TRANSIENT RADIATION FROM COAXIAL WAVEGUIDE AND CYLINDRICAL MONOPOLE ANTENNAS

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Norval L. Broome

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ABSTRACT

In this work the coaxial waveguide antenna is treated by the Wiener-Hopf technique and the transient radiation from a cylindrical monopole is developed in the light of the rigorous results obtained from the Wiener-Hopf analysis. Analytic expressions are derived for (1) the electromagnetic fields in the feed line and (2) the far zone radiation fields of the coaxial waveguide antenna, with time harmonic excitation voltage. Complete characterization of the transient behavior is also found for (1) the fields interior to the feed line and (2) the radiated fields for excitation voltages arbitrary in their time dependence to the extent that $kb, ka \ll 1$. This corresponds to the case of a thin antenna and excitation voltage with a non-zero rise time, specifically chosen so that frequencies violating the restriction $kb, ka \ll 1$ are negligible.

The transient radiation from the cylindrical monopole is developed in a closed analytic form which is relatively easy to interpret and apply. The expressions found offer an alternative to transient analysis by conventional methods requiring numerical techniques involving extensive computer calculations. They are also the basis for an uncomplicated procedure to synthesize a desired behavior of the transient radiation from cylindrical monopole antennas.

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1. INTRODUCTION

This dissertation presents an analysis of the coaxial waveguide antenna by the Wiener-Hopf technique and the determination of transient radiation from the cylindrical monopole antenna. This work is the first successful treatment of a semi-infinite coaxial structure by the method of Wiener-Hopf and the first rigorous determination of transient radiation emanating from an open ended semi-infinite waveguiding structure. In addition, a new and simple method for calculating the transient electromagnetic radiation of a cylindrical monopole antenna is presented.

The study of electromagnetic boundary-value problems by the Wiener-Hopf technique began in the early 1940's. Among the most notable of the early workers were Schwinger [1] and Copson [2]. The early applications of this powerful method were in the rigorous solution for the fields within a bifurcated parallel plate waveguide and diffraction of plane waves by a conducting half-plane. The method was later used to solve the problem of radiation impinging on or emerging from the open end of an unflanged circular pipe and a pair of parallel plates. Much of the early history of application of the Weiner-Hopf method is detailed by Bouwkamp [3] and an authoritative account of the method was published in 1958 by Noble [4]. In this work the integral equation approach of the early workers is not stressed. A more direct simple method due to Jones [5] is emphasized. It is the application of Wiener-Hopf technique due to Jones that will be employed within this report.

There have been several recent investigations of the transient behavior of antennas [6-15]. The focus of many of these investigations is on the time behavior of the radiated electromagnetic field. References [6,8] and [14] present theoretical as well as experimental results. In most of the prior research efforts, the problem is formulated in terms of circuit theory and the antenna is treated as a lumped circuit element. The feedline and the exciting sources are replaced by their Thevenin or Norton equivalents. The formulation of the problem in terms of circuit theory yields complex frequency domain fields which must be Fourier transformed if the time behavior is of interest. Such calculations require numerical methods due to their extreme complexity.

The work of Schmitt, Harrison and Williams [6] is a determination of the transient radiation and reception performance of a thin, finite length cylindrical monopole over a perfectly conducting infinite ground plane with a coaxial feed. In the circuit analysis formulation (frequency domain) the antenna was represented by an impedance and the radiated field related to the voltage across the antenna impedance through a proportionality factor, called effective height. The time dependent behavior is obtained by Fourier transforming the frequency domain field. A similar procedure was used to obtain the properties of the antenna in reception; the induced voltage on the receiving antenna was related to the incident electric field intensity through the effective height. The theoretical work of Schmitt, Harrison and Williams was supplemented in 1971 by Abo-Zena and Beam [9] to include the time behavior of near zone field intensities and in 1970 by Palciauskas and Beam [10] to include the far zone time behavior at latitudinal angles other than 90^{o}, which was the only angle considered by Schmitt et al.

King and Harrison [8] formulated a frequency averaged reflection coefficient to theoretically obtain the reflected electromagnetic fields on the coaxial feed line of an infinitely long cylindrical antenna or for the time interval before the excitation pulse can reach the tip of a finite length antenna. Use of frequency averaging is necessitated by the fact that the antenna input admittance is frequency dependent, which constrains the conventional transmission line reflection coefficient to be frequency dependent.

The transient current on a circular tubular infinitely long antenna that is excited by a voltage which is a step function of time was determined in 1961 by T. Wu. He later (1969) furnished a corrected analysis for this current in Reference [11]. An earlier work (1960) by Brundell [12] treated the more general problem of determining the space and time behavior of the electromagnetic field for such an antenna. This work was also supplemented in 1970 by Latham and Lee [13] who found the early and late time behavior of the field radiation from a hollow infinite cylindrical antenna excited by a voltage that is a step function of time.

Lamensdorf [14] determined that the time behavior of the coaxial cone antenna in reception is primarily a time derivative of the incident electric field intensity falling on the aperture. This work also contains a heuristic discussion of the time dependent electric field radiated by a cone antenna. Chang [15] analyzed the transient reception characteristics of the annular slot antenna using the circuit theory approach. References [14] and [15] are recent works, 1970 and 1971 respectively, and have application as part of a receiving system for measuring scattered impulse responses of various objects or as a receiving antenna on an aircraft or guided missile.

In the systematic study of radiation from the coaxial aperture antenna which follows, analytic expressions for the electromagnetic fields of the coaxial aperture antenna are given. These expressions account for waves traveling in both the forward and reverse directions on the feedline and the radiation fields for harmonic time dependent excitation. Complete characterization of the transient behavior is found for the fields interior to the feedline and the radiated fields, for excitation voltages which are arbitrary to the extent that $kb, ka \ll 1$. This corresponds to the case of a thin antenna and excitation voltages with non-zero rise times, specifically chosen so that frequencies which violate the restriction $kb, ka \ll 1$ are negligible.

The annular slot antenna is considered to be a special case of a cylindrical monopole antenna ($h = 0$). The fields radiated from the annular slot antenna are approximated by well known methods in which the incident field on the aperture is considered to be the aperture field. The radiated fields so determined are interpreted in light of the rigorous results obtained from the coaxial waveguide antenna and a model is obtained for a cylindrical monopole antenna in which the height of the radiating element is greater than zero ($h > 0$). The transient radiation field intensities are obtained from this antenna model. The determination of the transient radiation through the procedures developed in this research offer an alternative to computing transient behavior by numerical methods where extensive machine calculations are required. The procedures reported on herein can also be used as the basis of an uncomplicated synthesis procedure to produce a desired time behavior in fields radiated from the coaxial waveguide and the cylindrical monopole antennas.

The results of this research should prove to be very useful in the areas of 1) determination of antenna properties (driving point impedance and radiation pattern) through time domain measurements [16], 2) identification of and discrimination between conducting bodies through radiation scattering from their surfaces and 3) the calibration of equipment designed to measure the effectiveness of shields designed to protect equipment or installations from intense, short rise time electromagnetic pulses. In each of the above areas a precise knowledge of the time-behavior of the radiated electromagnetic field is required. Additionally the ability to synthesize a particular time behavior in the radiated field is highly desirable and very important.

2. THE COAXIAL WAVEGUIDE: AN APPLICATION OF THE WIENER-HOPF TECHNIQUE

In this chapter the frequency (spatial) domain electromagnetic intensities for an open ended coaxial waveguide will be derived. This will be done using the methods of Wiener and Hopf [4]. Use of this technique will provide a determination of transverse electromagnetic mode (TEM) as well as higher order mode intensities within the structure.

We take the coaxial waveguide structure to be that of Figure 2-1. The inner conductor is a hollow tube. Both outer and inner conductors are assumed to be semi-infinite in length, infinitesimally thin and perfectly conducting. The cross section is circular providing for a coaxial annular region between the conductors. We also assume that the waveguide is embedded in a homogeneous, isotropic, dielectric medium having the constitutive parameters of free space.

Specification of the Problem

We begin the specification of the problem by listing the Maxwell equations:

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\nabla \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t}$$

$$\nabla \cdot \vec{D} = \rho$$

$$\nabla \cdot \vec{B} = 0 \qquad (2.1)$$

The fields which satisfy the above equations must also satisfy the constitutive relations:

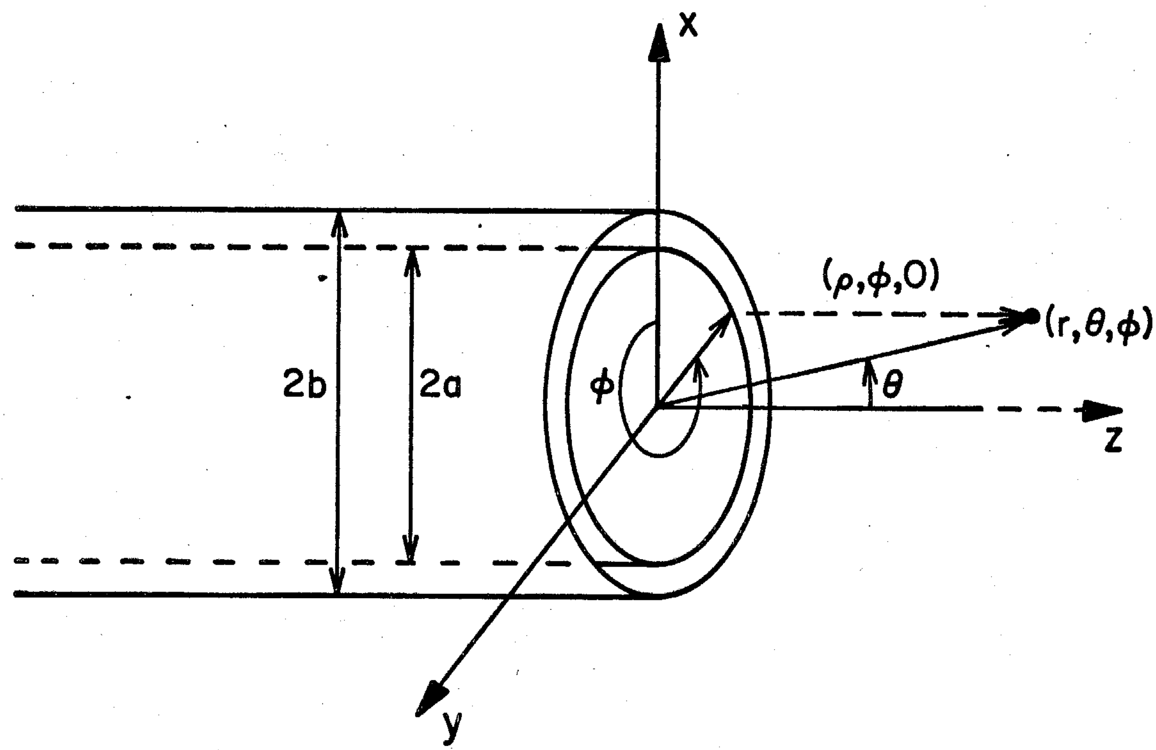


Figure 2-1. The coaxial waveguide antenna

$$\vec{B} = \mu_o \vec{H} \qquad , \qquad \vec{D} = \varepsilon_o \vec{E}$$

where

$$\frac{1}{c^2} = \mu_o \varepsilon_o$$

Under the assumption that the coaxial waveguide has been excited in TEM mode, there is a traveling electromagnetic disturbance proceeding in the positive z direction. The time origin is taken to be the point in time when the leading edge of the TEM mode exciting field is incident upon the aperture located in the $z = 0$ plane.

The specific method used to initially excite the coaxial structure and the field distribution prior to the time origin are not of interest in this work.

It is well known that the TEM mode field of a coaxial waveguide is independent of angular variation [17, p.326] in the cylindrical coordinate system. The coaxial structure also does not introduce any boundary conditions which have angular dependence. From these observations we conclude that the scattered field must also be characterized by the same symmetry (i.e., independent of angular variation in cylindrical coordinate system).

In cylindrical coordinates the field components can be written as

$$\vec{E} = E_\rho \vec{a}_\rho + E_\phi \vec{a}_\phi + E_z \vec{a}_z$$

and

$$\vec{H} = H_\rho \vec{a}_\rho + H_\phi \vec{a}_\phi + H_z \vec{a}_z$$

where $\vec{a}_\rho, \vec{a}_\phi, \vec{a}_z$ is the triad of unit vectors along the coordinate axes.

We now Fourier transform the Maxwell equations to change from a time domain to a frequency domain representation. The Fourier transform pair [18] is defined as follows:

$$F(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)\, e^{i\omega t}\, dt$$

$$f(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(\omega)\, e^{-i\omega t}\, d\omega$$

When the first two Maxwell equations are multiplied by $\frac{1}{\sqrt{2\pi}} e^{i\omega t}$ and operated on by the integral operator, the results are:

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \nabla \times \vec{E}\, e^{i\omega t}\, dt = -\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{\partial B}{\partial t}\, e^{i\omega t}\, dt$$

and

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \nabla \times \vec{H}\, e^{i\omega t}\, dt = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} [\vec{J} + \frac{\partial \vec{D}}{\partial t}]\, e^{i\omega t}\, dt$$

Interchanging the curl operator and the integral sign on the left hand side and integrating by parts on the right hand side yields equations in the frequency domain. To form these results, use has been made of the fact that the field intensities must be zero as $|t| \to \infty$.

$$\nabla \times \vec{E}(\omega) = + i\omega\, \vec{B}(\omega)$$

$$\nabla \times \vec{H}(\omega) = \vec{J}(\omega) - i\omega\, \vec{D}(\omega)$$

It similarly follows that

$$\nabla \cdot \vec{D}(\omega) = \rho(\omega)$$

$$\nabla \cdot \vec{B}(\omega) = 0$$

$$\vec{B}(\omega) = \mu_0 \vec{H}(\omega) \quad \text{and} \quad \vec{D}(\omega) = \varepsilon_0 \vec{E}(\omega)$$

When the field components in cylindrical coordinates are substituted into the frequency domain Maxwell equations for a source free region (i.e., $\vec{J}(\omega)$ is set to zero), we obtain the following relations between the field components:

$$i\omega B_\rho = -\frac{\partial E_\phi}{\partial z} \tag{2.1.1}$$

$$i\omega B_z = \frac{\partial E_\phi}{\rho} + \frac{\partial E_\phi}{\partial \rho} \tag{2.1.2}$$

$$i\omega B_\phi = \frac{\partial E_\rho}{\partial z} - \frac{\partial E_z}{\partial \rho} \tag{2.1.3}$$

and

$$-i\omega D_\rho = -\frac{\partial}{\partial z} H_\phi \tag{2.1.4}$$

$$-i\omega D_z = \frac{H_\phi}{\rho} + \frac{\partial H_\phi}{\partial \rho} \tag{2.1.5}$$

$$-i\omega D_\phi = \frac{\partial H_\rho}{\partial z} - \frac{\partial H_z}{\partial \rho} \tag{2.1.6}$$

By using the constitutive relations and combining equations (2.1.3), (2.1.4) and (2.1.5), we derive the scalar wave equation for H_ϕ,

$$i\omega \mu_0 H_\phi = \frac{\partial}{\partial z}\left[\frac{1}{i\omega\varepsilon_0}\frac{\partial}{\partial z} H_\phi\right] - \frac{\partial}{\partial \rho}\left[-\frac{1}{i\omega\varepsilon_0}\left(\frac{H_\phi}{\rho} + \frac{\partial H_\phi}{\partial \rho}\right)\right]$$

This equation, upon simplification, becomes:

$$\left[\frac{\partial^2}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial}{\partial\rho} - \frac{1}{\rho^2} + \frac{\partial^2}{\partial z^2} + \frac{\omega^2}{c^2}\right] H_\phi(\omega) = 0 \qquad (2.2)$$

where $\mu_0\varepsilon_0 = \frac{1}{c^2}$. Similarly equations (2.1.1), (2.1.2) and (2.1.6) produce

$$\left[\frac{\partial^2}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial}{\partial\rho} - \frac{1}{\rho^2} + \frac{\partial^2}{\partial z^2} + \frac{\omega^2}{c^2}\right] E_\phi(\omega) = 0 \qquad (2.3)$$

From the foregoing equations it is apparent that the solution of (2.2) and (2.3) for H_ϕ and E_ϕ is enough to completely determine all field components. The work incorporated in Appendix A shows that E_ϕ must be zero for the coaxial waveguide, therefore the only field intensity components present are H_ϕ, E_ρ and E_z . The rest are identically zero. The analysis has been reduced to solving the boundary value problem

$$\left[\frac{\partial^2}{\partial z^2} + \frac{\partial^2}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial}{\partial\rho} - \frac{1}{\rho^2} + k^2\right] H_\phi(\rho,z,\omega) = 0 \qquad (2.4)$$

$$H_\phi(0,z,\omega) = 0 \; ; \qquad -\infty < z < \infty \qquad (2.4.1)$$

$$E_z(a,z,\omega) = E_z(b,z,\omega) = 0 \; ; \quad z \leq 0 \qquad (2.4.2)$$

The Sommerfeld radiation condition,

$$\lim_{r\to\infty} r\left(\frac{\partial H_\phi}{\partial r} - ik\,H_\phi\right) = 0$$ must also be satisfied.

At this point it is useful to point out that in the above equations

$$k^2 = \frac{\omega^2}{c^2} \qquad (2.4.3)$$

is a scalar quantity and equation (2.1.5) can be written as

$$-i\omega\varepsilon_0 E_z = \frac{1}{\rho}\frac{\partial}{\partial\rho}[\rho\, H_\phi] \tag{2.5}$$

This will be of use later.

The well known expressions for the TEM mode fields for the coaxial waveguide [17] will be given here to set the notation. We take the field intensities incident on the aperture from the negative z direction to be

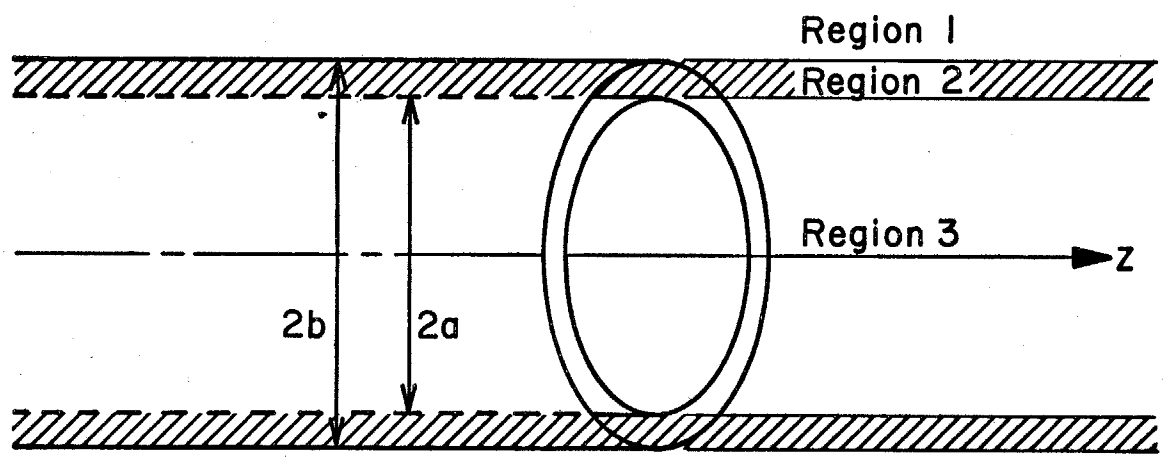
$$E^i_\rho = \frac{v(\omega)}{\ell n\ b/a}\ \frac{e^{ikz}}{\rho} \tag{2.6}$$

$$H^i_\phi = \sqrt{\frac{\varepsilon_0}{\mu_0}}\ \frac{v(\omega)}{\ell n\ b/a}\ \frac{e^{ikz}}{\rho} = M(\omega)\ \frac{e^{ikz}}{\rho} \tag{2.7}$$

$v(\omega)$ is the frequency domain voltage between the inner and outer conductors.

The space containing the waveguide is divided into three regions, portrayed in Figure 2-2. Equation (2.4) applies to each of these regions. The field satisfying this equation in region one is designated $H_{\phi_1}(\rho,z,\omega)$. Similarly we have $H_{\phi_2}(\rho,z,\omega)$ for region two and $H_{\phi_3}(\rho,z,\omega)$ for region three. The equation for each of the regions is a partial differential equation in the variables ρ and z . By Fourier transforming in the spatial dimension z , an ordinary differential equation in the variable ρ is obtained.

Following the notation of [4] , the spatial Fourier transform pair is defined as follows:



Region 1	$b \leq \rho$
Region 2	$a \leq \rho \leq b$
Region 3	$\rho \leq a$

Figure 2-2. Division of the space surrounding the coaxial waveguide antenna into regions

$$F(\alpha) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(z)\, e^{i\alpha z}\, dz \tag{2.8}$$

$$f(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(\alpha)\, e^{-i\alpha z}\, d\alpha \tag{2.8.1}$$

The "plus" and "minus" functions are defined as

$$F^{+}(\alpha) = \frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} f(z)\, e^{i\alpha z}\, dz \tag{2.8.2}$$

$$F^{-}(\alpha) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{0} f(z)\, e^{i\alpha z}\, dz \tag{2.8.3}$$

where

$$F(\alpha) = F^{+}(\alpha) + F^{-}(\alpha) \tag{2.8.4}$$

$\alpha = \sigma + i\tau$ is complex .

Using well known theorems involving Fourier transforms in the complex plane [4], the regions of analyticity of $F^{+}(\alpha)$ and $F^{-}(\alpha)$ will be found. Suppose that we are given $\tau_{-} < \tau < \tau_{+}$, all real numbers. In addition suppose that

$$|f(z)| < A \exp(\tau_{-} z) \quad \text{as} \quad z \to \infty$$

where A is a constant. Then

$$|F^{+}(\alpha)| < \frac{1}{\sqrt{2\pi}} \left| \int_{0}^{M} f(z)\, e^{i\alpha z}\, dz \right| + \frac{1}{\sqrt{2\pi}} \int_{M}^{\infty} A\, e^{(i\alpha + \tau_{-})z}\, dz$$

where $M < \infty$ is chosen large enough for the inequality to be valid.

The last integral of the above expression converges if and only if

$$e^{(i\alpha+\tau_-)z} \to 0 \qquad \text{as} \quad z \to \infty$$

or

$$e^{(i\sigma-\tau+\tau_-)z} \to 0 \qquad \text{as} \quad z \to \infty$$

This last expression is valid provided

$$\tau > \tau_-$$

By similar reasoning, it can be shown that if $|f(z)| < B \exp(\tau_+ z)$ as $z \to -\infty$ (B constant) then $|F^-(\alpha)|$ is convergent provided

$$\tau < \tau_+$$

Therefore $F^+(\alpha)$ and $F^-(\alpha)$ are convergent in the strip $\tau_- < \tau < \tau_+$ of the α plane.

We also note that if $|F^+(\alpha)|$ and $|F^-(\alpha)|$ are convergent in a region of the complex plane, then the magnitude of nth ordered derivative ($n < \infty$) of $F^+(\alpha)$ and $F^-(\alpha)$ with respect to α also exists in that region. In fact such functions are nth order continuously differentiable in the region.

$$\left|\frac{d^n}{d\alpha^n}\left(F^+(\alpha)\right)\right| = \left|\frac{d^n}{d\alpha^n}|F^+(\alpha)|\, e^{i\phi(\alpha)}\right| = \left|\frac{(i)^n}{\sqrt{2\pi}}\int_0^\infty z^n f(z) e^{i\alpha z}\, dz\right|$$

$$< \left|\frac{i^n}{\sqrt{2\pi}}\int_0^M z^n f(z)\, e^{i\alpha z}\, dz\right| + \frac{1}{\sqrt{2\pi}}\int_M^\infty z^n\, e^{(i\alpha+\tau_-)z}\, dz$$

where $M < \infty$ is properly chosen to ensure the inequality. Concentrating on the term of significance, we write

$$\int_M^\infty z^n e^{(i\alpha+\tau_-)z} dz = z^n e^{(i\alpha+\tau_-)z}\Big|_M^\infty - \frac{n}{i\alpha+\tau_-}\int_M^\infty z^{n-1} e^{(i\alpha+\tau_-)z} dz$$

After n integrations by parts, the above becomes

$$\sum_{P=1}^{n} \frac{n!}{P!} \frac{(z)^P(-1)^{n-P}}{(i\alpha+\tau_-)^{n-P}} e^{(i\alpha+\tau_-)z}\Big|_M^\infty + (-1)^n \frac{n(n-1)\cdots 1}{(i\alpha+\tau_-)^n} e^{(i\alpha+\tau_-)z}\Big|_M^\infty$$

which is finite for finite n provided

$$\tau > \tau_-$$

Again by a similar discussion, it can be shown that $F^-(\alpha)$ is nth ordered continuously differentiable in the open region $\tau < \tau_+$ provided $|f(z)| < Be^{\tau_+ z}$ as $z \to -\infty$.

Since all that is required for analyticity in a region of the complex plane is that the first derivative of the function exist in that region, we are assured that $F_+(\alpha)$ and $F^-(\alpha)$ are analytic in the strip $\tau_- < \tau < \tau_+$ of the α plane provided of course that

$$|f(z)| < A\, e^{\tau_- z} \qquad z \to \infty$$

and

$$|f(z)| < B\, e^{\tau_+ z} \qquad z \to -\infty$$

Upon multiplication of equation 2.4 by $\frac{1}{\sqrt{2\pi}} e^{i\alpha z}$ and operating with the integral $\int_{-\infty}^{\infty} dz$, the following equation in ρ emerges.

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{d^2}{dz^2} H_\phi(\rho,z,\omega) e^{i\alpha z} + [\frac{\partial^2}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial}{\partial\rho} - \frac{1}{\rho^2} + k^2] H_\phi(\rho,\alpha,\omega) = 0$$

Integrating by parts twice and imposing the conditions that $H_\phi(\rho,z,\omega)$ and $\frac{d}{dz} H_\phi(\rho,z,\omega) \to 0$ as $|z| \to \infty$, produces an ordinary differential equation in ρ,

$$[\frac{d^2}{d\rho^2} + \frac{1}{\rho}\frac{d}{d\rho} - \frac{1}{\rho^2} - \gamma^2] H_\phi(\rho,\alpha,\omega) = 0 \qquad (2.8.5)$$

$$\gamma^2 = \alpha^2 - k^2 \qquad (2.8.6)$$

It is noted at this point that the incident magnetic intensity does not satisfy the differential equation (2.8.5) since it does not converge in the limit as $z \to -\infty$. This will be explored in detail a little later.

For region 1:

$$[\frac{d^2}{d\rho^2} + \frac{1}{\rho}\frac{d}{d\rho} - \frac{1}{\rho^2} - \gamma^2] H_{\phi_1}(\rho,\alpha,\omega) = 0 \qquad (2.9)$$

$$H_{\phi_1}(\rho,\alpha,\omega) \to 0 \quad \text{as} \quad \rho \to \infty \qquad (2.9.1)$$

$$\frac{d}{d\rho} [\rho H^-_{\phi_1}(\rho,\alpha,\omega)]\Big|_{\rho=b} = 0 \qquad (2.9.2)$$

Equation (2.9.1) follows from the Sommerfeld radiation condition and equation (2.9.2) follows from equation (2.5) and the boundary condition (2.4.2).

For region 3:

$$\left[\frac{d^2}{d\rho^2} + \frac{1}{\rho}\frac{d}{d\rho} - \frac{1}{\rho^2} - \gamma^2\right] H_{\phi_3}(\rho,\alpha,\omega) = 0 \tag{2.10}$$

$$H_{\phi_3}(0,\alpha,\omega) = 0 \tag{2.10.1}$$

$$\frac{d}{d\rho}\left[\rho\, H^-_{\phi_3}(\rho,\alpha,\omega)\right]\Big|_{\rho=a} = 0 \tag{2.10.2}$$

Equation (2.10.1) and (2.10.2) result from imposing (2.4.1) and (2.4.2).

For region 2:

In this region the magnetic intensity will be written as a linear combination of the incident TEM magnetic intensity and another term $\psi_{\phi_2}(\rho,z,\omega)$.

$$H_{\phi_2}(\rho,z,\omega) = H^i_\phi + \psi_{\phi_2}(\rho,z,\omega) \tag{2.11}$$

The term $\psi_{\phi_2}(\rho,z,\omega)$ represents obviously all of the reflected fields that exist within the feedline, on $z < 0$. For the other extreme of $z > 0$, $\psi_{\phi_2}(\rho,z,\omega)$ must account for the fields which exist in free space from sources within the feedline. It must be representable as

$$\psi_{\phi_2}(\rho,z,\omega) = (-H^i_\phi + \text{other field terms}) \;,\quad z > 0$$

since H^i_ϕ cannot be present in the half space $z > 0$ because it violates required boundary conditions.

Since the intensity $H_{\phi_2}(\rho,z,\omega)$ must be a solution to the differential equation (2.4) the terms H^i_ϕ and $\psi_{\phi_2}(\rho,z,\omega)$ must satisfy the following:

$$[\frac{\partial^2}{\partial z^2} + \frac{\partial^2}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial}{\partial \rho} - \frac{1}{\rho^2} + k^2]\, H^i_\phi = 0$$

and

$$[\frac{\partial^2}{\partial z^2} + \frac{\partial^2}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial}{\partial \rho} - \frac{1}{\rho^2} + k^2]\, \psi_{\phi_2}(\rho,z,\omega) = 0$$

The incident field $H^i_\phi = \frac{M(\omega)}{\rho} e^{ikz}$ identically satisfies the first of the above equations. Therefore we only need to solve the second equation. The magnetic field intensity for region 2 will be completely known after this is done.

The above differential equation for $\psi_{\phi_2}(\rho,z,\omega)$ will now be transformed by multiplying through by $\frac{1}{\sqrt{2\pi}} e^{i\alpha z}$ and operating with $\int_{-\infty}^{\infty} dz$.

$$\frac{1}{\sqrt{2\pi}} \int_{\infty}^{\infty} \frac{\partial^2}{\partial z^2} \psi_{\phi_2}(\rho,z,\omega) e^{i\alpha z}\, dz + [\frac{\partial^2}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial}{\partial \rho} - \frac{1}{\rho^2} + k^2]\psi_{\phi_2}(\rho,\alpha,z) = 0$$

Integrating by parts twice and requiring that $\psi_{\phi_2}(\rho,z,\omega)$ and $\frac{d}{dz}\psi_{\phi_2}(\rho,z,\omega) \to 0$ as $|z| \to \infty$, we derive the differential equation for region 2.

$$[\frac{d}{d\rho^2} + \frac{1}{\rho}\frac{\partial}{\partial \rho} - \frac{1}{\rho^2} - \gamma^2]\, \psi_{\phi_2}(\rho,\alpha,\omega) = 0 \qquad (2.12)$$

$$\frac{d}{d\rho}(\rho\, \psi_{\phi_2}(\rho,\alpha,\omega))\Big|_{\rho=a,b} = 0 \qquad (2.12.1)$$

The condition expressed in (2.12.1) follows when it is recognized that

$$\frac{d}{d\rho}(\rho\, H^i_\phi) = 0 \tag{2.12.2}$$

and use is made of the boundary condition expressed in (2.4.2) and (2.11).

It was previously stated that the incident magnetic intensity does not satisfy equation (2.8.5). In the derivation of (2.8.5) it is essential that the function and its derivative vanish as $|z| \to \infty$. It is obvious for slightly lossy media that H^i_ϕ does not vanish as $z \to -\infty$, therefore it cannot satisfy equation (2.8.5).

Upon multiplication of equation (2.11) by $\frac{1}{\sqrt{2\pi}} e^{i\alpha z}$ and operating with $\int_{-\infty}^{\infty} dz$, the following results:

$$H_{\phi_2}(\rho,\alpha,\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} (H^i_\phi)\, e^{i\alpha z}\, dz + \psi_{\phi_2}(\rho,\alpha,\omega) \tag{2.12.3}$$

Substituting the value of H^i_ϕ in the above integral produces:

$$\frac{M(\omega)}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{e^{ikz}}{\rho}\, e^{i\alpha z}\, dz \tag{2.12.4}$$

This integral does not converge. If it is assumed that the medium is slightly lossy, we can write the quantity k as a complex number

$$k = k_1 + ik_2 \quad ; \quad k_1 >> k_2 \, ; \quad k_1, k_2 > 0$$

The loss term k_2 will be allowed to go to zero later in the problem to yield results for a lossless medium. The value k is specifically chosen in this form to ensure convergence of the TEM magnetic intensity as $z \to +\infty$. This choice of k ensures convergence of (2.11.4) at the upper limit ($z \to +\infty$) . Therefore it is concluded that we should operate on (2.11) with the integral $\int_0^\infty dz$ rather than $\int_{-\infty}^\infty dz$. When this is done, the following equation, which will be used later, results :

$$H_{\phi_2}^+(\rho,\alpha,\omega) = + \frac{iM(\omega)}{\sqrt{2\pi}\,\rho(\alpha+k)} + \psi_{\phi_2}^+(\rho,\alpha,\omega) \tag{2.12.5}$$

In the differential equations for the various regions of the problem we have made use of a quantity

$$\gamma^2 = \alpha^2 - k^2$$

The square root is

$$\gamma = \sqrt{\alpha^2 - k^2} \tag{2.13}$$

which is a multivalued function of the complex variable α defined on a two sheeted Riemann surface. The value of γ must be uniquely defined so that only the proper branch of this double valued function is used. The branch cuts for this function are determined in Appendix B and depicted in Figure 2-3.

The necessity of satisfying the Sommerfeld radiation condition requires that the asymptotic frequency domain behavior of the magnetic intensity for the space around the coaxial waveguide antenna (exclusive of regions 2 and 3 with $z < 0$) be that of an outward traveling or evanescent spherical wave.

$$H_\phi \sim \frac{e^{ik|\vec{r}|}}{|\vec{r}|}$$

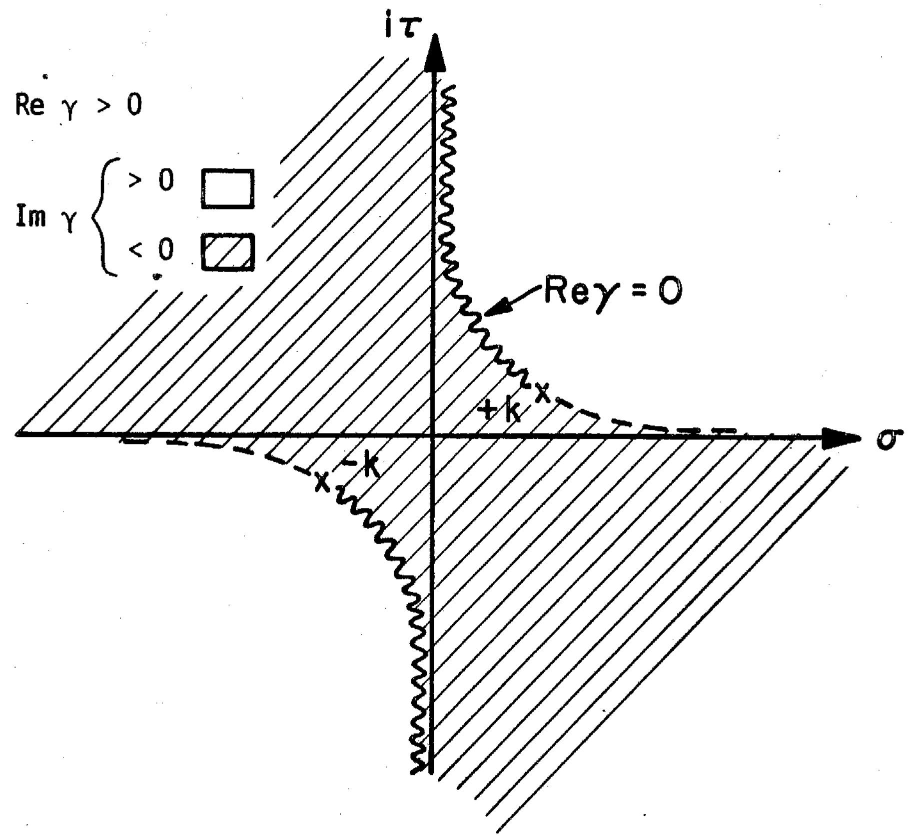


Figure 2-3. Branch cut for multivalued function γ

$$|\vec{r}| = \sqrt{\rho^2 + z^2}$$

If the value of z is such that $z \gg \rho$ the asymptotic behavior specializes to

$$H_{\phi_1}(\rho,z,\omega) \sim \frac{e^{ik|z|}}{|z|} \quad ; \quad z \to \pm\infty$$

$$\psi_{\phi_2}(\rho,z,\omega) \sim \frac{e^{ikz}}{z} \quad ; \quad z \to +\infty$$

$$H_{\phi_3}(\rho,z,\omega) \sim \frac{e^{ikz}}{z} \quad ; \quad z \to +\infty$$

In region 2 with $z < 0$, the magnetic intensity is known [17, p.327] to consist of the TEM mode and higher order E_{on} modes. Only E_{on} modes are present due to the symmetry requiring no variation with the angle ϕ. The only field intensity components allowed are H_ϕ, E_θ and E_z.

From equation (2.11) we have

$$\psi_{\phi_2}(\rho,z,\omega) = \frac{C e^{-ikz}}{\rho} +$$

$$\sum_{n=1}^{\infty} C_{on} \{J_1(\gamma'_{on}\rho)Y_o(\gamma'_{on}a) - Y_1(\gamma'_{on}\rho)J_o(\gamma'_{on}a)\}e^{-iz\sqrt{k^2-(\gamma'_{on})^2}}$$

where $\sqrt{k^2 - (\gamma'_{on})^2} = i\sqrt{(\gamma'_{on})^2 - k^2}$, $n=1,2,3,\cdots$

γ'_{on} is the nth ordered root of the auxiliary equation

$$J_o(\gamma'_{on}b)\, Y_o(\gamma'_{on}a) = Y_o(\gamma'_{on}b)\, J_o(\gamma'_{on}a)$$

C and C_{on} are constants relative to the spatial variables ρ and z. $J_\nu(z)$ and $Y_\nu(z)$ are the ordinary Bessel function of the first and second kind, respectively of the νth order.

In region 3 with $z < 0$, the magnetic intensity is also known [17, p.322] to be the E_{on} modes of a cylindrical waveguide.

$$H_{\phi_3}(\rho,z,\omega) = \sum_{n=1}^{\infty} C_{on} J_1\left(\frac{\rho\, u_{on}}{a}\right) e^{-iz\sqrt{k^2-\left(\frac{u_{on}}{a}\right)^2}}$$

where $\sqrt{k^2 - \left(\frac{u_{on}}{a}\right)^2} = i\sqrt{\left(\frac{u_{on}}{a}\right)^2 - k^2}$; $n=1,2,3,\cdots$, C_{on} is a constant relative to ρ and z and u_{on} is the nth ordered root of the auxiliary equation,

$$J_o(u) = 0$$

Given $k = k_1 + ik_2$, and the asymptotic behavior of the field in region 1, it is apparent from the earlier discussions that $H_{\phi_1}(\rho,\alpha,\omega)$ is analytic in the strip

$$-k_2 < \tau < k_2$$

of the complex α plane.

For regions 2 and 3 we must consider two cases, the first case is when the E_{on} modes within the region are evanescent and the second case is when they are not.

Case 1: $k < \gamma'_{on}$ region 2

$k < \frac{u_{on}}{a}$ region 3

Case 2: $k > \gamma'_{on}$ region 2

$k > \frac{u_{on}}{a}$ region 3

For Case 1 the E_{on} modes in regions 2 and 3 are evanescent, therefore the asymptotic value of the magnetic intensity as $z \to -\infty$ is the TEM mode portion of ψ_{ϕ_2} within region 2 and zero within region 3. We conclude that in the event of Case 1,

$$\psi^{\bullet}_{\phi_2}(\rho,\alpha,\omega) \quad \text{and} \quad H_{\phi_3}(\rho,\alpha,\omega)$$

are analytic with the strip $-k_2 < \tau < k_2$ of the α plane.

For Case 2, the E_{on} modes are not cut off and the asymptotic forms behave as

$$e^{-izk\sqrt{1 - \frac{\hat{\beta}^2}{k^2}}} = e^{-iz(k_1 + ik_2)\sqrt{1 - \frac{\hat{\beta}^2}{k^2}}} \quad , \quad z \to -\infty$$

This behavior produces a strip of analyticity that is at least given by

$$-k_2 < \tau < k_2 \operatorname{Re}\left(\sqrt{1 - \left(\frac{\hat{\beta}}{k}\right)^2}\right)$$

The symbol Re means the real part of the quantity that it precedes. Although at this point there is no apparent difficulty with this result, the inclusion of values of ka and kb such that

$$ka = u_{on}$$

and

$$kb = u_{om}$$

where u_{om} is the mth ordered root of $J_o(u) - J_o(\frac{a}{b}u) = 0$ requires special attention. It will be shown later, that the field intensity outside the waveguiding structure is zero at these frequencies.

In light of the above, we restrict ourselves to the fundamental mode by imposing the conditions

$$kb \ll 1 \qquad \text{and} \qquad ka \ll 1$$

when we are considering a time variation other than harmonic time dependence. Under this restriction Case 2 does not occur, and the functions

$$H_{\phi_1}(\rho,\alpha,\omega) \quad , \quad \psi_{\phi_2}(\rho,\alpha,\omega) \quad \text{and} \quad H_{\phi_3}(\rho,\alpha,\omega)$$

are analytic in the strip

$$-k_2 < \tau < k_2$$

of the complex α plane, see Fig. 2-4.

When we consider harmonic time dependence, the strip of analyticity is changed to

$$-k_2 < \tau < k_2 \, \mathrm{Re}\sqrt{1 - (\frac{\hat{\beta}}{k})^2}$$

where $\hat{\beta} = \gamma'_{on}$ or $\frac{u_{on}}{a}$ for region 2 or 3, respectively, $n=1,2,3,\cdots$. The value of $\hat{\beta}$ is the magnitude of the cutoff wave vector of the highest order E_{on} mode which is non-evanescent in region 2 or 3, as appropriate. Also $ka \neq u_{on}$ and $kb \neq u_{om}$.

To obtain a unique solution to the problem, it is necessary to determine the algebraic behavior (as $\alpha \to \infty$) of the factors in the completely factored and decomposed Wiener-Hopf equation [4, p.37]. This determination is made by considering the edge conditions [19].

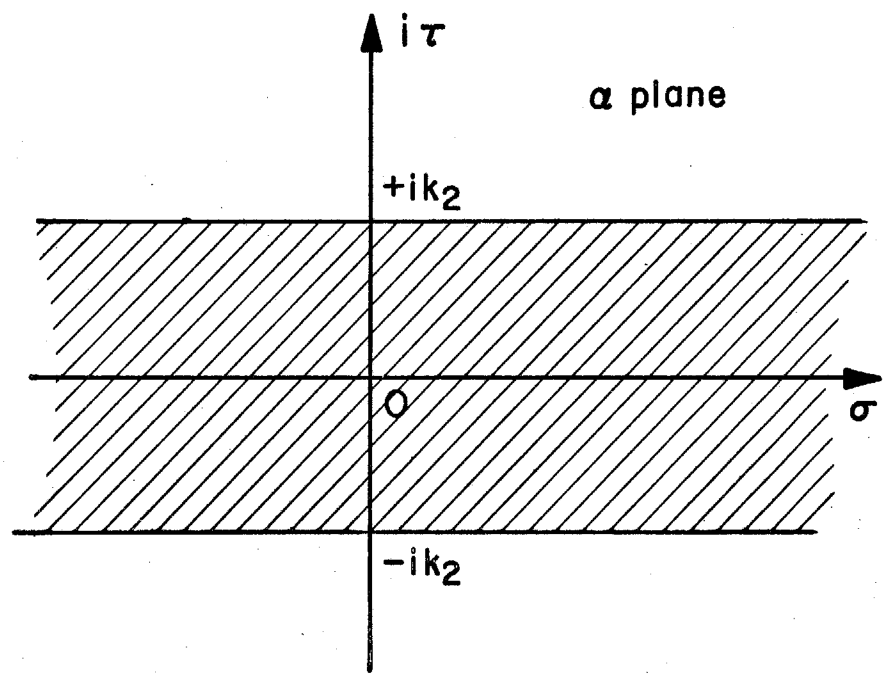


Figure 2-4. Strip of analyticity for $H_{\phi_1}(\rho,\alpha,\omega)$, $\psi_{\phi_2}(\rho,\alpha,\omega)$ and $H_{\phi_3}(\rho,\alpha,\omega)$.

For the infinitesimally thin conductors utilized in this problem, the edge conditions are

$$E_z(\rho,z,\omega) \propto z^{-1/2} \qquad \text{as} \qquad z \to 0^+ \text{ at } \rho = a \text{ and } b$$

Equation (2.5) states that

$$E_z(\rho,z,\omega) \propto \frac{1}{\rho}\frac{\partial}{\partial\rho}[\rho\, H_\phi(\rho,z,\omega)]$$

The point of interest is that

$$\frac{\partial}{\partial\rho}[\rho\, H_\phi(\rho,z,\omega)] \propto z^{-1/2} \tag{2.14}$$

as $z \to 0^+$ at $\rho = a$ or b . $z \to 0^+$ means that $z = 0$ is approached from the direction of positive values of z .

Development of the Wiener-Hopf Equations

In this section the Wiener-Hopf equations will be developed. The solution to these equations will lead to the spatial transform of the desired field intensities.

We begin by writing solutions to equations (2.9), (2.12) and (2.10) in that order.

$$H_{\phi_1}(\rho,\alpha,\omega) = A(\alpha,\omega)\, K_1(\gamma\rho) \tag{2.15}$$

$$\psi_{\phi_2}(\rho,\alpha,\omega) = B(\alpha,\omega)\, I_1(\gamma\rho) + C(\alpha,\omega)\, K_1(\gamma\rho) \tag{2.16}$$

$$H_{\phi_3}(\rho,\alpha,\omega) = D(\alpha,\omega)\, I_1(\gamma\rho) \tag{2.17}$$

The form of the solution to (2.9) expressed in (2.15) was chosen so that the asymptotic behavior would satisfy (2.9.1), thereby satisfying the Sommerfeld radiation condition.

The solution to (2.10) expressed in (2.17) does not include the first order modified Bessel function of the second kind since it becomes infinite as $\rho \to 0$ a condition that violates equation (2.10.1). The function $I_1(\gamma\rho) \to 0$ as $\rho \to 0$, so $H_{\phi_3}(0,\alpha,\omega) = 0$ and (2.10.1) is satisfied.

The coefficients $A(\alpha,\omega)$, $B(\alpha,\omega)$, $C(\alpha,\omega)$ and $D(\alpha,\omega)$ are unknown at this point. It is the task of finding these coefficients which leads to the Wiener-Hopf equations. Each of the coefficients will be expressed in terms of two heretofore unmentioned functions, $W^+(\alpha,\omega)$ and $H^+(\alpha,\omega)$, which will be found by employing the Wiener-Hopf procedure. To this end we will make use of the relationships which exist among the magnetic intensities and their derivatives (with respect to ρ) evaluated on the boundaries separating the three regions.

At $\rho = a$ we have

$$H_{\phi_2}(a,z,\omega) = H_{\phi_3}(a,z,\omega) \qquad \text{for} \qquad z \geq 0$$

Therefore,

$$H^+_{\phi_2}(a,\alpha,\omega) = H^+_{\phi_3}(a,\alpha,\omega) \tag{2.18}$$

From the analogous condition on the magnetic intensity at $\rho = b$ for $z \geq 0$, we have

$$H^+_{\phi_1}(b,\alpha,\omega) = H^+_{\phi_2}(b,\alpha,\omega) \tag{2.19}$$

These equations relate the magnetic intensities at the boundaries and will be used at a later stage in the development which leads to the Wiener-Hopf equations.

It was assumed earlier that the coaxial waveguide antenna structure was made of a perfectly conducting material. Such material does not support a tangential electric field, therefore

$$E_z(a,z,\omega) = 0$$

and

$$E_z(b,z,\omega) = 0 \quad \text{if} \quad z < 0$$

$E_z(\rho,z,\omega)$ is a continuous function of ρ, thus $E_z(\rho,z,\omega)$ has the same value at a given value of ρ regardless of the direction of approach, whether from smaller or larger values of ρ.

Equation (2.5) states that

$$E_z \propto \frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho H_\phi) \quad .$$

When this is combined with the preceding discussion about the tangential electric field, the results are

$$\frac{d}{d\rho}[\rho H_{\phi_3}(\rho,z,\omega)] = \frac{d}{d\rho}[\rho H_{\phi_2}(\rho,z,\omega)]$$

at $\rho = a$. By a spatial Fourier transform (see equation (2.8)) of these equations, it is found that

$$\frac{d}{d\rho}[\rho H_{\phi_3}(\rho,\alpha,\omega)]\Big|_{\rho=a} = \frac{d}{d\rho}[\rho H_{\phi_2}(\rho,\alpha,\omega)]\Big|_{\rho=a}$$

When we consider that $E_z(a,z,\omega) = 0$ for $z \leq 0$, this equation reduces to

$$\frac{d}{d\rho}[\rho H^{+}_{\phi_3}(\rho,\alpha,\omega)]\Big|_{\rho=a} = \frac{d}{d\rho}[\rho\psi^{+}_{\phi_2}(\rho,\alpha,\omega)]\Big|_{\rho=a} \tag{2.20}$$

Note that we have used equations (2.11) and (2.12.2) to obtain (2.20).

By very similar arguments with $\rho = b$ we also find that

$$\frac{d}{d\rho}[\rho H^{+}_{\phi_1}(\rho,\alpha,\omega)]\Big|_{\rho=b} = \frac{d}{d\rho}[\rho\psi^{+}_{\phi_2}(\rho,\alpha,\omega)]\Big|_{\rho=b} \tag{2.21}$$

We now define $W^{+}(\alpha,\omega)$ and $H^{+}(\alpha,\omega)$ as follows

$$W^{+}(\alpha,\omega) \triangleq \frac{d}{d\rho}[\rho H^{+}_{\phi_3}(\rho,\alpha,\omega)]\Big|_{\rho=a}$$

$$H^{+}(\alpha,\omega) \triangleq \frac{d}{d\rho}[\rho H^{+}_{\phi_1}(\rho,\alpha,\omega)]\Big|_{\rho=b}$$

These functions of α are analytic in the upper half plane given by $\tau > -k_2$.

Returning to the spatial frequency domain field solutions given by (2.15), (2.16) and (2.17), we multiply each of the equations by ρ and take the ρ derivative of the resulting equations [20, p.376].

$$\frac{d}{d\rho}[\rho H^{+}_{\phi_1}] + \frac{d}{d\rho}[\rho H^{-}_{\phi_1}] = -\gamma\rho\, A(\alpha,\omega)\, K_0(\gamma\rho) \tag{2.22.0}$$

$$\frac{d}{d\rho}[\rho\psi^{+}_{\phi_2}] + \frac{d}{d\rho}[\rho\psi^{-}_{\phi_2}] = \gamma\rho[B(\alpha,\omega)\, I_0(\gamma\rho) - C(\alpha,\omega)\, K_0(\gamma\rho)] \tag{2.22.1}$$

$$\frac{d}{d\rho}[\rho H^{+}_{\phi_3}] + \frac{d}{d\rho}[\rho H^{-}_{\phi_3}] = \gamma\rho\, D(\alpha,\omega)\, I_0(\gamma\rho) \tag{2.22.2}$$

By taking the last of the above equations and evaluating it at $\rho = a$, it is seen that

$$\frac{d}{d\rho}[\rho H^{+}_{\phi_3}]\Big|_{\rho=a} = W^{+}(\alpha,\omega) = \gamma a\ D(\alpha,\omega)\ I_0(\gamma a)$$

or

$$D(\alpha,\omega) = \frac{W^{+}(\alpha,\omega)}{\gamma a\ I_0(\gamma a)} \tag{2.23}$$

Note that

$$\frac{d}{d\rho}[\rho H^{-}_{\phi_3}]\Big|_{\rho=a} = 0$$

since,

$$E_z(a,z,\omega) = 0 \qquad \text{for} \qquad z \le 0$$

Recall that

$$\frac{d}{d\rho}[\rho H_{\phi_3}(\rho,z,\omega)] = \frac{d}{d\rho}[\rho H_{\phi_2}(\rho,z,\omega)] = \frac{d}{d\rho}[\rho\psi_{\phi_2}(\rho,z,\omega)]$$

at $\rho = a$. (This may be easily seen to stem from equation (2.12.2) and the continuity of the tangential electric field E_z). Since the tangential electric field is zero at $\rho = a$ or b for $z \le 0$,

$$\frac{d}{d\rho}[\rho\psi_{\phi_2}(\rho,z,\omega)]\Big|_{\rho=a,b} = 0 \qquad \text{for} \qquad z \le 0$$

and

$$\frac{d}{d\rho}[\rho\psi^{-}_{\phi_2}(\rho,\alpha,\omega)]\Big|_{\rho=a,b} = 0$$

Combining this result with equation (2.20) and equation (2.22.1) evaluated at $\rho = a$ leads to the following:

$$W^{+}(\alpha,\omega) = \gamma a[B(\alpha,\omega)\, I_0(\gamma a) - C(\alpha,\omega)\, K_0(\gamma a)]$$

or

$$C(\alpha,\omega) = -\frac{W^{+}(\alpha,\omega) - \gamma a B(\alpha,\omega)\, I_0(\gamma a)}{\gamma a\, K_0(\gamma a)} \qquad (2.24)$$

Substitution of the values of $D(\alpha,\omega)$ and $C(\alpha,\omega)$ given by equations (2.23) and (2.24) into equations (2.17) and (2.16), respectively, yields

$$H_{\phi_3}(\rho,\alpha,\omega) = \frac{W^{+}(\alpha,\omega)}{\gamma a\, I_0(\gamma a)}\, I_1(\gamma\rho)$$

and

$$\psi_{\phi_2}(\rho,\alpha,\omega) = B(\alpha,\omega)\, I_1(\gamma\rho) - \left[\frac{W^{+}(\alpha,\omega) - B(\alpha,\omega)\gamma a\, I_0(\gamma a)}{\gamma a\, K_0(\gamma a)}\right] K_1(\gamma\rho)$$

Evaluation of the immediately preceding equations at $\rho = a$ and use of the Wronskian relationship produces the following intermediate expressions:

$$H^{+}_{\phi_3}(a,\alpha,\omega) + H^{-}_{\phi_3}(a,\alpha,\omega) = \frac{W^{+}(\alpha,\omega)\, I_1(\gamma a)}{\gamma a\, I_0(\gamma a)}$$

$$\psi^{+}_{\phi_2}(a,\alpha,\omega) + \psi^{-}_{\phi_2}(a,\alpha,\omega) = \frac{B(\alpha,\omega)}{\gamma a\, K_0(\gamma a)} - \frac{W^{+}(\alpha,\omega)\, K_1(\gamma a)}{\gamma a\, K_0(\gamma a)}$$

By subtracting the lower of the two equations immediately above from the top equation, we find that

$$H^+_{\phi_3}(a,\alpha,\omega) - \psi^+_{\phi_2}(a,\alpha,\omega) + H^-_{\phi_3}(a,\alpha,\omega) - \psi^-_{\phi_2}(a,\alpha,\omega) =$$

$$\frac{W^+(\alpha,\omega)}{(\gamma a)^2 I_0(\gamma a)\ K_0(\gamma a)} - \frac{B(\alpha,\omega)}{\gamma a\ K_0(\gamma a)}$$

The Wronskian was used to achieve some simplification in the above equation.

If we evaluate (2.12.5) at $\rho = a$ we find

$$H^+_{\phi_2}(a,\alpha,\omega) - \psi^+_{\phi_2}(a,\alpha,\omega) = \frac{iM(\omega)}{\sqrt{2\pi}\ a(\alpha+k)}$$

But it is known from (2.18) that

$$H^+_{\phi_3}(a,\alpha,\omega) = H^+_{\phi_2}(a,\alpha,\omega)$$

Therefore,

$$H^+_{\phi_3}(a,\alpha,\omega) - \psi^+_{\phi_2}(a,\alpha,\omega) = \frac{iM(\omega)}{\sqrt{2\pi}\ a(\alpha+k)}$$

and it follows that

$$\frac{iM(\omega)}{\sqrt{2\pi}\ a(\alpha+k)} + H^-_{\phi_3}(a,\alpha,\omega) - \psi^-_{\phi_2}(a,\alpha,\omega) = \frac{W^+(\alpha,\omega)}{(\gamma a)^2 I_0(\gamma a) K_0(\gamma a)} - \frac{B(\alpha,\omega)}{\gamma a\ K_0(\gamma a)} \tag{2.25}$$

Evaluating (2.22.0) at $\rho = b$ we see that

$$A(\alpha,\omega) = -\frac{H^+(\alpha,\omega)}{\gamma b\ K_0(\gamma b)} \tag{2.26}$$

Note that the boundary condition $E_z(b,z,\omega) = 0$ for $z \leq 0$ was invoked. This allows substitution of

$$\frac{d}{d\rho}[\rho H^-_{\phi_1}]\Big|_{\rho=b} = 0$$

into equation (2.22.0).

When (2.22.1) is evaluated at $\rho = b$ it becomes

$$H^+(\alpha,\omega) = \gamma b[B(\alpha,\omega)\, I_0(\gamma b) - C(\alpha,\omega)\, K_0(\gamma b)]$$

or

$$C(\alpha,\omega) = -\frac{H^+(\alpha,\omega) - B(\alpha,\omega)\,\gamma b\, I_0(\gamma b)}{\gamma b\; K_0(\gamma b)} \tag{2.27}$$

If we replace the coefficients in (2.15) and (2.16) by the expressions given in (2.26) and (2.27) respectively, then it is apparent that

$$H_{\phi_1}(\rho,\alpha,\omega) = -\frac{H^+(\alpha,\omega)}{\gamma b\, K_0(\gamma b)}\; K_1(\gamma\rho)$$

and

$$\psi_{\phi_2}(\rho,\alpha,\omega) = B(\alpha,\omega)\, I_1(\gamma\rho) - [\frac{H^+(\alpha,\omega) - B(\alpha,\omega)\gamma b\; I_0(\gamma b)}{\gamma b\; K_0(\gamma b)}]K_1(\gamma\rho)$$

By evaluation of the preceding equations at $\rho = b$ and simplifying, we get

$$H^+_{\phi_1}(b,\alpha,\omega) + H^-_{\phi_1}(b,\alpha,\omega) = -\frac{H^+(\alpha,\omega)\; K_1(\gamma b)}{\gamma b\; K_0(\gamma b)}$$

$$\psi^+_{\phi_2}(b,\alpha,\omega) + \psi^-_{\phi_2}(b,\alpha,\omega) = \frac{B(\alpha,\omega)}{\gamma b\; K_0(\gamma b)} - \frac{H^+(\alpha,\omega)\; K_1(\gamma b)}{\gamma b\; K_0(\gamma b)}$$

Subtraction of the latter from the former yields

$$H^{+}_{\phi_1}(b,\alpha,\omega) - \psi^{+}_{\phi_2}(b,\alpha,\omega) + H^{-}_{\phi_1}(b,\alpha,\omega) - \psi^{-}_{\phi_2}(b,\alpha,\omega) = -\frac{B(\alpha,\omega)}{\gamma b\ K_0(\gamma b)}$$

It is seen that

$$H^{+}_{\phi_1}(b,\alpha,\omega) - \psi^{+}_{\phi_2}(b,\alpha,\omega) = \frac{iM(\omega)}{\sqrt{2\pi}\ b(\alpha+k)}$$

by evaluating (2.12.5) at $\rho = b$ and using the equality given by (2.19). Substitution of this result leads to

$$\frac{iM(\omega)}{\sqrt{2\pi}\ b(\alpha+k)} + H^{-}_{\phi_1}(b,\alpha,\omega) - \psi^{-}_{\phi_2}(b,\alpha,\omega) = -\frac{B(\alpha,\omega)}{\gamma b\ K_0(\gamma b)} \qquad (2.28)$$

Equation (2.28) is analogous to (2.25). These two equations are the result of matching the tangential electric intensity $(-\infty < z < \infty)$ and the magnetic intensity $(z \geq 0)$ at the common boundaries between the regions into which the space was earlier divided. These equations are also combined to form one of the Wiener-Hopf equations by eliminating $B(\alpha,\omega)$ between them. The resulting Wiener-Hopf equation involving $W^{+}(\alpha,\omega)$ is

$$\frac{i\gamma\, M(\omega)}{\sqrt{2\pi}\ (\alpha+k)}\,[K_0(\gamma a) - K_0(\gamma b)] = \frac{W^{+}(\alpha,\omega)}{\gamma a\ I_0(\gamma a)} + \gamma b\ K_0(\gamma b)$$

$$\times \quad [H^{-}_{\phi_1}(b,\alpha,\omega) - \psi^{-}_{\phi_2}(b,\alpha,\omega)] - \gamma a\ K_0(\gamma a)[H^{-}_{\phi_3}(a,\alpha,\omega) - \psi^{-}_{\phi_2}(a,\alpha,\omega)]$$

(2.29)

Since $C(\alpha,\omega)$ in this problem is unique, the two values--one given in expression (2.24) and the other given in (2.27) must be equivalent. When these expressions are equated the results relate the two variables $W^+(\alpha,\omega)$ and $H^+(\alpha,\omega)$,

$$W^+(\alpha,\omega) = \frac{a\,K_o(\gamma a)}{b\,K_o(\gamma b)}[H^+(\alpha,\omega)] - B(\alpha,\omega)\left[\frac{I_o(\gamma b)}{K_o(\gamma b)} - \frac{I_o(\gamma a)}{K_o(\gamma a)}\right]\gamma a\,K_o(\gamma a)$$

The terms $W^+(\alpha,\omega)$ and $B(\alpha,\omega)$ appearing in the above expression are eliminated by using expressions (2.28) and (2.29), giving a Wiener-Hopf equation involving the function $H^+(\alpha,\omega)$,

$$\frac{i\gamma\,M(\omega)}{\sqrt{2\pi}(\alpha+k)}[I_o(\gamma a) - I_o(\gamma b)] = \frac{H^+(\alpha,\omega)}{\gamma b\,K_o(\gamma b)}$$

$$+ \gamma b\,I_o(\gamma b)[H^-_{\phi_1}(b,\alpha,\omega) - \psi^-_{\phi_2}(b,\alpha,\omega)]$$

$$- \gamma a\,I_o(\gamma a)[H^-_{\phi_3}(a,\alpha,\omega) - \psi^-_{\phi_2}(a,\alpha,\omega)] \qquad (2.30)$$

In the coaxial waveguide antenna we usually have $b-a \ll \frac{a+b}{2}$ to ensure TEM mode excitation at microwave frequencies. For this case the field intensities will be rapidly varying when compared to ρ, and the fields in the annular region will approach those of a parallel plate waveguide [17, p.329] of width (b-a).

For the parallel plate waveguide it is well known from the preservation of symmetry that for a TEM mode excitation the current distribution on the two parallel plates are equal in magnitude and opposite in direction [21, p.126]. For excitation by E_{on} mode fields, the currents on the two parallel plates will be equal in magnitude and in the same direction for odd n, and in opposite directions for even n.

Up to this point no approximations have been made and equations (2.29) and (2.30) are exact. It is recognized, though, that the form of these equations is such that they cannot be decomposed into two expressions, one analytic in the upper half plane and the other analytic in the lower half plane with an overlapping strip in which both functions are analytic. This is made apparent by observing that the coefficients multiplying the "minus" functions on the right hand side of each equation are not identical. These same coefficients also contain a branch point and are therefore not of the form where an exact solution is possible [4, p.153].

We note, however, that if $b-a << \frac{a+b}{2}$ then the field intensities approach those of a parallel plate waveguide and if TEM mode excitation is impressed, the currents on the cylindrical waveguide conductors are approximately equal in magnitude and opposite in direction. The mathematical expression of this condition applicable to the structure under consideration is

$$2\pi b[H_{\phi_1}(b,z,\omega) - H_{\phi_2}(b,z,\omega)] \approx 2\pi a[H_{\phi_3}(a,z,\omega) - H_{\phi_2}(a,z,\omega)]$$

for $z \leq 0$. We now recall equations (2.11) and (2.7):

$$H_{\phi_2}(\rho,z,\omega) = H_\phi^i + \psi_{\phi_2}(\rho,z,\omega) \tag{2.11}$$

$$H_\phi^i = \frac{M(\omega)}{\rho} e^{ikz} \tag{2.7}$$

Substitutions of these quantities in the above equations yields

$$b[H_{\phi_1}(b,z,\omega) - \psi_{\phi_2}(b,z,\omega) - \frac{M(\omega)}{b} e^{ikz}]$$

$$\simeq a[H_{\phi_3}(a,z,\omega) - \psi_{\phi_2}(a,z,\omega) - \frac{M(\omega)}{a} e^{ikz}] \text{ for } z \leq 0$$

Under the assumption that the approximation is very good, we can equate the two currents and obtain

$$b[H_{\phi_1}(b,z,\omega) - \psi_{\phi_2}(b,z,\omega)] = a[H_{\phi_3}(a,z,\omega) - \psi_{\phi_2}(a,z,\omega)] \quad \text{for } z \leq 0$$

When this equation is spatially Fourier transformed the results are precisely what is needed to allow decomposition of the two Wiener-Hopf equations (2.29) and (2.30), i.e.,

$$b[H^-_{\phi_1}(b,\alpha,\omega) - \psi^-_{\phi_2}(b,\alpha,\omega)] = a[H^-_{\phi_3}(a,\alpha,\omega) - \psi^-_{\phi_2}(a,\alpha,\omega)] \tag{2.31}$$

when

$$b - a << \frac{a+b}{2} \tag{2.32}$$

It is also noted that the expression (2.31) is analytic in the lower half plane given by $\tau < k_2$ or $k_2 Re(\sqrt{1 - (\hat{\beta}/k)^2})$ depending on the excitation. It is convenient to define $S^-(\alpha)$ such that

$$S^-(\alpha) = b[H^-_{\phi_1}(b,\alpha,\omega) - \psi^-_{\phi_2}(b,\alpha,\omega)] = a[H^-_{\phi_3}(a,\alpha,\omega) - \psi^-_{\phi_2}(a,\alpha,\omega)] \tag{2.33}$$

To derive decomposable Wiener-Hopf equations we substitute (2.33) into (2.29) and (2.30). The resulting equations are:

$$\frac{iM(\omega)}{\sqrt{2\pi}(\alpha+k)} + \frac{W^{+}(\alpha,\omega)}{\gamma^2 a I_0(\gamma a)[K_0(\gamma b) - K_0(\gamma a)]} + S^{-}(\alpha) = 0 \qquad (2.34)$$

$$\frac{iM(\omega)}{\sqrt{2\pi}(\alpha+k)} + \frac{H^{+}(\alpha,\omega)}{\gamma^2 b K_0(\gamma b)[I_0(\gamma b) - I_0(\gamma a)]} + S^{-}(\alpha) = 0 \qquad (2.35)$$

It is interesting to note that if the radius of curvature is large we may imagine the conductors in Figure 2-1 to be cut along the z direction and flattened into parallel planes. Therefore it is reasonable to expect that equations (2.34) and (2.35) will become the Wiener-Hopf equations for two parallel planes in the limit as radii a and b approach infinity. The planes would be b-a distance apart and the axis of symmetry would be displaced a distance $a \simeq b \to \infty$ from the coordinate z axis. By taking the limit as a and $b \to \infty$ and using the asymptotic forms for the modified Bessel function, equations (2.34) and (2.35), respectively, become

$$\frac{iM(\omega)}{\sqrt{2\pi}(\alpha+k)} - \frac{W^{+}(\alpha,\omega)}{\gamma^2 a}\left[\frac{\gamma a}{e^{-\gamma\frac{(b-a)}{2}} \sinh \frac{\gamma(b-a)}{2}}\right] - S^{-}(\alpha) = 0$$

and

$$\frac{iM(\omega)}{\sqrt{2\pi}(\alpha+k)} + \frac{H^{+}(\alpha,\omega)}{\gamma^2 b}\left[\frac{\gamma b}{e^{-\gamma\frac{(b-a)}{2}} \sinh \frac{\gamma(b-a)}{2}}\right] - S^{-}(\alpha) = 0$$

It can also be shown from the definition of $W^{+}(\alpha,\omega)$ and $H^{+}(\alpha,\omega)$ that for a parallel plate case $H^{+}(\alpha,\omega) = -W^{+}(\alpha,\omega)$. Therefore the equations (2.34) and (2.35) become identical in the limit as the radii approach infinity and each is the Wiener-Hopf equation for the radiation from an open ended parallel plate waveguide with TEM mode excitation [4, p.107 or 21, p.128].

The solution of equations (2.34) and (2.35) for $W^+(\alpha,\omega)$ and $H^+(\alpha,\omega)$ allows a complete characterization of the magnetic intensity in the spatial frequency domain. The foregoing work already contains the equations relating $A(\alpha,\omega)$ and $D(\alpha,\omega)$ to $W^+(\alpha,\omega)$ and $H^+(\alpha,\omega)$. These coefficients are given by (2.26) and (2.23), respectively. To obtain the remaining coefficients $C(\alpha,\omega)$ and $D(\alpha,\omega)$ in terms of $W^+(\alpha,\omega)$ and $H^+(\alpha,\omega)$, we return to equation (2.22.1) and evaluate it at $\rho = a$ and b. When these results are considered with (2.20), (2.21) and the relation

$$\frac{d}{d\rho}[\rho\psi^-_{\phi_2}(\rho,\alpha,\omega)]\Big|_{\rho=a,b} = 0$$

we get

$$W^+(\alpha,\omega) = \gamma a[B(\alpha,\omega)I_0(\gamma a) - C(\alpha,\omega)K_0(\gamma a)]$$

$$H^+(\alpha,\omega) = \gamma b[B(\alpha,\omega)I_0(\gamma b) - C(\alpha,\omega)K_0(\gamma b)]$$

The solutions to these simultaneous equations yield the coefficients we seek,

$$B(\alpha,\omega) = \left[\frac{H^+(\alpha,\omega)K_0(\gamma a)}{\gamma b} - \frac{W^+(\alpha,\omega)K_0(\gamma b)}{\gamma a}\right]\left[\frac{1}{I_0(\gamma b)K_0(\gamma a) - I_0(\gamma a)K_0(\gamma b)}\right] \tag{2.36}$$

$$C(\alpha,\omega) = \left[\frac{H^+(\alpha,\omega)I_0(\gamma a)}{\gamma b} - \frac{W^+(\alpha,\omega)I_0(\gamma b)}{\gamma a}\right]\left[\frac{1}{I_0(\gamma b)K_0(\gamma a) - I_0(\gamma a)K_0(\gamma b)}\right] \tag{2.37}$$

To complete the list we add:

$$A(\alpha,\omega) = \frac{-H^{+}(\alpha,\omega)}{\gamma b K_o(\gamma b)} \tag{2.26}$$

$$D(\alpha,\omega) = \frac{W^{+}(\alpha,\omega)}{\gamma a I_o(\gamma a)} \tag{2.23}$$

We return to the earlier discussion that certain values of ka,kb are excluded. Namely

$$ka = u_{on}$$

and

$$kb = u_{om}$$

where u_{on} is the nth ordered root of $J_o(u) = 0$

and u_{om} is the mth ordered root of $J_o(u) - J_o(\frac{a}{b}u) = 0$

There is an important mathematical reason for excluding these particular values of ka and kb that shows up in the factorization procedure of Appendix D. The physical reason for their exclusion is that at these frequencies the magnetic intensity everywhere outside the structure is zero.

If $ka = u_{on}$, then the magnetic intensity in the region $\rho < a$, $z < 0$ must be zero. This value of ka corresponds to the cutoff wavelength for the E_{on} mode of a circular waveguide of radius a. Exactly at cutoff, the magnetic intensity for the E_{on} mode in the circular waveguide is zero [17, p.322].

Now considering that $b-a \ll \frac{a+b}{2}$ the structure has fields that behave approximately as those of a parallel plate waveguide. In this limit we have shown above that the field exterior to one of the plates is zero. By symmetry the field that is on the exterior side of the other plate must also vanish. By continuity it is concluded that the field is zero everywhere exterior to the waveguide channel.

For the other case, $kb = u_{om}$, we examine the asymptotic behavior of $|u_{om}|$ as $m \to \infty$

$$J_o(u) - J_o(\frac{a}{b} u) \simeq \sqrt{\frac{2}{\pi u}} \cos(u - \frac{\pi}{4}) - \sqrt{\frac{2}{\pi \frac{a}{b} u}} \cos(\frac{a}{b} u - \frac{\pi}{4})$$

$$\simeq \sqrt{\frac{2}{\pi u}} (\cos(u - \frac{\pi}{4}) - \cos(\frac{a}{b} u - \frac{\pi}{4}))$$

with a and b of the same order of magnitude as $u \to \infty$. Through the trigonometric identity

$$\cos x - \cos y = - 2 \sin(\frac{x+y}{2}) \sin \frac{x-y}{2}$$

it is determined that the zeros are approximately located at

$$|u_{om}| \simeq (m - \frac{1}{4})(\frac{2\pi b}{a+b}) \quad \text{or} \quad m(\frac{2\pi b}{a-b}) \quad , \quad m=1,2,3,\cdots$$

for $u_{om} \to \infty$ or $m \to \infty$.

The asymptotic values of u_{om} given above are recognized as the asymptotic values of the cutoff frequencies as $m \to \infty$ for the E_{om} modes of a circular waveguide of radius $\frac{a+b}{2}$ and the even ordered E_{om} modes of a parallel plate waveguide, with a channel width of b-a, respectively. The second value of u_{om} also corresponds to the asymptotic value of the even ordered E_{om} modes of a coaxial waveguide [22]. In the event $b-a << \frac{a+b}{2}$, we may approximate the structure as a circular waveguide of mean radius $\frac{a+b}{2}$ or a parallel plate waveguide, depending on the excitation. At the cutoff frequencies for such structures, the electromagnetic fields are identically zero. Therefore we have excluded the use of such frequencies, since they invoke no response.

Solution of the Wiener-Hopf Equations

The two Wiener-Hopf equations (2.34) and (2.35) are solvable in the conventional Wiener-Hopf fashion. The various terms are factored into the product of a "plus" and "minus" function, where required, and the resulting equation is decomposed into the sum of "plus" and "minus" functions. [The "plus" ("minus") functions are regular in the entire upper (lower) half plane.] Analytic continuation arguments are then invoked to yield the solution.

To preserve the continuity of the work, Appendix D was devoted to the factorization [23] of the kernels

$$\{I_0(\gamma a)[K_0(\gamma b) - K_0(\gamma a)]\}^{-1} \triangleq x(\alpha) = x^+(\alpha)\ x^-(\alpha)$$

and

$$\{K_0(\gamma b)[I_0(\gamma b) - I_0(\gamma a)]\}^{-1} \triangleq y(\alpha) = y^+(\alpha)\ y^-(\alpha)$$

In this section we will use the expressions $x^+(\alpha)$, $x^-(\alpha)$, $y^+(\alpha)$, $y^-(\alpha)$. Their specific functional form is found in Appendix D.

We rewrite (2.34) and (2.35) using the newly defined symbols for the kernels that must be factored,

$$\frac{iM(\omega)}{\sqrt{2\pi}(\alpha+k)} + \frac{W^+(\alpha,\omega)}{\gamma^2 a}[x^+(\alpha)\ x^-(\alpha)] + S^-(\alpha) = 0 \tag{2.38}$$

$$\frac{iM(\omega)}{\sqrt{2\pi}\ (\alpha+k)} + \frac{H^+(\alpha,\omega)}{\gamma^2 b}[y^+(\alpha)\ y^-(\alpha)] + S^-(\alpha) = 0 \tag{2.39}$$

Recall that the value of γ^2 is given as

$$\gamma^2 = \alpha^2 - k^2$$

When this value of γ^2 is substituted in the above equations, we can reduce them to

$$\frac{iM(\omega)(\alpha-k)}{\sqrt{2\pi}(\alpha+k)\ x^-(\alpha)} + \frac{W^+(\alpha,\omega)\ x^+(\alpha)}{a(\alpha+k)} + \frac{S^-(\alpha)(\alpha-k)}{x^-(\alpha)} = 0$$

and

$$\frac{iM(\omega)(\alpha-k)}{\sqrt{2\pi}(\alpha+k)\ y^-(\alpha)} + \frac{H^+(\alpha,\omega)\ y^+(\alpha)}{b(\alpha+k)} + \frac{S^-(\alpha)(\alpha-k)}{y^-(\alpha)} = 0$$

Examination of the region of analyticity of the terms in the above equations reveals that the middle term in each equation is analytic in the half plane $\tau > -k_2$ and the last term in each equation is analytic in the half plane $\tau < k_2$. The first term in each equation has a pole at $\alpha = -k$. If this pole were not present it would be analytic in the half plane $\tau < k_2$. By removal of the residue of this pole, we obtain the completely decomposed equations:

$$\frac{W^+(\alpha,\omega)\ x^+(\alpha)}{a(\alpha+k)} - i\sqrt{\frac{2}{\pi}}\frac{kM(\omega)}{(\alpha+k)\ x^+(k)}$$

$$= -\left\{\frac{S^-(\alpha)(\alpha-k)}{x^-(\alpha)} + \frac{iM(\omega)}{\sqrt{2\pi}(\alpha+k)}\left[\frac{\alpha-k}{x^-(\alpha)} + \frac{2k}{x^+(k)}\right]\right\} \quad (2.40)$$

$$\frac{H^+(\alpha,\omega)\ y^+(\alpha)}{b(\alpha+k)} - i\sqrt{\frac{2}{\pi}}\frac{kM(\omega)}{(\alpha+k)\ y^+(k)}$$

$$= -\left\{\frac{S^-(\alpha)(\alpha-k)}{y^-(\alpha)} + \frac{iM(\omega)}{\sqrt{2\pi}(\alpha+k)}\left[\frac{\alpha-k}{y^-(\alpha)} + \frac{2k}{y^+(k)}\right]\right\} \quad (2.41)$$

In the above expression we have used the result from Appendix D, that

$$x^+(\alpha) = x^-(-\alpha)$$

and

$$y^+(\alpha) = y^-(-\alpha)$$

The terms in (2.40) and (2.41) are arranged so that the left hand side is analytic in the half plane $\tau_2 > -k_2$ and the right hand side is analytic in the half plane $\tau_2 < k_2$ or $k_2\ \mathrm{Re}(\sqrt{1-(\hat{\beta}/k)^2}\ )$ depending on the type and frequency of the excitation. The important point in this discussion is that there is a strip of overlap in the regions of analyticity, that is, both sides of (2.40) and (2.41) are analytic in a common strip

$$-k_2 < \tau < k_2 \quad \text{or} \quad k_2\ \mathrm{Re}\sqrt{1-(\hat{\beta}/k)^2}$$

We therefore conclude that each side of the expressions (2.40) and (2.41) represents the analytic continuation of the other side. Since these expressions are regular throughout the complex plane they are an entire or integral function.

If we can show that each side of (2.40) and (2.41) is bounded as $|\alpha| \to \infty$ in the half plane for which that side of the expression is regular, then we can use Liouville's theorem to set each expression equal to a constant. Liouville's theorem states that the only bounded entire functions are constants.

To determine the limiting behavior as $|\alpha| \to \infty$ for the terms in (2.40) and (2.41), we examine the edge conditions given in (2.14) as

$$\frac{\partial}{\partial \rho}\left[\rho H_\phi(\rho,z,\omega)\right] \propto z^{-1/2}$$

as $z \to 0^+$ at $\rho = a$ or b . Taking a slightly more general case, the Fourier transforms

$$\frac{1}{\sqrt{2\pi}}\int_0^\infty z^n e^{i\alpha z}\,dz$$

and

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^0 z^n e^{i\alpha z}\,dz$$

determine the limiting behavior. In these integrals $n < 0$ to have absolute and uniform convergence of the improper integrals.

It is also well known that if $|f(z)| < e^{-k_2 z}$ as $z \to \infty$ then

$$\lim_{z \to 0^+} f(z) = \lim_{\alpha \to \infty} i\alpha F^+(\alpha) = (\lim_{\alpha \to \infty} i\alpha)(\lim_{\alpha \to \infty} F^+(\alpha))$$

for $\tau > -k_2$ where $\alpha = \sigma + i\tau$. Similarly, if $|f(z)| < e^{k_2 z}$ as $z \to -\infty$ then

$$\lim_{z \to 0^-} f(z) = (\lim_{\alpha \to -\infty} i\alpha) \; (\lim_{\alpha \to -\infty} F^-(\alpha))$$

for $\tau < k_2$. Thus the limiting behavior as $|\alpha| \to \infty$ in the region of regularity of the spatial transforms is determined by the behavior of the space domain function in the limit as the origin is approached.

Using the gamma function [18, p.183]

$$\Gamma(x) = \int_0^\infty e^{-t}\, t^{x-1} dt \quad ; \quad \mathrm{Re}\; x > 0$$

and contour integration, we find

$$\frac{1}{\sqrt{2\pi}} \int_0^\infty z^n \, e^{i\alpha z} \, dz = + \frac{1}{\sqrt{2\pi}} \frac{\Gamma(n+1)}{\alpha^{n+1}} \, e^{i\frac{\pi}{2}(n+1)}$$

and

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^0 z^n \, e^{i\alpha z} \, dz = - \frac{1}{\sqrt{2\pi}} \frac{\Gamma(n+1)}{\alpha^{n+1}} \, e^{i\frac{\pi}{2}(n+1)}$$

with $-1 < n < 0$ in both integrals. The lower extremity on the range of n is set at -1 to ensure the existence of the gamma function.

From the edge condition we have $n=-\frac{1}{2}$ and determine

$$W^+(\alpha,\omega) \sim \alpha^{-1/2}$$

and

$$H^+(\alpha,\omega) \sim \alpha^{-1/2} \qquad \text{as } |\alpha| \to \infty \text{ with } \tau > -k_2 .$$

From the definition of $S^{-}(\alpha)$, it can be seen that it is the spatial transform on negative values of z of the current normal to the edge at $z = 0$ on either the inner or outer conductor, exclusive of that current contributed by the incident field. The magnetic intensity is continuous at all values of ρ for $z = 0$, thus the current in each conductor must approach zero as $z \to 0^{-}$. This is apparent from the fact that the current in each conductor is proportional to the discontinuity in the magnetic intensity across the conductor. The discontinuity disappears at $z = 0$. Because the total current approaches zero as $z \to 0^{-}$, the quantity which transforms to $S^{-}(\alpha)$ must approach that of a TEM mode field intensity traveling in the minus z direction. This quantity should just cancel the incident mode field intensity at $z = 0$. Thus

$$2\pi a[H_{\phi_3}(a,z,\omega) - \psi_{\phi_2}(a,z,\omega)] = 2\pi b[H_{\phi_1}(b,z,\omega) - \psi_{\phi_2}(b,z,\omega)]$$

$$\Rightarrow M(\omega)\, e^{-ikz} \quad \text{as} \quad z \to 0^{-}$$

The transform of this quantity is

$$2\pi S^{-}(\alpha) \sim \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{0} M(\omega)\, e^{-ikz}\, e^{i\alpha z}\, dz = \frac{-iM(\omega)}{\sqrt{2\pi}} \left(\frac{1}{\alpha - k}\right)$$

From this result we conclude that $S^{-}(\alpha) \sim \left(\frac{1}{\alpha-k}\right)$ as $|\alpha| \to \infty$ with $\tau < +k_2$.

Collecting some results from Appendix D, we write

$$x^+(\alpha) \sim |\alpha|^{1/2}$$

and

$$y^+(\alpha) \sim |\alpha|^{1/2} \qquad \text{as } |\alpha| \to \infty \text{ with } \tau > -k_2$$

Also,

$$x^-(\alpha) \sim |\alpha|^{1/2}$$

and

$$y^-(\alpha) \sim |\alpha|^{1/2} \qquad \text{as } |\alpha| \to \infty \text{ with } \tau < k_2$$

An examination of (2.40) and (2.41) in the light of the asymptotic behavior of the terms shows that each side of these expressions goes to zero as $|\alpha| \to \infty$ in their half plane of regularity. Liouville's theorem is therefore applicable and each side of (2.40) and (2.41) must be identically zero. The quantities of interest are

$$\frac{W^+(\alpha,\omega)\; x^+(\alpha)}{a(\alpha+k)} - i\sqrt{\frac{2}{\pi}}\,\frac{k\; M(\omega)}{(\alpha+k)\; x^+(k)} \equiv 0$$

and

$$\frac{H^+(\alpha,\omega)\; y^+(\alpha)}{b(\alpha+k)} - i\sqrt{\frac{2}{\pi}}\,\frac{k\; M(\omega)}{(\alpha+k)\; y^+(k)} \equiv 0$$

Therefore

$$W^+(\alpha,\omega) = i\sqrt{\frac{2}{\pi}}\,\frac{ka\; M(\omega)}{x^+(\alpha)\; x^+(k)} \tag{2.44}$$

$$H^+(\alpha,\omega) = i\sqrt{\frac{2}{\pi}}\,\frac{kb\; M(\omega)}{y^+(\alpha)\; y^+(k)} \tag{2.45}$$

and the coefficients $A(\alpha,\omega)$, $B(\alpha,\omega)$, $C(\alpha,\omega)$ and $D(\alpha,\omega)$ have been determined, see equations (2.23), (2.26), (2.36), and (2.37). The spatial domain field intensities are now written:

$$H_{\phi_1}(\rho,\alpha,\omega) = -i\sqrt{\frac{2}{\pi}}\,\frac{k\,M(\omega)}{\gamma K_0(\gamma b)}\,\frac{K_1(\gamma\rho)}{y^+(k)\,y^+(\alpha)} \tag{2.46}$$

$$H_{\phi_3}(\rho,\alpha,\omega) = i\sqrt{\frac{2}{\pi}}\,\frac{k\,M(\omega)}{\gamma I_0(\gamma a)}\,\frac{I_1(\gamma\rho)}{x^+(k)\,x^+(\alpha)} \tag{2.47}$$

and

$$\psi_{\phi_2}(\rho,\alpha,\omega) = i\sqrt{\frac{2}{\pi}}\,\frac{kM(\omega)}{\gamma}\left[\frac{1}{I_0(\gamma b)K_0(\gamma a) - I_0(\gamma a)K_0(\gamma b)}\right]$$

$$\times\left\{\frac{I_1(\gamma\rho)K_0(\gamma a) + K_1(\gamma\rho)I_0(\gamma a)}{y^+(k)\,y^+(\alpha)} - \frac{I_1(\gamma\rho)K_0(\gamma b) + K_1(\gamma\rho)I_0(\gamma b)}{x^+(k)\,x^+(\alpha)}\right\} \tag{2.48}$$

3. Frequency Domain Description of Fields Interior to the Coaxial Waveguide

In this chapter we shall derive the frequency domain fields interior to the coaxial waveguide. We will completely characterize the TEM mode as well as the E_{0q} $(q=1,2,3,\cdots)$ mode fields. As previously stated no other modes are allowed due to the symmetry of the structure.

Let us begin by recalling equation (2.11)

$$H_{\phi_2}(\rho,z,\omega) = H_\phi^i + \psi_{\phi_2}(\rho,z,\omega)$$

The incident field is known to be

$$H_\phi^i = \frac{M(\omega)\, e^{ikz}}{\rho}$$

Our task is to find $\psi_{\phi_2}(\rho,z,\omega)$. From the definition of the spatial transform we write

$$\psi_{\phi_2}(\rho,z,\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \psi_{\phi_2}(\rho,\alpha,\omega)\, e^{-i\alpha z}\, d\alpha \tag{3.1}$$

Before attempting this integration we will examine the integrand and all singularities will be located. To do this we make use of the relations [20, p.375] between the modified and ordinary Bessel functions that follow:

$$I_\nu(z) = e^{-\frac{1}{2}\nu\pi i} J_\nu(z\, e^{\frac{1}{2}\pi i}) \;; \qquad (-\pi < \arg z \le \tfrac{\pi}{2})$$

$$K_\nu(z) = \frac{1}{2}\pi i\, e^{\frac{1}{2}\nu\pi i} H_\nu^{(1)}(z\, e^{\frac{1}{2}\pi i}); \quad (-\pi < \arg z \le \tfrac{\pi}{2})$$

We also make use of the analytic continuations [24, p.80]

$$I_\nu(z\, e^{m\pi i}) = e^{m\nu\pi i}\, I_\nu(z)$$

$$K_\nu(z\, e^{m\pi i}) = e^{-m\nu\pi i}\, K_\nu(z) - i\,\frac{\sin m\nu\pi}{\sin \nu\pi}\, I_\nu(z)$$

By these formulas it can be shown that

$$I_o(\gamma b)K_o(\gamma a) - I_o(\gamma a)K_o(\gamma b) = -\frac{\pi}{2}[J_o(\gamma' b)Y_o(\gamma' a) - J_o(\gamma' a)Y_o(\gamma' b)]$$

where $\gamma = -i\sqrt{k^2 - \alpha^2} = -i\gamma'$.

It is clear that expression (2.48) does not have a branch point at $\alpha = +k$, since we may replace γ by $-\gamma$ and not change the value of the expression. However, (2.48) has a branch singularity at $\alpha = -k$ due to the presence of $x^+(\alpha)$ and $y^+(\alpha)$. From the above, we conclude that the only singularities of $\psi_{\phi_2}(\rho,\alpha,\omega)$ which exist in the upper half plane are poles, located at $\alpha = k$ and $\alpha = \alpha_q$ $(q=1,2,3,\cdots)$ with

$$\alpha_q = \sqrt{k^2 - (\gamma'_{oq})^2} = i\sqrt{\gamma'^2_{oq} - k^2}$$

where γ'_{oq} is the qth ordered root of

$$J_o(\gamma' b)Y_o(\gamma' a) - J_o(\gamma' a)Y_o(\gamma' b) = 0 \qquad (3.2)$$

The γ'_{oq} determined by (3.2) are the eigenvalues determining the cut off frequencies of the E_{oq} modes of a coaxial waveguide [22].

To evaluate the integral (3.1), we integrate along the closed contour in the upper half plane as shown in Figure 3-1. On this

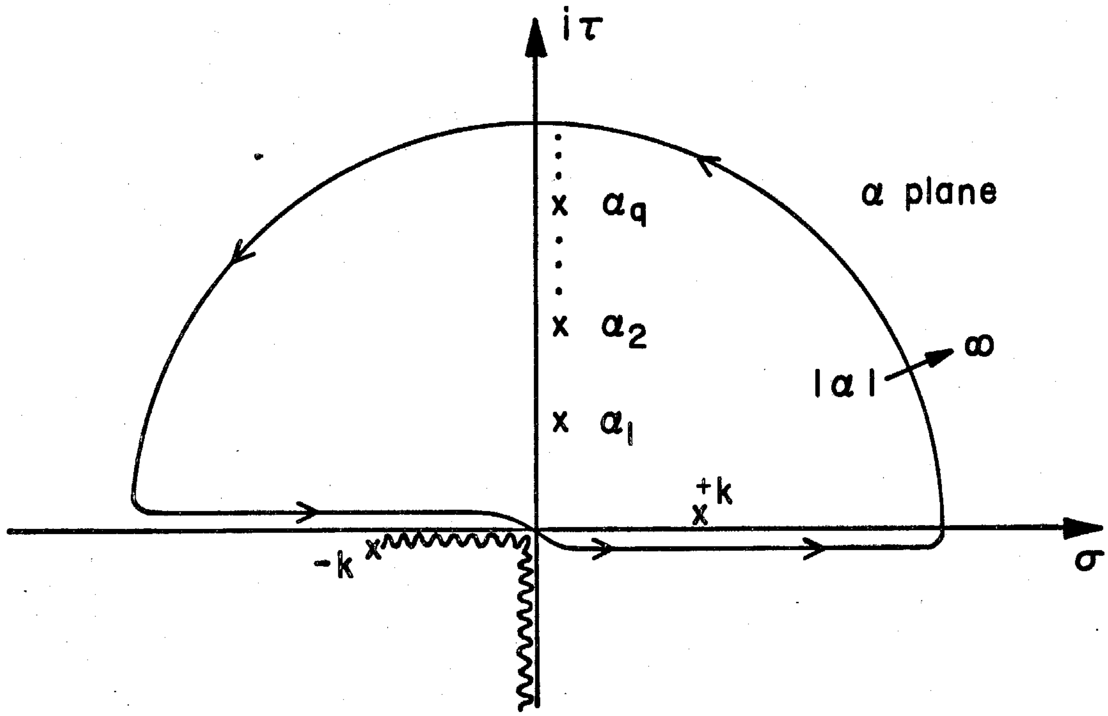


Figure 3-1. Integration Contour

contour the integral converges for $z < 0$ and the contribution along the semicircular path at $|\alpha| \to \infty$ is easily seen to be zero by replacing the modified Bessel functions in $\psi_{\phi_2}(\rho,\alpha,\omega)$ by their asymptotic values and using Jordan's lemma to evaluate the resultant integral over the portion of the contour at $|\alpha| \to \infty$.

By the residue theorem, we have

$$\psi_{\phi_2}(\rho,z,\omega) = 2\pi i \sum \text{ residues of } \{\frac{1}{\sqrt{2\pi}}\psi_{\phi_2}(\rho,\alpha,\omega)e^{-i\alpha z}\} \text{ at the poles interior to contour of Figure 3-1} \tag{3.3}$$

The residue of the pole at k is

$$\lim_{\alpha \to k}(\alpha-k)[\frac{ikM(\omega)e^{-i\alpha z}}{\pi\gamma}][\frac{1}{I_o(\gamma b)K_o(\gamma a) - I_o(\gamma a)K_o(\gamma b)}]$$

$$\{\frac{I_1(\gamma\rho)K_o(\gamma a) + K_1(\gamma\rho)I_o(\gamma a)}{y^+(k)\ y^+(\alpha)} - \frac{I_1(\gamma\rho)K_o(\gamma b) + K_1(\gamma\rho)I_o(\gamma b)}{x^+(k)\ x^+(\alpha)}\}$$

$$= \frac{ik\ M(\omega)e^{-ikz}}{\pi} \lim_{\alpha \to k}[\frac{\sqrt{\alpha-k}}{\sqrt{\alpha+k}}]$$

$$\times \frac{[y^+(k)]^{-2}(\frac{-\gamma\rho}{2}\ \ell n\ \gamma a + \frac{1}{\gamma\rho}) - [x^+(k)]^{-2}(-\frac{\gamma\rho}{2}\ \ell n\ \gamma b + \frac{1}{\gamma\rho})}{\ell n\ b/a}$$

$$= \frac{iM(\omega)}{2\pi\ \ell n\ b/a}\{[y^+(k)]^{-2} - [x^+(k)]^{-2}\}\ \frac{e^{-ikz}}{\rho} \tag{3.4}$$

where limiting forms of the Bessel functions have been used to characterize their behavior for small arguments.

The residue of the pole at α_q is

$$\lim_{\alpha\to\alpha_q} (\alpha-\alpha_q) \frac{ikM(\omega)\, e^{-i\alpha z}}{\pi\gamma\{-\frac{\pi}{2}[J_0(\gamma'b)Y_0(\gamma'a) - J_0(\gamma'a)Y_0(\gamma'b))]\}}$$

$$\{\frac{I_1(\gamma\rho)K_0(\gamma a) + K_1(\gamma\rho)J_0(\gamma a)}{y^+(k)\; y^+(\alpha)} - \frac{I_1(\gamma\rho)K_0(\gamma b) + K_1(\gamma\rho)\; J_0(\gamma b)}{x^+(k)\; x^+(\alpha)}\}$$

Using the L'Hôpital rule this becomes

$$\lim_{\alpha\to\alpha_q} [-\frac{kM(\omega)e^{-iz}}{\pi}]$$

$$\times\left\{\frac{i\,\frac{\pi}{2}[\frac{J_1(\gamma'\rho)Y_0(\gamma'a) - Y_1(\gamma'\rho)J_0(\gamma'a)}{y^+(k)\; y^+(\alpha)} - \frac{J_1(\gamma'\rho)Y_0(\gamma'b) - Y_1(\gamma'\rho)J_0(\gamma'b)}{x^+(k)\; x^+(\alpha)}]}{\frac{d}{d\alpha}[\gamma'(-\frac{\pi}{2})(J_0(\gamma'b)Y_0(\gamma'a) - J_0(\gamma'a)Y_0(\gamma'b))]}\right\} \tag{3.5}$$

Observe that $\gamma' = \sqrt{k^2 - \alpha^2}$ and $\gamma'_{oq} = \sqrt{k^2 - \alpha_q^2}$. By invoking relation (3.2) the derivative in the denominator of the above relation is readily reduced to

$$\frac{d}{d\alpha}\{\gamma'[-\frac{\pi}{2}][J_0(\gamma'b)Y_0(\gamma'a) - J_0(\gamma'a)Y_0(\gamma'b)]\}$$

$$= -\frac{\pi}{2}\gamma'[\frac{d}{d\alpha}(J_0(\gamma'b)Y_0(\gamma'a) - J_0(\gamma'a)Y_0(\gamma'b))]$$

$$= -\frac{\pi}{2}\gamma'[\frac{\alpha}{\gamma'}][b(J_1(\gamma'b)Y_0(\gamma'a) - J_0(\gamma'a)Y_0(\gamma'b))$$

$$+ a(J_0(\gamma'b)Y_1(\gamma'a) - J_1(\gamma'a)Y_0(\gamma'b))]$$

Putting this value in (3.5) yields the residue

$$\frac{i}{\pi}\ \frac{k\ M(\omega)\ e^{-iz\sqrt{k^2-(\gamma'_{oq})^2}}}{\sqrt{k^2-(\gamma'_{oq})^2}}\ [J_1(\gamma'_{oq}\rho)Y_o(\gamma'_{oq}a) - Y_1(\gamma'_{oq}\rho)J_o(\gamma'_{oq}a)]$$

$$\times \left\{ \frac{\dfrac{1}{y^+(k)y^+(\sqrt{k^2-(\gamma'_{oq})^2})} - \dfrac{\dfrac{J_o(\gamma'_{oq}b)}{J_o(\gamma'_{oq}a)}}{x^+(k)x^+(\sqrt{k^2-(\gamma'_{oq})^2})}}{b(J_1(\gamma'_{oq}b)Y_o(\gamma'_{oq}a) - J_o(\gamma'_{oq}a)Y_1(\gamma'_{oq}b)) + a(J_o(\gamma'_{oq}b)Y_1(\gamma'_{oq}a) - J_1(\gamma'_{oq}a)Y_o(\gamma'_{oq}b))} \right\} \tag{3.6}$$

The relation $\dfrac{J_o(\gamma'_{oq}b)}{J_o(\gamma'_{oq}a)} = \dfrac{Y_o(\gamma'_{oq}b)}{Y_o(\gamma'_{oq}a)}$ was also used to simplify (3.6). This relationship follows from (3.2).

$\psi_{\phi_2}(\rho,z,\omega)$ for $z \leq 0$ is found by substituting (3.4) and (3.6) into (3.3). The result is

$$\psi_{\phi_2}(\rho,z,\omega) = -M(\omega)\ \frac{R}{\rho}\ e^{-ikz} - \sum_{q=1}^{\infty} M(\omega)R_q f_q(\rho)e^{-iz\sqrt{k^2-(\gamma'_{oq})^2}}$$

$$z \leq 0$$

$$R = \frac{1}{\ln b/a}\ [[y^+(k)]^{-2} - [x^+(k)]^{-2}] \tag{3.7}$$

$$R_q = \frac{2k}{\sqrt{k^2 - (\gamma'_{oq})^2}}$$

$$\times \left\{ \frac{\dfrac{1}{y^+(k)y^+(\sqrt{k^2-(\gamma'_{oq})^2})} - \dfrac{J_o(\gamma'_{oq}b)}{J_o(\gamma'_{oq}a)\; x^+(k)\; x^+(\sqrt{k^2-(\gamma'_{oq})^2})}}{b(J_1(\gamma'_{oq}b)\,Y_o(\gamma'_{oq}a) - J_o(\gamma'_{oq}a)\,Y_1(\gamma'_{oq}b)) + a(J_o(\gamma'_{oq}b)\,Y_1(\gamma'_{oq}a) - J_1(\gamma'_{oq}a)\,Y_o(\gamma'_{oq}b))} \right\} \tag{3.8}$$

$$f_q(\rho) \doteq J_1(\gamma'_{oq}\rho)Y_o(\gamma'_{oq}a) - Y_1(\gamma'_{oq}\rho)J_o(\gamma'_{oq}a) \tag{3.9}$$

The magnetic intensity for the interior of the feedline is seen to be

$$H_{\phi_2}(\rho,z,\omega) = \frac{M(\omega)\, e^{ikz}}{\rho} + \psi_{\phi_2}(\rho,z,\omega) \quad , \quad z \leq 0 \tag{3.10}$$

4. The Radiated Electromagnetic Field

To find the magnetic intensity in region 1, we must take the inverse spatial Fourier transform of the spatial frequency domain magnetic intensity found in Chapter 2. The integral to be evaluated is

$$H_{\phi_1}(\rho,z,\omega) = -\frac{i}{\pi}\int_{-\infty}^{\infty} \frac{k\ M(\omega)\ K_1(\gamma\rho)e^{-i\alpha z}}{\gamma K_0(\gamma b)\ y^+(k)\ y^+(\alpha)}\,d\alpha \tag{4.1}$$

for $\rho \geq b$. This equation was obtained from equation (2.46).

Examination of the integrand of (4.1) using the analytic continuations of the Bessel functions and the value of γ on both sides of the branch cut reveals that the integrand has branch points at $\alpha=\pm k$. Any attempt to evaluate the above integral in closed form by contour integration would have to account for integration along a path alongside the branch cut. If the contour is closed in the upper half plane (for $z \leq 0$) then the contribution from the path alongside the branch cut in the upper half plane must be calculated. If the contour is closed in the lower half plane ($z \geq 0$) , then the contribution from the path alongside the branch cut in the lower half plane must be calculated. In the special case of $z = 0$ the integration around the branch cut in the lower half plane resulting from closing the contour in a negative sense equals the integration around the branch cut in the upper half plane which results from closing the contour in a positive sense. There are no poles located inside the contours under consideration.

The integrations around the branch cuts are quite complicated and are not evaluated in closed form. Since our primary interest is in the radiated fields (i.e., those with $1/r$ behavior), it is sufficient for our purposes to approximate the value of the integral (4.1) for $r \to \infty$,

$$r = \sqrt{\rho^2 + z^2}$$

In the limit as $r \to \infty$, we replace the Bessel function dependent on ρ by its asymptotic value for large arguments [20, p.378]. Using Figure 4-1, we readily have

$$\rho = r \cos \chi \qquad \text{and} \qquad z = r \sin \chi$$

where $|\chi| \leq \pi/2$. Using the asymptotic value

$$K_1(\gamma\rho) \sim \sqrt{\frac{\pi}{2\gamma\rho}}\; e^{-\gamma\rho}$$

the integral (4.1) becomes

$$H_{\phi_1}(r,\theta,\omega) \simeq \frac{-iM(\omega)k}{\sqrt{2\pi}\, y^+(k)} \int_{-\infty}^{\infty} \frac{e^{-(\gamma r \cos \chi + i\alpha r \sin \chi)}}{\gamma K_0(\gamma b) y^+(\alpha)[\sqrt{\gamma r \cos \chi}]}\, d\alpha \tag{4.2}$$

as $r \to \infty$, $\rho \geq b$.

The variable θ has been introduced at this point since this is the variable in which we wish to have the final expressions. It is obvious from Figure 4-1 that

$$\theta = \frac{\pi}{2} - \chi$$

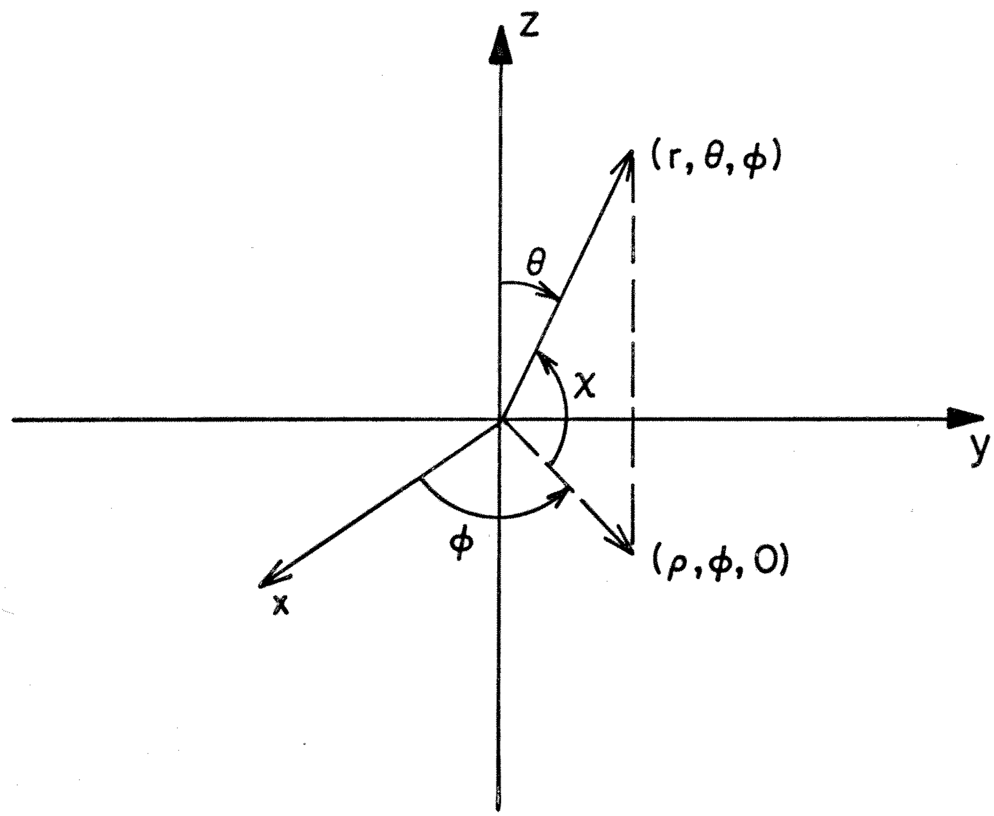


Figure 4-1. Coordinate Axes

Define a change of variable by introducing the transformation

$$\alpha = k \sin z \tag{4.3}$$

z is the complex variable $x+iy$. Under this change, the function γ becomes

$$\gamma = \sqrt{k^2 \sin^2 z - k^2} = -ik \cos z \tag{4.3.1}$$

It is necessary to specify precisely the value of z, since it is given by the multivalued function

$$z = \arcsin(\tfrac{\alpha}{k})$$

which has infinitely many branches. The precise specification will allow a one to one mapping from the complex α plane to the complex z plane. Since

$$\frac{\alpha}{k} = \sin z = \frac{e^{iz} - e^{-iz}}{2i}$$

we may write the equation

$$e^{2iz} - 2i\frac{\alpha}{k} e^{iz} - 1 = 0$$

This quadratic is readily solved for z

$$z = -i\,\ln[i\,\frac{\alpha}{k} + [1 - (\frac{\alpha}{k})^2]^{1/2}]$$

In this solution, the principal branch of the logarithm function has been used and the plus sign has been used with the square root function [25]. By the branch previously defined for γ, the above equation simplifies to

$$z = -i\ \ell n[i\ \frac{(\alpha+\gamma)}{k}] \tag{4.4}$$

The equation just developed is useful in determining how the two-sheeted α plane maps into the z plane.

Under the transformation (4.4) the numbered regions of Figure 4-2 map into the correspondingly numbered regions of Figure 4-3. The path of integration for the integral of (4.2), depicted in Figure 4-2, maps into the path shown in Figure 4-3.

As examples consider two points on the path of integration in the α plane

1) $\sigma \to -\infty$ with $|\tau| \to 0^+$

2) $\sigma \to +\infty$ with $|\tau| \to 0^-$

For case 1:

$$\gamma = \lim_{\substack{\sigma\to\infty \\ |\tau|\to 0^+}} [(-|\sigma| + i|\tau|)^2 - k^2]^{1/2} \simeq \lim_{\substack{\sigma\to\infty \\ |\tau|\to 0^+}} [|\sigma|(1 - \frac{k^2}{2|\sigma|^2}) - i|\tau|]$$

and

$$z = -i \lim_{\substack{|\sigma|\to\infty \\ |\tau|\to 0^+}} \ell n[i\frac{(-|\sigma| + i|\tau| + \gamma)}{k}] = -i \lim_{|\sigma|\to\infty} [\ell n\ \frac{-ik}{2|\sigma|}] = -\frac{\pi}{2} + i\infty$$

For case 2:

$$\gamma = \lim_{\substack{|\sigma|\to\infty \\ |\tau|\to 0^-}} [(|\sigma| - i|\tau|)^2 - k^2]^{1/2} \simeq \lim_{\substack{|\sigma|\to\infty \\ |\tau|\to 0^-}} [|\sigma|(1 - \frac{k^2}{2|\sigma|^2}) - i|\tau|]$$

and

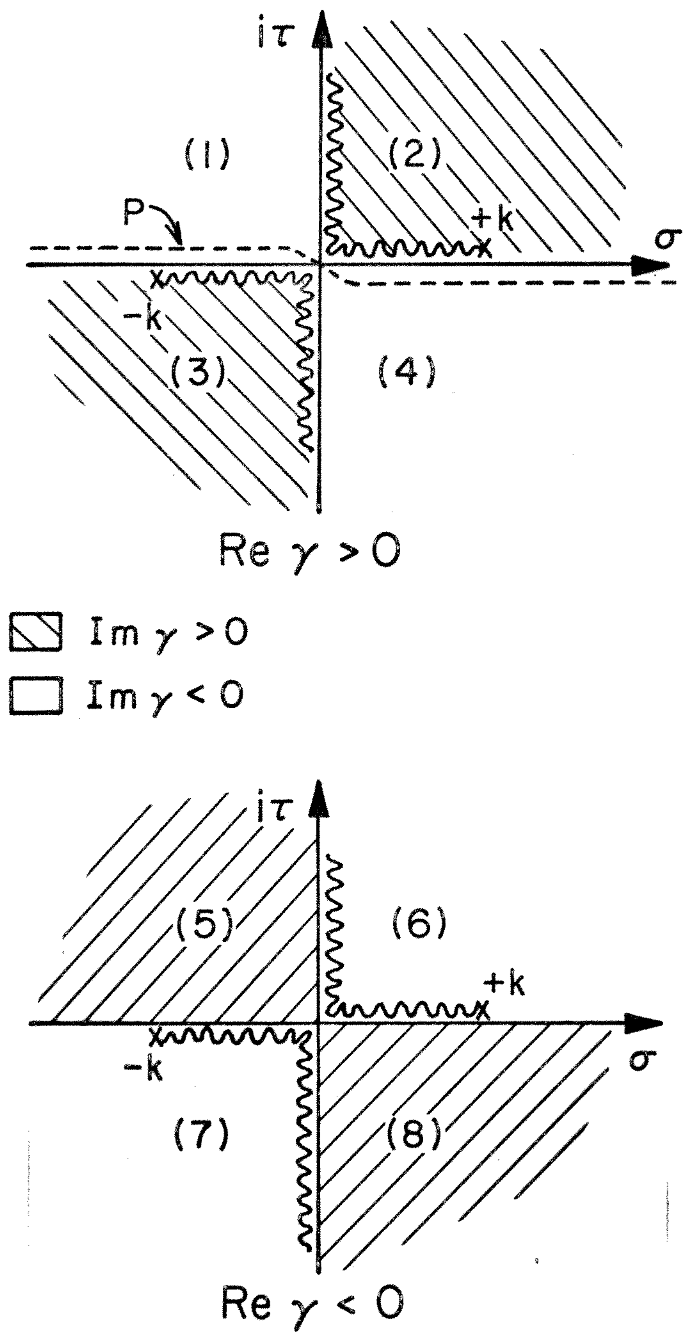


Figure 4-2. Integration path in α plane

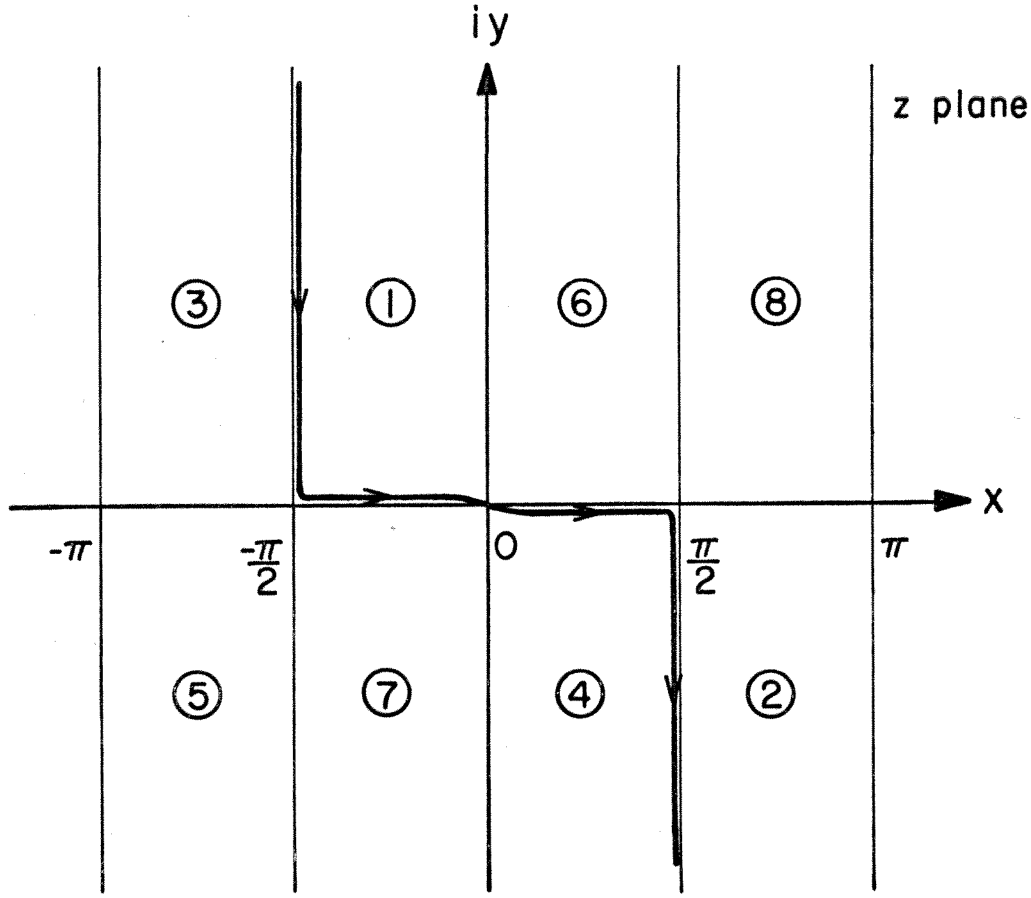


Figure 4-3. Integration path in z plane

$$z = -i \lim_{\substack{|\sigma|\to\infty \\ |\tau|\to 0^-}} \ln[i(\frac{|\sigma| - i|\tau| + \gamma}{k})] = -i \lim_{|\sigma|\to\infty} \ln[i \frac{2|\sigma|}{k}] = \frac{\pi}{2} - i\infty$$

Upon substituting (4.3) and (4.3.1) into (4.2), the integral under examination becomes

$$H_{\phi_1}(r,\theta,\omega) = \frac{M(\omega)\ k}{\sqrt{2\pi}\ y^+(k)}$$

$$\times \int_{-\infty}^{\infty} \frac{\exp(ikr(\cos\chi \cos z - \sin\chi \sin z))}{y^+(k \sin z)K_0(-ikb \cos z)\ \sqrt{-irk \cos\chi \cos z}}\, dz \qquad (4.5)$$

The integral is in the form amenable to the saddle point method of integration, since $r \to \infty$.

Define the z dependent quantity in the exponent as

$$g(z) = i(\cos\chi \cos z - \sin\chi \sin z)$$

$$= i \cos(z + \chi)$$

It follows that

$$\frac{d}{dz}\ g(z) = -i \sin(z + \chi)$$

The saddle points z_s are the points at which $g(z)$ is a maximum. Therefore from the above we know that

$$z_s = n\pi - \chi \qquad ; \qquad n=0, \pm 2, \pm 4 \cdots$$

Concentrating on the point $z_s = -\chi$, we note that $g(z_s) = i$. Since a steepest descent path is a path along which the imaginary part of $g(z)$ is constant, the steepest descent path through $z_s = -\chi$ is given by

$$\text{Im } g(z) = 1$$

Since $g(z) = i \cos(x+iy + \chi) = i(\cos(x+\chi)\text{ch } y - i \sin(x+\chi)\text{sh } y)$, the steepest descent path through $z_s = -\chi$ is given by

$$\cos(x+\chi)\text{ch } y = 1$$

This path is shown in Figure 4-4. We are assured that this is a path of steepest descent rather than one of steepest ascent, since the Re $g(z)$ monotonically decreases along this path as we move away from the saddle point.

By the residue theorem, we deform the path of integration of Figure 4-3 into the steepest descent path. The major contributions to an integral along a path of steepest descent come from points in the neighborhood of the saddle point. The slowly varying terms of the integral are replaced by their value at the saddle point and the function $g(z)$ is replaced by the leading terms of its Taylor series about the saddle point.

$$g(z) \simeq i - \frac{i}{2}(z+\chi)^2$$

$$H_{\phi_1}(r,\theta,\omega) \simeq \frac{M(\omega)k\, e^{ikr} \int_{SDP} \exp(-i\, \frac{kr}{2}(z+\chi)^2)\, dz}{\sqrt{2\pi}\; y^+(k)y^+(-k \sin\chi)\sqrt{-irk}\, \cos\chi\, K_0(-ikb \cos\chi)} \tag{4.5}$$

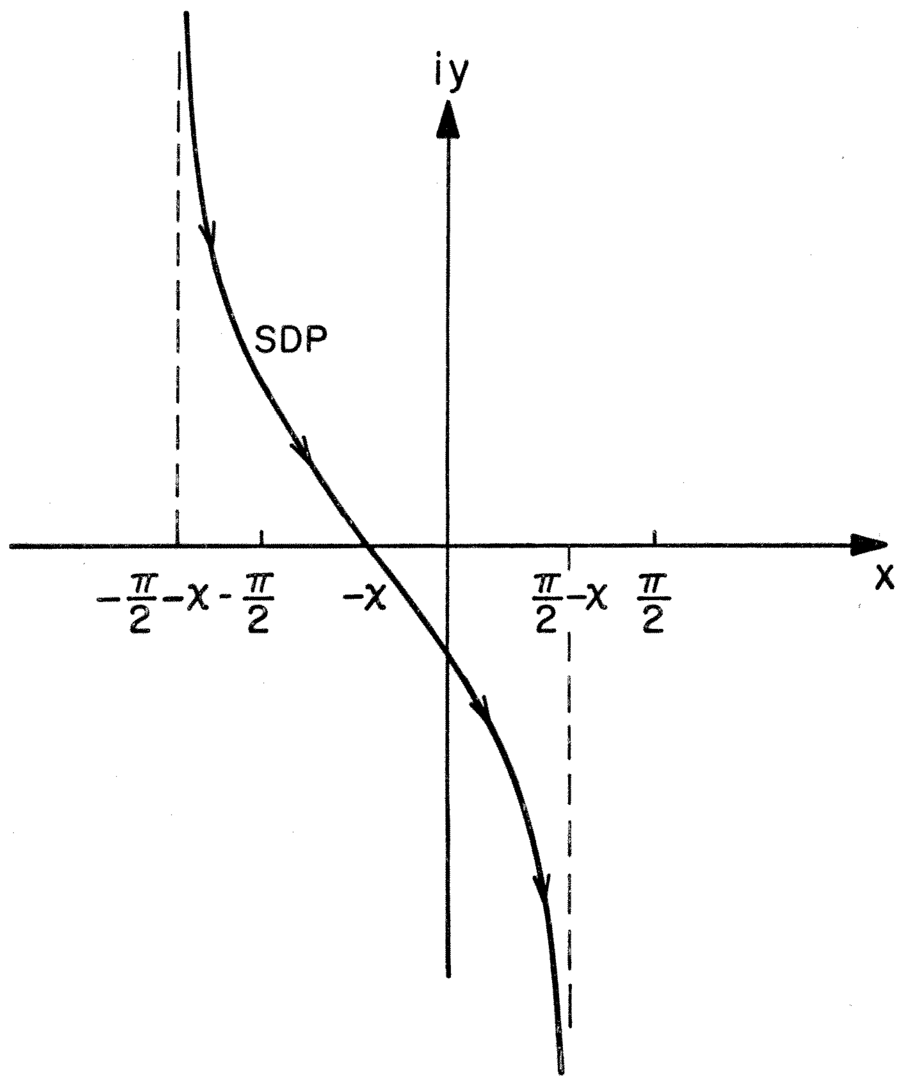


Figure 4-4. Steepest descent path .

ong the steepest descent path

$$\text{ch } y = \frac{1}{\cos(x+\chi)}$$

and

$$\frac{dy}{dx} = + \frac{\tan(x+\chi)}{\cos(x+\chi)\text{sh } y}$$

As we approach the saddle point in the manner $x \to -\chi$ and $y \to 0^+$, the slope is

$$\lim_{\substack{x \to -\chi \\ y \to 0^+}} \frac{dy}{dx} = -1$$

As we approach the saddle point in the manner $x \to -\chi$ and $y \to 0^-$, the slope is

$$\lim_{\substack{x \to -\chi \\ y \to 0^-}} \frac{dy}{dx} = -1$$

Therefore the angle that the SDP makes with the real axis at the saddlepoint is either

$$\frac{3\pi}{4} \quad \text{or} \quad -\frac{\pi}{4}$$

Examination of Figure 4-4 shows that at the saddle point the SDP and the negative real axis form the angle $3\pi/4$ in the second quadrant and $-\pi/4$ in the third quadrant. This observation permits us to express

$$z + \chi = u = \begin{cases} |u|\, e^{i\, 3\pi/4} & ; \quad u \in \text{SDP and } y > 0 \\ |u|\, e^{-i\, \pi/4} & ; \quad u \in \text{SDP and } y < 0 \end{cases}$$

It is also observed that

$$\frac{du}{dz} = \frac{d|u|}{dz} e^{-i\ \pi/4} \quad \text{at the saddle point}$$

Since equal contributions to the integral of (4.5) are received as we move away from the saddle point in the two directions of increasing u, we write the integral which appears in (4.5) as

$$\int_{SDP} \exp(-i\ \frac{kr}{2}(z+\chi)^2)dz \simeq 2\int_0^\infty e^{-\frac{kr}{2}|u|^2}\ e^{-i\frac{\pi}{4}}\ d|u| \tag{4.6}$$

Using the well known integral

$$\int_0^\infty e^{-a^2\alpha^2}\ dx = \frac{\sqrt{\pi}}{2a}$$

the value of (4.6) is seen to be

$$2\int_0^\infty (e^{-\frac{kr}{2}|u^2| - i\frac{\pi}{4}})\ d|u| = e^{-i\frac{\pi}{4}}\sqrt{\frac{2\pi}{kr}} \tag{4.7}$$

Equation (4.5) becomes

$$H_{\phi_1}(r,\theta,\omega) \simeq M(\omega)\ \frac{e^{ikr}}{r}\ \{\frac{1}{y^+(k)y^+(-k\ \sin\ \chi)K_o(-ikb\ \cos\ \chi)\cos\ \chi}\} \tag{4.8}$$

To simplify the above expression we use the identity relating modified Bessel functions with ordinary Bessel functions:

$$K_o(-ikb\ \cos\ \chi) = i\ \frac{\pi}{2}\ H_o^{(1)}(kb\ \cos\ \chi)$$

From Appendix D, the expression

$$y(\alpha) = \left[i \frac{\pi}{2} H_0^{(1)}(Kb)[J_0(Kb) - J_0(Ka)] \right]^{-1}$$

is formed, where $K = +i\gamma$. In expression (4.8) we have

$$i \frac{\pi}{2} H_0^{(1)}(kb \cos \chi) = i \frac{\pi}{2} H_0^{(1)}(kb \sin \theta)$$

If in the expression $y(\alpha)$, K is set equal to $k \sin \theta$ and α is set equal to $k \cos \theta$, which follows from the fact that $\gamma^2 = -K^2 = \alpha^2 - k^2$, then

$$y(k \cos \theta) = y^+(k \cos \theta)\, y^-(k \cos \theta)$$

$$= \left[i \frac{\pi}{2} H_0^{(1)}(kb \sin \theta)[J_0(kb \sin \theta) - J_0(ka \sin \theta)] \right]^{-1}$$

When this equation is solved for $i \frac{\pi}{2} H_0^{(1)}(kb \sin \theta)$ and the result substituted into (4.8), the far zone magnetic intensity is seen to be

$$H_{\phi_1}(r,\theta,\omega) \simeq M(\omega) \frac{e^{ikr}}{r} \left[\frac{J_0(kb \sin \theta) - J_0(ka \sin \theta)}{\sin \theta}\right]\left(\frac{y^+(k \cos \theta)}{y^+(k)}\right) \tag{4.10}$$

The relation (4.10) has the restriction that $\rho \geq b$. In the region $z < 0$ the equation (4.10) is only valid in the region $\rho \geq b$; however, for $z \geq 0$, the H_{ϕ_1} can be analytically continued from region 1 to 2 and from region 2 to 3. Therefore the restriction requiring $\rho \geq b$ can be removed in the region $z \geq 0$ and equation (4.10) becomes an expression for the magnetic intensity for all values of ρ in the far zone $(r \to \infty)$ exterior to the feedline.

It is recognized that equation (4.10), except for the factor $\frac{y^+(k\cos\theta)}{y^+(k)}$, exactly equals the results obtained from assuming that the incident field is the total aperture field and that no contribution is made to the radiated fields by currents which flow on the waveguide structure.

The new field factor contribution $\frac{y^+(k\cos\theta)}{y^+(k)}$ to the coaxial waveguide antenna first obtained in this work modifies the field pattern obtained by the conventional solution to the problem of radiation from this type of antenna [19,p.302]. It will be seen later that this factor has a significant effect at high frequencies and modifies the field pattern to direct the radiation in a more forward direction.

By equation (2.11) we expressed the magnetic intensity in region 2 for all z as the sum of the incident field and another term which represented the reflected and higher order mode field intensities. By the radiation condition, we know that the incident field cannot exist in region 2 for $z > 0$.

To show that the incident field intensity is canceled in region 2 for $z > 0$, we evaluate the integral

$$\psi_{\phi_2}(\rho,z,\omega) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \psi_{\phi_2}(\rho,\alpha,\omega)\, e^{-i\alpha z}\, d\alpha\,, \quad z > 0 \tag{4.11}$$

by using the contour of Figure 4-5, $\psi_{\phi_2}(\rho,\alpha,\omega)$ is given by (2.48). The value of the contour integral is calculated by the residue theorem. The value of the integral (4.11) is the negative of the contribution from the path around the branch cut and $-2\pi i$ times the summation of the

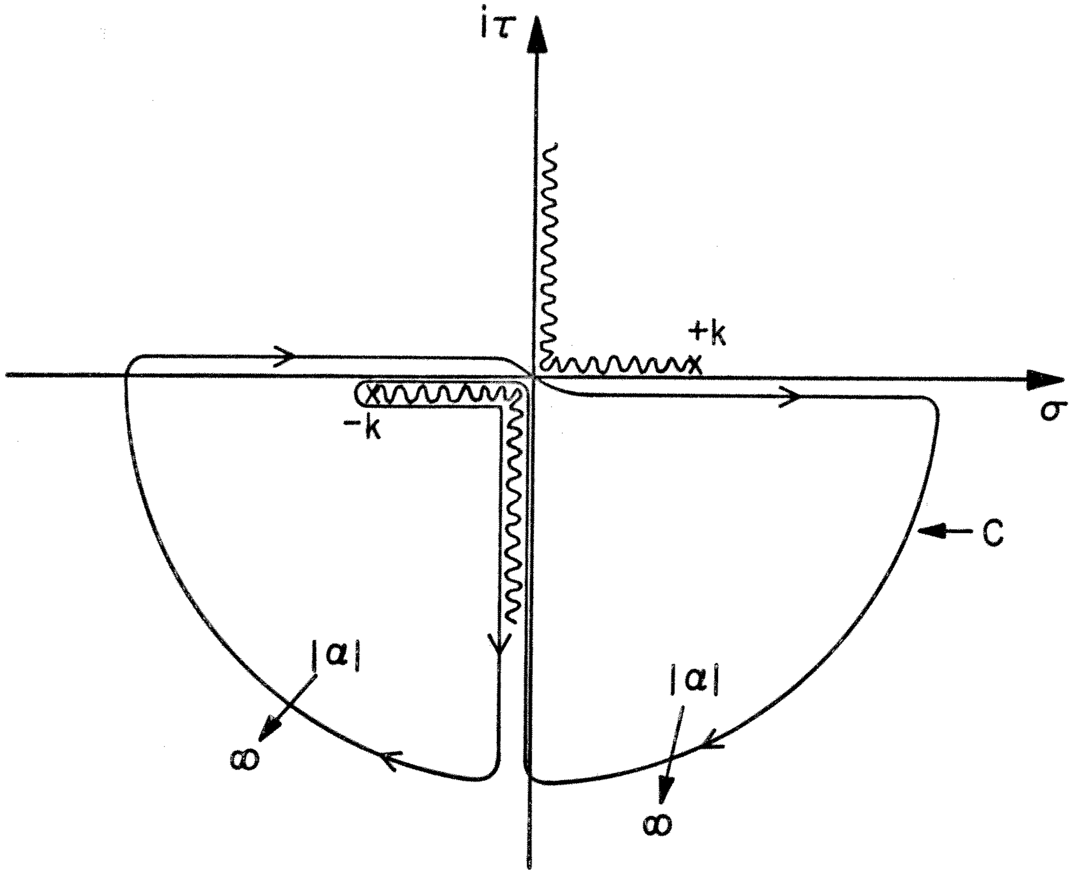


Figure 4-5. Integration contour

residues at the poles of the integrand located on the interior of the contour c. The contribution along the semicircular arc at $|\alpha| \to \infty$ is zero.

The integral (4.11) is not evaluated in closed form due to the complex integration on the path alongside the branch cut. However, we shall show that the contribution to the value of the integral from the circular path around the branch point (k) is just enough to cancel the incident field intensity.

On the circular path around the branch point $\alpha = -k + \varepsilon e^{i\theta}$, $\varepsilon \to 0$ and $0 \le \theta \le 2\pi$. The integral of interest becomes

$$\lim_{\varepsilon \to 0} \left[- \frac{1}{\sqrt{2\pi}} \int_{\circlearrowleft -k} \psi_2(\rho,\alpha,\omega)\, e^{-i\alpha z}\, d\alpha \right]$$

$$= \lim_{\varepsilon \to 0} \left\{ \frac{-ikM(\omega)\, e^{ikz}}{\pi\, \ell n \frac{b}{a}} \int_0^{2\pi} \left[\frac{-\gamma\rho\, \ell n\, \gamma a + \frac{1}{\gamma\rho}}{y^+(k)\, y^+(\alpha)} - \frac{-\gamma\rho\, \ell n\, \gamma b + \frac{1}{\gamma\rho}}{x^+(k)\, x^+(\alpha)} \right] \frac{d\alpha}{\gamma} \right\}$$

$$= \lim_{\varepsilon \to 0} \left\{ \frac{-ik\, M(\omega)\, e^{ikz}}{\pi\, \ell n \frac{b}{a}} \left[\frac{1}{y(k)} - \frac{1}{x(k)} \right] \int_0^{2\pi} \frac{d\alpha}{\gamma^2 \rho} \right\}$$

$$= - \frac{M(\omega)}{\rho\, \ell n \frac{b}{a}} \left[\frac{1}{y(k)} - \frac{1}{x(k)} \right] e^{ikz}$$

We have made use of the limiting forms of Bessel functions for small arguments and the following:

$$x^{+}(-k) = x^{-}(k) \quad ; \quad x^{+}(k)\, x^{-}(k) = x(k)$$

$$y^{+}(-k) = y^{-}(k) \quad ; \quad y^{+}(k)\, y^{-}(k) = y(k)$$

To see that this is just enough to cancel the incident magnetic intensity, note that

$$\frac{1}{y(k)} - \frac{1}{x(k)} = \ell n\, \frac{b}{a}$$

which is easily found from the definition of $y(k)$ and $x(k)$ given in Chapter 2 and the limiting forms of Bessel functions for small arguments.

5. Time Behavior of the Radiated and Reflected Electromagnetic Fields

The time behavior of the field intensities is found by computing the inverse time transform of the frequency domain expressions. To find the time behavior of the radiated magnetic intensity emanating from the waveguide antenna, the inverse transform of equation (4.10) must be obtained. Because of the complicated functional behavior of the factor $\frac{y^+(k \cos \theta)}{y^+(k)}$ the computation of the integral

$$H(r,\theta,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} M(\omega) \frac{e^{ikr}}{r} \left[\frac{J_o(kb \sin \theta) - J_o(ka \sin \theta)}{\sin \theta}\right] \times \frac{y^+(k \cos \theta)}{y^+(k)} e^{-i\omega t} \, d\omega \qquad (5.1)$$

is extremely difficult.

For harmonic time dependence the exciting voltage $V(t)$ is $\text{Re}(A\, e^{i\omega_o t})$ (see equation (2.7)). Then

$$M(\omega) = \text{Re}\left[\sqrt{2\pi} \sqrt{\frac{\varepsilon_o}{\mu_o}} \frac{A}{\ell n\, b/a} \delta(\omega - \omega_o)\right]$$

where $\delta(x)$ is the Dirac delta function. Therefore, the case of harmonic time dependence (5.1) reduces to

$$H(r,\theta,t) = \text{Re}\left[\sqrt{\frac{\varepsilon_o}{\mu_o}} \frac{A}{\ell n\, b/a} \frac{y^+(\frac{\omega_o}{c} \cos \theta)}{y^+(\frac{\omega_o}{c})} \frac{e^{-i[\omega_o(t - r/c)]}}{r \sin \theta} \times \left[J_o\left(\frac{\omega_o b}{c} \sin \theta\right) - J_o\left(\frac{\omega_o a}{c} \sin \theta\right)\right]\right] \qquad (5.2)$$

The value of k is ω/c as given by (2.4.3).

The factors of equation (5.2) except $\dfrac{y^{+}(\frac{\omega_o}{c}\cos\theta)}{y^{+}(\frac{\omega_o}{c})}$ are well understood, and are not in need of further simplification.

From Appendix D we transpose the function

$$\frac{y^{+}(k\cos\theta)}{y^{+}(k)} = \prod_{m=1}^{\infty}\left[\frac{1+\frac{k}{i|\alpha_m|}}{1+\frac{k\cos\theta}{i|\alpha_m|}}\right] e^{i\frac{(a+b)k}{2m}(1-\cos\theta)}\left\{e^{\frac{i\theta bk}{\pi}\sin\theta}\right\}$$

$$\times\left\{\exp\left(\int_{\delta\to 0^+}^{\infty} K^{(2)}(u)\,\ell n\left[\frac{1+\frac{k}{\sqrt{k^2-u^2}}}{1+\frac{k\cos\theta}{\sqrt{k^2-u^2}}}\right]du\right)\right\}\left\{e^{k(\frac{a+b}{i2\pi})(\cos\theta-1)(1-C+i\frac{\pi}{2})}\right\}$$

$$\times\left\{\exp\left(k(\frac{a+b}{i2})\cos\theta[-\ell n(k(\frac{a+b}{2\pi})\cos\theta)+\frac{2b}{a+b}\,\ell n(2\cos\theta)\right.\right.$$

$$\left.\left.-\,k\,\frac{(a+b)}{i2\pi}(-\ell n\frac{k(a+b)}{\pi}+\frac{2b}{a+b}\,\ell n\,2)\right)\right\} \tag{5.3}$$

In this formula we have used the fact that if $\alpha = k\cos\theta$, then

$$\ell n(\frac{\gamma+(\alpha-k)}{\gamma-(\alpha-k)}) = \ell n[\frac{-ik\sin\theta+k(\cos\theta-1)}{-ik\sin\theta-k(\cos\theta-1)}] = \ell n\left[\frac{e^{-i\theta}-1}{1-e^{i\theta}}\right] = -i\theta$$

using the principal branch of the logarithm function. Additionally,

$$K^{(2)}(u) = \frac{b}{\pi}\left[1-\frac{2}{\pi ub}\left(\frac{1}{J_o^2(ub)+Y_o^2(ub)}\right)\right]$$

It is also known from Appendix D that $y^+(\alpha) \sim |\alpha|^{1/2}$ as $\alpha \to \infty$. From this asymptotic behavior we see for very high frequencies ($\lambda << b-a$) that

$$\lim_{k \to \infty} \frac{y^+(k \cos \theta)}{y^+(k)} \sim |\cos \theta|^{1/2} \tag{5.4}$$

In Chapter 2, there was imposed a restriction that $kb, ka << 1$ for non-harmonic time dependence. In keeping with this restriction, equation (5.4) should only be used for the case of harmonic time dependent excitation. For values of $ka, kb << 1$ the value of $\frac{y_+(k \cos \theta)}{y_+(k)}$ is more complicated but instructive to determine. To determine the value of this field factor in the low frequency limit ($\lambda >> b-a$) we use the series expansion of the infinite product

$$\prod_{m=1}^{\infty} \left\{ \frac{1 + \frac{k}{i|\alpha_m|}}{1 + \frac{k \cos \theta}{i|\alpha_m|}} \right\} e^{i\frac{(a+b)k}{2m}(1 - \cos \theta)}$$

$$= 1 - ik\left(\frac{a+b}{2}\right)(1 - \cos \theta) \sum_{m=1}^{\infty} \left[\frac{1}{|\alpha_m|\left(\frac{a+b}{2}\right)} - \frac{1}{m\pi}\right] + O(ka)^2$$

The infinite series is the difference of two divergent series but is itself convergent [26, p.33]. This is apparent since

$$\frac{1}{|\alpha_m| \frac{a+b}{2}} - \frac{1}{m\pi} \simeq \frac{1}{(m - \frac{1}{4})\pi} - \frac{1}{m\pi} = \frac{1}{4\pi m^2}$$

as $m \to \infty$ and the infinite series $\sum_{m=1}^{\infty} \frac{1}{m^2}$ is known to be absolutely convergent. The following integral

$$\int_{\delta\to 0^+}^{\infty} K^{(2)}(u)\ \ln\left[\frac{1+\dfrac{k}{\sqrt{k^2+u^2}}}{1+\dfrac{k\cos\theta}{\sqrt{k^2-u^2}}}\right] du$$

also appears in $\dfrac{y^+(k\cos\theta)}{y^+(k)}$. This integral will now be shown to be convergent and its approximate value will be determined in the limit $kb, ka << 1$.

Upon substitution of the value of $K^{(2)}(u)$ and simplifying, the above integral becomes

$$\int_{\delta\to 0^+}^{\infty} \frac{b}{\pi}\left[1-\frac{2}{\pi ub}\left(\frac{1}{J_o^2(ub)-Y_o^2(ub)}\right)\right] \ln\left[1+\frac{1-\cos\theta}{\cos\theta+\sqrt{1-(\frac{u}{k})^2}}\right] du$$

Introducing a change of variable, $ub = z$, we get

$$\int_{\delta\to 0^+}^{\infty} \frac{1}{\pi}\left[1-\frac{2}{\pi z}\left[\frac{1}{J_o^2(z)+Y_o^2(z)}\right]\right] \ln\left[1+\frac{1-\cos\theta}{\cos\theta+\sqrt{1-(\frac{z}{kb})^2}}\right] dz \tag{5.5}$$

To examine the behavior of (5.5) for small values of z, we consider

$$\frac{1}{\pi}\int_{\delta\to 0^+}^{\varepsilon} \left[1-\frac{2}{\pi z}\frac{1}{1+\frac{4}{\pi^2}\ln^2 z}\right] \ln\left[1+\frac{1-\cos\theta}{\cos\theta+\sqrt{1+(\frac{z}{kb})^2}}\right] dz$$

This integral converges as $\int_0^{\varepsilon} [\frac{1}{z}(\frac{1}{1+\frac{4}{\pi^2}\ln^2 z})]\, dz$ converges.

Now

$$\int_0^{\varepsilon} [\frac{1}{z}(\frac{1}{1+\frac{4}{\pi^2}\ln^2 z})]\, dz \simeq \frac{\pi^2}{4}\int_0^{\varepsilon} \frac{dz}{z\,\ln^2(z)}$$

$$= \frac{\pi^2}{4}\int_0^{\varepsilon} d[-\frac{1}{\ln z}] = -\frac{\pi^2}{4}\frac{1}{\ln z}\Bigg|_0^{\varepsilon}$$

This quantity is seen to be convergent for all $\varepsilon \neq 1$. We can also remove the principal value sign and change the lower limit to zero without affecting the value of the integration.

At the other extreme for large values of z, we consider

$$\frac{1}{\pi}\int_M^{\infty} \left\{1 - \frac{2}{\pi z}[\frac{1}{|H_o^{(1)}(z)|^2}]\right\} \ln\left[1 + \frac{1-\cos\theta}{\cos\theta - i\sqrt{(\frac{z}{kb})^2 - 1}}\right] dz$$

for $M \gg 1$. The asymptotic value

$$H_o^{(1)}(z) \sim (\frac{2}{\pi z})^{1/2} e^{i(z-\frac{\pi}{4})}[1 + 0(\frac{1}{z})] \quad \text{for} \quad z \to \infty$$

is now substituted with the result

$$\frac{1}{\pi}\int_M^{\infty} [1 - \frac{1}{[1+0(\frac{1}{z})]}] \ln\left[1 + \frac{1-\cos\theta}{\cos\theta - i\sqrt{(\frac{z}{kb})^2 - 1}}\right] dz$$

$$= \frac{1}{\pi}\int_M^{\infty} 0(\frac{1}{z}) \ln\left[1 + \frac{1-\cos\theta}{\cos\theta - i\sqrt{(\frac{z}{kb})^2 - 1}}\right] dz$$

The logarithmic function $\ell n(1+x)$ can be expanded

$$\ell n(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots \quad , \quad |x| \leq 1 \; ; \quad x \neq -1$$

When this expansion is considered along with the fact that

$$1 + \frac{1 - \cos\theta}{\cos\theta - i\sqrt{(\frac{z}{kb})^2 - 1}} \Rightarrow 1 + \frac{A}{z}$$

as $z \to \infty$ (with z real and $|A|$ finite), then

$$\frac{1}{\pi}\int_M^\infty O(\frac{1}{z}) \, \ell n[\, 1 + \frac{1 - \cos\theta}{\cos\theta - i\sqrt{(\frac{z}{kb})^2 - 1}}\,] dz \simeq$$

$$\frac{1}{\pi}\int_M^\infty O(\frac{1}{z}) \, (\frac{A}{z}) \, dz$$

This integral converges.

The integrand of (5.5) has an integrable singularity at $z = kb$ when $\theta = \pi/2$. By the above we conclude that, for all values of θ and k the integral of (5.5) is convergent.

Since our interest is in the value of $\frac{y_+(k\cos\theta)}{y_+(k)}$ for $ka, kb << 1$, we next endeavor to find the value of (5.5) in the limit as $kb \to 0$. We will use this limiting value as an approximation for the integral appearing in equation (5.3).

$$\lim_{kb \to 0} \frac{1}{\pi} \int_0^\infty [1 - \frac{2}{\pi z} (\frac{1}{J_o^2(z) + Y_o^2(z)})] \ln\left(1 + \frac{1 - \cos\theta}{\cos\theta + \sqrt{1 - (\frac{z}{kb})^2}}\right) dz$$

$$= \lim_{kb \to 0} \frac{1}{\pi} \int_0^{kb} [1 - \frac{2}{\pi z} (\frac{1}{J_o^2(z) + Y_o^2(z)})] \ln\left(1 + \frac{1 - \cos\theta}{\cos\theta + \sqrt{1 - (\frac{z}{kb})^2}}\right) dz$$

$$+ \lim_{kb \to 0} \frac{1}{\pi} \int_{kb}^\infty [1 - \frac{2}{\pi z} (\frac{1}{J_o^2(z) + Y_o^2(z)})] \ln\left(1 + \frac{1 - \cos\theta}{\cos\theta - i\sqrt{(\frac{z}{kb})^2 - 1}}\right) dz$$

$$= 0$$

Note that the quantity

$$\lim_{kb \to 0} [\ln(1 + \frac{1 - \cos\theta}{\cos\theta - i\sqrt{(\frac{z}{kb})^2 - 1}})] \Longrightarrow 0$$

From the above work we conclude that as kb goes to zero, the integral (5.5) goes to zero. Thus, for low frequencies, we approximate the function

$$\exp\left\{ \int_{\delta \to 0^+}^{\infty} K^{(2)}(u) \ln \left[\frac{1 + \frac{k}{\sqrt{k^2 - u^2}}}{1 + \frac{k\cos\theta}{\sqrt{k^2 - u^2}}} \right] du \right\}$$

appearing in equation (5.3) by unity.

If in (5.3) we make use of the limit that $x \ln x \simeq x$ as $x \to 0$, then the field factor $\frac{y^+(k\cos\theta)}{y^+(k)}$ can be approximated as

$$\frac{y^+(k\cos\theta)}{y^+(k)} \simeq \left(1 - ik(\tfrac{a+b}{2})[1-\cos\theta]\sum_{m=1}^{\infty}(\frac{1}{|\alpha_m|\,\frac{a+b}{2}} - \frac{1}{m\pi}) + O(ka)^2\right)$$

$$\times \exp\left(i\,\frac{bk}{\pi}\sin\theta + ik(\tfrac{a+b}{2\pi})[C(1-\cos\theta)] - \frac{2b}{a+b}[\cos\theta\,\ell n(2\cos\theta) - \ell n\,2]\right)$$

$$\text{for } ka,kb \ll 1 \qquad (5.6)$$

The time behavior of the far zone radiated magnetic intensity for a non-harmonic time dependent incident field intensity with $ka,kb \ll 1$ is given by (5.1) when equation (5.6) is used for $\frac{y^+(k\cos\theta)}{y^+(k)}$. In the far zone the value of r in equation (5.1) is very much larger than b or $\frac{a+b}{2}$. To the approximation used, the exponential function in (5.6) is replaced by one and for that reason it does not contribute to the phase factors in integral (5.1). In addition the term of (5.6) containing the infinite sum can be dropped. This is apparent since $ka,kb \ll 1$ and the Fourier transform of this term will be of order $1/c$, which will make its contribution very much smaller than the contribution from the remaining term. In view of this

$$\frac{y^+(k\cos\theta)}{y^+(k)} \simeq 1$$

and

$$H_\phi(r,\theta,t) \simeq \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{M(\omega)e^{-i(\omega t - kr)}}{r\sin\theta}\,[J_0(kb\sin\theta) - J_0(ka\sin\theta)]\,d\omega$$

$$\text{for } ka,kb \ll 1 \qquad (5.7)$$

When the incident magnetic intensity is harmonically time dependent, then the value of (5.7) is easily obtained. In this case

$$M(\omega) = \mathrm{Re}\sqrt{2\pi}\sqrt{\frac{\varepsilon_o}{\mu_o}}\frac{A}{\ell n\ b/a}\,\delta(\omega-\omega_o)$$

and for low frequencies ($\lambda >> b-a$) the radiated fields are

$$H_\phi(r,\theta,t) \simeq A\sqrt{\frac{\varepsilon_o}{\mu_o}}\frac{\cos(\omega_o(t-\frac{r}{c}))}{r(\ell n\ b/a)\sin\theta}\left[J_o(\frac{\omega_o b}{c}\sin\theta) - J_o(\frac{\omega_o a}{c}\sin\theta)\right]$$

$$E_\theta(r,\theta,t) = \sqrt{\frac{\mu_o}{\varepsilon_o}}\,H_\phi(r,\theta,t) \tag{5.8}$$

For non-harmonic time dependence the radiated field can be found by direct integration of (5.7) or by a time domain convolution. The integrand of equation (5.7) can be considered as the product of the Fourier transforms of two time domain functions. Therefore this integral can be evaluated by the convolution theorem and the impulse response is readily identified from the result.

In applying the convolution theorem, use is made of the following well known integral [27]:

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} J_o(\alpha z)e^{-i\alpha t}\,d\alpha = \begin{cases} \dfrac{\sqrt{2/\pi}}{\sqrt{z^2-t^2}} & ; \quad 0 \le |t| < z \\ \infty & ; \quad |t| = z \\ 0 & ; \quad \text{otherwise} \end{cases} \tag{5.8.1}$$

Upon evaluation of equation (5.7) by the convolution theorem we get the result which follows:

$$H_\phi(r,\theta,t) \simeq \frac{1}{\pi r \sin\theta} \int_{-\infty}^{t'} M(t'-\tau)\, f(\tau)\, d\tau \qquad (5.9)$$

where $f(\tau)$ is

$$f(\tau) = \begin{cases} 0 & ; \quad \tau < -\frac{b}{c}\sin\theta \\ \dfrac{1}{\sqrt{(\frac{b}{c}\sin\theta)^2 - \tau^2}} & ; \quad -\frac{b}{c}\sin\theta \le \tau \le -\frac{a}{c}\sin\theta \\ \dfrac{1}{\sqrt{(\frac{b}{c}\sin\theta)^2 - \tau^2}} - \dfrac{1}{\sqrt{(\frac{a}{c}\sin\theta)^2 - \tau^2}} & ; \quad -\frac{a}{c}\sin\theta \le \tau \le \frac{a}{c}\sin\theta \\ \dfrac{1}{\sqrt{(\frac{b}{c}\sin\theta)^2 - \tau^2}} & ; \quad \frac{a}{c}\sin\theta \le \tau \le \frac{b}{c}\sin\theta \\ 0 & ; \quad \tau > \frac{b}{c}\sin\theta \end{cases} \qquad (5.9.1)$$

and

$$t' = t - \frac{r}{c}$$

The response to an impulse of voltage excitation is therefore

$$H_\phi(r,\theta,t) = \sqrt{\frac{\varepsilon_0}{\mu_0}}\, \frac{f(t - \frac{r}{c})}{\pi r \sin\theta\, \ell n\, b/a} \qquad (5.10)$$

From the character of the impulse response we see that at an observation point in the far zone, the radiation lasts a total time of $(2b/c)\sin\theta$. By ignoring for the moment the frequency restriction of ka,kb << 1 and allowing an excitation with zero rise time, a step function of voltage , then by integration of the impulse response we obtain the step response,

$$H_\phi(r,\theta,t) = \sqrt{\frac{\varepsilon_0}{\mu_0}}\left(\frac{1}{\pi r \sin\theta \, \ell n \frac{b}{a}}\right) \times$$

$$\begin{bmatrix} 0 & ; \ \tau \le -\frac{b}{c}\sin\theta \\ \frac{\pi}{2} + \sin^{-1}\frac{\tau c}{b\sin\theta} & ; \ -\frac{b}{c}\sin\theta \le \tau \le -\frac{a}{c}\sin\theta \\ \sin^{-1}\frac{\tau c}{b\sin\theta} - \sin^{-1}\frac{\tau c}{a\sin\theta} & ; \ -\frac{a}{c}\sin\theta \le \tau \le \frac{a}{c}\sin\theta \\ -\frac{\pi}{2} + \sin^{-1}\frac{\tau c}{b\sin\theta} & ; \ \frac{a}{c}\sin\theta \le \tau \le \frac{b}{c}\sin\theta \\ 0 & ; \ \tau \ge \frac{b}{c}\sin\theta \end{bmatrix} \tag{5.11}$$

where $\tau = t - \frac{r}{c}$.

It is advantageous to have the step response for numerical calculations since its magnitude is finite at all points and the far zone fields can be found by the convolution of the time derivative of the excitation and the step response. This procedure does not violate the low frequency restriction.

Because of the low frequency restriction the input excitation should not contain any high frequency components. To render the high frequency components in the excitation negligible, the rise time of the excitation pulses must be long compared with the time b/c . If the input is of the form $[1 - e^{-t/T}]\, u(t)$ then the time constant T must be chosen so that $T \gg b/c$. This will preclude any violation of the low frequency restriction.

For such inputs, equation (5.9) contains convolution integrals of the type

$$\int_{-\frac{b}{c}\sin\theta}^{t'} \frac{1 - e^{-\frac{|t'-\tau|}{T}}}{\sqrt{(\frac{b}{c}\sin\theta)^2 - \tau^2}}\, d\tau, \text{ with } -\frac{b}{c}\sin\theta \le \tau \le \frac{b}{c}\sin\theta$$

$$\text{and } \tau \le t'$$

$$\int_{-\frac{b}{c}\sin\theta}^{t'} \frac{(1 - e^{-\frac{|t'-\tau|}{T}})\, d\tau}{\sqrt{(\frac{b}{c}\sin\theta)^2 - \tau^2}} \approx \int_{-\frac{b}{c}\sin\theta}^{t'} \frac{(1 - e^{-\frac{|t'|}{T}})\, d\tau}{\sqrt{(\frac{b}{c}\sin\theta)^2 - \tau^2}}$$

$$+ \frac{e^{-|t'|/T}}{T} \int_{-\frac{b}{c}\sin\theta}^{t'} \frac{\tau\, d\tau}{\sqrt{(\frac{b}{c}\sin\theta)^2 - \tau^2}}$$

$$= (1 - e^{-|t'|/T}) \int_{-\frac{b}{c}\sin\theta}^{t'} \frac{d\tau}{\sqrt{(\frac{b}{c}\sin\theta)^2 - \tau^2}} - \frac{e^{-|t'|/T}}{T}\sqrt{(\frac{b}{c}\sin\theta)^2 - (t')^2}$$

$$0 \le |t'| \le \frac{b}{c}\sin\theta$$

Since $T \gg b/c$ we neglect the second term and conclude that the response to an input of finite rise time with time constant $T \gg b/c$ approximately equals the step response multiplied by $(1 - e^{-|t'|/T})$.

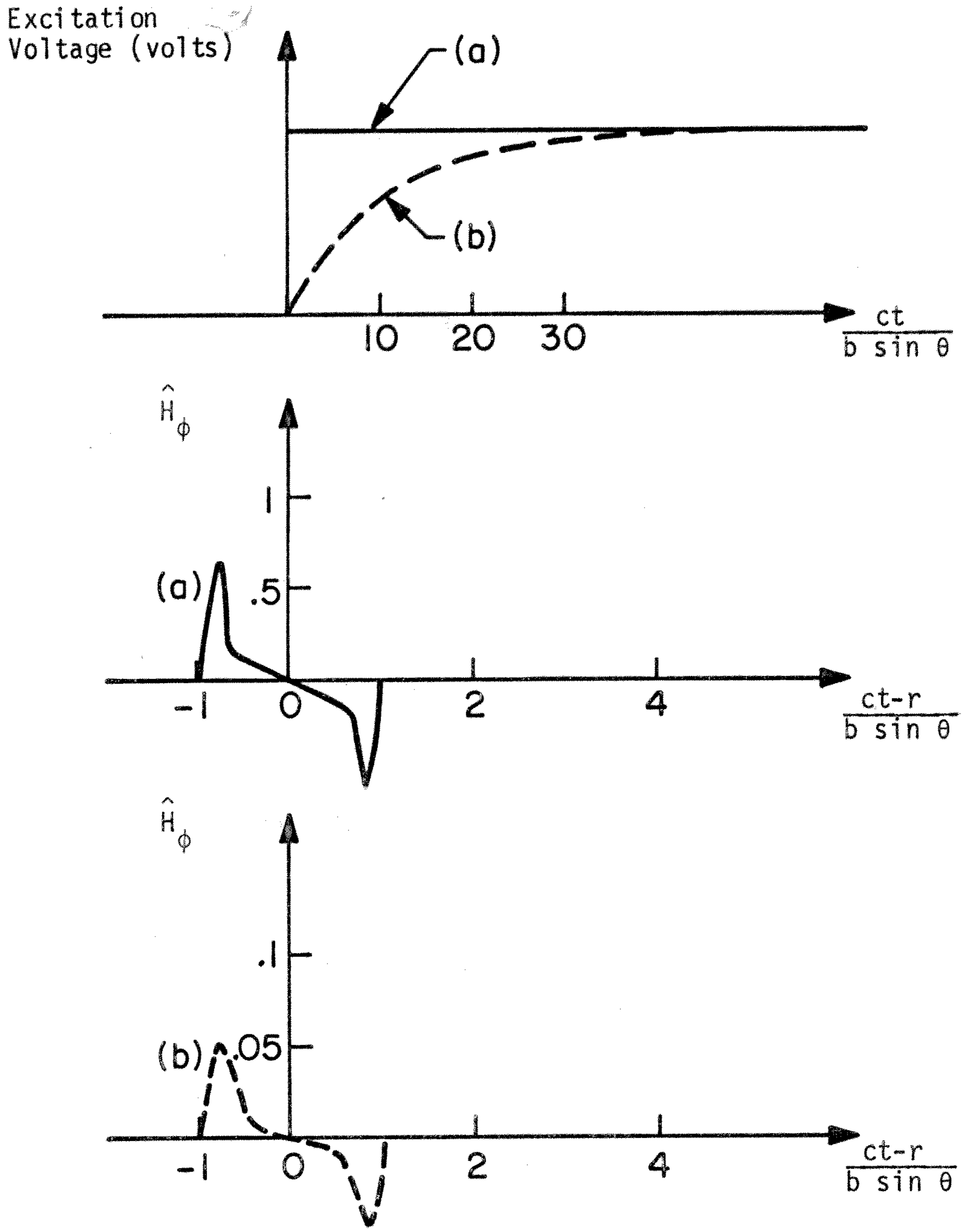


Figure 5-1. Radiated magnetic intensity of the coaxial waveguide antenna for (a) step function and (b) exponential excitation.
Ordinate: $\hat{H}_\phi = H_\phi(\sqrt{\mu_o/\varepsilon_o}\ r\ \ell n \frac{b}{a} \sin\theta)$ (volts)
Time constant for (b) is $cT/(b \sin\theta) = .1$
Antenna Dimensions: $a = .8b$

A basic assumption used in Chapter 2 is that the total current in the inner conductor is equal and opposite to that in the outer conductor. This assumption implies that the contribution to the radiated fields from all parts of the structure outside of the aperture are negligible. Examination of the impulse response shows that the radiation appears to originate from the edges of the open ended structure located in the aperture plane. The outer edge gives a positive contribution and the inner edge gives a negative contribution. The radiation starts when the incident current reaches the edge and lasts a time proportional to the time it takes light to traverse the diameter of the conductor. The difference in sign of the contributions is due to the difference in sign of the current on the conductors.

We now turn to the problem of determining the time behavior of the field intensities for $a \leq \rho \leq b;\ z \leq 0$. The magnetic intensity in this region is given by

$$H_{\phi_2}(\rho,z,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i\omega t} M(\omega) \left[\frac{e^{ikz}}{\rho} - \frac{Re^{-ikz}}{\rho} - \sum_{q=1}^{\infty} R_q f_q(\rho)\, e^{-iz\sqrt{k^2-(\gamma_q')^2}} \right] d\omega \qquad (5.12)$$

For harmonic time dependent excitation voltage $\mathrm{Re}(Ae^{-i\omega_o t})$,

$$M(\omega) = \mathrm{Re}[\sqrt{2\pi} \sqrt{\frac{\varepsilon_o}{\mu_o}} \frac{A}{\ln \frac{b}{a}} \delta(\omega-\omega_o)]$$

and (5.12) becomes:

$$H_{\phi_2}(\rho,z,t) = \sqrt{\frac{\varepsilon_o}{\mu_o}}\,\frac{A}{\ell n\,\frac{b}{a}}\Big[\frac{\cos[\omega_o(t - \frac{z}{c})]}{\rho} - \frac{\cos[\omega_o(t + \frac{z}{c})]}{\rho}\,R(\omega_o)$$

$$+ \sum_{q=1}^{\infty} f_q(\rho)\,R_q(\omega_o)e^{-iz\sqrt{(\frac{\omega_o}{c})^2 - (\gamma'_{oq})^2}}\Big] \tag{5.13}$$

where $R(\omega_o)$, $R_q(\omega_o)$ and $f_q(\rho)$ are the functions (3.7), (3.8) and (3.9) evaluated at $\omega = \omega_o$, if applicable. ω_o should not correspond to the cut off frequency of one of the E_{on} modes of the coaxial waveguide, since our formulation is not valid for these frequencies. We also note that as $\omega_o \to \infty$, $R(\omega_o) \to 0$, thus the TEM mode reflected field tends to vanish for very high frequency excitations. If $ka, kb \ll 1$, it is obvious that for large values of z the higher order E_{on} modes are negligible since they are evanescent in this frequency range. Since the time behavior of the incident field is given, we shall only be concerned with evaluating the integral for the reflected TEM mode fields or

$$H_{\phi_2}(\rho,z,t)_{\substack{\text{reflected}\\ \text{TEM}}} = -\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{M(\omega)}{\rho}\,Re^{-ikz - i\omega t}\,d\omega$$

$$R = \frac{[y^+(k)]^{-2} - [x^+(k)]^{-2}}{\ell n\,\frac{b}{a}} \tag{5.14}$$

From the values of $y^+(k)$ and $x^+(k)$ determined in Appendix D, it can be adduced that to the order of $(ka)^2$ or $(kb)^2$, $[y^+(k)]^{-2} = 0$ and

$$[x^+(k)]^{-2} \simeq -\ln \frac{b}{a}[J_0(ka)] \{1 - 2ika \sum_{n=1}^{\infty} (\frac{1}{|\alpha_n| a} - \frac{1}{n\pi})\}$$

$$\times \quad e^{-\frac{2ka}{i\pi}(1 - C + \ln 2)} \quad e^{-ika}$$

$$\times \quad \exp[2 \int_{\delta \to 0^+}^{\infty} K^{(1)}(u) \, \{\ln(1 + \frac{|k|}{\sqrt{k^2 - u^2}})\} du] + O(ka)^2 \quad (5.15)$$

By examination of the integral in (5.15) at the extremes of very small and very large values of u, it is shown that

$$\int_{\delta \to 0^+}^{\infty} K^{(1)}(u) \, \{\ln(1 + \frac{|k|}{\sqrt{k^2 - u^2}})\} \, du$$

is convergent for all values of k. Additionally, the integrable singularity at $u = k$ presents no difficulty to evaluation. After substitution and a change of variable the integral becomes:

$$\frac{1}{\pi} \int_0^{\infty} \{1 - \frac{\frac{4}{\pi x} + \frac{b}{a}[Y_1(\frac{b}{a}x)J_0(x) - J_1(\frac{b}{a}x)Y_0(x)] + Y_1(x)J_0(\frac{b}{a}x) - J_1(x)Y_0(\frac{b}{a}x)}{[J_0(\frac{b}{a}x) - J_0(x)]^2 + [Y_0(\frac{b}{a}x) - Y_0(x)]^2}\}$$

$$\times \{\ln(1 + \frac{1}{\sqrt{1 - (\frac{x}{ka})^2}})\} \quad dx \qquad (5.16)$$

For small values of x, integral (5.16) can be approximated by

$$\int_0^{\varepsilon} \{ 1 - \frac{\frac{4}{\pi x} - \frac{b}{a}[\frac{2}{\pi} \frac{a}{bx}] - \frac{1}{2}\frac{b}{a} x \frac{2}{\pi} \ell n\, x - \frac{2}{\pi x} - \frac{1}{2} x \frac{2}{\pi} \ell n \frac{b}{a} x}{(\ell n \frac{b}{a})^2} \} dx$$

$$= \int_0^{\varepsilon} (1 - \frac{-\frac{1}{\pi}[x\, \ell n\, x](\frac{b}{a} + 1) - \frac{x}{\pi} \ell n \frac{b}{a}}{(\ell n \frac{b}{a})^2}) \; dx$$

This last integral is finite, thus the integral (5.15) is convergent as $u \to 0$. For very large values of x , (5.16) is proportional to

$$\int_M^{\infty} 1 - \frac{\frac{4}{\pi x} + \left[\frac{b}{a} \sin(\frac{b}{a}x - \frac{3\pi}{4})\cos(x - \frac{\pi}{4}) + 0(\frac{1}{x}) + [\text{additional terms}]\right] \frac{2}{\pi x}}{\frac{2}{\pi x}\{[\cos(\frac{b}{a}x - \frac{\pi}{4}) - \cos(x - \frac{\pi}{4}) + 0(\frac{1}{x})]^2 + [\sin(\frac{b}{a}x - \frac{\pi}{4}) - \sin(x - \frac{\pi}{4}) + 0(\frac{1}{x})]^2\}}$$

$$\times \; \{\ell n(1 + \frac{1}{-i\sqrt{(\frac{x}{ka})^2 - 1}})\} \; dx$$

$$\leq \int_M^{\infty} [1 - \frac{\frac{4}{\pi x}}{\frac{2}{\pi x}(2 + 0(\frac{1}{x}))}] \; \{\ell n(1 + \frac{1}{-i\sqrt{(\frac{x}{ka})^2 - 1}})\} \; dx$$

$$\simeq \int_M^{\infty} [1 - \frac{\frac{4}{\pi x}}{\frac{4}{\pi x}(1 + 0(\frac{1}{x}))}] \; (\frac{A}{x}) \; dx$$

where we have used the expansion of $\ell n(1+z)$ with $z = A/x$; $z << 1$; $z \neq -1$. Additionally the convergence of

$$\int_M^{\infty} \sin mx \,(\frac{A}{x})\, dx \quad \text{and} \quad \int_M^{\infty} \cos mx (\frac{A}{x})\, dx$$

has been utilized. We are now able to conclude that the integral in (5.15) is convergent as $u \to \infty$.

Combining all the foregoing, it is apparent that the integral appearing in (5.15) is convergent for all values of k and it is apparent from taking the limit as $ka \to 0$ of (5.16) that

$$\lim_{ka\to 0} \int_0^\infty K^{(1)}(u) \{\ln(1 + \frac{|k|}{\sqrt{k^2 - u^2}})\} du = 0$$

Additionally, the term involving the summation in (5.15) is very much smaller than one and therefore ignorable. The exponential factors are also negligible when large values of z are concerned. Therefore $[x^+(k)]^{-2}$ in the low frequency limit becomes

$$[x^+(k)]^{-2} \simeq - \ln \frac{b}{a} [J_o(ka)]e^{-ika} \text{ for } ka,kb << 1 \tag{5.17}$$

The reflected TEM mode field is now evaluated by substituting (5.17) into (5.14). The results are

$$H_{\phi_2 \text{reflected TEM}} (\rho,z,t) \simeq - \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{M(\omega)}{\rho} J_o(ka) e^{-ik(z+a)-i\omega t} d\omega \tag{5.18}$$

By the convolution theorem and well known integral (5.8.1)

$$H_{\phi_2 \text{reflected TEM}} (\rho,z,t) \simeq - \frac{1}{\pi\rho} \int_{-\infty}^{t'} M(t'-\tau) g(\tau) d\tau \quad ; \quad z \leq 0 \tag{5.19}$$

where

$$g(\tau) = \begin{cases} \dfrac{1}{\sqrt{(\frac{a}{c})^2 - \tau^2}} & ; \quad 0 \leq |\tau| \leq \frac{a}{c} \\ \infty & ; \quad |\tau| = \frac{a}{c} \\ 0 & ; \quad \text{otherwise} \end{cases}$$

and

$$t' = t - \frac{|z|}{c} - \frac{a}{c}$$

If we again ignore for the moment the frequency restraints and allow a unit step function voltage excitation, then the reflected field is given by the integral of $g(\tau)$. Carrying out the integration produces the following:

$$H_{\phi_2}(\rho,z,t)_{\text{reflected TEM}} = -\sqrt{\frac{\varepsilon_o}{\mu_o}} \left(\frac{1}{\pi\rho \ \ell n \frac{b}{a}}\right) \begin{cases} 0 & ; \quad \tau \leq -\frac{a}{c} \\ \frac{\pi}{2} + \sin^{-1}\frac{c\tau}{a} & ; \quad -\frac{a}{c} \leq \tau \leq \frac{a}{c} \\ \pi & ; \quad \tau \geq \frac{a}{c} \end{cases} , \quad \tau = t - \frac{|z|}{c} - \frac{a}{c}$$

and the electric field is

$$E_{\rho_2}(\rho,z,t)_{\text{reflected TEM}} = -\sqrt{\frac{\mu_o}{\varepsilon_o}} \ H_{\phi_2}(\rho,z,t)_{\text{reflected TEM}}$$

In the case of a step input the reflected electric field is essentially the negative of the incident field for $t \geq \frac{2a}{c} + \frac{|z|}{c}$, (see equation (2.6)). The current reflection coefficient is $-J_0(ka)$ for $ka \ll 1$, which is consistent with transmission line theory. For allowable inputs of the form $(1 - e^{-t/T})$ with T large enough so that the low frequency restriction is not violated, the reflected fields are given by the above equations, multiplied by $(1 - e^{-|t'|/T})$, $t' = t - \frac{z}{c} - \frac{a}{c}$.

6. Radiation from the Pulse Excited Cylindrical Monopole Antenna

The objective of this chapter is to examine the radiation from a pulse excited cylindrical monopole antenna in the light of results obtained from the preceding study of a coaxial waveguide antenna. We have determined that in the far zone the radiation appears to originate from the edges of the open ended structure located in the aperture plane. The edge starts radiating when the leading edge of the incident current pulse on the feed line impinges upon it. Radiation from the inner edge is opposite in sign to radiation from the outer edge.

We begin by considering the approximate far zone fields from the annular slot antenna. This antenna has been the subject of several authors [28-32]. The structure of the annular slot antenna is that of a coaxial line terminated in an infinite ground plane as shown in Figure 6-1. Note that the center conductor is not hollow.

We will follow the approach of Ref. [32] and replace the electric field in the aperture by an equivalent magnetic current and develop the fields in the half space $z \geq 0$ from this equivalent source and image theory. The equivalent magnetic current is

$$\vec{J}_m = -\vec{a}_z \times E'_\rho \, \vec{a}'_\rho = -E'_\rho \, \vec{a}'_\phi$$

Since this source is just above a perfectly conducting ground plane, the electric vector potential is given by

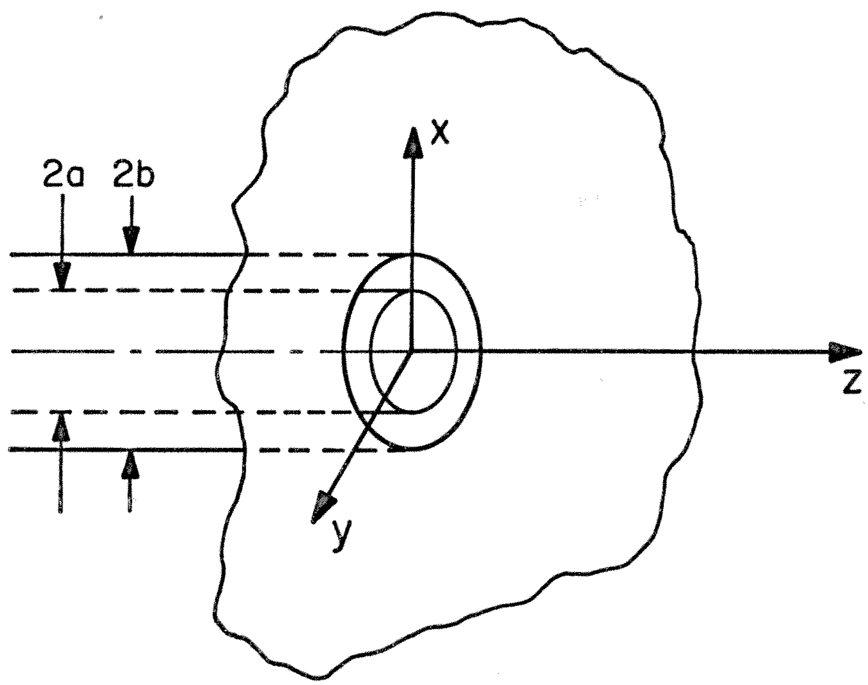


Figure 6-1. The annular slot antenna

$$\vec{F} = -\frac{\varepsilon_o}{4\pi}\int_0^{2\pi}\int_a^b \frac{2\vec{E}_\rho'\,\vec{a}_\phi'\,e^{ik|\vec{r}-\vec{r}'|}}{|\vec{r}-\vec{r}'|}\,\rho'd\rho'd\phi'$$

where

$$\varepsilon_o\vec{E} = -\nabla\times\vec{F}$$

and

$$|\vec{r}-\vec{r}'| = [\rho^2+\rho'^2-2\rho\rho'\cos(\phi-\phi')+(z-z')^2]^{1/2}$$

Now it is clear that the electric vector potential and the fields are independent of the coordinate ϕ, from the cylindrical symmetry of the structure. Therefore we can set $\phi = 0$ without loss of generality. We also know that the only nonzero component of $\vec{F}$ is the ϕ component due to the symmetry of the equivalent magnetic source. From geometry we find

$$\vec{a}_{\phi'} = \vec{a}_y\cos\phi' - \vec{a}_x\sin\phi'$$

since $\phi = 0$; $\vec{a}_y = \vec{a}_\phi$ for this formulation. Therefore

$$\vec{F} = -\frac{\varepsilon_o}{2\pi}\vec{a}_\phi\int_0^{2\pi}\int_a^b E_\rho'\cos\phi'\,\frac{e^{ikR}}{R}\,\rho'd\phi'd\rho'$$

with

$$R = (\rho^2+\rho'^2-2\rho\rho'\cos\phi'+(z-z')^2)^{1/2}$$

From Maxwell's equations and the Lorentz gauge it is an easy matter to show that

$$\vec{H} = \frac{i\omega}{k_o^2}[-\nabla(\nabla \cdot \vec{F}) + k_o^2 \vec{F}]$$

Since $\vec{F}$ has only a ϕ component which is independent of the angle ϕ

$$\nabla \cdot \vec{F} = 0$$

and

$$H_\phi = i\omega F_\phi = -\frac{i\omega\varepsilon_o}{2\pi}\int_a^b E_\rho' \rho' d\rho' \int_0^{2\pi} \cos\phi' \frac{e^{ikR}}{R} d\phi' \tag{6.1}$$

Equation (6.1) is exact--no approximations have been imposed. If E_ρ', the electric field intensity in the aperture, were known exactly, then the integration could be carried out and the result would be exact.

If we approximate the aperture field as being that of the incident field in the aperture, i.e.,

$$E_\rho'(\rho,0,\omega) = \frac{V(\omega)}{\rho' \ \ell n \frac{b}{a}}$$

then

$$H_\phi \simeq -\frac{i\omega\varepsilon_o}{2\pi} \frac{V(\omega)}{\ell n \frac{b}{a}} \int_a^b \int_0^{2\pi} \cos\phi' \frac{e^{ikR}}{R} d\rho' d\phi' \tag{6.2}$$

In the far zone the distance R becomes

$$R = (\rho^2 + \rho'^2 - 2\rho\rho' \cos\phi' + (z-z')^2)^{1/2}$$

$$= (r^2 + \rho'^2 - 2\rho' r \sin\theta \cos\phi' - 2rz' \cos\theta + z'^2)^{1/2}$$

$$= r[1 + \frac{\rho'^2}{r^2} - \frac{2\rho'}{r} \sin\theta \cos\phi' - \frac{2z'}{r} \cos\theta + \frac{z'^2}{r^2}]^{1/2}$$

$$\simeq r[1 - \frac{\rho'}{r} \sin\theta \cos\phi']$$

since $r >> \rho'$ and $r >> z'$,

$$\frac{e^{ikR}}{R} \simeq \frac{e^{ik(r - \rho'\sin\theta\cos\phi')}}{r}$$

We have retained more terms of the approximation to R in the phase factor than in the amplitude factor, since small variations in R cause large changes in the value of the exponent. The amplitude of the function is relatively insensitive to variations in R as $r \to \infty$ whereas the phase is not.

Upon incorporation of the well known expansion [28, p.230]

$$e^{ik\rho'\sin\theta\cos\phi'} = \sum_{n=-\infty}^{\infty} (-i)^n J_n(k\rho'\sin\theta)e^{in\phi'}$$

equation (6.2) becomes

$$H_\phi \simeq -\frac{i\omega\varepsilon_o}{2\pi} \frac{V(\omega)e^{ikr}}{(\ell n \frac{b}{a})r} \int_a^b \int_0^{2\pi} \cos\phi' \sum_{n=-\infty}^{\infty} (-i)^n J_n(k\rho'\sin\theta)e^{in\phi'} d\phi' d\rho'$$

$$= -\frac{\omega\varepsilon_o}{2\pi} \frac{V(\omega)}{\ell n \frac{b}{a}} \frac{e^{ikr}}{r} \int_a^b \int_0^{2\pi} 2\cos^2\phi' d\phi' \; J_1(k\rho'\sin\theta)d\rho'$$

$$= -\frac{\omega\varepsilon_o V(\omega)}{\ell n \frac{b}{a}} \frac{e^{ikr}}{r} \int_a^b J_1(k\rho'\sin\theta)d\rho'$$

$$= \frac{\omega\varepsilon_o V(\omega)}{\ell n \frac{b}{a}} \frac{e^{ikr}}{r} \int_a^b \frac{d(J_o(k\rho'\sin\theta))}{k\sin\theta}$$

$$= \sqrt{\frac{\varepsilon_o}{\mu_o}} \frac{V(\omega)}{\ell n \frac{b}{a}} \frac{e^{ikr}}{r} \left[\frac{J_o(kb\sin\theta) - J_o(ka\sin\theta)}{\sin\theta}\right] \qquad (6.3)$$

The approximate time behavior of the far zone field follows directly from the inverse Fourier transform of (6.3)

$$H_\phi(r,\theta,t) = \frac{1}{\sqrt{2\pi}}\sqrt{\frac{\varepsilon_0}{\mu_0}}\,\frac{1}{(\ln\frac{b}{a})r\sin\theta}\int_{-\infty}^{\infty} e^{-i\omega(t-\frac{r}{c})}\,V(\omega)[J_0(kb\sin\theta) - J_0(ka\sin\theta)]d\omega \quad (6.4)$$

When this integration is carried out by the convolution theorem, the impulse response of the annular slot antenna is found to be given by equation (5.10) and the step response is found to be given by equation (5.11).

From this result it is apparent that in the annular slot antenna the radiation also appears to emanate from the edges in the aperture plane. The radiation starts when the incident current on the conducting surface reaches the edge. One edge is located at $\rho = b$ and the other at $\rho = a$. For an incident impulse of voltage the outer edge contributes radiated magnetic intensity in the amount

$$\sqrt{\frac{\varepsilon_0}{\mu_0}}\,\frac{1}{\pi r\sin\theta\,\ln\frac{b}{a}}\begin{Bmatrix} 0 & ; \quad \tau \le -\frac{b}{c}\sin\theta \\ \dfrac{1}{\sqrt{(\frac{b}{c}\sin\theta)^2 - \tau^2}} & ; \quad -\frac{b}{c}\sin\theta \le \tau \le \frac{b}{c}\sin\theta \\ 0 & ; \quad \tau \ge \frac{b}{c}\sin\theta \end{Bmatrix} \quad (6.4.1)$$

and the inner edge contributes radiated magnetic intensity in the amount

$$\sqrt{\frac{\varepsilon_o}{\mu_o}} \frac{1}{\pi r \sin\theta \, \ln \frac{b}{a}} \left\{ \begin{array}{lcl} 0 & ; & \tau \leq -\frac{a}{c}\sin\theta \\ \frac{-1}{\sqrt{(\frac{a}{c}\sin\theta)^2 - \tau^2}} & ; & -\frac{a}{c}\sin\theta \leq \tau \leq \frac{a}{c}\sin\theta \\ 0 & ; & \tau \geq \frac{a}{c}\sin\theta \end{array} \right\}$$

with $$\tau = t - \frac{r}{c} \qquad (6.4.2)$$

We make the assertion at this point that the tip and the base of a cylindrical monopole antenna (Figure 6-2) give off radiation in a manner identical to that of the annular slot antenna. In this case, the radiation from the tip must be delayed in time by an amount which equals the time it takes light to travel the length of the antenna, h/c . We further assume that the currents on the conductors are reflected in the same manner as the currents on the conductors of the open ended coaxial waveguide. That is, the total current must go to zero at the tip and the base.

In Appendix E we develop the radiated fields of a very long hollow cylindrical antenna. Transcribing those results, we have that the magnetic intensity of such an antenna is

$$H_\phi(r,\theta,\omega) \simeq -\sqrt{\frac{\varepsilon_o}{\mu_o}} \frac{V(\omega)}{\sin\theta \, \ln \frac{b}{a}} \frac{e^{ikr}}{r} \frac{J_o(ka \sin\theta)}{\sin\theta}$$

This result is identical to the frequency domain radiation from the

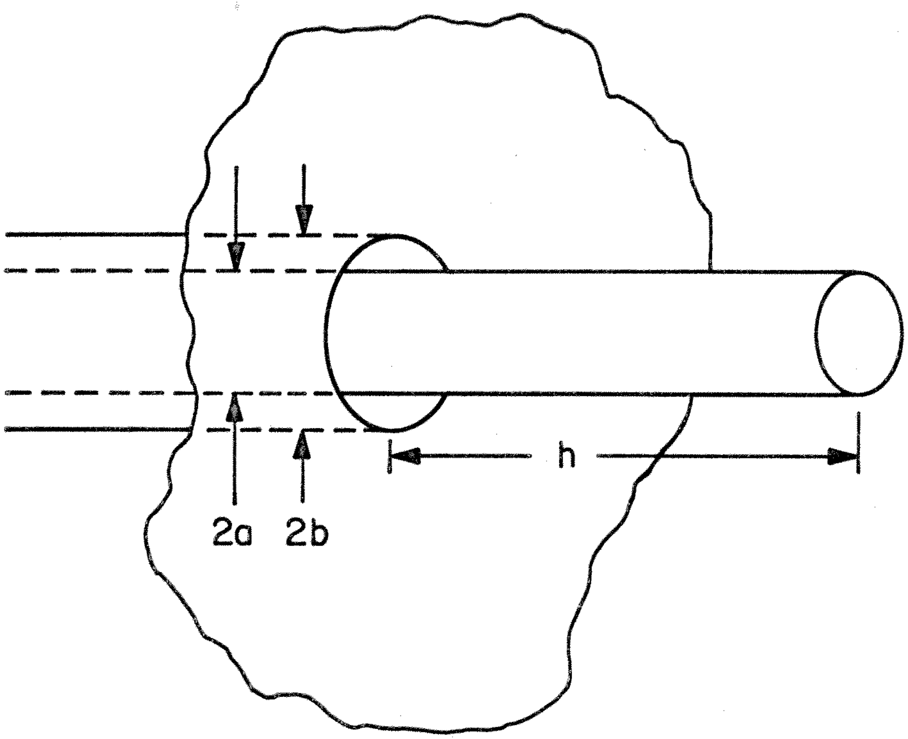


Figure 6-2. The cylindrical monopole antenna

tip of the center conductor of the annular slot antenna determined in (6.3) above. Since $ka << 1$, the particular form of the end does not have a significant effect on the field intensities. We therefore take these results to be valid for all thin cylindrical antennas regardless of the end configuration (with the obvious exception of those with end loading.)

From the above discussion, it is clear that the contribution to the radiation from the tip of the center conductor is identical for $h = 0$ and for extremely large values of h . We assume this result to be valid for all cylndrical monopoles with $h \geq 0$. Since the base of the antenna is excited at $t = 0$, when the incident excitation voltage reaches the ground plane, appropriate mathematical factors must be incorporated into the field intensity equations to account for the finite time h/c required for the excitation current to reach the tip of the antenna. (c is the speed of light).

Since the electric field at the junction of the feedline and the antenna must be continuous, when the flowing charges reflected from the tip reach the junction, charges of opposite sign must travel back along the antenna [8]. If we consider that the annular thickness $(b-a) << \lambda$, then image theory is applicable and it is readily seen that the current relaunched onto the antenna from the base is equal in magnitude and opposite in sign to the current incident from the antenna.

No radiation besides the initial radiation due to the current incident from the feedline originates from the base of the antenna. The current reflected from the antenna tip is cancelled by the image current at the $z = 0$ plane.

To determine the total contributions to the far zone radiation, use is made of Figure 6-3. Proper accounting must be made of the radiation from the tip of the antenna and that reflected from the ground plane. It is readily seen from this figure and the above discussion that the radiated magnetic intensity in the frequency domain is

$$H_\phi(r,\theta,\omega) = \sqrt{\frac{\varepsilon_o}{\mu_o}} \frac{V(\omega)\, e^{ikr}}{(\ln \frac{b}{a})\, r \sin\theta} [J_o(kb \sin\theta) - J_o(ka \sin\theta)$$

$$\times \{\frac{e^{ikh(1 - \cos\theta)} + e^{ikh(1 + \cos\theta)}}{1 + e^{2ikh}}\}]$$

$$kh \neq (2n+1)\frac{\pi}{2} \quad ; \quad n=0,1,2,3,\cdots \tag{6.6}$$

The case for $kh = (2n+1)\frac{\pi}{2}$ will be treated later.

The time behavior follows from the inverse Fourier transform of (6.6)

$$H_\phi(r,\theta,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} H_\phi(r,\theta,\omega)\, e^{-i\omega t}\, d\omega \tag{6.7}$$

If the excitation voltage is harmonic time dependent, then

$$V(\omega) = \mathrm{Re}[\sqrt{2\pi}\, A\, \delta(\omega - \omega_o)]$$

From (6.6) and (6.7), we have

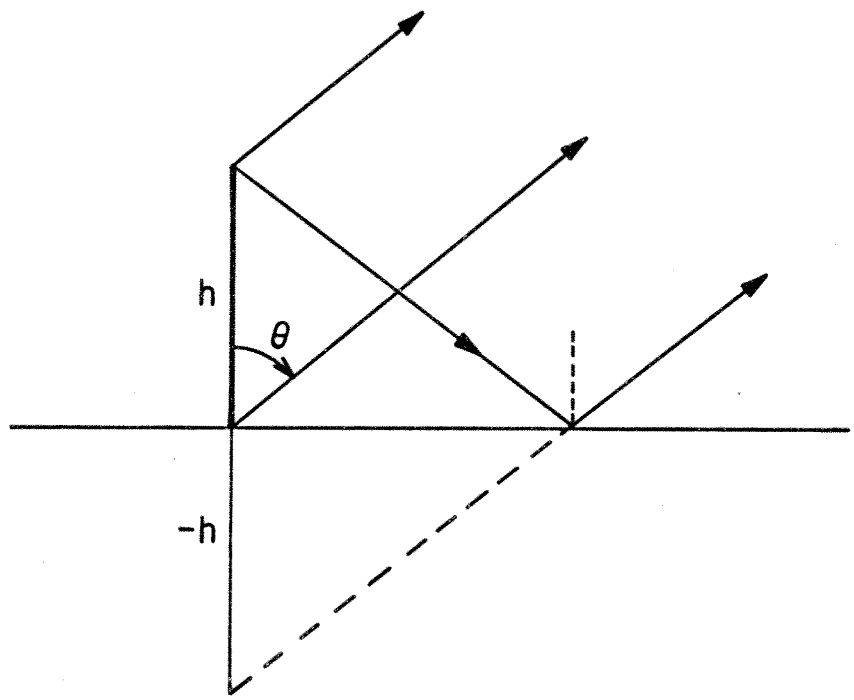


Figure 6-3. Effect of image plane

$$H_\phi(r,\theta,t) = \sqrt{\frac{\varepsilon_o}{\mu_o}}\ \frac{A}{(\ell n\ \frac{b}{a}) r \sin\theta}\ \mathrm{Re}\left\{ J_o(\frac{\omega_o b}{c} \sin\theta) - J_o(\frac{\omega_o a}{c}\sin\theta)\left[\frac{e^{i\frac{\omega_o h}{c}(1-\cos\theta)} + e^{i\frac{\omega_o h}{c}(1+\cos\theta)}}{1+e^{2i\frac{\omega_o h}{c}}}\right]\right\} \times \{e^{-i\omega_o(t-\frac{r}{c})}\}$$

Upon simplification this becomes

$$H_\phi(r,\theta,t) = \sqrt{\frac{\varepsilon_o}{\mu_o}}\ \frac{A\cos\omega_o(t-\frac{r}{c})}{(\ell n\ \frac{b}{a}) r\sin\theta}\left\{ J_o(\frac{\omega_o b}{c}\sin\theta) - \frac{[J_o(\frac{\omega_o a}{c}\sin\theta)][\cos(\frac{\omega_o h}{c}\cos\theta)]}{\cos\frac{\omega_o h}{c}}\right\}$$

$$b \neq a\ ;\quad \frac{\omega_o h}{c} \neq (2n+1)\,\frac{\pi}{2}\ ;\quad n=0,1,2,3,\cdots \tag{6.8}$$

This expression shows that the radiation pattern is

$$F(\theta) = \frac{J_o(\frac{\omega_o b}{c}\sin\theta)\cos\frac{\omega_o h}{c} - [J_o(\frac{\omega_o a}{c}\sin\theta)][\cos(\frac{\omega_o h}{c}\cos\theta)]}{\sin\theta} \tag{6.9}$$

The radiation pattern given by (6.9) can be compared with the pattern derived when the antenna is assumed to be very thin with a filamentary sinusoidal current distribution [33]. For that case

$$F(\theta) = \frac{\cos \frac{\omega_o h}{c} - \cos(\omega_o \frac{h}{c} \cos \theta)}{\sin \theta} \qquad (6.10)$$

In the limit when the feedline diameters are "thin" (i.e., $\lambda >> a,b$), the patterns given by (6.9) and (6.10) are identical. Equation (6.8) shows that the radiation along $\theta = 0$ is always zero, as it must be.

To get expressions for the cases where $kh = (2n+1)\frac{\pi}{2}$, $n=0,1,2,3,\cdots$, we must consider the fact that ohmic losses are present in any real system allowing us to use a complex value of k or we can take k to be complex for the sake of mathematical expediency and let its imaginary part vanish at a later stage in the analysis. Thus,

$$k = k_1 + ik_2 \quad ; \qquad k_2 << k_1 = \frac{\omega_o}{c}$$

With complex k the factor

$$\frac{e^{ikh(1 - \cos \theta)} + e^{ikh(1 + \cos \theta)}}{1 + e^{2ikh}} = \frac{\cos[kh \cos \theta]}{\cos kh}$$

$$= \frac{\cos[(k_1 + ik_2)h \cos \theta]}{\cos(k_1 + ik_2)h}$$

$$= \frac{\cos(k_1 h \cos \theta)\cosh(k_2 h \cos \theta) - i \sin(k_1 h \cos \theta)\sinh(k_2 h \cos\theta)}{\cos k_1 h \cosh(k_2 h) - i \sin k_1 h \sinh(k_2 h)}$$

In the limit as $k_1 h \rightarrow (2n+1)\frac{\pi}{2}$, $n=0,1,2,3,\cdots$, this expression is seen to be finite for $k_2 \neq 0$ and the magnitude is only limited by the fact that the conductivity of the material is not infinite

$$\lim_{k_1 h \to (2n+1)\frac{\pi}{2}} \left(\frac{\cos(kh\ \cos\ \theta)}{\cos\ kh}\right) \longrightarrow \frac{\cos[(2n+1)\frac{\pi}{2}\cos\ \theta]}{-i[\sin(2n+1)\frac{\pi}{2}]\sinh(k_2 h)}$$

$$n=0,1,2,3,\cdots \quad \text{and} \quad k_2 = \varepsilon > 0$$

The magnetic intensity becomes

$$H_\phi(r,\theta,t) \simeq \sqrt{\frac{\varepsilon_o}{\mu_o}} \; \frac{A\ \sin(\omega_o t - \frac{r}{c})}{(\ell n\ \frac{b}{a})r\ \sin\ \theta}$$

$$\times \; \frac{-J_o(\omega_o\ \frac{a}{c}\ \sin\ \theta)[\cos((2n+1)\frac{\pi}{2}\ \cos\ \theta)]}{\sin[(2n+1)\frac{\pi}{2}]\ \sinh(k_2 h)}$$

The input current to the antenna peaks at the values of kh given by $kh = (2n+1)\frac{\pi}{2}$, $n=0,1,2,3,\cdots$. The explanation for this is that the monopole can be considered as a simple extension of the feed-line. The tip of the antenna would correspond to the open end of a transmission line. Any odd number of quarter wavelengths from the tip along the line toward the source would correspond to a point where the impedance is a minimum (a point of series resonance) and the current is a maximum. When $kh = (2n+1)\frac{\pi}{2}$, the feed point is an odd number of quarter wavelengths from the open end of the extended transmission line.

Taking the inverse Fourier transform of (6.6), we find the radiated magnetic intensity for a voltage impulse of excitation

$$H_\phi(r,\theta,t)_{\substack{\text{impulse}\\\text{excitation}}} = \sqrt{\frac{\varepsilon_o}{\mu_o}}\left(\frac{1}{\pi(\ln\frac{b}{a})r\sin\theta}\right)\left\{\begin{array}{ll}\dfrac{1}{\sqrt{(\frac{b}{c}\sin\theta)^2-\tau^2}} ; & 0\le\tau\le\frac{b}{c}\sin\theta \\ 0 ; & \text{elsewhere}\end{array}\right\} \quad \tau = t-\frac{r}{c}$$

$$-\left\{\begin{array}{ll}\sum_{n=0}^{\infty}\dfrac{(-1)^n}{\sqrt{(\frac{a}{c}\sin\theta)^2-\Omega_n^2}} ; & 0\le\Omega_n\le\frac{a}{c}\sin\theta \\ 0 ; & \text{elsewhere}\end{array}\right\} \quad \Omega_n = t-\frac{r}{c}-\frac{h}{c}(1-\cos\theta)-\frac{2nh}{c}$$

$$-\left\{\begin{array}{ll}\sum_{n=0}^{\infty}\dfrac{(-1)^n}{\sqrt{(\frac{a}{c}\sin\theta)^2-\tau_n^2}} ; & 0\le\tau_n\le\frac{a}{c}\sin\theta \\ 0 ; & \text{elsewhere}\end{array}\right\} \quad \tau_n = t-\frac{r}{c}-\frac{h}{c}(1+\cos\theta)-\frac{2nh}{c} \qquad (6.12)$$

For a unit step of voltage excitation the radiated magnetic intensity is the integral of the preceding expression.

$H_\phi(r,\theta,t)$
unit step
excitation

$$= \sqrt{\frac{\varepsilon_o}{\mu_o}}\ \frac{1}{\pi(\ln \frac{b}{a}) r \sin\theta} \left[\left\{ \begin{array}{ll} 0 & ; \ \tau \le -\frac{b}{c}\sin\theta \\ \frac{\pi}{2} + \sin^{-1}\left(\frac{c\tau}{b\sin\theta}\right) & ; \ -\frac{b}{c}\sin\theta \le \tau \le \frac{b}{c}\sin\theta \\ \pi & ; \ \tau \ge \frac{b}{c}\sin\theta \end{array} \right\} \right.$$

$$\tau = t - \frac{r}{c}$$

$$- \left\{ \begin{array}{ll} 0 & ; \ \Omega_n \le -\frac{a}{c}\sin\theta \\ \sum_{n=0}^{\infty} (-1)^n \left(\frac{\pi}{2} + \sin^{-1}\frac{\Omega_n c}{a\sin\theta}\right) & ; \ -\frac{a}{c}\sin\theta \le \Omega_n \le \frac{a}{c}\sin\theta \\ \pi & ; \ \Omega_n \ge \frac{a}{c}\sin\theta \end{array} \right\}$$

$$\Omega_n = t - \frac{r}{c} - \frac{h}{c}(1 - \cos\theta) - \frac{2nh}{c}$$

$$\left. - \left\{ \begin{array}{ll} 0 & ; \ \tau_n \le -\frac{a}{c}\sin\theta \\ \sum_{n=0}^{\infty} (-1)^n \left(\frac{\pi}{2} + \sin^{-1}\frac{\tau_n c}{a\sin\theta}\right) & ; \ -\frac{a}{c}\sin\theta \le \tau_n \le \frac{a}{c}\sin\theta \\ \pi & ; \ \tau_n \ge \frac{a}{c}\sin\theta \end{array} \right\} \right]$$

$$\tau_n = t - \frac{r}{c} - \frac{h}{c}(1 + \cos\theta) - \frac{2nh}{c}$$

(6.13)

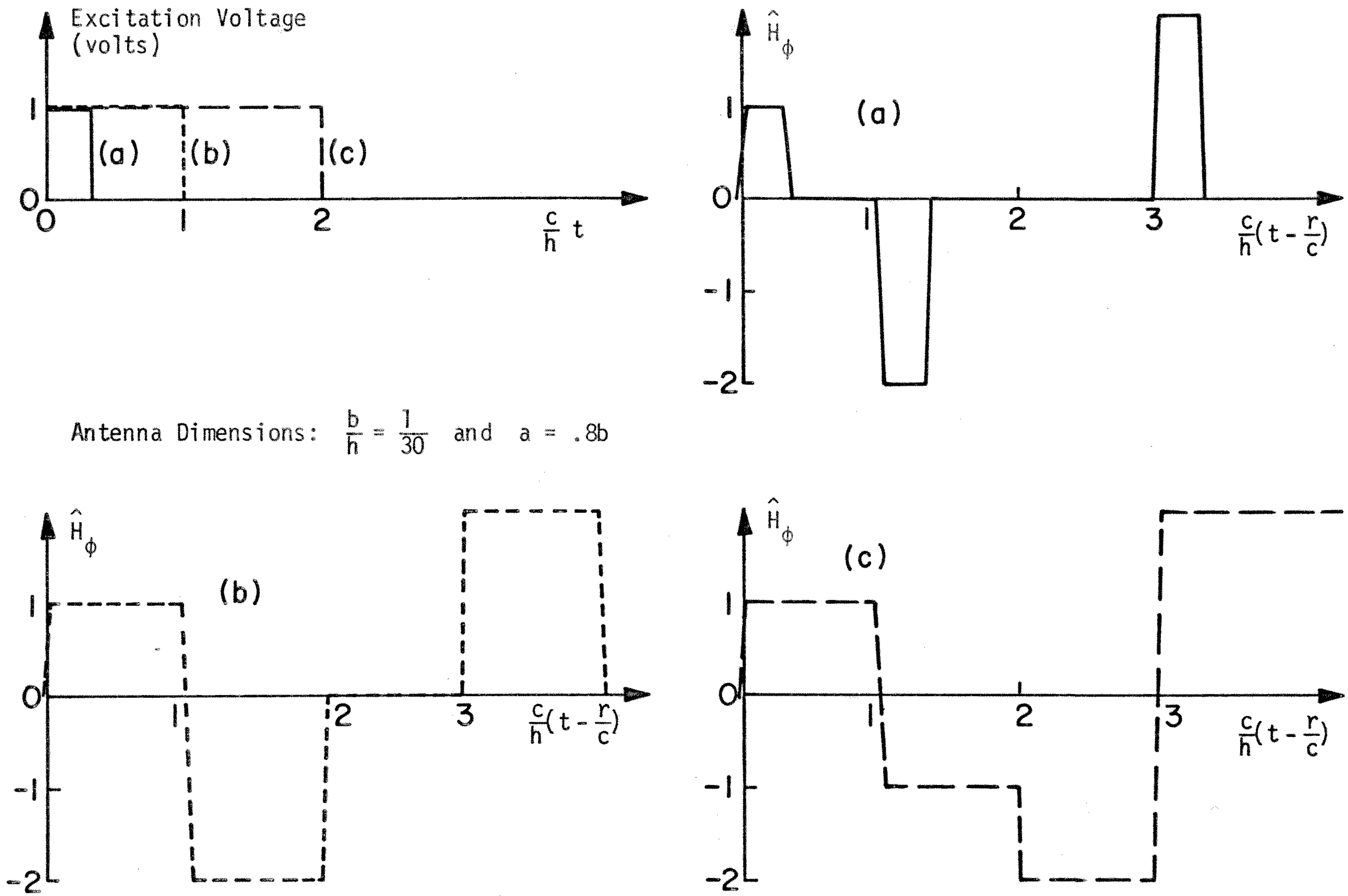
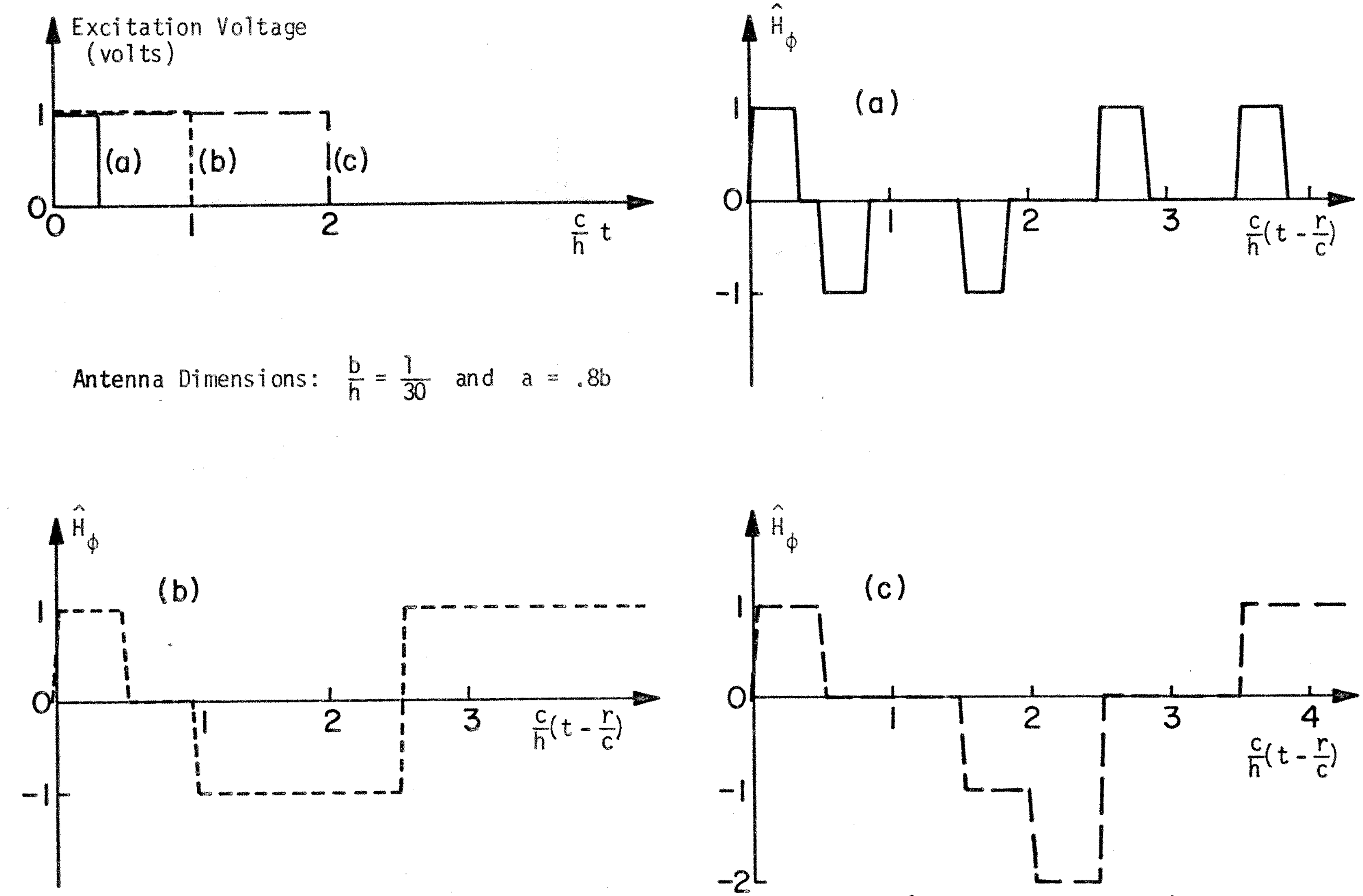


Figure 6-4. Radiated magnetic intensity at $\theta = \pi/2$. Ordinate: $\hat{H}_\phi = H_\phi(\sqrt{\mu_o/\varepsilon_o}\ r\ \ell n\ \frac{b}{a})$ (volts)



Antenna Dimensions: $\frac{b}{h} = \frac{1}{30}$ and $a = .8b$

Figure 6-5. Radiated magnetic intensity at $\theta = \pi/3$. Ordinate: $\hat{H}_\phi = H_\phi(\sqrt{\mu_o/\varepsilon_o}\ r\ \ell n\ \frac{b}{a}\ \sin\frac{\pi}{3})$ (volts)

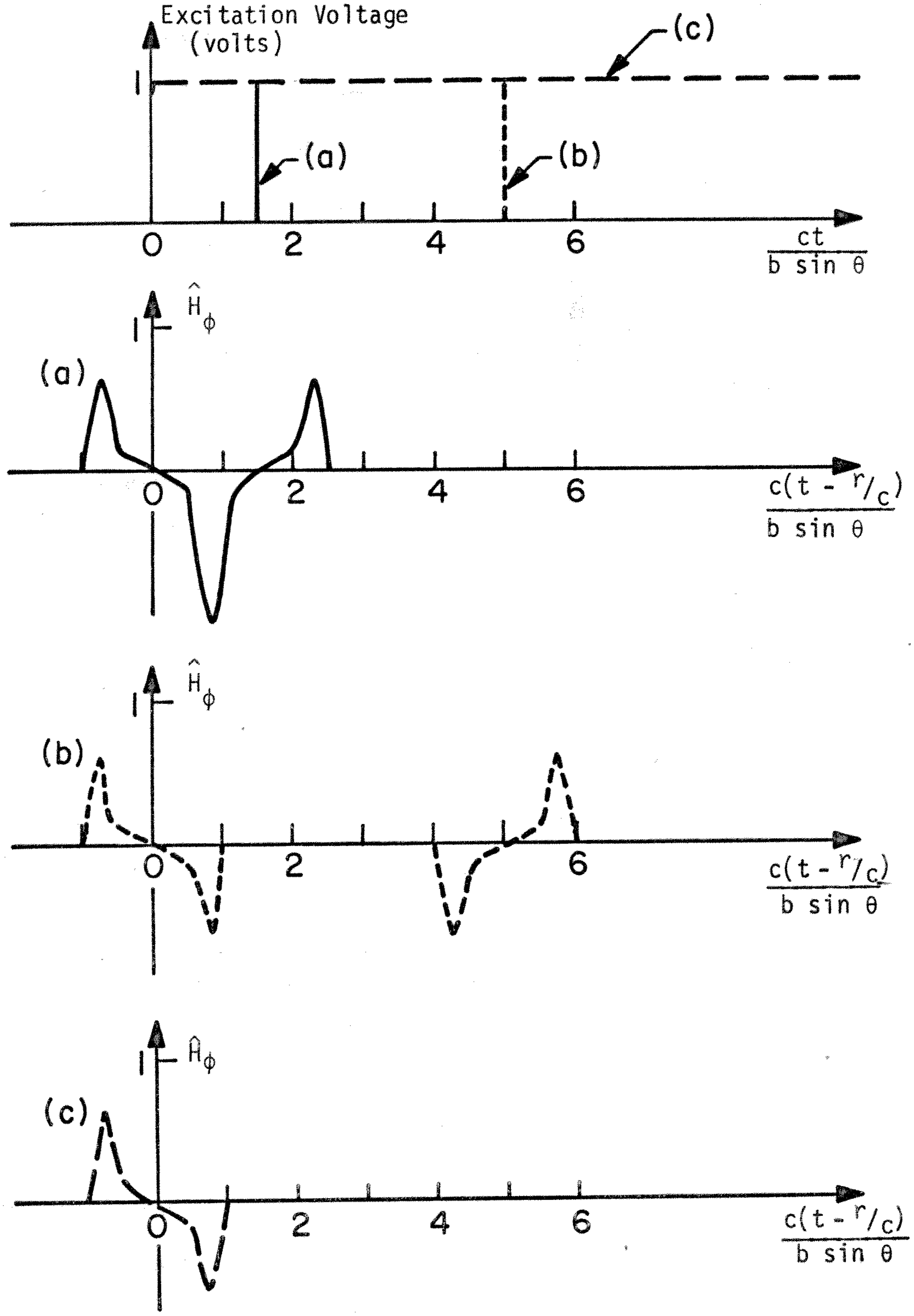


Figure 6-6. Radiated magnetic intensity of annular slot antenna

Ordinate: $\hat{H}_\phi = H_\phi(\sqrt{\mu_o/\varepsilon_o}\ r\ \ell n\ \frac{b}{a}\sin\theta)$(volts)

Antenna Dimensions: h = 0 and a = .8b

The magnetic intensity for arbitrary time dependence can be obtained by convolution of the input excitation voltage with the impulse response (6.12) or by convolution of the time derivative of the input voltage with the step response (6.13). As expected, the radiated field for the infinite monopole is the same as that for the finite length monopole for the time interval before the current pulse reaches the tip of the antenna. The results, equations (6.12) and (6.13), are easily extended to the infinite cylindrical antenna by letting $h \to \infty$. The radiated fields for the annular slot are found by letting $h \to 0$ in expression (6.6). It is also worth observing that the results (6.12) and (6.13) are based on the assumption of no ohmic losses as the current pulse travels back and forth between base and tip of the antenna. The attenuation can be found using the conventional transmission line approach for small attenuations on good conductors. The determination of the time domain fields in the case of a lossy conductor is complicated since the resistance per unit length is proportional to the square root of the frequency. The incident pulse supplies the total of the radiated energy, ohmic losses, reflected energy and energy stored in the induction fields. The stored energy of the induction fields is the source of energy for the second and subsequent pulses launched down the feedline toward the source, the energy radiated after time $2h/c$, and the ohmic losses which occur after time $2h/c$.

The theoretical results obtained in this chapter are in good agreement with the experimental and numerical work of Schmitt, et al [6], Palciauskas and Beam [10], King and Harrison [8], Burrell [34], and Lamensdorf [14].

Figure 6-7 is a comparison of the theoretical results obtained in this work with the experimental results of Schmitt, et al. [6]. Reference [6] states that the second pulse measured in the experiment should be moved .74 ns toward the origin and its amplitude should be about 3/2 as much as observed. This is necessary to account for experimental deviations associated with locating the receiving probe a finite distance from the antenna. In Figure 6-7 these corrections are applied to the second and subsequent pulses.

It is also constructive to include the following observation. When a cylindrical dipole antenna is infinitely thin and center fed, it is common to consider it as an open ended transmission line that has been spread or opened out. A sinusoidal antenna current distribution with current nodes at the ends is assumed and the radiated fields are obtained on that basis. If it is further assumed that the current at the antenna feed point is equal to the current existing at this point on an equivalent length of open ended transmission line, then the field expressions developed from the assumed sinusoidal current distribution are identical to the field expressions obtained in this work in the limit of very thin antennas.

To demonstrate this point, we first observe that the radiated magnetic intensity created by a filamentary sinusoidal current distribution of $I_o \sin k(h - |z|)$ is [33]

$$H_\phi = \frac{iI_o e^{ikr}}{2\pi r \sin\theta}[\cos kh - \cos(kh\cos\theta)]$$

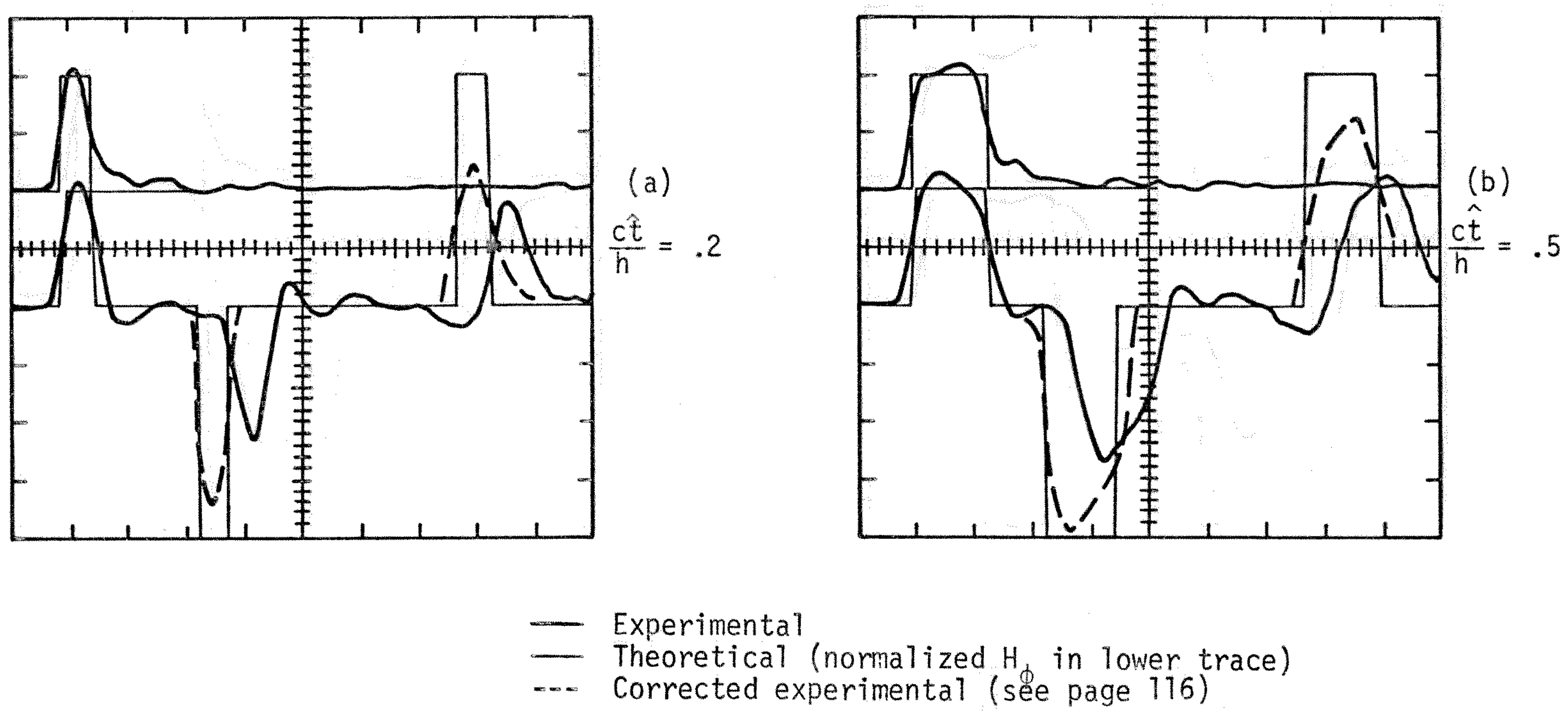


Figure 6-7 (a) and (b). Comparison of theoretical with experimental results of Schmitt, et al. [6]. Upper trace - excitation voltage (volts)
Lower trace - $H_\phi(r, \pi/2, t - r/c)$
Time scale 1.25 ns/division. $c\hat{t}/h$ denotes excitation pulse width.
(Comparison is to be made in shape due to lack of information in [6] on the scale used for the ordinate. For theoretical portion of plot it is assumed that 2 divisions = 1 volt.)

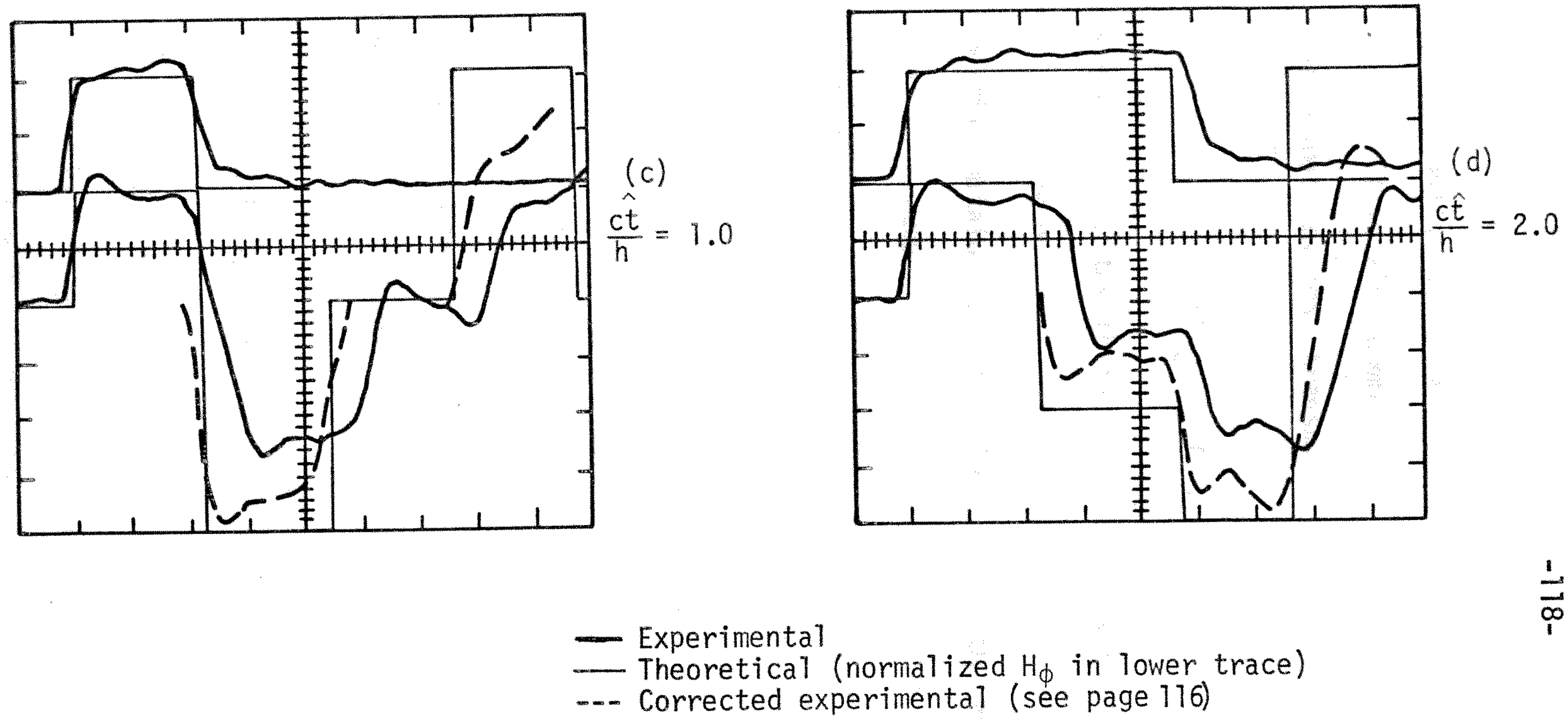


Figure 6-7 (c) and (d). Comparison of theoretical with experimental results of Schmitt, et al. [6].
Upper trace - excitation voltage (volts)
Lower trace - $H_\phi(r, \pi/2, t - r/c)$
Time scale 1.25 ns/division. $c\hat{t}/h$ denotes excitation pulse width.
(Comparison is to be made in shape due to lack of information in [6] on the scale used for the ordinate. For theoretical portion of plot it is assumed that 2 divisions = 1 volt.)

The driving point current is $I_o \sin kh$. The other symbols have the usual meanings consistent with their previous use within this chapter.

It is also common knowledge that the input impedance of an open ended lossless transmission line of length h is

$$Z = -iZ_o \cot kh$$

where Z_o is the characteristic impedance of the line. If we drive such a line by a perfect voltage source $V(\omega)$, the source current is readily obtained by Ohm's law. Since the antenna driving point current is assumed to be equal to the current which would exist at the input of an equivalent length of open ended transmission line, we equate the driving point current to the source current. Therefore,

$$I_o \sin kh = \frac{V(\omega)}{-iZ_o \cot kh}$$

and

$$I_o = \frac{V(\omega)}{-iZ_o \cos kh}$$

By substituting this value of I_o into the field intensity equation, we derive the general result for very thin antennas,

$$H_\phi(r,\theta,\omega) = \frac{-V(\omega)\, e^{ikr}}{2\pi Z_o r \sin\theta}\left[1 - \frac{\cos(kh \cos\theta)}{\cos kh}\right] \qquad (6.14)$$

Outside of a discrepancy in sign, it is an easy matter to show that this general result reduces to equation (6.6) when the limiting form for small arguments replaces the Bessel functions of equation (6.6) and the characteristic impedance of a lossless coaxial line is substituted, i.e.,

$$Z_o = \frac{1}{2\pi}\sqrt{\frac{\mu_o}{\varepsilon_o}}\ \ell n\ \frac{b}{a}$$

The discrepancy in sign is the result of not specifying a voltage reference point and it is not significant relative to the results.

The approximate transient behavior of a very thin dipole antenna fed by a transmission line with characteristic impedance Z_o is found by taking the inverse Fourier transform of equation (6.14).

7. Conclusions

The results presented in this report provide a comprehensive analysis of the open ended coaxial transmission line or coaxial waveguide antenna. The fields radiated from such structures can be computed without a priori assumption of the aperture field. The current reflection coefficient for the fundamental mode is found to be $-J_0(ka)$ for $ka << 1$.

The conventional approximation for the fields radiated from the coaxial aperture antenna gives good agreement with the results obtained through the Wiener-Hopf method for thin radii in the feedline ($kb, ka << 1$).

When the excitation is such that $\lambda << b-a$, the radiation pattern obtained by the conventional analysis method is shown to be valid for the predominantly forward direction only. We determine that the classical radiation pattern obtained by conventional methods should be multiplied by the factor $|\cos\theta|^{1/2}$ for $\lambda \to 0$. The resulting pattern is one in which the radiation at high frequencies is confined to the forward direction as expected from geometrical optics predictions.

A model for the cylindrical monopole or dipole antenna is developed based on the observation that radiation appears to emanate only from the base and tip of the structure in pulse excitation. This model is used to determine the radiated electromagnetic field of cylindrical monopole antennas for both harmonic time dependent and arbitrary time dependent excitation voltages.

The mathematical techniques required to obtain results using this model are extremely simple and in most cases hand calculations are sufficient. The results obtained show good agreement with experiment and are useful for both analysis of the radiation field when the driving voltage is given and for synthesis of a driving voltage to produce a specific radiation field.

Future research can be aimed at removing the restriction that the width of the annular region of the coaxial waveguide antenna is very much smaller than the wavelength of the exciting source. The results could also be extended to the thick antenna case (i.e., $kb, ka \geq 1$). Additional research effort on the transient behavior of the cylindrical monopole could be directed toward characterizing the effect produced by different end cap configurations, determination of frequency limitations, accounting for dispersion and describing the effect of higher order mode components of the aperture field on the transient radiation.

Appendix A - Analytical Determination that H Modes are not Present on Structure

In this appendix we will show that H modes are not present on the coaxial waveguide antenna excited by a TEM mode.
It is obvious that symmetry precludes excitation of all H modes except H_{on} $(n=1,2,3,\cdots)$ modes.

By the edge conditions [19] (see Figure 2-1) we know that the longitudinal current must approach zero. Thus the sum of the incident current and the reflected (longitudinal) current must be zero. Therefore the reflected current on each conductor has a longitudinal component equal and opposite to the incident current. All of the reflected current is longitudinal as it must be to completely cancel the incident current. To have H_{on} modes, a circumferential component of current should be present on the structure. Since no current source is available to launch a circumferential current, the H_{on} modes cannot be present.

The above paragraph notwithstanding, let us assume that some current is launched in the circumferential direction upon reflection of the incident current. It is shown in Chapter 2 that the longitudinal current in each conductor is equal and opposite. In keeping with this, we assume that the reflected currents on inner and outer conductors must also be equal in magnitude and directed in opposite circumferential directions. It will now be shown that $H_{on} \equiv 0$ $(n=1,2,3,\cdots)$. The applicable scalar Helmholtz equation for this problem is

$$\left[\frac{\partial^2}{\partial z^2} + \frac{\partial^2}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial}{\partial \rho} + k^2 - \frac{1}{\rho^2}\right] E_\phi(\rho,z,\omega) = 0 \qquad \text{(A.1)}$$

$$E_\phi(\rho,z,\omega) = 0 \quad \text{at} \quad \rho = a,b \; ; \quad z \leq 0$$

$$E_\phi(\rho,z,\omega) = 0 \quad \text{at} \quad \rho = 0 \quad ; \quad -\infty \leq z \leq \infty$$

with edge condition:

$$E_\phi(\rho,z,\omega) \to 0 \quad \text{at} \quad \rho = a,b \; ; \quad z \to 0^+$$

Fourier transforming (A.1) gives:

$$\left[\frac{\partial^2}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial}{\partial \rho} - \gamma^2 - \frac{1}{\rho^2}\right] E_\phi(\rho,\alpha,\omega) = 0 \qquad \text{(A.2)}$$

Using the regions of Figure 2-2, with H_ϕ replaced by E_ϕ and the asymptotic behavior of $E_\phi(\rho,z,\omega)$ as $z \to \pm\infty$, we find that $E_{\phi_1}(\rho,\alpha,\omega)$, $E_{\phi_2}(\rho,\alpha,\omega)$ and $E_{\phi_3}(\rho,\alpha,\omega)$ are analytic in the strip shown in Figure 2-4. Solving equation (A.2) with $E_\phi = E_{\phi_1}$, E_{ϕ_2}, or E_{ϕ_3} respectively, yields

$$E_{\phi_1} = AK_1(\gamma\rho) \qquad \text{(A.3)}$$

$$E_{\phi_2} = BI_1(\gamma\rho) + CK_1(\gamma\rho) \qquad \text{(A.4)}$$

$$E_{\phi_3} = DI_1(\gamma\rho) \qquad \text{(A.5)}$$

(A.3) satisfies the radiation condition and (A.5) goes to zero at $\rho = 0$. At $\rho = a$,

$$E^{+}_{\phi_3}(a,\alpha,\omega) = E^{+}_{\phi_2}(a,\alpha,\omega) \tag{A.6}$$

$$E^{-}_{\phi_3}(a,\alpha,\omega) = E^{-}_{\phi_2}(a,\alpha,\omega) \equiv 0 \tag{A.7}$$

$$\frac{d}{d\rho}[\rho E^{+}_{\phi_2}(\rho,\alpha,\omega)]\Big|_{\rho=a} = \frac{d}{d\rho}[\rho E^{+}_{\phi_3}(\rho,\alpha,\omega)]\Big|_{\rho=a} \tag{A.8}$$

$$E^{+}_{\phi_3}(a,\alpha,\omega) = BI_1(\gamma a) + CK_1(\gamma a)$$

or

$$C = \frac{E^{+}_{\phi_3}(a,\alpha,\omega) - B(I_1(\gamma a))}{K_1(\gamma a)} \tag{A.9}$$

$$\frac{d}{d\rho}[\rho E_{\phi_2}(\rho,\alpha,\omega)]\Big|_{\rho=a} = \frac{B}{K_1(\gamma a)} - \gamma a \, E^{+}_{\phi_3}(\rho,\alpha,\omega)\frac{K_o(\gamma a)}{K_1(\gamma a)} \tag{A.10}$$

From (A.5) we have

$$\frac{d}{d\rho}[\rho E_{\phi_3}(\rho,\alpha,\omega)]\Big|_{\rho=a} = \gamma a \, \frac{E^{+}_{\phi_3}(a,\alpha,\omega) \; I_o(\gamma a)}{I_1(\gamma a)} \tag{A.11}$$

where use was made of

$$D = \frac{E^{+}_{\phi_3}(a,\alpha,\omega)}{I_1(\gamma a)} \tag{A.11.1}$$

Subtracting (A.11) from (A.10) and using (A.8), it follows that

$$\frac{d}{d\rho}[\rho E^-_{\phi_3}(\rho,\alpha,\omega)]\Big|_{\rho=a} - \frac{d}{d\rho}[\rho E^-_{\phi_2}(\rho,\alpha,\omega)]\Big|_{\rho=a}$$

$$= -\frac{B}{K_1(\gamma a)} + \gamma\, a E^+_{\phi_3}(a,\alpha,\omega)\left[\frac{K_o(\gamma a)}{K_1(\gamma a)} + \frac{I_o(\gamma a)}{I_1(\gamma a)}\right]$$

Define the left hand side of the above equation as $G^-(\alpha)$

$$G^-(\alpha) = -\frac{B}{K_1(\gamma a)} + \frac{E^+_{\phi_3}(a,\alpha,\omega)}{K_1(\gamma a) I_1(\gamma a)} \qquad \text{(A.12)}$$

At $\rho = b$:

$$E^+_{\phi_1}(b,\alpha,\omega) = E^+_{\phi_2}(b,\alpha,\omega) \qquad \text{(A.13)}$$

$$E^-_{\phi_1}(b,\alpha,\omega) = E^-_{\phi_2}(b,\alpha,\omega) \equiv 0 \qquad \text{(A.14)}$$

$$\frac{d}{d\rho}[\rho E^+_{\phi_1}(\rho,\alpha,\omega)]\Big|_{\rho=b} = \frac{d}{d\rho}[\rho E^+_{\phi_2}(\rho,\alpha,\omega)]\Big|_{\rho=b} \qquad \text{(A.15)}$$

$$E^+_{\phi_1}(b,\alpha,\omega) = BI_1(\gamma b) + CK_1(\gamma b)$$

or

$$C = \frac{E^+_{\phi_1}(b,\alpha,\omega) = BI_1(\gamma b)}{K_1(\gamma b)} \qquad \text{(A.16)}$$

$$\frac{d}{d\rho}[\rho E_{\phi_2}(\rho,\alpha,\omega)]\Big|_{\rho=b} = \gamma b[BI_o(\gamma b) - CK_o(\gamma b)]$$

$$= \frac{B}{K_1(\gamma b)} - \gamma b E^+_{\phi_1}(b,\alpha,\omega)\,\frac{K_o(\gamma b)}{K_1(\gamma b)} \qquad \text{(A.17)}$$

$$\frac{d}{d\rho}[\rho E_{\phi_1}(\rho,\alpha,\omega)]\Big|_{\rho=b} = -\gamma b\ E^+_{\phi_1}(b,\alpha,\omega)\ \frac{K_o(\gamma b)}{K_1(\gamma b)} \tag{A.18}$$

where we have used

$$A = \frac{E^+_{\phi_1}(b,\alpha,\omega)}{K_1(\gamma a)} \tag{A.19}$$

Subtracting (A.17) from (A.18) and using (A.15) it follows that

$$\frac{d}{d\rho}[\rho E^-_{\phi_1}(\rho,\alpha,\omega)]\Big|_{\rho=b} - \frac{d}{d\rho}[\rho E^-_{\phi_2}(\rho,\alpha,\omega)]\Big|_{\rho=b} = -\frac{B}{K_1(\gamma b)}$$

Define the left hand side of the above as $E^-(\alpha)$. Then

$$E^-(\alpha) = \frac{-B}{K_1(\gamma b)} \tag{A.20}$$

If we substitute (A.20) into (A.12) we get

$$G^-(\alpha) - E^-(\alpha)\ \frac{K_1(\gamma b)}{K_1(\gamma a)} = \frac{E^+_{\phi_3}(a,\alpha,\omega)}{K_1(\gamma a)I_1(\gamma a)} \tag{A.21}$$

By equating the values of C given in (A.16) and (A.9), it follows that

$$E^+_{\phi_3}(a,\alpha,\omega) = \frac{K_1(\gamma a)}{K_1(\gamma b)}\ E^+_{\phi_3}(a,\alpha,\omega) - B\left[\frac{I_1(\gamma b)}{K_1(\gamma b)} - \frac{I_1(\gamma a)}{K_1(\gamma a)}\right] K_1(\gamma a)$$

Upon use of equation (A.20) the above relation reduces to

$$E^+_{\phi_3}(a,\alpha,\omega) - \frac{K_1(\gamma a)}{K_1(\gamma b)}\ E^+_{\phi_1}(a,\alpha,\omega) = E^-(\alpha)[K_1(\gamma a)I_1(\gamma b) - I_1(\gamma a)K_1(\gamma b)].$$

Substitution of this equation into (A.21) gives the companion equation:

$$G^-(\alpha) - E^-(\alpha)\,\frac{I_1(\gamma b)}{I_1(\gamma a)} = \frac{E^+_{\phi_1}(b,\alpha,\omega)}{K_1(\gamma b)I_1(\gamma a)} \tag{A.22}$$

From the definition of $G^-(\alpha)$ we note that it is proportional to $H_{z_3}(a,\alpha,\omega) - H_{z_2}(a,\alpha,\omega)$, and $E^-(\alpha)$ is proportional to $H_{z_1}(b,\alpha,\omega) - H_{z_2}(b,\alpha,\omega)$ with the same proportionality factors. By well known boundary conditions we know the surface currents on the outer conductors are, respectively:

$$-\vec{a}_\rho \times [H_{z_3}(a,\alpha,\omega) - H_{z_2}(a,\alpha,\omega)]\vec{a}_z = +I_{\phi_a}\vec{a}_\phi$$

$$\vec{a}_\rho \times [H_{z_1}(b,\alpha,\omega) - H_{z_2}(b,\alpha,\omega)]\vec{a}_z = -I_{\phi_b}\vec{a}_\phi$$

In this problem we have assumed that $b-a \ll \frac{a+b}{2}$ and therefore the parallel plate case is approached. Under these conditions with TEM excitation, the currents on the plates can be taken as equal in magnitude and opposite in direction, with a great degree of reliability. Therefore $G^-(\alpha) = -E^-(\alpha)$ and equation (A.21) becomes

$$G^-(\alpha) = \frac{E^+_{\phi_3}(a,\alpha,\omega)}{I_1(\gamma a)[K_1(\gamma b) + K_1(\gamma a)]} \tag{A.23}$$

Simultaneously (A.22) becomes

$$G^-(\alpha) = \frac{E^+_{\phi_1}(b,\alpha,\omega)}{K_1(\gamma b)[I_1(\gamma b) + I_1(\gamma a)]} \tag{A.24}$$

We may factor the denominator of the right hand member in (A.23) and

(A.24), using the techniques of Appendix D.

$$K_1(\gamma b)[I_1(\gamma b) + J_1(\gamma a)] = N^+(\alpha)N^-(\alpha)$$

with $\quad N^+(\alpha) = N^-(-\alpha) \sim (|\alpha|)^{-1/2} \quad$ as $\quad \alpha \to \infty$

with $\quad |\mathrm{Im}\ \alpha| > -k_2$

Similarly

$$I_1(\gamma a)[K_1(\gamma b) + K_1(\gamma a)] = M^+(\alpha)\ M^-(\alpha)$$

with $\quad M^+(\alpha) = M^-(-\alpha) \sim (|\alpha|)^{-1/2} \quad$ as $\quad \alpha \to \infty$

with $\quad |\mathrm{Im}\ \alpha| > -k_2$

It is also necessary to point out that $G^-(\alpha) \sim \alpha^{-1/2}$ as $\alpha \to \infty$. This is determinable from the edge condition that $H_z(\rho,z,\omega) \sim z^{-1/2}$ as $z \to 0^-$ at $\rho = a,b$.

Substitution of these factored forms into (A.24) and (A.23) and algebraic operation yield the entire or integral functions (A.25) and (A.26)

$$G^-(\alpha)\ M^-(\alpha) = \frac{E^+_{\phi_1}(b,\alpha,\omega)}{M^+(\alpha)} \tag{A.25}$$

$$G^-(\alpha)\ N^-(\alpha) = \frac{E^+_{\phi_3}(a,\alpha,\omega)}{N^+(\alpha)} \tag{A.26}$$

Each side of (A.25) or (A.26) is the analytic continuation of the other side. Both sides of (A.25) and (A.26) are analytic in the strip $-k_2 < \tau < k_2$. Since each side of (A.25) and (A.26) is bounded for all α , they must also be constant (Liouville theorem). Examination of the asymptotic behavior of the above expressions shows that each side is identically zero. $E^+_{\phi_3}(a,\alpha,\omega) \to 0$ as $|\alpha| \to \infty$, since $E_\phi(a,z,\omega) \to 0$ as $z \to 0^+$. Therefore $E^+_{\phi_1}(b,\alpha,\omega) = E^+_{\phi_3}(a,\alpha,\omega) = 0$. Each of the coefficients A,B,C,D can be expressed by a linear combination of $E^+_{\phi_1}(b,\alpha,\omega)$ and $E^+_{\phi_3}(a,\alpha,\omega)$. This conclusion is evident by observation of equations (A.11.1), (A.19), and the simultaneous solution of

$$E^+_{\phi_3}(a,\alpha,\omega) = BI_1(\gamma a) + CK_1(\gamma a)$$

$$E^+_{\phi_1}(b,\alpha,\omega) = BI_1(\gamma b) + CK_1(\gamma b)$$

Hence E_{ϕ_1}, E_{ϕ_2} and E_{ϕ_3} are identically zero and $E_\phi(\rho,z,\omega)$ does not appear in any region of the coaxial antenna structure.

Appendix B. Determination of Branch Cuts for γ

In this appendix we will develop the branch cuts in the complex α plane that define the region of analyticity of γ , a multi-valued function of α

$$\gamma = \sqrt{\alpha^2 - k^2}$$

Our starting point for the work in this appendix will be equations (2.9) and (2.9.1):

$$\left(\frac{d^2}{d\rho^2} + \frac{1}{\rho}\frac{d}{d\rho} - \frac{1}{\rho^2} - \gamma^2\right) H_{\phi_1}(\rho,\alpha,\omega) = 0 \tag{2.9}$$

$$H_{\phi_1}(\rho,\alpha,\omega) \to 0 \quad \text{as} \quad \rho \to \infty \tag{2.9.1}$$

The solution to equation (2.9) satisfying (2.9.1) was found in Chapter 2,

$$H_{\phi_1}(\rho,\alpha,\omega) = A(\alpha)\, K_1(\gamma\rho) \tag{2.15}$$

The asymptotic value of the modified Bessel function [20] is known to be

$$K_1(\gamma\rho) \propto \sqrt{\frac{\pi}{2\gamma\rho}}\; e^{-\gamma\rho} \quad \text{as} \quad \rho \to \infty$$

Hence

$$H_{\phi_1}(\rho,z,\omega) \propto \frac{1}{2\sqrt{\gamma\rho}} \int_{-\infty}^{\infty} A(\alpha)\, e^{-\gamma\rho}\, e^{-i\alpha z}\, d\alpha \qquad \text{as} \quad \rho \to \infty \tag{B.1}$$

From the Sommerfeld radiation condition we know that the field intensity must be an outward traveling or evanescent wave for large $r = \sqrt{\rho^2 + z^2}$.

Following the manner of Reference [21, p.20], we note that if we require that

$$\text{Re}\ \gamma \geq 0 \tag{B.2}$$

$$\text{Im}\ \gamma \leq 0 \tag{B.3}$$

then the radiation condition will be satisfied by $H_{\phi_1}(\rho,z,\omega)$. This may be verified by examination of (B.1). The branch cuts will be chosen in a manner which ensures that the conditions (B.2) and (B.3) are satisfied.

The α plane is viewed as a two-sheeted Riemann surface with the sheets connected along the branch cut. In each of the sheets the function γ is single valued. The sign of the function on one sheet is the negative of the sign on the other sheet. The value of γ becomes discontinuous only if a branch cut in the α plane is crossed.

Let us define the branch cut of γ such that $\text{Re}\ \gamma > 0$ in the top sheet and $\text{Re}\ \gamma < 0$ in the bottom sheet. Thus the two sheets are joined together by the curve given by $\text{Re}\ \gamma = 0$.

The branch point is obviously k and we will, for computational purposes, take k complex

$$k = k_1 + ik_2 \quad , \qquad k_1 \gg k_2 \ , \quad k_1, k_2 > 0$$

We may therefore write

$$\gamma^2 = \alpha^2 - k^2 = (\sigma + i\tau)^2 - (k_1 + ik_2)^2$$

$$= (\sigma^2 - \tau^2) - (k_1^2 - k_2^2) + 2i(\sigma\tau - k_1k_2) \qquad \text{(B.4)}$$

Note that $\text{Re}\ \gamma > 0$ in the entire top sheet, therefore if we specify $\gamma^2 = re^{i\theta}$ and $\gamma = r^{1/2}e^{i\theta/2}$ with $0 \leq \theta < 2\pi$, then $\text{Re}\ \gamma > 0$ only if $|\theta| < \pi$. Thus the branch cut must be given by $|\theta| = \pi$ or $\text{Re}\ \gamma^2 \leq 0$ and $\text{Im}\ \gamma^2 = 0$. The branch cuts which divide the α plane can be easily identified by examination of Figure B-1 and are seen to be a portion of the curve $\sigma\tau = k_1k_2$.

Since the branch cut is given by $\gamma^2 = re^{i\pi}$ it follows that on the branch cut $\gamma = r^{1/2}e^{i\pi/2}$ or $\text{Re}\ \gamma = 0$. The graph (B-2) depicts the branch cuts for the top sheet.

For the branch cuts in the bottom sheet, it is only necessary to use the opposite sign of γ everywhere in Figure B-2.

It is also worthwhile to state that for this branch definition

$$\gamma = \sqrt{r_1r_2}\, e^{i\theta} \quad \text{where} \quad \theta = \frac{\phi_1 + \phi_2}{2}$$

$$\frac{\pi}{2} \leq \phi_1 < -\frac{3\pi}{2} \;, \quad -\frac{\pi}{2} \leq \phi_2 < \frac{3\pi}{2} \quad \text{and} \quad -\frac{\pi}{2} < \theta < \frac{\pi}{2}$$

$$\gamma = -i\sqrt{k^2 - \alpha^2} \quad \text{or} \quad \sqrt{k^2 - \alpha^2} = i\gamma$$

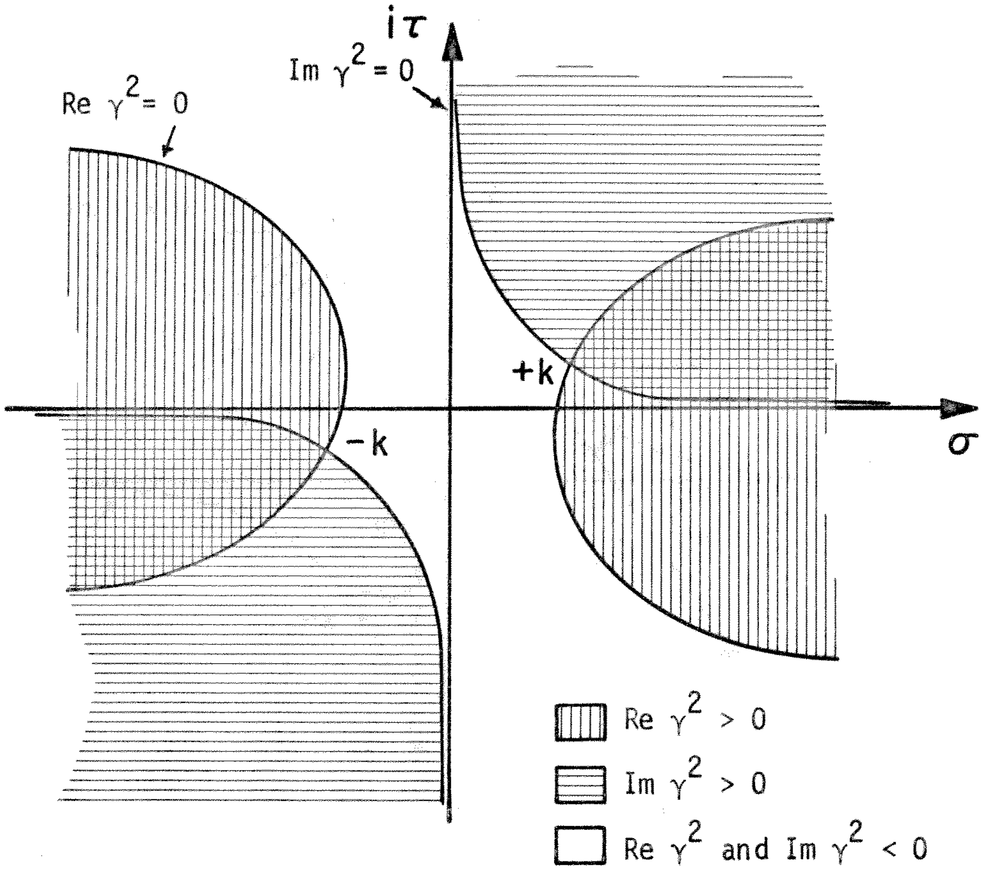
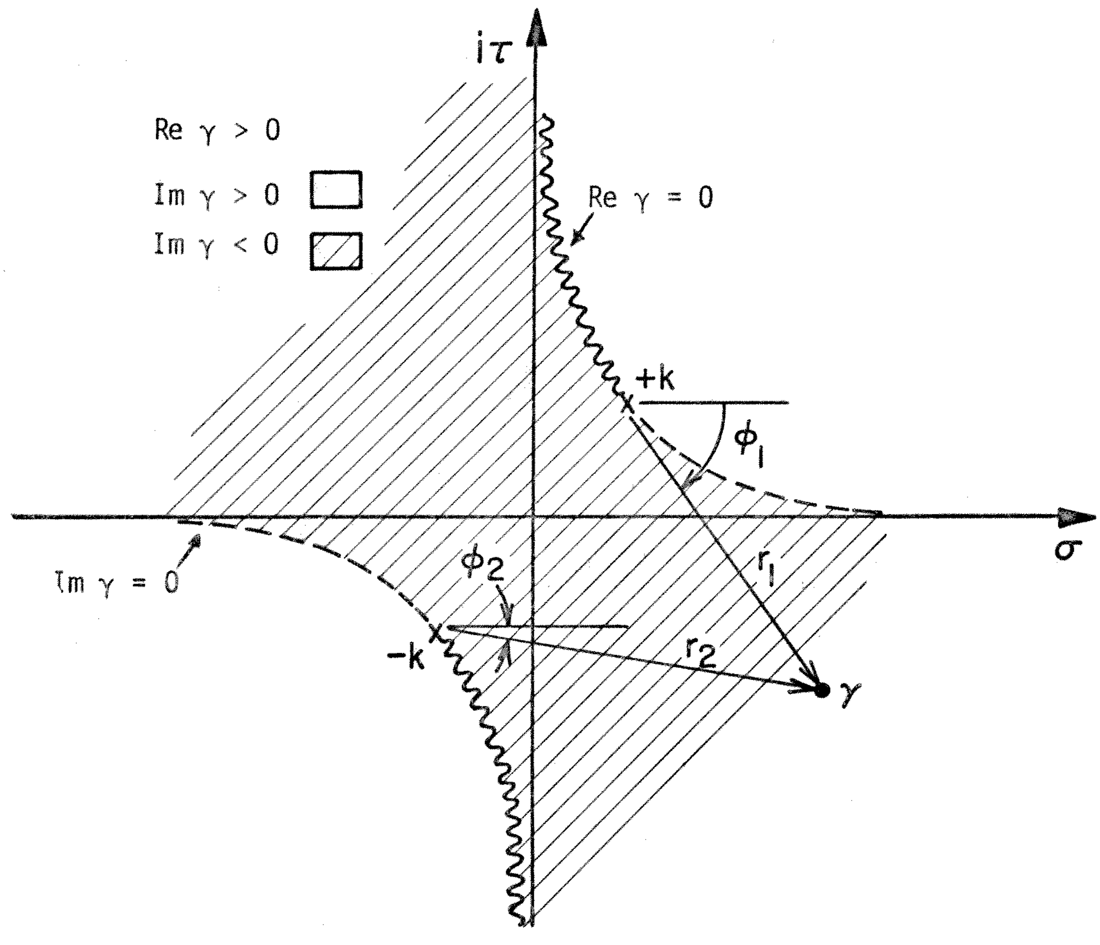


Figure B-1. Regions of complex α plane



ıre B-2. Branch cut for multivalued function γ

Appendix C. Useful Expressions and Expansions

In the main body of this work, extensive use is made of ordinary and modified Bessel functions and Hankel functions. Listed below are explicit expressions, asymptotic approximations and limiting forms for some of these functions [20,24] of integer order.

(i) Relations between modified and ordinary Bessel functions:

$$I_\nu(z) = e^{-\frac{1}{2}\nu\pi i} J_\nu(z\, e^{\frac{1}{2}\pi i}) \quad ; \qquad (-\pi < \arg z \leq \tfrac{\pi}{2})$$

$$K_\nu(z) = \frac{1}{2}\pi i\, e^{\frac{1}{2}\nu\pi i} H_\nu^{(1)}(z e^{\frac{1}{2}\pi i}); \qquad (-\pi < \arg z \leq \tfrac{\pi}{2})$$

$$H_\nu^{(1)}(z) = J_\nu(z) + i\, Y_\nu(z)$$

$$H_\nu^{(2)}(z) = J_\nu(z) - iY_\nu(z)$$

(ii) Wronskians:

$$I_\nu(z)\, K_{\nu+1}(z) + I_{\nu+1}(z)\, K_\nu(z) = \frac{1}{z}$$

$$J_{\nu+1}(z)\, Y_\nu(z) - J_\nu(z)\, Y_{\nu+1}(z) = \frac{2}{\pi z}$$

(iii) Series:

$$J_0(z) = 1 - \frac{\frac{1}{4} z^2}{(1!)^2} + \frac{(\frac{1}{4} z^2)^2}{(2!)^2} - \frac{(\frac{1}{4} z^2)^3}{(3!)^2} + \cdots$$

$$Y_o(z) = \frac{2}{\pi}\{(\ln \frac{z}{2}) + C\} J_o(z) + \frac{2}{\pi} \frac{\frac{1}{4} z^2}{(1!)^2} - \frac{3}{2} \frac{(\frac{1}{4} z^2)^2}{(2!)^2} + \cdots$$

(iv) Limiting forms for small arguments (when ν is fixed and $z \to 0$):

$$J_\nu(z) \sim (\frac{1}{2} z)^\nu / \Gamma(\nu+1) \quad ; \quad \nu \neq -1,-2,-3,\cdots$$

$$Y_o(z) \sim -iH_o^{(1)}(z) \sim iH_o^{(2)}(z) \sim \frac{2}{\pi} \ln z$$

$$Y_\nu(z) \sim -iH_\nu^{(1)}(z) \sim iH_\nu^{(2)}(z) \sim -\frac{\Gamma(\nu)}{\pi} (\frac{z}{2})^{-\nu}; \quad \mathrm{Re}(\nu) > 0$$

$$I_\nu(z) \sim (\frac{1}{2} z)^\nu / \Gamma(\nu+1) \quad ; \quad \nu \neq -1,-2,-3,\cdots$$

$$K_o(z) \sim -\ln z$$

$$K_\nu(z) \sim \frac{1}{2} \Gamma(\nu)(\frac{z}{2})^{-\nu}$$

(v) Asymptotic Expansions (when ν is fixed and $|z| \to \infty$)

$$J_\nu(z) = \sqrt{\frac{2}{\pi z}} \{\cos(z - \frac{1}{2} \nu\pi - \frac{\pi}{4})\} + O(\frac{1}{|z|}) \quad ; \quad (|\arg z| < \pi)$$

$$Y_\nu(z) = \sqrt{\frac{2}{\pi z}} \{\sin(z - \frac{1}{2} \nu\pi - \frac{\pi}{4})\} + O(\frac{1}{|z|}) \quad ; \quad (|\arg z| < \pi)$$

$$I_\nu(z) \sim \frac{e^z}{\sqrt{2\pi z}} \quad ; \quad (|\arg z| < \frac{\pi}{2})$$

$$K_\nu(z) \sim \sqrt{\frac{\pi}{2z}} e^{-z} \quad ; \quad (|\arg z| < \frac{3\pi}{2})$$

(vi) Derivatives:

$$\frac{d}{dz}[z^{\nu} I_{\nu}(z)] = z^{\nu} I_{\nu-1}(z)$$

$$\frac{d}{dz}[z^{\nu} K_{\nu}(z)] = -z^{\nu} K_{\nu-1}(z)$$

(vii) Recurrence Relations:

$$I_{\nu-1}(z) - I_{\nu+1}(z) = \frac{2\nu}{z} I_{\nu}(z)$$

$$K_{\nu-1}(z) - K_{\nu+1}(z) = -\frac{2\nu}{z} K_{\nu}(z)$$

(viii) Other Relations:

$$I_{-\nu}(z) = I_{\nu}(z)$$

$$K_{-\nu}(z) = K_{\nu}(z)$$

Appendix D. Factorization Procedure

The purpose of this appendix is to furnish a factorization procedure for the Wiener-Hopf kernels which arise throughout this work. The factorization procedure used extensively in this work and developed in this appendix was originated by Bates and Mittra [23]. The mathematical development of this procedure is first presented and then it is applied to the specific expressions to be factored.

We begin this appendix with a factorization of $e^{-\gamma h}$ with $\gamma = \sqrt{\alpha^2 - k^2}$ using the well known theorems of [4].

Theorem A: Let $f(\alpha)$ be an analytic function of $\alpha = \sigma + i\tau$ regular in the strip $\tau_- < \tau < \tau_+$, such that $f(\alpha) < C|\sigma|^{-P}$, $P > 0$ for $|\sigma| \to \infty$, the inequality holding uniformly for all τ in the strip. Then $f(a)$ can be decomposed such that

$$f(\alpha) = f^+(\alpha) + f^-(\alpha)$$

with

$$f^+(\alpha) = \frac{1}{2\pi i} \int_{-\infty+ic}^{\infty+ic} \frac{f(\beta)}{\beta - \alpha} d\beta \qquad ; \qquad \tau_- < c < \tau < \tau_+$$

$$f^-(\alpha) = -\frac{1}{2\pi i} \int_{-\infty+id}^{\infty+id} \frac{f(\beta)}{\beta-\alpha} d\beta \qquad ; \qquad \tau_- < \tau < d < \tau_+$$

where $f^+(\alpha)$ is analytic in the portion of the α plane defined by $\tau > \tau_-$ and $f^-(\alpha)$ is analytic in that portion of the α plane defined by $\tau < \tau_+$.

Theorem B: If $\ln g(\alpha)$ satisfies the conditions of Theorem A, implying that $g(\alpha)$ is an analytic function of α , which is regular and nonzero in the strip $\tau_- < \tau < \tau_+$ and $g(\alpha) \to 1$ uniformly as $\sigma \to \pm\infty$ in the strip, then $g(\alpha)$ can be factored such that

$$g(\alpha) = g^+(\alpha)\, g^-(\alpha)$$

with

$$g^+(\alpha) = \exp \frac{1}{2\pi i} \int_{-\infty+ic}^{\infty+ic} \frac{\ln g(\beta)}{\beta-\alpha}\, d\beta \quad ; \quad \tau_- < c < \tau < \tau_+$$

$$g^-(\alpha) = \exp - \frac{1}{2\pi i} \int_{-\infty+id}^{\infty+id} \frac{\ln g(\beta)}{\beta-\alpha}\, d\beta \; ; \; \tau_- < \tau < d < \tau_+$$

The function $g^+(\alpha)$ is analytic in that portion of the α plane defined by $\tau > \tau_-$ and $g^-(\alpha)$ is analytic in that portion of the plane defined by $\tau < \tau_+$. The principal branch of the logarithmic function is used throughout. It is also worthy to note that if $g(\alpha) \sim \frac{1}{|\alpha|}$ and $\ln g(\alpha) \sim -\ln|\alpha|$ as $|\alpha| \to \infty$, Theorem B can still be applied. The integrals are convergent in the sense

$$\lim_{T \to \infty} \int_{-T+i\tau}^{T+i\tau} \{ \quad \} \, d\beta$$

If $f(\alpha)$ and $g(\alpha)$ are even functions of α then

$$f^+(-\alpha) = f^-(\alpha) \qquad \text{or} \qquad f^+(\alpha) = f^-(-\alpha)$$

and

$$g^{+}(-\alpha) = g^{-}(\alpha) \qquad \text{or} \qquad g^{+}(\alpha) = g^{-}(-\alpha)$$

The function $e^{-\gamma h}$, which we wish to factor is interpreted as $e^{-h\sqrt{\alpha^2 - k^2}}$ for $\alpha^2 > k^2$ or $e^{+ih\sqrt{k^2 - \alpha^2}}$ for $k^2 > \alpha^2$. Examine

$$\ell n\, g(\alpha) = -\gamma h = -h(\alpha^2 - k^2)(\alpha^2 - k^2)^{-1/2}$$

The first factor is an entire or integral function of α . Therefore we only need to decompose the second factor $1/\gamma$. Since γ is multivalued, we will use the branch cuts for the top sheet of the two sheeted Riemann surface of the α plane (see Appendix B).

From Theorem A, we have

$$f^{+}(\alpha) = \frac{1}{2\pi i} \int_{-\infty+ic}^{\infty+ic} \frac{d\beta}{\sqrt{\beta^2 - k^2}(\beta-\alpha)} \;; \qquad -k_2 < c < \tau \tag{D.1}$$

The integration path for equation (D.1) is diagrammed in Figure D-1. The integral (D.1) will be evaluated by integration along a contour in the upper half plane, which does not cross the branch cut located in that half plane (see Figure D-2). By Cauchy's theorem the value of the integral (D.1) is

$$f^{+}(\alpha) = \frac{1}{\sqrt{\alpha^2 - k^2}} - \frac{1}{2\pi i} \int_{P_1 + P_2} \frac{d\beta}{\sqrt{\beta^2 - k^2}(\beta-\alpha)} \tag{D.2}$$

The contributions to the integral from the semicircular portions of the path at $|\beta| \to \infty$ are zero by Jordan's lemma.

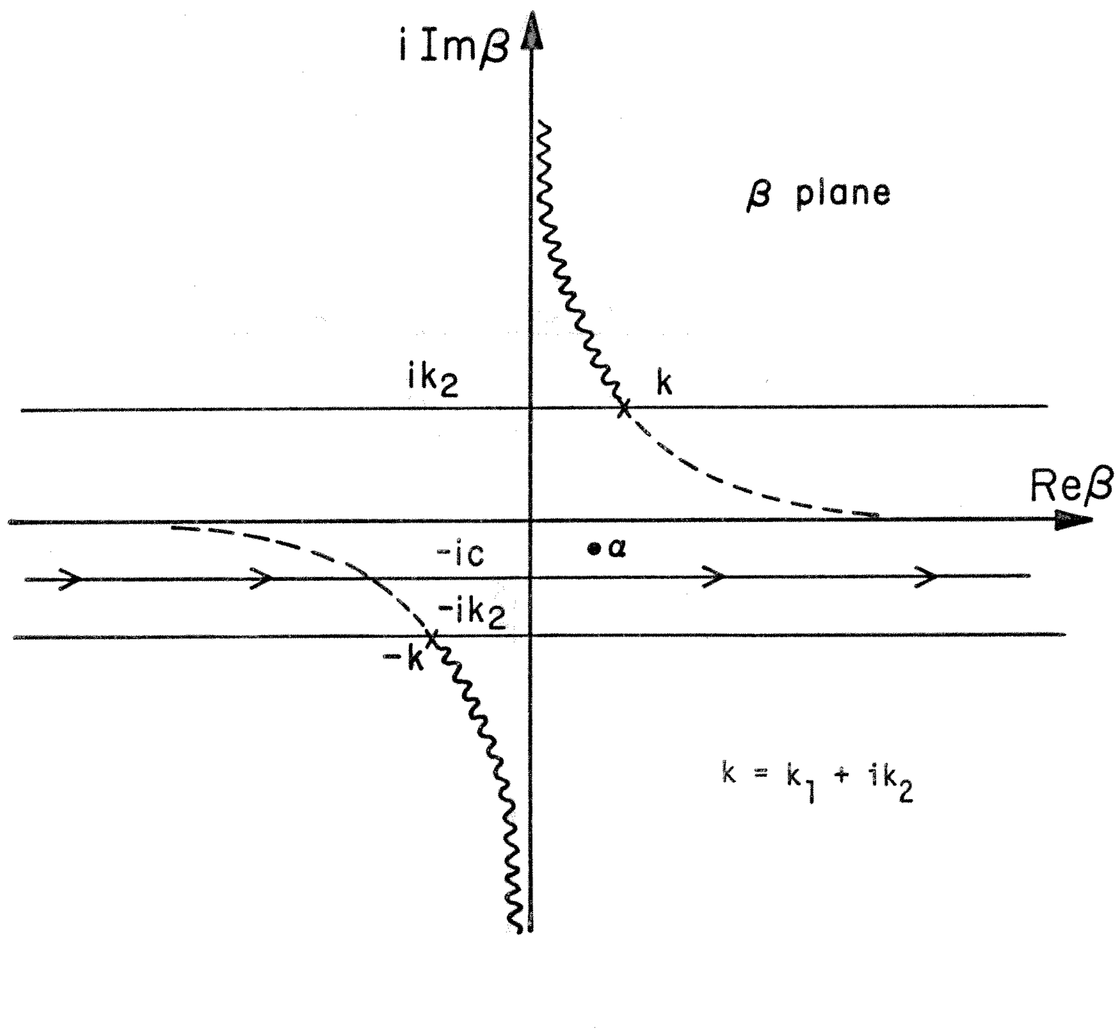


Figure D-1 .

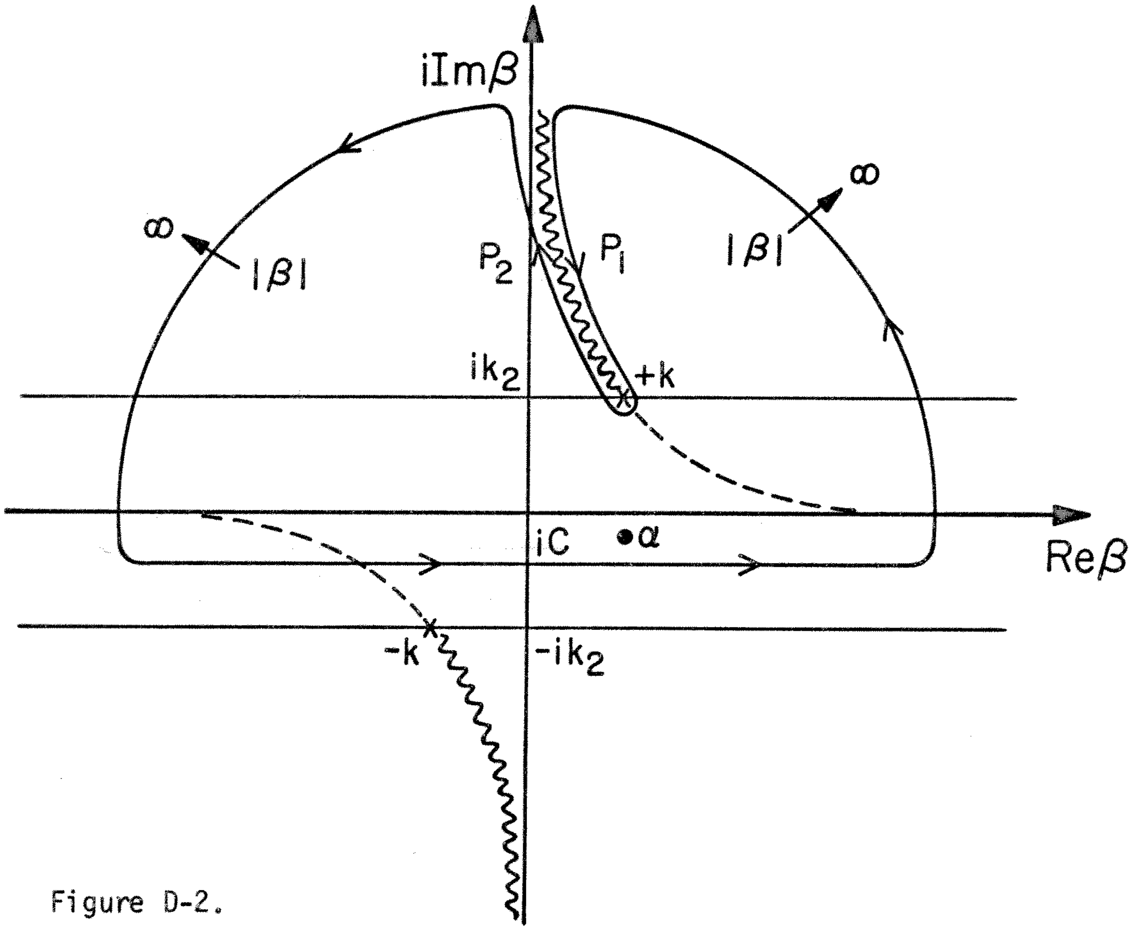


Figure D-2.

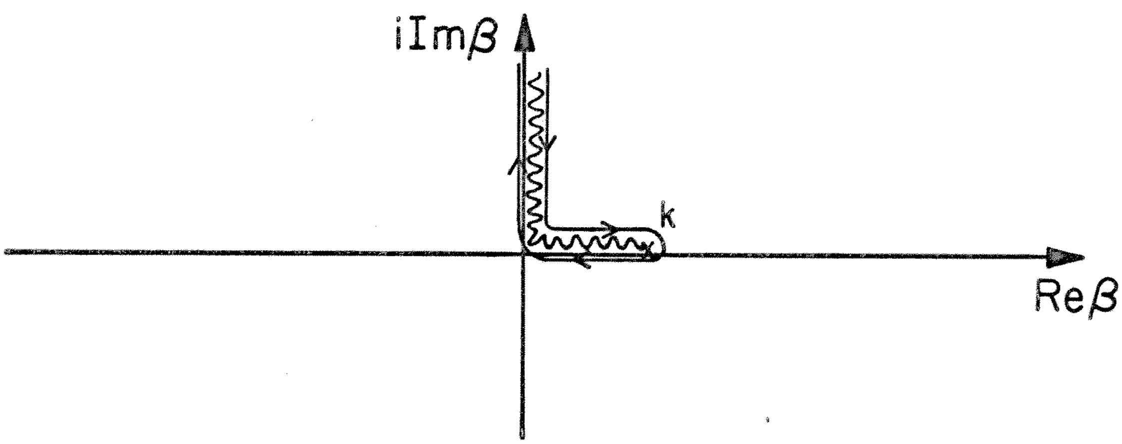


Figure D-3.

For the limiting case of no losses, $k_2 \to 0^+$, the path $P_1 + P_2$ is adjacent to the coordinates as shown in Figure D-3. Under these circumstances the remaining integral of (D.2) is expressed as the five separate integrals which follow:

$$\frac{1}{2\pi i}\left\{ \int_{\infty}^{0} \frac{i\,dy}{i\sqrt{y^2+k^2}\,(iy-\alpha)} + \int_{0}^{k} \frac{dx}{i\sqrt{k^2-x^2}\,(x-\alpha)} \right.$$

$$+ \lim_{r\to 0} \int_{\pi}^{-\pi} \frac{i r e^{i\theta}\, d\theta}{\sqrt{2rke^{i\theta} + r^2 e^{2i\theta}}\,(k + re^{i\theta} - \alpha)} + \int_{k}^{0} \frac{dx}{-i\sqrt{k^2-x^2}\,(x-a)}$$

$$\left. + \int_{0}^{\infty} \frac{i\,dy}{-i\sqrt{y^2+k^2}\,(iy-\alpha)} \right\}$$

By reversing the limits on the last two integrals and evaluation of the third, we can reduce the above set of integrals to:

$$-\frac{1}{\pi}\int_{0}^{k} \frac{dx}{(x-\alpha)\sqrt{k^2-x^2}} + \frac{1}{\pi}\int_{0}^{\infty} \frac{dy}{\sqrt{y^2+k^2}\,(y+i\alpha)}$$

A further reduction is achieved by a change of variable. Let $x = k \sin\delta$ and $y = ik \sin\delta$. Incorporation of this change yields:

$$-\frac{1}{\pi}\int_{0}^{\pi/2} \frac{d\delta}{k \sin\delta - \alpha} + \frac{1}{\pi}\int_{0}^{-i\infty} \frac{d\delta}{k \sin\delta - \alpha}$$

Note that use has been made of the identity, $\sin(-iy) = -i \sinh y$.

From Reference [20, p.78] we have the following integral:

$$\int \frac{dz}{a+b\sin z} = \begin{cases} \dfrac{2}{(a^2-b^2)^{1/2}} \arctan\left[\dfrac{a\tan\frac{z}{2}+b}{(a^2-b^2)^{1/2}}\right] & ;\ (a^2 > b^2) \\[2ex] \dfrac{1}{(b^2-a^2)^{1/2}} \ln\left[\dfrac{a\tan\frac{z}{2}+b-(b^2-a^2)^{1/2}}{a\tan\frac{z}{2}+b+(b^2-a^2)^{1/2}}\right] & ;\ (b^2 > a^2) \end{cases}$$

Through this integral we are able to complete the evaluation as follows:

$$\begin{cases} -\dfrac{2}{\pi(\alpha^2-k^2)^{1/2}} \left\{ \arctan\left[\dfrac{-\alpha\tan\frac{\delta}{2}+k}{(\alpha^2-k^2)^{1/2}}\right]_0^{\frac{\pi}{2}} - \arctan\left[\dfrac{-\alpha\tan\frac{\delta}{2}+k}{(\alpha^2-k^2)^{1/2}}\right]_0^{-i\infty} \right\} & ;\ (\alpha^2 > k^2) \\[3ex] -\dfrac{1}{\pi(k^2-\alpha^2)^{1/2}} \left\{ \ln\left[\dfrac{-\alpha\tan\frac{\delta}{2}+k-(k^2-\alpha^2)^{\frac{1}{2}}}{-\alpha\tan\frac{\delta}{2}+k+(k^2-\alpha^2)^{\frac{1}{2}}}\right]_0^{\frac{\pi}{2}} - \ln\left[\dfrac{-\alpha\tan\frac{\delta}{2}+k-(k^2-\alpha^2)^{\frac{1}{2}}}{-\alpha\tan\frac{\delta}{2}+k+(k^2-\alpha^2)^{\frac{1}{2}}}\right]_0^{-i\infty} \right\} & ;\ (k^2 > \alpha^2) \end{cases}$$

Since $\tan(-iy) = -i\tanh y$ and $\lim_{y\to\infty} \tanh y = 1$, the above relation simplifies to:

$$\begin{cases} \dfrac{2}{\pi(\alpha^2-k^2)^{1/2}} \left[\arctan\left(\dfrac{+i(\alpha+k)}{\sqrt{\alpha^2-k^2}}\right)\right] & ;\ (\alpha^2 > k^2) \\[3ex] \dfrac{1}{\pi(k^2-\alpha^2)^{1/2}} \left\{ \ln\left[\dfrac{+i\alpha+k-(k^2-\alpha^2)^{\frac{1}{2}}}{+i\alpha+k+(k^2-\alpha^2)^{\frac{1}{2}}}\right] - \ln\left[\dfrac{-\alpha+k-(k^2-\alpha^2)^{1/2}}{-\alpha+k+(k^2-\alpha^2)^{1/2}}\right] \right\} & ;\ (k^2 > \alpha^2) \end{cases}$$

The trigonometric identity

$$\arctan z_1 \pm \arctan z_2 = \arctan[\frac{z_1 \pm z_2}{1 \mp z_1 z_2}]$$

was applied to the top expression.

Recognizing the relation [25, p.62]

$$\tan^{-1} z = \frac{i}{2} \ln[\frac{1 - iz}{1 + iz}]$$

and applying it, we are able to achieve great simplification

$$\left\{ \begin{array}{ll} -\frac{i}{\pi\gamma} \ln[\frac{\gamma - (\alpha + k)}{\gamma + (\alpha + k)}] & ; \quad (\alpha^2 > k^2) \\ -\frac{i}{\pi\gamma} \ln[\frac{\gamma - (\alpha + k)}{\gamma + (\alpha + k)}] & ; \quad (k^2 > \alpha^2) \end{array} \right\}$$

Note that the same result is found for $\alpha^2 > k^2$ and $k^2 > \alpha^2$. When the above is reinserted into (D.2), the decomposition is completed and

$$f^+(\alpha) = \frac{1}{\gamma}[1 + \frac{i}{\pi} \ln[\frac{\gamma - (\alpha + k)}{\gamma + (\alpha + k)}]]$$

$$= \frac{-i}{\pi\gamma}[+\pi i - \ln[\frac{\gamma - (\alpha + k)}{\gamma + (\alpha + k)}]]$$

$$= -\frac{i}{\pi\gamma} \ln[\frac{(\alpha + k) + \gamma}{(\alpha + k) - \gamma}] = \frac{-i}{\pi\gamma} \ln[\frac{\gamma + (\alpha - k)}{\gamma - (\alpha - k)}]$$

The factorization of $g(\alpha) = e^{-\gamma h}$ can now be easily completed. From the above we have

$$\ln g^+(\alpha) = + \frac{ih\gamma}{\pi} \ln\left[\frac{\gamma + (\alpha - k)}{\gamma - (\alpha - k)}\right]$$

Because $g(\alpha)$ is even in α, it follows that

$$\ln g^+(-\alpha) = \ln g^-(\alpha)$$

and

$$\ln g^-(\alpha) = + \frac{ih\gamma}{\pi} \ln\left[\frac{\gamma - (\alpha + k)}{\gamma + (\alpha + k)}\right]$$

Finally

$$g^+(\alpha) = g^-(-\alpha) = e^{+\frac{ih\gamma}{\pi} \ln\left[\frac{\gamma + (\alpha - k)}{\gamma - (\alpha - k)}\right]} \tag{D.3}$$

This result is consistent with that of Noble [4, pp.20-21], and is shown to be correct by the following:

$$g(\alpha) = g^+(\alpha)\, g^-(\alpha) = e^{\frac{ih\gamma}{\pi}\left\{\ln\left[\frac{\gamma + (\alpha-k)}{\gamma - (\alpha-k)}\right] + \ln\left[\frac{\gamma - (\alpha+k)}{\gamma + (\alpha+k)}\right)\right]\right\}}$$

$$= e^{+\frac{ih\gamma}{\pi}(i\pi)} = e^{-h\gamma} \tag{D.3.1}$$

where $\ln z = \ln|z| + i\phi$

$$-\pi < \phi \le \pi$$

The only possible singularity of $g^+(\alpha)$ in the upper half plane is a branch point singularity. It is obvious that this singularity is not present and that $g^+(\alpha)$ is regular in the upper half plane, since substitution of $\pm i|\gamma|$ for γ has the same result. Similarly, for $g^-(\alpha)$,

$$e^{\pm ih\frac{|\gamma|}{\pi}\ \ell n[\frac{\pm i|\gamma| + (\alpha-k)}{\pm i|\gamma| - (\alpha-k)}]} = e^{\pm ih\frac{|\gamma|}{\pi}[\ell n\ e^{i\phi}]} = e^{-h\phi\frac{\gamma}{\pi}}$$

$$\phi = 2\ \tan^{-1}\frac{|\gamma|}{\alpha-k}$$

Theorem C: [23] Let $G(\alpha)$ be an analytic function of α in the region $\tau_- < \alpha < \tau_+$ with the following conditions:

1. $G(\alpha)$ is regular in the region $\tau_- < \alpha < \tau_+$.
2. $G(\alpha) \neq 0$; $G(\alpha) = G(-\alpha)$ (even).
3. $G(\alpha) \sim c\alpha^{\nu}\, e^{-h|\alpha|}$ as $|\mathrm{Re}\ \alpha| \to \infty$ where ν and h are real constants.

Then $G(\alpha)$ can be represented within the strip by

$$G(\alpha) = G^+(\alpha)\ G^-(\alpha)$$

where $G^+(\alpha)$ is analytic and nonzero in the half plane $\tau > \tau_-$ and $G^-(\alpha)$ is analytic and nonzero in the half plane $\tau < \tau_+$. The expressions for $G^+(\alpha)$ and $G^-(\alpha)$ are given by

$$G^+(\alpha) = G^-(-\alpha) = \{\sqrt{G(0)}\ (1+\frac{\alpha}{k})^{\nu/2}\}$$

$$\{B^+(\alpha)\quad \exp[\frac{ih\gamma}{\pi}\ \ell n\ [\frac{\gamma + (\alpha-k)}{\gamma - (\alpha-k)}] - \frac{ikh}{2}]\} \qquad (D.4)$$

with

(a)

$$B^{+}(\alpha) = \exp\left[\frac{1}{2\pi i}\int_{-\infty-id}^{\infty-id}\frac{\alpha F(\beta)}{\beta(\beta-\alpha)}\,d\beta\right]; \quad |\mathrm{Im}\ \alpha| < |d|;\ |\tau_{-}| > |d| \tag{D.4.1}$$

(b)

$$F(\beta) = h(\beta^2 - k^2)^{1/2} - \nu\,\ln(\beta^2 - k^2)^{1/2} - \ln c + \ln G(\beta) \tag{D.4.2}$$

(c)

$\gamma = (\alpha^2 - k^2)^{1/2}$ with branch of Appendix B.

(d)

$k = k_1 + ik_2 \quad ; \quad k_1 >> k_2 \ ; \quad k_2, k_1 > 0$ and $|\tau_{-}| \leq k_2 \ ; \quad \tau_{+} \leq k_2$

We will now show the validity of the above theorem. We begin by observing that

$$B^{+}(-\alpha) = \exp\left[\frac{1}{2\pi i}\int_{-\infty-id}^{\infty-id}\frac{-\alpha F(\beta)}{\beta(\beta+\alpha)}\,d\beta\right]$$

If β is replaced by $-\beta$ and use is made of the fact that $F(\beta) = F(-\beta)$ then $B^{+}(-\alpha)$ becomes:

$$B^{+}(-\alpha) = \exp\left[\frac{1}{2\pi i}\int_{\infty+id}^{-\infty+id}\frac{\alpha F(\beta)\,d\beta}{\beta(\beta-\alpha)}\right]$$

and

$$B^{+}(-\alpha) = B^{-}(\alpha)$$

We next find the product

$$B^{+}(\alpha)\ B^{-}(\alpha) = \exp\left[\frac{1}{2\pi i}\int_{c} \frac{\alpha F(\beta)\ d\beta}{\beta(\beta-\alpha)}\right]$$

where c is the contour shown in Figure D-4. Observe that the asymptotic behavior of the integrand can be used to show that there is no contribution to the integral from the vertical segments of the contour at $|\mathrm{Re}\ \beta| \Rightarrow \infty$. By Cauchy's theorem, the value of the integral exponent in the above expression is

$$2\pi i \sum \left(\text{residues of the integrand, from the poles at } \beta = 0 \text{ and } \beta = \alpha\right)$$

$$= -F(0) + F(\alpha)$$

$$= h\gamma + ihk + \ln\left[\frac{G(\alpha)}{G(0)(1-\frac{\alpha^2}{k^2})^{\nu/2}}\right]$$

Therefore

$$B^{+}(\alpha)\ B^{-}(\alpha) = \frac{G(\alpha)\exp[h\gamma + ihk]}{G(0)[1-\frac{\alpha^2}{k^2}]^{\nu/2}}$$

From (D.4), we formulate

$$G^{+}(\alpha)\ G^{-}(\alpha) = G(0)(1-\frac{\alpha^2}{k^2})^{\nu/2}\ B^{+}(\alpha)\ B^{-}(\alpha)$$

$$\exp\left[\frac{ih\gamma}{\pi}\ln\left[\frac{\gamma + (\alpha-k)}{\gamma - (\alpha-k)}\right] + \frac{ih\gamma}{\pi}\ln\left[\frac{\gamma - (\alpha+k)}{\gamma + (\alpha+k)}\right] - ikh\right]$$

Substituting for $B^{+}(\alpha)\ B^{-}(\alpha)$ and using equation (D.3.1), we find

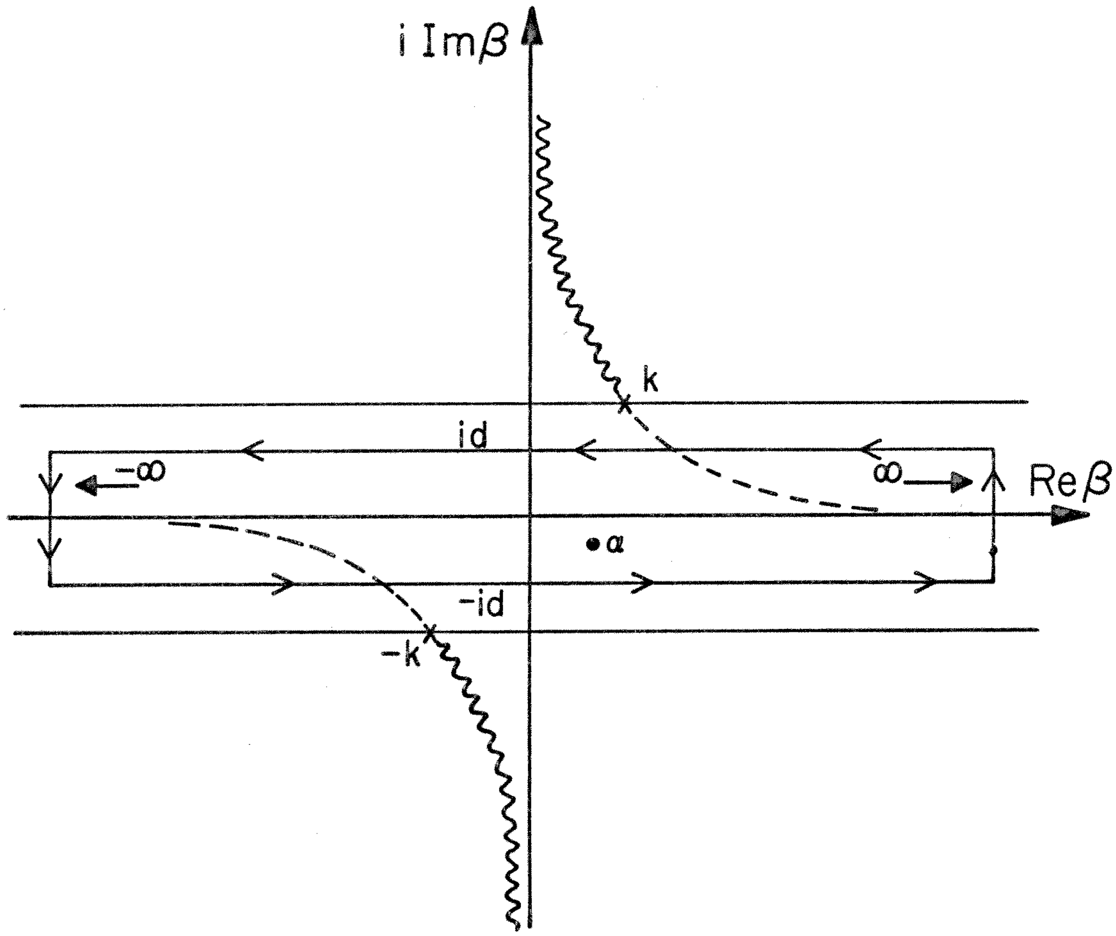


Figure D-4.

$G^{+}(\alpha)\ G^{-}(\alpha) = G(\alpha)$, which establishes Theorem C.

Precise knowledge of the asymptotic behavior of $G^{+}(\alpha) = G^{-}(-\alpha)$ as $|\alpha| \to \infty$ within $\tau > \tau_{-}$ is necessary for successful use of the Wiener-Hopf method. As our next task, we will determine this asymptotic behavior. Notice that $B^{+}(\alpha)$ is bounded as $|\alpha| \to \infty$, therefore by letting $|\alpha| \to \infty$ in (D.4), we find that

$$G^{+}(\alpha) = G^{-}(-\alpha) \sim (|\alpha|)^{\nu/2} \lim_{|\alpha| \to \infty} e^{i \frac{\gamma h}{\pi} \ell n \left| \frac{\gamma + (\alpha - k)}{\gamma - (\alpha - k)} \right|}$$

$$= (|\alpha|)^{\nu/2} e^{i \frac{\alpha h}{\pi} \ell n \left| \frac{2\alpha}{k} \right|} \tag{D.5}$$

for $|\alpha| \to \infty$ within $\tau > \tau_{-}$. In the manner of [23] the factorization formula of Theorem C will now be converted to a form convenient for numerical work. In this development, we concentrate on changing

$$B^{+}(\alpha) = \exp \left[\frac{1}{2\pi i} \int_{-\infty - id}^{+\infty - id} \frac{\alpha F(\beta)}{\beta(\beta - \alpha)} \, d\beta \right]$$

to a more tractable expression.

Taking the logarithm and integrating by parts yields

$$\ell n\ B^{+}(\alpha) = \frac{1}{2\pi i} \int_{-\infty - id}^{\infty - id} F(\beta)\ \left(\frac{\alpha}{\beta - \alpha}\right) \frac{d\beta}{\beta}$$

$$= F(\beta)\ \ell n\left(\frac{\beta - \alpha}{\beta}\right) - \int_{-\infty - id}^{\infty - id} A(\beta)\ \ell n\left(\frac{\beta - \alpha}{\beta}\right) d\beta$$

where

$$A(\beta) = \frac{\beta h}{2\pi(k^2-\beta^2)^{1/2}} + \frac{\beta\nu}{2\pi i(k^2-\beta^2)} + \frac{1}{2\pi i\ G(\beta)} \frac{dG(\beta)}{d\beta}$$

If we let $\beta \to -\beta$, then

$$\ell n\ B^+(\alpha) = \int_{\infty+id}^{-\infty+id} A(-\beta)\ \ell n(\frac{\beta+\alpha}{\beta})\ d\beta$$

This integral will be evaluated on a closed contour in the upper half plane by the residue theorem. Therefore

$$\ell n\ B^+(\alpha) = -\int_p A(-\beta)\ \ell n(1 + \frac{\alpha}{\beta})\ d\beta - 2\pi i \sum \text{(residues at the poles of integrand inside contour)}$$

The path p and the contour are given in Figure D-5. The contribution from the semicircular path at $|\beta| \to \infty$ is zero.

We now assume that

(1) $G(\alpha)$ has at most one branch singularity in the upper half plane located at $\alpha = +k$ in the form $\gamma = (\alpha^2 - k^2)^{1/2}$.

(2) $G(\alpha)$ has only simple poles in the upper half of the α plane, located at $\alpha = +p_n$ with $|\text{Im}\ p_n| > \tau_+$, n=1,2,3,...

(3) $G(\alpha)$ has only simple zeros in the upper half of the α plane located at $\alpha = z_m$ with $|\text{Im}\ z_m| > \tau_+$, m=1,2,3,... .

Examination of $A(\beta)$ shows that its only possible poles within the contour are those of $\frac{1}{2\pi i\ G(\beta)} \frac{dG(\beta)}{d\beta}$. The poles of this function occur at the poles and the zeros of $G(\beta)$. The residue contribution

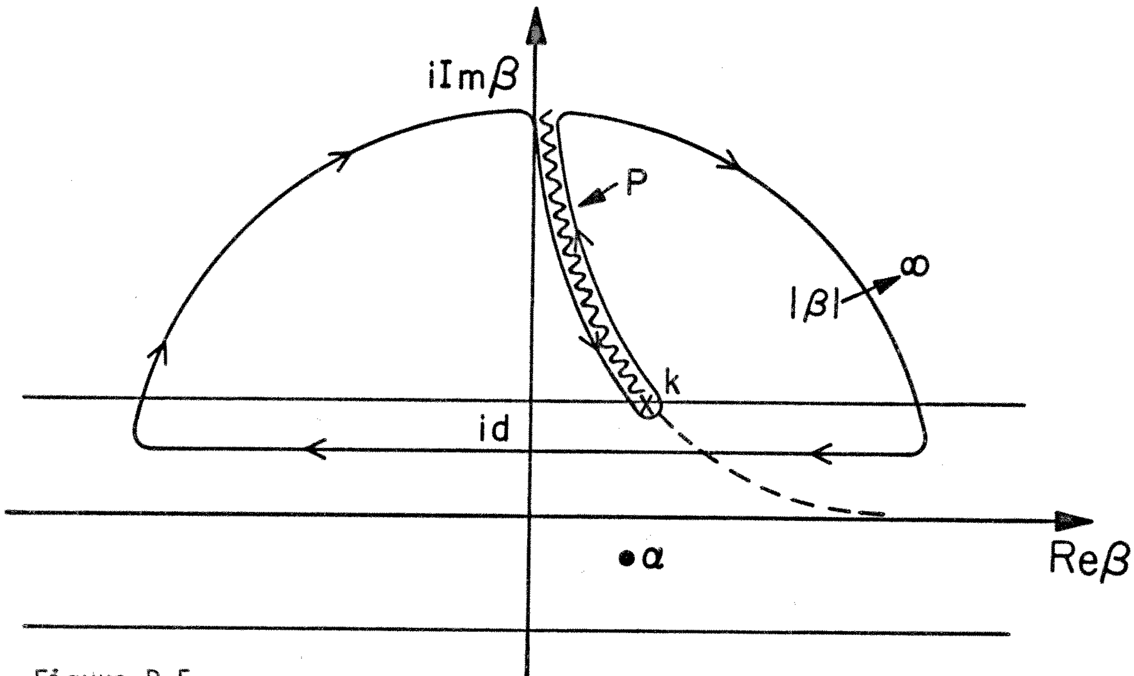


Figure D-5.

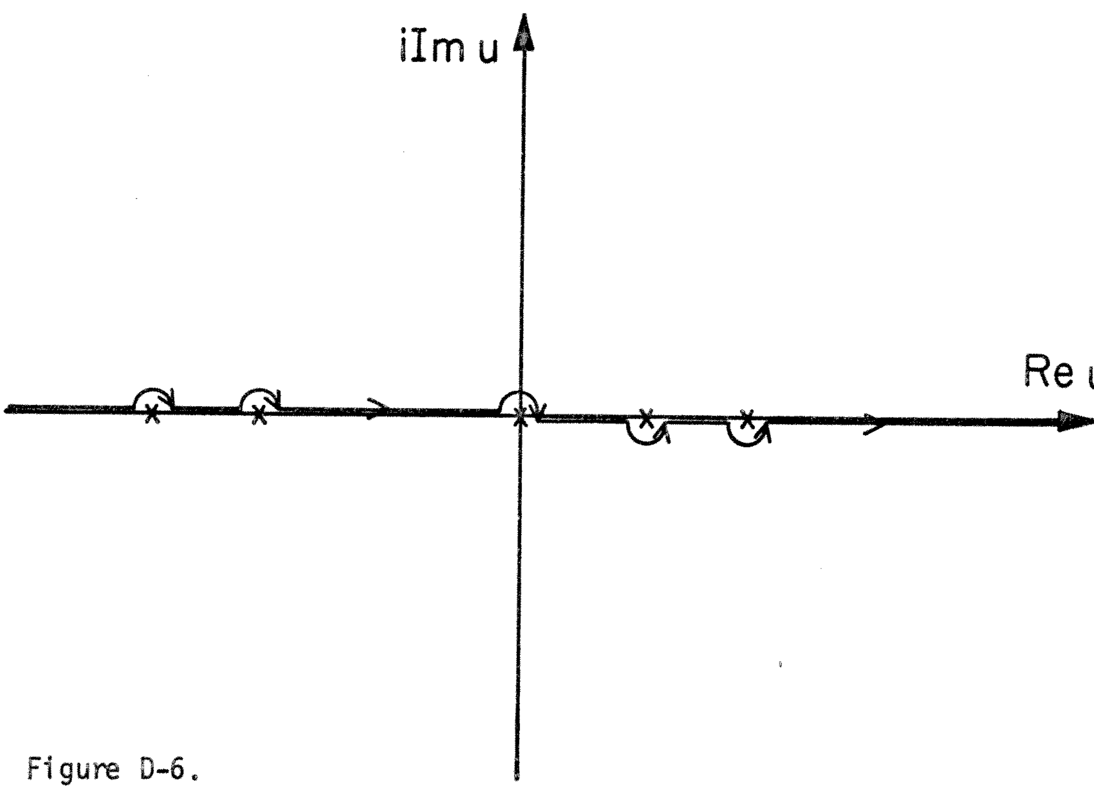


Figure D-6.

from the poles of $G(\beta)$ is $(\frac{-1}{2\pi i})$ and from the zeros of $G(\beta)$ it is $(\frac{1}{2\pi i})$. Since $G(\beta)$ is an even function, when β is replaced by $(-\beta)$ the contributions from the poles and zeros become the negative of the above. We may now write

$$\ell n\ B^{+}(\alpha) = -\int_{p} A(-\beta)\ \ell n(1+\frac{\alpha}{\beta})d\beta - \sum_{n=1}^{\infty}\ \ell n[1+\frac{\alpha}{p_n}] + \sum_{m=1}^{\infty}\ \ell n[1+\frac{\alpha}{z_m}]$$

Set

$$u = (k^2-\beta^2)^{1/2} = i(\beta^2-k^2)^{1/2}$$

where the branch on the upper Riemann sheet is used (see Appendix B)

$$\beta = i(u^2-k^2)^{1/2} = (k^2-u^2)^{1/2}$$

Using this substitution, we have

$$\ell n\ B^{+}(\alpha) = -\int_{p} A(-\beta)\ \ell n(1+\frac{\alpha}{\beta})d\beta = \int_{p'} C(u)\ \ell n[1-\frac{\alpha}{(k^2-u^2)^{1/2}}]du$$

where

$$C(u) = \frac{h}{2\pi} + \frac{\nu}{2\pi i u} - \frac{1}{2\pi i}\frac{d}{du}[\ell n\ G((k^2-u^2)^{1/2})] \qquad \text{(D.6)}$$

(see Figure D-6).

From Appendix B we know that on the branch cut $(\beta^2-k^2) = 0$, therefore (β^2-k^2) is pure imaginary and the variable u is pure real. So the introduction of the variable change maps the contour p into a contour along the real axis. The contour p' must be indented around any poles of the integrand which appear on the real axis. The indentations are consistent with pole movement from the real axis if the medium is assumed slightly lossy. Note also that we have accounted for a negative sign by taking p' in the opposite direction to the path which p maps into under the variable change.

Examining C(u), we note that the second term has a pole at the origin, and the third term

$$-\frac{1}{2\pi i}\frac{d}{du}[\ell n\ G((k^2-u^2)^{1/2})] = -\frac{1}{(2\pi i)\ G((k^2-u^2)^{1/2})}\frac{d}{du}G((k^2-u^2)^{1/2})$$

could possibly have a pole at the origin, and at other locations along the real axis, which follows from the fact that u is real. The contribution to $\ell n\ B^+(\alpha)$ from integrating over the semicircular path around the origin is

$$\frac{1}{2}[R_o - \nu]\ \ell n[1 + \frac{\alpha}{k}]$$

where R_o is the

$$\lim_{u \to 0} u\frac{d}{du}[\ell n\ G(k^2-u^2)^{1/2})] \tag{D.7}$$

The remaining poles will contribute $+\frac{1}{2}\sum_{n=1}^{N} R_n(\alpha)$, where

$$R_n(\alpha) = \lim_{u \to p_n} (u-p_n)\ \frac{d}{du}[\ell n\ G((k^2-u^2)^{1/2})]\ \{\ell n[1 + \frac{\alpha}{\sqrt{k^2-u^2}}]\}$$

where p_n is the nth ordered pole of C(u) on the real axis.

We may calculate the integral along the straight line portions of the path as

$$\underset{\delta\to 0^+}{\fint^{\infty}} K(u) \quad \ln[1 + \frac{\alpha}{\sqrt{k^2-u^2}}]\, du$$

where

$$K(u) = \frac{h}{\pi} - \frac{1}{2\pi i}\,[L(u) + L(ue^{i\pi})] \tag{D.8}$$

and

$$L(u) = \frac{d}{du}[\ln G((k^2-u^2)^{1/2})] \tag{D.9}$$

The integral is interpreted as a principal value type, and it is denoted by the bar on the integral sign. The principal value of the logarithmic function is understood to have been used throughout.

In summary, we have found that

$$\ln B^+(\alpha) = \underset{\delta\to 0^+}{\fint^{\infty}} K(u) \quad \ln[1 + \frac{\alpha}{\sqrt{k^2-u^2}}]\, du + \frac{1}{2}(R_o-\nu)\ \ln[1+\frac{\alpha}{k}]$$

$$+ \frac{1}{2}\sum^{n} R_n(\alpha) - \sum_{n=1}^{\infty} \ln[1 + \frac{\alpha}{p_n}] + \sum_{m=1}^{\infty} \ln[1 + \frac{\alpha}{z_m}]$$

and the value of $G^+(\alpha)$ can be written as

$$G^+(\alpha) = \sqrt{G(0)}\ (1 + \frac{\alpha}{k})^{R_o/2} \prod_{n=1}^{\infty} [1 + \frac{\alpha}{p_n}]^{-1} \prod_{m=1}^{\infty} [1 + \frac{\alpha}{z_m}]$$

$$\exp\left[\frac{ih\gamma}{\pi} \ln[\frac{\gamma + (\alpha-k)}{\gamma - (\alpha-k)}] - \frac{ikh}{2} + \underset{\delta\to 0^+}{\fint^{\infty}} K(u)\ \ln[1 + \frac{\alpha}{\sqrt{k^2- u^2}}]\, du\right.$$

$$\left. \times\ \frac{1}{2}\sum^{n} R_n(\alpha)\right] \tag{D.9.1}$$

If we perform a similar derivation in which we use $B^-(\alpha)$ instead of $B^+(\alpha)$, as the starting point, the result would be:

$$\ln B^-(\alpha) = \int_{\delta\to 0^+}^{\infty} K(u)\,\ln[1 + \frac{\alpha}{\sqrt{k^2-u^2}}]\,du + \frac{1}{2}(R_o-\nu)\quad \ln[1+\frac{\alpha}{k}]$$

$$-\frac{1}{2}\sum^{n} R_n(\alpha) + \sum_{n=1}^{\infty} \ln[1-\frac{\alpha}{p_n}] - \sum_{m=1}^{\infty} \ln[1-\frac{\alpha}{z_m}]$$

Similar to (D.9.1), it follows that

$$G^-(\alpha) = \sqrt{G(0)}\,(1+\frac{\alpha}{k})^{R_o/2} \prod_{n=1}^{\infty} [1-\frac{\alpha}{p_n}]^{-1} \prod_{m=1}^{\infty} (1-\frac{\alpha}{z_m})$$

$$\times\ \exp[\frac{ih\gamma}{\pi}\ln(\frac{\gamma-(\alpha+k)}{\gamma+(\alpha+k)}) - \frac{ikh}{2} + \int_{\delta\to 0^+}^{\infty} K(u)\,\ln[1+\frac{\alpha}{\sqrt{k^2-u^2}}]du$$

$$+\frac{1}{2}\sum^{n} R_n(\alpha) \qquad \text{(D.9.2)}$$

From (D.9.1) and (D.9.2), and knowing that $G^+(\alpha) = G^-(\alpha)$, we recognize that the factorization is:

$$G^+(\alpha) = G^-(-\alpha) = G(0)\,(1+\frac{\alpha}{k})^{R_o/2} \prod_{n=1}^{\infty} (1+\frac{\alpha}{p_n})^{-1}$$

$$\times \prod_{m=1}^{\infty} (1+\frac{\alpha}{z_m})\,\exp\left[\frac{ih\gamma}{\pi}\ln[\frac{\gamma+(\alpha-k)}{\gamma-(\alpha-k)}] - \frac{ikh}{2}\right.$$

$$\left. + \int_{\delta\to 0^+}^{\infty} K(u)\,\ln[1+\frac{|\alpha|}{\sqrt{k^2-a^2}}]\,du + \frac{1}{2}\sum^{n} R_n(|\alpha|)\right] \qquad \text{(D.10)}$$

To complete the theoretical portion of this appendix, we cite a theorem which provides for the expansion of a class of functions into infinite products [26,p.136].

Theorem: Let $f(\alpha)$ be a function with its only zeros at $a_1, a_2, a_3, \cdots$ where $\lim_{n \to \infty} a_n$ is infinite and let $f(\alpha)$ be analytic for all α. Then $f(\alpha)$ can be expressed

$$f(\alpha) = f(0)\, e^{\alpha \frac{f'(0)}{f(0)}} \prod_{n=1}^{\infty} \{1 - \frac{\alpha}{a_n}\} e^{\alpha/a_n} \quad ; \qquad a_n \neq 0 \qquad \text{(D.11)}$$

If $f(\alpha)$ is both even and entire with its only zeros occurring at $\pm a_1, \pm a_2, \pm a_3, \cdots$ and $\lim_{n \to \infty} \pm a_n = \pm \frac{n\pi}{a}$, then $f(\alpha)$ can be written

$$f(\alpha) = f(0)\, e^{\alpha \frac{f'(0)}{f(0)}} \prod_{n=1}^{\infty} \{1 - \frac{\alpha}{a_n}\} e^{\frac{\alpha a}{n\pi}} \prod_{n=1}^{\infty} (1 + \frac{\alpha}{a_n})\, e^{-\frac{\alpha a}{n\pi}} \qquad \text{(D.12)}$$

The exponential factors for the infinite products ensure convergence of the separate products.

Consider

$$(1 - \frac{z}{a_n})\, e^{\frac{za}{n\pi}} \sim (1 - \frac{za}{n\pi})\, e^{\frac{za}{n\pi}} \sim 1 - \frac{z^2 a^2}{n^2 \pi^2} \quad ; \qquad n \to \infty$$

Thus,

$$\prod_{n=1}^{\infty} (1 - \frac{z}{a_n})\, e^{\frac{za}{n\pi}} \sim 1 - \sum_{n=1}^{\infty} \frac{z^2 a^2}{n^2 \pi^2} \quad \text{as} \quad n \to \infty$$

Because $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges as $n \to \infty$, the infinite product also converges as $n \to \infty$.

We will now apply the foregoing factorization formulas to factor the expressions:

$$\{ I_o(\gamma a) \, [K_o(\gamma b) - K_o(\gamma a)] \}^{-1}$$

and

$$\{ K_o(\gamma b) \, [I_o(\gamma b) - I_o(\gamma a)] \}^{-1}$$

Taking the top expression first, we define

$$f^{(1)}(\alpha) = I_o(\gamma a) = J_o(a\sqrt{k^2 - \alpha^2})$$

$$G^{(1)}(\alpha) = K_o(\gamma b) - K_o(\gamma a) = i\,\frac{\pi}{2}[H_o^{(1)}(b\sqrt{k^2-\alpha^2}) - H_o^{(1)}(a\sqrt{k^2-\alpha^2})]$$

where use has been made of the well known relation between the modified and the ordinary Bessel functions [20,p.370].

It is readily seen from (D.12) that

$$f^{(1)}(\alpha) = J_o(ka) \prod_{n=1}^{\infty} \left(1 - \frac{\alpha}{i|\alpha_n|}\right) e^{-\frac{i\alpha a}{n\pi}} \prod_{n=1}^{\infty} \left(1 + \frac{\alpha}{i|\alpha_n|}\right) e^{+\frac{i\alpha a}{n\pi}}$$

where $a_n = \pm i|\alpha_n|$; $|\alpha_n| = \sqrt{(\frac{z_n}{a})^2 - k^2}$; z_n is the nth ordered zero of $J_o(z)$ and $z_n \sim (n - \frac{1}{4})\pi$ as $n \to \infty$. The value of ka must be restricted so that $|\alpha_n| \neq 0$, therefore take $ka << 1$.

Theorem C is used to factor $G^{(1)}(\alpha)$. From the asymptotic value of the modified Bessel functions, we have

$$G^{(1)}(\alpha) \sim \sqrt{\frac{\pi}{2\alpha b}}\, e^{-b|\alpha|} - \sqrt{\frac{\pi}{2\alpha a}}\, e^{-a|\alpha|}$$

$$= \alpha^{-1/2}\, e^{-a|\alpha|} \sqrt{\frac{\pi}{2}}\, [(\tfrac{1}{b})^{1/2}\, e^{-b/a} - (\tfrac{1}{a})^{1/2}] \quad \text{as} \quad \alpha \to \infty$$

Thus $\nu = -\frac{1}{2}$ and $h = a$. From (D.9),

$$L^{(1)}(u) = \frac{d}{du}[\ell n\, G((\sqrt{k^2-u^2})^{1/2})] = \frac{d}{du}[\ell n\; i\, \frac{\pi}{2}[H_0^{(1)}(ub) - H_0^{(1)}(ua)]]$$

$$= \frac{-b\, H_1^{(1)}(ub) + a\, H_1^{(1)}(ua)}{H_0^{(1)}(ub) - H_0^{(1)}(ua)}$$

From (D.7)

$$R_o = \lim_{u \to 0} u \frac{d}{du} L^{(1)}(u) = 0 \quad ,$$

$L(u)$ has no other poles on the real axis, so $R_n = 0$ for all n .

By (D.8) and the analytic continuation $H_\nu^{(2)}(ze^{-\pi i}) = -e^{i\nu\pi i}\, H_\nu^{(1)}(z)$

$$K^{(1)}(u) = \frac{a}{\pi} - \frac{1}{2\pi i}\left\{\frac{-bH_1^{(1)}(ub) + aH_1^{(1)}(ua)}{H_0^{(1)}(ub) - H_0^{(1)}(ua)}\right.$$

$$\left. - \frac{-bH_1^{(2)}(ub) + aH_1^{(2)}(ua)}{H_0^{(2)}(ub) - H_0^{(2)}(ua)}\right\} = \frac{a}{\pi} - \frac{1}{2\pi}$$

$$\times \frac{\frac{8}{\pi u} + 2b[Y_1(ub)J_0(ua) - J_1(ub)Y_0(ua)] + 2a[Y_0(ua)J_0(ub) - J_1(ua)Y_0(ub)]}{[J_0(ub) - J_0(ua)]^2 + [Y_0(ub) - Y_0(ua)]^2}$$

Factorization formula (D.10) gives

$$G^{(1)+}(\alpha) = \{\frac{\pi}{2} i[H_0^{(1)}(kb) - H_0^{(1)}(ka)]\}^{1/2} \exp[\frac{ia\gamma}{\pi} \ln[\frac{\gamma + (\alpha - k)}{\gamma - (\alpha - k)}]$$

$$- \frac{ika}{2} + \int_{\delta \to 0+}^{\infty} K^{(1)}(u) \ln[1 + \frac{|\alpha|}{\sqrt{k^2 - u^2}}] du] \qquad (D.13)$$

$$X(\alpha) = X^+(\alpha) X^-(\alpha) = \{I_0(\gamma a)[K_0(\gamma b) - K_0(\gamma a)]\}^{-1}$$

Thus

$$X^+(\alpha) = \frac{e^{\chi(\alpha)}}{G^{(1)+}(\alpha) \sqrt{J_0(ka)} \prod_{n=1}^{\infty} (1 + \frac{\alpha}{i|\alpha_n|})e^{i\alpha a/n\pi}} \qquad (D.14)$$

$e^{\chi(\alpha)}$ is introduced in the "plus" function and $e^{-\chi(\alpha)}$ in the "minus" function to ensure algebraic behavior of each of these functions as $|\alpha| \to \infty$. Note also that $\chi(-\alpha) = -\chi(\alpha)$ since $\chi^+(-\alpha) = \chi^-(\alpha)$. We determine the value of $\chi(\alpha)$ from the asymptotic behavior of the denominator of equation (D.14).

By (D.5) we know the asymptotic behavior of $G^{(1)+}(\alpha)$ is

$$G^{(1)+}(\alpha) \sim |\alpha|^{-1/4} e^{i \frac{\alpha a}{\pi} \ln\left|\frac{2\alpha}{k}\right|} \quad \text{as} \quad |\alpha| \to \infty$$

in the upper half plane. The asymptotic behavior of the infinite product in the denominator of (D.14) can be found from the well known property of the gamma function [4,p.41],

$$\prod_{n=1}^{\infty} [1 + \frac{\alpha}{an+b}] e^{-\alpha/an} = e^{-C\alpha/a} \frac{\Gamma(\frac{b}{a}+1)}{\Gamma(\frac{\alpha}{a}+\frac{b}{a}+1)}$$

Introduce Sterling's formula,

$$\Gamma(\alpha) \sim e^{-\alpha} \alpha^{\alpha - \frac{1}{2}} \sqrt{2\pi} \{1 + (12\alpha)^{-1} + \cdots\} \text{ as } \alpha \to \infty , \quad |\arg \alpha| < \pi$$

Combining these two relations gives the useful result

$$\lim_{|\alpha|\to\infty} \prod_{n=1}^{\infty} [1 + \frac{\alpha}{an+b}] e^{-\frac{\alpha}{an}} \sim \frac{\Gamma(\frac{b}{a}+1)}{\sqrt{2\pi}} (\frac{|\alpha|}{a})^{-[\frac{1}{2}+\frac{b}{a}]} \times \exp \{\frac{\alpha}{a}[1 - C - \ell n \frac{|\alpha|}{a}]\} \quad (D.15)$$

In (D.15) C is Euler's constant (0.5772 . . .). Applying this result we find the asymptotic behavior of the infinite product in (D.14) to be

$$\lim_{\substack{\alpha\to\infty \\ |\arg \alpha|<\pi}} \prod_{n=1}^{\infty} (1 + \frac{\alpha}{i|\alpha_n|}) e^{i\frac{\alpha a}{n\pi}} \sim |\alpha|^{-\frac{1}{4}} \exp\{\frac{\alpha a}{i\pi}[1 - C - \ell n |\frac{\alpha a}{\pi}| + \frac{i\pi}{2}]\}$$

where $|\alpha_n| \to (n - \frac{1}{4}) \frac{\pi}{a}$ as $n \to \infty$.

By collecting the above results, we determine that

$$x(\alpha) = \frac{\alpha a}{i\pi} \{1 - C - \ell n \left|\frac{\alpha a}{\pi}\right| + \frac{i\pi}{2} + \ell n \left|\frac{2\alpha}{k}\right|\} \quad (D.16)$$

and that $x^+(\alpha) = x^-(-\alpha)$ behaves as $|\alpha|^{1/2}$ as $|\alpha| \to \infty$ in the region of the α plane with $\tau > -k_2$.

Turning to the other expression to be factored,

$$K_0(\gamma b)[I_0(\gamma b) - I_0(\gamma a)]$$

we define

$$f^{(2)}(\alpha) = I_0(\gamma b) - I_0(\gamma a) = J_0(b\sqrt{k^2-\alpha^2}) - J_0(a\sqrt{k^2-\alpha^2})$$

$$G^{(2)}(\alpha) = K_0(\gamma b) = i\,\frac{\pi}{2}\,[H_0^{(1)}(b\sqrt{k^2-\alpha^2})]$$

Using (D.12), we have

$$f^{(2)}(\alpha) = [J_0(kb) - J_0(ka)] \prod_{m=1}^{\infty} \left(1 - \frac{\alpha}{i|\alpha_m|}\right) e^{\frac{i\alpha}{m\pi}\left(\frac{a+b}{2}\right)}$$
$$\times \prod_{m=1}^{\infty} \left(1 + \frac{\alpha}{i|\alpha_m|}\right) e^{\frac{i\alpha}{m\pi}\left(\frac{a+b}{2}\right)}$$

where $a_m = \pm i|\alpha_m|$,

$|\alpha_m| = \sqrt{\left(\frac{z_m}{b}\right)^2 - k^2}$ and z_m is the mth ordered zero of $J_0(z) - J_0(\frac{a}{b}z)$

The value of kb is restricted to $kb \ll 1$, so that $|\alpha_m| \neq 0$.

The asymptotic behavior of the zeros of $[J_0(z) - J_0(\frac{a}{b}z)]$ are determinable by considering the asymptotic behavior of the Bessel function for large arguments and considering that the difference between the values of a and b does not affect the magnitude. Differences between the values of a and b , however, cannot be neglected in phase factors. Through these considerations and trigonometry, we find that the most rapidly varying roots are those given by

$$z_m \to (m - \tfrac{1}{4}) \frac{2\pi b}{a+b} \quad \text{as} \quad m \to \infty \quad ;$$

$$m = 1,2,3,\cdots$$

Since

$$G^{(2)}(\alpha) \sim \sqrt{\frac{\pi}{2\alpha b}}\; e^{-b|\alpha|} \qquad \text{as} \quad \alpha \to \infty$$

then $\nu = -\frac{1}{2}$ and $h = b$

$$L^{(2)}(u) = \frac{d}{du}\,[\ell n(i\,\tfrac{\pi}{2})\,H_0^{(1)}(ub)] = \frac{-bH_1^{(1)}(ub)}{H_0^{(1)}(ub)}$$

$$R_o = \lim_{u\to 0}\; u\,\frac{d}{du}\,L^{(2)}(u) = 0$$

$$K^{(2)}(u) = \frac{b}{\pi} + \frac{b}{2\pi i}\Big[\frac{H_1^{(1)}(ub)}{H_0^{(1)}(ub)} - \frac{H_1^{(2)}(ub)}{H_0^{(2)}(ub)}\Big]$$

$$= \frac{b}{\pi}\,\Big[1 - \frac{2}{\pi ub}\;\frac{1}{J_0^2(ub) + Y_0^2(ub)}\Big]$$

$$G^{(2)+}(\alpha) = [\tfrac{\pi}{2}\,i\,H_0^{(1)}(kb)]^{1/2}\,\exp\Big[\frac{ib\gamma}{\pi}\,\ell n\Big[\frac{\gamma + (\alpha-k)}{\gamma - (\alpha-k)}\Big] - \frac{ikb}{2}$$

$$+ \int_{\delta\to 0^+}^{\infty} K^{(2)}(u)\;\ell n\Big[1 + \frac{|\alpha|}{\sqrt{k^2 - u^2}}\Big]\;du\Big] \qquad \text{(D.17)}$$

Collecting the results, we get

$$Y(\alpha) = Y^{+}(\alpha)\, Y^{-}(\alpha) = \{K_o(\gamma b)[I_o(\gamma b) - I_o(\gamma a)]\}^{-1}$$

where

$$Y^{+}(\alpha) = \frac{e^{\phi(\alpha)}}{G^{(2)+}(\alpha)\sqrt{J_o(kb) - J_o(ka)}\ \prod_{m=1}^{\infty} (1 + \frac{\alpha}{i|\alpha_m|})\, e^{\frac{i\alpha(a+b)}{m\pi 2}}} \tag{D.18}$$

By (D.5), we know that

$$G^{(2)+}(\alpha) \sim |\alpha|^{-1/4}\, e^{-\frac{i\alpha b}{\pi}\ \ell n|\frac{2\alpha}{k}|} \quad \text{as } |\alpha| \to \infty$$

in the upper half plane and from the asymptotic behavior of infinite products (D.15), we know

$$\lim_{\substack{|\alpha|\to\infty \\ |\arg \alpha|<\pi}} \prod_{m=1}^{\infty} (1 + \frac{\alpha}{i|\alpha_m|}) e^{\frac{i}{m\pi}(\frac{a+b}{2})} \sim |\alpha|^{-1/4} \exp\{\frac{\alpha(a+b)}{i2\pi}[1 - C - \ell n\left|\frac{\alpha(a+b)}{2\pi}\right| + i\,\frac{\pi}{2}]\}$$

$$|\alpha_m| \to (m - \frac{1}{4})\, \frac{2\pi}{a+b} \qquad \text{as } m \to \infty$$

Therefore

$$\phi(\alpha) = \frac{\alpha(a+b)}{i2\pi}\, [1 - C - \ell n\left|\frac{\alpha(a+b)}{2\pi}\right| + \frac{i\pi}{2} + \frac{2b}{a+b}\, \ell n|\frac{2\alpha}{k}|] \tag{D.19}$$

and $Y^{+}(\alpha) = Y^{-}(-\alpha) \sim |\alpha|^{1/2}$ as $|\alpha| \to \infty$ in $\tau > -k_2$.

Appendix E - Radiation from the Infinite Cylindrical Antenna

In this appendix we shall determine the radiation from a hollow infinite cylindrical antenna. An integral equation for the current on such an antenna will be formulated and solved. The fields will then be determined from the current. The integral equation considered in this appendix was first formulated and solved by Hallén [35,36]. He was interested in the reflected current on a cylindrical antenna, when the incident current wave comes from a source that appears to be an infinite distance from the end. Although the integral equation is the same as that found by Hallén, the application used in this appendix is novel.

Consider Figure E-1, which is an illustration of the actual geometry and its equivalent representation, which is amenable to analysis. Since the antenna is made of perfectly conducting material, its surface is an equipotential and

$$E_z(\vec{r},t) = -v(t)\ \delta(z) \qquad \text{when } \rho = a,\ -\infty < z < \infty$$

In the frequency domain this becomes

$$E_z(\vec{r},\omega) = -v(\omega)\ \delta(z)$$

In view of the rotational symmetry, it can easily be shown that

$$E_z(\vec{r},\omega) = +i\omega[A_z(\vec{r}) + \frac{1}{k^2}\frac{\partial^2 A_z(\vec{r})}{\partial z^2}] = -v(\omega)\ \delta(z)$$

$\vec{r}$ is the vector from the origin to a point on the surface of the

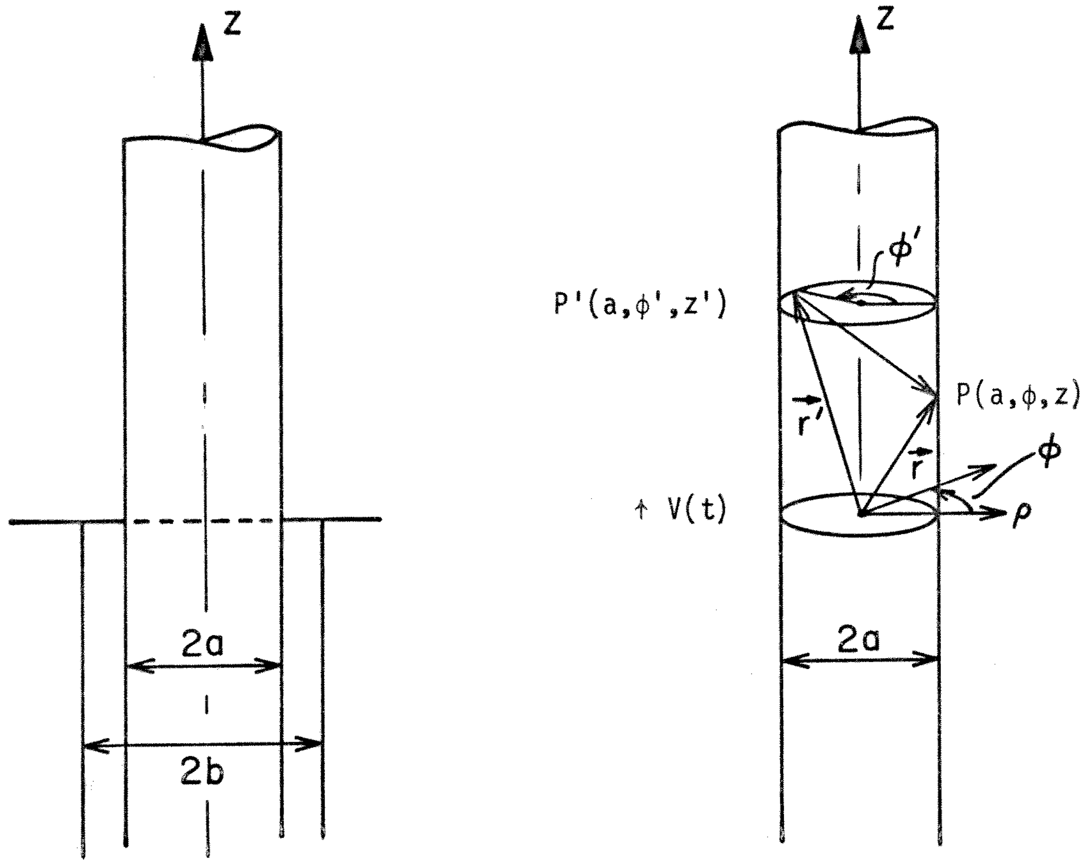


Figure E-1. Infinite cylindrical antenna and equivalent representation using gap generator as source antenna and $A_z(\vec{r})$ is the well known vector magnetic potential, which satisfies the relation $\vec{B} = \nabla \times \vec{A}$.

The symmetry of the structure also requires that $A_z(\vec{r}) = A_z(z)$, thus the axial component of the vector magnetic potential satisfies the differential equation

$$\frac{\partial^2 A_z(z)}{\partial z^2} + k^2 A_z(z) = + \frac{ik^2}{\omega} v(\omega) \quad \delta(z) \tag{E.1}$$

We also have from symmetry that $A_z(z) = A_z(-z)$.

The solution to (E.1) obtained in a straightforward fashion is

$$A_z(z) = C_1 \cos kz + C_2 \sin kz + \frac{kv(\omega)e^{ik|z|}}{2\omega}$$

From the symmetry condition that $A(z) = A(-z)$ and the requirement for only outgoing waves on the infinite structure, $C_1 = C_2 = 0$ and

$$A_z(z) = \frac{kv(\omega)e^{ik|z|}}{2\omega} \tag{E.2}$$

The vector potential will now be determined by an alternative fashion, whereby use is made of the free space Green's function and the surface current. The general formula is:

$$\vec{A}(\vec{r}) = \frac{\mu_o}{4\pi} \int_{S'} \vec{K}(\vec{r}\,') \frac{e^{ik|\vec{r}-\vec{r}\,'|}}{|\vec{r}-\vec{r}\,'|} \, dS' \quad .$$

with

$$|\vec{r} - \vec{r}\,'| = \sqrt{(z-z')^2 + \rho^2 + \rho'^2 - 2\rho\rho' \cos(\phi-\phi')}$$

In the case of the infinite cylindrical antenna with the observation point on the cylindrical surface, this general formula becomes

$$A_z(z) = \frac{\mu_0}{4\pi} \int_{-\infty}^{\infty} \frac{I(z')}{2\pi} \int_0^{2\pi} \frac{e^{ikr}}{r} \, d\phi' dz' \qquad (E.3)$$

where

$$r = \sqrt{(z-z')^2 + 4a^2 \sin^2(\tfrac{\phi'}{2})}$$

and ϕ is taken as zero. Since the vector potential is independent of ϕ , the particular choice of ϕ is immaterial. Since (E.2) and (E.3) are both equivalent expressions for the vector potential, they may be set equal to each other. The results are

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} dz' \, I(z') \int_0^{2\pi} \frac{e^{ikr}}{r} \, d\phi' = \frac{2\pi k \, v(\omega) \, e^{ik|z|}}{\mu_0 \omega}$$

$$= 2\pi \sqrt{\frac{\varepsilon_0}{\mu_0}} \quad v(\omega) \, e^{ik|z|} \qquad (E.4)$$

where $-\infty < z < \infty$.

The equation (E.4) is valid for all values of z , since it is based on the mathematical model of the actual antenna structure. However, in the actual antenna we observe that the current on the feedline after $t = 0$, the time when the incident excitation reaches the $z = 0$ plane, is a TEM mode wave. The current wave can be expressed as:

$$I(z) = I(0) \, e^{-ikz} \quad , \qquad z \leq 0$$

$I(0)$ is the magnitude of the current at the plane $z = 0$. In view of the above, it is justified to represent the antenna feedline system with the following set of equations:

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} dz' \; I(z') \int_{0}^{2\pi} \frac{e^{ikr}}{r} \, d\phi' = \begin{cases} 2\pi \sqrt{\frac{\varepsilon_0}{\mu_0}} \, v(\omega) e^{ikz} & ; \quad z > 0 \\ g(z) & ; \quad z < 0 \end{cases}$$

$$I(z) = I(0) \, e^{-ikz} \qquad ; \quad z < 0 \qquad (E.5)$$

We restrict the analysis to the consideration of TEM mode feedline current only. The function $g(z)$ for $z < 0$ is unknown. The solution to the integral equation (E.5) is the current on the infinite cylindrical antenna when there is an outward (from the origin) travelling current wave on the feedline.

The current is now expressed in terms of its spatial Fourier inverse

$$I(z') = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i\alpha z'} \, I(\alpha) \, d\alpha$$

$$I(\alpha) = I_+(\alpha) + I_-(\alpha)$$

with

$$I^+(\alpha) = \frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} I(z') \, e^{i\alpha z'} \, dz'$$

$$I^-(\alpha) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{0} I(z') \, e^{i\alpha z'} \, dz'$$

Note that $|I(z')| \leq A\, e^{-k_2 z}$ as $z \to \infty$ and $|I(z')| \leq B\, e^{k_2 z}$ as $z \to -\infty$ where k_2 is the imaginary part of the propagation constant. $I(\alpha)$ is obviously analytic in the strip $-k_2 < \tau < k_2$ of the $\alpha = \sigma + i\tau$ plane. $I^+(\alpha)$ is analytic in the half plane $\tau > -k_2$ and $I^-(\alpha)$ is analytic in the half plane $\tau < k_2$.

From (E.5) it follows that

$$I^-(\alpha) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{0} I(0)\, e^{-ikz}\, e^{+i\alpha z}\, dz = -\, i\, \frac{I(0)}{\sqrt{2\pi}\, (\alpha - k)}$$

for $\tau < k_2$. Additionally,

$$G^+(\alpha) = \frac{1}{\sqrt{2\pi}} \int_0^{\infty} \frac{2\pi}{\sqrt{\frac{\mu_0}{\varepsilon_0}}}\, v(\omega)\, e^{ikz}\, e^{i\alpha z}\, dz$$

$$= i \sqrt{\frac{2\pi\varepsilon_0}{\mu_0}}\, \frac{v(\omega)}{(\alpha + k)} \qquad \text{for } \tau > -k_2$$

Upon putting these results back into (E.5), we obtain

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} dz' \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i\alpha z'}\, I(\alpha)\, d\alpha \int_0^{2\pi} \frac{e^{ikr}}{r}\, d\phi' = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i\alpha z}\, G(\alpha)\, d\alpha$$

or equivalently

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} dz' \int_{-\infty}^{\infty} e^{-i\alpha z'} [I^+(\alpha) - \frac{i I(0)}{\sqrt{2\pi}\, (\alpha - k)}] d\alpha \int_0^{2\pi} \frac{e^{ikr}}{r}\, d\phi'$$

$$= \int_{-\infty}^{\infty} e^{-i\alpha z} [\sqrt{\frac{2\pi\varepsilon_0}{\mu_0}}\, \frac{i v(\omega)}{(\alpha + k)} + G^-(\alpha)]\, d\alpha \qquad \text{(E.6)}$$

We digress at this point to find an equivalent form for $\int_0^{2\pi} \frac{e^{ikr}}{r}\, d\phi'$. It is well known that the free space scalar Green function in spherical coordinates satisfies $(\nabla^2 + k^2)\, G(\vec{r},\vec{r}\,') = -\delta(\vec{r}-\vec{r}\,')$, and

$$G(\vec{r},\vec{r}\,') = \frac{1}{4\pi} \frac{e^{ik|\vec{r}-\vec{r}\,'|}}{|\vec{r}-\vec{r}\,'|}$$

In cylindrical coordinates the Green function satisfies

$$(\nabla^2 + k^2)\, G(\rho,\rho',\phi-\phi',z-z') = -\frac{\delta(\rho-\rho')}{\rho}\, \delta(\phi-\phi')\, \delta(z-z')$$

and

$$G(\rho,\rho',\phi-\phi',z-z') = \frac{1}{4\pi^2} \sum_{m=-\infty}^{\infty} e^{im(\phi-\phi')}$$

$$\times \int_{\infty}^{\infty} e^{-i\alpha(z-z')}\, I_m(\gamma\rho_<)\, K_m(\gamma\rho_>)\, d\alpha$$

In the above equations $\gamma = (\alpha^2 - k^2)^{1/2}$ and $\rho_{\substack{<\\>}}$ is the $\binom{\text{lesser}}{\text{greater}}$ of ρ and ρ' . These two Green's functions are equivalent three-dimensional representations. Therefore,

$$\frac{e^{ik|\vec{r}-\vec{r}\,'|}}{|\vec{r}-\vec{r}\,'|} = \frac{1}{\pi} \sum_{m=-\infty}^{\infty} e^{im(\phi-\phi')} \int_{-\infty}^{\infty} e^{-i\alpha(z-z')}\, I_m(\gamma\rho_<) K_m(\gamma\rho_>)\, d\alpha$$

and

$$\int_0^{2\pi} \frac{e^{ik|\vec{r}-\vec{r}\,'|}}{|\vec{r}-\vec{r}\,'|}\, d\phi' = 2 \int_{-\infty}^{\infty} e^{-i\alpha(z-z')}\, I_o(\gamma\rho_<)\, K_o(\gamma\rho_>)\, d\alpha$$

We now evaluate the above equation at $\rho = \rho' = a$ and substitute it into (E.6).

$$\frac{1}{\pi} \int_{-\infty}^{\infty} dz' \int_{-\infty}^{\infty} e^{-i\alpha z'}[I^+(\alpha) - \frac{iI(0)}{\sqrt{2\pi}(\alpha-k)}] \, d\alpha \int_{-\infty}^{\infty} e^{-i\beta(z-z')} I_0(\gamma a) K_0(\gamma a) d\beta$$

$$= \int_{-\infty}^{\infty} e^{-i\alpha z} [\sqrt{\frac{2\pi\varepsilon_0}{\mu_0}} \frac{iv(\omega)}{(\alpha+k)} + G^-(\alpha)] \, d\alpha \qquad (E.7)$$

Interchanging the order of the integrations on the left hand side of the above equation, and recognizing that

$$\int_{-\infty}^{\infty} e^{-i(\alpha-\beta)z'} dz' = 2\pi\delta(\alpha-\beta)$$

equation (E.7) transforms into:

$$2 \int_{-\infty}^{\infty} d\alpha \int_{-\infty}^{\infty} e^{-i\beta z}[I^+(\alpha) - \frac{iI(0)}{\sqrt{2\pi}(\alpha-k)}] \, I_0(\gamma a) K_0(\gamma a) \, \delta(\alpha-\beta) \, d\beta$$

$$= \int_{-\infty}^{\infty} e^{-i\alpha z} [\sqrt{\frac{2\pi\varepsilon_0}{\mu_0}} \frac{iv(\omega)}{(\alpha+k)} + G^-(\alpha)] \, d\alpha$$

Integration over the β variable produces the result:

$$2 \int_{-\infty}^{\infty} d\alpha \, e^{-i\alpha z} \{[I^+(\alpha) - \frac{iI(0)}{\sqrt{2\pi}(\alpha-k)}] \, [I_0(\gamma a) \, K_0(\gamma a)]$$

$$- [i\sqrt{\frac{\pi\varepsilon_0}{2\mu_0}} \, \frac{v(\omega)}{(\alpha+k)} + \frac{G^-(\alpha)}{2}] \} = 0 \qquad (E.8)$$

Let $I_0(\gamma a)\ K_0(\gamma a)$ be factored into the product $L^+(\alpha)\ L^-(\alpha)$, where $L^+(\alpha)$ is analytic in the half plane $\tau > -k$, and $L^-(\alpha)$ is analytic in the half plane $\tau < k_2$.

After substituting the factored form into (E.8) we derive the following result:

$$2\int_{-\infty}^{\infty} d\alpha\ e^{-i\alpha z}\ L^-(\alpha)\ \Big\{\Big[I^+(\alpha) - \frac{iI(0)}{\sqrt{2\pi}(\alpha-k)}\Big]\ L^+(\alpha) - \frac{1}{L^-(\alpha)}\Big[i\sqrt{\frac{\pi\varepsilon_0}{2\mu_0}}\ \frac{v(\omega)}{(\alpha+k)} + G^-(\alpha)\Big]\Big\} = 0$$

The integrand is regular in the strip $-k_2 < \tau < k_2$ of the plane. Since it must vanish, it gives the result

$$\Big[I^+(\alpha) - \frac{iI(0)}{\sqrt{2\pi}(\alpha-k)}\Big]\ L^+(\alpha) = \frac{G^-(\alpha)}{2L^-(\alpha)} + i\sqrt{\frac{\pi\varepsilon_0}{2\mu_0}}\ \frac{v(\omega)}{(\alpha+k)L^-(\alpha)}$$

Multiplication of both sides by $\alpha^2 - k^2$ yields the integral or entire function

$$L^+(\alpha)\ I^+(\alpha)(\alpha^2-k^2) - \frac{iI(0)(\alpha+k)}{\sqrt{2\pi}}\ L^+(\alpha) = \frac{G^-(\alpha)}{2L^-(\alpha)}\ (\alpha^2-k^2) + i\sqrt{\frac{\pi\varepsilon_0}{2\mu_0}}\ \frac{v(\omega)(\alpha-k)}{L^-(\alpha)} = P(\alpha) \qquad \text{(E.9)}$$

The left hand side is analytic in the upper half plane ($\tau > -k_2$), the right hand side is analytic in the lower half plane ($\tau < k_2$) and each side is the analytic continuation of the other. We divert our attention from (E.9) for the moment to consider $L^+(\alpha)$, $L^-(\alpha)$ and their asymptotic behavior as $\alpha \to \infty$ in their respective half planes of regularity.

Earlier, we defined the functions

$$L^+(\alpha)\, L^-(\alpha) = I_o(\gamma a)\, K_o(\gamma a)$$

By the methods of Appendix D, this factorization can be carried out with the results

$$L^+(\alpha) = L^-(-\alpha) = \frac{G^{+(3)}(\alpha)\, \sqrt{J_o(ka)}\, \prod_{m=1}^{\infty} \left(1 + \frac{\alpha}{i|\beta_m|}\right) e^{\frac{i\alpha a}{m\pi}}}{e^{\psi(\alpha)}}$$

$$G^{+(3)}(\alpha) = \left[\frac{\pi}{2}\, iH_o^{(1)}(ka)\right]^{1/2} \exp\left[\frac{ia\gamma}{\pi}\, \ln\left[\frac{\gamma + (\alpha-k)}{\gamma - (\alpha-k)}\right] - \frac{ika}{2} + \int_{\delta\to 0^+}^{\infty} K^{(2)}(u)\, \ln\left[1 + \frac{|\alpha|}{\sqrt{k^2 - u^2}}\right] du\right]$$

$$K^{(2)}(u) = \frac{a}{\pi}\left[1 - \frac{2}{\pi u a}\, \frac{1}{J_o^2(ua) + Y_o^2(ua)}\right]$$

$$\psi(\alpha) = \frac{\alpha a}{i\pi}\left[1 - C - \ln\left|\frac{\alpha a}{\pi}\right| + \ln\left|\frac{2\alpha}{k}\right| + i\,\frac{\pi}{2}\right]$$

$$|\beta_m| = \sqrt{\left(\frac{z_m}{a}\right)^2 - k^2}$$ where z_m is the mth ordered zero of $J_o(z)$

$$L^+(\alpha) \sim |\alpha|^{-1/2} \quad \text{as} \quad |\alpha| \to \infty \quad \text{in half plane} \quad \tau > -k_2 .$$

$$L^-(\alpha) \sim |\alpha|^{-1/2} \quad \text{as} \quad |\alpha| \to \infty \quad \text{in half plane} \quad \tau < k_2 .$$

Before returning to (E.9), we note from the edge conditions [21] that the current goes to zero at the terminus of the hollow conductor. As $I^+(\alpha)$ and $G^-(\alpha)$ represent transforms of current we can deduce the asymptotic behavior of these functions for $\alpha \to \infty$ by examining the behavior of the respective currents as we approach the ends of the antenna. By considering the antenna length to be finite and changing the variable to place the origin at the terminal ends, it is easily shown that $I^+(\alpha)$ and $G^-(\alpha)$ vanish as $|\alpha| \to \infty$ in their respective half planes of analyticity.

Application of the asymptotic behavior of the various functions to (E.9) shows that

$$\left| L^+(\alpha)\, I^+(\alpha)\, (\alpha^2-k^2) - \frac{iI(0)(\alpha+k)}{\sqrt{2\pi}}\, L^+(\alpha) \right| < |\alpha| \quad \text{as} \quad \alpha \to \infty \ , \quad \tau < -k_2$$

and

$$\left| \frac{G^-(\alpha)}{L^-(\alpha)}\, \frac{(\alpha^2-k^2)}{2} + i\sqrt{\frac{\pi\varepsilon_0}{2\mu_0}}\, \frac{v(\omega)(\alpha-k)}{L^-(\alpha)} \right| < |\alpha|^2 \quad \text{as} \quad \alpha \to \infty \ , \quad \tau < k_2$$

From the extended form of Liouville's theorem, $P(\alpha)$ is a polynomial of degree less than or equal to one [4, p.38]. We may therefore write $P(\alpha) = C_1 + C_2(\alpha-k)$, where C_1 and C_2 are constants to be determined. By setting $\alpha = k$ in the left side of equation (E.9), we get

$$C_1 = -\frac{i2I(0)k\ L^+(k)}{\sqrt{2\pi}}$$

Additionally,

$$\frac{C_1}{\alpha-k} + C_2 = L^+(\alpha)\ I^+(\alpha)(\alpha+k) + \frac{iI(0)\ L^+(\alpha)(\alpha+k)}{\sqrt{2\pi}\ (\alpha-k)}$$

By letting $\alpha \to \infty$ in the above expression, we find $C_2 = 0$. Thus $P(\alpha)$ is a constant.

From (E.9) and the value of $P(\alpha)$ just determined, we have

$$I^+(\alpha) = -\frac{iI(0)}{\sqrt{2\pi}\ (\alpha-k)} - \frac{i2I(0)k\ L^+(k)}{\sqrt{2\pi}(\alpha^2-k^2)L^+(\alpha)}$$

Recall that $I(\alpha) = I^+(\alpha) + I^-(\alpha)$, so the transform of the current is

$$I(\alpha) = -\frac{i2I(0)k\ L^+(k)}{\sqrt{2\pi}(\alpha^2-k^2)L^+(\alpha)}$$

$$= -\frac{i2I(0)k\ L^+(k)\ L^-(\alpha)}{\sqrt{2\pi}\ (\alpha^2-k^2)I_0(\gamma a)K_0(\gamma a)} \qquad \text{(E.10)}$$

Inverse Fourier transformation of the above yields

$$I(z) = -\frac{iI(0)k\ L^+(k)}{\pi}\int_{-\infty}^{\infty}\frac{e^{-i\alpha z}\ L^-(\alpha)\ d\alpha}{(\alpha^2-k^2)I_0(\gamma a)K_0(\gamma a)} \qquad \text{(E.11)}$$

It is also clear that if we close the contour in the upper half plane, the integration produces $I(z) = I(0)e^{-ikz}$; $z < 0$, as required.

The vector magnetic potential can now be found from the general formula,

$$A_z = \frac{\mu_o}{4\pi} \int_{-\infty}^{\infty} \frac{I(z')}{2\pi} \int_0^{2\pi} \frac{e^{ik|\vec{r}-\vec{r}'|}}{|\vec{r}-\vec{r}'|} d\phi' \, dz'$$

$$A_z = -\frac{i\mu_o I(0)k \, L^+(k)}{4\pi^2} \frac{1}{2\pi} \int_{-0}^{2\pi} d\phi' \int_{-\infty}^{\infty} dz' \frac{e^{ik|\vec{r}-\vec{r}'|}}{|\vec{r}-\vec{r}'|}$$

$$\times \int_{-\infty}^{\infty} \frac{e^{-i\alpha z'} \, L^-(\alpha)}{(\alpha^2-k^2) \, J_o(\gamma a) K_o(\gamma a)} \, d\alpha$$

$$= -\frac{i\mu_o I(0)k \, L^+(k)}{4\pi^3} \int_{-\infty}^{\infty} dz' \int_{-\infty}^{\infty} d\beta [e^{-i\beta(z-z')} I_o(a\sqrt{\beta^2-k^2}\,)$$

$$\times \; K_o(\rho\sqrt{\beta^2-k^2}\,)] \int_{-\infty}^{\infty} \frac{e^{-i\alpha z'} \, L^-(\alpha) \, d\alpha}{(\alpha^2-k^2) I_o(\gamma a) K_o(\gamma a)}$$

$$= -\frac{i\mu_o I(0)k \, L^+(k)}{2\pi^2} \int_{-\infty}^{\infty} d\beta \int_{-\infty}^{\infty} \frac{d\alpha \; I_o(a\sqrt{\beta^2-k^2}) K_o(\rho\sqrt{\beta^2-k^2}) L^-(\alpha) e^{-i\beta z}}{(\alpha^2-k^2) I_o(\gamma a) \; K_o(\gamma a)}$$

$$\times \; \delta(\alpha-\beta)$$

$$= -\frac{i\mu_o I(0)k \, L^+(k)}{2\pi^2} \int_{-\infty}^{\infty} e^{-i\alpha z} \frac{K_o(\gamma\rho) \; L^-(\alpha)}{(\alpha^2-k^2) \; K_o(\gamma a)} \, d\alpha \, .$$

This equation is valid for $\rho > a$, $z > 0$. Note that the magnetic flux density is given by

$$B_\phi = - \frac{\partial A_z}{\partial \rho}$$

therefore the magnetic intensity is

$$H_\phi(\rho,z,\omega) = - \frac{iI(0)k\, L^+(k)}{2\pi^2} \int_{-\infty}^{\infty} \frac{e^{-i\alpha z}\, K_1(\gamma\rho)\, L^-(\alpha)\, d\alpha}{\gamma\, K_o(\gamma a)} \qquad (E.12)$$

Equation (E.12) is exact for an infinite length antenna. If we assume that it also holds for an extremely long antenna and employ the methods of steepest descent as was done in Chapter 4, the resulting equation for the magnetic intensity is given by

$$H_\phi(\rho,z,\omega) \simeq \frac{I(0)e^{ikr}}{2\pi r} \; \frac{J_o(ka \sin\theta)}{\sin\theta} \; \frac{L^+(k)}{L^-(k \cos\theta)} \qquad (E.13)$$

Earlier we assumed that $I(0)$ was the incident current which is reflected from the discontinuity at the $z = 0$ plane [8]. The reflected current on the feedline from pulse excitation is known through experiment to approximate a reflected incident current pulse. From the edge conditions for this structure it is clear that the current on the outer conductor vanishes at $z = 0$. If the current on the outer and inner conductors are to be equal in magnitude and opposite in direction, then the current on the inner conductor must also vanish at $z = 0$, the point of discontinuity. We therefore conclude that a pulse of current is launched onto the antenna and an oppositely traveling pulse is launched back upon the center conductor of the feed line. The magnitude of this current is assumed to equal the magnitude of the incident pulse.

If the incident field is given by equation (2.7) of Chapter 2, then

$$I(0) = -2\pi\sqrt{\frac{\varepsilon_o}{\mu_o}}\ \frac{v(\omega)}{\ell n \frac{b}{a}}$$

The negative sign is required since the reflected current travels in a direction opposite to that of the incident current. Incorporation of the above into (E.13) gives the final form of the magnetic intensity,

$$H_\phi(\rho,z,\omega) = -\sqrt{\frac{\varepsilon_o}{\mu_o}}\ \frac{v(\omega)}{\ell n \frac{b}{a}}\ \frac{e^{ikr}}{r}\ \frac{J_o(ka \sin\theta)}{\sin\theta}\ \frac{L^+(k)}{L^-(k\cos\theta)} \tag{E.14}$$

If

$$\lim_{ka\to 0} \frac{L^+(k)}{L^-(k\cos\theta)} = 1$$

then (E.14) obviously reduces to

$$H_\phi(\rho,z,\omega) = -\sqrt{\frac{\varepsilon_o}{\mu_o}}\ \frac{v(\omega)}{\ell n \frac{b}{a}}\ \frac{e^{ikr}}{r}\ \frac{J_o(ka \sin\theta)}{\sin\theta} \tag{E.15}$$

for an antenna with $kh >> 1$ and $ka << 1$.

To show the validity of this limit we rewrite it from the expressions for $L^+(k)$ and $L^-(k\cos\theta)$,

$$\lim_{ka\to 0} \frac{\prod_{m=1}^{\infty}\left(1 + \frac{k}{i|\beta_m|}\right)e^{\frac{ika}{m\pi}}}{\prod_{m=1}^{\infty}\left(1 - \frac{k\cos\theta}{i|\beta_m|}\right)e^{(-ika\cos\theta)/(m\pi)}}\ \frac{G^{+(3)}(k)\ e^{\psi(-k\cos\theta)}}{G^{+(3)}(-k\cos\theta)\ e^{\psi(k)}} \simeq$$

$$\simeq \lim_{ka \to 0} \frac{\exp\left[\int\limits_{\delta=0^+}^{\infty} K^{(2)}(u) \quad \ell n[1 + \frac{k}{\sqrt{k^2-u^2}}]du\right]}{\exp\left[\int\limits_{\delta\to 0^+}^{\infty} K^2(u) \quad \ell n[1 + \frac{k \cos \theta}{\sqrt{k^2 - u^2}}]du\right]}$$

$$= \lim_{ka \to 0} \quad \exp\left[\int\limits_{\delta\to 0^+}^{\infty} K^2(u) \, \ell n[1 + \frac{1 - \cos \theta}{\cos \theta + \sqrt{1 - (u/k)^2}}] \, du\right]$$

This final limit was investigated in Chapter 5 where it was shown to be unity.

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