \tau_c \approx \frac{\tau_0}{2} + \sqrt{\left(\frac{\tau_0}{2}\right)^2 + 4\xi_0\tau_{ss}}$$

or

$$1 \geq \frac{\tau_0}{\tau_{ss}} + \frac{4\xi_0}{\tau_{ss}} \tag{2.27}$$

Since the assumption that $\tau_{ss} \ll 1$ is used to derive the above equations, the damping factor, $\xi_0$, can be seen from eqn. (2.27) to be very small (i.e. $\xi_0 < .025$ for $\tau_{ss} = .1$) even for small values of $\tau_0$.

The equilibrium voltage is given by eqn. (2.24) where $\tau_c$ is a function of $\tau_0$

$$x_{sf} = \frac{\kappa E \tau_0}{\tau_c(\tau_0)} \tag{2.24}$$

and

$$\tau_0 = \tau_{00} + a_1(x_{ss} - x_{sf})$$

If the right hand side of eqn. (2.24) is expanded out in a Taylor Series, it becomes

$$x_{sf} = \frac{\kappa E \tau_{00}}{\tau_c(\tau_{00})} + \kappa E \left( \frac{1}{\tau_c} - \frac{\tau_0}{\tau_c^2} \frac{\partial \tau_c}{\partial \tau_0} \right) \frac{\partial \tau_0}{\partial x_{sf}} (x_{sf} - x_{ss}) + \cdots$$

let

$$S = -\kappa E \left( \frac{1}{\tau_c} - \frac{\tau_0}{\tau_c^2} \frac{\partial \tau_c}{\partial \tau_0} \right) \frac{\partial \tau_0}{\partial x_{sf}}$$

but

$$\frac{\partial \tau_0}{\partial x_{sf}} = -a_1$$

and from eqn. (2.26)

$$\frac{\partial \tau_c}{\partial \tau_0} = \frac{\tau_c}{2\tau_c - \tau_0}$$

so

$$S = \frac{a_1 \kappa E}{\tau_c} \frac{2(1 - \tau_0/\tau_c)}{(2 - \tau_0/\tau_c)}$$

The approximate solution for the voltage is then

$$x_{sf} = \frac{x_{ss} + \frac{\kappa E \tau_{00}}{\tau_c}/S}{1 + 1/S} \qquad (2.25)$$

where

$$S = \frac{a_1 \kappa E}{\tau_c} \frac{2(1 - \tau_0/\tau_c)}{(2 - \tau_0/\tau_c)}$$

The form of eqn. (2.25) is the same as eqn. (2.5), but the closed loop gain is defined differently. The closed loop gain defined by eqn. (2.25) will be used in the stability analysis as a figure of merit for the discontinuous P.W.M.

The stability analysis begins by perturbing the voltage about the equilibrium point, $x_{sf}$ on the $n^{th}$ switching period.

$$\delta x_{n+1} = e^{-2\xi_0(\tau_{ss}-\tau_c)} \; 2\xi_0 \{[y_{11}(\tau_c) - 2\xi_0 y_{12}(\tau_c)]x_{sf} + \kappa E[y_{11}(\tau_c-\tau_0) - y_{11}(\tau_c)]\}$$

$$+ [y_{21}(\tau_c) - 2\xi_0 y_{22}(\tau_c)]x_{sf} + \kappa E[y_{21}(\tau_c-\tau_0) - y_{21}(\tau_c)] \; \frac{\partial \tau_c}{\partial x_n} \delta x_n$$

$$+ \kappa E e^{-2\xi_0(\tau_{ss}-\tau_c)} \; y_{12}(\tau_c-\tau_0) \; \frac{\partial \tau_0}{\partial x_n} \; \delta x_n$$

$$+ e^{-2\xi_0(\tau_{ss}-\tau_c)}[y_{11}(\tau_c) - 2\xi_0 y_{12}(\tau_c)] \; \delta x_n$$

The coefficient of $\frac{\partial \tau_c}{\partial x_n}$ is zero by eqn. (2.22)

$$2\xi_0 x(\tau_c) + \dot{x}(\tau_c) = 0$$

and from eqn. (2.25)

$$\frac{\partial \tau_0}{\partial x_n} = -a_1$$

The variational equation for the discontinuous P.W.M. is then

$$\delta x_{n+1} = e^{-2\xi_0(\tau_{ss}-\tau_c)}[y_{11}(\tau_c) - 2\xi_0 y_{12}(\tau_c) - a_1 \kappa E y_{12}(\tau_c - \tau_0)]\,\delta x_n \qquad (2.26)$$

The system will be stable if the magnitude of the variation for the $(n+1)^{st}$ period is less than the variation of the $n^{th}$ period. Since the feedback is negative, the stability criterion is

$$-1 < e^{-2\xi_0(\tau_{ss}-\tau_c)}[y_{11}(\tau_c) - 2\xi_0 y_{12}(\tau_c) - a_1 \kappa E y_{12}(\tau_c - \tau_0)]$$

or

$$S < \frac{2(1-\tau_0/\tau_c)}{\tau_c(2-\tau_0/\tau_c)} \; \frac{[e^{2\xi_0(\tau_{ss}-\tau_c)} + y_{11}(\tau_c) - 2\xi_0 y_{12}(\tau_c)]}{y_{12}(\tau_c - \tau_0)} = S_{max} \qquad (2.27a)$$

If the assumption is made that $\tau_s \ll 1$, eqn. (2.27a) reduces to

$$S < \frac{4}{\tau_c^2(2-\tau_0/\tau_c)} \approx S_{max} \qquad (2.27b)$$

The value of the maximum loop gain for the discontinuous P.W.M. is higher than all the others except for the dither stabilized P.W.M.

Since there is only one degree of freedom for the discontinuous P.W.M., simple proportional control is all that is needed for good regulation.

A regulator which converges in one step, similar to the zero matrix P.W.M., is obtained by setting

$$a_1 = \frac{y_{11}(\tau_c) - 2\xi_0 y_{12}(\tau_c)}{\kappa E y_{12}(\tau_c - \tau_0)} \qquad (2.28a)$$

this makes the variation in the $(n+1)^{st}$ iteration equal to zero, see eqn. (2.26). If it is assumed that $\tau_{ss} \ll 1$, then the feedback constant is approximated by

$$a_1 \approx \frac{1}{\kappa E(\tau_c - \tau_0)} \qquad (2.29b)$$

The loop gain of this regulator is one-half the maximum loop gain so that the margin of stability is good.

### 2.3.7 Minus One Eigenvalue P.W.M.

The discontinuous P.W.M., because its recursion formula is a scalar with negative feedback, was unstable when the $(n+1)^{st}$ variation of the state, $\delta x_{n+1}$, became greater in magnitude but opposite in sign to the $n^{th}$ variation. The variations oscillate from one side of the equilibrium point to the other while increasing in magnitude. The matrix equivalent of the scalar case occurs when the eigenvalues of the P-matrix are minus one. The control law for the minus one eigenvalue, M.O.E., P.W.M. is found by equating the determinate of the P-matrix to plus one and its trace to minus two.

$$\text{Det.}(P) = e^{-2\xi\tau_{ss}} + a_1\kappa E e^{-2\xi(\tau_{ss}-\tau_0)} y_{12}(\tau_0) - b_1\kappa E e^{-2\xi(\tau_{ss}-\tau_0)} y_{11}(\tau_0) = 1$$

$$TR(P) = y_{11}(\tau_s) + y_{22}(\tau_s) - a_1\kappa E y_{12}(\tau_{ss}-\tau_0) - b_1\kappa E y_{22}(\tau_s-\tau_0) = -2$$

The solution to the above equations for the feedback constants, $a_1$ and $b_1$, is

$$a_1 = \frac{1}{\kappa E y_{12}(\tau_{ss})}\left\{2y_{11}(\tau_0)(1+e^{-\xi\tau_{ss}}\cos\omega_d\tau_{ss}) - y_{22}(\tau_{ss}-\tau_0)\left[e^{-2\xi\tau_0} - e^{2\xi(\tau_s-\tau_0)}\right]\right\}$$

$$b_1 = \frac{1}{\kappa E y_{12}(\tau_{ss})}\left\{2y_{12}(\tau_0)(1+e^{-\xi\tau_{ss}}\cos\omega_d\tau_{ss}) - y_{12}(\tau_{ss}-\tau_0)\left[e^{2\xi(\tau_{ss}-\tau_0)} - e^{-2\xi\tau_0}\right]\right\} \tag{2.30a}$$

and when $\tau_{ss} \ll 1$

$$\left.\begin{aligned} a_1 &\approx \frac{4}{\kappa E\tau_{ss}} \\ b_1 &\approx \frac{4\tau_0}{\kappa E\tau_{ss}} \end{aligned}\right\} \quad \frac{b_1}{a_1} = \tau_0 \tag{2.30b}$$

The maximum closed loop gain for a P.W.M. whose feedback constants are in the ratio $b_1/a_1 = \tau_0$ can be obtained from eqn. (2.4b)

$$1 + e^{-\xi\tau_{ss}}\left[2\cos(\omega_d\tau_{ss}) + e^{-\xi\tau_{ss}}\right] + a_1\kappa E\left\{e^{-2\xi(\tau_{ss}-\tau_0)} y_{12}(\tau_{ss}-\tau_0)\right.$$

$$\left. - \frac{b_1}{a_1}\left[e^{-2\xi(\tau_{ss}-\tau_0)} y_{11}(\tau_0) + y_{22}(\tau_s-\tau_0)\right]\right\} > 0$$

or

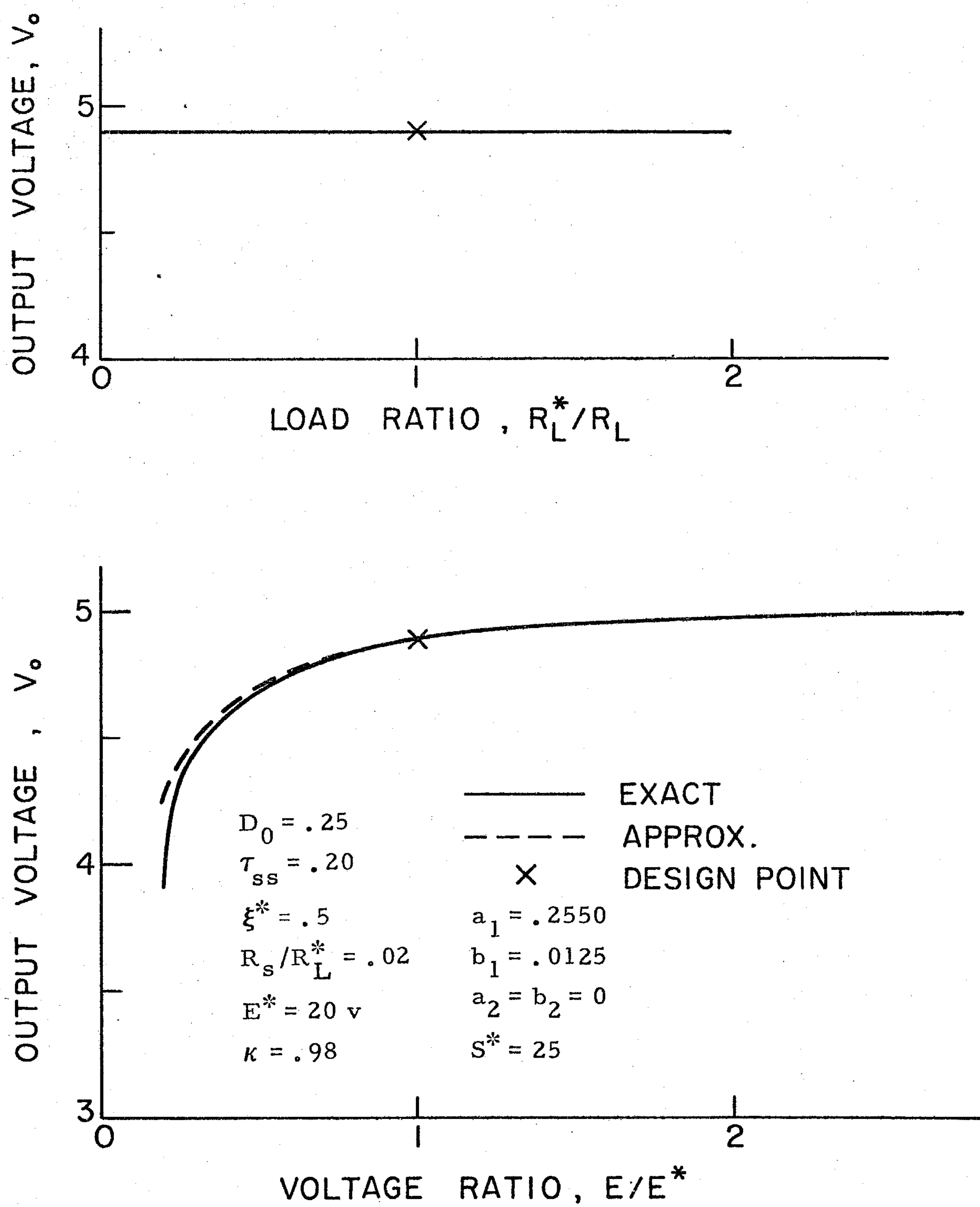



Fig. 2.10. Voltage Regulation for M.O.E. P.W.M.

$$S < \frac{1 + e^{-\xi\tau_{ss}}[2\cos(\omega_d\tau_{ss}) + e^{-\xi\tau_{ss}}]}{\tau_{ss}\{y_{12}(\tau_{ss}-\tau_o) + \tau_0 y_{22}(\tau_s-\tau_0) - e^{-2\xi(\tau_{ss}-\tau_0)}[y_{12}(\tau_0) - \tau_0 y_{11}(\tau_0)]\}} \quad (2.31a)$$

and when $\tau_{ss} \ll 1$

$$S < \frac{4}{\tau_{ss}^2} \approx S_{max} \quad (2.31b)$$

This loop gain, eqn. (2.31b), is the same as the one derived for the dither stabilized P.W.M., eqn. (2.15b). In fact the ratio of the feedback constants for the dither stabilized P.W.M. is also the on-time (i.e. $b_1/a_1 \approx \tau_0$). When $\tau_{ss} \ll 1$, the dither stabilized P.W.M. approximates the M.O.E. P.W.M.

In fig. 2.10 the equilibrium voltage is plotted against the input parameters. The load ratio is limited to those values, $R_L^*/R_L \leq 2$, for which the damping factor is less than one. At the lower voltage ratios the P.W.M. saturates, and the on-time equals the switching period. The range of the voltage ratio for which the system is locally stable is large, $E/E^* = 3.0$, when compared to the other P.W.M.'s investigated in this chapter. The system converges to the equilibrium voltage from zero initial conditions for all values shown in fig. 2.10 except when the load ratio is less than point five, $R_L^*/R_L < .5$.

## 2.4 Discussion of Results

The general expression for the local stability and regulation, derived in App. A and B, are used to evaluate the various

P.W.M. s. The P.W.M.'s ability to regulate is summarized in the loop gain, S, such that a large value for the loop gain means good regulation. The analysis shows that requirements for good regulation and stability are contradictory, at least for the P.W.M. s analyzed in the previous section. The loop gain portrays this conflict by showing that stability considerations limit the maximum loop gain, thus limiting the maximum regulation possible. The maximum loop gain therefore makes a good figure of merit in comparing the stability of the different P.W.M. s. The ratio of the loop gain to the maximum loop gain gives a measure of how stable the regulator is.

The approximate and exact solution for the equilibrium voltage are compared for a few of the P.W.M. s. The agreement between the solutions is remarkable considering the assumptions made in arriving at the approximate solution. For both the approximate and exact solutions, the dependence on the load is very small and is mainly a function of the parameter $\kappa = 1/(1 + R_s/R_L)$. The reason for this lack of dependence is that switching regulators are designed to be very efficient so that the ratio of the series resistance, $R_s$, to the load resistance, $R_L$, is very small. In the examples of this chapter the ratio was two percent. The output voltage is much more sensitive to changes in the input voltage than to changes in the load. It is apparent that the main function of the buck regulators is to regulate against input voltage variations. An open loop controller which would set the on-time, $\tau_{00}$, and the switching period, $\tau_{ss}$, according to the current value of the input voltage would make a more effective regulator. Such a controller would, in effect, continually update the design point with respect to the latest information on the input voltage.

The exact expressions for the local stability of the various P.W.M.s can be adequately approximated by simple formulas when $\tau_{ss} \ll 1$. These simple formulas contain much useful information. They show that the uniformly sampled voltage P.W.M. does not regulate as well as the other P.W.M.s. The reason for its inability to regulate is that it uses only simple proportional control on the voltage. At very low damping factors, when the current in the inductor becomes zero, the matrix equation reduces to a scalar and proportional control is again effective, eqn. (2.27b). The error integrating and dither stabilized P.W.M., although they only sample the voltage, have proportional plus rate control since the control law contains information on both the voltage and its derivative.

The stability of the dither stabilized P.W.M. depends mainly on the switching period, $\tau_{ss}$, and is almost independent of the damping factor and on-time. The input voltage is the only parameter the dither stabilized P.W.M. is dependent on, and that is because the definition of the loop gain contains the input voltage, E. If the input voltage could be sampled and the feedback constants modified to make the loop gain independent of the input voltage, then the local stability would be independent of all the parameters. Making the local stability independent of the input voltage would greatly improve a number of P.W.M.s, such as the zero eigenvalue P.W.M., whose range of input voltage is limited solely by stability considerations.

One way of modifying the design on-time, $\tau_{00}$, in response to changes in the average input voltage, $\overline{E}$, is illustrated in fig. 2.11. The on-time is defined as the time it takes the control voltage, $V_c$, to reach a specified level, $V_s$. The control voltage is a linear function of the average input voltage so that

$$V_c(\tau) = \kappa\overline{E}\tau$$

and

$$V_c(\tau_0) = \kappa\overline{E}\tau_0$$

The product of the average input voltage and on-time is constant, and this is the control needed to maintain the output voltage at the desired reference voltage, $V_R$ (i.e. $V_R = \frac{1}{\tau_{ss}} \tau_0 E$). The control voltage can be produced by using a current Generator, a comparator, and a capacitor as shown in fig. 2.12. The current, $I_c$, is proportional to the average input voltage so that the voltage across the capacitor is the control voltage, $V_c(\tau)$. The comparator turns the switch off when the control voltage reaches the specified value $V_s$ so that

$$V_c(\tau_0) = \frac{I_c\tau_0}{C} = V_s$$

It is also possible to regulate against changes in the load by letting

$$V_s(\tau) = V_R + [V_R - V_0(\tau)]$$

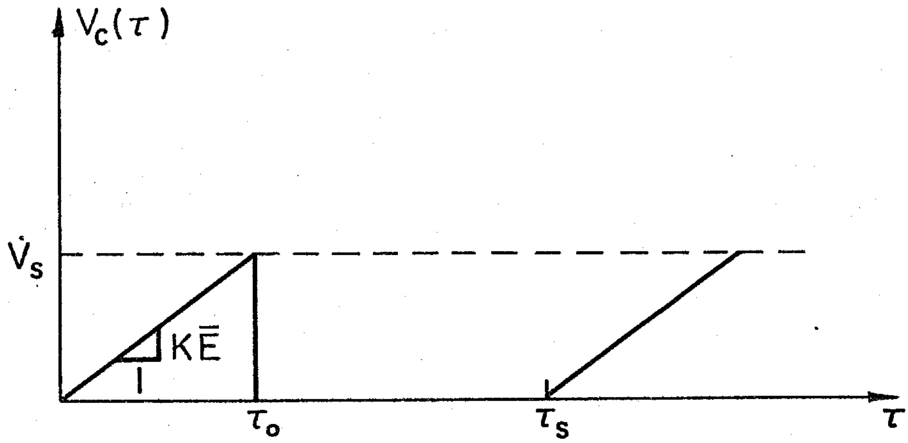



Fig. 2.11. Control Voltage

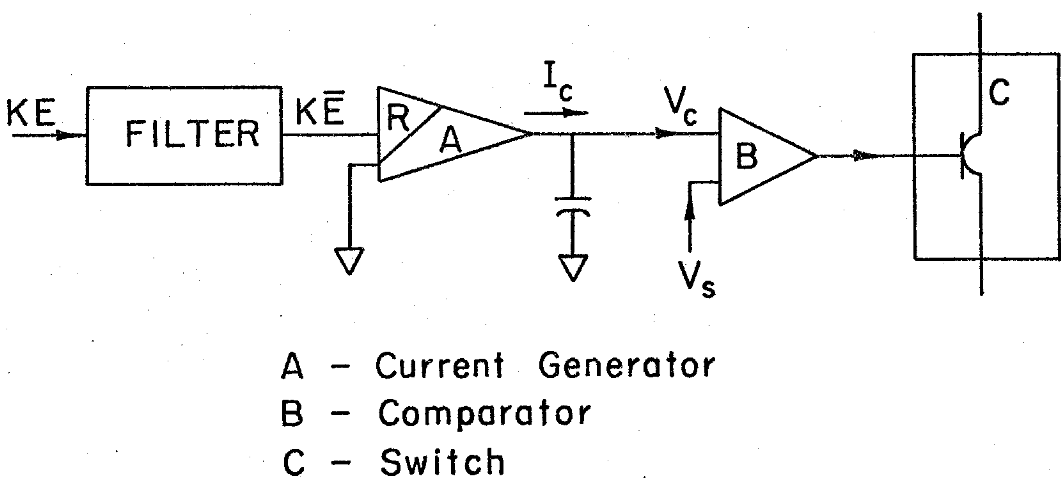



Fig. 2.12. Control Circuit

then

$$\frac{I_c \tau_0}{C} = 2V_R - V_0(\tau)$$

but

$$I_c = \kappa k \overline{E}$$

so

$$2V_R = \frac{\kappa k \overline{E}}{C} \tau_0 + V_0(\tau) \tag{2.32}$$

This equation, eqn. (2.32), is the same equation as the control law for the dither stabilized P.W.M. except $E_b$ is replaced by $\overline{E}$. The feedback constant $a_1$ will be the same as the dither stabilized one except for $\overline{E}$ being substituted for $E_b$

$$a_1 = \frac{y_{11}(\tau_0)}{\frac{k\overline{E}}{C} + \dot{x}(\tau_0)}$$

and when $\qquad \dot{x}(\tau_0) \ll \frac{k\overline{E}}{C}$

$$a_1 \approx \frac{y_{11}(\tau_0)}{\frac{k\overline{E}}{C}}$$

The closed loop gain for this dither stabilized P.W.M. is

$$S = \frac{y_{11}(\tau_0)}{k/C} \tag{2.33}$$

which is independent of the input voltage. This is one way to make the stability of the dither stabilized P.W.M. independent of the input parameters. This type of control also achieves the desirable characteristic of modifying the design point on-time, $\tau_{00}$, according to the value of the average input voltage, $\overline{E}$.

# CHAPTER III - BOOST REGULATOR

## 3.1 Recursion Formula

The output voltage of a boost regulator is higher than the input voltage. The voltage increase is achieved by first charging an inductor and then discharging it into the load. The circuit configuration is different during the charging and discharging part of the switching period. The charging part occurs when the switch of fig. 3.1 is closed. The circuit is then modeled as two first order differential equations. When the switch is open, the circuit is the same as that of the buck regulator during its duty cycle. The duty cycle, D, of the boost regulator is defined as the ratio of the charging time to the switching period.

The state variables of the boost regulator are chosen to be the output voltage and the current. The choice of state variables was made because the voltage across the capacitor and the current through the inductor are continuous, whereas the derivative of the output voltage is not. The state variables are defined as

$$\underline{z}(\tau) = \begin{pmatrix} z^{(1)}(\tau) \\ \\ z^{(2)}(\tau) \end{pmatrix} = \begin{pmatrix} V_0(\tau) \\ \\ \dfrac{i_k(\tau)}{\omega_k C} \end{pmatrix}$$

The derivation of the recursion formula is divided into two parts depending on whether the switch is closed or open.

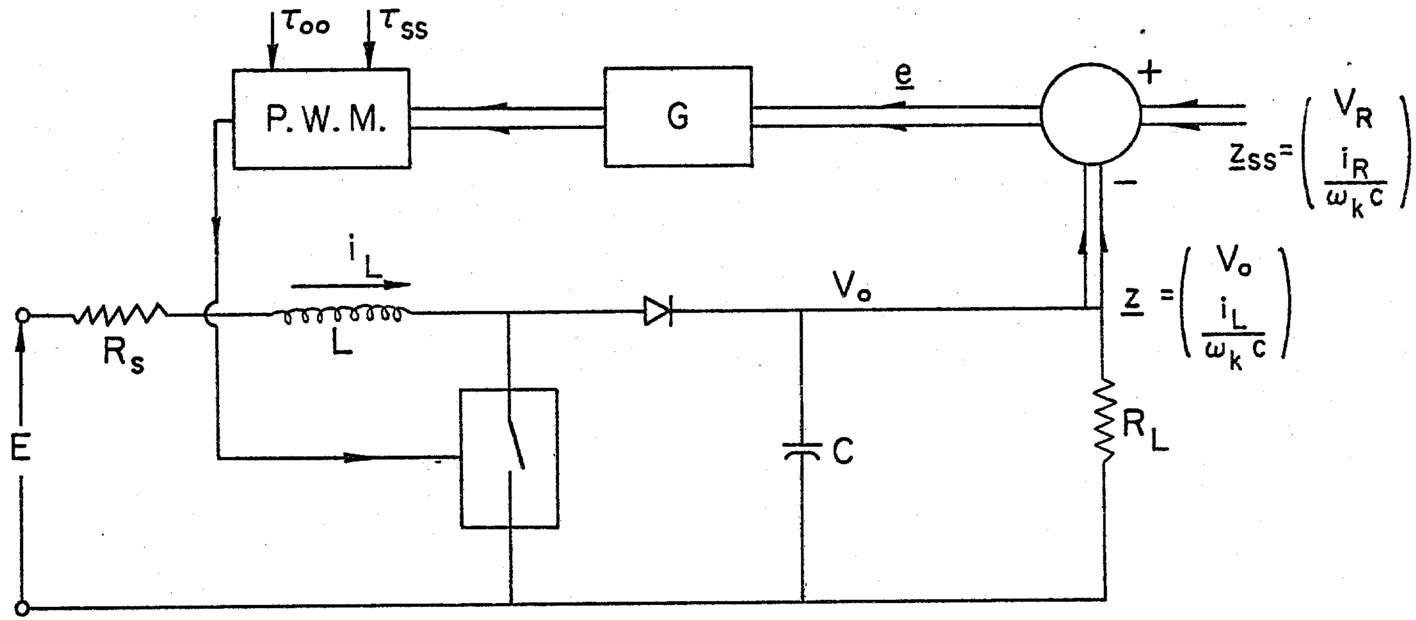



Fig. 3.1. Boost Regulator

Switch Closed:

The on-time, $\tau_0$, is defined as the time during which the switch is closed. In this configuration the inductor is charging and the capacitor is discharging.

$$E = i_L R_s + L\frac{di_L}{dt} \qquad C\frac{dV_0}{dt} + \frac{V_0}{R_L} = 0$$

or

$$\frac{dz^{(2)}}{d\tau} + \frac{R_s}{\omega_k L} z^{(2)} = \kappa E \qquad \frac{dz^{(1)}}{d\tau} + \frac{z^{(1)}}{\omega_k R_L C} = 0$$

let

$$2\xi_0 = \frac{1}{\omega_k R_L C} \quad \text{and} \quad \mu_s = R_s C$$

then

$$\underline{z}(\tau_0) = \begin{pmatrix} e^{-2\xi_0\tau_0} & 0 \\ 0 & e^{-\kappa\omega_k\mu_s\tau_0} \end{pmatrix}\underline{z}(0) + \frac{E}{\omega_k\mu_s}(1 - e^{-\kappa\ \omega_k\mu_s\tau_0})\begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{3.1}$$

Switch Open:

This configuration is the same configuration the buck regulator is in during its duty cycle. The output voltage and its derivative at $\tau = \tau_s$ is

$$\underline{x}(\tau_s) = Y(\tau_s - \tau_0)\underline{x}(\tau_0) + \kappa E\begin{pmatrix} 1 - y_{11}(\tau_s - \tau_0) \\ y_{12}(\tau_s - \tau_0) \end{pmatrix}$$

The transformation which relates the state variables, $\underline{z}(\tau)$, to the vector $\underline{x}(\tau)$ is

$$\underline{z}(\tau) = \begin{pmatrix} 1 & 0 \\ 2\xi_0 & 1 \end{pmatrix} \underline{x}(\tau)$$

so

$$\underline{z}(\tau_s) = \begin{pmatrix} 1 & 0 \\ 2\xi_0 & 1 \end{pmatrix} Y(\tau_s - \tau_0) \begin{pmatrix} 1 & 0 \\ -2\xi_0 & 1 \end{pmatrix} \underline{z}(\tau_0)$$

$$+ \kappa E \begin{pmatrix} 1 & 0 \\ 2\xi_0 & 1 \end{pmatrix} \begin{pmatrix} 1 - y_{11}(\tau_s - \tau_0) \\ y_{12}(\tau_s - \tau_0) \end{pmatrix}$$

or

$$\underline{z}(\tau_s) =$$

$$\begin{pmatrix} \{y_{11}(\tau_s-\tau_0) - 2\xi_0 y_{12}(\tau_s-\tau_0)\} & \{y_{12}(\tau_s-\tau_0)\} \\ \{y_{21}(\tau_s-\tau_0)+2\xi_0[y_{11}(\tau_s-\tau_0)-y_{22}(\tau_s-\tau_0)]-4\xi_0^2 y_{12}(\tau_s-\tau_0)\} & \{y_{22}(\tau_s-\tau_0)+2\xi_0 y_{12}(\tau_s-\tau_0)\} \end{pmatrix} \underline{z}(\tau_0)$$

$$+ \kappa E \begin{pmatrix} 1-y_{11}(\tau_s-\tau_0) \\ y_{12}(\tau_s-\tau_0)+2\xi_0[1-y_{11}(\tau_s-\tau_0)] \end{pmatrix}$$

substituting eqn. (3.1) for $\underline{z}(\tau_0)$ and simplifying

$$\underline{z}(\tau_s) =$$

$$\begin{pmatrix} e^{-2\xi_0\tau_0}\{y_{22}(\tau_s-\tau_0)+2(\xi-\xi_0)y_{12}(\tau_s-\tau_0)\} & e^{-\kappa\omega_k\mu_s\tau_0}\, y_{12}(\tau_s-\tau_0) \\ e^{-2\xi_0\tau_0}\{y_{21}(\tau_s-\tau_0)+4\xi_0(\xi-\xi_0)y_{12}(\tau_s-\tau_0)\} & e^{-\kappa\omega_k\mu_s\tau_0}\{y_{11}(\tau_s-\tau_0)+2(\xi_0-\xi)y_{12}(\tau_s-\tau_0)\} \end{pmatrix} z(0)$$

$$+ \kappa E \frac{(1-e^{-\kappa\omega_k\mu_s\tau_0})}{\kappa\omega_k\mu_s} \begin{pmatrix} y_{12}(\tau_s-\tau_0) \\ y_{11}(\tau_s-\tau_0)+2(\xi_0-\xi)y_{12}(\tau_s-\tau_0) \end{pmatrix}$$

$$+ \kappa E \begin{pmatrix} 1-y_{11}(\tau_s-\tau_0) \\ y_{12}(\tau_s-\tau_0)+2\xi_0[1-y_{11}(\tau_s-\tau_0)] \end{pmatrix}$$

The relation between the two damping factors, $\xi$ and $\xi_0$, is

$$\xi = \xi_0 + \kappa\omega_k\mu_s/2$$

If the assumption is made that $\omega_k\mu_s \ll \xi$ and $\omega_k\mu_s\tau_0 \ll 1$, then the expression for the state vector at the end of the switching period can be greatly simplified.

$$\underline{z}_{n+1} = \begin{pmatrix} e^{-2\xi_0\tau_0}\, y_{22}(\tau_s-\tau_0) & y_{12}(\tau_s-\tau_0) \\ e^{-2\xi_0\tau_0}\, y_{21}(\tau_s-\tau_0) & y_{11}(\tau_s-\tau_0) \end{pmatrix} \underline{z}_n \ .$$

$$+ \kappa E \begin{pmatrix} \tau_0 y_{12}(\tau_s-\tau_0) + [1-y_{11}(\tau_s-\tau_0)] \\ \tau_0 y_{11}(\tau_s-\tau_0) + y_{12}(\tau_s-\tau_0) + 2\xi_0[1-y_{11}(\tau_s-\tau_0)] \end{pmatrix} \qquad (3.2)$$

Eqn. (3.2) is the recursion formula for the boost regulator. A control law must be given before the state of the system is completely described.

The control laws used with the boost regulator, like those of the buck regulator, are either linear or can be approximated adequately by their linear part. The control laws are given by

$$\tau_0(\underline{z}_n) = \tau_{00} + a_1[z_{ss}^{(1)} - z_n^{(1)}] + b_1[z_{ss}^{(2)} - z_n^{(2)}] \qquad (3.3a)$$

and

$$\tau_s(\underline{z}_n) = \tau_{ss} + a_2[z_n^{(1)} - z_{ss}^{(1)}] + b_2[z_n^{(2)} - z_{ss}^{(2)}] \qquad (3.3b)$$

where the feedback is on the state variables of output voltage and inductor current. The feedback constants of the general control laws will be specialized when analyzing the different P.W.M.s.

## 3.2 Regulation and Local Stability

The recursion formula and the control law enables the state to be calculated for any switching period. The steady-state, $\underline{z}_{ss}$, can be found by solving the recursion formula, eqn. (3.2), with the feedback constants of the control law, eqn. (3.3), set equal to zero. The solution for the steady-state without feedback, $\underline{z}_{ss}$, is more easily carried out if the assumption is made that $\tau_{ss} \ll 1$, then

$$\underline{z}_{ss} \approx \begin{pmatrix} \{1 - 2\xi\tau_{ss} - \frac{1}{2}(\tau_{ss}-\tau_{00})^2\} & \{\tau_{ss}-\tau_{00}\} \\ -\{\tau_{ss}-\tau_{00}\} & \{1 - \frac{1}{2}(\tau_{ss}-\tau_{00})^2\} \end{pmatrix} \underline{z}_{ss}$$

$$+ \kappa E \begin{pmatrix} \frac{1}{2}(\tau_{ss}^2 - \tau_{00}^2) \\ \tau_{ss} \end{pmatrix}$$

so that

$$z_{ss}^{(1)} \approx \kappa E / D_0'$$

$$z_{ss}^{(2)} \approx \kappa E \left( \frac{2\xi}{D_0'^2} - \frac{\tau_{00}}{2} \right) \qquad (3.4a)$$

where

$$D_0 = \tau_{00}/\tau_{ss}$$

Eqn. (3.4) gives the steady-state for an uncontrolled boost regulator. When control is added, the steady-state solution without feedback, $\underline{z}_{ss}$, is assumed to be the reference vector.

The equilibrium point with feedback, $\underline{z}_{sf}$, cannot be solved analytically, but it can be solved numerically. If the assumption is made that $\tau_s \ll 1$, then the steady-state with feedback, $\underline{z}_{sf}$, is given by eqn. (3.4a) with the on-time and switching period now functions of the state.

$$z_{sf}^{(1)} \approx \frac{\kappa E \tau_s}{(\tau_s - \tau_0)}$$

$$z_{sf}^{(2)} \approx \kappa E \left[ 2\xi \frac{\tau_s^{\,2}}{(\tau_s - \tau_0)^2} - \frac{\tau_0}{2} \right] \qquad (3.4b)$$

In Appendix (II. A) eqn. (3.4b) is linearized about the design point to give

$$z_{sf}^{(1)} = \Big\{ z_{ss}^{(1)} + \frac{\kappa E / D_0'}{S} \Big[ 1 + \kappa E \frac{2\xi}{\tau_{00} D_0'^{\,3}} (b_1 + D_0 b_2) + \frac{\kappa E}{2D_0'} \big[ (D_0 - D_0') b_1 + D_0^2 b_2 \big] + \frac{(b_1 + D_0 b_2)}{\tau_{ss} D_0'} z_{ss}^{(2)} \Big] \Big\} \Big/ \Big\{ 1 + \frac{1}{S} \Big[ 1 + \kappa E \frac{4\xi}{\tau_{ss} D_0'^{\,3}} (b_1 + D_0 b_2) - \kappa E \frac{b_1}{2} \Big] \Big\}$$

$$z_{sf}^{(2)} = \Big\{ S1 z_{ss}^{(2)} + \kappa E \left( \frac{2\xi}{D_0'^{\,2}} - \frac{\tau_{00}}{2} \right) - \kappa^2 E^2 \frac{2\xi}{\tau_{ss} D_0'^{\,4}} (a_1 + D_0 a_2) - \frac{\kappa^2 E^2}{2 D_0'^{\,2}} \big[ a_1 (D_0 - D_0') + D_0^2 a_2 \big] + \kappa E \Big[ \frac{4\xi}{\tau_{ss} D_0'^{\,3}} (a_1 + D_0 a_2) - \frac{a_1}{2} \Big] z_{ss}^{(1)} \Big\} \Big/ \Big\{ S1 + \Big[ 1 + \frac{\kappa E}{\tau_{ss} D_0'^{\,2}} (a_1 + D_0 a_2) \Big] \Big\} \qquad (3.5)$$

$$S = \frac{\kappa E}{\tau_{ss} D_0'^{\,2}} (a_1 + D_0 a_2) + \frac{\kappa^2 E^2 D_0}{2 \tau_{ss} D_0'^{\,2}} (a_1 b_2 - b_1 a_2)$$

$$S1 = \kappa E \frac{4\xi}{\tau_{ss} D_0'^3}(b_1 + D_0 b_2) - \kappa E \frac{b_1}{2} + \frac{\kappa^2 E^2 D_0}{2\tau_{ss} D_0'^2}(a_1 b_2 - b_1 a_2)$$

The derivation of the equilibrium point with feedback, $\underline{z}_{sf}$, was done mainly to define the loop gain, S. The loop gain, like that of the buck regulator, indicates the regulator's ability to regulate. The boost regulator exhibits a greater dependence on the load than did the buck regulator because the damping factor, $\xi$, appears explicitly in the expression of the boost regulator's output voltage, $z_{sf}^{(1)}$.

The local stability is obtained by perturbing the recursion formula about the equilibrium point, $\underline{z}_{sf}$. The variation of state of the $(n+1)^{st}$ iteration is related through the perturbation matrix, P, to the variation of state for the $n^{th}$ iteration. The variational equation, see Appendix (II. B) for the derivation, is

$$\delta \underline{z}_{n+1} = P\, \delta \underline{z}_n \tag{3. 6}$$

where

$$P = \begin{pmatrix} \{a_{11} - a_1 g_1 + a_2 h_1\} & \{a_{12} - b_1 g_1 + b_2 h_1\} \\ \{a_{21} - a_1 g_2 + a_2 h_2\} & \{a_{22} - b_1 g_2 + b_2 h_2\} \end{pmatrix}$$

and

$$g = \begin{pmatrix} e^{-2\xi_0 \tau_0}\, y_{12}(\tau_s - \tau_0) z_{sf}^{(1)} - y_{22}(\tau_s - \tau_0) z_{sf}^{(2)} - \kappa E \tau_0 y_{22}(\tau_s - \tau_0) \\ e^{-2\xi_0 \tau_0}\, y_{11}(\tau_s - \tau_0) z_{sf}^{(1)} + y_{12}(\tau_s - \tau_0) z_{sf}^{(2)} + \kappa E \tau_0 y_{12}(\tau_s - \tau_0) \end{pmatrix}$$

with

$\underline{h}$ =

$$\begin{pmatrix} -e^{-2\xi_0\tau_0}[y_{12}(\tau_s-\tau_0)+2\xi_0 y_{22}(\tau_s-\tau_0)]\, z_{sf}^{(1)}+y_{22}(\tau_s-\tau_0)\, z_{sf}^{(2)}+\kappa E[\tau_0 y_{22}(\tau_s-\tau_0)+y_{12}(\tau_s-\tau_0)] \\ -e^{-2\xi_0\tau_0}\, y_{22}(\tau_s-\tau_0)\, z_{sf}^{(1)}+y_{21}(\tau_s-\tau_0)\, z_{sf}^{(2)}+\kappa E[\tau_0 y_{21}(\tau_s-\tau_0)+y_{11}(\tau_s-\tau_0)] \end{pmatrix}$$

The boost regulator will be locally stable if the modulus of the eigenvalues of the P matrix are less than one in magnitude. Before the local stability can be evaluated, the steady-state with feedback, $\underline{z}_{sf}$, along with the on-time, $\tau_0$, and switching period, $\tau_s$, must be solved for from the recursion formula, eqn. (3.2), and the control laws, eqn. (II.A.1a). The general solution for the local stability of a boost regulator is obtained easiest with the use of a computer.

It is possible to simplify the form of the perturbation matrix if it is assumed that $\tau_s \ll 1$. The elements of the P matrix, see eqn. (II.B.5a), are then

$$p_{11} = a_{11} - a_1\{(\tau_s-\tau_0)\, z_{sf}^{(1)} - z_{sf}^{(2)} - \kappa E \tau_0\} - a_2\{[(\tau_s-\tau_0) + 2\xi]\, z_{sf}^{(1)} - z_{sf}^{(2)} - \kappa E \tau_s\}$$

$$p_{12} = a_{12} - b_1\{(\tau_s-\tau_0)\, z_{sf}^{(1)} - z_{sf}^{(2)} - \kappa E\, \tau_0\} - b_2\{[(\tau_s-\tau_0) + 2\xi]\, z_{sf}^{(1)} - z_{sf}^{(2)} - \kappa E \tau_s\}$$

$$p_{21} = a_{21} - a_1\{z_{sf}^{(1)} + (\tau_s-\tau_0)\, z_{sf}^{(2)}\} - a_2\{z_{sf}^{(1)} + (\tau_s-\tau_0)\, z_{sf}^{(2)} - \kappa E\}$$

$$p_{22} = a_{22} - b_1\{z_{sf}^{(1)} + (\tau_s-\tau_0)\, z_{sf}^{(2)}\} - b_2\{z_{sf}^{(1)} + (\tau_s-\tau_0)\, z_{sf}^{(2)} - \kappa E\} \qquad (3.7a)$$

If the system is being evaluated at the design point (i.e. $\underline{z}_{sf} = \underline{z}_{ss}$), then the perturbation matrix is

$$p_{11} = a_{11} + a_1\kappa^*E^*\left\{\frac{2\xi^*}{D_0'^2} - \frac{2\tau_{ss}-\tau_{00}}{2}\right\} + a_2\kappa^*E^*D_0\left\{\frac{2\xi^*}{D_0'^2} - \frac{\tau_{ss}}{2}\right\}$$

$$p_{12} = a_{12} + b_1\kappa^*E^*\left\{\frac{2\xi^*}{D_0'^2} - \frac{2\tau_{ss}-\tau_{00}}{2}\right\} + b_2\kappa^*E^*D_0\left\{\frac{2\xi^*}{D_0'^2} - \frac{\tau_{ss}}{2}\right\}$$

$$p_{21} = a_{21} - a_1\kappa^*E^*/D_0' - a_2\kappa^*E^*D_0/D_0'$$

$$p_{22} = a_{22} - b_1\kappa^*E^*/D_0' - b_2\kappa^*E^*D_0/D_0' \qquad (3.7b)$$

The asterisk is used in eqn. (3.7b) to indicate that the parameters are the design parameters. The approximate stability criterion for a boost regulator, eqn. (II.B.7), is

$$a_1[z_{sf}^{(2)} + \kappa E\tau_0] + a_2[-2\xi z_{sf}^{(1)} + z_{sf}^{(2)} + \kappa E\tau_0] - (b_1+b_2)z_{sf}^{(1)}$$

$$+ b_2\kappa E + \kappa^2E^2(a_1b_2-b_1a_2)\left\{\left(\frac{z_{sf}^{(2)}}{\kappa E} + \tau_0\right) - 2\xi\left(\frac{z_{sf}^{(1)}}{\kappa E}\right)^2\right\}$$

$$< 2\xi\tau_s \qquad (3.8a)$$

when the eigenvalues of the perturbation matrix are complex, and

$$4 + a_1\{2[z_{sf}^{(2)} + \kappa E\tau_0] - (\tau_s-\tau_0)z_{sf}^{(1)}\}$$

$$+ a_2\{-[(\tau_s-\tau_0) + 4\xi]z_{sf}^{(1)} + 2z_{sf}^{(2)} + \kappa E(\tau_s+\tau_0)\}$$

$$- b_1\{2z_{sf}^{(1)} + (\tau_s-\tau_0)z_{sf}^{(2)}\} - b_2\{2[z_{sf}^{(1)} - \kappa E] + (\tau_s-\tau_0)z_{sf}^{(2)}\}$$

$$+ \kappa^2E^2(a_1b_2-b_1a_2)\left\{\left(\frac{z_{sf}^{(2)}}{\kappa E} + \tau_0\right) - 2\xi\left(\frac{z_{sf}^{(1)}}{\kappa E}\right)^2\right\} > 0 \qquad (3.8b)$$

when the eigenvalues are real. The stability criterion, eqn. (3.8), has been derived, see App. (II. B), by neglecting terms of order $\tau_s$ multiplied by the damping factor, $\xi$. The stability criterion is valid to second order in $\tau_s$ only when $\xi \leq \tau_s$. In the next section the general expression for stability will be used in evaluating some P.W.M.s.

## 3.3 Comparison of P.W.M.s

### 3.3.1 Uniformly Sampled Voltage P.W.M.

The feedback constants of the uniformly sampled voltage P.W.M. are all zero except one. The control law, when only the on-time is varied, has the form

$$\tau_0(\underline{z}_n) = \tau_{00} + a_1[z_{ss}^{(1)} - z_n^{(1)}]$$

and the closed loop gain is defined as

$$S = \frac{a_1\kappa E}{\tau_{ss}D_0'^{2}}$$

The maximum closed loop gain for which the regulator is locally stable is given by eqn. (3.8a) which reduces, for this P.W.M., to

$$S < \frac{2\xi}{D_0'^{\,2}\left[\frac{z_{sf}^{(2)}}{\kappa E} + \tau_0\right]} = S_{max} \qquad (3.9a)$$

If the system is evaluated at the design point, $\underline{z}_{ss} = \underline{z}_{sf}$, then eqn. (3.9a) becomes

$$S < \frac{1}{1 + \frac{D_0 D_0'^{\,2}\,\tau_{ss}}{4\xi^*}} = S_{max} \qquad (3.9b)$$

The maximum loop gain is small and it can never become larger than one (i.e. $\tau_{ss}/\xi^* \approx 0$). The buck regulator, in contrast, exhibits much better stability, eqn. (2.5).

When only the switching period is varied, the control law is

$$\tau_s(\underline{z}_n) = \tau_{ss} + a_2[z_n^{(1)} - z_{ss}^{(1)}]$$

and so the loop gain is

$$S = \frac{a_2 \kappa D_0 E}{\tau_{ss} D_0'^{\,2}}$$

The maximum loop gain for the variable switching period P.W.M. is

$$S < \frac{2\xi D_0}{D_0'^{\,2}\left[\frac{z_{sf}^{(2)}}{\kappa E} - 2\xi\,\frac{z_{sf}^{(1)}}{\kappa E} + \tau_0\right]} \approx S_{max} \qquad (3.10a)$$

and when $\underline{z}_{ss} = \underline{z}_{sf}$ eqn. (3.10a) reduces to

$$S < \frac{1}{1 + \frac{D_0'^{\,2}\,\tau_{ss}}{4\xi^*}} \qquad (3.10b)$$

The stability of the V.S.P. controlled regulator is no better than that of the V.O.T. regulator.

The uniformly sampled boost regulator is more unstable than the uniformly sampled buck regulator. The reason for this can be seen by comparing the perturbation matrix of the two regulators at the design point with $\tau_s \ll 1$. For variable on-time control, the perturbation matrix of the boost regulator, eqn. (3.7b), is

$$P = \begin{pmatrix} \left\{a_{11} + a_1\kappa^*E^*\left(\frac{2\xi^*}{D_0'^{\,2}} - \frac{2\tau_{ss} - \tau_{00}}{2}\right)\right\} & a_{12} \\ \{a_{21} - a_1\kappa^*E^*/D_0'\} & a_{22} \end{pmatrix}$$

and for the buck regulator, eqn. (2.3), it is

$$P = \begin{pmatrix} \{y_{11}(\tau_s) - a_1\kappa^*E^*(\tau_s - \tau_0)\} & y_{12}(\tau_s) \\ \{y_{21}(\tau_s) - a_1\kappa^*E^*\} & y_{22}(\tau_s) \end{pmatrix}$$

The boost regulator has, for the damping factor, $\xi^*$, sufficiently large, positive feedback on the voltage whereas the buck regulator always has negative feedback. The positive feedback results from the current buildup in the inductor, $z_{sf}^{(2)}$, as shown by eqn. (3.7a). If the voltage is high at the beginning of a switching period, the on-time is decreased so that the time the inductor is charging the capacitor must increase. The result of this increase in charging time is to increase, not decrease, the output voltage at the beginning of the next switching period. It seems feasible that making the switching period proportional to the change in on-time might solve the problem. In fact, if the feedback constants were

$$a_2 = \frac{a_1}{D_0} \tag{3.11}$$

the dependence of the perturbation matrix on the damping factor would be eliminated completely and the stability improved. However, with the feedback constants related by eqn. (3.11), the loop gain, eqn. (3.5), is zero and the regulator does not regulate. As in the case of the buck regulator, it is necessary to go to proportional plus rate control to improve the stability of the boost regulator. The rate control is obtained by using feedback on the inductor current.

### 3.3.2 Zero Eigenvalue P.W.M.

The zero eigenvalue P.W.M. utilizes proportional plus rate control, and it achieves better regulation than the uniformly sampled voltage P.W.M. The two equations needed to solve for the feedback constants, $a_1$ and $b_1$, are obtained by setting the determinate and the trace of the P matrix to zero,

$$\text{Det.}(P) = 1 + a_1 \kappa^* E^* \left( \frac{2\xi^*}{D_0'^2} + \frac{\tau_{00}}{2} \right) - b_1 \kappa^* E^* / D_0' = 0$$

$$\text{TR}(P) = 2 + a_1 \kappa^* E^* \left( \frac{2\xi^*}{D_0'^2} - \frac{2\tau_{ss} - \tau_{00}}{2} \right) - b_1 \kappa^* E^* / D_0' = 0 \qquad (3.12)$$

The solution of eqn. (3.12) for the feedback constants is

$$a_1 = \frac{1}{\kappa^* E^* \tau_{ss}}$$

$$b_1 = \frac{2\xi^* + D_0'^2 (\tau_{ss} + \tau_{00}/2)}{\kappa^* E^* D_0' \tau_{ss}}$$

Unlike the zero eigenvalue P.W.M. for the buck regulator, the feedback constant, $b_1$, of the boost regulator depends explicitly on the damping coefficient, $\xi^*$. The closed loop gain, eqn. (3.5), for this P.W.M. is

$$S = \frac{a_1 \kappa E}{\tau_{ss} D_0'^2}$$

and at the design point this becomes

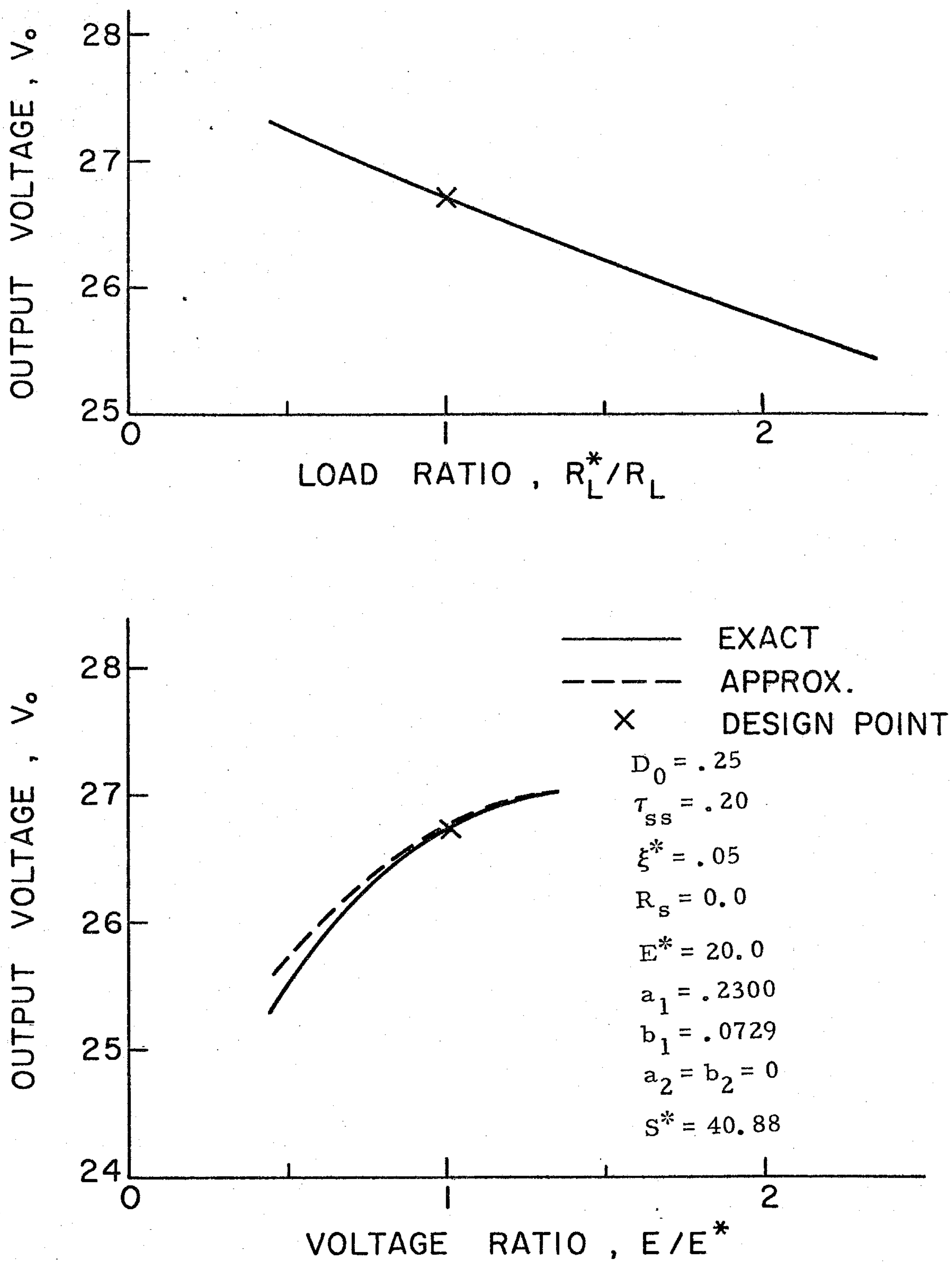



Fig. 3.2. Voltage Regulation for Zero Eigenvalue P.W.M.

$$S = \frac{1}{\tau_{ss}^{2} D_{0}'^{2}} \tag{3.13}$$

This closed loop gain is very similar to that of the buck regulator's (i.e. $S = 1/\tau_{ss}^2$).

The zero eigenvalue P.W.M. is locally stable at the design point, $\underline{z}_{ss}$. If the parameters change, the equilibrium point will change as will the steady-state on-time, $\tau_0$, and switching period, $\tau_s$. The local stability is given by eqn. (3.8) when $\tau_s \ll 1$ and $\xi \leqslant \tau_s$ for any equilibrium point, but it is not easy to see what changes in the parameters will cause the system to become locally unstable.

In fig. (3.2) the equilibrium voltage of a zero eigenvalue P.W.M. is plotted against changes in the parameters. The system is locally stable and converges from zero initial conditions to the equilibrium voltage shown. The upper limit to the voltage ratio results when the input voltage exceeds the desired output voltage, but all other limits are due to the system being unstable. The dependence of the boost regulator on the load is much more pronounced than that shown by the buck regulator.

### 3.3.3 Minus One Eigenvalue P.W.M.

The feedback constants, $a_1$ and $b_1$, for the M.O.E. P.W.M. can be solved for by equating the trace of the perturbation matrix to minus two and its determinate to plus one. The equations to be solved are

$$\text{DET.}(P) = 1 + a_1\kappa^*E^*\left(\frac{2\xi^*}{D_0'^2} + \frac{\tau_{00}}{2}\right) - b_1\kappa^*E^*/D_0' = 1$$

$$\text{TR}(P) = 2 + a_1\kappa^*E^*\left(\frac{2\xi^*}{D_0'^2} - \frac{2\tau_{ss} - \tau_{00}}{2}\right) - b_1\kappa^*E^*/D_0' = -2$$

and the solution is

$$a_1 = \frac{4}{\kappa^*E^*\tau_{ss}}$$

$$b_1 = \frac{2(4\xi^* + D_0D_0'^2\tau_{ss})}{\kappa^*E^*D_0'\tau_{ss}} \qquad (3.14)$$

so

$$\frac{b_1}{a_1} = \frac{2\xi^*}{D_0'} + \frac{D_0D_0'\tau_{ss}}{2}$$

The feedback constant, $b_1$, is dependent on the damping factor, $\xi^*$. The closed loop gain for this P.W.M. evaluated at the design point is

$$S = \frac{4}{\tau_{ss}^2 D_0'^2} = S_{max} \qquad (3.15)$$

The form of the loop gain for the boost regulator is very similar to that of the buck regulator (i.e. $S = 4/\tau_{ss}^2$). The loop gain derived is the maximum loop gain so that the regulator is only marginally stable. The feedback constants must be decreased in order for the regulator to operate effectively.

The closed loop gain, S, was derived by assuming that $\xi \leq \tau_s$ making the $\xi\tau_s$ terms second order. This assumption is partly justified by the fact that the $p_{11}$ and $p_{12}$ components of the perturbation matrix, see eqn. (3.16), exhibit positive feedback when the damping factor is large.

$$p_{11} = a_{11} + a_1\kappa^* E^* \left\{ \frac{2\xi^*}{D_0'^2} - \frac{2\tau_{ss} - \tau_{00}}{2} \right\}$$
$$p_{12} = a_{12} + b_1\kappa^* E^* \left\{ \frac{2\xi^*}{D_0'^2} - \frac{2\tau_{ss} - \tau_{00}}{2} \right\} \qquad (3.16)$$

In fig. (3.3) the maximum closed loop gain, $S_{max}$, does drop off as the damping factor, $\xi^*$, increases.

In fig. 3.3 the approximate and exact expression for the maximum loop gain is shown. These expressions diverge when the damping factor, $\xi$, becomes comparable to the switching periods (i.e. when $\xi \rightarrow \tau_{ss} = .2$) so that the higher order terms are not negligible. The exact solution for the maximum loop gain is obtained using both the approximate, eqn. (3.14), and correct values for the feedback constants. The maximum loop gain for both sets of feedback constants drop off for the higher damping factors. This decrease in the maximum loop gain for the higher

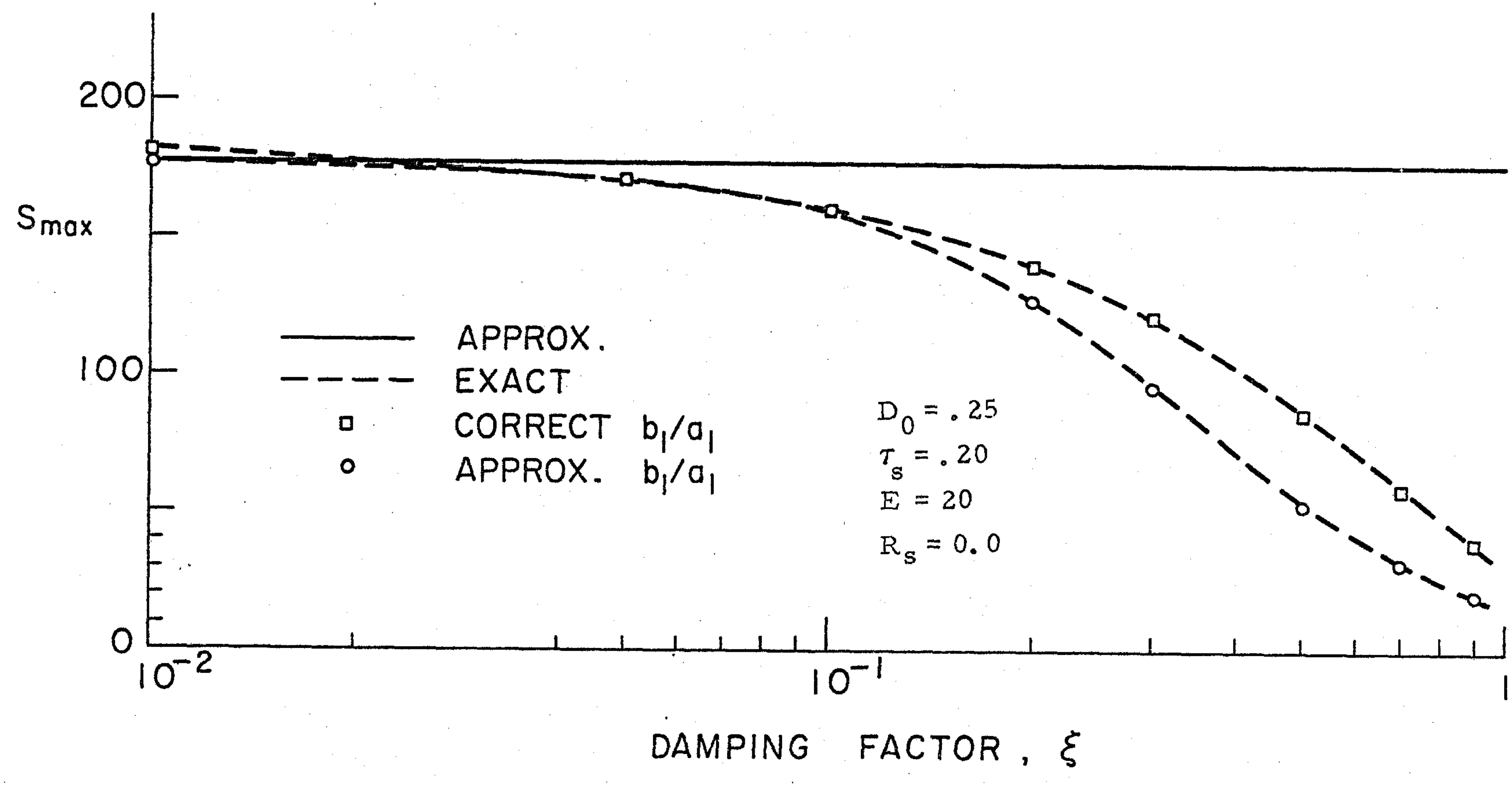



Fig. 3.3. Maximum Loop Gain

damping factors (i.e. $\xi \geqslant \tau_{ss}$) is attributed to the problem of positive feedback as discussed in Sec. 3.3.1. The series resistance, $R_s$, is made zero to make certain that $\omega_k \mu_s \ll \xi$.

The local stability of this P.W.M. is very sensitive to changes in parameters. Small changes in the parameters from design conditions causes the system to become unstable. This behavior is in contrast to the M.O.E. P.W.M. of the buck regulator which was stable over a wide range of input parameters, see fig. 2.10.

3.3.4 Zero Matrix P.W.M.

The feedback constants for the zero matrix P.W.M. can be solved for by making the perturbation matrix equal to zero. If it is assumed that $\tau_s \ll 1$ and $\xi \leqslant \tau_s$, then the elements of the P-matrix given by eqn. (3.7b) can be used to give an approximate value for the constants. The feedback constants are

$$a_1 \approx \frac{2}{\kappa^* E^* D_0' \tau_{ss}}$$

$$a_2 \approx - \frac{2}{\kappa^* E^* D_0 D_0' \tau_{ss}} \qquad (3.17)$$

$$b_1 \approx \frac{4\xi^* + D_0'^2 \tau_{ss}}{\kappa^* E^* D_0'^2 \tau_{ss}}$$

and

$$b_2 \approx - \frac{4\xi^* + D_0 D_0'^2 \tau_{ss}}{\kappa^* E^* D_0 D_0'^2 \tau_{ss}} \qquad (3.17 \text{ cont.})$$

where the asterisk denotes the design parameters. The loop gain is given by eqn. (3.5), and its value is

$$S = (a_1 b_2 - b_1 a_2) \frac{\kappa^{*2} E^{*2} D_0}{2\tau_{ss} D_0'} \qquad (3.18a)$$

since

$$a_1 + D_0 a_2 \approx 0$$

At the design point the loop gain simplifies to

$$S = \frac{1}{\tau_{ss}^2 D_0'^2} \qquad (3.18b)$$

The behavior of the zero matrix P.W.M. for the boost regulator is very similar to that of the buck regulator. In fact, the feedback constants on the voltage, $a_1$ and $a_2$, of the two regulators are the same, compare eqn. (2.19b) with eqn. (3.17). The regulators also have a loop gain which is dependent only on the product of the feedback constants, compare eqn. (2.20a) with eqn. (3.18a), and is equal to the loop gain of the zero eigenvalue P.W.M.

The local stability of the zero matrix P.W.M. is very sensitive to parameter changes, see fig. (3.4). The system is also globally unstable at the design point. In addition to these drawbacks the on-time and switching period undergo large variations for small

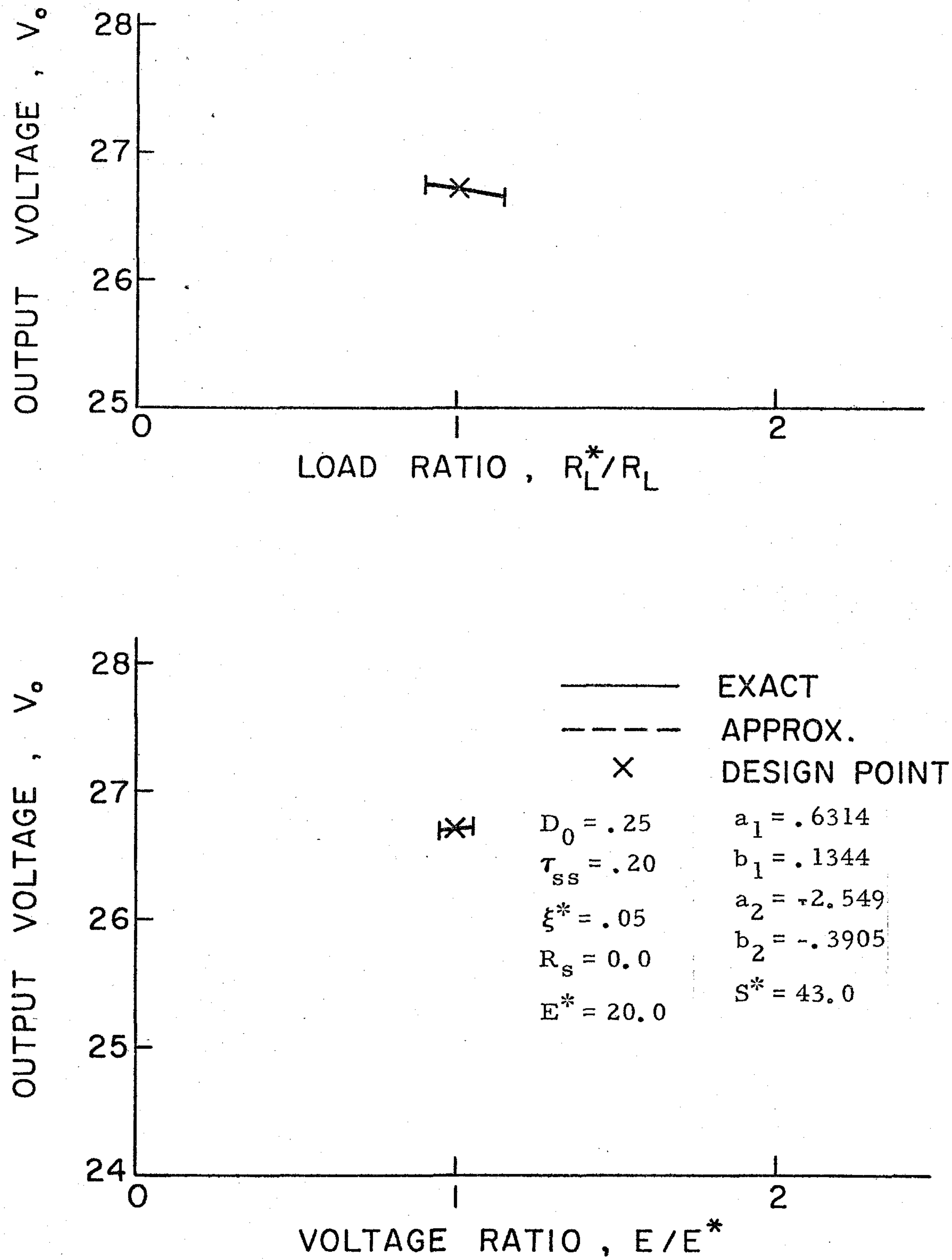



Fig. 3.4. Voltage Regulation for Zero Matrix P.W.M.

changes in the equilibrium state. This behavior is similar to that found for the Zero Matrix P.W.M. of the buck regulator.

3.3.5 Discontinuous P.W.M.

In ref. [10] the operation of a discontinuous P.W.M. for a boost regulator is explained. The inductor current waveform of this P.W.M., fig. (3.5), shows that the end of one switching period and the beginning of the next one occurs when the current becomes zero. The switching period, $\tau_s$, is therefore not constant and depends on the on-time, $\tau_0$. This situation is different from that of the discontinuous P.W.M. for the buck regulator whose switching period was constant, see fig. (2.9). Since the inductor current at the beginning of each iteration is always zero, the recursion formula, eqn. (3.2), can be modified for the discontinuous P.W.M. to give

$$z_{n+1} = e^{-2\xi_0\tau_0}\, y_{22}(\tau_s - \tau_0)\, z_n + \kappa E\left\{\tau_0 y_{12}(\tau_s - \tau_0) + [1 - y_{11}(\tau_s - \tau_0)]\right\} \qquad (3.19a)$$

$$0 = e^{-2\xi_0\tau_0}\, y_{21}(\tau_s - \tau_0)\, z_n + \kappa E\left\{\tau_0 y_{11}(\tau_s - \tau_0) + y_{12}(\tau_s - \tau_0) + 2\xi_0[1 - y_{11}(\tau_s - \tau_0)]\right\} \qquad (3.19b)$$

with the on-time given as

$$\tau_0 = \tau_{00} + a_1(z_{ss} - z_n) \qquad (3.19c)$$

The first equation, eqn. (3.19a), is the recursion formula for the output voltage, $z_n$. The second equation, eqn. (3.19b), defines the switching period, $\tau_s$, while the on-time is given by eqn. (3.19c).

The equilibrium voltage, $z_{sf}$, along with the switching period and on-time can be solved for from eqn. (3.19). If the assumption is made that $\tau_s \ll 1$, then the above equations can be reduced to

$$z_{sf} = \frac{\kappa E \tau_s}{(\tau_s - \tau_0)} \tag{3.20a}$$

$$\tau_0(\tau_s - \tau_0)^2 - 4\xi\tau_s^2 = 0 \tag{3.20b}$$

and

$$\tau_0 = \tau_{00} + a_1(z_{ss} - z_{sf}) \tag{3.20c}$$

Since eqn. (3.20) is valid for any equilibrium point including the design point, the equilibrium voltage without feedback is

$$z_{ss} = \frac{\kappa E \tau_{ss}}{\tau_{ss} - \tau_{00}} = \kappa E / D_0'$$

When feedback is used, the steady-state on-time and switching period become a function of the equilibrium voltage. The linear part of eqn. (3.20) is

$$z_{sf} = \kappa E/D_0' + \kappa E \left[ \frac{\tau_{ss}}{(\tau_{ss} - \tau_{00})^2} - \frac{\tau_{00}}{(\tau_{ss} - \tau_{00})^2} \frac{\partial \tau_s}{\partial \tau_0} \right] \frac{\partial \tau_0}{\partial z_{sf}} (z_{sf} - z_{ss})$$

and from eqn. (3.20b)

$$\frac{\partial \tau_s}{\partial \tau_0} = - \frac{(\tau_{ss} - \tau_{00}) - \tau_{00}}{2\tau_{00} - 8\xi/D_0'}$$

with

$$\frac{\partial \tau_0}{\partial z_{sf}} = - a_1$$

The approximate equilibrium voltage can now be solved for, and it is

$$z_{sf} = \frac{z_{ss} + \frac{\kappa E/D_0'}{S}}{1 + \frac{1}{S}}$$

where

$$S = \frac{a_1 \kappa E}{\tau_{ss} D_0'^2} \left[ \frac{3D_0 D_0'^2 \tau_{ss} - 8\xi}{2D_0 D_0' \tau_{ss} - 8\xi} \right] \tag{3.21a}$$

The closed loop gain, S, is a function of both the input voltage, E, and the damping factor, $\xi$. At the design point eqn. (3.20b) can be used to eliminate the damping factor, $\xi^*$, so that

$$S = \frac{a_1 \kappa^* E^*}{2D_0 D_0' \tau_{ss}} \tag{3.21b}$$

The closed loop gain, eqn. (3.21), will be used in the stability analysis as a figure of merit for the discontinuous P.W.M.

The stability is determined by slightly disturbing the voltage on the $n^{th}$ switching period. The variation in the voltage at the beginning of the $(n+1)^{st}$ switching period is, from eqn. (3.19a),

$$\delta z_{n+1} = \left( a_{11} + g_1 \frac{\partial \tau_0}{\partial z_n} + h_1 \frac{\partial \tau_s}{\partial z_n} \right) \delta z_n \tag{3.22a}$$

where $g_1$ and $h_1$ are given by eqn. (3.6) with the current, $z_{sf}^{(2)}$, set equal to zero. The partial derivatives are determined from eqn. (3.19a & b)

$$\frac{\partial \tau_s}{\partial z_n} = - \frac{a_{21}}{h_2} - \frac{g_2}{h_2} \frac{\partial \tau_0}{\partial z_n}$$

and

$$\frac{\partial \tau_0}{\partial z_n} = - a_1$$

The variational equation, eqn. (3.22a), reduces to

$$\delta z_{n+1} = \frac{1}{h_2} \{(a_{11}h_2 - a_{21}h_1) - a_1(g_1h_2 - g_2h_1)\} \delta z_n \qquad (3.22b)$$

If the magnitude of the variation in the voltage for the $(n+1)^{st}$ switching period is less than that of the $n^{th}$ switching period, the system is locally stable. The system will be stable if

$$-1 < \frac{1}{h_2} \{(a_{11}h_2 - a_{21}h_1) - a_1(g_1h_2 - g_2h_1)\}$$

or

$$a_1 < \frac{(a_{11}h_2 - a_{21}h_1) + h_2}{(g_1h_2 - g_2h_1)} \qquad (3.23a)$$

If the assumption is made that $\tau_s \ll 1$, then the terms of eqn. (3.23) are approximated in App. (II.B), and the stability criterion is

$$a_1 < \frac{2[1 - 2\xi(2\tau_s + \tau_0)] z_{sf} - 2(1 - \xi\tau_s)\kappa E}{\kappa^2 E^2 \left\{\tau_0 - 2\xi[1 - 2\xi(\tau_s + \tau_0)] \left(\frac{z_{sf}}{\kappa E}\right)^2\right\}} \qquad (3.23b)$$

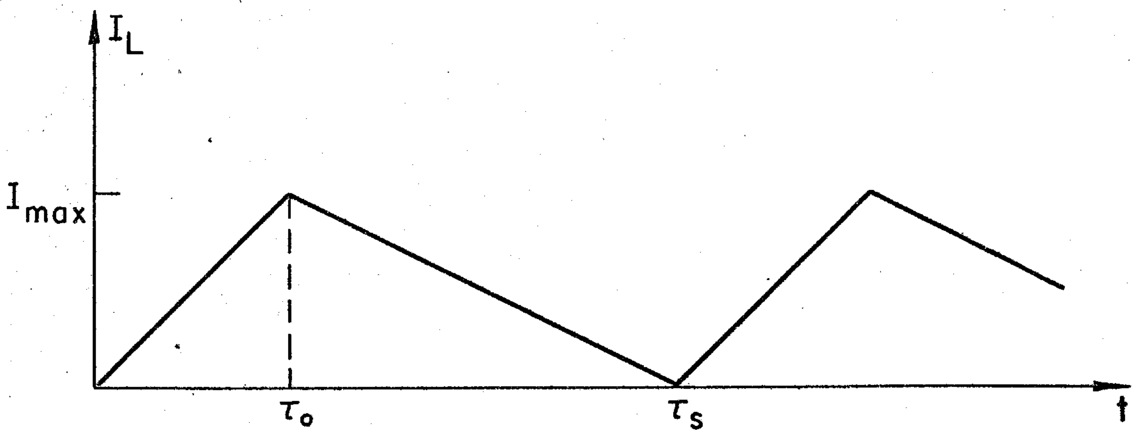



Fig. 3.5. Inductor Current Waveform

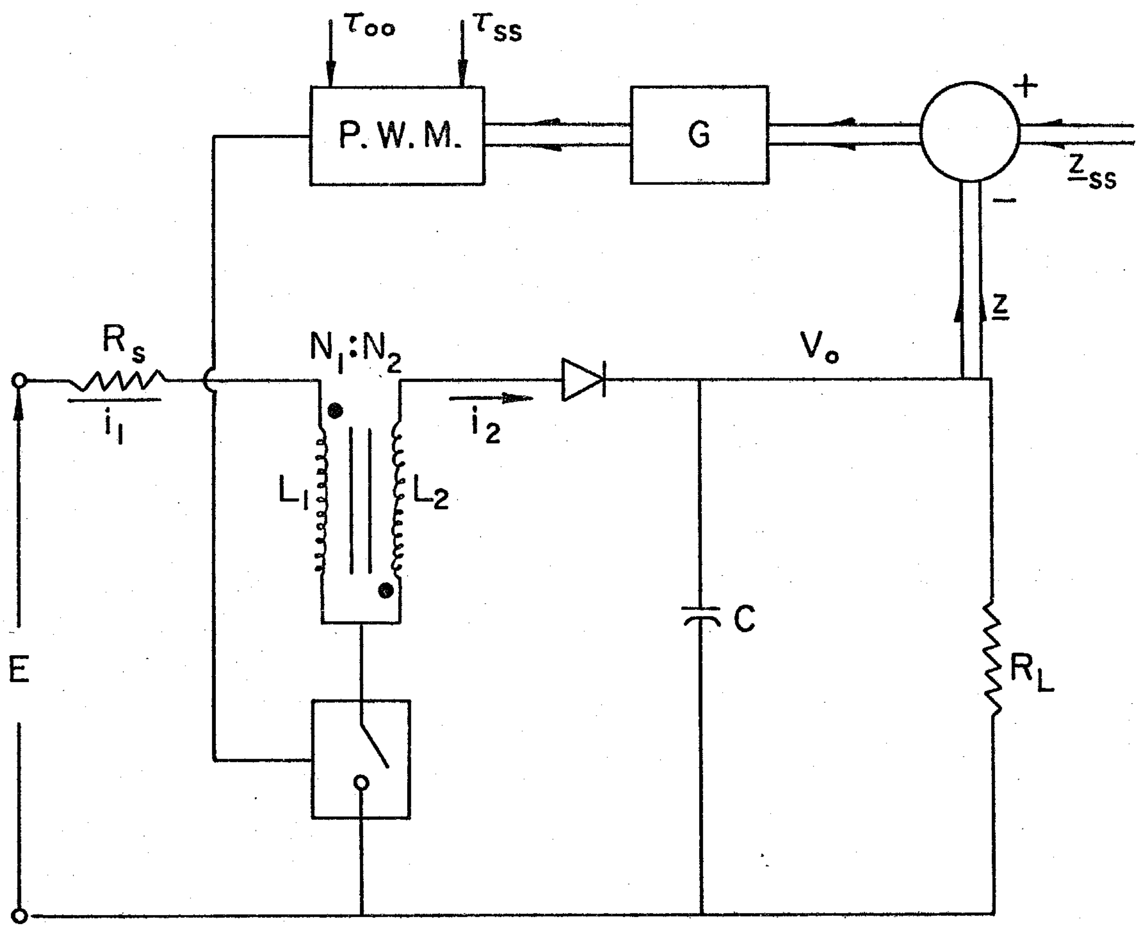



Fig. 3.6. Transformer Coupled Boost Regulator

At the design point the stability criterion reduces to

$$S < \frac{1}{\tau_{ss}^{2} D_0'^{2} \left[ D_0 - 2\xi^* / (D_0'^{2} \tau_{ss}) \right]}$$

and after eqn. (3.20b) is substituted for the damping factor, $\xi^*$, it becomes

$$S < \frac{2}{\tau_{ss}^{2} D_0 D_0'^{2}} \approx S_{max} \tag{3.23c}$$

The maximum closed loop gain for the discontinuous P.W.M. compares favorably with the other P.W.M.'s. Since the recursion formula reduces to a scalar for this P.W.M., good regulation is achieved with only proportional control.

## 3.4 Transformer Coupled Regulator

In some practical applications the input and output voltage of a boost regulator are coupled through a transformer as shown in fig. (3.6). G. W. Wester analyzes a transformer coupled boost regulator in ref. [13]. The current in the transformer, as can be seen in fig. (3.7), is discontinuous but operates in a continuous mode.

The ratio of the currents before and after the jump can be determined from the laws of conservation of flux and charge. The following definitions are needed for the analysis.

$\phi$ = total magnetic flux associated with the transformer

$\phi_1$ = magnetic flux associated with $L_1$

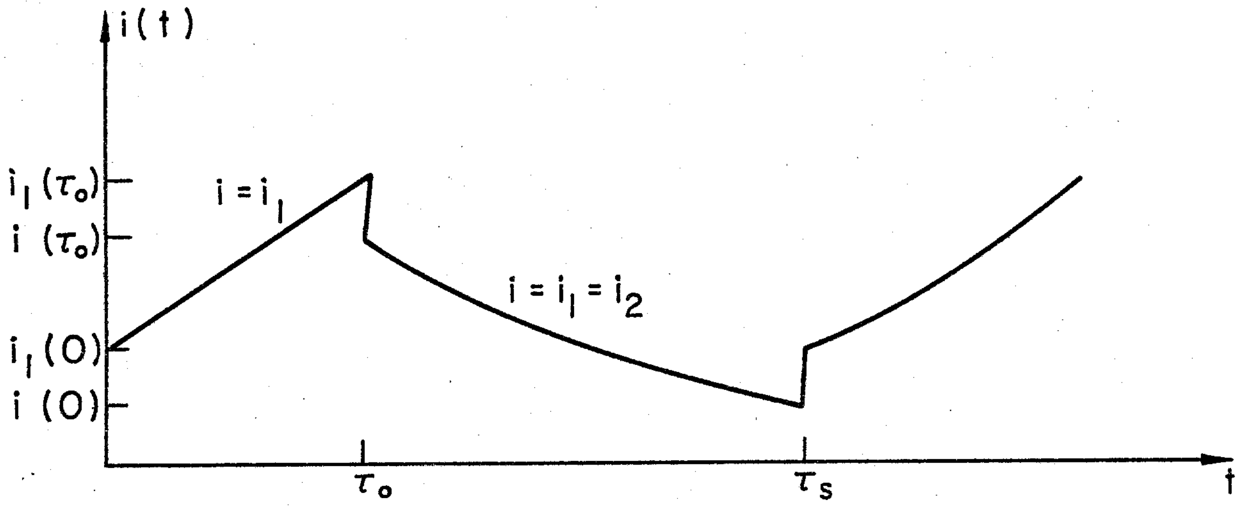



Fig. 3.7. Current Waveform

$\phi_2$ = magnetic flux associated with $L_2$

M = $r\sqrt{L_1 L_2}$ = mutual inductance

r = ratio of flux reaching $L_2$ from $L_1$ to flux generated in $L_1$. The transformer is said to be perfectly coupled if r = 1.

L = $L_1 + L_2 + 2M$

Switch-on: $\phi = \phi_1 = \frac{L_1}{N_1} i_1 \qquad i = i_1 \qquad i_2 = 0$

Switch-off: $\phi = \phi_1 + \phi_2 \qquad i = i_1 = i_2$

$$\phi = \frac{L_1}{N_1} i_1 + \frac{L_2}{N_2} i_2$$

$$\phi = \left(\frac{L_1}{N_1} + \frac{L_2}{N_2}\right) i$$

It is easy to show that if the magnetic flux is conserved (i. e. $\phi$ is constant), then the following relations must be true,

$$i(\tau_0) = \frac{N_2 L_1}{N_2 L_1 + N_1 L_2} i_1(\tau_0)$$

and

$$i_1(0) = \frac{N_2 L_1 + N_1 L_2}{N_2 L_1} i(0)$$

If the state variables, $\underline{z}_n$, are defined to be the current and voltage at the beginning of the on-time, $\tau_0$, just prior to closing the switch, then

$$z_n^{(2)}(\tau_0) = e^{-\kappa\omega_k\mu_s\mu_L\tau_0}\, z_n^{(2)} + \frac{\gamma' E}{\omega_k\mu_s}(1 - e^{-\kappa\omega_k\mu_s\mu_L\tau_0})$$

where

$$\mu_L = \frac{L}{L_1}$$

and

$$\gamma' = \frac{N_2 L_1}{N_2 L_1 + N_1 L_2}$$

The recursion formula for the transformer coupled regulator is slightly different from that given by eqn. (3.2).

$$\underline{z}_{n+1} = \begin{pmatrix} e^{-2\xi_0\tau_0}\, y_{22}(\tau_s - \tau_0) & y_{12}(\tau_s - \tau_0) \\ e^{-2\xi_0\tau_0}\, y_{21}(\tau_s - \tau_0) & y_{11}(\tau_s - \tau_0) \end{pmatrix} \underline{z}_n \tag{3.24a}$$

$$+ \kappa E \begin{pmatrix} \mu_L\gamma'\tau_0 y_{12}(\tau_s - \tau_0) + [1 - y_{11}(\tau_s - \tau_0)] \\ \mu_L\gamma'\tau_0 y_{11}(\tau_s - \tau_0) + y_{12}(\tau_s - \tau_0) + 2\xi_0[1 - y_{11}(\tau_s - \tau_0)] \end{pmatrix}$$

let

$$\gamma = \mu_L\gamma' = \frac{N_2 L}{N_2 L_1 + N_1 L_2}$$

so $\gamma = (N_1 + N_2)/N_1$ for a perfectly coupled transformer (i.e. $r = 1$). The recursion formula can be written in a different and interesting way as

$$\underline{z}_{n+1} = \begin{pmatrix} e^{-2\xi_0\tau_0}\, y_{22}(\tau_s - \tau_0) & y_{12}(\tau_s - \tau_0) \\ e^{-2\xi_0\tau_0}\, y_{21}(\tau_s - \tau_0) & y_{11}(\tau_s - \tau_0) \end{pmatrix} \underline{z}_n$$

$$+ \kappa E \begin{pmatrix} \tau_0 y_{12}(\tau_s - \tau_0) + [1 - y_{11}(\tau_s - \tau_0)] \\ \tau_0 y_{11}(\tau_s - \tau_0) + y_{12}(\tau_s - \tau_0) + 2\xi_0[1 - y_{11}(\tau_s - \tau_0)] \end{pmatrix}$$

$$+ (\gamma - 1)\kappa E \tau_0 \begin{pmatrix} y_{12}(\tau_s - \tau_0) \\ y_{11}(\tau_s - \tau_0) \end{pmatrix} \qquad (3.24b)$$

When the parameter $\gamma$ is equal to one, eqn. (3.24) is the same as the recursion formula for the regular boost regulator, eqn. (3.2). However, when $\gamma$ is not equal to one, the added forcing term is like that of the buck-boost regulator, eqn. (4.3). Since the analysis has already been done for the boost and buck-boost regulators, the analysis for this regulator is made much simpler.

The approximate steady-state vector without feedback, $\underline{z}_{ss}$, can be solved for from the recursion formula

$$z_{ss}^{(1)} = \frac{\kappa E}{D_0'} + (\gamma - 1)\frac{\kappa E D_0}{D_0'}$$

$$z_{ss}^{(2)} = \kappa E\left(\frac{2\xi}{D_0'^{\,2}} - \frac{\tau_{00}}{2}\right) + (\gamma - 1)\kappa E D_0\left(\frac{2\xi}{D_0'^{\,2}} - \frac{\tau_{ss}}{2}\right) \qquad (3.25)$$

If $\gamma$ is set equal to one in eqn. (3.25), the steady-state vector for the boost regulator, eqn. (3.4a), is recovered. When $\gamma$ is greater than one, the added term is the steady-state vector of the buck-boost regulator, eqn. (4.5), multiplied by $(\gamma-1)$. The steady-state voltage given above, for a perfectly coupled transformer, can be rearranged to have the same form as that derived by G. W. Wester in ref. [13]. If the recursion formula is linearized, and the assumption made that $\tau_s \ll 1$, then the closed loop gain for this regulator is

$$S = \frac{\gamma\kappa E}{\tau_{ss} D_0'^2}(a_1 + D_0 a_2) + \frac{\gamma^2\kappa^2 E^2 D_0}{2\tau_{ss} D_0'^2}(a_1 b_2 - b_1 a_2)$$

and

$$S1 = \kappa E \frac{2\xi}{\tau_{ss} D_0'^2}(b_1 + D_0 b_2) + \gamma\kappa E\left[\frac{2\xi(1+D_0)}{\tau_{ss} D_0'^3}(b_1 + D_0 b_2) - \frac{b_1}{2}\right]$$

$$+ \frac{\gamma^2\kappa^2 E^2 D_0}{2\tau_{ss} D_0'^2}(a_1 b_2 - b_1 a_2) \qquad (3.26)$$

The expression for the closed loop gain of the regulator, eqn. (3.26), is very similar to both the boost and buck-boost regulator, eqn. (3.5) or eqn. (4.6).

The variational equation for the transformer coupled boost regulator is

$$\delta \underline{z}_{n+1} = P\, \delta \underline{z}_n \qquad (3.27)$$

where

$$P = \begin{pmatrix} \{a_{11} - a_1 g_1 + a_2 h_1\} & \{a_{12} - b_1 g_1 + b_2 h_1\} \\ \{a_{21} - a_1 g_2 + a_2 h_2\} & \{a_{22} - b_1 g_2 + b_2 h_2\} \end{pmatrix}$$

It is relatively easy to form the vectors $\underline{g}$ and $\underline{h}$ from eqn. (3.6) of the boost regulator, and eqn. (4.7) of the buck-boost regulator; so

$$\underline{g} = \begin{pmatrix} e^{-2\xi_0\tau_0} y_{12}(\tau_s-\tau_0) z_{sf}^{(1)} - y_{22}(\tau_s-\tau_0) z_{sf}^{(2)} - \gamma\kappa E\tau_0 y_{22}(\tau_s-\tau_0) + (\gamma-1)\kappa E y_{12}(\tau_s-\tau_0) \\ e^{-2\xi_0\tau_0} y_{11}(\tau_s-\tau_0) z_{sf}^{(1)} + y_{12}(\tau_s-\tau_0) z_{sf}^{(2)} + \gamma\kappa E\tau_0 y_{12}(\tau_s-\tau_0) + (\gamma-1)\kappa E y_{11}(\tau_s-\tau_0) \end{pmatrix}$$

and

$$\underline{h} = \begin{pmatrix} -e^{-2\xi_0\tau_0}[y_{12}(\tau_s-\tau_0) + 2\xi_0 y_{22}(\tau_s-\tau_0)] z_{sf}^{(1)} + y_{22}(\tau_s-\tau_0) z_{sf}^{(2)} + \kappa E[\gamma\tau_0 y_{22}(\tau_s-\tau_0) + y_{12}(\tau_s-\tau_0)] \\ -e^{-2\xi_0\tau_0} y_{22}(\tau_s-\tau_0) z_{sf}^{(1)} + y_{21}(\tau_s-\tau_0) z_{sf}^{(2)} + \kappa E[\gamma\tau_0 y_{21}(\tau_s-\tau_0) + y_{11}(\tau_s-\tau_0)] \end{pmatrix}$$

If it is assumed that $\tau_s \ll 1$, then the above expressions simplify to

$$\underline{g} = \begin{pmatrix} (\tau_s - \tau_0) z_{sf}^{(1)} - [1 - 2\xi(\tau_s - \tau_0)] z_{sf}^{(2)} - \kappa E(\tau_s - \tau_0) + \gamma\kappa E(\tau_s - 2\tau_0) \\ (1 - 2\xi\tau_0) z_{sf}^{(1)} + (\tau_s - \tau_0) z_{sf}^{(2)} + (\gamma - 1)\kappa E \end{pmatrix}$$

and

$$\underline{h} = \begin{pmatrix} -[(\tau_s - \tau_0) + 2\xi(1 - 2\xi\tau_s)] z_{sf}^{(1)} + [1 - 2\xi(\tau_s - \tau_0)] z_{sf}^{(2)} + \kappa E[\tau_s + (\gamma-1)\tau_0] \\ -(1 - 2\xi\tau_s) z_{sf}^{(1)} - (\tau_s - \tau_0) z_{sf}^{(2)} + \kappa E \end{pmatrix}$$

At the design point, $\underline{z}_{sf} = \underline{z}_{ss}$, and for $\tau_s \ll 1$, the elements of the perturbation matrix are

$$p_{11} = a_{11} + a_1\kappa^*E^*\left\{\frac{2\xi^*}{D_0'^2}\left[1+(\gamma-1)D_0\right] - \gamma\,\frac{(2\tau_{ss}-\tau_{00})}{2}\right\}$$

$$+ a_2\kappa^*E^*D_0\left\{\frac{2\xi^*}{D_0'^2}\left[1+(\gamma-1)D_0\right] - \frac{\gamma\tau_{ss}}{2}\right\}$$

$$p_{12} = a_{12} + b_1\kappa^*E^*\left\{\frac{2\xi^*}{D_0'^2}\left[1+(\gamma-1)D_0\right] - \gamma\,\frac{(2\tau_{ss}-\tau_{00})}{2}\right\}$$

$$+ b_2\kappa^*E^*D_0\left\{\frac{2\xi^*}{D_0'^2}\left[1+(\gamma-1)D_0\right] - \frac{\gamma\tau_{ss}}{2}\right\}$$

$$p_{21} = a_{21} - a_1\kappa^*E^*\gamma/D_0' - a_2\kappa^*E^*\gamma D_0/D_0'$$

$$p_{22} = a_{22} - b_1\kappa^*E^*\gamma/D_0' - b_2\kappa^*E^*\gamma D_0/D_0'$$

The above expressions always reduce to the corresponding quantity of the boost regulator when $\gamma$ is set equal to one.

The transformer coupled boost regulator will be locally stable about the equilibrium, $\underline{z}_{sf}$, if and only if the moduli of the eigenvalues of the P-matrix are less than one. The determinate of the P-matrix is

$$\text{Det.}(P) = \text{Det.}(A) + a_1(a_{12}g_2 - a_{22}g_1) + a_2(a_{22}h_1 - a_{12}h_2)$$

$$+ b_1(a_{21}g_1 - a_{11}g_2) + b_2(a_{11}h_2 - a_{21}h_1)$$

$$+ (a_1b_2 - b_1a_2)(h_1g_2 - g_1h_2)$$

If the assumption is made that $\tau_{ss} \ll 1$, then the coefficients of the feedback constants in the previous equation become

$$a_{12}g_2 - a_{22}g_1 = [1 - 2\xi(\tau_s - \tau_0)]z_{sf}^{(2)} + \gamma\kappa E\tau_0$$

$$a_{22}h_1 - a_{12}h_2 = -2\xi(1 - 2\xi\tau_s)z_{sf}^{(1)} + [1 - 2\xi(\tau_s - \tau_0)]z_{sf}^{(2)} + \gamma\kappa E\tau_0$$

$$a_{21}g_1 - a_{11}g_2 = -[1 - 2\xi(\tau_s + \tau_0)]z_{sf}^{(1)} - (1 - 2\xi\tau_s)(\gamma - 1)\kappa E$$

$$a_{11}h_2 - a_{21}h_1 = -[1 - 2\xi(\tau_s + \tau_0)]z_{sf}^{(1)} + (1 - 2\xi\tau_s)\kappa E$$

and

$$h_1g_2 - g_1h_2 = \kappa^2E^2\left\{\gamma^2\tau_0 + \gamma[1 - 2\xi(\tau_s - \tau_0)]\left(\frac{z_{sf}^{(2)}}{\kappa E}\right) - 2\xi[1 - 2\xi(\tau_s + \tau_0)]\left(\frac{z_{sf}^{(1)}}{\kappa E}\right)^2\right\}$$

The stability criterion for the transformer coupled boost regulator when the eigenvalues are complex is

$$\begin{aligned} &a_1[z_{sf}^{(2)} + \gamma\kappa E\tau_0] + a_2[-2\xi z_{sf}^{(1)} + z_{sf}^{(2)} + \gamma\kappa E\tau_0] \\ &- (b_1 + b_2)z_{sf}^{(1)} + \kappa E[b_2 - (\gamma - 1)b_1] \\ &+ \kappa^2E^2(a_1b_2 - b_1a_2)\left\{\gamma\left[\gamma\tau_0 + \frac{z_{sf}^{(2)}}{\kappa E}\right] - 2\xi\left[\frac{z_{sf}^{(2)}}{\kappa E}\right]^2\right\} \\ &< 2\xi\tau_s \end{aligned} \tag{3.28a}$$

and when the eigenvalues are real the stability criterion is

$$4 + a_1\{2[z_{sf}^{(2)} + \gamma\kappa E\tau_0] - (\tau_s - \tau_0)z_{sf}^{(1)} - (\gamma-1)\kappa E(\tau_s - \tau_0)\}$$

$$+ a_2\{-[(\tau_s - \tau_0) + 4\xi]z_{sf}^{(1)} + 2z_{sf}^{(2)} + \kappa E[\tau_s + (2\gamma-1)\tau_0]\}$$

$$- b_1\{2[z_{sf}^{(1)} + (\gamma-1)\kappa E] + (\tau_s - \tau_0)z_{sf}^{(2)}\}$$

$$- b_2\{2[z_{sf}^{(1)} - \kappa E] + (\tau_s - \tau_0)z_{sf}^{(2)}\}$$

$$+ \kappa^2 E^2(a_1 b_2 - b_1 a_2)\left\{\gamma\left[\gamma\tau_0 + \frac{z_{sf}^{(2)}}{\kappa E}\right] - 2\xi\left[\frac{z_{sf}^{(1)}}{\kappa E}\right]^2\right\} > 0 \quad (3.28b)$$

The stability criterion given by eqn. (3.28) will reduce to that given by eqn. (3.8) when $\gamma$ is equal to one. This stability criterion, like that of the boost regulator, is only valid to second order in $\tau_s$ when $\xi \leqslant \tau_s$.

## 3.5 Discussion of Results

In this chapter the same analysis which was done for the buck regulator is done for the boost regulator. The recursion formula is derived, and then the general expressions for the regulation and local stability are obtained. It is found that a closed loop gain can be defined for the boost regulator which plays the same role as the one defined for the buck regulator. The actual definitions of these two loop gains are very similar.

Comparisons between the buck and boost regulators are also possible when examining the different P.W.M.s. The loop gains of the zero eigenvalue, zero matrix, and M.O.E. P.W.M.s of the buck regulator differ only by a constant, $D_0'^2$, from the corresponding quantities of the boost regulator. The loop gain of the zero eigenvalue and zero matrix P.W.M. for the boost regulator are identical as they also are for the buck regulator. The fact that they are equal is surprising since the feedback constants of the two P.W.M.s are different. The feedback constants on the voltage of the three P.W.M.s mentioned above for the boost regulator are identical to those of the buck regulator.

The similarities between the two regulators are many, but there are also a number of contrasts. The equilibrium voltage of the boost regulator showed much more dependence on variations in the load than did the buck regulator. The dependence of stability on the input parameters was much more sensitive for the boost regulator than was observed for the buck regulator. This difference in sensitivity is dramatized by the M.O.E. P.W.M. which was stable over a wide range of input parameters for the buck regulator but showed virtually no range for the boost regulator. The biggest difference between the buck and boost regulator is that the boost regulator exhibits positive feedback for the higher damping factors. In fact, the simple expressions derived in this chapter are valid only for small damping factors (i.e. the assumptions are $\tau_{ss} \ll 1$ and $\xi \leq \tau_{ss}$). If the damping factor is made large in comparison with the switching period, the expressions derived are not valid, and the stability is decreased.

# CHAPTER IV - BUCK-BOOST REGULATOR

## 4.1 Recursion Formula

The buck-boost regulator can regulate an output voltage which is either lower or higher than the input voltage. It therefore acts both as a buck and a boost regulator. When the switches of fig. 4.1 are closed, the buck-boost regulator is in the same configuration that the boost regulator is in during its duty cycle, and when the switches are open, the configuration is that of a buck regulator not in its duty cycle. The derivation of the recursion formula proceeds in two parts depending on whether the switch is closed or open. The state variables, like those of the boost regulator, are taken to be the output voltage and inductor current.

Switches Closed:

The time the switches are closed is denoted as the on-time, $\tau_0$, and the duty cycle is defined as the ratio of the on-time to the switching period. The differential equations are the same as the boost regulator during its duty cycle, so the solution is

$$\underline{z}(\tau_0) = \begin{pmatrix} e^{-2\xi_0\tau_0} & 0 \\ 0 & e^{-\kappa\omega_k\mu_s\tau_0} \end{pmatrix} \underline{z}(0) + \frac{E}{\omega_k\mu_s}(1-e^{-\kappa\omega_k\mu_s\tau_0}) \begin{pmatrix} 0 \\ 1 \end{pmatrix} \qquad (4.1)$$

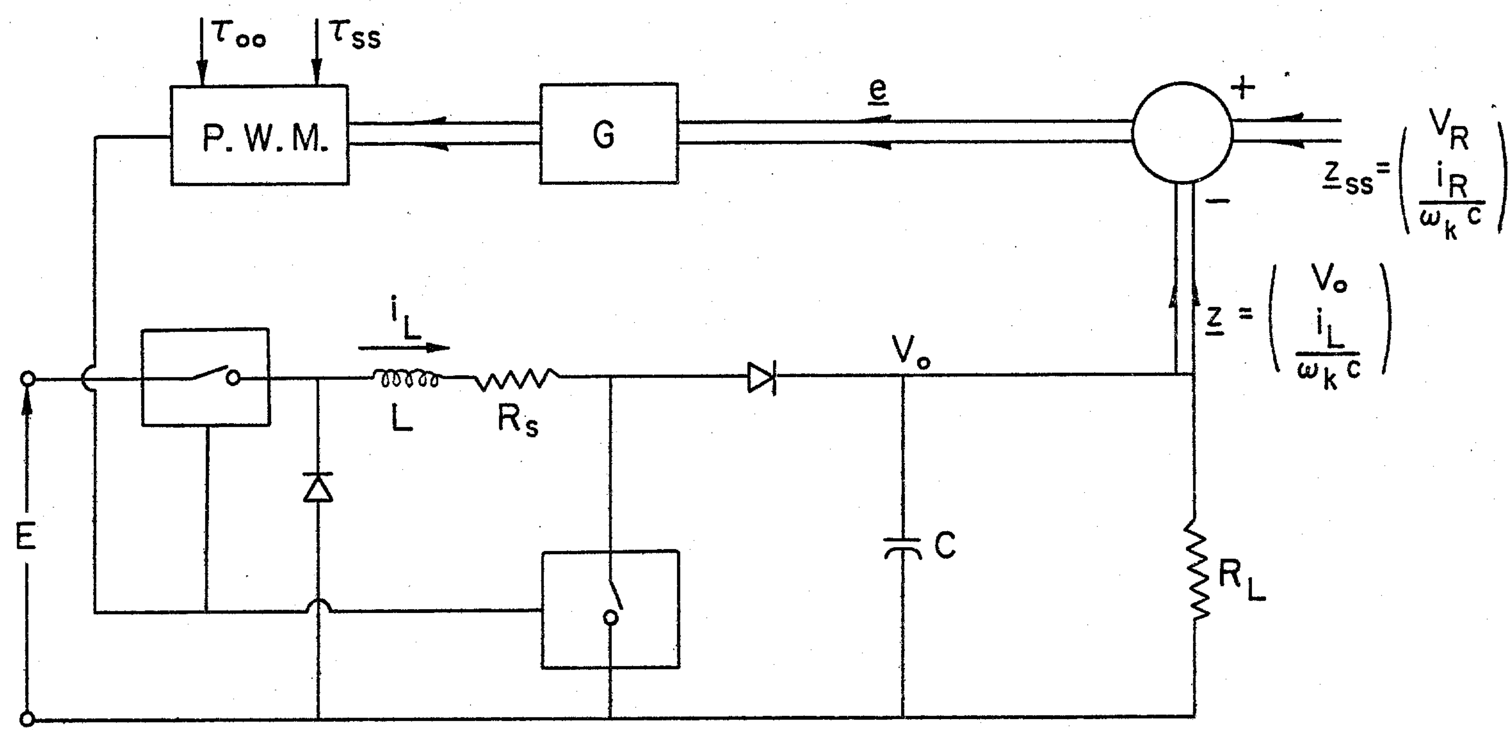



Fig. 4.1. Buck-Boost Regulator

Switches Open:

The output voltage and its derivative for this configuration is the same as that of the buck regulator with no input voltage.

$$\underline{x}(\tau_s) = Y(\tau_s - \tau_0)\,\underline{x}(\tau_0) \tag{4.2}$$

The transformation which relates the state variables, $\underline{z}(\tau)$, to the vector $\underline{x}(\tau)$ is

$$\underline{z}(\tau) = \begin{pmatrix} 1 & 0 \\ 2\xi_0 & 1 \end{pmatrix} \underline{x}(\tau)$$

so

$$\underline{z}(\tau_s) = \begin{pmatrix} 1 & 0 \\ 2\xi_0 & 1 \end{pmatrix} Y(\tau_s - \tau_0) \begin{pmatrix} 1 & 0 \\ -2\xi_0 & 1 \end{pmatrix} \underline{z}(\tau_0)$$

and after eqn. (4.1) is substituted for $\underline{z}(\tau_0)$

$$\underline{z}(\tau_s) =$$

$$\begin{pmatrix} e^{-2\xi_0\tau_0}\{y_{22}(\tau_s-\tau_0)+2(\xi-\xi_0)\,y_{12}(\tau_s-\tau_0)\} & e^{-\kappa\omega_k\mu_s\tau_0}\,y_{12}(\tau_s-\tau_0) \\ e^{-2\xi_0\tau_0}\{y_{21}(\tau_s-\tau_0)+4\xi_0(\xi-\xi_0)y_{12}(\tau_s-\tau_0)\} & e^{-\kappa\omega_k\mu_s\tau_0}\{y_{11}(\tau_s-\tau_0)+2(\xi_0-\xi)y_{12}(\tau_s-\tau_0)\} \end{pmatrix} \underline{z}(0)$$

$$+ \frac{E}{\omega_k\mu_s}(1 - e^{-\kappa\omega_k\mu_s\tau_0}) \begin{pmatrix} y_{12}(\tau_s-\tau_0) \\ y_{11}(\tau_s-\tau_0)+2(\xi_0-\xi)\,y_{12}(\tau_s-\tau_0) \end{pmatrix}$$

If the assumptions are made that $\omega_k\mu_s \ll \xi$ and $\omega_k\mu_s\tau_0 \ll 1$, which are the same assumptions made for the boost regulator, then the above equation reduces to

$$\underline{z}_{n+1} = \begin{pmatrix} e^{-2\xi_0\tau_0}\, y_{22}(\tau_s-\tau_0) & y_{12}(\tau_s-\tau_0) \\ e^{-2\xi_0\tau_0}\, y_{21}(\tau_s-\tau_0) & y_{11}(\tau_s-\tau_0) \end{pmatrix} \underline{z}_n + \kappa E \tau_0 \begin{pmatrix} y_{12}(\tau_s-\tau_0) \\ y_{11}(\tau_s-\tau_0) \end{pmatrix} \qquad (4.3)$$

Eqn. (4. 3) is the recursion formula for the buck-boost regulator. It is very similar to the recursion formula of the boost regulator, eqn. (3. 2).

The form of the control laws used with the buck-boost regulator will be the same as the control laws used with the boost regulator. These linear control laws are

$$\tau_0(\underline{z}_n) = \tau_{00} + a_1[z_{ss}^{(1)} - z_n^{(1)}] + b_1[z_{ss}^{(2)} - z_n^{(2)}] \qquad (4.4a)$$

and

$$\tau_s(\underline{z}_n) = \tau_{ss} + a_2[z_n^{(1)} - z_{ss}^{(1)}] + b_2[z_n^{(2)} - z_{ss}^{(2)}] \qquad (4.4b)$$

The feedback constants in these control laws will be specialized later for the various P. W. M. s analyzed.

## 4. 2 Regulation and Local Stability

The steady-state with feedback, $\underline{z}_{ss}$, can be calculated from the recursion formula, eqn. (4. 3), with the feedback constants set equal to zero. If the assumption is made that $\tau_s \ll 1$, then the recursion formula can be approximated as

$$\underline{z}_{ss} = \begin{pmatrix} \{1-2\xi\tau_{ss} - \frac{1}{2}(\tau_{ss}-\tau_{00})^2\} & \{\tau_{ss}-\tau_{00}\} \\ -\{\tau_{ss}-\tau_{00}\} & \{1-\frac{1}{2}(\tau_{ss}-\tau_{00})^2\} \end{pmatrix} \underline{z}_{ss} + \kappa E \tau_0 \begin{pmatrix} (\tau_s-\tau_0) \\ 1 \end{pmatrix}$$

so that

$$z_{ss}^{(1)} \approx \kappa E D_0 / D_0' \qquad (4.5)$$

$$z_{ss}^{(2)} \approx \kappa E D_0 \left( \frac{2\xi}{D_0'^{\,2}} - \frac{\tau_{ss}}{2} \right)$$

where

$$D_0 = \tau_{00} / \tau_{ss}$$

Eqn. (4.5) is the steady-state for an uncontrolled regulator when $\tau_s \ll 1$. The reference vector for the controlled regulator is taken to be the steady-state of the uncontrolled regulator, $\underline{z}_{ss}$.

The steady-state with feedback, $\underline{z}_{sf}$, can only be solved for with the aid of a computer. An approximate solution, eqn. (4.6), is derived in Appendix (III.A) for small changes in the equilibrium on-time or switching period and when $\tau_s \ll 1$.

$$z_{sf}^{(1)} = \left\{ z_{ss}^{(1)} + \frac{\kappa E D_0 / D_0'}{S} \left[ 1 + \kappa E D_0 \frac{2\xi}{\tau_{ss} D_0'^{\,3}} (b_1 + D_0 b_2) + \frac{\kappa E D_0}{D_0'} \left( \frac{b_1 + b_2}{2} \right) + \frac{(b_1 + D_0 b_2)}{\tau_{ss} D_0 D_0'} z_{ss}^{(2)} \right] \right\} \Big/ \left\{ 1 + \frac{1}{S} \left[ 1 + \kappa E \left( \frac{2\xi (1 + D_0)}{\tau_{ss} D_0'^{\,3}} (b_1 + D_0 b_2) - \frac{b_1}{2} \right) \right] \right\}$$

$$z_{sf}^{(2)} = \left\{ S1 z_{ss}^{(2)} + \kappa E D_0 \left( \frac{2\xi}{D_0'^2} - \frac{\tau_{ss}}{2} \right) - \kappa^2 E^2 \frac{2\xi D_0}{\tau_{ss} D_0'^4} (a_1 + D_0 a_2) \right.$$

$$\left. - \frac{\kappa^2 E^2 D_0^2}{2 D_0'^2} (a_1 + a_2) \right\} \Big/ \left\{ S1 + \left[ 1 + \kappa E \left( \frac{a_1 + D_0 a_2}{\tau_{ss} D_0'^2} \right) \right] \right\}$$

$$+ \frac{\kappa E \left( \frac{2\xi(1+D_0)}{\tau_{ss} D_0'^3} (a_1 + D_0 a_2) - \frac{a_1}{2} \right) z_{ss}^{(1)}}{S1 + \left[ 1 + \kappa E \left( \frac{a_1 + D_0 a_2}{\tau_{ss} D_0'^2} \right) \right]} \tag{4.6}$$

$$S = \frac{\kappa E}{\tau_{ss} D_0'^2} (a_1 + D_0 a_2) + \frac{\kappa^2 E^2 D_0}{2 \tau_{ss} D_0'^2} (a_1 b_2 - b_1 a_2)$$

$$S1 = \kappa E \left[ \frac{2\xi(1+D_0)}{\tau_{ss} D_0'^3} (b_1 + D_0 b_2) - \frac{b_1}{2} \right] + \frac{\kappa^2 E^2 D_0}{2 \tau_{ss} D_0'^2} (a_1 b_2 - b_1 a_2)$$

The closed loop gain, S, of the buck-boost regulator is defined the same as the boost regulator's. The buck-boost regulator also shows the same sort of dependence on the load as the boost regulator does. In fact, all the comments made about the equilibrium voltage of the boost regulator can also be made about the buck-boost regulator.

The variational equation of the buck-boost regulator, see Appendix (III. B) for derivation, is

$$\delta \underline{z}_{n+1} = P \delta \underline{z}_n \tag{4.7}$$

where

$$P = \begin{pmatrix} \{a_{11} - a_1 g_1 + a_2 h_1\} & \{a_{12} - b_1 g_1 + b_2 h_1\} \\ \{a_{21} - a_1 g_2 + a_2 h_2\} & \{a_{22} - b_1 g_2 + b_2 h_2\} \end{pmatrix}$$

and

$$\underline{g} = \begin{pmatrix} e^{-2\xi_0 \tau_0} y_{12}(\tau_s - \tau_0) z_{sf}^{(1)} - y_{22}(\tau_s - \tau_0) z_{sf}^{(2)} + \kappa E[y_{12}(\tau_s - \tau_0) - \tau_0 y_{22}(\tau_s - \tau_0)] \\ e^{-2\xi_0 \tau_0} y_{11}(\tau_s - \tau_0) z_{sf}^{(1)} + y_{12}(\tau_s - \tau_0) z_{sf}^{(2)} + \kappa E[y_{11}(\tau_s - \tau_0) + \tau_0 y_{12}(\tau_s - \tau_0)] \end{pmatrix}$$

$$\underline{h} = \begin{pmatrix} -e^{-2\xi_0 \tau_0}[y_{12}(\tau_s - \tau_0) + 2\xi_0 y_{22}(\tau_s - \tau_0)] z_{sf}^{(1)} + y_{22}(\tau_s - \tau_0) z_{sf}^{(2)} + \kappa E \tau_0 y_{22}(\tau_s - \tau_0) \\ -e^{-2\xi_0 \tau_0} y_{22}(\tau_s - \tau_0) z_{sf}^{(1)} + y_{21}(\tau_s - \tau_0) z_{sf}^{(2)} + \kappa E \tau_0 y_{21}(\tau_s - \tau_0) \end{pmatrix}$$

The local stability of the buck-boost regulator will be guaranteed if the modulus of the eigenvalues of the P matrix are less than one.

The elements of the P matrix can be simplified, see eqn. (III.B.5a), if it is assumed that $\tau_s \ll 1$,

$$\begin{aligned} p_{11} &= a_{11} - a_1\{(\tau_s - \tau_0) z_{sf}^{(1)} - z_{sf}^{(2)} + \kappa E(\tau_s - 2\tau_0)\} \\ &\quad - a_2\{[(\tau_s - \tau_0) + 2\xi] z_{sf}^{(1)} - z_{sf}^{(2)} - \kappa E \tau_0\} \\ p_{12} &= a_{12} - b_1\{(\tau_s - \tau_0) z_{sf}^{(1)} - z_{sf}^{(2)} + \kappa E(\tau_s - 2\tau_0)\} \\ &\quad - b_2\{[(\tau_s - \tau_0) + 2\xi] z_{sf}^{(1)} - z_{sf}^{(2)} - \kappa E \tau_0\} \end{aligned} \qquad (4.8a)$$

$$p_{21} = a_{21} - a_1\{z_{sf}^{(1)} + (\tau_s - \tau_0) z_{sf}^{(2)} + \kappa E\} - a_2\{z_{sf}^{(1)} + (\tau_s - \tau_0) z_{sf}^{(2)}\}$$

$$p_{22} = a_{22} - b_1\{z_{sf}^{(1)} + (\tau_s - \tau_0) z_{sf}^{(2)} + \kappa E\} - b_2\{z_{sf}^{(1)} + (\tau_s - \tau_0) z_{sf}^{(2)}\}$$

When the system is evaluated at the design point (i.e. $\underline{z}_{sf} = \underline{z}_{ss}$), the perturbation matrix is

$$p_{11} = a_{11} + a_1 \kappa^* E^* \left[ \frac{2\xi^* D_0}{D_0'^{\,2}} - \frac{(2-D_0)\tau_{ss}}{2} \right] + a_2 \kappa^* E^* D_0 \left[ \frac{2\xi^* D_0}{D_0'^{\,2}} - \frac{\tau_{ss}}{2} \right]$$

$$p_{12} = a_{12} + b_1 \kappa^* E^* \left[ \frac{2\xi^* D_0}{D_0'^{\,2}} - \frac{(2-D_0)\tau_{ss}}{2} \right] + b_2 \kappa^* E^* D_0 \left[ \frac{2\xi^* D_0}{D_0'^{\,2}} - \frac{\tau_{ss}}{2} \right]$$

$$p_{21} = a_{21} - a_1 \kappa^* E^* / D_0' - a_2 \kappa^* E^* D_0 / D_0' \qquad (4.8b)$$

$$p_{22} = a_{22} - b_1 \kappa^* E^* / D_0' - b_2 \kappa^* E^* D_0 / D_0'$$

The asterisk in eqn. (4.8b) indicates that the parameters are the design parameters. The approximate stability criterion for the buck-boost regulator, eqn. (III.B.7), is

$$a_1 [ z_{sf}^{(2)} + \kappa E \tau_0 ] + a_2 [ -2\xi z_{sf}^{(1)} + z_{sf}^{(2)} + \kappa E \tau_0 ]$$

$$- (b_1 + b_2) z_{sf}^{(1)} - b_1 \kappa E$$

$$+ \kappa^2 E^2 (a_1 b_2 - b_1 a_2) \left\{ \left( \frac{z_{sf}^{(2)}}{\kappa E} + \tau_0 \right) - 2\xi \frac{z_{sf}^{(1)}}{\kappa E} \left( 1 + \frac{z_{sf}^{(1)}}{\kappa E} \right) \right\}$$

$$< 2\xi \tau_s \qquad (4.9a)$$

when the eigenvalues are complex, and

$$4 + a_1[2z_{sf}^{(2)} - (\tau_s - \tau_0)z_{sf}^{(1)} - \kappa E(\tau_s - 3\tau_0)]$$

$$+ a_2\{-[(\tau_s - \tau_0) + 4\xi]z_{sf}^{(1)} + 2z_{sf}^{(2)} + 2\kappa E\tau_0\}$$

$$- b_1[2z_{sf}^{(1)} + 2\kappa E + (\tau_s - \tau_0)z_{sf}^{(2)}] - b_2[2z_{sf}^{(1)} + (\tau_s - \tau_0)z_{sf}^{(2)}]$$

$$+ \kappa^2E^2(a_1b_2 - b_1a_2)\left\{\left(\frac{z_{sf}^{(2)}}{\kappa E} + \tau_0\right) - 2\xi\frac{z_{sf}^{(1)}}{\kappa E}\left(1 + \frac{z_{sf}^{(1)}}{\kappa E}\right)\right\}$$

$$> 0 \tag{4.9b}$$

when the eigenvalues are real. The stability criterion given by eqn. (4.9) is only valid when $\tau_s \ll 1$. The stability criterion for the buck-boost and the boost regulators are remarkably similar [i.e. compare eqn. (3.8) and eqn. (4.9)]. The equilibrium point, $\underline{z}_{sf}$, is of course different for the two regulators. The stability criterion, eqn. (4.9), is valid to second order in $\tau_s$ only when $\xi \leqslant \tau_s$. In the next section the general stability equation will be used to evaluate some specific P.W.M.s.

## 4.3 Comparison of P.W.M.s

### 4.3.1 Uniformly Sampled Voltage P.W.M.

The control law for the uniformly sampled voltage P.W.M., with only the on-time varied, is

$$\tau_0(\underline{z}_n) = \tau_{00} + a_1[z_{ss}^{(1)} - z_n^{(1)}]$$

The closed loop gain defined by eqn. (4.6) is

$$S = \frac{a_1 \kappa E}{\tau_{ss} D_0'^2}$$

The maximum closed loop gain is given by eqn. (4.9a) which reduces to

$$S < \frac{2\xi}{D_0'^2 \left( \frac{z_{sf}^{(2)}}{\kappa E} + \tau_0 \right)} \approx S_{max} \qquad (4.10a)$$

When the stability is evaluated at the design point, $\underline{z}_{ss} = \underline{z}_{sf}$, the maximum loop gain becomes

$$S < \frac{1}{D_0 \left( 1 + \frac{D_0'^2 \tau_{ss}}{4\xi^*} \right)} \approx S_{max} \qquad (4.10b)$$

The maximum closed loop gain of the buck-boost regulator in eqn. (4.10a) is the same as that of the boost regulator given by eqn. (3.9a). The equilibrium current, $z_{sf}^{(2)}$, however, is different for the two regulators and that is why eqn. (4.10b) and eqn. (3.9b) are not the same.

The control law for this P.W.M., when only the switching period is varied, is

$$\tau_s(\underline{z}_n) = \tau_{ss} + a_2[z_n^{(1)} - z_{ss}^{(1)}]$$

and the loop gain, eqn. (4.6), is

$$S = \frac{a_2 \kappa D_0 E}{\tau_{ss} D_0'^2}$$

The maximum closed loop gain for the buck-boost regulator is

$$S < \frac{2\xi D_0}{D_0'^2 \left( \frac{z_{sf}^{(2)}}{\kappa E} - 2\xi \frac{z_{sf}^{(1)}}{\kappa E} + \tau_0 \right)} \approx S_{max} \qquad (4.11a)$$

which is the same form as the boost regulator, eqn. (3.10a). At the design point the maximum closed loop gain can be written as

$$S < \frac{1}{D_0 + \frac{D_0'^2 \tau_{ss}}{4\xi^*}} \approx S_{max} \qquad (4.11b)$$

The maximum closed loop gain is small for either a V.O.T. or V.S.P. controlled buck-boost regulator, and the expressions are similar to those derived for the boost regulator.

The reason the maximum loop gain is small for the buck-boost regulator is that the perturbation matrix, eqn. (4.8b), exhibits positive feedback for sufficiently large damping factors, $\xi^*$. The explanation for this positive feedback is the same as it was for the boost regulator. A decrease in the on-time increases the charging time of the capacitor. When feedback is used on both state variables, the stability of the buck-boost regulator can be greatly improved.

4.3.2 Zero Eigenvalue P.W.M.

The zero eigenvalue P.W.M. has feedback on both the state variables, and it does have better stability than the uniformly sampled voltage P.W.M. The feedback constants, $a_1$ and $b_1$, can be solved approximately from the following two equations

$$\mathrm{Tr}(P) = 2 + a_1 \kappa^* E^* D_0 \left[ \frac{2\xi^*}{D_0'^{\,2}} - \frac{(2-D_0)\tau_{ss}}{2D_0} \right] - b_1 \kappa^* E^* / D_0' = 0$$

$$\mathrm{Det.}(P) = 1 + a_1 \kappa^* E^* D_0 \left[ \frac{2\xi^*}{D_0'^{\,2}} + \frac{\tau_{ss}}{2} \right] - b_1 \kappa^* E^* / D_0' = 0 \qquad (4.12)$$

The solution of eqn. (4.12) gives the following values for the feedback constants

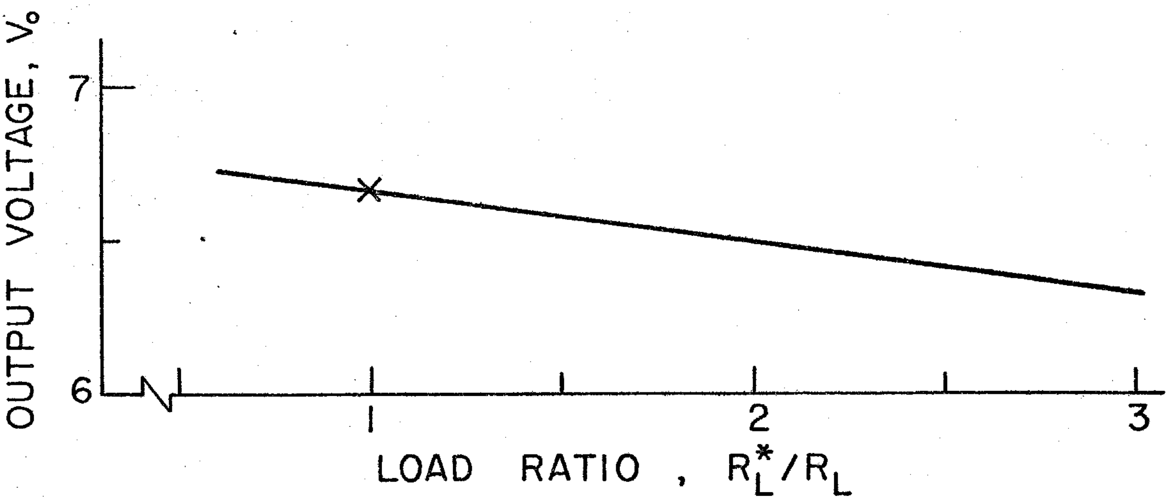
$$a_1 \approx \frac{1}{\kappa^* E^* \tau_{ss}}$$

$$b_1 \approx \frac{2D_0 \xi^* + D_0'^{\,2} \left(\tau_{ss} + \frac{\tau_{00}}{2}\right)}{\kappa^* E^* D_0' \tau_{ss}}$$

The closed loop gain evaluated at the design point is

$$S \approx \frac{1}{\tau_{ss}^2 D_0'^{\,2}} \qquad (4.13)$$

The feedback constant on the voltage, $a_1$, is the same for both the buck-boost and boost regulators, and therefore the loop gains are



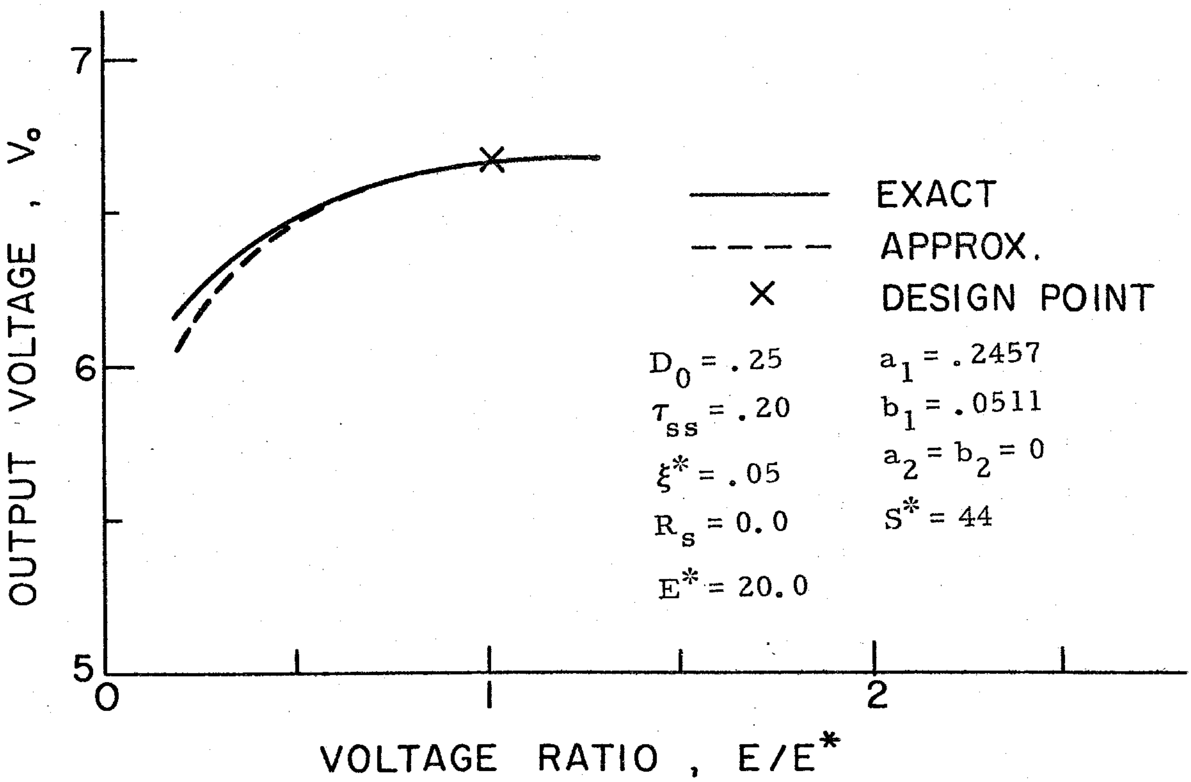




Fig. 4.2. Voltage Regulation for Zero Eigenvalue P.W.M.

also the same [see eqn. (3.13)]. The same comments made about the zero matrix P.W.M. for the boost regulator are applicable to the buck-boost regulator.

In fig. 4.2 the equilibrium voltage is plotted against changes in the input parameters. All the points shown are locally stable, and the system also converges to them from zero initial conditions. The load ratio can be increased past three and the regulator will still be stable. At the lower load ratio the current becomes zero, and the equations are no longer applicable. As in the case of the boost regulator, the buck-boost regulator shows more dependence on the load ratio than did the buck regulator.

4.3.3 Minus One Eigenvalue P.W.M.

The feedback constants, $a_1$ and $b_1$, which will make both eigenvalues of the perturbation matrix equal to a minus one can be solved for from the following two equations.

$$\mathrm{Tr}(P) = 2 + a_1 \kappa^* E^* D_0 \left[ \frac{2\xi^*}{D_0'^{\,2}} - \frac{(2-D_0)\tau_{ss}}{2D_0} \right] - b_1 \kappa^* E^*/D_0' = -2$$

$$\mathrm{Det}(P) = 1 + a_1 \kappa^* E^* D_0 \left[ \frac{2\xi^*}{D_0'^{\,2}} + \frac{\tau_{ss}}{2} \right] - b_1 \kappa^* E^*/D_0' = 1 \qquad (4.14)$$

The feedback constants for the M.O.E. P.W.M. are

$$a_1 \approx \frac{4}{\kappa^* E^* \tau_{ss}}$$

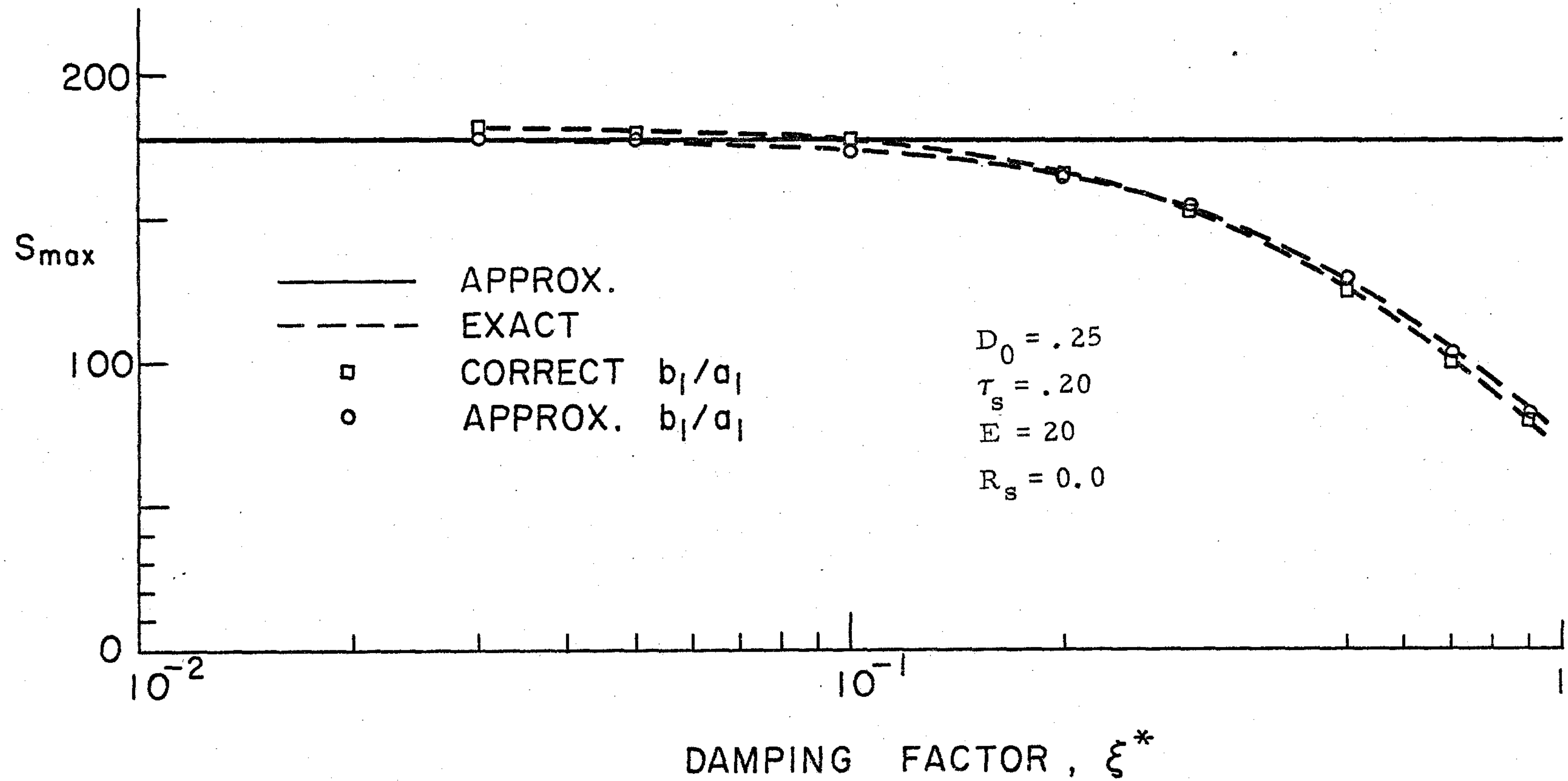



Fig. 4.3. Maximum Loop Gain

$$b_1 \approx \frac{2D_0(4\xi^* + D_0'^2 \tau_{ss})}{\kappa^* E^* D_0' \tau_{ss}}$$

The closed loop gain of this P.W.M. evaluated at the design point is

$$S < \frac{4}{\tau_{ss}^2 D_0'^2} \approx S_{max} \qquad (4.15)$$

The loop gain so derived is the maximum loop gain, and it is equal to the loop gain of the boost regulator with a M.O.E. P.W.M. Since eqn. (4.14) is valid to second order in $\tau_s$ only when $\xi \leq \tau_s$, the maximum closed loop gain is valid, eqn. (4.15), only when the damping factor is small. This fact is illustrated in fig. (4.3) where the approximate and exact curves for the maximum loop gain diverge when the damping factor equals the switching period (i.e., $\xi = \tau_{ss} = .2$). The drop in the stability for the higher damping factors is attributed to the positive feedback exhibited by the buck-boost regulator.

4.3.4 Zero Matrix P.W.M.

The feedback constants for a buck-boost regulator with a zero matrix P.W.M. are

$$a_1 = \frac{2}{\kappa^* E^* D_0' \tau_{ss}}$$

$$a_2 = -\frac{2}{\kappa^* E^* D_0 D_0' \tau_{ss}}$$

$$b_1 = \frac{4\xi^* D_0 + D_0'^2 \tau_{ss}}{\kappa^* E^* D_0'^2 \tau_{ss}}$$

and

$$b_2 = -\frac{4\xi^* + D_0'^2 \tau_{ss}}{\kappa^* E^* D_0'^2 \tau_{ss}}$$

The loop gain for this P.W.M., see eqn. (4.6), is

$$S = (a_1 b_2 - b_1 a_2) \frac{\kappa^2 E^2 D_0}{2 D_0'^2 \tau_{ss}} \qquad (4.16a)$$

because

$$a_1 + D_0 a_2 = 0$$

The loop gain given by eqn. (4.16a) reduces, at the design point, to

$$S = \frac{1}{\tau_{ss}^2 D_0'^2} \qquad (4.16b)$$

The loop gain of the buck-boost regulator with a zero matrix P.W.M. is the same as that of the boost regulator. In fact, the feedback constants on the voltage, $a_1$ and $a_2$, for the zero matrix P.W.M. are the same for the buck, boost, and buck-boost regulators. The loop gain of the zero matrix and the zero eigenvalue P.W.M. are also the same for each regulator.

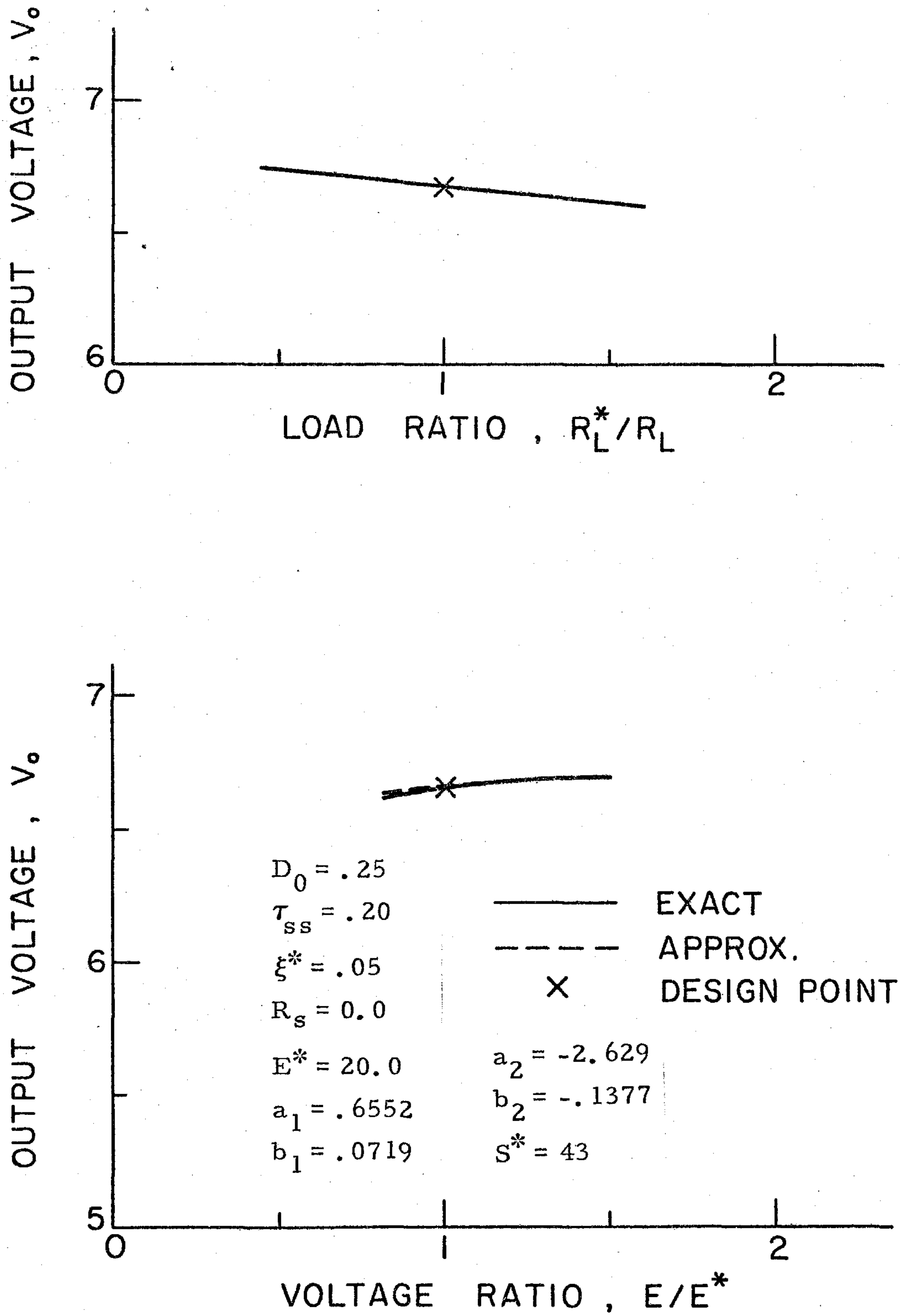



Fig. 4.4. Voltage Regulation for Zero Matrix P.W.M.

In fig. (4.4) the equilibrium voltage is plotted against the input parameters. The lower load ratio results when the current becomes zero, but the other limits to the parameters are due to the system being locally unstable. The design point of the example is not globally stable since the system does not converge to it from zero initial conditions. The regulation is good over a limited range of parameters, but the system, although locally stable for the points shown, is not globally stable even at the design point.

## 4.4 Discussion of Results

The results of this chapter are almost identical to those of the previous chapter. Most of the loop gains of the various P.W.M.s analyzed are the same for both the buck-boost and boost regulators, and even the feedback constants on the voltage are identical. The equilibrium voltage of the buck-boost regulator is dependent on variations in the load. Its stability is very sensitive to changes in both the load and input voltage. The problem of positive feedback occurs at the higher damping factors in both the buck-boost and boost regulators.

# CHAPTER V - GLOBAL STABILITY AND CONVERGENCE

## 5.1 Introduction

In the stability analysis the system is assumed to be at its equilibrium point, $\underline{x}_{sf}$. If the system returns to its equilibrium point after being disturbed, it is said to be asymptotically stable. For small disturbances, the linear part of the recursion formula is used to determine if the system is locally stable. The exact equations are needed to solve the large disturbance stability problem, and if the system is asymptotically stable for all disturbances, it is globally stable.

In the previous chapters the local stability was examined for some switching regulators. The conditions for which the system is locally stable are also the necessary conditions for global stability. In this chapter, a method is derived which gives sufficient conditions for global stability. The conditions for local stability do not correspond to the sufficient conditions for global stability which means a system could be locally stable but globally unstable.

Sufficient conditions for global stability are obtained by translating the origin to the point being investigated so that the new coordinates are

$$\underline{\zeta}_n = \underline{x}_n - \underline{x}_{sf}$$

The recursion formula before the transformation is

$$\underline{x}_{n+1} = A_n \underline{x}_n + \underline{b}_n \qquad (5.1)$$

where the subscript n denotes a function of $\underline{x}_n$. The equilibrium point, $\underline{x}_{sf}$, is defined by the following equation

$$\underline{x}_{sf} = A_{sf}\underline{x}_{sf} + \underline{b}_{sf} \qquad (5.2)$$

where the subscript sf denotes a function of $\underline{x}_{sf}$. The recursion formula relative to the new origin is obtained by subtracting eqn. (5.2) from eqn. (5.1) to give

$$\underline{\zeta}_{n+1} = \underline{x}_{n+1} - \underline{x}_{sf} = A_n\underline{\zeta}_n + \Delta A_n\underline{x}_{sf} + \Delta\underline{b}_n$$

where

$$\Delta A_n = A_n - A_{sf}$$

$$\Delta\underline{b}_n = \underline{b}_n - \underline{b}_{sf}$$

It is convenient and instructive to rewrite the above recursion formula in the following form

$$\underline{\zeta}_{n+1} = P_n\underline{\zeta}_n \qquad (5.3)$$

Eqn. (5.3) is the perturbation equation, and $P_n$ is the nonlinear perturbation matrix. The nonlinear perturbation matrix, $P_n$, is related to the perturbation matrix, P, in the following manner

$$\lim_{\|\underline{\zeta}_n\| \to 0} P_n = P \qquad (5.4)$$

It is always possible to construct the matrix $P_n$ if the system is locally stable since eqn. (5.4) is simply the nonlinearity condition of the proof, see ref. [5]. The system must of course be locally stable if it is to have a chance at being globally stable.

The method for determining the form of the nonlinear perturbation matrix, $P_n$, is best illustrated with an example. The recursion formula for a buck regulator, eqn. (2.1), is

$$\underline{x}_{n+1} = Y(\tau_s)\underline{x}_n + \kappa E \begin{pmatrix} y_{11}(\tau_s - \tau_0) - y_{11}(\tau_s) \\ y_{12}(\tau_s) - y_{12}(\tau_s - \tau_0) \end{pmatrix}$$

If only V.O.T. control is used, the control laws become

$$\tau_0(\underline{x}_n) = \tau_{00} + a_1(x_{ss} - x_n) + b_1(\dot{x}_{ss} - \dot{x}_n)$$

$$\tau_s(\underline{x}_n) = \tau_{ss}$$

After the origin is translated to the equilibrium point, the recursion formula and control laws become

$$\underline{\zeta}_{n+1} = Y(\tau_{ss})\underline{\zeta}_n + \kappa E \begin{pmatrix} y_{11}[\tau_{ss} - \tau_0(\underline{\zeta}_n)] - y_{11}(\tau_{ss} - \tau_0) \\ y_{12}(\tau_{ss} - \tau_0) - y_{12}[\tau_{ss} - \tau_0(\underline{\zeta}_n)] \end{pmatrix} \qquad (5.5)$$

and

$$\tau_0(\underline{\zeta}_n) = \tau_0 - a_1\zeta_n - b_1\dot{\zeta}_n$$

where

$$\tau_0 = \tau_{00} + a_1(x_{ss} - x_{sf}) + b_1(\dot{x}_{ss} - \dot{x}_{sf})$$

If the following transformation is made,

$$\eta_n = \zeta_n + \frac{b_1}{a_1}\dot{\zeta}_n$$

$$\dot{\eta}_n = \dot{\zeta}_n$$

then the recursion formula can be written as

$$\underline{\eta}_{n+1} = T\,Y(\tau_{ss})T^{-1}\underline{\eta}_n + T\underline{f}(\eta_n)$$

where

$$T = \begin{pmatrix} 1 & b_1/a_1 \\ 0 & 1 \end{pmatrix}$$

It is now possible to divide the forcing term, $\underline{f}(\eta_n)$, by $\eta_n$ because

$$\lim_{\eta_n \to 0} \left|\underline{f}(\eta_n)\right|/\eta_n < \infty$$

so that

$$\underline{\eta}_{n+1} = T\left\{Y(\tau_{ss}) + \left[\underline{f}(\eta_n)/\eta_n, \underline{0}\right]T\right\}T^{-1}\underline{\eta}_n$$

or

$$\underline{\eta}_{n+1} = T P_n T^{-1}\underline{\eta}_n \Rightarrow \underline{\zeta}_{n+1} = P_n\underline{\zeta}_n$$

where

$$P_n = \begin{pmatrix} \{y_{11}(\tau_{ss}) + f_1(\eta_n)/\eta_n\} & \{y_{12}(\tau_{ss}) + \frac{b_1}{a_1} f_1(\eta_n)/\eta_n\} \\ \\ \{y_{21}(\tau_{ss}) + f_2(\eta_n)/\eta_n\} & \{y_{22}(\tau_{ss}) + \frac{b_1}{a_1} f_2(\eta_n)/\eta_n\} \end{pmatrix}$$

The nonlinear perturbation matrix, $P_n$, given by eqn. (5.6) does reduce to the perturbation matrix, P, given by eqn. (2.3) when $\eta_n$ becomes zero.

Liapunov's direct method for determining stability can be extended to difference equations, see ref. [6]. The method involves defining a Liapunov function

$$V_n(\underline{\zeta}_n) = \underline{\zeta}_n^* L \underline{\zeta}_n$$

where the asterisk denotes the conjugate transpose. The Liapunov matrix, L, must be positive definite and Hermitian. If the Liapunov function always decreases while inside some domain defined by the relation $V_n = \text{const.}$, then the system is stable in this domain. The change in the Liapunov function is

$$\Delta V_n = V_{n+1} - V_n$$

$$= \underline{\zeta}_{n+1}^* L \underline{\zeta}_{n+1} - \underline{\zeta}_n^* L \underline{\zeta}_n$$

but

$$\underline{\zeta}_{n+1} = P_n \underline{\zeta}_n$$

$$\therefore \qquad \Delta V_n = \underline{\zeta}_n^* [(P_n)^* L P_n - L] \underline{\zeta}_n < 0 \qquad \text{for stability}$$

let

$$Q_n = L - (P_n)^* L P_n$$

The condition for stability is that the Hermitian matrix $Q_n$ be positive definite. The difficulty with using this method is in choosing the Liapunov function. The proper choice for the Liapunov function can greatly increase the domain of stability, and it is therefore worthwhile to investigate the Liapunov function more closely.

The Liapunov function is actually a way of defining a norm, or, to be more precise, the square of a norm.

If L is set equal to I, then

$$V_n = \underline{x}_n^* \underline{x}_n$$

which is the Euclidean vector norm squared. The Euclidean vector norm is compatible with the spectral matrix norm. The curves of constant $V_n$ are represented by circles in the two-dimensional space $\underline{x}_n$. If the following transformation is made,

$$\underline{y}_n = N \underline{x}_n$$

then the square of the Euclidean norm of $\underline{y}_n$ is

$$(\underline{y}_n, \underline{y}_n) = \underline{y}_n^* \underline{y}_n = \underline{x}_n^* N^* N \underline{x}_n$$

The matrix $N^*N$ is a positive definite Hermitian matrix and so a Liapunov function can be defined such that

$$V_n = \underline{x}_n^* N^* N \underline{x}_n = \underline{x}_n^* L \underline{x}_n$$

The curves of constant $V_n$ are circles in the two dimensional $\underline{y}_n$ space and ellipses in the $\underline{x}_n$ space. The system will be stable if some norm in $\underline{x}_n$ space can be found which decreases after each step. Alternatively, the system will be stable if the Euclidean norm relative to some basis, $\underline{y}_n$, decreases at each step. The selection of a norm, a basis, or a Liapunov function are equivalent procedures for determining stability.

The easiest application of Liapunov's method occurs when the recursion formula is a scalar

$$\zeta_{n+1} = a_n \zeta_n$$

where the subscript n denotes a function of $\zeta_n$. This is the form of the recursion formula of the discontinuous P.W.M. for either the buck or boost regulator.

Let

$$V_n = \zeta_n^2$$

so

$$\Delta V_n = V_{n+1} - V_n$$
$$= \zeta_{n+1}^2 - \zeta_n^2$$

$$\Delta V_n = (a_n^2 - 1)\zeta_n^2 < 0 \quad \text{for stability}$$

The Liapunov matrix, L, reduces to a scalar in this case and is equal to one. The domain of stability is determined by finding the smallest $V_n$ for which the Liapunov function increases. Any point whose Liapunov function is less than this will be stable. The domain of stability in this case is a line segment centered at the origin. Since only sufficient conditions for stability are given, a point not on the line segment is not necessarily unstable.

If the perturbation matrix is constant, then the recursion formula is written as

$$\underline{\zeta}_{n+1} = P\,\underline{\zeta}_n \tag{5.6}$$

The form of eqn. (5.6) is familiar since the local stability is determined from a similar type of equation. The necessary and sufficient conditions for stability are that the modulus of the eigenvalues of the P matrix be less than one. In order to derive these same results using Liapunov's method requires the selection of the proper Liapunov function.

Let

$$V_n = \underline{\zeta}_n^* L \underline{\zeta}_n$$

where

$$L = (T^{-1})^* T^{-1}$$

and

$$\Lambda = T^{-1} P T \text{ - diagonal matrix (ie. assumes P diagonalizable)}$$

The change in the Liapunov function is

$$\Delta V_n = \underline{\zeta}_{n+1}^* L \underline{\zeta}_{n+1} - \underline{\zeta}_n^* L \underline{\zeta}_n$$

$$= \underline{\zeta}_n^* (P^* L P - L) \underline{\zeta}_n$$

let

$$\underline{\zeta}_n = T \underline{\eta}_n$$

so

$$\Delta V_n = \underline{\eta}_n^* [(T^{-1}PT)^* (T^{-1}PT) - I] \underline{\eta}_n$$

$$= \underline{\eta}_n^* (\Lambda^* \Lambda - I) \underline{\eta}_n < 0 \quad \text{for stability}$$

The system is stable if the modulus of the eigenvalues of the P matrix are less than one. The Liapunov matrix used to obtain the necessary and sufficient conditions for stability is related to the eigenvectors of the perturbation matrix.

It is informative to investigate the stability of eqn. (5.6) in a different but equivalent way. For any initial conditions, $\underline{\zeta}_n$, the value of the state after m iterations is

$$\underline{\zeta}_{n+m} = P \cdot P \cdot P \cdots P\,\underline{\zeta}_n = P^m \underline{\zeta}_n$$

If some matrix norm for P can be shown to be less than one, then

$$|\underline{\zeta}_{n+m}| < |P|^m |\underline{\zeta}_n|$$

and

$$|\underline{\zeta}_{n+m}| < |\underline{\zeta}_n|$$

If the following transformation is made, the choice of the proper norm to use is simplified.

Let

$$\underline{\eta}_n = T^{-1} \underline{\zeta}_n$$

then

$$\underline{\eta}_{n+m} = T^{-1} P^m T \underline{\eta}_n$$

or

$$\underline{\eta}_{n+m} = T^{-1} P \underbrace{T T^{-1}}_{I} P T \cdots T^{-1} P T \underline{\eta}_n$$

Since the P matrix is constant, it can be diagonalized by a similarity transformation so that

$$\underline{\eta}_{n+m} = \Lambda^m \underline{\eta}_n$$

or

$$|\underline{\eta}_{n+m}| < |\underline{\eta}_n| \quad \text{if} \quad |\Lambda| < 1$$

If the matrix norm used is the spectral norm, this result is the same one derived by using a Liapunov function.

When the perturbation matrix is symmetric and a nonlinear function of the state, $\underline{\zeta}_n$, the sufficient conditions for stability are the same as the ones derived for the constant matrix case. The recursion formula is

$$\underline{\zeta}_{n+1} = P_n \underline{\zeta}_n$$

where

$$P_n = (P_n)^T$$

The Liapunov function chosen to show stability is

$$V_n = \underline{\zeta}_n^* \underline{\zeta}_n$$

so

$$\Delta V_n = \underline{\zeta}_n^* (P_n^* P_n - I) \underline{\zeta}_n$$

let

$$\underline{\zeta}_n = T \, \underline{\eta}_n$$

where

$$\Lambda = T^{-1} P_n T = T^* P_n T$$

then

$$\Delta V_n = \underline{\eta}_n^* (T^* P_n^* T T^* P_n T - T^* T) \underline{\eta}_n$$

$$= \underline{\eta}_n^* (\Lambda_n^* \Lambda_n - I) \underline{\eta}_n$$

The fact that the perturbation matrix is symmetric makes it possible to diagonalize it with an orthogonal transformation, T. Sufficient conditions for global stability are that the modulus of the eigenvalues of the $P_n$ matrix never exceed one for any point in the phase plane. If at some point the modulus of an eigenvalue does exceed one, the domain of stability will consist of a circle centered at the origin with the above mentioned point on its circumference.

The system will probably be stable for a larger domain, but the stability criterion does not guarantee it. The sufficient conditions are very stringent because it requires that the system decreases relative to some norm at every step. When the perturbation matrix was constant the system could be shown to decrease at every step if it decreased one step, and hence necessary as well as sufficient conditions were obtained. When the perturbation matrix is a non-linear function of the state, the system cannot increase in the norm of interest for any step without nullifying the sufficient conditions.

It is instructive to look at the system after an arbitrary number of iterations.

$$\underline{\zeta}_{n+m} = P_{n+m} P_{n+m-1} \cdots P_n \underline{\zeta}_n \qquad (5.8)$$

The subscripts on the perturbation matrices are used to indicate that the matrices are dependent on the state of the particular iteration. The previous stability criterion is easily obtained since

$$|\underline{\zeta}_{n+m}| \leqslant |P_{n+m}|\ |P_{n+m-1}| \cdots |P_n|\ |\underline{\zeta}_n|$$

if

$$|P_j| < 1 \quad \text{for} \quad n \leqslant j \leqslant n+m$$

then

$$|\underline{\zeta}_{n+m}| < |\underline{\zeta}_n|$$

If the Euclidean vector norm and spectral matrix norm are chosen for the above norms, then sufficient conditions for stability are that the modulus of the eigenvalues of each matrix be less than one.

This result is only true for symmetric matrices since only then will the spectral norm be equal to the modulus of the largest eigenvalue. The sufficient conditions so derived are violated if the norm of one of the matrices in the long chain of matrices is greater than one.

It is difficult to obtain worthwhile sufficient conditions when the perturbation matrix is nonsymmetric and a function of the state. The recursion formula for this example is

$$\underline{\zeta}_{n+1} = P_n \underline{\zeta}_n \tag{5.9}$$

where

$$P_n \neq P_n^T$$

let

$$V_n = \underline{\zeta}_n^* \underline{\zeta}_n$$

then

$$\Delta V_n = \underline{\zeta}_n^* (P_n^* P_n - I) \underline{\zeta}_n < 0 \quad \text{for stability}$$

The stability condition so derived is the same as that of the symmetric matrix. However, when the perturbation matrix is nonsymmetric the modulus of the largest eigenvalue of the perturbation matrix does not correspond to its spectral norm. The spectral norm of a nonsymmetric matrix is the square root of the modulus of the largest eigenvalue of the matrix $P_n^* P_n$. Unfortunately, these sufficient conditions are not very good for the switching regulators investigated in this thesis.

It should be possible to find a Liapunov function which is better than the simple Euclidean norm used above. If the recursion formula, eqn. (5.9), is written in terms of its linear and nonlinear parts it becomes

$$\underline{\zeta}_{n+1} = P\underline{\zeta}_n + \Delta P_n \underline{\zeta}_n \qquad (5.10)$$

where

$$\Delta P_n = P_n - P$$

and

$$P = \text{constant matrix}$$

For values of the state close to the origin (i.e., $|\underline{\zeta}_n| \ll 1$), the nonlinear part of eqn. (5.10) will be negligible so that only the linear part is left. It is the linear part of the equation which is used to determine the local stability, and the Liapunov function used to derive necessary and sufficient conditions for local stability is known. If the nonlinear part of the recursion formula is not significant, then the Liapunov function associated with the linear part will be a good one to use in trying to establish sufficient conditions for global stability.

Let

$$L = (T^{-1})^* C T^{-1}$$

where

$$\Lambda = T^{-1} P T \text{ - diagonal matrix}$$

and

$$C = \begin{pmatrix} 1/(1-|\lambda_1|^2) & 0 \\ 0 & 1/(1-|\lambda_n|^2) \end{pmatrix}$$ - diagonal matrix where the $\lambda_i$s are the eigenvalues of the P matrix

then

$$\Delta V_n = \underline{\zeta}_n^*(P + \Delta P_n)^* L (P + \Delta P_n)\underline{\zeta}_n - \underline{\zeta}_n^* L \underline{\zeta}_n$$

$$= \underline{\zeta}_n^*(P^*LP - L)\underline{\zeta}_n + \underline{\zeta}_n^*(P^*L\Delta P_n + \Delta P_n^* LP)\underline{\zeta}_n$$

$$+ \underline{\zeta}_n^* \Delta P_n^* L \Delta P_n \underline{\zeta}_n$$

The first term on the right side of the equation is that of the linear regulator, the second involves both the linear and nonlinear part of the recursion formula, and the third contains only the nonlinear part. If the transformation is made so that

$$\underline{\zeta}_n = T\,\underline{\eta}_n$$

then the change in the Liapunov function can be written as

$$\Delta V_n = -\underline{\eta}_n^* \underline{\eta}_n + \underline{\eta}_n^*(\Lambda^* C R_n + R_n^* C \Lambda)\underline{\eta}_n$$

$$+ \underline{\eta}_n^* R_n^* C R_n \underline{\eta}_n < 0 \quad \text{for stability} \qquad (5.11)$$

where

$$R_n = T^{-1} \Delta P_n T$$

The local stability criterion can be recovered from eqn. (5.11) since for small values of state the last two terms on the right hand side become negligible. The reason for choosing the above Liapunov function is that it gives the most negative value to the first term of eqn. (5.11). This function will not necessarily be the best one to use because the nonlinear terms can be more important than the linear terms.

It would seem possible that if the Liapunov matrix, L, were made a function of the state the domain of stability could be increased. In the previous examples of the constant and symmetric perturbation matrix, the stability criterion was that the modulus of the largest eigenvalue of the perturbation matrix be less than one. It seems reasonable that this same result could be shown for the nonsymmetric case. In the many attempts to show this result, the one which illustrates most clearly the problems involved is based on the following Liapunov function.

Let

$$V_n = \underline{\zeta}^*_{n+1} L_n \underline{\zeta}_{n+1} = \underline{\zeta}^*_n (P^*_n L_n P_n) \underline{\zeta}_n$$

where

$$L_n = (T_n^{-1})^* T_n^{-1}$$

and

$$\Lambda_n = T_n^{-1} P_n T_n$$

The Liapunov function associated with the state at the $n^{th}$ iteration depends on the perturbation matrix, and therefore the states of the $n^{th}$ iteration. The change in the Liapunov function from the $n^{th}$ to the $(n+1)^{st}$ iteration is

$$\Delta V_n = \underline{\zeta}^*_{n+1} (P^*_{n+1} L_{n+1} P_{n+1} - L_n) \underline{\zeta}_{n+1}$$

let

$$\underline{\zeta}_{n+1} = T_{n+1} \underline{\eta}_{n+1}$$

then

$$\Delta V_n = \underline{\eta}^*_{n+1}[\Lambda^*_{n+1}\Lambda_{n+1} - (T_n^{-1}T_{n+1})^*(T_n^{-1}T_{n+1})]\,\underline{\eta}_{n+1} < 0 \quad \text{for stability}$$

The stability criterion depends not only on the modulus of the eigenvalues of the perturbation matrix but also on the transformation matrices. When the transformation matrix is orthogonal, as they are when the perturbation matrix is symmetric, the complex conjugate transpose of it is also the inverse

$$(T_n^{-1}T_{n+1})^*(T_n^{-1}T_{n+1}) = I$$

The product of the transformations shown above is also equal to the identity matrix when the perturbation matrix is constant (i.e., $T_n = T_{n+1}$). The stability criterion of eqn. (5.12) thus reduces to the previous stability criterions for the case of the constant or symmetric perturbation matrix.

If a Liapunov norm is associated with each step, then the role the transformation matrices play in the stability criterion of eqn. (5.12) can be clearly shown. When the state of the system moves from the $n^{th}$ to the $(n+1)^{st}$ position, the square of the Liapunov norm used to measure the change is defined to be

$$V_n = \underline{\zeta}^*_{n+1} L_n \underline{\zeta}_{n+1}$$

where

$$L_n = (T_n^{-1})^* T_n^{-1} \qquad \text{and} \qquad \Lambda_n = T_n^{-1} P_n T_n$$

then

$$\Delta V_n = \underline{\zeta}^*_{n+1}(P^*_{n+1} L_{n+1} P_{n+1} - L_{n+1})\underline{\zeta}_{n+1} + \underline{\zeta}^*_{n+1}(L_{n+1} - L_n)\underline{\zeta}_{n+1}$$

let

$$\underline{\zeta}_{n+1} = T_{n+1}\, \underline{\eta}_{n+1}$$

so

$$\Delta V_n = \underline{\eta}^*_{n+1}(\Lambda^*_{n+1}\Lambda_{n+1} - I)\,\underline{\eta}_{n+1} + \underline{\zeta}^*_{n+1}(L_{n+1} - L_n)\underline{\zeta}_{n+1}$$

The system does decrease relative to some defined Liapunov norm each step if the modulus of the largest eigenvalue of the perturbation matrix is less than one. However, it is necessary to take into account the change in the Liapunov function which results from the change in norms. The second term on the right hand side of the expression for the change in Liapunov function does exactly that. In fig. (5.1) the dilemma described above for the nonsymmetric matrix is illustrated by showing a system which decreases relative to some norm every step and yet is still unstable. In the case of the symmetric matrix, curves of constant $V_n$ are circular so that if the system decreases relative to one norm it must decrease relative to all the norms. When the perturbation matrix is constant, only one norm needs to be considered. The stability criterion of eqn. (5.12) requires that the system truly decreases and not just decrease relative to some norm conveniently chosen for that particular step.

If the system is two-dimensional and the eigenvalues of the perturbation matrix are complex, the stability criterion given by eqn. (5.12) can be simplified since

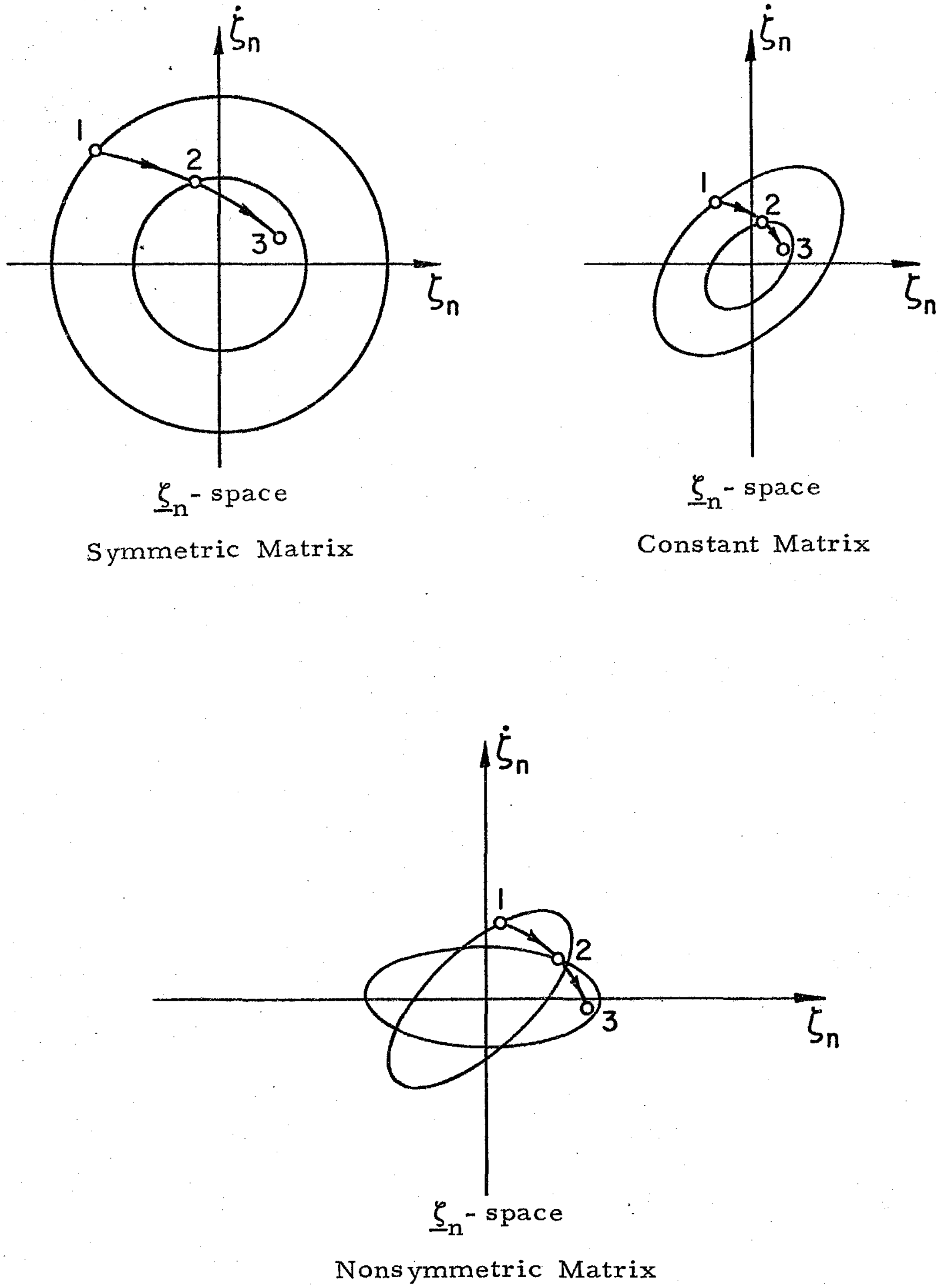



Fig. 5.1. Liapunov Norms

$$\Lambda^*_{n+1}\Lambda_{n+1} = |\lambda_{n+1}| I$$

where

$|\lambda_{n+1}|$ = modulus of either eigenvalue of the perturbation matrix

If the following transformation is made

$$\underline{\zeta}_{n+1} = \theta_n \underline{\eta}_n$$

where

$$\phi_n = \theta_n^{-1} S_n \theta_n \text{ - diagonal matrix}$$

and

$$S_n = (T_n^{-1} T_{n+1})^*(T_n^{-1} T_{n+1}) \text{ - positive definite Hermitian matrix}$$

then the stability criterion of eqn. (5.12) becomes

$$\Delta V_n = \underline{\eta}_n^*(|\lambda_{n+1}|^2 \theta_n^* \theta_n - \theta_n^* S_n \theta_n)\underline{\eta}_n$$

now

$$\theta_n^* = \theta_n^{-1} \qquad \text{(i.e., } S_n \text{ is a Hermitian matrix so that } \theta_n \text{ is orthogonal)}$$

so

$$\Delta V_n = \underline{\eta}_n^*(|\lambda_{n+1}|^2 I - \phi_n)\underline{\eta}_n$$

For this special case, if the modulus of the eigenvalues of the perturbation matrix are less than the smallest eigenvalue of the product of the transformation matrices, $\phi_n$, the system will be stable. This condition for stability should be contrasted with that of the symmetric or constant perturbation matrix which required only that the modulus of the largest eigenvalue be less than one. Although the matrix $S_n$ is a function of the two states $\underline{\zeta}_n$ and $\underline{\zeta}_{n+1}$, the subscript n is used because $\underline{\zeta}_{n+1}$ can always be related to $\underline{\zeta}_n$ by the perturbation matrix $P_n$.

The method which will be used to obtain sufficient conditions for global stability is based on the phase plane. The phase plane approach for discrete systems is not as powerful as the techniques developed for continuous systems. The trajectories of the continuous system naturally dissect the phase plane into regions which can then be classified for stability. The difference equation transforms one point of the phase plane into another and no curves are identified with it.

R. E. Kalman in ref. [8] uses the idea of paired systems to enable techniques developed for continuous systems to be used for discrete systems. The basis for the method is that the stability of the discrete system can be determined from the continuous system if the ordered points of the discrete system $(\underline{x}_0, \underline{x}_1, \ldots, \underline{x}_n)$ lie on the trajectory of the continuous system. It is always possible to pair a discrete system with a continuous system, but it is not always possible to pair a continuous system with a discrete system. It is, however, always possible to pair a linear differential equation with constant coefficients to a linear difference equation with constant coefficient, $\underline{x}_{n+1} = P\,\underline{x}_n$, if and only if none of the characteristic roots of P is real and negative. In the next section an example is given which illustrates how the preceding techniques are used to guarantee global stability for a buck regulator.

5.2 Example

In this example a buck regulator with a zero eigenvalue P.W.M. is shown to be globally stable. The method used to show stability is basically Liapunov's, and although the system is not shown to decrease relative to a Liapunov norm every step, it is always shown to decrease after a number of steps. It is possible to keep track of the discrete system for the required number of steps by pairing a continuous system to the discrete system in the saturated regions of the phase plane, see fig. 5.2. The saturated regions are so named because the forcing vector, $\underline{f}$, of the recursion formula, eqn. (5.5), is saturated in these regions. The forcing function saturates because the on-time must be greater than zero but less than the switching period, $\tau_{ss}$.

When the forcing vector saturates, the recursion formula takes the form

$$\underline{\zeta}_{n+1} = Y(\tau_{ss})\underline{\zeta}_n + \underline{b}$$

where

$$\underline{b} = \underline{b}_1 = \kappa E \begin{pmatrix} y_{11}(\tau_{ss}) - y_{11}(\tau_{ss} - \tau_0) \\ y_{12}(\tau_{ss} - \tau_0) - y_{12}(\tau_{ss}) \end{pmatrix} \qquad \text{when } \tau_0(\underline{\zeta}_n) = 0$$

and

$$\underline{b} = \underline{b}_2 = \kappa E \begin{pmatrix} -y_{11}(\tau_{ss} - \tau_0) \\ y_{12}(\tau_{ss} - \tau_0) \end{pmatrix} \qquad \text{when } \tau_0(\underline{\zeta}_n) = \tau_{ss}$$

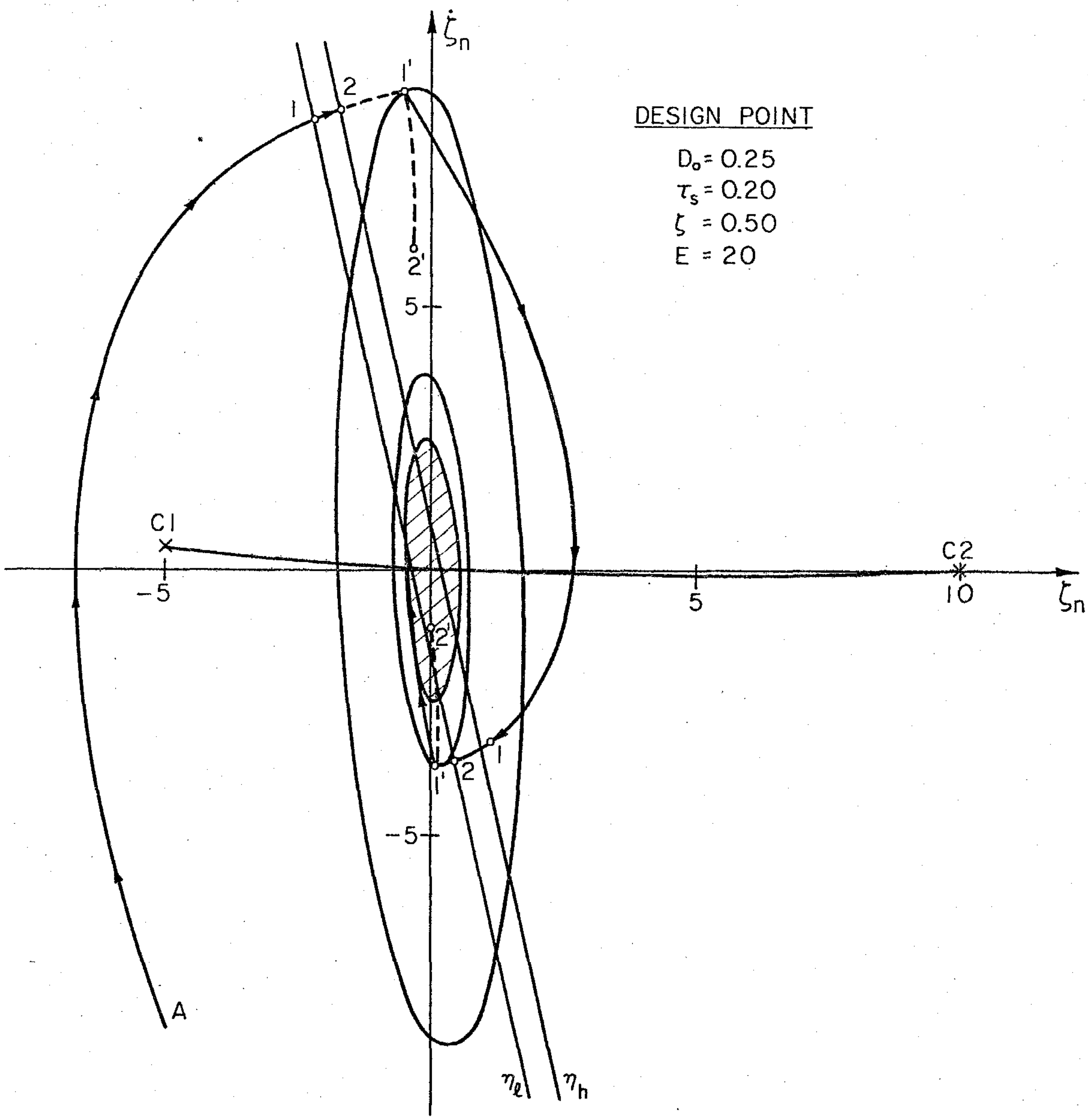



Fig. 5.2. Phase Plane

Besides the origin, the system will have two equilibrium points which result because of the forcing vector saturating. These equilibrium points are

$$\underline{\zeta}_e = Y^{-1}(\tau_{ss})\underline{b}$$

If the origin is translated to one of these equilibrium points, the recursion formula is

$$\underline{u}_{n+1} = \underline{\zeta}_{n+1} - \underline{\zeta}_e = Y(\tau_{ss})\underline{\zeta}_n + \underline{b} - \left[ Y(\tau_{ss})\underline{\zeta}_e - \underline{b} \right]$$

or

$$\underline{u}_{n+1} = Y(\tau_{ss})\underline{u}_n \qquad (5.13)$$

when the forcing vector is saturated. The matrix $Y(\tau_{ss})$ is a constant matrix with complex eigenvalues, and it is therefore possible to pair a continuous system to the discrete system.

The perturbation matrix of the buck regulator is given by eqn. (5.6), and it is a function of only one variable, $\eta_n$. If this variable is below a certain value, $\eta_\ell$, the P.W.M. keeps the voltage on for the entire switching period while if it's higher than $\eta_h$ the voltage is turned off. The region in the phase plane which corresponds to the voltage always being turned on is to the left of the line

$$\eta_\ell = \zeta_n + \frac{b_1}{a_1}\dot{\zeta}_n$$

and the region where the voltage is off is to the right of the line

$$\eta_h = \zeta_n + \frac{b_1}{a_1}\dot{\zeta}$$

In fig. 5.2 these lines are labeled $\eta_\ell$ and $\eta_h$ respectively. The recursion formula in the saturated regions is given by eqn. (5.13)

$$\underline{u}_{n+1} = Y(\tau_{ss})\underline{u}_n$$

where the origin of the new coordinate system is at the equilibrium point $\underline{\zeta}_e$ of the old. The origins for the saturated regions will be called centers. The center of the saturated region located to the left of the origin in the phase plane shown in fig. 5.2 is C2, and the center of the saturated region located to the right is C1. In these regions a continuous system whose origin is at the proper center can be paired to the discrete system. Trajectories can then be drawn in these regions. It is possible to find a center for every forcing vector of the system. The line connecting the two centers C1 and C2 is called the line of centers. Every point in the phase plane has a center on the line of centers where its recursion formula is given by eqn. (5.13).

The entire nonlinear region of the phase plane is contained in the thin strip between the saturated regions. The strategy of this analysis is to first find a Liapunov function which will decrease for any point in the nonlinear region; then it is only necessary to show that the Liapunov function decreases in the saturated region. The Liapunov matrix chosen is

$$L = (T^{-1})^* \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} T^{-1} \tag{5.14}$$

where

$$J = T^{-1} P T \text{ - Jordan form}$$

The T matrix is composed of the generalized eigenvectors of the linear perturbation matrix P. The change in Liapunov function for the linear system is

$$\Delta V_n = \underline{\zeta}_n^*(P^* L P - L)\underline{\zeta}_n$$

let

$$\underline{\eta}_n = T\underline{\zeta}_n$$

then

$$\Delta V_n = \underline{\eta}_n^* \left( J^* \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} J - \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \right) \underline{\eta}_n$$

but

$$J = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

so

$$\Delta V_n = -\underline{\eta}_n^* \underline{\eta}_n$$

The change in the Liapunov function for the nonlinear system is

$$\Delta V_n = \underline{\zeta}_n^*(P_n^* L P_n - L)\underline{\zeta}_n$$

The system will decrease at $\underline{\zeta}_n$ relative to the given Liapunov norm if the matrix $Q_n$, which is defined below, is positive definite.

$$Q_n = L - P_n^* L P_n$$

The nonlinear perturbation matrix, $P_n$, is a function of only one variable, $\eta_n$, and if the Liapunov function decreases for those values of $\eta_n$ between $\eta_\ell$ and $\eta_h$, then the norm of all points in the nonlinear region will decrease.

In the example chosen, a Liapunov function was not found which showed that the norm of all points in the nonlinear region decreased. It was necessary to limit the amount of time the switch was on thereby decreasing the magnitude of the maximum forcing vector, $\underline{f}$. When the on-time is limited to two-thirds the switching period, a Liapunov function is found which gives stability for the shaded region shown in fig. 5.2. This decrease in the forcing vector also causes the center C2 to move closer to the origin. The Liapunov function used for this example is

$$V_n = \underline{\zeta}_n^* \begin{pmatrix} 28.01 & 1.123 \\ 1.123 & 1.055 \end{pmatrix} \underline{\zeta}_n$$

The Liapunov matrix used above is that given by eqn. (5.14). However, the linear perturbation matrix from which the Liapunov matrix was calculated is dimensional in time. It was found that the domain of stability could be greatly increased by varying the frequency $\omega_k$ and switching period $T_s$ while maintaining $\omega_k T_s = \tau_s$. Stability depends only on the parameter $\tau_s$, but the Liapunov matrix generated by eqn. (5.14) depends on $\omega_k$ and $T_s$ individually. As was mentioned before, the Liapunov function which gives the greatest decrease for the linear system is not necessarily the best one to use for the nonlinear system.

The phase plane of fig. 5.2 is divided into three regions, the nonlinear region, which is the center region located between the two straight lines, and the saturated regions, with the Liapunov curves superimposed on these regions. The continuous system whose trajectories are associated with one of the saturated centers, C1 or C2, is paired to the discrete system $\underline{\eta}_{n+1} = Y(\tau_{ss})\underline{\eta}_n$. The system, in one step, will decrease relative to the Liapunov norm from any point in the nonlinear region. If the trajectories in the saturated regions are followed, the system can also be shown to decrease relative to the Liapunov norm from any point in the saturated region. Global stability is therefore guaranteed.

In fig. 5.2 a trajectory is followed into the shaded region of the phase plane from the initial point (A). The trajectory is easily followed until it enters or jumps across the nonlinear region. The nonlinear region acts like a switching line of a continuous system in as much as the system switches from one set of trajectories to another. Since there are identified with one continuous trajectory a number of discrete trajectories depending on the initial conditions, the exact point and manner in which the switch is made is not clear.

If the system jumps across the nonlinear region, the new trajectory will begin on the dashed line whose ends are marked (2) and (1′). If the point trajectory lands in the nonlinear region, the new trajectory will begin somewhere on the line segment (1′)

to (2′) . The worst trajectory as far as the stability analysis is concerned occurs when the system lands at (1′) . The worst trajectory is always used as the new trajectory, and in this way the worst possible trajectory is obtained for the system. As can be seen in fig. 5.2, each time the system crosses the nonlinear region the new trajectory is always closer to the origin than the trajectory of the previous cycle. If this were not the case, the system could be unstable. The system can only be shown to be unstable if the best trajectory for stability, not the worst one, is found to be farther away from the origin than the previous trajectory. If neither of these conditions hold, the system could be either stable or unstable. In ref. [8] R. E. Kalman notes that even though the discrete systems are completely deterministic, it is sometimes necessary to use a probabilistic approach to deal with the nonlinearities.

The two most important parameters in the global stability analysis are the switching period, $\tau_{ss}$, and the damping factor, $\xi$. The damping factor is important because it determines the shape of the trajectories in the saturated regions. If there is a lot of damping, the trajectories will decrease rapidly relative to the centers while if there is no damping, the trajectories will be circular. The global stability is improved for large damping factors. The switching period can be viewed as the step size of the system. The larger the switching period the larger will be the distance between successive points in the discrete trajectory. The worst trajectory occurs when the system steps from (1) to

(1') in fig. 5.2. If the switching period is small, then the worst trajectory will be close to the origin, and the stability will be improved. The dependence of the global stability on these parameters is analogous to the local stability dependence.

## 5.3 Convergence

In the previous section it was mentioned that the nonlinear region of fig. 5.2 approximates a switching line of a continuous system. A buck regulator is actually a continuous system in the saturated regions because the voltage is either on or off all the time. If the nonlinear region was replaced by a switching line, the continuous trajectory would have to switch at the line, and the exact trajectory could be determined. It might be desirable to replace only the nonlinear region outside the shaded, stable region of fig. 5.2 with a switching line. This system would involve two levels of control with the discrete control only operating when the Liapunov function formed from the current value of state became less than the value of the stable Liapunov curve. With this type of control, the system would converge to the origin and act as a buck regulator in the shaded region around the origin. The rate of convergence of the system from a point in the phase plane can be manipulated by changing the slope of the switching line.

In the case of the buck regulator it is possible to define an optimal switching curve. The curve is optimal in the sense that the system will switch trajectories so that the time it takes to reach the origin is a minimum. In ref. [1] the minimum time problem with "bang-bang" control is discussed. The optimal switching line is found by minimizing the following performance index

$$\text{minimize } J = \int_0^{\tau_f} d\tau$$

with the constraints

$$\dot{\underline{\zeta}} = F\underline{\zeta} + \underline{g}\,u(\tau)$$

$$\underline{\zeta}(\tau_f) = \underline{0}$$

where

$$F = \begin{pmatrix} 0 & 1 \\ -1 & -2\zeta \end{pmatrix}$$

$$\underline{g} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

and

$$-(x_{sf} + 2\zeta\dot{x}_{sf}) \leq u(\tau) \leq (\kappa E - x_{sf} - 2\zeta\dot{x}_{sf})$$

or

$$-u_\ell \leq u(\tau) \leq u_h$$

The control function, $u(\tau)$, is constrained to lie between an upper and a lower value. These values are functions of the equilibrium state because the system's origin has been translated to the equilibrium point. When the constraints are adjoined to the performance index, the Hamiltonian, H, which must be minimized with respect to u is

$$H = 1 + \underline{\lambda}^T(F\underline{\zeta} + \underline{g}u)$$

where

$\underline{\lambda}$ = Lagrange multipliers used to adjoin the constraints to the performance index

The Hamiltonian is linear in u so that to minimize H with respect to u requires that the product of $\underline{\lambda}^T\underline{g}u$ be as negative as possible.

$$u = \begin{cases} u_h & \lambda^T g < 0 \\ -u_\ell & \lambda^T g > 0 \end{cases}$$

The optimal control operates with either the voltage full on or off. This predicament is not uncommon for linear systems since a true minimum does not exist, and the optimal control is the maximum control.

The trajectory to the origin must be with either the maximum positive value for control, $u_h$, or the maximum negative value, $-u_\ell$. The differential equation is then

$$\dot{\underline{\zeta}} = F\underline{\zeta} - u_\ell \underline{g}$$

or

$$\dot{\underline{\zeta}} = F\underline{\zeta} + u_h \underline{g}$$

These equations can be transformed to new origins such that only the homogeneous part remains.

$$\dot{\underline{\eta}} = F \underline{\eta} \quad \text{relative to C1 } (-u_\ell \text{ control})$$

and

$$\dot{\underline{\eta}} = F \underline{\eta} \quad \text{relative to C2 } (u_h \text{ control})$$

The centers C1 and C2 are the centers of the saturated regions of the discrete system. If it is assumed that only one switch is made before the system reaches the origin, then the switching line can easily be drawn. The switching line for the undamped case is shown in fig. 5.3. The switching line is composed of the natural trajectories relative to the saturated centers which pass through the origin, and for the undamped case these trajectories are circular arcs.

In the present example the trajectories of the continuous and discrete systems are the same. The reason the systems are identical is that the voltage is either on or off the entire switching period, and when the switching line is reached, the discrete system changes control no matter where it is in the switching period. If the voltage were only on during part of the switching period, the true trajectory of the discrete system would not follow that of the associated continuous system. The trajectories in the left hand saturated region of fig. 5.2 are not the true trajectories of the discrete system since the voltage is only on two-thirds of the switching period. The optimal switching curve for the discrete system shown in fig. 5.2 is not easy to find. However, the problem of finding the switching line for a continuous system using a discontinuous forcing term has been solved.

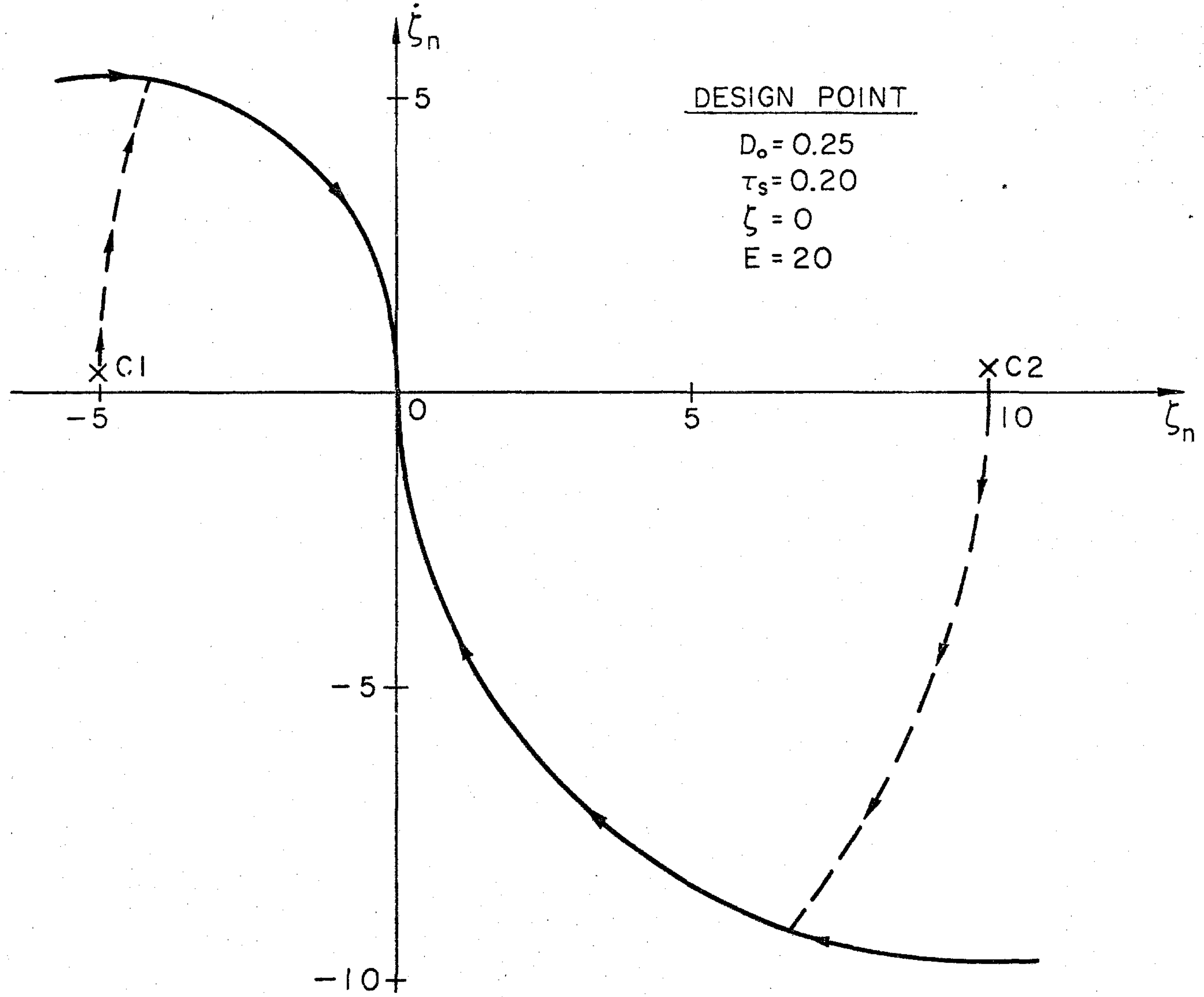



Fig. 5.3. Switching Line

D. W. Bushaw in ref. [2] solves the above problem for the differential equation

$$m \frac{d^2x}{dt^2} + c\frac{dx}{dt} + Kx = \pm D$$

The method he uses in arriving at the switching curve is quite elegant, and it is based on the idea that because the trajectory is composed of only two sets of arcs, the arcs which pass through the region of higher velocities will minimize the time. H. S. Tsien in ref. [11] gives a simplified explanation of the procedure used to choose these arcs. The switching curve is again composed of the two trajectories which pass through the origin. As has been shown, if the forcing terms $\pm D$ are the maximum and minimum possible, the linear system will converge to the origin in minimum time.

The problem of finding the optimal switching line for the continuous system associated with the discrete system of fig. 5.2 can be solved. It is not clear, however, that the switching line associated with the continuous system is the optimal switching line of the discrete system. Since with linear systems the optimal strategy is to use the maximum control available, the maximum use of the voltage, even if it is only for two-thirds of the switching period, appears to be a plausible control scheme. The real difficulty in the analysis is that the continuous and discrete trajectories only correspond at the sampling instants. Only a plausibility argument has been made to indicate that the optimal switching line of the associated continuous system can be used for the discrete system. In the case of the boost and buck-boost regulators, the continuous systems associated with the saturated regions are different so that solving for the switching line of the associated continuous system is made more difficult.

The switching line derived for the associated continuous system can be viewed as only an approximation to the switching line for the discrete system. An approximation to the switching line is usually necessary since it is difficult to store the exact curve and make decisions from that knowledge. A polynomial can be used to approximate the switching line so that near optimal convergence is achieved. In engineering applications the near optimal trajectories are good enough to eliminate the complexities of using the exact curves.

## 5.4 Summary

Liapunov's method is used to obtain sufficient conditions for global stability. The method works very well when the system is a scalar or when the perturbation matrix is either constant or symmetric. For these cases the stability criterion is that the modulus of the eigenvalues of the perturbation matrix be less than one. When the perturbation matrix is nonsymmetric and not constant, the sufficient conditions for stability obtained by using Liapunov's method are not very good. The concept of paired systems is introduced so that better conditions for global stability can be derived.

An example is worked to illustrate how Liapunov's method along with the method of paired systems can be used to evaluate the stability of a buck regulator. It was found that the nonlinear region of the phase plane is a small part of the total area. The rest of the phase plane is composed of two saturated regions where continuous systems can be paired to the discrete system. Stability is shown by finding a Liapunov function which decreases for any point in the nonlinear region and also decreases along the trajectories after a few steps in the saturated regions. In this way it is shown that the Liapunov function will eventually decrease although it does not necessarily decrease each step.

The nonlinear region of the phase plane in some ways resembles a switching line. This resemblance is due to the fact that the system changes trajectories from one saturated region to another when it crosses the nonlinear region. It is possible to monitor the discrete system continuously so that it always switches exactly at a switching line. Two levels of control could be used with the system being brought close to the origin by using a switching line where it would then revert to the usual discrete regulator.

A desirable characteristic of any regulator is rapid convergence. For a buck regulator where the on-time is allowed to vary between zero and the switching period, an optimal switching line can be found. For the particular case mentioned, the trajectory of the associated continuous system and the discrete system are exactly the same. This fortuitous situation allows the analysis of the discrete system to be carried out exactly as a continuous system. The optimal control strategy for a linear system is to use the maximum control possible. This type of control is called "bang-bang" control.

When the trajectories of the discrete and associated continuous systems do not coincide, the analysis of the discrete switching strategy of the associated continuous system, which can usually be solved for, can be used for the discrete system. The idea for doing this is that the optimal switching strategy for the associated system should, in some sense, approximate that of the discrete system. In fact, even when the exact switching curve is known, it is usually necessary to approximate it with a polynomial so that the implementation of the control is simplified.

# CHAPTER VI - SUMMARY AND CONCLUSIONS

The regulation properties of the linear switching regulator can be effectively characterized by the quantity called the closed loop gain. The definitions of the closed loop gain for the different regulators are very similar, and the definitions of the boost and buck-boost regulators are identical. The closed loop gain is a function of the feedback constants and is also dependent on the design parameters. It is therefore possible to modify the closed loop gain by either changing the feedback constants or design point. The regulation properties of the regulator are improved by increasing the closed loop gain, but the regulator is made less stable. The closed loop gain makes a good figure of merit for the stability analysis because it does portray this conflict between good regulation and stability. The major difference found between the regulators in performing the regulation analysis is that the equilibrium voltage of the buck regulator is not as sensitive to the damping factor, and therefore the load, as the other two regulators.

The exact expressions for the local stability of the buck regulator are derived. These expressions are then simplified by assuming that $\tau_s \ll 1$. Simplified expressions can also be derived for the boost and buck-boost regulators with the additional assumption that $\xi \leq \tau_s$. This added assumption is usually valid because the boost and buck-boost regulators exhibit positive feedback when the damping factor is made too large. The positive feedback results from the increase in charging time of the capacitor, $(\tau_s - \tau_0)$, when the on-time, $\tau_0$, is decreased to compensate for a high output voltage.

After the stability criteria are derived, the different P.W.M.s can be analyzed. The P.W.M.s are classified by their feedback constants which determine the properties of the constant perturbation matrix. When the regulators are classified in this way, many similarities between them become apparent. The feedback constants on the voltage of a P.W.M., such as the zero eigenvalue P.W.M., are identical for the three regulators. The closed loop gains of the three regulators are very similar, actually identical for the boost and buck-boost regulators, for a specified P.W.M. A comparison between the P.W.M.s of a regulator also shows some similarities. The zero eigenvalue and zero matrix P.W.M.s of a given regulator have the same loop gain although the feedback constants are different. The closed loop gain is an effective and simple way of comparing P.W.M.s, but the stability expressions are valid only for the assumptions for which they were derived.

For all the regulators analyzed the stability appears to be improved by decreasing the switching period, however, in the case of the boost and buck-boost regulator the assumption that $\xi \leqslant \tau_s$ means that when the switching period is decreased beyond a certain value, the stability expression is no longer valid. Since positive feedback occurs when the damping factor becomes larger than the switching period, the system is more unstable than indicated by the derived expressions for large damping factors. The assumptions made in deriving the recursion formula must also be considered in interpreting the final expressions for stability. These assumptions will be valid when the parasitic resistances of the circuit are negligible. When the damping factor is small, the parasitic resistances, even though they are also small, may still not be negligible.

The above assumptions, and the simplifications which these assumptions make possible, are not necessary in evaluating the local stability of the regulator. The local stability can be easily evaluated with the use of a computer. Computer techniques, however, while accurately predicting the local stability of these regulators, do not readily yield insight to the regulator's design. The simple relationships developed in this thesis could be used in the first stages of a design to compare different control laws.

The local stability of switching regulators not operating at the design point was investigated. It was found that the stability of some of the P.W.M.s are very sensitive to changes in the input parameters. An important consideration in the design of a regulator is how sensitive the stability is to changes in the parameters. For the buck regulator, a "feedforward" type of control system which changes the design on-time, $\tau_{00}$, in relation to the current values of parameters can be used to minimize the stability dependence on changes in parameters. This type of control can also be used for the boost or buck-boost regulators.

The discrete phase plane is used to analyze the global properties of the switching regulators. Liapunov's direct method for determining stability as applied to discrete systems along with paired continuous systems are used to obtain sufficient conditions for global stability. The necessary conditions for global stability are actually those obtained from the local stability analysis. A buck regulator is shown to be globally stable to illustrate the phase plane techniques mentioned above. The techniques are thus used to guarantee that the system converges globally.

In the stability analysis a continuous system and its trajectories are associated with the discrete system in the saturated regions of the phase plane. The saturated regions are separated by the nonlinear region which acts somewhat like the switching line of a continuous system. The discrete system for a buck regulator can actually be made into a continuous system with a switching line. The analysis in this case can be carried out in the continuous phase plane, and the switching line for which the system converges to the equilibrium point in minimum time can be determined. The optimal switching line can also be found for the paired continuous systems of the boost and buck-boost regulators. This switching line is not the optimal switching line of the discrete system because the associated continuous trajectories in the saturated regions are not identical to the trajectories of the discrete system. The optimal switching line of the paired continuous system does, however, approximate the one for the discrete system.

References


1. A.E. Bryson, Jr., Yu-Chi Ho, "Applied Optimal Control," Section 3.9, pp. 110-117, Ginn & Company, Waltham, Mass., 1969.

2. D. W. Bushaw, "Differential Equations with a Discontinuous Forcing Term," Report No. 469, Stevens Institute of Technology, January, 1953.

3. A. Capel, J.G. Ferrante, R. Prajoux, "Dynamic Behavior of z-Transform Stability Analysis of DC/DC Regulators with a Non-Linear P.W.M. Control Loop," P.E.S.C., 1973, Pasadena, Calif.

4. A. Capel, J.G. Ferrante, R. Prajoux, "State Variable Stability Analysis of Multi-Loop P.W.M. Controlled DC/DC Regulators in Light and Heavy Mode," P.E.S.C., 1975, Los Angeles, Calif.

5. T. K. Caughey, S. F. Masri, "On the Stability of the Impact Damper," Journal of Applied Mechanics, Paper No. 66-APM-CC, February 21, 1966.

6. Wolfgang Hahn, "Theory and Application of Liapunov's Direct Method," Section 39, Prentice-Hall, Englewood Cliffs, New Jersey, 1963.

7. R. P. Iwens, Y. Yu, J. E. Triner, "Time Domain Modelling and Stability Analysis of an Integral Pulse Frequency Modulated DC to DC Power Converter," P.E.S.C., 1975, Los Angeles, California.

8. R. E. Kalman, "Nonlinear Aspects of Sampled-Data Control Systems," Proceedings of the Second Symposium on Nonlinear Circuit Analysis, Polytechnic Institute of Brooklyn, April 25-27, 1956.

9. R. E. Kalman, J. E. Bertram, "A Unified Approach to the Theory of Sampling Systems," J. Franklin Institute, vol. 267, 1959, pp. 405-436.

10. C.W.T. McLyman, K. Willner, "Boost Regulator Design and Development," Tech. Report No. TR-1001, JPL Contract No. 952293.

11. H. S. Tsien, "Engineering Cybernetics," Section 10.7, pp. 150-159, McGraw-Hill, New York, 1954.

12. G. W. Wester, R. D. Middlebrook, "Low Frequency Characterization of Switched DC/DC Converters," P.E.S.C., 1972, Atlantic City, New Jersey.

13. G. W. Wester, "Linearized Stability Analysis and Design of a Flyback DC-DC Boost Regulator," P.E.S.C., 1973, Pasadena, Calif.

14. G. W. Wester, "Stability Analysis and Compensation of a Boost Regulator with Two-Loop Control," P.E.S.C., 1974, Murray Hill, New Jersey.

15. Norbert Wiener, "Cybernetics: or Control and Communication in the Animal and the Machine," The M.I.T. Press, Cambridge, Massachusetts, 1961.

16. Norbert Wiener, "The Human Use of Human Beings," Avon Books, New York, New York, 1954.

17. J. M. Yuh, "Analysis and Comparative Study of Buck Regulators," Doctoral Thesis, Calif. Inst. of Tech., April, 1973.

Appendix I:

## A. Regulation Analysis for Buck Regulator

The derivation of the regulation equation begins by substituting the control laws, eqn. (2.2), into eqn. (2.4b)

$$x_{sf} = \kappa E\left(\frac{\tau_{00} + \Delta\tau_0}{\tau_{ss} + \Delta\tau_s}\right)$$

$$\dot{x}_{sf} = -\tfrac{1}{2}\kappa E(\tau_{00} + \Delta\tau_0)\left(1 - \frac{\tau_{00} + \Delta\tau_0}{\tau_{ss} + \Delta\tau_s}\right)$$

where

$$\Delta\tau_0 = a_1(x_{ss} - x_{sf}) + b_1(\dot{x}_{ss} - \dot{x}_{sf})$$

$$\Delta\tau_s = a_2(x_{sf} - x_{ss}) + b_2(\dot{x}_{sf} - \dot{x}_{ss})$$

If the changes in the on-time and switching period are small in comparison with the switching period (i.e., $\Delta\tau_0/\tau_{ss} \ll 1$ and $\Delta\tau_s/\tau_{ss} \ll 1$), then the above equations can be expanded out and linearized. The final results can be thought of as the local regulation and will be valid only for small changes from the reference state, $\underline{x}_{ss}$. This linearization is being done to arrive at a figure of merit to compare the different types of regulators and not for accurately determining $\underline{x}_{sf}$. The range of validity for these assumptions can be illustrated by noting that regulation over twenty percent of the switching period corresponds to $\Delta\tau_s/\tau_{ss} = \pm\frac{1}{10}$ and $(\Delta\tau_s/\tau_{ss})^2 = \frac{1}{100}$. The linearization begins with

$$x_{sf} \approx \kappa E\left(\frac{\tau_{00}}{\tau_{ss}} + \frac{\Delta\tau_0}{\tau_{ss}}\right)\left(1 - \frac{\Delta\tau_s}{\tau_{ss}}\right)$$

$$\dot{x}_{sf} \approx -\tfrac{1}{2}\kappa E\tau_{ss}\left(\frac{\tau_{00}}{\tau_{ss}} + \frac{\Delta\tau_0}{\tau_{ss}}\right)\left[1 - \left(\frac{\tau_{00}}{\tau_{ss}} + \frac{\Delta\tau_0}{\tau_{ss}}\right)\left(1 - \frac{\Delta\tau_s}{\tau_{ss}}\right)\right]$$

multiplying out and neglecting higher order terms

$$x_{sf} = \kappa E\left(D_0 + \frac{\Delta\tau_0 - D_0\Delta\tau_s}{\tau_{ss}}\right)$$

$$\dot{x}_{sf} = -\tfrac{1}{2}\kappa\tau_{ss}E\left[D_0D_0' + \frac{(D_0' - D_0)\Delta\tau_0 + D_0^2\Delta\tau_s}{\tau_{ss}}\right]$$

since

$$D_0 = \frac{\tau_{00}}{\tau_{ss}}$$

The equations have now been completely linearized, and the steady-state with feedback, $\underline{x}_{sf}$, can be solved for.

After the expressions for $\Delta\tau_0$ and $\Delta\tau_s$ are substituted into the above equations, they become

$$x_{sf} = \kappa D_0E + \frac{\kappa E}{\tau_{ss}}(a_1 + D_0a_2)(x_{ss} - x_{sf}) + \frac{\kappa E}{\tau_{ss}}(b_1 + D_0b_2)(\dot{x}_{ss} - \dot{x}_{sf})$$

$$\dot{x}_{sf} = -\frac{\kappa\tau_{ss}}{2}D_0D_0'E + \frac{\kappa E}{2}\left[(D_0 - D_0')a_1 + D_0^2a_2\right](x_{ss} - x_{sf})$$

$$+ \frac{\kappa E}{2}\left[(D_0 - D_0')b_1 + D_0^2b_2\right](\dot{x}_{ss} - \dot{x}_{sf})$$

in matrix notation

$$\begin{pmatrix} \{1 + \frac{\kappa E}{\tau_{ss}}(a_1 + D_0 a_2)\} & \frac{\kappa E}{\tau_{ss}}(b_1 + D_0 b_2) \\ \frac{\kappa E}{2}[(D_0 - D_0')a_1 + D_0^2 a_2] & \{1 + \frac{\kappa E}{2}[(D_0 - D_0')b_1 + D_0^2 b_2]\} \end{pmatrix} \underline{x}_{sf} =$$

$$\begin{pmatrix} \kappa D_0 E + \frac{\kappa E}{\tau_{ss}}(a_1 + D_0 a_2)x_{ss} + \frac{\kappa E}{\tau_{ss}}(b_1 + D_0 b_2)\dot{x}_{ss} \\ -\frac{\kappa \tau_{ss}}{2} D_0 D_0' E + \frac{\kappa E}{2}[(D_0 - D_0')a_1 + D_0^2 a_2]x_{ss} + \frac{\kappa E}{2}[(D_0 - D_0')b_1 + D_0^2 b_2]\dot{x}_{ss} \end{pmatrix}$$

let

$$C_1 = \frac{\kappa E}{\tau_{ss}}(a_1 + D_0 a_2) \qquad C_3 = \frac{\kappa E}{\tau_{ss}}(b_1 + D_0 b_2)$$

$$C_2 = \frac{\kappa E}{2\tau_{ss}}[(D_0 - D_0')a_1 + D_0^2 a_2] \qquad C_4 = \frac{\kappa E}{2\tau_{ss}}[(D_0 - D_0')b_1 + D_0^2 b_2]$$

then

$$\begin{pmatrix} \{1 + C_1\} & C_3 \\ \tau_{ss} C_2 & \{1 + \tau_{ss} C_4\} \end{pmatrix} \underline{x}_{sf} = \begin{pmatrix} \kappa D_0 E + C_1 x_{ss} + C_3 \dot{x}_{ss} \\ -\frac{\kappa \tau_{ss}}{2} D_0 D_0' E + \tau_{ss} C_2 x_{ss} + \tau_{ss} C_4 \dot{x}_{ss} \end{pmatrix}$$

The solution for the state vector with feedback is then

$$\underline{x}_{sf} = \frac{1}{\text{Det.}} \begin{pmatrix} (1 + \tau_{ss} C_4) & -C_3 \\ -\tau_{ss} C_2 & (1 + C_1) \end{pmatrix} \begin{pmatrix} \kappa D_0 E + C_1 x_{ss} + C_3 \dot{x}_{ss} \\ -\frac{\kappa \tau_{ss}}{2} D_0 D_0' E + \tau_{ss} C_2 x_{ss} + \tau_{ss} C_4 \dot{x}_{ss} \end{pmatrix}$$

where

$$\begin{aligned} \text{Det.} &= (1 + C_1)(1 + \tau_{ss} C_4) - \tau_{ss} C_2 C_3 \\ &= (1 + \tau_{ss} C_4) + C_1 + \tau_{ss}(C_1 C_4 - C_2 C_3) \end{aligned}$$

but

$$C_1C_4 - C_2C_3 = \frac{\kappa^2E^2}{2\tau_{ss}^2}\{(a_1+D_0a_2)[(D_0-D_0')b_1+D_0^2b_2]$$

$$- (b_1+D_0b_2)[(D_0-D_0')a_1+D_0^2a_2]\}$$

$$= \frac{\kappa^2E^2}{2\tau_{ss}^2}\{(D_0-D_0')a_1b_1+D_0^2a_1b_2+D_0(D_0-D_0')b_1a_2$$

$$+ D_0^3a_2b_2-(D_0-D_0')a_1b_1-D_0^2b_1a_2-D_0(D_0-D_0')a_1b_2$$

$$- D_0^3a_2b_2\}$$

$$= \frac{\kappa^2E^2D_0D_0'}{2\tau_{ss}^2}(a_1b_2-b_1a_2)$$

define

$$S = C_1 + \tau_{ss}(C_1C_4 - C_2C_3)$$

$$S = \frac{\kappa E}{\tau_{ss}}(a_1+D_0a_2) + \frac{\kappa^2E^2D_0D_0'}{2\tau_{ss}}(a_1b_2 - b_1a_2) \qquad \text{(I.A.1)}$$

The parameter S is important to the regulation problem and is called the loop gain.

The steady-state vector with feedback is determined by multiplying the matrices out and simplifying

$$x_{sf} = \frac{Sx_{ss} + (1+\tau_{ss}C_4)\kappa D_0E + C_3\frac{\kappa\tau_{ss}}{2}D_0D_0'E + C_3\dot{x}_{ss}}{S + (1+\tau_{ss}C_4)}$$

$$\dot{x}_{sf} = \frac{S1\dot{x}_{ss} - \frac{\kappa\tau_{ss}}{2} D_0 D_0' E - \kappa D_0 \tau_{ss} E \left(C_2 + \frac{C_1 D_0'}{2}\right) + \tau_{ss} C_2 x_{ss}}{S1 + (1 + C1)}$$

where

$$S1 = \tau_{ss} C_4 + \tau_{ss}(C_1 C_4 - C_2 C_3)$$

$$S1 = \frac{\kappa E}{2}\left[(D_0 - D_0')b_1 + D_0^2 b_2\right] + \frac{\kappa^2 E^2 D_0 D_0'}{2\tau_{ss}}(a_1 b_2 - b_1 a_2)$$

but

$$\tau_{ss} C_4 \kappa D_0 E + C_3 \frac{\kappa\tau_{ss}}{2} D_0 D_0' E = \frac{\kappa^2 D_0^2 E^2}{2}(b_1 + b_2)$$

and

$$C_2 + \frac{C_1 D_0'}{2} = \frac{\kappa D_0 E}{2\tau_{ss}}(a_1 + a_2)$$

The final equations can be written as

$$x_{sf} = \frac{x_{ss} + \frac{\kappa D_0 E}{S}\left[1 + \frac{\kappa D_0 E}{2}(b_1 + b_2) + \frac{\dot{x}_{ss}}{D_0 \tau_{ss}}(b_1 + D_0 b_2)\right]}{1 + \frac{1}{S}\left\{1 + \frac{\kappa E}{2}\left[(D_0 - D_0')b_1 + D_0^2 b_2\right]\right\}} \tag{I.A.2}$$

$$\dot{x}_{sf} = \frac{S1\dot{x}_{ss} - \frac{1}{2}\kappa\tau_{ss} D_0 D_0' E\left[1 + \frac{\kappa D_0 E}{\tau_{ss}}\left(\frac{a_1 + a_2}{D_0'}\right) - \frac{x_{ss}}{\tau_{ss}}\left(\frac{D_0 - D_0'}{D_0 D_0'}\right)a_1 + \frac{D_0}{D_0'}a_2\right]}{S1 + \left[1 + \frac{\kappa E}{\tau_{ss}}(a_1 + D_0 a_2)\right]}$$

for

$$S \neq 0$$

It can be seen from eqn. (I. A. 2) that in some sense the loop gain, S, is a measure of the regulator's ability to regulate, for as the loop gain becomes large, the value of $x_{sf}$ will approach $x_{ss}$. The loop gain is used as a figure of merit in the stability analysis.

Appendix I:

B. Stability Analysis for a Buck Regulator

The variational equation, eqn. (2.2), can be written as

$$\delta \underline{x}_{n+1} = \frac{\partial g}{\partial \underline{x}_n}\Big|_{\underline{x}_{sf}} \delta \underline{x}_n + \left(\frac{\partial g}{\partial \tau_0}\ \frac{\partial \tau_0(\underline{x}_n)}{\partial \underline{x}_n}\right)\Big|_{\underline{x}_{sf}} \delta \underline{x}_n + \left(\frac{\partial g}{\partial \tau_s}\ \frac{\partial \tau_s(\underline{x}_n)}{\partial \underline{x}_n}\right)\Big|_{\underline{x}_{sf}} \delta \underline{x}_n$$

since

$$\tau_0(\underline{x}_n) = \tau_{00} + a_1(x_{ss} - x_n) + b_1(\dot{x}_{ss} - \dot{x}_n)$$

and

$$\tau_s(\underline{x}_n) = \tau_{ss} + a_2(x_n - x_{ss}) + b_2(\dot{x}_n - \dot{x}_{ss})$$

When these partial derivatives are evaluated at the equilibrium point, $\underline{x}_{sf}$, they become

$$\frac{\partial g}{\partial \underline{x}_n}\Big|_{\underline{x}_{sf}} \delta \underline{x}_n = Y(\tau_s)\delta \underline{x}_n$$

$$\left(\frac{\partial g}{\partial \tau_0}\ \frac{\partial \tau_0}{\partial \underline{x}_n}\right)\Big|_{\underline{x}_{sf}} \delta \underline{x}_n = -\kappa E \begin{pmatrix} y_{12}(\tau_s - \tau_0) \\ y_{22}(\tau_s - \tau_0) \end{pmatrix} (a_1 \delta x_n + b_1 \delta \dot{x}_n)$$

and

$$\left(\frac{\partial g}{\partial \tau_s}\ \frac{\partial \tau_s}{\partial \underline{x}_n}\right)\Big|_{\underline{x}_{sf}} \delta \underline{x}_n = \left(\frac{\partial Y(\tau_s)}{\partial \tau_s}\, \underline{x}_{sf} - \kappa E \begin{pmatrix} y_{12}(\tau_s - \tau_0) - y_{12}(\tau_s) \\ y_{22}(\tau_s - \tau_0) - y_{22}(\tau_s) \end{pmatrix}\right) (a_2 \delta x_n + b_2 \delta \dot{x}_n)$$

where

$$\frac{\partial Y}{\partial \tau_s} = \begin{pmatrix} -y_{12}(\tau_s) & y_{22}(\tau_s) \\ -y_{22}(\tau_s) & -[y_{12}(\tau_s) + 2\xi y_{22}(\tau_s)] \end{pmatrix}$$

with

$$\tau_0 = \tau_{00} + a_1(x_{ss} - x_{sf}) + b_1(\dot{x}_{ss} - \dot{x}_{sf})$$

$$\tau_s = \tau_{ss} + a_2(x_{sf} - x_{ss}) + b_2(\dot{x}_{sf} - \dot{x}_{ss})$$

These terms can now be combined to form the P-matrix

$$P = \left( \frac{\partial g}{\partial \underline{x}_n} + \frac{\partial g}{\partial \tau_0} \frac{\partial \tau_0(\underline{x}_n)}{\partial \underline{x}_n} + \frac{\partial g}{\partial \tau_s} \frac{\partial \tau_s(\underline{x}_n)}{\partial \underline{x}_n} \right) \Bigg|_{\underline{x}_n = \underline{x}_{sf}} \qquad \text{(I.B.1)}$$

or

$$P = \begin{pmatrix} [y_{11}(\tau_s) - a_1 \kappa E y_{12}(\tau_s - \tau_0) - a_2 h_1] & [y_{12}(\tau_s) - b_1 \kappa E y_{12}(\tau_s - \tau_0) - b_2 h_1] \\ [y_{21}(\tau_s) - a_1 \kappa E y_{22}(\tau_s - \tau_0) - a_2 h_2] & [y_{22}(\tau_s) - b_1 \kappa E y_{22}(\tau_s - \tau_0) - b_2 h_2] \end{pmatrix}$$

where

$$h_1 = y_{12}(\tau_s) x_{sf} - y_{22}(\tau_s) \dot{x}_{sf} + \kappa E[y_{12}(\tau_s - \tau_0) - y_{12}(\tau_s)]$$

$$h_2 = y_{22}(\tau_s) x_{sf} + [y_{12}(\tau_s) + 2\xi y_{22}(\tau_s)] \dot{x}_{sf} + \kappa E[y_{22}(\tau_s - \tau_0) - y_{22}(\tau_s)]$$

The P-matrix consists of a natural part, $Y(\tau_s)$, and a control part. Because negative feedback is used on the state, most of the control terms are negative. For the buck regulator, increasing the feedback gains will decrease the trace of the P-matrix.

The eigenvalues of the P-matrix can be written as

$$\lambda_{1,2} = \frac{TR(P)}{2} \pm \sqrt{\left[\frac{TR(P)}{2}\right]^2 - \text{Det.}(P)}$$

where

$$TR(P) = p_{11} + p_{22} \text{ - Trace of the P-matrix}$$

$$Det.(P) = p_{11}p_{22} - p_{12}p_{21} \text{ - Determinate of the P-matrix}$$

The system will be locally stable if and only if all the eigenvalues of P have modulus less than one. For complex eigenvalues of a two by two matrix, the determinate is equal to the modulus of the eigenvalues squared so that the condition for stability is

$$|\lambda|^2 = Det.(P) < 1 \qquad \text{(B. 2a)}$$

When the eigenvalues are real, the magnitude of the largest eigenvalue must be less than one for stability. If the trace of the P-matrix is negative, then the stability criterion is that

$$-1 \leqslant \frac{TR(P)}{2} - \sqrt{\left[\frac{TR(P)}{2}\right]^2 - Det.(P)}$$

or

$$1 + Tr(P) + Det.(P) \geqslant 0 \qquad \text{(B. 2b)}$$

The instability associated with real eigenvalues usually occur at high gains where the trace of the P-matrix is negative. These two stability criterions, eqn. (I. B. 2a&b), are used to evaluate the stability of various P. W. M. s.

The easiest way of deriving the structure of eqn. (I. B. 2) for a buck regulator is to change eqn. (2. 1) to its canonical form.

$$\underline{y}_{n+1} = \begin{pmatrix} e^{\lambda_1 \tau_s} & 0 \\ 0 & e^{\lambda_2 \tau_s} \end{pmatrix} \underline{y}_n + \frac{\gamma_i}{2} \begin{pmatrix} \dfrac{e^{\lambda_1(\tau_s - \tau_0)}}{\lambda_1} \\ -\dfrac{e^{\lambda_2(\tau_s - \tau_0)}}{\lambda_2} \end{pmatrix} - \frac{\gamma_i}{2} \begin{pmatrix} \dfrac{e^{\lambda_1 \tau_s}}{\lambda_1} \\ -\dfrac{e^{\lambda_2 \tau_s}}{\lambda_2} \end{pmatrix}$$

where

$$\underline{x}_n = T\underline{y}_n \quad \text{and} \quad T = \begin{pmatrix} 1 & 1 \\ \lambda_1 & \lambda_2 \end{pmatrix}$$

with

$$\lambda_1 = \xi + i\sqrt{1 - \xi^2} \qquad \gamma = \frac{E}{\sqrt{1 - \xi^2}}$$

$$\lambda_2 = \xi - i\sqrt{1 - \xi^2}$$

The variation of the state at the $(n+1)^{st}$ iteration, $\delta\underline{y}_{n+1}$, is related to the variation in the state at the $n^{th}$ iteration, $\delta\underline{y}_n$, by

$$\delta\underline{y}_{n+1} = \begin{pmatrix} e^{\lambda_1 \tau_s} & 0 \\ 0 & e^{\lambda_2 \tau_s} \end{pmatrix} \delta\underline{y}_n - \frac{\gamma_i}{2} \begin{pmatrix} e^{\lambda_1(\tau_s - \tau_0)} \\ -e^{\lambda_2(\tau_s - \tau_0)} \end{pmatrix} \frac{\partial \tau_0}{\partial \underline{y}_n} \delta\underline{y}_n$$

$$+ \left\{ \begin{pmatrix} \lambda_1 e^{\lambda_1 \tau_s} & 0 \\ 0 & \lambda_2 e^{\lambda_2 \tau_s} \end{pmatrix} \underline{y}_{sf} + \frac{\gamma_i}{2} \begin{pmatrix} e^{\lambda_1(\tau_s - \tau_0)} \\ -e^{\lambda_2(\tau_s - \tau_0)} \end{pmatrix} \right.$$

$$\left. - \frac{\gamma_i}{2} \begin{pmatrix} e^{\lambda_1 \tau_s} \\ -e^{\lambda_2 \tau_s} \end{pmatrix} \right\} \frac{\partial \tau_s}{\partial \underline{y}_n} \delta\underline{y}_n$$

where

$$\underline{x}_{sf} = T\underline{y}_{sf}$$

The variational equation can be rewritten as

$$\delta \underline{y}_{n+1} = \begin{pmatrix} \left\{ e^{\lambda_1 \tau_s} + d_1 \frac{\partial \tau_s}{\partial y_n} - c_1 \frac{\partial \tau_0}{\partial y_n} \right\} & \left\{ d_1 \frac{\partial \tau_s}{\partial \dot{y}_n} - c_1 \frac{\partial \tau_0}{\partial \dot{y}_n} \right\} \\ \left\{ d_2 \frac{\partial \tau_s}{\partial y_n} - c_2 \frac{\partial \tau_0}{\partial y_n} \right\} & \left\{ e^{\lambda_2 \tau_s} + d_2 \frac{\partial \tau_s}{\partial \dot{y}_n} - c_2 \frac{\partial \tau_0}{\partial \dot{y}_n} \right\} \end{pmatrix} \delta \underline{y}_n$$

where

$$d_1 = e^{\lambda_1 \tau_s} \left( \lambda_1 y_{sf} + \frac{\gamma_i}{2} e^{-\lambda_1 \tau_0} - \frac{\gamma_i}{2} \right)$$

$$d_2 = e^{\lambda_2 \tau_s} \left( \lambda_2 \dot{y}_{sf} - \frac{\gamma_i}{2} e^{-\lambda_2 \tau_0} + \frac{\gamma_i}{2} \right)$$

$$c_1 = -e^{\lambda_1 \tau_s} \frac{\gamma_i}{2} e^{-\lambda_1 \tau_0}$$

$$c_2 = e^{\lambda_2 \tau_s} \frac{\gamma_i}{2} e^{-\lambda_2 \tau_0}$$

The determinate of the perturbation matrix is

$$\text{Det.}(P) = e^{(\lambda_1 + \lambda_2)\tau_s} + \left( e^{\lambda_1 \tau_s} d_2 \frac{\partial \tau_s}{\partial \dot{y}_n} + e^{\lambda_2 \tau_s} d_1 \frac{\partial \tau_s}{\partial y_n} \right)$$

$$- \left( e^{\lambda_1 \tau_s} c_2 \frac{\partial \tau_0}{\partial \dot{y}_n} + e^{\lambda_2 \tau_s} c_1 \frac{\partial \tau_0}{\partial y_n} \right)$$

$$+ (c_1 d_2 - d_1 c_2) \left( \frac{\partial \tau_0}{\partial \dot{y}_n} \frac{\partial \tau_s}{\partial y_n} - \frac{\partial \tau_0}{\partial y_n} \frac{\partial \tau_s}{\partial \dot{y}_n} \right)$$

now

$$\frac{\partial \tau_0}{\partial y_n} = \frac{\partial \tau_0}{\partial x_n}\frac{\partial x_n}{\partial y_n} + \frac{\partial \tau_0}{\partial \dot{x}_n}\frac{\partial \dot{x}_n}{\partial y_n} \qquad \frac{\partial \tau_0}{\partial \dot{y}_n} = \frac{\partial \tau_0}{\partial x_n}\frac{\partial x_n}{\partial \dot{y}_n} + \frac{\partial \tau_0}{\partial \dot{x}_n}\frac{\partial \dot{x}_n}{\partial \dot{y}_n}$$

$$\frac{\partial \tau_s}{\partial y_n} = \frac{\partial \tau_s}{\partial x_n}\frac{\partial x_n}{\partial y_n} + \frac{\partial \tau_s}{\partial \dot{x}_n}\frac{\partial \dot{x}_n}{\partial y_n} \qquad \frac{\partial \tau_s}{\partial \dot{y}_n} = \frac{\partial \tau_s}{\partial x_n}\frac{\partial x_n}{\partial \dot{y}_n} + \frac{\partial \tau_s}{\partial \dot{x}_n}\frac{\partial \dot{x}_n}{\partial \dot{y}_n}$$

with

$$\frac{\partial \tau_0}{\partial x_n} = -a_1 \qquad \frac{\partial \tau_0}{\partial \dot{x}_n} = -b_1$$

$$\frac{\partial \tau_s}{\partial x_n} = a_2 \qquad \frac{\partial \tau_s}{\partial \dot{x}_n} = b_2$$

and

$$\underline{x}_n = T\underline{y}_n = \begin{pmatrix} 1 & 1 \\ \lambda_1 & \lambda_2 \end{pmatrix} \underline{y}_n$$

$$\frac{\partial x_n}{\partial y_n} = 1 \qquad \frac{\partial x_n}{\partial \dot{y}_n} = 1$$

$$\frac{\partial \dot{x}_n}{\partial y_n} = \lambda_1 \qquad \frac{\partial \dot{x}_n}{\partial \dot{y}_n} = \lambda_2$$

combining terms

$$\frac{\partial \tau_0}{\partial y_n} = -a_1 - \lambda_1 b_1 \qquad \frac{\partial \tau_0}{\partial \dot{y}_n} = -a_1 - \lambda_2 b_1$$

$$\frac{\partial \tau_s}{\partial y_n} = a_2 + \lambda_1 b_2 \qquad \frac{\partial \tau_s}{\partial \dot{y}_n} = a_2 + \lambda_2 b_2$$

When these quantities are substituted into eqn. (B. 5), it becomes

$$\text{Det.}(P) = e^{(\lambda_1+\lambda_2)\tau_s} + a_2(d_2 e^{\lambda_1\tau_s} + d_1 e^{\lambda_2\tau_s})$$

$$+ b_2(\lambda_1 d_1 e^{\lambda_2\tau_s} + \lambda_2 d_2 e^{\lambda_1\tau_s})$$

$$- a_1(c_2 e^{\lambda_1\tau_s} + c_1 e^{\lambda_2\tau_s}) - b_1(\lambda_1 c_1 e^{\lambda_2\tau_s} + \lambda_2 c_2 e^{\lambda_1\tau_s})$$

$$+ (c_1 d_2 - d_1 c_2)(\lambda_2 - \lambda_1)(a_1 b_2 - a_2 b_1)$$

Evaluating the terms on the right

$$e^{(\lambda_1+\lambda_2)\tau_s} = e^{-2\xi\tau_s}$$

$$a_2(d_2 e^{\lambda_1\tau_s} + d_1 e^{\lambda_2\tau_s}) = a_2 e^{-2\xi\tau_s}[\dot{x}_{sf} + \kappa E e^{2\xi\tau_0}\, y_{12}(\tau_0)]$$

$$b_2(\lambda_1 d_1 e^{\lambda_2\tau_s} + \lambda_2 d_2 e^{\lambda_1\tau_s}) = b_2 e^{-2\xi\tau_s}[\kappa E(1 - e^{2\xi\tau_0}\, y_{11}(\tau_0)] - x_{sf} - 2\xi\dot{x}_{sf}]$$

$$a_1(c_2 e^{\lambda_1\tau_s} + c_1 e^{\lambda_2\tau_s}) = a_1\kappa E e^{-2\xi(\tau_s-\tau_0)}\, y_{21}(\tau_0)$$

$$b_1(\lambda_1 e^{\lambda_2\tau_s} + \lambda_2 c_2 e^{\lambda_1\tau_s}) = b_1\kappa E e^{-2\xi(\tau_s-\tau_0)}\, y_{11}(\tau_0)$$

and

$$(c_1 d_2 - d_1 c_2)(\lambda_2 - \lambda_1)(a_1 b_2 - a_2 b_1) =$$

$$\kappa E e^{-2\xi(\tau_s-\tau_0)}\,[(x_{sf} - E)\,y_{21}(\tau_0) + \dot{x}_{sf} y_{22}(\tau_0)]\,(a_1 b_2 - a_2 b_1)$$

The stability criterion for complex eigenvalues is

$$\text{Det.}(P) < 1$$

or

$$(a_2 - 2\xi b_2)\dot{x}_{sf} + b_2(\kappa E - x_{sf})$$

$$+ (a_1 + a_2)\kappa E e^{2\xi\tau_0}\, y_{12}(\tau_0) - (b_1 + b_2)\kappa E e^{2\xi\tau_0} y_{11}(\tau_0)$$

$$+ (a_1 b_2 - a_2 b_1)\kappa E e^{2\xi\tau_0}\left[(\kappa E - x_{sf})y_{12}(\tau_0) + \dot{x}_{sf}y_{22}(\tau_0)\right]$$

$$< (e^{2\xi\tau_s} - 1) \qquad \text{(I.B.3a)}$$

The stability criterion for real eigenvalues is

$$1 + \mathrm{Tr}(P) + \mathrm{Det}(P) > 0$$

which can be written, after the trace of the P-matrix from eqn. (I. B. 1) is determined, as

$$1 + e^{-\xi\tau_s}(2\cos\omega_d\tau_s + e^{-\xi\tau_s}) + (a_1 + a_2)\kappa E\left[e^{-2\xi(\tau_s - \tau_0)}y_{12}(\tau_0) - y_{12}(\tau_s - \tau_0)\right]$$

$$- (b_1 + b_2)\kappa E\left[e^{-2\xi(\tau_s - \tau_0)}y_{11}(\tau_0) + y_{22}(\tau_s - \tau_0)\right]$$

$$- a_2\left[y_{12}(\tau_s)x_{sf} - \mu\dot{x}_{sf} - \kappa E y_{12}(\tau_s)\right]$$

$$- b_2\left[\mu(x_{sf} - \kappa E + 2\xi\dot{x}_{sf}] + y_{12}(\tau_s)\dot{x}_{sf}\right]$$

$$+ (a_1 b_2 - a_2 b_1)\kappa E e^{-2\xi(\tau_s - \tau_0)}\left[(\kappa E - x_{sf})y_{12}(\tau_0) + \dot{x}_{sf}y_{22}(\tau_0)\right]$$

$$\geqslant 0 \qquad \text{(I.B.3b)}$$

where

$$\mu = e^{-2\xi\tau_s} + y_{22}(\tau_s)$$

The stability criterions, eqn. (I. B. 3a&b), are used to evaluate the local stability of the buck regulators.

Appendix II:

A. Regulation Analysis for Boost Regulator

The procedure for deriving the regulation of a boost regulator is the same as that of the buck regulator. When the control laws, eqn. (3.3), are substituted into eqn. (3.4b), it becomes

$$z_{sf}^{(1)} = \kappa E \frac{(\tau_{ss} + \Delta\tau_s)}{(\tau_{ss} - \tau_{00}) + (\Delta\tau_s - \Delta\tau_0)}$$

(II.A.1a)

$$z_{sf}^{(2)} = \kappa E \left\{ 2\xi \frac{(\tau_{ss} + \Delta\tau_s)^2}{[(\tau_{ss} - \tau_{00}) + (\Delta\tau_s - \Delta\tau_0)]^2} - \frac{(\tau_{00} + \Delta\tau_0)}{2} \right\}$$

where

$$\Delta\tau_0 = a_1[z_{ss}^{(1)} - z_{sf}^{(1)}] + b_1[z_{ss}^{(2)} - z_{sf}^{(2)}]$$

$$\Delta\tau_s = a_2[z_{sf}^{(1)} - z_{ss}^{(1)}] + b_2[z_{sf}^{(2)} - z_{ss}^{(2)}]$$

When the changes in the on-time and switching period are small in comparison with the difference between the switching period and on-time (i.e. $\Delta\tau_0/(\tau_{ss} - \tau_{00}) \ll 1$ and $\Delta\tau_s/(\tau_{ss} - \tau_{00}) \ll 1$), then eqn. (C.1) can be linearized so that

$$z_{sf}^{(1)} = \frac{\kappa E}{D_0'} + \frac{\kappa E}{\tau_{ss} D_0'^2} \Delta\tau_0 - \frac{D_0 \kappa E}{\tau_{ss} D_0'^2} \Delta\tau_s$$

(II.A.1b)

$$z_{sf}^{(2)} = \kappa E \left( \frac{2\xi}{D_0'^2} - \frac{\tau_{00}}{2} \right) + \kappa E \left( \frac{4\xi}{\tau_{ss} D_0'^3} - \frac{1}{2} \right) \Delta\tau_0 - \frac{4\xi\kappa D_0 E}{\tau_{ss} D_0'^3} \Delta\tau_s$$

Since the change in on-time and switching period are a function of the state, $\underline{z}_{sf}$, eqn. (II. A. 1b) can be rewritten as

$$\underline{z}_{sf} = (I+G)^{-1}\left\{\begin{pmatrix} \kappa E/D_0' \\ \kappa E\left[\dfrac{2\xi}{D_0'^{\,2}} - \dfrac{\tau_{00}}{2}\right]\end{pmatrix} + G\underline{z}_{ss}\right\}$$

where

$$G = \begin{pmatrix} \dfrac{\kappa E}{\tau_{ss}D_0'^{\,2}}(a_1 + D_0 a_2) & \dfrac{\kappa E}{\tau_{ss}D_0'^{\,2}}(b_1 + D_0 b_2) \\ \left[\dfrac{4\xi\kappa E}{\tau_{ss}D_0'^{\,3}}(a_1 + D_0 a_2) - \dfrac{a_1\kappa E}{2}\right] & \left[\dfrac{4\xi\kappa E}{\tau_{ss}D_0'^{\,3}}(b_1 + D_0 b_2) - \dfrac{b_1\kappa E}{2}\right]\end{pmatrix}$$

and

$$(I + G)^{-1} = \frac{1}{\left[(1+g_{11})(1+g_{22}) - g_{12}g_{21}\right]}\begin{pmatrix} (1+g_{22}) & -g_{12} \\ -g_{21} & (1+g_{11})\end{pmatrix}$$

let

$$S = g_{11} + g_{11}g_{22} - g_{12}g_{21} = \frac{\kappa E}{\tau_{ss}D_0'^{\,2}}(a_1 + D_0 a_2) + \frac{\kappa^2E^2D_0}{2\tau_{ss}D_0'^{\,2}}(a_1b_2 - b_1a_2)$$

$$S1 = g_{22} + g_{11}g_{22} - g_{12}g_{21} = \frac{4\xi\kappa E}{\tau_{ss}D_0'^{\,3}}(b_1 + D_0 b_2) - \frac{b_1\kappa E}{2} + \frac{\kappa^2E^2D_0}{2\tau_{ss}D_0'^{\,2}}(a_1b_2 - b_1a_2)$$

so

$$z_{sf}^{(1)} = \frac{(1+g_{22})\kappa E/D_0' - g_{12}\kappa E\left(\frac{2\xi}{D_0'^{2}} - \frac{\tau_{00}}{2}\right) + Sz_{ss}^{(1)} + z_{ss}^{(2)}\, g_{12}}{\left\{1 + \frac{4\xi\kappa E}{\tau_{ss}D_0'^{3}}(b_1 + D_0 b_2) - \frac{b_1\kappa E}{2}\right\} + S}$$

$$z_{sf}^{(2)} = \frac{-g_{21}\kappa E/D_0' + (1+g_{11})\kappa E\left(\frac{2\xi}{D_0'^{2}} - \frac{\tau_{00}}{2}\right) + g_{21}z_{ss}^{(1)} + S1\, z_{ss}^{(2)}}{\left\{1 + \frac{\kappa E}{\tau_{ss}D_0'^{2}}(a_1 + D_0 a_2)\right\} + S1}$$

now

$$(1+g_{22})\kappa E/D_0' - g_{12}\kappa E\left(\frac{2\xi}{D_0'^{2}} - \frac{\tau_{00}}{2}\right) + g_{12}z_{ss}^{(2)} =$$

$$\frac{\kappa E}{D_0'}\left\{1 + \frac{2\xi\kappa E}{\tau_{ss}D_0'^{3}}(b_1 + D_0 b_2) + \frac{\kappa E}{2D_0'}\left[(D_0 - D_0')b_1 + D_0^2 b_2\right] + \frac{(b_1 + D_0 b_0)}{\tau_{ss}D_0'}\, z_{ss}^{(2)}\right\}$$

with

$$-g_{21}\kappa E D_0' + (1+g_{11})\kappa E\left(\frac{2\xi}{D_0'^{2}} - \frac{\tau_{00}}{2}\right) + g_{21}z_{ss}^{(1)} =$$

$$\kappa E\left(\frac{2\xi}{D_0'^{2}} - \frac{\tau_{00}}{2}\right) - \kappa^2 E^2\, \frac{2\xi}{\tau_{ss}D_0'^{4}}(a_1 + D_0 a_2) - \frac{\kappa^2 E^2}{2D_0'^{2}}\left[a_1(D_0 - D_0') + D_0^2 a_2\right]$$

$$+ \kappa E\left[\frac{4\xi}{\tau_{ss}D_0'^{3}}(a_1 + D_0 a_2) - \frac{a_1}{2}\right] z_{ss}^{(1)}$$

The approximate expression for the state with feedback, $\underline{z}_{sf}$, is

$$z_{sf}^{(1)} = \left\{ z_{ss}^{(1)} + \frac{\kappa E/D_0'}{S} \left[ 1 + \kappa E \frac{2\xi}{\tau_{ss} D_0'^{3}} (b_1 + D_0 b_2) \right.\right.$$

$$\left.\left. + \frac{\kappa E}{2D_0'} \left[ (D_0 - D_0') b_1 + D_0^2 b_2 \right] + \frac{(b_1 + D_0 b_2)}{\tau_{ss} D_0'} z_{ss}^{(2)} \right] \right\} /$$

$$/ \left\{ 1 + \frac{1}{S} \left[ 1 + \kappa E \frac{4\xi}{\tau_{ss} D_0'^{3}} (b_1 + D_0 b_2) - \kappa E \frac{b_1}{2} \right] \right\}$$

$$z_{sf}^{(2)} = S1 z_{ss}^{(2)} + \kappa E \left( \frac{2\xi}{D_0'^{2}} - \frac{\tau_{00}}{2} \right) - \kappa^2 E^2 \frac{2\xi}{\tau_{ss} D_0'^{4}} (a_1 + D_0 a_2)$$

$$- \frac{\kappa^2 E^2}{2D_0'^{2}} \left[ a_1 (D_0 - D_0') + D_0^2 a_2 \right] + \kappa E \left[ \frac{4\xi}{\tau_{ss} D_0'^{3}} (a_1 + D_0 a_2) - \frac{a_1}{2} \right] \cdot$$

$$\cdot \; z_{ss}^{(1)} \} / \left\{ S1 + \left[ 1 + \frac{\kappa E}{\tau_{ss} D_0'^{2}} (a_1 + D_0 a_2) \right] \right\}$$

(II.A.2)

$$S = \frac{\kappa E}{\tau_{ss} D_0'^{2}} (a_1 + D_0 a_2) + \frac{\kappa^2 E^2 D_0}{2 \tau_{ss} D_0'^{2}} (a_1 b_2 - b_1 a_2)$$

$$S1 = \kappa E \frac{4\xi}{\tau_{ss} D_0'^{3}} (b_1 + D_0 b_2) - \kappa E \frac{b_1}{2} + \frac{\kappa^2 E^2 D_0}{2 \tau_{ss} D_0'^{2}} (a_1 b_2 - b_1 a_2)$$

The closed loop gain, S, will act as a figure of merit in the stability analysis of the boost regulator.

Appendix II:

B. Stability Analysis for a Boost Regulator

The recursion formula for the boost regulator, eqn. (3.2), is

$$\underline{z}_{n+1} = A\underline{z}_n + \underline{b} \tag{II.B.1}$$

where

$$A = \begin{pmatrix} e^{-2\xi_0\tau_0}\, y_{22}(\tau_s - \tau_0) & y_{12}(\tau_s - \tau_0) \\ e^{-2\xi_0\tau_0}\, y_{21}(\tau_s - \tau_0) & y_{11}(\tau_s - \tau_0) \end{pmatrix}$$

and

$$\underline{b} = \kappa E \begin{pmatrix} \tau_0 y_{12}(\tau_s - \tau_0) + [1 - y_{11}(\tau_s - \tau_0)] \\ \tau_0 y_{11}(\tau_s - \tau_0) + y_{12}(\tau_s - \tau_0) + 2\xi_0[1 - y_{11}(\tau_s - \tau_0)] \end{pmatrix}$$

The variation of the state at the $(n+1)^{st}$ iteration is related to the variation in state of the $n^{th}$ iteration by

$$\delta\underline{z}_{n+1} = A\delta\underline{z}_n + \left(\frac{\partial A}{\partial \tau_0}\underline{z}_{sf} + \frac{\partial \underline{b}}{\partial \tau_0}\right)\frac{\partial \tau_0}{\partial \underline{z}_n}\delta\underline{z}_n + \left(\frac{\partial A}{\partial \tau_s}\underline{z}_{sf} + \frac{\partial \underline{b}}{\partial \tau_s}\right)\frac{\partial \tau_0}{\partial \tau_s}\delta\underline{z}_n \tag{II.B.2}$$

where

$$\frac{\partial A}{\partial \tau_0} = \begin{pmatrix} e^{-2\xi_0\tau_0}\, y_{12}(\tau_s - \tau_0) & -y_{22}(\tau_s - \tau_0) \\ e^{-2\xi_0\tau_0}\, y_{11}(\tau_s - \tau_0) & y_{12}(\tau_s - \tau_0) \end{pmatrix}$$

$$\frac{\partial A}{\partial \tau_s} = \begin{pmatrix} -e^{-2\xi_0\tau_0}\left[y_{12}(\tau_s-\tau_0) + 2\xi_0 y_{22}(\tau_s-\tau_0)\right] & y_{22}(\tau_s-\tau_0) \\ -e^{-2\xi_0\tau_0}\, y_{22}(\tau_s-\tau_0) & y_{21}(\tau_s-\tau_0) \end{pmatrix}$$

$$\frac{\partial \underline{b}}{\partial \tau_0} = \kappa E \tau_0 \begin{pmatrix} -y_{22}(\tau_s-\tau_0) \\ y_{12}(\tau_s-\tau_0) \end{pmatrix}$$

and

$$\frac{\partial \underline{b}}{\partial \tau_s} = \kappa E \begin{pmatrix} \tau_0 y_{22}(\tau_s-\tau_0) + y_{12}(\tau_s-\tau_0) \\ \tau_0 y_{21}(\tau_s-\tau_0) + y_{11}(\tau_s-\tau_0) \end{pmatrix}$$

The variational equation, eqn. (II. B. 2), can now be written as

$$\delta \underline{z}_{n+1} = P\, \delta \underline{z}_n \qquad \text{(II.B.2)}$$

where

$$P = \begin{pmatrix} \left\{a_{11} + g_1 \frac{\partial \tau_0}{\partial z_n^{(1)}} + h_1 \frac{\partial \tau_s}{\partial z_n^{(1)}}\right\} & \left\{a_{12} + g_1 \frac{\partial \tau_s}{\partial z_n^{(2)}} + h_1 \frac{\partial \tau_s}{\partial z_n^{(2)}}\right\} \\ \left\{a_{21} + g_2 \frac{\partial \tau_0}{\partial z_n^{(1)}} + h_2 \frac{\partial \tau_s}{\partial z_n^{(1)}}\right\} & \left\{a_{22} + g_2 \frac{\partial \tau_s}{\partial z_n^{(2)}} + h_2 \frac{\partial \tau_s}{\partial z_n^{(2)}}\right\} \end{pmatrix}$$

with

$$\underline{g} = \begin{pmatrix} g_1 \\ g_2 \end{pmatrix} = \left( \frac{\partial A}{\partial \tau_0} \underline{z}_{sf} + \frac{\partial \underline{b}}{\partial \tau_0} \right)$$

$$\underline{h} = \begin{pmatrix} h_1 \\ h_2 \end{pmatrix} = \left( \frac{\partial A}{\partial \tau_s} \underline{z}_{sf} + \frac{\partial \underline{b}}{\partial \tau_s} \right)$$

The on-time and switching period are related to the state by the control law, eqn. (3.3), so that

$$\frac{\partial \tau_0}{\partial z_n^{(1)}} = -a_1 \qquad \frac{\partial \tau_s}{\partial z_n^{(1)}} = a_2$$

$$\frac{\partial \tau_0}{\partial z_n^{(2)}} = -b_1 \qquad \frac{\partial \tau_s}{\partial z_n^{(2)}} = b_2$$

Substituting the above expressions into the perturbation matrix gives

$$P = \begin{pmatrix} \{a_{11} - a_1 g_1 + a_2 h_1\} & \{a_{12} - b_1 g_1 + b_2 h_1\} \\ \{a_{21} - a_1 g_2 + a_2 h_2\} & \{a_{22} - b_1 g_2 + b_2 h_2\} \end{pmatrix} \quad \text{(II.B.3)}$$

where

$$\underline{g} = \begin{pmatrix} e^{-2\xi_0\tau_0}\, y_{12}(\tau_s - \tau_0) z_{sf}^{(1)} - y_{22}(\tau_s - \tau_0) z_{sf}^{(2)} - \kappa E \tau_0 y_{22}(\tau_s - \tau_0) \\ e^{-2\xi_0\tau_0}\, y_{11}(\tau_s - \tau_0) z_{sf}^{(1)} + y_{12}(\tau_s - \tau_0) z_{sf}^{(2)} + \kappa E \tau_0 y_{12}(\tau_s - \tau_0) \end{pmatrix}$$

$$\underline{h} = \begin{pmatrix} -e^{-2\xi_0\tau_0}[y_{12}(\tau_s-\tau_0) + 2\xi_0 y_{22}(\tau_s-\tau_0)]\, z_{sf}^{(1)} + y_{22}(\tau_s-\tau_0)\, z_{sf}^{(2)} + \kappa E[\tau_0 y_{22}(\tau_s-\tau_0) + y_{12}(\tau_s-\tau_0)] \\ -e^{-2\xi_0\tau_0}\, y_{22}(\tau_s-\tau_0)\, z_{sf}^{(1)} + y_{21}(\tau_s-\tau_0)\, z_{sf}^{(2)} + \kappa E[\tau_0 y_{21}(\tau_s-\tau_0) + y_{11}(\tau_s-\tau_0)] \end{pmatrix}$$

The structure of the perturbation equation for the boost regulator is more complicated than that of the buck regulator. It is therefore convenient to make the assumption that $\tau_s \ll 1$ in order to simplify the calculations. If $\tau_{ss} \ll 1$, then

$$A \approx \begin{pmatrix} (1 - 2\xi\tau_s) & (\tau_s - \tau_0) \\ -(\tau_s - \tau_0) & 1 \end{pmatrix}$$

and

$$\underline{g} \approx \begin{pmatrix} (\tau_s - \tau_0) z_{sf}^{(1)} - [1 - 2\xi(\tau_s - \tau_0)] z_{sf}^{(2)} - \kappa E \tau_0 \\ (1 - 2\xi\tau_0) z_{sf}^{(1)} + (\tau_s - \tau_0) z_{sf}^{(2)} \end{pmatrix}$$

with (II.B.4a)

$$\underline{h} \approx \begin{pmatrix} -[(\tau_s - \tau_0) + 2\xi(1 - 2\xi\tau_s)] z_{sf}^{(1)} + [1 - 2\xi(\tau_s - \tau_0)] z_{sf}^{(2)} + \kappa E \tau_s \\ -(1 - 2\xi\tau_s) z_{sf}^{(1)} - (\tau_s - \tau_0) z_{sf}^{(2)} + \kappa E \end{pmatrix}$$

where terms of order $\tau_s^2$ have been neglected. Some of the terms of order $\tau_s$ in eqn. (II.B.4) serve to only slightly modify a coefficient of order one, and these terms can adequately be approximated by one (i.e. $1 - 2\xi\tau_s \approx 1$) so that

$$\underline{g} \approx \begin{pmatrix} (\tau_s - \tau_0) z_{sf}^{(1)} - z_{sf}^{(2)} - \kappa E \tau_0 \\ z_{sf}^{(1)} + (\tau_s - \tau_0) z_{sf}^{(2)} \end{pmatrix}$$

and (II.B.4b)

$$\underline{h} \approx \begin{pmatrix} -[(\tau_s - \tau_0) + 2\xi] z_{sf}^{(1)} + z_{sf}^{(2)} + \kappa E \tau_s \\ -z_{sf}^{(1)} - (\tau_s - \tau_0) z_{sf}^{(2)} + \kappa E \end{pmatrix}$$

The elements of the perturbation matrix can now be approximated as

$$p_{11} = a_{11} - a_1\{(\tau_s - \tau_0) z_{sf}^{(1)} - z_{sf}^{(2)} - \kappa E \tau_0\} - a_2\{[(\tau_s - \tau_0) + 2\xi] z_{sf}^{(1)} - z_{sf}^{(2)} - \kappa E \tau_s\}$$

$$p_{12} = a_{12} - b_1\{(\tau_s - \tau_0) z_{sf}^{(1)} - z_{sf}^{(2)} - \kappa E \tau_0\} - b_2\{[(\tau_s - \tau_0) + 2\xi] z_{sf}^{(1)} - z_{sf}^{(2)} - \kappa E \tau_s\}$$

(II.B.5a)

$$p_{21} = a_{21} - a_1\{z_{sf}^{(1)} + (\tau_s - \tau_0) z_{sf}^{(2)}\} - a_2\{z_{sf}^{(1)} + (\tau_s - \tau_0) z_{sf}^{(2)} - \kappa E\}$$

$$p_{22} = a_{22} - b_1\{z_{sf}^{(1)} + (\tau_s - \tau_0) z_{sf}^{(2)}\} - b_2\{z_{sf}^{(1)} + (\tau_s - \tau_0) z_{sf}^{(2)} - \kappa E\}$$

If the system is being evaluated at the design point (i.e. $\underline{z}_{sf} = \underline{z}_{ss}$), then the perturbation matrix is

$$p_{11} = a_{11} + a_1 \kappa^* E^* \left( \frac{2\xi^*}{D_0'^2} - \frac{2\tau_{ss} - \tau_{00}}{2} \right) + a_2 \kappa^* E^* D_0 \left( \frac{2\xi^*}{D_0'^2} - \frac{\tau_{ss}}{2} \right)$$

$$p_{12} = a_{12} + b_1 \kappa^* E^* \left( \frac{2\xi^*}{D_0'^2} - \frac{2\tau_{ss} - \tau_{00}}{2} \right) + b_2 \kappa^* E^* D_0 \left( \frac{2\xi^*}{D_0'^2} - \frac{\tau_{ss}}{2} \right)$$

(II.B.5b)

$$p_{21} = a_{21} - a_1 \kappa^* E^* / D_0' - a_2 \kappa^* E^* D_0 / D_0'$$

$$p_{22} = a_{22} - b_1 \kappa^* E^* / D_0' - b_2 \kappa^* E^* D_0 / D_0'$$

The above equations, eqn. (II.B.5a&b), give only the approximate values for the elements of the P-matrix when the switching period is small (i.e. $\tau_{ss} \ll 1$). The asterisk is used in eqn. (II.B.5b) to indicate that the parameters are the design parameters, and to emphasize that the equation is only valid for the design point.

The stability criterion for the buck regulator is applicable to the boost regulator. If the eigenvalues of the P matrix are complex, then the stability criterion is given by eqn. (I. B. 2a).

$$\begin{aligned}\text{Det.}(P) = \text{Det.}(A) &+ a_1(a_{12}g_2 - a_{22}g_1) + a_2(a_{22}h_1 - a_{12}h_2) \\ &+ b_1(a_{21}g_1 - a_{11}g_2) + b_2(a_{11}h_2 - a_{21}h_1) \\ &+ (a_1b_2 - b_1a_2)(h_1g_2 - g_1h_2) < 1 \qquad \text{(II.B.6)}\end{aligned}$$

If the assumption is made that $\tau_s \ll 1$, then the coefficients of the feedback constants in eqn. (II. B. 6) become

$$a_{12}g_2 - a_{22}g_1 = [1 - 2\xi(\tau_s - \tau_0)]z_{sf}^{(2)} + \kappa E\tau_0$$

$$a_{22}h_1 - a_{12}h_2 = -2\xi(1 - 2\xi\tau_s)z_{sf}^{(1)} + [1 - 2\xi(\tau_s - \tau_0)]z_{sf}^{(2)} + \kappa E\tau_0$$

$$a_{21}g_1 - a_{11}g_2 = -[1 - 2\xi(\tau_s + \tau_0)]z_{sf}^{(1)}$$

$$a_{11}h_2 - a_{21}h_1 = -[1 - 2\xi(\tau_s + \tau_0)]z_{sf}^{(1)} + (1 - 2\xi\tau_s)\kappa E$$

and

$$\begin{aligned}h_1g_2 - g_1h_2 = \kappa^2E^2 \Bigg\{\tau_0 &+ [1 - 2\xi(\tau_s - \tau_0)]\frac{z_{sf}^{(2)}}{\kappa E} \\ &- 2\xi[1 - 2\xi(\tau_s + \tau_0)]\left(\frac{z_{sf}^{(1)}}{\kappa E}\right)^2\Bigg\}\end{aligned}$$

The stability criterion for complex eigenvalues when $\tau_s \ll 1$ is

$$\begin{aligned}&a_1[z_{sf}^{(2)} + \kappa E\tau_0] + a_2[-2\xi z_{sf}^{(1)} + z_{sf}^{(2)} + \kappa E\tau_0] - (b_1 + b_2)z_{sf}^{(1)} \\ &\quad + b_2\kappa E + \kappa^2E^2(a_1b_2 - b_1a_2)\left\{\left(\frac{z_{sf}^{(2)}}{\kappa E} + \tau_0\right) - 2\xi\left(\frac{z_{sf}^{(1)}}{\kappa E}\right)^2\right\} \\ &\quad < 2\xi\tau_s \qquad \text{(II.B.7a)}\end{aligned}$$

and when the eigenvalues are real it is

$$4 + a_1\{2[z_{sf}^{(2)} + \kappa E\tau_0] - (\tau_s - \tau_0)z_{sf}^{(1)}\}$$

$$+ a_2\{-[(\tau_s - \tau_0) + 4\xi]z_{sf}^{(1)} + 2z_{sf}^{(2)} + \kappa E(\tau_s + \tau_0)\}$$

$$- b_1\{2z_{sf}^{(1)} + (\tau_s - \tau_0)z_{sf}^{(2)}\} - b_2\{2[z_{sf}^{(1)} - \kappa E] + (\tau_s - \tau_0)z_{sf}^{(2)}\}$$

$$+ \kappa^2E^2(a_1b_2 - b_1a_2)\left\{\left(\frac{z_{sf}^{(2)}}{\kappa E} + \tau_0\right) - 2\xi\left(\frac{z_{sf}^{(1)}}{\kappa E}\right)^2\right\} > 0 \qquad \text{(II.B.7b)}$$

The above equation, eqn. (II. B. 7), gives the approximate local stability for a boost regulator. Since the terms multiplying the feedback constants have different signs, it is possible they could cancel, thereby making terms of order $\tau_s$ multiplied by the damping factor, $\xi$, important. If the damping factor, $\xi$, is less than or equal to the switching period, $\tau_s$, then eqn. (II. B. 7) is valid to second order (i. e. $\tau_s^2$, $\xi\tau_s$, or $\xi^2$). A certain amount of care should always be used when applying eqn. (II. B. 7).

Appendix III:

A. Regulation Analysis for a Buck-Boost Regulator

The steady-state with feedback, $\underline{z}_{sf}$, when $\tau_s \ll 1$ is given by eqn. (4.5) where the on-time and switching period are now a function of the state.

$$z_{sf}^{(1)} = \kappa E \frac{\tau_0}{(\tau_s - \tau_0)}$$

$$z_{sf}^{(2)} = \kappa E \tau_0 \left[ \frac{2\xi\tau_s}{(\tau_s - \tau_0)^2} - \frac{1}{2} \right] \tag{III.A.1a}$$

where

$$\tau_0 = \tau_{00} + \Delta\tau_0 = \tau_{00} + a_1[z_{ss}^{(1)} - z_{sf}^{(1)}] + b_1[z_{ss}^{(2)} - z_{sf}^{(2)}]$$

$$\tau_s = \tau_{ss} + \Delta\tau_s = \tau_{ss} + a_2[z_{sf}^{(1)} - z_{ss}^{(1)}] + b_2[z_{sf}^{(2)} - z_{ss}^{(2)}]$$

When the changes in the on-time and switching period are small in comparison with the difference between the switching period and on-time (i.e. $\Delta\tau_0/(\tau_s - \tau_0) \ll 1$ and $\Delta\tau_s/(\tau_s - \tau_0) \ll 1$), then eqn. (III.A.1a) can be linearized to give

$$z_{sf}^{(1)} = \frac{\kappa E D_0}{D_0'} + \frac{\kappa E}{\tau_{ss} D_0'^2} \Delta\tau_0 - \frac{\kappa E D_0}{\tau_{ss} D_0'^2} \Delta\tau_s \tag{III.A.1b}$$

$$z_{sf}^{(2)} = \kappa E D_0 \left( \frac{2\xi}{D_0'^2} - \frac{\tau_{ss}}{2} \right) + \kappa E \left[ \frac{2\xi(1+D_0)}{\tau_{ss} D_0'^3} - \frac{1}{2} \right] \Delta\tau_0 - \kappa E \frac{2\xi D_0(1+D_0)}{\tau_{ss} D_0'^3} \Delta\tau_s$$

When the control laws of eqn. (III. A. 1a) are substituted into eqn. (III. A. 1b), the solution for the steady-state with feedback can be written as

$$\underline{z}_{sf} = (I + G)^{-1} \left\{ \kappa E D_0 \begin{pmatrix} 1/D_0' \\ \dfrac{2\xi}{D_0'^{2}} - \dfrac{\tau_{ss}}{2} \end{pmatrix} + G\underline{z}_{ss} \right\}$$

where

$$G = \kappa E \begin{pmatrix} \left(\dfrac{a_1 + D_0 a_2}{\tau_{ss} D_0'^{2}}\right) & \left(\dfrac{b_1 + D_0 b_2}{\tau_{ss} D_0'^{2}}\right) \\ \left(\dfrac{2\xi(1+D_0)}{\tau_{ss} D_0'^{3}}(a_1 + D_0 a_2) - \dfrac{a_1}{2}\right) & \left(\dfrac{2\xi(1+D_0)}{\tau_{ss} D_0'^{3}}(b_1 + D_0 b_2) - \dfrac{b_1}{2}\right) \end{pmatrix}$$

now

$$(I + G)^{-1} = \frac{1}{[(1+g_{11})(1+g_{22}) - g_{12}g_{21}]} \begin{pmatrix} (1+g_{22}) & -g_{12} \\ -g_{21} & (1+g_{11}) \end{pmatrix}$$

with

$$(1+g_{22})\kappa E D_0/D_0' - g_{12}\kappa E D_0 \left(\frac{2\xi}{D_0'^{2}} - \frac{\tau_{ss}}{2}\right) + g_{12} z_{ss}^{(2)} =$$

$$\frac{\kappa E D_0}{D_0'} \left[ 1 + \kappa E D_0 \frac{2\xi}{\tau_{ss} D_0'^{3}} (b_1 + D_0 b_2) + \frac{\kappa E D_0}{D_0'} \left(\frac{b_1 + b_2}{2}\right) + \frac{(b_1 + D_0 b_2)}{\tau_{ss} D_0 D_0'} z_{ss}^{(2)} \right]$$

and

$$-g_{21}\kappa E D_0/D_0' + (1+g_{11})\kappa E D_0\left(\frac{2\xi}{D_0'^2} - \frac{\tau_{ss}}{2}\right) + g_{21}z_{ss}^{(1)} =$$

$$\kappa E D_0\left(\frac{2\xi}{D_0'^2} - \frac{\tau_{ss}}{2}\right) - \kappa^2E^2\,\frac{2\xi D_0^2}{\tau_{ss}D_0'^4}(a_1+D_0a_2) - \frac{\kappa^2E^2D_0^2}{2D_0'^2}(a_1+a_2)$$

$$+ \kappa E\left[\frac{2\xi(1+D_0)}{\tau_{ss}D_0'^3}(a_1+D_0a_2) - \frac{a_1}{2}\right]z_{ss}^{(1)}$$

The approximate steady-state with feedback, $\underline{z}_{sf}$, is then

$$z_{sf}^{(1)} = \left\{z_{ss}^{(1)} + \frac{\kappa E D_0/D_0'}{S}\left[1 + \kappa E D_0\,\frac{2\xi}{\tau_{ss}D_0'^3}(b_1+D_0b_2)\right.\right.$$

$$\left.\left. + \frac{\kappa E D_0}{D_0'}\left(\frac{b_1+b_2}{2}\right) + \frac{(b_1+D_0b_2)}{\tau_{ss}D_0D_0'}\,z_{ss}^{(2)}\right]\right\} \Big/ \left\{1 + \right.$$

$$\left.\frac{1}{S}\left[1 + \kappa E\left(\frac{2\xi(1+D_0)}{\tau_{ss}D_0'^3}(b_1+D_0b_2) - \frac{b_1}{2}\right)\right]\right\}$$

$$z_{sf}^{(2)} = \left\{ S1 z_{ss}^{(2)} + \kappa E D_0 \left( \frac{2\xi}{D_0'^{2}} - \frac{\tau_{ss}}{2} \right) - \kappa^2 E^2 \frac{2\xi D_0^2}{\tau_{ss} D_0'^{4}} (a_1 + D_0 a_2) \right.$$

$$\left. - \frac{\kappa^2 E^2 D_0^2}{2 D_0'^{2}} (a_1 + a_2 \right\} \Big/ \left\{ S1 + \left[ 1 + \kappa E \left( \frac{a_1 + D_0 a_2}{\tau_{ss} D_0'^{2}} \right) \right] \right\}$$

$$+ \frac{\kappa E \left[ \dfrac{2\xi(1+D_0)}{\tau_{ss} D_0'^{3}} (a_1 + D_0 a_2) - \dfrac{a_1}{2} \right] z_{ss}^{(1)}}{S1 + \left[ 1 + \kappa E \left( \dfrac{a_1 + D_0 a_2}{\tau_{ss} D_0'^{2}} \right) \right]} \qquad \text{(III.A.2)}$$

$$S = \frac{\kappa E}{\tau_{ss} D_0'^{2}} (a_1 + D_0 a_2) + \frac{\kappa^2 E^2 D_0}{2\tau_{ss} D_0'^{2}} (a_1 b_2 - b_1 a_2)$$

$$S1 = \kappa E \left[ \frac{2\xi(1+D_0)}{\tau_{ss} D_0'^{3}} (b_1 + D_0 b_2) - \frac{b_1}{2} \right] + \frac{\kappa^2 E^2 D_0}{2\tau_{ss} D_0'^{2}} (a_1 b_2 - b_1 a_2)$$

The closed loop gain, S, of the buck-boost regulator is identical to the closed loop gain of the boost regulator.

Appendix III:

B. Stability Analysis for a Buck-Boost Regulator

The recursion formula for the boost regulator, eqn. (4.3), is

$$\underline{z}_{n+1} = A\underline{z}_n + c \qquad \text{(III.B.1)}$$

where

$$\underline{c} = \kappa E \tau_0 \begin{pmatrix} y_{12}(\tau_s - \tau_0) \\ y_{11}(\tau_s - \tau_0) \end{pmatrix}$$

and the matrix A is the same one used for the boost regulator, see eqn. (II. B. 1). The variation of eqn. (III. B. 1) is

$$\delta\underline{z}_{n+1} = A\delta\underline{z}_n + \left(\frac{\partial A}{\partial \tau_0}\underline{z}_{sf} + \frac{\partial \underline{c}}{\partial \tau_0}\right)\frac{\partial A}{\partial \underline{z}_n} + \left(\frac{\partial A}{\partial \tau_s}\underline{z}_{sf} + \frac{\partial \underline{c}}{\partial \tau_s}\right)\frac{\partial \tau_s}{\partial \underline{z}_n}\,\delta\underline{z}_n \qquad \text{(III.B.2)}$$

where

$$\frac{\partial A}{\partial \tau_0} = \begin{pmatrix} e^{-2\xi_0\tau_0}\, y_{12}(\tau_s - \tau_0) & -y_{22}(\tau_s - \tau_0) \\ e^{-2\xi_0\tau_0}\, y_{11}(\tau_s - \tau_0) & y_{12}(\tau_s - \tau_0) \end{pmatrix}$$

$$\frac{\partial A}{\partial \tau_s} = \begin{pmatrix} -e^{-2\xi_0\tau_0}\,[y_{12}(\tau_s - \tau_0) + 2\xi_0 y_{22}(\tau_s - \tau_0)] & y_{22}(\tau_s - \tau_0) \\ -e^{-2\xi_0\tau_0}\, y_{22}(\tau_s - \tau_0) & y_{21}(\tau_s - \tau_0) \end{pmatrix}$$

$$\frac{\partial \underline{c}}{\partial \tau_0} = \kappa E \begin{pmatrix} y_{12}(\tau_s - \tau_0) - \tau_0 y_{22}(\tau_s - \tau_0) \\ y_{11}(\tau_s - \tau_0) + \tau_0 y_{12}(\tau_s - \tau_0) \end{pmatrix}$$

and

$$\frac{\partial \underline{c}}{\partial \tau_s} = \kappa E \tau_0 \begin{pmatrix} y_{22}(\tau_s - \tau_0) \\ \\ y_{21}(\tau_s - \tau_0) \end{pmatrix}$$

The perturbation matrix is then

$$P = \begin{pmatrix} \left\{ a_{11} + g_1 \frac{\partial \tau_0}{\partial z_n^{(1)}} + h_1 \frac{\partial \tau_s}{\partial z_n^{(1)}} \right\} & \left\{ a_{12} + g_1 \frac{\partial \tau_s}{\partial z_n^{(2)}} + g_1 \frac{\partial \tau_s}{\partial z_n^{(2)}} \right\} \\ \left\{ a_{21} + g_2 \frac{\partial \tau_0}{\partial z_n^{(1)}} + h_2 \frac{\partial \tau_s}{\partial z_n^{(1)}} \right\} & \left\{ a_{22} + g_2 \frac{\partial \tau_s}{\partial z_n^{(2)}} + h_2 \frac{\partial \tau_s}{\partial z_n^{(2)}} \right\} \end{pmatrix}$$

where

$$\underline{g} = \begin{pmatrix} g_1 \\ g_2 \end{pmatrix} = \left( \frac{\partial A}{\partial \tau_0} \underline{z}_{sf} + \frac{\partial \underline{c}}{\partial \tau_0} \right)$$

$$\underline{h} = \begin{pmatrix} h_1 \\ h_2 \end{pmatrix} = \left( \frac{\partial A}{\partial \tau_s} \underline{z}_{sf} + \frac{\partial \underline{c}}{\partial \tau_s} \right)$$

After the feedback constants are substituted for the partials of the on-time and switching period with respect to the state, $\underline{z}_n$, the perturbation matrix for the buck-boost regulator assumes the same form as the boost regulator, see eqn. (II. B. 3),

$$P = \begin{pmatrix} \{a_{11} - a_1 g_1 + a_2 h_1\} & \{a_{12} - b_1 g_1 + b_2 h_1\} \\ \{a_{21} - a_1 g_2 + a_2 h_2\} & \{a_{22} - b_1 g_2 + b_2 h_2\} \end{pmatrix} \qquad \text{(III.B.3a)}$$

where

$$\underline{g} = \begin{pmatrix} e^{-2\xi_0\tau_0} y_{12}(\tau_s-\tau_0) z_{sf}^{(1)} - y_{22}(\tau_s-\tau_0) z_{sf}^{(2)} + \kappa E[y_{12}(\tau_s-\tau_0) - \tau_0 y_{22}(\tau_s-\tau_0)] \\ e^{-2\xi_0\tau_0} y_{11}(\tau_s-\tau_0) z_{sf}^{(1)} + y_{12}(\tau_s-\tau_0) z_{sf}^{(2)} + \kappa E[y_{11}(\tau_s-\tau_0) + \tau_0 y_{12}(\tau_s-\tau_0)] \end{pmatrix}$$

$$\underline{h} = \begin{pmatrix} -e^{-2\xi_0\tau_0}[y_{12}(\tau_s-\tau_0) + 2\xi_0 y_{22}(\tau_s-\tau_0)] z_{sf}^{(1)} + y_{22}(\tau_s-\tau_0) z_{sf}^{(2)} + \kappa E\tau_0 y_{22}(\tau_s-\tau_0) \\ -e^{-2\xi_0\tau_0} y_{22}(\tau_s-\tau_0) \quad z_{sf}^{(1)} + y_{21}(\tau_s-\tau_0) z_{sf}^{(2)} + \kappa E\tau_0 y_{21}(\tau_s-\tau_0) \end{pmatrix}$$

It is convenient, as it was with the boost regulator, see eqn. (II.B.4b), to make the assumption that $\tau_s \ll 1$ in order to simplify eqn. (III.B.3a); so

$$\underline{g} \approx \begin{pmatrix} (\tau_s - \tau_0) z_{sf}^{(1)} - [1 - 2\xi(\tau_s - \tau_0)] z_{sf}^{(2)} + \kappa E(\tau_s - 2\tau_0) \\ (1 - 2\xi\tau_0) z_{sf}^{(1)} + (\tau_s - \tau_0) z_{sf}^{(2)} + \kappa E \end{pmatrix}$$

and (III.B.4)

$$\underline{h} \approx \begin{pmatrix} -[(\tau_s - \tau_0) + 2\xi(1 - 2\xi\tau_s)] z_{sf}^{(1)} + [1 - 2\xi(\tau_s - \tau_0)] z_{sf}^{(2)} + \kappa E\tau_0 \\ -(1 - 2\xi\tau_s) z_{sf}^{(1)} - (\tau_s - \tau_0) z_{sf}^{(2)} \end{pmatrix}$$

The elements of the perturbation matrix can be approximated as

$$p_{11} = a_{11} - a_1\{(\tau_s - \tau_0) z_{sf}^{(1)} - z_{sf}^{(2)} + \kappa E(\tau_s - 2\tau_0)\}$$
$$- a_2\{[(\tau_s - \tau_0) + 2\xi] z_{sf}^{(1)} - z_{sf}^{(2)} - \kappa E\tau_0\}$$

$$p_{12} = a_{12} - b_1\{(\tau_s - \tau_0) z_{sf}^{(1)} - z_{sf}^{(2)} + \kappa E(\tau_s - 2\tau_0)\}$$
$$- b_2\{[(\tau_s - \tau_0) + 2\xi] z_{sf}^{(1)} - z_{sf}^{(2)} - \kappa E\tau_0\}$$

(III.B.5a)

$$p_{21} = a_{21} - a_1\{z_{sf}^{(1)} + (\tau_s - \tau_0)z_{sf}^{(2)} + \kappa E\} - a_2\{z_{sf}^{(1)} + (\tau_s - \tau_0)z_{sf}^{(2)}\}$$

$$p_{22} = a_{22} - b_1\{z_{sf}^{(1)} + (\tau_s - \tau_0)z_{sf}^{(2)} + \kappa E\} - b_2\{z_{sf}^{(1)} + (\tau_s - \tau_0)z_{sf}^{(2)}\}$$

(III.B.5a)
cont.

When the system is evaluated at the design point (i.e., $\underline{z}_{sf} = \underline{z}_{ss}$), then the perturbation matrix becomes

$$p_{11} = a_{11} + a_1\kappa^*E^*\left(\frac{\frac{2\xi^*D_0}{2}}{D_0'} - \frac{(2-D_0)\tau_{ss}}{2}\right) + a_2\kappa^*E^*D_0\left(\frac{\frac{2\xi^*D_0}{2}}{D_0'} - \frac{\tau_{ss}}{2}\right)$$

$$p_{12} = a_{12} + b_1\kappa^*E^*\left(\frac{\frac{2\xi^*D_0}{2}}{D_0'} - \frac{(2-D_0)\tau_{ss}}{2}\right) + b_2\kappa^*E^*D_0\left(\frac{\frac{2\xi^*D_0}{2}}{D_0'} - \frac{\tau_{ss}}{2}\right)$$

(III.B.5b)

$$p_{21} = a_{21} - a_1\kappa^*E^*/D_0' - a_2\kappa^*E^*D_0/D_0'$$

$$p_{22} = a_{22} - b_1\kappa^*E^*/D_0' - b_2\kappa^*E^*D_0/D_0'$$

The asterisk in eqn. (III. B. 5b) indicates that the input parameters are the design parameters.

The stability criterion of the buck-boost regulator when the eigenvalues are complex is given by eqn. (II. B. 6)

$$\text{Det.}(P) = \text{Det.}(A) + a_1(a_{12}g_2 - a_{22}g_1) + a_2(a_{22}h_1 - a_{12}h_2)$$
$$+ b_1(a_{21}g_1 - a_{11}g_2) + b_2(a_{11}h_2 - a_{21}h_1)$$
$$+ (a_1b_2 - b_1a_2)(h_1g_2 - g_1h_2) < 1 \qquad \text{(III.B.6)}$$

When the assumption is made that $\tau_s \ll 1$, the A matrix is approximated by eqn. (II.B.4a), and the vectors, g and h, are approximated by eqn. (III.B.4) so that

$$a_{12}g_2 - a_{22}g_1 = [1-2\xi(\tau_s-\tau_0)]z_{sf}^{(2)} + \kappa E\tau_0$$

$$a_{22}h_1 - a_{12}h_2 = -2\xi(1-2\xi\tau_s)z_{sf}^{(1)} + [1-2\xi(\tau_s-\tau_0)]z_{sf}^{(2)} + \kappa E\tau_0$$

$$a_{21}g_1 - a_{11}g_2 = -[1-2\xi(\tau_s+\tau_0)]z_{sf}^{(1)} - (1-2\xi\tau_s)\kappa E$$

$$a_{11}h_2 - a_{21}h_1 = -[1-2\xi(\tau_s+\tau_0)]z_{sf}^{(1)}$$

$$h_1g_2 - g_1h_2 = \kappa^2E^2\left\{\tau_0 + [1-2\xi(\tau_s-\tau_0)]\frac{z_{sf}^{(2)}}{\kappa E} - 2\xi\frac{z_{sf}^{(1)}}{\kappa E}\left[(1-2\xi\tau_s) + [1-2\xi(\tau_s+\tau_0)]\frac{z_{sf}^{(1)}}{\kappa E}\right]\right\}$$

The stability criterion for the buck-boost regulator when the eigenvalues of the P-matrix are complex is

$$a_1[z_{sf}^{(2)} + \kappa E\tau_0] + a_2[-2\xi z_{sf}^{(1)} + z_{sf}^{(2)} + \kappa E\tau_0] - (b_1+b_2)z_{sf}^{(1)} - b_1\kappa E + \kappa^2E^2(a_1b_2-b_1a_2)\left\{\left(\frac{z_{sf}^{(2)}}{\kappa E}+\tau_0\right) - 2\xi\frac{z_{sf}^{(1)}}{\kappa E}\left(1+\frac{z_{sf}^{(1)}}{\kappa E}\right)\right\} < 2\xi\tau_s \qquad \text{(III.B.7a)}$$

and when the eigenvalues are real the stability criterion is

$$4 + a_1[2z_{sf}^{(2)} - (\tau_s - \tau_0)z_{sf}^{(1)} - \kappa E(\tau_s - 3\tau_0)]$$

$$+ a_2\{-[(\tau_s - \tau_0) + 4\xi]\, z_{sf}^{(1)} + 2z_{sf}^{(2)} + 2\kappa E\tau_0\}$$

$$- b_1[2z_{sf}^{(1)} + (\tau_s - \tau_0)z_{sf}^{(2)} + 2\kappa E] - b_2[2z_{sf}^{(1)} + (\tau_s - \tau_0)z_{sf}^{(2)}]$$

$$+ \kappa^2E^2(a_1b_2 - b_1a_2)\left\{\left(\frac{z_{sf}^{(2)}}{\kappa E} + \tau_0\right) - 2\xi\,\frac{z_{sf}^{(1)}}{\kappa E}\left(1 + \frac{z_{sf}^{(1)}}{\kappa E}\right)\right\}$$

$$> 0 \qquad \text{(III.B.7b)}$$

The local stability of the buck-boost regulator is given by eqn. (III. B. 7). The form of the stability equation for a buck-boost regulator is very similar to the stability criterion for the boost regulator, and like the boost regulator, care should be taken in applying eqn. (III. B. 7).