Singular Perturbations of a Boundary-Value Problem for a System of Nonlinear Differential Equations

Thesis by

Jack William Macki

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Abstract

The nonlinear boundary-value problem

$$\frac{dx}{dt} = f(x,y,t,\epsilon),$$

$$\epsilon\frac{dy}{dt} = g(x,y,t,\epsilon),$$

$$a_1x(0,\epsilon) + a_2y(0,\epsilon) = \alpha(\epsilon),$$

$$b_1x(1,\epsilon) + b_2y(1,\epsilon) = \beta(\epsilon),$$

is examined, under the hypothesis that the degenerate problem

$$\frac{dx_o}{dt} = f(x_o,y_o,t,0),$$

$$0 = g(x_o,y_o,t,0),$$

$$b_1x_o(1) + b_2y_o(1) = \beta(0),$$

has a continuously differentiable solution. Under a series of assumptions concerned, for the most part, with the smoothness of the functions f and g, it is proved that, for ϵ restricted to a small enough interval of the form $0 < \epsilon < \epsilon_o$, the above boundary-value problem has a solution of the form

$$x(t,\epsilon) = x_o(t) + \epsilon p(t,\epsilon) + \epsilon\rho(t,\epsilon),$$

$$y(t,\epsilon) = y_o(t) + \epsilon q(t,\epsilon) + \tau(t,\epsilon),$$

where p and q are both O(1) uniformly in t as ϵ goes to zero, while ρ and τ exhibit a boundary-layer type of behavior.

0. INTRODUCTION

Perturbation methods have long been of importance in treating boundary-value problems which involve a small parameter, that is, boundary-value problems based on an equation of the form

$$\frac{dx}{dt} = F(x,t,\epsilon),$$

with x and F vectors, and ϵ a small parameter. If F depends regularly on ϵ near $\epsilon = 0$, then one would expect that, under relatively mild conditions, the solution to a boundary-value problem based on the above equation would converge, as ϵ goes to zero, to a solution of the degenerate ($\epsilon = 0$) boundary-value problem. The requirement that F depend regularly on ϵ near $\epsilon = 0$, however, is not met for a large class of important equations. Of particular interest are those problems in which one or more of the components of F are of the form

$$F_k(x,t,\epsilon)/\epsilon,$$

with F_k a regular function of ϵ near $\epsilon = 0$. The k^{th} equation in the system can then be written

$$\epsilon x_k' = F_k(x,t,\epsilon),$$

so that the degenerate system is of lower order than the given system. This means that no solution to the degenerate system will in general satisfy all of the boundary conditions in the given problem; consequently, we cannot expect a solution to the given problem to converge uniformly to some solution of the degenerate system as ϵ goes to zero. In the type of problem with which we shall be concerned, the solution does converge uniformly to a solution of the degenerate system on every closed subset of the basic interval that does not contain a certain point, t_o. This point of non-uniformity is determined by the structure of the given system. Because of this non-uniform behavior, problems based on systems of the type described above must be treated by special means - the methods of the theory of singular perturbations.

An extensive literature has developed concerning the existence of, and expansions for, solutions of singular perturbations of ordinary linear differential equations whose coefficients are analytic in ϵ in some neighborhood of $\epsilon = 0$. Formal series solutions are constructed by substitution, setting the coefficients of each power of ϵ to zero and recursively solving the resulting differential equations for the coefficients appearing in the formal series. True solutions are then constructed, and shown to be represented asymptotically by the formal series. For rigorous expositions of the theory, with extensive bibliographies, see Turrittin (1) and Wasow (2).

In recent years, a great deal of attention has been directed towards nonlinear problems. Those investigations to which this paper is related have concerned themselves with either the problem P_ϵ:

$$\epsilon x'' = g(x,x',t,\epsilon),\ x(0) = \alpha,\ x(1) = \beta;$$

or the slightly more general problem Q_ϵ:

$$x' = f(x,y,t,\epsilon)$$
$$\epsilon y' = g(x,y,t,\epsilon),\ x(0) = \alpha,\ x(1) = \beta.$$

Since P_ϵ can be interpreted as the special case of Q_ϵ with $f(x,y,t,\epsilon) = y$, we shall use the notation of Q_ϵ for discussing both problems. It is assumed that the degenerate system has a solution, $\begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$, satisfying one of the two boundary conditions, with

$g_y(x_o(t),y_o(t),t,0)$ non-zero and of constant sign on $[0,1]$. All investigations to date have indicated that the function $g_y(x_o(t),y_o(t),t,0)$ determines a single point of non-uniformity, t_o - if $g_y(x_o(t),y_o(t),t,0) < 0$, then $t_o = 0$, while if $g_y(x_o(t),y_o(t),t,0) > 0$, then $t_o = 1$. Since we want the solution to P_ϵ or Q_ϵ to converge to the hypothesized degenerate solution at every point except t_o, it is clear that if we assume $g_y(x_o(t),y_o(t),t,0) < 0$, then we must assume $x_o(1) = \beta$; while if we assume $g_y(x_o(t),y_o(t),t,0) > 0$, we must assume $x_o(0) = \alpha$.

The question of the existence of solutions to the problems P_ϵ and Q_ϵ under even relatively simple conditions is not a trivial one. For example, Coddington and Levinson (3) have shown that the problem P_ϵ based on the equation

$$\epsilon x'' = -x' - (x')^3$$

will in general fail to have a solution for small ϵ. This has led most investigators to assume that $g(x,y,t,\epsilon)$ is linear in y.

The methods of attacking the problems P_ϵ

and Q_ϵ can be roughly classified into two groups. In order to simplify the discussion, let us consider the problem P_ϵ, and suppose that x_o satisfies the boundary condition at $t = 1$ so that $t_o = 0$ is our point of non-uniformity.

The first method is characterized by the assumption that $g(x,y,t,\epsilon)$ is analytic in x and ϵ, linear in y, and differentiable to a certain degree in t, all within a certain region of (x,y,t,ϵ)-space containing the curve $\{(x_o(t),x_o'(t),t,o):0\leq t\leq 1\}$. The first step in this method is to construct a solution, $x^*(t)$, to the equation in P_ϵ, which differs little from x_o and satisfies the boundary condition at $t = 1$. The remaining boundary condition on x^* is taken so as to keep x^* close to x_o (e.g., $x^{*\prime}(o) = x_o'(o)$). To construct the function $x^*(t)$, one "linearizes" the equation about x_o by writing $x^* = x_o + \phi$, and then solves the resulting nonlinear problem in ϕ. In solving the problem for ϕ, use is made of the results for linear equations with analytic coefficients as described above. The function ϕ, called the outer correction, is invariably found to converge uniformly to zero as ϵ goes to zero. This approach, because of the powerful assumptions made, does a great deal more than show that $\emptyset$ goes uniformly to zero - it gives an asymptotic expansion in powers of ϵ for $\emptyset$, uniformly valid for $0 \leq t \leq 1$.

The next step is to again "linearize" the equation, this time about x^*, by writing the solution to the problem P_ϵ in the form

$$x = x^* + \psi .$$

The resulting nonlinear problem for ψ is solved by assuming that $\psi(t,\epsilon)$ has a formal expansion in powers of

$$\mu = \alpha - x^*(o),$$

and applying the same type of techniques used in determining the outer correction. Then it is shown that, for μ sufficiently small, the formal series actually represents a solution of the problem for ψ. The function $\psi(t,\epsilon)$ is entirely a boundary-layer type of function, being significant only near $t = 0$, and converging exponentially to zero, at every point except $t = 0$, as ϵ goes to zero.

Wasow (4) has given a complete constructional procedure for obtaining the solution to P_ϵ by the above method, and Harris (5) has done the same for Q_ϵ. It should be remarked that the technique of linearization leads to the necessity of applying the previously developed theory of singular perturbations of <u>linear</u> equations with <u>analytic coefficients</u>.

The second method of attack - of which this paper is an example - is typified by much more limited assumptions on g, usually consisting of differentiability to a certain order in t, x, and y, and continuity in ϵ at $\epsilon = 0$. These comparatively weak assumptions, of course, give less complete results - in general only one or two terms of the expansion of the solution. For examples of this approach, see Nagumo (6), von Mises (7), Levinson (8), Coddington and Levinson (9), Briš (10), and Erdélyi (11, 12, 13). The specific assumptions and techniques used by different investigators are quite diverse, and the results are just as much so. The investigations of Erdélyi are most closely related to this paper, especially (12), where he considers the problem P_ϵ. One remarkable result of his investigation is that the condition that g be linear in y may be replaced by the condition $g_{yy} = O(\epsilon)$.

This paper is concerned with the problem Q_ϵ, under the slightly more general boundary conditions: $a_1 x(0) + a_2 y(0) = \alpha(\epsilon), b_1 x(1) + b_2 y(1) = \beta(\epsilon)$. The technique of linearization described above is used, which means that singular perturbations of linear systems must be examined, since our assumptions are not strong enough to allow us to use the theory of linear equations with analytic coefficients. This is done in sections 1, 2, and 3. In section 4, the nonlinear boundary-value problem is stated, along with the first assumptions on f and g, and the outer correction is obtained. In section 5 we derive the boundary-value problem for the inner correction, state the additional assumptions that we require to solve this problem, and obtain the inner correction. The boundary-layer behavior of the inner correction is clearly exhibited. In section 6 we summarize our results, discuss some of their more important features, and give explicit formulas for the leading terms in the expansions of the inner and outer corrections.

Notation

The independent variable t will always range over [0,1]. If f(t) is a bounded function of t on [0,1], we set $\|f\| = \sup_{0\leq t\leq 1} |f(t)|$. If $A(t) = [a_{ij}(t)]$ is a matrix, we set $|A(t)| = \sum_{i,j} |a_{ij}(t)|$, $\|A\| = \sum_{i,j} \|a_{ij}\|$. This definition, of course, includes column matrices, which we shall call vectors. A column matrix of k elements will be called a k-vector. The transpose of a matrix A(t) will be denoted by $A^t(t)$. Any matrix consisting solely of zeros will be simply written 0. For any square matrix, A(t), we will denote the determinant by det A(t).

We shall say that the k-vector, $\rho(t)$, is integrable if each component of $\rho(t)$ is integrable on [0,1], and $\int_a^b \rho(s)ds$ will denote that k-vector, each of whose components is the integral from a to b of the corresponding component of $\rho(t)$. The collection of integrable k-vectors will be denoted by $L_1(k)$, and for $\rho \epsilon L_1(k)$ we define $\|\rho\|_1 = \int_0^1 |\rho(s)|ds$.

The parameter ϵ will always be restricted to an open interval of the form $0 < \epsilon < \epsilon_0$. The statement $m(t,\epsilon) = 0(\alpha(\epsilon))$, $m(t,\epsilon)$ a matrix or scalar, will mean that there exists a constant, K, such that $|m(t,\epsilon)| \leq K\alpha(\epsilon)$ for $0 \leq t \leq 1$, $0 < \epsilon < \epsilon_0$. For two scalars, $f(t,\epsilon)$ and $g(t,\epsilon)$, the statement $g(t,\epsilon)=[[f(t,\epsilon)]]$ means that $g(t,\epsilon)=f(t,\epsilon)[1+0(\epsilon)]$.

Unless explicitly stated otherwise, $z(t)$ and $h(t,\epsilon)$ will denote the 2-vectors $\begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$ and $\begin{bmatrix} f(t,\epsilon) \\ g(t,\epsilon) \end{bmatrix}$, respectively, and subscripted or superscripted versions of $z(t)$ and $h(t,\epsilon)$ will denote the same 2-vectors with the components sub- or superscripted. The symbols a and b will denote the constant 2-vectors $\begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$ and $\begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$, respectively, while $\Delta(\epsilon)$ will stand for the matrix $\begin{bmatrix} 1 & 0 \\ 0 & \epsilon \end{bmatrix}$.

In cases where it leads to no confusion, variables of integration have not been written - for example, $\int_0^1 h$ in place of $\int_0^1 h(s,\epsilon)ds$.

For the convenience of the reader, a table of the more important notations and formulas used in this work follows.

Symbol	Formula (or Asymptotic Form)
$\Delta(\epsilon)$	$\begin{bmatrix} 1 & 0 \\ 0 & \epsilon \end{bmatrix}$
$L(s,t,\epsilon)$	$\exp\left[\int_s^t (\ell + p_o c_o)\right]$
$P(s,t,\epsilon)$	$\exp\left[\epsilon^{-1}(s-t) - \int_s^t p_o c_o\right]$
$W(s,t,\epsilon)$	$\exp\left[\int_s^t (w - p_o c_o)\right]$
$z_\rho(t,\epsilon)$	$\begin{bmatrix} 1 \\ c_o(t,\epsilon) \end{bmatrix} L(0,t,\epsilon) + O(\epsilon)$
$z_\tau(t,\epsilon)$	$\begin{bmatrix} -\epsilon p_o(t,\epsilon) + O(\epsilon^2) \\ 1 \quad + O(\epsilon) \end{bmatrix} W(1,t,\epsilon) e^{-t/\epsilon}$
s	$\lvert f_y(x_o(0), y_o(0), 0, \epsilon) \rvert^{-1} = O(\epsilon^{-s})$
m	0, if $a_2 \neq 0$; $-1-s$, if $a_2 = 0$.
$h_z(z,t,\epsilon)$	$\begin{bmatrix} f_x(x,y,t,\epsilon), & f_y(x,y,t,\epsilon) \\ g_x(x,y,t,\epsilon), & g_y(x,y,t,\epsilon) \end{bmatrix}$
$H(X,t,\epsilon)$	$h(z_o(t) + X,t,\epsilon) - \Delta(\epsilon) z_o'(t) - h_z(z_o(t),t,\epsilon)X$
$H^*(X,t,\epsilon)$	$h(z^*(t) + X,t,\epsilon) - h(z^*(t),t,\epsilon) - h_z(z^*(t),t,\epsilon)X$
$\mu(\epsilon)$	$\alpha(\epsilon) - a^t z^*(0)$
$q(b,t,\epsilon)$	$\begin{cases} e^{-t/\epsilon} \text{ if } b_2 = 0, \\ e^{-t/\epsilon} + \epsilon^{-1} e^{-1/\epsilon} \text{ if } b_2 \neq 0. \end{cases}$

1. THE LINEAR SYSTEM

We are concerned with the existence and asymptotic behavior of solutions to initial-value and boundary-value problems connected with the linear system

$$\begin{bmatrix} 1 & 0 \\ 0 & \epsilon \end{bmatrix} z' = \Delta(\epsilon) z' = \begin{bmatrix} \ell(t,\epsilon), & p(t,\epsilon) \\ c(t,\epsilon), & -1+\epsilon w(t,\epsilon) \end{bmatrix} z. \qquad (1.1)$$

<u>Assumption A</u>

There exists $\epsilon_1 > 0$ such that, for $0 < \epsilon < \epsilon_1$:

(a) $\begin{bmatrix} \ell(t,\epsilon), & p(t,\epsilon) \\ c(t,\epsilon), & w(t,\epsilon) \end{bmatrix} = 0(1)$,

(b) $\ell(t,\epsilon)$ and $w(t,\epsilon)$ are measurable,

(c) $\begin{bmatrix} c(t,\epsilon) \\ p(t,\epsilon) \end{bmatrix} = \begin{bmatrix} c_o(t,\epsilon) \\ p_o(t,\epsilon) \end{bmatrix} + \epsilon \begin{bmatrix} c_1(t,\epsilon) \\ p_1(t,\epsilon) \end{bmatrix}$, where

c_o and p_o are differentiable functions of t, and p_1, c_1, p_o', c_o' are $0(1)$.

Assumption A will be assumed to hold throughout sections 1, 2, and 3.

In order to convert the equation 1.1 into a useful integral equation, we shall construct matrices $F(t,\epsilon)$, $G(t,\epsilon)$, and $E(t,\epsilon)$, such that

$$F(t,\epsilon)\left\{\Delta(\epsilon)z' - \begin{bmatrix} \ell, p_o \\ c_o, -1 \end{bmatrix} z\right\} = \left[G(t,\epsilon)z\right]' + E(t,\epsilon)z, \qquad (1.2)$$

with $E(t,\epsilon)$ small in a sense which will become apparent below. We introduce the notation

$$L(s,t,\epsilon) = \exp\left[\int_s^t (\ell + p_o c_o)\right], \quad P(s,t,\epsilon) = \exp\left[\epsilon^{-1}\left(s-t-\int_s^t \epsilon p_o c_o\right)\right].$$

If we take

$$F(t,\epsilon) = \begin{bmatrix} L(t,0,\epsilon), & p_o(t,\epsilon)L(t,0,\epsilon) \\ -\epsilon c_o(t,\epsilon)P(t,0,\epsilon), & P(t,0,\epsilon) \end{bmatrix},$$

and note that we must have

$$G(t,\epsilon) = F(t,\epsilon)\Delta(\epsilon),$$

then

$$E(t,\epsilon) = \epsilon\begin{bmatrix} 0 & , [-p_o' + p_o(\ell + p_o c_o)]L(t,0,\epsilon) \\ [c_o' + c_o(\ell + p_o c_o)]\ P(t,0,\epsilon), & 0 \end{bmatrix}.$$

We take

$$0 < \epsilon_2 < \min\left\{\epsilon_1, \sup_{0<\epsilon<\epsilon_1} \|p_o(t,\epsilon)c_o(t,\epsilon)\|^{-1}\right\},$$

and restrict the number ϵ to $0 < \epsilon < \epsilon_2$. Since

$$\det G(t,\epsilon) = \epsilon L(t,0,\epsilon)P(t,0,\epsilon)[1 + \epsilon p_o(t,\epsilon)c_o(t,\epsilon)],$$

it is clear that $G(t,\epsilon)$ is non-singular for $0 < \epsilon < \epsilon_2$, $0 \leq t \leq 1$.

If we now multiply equation 1.1 by $F(t,\epsilon)$ and use 1.2 to integrate the resulting equation, we get - upon multiplication by $G^{-1}(t,\epsilon)$ - the following integral equation for z:

$$z(t) = h(t,\epsilon) + \epsilon\begin{bmatrix} 1 & , p_o(t,\epsilon) \\ c_o(t,\epsilon), & -1/\epsilon \end{bmatrix}\begin{bmatrix} T_1(z) \\ T_2(z) \end{bmatrix}, \quad (1.3)$$

where

$$h(t,\epsilon) = [1+\epsilon p_o(t,\epsilon)c_o(t,\epsilon)]^{-1}\left\{\begin{bmatrix} 1 & , & \epsilon p_o(0,\epsilon) \\ c_o(t,\epsilon), & \epsilon p_o(0,\epsilon)c_o(t,\epsilon) \end{bmatrix} L(0,t,\epsilon) + \right.$$

$$\left. + \begin{bmatrix} \epsilon c_o(0,\epsilon)p_o(t,\epsilon), & -\epsilon p_o(t,\epsilon) \\ -c_o(0,\epsilon) & , \quad 1 \end{bmatrix} P(0,t,\epsilon)\right\} z(0),$$

$$T_1[z] = [1+\epsilon p_o(t,\epsilon)c_o(t,\epsilon)]^{-1}\int_o^t \begin{bmatrix} p_o c_1 \\ p_o'+p_1+p_o(w-\ell-p_o c_o) \end{bmatrix}^t z(s)L(s,t,\epsilon)ds,$$

$$T_2[z] = [1+\epsilon p_o(t,\epsilon)c_o(t,\epsilon)]^{-1}\int_o^t \begin{bmatrix} c_o'-c_1+c_o(\ell+p_o c_o) \\ \epsilon c_o p_1 - w \end{bmatrix}^t z(s)P(s,t,\epsilon)ds.$$

We shall presently show that our choice of F, G, and E has given an integral equation which, for given $z(0)$, has a unique solution of the form:

$$z(t) = h(t,\epsilon) + O(\epsilon|h(t,\epsilon)|).$$

The following bounds are easily seen to hold for $0 \leq t \leq 1$, $0 < \epsilon < \epsilon_2$:

$$|T_k(z)| \leq A_k \int_o^t |z(s)|ds, \quad k = 1,2,$$

$$|T_1(z)| \leq B_1\|z\|, \tag{1.4}$$

$$|T_2(z)| \leq \epsilon B_2\|z\|.$$

Then, if we rewrite equation 1.3 as

$$z(t) = h(t,\epsilon) + \begin{bmatrix} T_A(z) \\ T_B(z) \end{bmatrix}, \qquad (1.5)$$

the following bounds are immediate:

$$|T_A(z)| \leq \epsilon A \int_0^t |z(s)|ds,$$

$$|T_B(z)| \leq B \int_0^t |z(s)|ds, \qquad (1.6)$$

$$|T_B(z)| \leq \epsilon C \|z\|,$$

for $0 \leq t \leq 1$, $0 < \epsilon < \epsilon_2$.

In the lemma and corollary below, we make a brief departure from our usual notation, in that z(t) and h(t) will represent the k-vectors $\begin{bmatrix} x_1(t) \\ \vdots \\ x_k(t) \end{bmatrix}$, $\begin{bmatrix} f_1(t) \\ \vdots \\ f_k(t) \end{bmatrix}$, respectively, and we shall use the notation $\|z\|_1 = \int_0^1 |z(s)|ds$. This lemma and its corollary are consequences of well-known results in the theory of integral equations, and are included here to keep this work as self-contained as possible.

LEMMA 1.1

Let T be a linear mapping of $L_1(k)$ into $L_1(k)$, such that

$$|T(z)| \leq A^*\int_0^t |z(s)|ds.$$

Then, given any finite-valued integrable k-vector, $h(t)$, the equation $z(t) = h(t) + T(z)$ has a unique solution $z \epsilon L_1(k)$, and

$$|z(t) - h(t)| \leq A^* \int_0^t |h(s)| \exp[A^*(t-s)]ds.$$

Proof:

Set $z_{-1}(t) = 0$, $z_{n+1}(t) = h(t) + T(z_n)$.

It is clear that $z_n \epsilon L_1(k)$ for all $n \geq -1$. The assumed bound on T, coupled with the linearity of T, gives $|z_1 - z_o| \leq A^*\|h\|_1$, and an easy induction then shows that

$$|z_{n+1}(t)-z_n(t)| \leq \frac{A^*(A^*t)^n\|h\|_1}{n!} \leq \frac{(A^*)^{n+1}\|h\|_1}{n!} .$$

Thus the sequence $\{z_n(t)\}$ is Cauchy for fixed t, which implies that the pointwise limit, $z(t) = \lim z_n(t)$, exists for $0 \leq t \leq 1$. Furthermore,

$$|z(t) - z_n(t)| \leq |\sum_n^\infty (z_{k+1}-z_k)| \leq \|h\|_1 \sum_n^\infty \frac{(A^*)^{k+1}}{k!} ,$$

so the convergence is uniform, and $z \epsilon L_1(k)$.

From the inequality

$$|T(z_n) - T(z)| \leq A^* \|z_n - z\|,$$

it is clear that $\lim T(z_n)$ exists and is in fact equal to $T(\lim z_n)$. Taking the limit of the relation $z_{n+1} = h + T(z_n)$, we see that z satisfies the given equation.

A straightforward induction shows that

$$|z_{n+1}(t) - z_n(t)| \leq \frac{(A^*)^{n+1}}{n!} \int_0^t (t-s)^n |h(s)| ds,$$

which implies

$$|z(t)-h(t)| \leq \sum_0^\infty |z_{n+1}-z_n| \leq A^* \int_0^t |h(s)| \exp[A^*(t-s)] ds.$$

Finally, if z and z* are two solutions to the equation, then

$$|z^*(t) - z(t)| \leq A^* \int_0^t |z(s) - z^*(s)| ds.$$

By induction one can now show that

$$|z^*(t)-z(t)| \leq \frac{(A^*)^{n+1}}{n!} \int_0^t (t-s)^n |z^*(s)-z(s)| ds \leq$$

$$\leq \frac{(A^*)^{n+1}}{n!} \|z^*-z\|_1,$$

for all $n \geq 0$. Hence $z^* = z$.

COROLLARY 1.1

If, in addition to the hypotheses of the lemma, we assume:

(a) $|h(t)|$ is bounded for $0 \leq t \leq 1$,

(b) there exists $0 < B^* < 1$, such that

$$|T(z)| \leq B^* \|z\|,$$

then the solution to $z = h + T(z)$ satisfies the inequality

$$|z(t) - h(t)| \leq B^* \|h\| (1 - B^*)^{-1}.$$

Proof:

Since $|T(z)| \leq A^* \|z\|_1$, the solution is bounded, its sup norm exists, and $\|z\| \leq \|h\| + B^* \|z\|$, so that $\|z\| \leq \|h\| (1-B^*)^{-1}$. Then $|z-h| \leq B^* \|z\| \leq$ $\leq B^* \|h\| (1-B^*)^{-1}$.

2. BASIC SOLUTIONS AND THE FUNDAMENTAL MATRIX

Using the bounds 1.6, and writing $A^* = B + \epsilon A$, $B^* = \epsilon(A + C)$, we see that we can apply the above lemma and corollary to equation 1.5. In order to use the corollary, we must have $\epsilon(A + C) < 1$, so we pick $\epsilon_3 < \min \left\{\epsilon_2, (A + C)^{-1}\right\}$, and restrict ϵ to $0 < \epsilon < \epsilon_3$.

The Basic Regular Solution

Setting $z(0) = \begin{bmatrix} 1 \\ c_o(0,\varepsilon) \end{bmatrix}$ in 1.3 and applying the lemma and corollary, we get the basic regular solution for the system 1.1, $z_\rho(t,\varepsilon)$, where z_ρ has the form:

$$z_\rho(t,\varepsilon) = \begin{bmatrix} 1 \\ c_o(t,\varepsilon) \end{bmatrix} L(0,t,\varepsilon) + O(\varepsilon),$$

$$z_\rho(0,\varepsilon) = \begin{bmatrix} 1 \\ c_o(0,\varepsilon) \end{bmatrix}.$$

The Basic Boundary-Layer Solution

The transformation

$$z(t) = \begin{bmatrix} \varepsilon u(1-t) \\ v(1-t) \end{bmatrix} e^{-t/\varepsilon},$$

coupled with the change of variable $r = 1 - t$, transforms the system 1.1 into the system

$$\Lambda(\varepsilon)\frac{dX}{dr} = \begin{bmatrix} -w(1-r,\varepsilon), & -c(1-r,\varepsilon) \\ -p(1-r,\varepsilon), & -1 - \varepsilon\,\ell(1-r,\varepsilon) \end{bmatrix} X(r),$$

where $X = \begin{bmatrix} v \\ u \end{bmatrix}$.

This system is of the same form as 1.1, and its coefficients satisfy the assumptions made above, so it can be transformed into an integral equation of the form of 1.5 with bounds of the form 1.6. We denote the constants appearing in these bounds by A', B', and C', respectively, choose $\epsilon_4 < \min\left\{\epsilon_3, (A' + C')^{-1}\right\}$, and restrict ϵ to $0 < \epsilon < \epsilon_4$. Then it is clear that this system has a basic regular solution of the form:

$$X(r,\epsilon) = \begin{bmatrix} 1 \\ -p_o(1-r,\epsilon) \end{bmatrix} \exp\left[\int_o^r (p_o(1-s,\epsilon)c_o(1-s,\epsilon) - w(1-s,\epsilon)ds\right] + O(\epsilon),$$

$$X(0,\epsilon) = \begin{bmatrix} 1 \\ -p_o(1,\epsilon) \end{bmatrix}.$$

Expressing the above solution in terms of z(t) and t, we get the basic boundary-layer solution for the system 1.1, $z_\tau(t,\epsilon)$, of the form:

$$z_\tau(t,\epsilon) = \begin{bmatrix} -\epsilon p_o(t,\epsilon) \\ 1 \end{bmatrix} W(1,t,\epsilon)e^{-t/\epsilon} + \begin{bmatrix} O(\epsilon^2) \\ O(\epsilon) \end{bmatrix} e^{-t/\epsilon},$$

$$z_\tau(1,\epsilon) = \begin{bmatrix} -\epsilon p_o(1,\epsilon) \\ 1 \end{bmatrix} e^{-1/\epsilon},$$

where we have introduced the convenient notation

$$W(s,t,\epsilon) = \exp\int_s^t (w - p_o c_o) .$$

From z_ρ and z_τ we can form the fundamental matrix

$$U(t,\epsilon) = \left[z_\rho(t,\epsilon),\ z_\tau(t,\epsilon)\right] =$$

$$= \begin{bmatrix} L(0,t,\epsilon)+O(\epsilon), & [-\epsilon p_o(t,\epsilon)W(1,t,\epsilon)+O(\epsilon^2)]e^{-t/\epsilon} \\ c_o(t,\epsilon)L(0,t,\epsilon)+O(\epsilon), & [W(1,t,\epsilon)+O(\epsilon)]\,e^{-t/\epsilon} \end{bmatrix}. \quad (2.1)$$

Noting that

$$\det U(t,\epsilon) = [L(0,t,\epsilon)W(1,t,\epsilon) + O(\epsilon)]e^{-t/\epsilon},$$

we take ϵ^* so that $\det U(t,\epsilon) \neq 0$ for $0 \leq t \leq 1$, $0 < \epsilon < \epsilon^*$, then set $\epsilon_5 \leq \min(\epsilon_4,\epsilon^*)$, and restrict ϵ to $0 < \epsilon < \epsilon_5$.

Using 2.1, it is easy to show that, for $0 \leq s \leq 1$, $0 \leq t \leq 1$, $0 < \epsilon < \epsilon_5$, the matrix $U(t,\epsilon)U^{-1}(s,\epsilon)$ has the form:

$$U(t,\epsilon)U^{-1}(s,\epsilon)=\begin{bmatrix} 1 & ,\epsilon p_o(s,\epsilon) \\ c_o(t,\epsilon) & ,\epsilon p_o(s,\epsilon)c_o(t,\epsilon) \end{bmatrix}[[L(s,t,\epsilon]]+$$

$$+\begin{bmatrix} \epsilon p_o(t,\epsilon)c_o(s,\epsilon) & ,-\epsilon p_o(t,\epsilon) \\ -c_o(s,\epsilon) & , \quad 1 \end{bmatrix}[[W(s,t,\epsilon)]]e^{(s-t)/\epsilon}, \quad (2.2)$$

where, given a bounded function, $f(s,t,\epsilon)$, we say that a function $g(s,t,\epsilon)$ is of the form $[[f(s,t,\epsilon]]$ if $g(s,t,\epsilon) = f(s,t,\epsilon)(1+O(\epsilon))$ for $0 < \epsilon < \epsilon_5$, $0 \leq s \leq 1$, $0 \leq t \leq 1$.

We point out that, if $\ell(t,\epsilon)$ and $w(t,\epsilon)$ can be written

$$\ell(t,\epsilon) = \ell_o(t,\epsilon) + O(\epsilon), \quad w(t,\epsilon) = w_o(t,\epsilon) + O(\epsilon),$$

then in the formulas 2.1 and 2.2 we may replace ℓ and w by ℓ_o and w_o.

Henceforth, in order to obtain a certain amount of economy of expression, we shall simply write $U(t)$ for the matrix $U(t,\epsilon)$.

3. INITIAL-VALUE AND BOUNDARY-VALUE PROBLEMS

The Initial-Value Problem

We consider the problem of obtaining solutions to the system 1.1 with the initial conditions $a^t z(0) = \alpha(\epsilon)$, $b^t z(0) = \beta(\epsilon)$, where a and b are given constant vectors, and α and β are given functions of ϵ. We assume that

$$\det\begin{bmatrix} a^t \\ b^t \end{bmatrix} \neq 0.$$

Then it is clear that, for $0 < \epsilon < \epsilon_5$,

$$z(t) = U(t)U^{-1}(0)\begin{bmatrix} a^t \\ b^t \end{bmatrix}^{-1}\begin{bmatrix} \alpha(\epsilon) \\ \beta(\epsilon) \end{bmatrix}$$

solves the problem. Formula 2.2 can be used to get asymptotic estimates for z.

The Homogeneous Boundary-Value Problem

We are now interested in solving the system 1.1 under the boundary conditions:

$$a^t z(0) = \alpha(\epsilon), \; b^t z(1) = \beta(\epsilon),$$

with $a, b, \alpha(\epsilon)$ and $\beta(\epsilon)$ as before, except that we do not require $\det\begin{bmatrix} a^t \\ b^t \end{bmatrix} \neq 0$. Writing $z(t) = U(t)c$, with the 2-vector c to be determined, and applying the boundary conditions, we get the following equation:

$$\begin{bmatrix} a^t U(0) \\ b^t U(1) \end{bmatrix} c = \begin{bmatrix} \alpha(\epsilon) \\ \beta(\epsilon) \end{bmatrix}.$$

This equation has a unique solution for c if and only if

$$\det \begin{bmatrix} a^t U(0) \\ b^t U(1) \end{bmatrix} =$$

$$= [[\epsilon a_1 p_o(0,\epsilon) - a_2]][[b_1 + b_2 c_o(1,\epsilon)]] L(0,1,\epsilon) W(1,0,\epsilon)$$

is non-zero.

<u>Assumption</u> B

There exists $\epsilon' > 0$ such that, for $0 < \epsilon < \epsilon'$:

(a) $b_1 + b_2 c_o(1,\epsilon) \neq 0,$

(b) $\epsilon a_1 p_o(0,\epsilon) - a_2 \neq 0.$

We remark that, since $p(t,\epsilon) = 0(1)$, if $a_2 \neq 0$ part (b) of the above assumption can always be satisfied by suitably restricting ϵ. If $a_2 = 0$, then part (b) is equivalent to assuming that $p_o(0,\epsilon) \neq 0$ for ϵ suitably restricted.

We set $\epsilon_6 = \min(\epsilon', \epsilon_5)$. Then it is clear that, for $0 < \epsilon < \epsilon_6$, the matrix

$$\begin{bmatrix} a^t U(0) \\ b^t U(1) \end{bmatrix}$$

is non-singular, so that

$$z(t) = U(t)\begin{bmatrix} a^t U(0) \\ b^t U(1) \end{bmatrix}^{-1}\begin{bmatrix} \alpha(\epsilon) \\ \beta(\epsilon) \end{bmatrix}$$

solves the problem.

<u>The Non-Homogeneous Boundary-Value Problem</u>

We consider the problem

$$\Delta(\epsilon)z' = \begin{bmatrix} \ell(t,\epsilon) & p(t,\epsilon) \\ c(t,\epsilon), & -1+\epsilon w(t,\epsilon) \end{bmatrix} z + h(t,\epsilon),$$

$$a^t z(0) = \alpha(\epsilon),\ b^t z(1) = \beta(\epsilon), \tag{3.1}$$

under the hypothesis that assumption B holds. We assume that $h \epsilon L_1(2)$ for $0 < \epsilon < \epsilon_5$. Using the method of variation of parameters, we get the following expression for z:

$$z(t) = U(t)c + \int_0^t U(t)U^{-1}(s)\Delta(1/\epsilon)h(s,\epsilon)ds, \tag{3.2}$$

where the constant vector c is to be determined. Here we have used the fact that $\Delta^{-1}(\epsilon) = \Delta(1/\epsilon)$. From equation 3.2 we see that the vector

$$X(t) = z(t) - \int_0^t U(t)U^{-1}(s)\Delta(1/\epsilon)h(s,\epsilon)ds$$

is a solution of the homogeneous system, satisfying the boundary conditions:

$$a^t X(0) = \alpha(\varepsilon),$$

$$b^t X(1) = \beta(\varepsilon) - \int_0^1 b^t U(1)U^{-1}(s)\Delta(1/\varepsilon)h(s,\varepsilon)ds.$$

On the basis of assumption B, we can apply the theory developed above for the homogeneous problem. The result is the following solution, for $0 < \varepsilon < \varepsilon_6$, to the non-homogeneous problem:

$$z(t) = U(t)\begin{bmatrix} a^t U(0) \\ b^t U(1) \end{bmatrix}^{-1} \gamma(\varepsilon) + $$

$$+ \int_0^t U(t)U^{-1}(s)\Delta(1/\varepsilon)h(s,\varepsilon)ds, \qquad (3.3)$$

where

$$\gamma(\varepsilon) = \begin{bmatrix} \alpha(\varepsilon) \\ \beta(\varepsilon) - \int_0^1 U(1)U^{-1}(s)\Delta(1/\varepsilon)h(s,\varepsilon)ds \end{bmatrix}.$$

It is easy to verify that 3.3 can also be written in the form:

$$z(t) = U(t)\begin{bmatrix} a^t U(0) \\ b^t U(1) \end{bmatrix}^{-1} \gamma^*(\varepsilon) - $$

$$- \int_t^1 U(t)U^{-1}(s)\Delta(1/\varepsilon)h(s,\varepsilon)ds, \qquad (3.4)$$

where

$$\gamma^*(\epsilon) = \begin{bmatrix} \alpha(\epsilon)+\int_0^1 a^t U(0)U^{-1}(s)\Delta(1/\epsilon)h(s,\epsilon)ds \\ \beta(\epsilon) \end{bmatrix}.$$

It will be useful later to have for reference the formula:

$$\begin{bmatrix} a^t U(0) \\ b^t U(1) \end{bmatrix}^{-1} = \begin{bmatrix} b^t z_\tau(1,\epsilon), -a^t z_\tau(0,\epsilon) \\ -b^t z_\rho(1,\epsilon), a^t z_\rho(0,\epsilon) \end{bmatrix} \times$$

$$\times \frac{L(1,0,\epsilon)W(0,1,\epsilon)}{[[\epsilon a_1 p_o(0,\epsilon)-a_2]][[b_1+b_2 c_o(1,\epsilon)]]}. \qquad (3.5)$$

Note that if $b_1+b_2c_o(1,\epsilon)$ is bounded away from zero for ϵ restricted to a small enough interval, then the last factor is $O(1)$ for $a_2 \neq 0$, while it is $O(1/\epsilon p_o(0,\epsilon))$ for $a_2 = 0$.

4. THE NONLINEAR BOUNDARY-VALUE PROBLEM

We are interested in the existence of, and asymptotic estimates for, solutions to the problem:

$$\Delta(\epsilon)z' = h(z,t,\epsilon), \qquad (4.1)$$
$$a^t z(0) = \alpha(\epsilon),\ b^t z(1) = \beta(\epsilon)$$

when we are given a continuously differentiable solution, $z_o(t)$, to the degenerate problem:

$$\Delta(0)z' = h(z,t,0),\ b^t z(1) = \beta(0). \qquad (4.2)$$

For simplicity, we shall assume a and b independent of ϵ. We write

$$h(z,t,\epsilon) = \begin{bmatrix} f(x,y,t,\epsilon \\ g(x,y,t,\epsilon) \end{bmatrix},$$

and introduce the notation for the matrix of first partial derivatives:

$$h_z(z,t,\epsilon) = \begin{bmatrix} f_x(x,y,t,\epsilon), & f_y(x,y,t,\epsilon) \\ g_x(x,y,t,\epsilon), & g_y(x,y,t,\epsilon) \end{bmatrix}.$$

Assumption 1

The degenerate problem 4.2 has a continuously differentiable solution, $z_o(t)$, and $h_z(z_o(t),t,\epsilon) = = O(1)$.

We assume that $\epsilon_1 > 0$ can be found so that the following three assumptions hold.

Assumption 2

There exists a region $\Omega = \{(z,t,\epsilon): 0 \leq t \leq 1, 0 < \epsilon < \epsilon_1, |z - z_o(t)| \leq \gamma\}$, with $\gamma > 0$, in which:

(a) f and g, along with all of their first and second partial derivatives with respect to x and y, exist and are continuous functions of x, y, and t;

(b) all of the second partials of f and g with respect to x and y are O(1).

<u>Assumption</u> 3

$g_y(x_o(t),y_o(t),t,\epsilon) = -1 + O(\epsilon)$, for $0 \leq t \leq 1$, $0 < \epsilon < \epsilon_1$.

<u>Assumption</u> 4

$f_y(x_o(t),y_o(t),t,\epsilon)$ and $g_x(x_o(t),y_o(t),t,\epsilon)$ can be written in the form:

$$f_y(x_o(t),y_o(t),t,\epsilon) = f_{yo}(t,\epsilon) + O(\epsilon),$$

$$g_x(x_o(t),y_o(t),t,\epsilon) = g_{xo}(t,\epsilon) + O(\epsilon),$$

where f_{yo} and g_{xo} are differentiable functions of t for $0 \leq t \leq 1$, $0 < \epsilon < \epsilon_1$, while f'_{yo} and g'_{xo} are $O(1)$.

On the basis of the results of sections 1 and 2, the above assumptions imply that there exists $\epsilon_2 > 0$ such that the linear system

$$\Delta(\epsilon)\ z' = h_z(z_o(t),t,\epsilon)z \tag{4.3}$$

has a non-singular fundamental matrix, U(t), for $0 < \epsilon < \epsilon_2$, $0 \leq t \leq 1$. The form of this matrix is given by equation 2.1, with the obvious identification.

<u>Assumption</u> 5

There exist $\epsilon_3 > 0$ and $s \geq 0$ such that, for

$0 < \varepsilon < \varepsilon_3$, we have:

$$|b_1 + b_2 g_{xo}(1,\varepsilon)|^{-1} = O(1),$$

$$|f_{yo}(0,\varepsilon)|^{-1} = O(\varepsilon^{-s}) .$$

Assumption 6

There exists $\varepsilon_4 > 0$, such that, for $0 < \varepsilon < \varepsilon_4$, $0 \leq t \leq 1$, we have:

(a) $\beta(\varepsilon) = \beta(0) + O(\varepsilon)$.

(b) $h(z_o(t),t,\varepsilon) = h(z_o(t),t,o) + O(\varepsilon)$.

Assumption 5 should be compared with Assumption B. The outer correction, $\phi(t,\varepsilon)$, is defined by the solution $z^*(t,\varepsilon) = z_o(t) + \phi(t,\varepsilon)$ to the problem:

$$\begin{aligned} &\Delta(\varepsilon)z^{*\prime} = h(z^*,t,\varepsilon), \\ &c^t z^*(0) = c^t z_o(0),\ b^t z^*(1) = \beta(\varepsilon), \end{aligned} \tag{4.4}$$

where

$$c = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \text{ if } a_2 = 0,$$

$$c = a \text{ if } a_2 \neq 0.$$

In order to be concise, we shall not continue to explicitly denote the ε-dependence of ϕ, but shall write $\phi(t)$ for $\phi(t,\varepsilon)$. The above problem can be written in terms of ϕ as follows:

$$\Delta(\epsilon)\phi' = h_z(z_o(t),t,\epsilon)\phi + H(\phi(t),t,\epsilon),$$
$$c^t\phi(0) = 0, \; b^t\phi(1) = \beta(\epsilon) - \beta(0), \tag{4.5}$$

where

$$H(\phi(t),t,\epsilon) = h(z_o(t) + \phi(t),t,\epsilon) - $$
$$- h_z(z_o(t),t,\epsilon)\phi(t) - \Delta(\epsilon)z_o'(t).$$

We shall transform the problem 4.5 into an integral equation and apply the method of successive approximations to show that the integral equation has a solution.

On the basis of Assumptions 1 through 5, we can, for ϵ restricted to a small enough interval - say $0 < \epsilon < \epsilon_5$, apply the results of section 3 to obtain an integrating factor for the problem 4.5. Using formula 3.3, we get the following integral equation for ϕ:

$$\phi(t) = T[\phi], \tag{4.6}$$

where

$$T[\phi] = \int_o^t U(t)U^{-1}(s)\Delta(1/\epsilon)H(\phi(s),s,\epsilon)ds +$$

$$+ U(t)\begin{bmatrix} c^tU(0) \\ b^tU(1) \end{bmatrix}^{-1} \times$$

$$\times \begin{bmatrix} 0 \\ \beta(\epsilon)-\beta(0)-\int_o^1 b^tU(1)U^{-1}(s)\Delta(1/\epsilon)H(\phi(s),s,\epsilon)ds \end{bmatrix}.$$

We remark that, on the basis of Assumption 5 and formula 3.5, both of the matrices

$$\begin{bmatrix} c^t U(0) \\ b^t U(1) \end{bmatrix}, \begin{bmatrix} a^t U(0) \\ b^t U(1) \end{bmatrix}$$

are non-singular for $0 < \epsilon < \epsilon_5$.

Using formulas 2.1, 2.2, and 3.5 (with the fact that the second component of c is non-zero), we obtain, for $0 < \epsilon < \epsilon_5$, $0 \leq t \leq 1$, the following bounds:

$$\left|\int_0^t U(t)U^{-1}(s)\Delta(1/\epsilon)r(s)ds\right| \leq K_1\|r\|,$$

$$\left|U(t)\begin{bmatrix} c^t U(0) \\ b^t U(1) \end{bmatrix}^{-1}\int_0^1 b^t U(1)U^{-1}(s)\Delta(1/\epsilon)r(s)ds\right| \leq K_2\|r\|, \tag{4.7}$$

where r(t) is any measurable, bounded two-vector. Since Assumption 6(b) implies that $H(0,t,\epsilon) = O(\epsilon)$, the inequalities 4.7 may be applied to show that, for $0 \leq t \leq 1$, $0 < \epsilon < \epsilon_5$,

$$|T[H(0,t,\epsilon)]| \leq R\epsilon \ . \tag{4.8}$$

We now set

$$\epsilon_6 = \min\{\epsilon_5, \gamma/2R, [3BR(K_1+K_2)]^{-1}\},$$

and restrict ϵ to $0 < \epsilon < \epsilon_6$.

We apply the method of successive approximations to equation 4.6, setting $\phi_{-1}(t) = 0$, $\phi_{n+1}(t) = T[\phi_n]$. From the inequality 4.8, we see that, for $0 \leq t \leq 1$, $0 < \epsilon < \epsilon_6$, we have

$$|\phi_0(t)| \leq R\epsilon \leq \gamma/2,$$

so that $(z_0(t)+\phi(t),t,\epsilon)$ is in Ω for $0 \leq t \leq 1$, $0 < \epsilon < \epsilon_6$. Henceforth, for simplicity, we shall write "f is in Ω" in place of "$(f(t),t,\epsilon)$ is in Ω for $0 \leq t \leq 1$, $0 < \epsilon < \epsilon_6$".

Before we can proceed, we must have an estimate for $H(X(t),t,\epsilon) - H(Y(t),t,\epsilon)$ when X and Y are bounded, measurable vectors, with $z_0 + X$ and $z_0 + Y$ in Ω. Applying the mean-value theorem twice, we get:

$$|H(X(t),t,\epsilon) - H(Y(t),t,\epsilon)| \leq$$
$$\leq B|X(t) - Y(t)|\max[|X(t)|,|Y(t)|], \qquad (4.9)$$

where B is the constant, which exists on the basis of Assumption 2(b), bounding the second partials of f and g with respect to x and y in Ω. Then

$$|H(\phi_0(t),t,\epsilon) - H(\phi_{-1}(t),t,\epsilon)| \leq BR^2\epsilon^2,$$

and so, using equations 4.6 and 4.7, we have:

$$|\phi_1(t) - \phi_0(t)| \leq B^*R\epsilon, \quad |\phi_1(t)| \leq R\epsilon(1+B^*),$$

where we have set

$$B^* = B(K_1+K_2)R\epsilon .$$

Since

$$B^* \leq B(K_1+K_2)R\epsilon_6 \leq 1/3,$$

we have

$$1 + \sum_{0}^{\infty} 2^k (B^*)^{k+1} \leq 2,$$

so that $|\emptyset_1(t)| \leq \gamma$ for $0 < \epsilon < \epsilon_6$. This implies that $z_o + \emptyset_1$ is in Ω, so we may continue. A straightforward induction now shows that

$$|\emptyset_{k+1}(t) - \emptyset_k(t)| \leq 2^k (B^*)^{k+1} R\epsilon, \; |\emptyset_k| \leq 2R\epsilon \leq \gamma,$$

for all $k \geq 0$, $0 < \epsilon < \epsilon_6$. In exactly the same way as with the lemma of section 1, we can argue the uniform convergence of the sequence $\{\emptyset_k(t)\}$ to a limit function, $\emptyset(t)$, satisfying 4.6, with

$$|\emptyset(t)-\emptyset_k(t)| \leq (1-2B^*)^{-1} 2^k (B^*)^{k+1} R\epsilon \leq (2/3)^k R\epsilon,$$

$$|\emptyset| \leq 2R\epsilon.$$

Since all of the functions in the matrix U(t) are continuous functions of t (see formula 2.1) on the basis of Assumption 2, it is evident from equation 4.6

that $\phi_k(t)$ is continuous for $k \geq -1$, hence ϕ is continuous. In fact, it is not difficult to show, using equations 2.1 and 4.6, along with Assumptions 2 and 4, that $\phi(t)$ is differentiable on $[0,1]$ for $0 < \epsilon < \epsilon_6$.

5. THE INNER CORRECTION

If $X(t)$ is a two-vector, we define

$H^*(X,t,\epsilon) = h(z^*+X,t,\epsilon)-h(z^*,t,\epsilon)-h_z(z^*,t,\epsilon)X(t)$,

where $z^* = z_o + \phi$ is the solution to the problem 4.4 (the existence of z^* for $0 < \epsilon < \epsilon_6$ was proved in the preceding section).

The inner correction, $\psi(t,\epsilon)$, is defined by writing the solution, $z(t,\epsilon)$, to the nonlinear boundary-value problem 4.1 in the form $z = z^* + \psi$. This results in the following problem for ψ:

$$\begin{aligned} \Delta(\epsilon)\psi' &= h_z(z^*(t,\epsilon),t,\epsilon)\psi + H^*(\psi,t,\epsilon), \\ a^t\psi(0,\epsilon) &= \alpha(\epsilon) - a^t z^*(0,\epsilon), \quad b^t\psi(1,\epsilon) = 0. \end{aligned} \tag{5.1}$$

In order to be concise, we shall - as we did with the outer correction - cease referring explicitly to the ϵ-dependence of ψ, and shall write $\psi(t)$ for $\psi(t,\epsilon)$. We choose to write $\psi(t)$ in the form:

$$\psi(t) = \begin{bmatrix} \epsilon\rho(t) \\ \tau(t) \end{bmatrix},$$

and introduce the notation $\mu(\epsilon) = \alpha(\epsilon) - a^t z^*(0,\epsilon)$.

Applying the mean-value theorem, we see that

$$h_z(z^*(t),t,\epsilon) = h_z(z_o(t),t,\epsilon) + h^*(t,\epsilon),$$

where, by Assumption 2 and the fact that $\emptyset(t)$ is $O(\epsilon)$ and continuous, $h^*(t,\epsilon)$ is $O(\epsilon)$ and continuous. Taking cognizance of the remark concerning the functions $\ell(t,\epsilon)$ and $w(t,\epsilon)$ that follows formula 2.2, we see that, on the basis of Assumptions 1 through 6 and the above expansion for $h_z(z^*(t),t,\epsilon)$, the system

$$\Delta(\epsilon)z' = h_z(z^*(t),t,\epsilon)z \tag{5.2}$$

has a non-singular fundamental matrix for ϵ suitably restricted, say $0 < \epsilon < \epsilon_7$. There will be no confusion if we call this matrix $U(t)$. We point out that $U(t)$ is given by formula 2.1 with the identification:

$$\begin{bmatrix} \ell_o(t,\epsilon), p_o(t,\epsilon) \\ c_o(t,\epsilon), w_o(t,\epsilon) \end{bmatrix} = \begin{bmatrix} f_x(x_o(t),y_o(t),t,\epsilon), & f_{yo}(t,\epsilon) \\ g_{xo}(t,\epsilon), & \epsilon^{-1}[1+g_y(x^*(t),y^*(t),t,\epsilon)] \end{bmatrix},$$

where f_{yo} and g_{xo} are defined in Assumption 4.

Just as we did for the outer correction, we can use the results of section 3 to restate the problem 5.1 as an integral equation. Using formula 3.4, we get the following equation for the inner correction:

$$\psi(t) = z_1(t,\epsilon) + T[H^*(\psi,s,\epsilon)], \tag{5.3}$$

where

$$z_1(t,\epsilon) = U(t)\begin{bmatrix} a^t U(0) \\ b^t U(1) \end{bmatrix}^{-1} \begin{bmatrix} \mu(\epsilon) \\ 0 \end{bmatrix}, \tag{5.4}$$

$$T[z] = U(t)\begin{bmatrix} a^t U(0) \\ b^t U(1) \end{bmatrix}^{-1} \begin{bmatrix} \int_0^1 a^t U(0) U^{-1}(s)\Delta(1/\epsilon) z(s)ds \\ 0 \end{bmatrix} - \int_t^1 U(t)U^{-1}(s)\Delta(1/\epsilon)z(s)ds. \tag{5.5}$$

We again point out that the matrix

$$\begin{bmatrix} a^t U(0) \\ b^t U(1) \end{bmatrix}^{-1}$$

exists for $0 < \epsilon < \epsilon_7$, as a consequence of Assumption 5 and formula 3.5.

We shall attempt to solve equation 5.3 by the method of successive approximations, setting $\psi_{-1}(t) = 0$,

$$\psi_o(t) = z_1(t,\epsilon) + T[H^*(0,s,\epsilon)] = z_1(t,\epsilon),$$

and in general,

$$\psi_{k+1}(t) = z_1(t,\epsilon) + T[H^*(\psi_k,s,\epsilon)].$$

We define the number m as follows:

if $a_2 = 0$, $m = -1 - s$;

if $a_2 \neq 0$, $m = 0$;

where the number s is defined in Assumption 5(b).

Then, using formulas 2.1 and 3.5, we can show that

$$|\rho_o(t)| = \epsilon^{-1}|x_1(t,\epsilon)| \leq \epsilon^m\mu(\epsilon)Mq(b,t,\epsilon), \qquad (5.6)$$
$$|\tau_o(t)| = |y_1(t,\epsilon)| \leq \epsilon^m\mu(\epsilon)Me^{-t/\epsilon},$$

where M is a constant, and we have introduced the function:

$$q(b,t,\epsilon) = e^{-t/\epsilon} \text{ if } b_2 = 0,$$
$$q(b,t,\epsilon) = e^{-t/\epsilon} + \epsilon^{-1}e^{-1/\epsilon} \text{ if } b_2 \neq 0.$$

The difference in the bound on $\rho_o(t)$ for the cases $b_2 = 0$ and $b_2 \neq 0$ is due to the term $b^t z_\tau(1,\epsilon)$ occurring in the matrix on the right-hand side of formula 3.5.

<u>Assumption</u> 7

There exists a region

$$\Omega_o = \{(z,t,\epsilon): 0 < \epsilon < \epsilon', |x-x_o(t)| \leq 2R\epsilon + P\epsilon^{1+m}q(b,t,\epsilon),$$
$$|y-y_o(t)| \leq 2R\epsilon + P\epsilon^m e^{-t/\epsilon}, 0 \leq t \leq 1\},$$

with P > 0, in which:

(a) f, g, and all of their first and second partials with respect to x and y exist, with f, g, and their first partials continuous in x and y.

(b) The functions $\epsilon^2 f_{xx}$, $\epsilon^2 g_{xx}$, ϵf_{xy}, ϵg_{xy}, f_{yy}, g_{yy} are all $O(\epsilon^{-m})$.

We shall simply say $z^* + \psi_k$ is in Ω_o if $(z^*(t) + \psi_k(t), t, \epsilon)$ is in Ω_o for $0 \leq t \leq 1$, $0 < \epsilon < \epsilon'$, and shall write:

$$H^*(\psi, t, \epsilon) = H^*(\psi) = \begin{bmatrix} F^*(\psi) \\ G^*(\psi) \end{bmatrix}.$$

On the basis of the above assumption, if $z^* + \psi_k$ and $z^* + \psi_{k-1}$ are in Ω_o, we can apply the mean-value theorem twice to obtain:

$$|F^*(\psi_k) - F^*(\psi_{k-1})|, \ |G^*(\psi_k) - G^*(\psi_{k-1})| \leq AM(t), \quad (5.7)$$

where

$$\epsilon^m M(t) = \max\Big\{ |\bar{\rho}(t)| |\rho_1(t) - \rho_2(t)|, |\bar{\rho}(t)| |\tau_1(t) - \tau_2(t)|, \\ |\bar{\tau}(t)| |\rho_1(t) - \rho_2(t)|, |\bar{\tau}(t)| |\tau_1(t) - \tau_2(t)| \Big\},$$

with

$$|\bar{\tau}(t)| = \max(|\tau_1(t)|, |\tau_2(t)|), \ |\bar{\rho}(t)| = \max(|\rho_1(t)|, |\rho_2(t)|).$$

If we write the transformation $T[z]$ appearing in equation 5.3 in the form

$$T[z] = \begin{bmatrix} T_1[z] \\ T_2[z] \end{bmatrix},$$

then a very long calculation, using formulas 2.1, 2.2, and 3.5, shows that, for any bounded, measurable two-vector, $z(t)$, the following bounds hold for $0 \leq t \leq 1$, $0 < \epsilon < \epsilon_7$:

$$|T_1[z(s)q^2(b,s,\epsilon)]| \leq K\epsilon\|z\|q(b,t,\epsilon), \qquad (5.8)$$
$$|T_2[z(s)q^2(b,s,\epsilon)]| \leq K\|z\|e^{-t/\epsilon}.$$

We remark that K depends on b_2, being in general considerably smaller for $b_2 = 0$ than for $b_2 \neq 0$.

Assumption 8

There exists $\epsilon_8 > 0$ such that, for $0 < \epsilon < \epsilon_8$:

(a) $\mu(\epsilon)M \leq P/2$,

(b) $2\mu(\epsilon)AKM \leq 1/3$.

We set $\epsilon_o = \min[\epsilon_7, \epsilon_8, \epsilon']$ and restrict ϵ to $0 < \epsilon < \epsilon_o$. We define Ω_o' to be the region obtained from Ω_o by restricting ϵ to $0 < \epsilon < \epsilon_o$, and remark that, since Ω_o' is contained in Ω_o, the bound 5.7 is valid for $z^* + \psi_k$, $z^* + \psi_{k-1}$ in Ω_o'. We also point out that assumption 9(b) implies that, for $0 < \epsilon < \epsilon_o$:

$$\sum_{o}^{\infty} 2^n[2\mu(\epsilon)AKM]^{n+1} \leq 1.$$

Under assumption 8(a), $z^* + \psi_o$ is in Ω_o', so we may apply the bounds 5.6, 5.7, and 5.8 to obtain:

$$|\rho_1-\rho_0| \leq M^*\epsilon^m\mu(\epsilon)Mq(b,t,\epsilon),$$
$$|\tau_1-\tau_0| \leq M^*\epsilon^m\mu(\epsilon)Me^{-t/\epsilon},$$
$$|\rho_1| \leq \epsilon^m(1+M^*)\mu(\epsilon)Mq(b,t,\epsilon),$$
$$|\tau_1| \leq \epsilon^m(1+M^*)\mu(\epsilon)Me^{-t/\epsilon},$$

for $0 < \epsilon < \epsilon_0$, $0 \leq t \leq 1$, where we have written

$$M^* = 2\mu(\epsilon)AKM.$$

Assumption 8(b) implies that $z^* + \psi_1$ is in Ω_0', so we may continue. An easy induction now shows that

$$|\rho_{n+1}-\rho_n|[q(b,t,\epsilon)]^{-1},\ |\tau_{n+1}-\tau_n|e^{t/\epsilon} \leq 2^n(M^*)^{n+1}\epsilon^m\mu(\epsilon)M,$$
$$|\rho_{n+1}|[q(b,t,\epsilon)]^{-1},\ |\tau_{n+1}|e^{t/\epsilon} \leq 2\epsilon^m\mu(\epsilon)M.$$

It is now a straightforward matter to prove that

$$\psi(t) = \lim \psi_n(t) = \psi_0(t) + \sum_0^\infty [\psi_{n+1}(t)-\psi_n(t)]$$

is a solution to equation 5.3, with

$$|\rho(t)|[q(b,t,\epsilon)]^{-1},\ |\tau(t)|e^{t/\epsilon} \leq 2\epsilon^m\mu(\epsilon)M;$$
$$|\rho(t)-\rho_n(t)|[q(b,t,\epsilon)]^{-1},\ |\tau(t)-\tau_n(t)|e^{t/\epsilon} \leq \qquad (5.9)$$
$$\leq (1-2M^*)^{-1}2^n(M^*)^{n+1}\epsilon^m\mu(\epsilon)M \leq (2/3)^n\epsilon^m\mu(\epsilon)M.$$

6. CONCLUSIONS

We can summarize the results of sections 4 and 5 in the following theorem.

THEOREM 6.1

Under Assumptions 1 through 8, there exists $\epsilon_o > 0$ such that the nonlinear boundary-value problem

$$\Delta(\epsilon)z' = h(z,t,\epsilon),$$
$$a^t z(0) = \alpha(\epsilon),\ b^t z(1) = \beta(\epsilon), \tag{6.1}$$

has, for $0 < \epsilon < \epsilon_0$, a solution of the form

$$z(t) = z_o(t) + \emptyset(t,\epsilon) + \psi(t,\epsilon),$$

with

$$\emptyset(t,\epsilon) = O(\epsilon),$$

and (writing $\psi = \begin{bmatrix} \epsilon\rho \\ \tau \end{bmatrix}$)

$$\rho(t,\epsilon) = O(\epsilon^m \mu(\epsilon))q(b,t,\epsilon),\ \tau(t,\epsilon) = O(\epsilon^m \mu(\epsilon))e^{-t/\epsilon},$$

where

$$m = 0 \text{ if } a_2 \neq 0,\ m = -1 - s \text{ if } a_2 = 0,$$
$$\mu(\epsilon) = \alpha(\epsilon) - a^t[z_o(0) + \emptyset(0,\epsilon)],$$
$$q(b,t,\epsilon) = e^{-t/\epsilon} \text{ if } b_2 = 0,$$
$$q(b,t,\epsilon) = e^{-t/\epsilon} + \epsilon^{-1}e^{-1/\epsilon} \text{ if } b_2 \neq 0,$$
$$|f_{yo}(0,\epsilon)|^{-1} = O(\epsilon^{-s}).$$

Using the fact that

$$\phi(t,\epsilon) = \phi_o(t,\epsilon) + O(\epsilon^2), \quad \phi_o(t,\epsilon) = T[0],$$

where T[z] is defined by equation 4.6, we can show that

$$\phi(t,\epsilon) = I(t) + \frac{\beta(\epsilon) - \beta(0) - b^t I(1)}{b_1 + b_2 g_{xo}(1,\epsilon)} \times$$

$$\times \begin{bmatrix} L(1,t,\epsilon) \\ g_{xo}(t,\epsilon)L(1,t,\epsilon)-[\gamma+g_{xo}(0,\epsilon)]W(0,t,\epsilon)L(1,0,\epsilon)e^{-t/\epsilon} \end{bmatrix} + O(\epsilon^2),$$

where

$$I(t) = \int_o^t \Big\{ L(s,t,\epsilon)\begin{bmatrix} 1 \\ g_{xo}(t,\epsilon) \end{bmatrix}\begin{bmatrix} 1 \\ f_{yo}(t,\epsilon) \end{bmatrix}^t [h(z_o(s),s,\epsilon)-\Delta(\epsilon)z_o'(s)] +$$

$$+ \epsilon^{-1}W(s,t,\epsilon)e^{(s-t)/\epsilon}\begin{bmatrix} 0 \\ g(x_o(s),y_o(s),s,\epsilon)-\epsilon y_o'(s) \end{bmatrix} \Big\} \, ds ,$$

$$L(s,t,\epsilon) = \exp\Big[\int_s^t \{ f_x(x_o(u),y_o(u),u,\epsilon) + f_{yo}(u,\epsilon)g_{xo}(u,\epsilon)\} du \Big] ,$$

$$W(s,t,\epsilon) = \exp\Big[\epsilon^{-1}\int_s^t \{1+g_y(x_o(u),y_o(u),u,\epsilon)-\epsilon f_{yo}(u,\epsilon)g_{xo}(u,\epsilon)\} du\Big] ,$$

$\gamma = a_1/a_2$ if $a_2 \neq 0$, $\gamma = 0$ if $a_2 = 0$.

Similarly, noting that

$$\psi(t,\epsilon) = \psi_o(t,\epsilon) + \begin{bmatrix} O(\epsilon^{m+1}\mu^2(\epsilon))q(b,t,\epsilon) \\ O(\epsilon^m\mu^2(\epsilon))e^{-t/\epsilon} \end{bmatrix},$$

$$\psi_o(t,\epsilon) = z_1(t,\epsilon),$$

where $z_1(t,\epsilon)$ is defined by equation 5.4, we can show that:

$$\psi(t,\epsilon) = \frac{\mu(\epsilon)W^*(0,t,\epsilon)e^{-t/\epsilon}}{\epsilon a_1 f_{yo}(0,\epsilon)-a_2}\begin{bmatrix}\epsilon f_{yo}(t,\epsilon)\\ -1\end{bmatrix} +$$

$$+\frac{\mu(\epsilon)L(1,t,\epsilon)W^*(0,1,\epsilon)e^{-1/\epsilon}[b_2-\epsilon b_1 f_{yo}(1,\epsilon)]}{[\epsilon a_1 f_{yo}(0,\epsilon)-a_2][b_1+b_2 g_{xo}(1,\epsilon)]}\begin{bmatrix}1\\ g_{xo}(t,\epsilon)\end{bmatrix} +$$

$$+\begin{bmatrix}O(\epsilon^{m+1}\mu^2(\epsilon))q(b,t,\epsilon)\\ O(\epsilon^m\mu^2(\epsilon))e^{-t/\epsilon}\end{bmatrix},$$

where $L(s,t,\epsilon)$ is defined above and

$$W^*(s,t,\epsilon) = \exp\left[\epsilon^{-1}\int_s^t\{1+g_y(x^*(u,\epsilon),y^*(u,\epsilon),u,\epsilon) - \right.$$

$$\left. - \epsilon f_{yo}(u,\epsilon)g_{xo}(u,\epsilon)\}du\right],$$

$$z^*(t,\epsilon) = z_o(t) + \emptyset(t,\epsilon).$$

Note that the behavior of $\psi(t,\epsilon)$ is significantly different in each of the four cases described by the possibilities $a_2 = 0$, $a_2 \neq 0$; $b_2 = 0$, $b_2 \neq 0$.

Our assumptions, for the most part, concern the smoothness of f and g and the possibility of expansion of $h(z_o(t),t,\epsilon)$ about $\epsilon = 0$ to terms of order ϵ. Briefly, we assume that f and g are twice continuously differentiable with respect to x and y, that f and g and their first and second partials are $O(1)$ in a region Ω (see Assumption 2) of (z,t,ϵ)-space about the curve $\{(z_o(t),t,0): 0 \le t \le 1\}$, and that the second partials of f and g satisfy certain inequalities in a region Ω_o (see Assumption 7) of (z,t,ϵ)-space. The region Ω_o is designed to contain the inner correction, and consequently is unbounded. In addition, we require that $f_y(x_o(t),y_o(t),t,\epsilon)$ and $g_x(x_o(t),y_o(t),t,\epsilon)$ differ by at most $O(\epsilon)$ from differentiable functions with uniformly bounded derivatives, and assume that $g_y(x_o(t),y_o(t),t,\epsilon)$ can be expanded about $\epsilon = 0$ to order ϵ, with leading term -1.

Although the requirement that $g_y(x_o(t),y_o(t),t,\epsilon)$ have leading term -1 appears rather restrictive, we can reduce a system of the form of 6.1 in which

$$g_y(x_o(s),y_o(s),s,\epsilon) = k(s) + O(\epsilon)$$

to one in which $g_y(x_o(t),y_o(t),t,\epsilon)$ has leading term -1, if $k(s)$ is different from zero and of constant sign on $[0,1]$. This reduction is accomplished by the change of variable

$$t = \int_o^s k(u)du\left[\int_o^1 k(u)du\right]^{-1} \text{ if } k < 0,$$

$$t = \int_s^1 k(u)du\left[\int_o^1 k(u)du\right]^{-1} \text{ if } k > 0,$$

coupled with the replacement of ϵ by

$$\epsilon' = \epsilon\left[\int_0^1 |k(u)|du\right]^{-1} .$$

If, in addition, we assume that $k(s)$ is continuously differentiable, then the integrability and differentiability of the first and second partials of f and g are unaffected by the transformation.

Of the remaining assumptions, one is especially worth discussing. In Assumption 9, we required that the quantity

$$\mu(\epsilon) = \alpha(\epsilon) - a^t[z_o(0) + \emptyset(0,\epsilon)]$$

be sufficiently small. This means, roughly speaking, that the boundary condition which we impose at $t = 0$ ($a^t z(0) = \alpha(\epsilon)$) must be sufficiently close to that satisfied by the degenerate solution. To be more precise, the restrictions on $\mu(\epsilon)$ are of the form:

$$\begin{aligned} &(a) \quad \mu(\epsilon)M \leq P/2 \\ &(b) \quad 2\mu(\epsilon)AKM \leq 1/3, \end{aligned} \qquad (6.2)$$

where (in Assumption 7) we had required that:

$$\epsilon^2|f_{xx}|, \epsilon^2|g_{xx}|, \epsilon|f_{xy}|, \epsilon|g_{xy}|, |f_{yy}|, |g_{yy}| \leq A\epsilon^{-m}$$

in the region Ω_o. It is clear that, though we can do little about the restriction 6.2(a), we could reduce the severity of 6.2(b) by replacing the constant A by a function of ϵ, $A(\epsilon)$, and requiring that $A(\epsilon) = o(1)$. We also remark that, since $m = -1 - s$ for $a_2 = 0$, while $m = 0$ for $a_2 \neq 0$, the above bound on the second partials is much more stringent - by a factor of ϵ^{s+1} - if $a_2 = 0$.

We can specialize the problem 6.1 to the problem considered by Erdélyi (12) by taking

$$f(x,y,t,\epsilon) = y, \ a_2 = b_2 = 0, \ m = -1.$$

A point by point comparison of assumptions reveals that he has assumed all of the hypotheses of Assumptions 2 and 7 hold in a single region D, where D can be interpreted as the union of Ω and Ω_o. In D, he has assumed that

$$g_{xx} = O(1), \ g_{xy} = O(1), \ g_{yy} = O(\epsilon),$$

while we have assumed that, in Ω (a bounded region)

$$g_{xx}, \ g_{xy}, \ g_{yy} = O(1)$$

and in Ω_o (an unbounded region)

$$g_{xx} = O(\epsilon^{-1}), \ g_{xy} = O(1), \ g_{yy} = O(\epsilon).$$

On the other hand, we have assumed (Assumption 4) that $g_x(x_o(t),y_o(t),t,\epsilon)$ differs by at most $O(\epsilon)$

from a differentiable function with uniformly bounded derivative, which has no counterpart in Erdélyi's hypotheses. Since this assumption is a direct consequence of our technique for converting a linear system into an integral equation, it is apparent that our methods are not as effective as those of Erdélyi for the particular case of a second order equation with boundary conditions of the form $x(o) = \alpha(\epsilon)$, $x(1) = \beta(\epsilon)$. With the exception of these two relatively minor differences, our assumptions and results are equivalent to those of Erdélyi.

Harris (5) treats a problem of the form 6.1 with $a_2 = b_2 = 0$, assuming that f and g are analytic in x and ϵ, linear in y, and C^∞ in t. As far as they can be compared, the only important difference in our hypotheses is our replacement of linearity of f and g in y by the requirement that f_{yy} and g_{yy} be $O(\epsilon)$. Harris obtains a convergent expansion for the solution, whereas, because of our less stringent assumptions, we have obtained only the leading terms. With the exception of the formula for the second component of the outer correction, our expressions for the leading terms of the inner and outer corrections can be reduced to his.

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