I. PROPAGATING AND WAITING FRONTS IN NONLINEAR DIFFUSION

II. SUSTAINED REENTRY ROLL RESONANCE

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ABSTRACT

Part I

We examine a nonlinear diffusion equation that arises in the study of a number of physical problems, where the equation is nonlinear because the diffusion coefficient is proportional to a power of the concentration. Previous authors have proven, using similarity solutions, that this dependence produces fronts (interfaces between regions of zero and nonzero concentration) which propagate with finite speed, as well as waiting-time behavior, where the fronts remain stationary for a finite amount of time before beginning to move. These similarity solutions provide limited information about the solution for general initial conditions, however.

To alleviate this deficiency, we construct approximate solutions for the above nonlinear diffusion equation using singular perturbation theory. We do so by considering the equation in the limit of nearly linear diffusion, but the analysis reveals the basic qualitative behavior outside this limit as well.

The basic behavior follows from the leading-order approximation of a transformed equation, and propagating and waiting fronts are due to the formation (in this approximation) of what we call corner shocks. This enables us to determine for which initial conditions waiting-time behavior will occur.

The transformed equation must be solved to first order to find the solution of the original equation to leading order, and when

corner shocks occur at a point of nonzero concentration, this first-order analysis shows that they become rounded (which we call a corner layer). When a corner shock occurs at a point of zero concentration, this rounding does not take place, and the corner shock remains sharp. This allows us to give a simple procedure for constructing approximate solutions of the nonlinear diffusion equation when corner shocks occur only at points of zero concentration.

Part II

We study a model of reentry roll resonance, a situation encountered when an almost axially symmetric vehicle reenters the earth's atmosphere, using the method of near identity transformations. This method allows us to extend previous results for the case of sustained resonance, when roll buildup occurs.

In particular, we give necessary conditions both for entrainment to sustained resonance and for sustained resonance to continue. These conditions imply that it is possible for sustained resonance to last for a finite time and then for unlocking of the resonance to occur. In addition, from the analysis we make a conjecture concerning sufficient conditions for sustained resonance.

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SUSTAINED REENTRY ROLL RESONANCE

PART I

PROPAGATING AND WAITING FRONTS IN NONLINEAR DIFFUSION

1. INTRODUCTION

1.1 Derivation of the Equations

We shall study the nonlinear diffusion equation

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(u^n \frac{\partial u}{\partial x}\right) ,$$

which arises naturally in a number of physical problems. For example, consider the flow of moisture in an unsaturated porous medium (see Philip [20]). Take θ to be the volumetric moisture content, the average fraction of water in a small region (but large compared to the pore size), and define $\underline{U}$ to be the average velocity. The equation for the conservation of mass is then

$$\frac{\partial \theta}{\partial t} + \nabla \cdot \underline{U} = 0 . \tag{1.1}$$

An experimental relation known as Darcy's law is used to complete the system of equations for θ and $\underline{U}$. Darcy's law states that

$$\underline{U} = -K(\theta)\nabla \Phi , \tag{1.2}$$

where K is the hydraulic conductivity and Φ is the total potential. In addition, Φ is assumed to have the form

$$\Phi(\theta) = \Psi(\theta) + z \ , \tag{1.3}$$

where Ψ is the moisture potential and z is the scaled gravitational potential.

With these assumptions $\underline{U}$ can be eliminated, and we find that θ satisfies

$$\frac{\partial \theta}{\partial t} = \nabla \cdot (D(\theta) \nabla \theta) + \frac{dK}{d\theta} \frac{\partial \theta}{\partial z} \ , \tag{1.4}$$

where $D(\theta) \equiv K(\theta)\, d\Psi/d\theta$ is the moisture diffusivity. Equation (1.4) is a diffusion equation in which the diffusion coefficient depends nonlinearly upon the concentration (there is also a convection term due to gravity). An example of the dependence of D upon θ for a particular soil is given in Fig. 1.1.

Our interest in the study of equations such as (1.4) was prompted by reports of an unusual effect in physical processes for which an equation of the form (1.4) is a good approximation. One such instance is the study of the spread of water in columns of sand [6]. Water is introduced into a container of sand (a relatively common porous medium), and the subsequent behavior of the distribution of water is observed; with care this can be done reproducibly.

If the water is introduced correctly, then it is found that the boundary between wet and dry sand does not begin to move immediately: the interface waits for a finite time before it begins

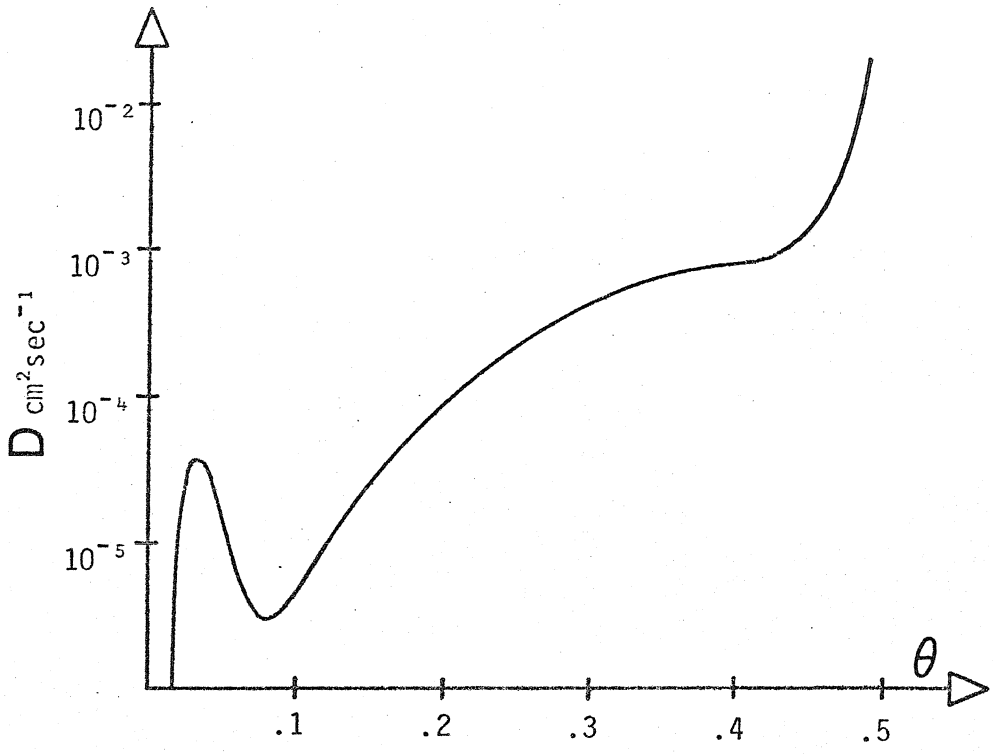

Figure 1.1. Relationship between D and θ for Yolo light clay [20].

to move. In certain instances this waiting time may be quite long, as long as 30 minutes to an hour. The water is not necessarily stationary during this time, however. The interface remains stationary, but the water can move behind the interface, redistributing itself.

It is possible that important physical effects have been neglected from the model (1.4), such as surface tension, absorption by particles, etc. Nevertheless, we shall show that equations of the form (1.4) do possess the waiting time solutions discussed above.

A complete discussion of (1.4) would be difficult, and therefore we shall make a number of simplifying assumptions: we shall study the flow in only one dimension, neglect the effects due to gravity, and assume a special form for the diffusion coefficient,

$$D(\theta) \equiv \theta^n , \qquad n > 0 , \tag{1.5}$$

so that $D(0) = 0$. This zero of the diffusion coefficient is the important factor for waiting-time solutions, and results in the case known as slow diffusion.

With these assumptions (1.4) becomes

$$\frac{\partial \theta}{\partial t} = \frac{\partial}{\partial x}\left(\theta^n \frac{\partial \theta}{\partial x}\right) . \tag{1.6}$$

This equation has been called the porous medium equation, although it clearly cannot accurately describe all aspects of flow in porous media.

An analysis of equation (1.6) is also of interest because it arises in the study of a number of other physical problems:

1. Thin saturated regions in porous media, where $n = 1$ [21].
2. Percolation of gas through porous media, where $n \geq 1$ [18].
3. Thin liquid films spreading under gravity, where $n = 3$ [7].
4. Radiative heat transfer by Marshak waves, where $n = 6$ [14].

The first three examples are also discussed by Lacey, Ockendon and Taylor [15]; the last is also discussed by Zel'dovich and Raizer [23].

1.2 Discussion of Previous Results

It is possible to regard any of the above physical problems as the particular one of interest. Rather than singling out any one example, however, we shall study the equation

$$u_{t^*} = \left(u^n u_x\right)_x , \tag{1.7}$$

where u is simply considered to be a concentration. Therefore, we shall think of (1.7) as a nonlinear diffusion equation, where the nonlinearity arises because the diffusion coefficient depends upon the concentration through the relation $D(u) = u^n$.

Existence and uniqueness of weak solutions to the above equation have been proved by Oleinik, et al., [19]. Their proof involves the use of a maximum principle which has been valuable for obtaining other results for (1.7). The maximum principle states that if u_1 and u_2 are two solutions to (1.7) for which $u_1 \le u_2$ at $t^* = 0$ (and for all x), then $u_1 \le u_2$ for all $t^* > 0$ as long as both solutions continue to exist.

It is possible to determine some exact solutions of (1.7) by the use of similarity solutions; one example is (see Fig. 1.2)

$$u = \begin{cases} t^{* \frac{-1}{2+n}} \left[a - \frac{n}{4+2n} \frac{x^2}{t^{* \frac{2}{2+n}}} \right]^{\frac{1}{n}} & \text{for } [\,] > 0 , \\ 0 & \text{otherwise.} \end{cases} \tag{1.8}$$

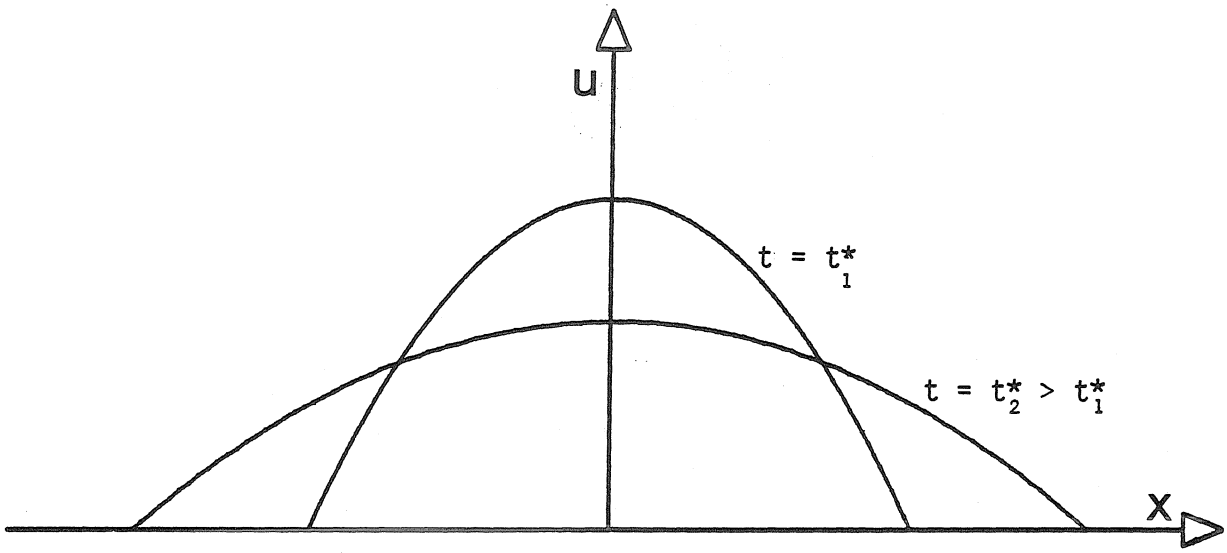

Figure 1.2. The similarity solution (1.8).

Notice from (1.8) that u becomes zero at a finite distance away from the origin for all $t^* > 0$; thus, there is an interface separating the region where $u = 0$ from the region where $u > 0$. This behavior is qualitatively different from that of linear diffusion, where there are no such interfaces. (Note also that the use of the maximum principle and the solution (1.8) shows that if any solution is zero outside a finite interval at $t^* = 0$, then it will possess these interfaces for all $t^* \geq 0$. Therefore, this behavior is not limited to the similarity solution.)

Knerr [13] has examined the behavior near the interface and has shown by careful estimation that the position of the interface $x = \zeta(t^*)$ satisfies

$$\frac{d\zeta}{dt^*} = -\frac{1}{n} \lim_{x \to \zeta} \frac{\partial}{\partial x} (u^n) , \qquad (1.9)$$

where the limit is from the side on which $u > 0$. This interface equation immediately gives the instantaneous velocity of the propagating front that separates the region where $u = 0$ from the region where $u > 0$.

In addition to (1.8), another similarity solution of interest is (see Fig. 1.3)

$$u = \begin{cases} \left[\frac{n}{4+2n} \frac{x^2}{(t_0^* - t^*)} \right]^{\frac{1}{n}} & \text{for } x \geq 0 , \\ 0 & \text{for } x < 0 . \end{cases} \tag{1.10}$$

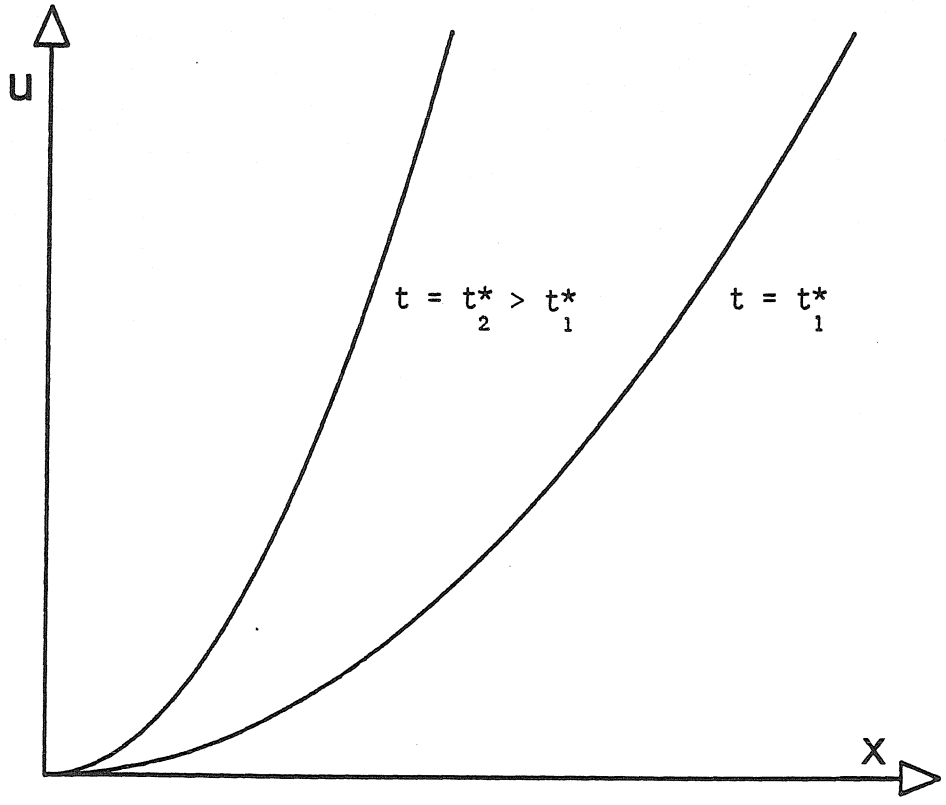

Figure 1.3. The similarity solution (1.10).

For this solution application of (1.9) gives $d\zeta/dt^* = 0$ for $t^* < t_0^*$, and this means that the interface remains stationary, at least until $t^* = t_0^*$. When combined with this similarity solution, the maximum principle can be used to prove the existence of waiting-time solutions to (1.7) (see [1], [12], and [13]; for related results see [3] and [15]). The basic idea is to bound the initial data for u by the similarity solution (1.10) at $t^* = 0$ by choosing t_0^* appropriately, so that the similarity solution bounds the solution u for $0 \leq t^* < t_0^*$ by the maximum principle. Since the interface of the similarity solution remains stationary during this time interval, the same must be true for the solution u.

1.3 Discussion of Present Results

The similarity solutions presented above show simply the qualitative behavior of solutions of (1.7), and also enable bounds on the waiting time to be determined. They do not give complete information about the solutions, however. For example, the similarity solution (1.10) ceases to exist at $t^* = t_0^*$. This means that even though this similarity solution shows that the interface does not move for $0 \leq t^* < t_0^*$ (for correct initial conditions), it gives no information about the interface for $t^* \geq t_0^*$. Also, we would like to obtain solutions for initial conditions which cannot be solved by similarity methods (although we can bound the true solution by using these similarity solutions, this bound may not give useful quantitative information).

In order to obtain information about solutions in the above cases, we shall use singular perturbation theory to construct

approximate solutions of (1.7) which are valid for small time, for all x, and for general initial conditions not solvable by similarity techniques. These formal perturbation methods have been used with much success to obtain approximate solutions and reveal the underlying structure of many problems.

We shall construct these approximate solutions of equation (1.7) by considering it in the limit $n \to 0$ (examining this limit enables us to determine the behavior of the solutions of (1.7) which is true for any $n > 0$). A leading-order approximation for u $\left(\text{error } 0(n)\right)$ will then be shown to follow from the first-order approximation $\left(\text{error } 0(n^2)\right)$ to the solution of the transformed equation

$$v_t = v_x^2 + n v v_{xx} ,$$

where $t^* = nt$ and $v = u^n$ (this equation has various physical interpretations, depending on the problem being considered).

The leading-order approximation for v will be shown to contain the basic behavior discussed above, including the finite speed of propagation of interfaces and the presence of waiting-time solutions. We shall also demonstrate that the waiting time is related to the time of formation of corner shocks (a corner shock is a discontinuous change in the first derivative v_x).

The limit $n \to 0$, the reasons for studying the transformed equation, and the information contained in its leading-order approximation will be discussed in Chapter 2.

To find the first-order approximation for v (which will be necessary to obtain the leading-order approximation for u), the region near the corner shock will be examined more carefully using a more sophisticated perturbation expansion, and we shall see that there are two cases.

If $v \neq 0$ at the corner shock, then we shall show that the corner shock is rounded in the first-order approximation, so that it becomes a corner layer, and shall give the structure of this layer. In addition, we shall show that when the corner layer is not present initially, the solution must be obtained by matching a transition solution valid near the time of shock formation with a regular expansion valid for earlier times, and with a corner layer expansion valid for later times. We shall not perform this matching, however. This analysis of the first-order approximation when no corners occur at $v = 0$ constitutes Chapter 3.

If $v = 0$ at the corner shock (as is the case at a moving interface), then we shall demonstrate that to the order of our expansion the corner is not rounded, but remains sharp. Moreover, in this case only minor modifications will be needed to obtain valid first-order approximations before, during, and after the time of corner shock formation. Therefore, we shall be able to give a simple procedure for constructing approximate solutions when corner shocks occur only at $v = 0$. We shall present a number of examples using this method, and also perform calculations to check the consistency of our expansion; it will be shown to be valid as long as the scaled time, t, does not become too large. This second case, where $v = 0$ at the corner shock, will comprise Chapter 4.

2. THE LEADING-ORDER APPROXIMATION FOR THE TRANSFORMED EQUATION

2.1 Introduction

Although using singular perturbation theory for (1.7) requires that several different kinds of behavior be analyzed, much of the basic structure can be discovered by relatively simple approximations. In order to clearly show this structure, we postpone a discussion of the more complicated methods that are used in this problem; here we restrict our attention to a discussion of the singular limit we consider for (1.7) and the basic behavior a leading-order approximation reveals.

2.2 The Singular Limit

Recall that for our choice of diffusion coefficient we have the nonlinear equation

$$u_{t^*} = \left(u^n u_x\right)_x . \tag{2.1}$$

We consider only the case $n > 0$; this is the case known as slow diffusion. When $n = 0$ (2.1) reduces to the linear diffusion equation.

To discover the difference between classical linear diffusion and the situation when $n > 0$, we shall take

$$n = \epsilon , \tag{2.2}$$

where $0 < \epsilon << 1$. Although this is unphysical for many problems, it does reveal the basic behavior of solutions to (2.1) true for any $n > 0$. With this assumption (2.1) becomes

$$u_{t^*} = \left(u^{\epsilon} u_x\right)_x . \tag{2.3}$$

Thus the diffusion coefficient, which is given by $D(u) = u^{\epsilon}$, is near 1 for u away from zero, but drops to zero quickly when u goes to zero.

The singular nature of this equation as $\epsilon \to 0$ is immediately evident; if we expand $u(x, t^*; \epsilon)$ in a series in ϵ,

$$u(x, t^*; \epsilon) = u_0(x, t^*) + \epsilon u_1(x, t^*) + \epsilon^2 u_2(x, t^*) + \cdots, \tag{2.4}$$

and also expand

$$u^{\epsilon} = 1 + \epsilon \ell n u + \frac{1}{2} \epsilon^2 \ell n u + \cdots, \tag{2.5}$$

we obtain the sequence of equations,

$$u_{0t^*} = u_{0xx} , \tag{2.6}$$

$$u_{1t^*} = u_{1xx} + (\ell n\, u_0 u_{0x})_x , \tag{2.7}$$

$$\vdots$$

This expansion is valid as long as u_0 remains O(1), but due to the presence of $\ell n\, u_0$ breaks down if u_0 goes to zero.

The trouble comes from the expansion of u^ϵ, (2.5). It breaks down when u goes to zero, and in particular when u is of order $e^{-1/\epsilon}$.

To eliminate the difficulty it is best to eliminate the occurrence of the power ϵ in the equation, which suggests the transformation

$$v = u^\epsilon \ . \tag{2.8}$$

Substituting this into (2.3) we have

$$v_{t^*} = \frac{1}{\epsilon} v_x^2 + v v_{xx} \ . \tag{2.9}$$

We also scale the time by taking

$$t^* = \epsilon t \ , \tag{2.10}$$

so that the resulting equation for $v(x, t; \epsilon)$ is

$$v_t = v_x^2 + \epsilon v v_{xx} \ . \tag{2.11}$$

This equation, in which no approximations have been made, has been used to prove many results for equation (2.1) (see Knerr [13], for example). It is also important for determining the structure of the solutions, as we shall see.

We now have an equation for which an expansion will be valid when u is O(1), (2.3), and an equation for which an expansion will be valid when u is small, (2.11). It would seem possible to combine these two expansions to obtain a uniformly valid approximation using the method of matched asymptotic expansions. This is extremely difficult, however.

Normally the method is used to match two expansions which are asymptotically valid for overlapping regions of the independent variable (as in boundary-layer theory); here we wish to match two expansions which are valid for overlapping regions of the dependent variable. This means the matching region will not be fixed in the x-t plane, but will rather depend upon the solution. Therefore, we will have a problem analagous to one in which there is a boundary condition on a free surface, since the matching can be thought of as supplying needed information from other regions in a form similar to a boundary condition. This variety of nonlinearity adds greatly to the complexity of the problem.

The difficulty can be avoided by more accurately approximating the solutions to (2.11). This is reminiscent of an alternate method used in boundary-layer theory, where rather than matching an inner expansion (near the boundary) to an outer expansion, the exact inner equation is more accurately approximated using another singular perturbation technique, the method of multiple scales. The resulting expansion of the inner equation is found to include the outer approximation (see Bender and Orszag [4], p. 559).

In our case we solve for v using a perturbation expansion as $\epsilon \to 0$, such as

$$v(x, t; \epsilon) \sim v_0(x, t) + \epsilon v_1(x, t) + \epsilon^2 v_2(x, t) + \cdots, \tag{2.12}$$

and transform back to find u,

$$u = v^{\frac{1}{\epsilon}} \sim (v_0 + \epsilon v_1 + \epsilon^2 v_2 + \cdots)^{\frac{1}{\epsilon}} . \tag{2.13}$$

For $v_0 = O(1)$ and ϵ small, we can approximate this as

$$u \sim v_0^{\frac{1}{\epsilon}} e^{\frac{v_1}{v_0}} \left[1 + \epsilon \left(\frac{v_2}{v_0} - \frac{1}{2} \frac{v_1^2}{v_0^2} \right) + \cdots \right] . \tag{2.14}$$

We find u accurate to O(1) by calculating v accurately to $O(\epsilon)$. This avoids the problem of matching the two expansions, but we pay the price of having to calculate an extra order in the expansion for v to get the desired accuracy for u; this is due to the nature of the transformation (2.8).

In addition, the initial condition for u is also affected by the transformation. For now we ignore any effect this produces and use the initial condition

$$v(x, 0; \epsilon) = f(x). \tag{2.15}$$

2.3 The Leading-Order Approximation

Thus, we consider

$$v_t = v_x^2 + \epsilon v v_{xx} \tag{2.16}$$

with initial condition

$$v(x, 0; \epsilon) = f(x) , \tag{2.17}$$

for $0 < \epsilon << 1$. Once we have the solution for v to $O(\epsilon)$, we calculate u to O(1) by using

$$u(x, t^*; \epsilon) = \left[v\left(x, \frac{t^*}{\epsilon}; \epsilon\right) \right]^{\frac{1}{\epsilon}} . \tag{2.18}$$

To study the behavior of solutions to (2.16) for small ϵ, we make the assumption that vv_{xx} remains O(1) everywhere. For a leading-order approximation we therefore neglect this term and study the solutions of

$$v_t = v_x^2 . \tag{2.19}$$

This equation is solved by the standard theory of first-order nonlinear partial differential equations ([10] or [22], p. 65) as follows. Write (2.19) as

$$H(x, t, v, p, q) \equiv q - p^2 = 0 ,$$

where $v_x \equiv p$, $v_t \equiv q$. Then solving (2.19) is equivalent to solving the system of ordinary differential equations,

$$\frac{dx}{ds} = H_p , \qquad \frac{dt}{ds} = H_q , \qquad \frac{dv}{ds} = pH_p + qH_q ,$$

$$\frac{dp}{ds} = -H_x - pH_v , \qquad \frac{dq}{ds} = -H_t - qH_v .$$

We find that the solution of (2.19) subject to (2.17) can be written implicitly as

$$v(\xi, t) = f(\xi) - f'^2(\xi)t , \tag{2.20}$$

$$x(\xi, t) = \xi - 2f'(\xi)t . \tag{2.21}$$

Elimination of ξ between (2.20) and (2.21) gives $v(x, t)$.

In Fig. 2.1 the leading-order solution for v is plotted, where we take

$$f(\xi) = \begin{cases} \dfrac{1 - \cos\pi\xi}{\pi^2} & \text{for } |\xi| < 2 , \\ \\ 0 & \text{for } |\xi| \geq 2 . \end{cases} \tag{2.22}$$

The solution is smooth and single-valued until $t = 0.5$, after which it becomes multivalued and, even worse, negative. This is impossible since it is known that $u \geq 0$ for all t if $u \geq 0$ at $t = 0$ [3].

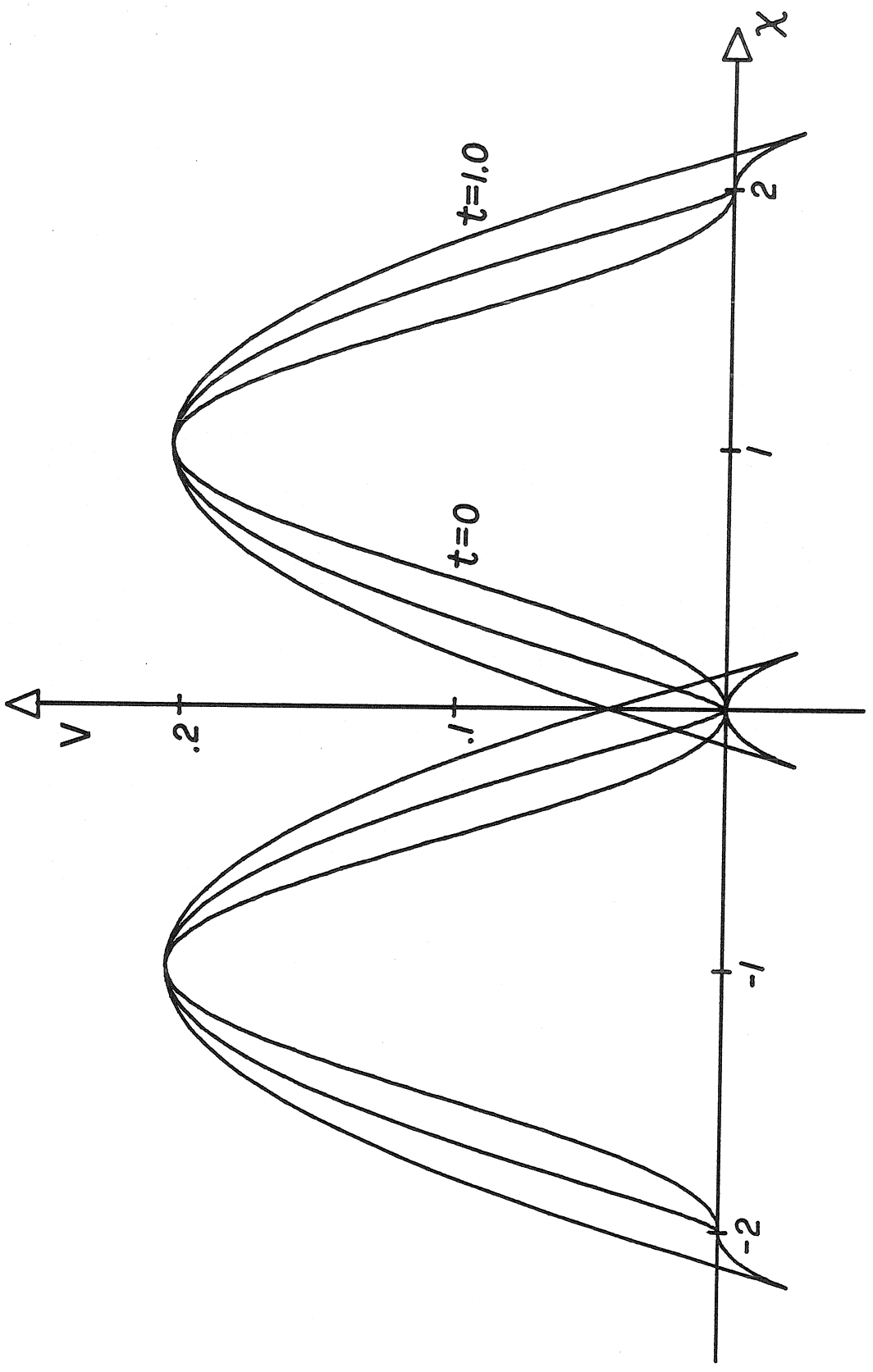

Figure 2.1. Solution of the leading-order equation for v, (2.19).

These defects are clearly due to the approximation we have made in (2.19). Although a complete remedy involves a detailed perturbation expansion (which is given in Chapter 3), to leading order the difficulties can be corrected by suitable placement of corner shocks in the solution. (We use the term corner shock to indicate a discontinuous jump in the first derivative.)

These corner shocks are used at any position where, according to the implicit solution (2.20) and (2.21),

$$v(\xi_1, t) = v(\xi_2, t) \tag{2.23}$$

and

$$x(\xi_1, t) = x(\xi_2, t) \tag{2.24}$$

but

$$v_x(\xi_1, t) \neq v_x(\xi_2, t) \ . \tag{2.25}$$

We see that with corner shocks included Fig. 2.1 becomes Fig. 2.2.

When ϵ is small but not zero we shall show that the corner shocks do not stay sharp. At such a shock $v_{xx} = \infty$, and the $\epsilon v v_{xx}$ term is not expected to remain small; the effect is to change the corner shock into a smooth corner layer of width $O(\epsilon)$ about the shock position. The solution is only modified to $O(\epsilon)$, however, because v and v_{xx} are O(1) away from the shock, and because the

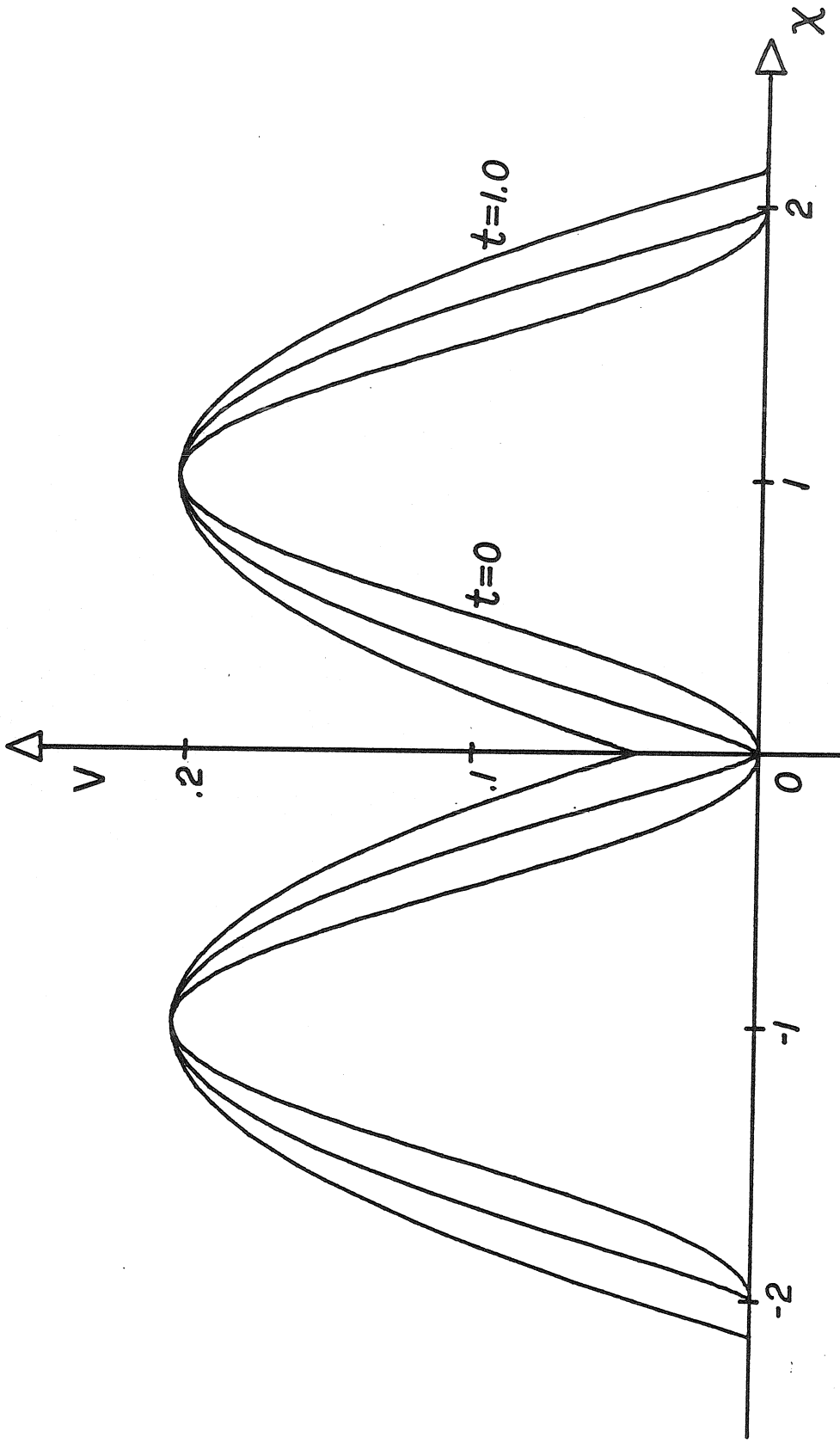

Figure 2.2. The leading-order approximation for v (with corner shocks).

layer is too narrow to cause any larger change. The effects of including the $\epsilon v v_{xx}$ term are discussed in detail in Chapter 3.

Note that the leading-order approximation for v shown in Fig. 2.2 does indeed possess interfaces separating the region where $v > 0$ from the regions where $v = 0$, and these interfaces propagate with finite speed, as mentioned in Chapter 1.

The conditions (2.23) and (2.24) are perhaps more familiar if we consider the equation for the derivative, $v_x \equiv w$:

$$w_t = 2ww_x \ . \tag{2.26}$$

This is an example of the first-order nonlinear wave equation discussed by Whitham [22, p. 19]. An illustrative initial condition and its subsequent behavior is shown in Fig. 2.3.

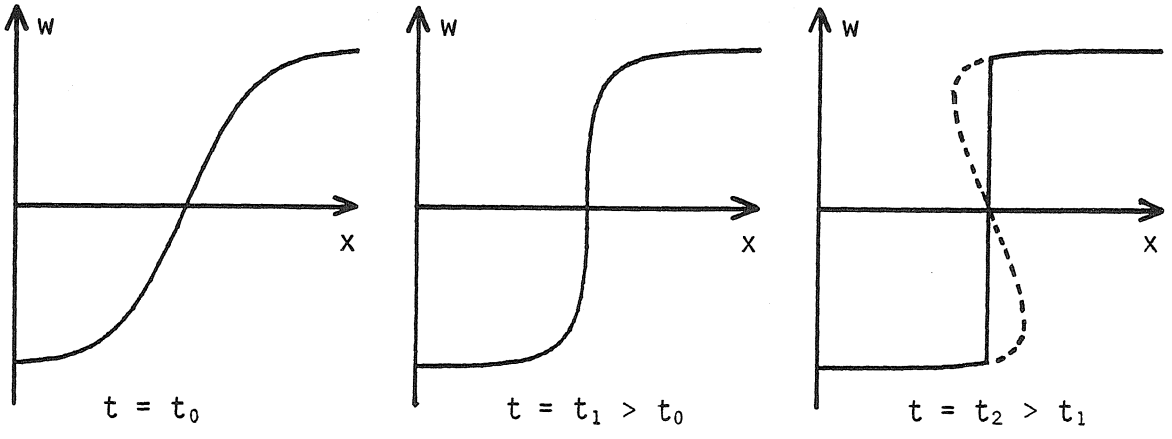

Figure 2.3. Formation of a shock in the derivative $v_x \equiv w$.

Shocks are fitted for (2.26) according to an equal area rule (dotted lines in Fig. 2.3 show parts of the implicit solution that are discarded), namely

$$\frac{1}{2}\left[F(\xi_1) + F(\xi_2)\right](\xi_1 - \xi_2) = \int_{\xi_2}^{\xi_1} F(\xi)\,d\xi, \tag{2.27}$$

where $w(x, 0) = F(x)$. In this case $F(\xi) = f'(\xi)$ and the integral can be evaluated:

$$\frac{1}{2}\left[f'(\xi_1) + f'(\xi_2)\right](\xi_1 - \xi_2) = f(\xi_1) - f(\xi_2)\ . \tag{2.28}$$

When this equation is combined with the characteristic condition, (2.24), the first requirement, (2.23), is obtained. The behavior of v is thus given by the integral of w with respect to x (Fig. 2.4).

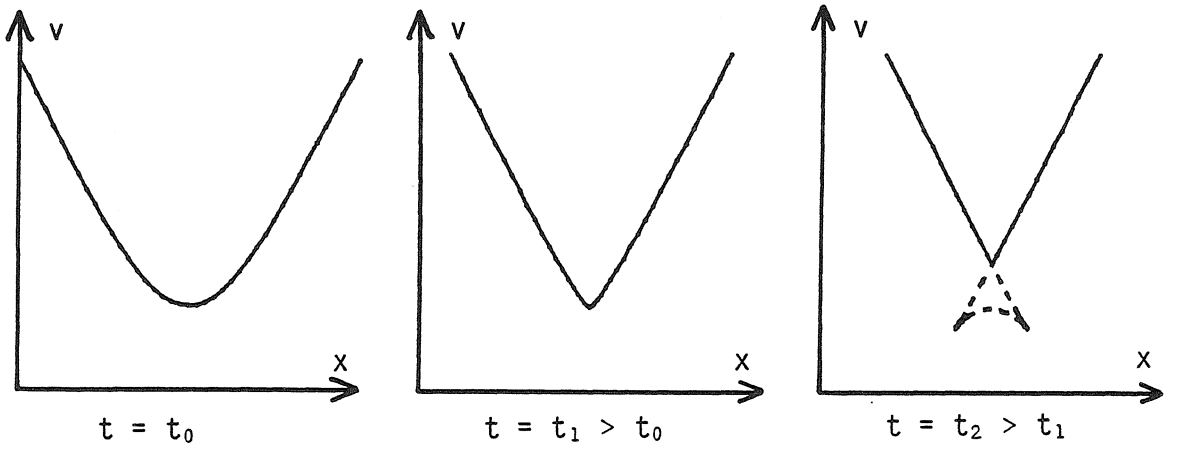

Figure 2.4. Formation of a corner shock.

The theory also gives the time of corner shock formation. It is the minimum time at which characteristics intersect, which satisfies

$$\frac{\partial x}{\partial \xi} = 1 - 2f''(\xi)t = 0 \ . \tag{2.29}$$

Any positive local maximum of $f''(x)$, provided it is separated sufficiently from other maxima, will generate a corner shock, and it will form at the time

$$t_s = \frac{1}{2[f''(\xi)]} \tag{2.30}$$

(here $f''(\xi)$ is evaluated at its local maximum). For the example shown in Fig. 2.2 corner shocks form at $t = 0.5$, which is the middle plot.

The shock-fitting conditions (2.23) and (2.24) cannot be solved explicitly to give the position of the corner shock, but its speed can be determined. Letting $\zeta(t)$ be the position of the shock and $v_s(t)$ be $v(\zeta(t), t)$, we have from (2.23) and (2.24)

$$\zeta = \xi_1 - 2f'(\xi_1)t = \xi_2 - 2f'(\xi_2)t \ , \tag{2.31}$$

$$v_s = f(\xi_1) - f'^2(\xi_1)t = f(\xi_2) - f'^2(\xi_2)t \ . \tag{2.32}$$

Thus

$$\zeta' = \left(1 - 2f''(\xi_i)t\right) \xi_i' - 2f'(\xi_i) , \tag{2.33}$$

$$v_s' = \left(1 - 2f''(\xi_i)t\right) f'(\xi_i)\xi_i' - f'^2(\xi_i) , \tag{2.34}$$

for i = 1, 2). Eliminating ξ_1' and ξ_2' from the above equations, we find

$$\zeta'(t) = -\left[f'(\xi_1) + f'(\xi_2)\right] , \tag{2.35}$$

$$v_s'(t) = -f'(\xi_1) f'(\xi_2) . \tag{2.36}$$

Thus, the speed of the corner shock depends only on the values of v_x on each side, since $v_x = f'(\xi)$.

Particularly interesting is the case when $f(\xi) \equiv 0$ on one side of the shock (the left, for example). Then the above equations become

$$\begin{aligned} \zeta'(t) &= -f'(\xi_1) , \\ v_s'(t) &= 0 . \end{aligned} \tag{2.37}$$

Equation (2.37) can also be written

$$\zeta'(t) + v_x\left(\zeta(t)^+, t\right) = 0 , \tag{2.38}$$

which is the interface equation (1.9). Since the interface equation

is true for any value of ϵ, this suggests that the modification due to the $\epsilon v v_{xx}$ term is negligible when the corner shock occurs at $v = 0$.

2.4 Wavefront Expansion Near a Corner Shock

To check the above conjecture, we examine (2.16) in the neighborhood of a corner shock by using a wavefront expansion [22, p.130]. We assume

$$v = v_0(t) + \left[v_1^-(t)\eta + \left\{ v_1^+(t) - v_1^-(t) \right\} H_1(\eta) \right]$$

$$+ \left[v_2^-(t) \cdot \frac{1}{2} \eta^2 + \left\{ v_2^+(t) - v_2^-(t) \right\} H_2(\eta) \right] + \cdots , \tag{2.39}$$

where $H_n(\eta)$ are generalizations of the standard Heaviside function

$$H_n(\eta) = \begin{cases} \frac{1}{n!} \eta^n & \text{for} \quad \eta \geq 0 , \\ 0 & \text{for} \quad \eta < 0 , \end{cases} \tag{2.40}$$

and where we take $\eta = x - \zeta(t)$ to measure the distance from the corner shock. If derivatives are taken in the generalized sense, we can also define $H_n(\eta)$ for negative n:

$$H_{-1}(\eta) = H_0'(\eta) = \delta(\eta) , \quad \text{etc.} \tag{2.41}$$

With these definitions, and writing Δv_i for $v_i^+ - v_i^-$, we calculate

$$v_t = v_0' - \zeta' \left[v_1^- + \Delta v_1 H_0 \right]$$

$$+ \left[\left(v_1^{-\prime} - \zeta' v_2^- \right) \eta + \left(\Delta v_1' - \zeta' \Delta v_2 \right) H_1 \right] + \cdots , \qquad (2.42)$$

$$v_x = v_1^- + \Delta v_1 H_0 + \left[v_2^- \eta + \Delta v_2 H_1 \right] + \cdots , \qquad (2.43)$$

$$v_{xx} = \Delta v_1 H_{-1} + \left[v_2^- + \Delta v_2 H_0 \right] + \left[v_3^- \eta + \Delta v_3 H_1 \right] + \cdots . \qquad (2.44)$$

In this expansion v is continuous, but its derivatives are discontinuous, as is true at a corner shock.

We substitute (2.42) - (2.44) into (2.16) and equate like powers of η on each side of the corner shock; equivalently, we equate like powers of η and also require the coefficients of the H_n to be equal. We find

$$H_{-1} : \qquad \epsilon v_0 \Delta v_1 = 0 , \qquad (2.45)$$

$$\eta^0 : \qquad v_0' - \zeta' v_1^- = \epsilon v_0 v_2^- + v_1^{-2} , \qquad (2.46)$$

$$H_0 : \qquad -\zeta' \Delta v_1 = \epsilon v_0 \Delta v_2 + \Delta (v_1^2) , \qquad (2.47)$$

and so on, where $\Delta(v_1^2) = v_1^{+2} - v_1^{-2}$.

The crucial equation is (2.45), the coefficient of H_{-1}, which comes from the $\epsilon v v_{xx}$ term. If $\epsilon \neq 0$ and $v_0 \neq 0$ then (2.45) gives $\Delta v_1 = 0$, so that no discontinuous jump in the derivatives of v is possible, and any corners that are present are smooth. For a

sharp corner (a corner shock) we must have $\Delta v_1 \neq 0$.

If $\epsilon = 0$ then (2.46) and (2.47) reduce to

$$v_0' = -v_1^+ v_1^- ,$$

$$\zeta' = -\left(v_1^+ + v_1^-\right) ,$$

which are merely (2.36) and (2.35) rederived.

If $\epsilon \neq 0$, however, and $v_0 = 0$ (which means either $v_1^+ = 0$ or $v_1^- = 0$), then

$$\Delta v_1 \neq 0$$

and either

$$\zeta' = -v_1^+ \quad \text{or} \quad \zeta' = -v_1^- \quad .$$

Either of these last two equations is again the interface equation, (1.9).

Since $\Delta v_1 \neq 0$, this means the corner is sharp when it occurs at $v = 0$. Apparently the zero of v cancels any effect due to v_{xx} being large, rendering the $\epsilon v v_{xx}$ term unimportant in the determination of the corner shock structure (even when ϵ is not small). This result will be examined in greater detail in Chapter 4.

2.5 Waiting-Time Solutions

The above discussion suggests that we should examine the region near $v = 0$ more closely. Fig. 2.5 is a greatly expanded view of Fig. 2.2 in the region near $x = -2$ for equally spaced times between 0.495 and 0.505.

For $t > 0.5$ there is a corner shock which has $v > 0$ to the right, and which propagates into the region where $v = 0$ as expected.

The behavior is different, however, for $t < 0.5$; in this case $v_x = 0$ at the interface, so that by (2.38) $\zeta' = 0$ and the interface remains stationary. The solution rearranges itself behind the motionless front, and at the time of shock formation, $t = 0.5$, has developed sufficiently to start the interface moving (note that the shock forms at the interface, where $v = 0$). This is an example of the waiting-time solutions discussed in Chapter 1, solutions with interfaces which wait a finite time before beginning to move.

There are other ways in which the interface can begin to move, and these are shown in Fig. 2.6. (We take

$$f(\xi) = \begin{cases} \frac{1}{12}(4-\xi)\xi^3 & \text{for } 0 \le \xi \le 4 , \\ \\ 0 & \text{otherwise.} \end{cases}$$

The interface on the right has $v_x < 0$ at $t = 0$, which implies $\zeta'(0) > 0$. It begins to move immediately, and the waiting time for this interface is zero.

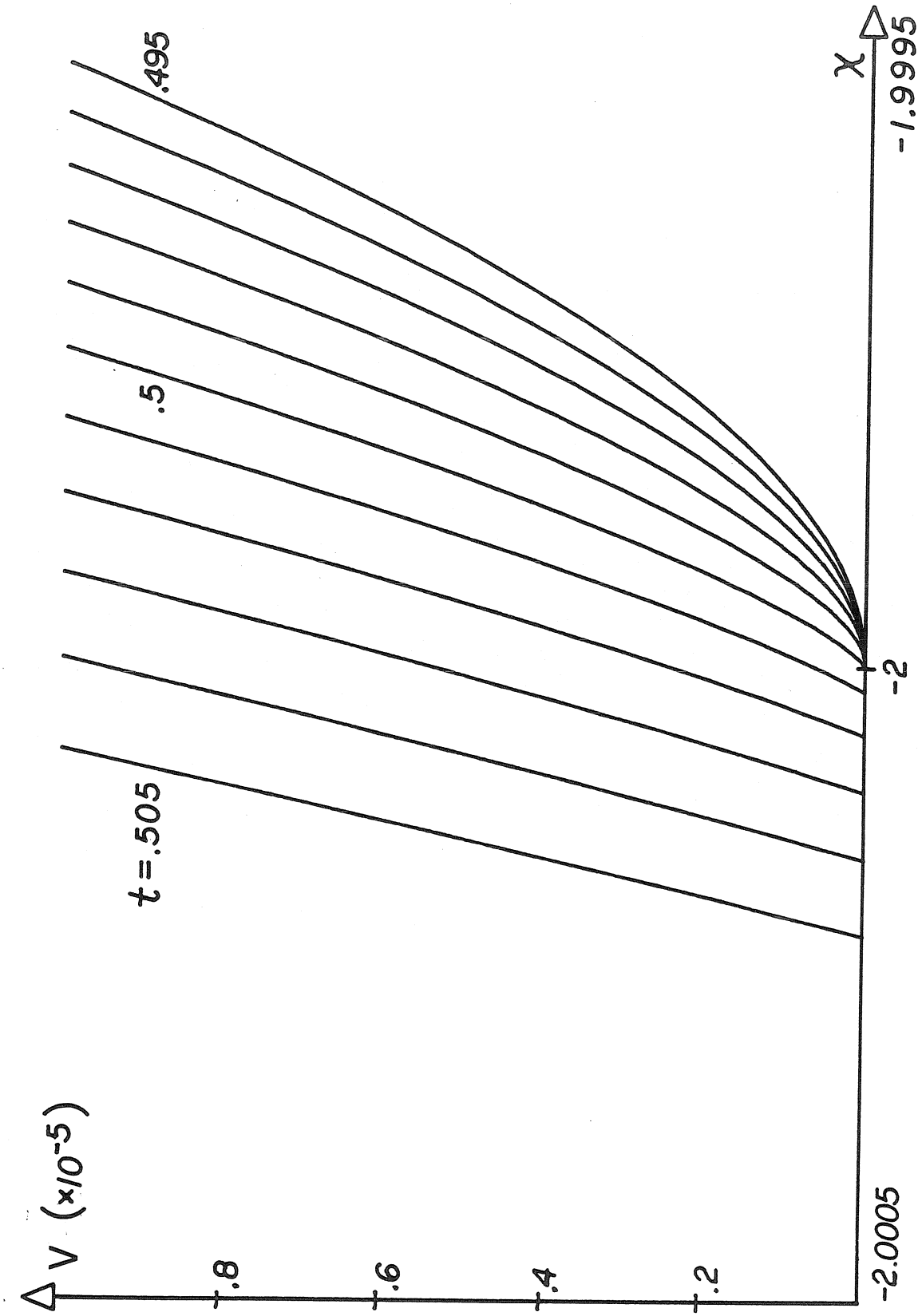

Figure 2.5. Leading-order approximation near the interface. A waiting-time solution.

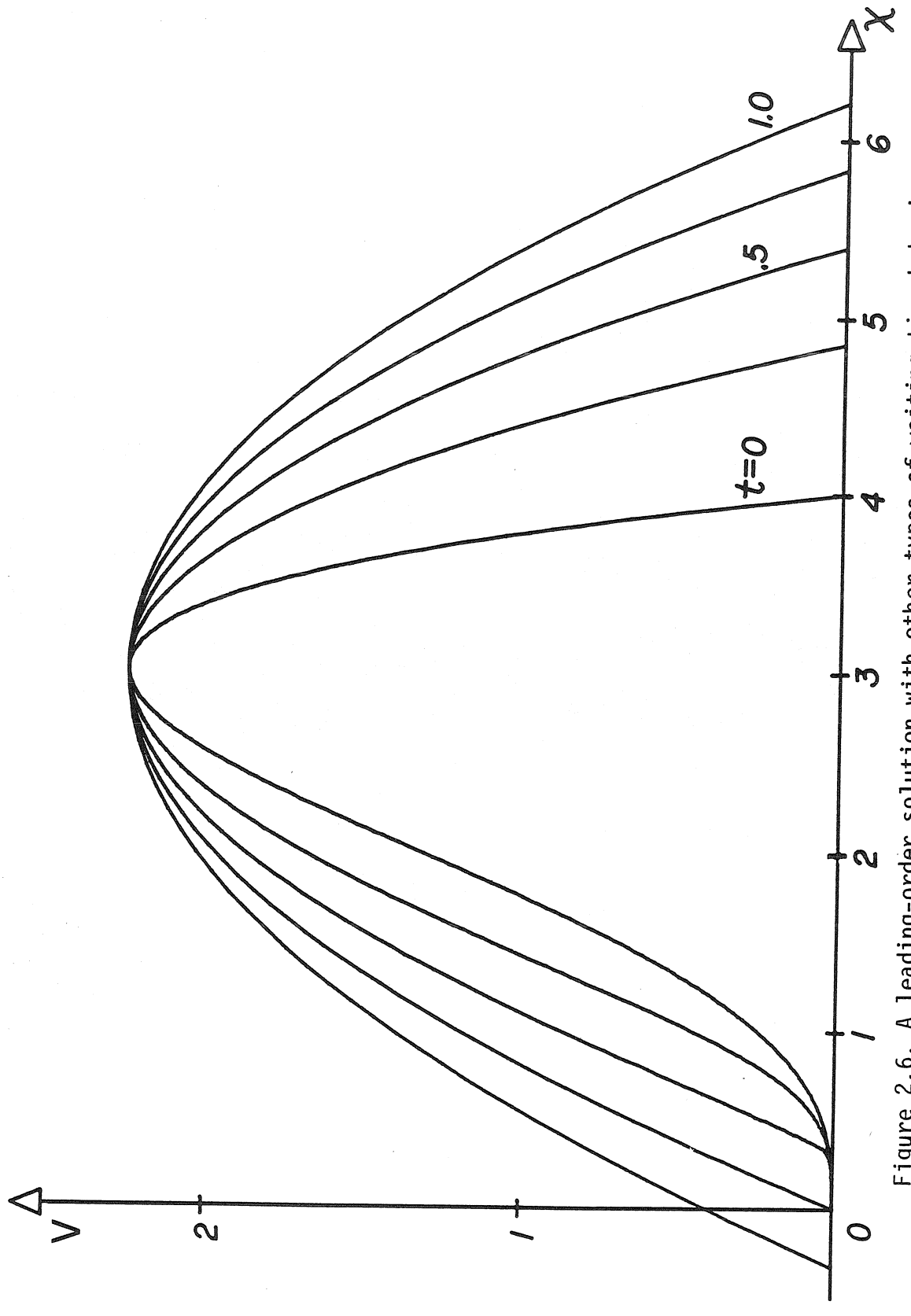

Figure 2.6. A leading-order solution with other types of waiting-time behavior.

The interface on the left is more interesting, and for clarity an expanded view is given in Fig. 2.7. The time of shock formation is again 0.5, but now the corner shock forms to the right of the interface at a point where $v > 0$ (because $f''(x)$ is maximum to the right of the interface for this example).

The corner shock has $v_x > 0$ on both sides, so from (2.35) and (2.36) $\zeta' < 0$ and $v_s' < 0$. Thus the corner shock moves toward the interface. At $t = 0.75$ the shock reaches $x = 0$, $v = 0$ and the interface begins to move. This rearrangement of the solution behind the interface is seen to be qualitatively different than that shown in Fig. 2.5.

When we include the effect of the $\epsilon v v_{xx}$ term we expect this description to be slightly modified, due to the rounding of the corner when $v > 0$. The basic behavior, however, will be the same.

2.6 A Leading-Order Approximation for the Waiting Time

A leading-order approximation to the time at which the interface begins to move can be obtained from the solution to (2.19); this is done by finding the time at which $v = 0$ and $x = \zeta_0$, the initial position of the interface:

$$f(\xi) - f'^2(\xi)t_c = 0 , \tag{2.48}$$

$$\xi - 2f'(\xi)t_c = \zeta_0 . \tag{2.49}$$

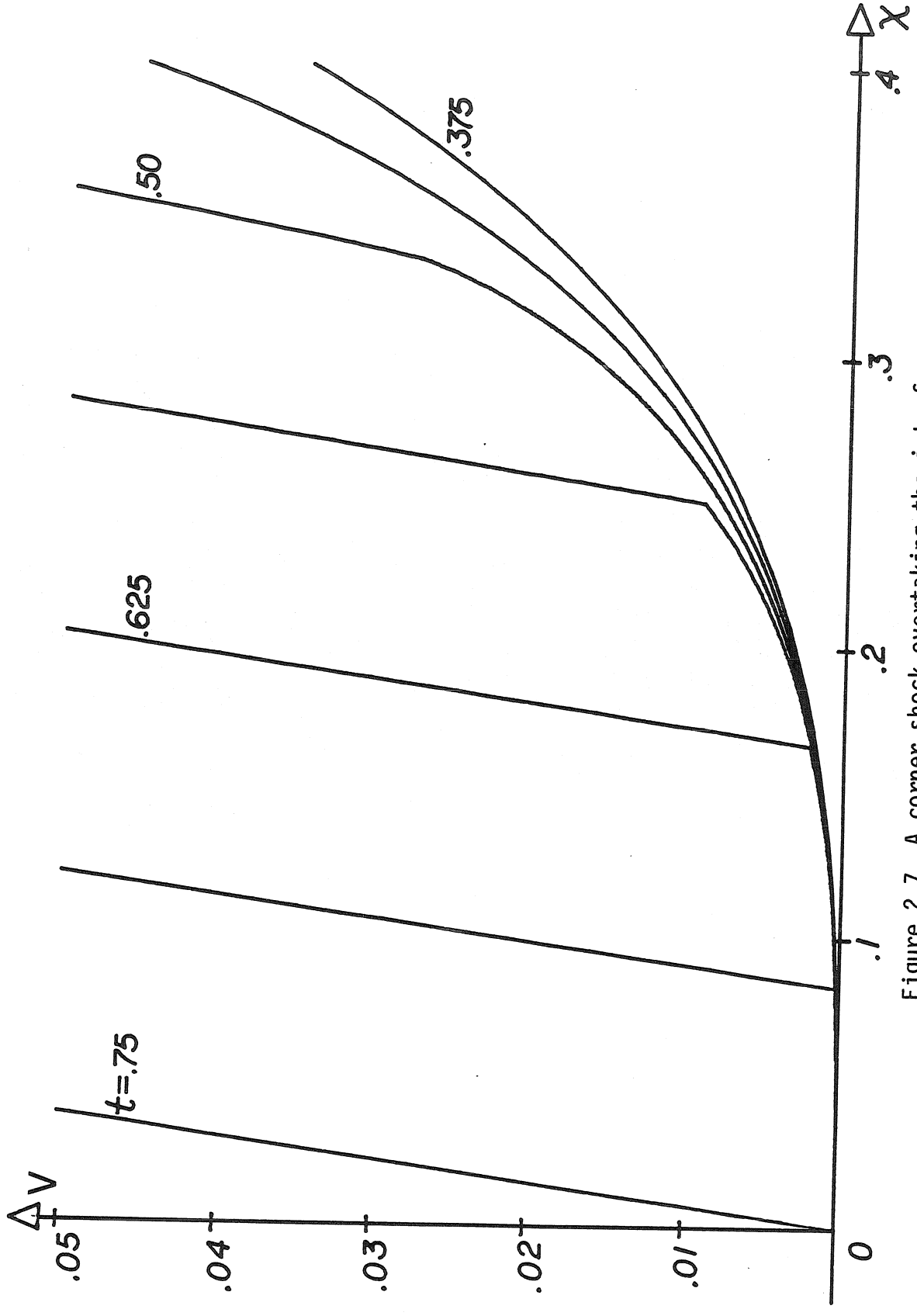

Figure 2.7. A corner shock overtaking the interface.

Solving for t_c ,

$$t_c = \frac{f(\xi)}{f'^2(\xi)} \quad , \tag{2.50}$$

where ξ satisfies

$$f'(\xi)(\xi - \zeta_0) = 2f(\xi) \ . \tag{2.51}$$

The appropriate solution to (2.51) which gives the minimum t_c when substituted in (2.50) is the one used to calculate the waiting time. (Although the case where $f'(\xi) = 0$ must be excluded to arrive at (2.50), it in fact gives the correct leading-order waiting time in this case upon application of l'Hospital's rule.)

If

$$f(\xi) \sim k(\xi - \zeta_0)^{\alpha} \quad \text{as} \quad \xi \to \zeta_0 \ ,$$

then $\xi = \zeta_0$ is a solution to (2.51) for $\alpha > 0$; the corresponding time is

$$t_c = \lim_{\xi \to \zeta_0} \frac{1}{k\alpha^2} (\xi - \zeta_0)^{2-\alpha} \ . \tag{2.52}$$

This local analysis shows there are three cases:

1) If $0 < \alpha < 2$ then $t_c = 0$, and the interface begins to move immediately.

2) If $\alpha = 2$ then t_c is finite, and is given by

$$t_c = \frac{1}{2f''(\zeta_0)} \tag{2.53}$$

upon application of l'Hospital's rule to (2.50).

3) If $\alpha > 2$ then $t_c = \infty$, and the interface begins to move when it is overtaken by a corner shock as in Fig. 2.7 (in this case the waiting time is found from other solutions of (2.51)).

When $\alpha \geq 2$ all appropriate solutions of (2.51) must be examined in order to determine the waiting time. For example, if $\alpha = 2$ we know that, according to the above local analysis, a corner shock will form at $x = \zeta_0$ and the interface will begin to move at $t = [2f''(\zeta_0)]^{-1}$. It is possible, however, that before this shock can form another shock will overtake the interface and it will begin to move at an earlier time. This will be predicted by a solution of (2.51) that gives a value of t_c smaller than $[2f''(\zeta_0)]^{-1}$.

When $\alpha = 2$ it is not always necessary to look for other solutions of (2.51) to obtain the waiting time, however; we can avoid doing so if $f''(x)$ is maximum at $x = \zeta_0$. Since we know that corner shocks form due to local maxima of $f''(x)$, in this case a shock will form at $x = \zeta_0$ before any others have had sufficient time to form. Thus the interface will begin to move at the time of shock formation, so that

$$t_c = \frac{1}{2f''(\zeta_0)} \quad .$$

It is also useful to note that the waiting time for the left interface is independent of the data to the right of the largest maximum of v, and vice versa (the characteristics always move away from this largest maximum). This fact may reduce the number of solutions to (2.51) that need to be considered.

In addition, it must be remembered that the waiting time is slightly modified due to the $\epsilon v v_{xx}$ term. This effect will be small, however, when ϵ is small.

In Table 2.1 we give the leading-order approximations to the waiting times for an example due to Aronson (see [2] and [3]),

$$f(\xi) = \begin{cases} (1-\theta)\cos^2\xi + \theta\cos^4\xi & \text{for } |\xi| \le \frac{\pi}{2}, \\ 0 & \text{for } |\xi| > \frac{\pi}{2}, \end{cases} \tag{2.54}$$

for values of θ between 0 and 1. Aronson gives the exact value of the waiting time for $0 \le \theta \le 1/4$, but no information for $\theta > 1/4$.

The function is symmetric so the waiting times for both interfaces are identical; we will do the calculations for the interface on the right. For $0 \le \theta \le 1/4$ $f''(x)$ is maximum at the interface, and the leading-order waiting time is given by $[2f''(\zeta_0)]^{-1}$, but for $1/4 < \theta \le 1$ (2.50) and (2.51) must be used.

In Table 2.1, the approximate waiting time is given in the scaled variable. In terms of the original time variable, t^*, the waiting time is

$$t^* = \epsilon t_c + o(\epsilon) .$$

TABLE 2.1

Leading-Order Waiting Time for (2.54)

θ	ξ	t_c
0.000	1.571	0.250
0.125	1.571	0.286
0.250	1.571	0.333
0.375	0.883	0.378
0.500	0.693	0.409
0.625	0.582	0.432
0.750	0.505	0.450
0.875	0.449	0.464
1.000	0.405	0.476

3. THE FIRST-ORDER APPROXIMATION FOR v WHEN CORNER SHOCKS DO NOT OCCUR AT v = 0

3.1 The Corner Layer

Although the leading-order approximation to v, (2.20) and (2.21), yields much information about the solutions of our nonlinear diffusion equation, it does not give a leading-order approximation for u. To obtain this we must determine v accurately to $O(\epsilon)$ everywhere, as stated in Chapter 2.

If we try to correct (2.20) by using the next order in a regular expansion of (2.16), it is clear that near a corner shock we will have a region of nonuniformity. The regular expansion is not general enough to give more than the leading-order approximation.

To see what features a successful expansion of (2.16) should include, let us examine the equation in the neighborhood of a corner shock. First, transform to a moving coordinate system following the shock by letting

$$\eta = x - \zeta(t) \; ; \tag{3.1}$$

equation (2.16) then becomes

$$v_t - \zeta' v_\eta = v_\eta^2 + \epsilon v v_{\eta\eta} \; . \tag{3.2}$$

Next, rescale near the shock by letting

$$v = v_0(t) + \delta v_1 \quad \text{and} \quad \eta = \beta\xi \ , \tag{3.3}$$

so that (3.2) becomes

$$v_0' + \delta v_{1t} - \zeta' \frac{\delta}{\beta} v_{1\xi} = \frac{\delta^2}{\beta^2} v_{1\xi}^2 + \frac{\epsilon\delta}{\beta^2}(v_0 + \delta v_1) v_{1\xi\xi} \ . \tag{3.4}$$

The appropriate distinguished limit is

$$\delta = \beta = \epsilon \ , \tag{3.5}$$

which gives the equation for the corner layer when $v_0 \neq 0$ (neglecting terms of $O(\epsilon)$),

$$v_0' - \zeta' v_{1\xi} = v_{1\xi}^2 + v_0 v_{1\xi\xi} \ . \tag{3.6}$$

Notice that now v_1 depends only parametrically on t.

To simplify (3.6), write it in the form

$$v_0 v_{1\xi\xi} = -\left(v_{1\xi}^2 + \zeta' v_{1\xi} - v_0' \right) \ , \tag{3.7}$$

and let the zeros of the right hand side be $w_1(t)$ and $w_2(t)$, where $w_1 \geq w_2$ (they must be real); then (3.7) can be written as

$$v_0 v_{1\xi\xi} = -(v_{1\xi} - w_1)(v_{1\xi} - w_2) \ . \tag{3.8}$$

With the equation in this form we see that

$$v_{1\xi} \sim w_1, w_2 \qquad \text{as } \xi \to \pm \infty ,$$

so that w_1 and w_2 are the values of v_x just outside the corner layer. (As $\xi \to +\infty$, $v_1 \sim w_1\xi$, so that $v \sim v_0(t) + \epsilon v_1 \sim v_0 + \epsilon w_1 \xi \sim v_0 + w_1\eta$; for this to be a good approximation to $v(\eta, t)$, η must be small, and therefore w_1 is the value of v_x just to the right of the corner layer. Similarly w_2 is the value of v_x just to the left of the corner layer.)

In terms of w_1 and w_2, ζ' and v_0' are determined by

$$\zeta' = -(w_1 + w_2) , \tag{3.9}$$

$$v_0' = -w_1 w_2 , \tag{3.10}$$

which are the shock conditions (2.35) and (2.36). This shows that the behavior is basically the same as in the case $\epsilon = 0$, although the corner is rounded: the speed of the corner still depends only on the values of v_x on each side of the layer.

With the arbitrary condition that $v_1 = 0$ when $\xi = 0$, the solution to (3.8) is

$$v_1 = \frac{w_1+w_2}{2}\,\xi + v_0 \ln \cosh\left[\frac{w_1-w_2}{2v_0}\,\xi\right] , \tag{3.11}$$

and in terms of our unscaled function $v(\eta, t)$ this is

$$v \approx v_0 + \frac{w_1+w_2}{2}\eta + \epsilon v_0 \ell n \cosh\left[\frac{w_1-w_2}{2\epsilon v_0}\eta\right] . \qquad (3.12)$$

The integration constants we have neglected serve merely to correct the position of the corner layer, etc., to the next order in ϵ.

The form of the solution (3.12) is shown in Fig. 3.1. As in the above discussion η must be small, but $\eta/\epsilon = \xi$ can be large (more precisely, we must require $\eta << 1$ as $\epsilon \to 0$, which means $\xi << \epsilon^{-1}$).

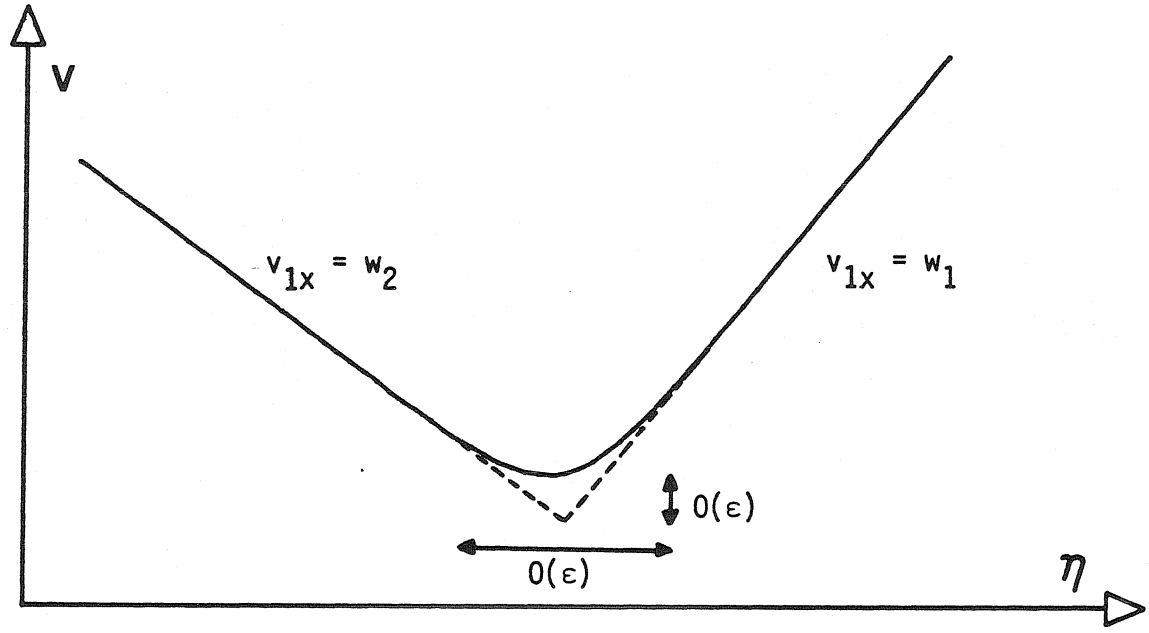

Figure 3.1. Corner layer structure.

The approximate solution (3.12) describes the basic character of solutions to (2.16) in the neighborhood of a corner (i.e., in the corner layer). If $\epsilon \to 0$ then the corner becomes more and more sharp, until finally when $\epsilon = 0$ we have a corner shock given by the limit of (3.12),

$$v \approx v_0 + (w_1 - w_2)\, H_1(\eta) + w_2 \eta . \qquad (3.13)$$

In addition, we see from (3.12) that the same sharpening of the corner occurs if $v_0 \to 0$, as was suggested by the wavefront expansion in Section 2.4. It is not conclusive from this analysis, however, that we have a corner shock if $v_0 = 0$; this is because one of the approximations used to obtain (3.6) from (3.4) was to neglect ϵv_1 in comparison with v_0. If $v_0 \to 0$ we cannot make this approximation.

In fact, this approximation is also not valid for ξ too large, because ϵv_1 becomes of the same order as v_0. A more detailed expansion (given in the next section) shows that our approximation for v, (3.12), is basically correct, however. Although ϵv_1 becomes of the same order as v_0, the second derivative $v_{1\xi\xi} \to 0$ fast enough to make the entire term unimportant. This more detailed expansion will also be used in Chapter 4 to examine the case when $v_0 \to 0$.

3.2 A Uniform First-Order Expansion

Although the preceding analysis gives the correct structure of the corner layer, the resulting approximation for v is not valid outside a region in x of width $O(\epsilon)$. We have two choices for completing the solution outside of this region: use the method of matched asymptotic expansions with an appropriate outer approximation, or construct a uniform expansion that includes both the inner and the outer approximations. For our problem we choose the latter method. Although it requires more work to derive, a uniform

expansion possesses the advantage that there is no need to do any matching, which for this problem is an even greater advantage since the matching region is moving.

To simplify the expansion let us examine the case where there is only one corner; the method can be extended to include more. The basic concept we employ was conceived of by Luke [17] to study nonlinear dispersive wave problems, and has been utilized by Berman [5] to examine combustion waves in slowly varying media.

Specifically, we assume

$$v(x, t;\epsilon) \sim v_0(x, t) + \epsilon v_1(\xi, x, t) + \epsilon^2 v_2(\xi, x, t) + \cdots \tag{3.14}$$

as $\epsilon \to 0$, where

$$\xi = \frac{1}{\epsilon}\zeta(x, t). \tag{3.15}$$

We see that ξ is a scaled variable that describes changes in the corner layer, and that this scaling is basically the same as in (3.3). The difference is that we are now including higher order corrections and the dependence upon x and t, which describes the behavior away from the corner layer.

Following the standard method of multiple-scale analysis [9, p. 79] we now regard $\xi, x,$ and t as independent variables. Thus

$$\left(\frac{\partial}{\partial x}\right)_t = \frac{1}{\epsilon}\zeta_x\left(\frac{\partial}{\partial \xi}\right)_{x,\,t} + \left(\frac{\partial}{\partial x}\right)_{\xi,\,t}, \tag{3.16}$$

$$\left(\frac{\partial}{\partial t}\right)_x = \frac{1}{\epsilon}\zeta_t\left(\frac{\partial}{\partial \xi}\right)_{x,\,t} + \left(\frac{\partial}{\partial t}\right)_{\xi,\,x}. \tag{3.17}$$

(From (3.16) we can see that if we had let v_0 depend on ξ in the expansion (3.14), then we would have been led to conclude immediately that $v_{0\xi}^2 = 0$, since this would have been the only term of order ϵ^{-2} in (2.16).)

Upon substituting (3.14) into (3.16) and (3.17), we find

$$v_x \sim (v_{0x} + \zeta_x v_{1\xi}) + \epsilon(v_{1x} + \zeta_x v_{2\xi}) + \cdots, \tag{3.18}$$

$$v_t \sim (v_{0t} + \zeta_t v_{1\xi}) + \epsilon(v_{1t} + \zeta_t v_{2\xi}) + \cdots. \tag{3.19}$$

and

$$v_{xx} \sim \frac{1}{\epsilon}\zeta_x^2 v_{1\xi\xi} + \left(v_{0xx} + 2\zeta_x v_{1x\xi} + \zeta_{xx} v_{1\xi} + \zeta_x^2 v_{2\xi\xi}\right) + \cdots. \tag{3.20}$$

We now substitute (3.14) and (3.18)-(3.20) into (2.16) and equate like powers of ϵ, with one important exception: we treat ϵv_1 as if it were a term of O(1). The reason for this is clear from (3.12), which shows that although ϵv_1 is $O(\epsilon)$ in the center of the corner layer, near the edges it becomes O(1). Thus, we cannot neglect ϵv_1 in comparison with v_0 if we wish to have a uniform expansion. (Note it is true that $v_{1\xi}$ remains O(1) uniformly in ξ, however.) This is the only term of which we must be careful,

because it will be shown that v_2 remains O(1) when the solution for v_1 is determined correctly.

With this in mind the substitution is performed, resulting in the sequence of equations,

$$v_{0t} + \zeta_t v_{1\xi} = (v_{0x} + \zeta_x v_{1\xi})^2 + \zeta_x^2 (v_0 + \epsilon v_1) v_{1\xi\xi} , \tag{3.21}$$

$$v_{1t} + \zeta_t v_{2\xi} = 2(v_{0x} + \zeta_x v_{1\xi})(v_{1x} + \zeta_x v_{2\xi})$$

$$+ (v_0 + \epsilon v_1)(v_{0xx} + 2\zeta_x v_{1\xi x} + \zeta_{xx} v_{1\xi} + \zeta_x^2 v_{2\xi\xi}), \tag{3.22}$$

Equation (3.21) is solved first. Write it as in (3.7),

$$(v_0 + \epsilon v_1) v_{1\xi\xi} = -\left[v_{1\xi}^2 - \frac{1}{\zeta_x^2}(\zeta_t - 2\zeta_x v_{0x}) v_{1\xi} - \frac{1}{\zeta_x^2}(v_{0t} - v_{0x}^2) \right], \tag{3.23}$$

and let

$$\frac{1}{\zeta_x^2}(\zeta_t - 2\zeta_x v_{0x}) = w_1 + w_2 , \tag{3.24}$$

$$\frac{1}{\zeta_x^2}(v_{0t} - v_{0x}^2) = -w_1 w_2 , \tag{3.25}$$

where $w_1 \geq w_2$. With these simplifications (3.23) can be written

$$(v_0 + \epsilon v_1) v_{1\xi\xi} + (v_{1\xi} - w_1)(v_{1\xi} - w_2) = 0 . \tag{3.26}$$

Since w_1 and w_2 are independent of ξ, this equation is almost identical with (3.8); the main difference is the presence of the ϵv_1

term, which is included to ensure the uniformity of the expansion.

The requirement that (3.18) and (3.19) be uniform expansions for v_x and v_t imposes severe restrictions on w_1 and w_2. To see this, note that for uniform expansions v_{1x} and v_{1t} must be bounded for all ξ. Consider first the case $\xi \to +\infty$; (3.26) then shows that $v_1 \sim w_1\xi$. Thus

$$v_{1x} \sim w_{1x}\xi \ ,$$

$$v_{1t} \sim w_{1t}\xi \ .$$

We see that w_1 must be a constant for the expansion to be uniform. Similarly w_2 must be a constant by considering the case $\xi \to -\infty$.

Once it is known that w_1 and w_2 are constants, it is possible to solve (3.24) and (3.25) for v_0 and ζ. Write (3.24) and (3.25) as

$$v_{0t} = v_{0x}^2 - w_1 w_2 \zeta_x^2 \ , \tag{3.27}$$

$$\zeta_t = 2\zeta_x v_{0x} + (w_1 + w_2)\zeta_x^2 \ . \tag{3.28}$$

These equations are equivalent to

$$(v_0 + w_1\zeta)_t = (v_0 + w_1\zeta)_x^2 \ , \tag{3.29}$$

$$(v_0 + w_2\zeta)_t = (v_0 + w_2\zeta)_x^2 \ , \tag{3.30}$$

so that the equations decouple.

Equations (3.29) and (3.30) are in fact the leading-order approximations to (2.16) away from the corner layer. For example, as $\xi \to +\infty$ (3.26) shows that $v_1 \sim w_1 \xi$, so that $v \sim v_0 + \epsilon v_1 \sim v_0 + \epsilon w_1 \xi \sim v_0 + w_1 \zeta$. Away from the corner layer we know that to leading order v satisfies $v_t = v_x^2$, and this is precisely (3.29). Similarly, equation (3.30) is the result when the limit $\xi \to -\infty$ is considered.

To make these leading-order approximations clearer, let

$$v_0 + w_i \zeta = \tilde{v}_i \,, \tag{3.31}$$

so that

$$\tilde{v}_{it} = \tilde{v}_{ix}^2 \,, \tag{3.32}$$

for $i = 1, 2$. Then v_0 and ζ are given by

$$\zeta = \frac{\tilde{v}_1 - \tilde{v}_2}{w_1 - w_2} \,, \tag{3.33}$$

$$v_0 = \frac{w_1 \tilde{v}_2 - w_2 \tilde{v}_1}{w_1 - w_2} \,.$$

The choice of the constants w_1 and w_2 is somewhat arbitrary, since we will have the same leading-order approximations away from the corner layer if $\tilde{v}_1$ and $\tilde{v}_2$ are fixed. If this is true (and if the arbitrary functions of x and t from integrating (3.26) are

chosen correctly, as done below), the effect of varying w_1 and w_2 is merely to change the form of v_0 and v_1; the approximation to v, $v_0 + \epsilon v_1$, is invariant to $O(\epsilon)$, however. Therefore w_1 and w_2 should be chosen to make the solution of (3.26) more convenient. We do so by choosing

$$w_1 = 1 , \qquad w_2 = -1. \tag{3.35}$$

This choice is most useful to determine the general structure of our expansion, but it is not the best choice when the corner occurs at $v = 0$. In Chapter 4 we shall make another choice to examine that case.

With w_1 and w_2 chosen as in (3.25) the expressions for ζ and v_0 become

$$\zeta = \frac{\tilde{v}_1 - \tilde{v}_2}{2} , \tag{3.36}$$

$$v_0 = \frac{\tilde{v}_1 + \tilde{v}_2}{2} , \tag{3.37}$$

and the differential equation for v_1 is

$$(v_0 + \epsilon v_1) v_{1\xi\xi} + v_{1\xi}^2 = 1 . \tag{3.38}$$

This equation cannot be solved in closed form for arbitrary ϵ. We expect, however, that because v_1 is $O(1)$ in the center of the corner layer (and because $v_0 \neq 0$), we can obtain a good approximation to (3.38) by examining the case $\epsilon = 0$. Knowledge of how the solution

for v_1 behaves when $\epsilon = 0$ will make it simpler to assess any differences that result by taking $\epsilon > 0$.

When $\epsilon = 0$ we have immediately that

$$v_1(\xi, x, t) = v_0(x, t) \ell n \cosh \left[\frac{\xi + \eta(x, t)}{v_0(x, t)} \right] + \nu(x, t) , \tag{3.39}$$

where $\eta(x, t)$ and $\nu(x, t)$ are arbitrary functions from the integration of (3.38). These functions will be determined at a later stage in the analysis by the requirement that v_2 is bounded as $\xi \to \pm \infty$, and once they are known the solution for v_1 will be determined uniquely.

Since we expect this approximation for v_1 to break down when ξ becomes large, let us expand (3.39) in the limit $\xi \to \pm \infty$. This gives

$$v_1 \sim \xi + \eta(x, t) + \nu(x, t) + \cdots \text{ as } \xi \to + \infty , \tag{3.40}$$

$$v_1 \sim -\xi - \eta(x, t) + \nu(x, t) + \cdots \text{ as } \xi \to -\infty , \tag{3.41}$$

where the neglected terms are in fact exponentially small in ξ. We will use this expansion to examine the differences between the two cases $\epsilon = 0$ and $\epsilon > 0$.

We have excluded the case $v_0 = 0$, so let us rescale (3.38) $\epsilon > 0$ by letting

$$v_1 = v_0 F(z) , \tag{3.42}$$

where

$$z = \frac{\xi}{v_0} \quad ; \tag{3.43}$$

(3.38) then becomes

$$F_z^2 + (1 + \epsilon F)F_{zz} = 1 \ . \tag{3.44}$$

To examine the behavior as $z \to \infty$, let

$$F(z) = z + \alpha + f(z) \ , \tag{3.45}$$

where α is independent of z and f(z) is o(1) as $z \to \infty$, so that

$$(1 + f_z)^2 + (1 + \epsilon z + \epsilon\alpha + \epsilon f)f_{zz} = 1 \ . \tag{3.46}$$

Neglecting f_z^2 in comparison with f_z and ϵf in comparison with $\epsilon(z+\alpha)$, f approximately satisfies

$$2f_z \sim -(1+\epsilon z+\epsilon\alpha)f_{zz} \quad \text{as} \quad z \to \infty \ . \tag{3.47}$$

Since we have assumed f(z) is o(1) as $z \to \infty$, the constant part of f must be zero and the solution is

$$f(z) \sim c(1 + \epsilon z + \epsilon\alpha)^{1-2/\epsilon} \quad \text{as} \quad z \to \infty \ . \tag{3.48}$$

From (3.48) the change in behavior as $\epsilon \to 0$ becomes clear. If $z << \epsilon^{-1}$ as $\epsilon \to 0$, then

$$f(z) \sim c\, e^{-2(z+\alpha)} \quad \text{as} \quad z \to \infty, \quad \epsilon z \to 0 \,, \tag{3.49}$$

which is the form observed for the exact solution in the case $\epsilon = 0$, (3.39). This change in behavior is clearly not uniform in z, for if we try to expand (3.48) in powers of ϵ the resulting series will not converge for all z.

The asymptotic solution (3.48) also shows why (3.39) is a good approximation to v_1 for ϵ small, even though the limit is non-uniform in z. It is true that the behavior of f(z) is different in the cases $\epsilon = 0$ and $\epsilon > 0$, but the difference is only important when z is $O(\epsilon^{-1})$, where f(z) is extremely small. Thus (3.39) approximates the true solution of (3.38) to O(1) everywhere as $\epsilon \to 0$. If we wish to go to higher orders in our uniform expansion, however, then we must use the full solution of (3.38).

Independent of whether $\epsilon = 0$ or $\epsilon > 0$, we do know that

$$v_1 \sim \xi + \alpha_1(x, t) \qquad \text{as} \quad \xi \to +\infty \,, \tag{3.50}$$

$$v_1 \sim -\xi + \alpha_2(x, t) \qquad \text{as} \quad \xi \to -\infty \,. \tag{3.51}$$

It is also true independent of ϵ that if we find α_1 and α_2 then the solution for v_1 is determined uniquely.

To see this, integrate (3.44) to

$$z + \beta = \pm \int_{1/k}^{F} \frac{dh}{\sqrt{1-(kh)^{-2/\epsilon}}} ,$$

where β and k depend on x and t. The + or - sign is chosen depending on whether we want $z + \beta > 0$ or $z + \beta < 0$. We can write this as

$$z + \beta = \pm\left[F - \frac{1}{k} + \int_{1/k}^{F} \left\{ \frac{1}{\sqrt{1-(kh)^{-2/\epsilon}}} - 1 \right\} dh\right] .$$

The integral is now finite as $F \to \infty$, so that

$$z + \beta \sim F + I(k) \quad \text{as} \quad z \to +\infty ,$$

$$z + \beta \sim -F - I(k) \quad \text{as} \quad z \to -\infty ,$$

where

$$I(k) = \int_{1/k}^{\infty} \left\{ \frac{1}{\sqrt{1-(kh)^{-2/\epsilon}}} - 1 \right\} dh - \frac{1}{k} .$$

These two asymptotic equations are equivalent to (3.50) and (3.51). Therefore if we know α_1 and α_2, we know k and β and the solution for F is determined uniquely.

The two functions α_1 and α_2 are found by considering the equation for v_2, (3.22), and requiring that v_2 be bounded as $\xi \to \pm \infty$. Equation (3.22) can be written

$$-\zeta_x^2\left[2v_{1\xi}\, v_{2\xi} + (v_0 + \epsilon v_1) v_{2\xi\xi}\right] =$$

$$-v_{1t} + 2v_{0x} v_{1x} + 2\zeta_x v_{1x} v_{1\xi}$$

$$+ (v_0 + \epsilon v_1)(2\zeta_x v_{1\xi x} + \zeta_{xx} v_{1\xi} + v_{0xx}) \ . \qquad (3.52)$$

Since $v_{1\xi} \to \pm 1$ as $\xi \to \pm \infty$, the right-hand side of (3.52) must go to zero as $\xi \to \pm \infty$ in order for v_2 to be bounded. Using (3.15), (3.36), (3.37), and (3.50), we have that as $\xi \to +\infty$ the right-hand side is asymptotic to

$$-\alpha_{1t} + 2\tilde{v}_{1x}\alpha_{1x} + \tilde{v}_1 \tilde{v}_{1xx} + \cdots$$

Similarly as $\xi \to -\infty$ the right-hand side is asymptotic to

$$-\alpha_{2t} + 2\tilde{v}_{2x}\alpha_{2x} + \tilde{v}_2 \tilde{v}_{2xx} + \cdots$$

The stipulation that v_2 be bounded (for a uniform expansion) requires that these expressions be zero, so that

$$\alpha_{it} = 2\tilde{v}_{ix}\alpha_{ix} + \tilde{v}_i \tilde{v}_{ixx} \ , \qquad (3.53)$$

for $i = 1, 2$. These equations determine the two functions α_1 and α_2, which completes the solution for v_1. As should have been expected, equation (3.53) is merely the next equation after (3.32) in a regular expansion of (2.16).

This shows that a regular expansion of (2.16) will be valid as long as it is not used near a corner shock. The basic structure of our corner layer expansion consists of two regular expansions, each of which is valid on one side of the corner, and which are joined together in a region where v_x changes rapidly by a functional relation of the form (3.39).

3.3 Discussion of the Uniform Expansion

We have not yet discussed some of the significant consequences of our corner layer expansion. To do so it is easiest to consider v_1 as being given by (3.39), so that we can write the solution explicitly.

Using (3.14)-(3.15), (3.36)-(3.37), and (3.39) results in the approximation for v,

$$v \sim \frac{\tilde{v}_1+\tilde{v}_2}{2} + \epsilon \frac{\tilde{v}_1+\tilde{v}_2}{2} \ell n \cosh\left[\frac{\tilde{v}_1-\tilde{v}_2+\epsilon(\alpha_1-\alpha_2)}{\epsilon(\tilde{v}_1+\tilde{v}_2)}\right] + \epsilon\frac{\alpha_1+\alpha_2}{2} + \cdots \tag{3.54}$$

as $\epsilon \to 0$, where we have also used

$$\alpha_{1,2} = \nu \pm \eta .$$

Notice that if $\tilde{v}_1$ and $\tilde{v}_2$ are expanded in a Taylor series about the (moving) position of the corner, then the approximation (3.12) is obtained. Therefore, we see that (3.54) is a generalization of (3.12). The form of this new corner layer solution is shown in Fig. 3.2.

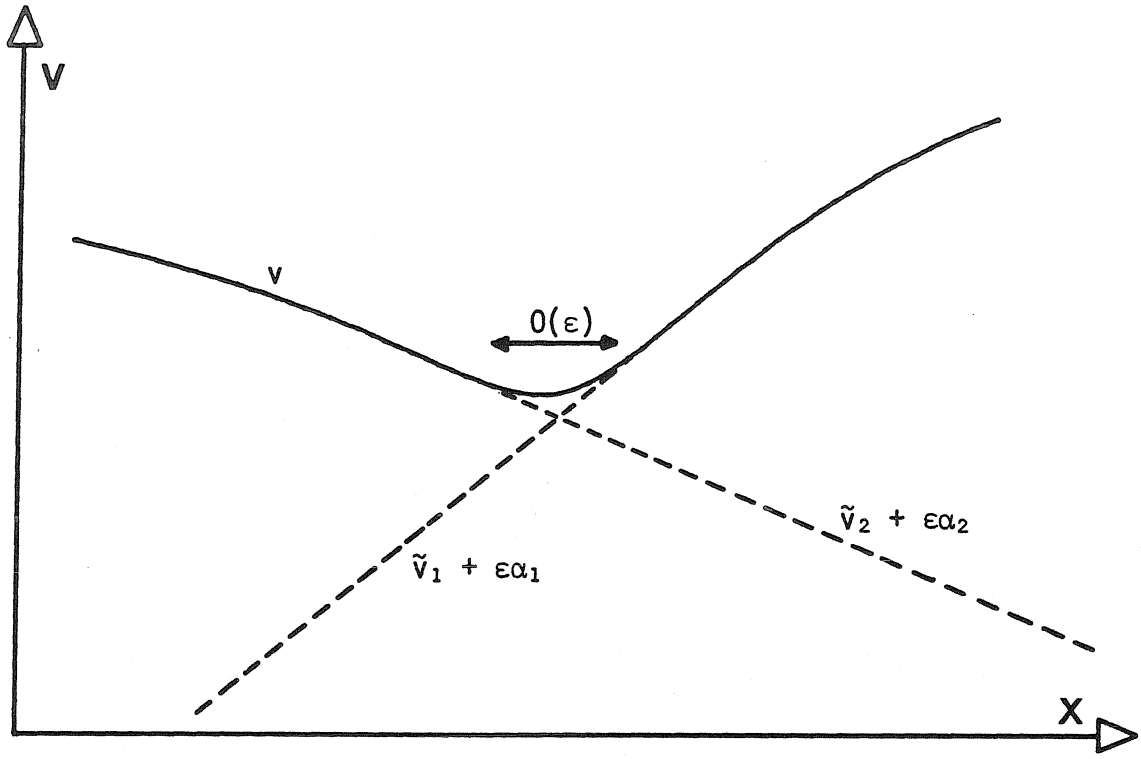

Figure 3.2. Structure of the uniform expansion.

To see how the expansion (3.54) should be used, suppose that $\tilde{v}_i(x, t)$ and $\alpha_i(x, t)$ are known ($i = 1, 2$). Since it has been assumed that there is only one corner layer, no shocks can develop in the solutions for $\tilde{v}_1$ and $\tilde{v}_2$ (which means that $\tilde{v}_{ixx} \leq 0$, from (2.30)). This is necessary for a uniform expansion.

When $\tilde{v}_i$ and α_i are known, the center of the corner layer, to $O(\epsilon)$, is given by the position where

$$\tilde{v}_1(x, t) - \tilde{v}_2(x, t) + \epsilon\left[\alpha_1(x, t) - \alpha_2(x, t)\right] = 0 ,$$

or

$$\tilde{v}_1(x, t) + \epsilon\alpha_1(x, t) = \tilde{v}_2(x, t) + \epsilon\alpha_2(x, t) . \qquad (3.55)$$

Therefore, the corner layer is placed at the position where the two outer expansions agree to $O(\epsilon)$. This gives the $O(\epsilon)$ correction to the leading-order position given by (2.23)-(2.25).

The functions $\tilde{v}_i(x, t)$ and $\alpha_i(x, t)$ must be defined on both sides of the corner layer; accordingly, we must think of them as being extended through the corner. This presents no real problem, however.

To the right of the corner layer the expansions (3.45) and (3.48) show that v only depends on $\tilde{v}_2$ through a term of the form

$$\left(1 + \frac{\tilde{v}_1 - \tilde{v}_2}{\tilde{v}_1 + \tilde{v}_2} + O(\epsilon)\right)^{1-\frac{2}{\epsilon}} .$$

Since $\tilde{v}_1 - \tilde{v}_2 > 0$ to the right of the corner layer, this is smaller than any fixed power of ϵ as $\epsilon \to 0$. Thus, $\tilde{v}_2$ only occurs in transcendentally small terms to the right of the corner layer. Similarly, $\tilde{v}_1$ only occurs in transcendentally small terms to the left of the corner layer.

Also, the leading-order characteristics always carry information into the corner layer. To see this, recall from (2.21) that the characteristic velocity is

$$\frac{dx}{dt} = -2f'(\xi) = -2\tilde{v}_x ;$$

also recall that the speed of the corner is given by (3.9). Since $\tilde{v}_1 - \tilde{v}_2 > 0$ to the right of the corner, we must have

$$\tilde{v}_{1x} - \tilde{v}_{2x} > 0$$

at the corner, and thus

$$-2\tilde{v}_{1x} < -(\tilde{v}_{1x} + \tilde{v}_{2x}) < -2\tilde{v}_{2x} ,$$

or

$$\left(\frac{dx}{dt}\right)_{right} < \zeta' < \left(\frac{dx}{dt}\right)_{left} .$$

Therefore, the characteristics point into the corner layer.

If we alter $\tilde{v}_2$ to the right of the corner, this change can never propagate back through the corner layer to influence the solution to the left. (Similarly, any change in $\tilde{v}_1$ to the left of the corner cannot propagate back through the layer to influence the solution for $\tilde{v}_1$ to the right.)

When combined with the transcendentally small dependence upon $\tilde{v}_2$ to the right of the corner layer, this means that the solution is independent of the value of $\tilde{v}_2$ to the right of the layer to any order in ϵ. Similarly, the solution is independent of the value of $\tilde{v}_1$ to the left of the corner layer to any order in ϵ. (There is a small region of width $O(\epsilon)$, however, where both $\tilde{v}_1$ and $\tilde{v}_2$ are important.)

Although equation (2.16) is indeed parabolic when $v > 0$, so that a disturbance at any one position immediately influences the solution everywhere, this influence is transcendentally small in ϵ if

we are any substantial distance away from the position of the initial disturbance. Therefore, when we construct approximate solutions to (2.16) which are of the form (2.12) or (3.14) (i.e., a power series in ϵ), the leading-order characteristics given by (2.19) do indeed determine the behavior of the solution (see [9]. p. 120).

This transcendentally small dependence upon $\tilde{v}_2$ to the right and $\tilde{v}_1$ to the left of the corner layer explains an apparent difficulty with the expansion. Originally only the initial condition $v(x, 0)$ was needed, but with the expansion it appears that both $\tilde{v}_1(x, 0)$ and $\tilde{v}_2(x, 0)$ are needed ($\alpha_1(x, 0)$ and $\alpha_2(x, 0)$ follow from the $O(\epsilon)$ terms in an expansion of the initial conditions, once the difficulty with $\tilde{v}_1$ and $\tilde{v}_2$ is resolved). From the above discussion, however, we see that the solution is unaffected to any order in ϵ by the values of $\tilde{v}_1(x, 0)$ to the left and $\tilde{v}_2(x, 0)$ to the right of the corner layer. Thus, we may use any initial condition in these regions. (A simple choice is to merely extend $\tilde{v}_1$ and $\tilde{v}_2$ as straight lines, in such a way that $\tilde{v}_{1x}$ and $\tilde{v}_{2x}$ are continuous.)

3.4 The Shock-Formation Transition Solution

The corner layer expansion (3.54) is designed to describe the solutions of (2.16) which have a region of rapid variation of v_x (i.e., a corner layer). For many initial conditions, however, a corner layer does not form until a finite time has elapsed (the shock formation time given in Chapter 2 is the leading-order approximation). This expansion cannot be used when these corner layers are not present, and a regular expansion should be used in this case. Only after the corner layers have had sufficient time to form is it possible to use (3.54).

When we have such an initial condition, we find that for times near the time of shock formation neither expansion is valid. The regular expansion develops a logarithmic singularity at the time of shock formation (see Chapter 4), and v_{xx} is not yet $O(\epsilon^{-1})$, so that the corner layer expansion cannot be used.

Thus, the solution has a different behavior for times near this shock-formation time, and it would seem that this transition solution must be determined before the corner layer expansion can be used. We know, however, that any information in the solution near the position of the corner layer will be carried into the layer by the leading-order characteristics. Thus, we conjecture that this shock-formation solution will only be needed near the time and position of shock formation, and that all of the information necessary for determining the solution after the corner layer has fully formed will come from outside this region.

To determine this transition solution, we must rescale the equation (2.16) in the shock-formation region. First we let

$$v = v_0 + c_0(x-x_s) + c_0^2(t-t_s) + \tilde{v} , \tag{3.56}$$

where x_s and t_s are the position and time of shock formation, $v_0 = v_0(x_s, t_s)$, and $c_0 = v_{0x}(x_s, t_s)$. We also let

$$x - x_s + 2c_0(t - t_s) = \tilde{x} \quad \text{and} \quad t - t_s = \tilde{t} ; \tag{3.57}$$

then (2.16) becomes

$$\tilde{v}_{\tilde{t}} = \tilde{v}^2_{\tilde{x}} + \epsilon(v_0 + c_0\tilde{x} - c_0^2\tilde{t} + \tilde{v})\tilde{v}_{\tilde{x}\tilde{x}} . \tag{3.58}$$

The transformation (3.56) ensures that $\tilde{v}_{\tilde{x}}$ is small in the center of the shock-formation region.

Following the analysis of Lighthill [16], we now rescale

$$\tilde{t} = \epsilon^{1/2} t^*, \qquad \tilde{x} = \epsilon^{3/4} x^* , \qquad \text{and} \qquad \tilde{v} = \epsilon v^* . \tag{3.59}$$

With this scaling (3.58) becomes

$$v^*_{t^*} = v^{*2}_{x^*} + (v_0 + \epsilon^{3/4} c_0 x^* - \epsilon^{1/2} c_0^2 t^* + \epsilon v^*)\, v^*_{x^* x^*} , \tag{3.60}$$

or to leading order

$$v^*_{t^*} = v^{*2}_{x^*} + v_0 v^*_{x^* x^*} , \tag{3.61}$$

which is Burgers' equation. This equation can be solved exactly by using a transformation due to Cole [8].

Although the exact solution to (3.61) is known, the matching of this shock-formation solution is difficult (due to the complicated representations of the various solutions) and has not been completed. In addition, there is some question as to what initial and boundary (at infinity) conditions to use for (3.61) in order to do the matching with the regular expansion (which is valid for times sufficiently smaller than the shock formation time).

Many of these difficulties arise from the necessity of including the time derivative in (3.61); when the shock is fully developed this derivative can be eliminated by the perturbation expansion, so that the equation depends only implicitly on time. In this case the structure of the corner layer is determined entirely by the conditions on each side of the layer.

One possibility that would enable us to avoid these difficulties would be to develop a uniform expansion using appropriately modified solutions of Burgers' equation. This possibility has not been adequately examined and requires further study.

3.5 Solutions with More Than One Corner Layer

Finally, we give two methods for including more than one corner layer in the solution. Both methods can easily be used to include as many corners as desired.

The simplest procedure is merely to use two expansions of the form (3.54) and match them together. This will clearly work as long as the two corner layers do not get too close to each other.

Another method, which is perhaps better, is to note that when we say $\tilde{v}_1$ satisfies

$$\tilde{v}_{1t} = \tilde{v}_{1x}^2 \quad ,$$

what we really mean is that this equation is a good approximation to the true equation for $\tilde{v}_1$,

$$\tilde{v}_{1t} = \tilde{v}_{1x}^2 + \epsilon \tilde{v}_1 \tilde{v}_{1xx} \quad .$$

Therefore, when we wish to have another corner layer in the region where $\tilde{v}_1$ is the leading-order approximation, we should continue to use (3.54) for the original corner layer, but now require that $\tilde{v}_1$ also be represented by an expansion of the form (3.54). With this nesting of corner layer expansions there seems to be no difficulty in the case where one corner layer overtakes another.

4. THE FIRST-ORDER APPROXIMATION FOR v WITH CORNER SHOCKS AT THE INTERFACE

4.1 The Corner Layer at v = 0

As we have seen, the behavior of the corner layer near $v = 0$ appears to differ markedly from its behavior away from this point. Since we wish to include the case when $v(x, 0)$ is allowed to be zero outside a finite interval (this is necessary to obtain both interfaces and waiting-time solutions), this behavior must be resolved in detail to complete our expansion of v to $O(\epsilon)$, and therefore of u to $O(1)$.

To examine the corner layer when it occurs at $v = 0$, we use the uniform expansion developed in Section 3.2. For that analysis the most convenient choice of w_1 and w_2 was (3.35), but now it is better to take

$$w_1 = 1 , \qquad w_2 = 0 , \tag{4.1}$$

which will be used to study the case when $v = 0$ to the left of the interface. A similar choice can be made to study the case when $v = 0$ to the right of the interface.

Upon using (4.1) in the appropriate equations from Section 3.2, we find that (3.33) and (3.34) are replaced by

$$\zeta = \tilde{v}_1 - \tilde{v}_2 ,$$

$$v_0 = \tilde{v}_2 ,$$

and that (3.26) becomes

$$(\tilde{v}_2 + \epsilon v_1)v_{1\xi\xi} + (v_{1\xi}-1)v_{1\xi} = 0 .$$

We know that the results of Chapter 3 can be rederived from these equations because of the arbitrariness of w_1 and w_2, so to avoid this duplication we make the additional assumption that $\tilde{v}_2 \equiv 0$. This means we are restricting ourselves to the case where to leading order the corner shock occurs at $v = 0$.

With this assumption the above equations become

$$\zeta = \tilde{v}_1 , \tag{4.2}$$

$$v_0 \equiv 0 , \tag{4.3}$$

and

$$\epsilon v_1 v_{1\xi\xi} + (v_{1\xi}-1)v_{1\xi} = 0 . \tag{4.4}$$

Note that now we cannot assume the effects due to the ϵv_1 term will be small.

This last equation can be integrated easily, so that the solution is given by

$$\xi = \int_k^{v_1} \frac{dh}{1 - (\frac{h}{\alpha_2})^{-1/\epsilon}} \quad , \tag{4.5}$$

where k and α_2 both depend on x and t. From this solution (or from asymptotic analysis of (4.4)), we find the behavior of v_1 away from the corner layer,

$$v_1 \sim \xi + \alpha_1(x, t) \qquad \text{as } \xi \to \infty \ , \tag{4.6}$$

$$v_1 \sim \alpha_2(x, t) \qquad \text{as } \xi \to -\infty \ , \tag{4.7}$$

where α_1 depends on both α_2 and k. This behavior is similar to that determined for v_1 and Chapter 3.

As before, we obtain equations for α_1 and α_2 by requiring that the solution for v_2 be bounded for all ξ. The analysis is similar to that of Section 3.2 and will not be repeated here. As expected, we find that α_1 and α_2 satisfy

$$\alpha_{it} = 2\tilde{v}_i\alpha_{ix} + \tilde{v}_i\tilde{v}_{ixx} \ , \tag{4.8}$$

for i = 1, 2.

Because we have assumed $\tilde{v}_2 \equiv 0$, we can solve the equation for α_2, which is

$$\alpha_{2t} = 0 \ .$$

Therefore, α_2 is given immediately by the $O(\epsilon)$ term of the initial data to the left of the corner layer and a suitable extension of it to the right. Since we are examining the case when v is initially zero to the left of the corner layer, we therefore take $\alpha_2 \equiv 0$.

If $\alpha_2 = 0$, we cannot immediately use (4.5) to determine the solution for v_1; thus, the limit $\alpha_2 \to 0$ must be examined more carefully. To do this, substitute

$$h = (1 + z)\alpha_2 \tag{4.9}$$

in (4.5), so that the integral becomes

$$\xi = \alpha_2 \int_{\frac{k}{\alpha_2}-1}^{\frac{v_1}{\alpha_2}} \frac{dz}{1-(1+z)^{-1/\epsilon}} \quad . \tag{4.10}$$

To quickly determine the behavior of (4.10) in the limit $\alpha_2 \to 0$, let v_1 (and k) be fixed as $\alpha_2 \to 0$. Then it is easy to see that

$$v_1 \sim \xi + k \qquad \text{as } \alpha_2 \to 0 \ . \tag{4.11}$$

This does not determine the behavior for $\xi + k \leq 0$ as $\alpha_2 \to 0$, however.

For a more careful analysis, let $v_1/\alpha_2 - 1$ be fixed as $\alpha_2 \to 0$ and write the integral in the form

$$\xi + k = \alpha_2\left(\frac{v_1}{\alpha_2}\right) + \alpha_2 \int_{\frac{k}{\alpha_2}-1}^{\frac{v_1}{\alpha_2}-1} \left[\frac{1}{1-(1+z)^{-1/\epsilon}} - 1\right] dz \ . \tag{4.12}$$

From (4.12) we see that $(\xi+k)/\alpha_2$ remains fixed as $\alpha_2 \to 0$, if v_1/α_2 remains fixed, and that this is also true if $\xi + k \leq 0$.

Note in particular this means $v_1 = O(\alpha_2)$ when $\xi + k = 0$. Since $v_1(\xi_1) < v_1(\xi_2)$ if $\xi_1 < \xi_2$ (easily seen from (4.5)), we can bound $v_1(\xi)$ for $\xi < -k$ by $v_1(-k)$, which is $O(\alpha_2)$. Therefore, $v_1(\xi) = O(\alpha_2)$ as $\alpha_2 \to 0$ for $\xi < -k$.

We also know $v_1 \geq \alpha_2^{-1}$ from the solution (4.5), so that we may summarize the above results as

$$v_1 \to (\xi+k) \qquad \text{for } \xi + k \geq 0 , \tag{4.13}$$

$$v_1 \to 0 \qquad \text{for } \xi + k < 0 , \tag{4.14}$$

in the limit $\alpha_2 \to 0$. In other words,

$$\lim_{\alpha_2 \to 0} v_1(\xi, x, t) = \frac{1}{2}\left[(\xi + \alpha_1) + |\xi + \alpha_1|\right] , \tag{4.15}$$

where now we have replaced k by $\alpha_1(x, t)$.

Recall from (4.3) that $v_0 \equiv 0$, which implies $v \sim \epsilon v_1 + \cdots$. Since (4.2) also shows that $\epsilon \xi \equiv \zeta = \tilde{v}_1$, the first-order uniform approximation for v is therefore

$$v \sim \frac{1}{2}\left[\left(\tilde{v}_1 + \epsilon\alpha_1\right) + |\tilde{v}_1 + \epsilon\alpha_1|\right] + \cdots . \tag{4.16}$$

This shows that to $O(\epsilon)$ a corner layer at $v = 0$ is in fact a corner shock: although we have both $v \to 0$ and $v_{xx} \to \infty$ in the $\epsilon v v_{xx}$ term,

the former is the more important limit.

Equation (4.16) gives an easy method for calculating v to $O(\epsilon)$ everywhere when corners occur only at $v = 0$. From (4.16) it is clear that to do so all that is needed is to calculate $\tilde{v}_1$ and α_1, the first two terms in a regular expansion for v, and to use $v = \tilde{v}_1 + \epsilon\alpha_1$ when $\tilde{v}_1 + \epsilon\alpha_1$ is positive; otherwise, take $v = 0$.

4.2 When the Waiting Time is Zero

The above expansion can be used immediately when $v(x, 0) = f(x)$ satisfies

$$f(x) = 0 \qquad \text{for } x \leq a \text{ and } x \geq b ,$$

$$f(x) > 0 \text{ and } f''(x) \leq 0 \qquad \text{for } a < x < b .$$

The crucial condition is $f''(x) \leq 0$; this assures that no corner layers will form where $v > 0$, as shown by (2.30). The other conditions ensure that there will only be two interfaces (that is, corner shocks at $v = 0$), so that we need not worry about fronts interacting.

To calculate the uniform approximation to (2.16) in this case, we begin by calculating the first two terms in a regular expansion of v as $\epsilon \to 0$. Since we will not be using $\tilde{v}_2$ and α_2, let us change notation and write the regular expansion as

$$v(x, t;\epsilon) \sim v_0(x, t) + \epsilon v_1(x, t) + \epsilon^2 v_2(x, t) + \cdots . \qquad (4.17)$$

As in Chapter 2, v_0 satisfies

$$v_{0t} = v_{0x}^2 ,$$

and with the condition $v_0(x, 0) = f(x)$, the solution is given implicitly by

$$v_0(\xi, t) = f(\xi) - f'^2(\xi)t, \tag{4.18}$$

$$x(\xi, t) = \xi - 2f'(\xi)t . \tag{4.19}$$

At the next order of the expansion, we find v_1 satisfies

$$v_{1t} = 2v_{0x}v_{1x} + v_0 v_{0xx} ,$$

with initial condition $v_1(x, 0) = 0$. The solution is

$$v_1(\xi, t) = \left[\frac{1}{4}\frac{f'^2(\xi)}{f''(\xi)} - \frac{1}{2}f(\xi)\right] \ell n\left|1-2f''(\xi)t\right| + \frac{1}{2} f'^2(\xi)t, \tag{4.20}$$

which is obtained by transforming to the characteristic variables (ξ, t) in the equation for v_1. As $\epsilon \to 0$, the regular expansion is therefore

$$v(x, t'\epsilon) \sim v_0(\xi, t) + \epsilon v_1(\xi, t) + \cdots, \tag{4.21}$$

where $\xi(x, t)$ is determined by (4.19).

The uniform approximation of v to $O(\epsilon)$ is obtained by using only the positive part of the expansion (4.21),

$$v(x, t;\epsilon) \sim \frac{1}{2}\left[v_0(\xi, t) + \epsilon v_1(\xi, t) + \left| v_0(\xi, t) + \epsilon v_1(\xi, t) \right| \right], \qquad (4.22)$$

with $\xi(x, t)$ given by the implicit solution of (4.19). An example of the use of (4.22) is shown in Fig. 4.1, where we have taken

$$f(\xi) = \begin{cases} \frac{3}{2}\sqrt{3}(\xi - \xi^3) & \text{for } 0 \le \xi \le 1 , \\ \\ 0 & \text{otherwise,} \end{cases}$$

and $\epsilon = 0.25$. Note the difference between the behavior of the leading-order and first-order approximations, shown in Figs. 2.6 and 4.1, near the maximum of v : for the leading-order approximation the maximum does not decrease with time, but remains fixed. The correction to this behavior that the first-order approximation supplies is important in the leading-order solution for u.

Once v is known uniformly to $O(\epsilon)$, the leading-order approximation for u is calculated by using (2.18). For our example the resulting approximation is shown in Fig. 4.2. (We plot u for the same values of the scaled time, t, as in Fig. 4.1). Notice that the $O(\epsilon)$ decrease in the maximum of v is amplified by the transformation to become an O(1) decrease in the maximum of u.

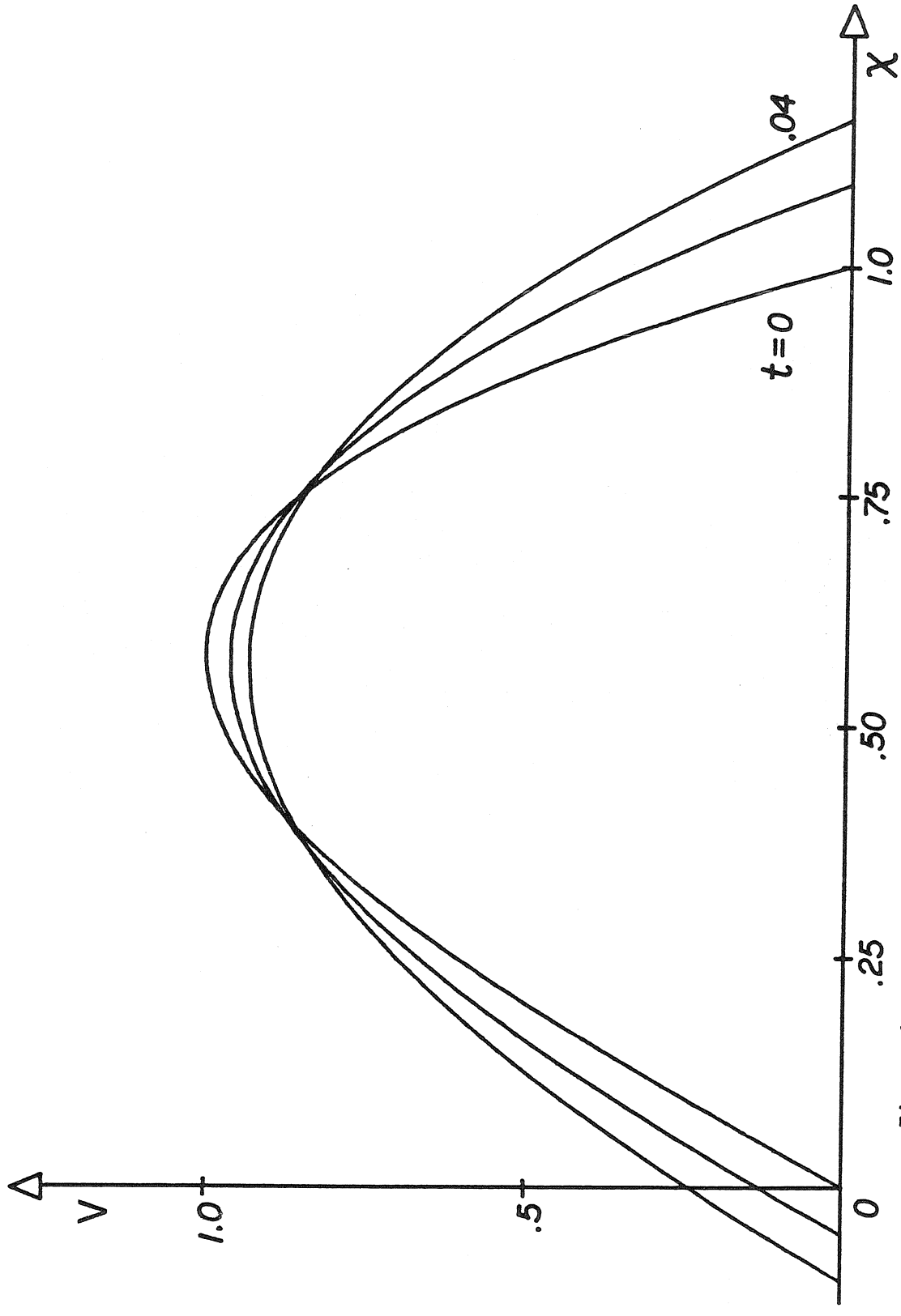

Figure 4.1. First-order solution for v with zero waiting time. $\varepsilon = 0.25$.

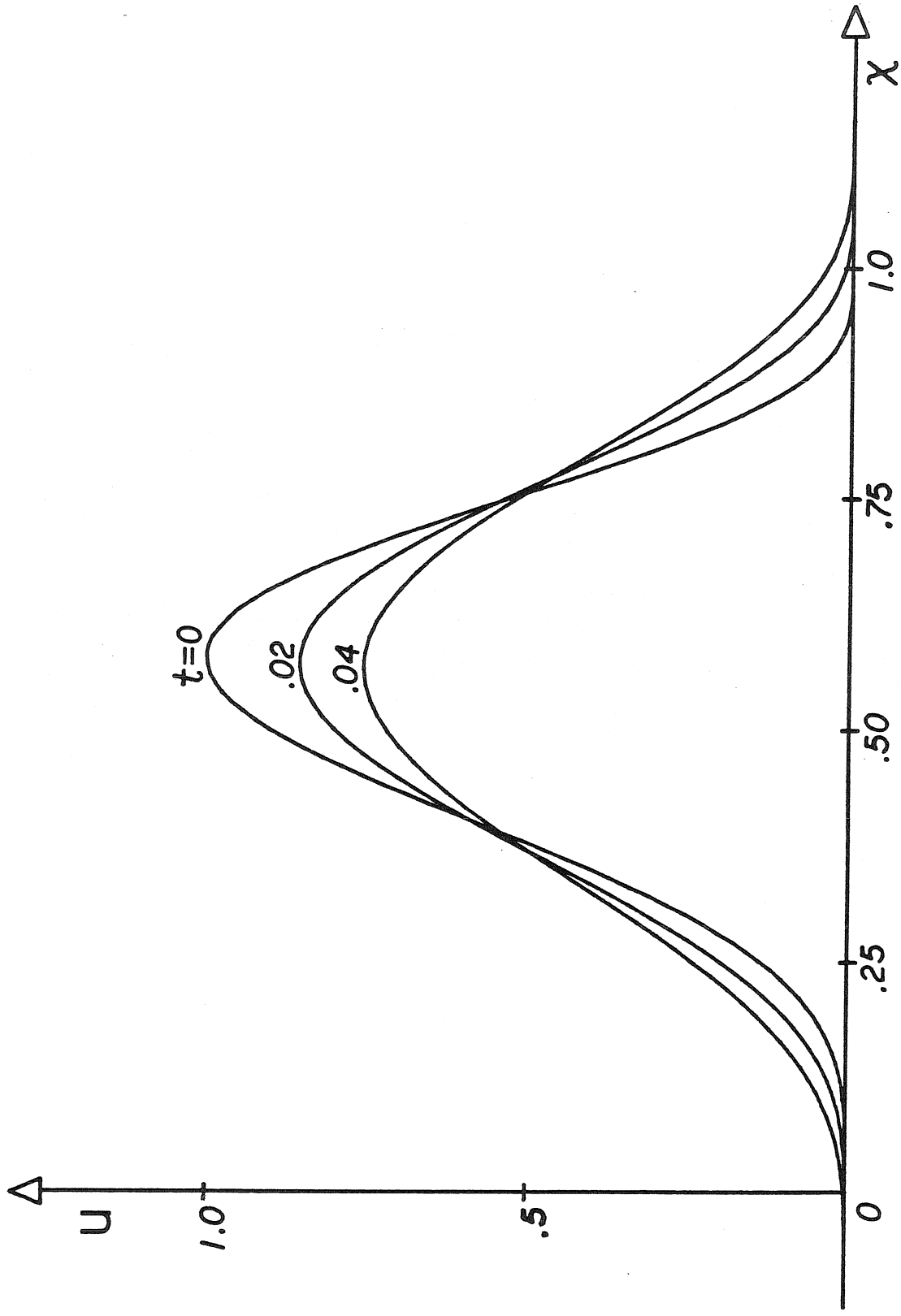

Figure 4.2. Leading-order approximation for u with zero waiting time. $\varepsilon = 0.25$.

4.3 When the Waiting Time is Nonzero--Formation of Corner Shocks at v = 0

Since the basic requirement for using the expansion (4.22) is that corner shocks only occur at v = 0, one would also expect the above method to work when $f''(x)$ has a positive maximum where f(x) = 0 and no (local) positive maxima when f(x) > 0. These assumptions ensure that the only corner shocks forming will be those occurring at v = 0 (again, we must also assume there is only one interval where f(x) > 0).

This is similar to the previous case. The difference is that now the corner shocks at v = 0 are not present initially, but take a finite time to form. Thus, the interface remains stationary until this happens.

The only possible difficulty with using the expansion (4.22) in this case would be some nonuniformity developing near the time of shock formation, since it is valid for earlier and later times. To check this, we take

$$f(\xi) = \begin{cases} \dfrac{1-\cos\pi\xi}{2} & \text{for } 0 \leq \xi \leq 2\ , \\ \\ 0 & \text{otherwise,} \end{cases}$$

and examine the approximate solution for v carefully near the time and position of shock formation (Fig. 4.3).

We see that there is indeed some difficulty just after the interface begins to move. It appears that just after this shock-

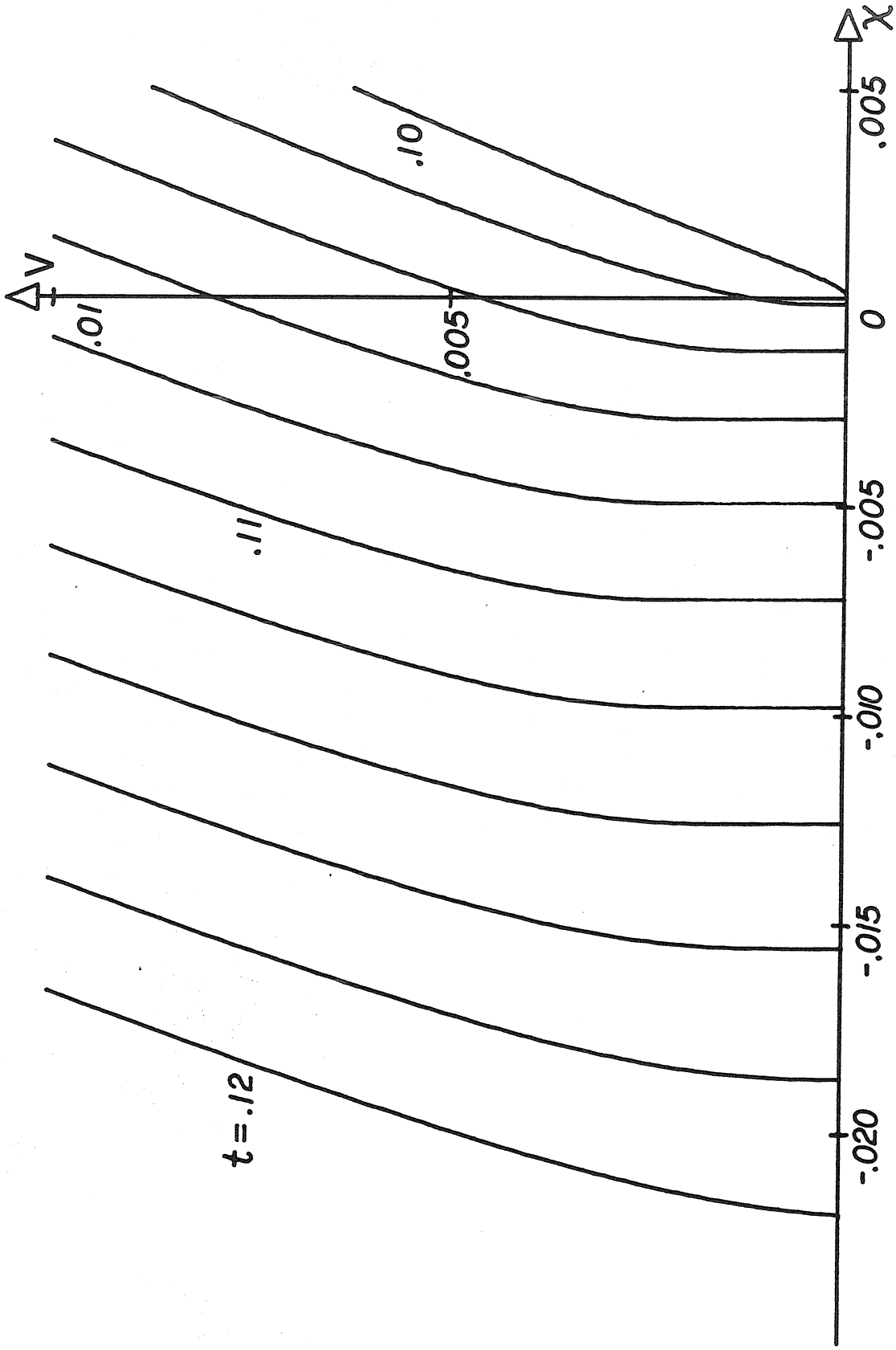

Figure 4.3. The trouble with the expansion just after the shock-formation time. $\varepsilon = 0.25$.

formation time there is a region near the interface where v_x is too large, as if the first-order term in the approximate solution were overcorrecting for the position of the interface. This trouble does not persist, however, and after a certain amount of time disappears.

By simply reducing the value of ϵ by one-half it is found that the difficulty lasts only half as long. Therefore, the error must be in the $O(\epsilon)$ term of the expansion.

We have already shown that when the corner layer occurs at $v = 0$, it is, in fact, sharp; hence, this error cannot be corrected by rescaling in the corner region. There is one effect that has been excluded from our perturbation expansion, however. Since we have been using the leading-order characteristics, there has been no way to correct the time at which the interface begins to move.

To do this we merely rescale the time by assuming

$$\tilde{t} = (1 + \epsilon\omega_1 + \epsilon^2\omega_2 + \cdots)t \ , \tag{4.23}$$

where $\omega_1, \omega_2, \cdots$ are chosen to correct the difficulties in the perturbation expansion (see [9], p. 82). With this assumption v satisfies

$$v_{\tilde{t}} = v_x^2 + \epsilon(vv_{xx} - \omega_1 v_{\tilde{t}}) + \cdots . \tag{4.24}$$

We solve (4.24) using a regular perturbation expansion, and find

$$v_0(\xi, \tilde{t}) = f(\xi) - f'^2(\xi)\tilde{t} \ , \tag{4.25}$$

$$x(\xi, \tilde{t}) = \xi - 2f'(\xi)\tilde{t} \ , \tag{4.26}$$

$$v_1(\xi, \tilde{t}) = \left[\frac{1}{4}\frac{f'^2(\xi)}{f''(\xi)} - \frac{1}{2}f(\xi)\right] \ell n\left|1-2f''(\xi)\tilde{t}\right| + (\tfrac{1}{2} - \omega_1) f'^2(\xi)\tilde{t} \ . \tag{4.27}$$

The constant ω_1 is determined by examining the approximate solution near the time and position of shock formation, i.e., for t near $[2f''(0)]^{-1}$ and ξ small. If we assume

$$f(\xi) \sim a\xi^2 + o(\xi^2) \quad \text{as} \quad \xi \to 0 \ ,$$

then $[2f''(0)]^{-1} = (4a)^{-1}$ and

$$v_0 + \epsilon v_1 \sim a\xi^2[1-4at + \epsilon(\tfrac{1}{2} - \omega_1)4at] + o(\xi^2) \quad \text{as} \quad \xi \to 0 \ .$$

From this expansion we can see that the difficulty arises due to the dominance of the $O(\epsilon)$ term over the $O(1)$ term when ξ is small and t is near $(4a)^{-1}$. We correct this by requiring that ϵv_1 remains small in comparison with v_0, which means that we must take

$$\omega_1 = \frac{1}{2} \ . \tag{4.28}$$

Therefore, we use (4.25), (4.26),

$$v_1(\xi, \tilde{t}) = \left[\frac{1}{4}\frac{f'^2(\xi)}{f''(\xi)} - \frac{1}{2}f(\xi)\right] \ell n\left|1-2f''(\xi)\tilde{t}\right|, \tag{4.29}$$

and

$$\tilde{t} = (1 + \tfrac{1}{2}\epsilon + \cdots)t \qquad (4.30)$$

as the corrected regular expansion for v to $O(\epsilon)$.

In Fig. 4.4 we show the result of using this corrected expansion, using the same initial condition as for Fig. 4.3. This demonstrates quite effectively that the difficulties from using the previous expansion have been eliminated to $O(\epsilon)$ by this rescaling of the time.

With the expansion corrected in this manner, we can accurately approximate to $O(\epsilon)$ the solutions of (2.16) which have quadratic behavior near the boundary, and which therefore begin to move when a corner shock forms at that boundary. A typical example is shown in Fig. 4.5 (again we use the same initial condition as for Fig. 4.3 but now include the entire domain over which $v > 0$).

As before, once v is known everywhere to $O(\epsilon)$, the transformation (2.18) is used to calculate u to $O(1)$. The result for this case is given in Fig. 4.6.

The new time-scaling (4.30) also provides a first-order correction to the leading-order waiting time in the case where corner shocks form only at the interface. In this case, the interface begins to move when

$$\tilde{t}_c = \frac{1}{2f''(\zeta_0)} ,$$

as can be seen from (4.25) and (4.26). In terms of the original

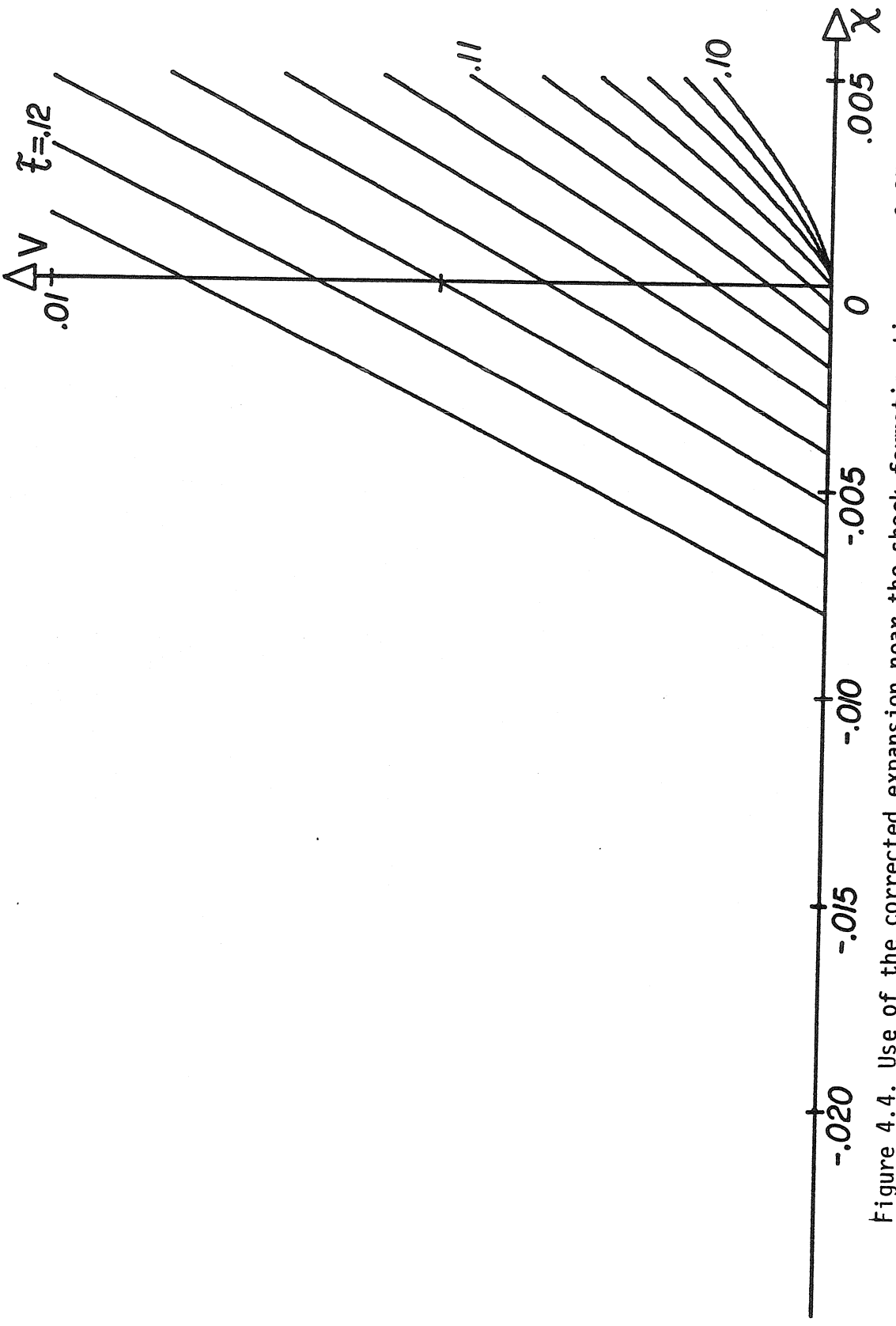

Figure 4.4. Use of the corrected expansion near the shock-formation time. $\varepsilon = 0.25$.

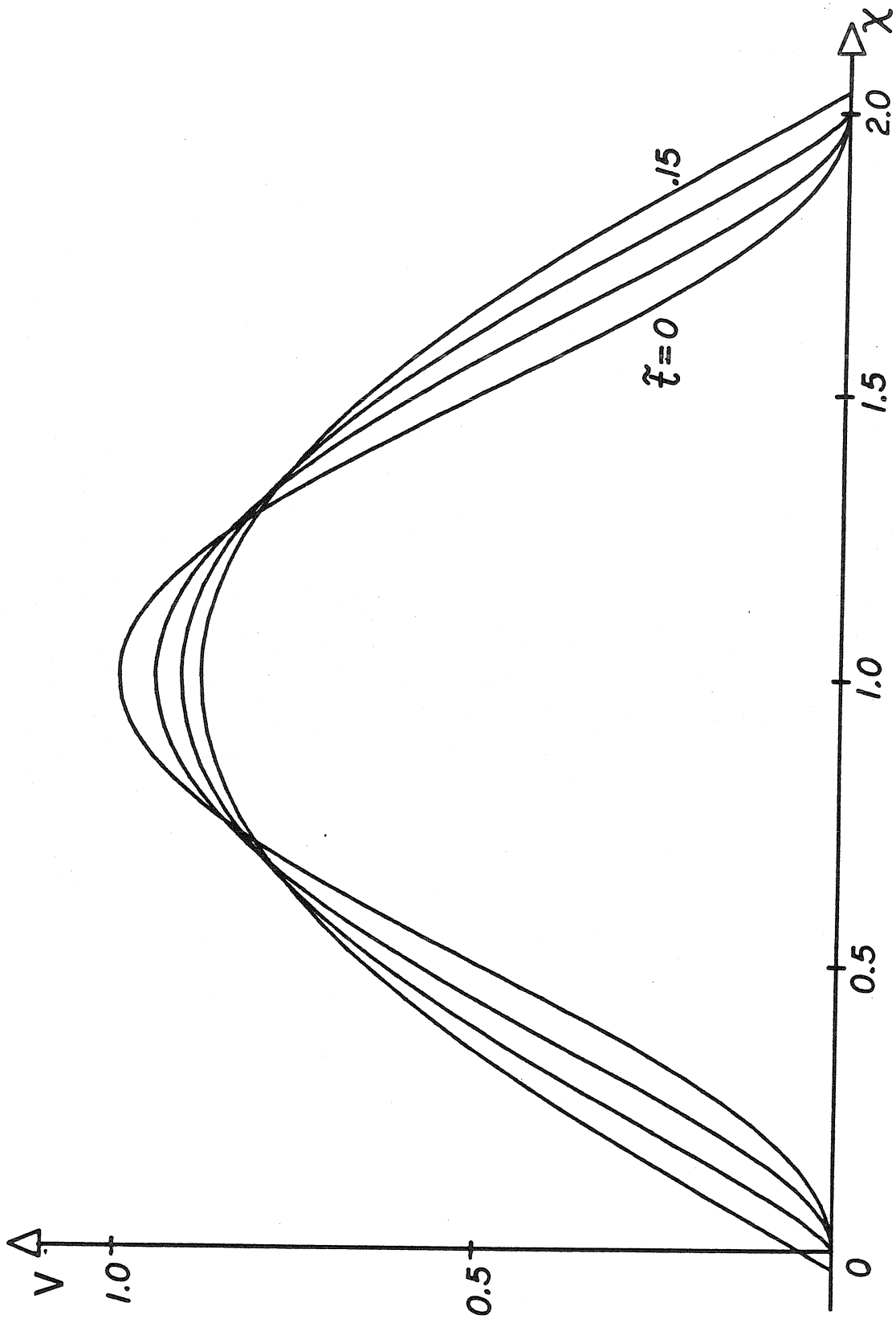

Figure 4.5. First-order solution for v with non-zero waiting time. $\varepsilon = 0.25$.

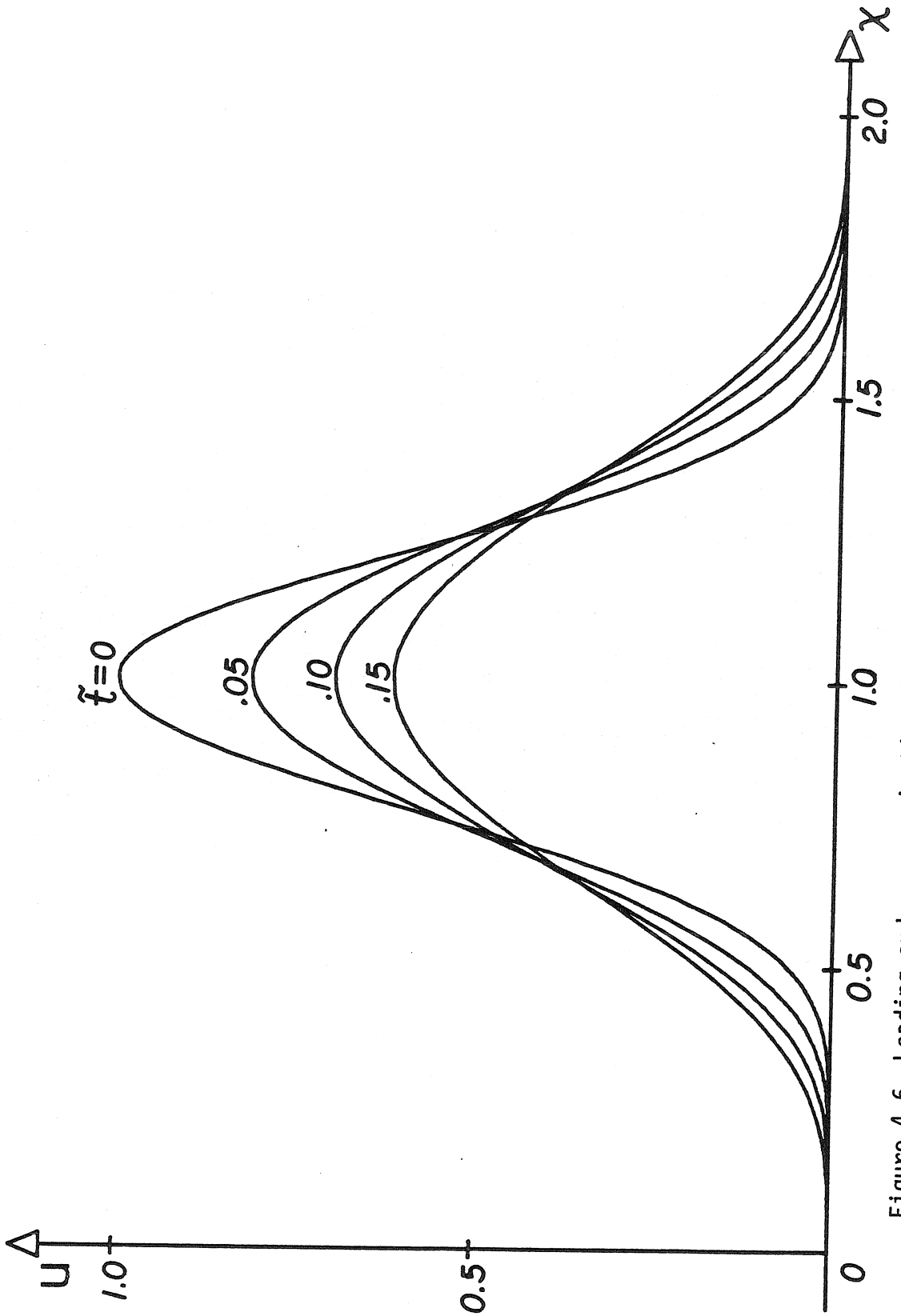

Figure 4.6. Leading-order approximation for u with non-zero waiting time. $\epsilon = 0.25$.

unscaled time variable,

$$t_c^* = \epsilon t_c = \frac{\epsilon}{1 + \frac{1}{2}\epsilon + \cdots} \tilde{t}_c ,$$

this gives

$$t_c^* = \frac{\epsilon}{2 + \epsilon + \cdots} \frac{1}{f''(\zeta_0)} \tag{4.31}$$

as the resulting approximation for the waiting time.

For comparison, note that the answer given by the use of the similarity solution (1.10) and the maximum principle is (see [1], [12], or [13])

$$t_c^* = \frac{\epsilon}{2+\epsilon} \frac{1}{f''(\zeta_0)} .$$

Therefore, our expansion correctly approximates the waiting time to the next order in ϵ.

4.4 Checks on the Validity of the Expansion

The above comparison of the approximate waiting time with the waiting time determined by similarity solutions provides one check on the consistency of our expansion procedure. In addition, there are other ways in which we can check this consistency.

One method is to compare our approximate solution with an exact solution of (2.16), such as the similarity solution

$$v = (t_0 + t)^{-\frac{\epsilon}{2+\epsilon}} \left[a - \frac{1}{4+2\epsilon} \frac{x^2}{(t_0+t)^{\frac{2}{2+\epsilon}}} \right]. \tag{4.32}$$

Here a is a constant which may depend on ϵ, and (4.32) is only used where it gives $v \geq 0$ (otherwise take $v = 0$).

If we assume that a does not depend on ϵ and expand (4.32) as $\epsilon \to 0$, we find

$$v \sim a - \frac{1}{4}\frac{x^2}{\tau} + \epsilon\left[-\frac{a}{2} \ell n\, \tau + \frac{1}{8}\frac{x^2}{\tau} \right] + \cdots, \tag{4.33}$$

where $\tau \equiv t_0 + t$. When evaluated at $t=0$ this asymptotic solution gives the initial conditions for our expansion. Therefore, we take

$$v_0(x, 0) = a - \frac{1}{4}\frac{x^2}{t_0},$$

$$v_1(x, 0) = -\frac{a}{2} \ell n\, t_0 + \frac{1}{8}\frac{x^2}{t_0},$$

.
.
.

The solutions having these initial conditions are easily found to be

$$v_0(\xi, \tilde{t}) = a - \frac{1}{4}\frac{\xi^2}{t_0} - \frac{1}{4}\frac{\xi^2}{t_0}\tilde{t} = a - \frac{1}{4}\left(\frac{\xi}{t_0}\right)^2 (\tilde{t}+t_0),$$

$$v_1(\xi, \tilde{t}) = \left[\frac{1}{4}\left(-\frac{1}{2}\frac{\xi^2}{t_0}\right) - \frac{1}{2}\left(a - \frac{1}{4}\frac{\xi^2}{t_0}\right) \right] \ell n \left| 1 + \frac{\tilde{t}}{t_0} \right| - \frac{a}{2} \ell n\, t_0 + \frac{1}{8}\frac{\xi^2}{t_0}$$

$$= -\frac{a}{2} \ell n(\tilde{t}+t_0) + \frac{1}{8}\frac{\xi^2}{t_0},$$

where

$$\tilde{t} = \left(1 + \frac{1}{2}\epsilon + \cdots\right) t ,$$

$$x(\xi, \tilde{t}) = \xi + \frac{\xi}{t_0}\tilde{t} = \frac{\xi}{t_0}(\tilde{t} + t_0) .$$

When we combine the first two terms of the perturbation series solution and revert to the original variables x and t, the approximation for v becomes

$$v \sim v_0(\xi, \tilde{t}) + \epsilon v_1(\xi, \tilde{t}) + \cdots$$

$$\sim a - \frac{1}{4}\frac{x^2}{(t+t_0)} + \epsilon\left[-\frac{a}{2}\ell n(t+t_0) + \frac{1}{8}\frac{x^2}{(t+t_0)}\right] + \cdots,$$

which agrees with the expansion of the exact solution (4.33).

Another method which we can use to check the consistency of our expansion is to see if the total concentration,

$$\int_{-\infty}^{\infty} u(x, t^*; \epsilon)\, dx, \tag{4.34}$$

is conserved. From equation (2.1) it is easy to see that this integral must be independent of the time. Our expansion procedure, however, makes no special assumptions to ensure that (4.34) remains constant. Therefore, we can use our leading-order approximation for u and see if the integral is also constant to leading order.

The results of this check are shown in Table 4.1, where we have used the initial condition

TABLE 4.1

Perturbation Result for Total Concentration

$\tilde{t}$	$\epsilon = 0.25$		$\epsilon = 0.125$		$\epsilon = 0.0625$	
	I	Δ	I	Δ	I	Δ
0.00	1.335	0	1.008	0	0.738	0
0.04	1.293	-0.126	0.992	-0.122	0.733	-0.120
0.08	1.247	-0.264	0.976	-0.251	0.727	-0.244
0.12	1.205	-0.390	0.961	-0.368	0.722	-0.357
0.16	1.168	-0.502	0.948	-0.472	0.717	-0.459
0.20	1.135	-0.601	0.936	-0.566	0.713	-0.550

$$v(x, 0; \epsilon) \equiv f(x) = \begin{cases} \frac{1-\cos \pi x}{2} & \text{for } 0 \leq x \leq 2, \\ 0 & \text{otherwise.} \end{cases}$$

In the table, I and Δ are defined by

$$I(\tilde{t}; \epsilon) \equiv \int_{-\infty}^{\infty} \frac{1}{2}\left[v_0 + \epsilon v_1 + |v_0 + \epsilon v_1|\right]^{\frac{1}{\epsilon}} dx,$$

$$\Delta(\tilde{t}; \epsilon) \equiv \frac{I(\tilde{t}; \epsilon) - I(0; \epsilon)}{\epsilon I(0; \epsilon)}.$$

We have taken $v(x, 0)$ to be independent of ϵ in order to make the comparison between different values of ϵ more convenient; note, however, that the value of (4.34) at $t = 0$ then depends on ϵ. The effect of taking $u(x, 0)$ to be independent of ϵ will be discussed below.

Because u is only accurate to O(1) in this approximation, we therefore only expect $I(\tilde{t};\epsilon)$ to be independent of $\tilde{t}$ to O(1) as $\epsilon \to 0$. Also, (2.14) shows the expected error in u to be $O(\epsilon)$ so that the error in I should be $O(\epsilon)$ as well. The function $\Delta(\tilde{t};\epsilon)$ is a measure of the relative error in I of $O(\epsilon)$.

As $\epsilon \to 0$ we see that $\Delta(\tilde{t};\epsilon)$ approaches a limiting value for fixed $\tilde{t}$. This agrees with our expectation that the error made by using our approximation in the integral (4.34) is of $O(\epsilon)$ and is a check that we have indeed calculated u correctly to O(1).

4.5 The Effect of the Nonlinear Transformation upon the Initial Conditions

In Chapter 2 we ignored the effect of the transformation (2.8) upon the initial condition, and for simplicity assumed that v(x, 0) was independent of ϵ (as in (2.15)). This assumption allowed us to easily determine the behavior of solutions to (2.16), and therefore, of solutions to (2.3).

More precisely, however, we should take the initial condition for u to be

$$u(x, 0;\epsilon) = g(x) , \tag{4.35}$$

so that the initial condition for v is

$$v(x, 0;\epsilon) \equiv f(x;\epsilon) = [g(x)]^{\epsilon} . \tag{4.36}$$

To determine the consequences of requiring the initial condition for v to depend on ϵ in this manner, we will again separate the analysis into two different parts: near v = 0 and away from v = 0.

To examine what happens near the interface, let us for simplicity assume that

$$g(x) = \begin{cases} x & \text{for} \quad x \geq 0 \ , \\ 0 & \text{for} \quad x < 0 \ . \end{cases}$$

This can be thought of as the first term in a Taylor series near the position where $u(x, 0;\epsilon) = 0$ (the initial position of the interface); then

$$f(x;\epsilon) = \begin{cases} x^{\epsilon} & \text{for} \quad x \geq 0 \ , \\ 0 & \text{for} \quad x < 0 \ . \end{cases} \tag{4.37}$$

It is clear that we cannot construct an expansion in ϵ for (4.37) that will be uniform both for x small and x O(1). What we do instead is use the initial condition (4.37) in our expansion ((4.25), (4.26), (4.29), and (4.30)) as if it did not depend on ϵ, and then see if any difficulties arise by making this supposition.

With this assumption, we find from (4.25) and (4.26) that the approximate position of the interface is

$$\zeta(\tilde{t};\epsilon) = -\frac{2-\epsilon}{\epsilon} (\epsilon^2 \tilde{t})^{\frac{1}{2-\epsilon}} \ . \tag{4.38}$$

If we now take the limit of (4.38) as $\epsilon \to 0$, we find

$$\lim_{\epsilon \to 0} \zeta(\tilde{t}; \epsilon) = -2\tilde{t}^{1/2} , \tag{4.39}$$

which is finite. Although we have made a number of scalings and nonlinear transformations, no apparent difficulty arises in our approximation to the position of the interface. Our expansion has indeed determined the basic behavior of the interface, even though the initial condition for v depends on ϵ as in (4.36).

It is important to note that

$$\frac{d\zeta}{d\tilde{t}} = O(1) \quad \text{as} \quad \epsilon \to 0, \ \tilde{t} \text{ fixed},$$

so that in terms of the scaled time the speed of the interface remains finite as $\epsilon \to 0$. In terms of the original time t^*,

$$\frac{d\zeta}{dt^*} = O(\frac{1}{\epsilon}) \quad \text{as} \quad \epsilon \to 0, \ t^* \text{ fixed},$$

and the true speed of the interface becomes infinite as $\epsilon \to 0$. This is consistent with the fact that the linear diffusion equation, to which (2.3) reduces in the limit $\epsilon \to 0$, possesses an infinite signal velocity.

Notice also that near the interface $v \sim \text{const.}(x-\zeta)$, or $u \sim \text{const.}(x-\zeta)^{1/\epsilon}$. Thus, for ϵ finite the behavior of u near the interface is algebraic. In the limit $\epsilon \to 0$, however, the behavior does not remain so, which shows the approach to the exponential decay characteristic of the linear diffusion equation.

We must also examine the effect of the initial condition (4.36) on the solution away from $v=0$. In this case, take as an example

$$g(x) = \begin{cases} \cos\pi x & \text{for} \quad |x| \le \frac{1}{2}, \\ 0 & \text{for} \quad |x| > \frac{1}{2}, \end{cases} \tag{4.40}$$

so that

$$f(x;\epsilon) = \begin{cases} (\cos\pi x)^{\epsilon} & \text{for} \quad |x| \le \frac{1}{2}, \\ 0 & \text{for} \quad |x| > \frac{1}{2}. \end{cases} \tag{4.41}$$

We will examine the solution and its behavior near $x=0$. Again, it is not possible to expand the initial condition (4.41) in a series in ϵ valid for all x, so we solve the problem for v by using the initial condition in our expansion ((4.25), 4.26), (4.29), and (4.30)) as if it did not depend on ϵ.

To check the validity of the resulting approximation, we examine the value at x = 0,

$$v(0, \tilde{t};\epsilon) \sim 1 - \frac{\epsilon}{2} \ell n(1 + 2\epsilon\pi^2\tilde{t}) + \cdots, \tag{4.42}$$

which yields for u, upon using (2.18),

$$u(0,t^*;\epsilon) \sim \left[1 - \frac{\epsilon}{2}\ell n\left(1+2\pi^2(1+\tfrac{1}{2}\epsilon+\cdots)t^*\right)\right]^{\frac{1}{\epsilon}}+\cdots. \qquad (4.43)$$

If we now keep t^* fixed and examine the limit $\epsilon \to 0$, we find (for this approximation)

$$\lim_{\epsilon\to 0} u(0,t^*;\epsilon) = e^{-\frac{1}{2}\ell n(1+2\pi^2 t^*)} = (1+2\pi^2 t^*)^{-\frac{1}{2}}, \qquad (4.44)$$

and for small t^* this gives

$$\lim_{\epsilon\to 0} u(0,t^*;\epsilon) = 1 - \pi^2 t^* + \frac{3}{2}\pi^4 t^{*2} + O(t^{*3}) . \qquad (4.45)$$

For ϵ small, however, we also expect the solution of the linear diffusion equation to give a good approximation near x = 0, since we are not near the position where u = 0. The solution with initial condition (4.40) is easily found,

$$u_D(x,t^*) = \frac{1}{\sqrt{4\pi t^*}} \int_{-\frac{1}{2}}^{\frac{1}{2}} e^{\frac{-(x-\xi)^2}{4t^*}} \cos\pi\xi d\xi . \qquad (4.46)$$

Standard approximation techniques can be used to find u_D for x = 0 and small t^*, and the result is

$$u_D(0,t^*) = 1 - \pi^2 t^* + \frac{1}{2}\pi^4 t^{*2} + O(t^{*3}). \qquad (4.47)$$

Clearly, the two expansions (4.45) and (4.47) agree only in the limit of small t^*. Therefore, we see that the result of using the

initial condition for v as if it does not depend on ϵ limits the validity of our expansion to small t^*.

More precisely, the expansion procedure we have used to obtain (4.45) will be valid if $t^* \to 0$ as $\epsilon \to 0$ or, in terms of the scaled variable $\tilde{t}$, if $\tilde{t} << 1/\epsilon$ as $\epsilon \to 0$ (i.e., for times that are not too long). Thus, we can be assured of the validity of our expansion when $\tilde{t}$ remains fixed as $\epsilon \to 0$. For the extension to longer times, however, further approximations must be made.

The above discussion is summarized in Figs. 4.7 and 4.8, which are computer generated plots of the approximate solution for u when $\epsilon = 0.25$ and 0.125; in both u(x, 0) is given by (4.40). The two figures are for the same values of the scaled time, namely $\tilde{t} = 0$, 0.05, and 0.10.

Of particular interest is comparison of the solutions for the two different values of ϵ. Although ϵ is reduced by a factor of 2 in the second figure, the position of the interface remains roughly the same. This is what is expected from the above analysis, since we are keeping $\tilde{t}$ fixed as $\epsilon \to 0$.

Near x = 0 we see that reducing ϵ does have a marked effect upon the solution. This also is expected, because if we rescale the time in (4.45), we obtain

$$u(0, \tilde{t}; \epsilon) \sim 1 - \epsilon \pi^2 \tilde{t} + \cdots .$$

Therefore, if we keep $\tilde{t}$ fixed and reduce ϵ by a factor of 2, the decrease from the initial value will also be reduced by a factor of 2,

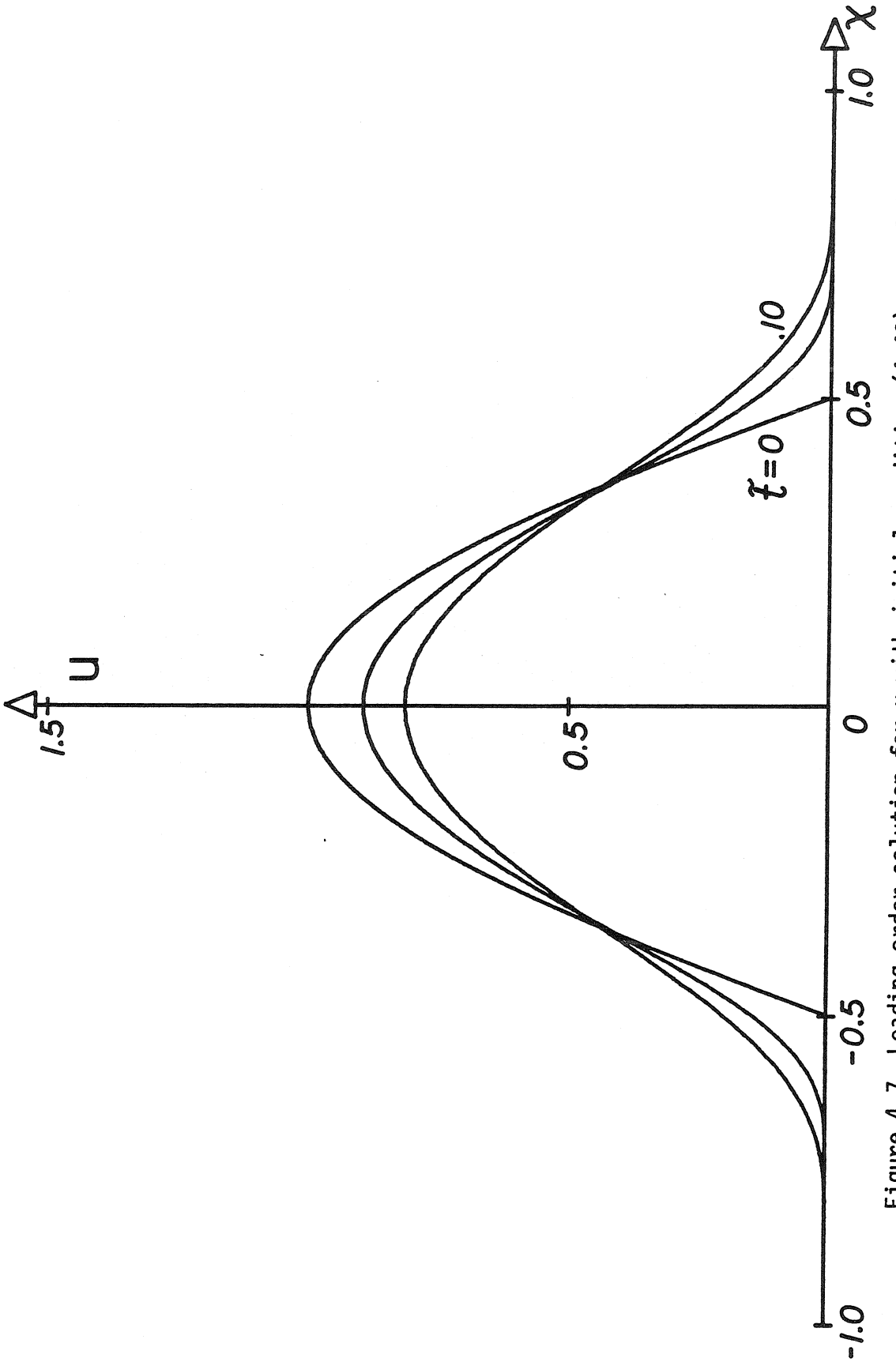

Figure 4.7. Leading-order solution for u with initial condition (4.40). $\varepsilon = 0.25$.

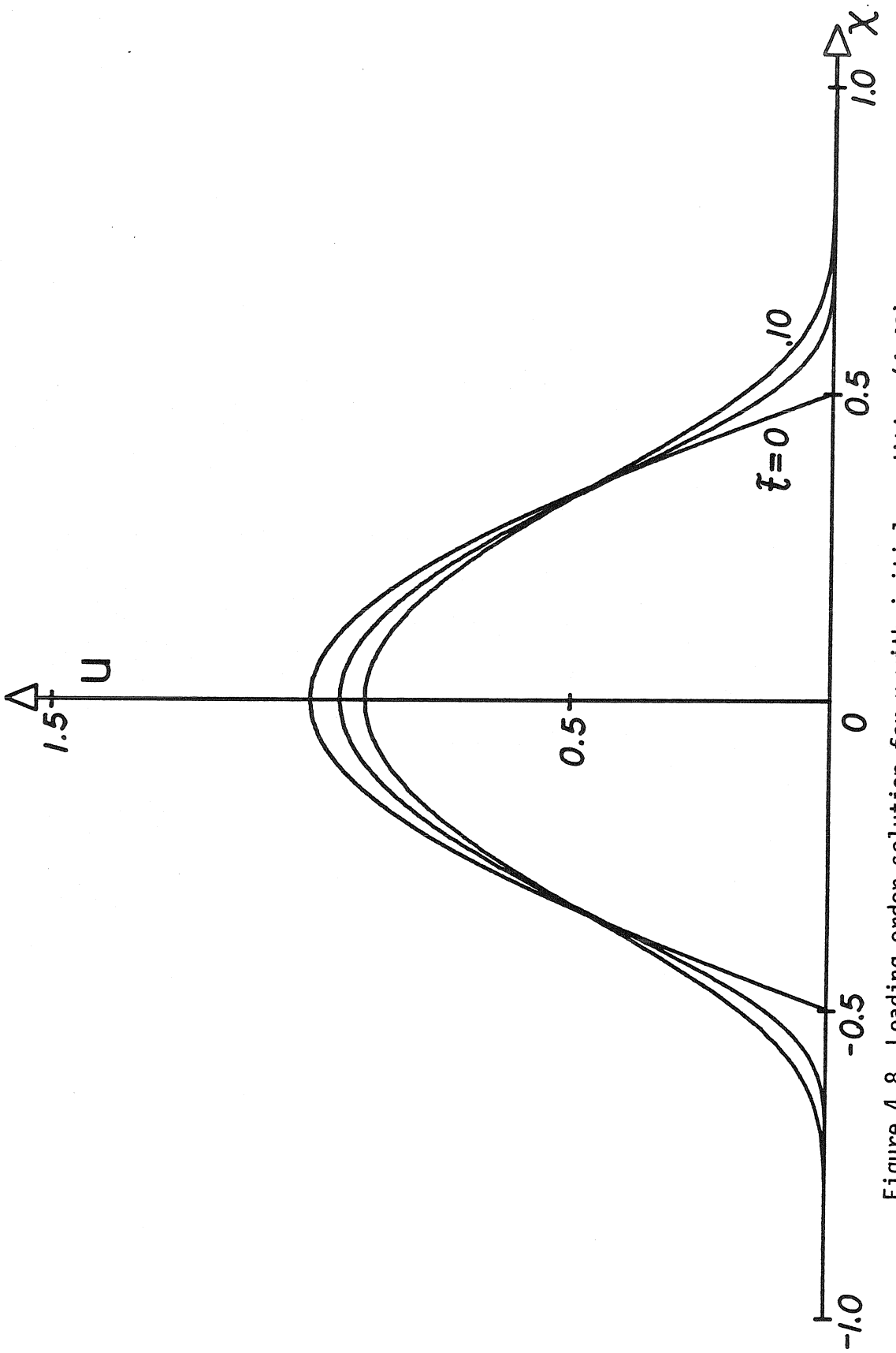

Figure 4.8. Leading-order solution for u with initial condition (4.40). $\varepsilon = 0.125$.

as is shown to be the case.

Table 4.2 gives the values of I and Δ for these two approximate solutions as another check.

TABLE 4.2

Area Calculations for Figs. 4.7 and 4.8

	$\epsilon = 0.25$		$\epsilon = 0.125$	
$\tilde{t}$	I	Δ	I	Δ
0.00	0.637	0	0.637	0
0.05	0.617	-0.124	0.634	-0.030
0.10	0.611	-0.162	0.636	-0.010

As expected, the error in the area is reduced by halving ϵ. Note, however, that in this case it does not appear that Δ is approaching a nonzero limit as $\epsilon \to 0$.

This can be explained by examining the $O(\epsilon)$ term in the expansion of u, (2.14), which depends on v_0, v_1, and v_2. When we take $v(x, 0)$ to be independent of ϵ, so are v_0, v_1, and v_2. Performing the integration in x, this means that the relative error in the integral (4.34) will be roughly ϵ times some function of $\tilde{t}$, especially for $\tilde{t}$ small. This is what is seen in Table 4.1.

Now, however, the initial condition for v does depend on ϵ, and therefore so must v_0, v_1, and v_2. Thus, the limit as $\epsilon \to 0$ of

the integral must be more complicated.

There is another, more subtle, reason why our expansion should not be expected to be valid for large times, no matter which initial condition is independent of ϵ. Kamenomostskaya [11] has proved that all solutions of equation (2.1) (with appropriate initial conditions) are asymptotic to the similarity solution

$$u_s = t^{*-\frac{1}{2+n}} \left[E - \frac{n}{4+2n} \frac{x^2}{t^{*2/(2+n)}} \right]^{\frac{1}{n}} \tag{4.48}$$

in the limit of large time, where the constant E depends upon the total initial concentration (4.34). The dependence upon the total initial area of u is important, because our expansion procedure does not incorporate this dependence. Therefore, it cannot be valid in the limit $t^* \to \infty$.

It may be possible, however, to use our approximate solution for small times, the similarity solution (4.48) for large times, and to match them with some other solution for intermediate times. At present, this possibility has not been adequately examined.

5. DISCUSSION OF THE RESULTS

We have constructed approximate solutions of the equation (1.7) in the limit of small n. These approximate solutions, although valid only in this limit, reveal clearly the behavior of the solutions of (1.7) which is true for all $n > 0$.

The solutions are constructed through the use of (2.16), the transformed equation for v. The leading-order approximation to this equation, (2.19), is found to contain all of the basic effects in which we are interested: we obtain solutions with propagating interfaces, beyond which the solution is identically zero, and for the correct initial conditions these interfaces remain stationary for a finite time before beginning to move. The approximate equation (2.19) also enables us to determine the manner in which the interface begins to move, and to give a leading-order approximation for the waiting time. This waiting time is zero unless the initial condition in v grows quadratically or more slowly near the interface.

To determine the solution for u to leading order, we find we must calculate v to first order for all x. This is straightforward except in regions where v_x changes rapidly (corner shocks, in the leading-order approximation), where a more sophisticated perturbation technique is used, and we see that there are two cases.

If $v \neq 0$ at a corner shock, then it becomes rounded in the next approximation (a corner layer), and the structure of this layer

is given. When this corner layer is not present initially, but develops at a later time, there is a transition solution near the time of shock formation from which our corner layer solution must be matched; we do not perform this matching, however.

If $v = 0$ at a corner shock, then to the order of our expansion it is not rounded, but remains sharp. In addition, in this case only minor modifications are needed to obtain approximate solutions which are valid through the shock formation region. Therefore, we can give a simple procedure for constructing approximate solutions when corner shocks occur only at $v = 0$, and a number of examples are given using this method.

Several checks are made on the consistency of the expansion, and there is good agreement with results from other methods as long as the scaled time, $\tilde{t}$, does not become too large. Thus, the approximations that are made limit the validity expansion to when $\tilde{t}$ remains $O(1)$ as $\epsilon \to 0$, or in other words to when t^* is small. This in no way invalidates our results (waiting-time estimates and solutions, etc.), because these interesting effects all occur for t^* small, well within the usable range of our expansion.

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PART II

SUSTAINED REENTRY ROLL RESONANCE

1. INTRODUCTION

Kevorkian [4] and Lewin and Kevorkian [5] have presented and analyzed a simplified mathematical model to explain the problem of roll buildup which occurs when a spacecraft with slight aerodynamic and mass asymmetries reenters the earth's atmosphere. This reentry vehicle is assumed to be almost axially symmetric and spinning about this axis of symmetry as it descends. Kevorkian's model for this situation is then

$$\begin{aligned} \ddot{x} + (p^2 + \omega^2)x &= 0 , \\ \dot{p} &= \epsilon\omega^2 x \sin\psi , \\ \dot{\psi} &= \sqrt{2}\,p . \end{aligned} \tag{1.1}$$

In the above equations x represents either a pitch or yaw angle (only one is included for simplicity) relative to rotating axes, p is the roll rate, and ψ is the roll angle. The small constant ϵ measures the amount of asymmetry present in the reentry vehicle (ϵ also measures the amount of coupling between the pitching and rolling motions). In addition, ω represents the natural frequency of pitching oscillations and is related to the density of the atmosphere and the velocity of descent by

$$\omega^2 = (\text{const.})\,\rho v^2 . \tag{1.2}$$

If we assume an exponential distribution of density in the atmosphere, a constant descent velocity, and a constant flight path

angle γ, then

$$\rho = \rho_0 \exp\left(\frac{-v t \sin\gamma}{H}\right) \equiv \rho_0 e^{kt} , \tag{1.3}$$

where H is the scale height of the atmosphere (note for reentry γ is negative so that k is positive). Kevorkian also assumes that the time for one oscillation in x is small compared to the time required for the density to change appreciably, and for definiteness takes $k = O(\epsilon)$, so that

$$\omega^2 = \omega_0^2 e^{\epsilon t} . \tag{1.4}$$

For our analysis we shall also assume that this oscillation frequency is a slowly varying function of the time (i.e., $\omega = \omega(\epsilon t)$), but shall not assume that it depends upon the slow time as in (1.4) (this allows either the descent velocity or the flight path angle to change during reentry). We shall see from the analysis that although the exponential dependence in (1.4) is indeed the simplest choice to make, it would cause us to miss some interesting and potentially important behavior of the solutions of (1.1).

Lewin and Kevorkian [5] show that roll buildup is essentially a case of sustained resonance, where the two basic frequencies of the model, ω and p, remain close for long periods of time. This corresponds to a sustained resonant coupling between the pitching and rolling motions of the reentering vehicle. They also show that sustained resonance occurs only for a narrow range of initial conditions. The initial roll rate must be larger than the initial value

of the frequency ω, which is increasing slowly with time. The roll rate remains roughly constant until ω reaches this value and resonance occurs. If conditions are correct, then sustained resonance and roll buildup follow, but in most cases there is passage through resonance without a substantial buildup in the roll rate.

We shall reexamine this model of reentry roll resonance, with particular emphasis upon the case of sustained resonance, by a method different from the multiple-scale expansion used by Lewin and Kevorkian. This method has been used recently under the name of near-identity transformations (see Neu, [6] and [7]), but has also been known as von Zeipel's method (see [3] and [8]) and as generalized averaging (see [2], p. 412). We shall show that this method possesses a number of advantages when used to study the problem of sustained resonance, and that these advantages allow us to extend the results of [5].

We shall also show the significance of not requiring the oscillation frequency $\omega(\epsilon t)$ to have an exponential dependence. If ω behaves as in (1.4), then once sustained resonance is started it will continue indefinitely. If ω has some other dependence, however, then it is possible for sustained resonance to begin, continue for a certain amount of time, and then cease: the changes in ω on the slow time scale can cause the sustained resonance between the pitching and rolling motions to unlock. We shall give a simple condition for ω which will tell when sustained resonance is possible, and which can be used to tell when this unlocking occurs.

2. THE NEAR-IDENTITY TRANSFORMATION

We study the system

$$\begin{aligned} \ddot{x} + (p^2 + \omega^2)x &= 0 , \\ \dot{p} &= \epsilon\omega^2 x \sin\psi , \\ \dot{\psi} &= \sqrt{2}\, p , \end{aligned} \tag{2.1}$$

where $\omega = \omega(\tilde{t})$, $\tilde{t} = \epsilon t$, and $0 < \epsilon << 1$. To use the near-identity transformation in its simplest form, we shall first make a number of changes of variable based upon the solution of (2.1) when $\epsilon = 0$.

Write the first equation as a system

$$\begin{aligned} \dot{x} &= y , \\ \dot{y} &= -(p^2 + \omega^2)x , \end{aligned} \tag{2.2}$$

and change from (x, y) to (r, φ) using

$$\begin{aligned} x &= r\cos\varphi , \\ y &= -(p^2 + \omega^2)^{1/2} r\sin\varphi . \end{aligned} \tag{2.3}$$

In terms of these polar coordinates (2.1) then becomes

$$\dot{r} = -\frac{1}{2} r \frac{p\dot{p} + \omega\dot{\omega}}{p^2+\omega^2} + \frac{1}{2} r \frac{p\dot{p}+\omega\dot{\omega}}{p^2+\omega^2} \cos 2\varphi ,$$

$$\dot{\varphi} = (p^2+\omega^2)^{1/2} - \frac{1}{2} \frac{p\dot{p}+\omega\dot{\omega}}{p^2+\omega^2} \sin 2\varphi , \tag{2.4}$$

$$\dot{p} = \frac{1}{2} \epsilon \omega^2 r \left[\sin(\psi+\varphi) + \sin(\psi-\varphi) \right] ,$$

$$\dot{\psi} = \sqrt{2}\, p .$$

The first equation can be written

$$\frac{d}{dt}\left[r(p^2+\omega^2)^{\frac{1}{4}} \right] = \frac{1}{2} r (p^2+\omega^2)^{\frac{1}{4}} \frac{p\dot{p}+\omega\dot{\omega}}{p^2+\omega^2} \cos 2\varphi,$$

which suggests the further transformation

$$a = r(p^2+\omega^2)^{\frac{1}{4}} . \tag{2.5}$$

If p were a known function of the slow time $\tilde{t}$, then a would be an adiabatic invariant for the x oscillation. When we also use

$$\dot{\omega}(\tilde{t}) = \epsilon \omega'(\tilde{t})$$

and the expression for p to simplify (2.4), it becomes (various trigonometric identities are also needed)

$$\dot{a} = \frac{\epsilon}{2} a \frac{\omega\omega'}{p^2+\omega^2} \cos 2\varphi + \epsilon \frac{pa^2\omega^2}{8(p^2+\omega^2)^{5/4}} \Big[\sin(\psi+3\varphi) + \sin(\psi-\varphi) + \sin(\psi+\varphi) + \sin(\psi-3\varphi) \Big] ,$$

$$\dot{\varphi} = (p^2+\omega^2)^{\frac{1}{2}} - \frac{\epsilon}{2} \frac{\omega\omega'}{p^2+\omega^2} \sin 2\varphi - \epsilon \frac{pa\omega^2}{8(p^2+\omega^2)^{5/4}} \Big[\cos(\psi-\varphi) - \cos(\psi+3\varphi) + \cos(\psi-3\varphi) - \cos(\psi+\varphi) \Big] , \tag{2.6}$$

$$\dot{p} = \epsilon \frac{a\omega^2}{2(p^2+\omega^2)^{1/4}} \Big[\sin(\psi+\varphi) + \sin(\psi-\varphi) \Big] ,$$

$$\dot{\psi} = \sqrt{2}\, p .$$

Although the system (2.6) may look algebraically complicated, it is in fact in a very simple form for further analysis. The reason for this is that in (2.6) only certain terms are important for determining the long-term behavior of (2.1); the other terms are inconsequential and yield small oscillations on the fast time scale. The method of near-identity transformations gives a systematic way of separating these important terms from these small, fast oscillations.

To clearly show how the method works, let us avoid the algebraic complexity of (2.6) by writing the system in vector form,

$$\dot{\vec{z}} = \vec{\Lambda}(\vec{z}) + \epsilon \vec{f}(\vec{z}, \tilde{t}) \ . \tag{2.7}$$

Here

$$\vec{z} = (a, \varphi, p, \psi)^T \ , \tag{2.8}$$

$$\vec{\Lambda}(\vec{z}) = \left(0, (p^2 + \omega^2)^{\frac{1}{2}}, \ 0, \sqrt{2}\, p\right)^T \ , \tag{2.9}$$

and $\vec{f}(\vec{z}, \tilde{t})$ includes all of the $O(\epsilon)$ terms in (2.6).

The method of near-identity transformations consists of examining the above equations for dependent variables that are nearby the original ones, by assuming

$$\begin{aligned} a &= A + \epsilon a_1(A, \Phi, P, \Psi, \tilde{t}) + \cdots \\ \varphi &= \Phi + \epsilon \varphi_1(A, \Phi, P, \Psi, \tilde{t}) + \cdots \\ p &= P + \epsilon p_1(A, \Phi, P, \Psi, \tilde{t}) + \cdots \\ \psi &= \Psi + \epsilon \psi_1(A, \Phi, P, \Psi, \tilde{t}) + \cdots \ , \end{aligned} \tag{2.10}$$

or in vector form

$$\vec{z} = \vec{Z} + \epsilon \vec{z}_1(\vec{Z}, \tilde{t}) + \cdots . \tag{2.11}$$

This shows the reason for the use of the term "near-identity transformation". We shall see that a judicious choice of a_1, φ_1, etc., yields much

simpler equations for $A, \Phi, P,$ and Ψ.

From (2.11) we find

$$\dot{\vec{z}} = [I + \epsilon H_1 + \cdots]\dot{\vec{Z}} + \cdots,$$

where

$$H_1 = \frac{\partial \vec{z}_1}{\partial \vec{Z}} \quad .$$

When this expression is inverted, and the near-identity transformation (2.11) is substituted in the equation for $\dot{\vec{z}}$, (2.7), we obtain

$$\dot{\vec{Z}} = \vec{\Lambda}(\vec{Z}) + \epsilon\left[\vec{f}(\vec{Z}, \tilde{t}) - H_1\vec{\Lambda}(\vec{Z}) + \frac{\partial \vec{\Lambda}}{\partial \vec{Z}}\vec{z}_1\right] + \cdots. \qquad (2.12)$$

The vector function $\vec{z}_1$ (the functions $a_1, \varphi_1, p_1,$ and ψ_1) is chosen to simplify the resulting equation for $\vec{Z}$ as much as possible, but in such a way that all of the components of $\vec{z}_1$ remain O(1) at all times.

For example, consider the equation for P, which is

$$\dot{P} = \epsilon\left[\frac{A\omega^2}{2(P^2+\omega^2)^{\frac{1}{4}}}\left\{\sin(\Psi + \Phi) + \sin(\Psi - \Phi)\right\}\right.$$

$$\left. -(P^2+\omega^2)^{\frac{1}{2}}\frac{\partial p_1}{\partial \Phi} - \sqrt{2}\, P\frac{\partial p_1}{\partial \Psi}\right] + \cdots. \qquad (2.13)$$

If we choose p_1 so that $\dot{P} = O(\epsilon^2)$ (the most we can simplify the equation), then p_1 must satisfy

$$(P^2+\omega^2)\frac{\partial p_1}{\partial \Phi} + \sqrt{2}\,P\frac{\partial p_1}{\partial \Psi} = \frac{A\omega^2}{2(P^2+\omega^2)^{\frac{1}{4}}}\left\{\sin(\Psi+\Phi)+\sin(\Psi-\Phi)\right\}. \qquad (2.14)$$

This equation has the particular solution

$$P_1 = -\frac{A\omega^2}{2(P^2+\omega^2)^{\frac{1}{4}}}\left[\frac{\cos(\Psi+\Phi)}{(P^2+\omega^2)^{\frac{1}{2}}+\sqrt{2}\,P} - \frac{\cos(\Psi-\Phi)}{(P^2+\omega^2)^{\frac{1}{2}}-\sqrt{2}\,P}\right], \qquad (2.15)$$

as can be verified easily by either direct calculation or substitution into (2.14).

The first term of the particular solution remains O(1) as long as P is not near $-\omega$, and the second term remains O(1) as long as P is not near ω. Since we wish to examine the case of sustained resonance, where P is close to ω for long periods of time, this second term clearly cannot be included in p_1, and therefore the $\sin(\Psi-\Phi)$ term in (2.13) must remain in the equation for P.

The other equations are examined in a like manner to determine a_1, φ_1, and ψ_1, and only terms depending upon the phase difference $\Psi-\Phi$ must remain in (2.12). (From (2.6) it can be seen that the only other trouble could come from terms depending on $\Psi \pm 3\Phi$, but since $3(P^2+\omega^2)^{\frac{1}{2}} \pm \sqrt{2}\,P$ is never equal to zero, no difficulty arises.) The solutions for a_1, φ_1, p_1, and ψ_1 are given in the Appendix.

The most interesting part of leaving only terms that depend upon $\Psi-\Phi$ is the equations for A, Φ, P, and Ψ that result. They are

$$\dot{A} = \epsilon \frac{PA^2\omega^2}{8(P^2+\omega^2)^{5/4}} \sin(\Psi-\Phi) + \cdots,$$

$$\dot{\Phi} = (P^2+\omega^2)^{\frac{1}{2}} - \epsilon \frac{AP\omega^2}{8(P^2+\omega^2)^{5/4}} \cos(\Psi-\Phi) + \cdots, \qquad (2.16)$$

$$\dot{P} = \epsilon \frac{A\omega^2}{2(P^2+\omega^2)^{\frac{1}{4}}} \sin(\Psi-\Phi) + \cdots,$$

$$\dot{\Psi} = \sqrt{2}\,P + \cdots,$$

all accurate to $O(\epsilon^2)$. We neglect the $O(\epsilon^2)$ terms and define

$$\theta = \Psi - \Phi ,$$

so that (2.16) becomes

$$\dot{A} = \epsilon \frac{A^2P\omega^2}{8(P^2+\omega^2)^{5/4}} \sin\theta ,$$

$$\dot{P} = \epsilon \frac{A\omega^2}{2(P^2+\omega^2)^{1/4}} \sin\theta , \qquad (2.17)$$

$$\dot{\theta} = \sqrt{2}P - (P^2+\omega^2)^{\frac{1}{2}} + \epsilon \frac{AP\omega^2}{8(P^2+\omega^2)^{5/4}} \cos\theta .$$

If $\sqrt{2}\,P - (P^2+\omega^2)^{\frac{1}{2}} = O(1)$, then $\theta = O(1)$, and we can show that

$$A = \text{const.} + O(\epsilon) ,$$

$$P = \text{const.} + O(\epsilon) ,$$

where the $O(\epsilon)$ terms are oscillatory on the fast time scale (this can

also be seen from the near-identity transformation, because if P is not near ω, then the terms that depend upon $\Psi-\Phi$ can be included in $\vec{z}_1$ as fast oscillatory terms). This corresponds to Kevorkian's outer solution (see [4]), which is valid away from resonance.

If $P-\omega$ is small, however, then $\dot{\theta}$ is small, and we can show that the leading order values of A and P change slowly. This is the case of resonance, and it explains why terms that depend on $\Psi-\Phi$ must be included in the analysis: near resonance $\Psi-\Phi$ is a slowly varying function, and therefore a term like $\sin(\Psi-\Phi)$ does not represent a fast oscillation in this case.

The near-identity transformation has eliminated the unimportant terms from (2.6), so that the approximate system (2.17) includes only the terms necessary for determining the behavior of the model problem. Note that (2.17) is valid both away from and during resonance. This means that solutions of (2.17) will include the initial behavior, the behavior during resonance, and the behavior during sustained resonance, if it occurs. Unfortunately, (2.17) is too complicated to solve analytically, and therefore these uniform solutions must be determined from numerical integration of (2.17).

The approximate system (2.17) can be used to examine the sustained resonance solutions in more detail, however, and this can be done analytically. To do so rescale in the resonance region by assuming

$$P - \omega = \mu\lambda \,, \tag{2.18}$$

where $\mu = \epsilon^{\frac{1}{2}}$ and $\lambda = O(1)$. The time is also rescaled by assuming

$$\tau = \mu t \ . \tag{2.19}$$

When (2.18) and (2.19) are substituted into (2.17) and the system is expanded in powers of μ, we find

$$\begin{aligned}
\frac{dA}{d\tau} &= \mu \frac{A^2\omega^{1/2}}{2^{17/4}} \sin\theta + O(\mu^2) \ , \\
\frac{d\lambda}{d\tau} &= \frac{A\omega^{3/2}}{2^{5/4}} \sin\theta - \omega' - \mu \frac{A\omega^{1/2}}{2^{13/4}} \lambda\sin\theta + O(\mu^2) \ , \\
\frac{d\theta}{d\tau} &= \frac{\sqrt{2}}{2}\lambda - \mu \frac{\lambda^2}{2^{5/2}\omega} + \mu \frac{A\omega^{1/2}}{2^{17/4}} \cos\theta + O(\mu^2) \ ,
\end{aligned} \tag{2.20}$$

where $\omega(\tilde{t}) = \omega(\mu^2 t) = \omega(\mu\tau) \equiv \omega(\tilde{\tau})$. These equations are sufficient to determine the behavior of (2.1) in the resonance region for times up to $\tau = O(\mu^{-1})$, or rather $t = O(\epsilon^{-1})$.

From (2.20) we can see that the basic behavior in the resonance region is governed by the leading-order system

$$\begin{aligned}
\frac{d\lambda}{d\tau} &= \frac{A\omega^{3/2}}{2^{5/4}} \sin\theta - \omega' \\
\frac{d\theta}{d\tau} &= \frac{\sqrt{2}}{2}\lambda \ ,
\end{aligned} \tag{2.21}$$

or in terms of θ alone

$$\frac{d^2\theta}{d\tau^2} = \frac{A\omega^{3/2}}{2^{7/4}} \sin\theta - \frac{\omega'}{2^{1/2}} \ . \tag{2.22}$$

(This is the equation Lewin and Kevorkian obtain governing solutions in the resonance region, where they take $\omega' = \frac{1}{2}\omega$ and use a different variable in place of A.)

If we ignore the dependence of A and ω upon the time for the moment, then we can use phase-plane arguments to determine the basic behavior of (2.21). The critical points are easily found to occur when

$$\lambda = 0 ,$$

$$\sin\theta = \frac{2^{5/4}\omega'}{A\omega^{3/2}} < 1$$

(we assume $\omega' > 0$ for simplicity). Because the system (2.21) is periodic in θ, we can restrict θ so that $0 \leq \theta \leq 2\pi$. In this interval there will be two critical points when the above condition is satisfied; one will be a center, and the other a saddle point. The phase plane for (2.21) is easily constructed, and when the critical points are present it has the form shown in Fig. 2.1.

Lewin and Kevorkian have shown that the separatrix emerging from the saddle point divides the behavior into two distinct types. Inside this closed curve the solutions are oscillations, and these correspond to sustained resonance solutions because λ remains O(1). Outside the separatrix, the solutions start with λ large and positive and finish with λ large and negative. These correspond to solutions which pass through the resonance region without experiencing sustained resonance.

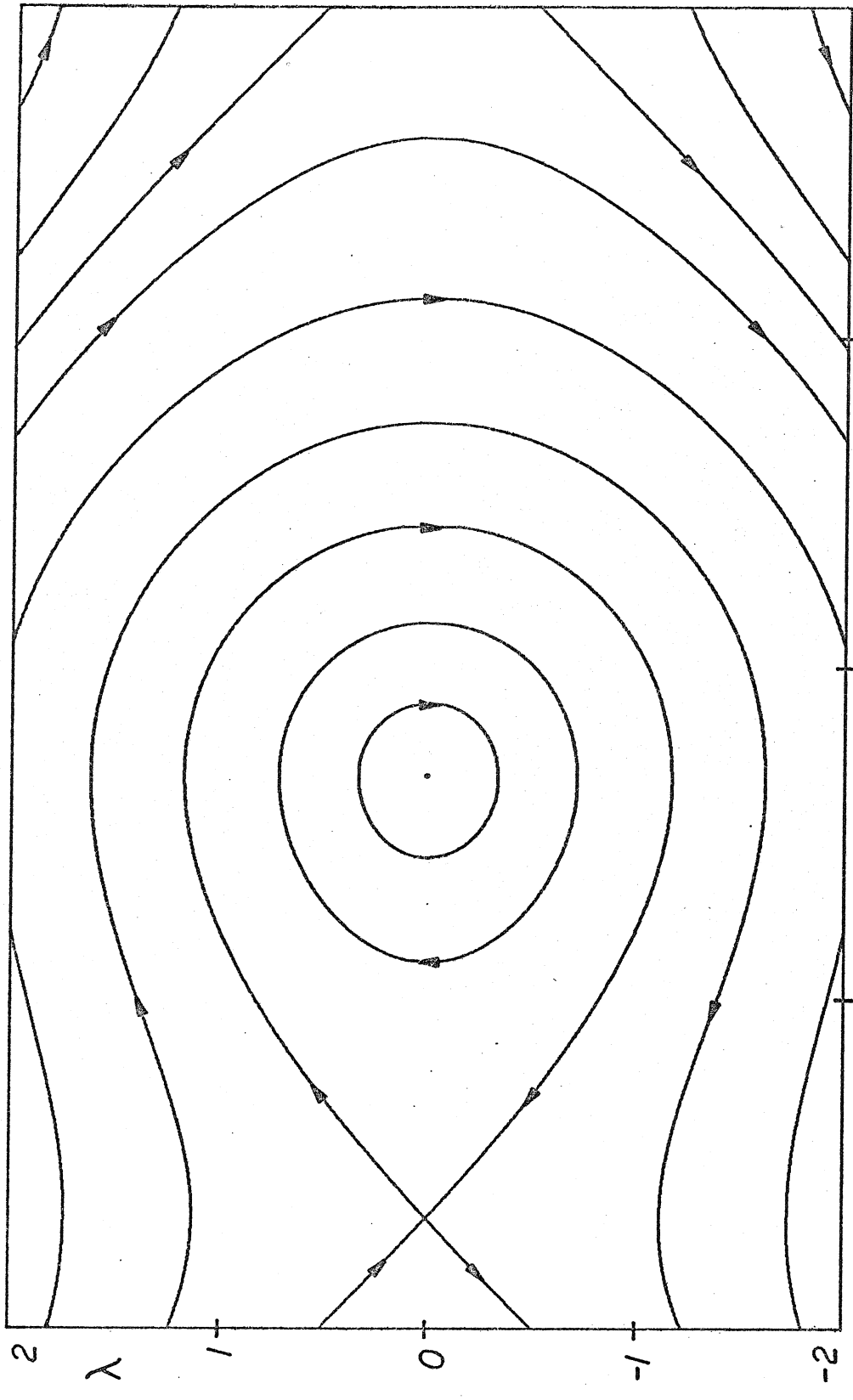

Figure 2.1. Leading-order phase-plane solution in the resonance region.

The actual system, (2.20), is not autonomous, however, because A and ω change on the slow time scale. This means that curves in the (λ, θ) phase plane should not expect to be closed. In addition, this means that the separatrix itself is not fixed in the phase plane, but also changes slowly with time. Thus, we have an explanation as to how solutions entering the resonance region can be entrained and become of the sustained resonance type: if the leading-order solution passes close enough to the separatrix, it is possible for it to be drawn inside by these slow changes.

Although a complete description of how solutions are entrained by the center seems possible only through numerical integration of (2.20), it is possible to give a necessary condition for sustained resonance solutions. For these solutions to be possible the critical points must be present, which means we must have

$$\frac{2^{5/4}\,\omega'}{A\,\omega^{3/2}} < 1 \,. \tag{2.23}$$

We can say more than this, however. We know that outside the resonance region A = const. + $O(\epsilon)$, and that in the resonance region $dA/d\tau = O(\mu)$. Since entrainment must occur on the first (and only) pass by the separatrix, this implies there will not be sufficient time for A to change substantially from its initial value. Thus, we can say that for entrainment to sustained resonance, we must have

$$\frac{2^{5/4}\,\omega'}{A_0\,\omega^{3/2}} < 1 \tag{2.24}$$

at the time of resonance, where A_0 is the initial value of A (it is

also the leading-order value of A before entering the resonance region, as mentioned above).

Resonance occurs when $\omega = P$ to leading order, or since P is also a constant to leading order away from the resonance region, when $\omega = P_0$ to leading order (P_0 being the initial value of P). Therefore, for entrainment to sustained resonance it must be true that

$$\omega'(\tilde{t}_R) < \frac{A_0 P_0^{3/2}}{2^{5/4}} \quad , \tag{2.25}$$

where $\tilde{t}_R$ satisfies

$$\omega(\tilde{t}_R) = P_0 \ .$$

3. THE SUSTAINED RESONANCE REGION

Determining exactly how sustained resonance solutions are entrained by the center is difficult, but we can extract more information from (2.20) about the behavior of these solutions once they are inside the separatrix. We shall see that all of the $O(\mu)$ terms in (2.20) are important for determining this behavior and the slow stability (whether they move slowly towards the center or not) of the sustained resonance solutions. The leading-order equation for θ, (2.22), does determine the basic structure of the phase plane, but it cannot determine the long-time stability of sustained resonance solutions, as Lewin and Kevorkian suggest. To do this the $O(\mu)$ terms must be included.

We shall employ the method of near-identity transformations to study the solutions of (2.20) in the sustained resonance region. First, write the system in the symbolic form

$$\begin{aligned} \dot{A} &= \mu f_1(A, \lambda, \theta, \tilde{\tau}) , \\ \dot{\lambda} &= -\frac{\partial H}{\partial \theta} + \mu f_2(A, \lambda, \theta, \tilde{\tau}) , \\ \dot{\theta} &= \frac{\partial H}{\partial \lambda} + \mu f_3(A, \lambda, \theta, \tilde{\tau}) , \end{aligned} \tag{3.1}$$

where now $\dot{} \equiv d/d\tau$. The function $H(A, \lambda, \theta, \tilde{\tau})$, which is

$$H = \frac{\sqrt{2}}{4}\lambda^2 + \frac{A\omega^{3/2}}{2^{5/4}}\cos\theta + \omega'\theta , \tag{3.2}$$

can be thought of as the Hamiltonian of the leading-order system. This problem is a generalization of a nonlinear oscillation problem

introduced by Kuzmak (see [1], p. 111).

As before, we facilitate the use of the near identity transformation by first making a change of variable using the exact solution when $\mu = 0$. Specifically, we change from (λ, θ) to (R, Γ) using

$$\begin{aligned} \lambda &= \Lambda(A, R, \Gamma, \tilde{t}) \ , \\ \theta &= \Theta(A, R, \Gamma, \tilde{t}) \ , \end{aligned} \tag{3.3}$$

where Λ and Θ give the exact solution of (3.1) when $\mu = 0$ (unfortunately, these solutions are not known in closed form). We choose R and Γ such that when $\mu = 0$, R is the constant of integration not associated with a shift in the origin of the time, and Γ is the time variable τ normalized so that the period in Γ is one. This normalization means that Λ and Θ satisfy the system

$$\begin{aligned} \Lambda_\Gamma &= -T\hat{H}_\Theta \ , \\ \Theta_\Gamma &= T\hat{H}_\Lambda \ , \end{aligned} \tag{3.4}$$

where $T = T(A, R, \tilde{\tau})$ is the period in τ and $\hat{H} \equiv H(A, \Lambda, \Theta, \tilde{\tau})$.

The period in Γ must be normalized in order for the perturbation expansion to be uniform. For example, suppose the period in Γ were $\zeta(\tilde{\tau})$. Thus, we would have

$$\Lambda(\Gamma + n\zeta, \tilde{\tau}) = \Lambda(\Gamma, \tilde{\tau})$$

(for the moment ignore any dependence upon A and R). From this it would follow that

$$\Lambda_\Gamma(\Gamma + n\zeta, \tilde{\tau}) \cdot n\underset{\sim}{\zeta}_\tau + \underset{\sim}{\Lambda}_\tau(\Gamma + n\zeta, \tilde{\tau}) = \underset{\sim}{\Lambda}_\tau(\Gamma, \tilde{\tau}) ,$$

or since $\Lambda_\Gamma(\Gamma + n\zeta, \tilde{\tau}) = \Lambda_\Gamma(\Gamma, \tilde{\tau})$, that

$$\underset{\sim}{\Lambda}_\tau(\Gamma + n\zeta, \tilde{\tau}) = \underset{\sim}{\Lambda}_\tau(\Gamma, \tilde{\tau}) + n\underset{\sim}{\zeta}_\tau\Lambda_\Gamma(\Gamma, \tilde{\tau}) .$$

We see that $\underset{\sim}{\Lambda}_\tau$ is unbounded as $n \to \infty$ unless ζ is independent of $\tilde{\tau}$. Similarly, it is easy to show that ζ must also be independent of A and R. The period ζ must therefore be a constant for the expansion to be uniform, and can be normalized to one.

From (3.4) we also see that

$$\hat{H}_\Lambda\Lambda_\Gamma + \hat{H}_\Theta\Theta_\Gamma = \hat{H}_\Gamma = 0 .$$

This can be integrated once to give

$$\hat{H}(A, \Lambda, \Theta, \tilde{\tau}) = E(A, R, \tilde{\tau}) . \qquad (3.5)$$

Using the change of variables (3.3) and writing

$$f_i(A, \lambda, \theta, \tilde{\tau}) = F_i(A, R, \Gamma, \tilde{\tau})$$

for i = 1, 2, 3, the system (3.1) becomes

$$\dot{A} = \mu F_1(A, R, \Gamma, \tilde{\tau}) ,$$

$$\begin{pmatrix} \dot{R} \\ \dot{\Gamma} \end{pmatrix} = \begin{pmatrix} 0 \\ \Omega \end{pmatrix} + \frac{\mu}{E_R} \begin{bmatrix} \hat{H}_\Lambda & \hat{H}_\Theta \\ -\Omega\Theta_R & \Omega\Lambda_R \end{bmatrix} \begin{pmatrix} F_2 - \Lambda_{\tilde{\tau}} - \Lambda_A F_1 \\ F_3 - \Theta_{\tilde{\tau}} - \Theta_A F_1 \end{pmatrix} , \tag{3.6}$$

where

$$\Omega(A, R, \tilde{\tau}) = \frac{1}{T(A, R, \tilde{\tau})} .$$

This can be further simplified to

$$\begin{aligned} \dot{A} &= \mu F_1(A, R, \Gamma, \tilde{\tau}) , \\ \dot{R} &= -\mu \frac{E_{\tilde{\tau}}}{E_R} - \mu \frac{E_A}{E_R} F_1 + \frac{\mu}{E_R}\left[\hat{H}_{\tilde{\tau}} + \hat{H}_A F_1 + \hat{H}_\Lambda F_2 + \hat{H}_\Theta F_3\right] \\ \dot{\Gamma} &= \Omega(A, R, \tilde{\tau}) + \mu \frac{\Omega}{E_R}\left[\Lambda_R F_3 - \Theta_R F_2 + \Theta_R \Lambda_{\tilde{\tau}} - \Lambda_R \Theta_{\tilde{\tau}} + F_1(\Theta_R \Lambda_A - \Lambda_R \Theta_A)\right] . \end{aligned} \tag{3.7}$$

Although any integration constant R can be used, a particularly convenient form results if we take it to be the energy E. With this choice

$$E_{\tilde{\tau}} = E_A = 0, \qquad \text{and} \qquad E_R = 1 .$$

Therefore, the above system of equations becomes

$$\dot{A} = \mu F_1(A, E, \Gamma, \tilde{\tau}),$$

$$\dot{E} = \mu\left[\hat{H}_{\tilde{\tau}} + \hat{H}_A F_1 + \hat{H}_\Lambda F_2 + \hat{H}_\Theta F_3\right], \tag{3.8}$$

$$\dot{\Gamma} = \Omega(A, E, \tilde{\tau}) + \mu\Omega\left[\Lambda_E F_3 - \Theta_E F_2 + \Theta_E \Lambda_{\tilde{\tau}} - \Lambda_E \Theta_{\tilde{\tau}} + F_1(\Theta_E \Lambda_A - \Lambda_E \Theta_A)\right].$$

(There is no inconsistency with using the energy E. Equation (3.5) says that E is independent of Γ only to leading-order, which is seen to be true from (3.8).) Note that no approximations have been made in arriving at the system (3.8) from its original form (3.1).

With the change of variables complete, the near-identity transformation can now be constructed for (3.8). We assume

$$\begin{aligned} A &= \alpha + \mu A_1(\alpha, e, \gamma) + \cdots, \\ E &= e + \mu E_1(\alpha, e, \gamma) + \cdots, \\ \Gamma &= \gamma + \mu \Gamma_1(\alpha, e, \gamma) + \cdots, \end{aligned} \tag{3.9}$$

where the functions A_1, E_1, and Γ_1 are chosen to simplify the resulting equations for α, e, and γ as much as possible. This is similar to the earlier discussion of the method, and in fact the analysis is even simpler in this case, because there is only one time-like variable.

For example, upon substitution of (3.9) into (3.8), we find that α satisfies

$$\frac{d\alpha}{d\tau} = \mu F_1(\alpha, e, \gamma, \tilde{\tau}) - \mu\Omega \frac{\partial A_1}{\partial \gamma} + \cdots. \tag{3.10}$$

A_1 must remain bounded, which means that we should take

$$\Omega \frac{\partial A_1}{\partial \gamma} = F_1 - \int_0^1 F_1(\alpha, e, \zeta, \tilde{\tau})\, d\zeta \tag{3.11}$$

($\partial A_1/\partial\gamma$ must have a zero average over one cycle if A_1 is to be bounded). The functions E_1 and Γ_1 are determined in a similar manner.

Since α, e, and γ represent the values of A, E, and Γ without the small oscillations on the τ time scale (which are included in A_1, E_1, and Γ_1), we are primarily interested in the equations that result for these leading-order variables once A_1, E_1, and Γ_1 are chosen. These equations are

$$\frac{d\alpha}{d\tau} = \mu \int_0^1 F_1(\alpha, e, \zeta, \tilde{\tau}) d\zeta,$$

$$\frac{de}{d\tau} = \mu \int_0^1 \left[\hat{H}_{\tilde{\tau}} + \hat{H}_A F_1 + \hat{H}_\Lambda F_2 + \hat{H}_\Theta F_3 \right] d\zeta ,$$

$$\frac{d\gamma}{d\tau} = \Omega(\alpha, e, \tilde{\tau}) + \mu\Omega \int_0^1 \left[\Lambda_E F_3 - \Theta_E F_2 + \Theta_E \Lambda_{\tilde{\tau}} - \Lambda_E \Theta_{\tilde{\tau}} + F_1(\Theta_E \Lambda_A - \Lambda_E \Theta_A) \right] d\zeta + \mu \int_0^1 \left[\Omega_A A_1 + \Omega_E E_1 \right] d\zeta ,$$

where all functions are evaluated at $A = \alpha$, $E = e$, and $\Gamma = \gamma$.

The first two equations are coupled nonlinear ordinary differential equations for α and e, since each of the integrals in (3.12) depends only on α, e, and $\tilde{\tau}$. They also show that α and e are functions of the slow time $\tilde{\tau}$, as expected. In terms of this time variable the equations have the form

$$\frac{d\alpha}{d\tilde{\tau}} = G_1(\alpha, e, \tilde{\tau}) \ ,$$
$$\frac{de}{d\tilde{\tau}} = G_2(\alpha, e, \tilde{\tau}) \ , \tag{3.13}$$

where G_1 and G_2 represent the integrals in the first two equations of (3.12). In addition, the above two equations are the important ones, because once we have solved for α and e we need merely integrate the last equation in (3.12) with respect to time to find γ.

It does not appear that much progress has been made due to the complexity of (3.13). For our problem, however, it is fortuitous that these equations decouple. When we substitute into (3.12) the specific forms for $\hat{H}$ and the F_i (i = 1, 2, 3) determined by (3.2) and (2.20), the equations for α and e become

$$\frac{d\alpha}{d\tilde{\tau}} = \frac{\alpha^2 \omega^{1/2}}{2^{17/4}} \langle \sin\Theta \rangle \ ,$$
$$\frac{de}{d\tilde{\tau}} = \frac{13\alpha\omega^{1/2}\omega'}{2^{17/4}} \langle \cos\Theta \rangle + \omega'' \langle \Theta \rangle - \frac{\omega'}{2^{5/2}\omega} \langle \Lambda^2 \rangle \ , \tag{3.14}$$

where $\langle \cdots \rangle$ means the average over one full cycle. From (3.4) we see that

$$\Omega \Lambda_\Gamma = \frac{A\omega^{3/2}}{2^{5/4}} \sin\Theta - \omega' \ ,$$

and since

$$\langle \Lambda_\Gamma \rangle = 0 \ ,$$

it follows that

$$\langle \sin\Theta \rangle = \frac{2^{5/4}\,\omega'}{\alpha\,\omega^{3/2}} \tag{3.15}$$

(remember that we must evaluate this at $A = \alpha$).

Therefore, the equation for α is

$$\frac{d\alpha}{d\tilde{\tau}} = \frac{\alpha}{8}\frac{\omega'}{\omega} , \tag{3.16}$$

which can be easily integrated to yield

$$\frac{\alpha(\tilde{\tau})}{\alpha(\tilde{\tau}_0)} = \left[\frac{\omega(\tilde{\tau})}{\omega(\tilde{\tau}_0)}\right]^{1/8} . \tag{3.17}$$

With α determined from (3.17), it can be eliminated from the equation for e. Unfortunately, none of the averages in the second equation of (3.14) can be evaluated in closed form, and therefore we cannot obtain a simple expression for e.

Although we cannot solve for e exactly, equation (3.17) does enable us to give a necessary condition for sustained resonance to continue. When we substitute (3.17) into (3.15) and use the fact that $|\sin\Theta| < 1$, we find that ω must satisfy

$$\frac{2^{5/4}\;\omega(\tilde{\tau}_0)^{1/8}\;\omega'}{\alpha(\tilde{\tau}_0)\;\omega^{13/8}} < 1 .$$

The two constants $\alpha(\tilde{\tau}_0)$ and $\omega(\tilde{\tau}_0)$ are the values of α and ω at the start of sustained resonance, which in Chapter 2 were shown to be

A_0 and P_0 to leading order. Thus, the necessary condition for the continuation of sustained resonance can be written

$$\frac{\omega'(\tilde{\tau})}{[\omega(\tilde{\tau})]^{13/8}} < \frac{A_0}{2^{5/4} P_0^{1/8}} \quad . \tag{3.18}$$

Note that (3.18) is an improvement upon the condition given by Lewin and Kevorkian, because all of the functions and constants in (3.18) are known explicitly.

The condition (3.18) also predicts an effect not discussed by Lewin and Kevorkian. Suppose that the entrainment condition (2.25) is satisfied, and that at the time of resonance solutions which pass near the separatrix are drawn inside it and undergo sustained resonance. Suppose also that at a still later time condition (3.18) is violated, so that sustained resonance cannot continue. Thus, we can have solutions that are drawn into the sustained resonance region, remain there for a finite time, and then become unlocked and pass out of the resonance region. This means it is possible for sustained resonance to last for only a finite time. Note also that (3.18) need only be violated briefly, because once the solution is out of the resonance region it does not enter again.

For this unlocking to occur, (3.18) shows that the velocity of descent or the flight path angle must depend upon the time. If we assume they are constant, as in [4] and [5], then ω is an exponential function of the slow time, and

$$\frac{\omega'}{\omega^{13/8}} = \frac{\text{const.}}{\omega^{5/8}} \quad .$$

This is an exponentially decreasing function, so that if (3.18) is satisfied at the time of resonance, it will be satisfied at all later times. Thus, if entrainment to sustained resonance occurs, it will continue indefinitely.

If, on the other hand, the descent velocity increases with time or the flight path angle decreases with time (so that the descent perpendicular to the surface of the earth is made more rapid), then $\omega'(\tilde{\tau})$ becomes larger, and the torque required to maintain sustained resonance increases. If this required torque becomes larger than the maximum roll torque that can be supplied by the aerodynamic and mass asymmetries, then the sustained resonance unlocks. The inequality (3.18) quantifies this condition.

This analysis suggests that roll resonance may be halted (after it has begun) by causing the rate of descent to be increased sufficiently. This can be accomplished either by increasing the descent velocity or by decreasing the flight path angle. (Slowing the descent would have the opposite effect, that of aiding sustained resonance. Only after the resonance region has been passed should the descent be slowed.) This conclusion, it must be emphasized, is only true for the model problem proposed by Kevorkian. For an actual reentry vehicle it is possible that increasing the rate of descent could promote roll resonance. An analysis of the full problem similar to that presented here should resolve this question.

Although we have so far only given necessary conditions for sustained resonance, it is possible to give some idea of sufficient conditions for this to occur. We do this by linearizing the system

(2.20) about the center. Thus, we assume

$$\theta = \theta_0 + \theta_1 \ ,$$
$$\lambda = \lambda_1 \ , \tag{3.19}$$

where λ_1 and θ_1 are small, and where

$$\frac{A\omega^{3/2}}{2^{5/4}} \sin\theta_0 = \omega' .$$

A method equivalent to the above is to keep only quadratic or smaller order terms in the Hamiltonian $\hat{H}$ and in the integrals in (3.12), discarding higher order terms. This procedure enables us to use the results that have already been derived. Thus, we expand

$$\hat{H} = \frac{A\omega^{3/2}}{2^{5/4}} \cos\Theta_0 + \omega'\Theta_0 + \frac{\sqrt{2}}{4}\Lambda_1^2 + \left(-\frac{A\omega^{3/2}}{2^{5/4}}\cos\Theta_0\right)\cdot\frac{1}{2}\Theta_1^2 + \cdots$$
$$= \hat{H}_0(A, \tilde{\tau}) + \hat{H}_2(A, \Lambda_1, \Theta_1, \tilde{\tau}) + \cdots . \tag{3.20}$$

The equations for Λ_1 and Θ_1 follow from (3.4), and they are

$$\Lambda_{1\Gamma} = -T\,\frac{A\omega^{3/2}}{2^{5/4}}(-\cos\Theta_0)\Theta_1 \ ,$$
$$\Theta_\Gamma = T\,\frac{\sqrt{2}}{2}\Lambda_1 \ . \tag{3.21}$$

Solving these equations (and requiring the solution to have a period of one in Γ), we find

$$\Theta_1 = \beta \sin 2\pi\Gamma \ ,$$

$$\Lambda_1 = \frac{A^{1/2}\omega^{3/4}(-\cos\Theta_0)^{1/2}}{2^{3/8}}\,\beta\cos 2\pi\Gamma \ , \tag{3.22}$$

$$T = 2\pi \cdot \frac{2^{7/8}}{A^{1/2}\,\omega^{3/4}}(-\cos\Theta_0)^{-1/2} \ .$$

From these solutions we can substitute back into (3.20) to determine that

$$\hat{H}_2 = \frac{A\omega^{3/2}}{2^{9/4}}\,(-\cos\Theta_0)\beta^2. \tag{3.23}$$

The equation for α in (3.12) has already been solved in general, so we need only focus our attention upon the equation for e. To eliminate the effects due to the slow-time dependence of the critical point, we let

$$e = e_0 + e_2 + \cdots, \tag{3.24}$$

where

$$e_0 = \frac{\alpha\omega^{3/2}}{2^{5/4}}\cos\Theta_0 + \omega'\Theta_0 = \hat{H}_0(\alpha, \tilde{\tau}) \ . \tag{3.25}$$

Since

$$\frac{de_0}{d\tilde{\tau}} = \hat{H}_{0A}\frac{d\alpha}{d\tilde{\tau}} + \hat{H}_{0\tilde{\tau}} \ ,$$

this can be subtracted from the equation for e, so that e_2 satisfies

$$\frac{de_2}{d\tilde{\tau}} = \int_0^1 \left[\hat{H}_{2\tilde{\tau}} + \hat{H}_{2A} F_1 + \hat{H}_{2\Lambda_1} F_2 + \hat{H}_{2\Theta_1} F_3 \right] d\zeta \, . \tag{3.26}$$

According to our procedure, we discard terms of order higher than quadratic in the above integral. The averages are straightforward, and the result is

$$\frac{de_2}{d\tilde{\tau}} = -\frac{1}{4} \frac{\omega'}{\omega} e_2 \, , \tag{3.27}$$

which can be integrated to

$$e_2 = (\text{const.}) \, \omega^{-1/4} \, . \tag{3.28}$$

Therefore, we see that e_2 decreases as ω increases, so that when the sustained resonance solution is close to the center and ω is increasing, it will spiral in slowly. Note that condition (3.18) must continue to be satisfied.

Although this is only a rough analysis, because we have not specified how close to the center it must be, it does suggest that a sufficient condition for the sustained resonance solutions to be drawn slowly toward the center is for (3.18) to be satisfied and for ω to be an increasing function of the slow time. From this information we can also make the conjecture that if (2.25) is satisfied and ω is increasing, then entrainment to sustained resonance will occur (for solutions which pass sufficiently close to the separatrix).

4. DISCUSSION

We have examined the simplified model of reentry roll resonance presented by Kevorkian, and have shown that the method of near identity transformations enables the basic behavior of the model to be simply determined. In particular, we have demonstrated that a necessary condition for entrainment to sustained resonance is

$$\omega'(\tilde{t}_R) < \frac{A_0 P_0^{3/2}}{2^{5/4}} \quad ,$$

where $\tilde{t}_R$ satisfies $\omega(\tilde{t}_R) = P_0$. These sustained resonance solutions are solutions where the roll rate p remains within $O(\epsilon^{1/2})$ of the frequency $\omega(\tilde{t})$ for long periods of time.

These solutions correspond to slowly varying oscillations inside the separatrix of Fig. 2.1, and by using the method of near identity transformations a second time, we have shown that a necessary condition for sustained resonance to continue is

$$\frac{\omega'(\tilde{t})}{[\omega(\tilde{t})]^{13/8}} < \frac{A_0}{2^{5/4} P_0^{1/8}} \quad .$$

This inequality implies that it is possible for solutions to enter into sustained resonance, remain in resonance for a finite time, and then break out of resonance when the condition is violated.

In addition, we have presented an analysis which indicates that sustained resonance solutions near the center spiral in slowly when ω increases. Therefore, we conjecture that a sufficient condition for sustained resonance to occur (when appropriate initial conditions are

chosen) is for the above two conditions to be satisfied and for ω to be an increasing function of the slow time.

APPENDIX

As discussed in the text, we choose the equations for a_1, φ_1, p_1, and ψ_1 so that only terms which depend upon phase difference $\Psi-\Phi$ remain in the equations for A, Φ, P, and Ψ. From the components of (2.12) we then find

$$(P^2+\omega^2)^{1/2}\frac{\partial a_1}{\partial \Phi}+\sqrt{2}\,P\frac{\partial a_1}{\partial \Psi}\equiv L\alpha_1=\frac{1}{2}\frac{A\omega\omega'}{P^2+\omega^2}\cos 2\Phi$$

$$+\frac{PA^2\omega^2}{8(P^2+\omega^2)^{5/4}}\left\{\sin(\Psi+3\Phi)+\sin(\Psi+\Phi)+\sin(\Psi-3\Phi)\right\},$$

$$L\varphi_1=-\frac{1}{2}\frac{\omega\omega'}{P^2+\omega^2}\sin 2\Phi+\frac{PA\omega^2}{8(P^2+\omega^2)^{5/4}}\left\{\cos(\Psi+3\Phi)\right.$$

$$\left.-\cos(\Psi-3\Phi)+\cos(\Psi+\Phi)\right\}+\frac{P}{(P^2+\omega^2)^{1/2}}\,p_1\,,$$

$$Lp_1=\frac{A\omega^2}{2(P^2+\omega^2)^{1/4}}\sin(\Psi+\Phi)\,,$$

$$L\psi_1=\sqrt{2}\,p_1\,.$$

These equations have the particular solutions

$$a_1 = \frac{1}{4}\frac{A\omega\omega'}{(P^2+\omega^2)^{3/2}}\sin 2\Phi$$

$$- \frac{PA^2\omega^2}{8(P^2+\omega^2)^{5/4}}\left\{\frac{\cos(\Psi+3\Phi)}{\sqrt{2}P+3(P^2+\omega^2)^{1/2}} + \frac{\cos(\Psi+\Phi)}{\sqrt{2}P+(P^2+\omega^2)^{1/2}} + \frac{\cos(\Psi-3\Phi)}{\sqrt{2}-3(P^2+\omega^2)^{\frac{1}{2}}}\right\},$$

$$\varphi_1 = \frac{1}{4}\frac{\omega\omega'}{(P^2+\omega^2)^{3/2}}\cos 2\Phi$$

$$+ \frac{PA\omega^2}{8(P^2+\omega^2)^{5/4}}\left\{\frac{\sin(\Psi+3\Phi)}{\sqrt{2}P+3(P^2+\omega^2)^{\frac{1}{2}}} - \frac{\sin(\Psi-3\Phi)}{\sqrt{2}P-3(P^2+\omega^2)^{\frac{1}{2}}} + \frac{\sin(\Psi+\Phi)}{\sqrt{2}P+(P^2+\omega^2)^{\frac{1}{2}}}\right\}$$

$$- \frac{PA\omega^2}{2(P^2+\omega^2)^{3/4}} \cdot \frac{\sin(\Psi+\Phi)}{[\sqrt{2}P + (P^2+\omega^2)^{\frac{1}{2}}]^2},$$

$$p_1 = \frac{-A\omega^2}{2(P^2+\omega^2)^{1/4}} \cdot \frac{\cos(\Psi+\Phi)}{\sqrt{2}P + (P^2+\omega^2)^{1/2}},$$

$$\psi_1 = - \frac{A\omega^2}{\sqrt{2}(P^2+\omega^2)^{1/4}} \cdot \frac{\sin(\Psi+\Phi)}{[\sqrt{2}P + (P^2+\omega^2)^{1/2}]^2}.$$

To each of these particular solutions an arbitrary function of $A, P, \tilde{t}$, and $\sqrt{2}P\Phi - (P^2+\omega^2)^{1/2}\Psi$ must be added to obtain the general solution. We expect these arbitrary functions to be determined by later states of the analysis. (For example, if the solution depends upon $\sqrt{2}P\,\Phi - (P^2+\omega^2)^{1/2}\Psi$, then differentiating with respect to P or $\tilde{t}$ yields linear terms in Φ and Ψ. These terms must not appear in the expansion, since Φ and Ψ are unbounded. Therefore, we cannot allow this dependence.)

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The end of a matter is better
than its beginning...

Ecclesiastes 7:8