A SINGULAR PERTURBATION METHOD FOR NON-LINEAR WATER WAVES PAST AN OBSTACLE

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ABSTRACT

The method of matched singular perturbation expansions is used to solve the problem of a steady two-dimensional flow of a perfect fluid with a free surface under the influence of gravity. A flat plate of length ℓ is inclined at an angle α to the horizontal and its trailing edge is immersed to a depth h below the surface of an otherwise uniform stream of infinite depth, the velocity at upstream infinity being U. A parameter $\beta = g\ell/U^2$ (Froude number $F = \beta^{-1/2}$) is assumed small so that the flow separates smoothly at the leading and trailing edges, giving rise to an upward jet and gravity waves in the downstream. An inner solution for the velocity field is obtained which is valid near the plate and an outer solution which holds far away. These are determined through the orders 1, $\beta \log \beta$, β, $\beta^2 \log^2 \beta$, $\beta^2 \log \beta$ up to order β^2, and are matched with one another to these orders. In contrast with linearized planing theory, the depth of submergence can be prescribed as a parameter. The lift coefficient is calculated for several values of α, h/ℓ and β. The results reduce to known ones in certain limits.

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1. INTRODUCTION

Even with the simplifying assumptions usually introduced for irrotational flows of incompressible, inviscid fluids, the exact treatment of free surface flows with gravity is quite difficult. Although these simplifications reduce the governing differential equation to Laplace's, there remain in the problem two fundamental non-linearities. The first concerns the fact that the free surface boundary conditions are given on a surface which itself is to be determined from the solution of the problem. The second non-linearity arises from non-linear terms in the boundary conditions themselves.

Treatment of non-linear free surface flow problems has been limited to a relatively small number of cases. Sautreaux(1) has constructed a relationship by means of which a free surface can be found corresponding to some solution of the exact problem. This is an inverse method, in that the problem to be solved cannot be chosen a priori, but where the boundary is found from the solution instead of vice versa. Examples of flows constructed by this method are given by Richardson(2) and Vitousek(3). An interesting example of an exact, though rotational, solution is that of Gerstner's trochoidal wave. In addition a number of general theorems on exact waves, including some existence theorems, have been proved.

Stokes and Rayleigh considered the problem of progressive and standing waves by expanding in terms of a small parameter. This expansion has been carried out as far as the fifth order. One result is that the crests are raised and sharpened while the troughs are raised and broadened, compared with first-order linear theory.

If gravity is excluded from the problem and the flow is two-dimensional, the powerful methods of free-streamline and complex variable theory can be used, as for example in cavity flows. One problem of this type which has particular relevance has been solved by Wagner(4) and Green(5). This is the problem of a jet, bounded below by an infinite straight line, impinging on a flat plate held at an angle to the stream. This may be regarded also as the flow due to a plate immersed in a moving stream of finite or infinite depth.

When gravity enters, recourse is usually made to one of two approximate theories. Shallow water theory deals with problems where the depth of the water is small compared with some characteristic length. This has little relevance to the problem at hand and will not be considered further.

The linearized theory of infinitesimal waves is obtained as a perturbation of a known flow. The flow quantities are formally expanded in terms of a small parameter. The boundary conditions are linear to each order and can be applied on the undisturbed free surface, thus removing the non-linearities which were mentioned above. A large variety of problems of practical interest has been solved using this approximation. Among them are infinitesimal standing and progressing waves in regions of bounded and unbounded extent, initial value problems, waves on beaches, waves due to moving disturbances, ship theory, etc. Most of the results of classical surface wave theory have been obtained by use of the linearized theory.

One such problem is that of a thin lamina planing on the surface of an infinite fluid. The steady state two-dimensional case has been treated by a number of authors, Sretenskii[6], Sedov[7], Chaplygin[8], Maruo[9], Cumberbatch[10]. Due to the limitations of linearized theory only small angles of attack could be considered. The present work removes this restriction and also allows the depth of submersion of the plate to be used as a parameter. This has not been possible in linearized theory.

In essence, the approach of the linearized theory has been to find first the flow due to a moving point pressure on the surface of the ocean. The plate is replaced by a distribution of such pressure points with the condition that the flow at the plate be tangential to its surface. This gives rise to an integral equation, which, for high speeds, is similar to the airfoil equation. This integral equation has been solved in a number of ways. Cumberbatch, for example, uses an iterative approximation for large Froude number. Such linearized solutions give a good approximation to the expected physical behavior, except in the neighborhood of the plate, where, for example, the flow near the stagnation point and the spray sheet cannot be regarded as small perturbations of the uniform stream.

On the other hand, Green's free-streamline solution gives the expected behavior near the body but breaks down at large distance, where the free surface drops away as the logarithm of the distance from the plate. The two solutions can be regarded as limiting cases of the exact solution which is supposed to exist. A method might be found by which these two solutions can be joined together.

A somewhat similar situation occurs in viscous flow theory. Examples are the Oseen and Stokes solutions for flow past a sphere for small Reynolds numbers, and the existence of boundary layers for large Reynolds numbers. To deal with such problems the idea of matched singular perturbation expansions has been introduced by Kaplun(11), Kaplun and Lagerstrom(12), and Lagerstrom and Cole(13). It has since found wide applications in viscous flow theory and in airfoil theory.

The need for a singular perturbation expansion is suggested when two or more physical processes are at work in a given flow. In boundary layer theory, for example, the outer flow can be taken to be inviscid and solved by potential theory. However, no matter how small the viscosity is, it cannot be neglected in the neighborhood of the wall. Thus the initial approximation of zero viscosity cannot be uniformly valid since the no-slip condition at the wall would then be absent. Thus there are two regions, in one of which viscosity can be neglected to a first approximation and in the other it dominates. Two expansions of the exact solution must be used: one in the potential or outer region, the other in the viscous or inner region.

Being expansions of the assumed exact solution, they must merge into one another in some sense. This will be accomplished by the matching principle which is mainly due to Kaplun[11]. In some cases more than two regions exist and extra expansions must be used.

In the present case, gravity takes the place of viscosity in an analogous manner. For high speeds inertial effects are much greater than those due to gravity in the neighborhood of the body. But for any gravitational effect, no matter how small, a gravity wave region exists where the main physical mechanism lies in the interchange of kinetic and potential energy and where inertia and gravity have effects of equal importance. Therefore analogy with the viscous case suggests that near the body an inner expansion, somehow based on Green's solution, exists while far away gravity waves will be represented by an outer expansion. Such, indeed, will be found to be the case, no intermediate expansion being necessary.

The problem is stated in mathematical form in sections 2 and 3. By use of potential theory and conformal mapping the problem is transferred to the lower half of a parametric ζ-plane. The unknown free surface is thus replaced by the part of the real ζ-axis outside the segment $(-1, 1)$, while the remainder of the axis represents the plate. The direct approach is to use the length of the plate, angle of attack and depth of submergence as primary parameters. However if this is done the solution becomes quite complicated since, for example, the stagnation point on the plate would change with each added order in the solution. A semi-inverse method which allows the direct parameters to specify the solution indirectly is outlined in section 5.

The selection of appropriate inner and outer expansions is next dealt with. The corresponding solutions are then found and matched to order β. The expected behavior is found: the inner solution is the free streamline flow together with a correction due to gravity. The outer solution represents a uniform flow disturbed by a point singularity on the surface. The depth of submergence is found to have a leading term of order $\log \beta$. The length, angle of attack and depth of submergence are found in terms of the new parameters so that inversion of the relations allows the calculation of the thrust on the plate in terms of ℓ, α and h. The results are then extended to order β^2 and are followed by a description of a scheme for calculating the thrust.

When the flow as shown in Figure 1 exists the Froude number must be relatively large since otherwise, for example, the fluid might not rise to the leading edge or separate clearly from the trailing edge. Thus β must be taken to be small. The condition is necessary, but may not be sufficient for the convergence of the resulting series. Another condition that seems to be required is that the depth of submergence be not too large. This is implied by the neglect of detailed study of the jet region.

The analysis is greatly facilitated by the restriction to two-dimensional flow which allows use of complex variable theory. A reasonably clear view of the principles used to solve the problem is obtained by the choice of a flat plate and an infinitely deep ocean.

Extension to finite depth and a cambered plate would not change the general outline of the method, but would only complicate the details of calculation.

2. STATEMENT OF PROBLEM

The problem treated here is that of the two-dimensional flow produced by a flat plate held at an arbitrary angle on the free surface of an otherwise uniform stream of an infinitely deep liquid under gravity (see Figure 1). The flow is assumed to be incompressible, inviscid and irrotational. The plate is of length ℓ and the trailing edge B is at a depth h below the free surface at upstream infinity. There is no angular or vertical motion of the plate so that the flow is steady. Then far upstream and at large depths the flow tends towards a uniform stream of velocity U. The plate AB is inclined at an angle α to the upstream horizontal. The flow comes from upstream infinity and divides along the stagnation streamline which branches off at the stagnation point C. Above this streamline the flow shoots off as a jet or bow plume J if, as assumed here, the Froude number is sufficiently large. This spray sheet may return to the main flow either upstream or downstream depending on U and α but can be appropriately removed from the flow. Hence its possible interaction with the main flow, such as rejoining the stream, will be neglected.

The trailing edge B is taken to be the origin of the z-plane, with y vertically upwards. Thus $y = h$ on the free surface at upstream infinity. For sufficiently large Froude number the flow is assumed to detach smoothly from the leading and trailing edges. Downstream, the inertial effects due to the presence of the plate gradually become less dominant and the effect of gravity begins to play an equally important role in producing a train of gravity waves, assumed to be in a permanent form, at large distances.

3. BOUNDARY VALUE PROBLEM

Let $u(x, y)$, $v(x, y)$ be the components of the velocity field in the x, y directions respectively. Since the flow is assumed to be irrotational and the fluid perfect, a potential function $\varphi(x, y)$ exists in the fluid. $\varphi(x, y)$ is harmonic and its conjugate $\psi(x, y)$ is the stream function.

The analytic function $f(z)$ is called the complex potential

$$f(z) = \varphi(x, y) + i\psi(x, y) \tag{1}$$

$$\frac{df}{dz} = \varphi_x + i\psi_x = \varphi_x - i\varphi_y = w(z) \tag{2}$$

The analytic function $w(z) = u - iv$ is the complex velocity. Let p and ρ be the pressure and density respectively.

Since the flow is steady and irrotational

$$p/\rho + \frac{1}{2}(u^2 + v^2) + gy = C \tag{3}$$

where C is a constant throughout and is found as follows. S_f is the free surface

$$p = 0 \quad \text{on } S_f \tag{4}$$

$$u = U, \quad v = 0 \quad \text{at } x = -\infty, \; y = h \tag{5}$$

since

$$y_s = h \quad \text{at} \quad x = -\infty \tag{6}$$

Hence

$$U^2 - (u^2 + v^2) = 2g\,(y_s - h) \tag{7}$$

where y_s is the value of y on the free surface S_f.

This is one of the conditions which holds on the free surface. The other is the kinematic condition which states that once located on the surface a fluid particle always remains there

$$\frac{dy_s}{dx} = \frac{v}{u} = \varphi_y / \varphi_x \tag{8}$$

Conditions (7) and (8) can be combined to give

$$\varphi_x^2 \varphi_{xx} + 2\varphi_x \varphi_y \varphi_{xy} + \varphi_y^2 \varphi_{yy} + g\varphi_y = 0 \text{ on } S_f \tag{9}$$

The boundary condition on the plate is

$$u \tan \alpha + v = 0 \tag{10}$$

These are the full non-linear conditions for steady flow, to which must be added conditions at infinity.

4. NON-DIMENSIONAL PROBLEM

The characteristic velocity is taken to be U, the velocity of the undisturbed stream. The wetted length of the plate ℓ is used as the characteristic length. The governing parameter is then

$$\beta = \frac{g\ell}{U^2} = 1/F^2 \tag{11}$$

and is assumed small enough to make the previous assumptions valid. Here F denotes the Froude number of the flow.

Let the non-dimensional variables be

$$z^* = z/\ell \qquad w^* = w/U \qquad p^* = p/\rho U^2 \tag{12}$$

As a consequence $\varphi^* = \varphi/\ell U$. In this way $w^*(z) \to 1$ at upstream infinity and the following conditions are obtained.

In the fluid:

$$\begin{aligned} &w^*(z^*) \text{ is an analytic function of } z^* \\ &\varphi^*(x^*, y^*) \text{ is harmonic.} \end{aligned} \tag{13}$$

On the free surface

$$1 - (u^{*2} + v^{*2}) = 2\beta(y^* - h^*) \tag{14}$$

$$v^* = u^* \frac{dy_s^*}{dx^*} \tag{15}$$

$$\varphi_{x^*}^{*2}\, \varphi^*_{x^*x^*} + 2\varphi^*_{x^*}\, \varphi^*_{y^*}\, \varphi^*_{x^*y^*} + \varphi_{y^*}^{*2}\, \varphi^*_{y^*y^*} + \beta\varphi^*_{y^*} = 0 \tag{16}$$

On the plate

$$u^* \tan \alpha + v^* = 0 \tag{17}$$

The behavior far upstream and at great depths should be that of a uniform stream

$$w^* \sim 1 \qquad x^* \to -\infty \qquad \text{or} \qquad y^* \to -\infty \tag{18}$$

This condition on the velocity at upstream infinity may be referred to as the 'Radiation Condition'. Far downstream gravity waves are expected to form on the free surface. In what follows the (*) on the dimensionless quantities will be omitted for simplicity, a reversion to physical quantities being understood at the end.

It will be convenient to solve the problem using alternately two different dependent variables w and Ω.

Let $$\Omega = \tau + i\theta = -\log w \tag{19}$$

i.e. $$\tau + i\theta = \log \frac{1}{|w|} - i \arg w \tag{20}$$

$$= \log \frac{1}{q} + i\theta$$

Then on the free surface

$$1 - e^{-2\tau} = 2\beta\,(y - h) \quad \text{on } S_f \tag{21}$$

And on the plate

$$\begin{aligned} \theta &= -\alpha \quad \text{on CB} \\ \theta &= \pi - \alpha \quad \text{on AC} \end{aligned} \tag{22}$$

Another way of writing the free surface condition (16) is

$$\mathrm{Re}\, w^2\, \overline{w}' = \beta\, \mathrm{Im}\, w \tag{23}$$

where the prime denotes differentiation with respect to z.

Consider next the potential plane or f-plane. IJ is the upstream free surface on which $\psi = a$, JA the other side of the leading edge jet. The plate is represented by ACB with the stagnation point C as the origin of the f-plane. BI is the downstream free surface. On IJ, $\psi = a$ represents the amount of fluid going into the jet per unit time per unit width perpendicular to the flow plane.

This polygon in the f-plane is now mapped onto the lower half of the auxiliary ζ-plane. The transformed problem will be solved in this ζ-plane and then transformed back to the physical z-plane. The plate is represented by the segment $|\xi| < 1$ on the real axis $\eta = 0$. Let the point J map onto $\zeta = -b$ and the stagnation point onto $\zeta = c$.

Since the points at infinity in the two planes correspond and since the leading and trailing edges map onto $\zeta = -1, 1$ the transformation is determined and the real constants b and c are as yet unknown aside from

$$b > 1, \quad |c| < 1. \tag{24}$$

The mapping is given by

$$\frac{df}{d\zeta} = \frac{a}{\pi(b+c)} \frac{\zeta - c}{\zeta + b} = H(\zeta) \tag{25}$$

Integration gives

$$\frac{\pi}{a} f(\zeta) = \frac{\zeta - c}{b + c} - \log \frac{\zeta + b}{b + c} .$$

The boundary conditions will now be transformed to the ζ-plane in which they are given on the known curve $\eta = 0$.

The velocity $w(z) = w(z(\zeta)) = w(\zeta)$ is taken to be invariant under the transformation. w will be written as $w(\zeta)$ as the problem is to be solved in the ζ-plane. Only when inversion is made to the z-plane will the distinction between $w(z(\zeta))$ and $w(\zeta)$ be required.

Now

$$w = \frac{df}{dz} = \frac{d\zeta}{dz} \cdot \frac{df}{d\zeta}$$

$$= H(\zeta) \frac{d\zeta}{dz} ,$$

therefore

$$dz = \frac{H(\zeta)}{w(\zeta)} d\zeta , \tag{26}$$

also

$$\frac{dw}{dz} = \frac{d\zeta}{dz}\,\frac{dw}{d\zeta}$$

$$= \frac{w(\zeta)}{H(\zeta)}\,w'(\zeta) \quad , \qquad (27)$$

where

$$w'(\zeta) = \frac{d}{d\zeta}\,w(\,z(\zeta)) = \frac{d}{d\zeta}\,w(\zeta) \,.$$

Integrating equation (26) gives

$$z = \int_1^{\zeta} \frac{H(\zeta)}{w(\zeta)}\,d\zeta \qquad (28)$$

The two forms of the free surface conditions equations (21) and (23) can thus be written

$$\text{(i)} \qquad 1 - e^{-2\tau(\xi)} = 2\beta \left\{ \text{Im} \int_1^{\xi} \frac{H(\zeta)}{w(\zeta)}\,d\zeta - h \right\}$$

$$(\,\eta = 0 \qquad |\xi| > 1\,) \qquad (29a)$$

$$\text{(ii)} \qquad w\,\bar{w}\,(w\,\bar{w})' = 2\beta\,H(\xi)\,\text{Im}\,w$$

$$(\,\eta = 0 \qquad |\xi| > 1\,) \qquad (29b)$$

where the prime denotes differentiation with respect to ζ .

The condition on the plate is now

$$\theta(\xi) = \begin{cases} -\,\alpha & (\,\eta = 0, \quad c < \xi < 1\,) \\ \pi - \alpha & (\,\eta = 0, \quad -1 < \xi < c\,) \end{cases} \qquad (30)$$

An appropriate radiation condition must be added to complete the formulation of the problem in the ζ-plane. Of course since $H(\zeta)$ is analytic, so are $\Omega(\zeta)$ and $w(\zeta)$.

It may be noted from equation (28) that since

$$H(\zeta) = \frac{a}{\pi(b + c)} \frac{\zeta - c}{\zeta + b} \sim A \quad \text{as} \quad \zeta \to \infty \tag{31}$$

where

$$A = \frac{a}{\pi(b + c)} \tag{32}$$

and since

$$w(\zeta) \sim 1 \quad \text{as} \quad \zeta \to \infty ,$$

hence

$$z \sim A\zeta \quad \text{as} \quad z, \zeta \to \infty . \tag{33}$$

Thus to within a real scale-factor the z- and ζ-planes correspond at infinity and there is a linear relationship between the flow pictures in the two planes for large z, ζ except in the flow region of the jet.

The problem has now been fully reduced in the ζ-plane and the inversion integral (28) written down. The solution will now be found using a singular perturbation expansion for small β.

5. THE INDIRECT PROBLEM

If α and h are given in the physical problem the parameters a, b, c must be found as functions of β, since for example the position of the stagnation point will change with velocity. Only two of the parameters a, b, c are independent since integration of equation (28) for $\zeta = -1$ will give a in terms of b and c on noting that the length of the plate has been normalized to unity.

Let the complete solution of the direct problem, where α and h are given, be represented symbolically by

$$P\ (\zeta; \alpha,\ h; \beta)$$

An indirect approach is to take the parameters b and c as fixed and then determine the α and h to which they correspond. Thus

a and h are regarded as functions of β . The complete solution of this indirect problem can be represented by

$$P'(\zeta; b, c; \beta)$$

An advantage of using the indirect method is that the mapping function $H(\zeta)$ becomes independent of β . Also the stagnation point will not change in the ζ-plane as higher order expansions are introduced. The direct solution can be recovered from the indirect solution as will be shown later.

6. SINGULAR PERTURBATION EXPANSION

If, as a first approximation, β is set to zero the problem becomes that of a gravity-free jet impinging on a plate. This has been solved by Green(5). This free-streamline solution does not satisfy the conditions at infinity. Instead of a uniform flow far upstream the free-streamline solution has a logarithmic singularity at infinity where $y_s \sim -\log |x|$. But when the gravity effect enters, for any value of $\beta > 0$, no matter how small, $|z|$ can be chosen large enough so that gravity waves can occur downstream. Setting $\beta = 0$ gives a good approximation near the plate but is completely incorrect far away.

The situation is similar to the one which occurs for viscous flow past a finite body at a small Reynolds number. For the viscous case the method of matched singular perturbation expansions has been developed by Kaplun(11). See also Kaplun and Lagerstrom(12), Lagerstrom and Cole[13]. A similar type of approach is used here, except that the singularity of the problem comes from the boundary conditions on the free surface.

The singular perturbation technique[11-13, 14] consists of choosing two or more perturbation expansions each valid in a certain region. These regions of validity do not completely overlap, otherwise at least one expansion would be redundant. However, each expansion represents the full solution within its valid region and has therefore definite relationships with its neighboring expansion. These relationships are expressed by the matching principle. In this problem only two expansions are necessary, the inner holding near the plate and the outer holding far away from it.

The simplest choice for an inner expansion is to retain the non-dimensional variables and hold them fixed as $\beta \longrightarrow 0$. This gives rise to the Inner Limit, for example

$$\lim_{\beta \longrightarrow 0} w(\zeta; \beta) \text{ with } \zeta \text{ fixed} . \tag{34}$$

The outer variables are chosen so as to characterize the flow at infinity. The outer solution is expected to have gravity waves downstream. Such waves normally contain terms like $e^{-i\beta z}$ and since for large $|z|$ the two variables z and ζ correspond to within a real scale factor, a favorable choice of independent outer variable is

$$\tilde{\zeta} = \beta \zeta \quad . \tag{35}$$

Note that there is no relative distortion, simply a magnification. The independent variable is still

$$w(\zeta;\ \beta) = \tilde{w}(\tilde{\zeta};\ \beta)\ . \qquad (36)$$

The outer limit is given by

$$\lim_{\beta \longrightarrow 0} \tilde{w}(\tilde{\zeta};\ \beta) \text{ with } \tilde{\zeta} \text{ fixed .} \qquad (37)$$

This essentially is an expansion for large ζ . The $\tilde{\zeta}$ and βz planes correspond except in the neighborhood of the origin of the $\tilde{\zeta}$-plane which represents the finite part of the z and ζ planes.

The condition at infinity affects the inner solution through the matching principle. This is also the means by which the conditions at the plate help determine the outer solution.

7. INNER EXPANSION

The variables are ζ and $\Omega(\zeta;\ \beta)$; the inner limit is to let β tend to zero for fixed ζ . The inner problem is

$$\Omega(\zeta) = \tau + i\theta \qquad (\eta < 0) \qquad (19)$$

$$\begin{cases} 1 - e^{-2\tau} = 2\beta\ (y_s - h) & (\eta = 0,\ |\xi| > 1) \qquad (29a) \\ \theta = \begin{cases} -\alpha & (\eta = 0,\ c < \xi < 1) \qquad (30) \\ \pi - \alpha & (\eta = 0,\ -1 < \xi < c) \end{cases} \end{cases}$$

The radiation condition is now replaced by the requirement that the inner solution match with the outer solution, while the latter retains the radiation condition.

Let the inner expansion be

$$\Omega(\zeta;\ \beta) = \Omega_0(\zeta) + \epsilon_1(\beta)\Omega_1(\zeta) + \epsilon_2(\beta)\Omega_2(\zeta) + \ . \ . \ . \qquad (38)$$

The ϵ_i are functions of β, which form an asymptotic series in such a way that

$$\lim_{\beta \to 0} \epsilon_{i+1}/\epsilon_i = 0, \quad \epsilon_o = 1 \tag{39}$$

The exact form of the ϵ_i will be determined by matching requirements.

The parameters a, h, ℓ must also be expanded for the indirect solution.

$$\begin{aligned} a &= a_o + \epsilon_1 a_1 + \epsilon_2 a_2 + \cdots \\ h &= \epsilon_{-1} h_1 + h_2 + \epsilon_* h_3 + \epsilon_1 h_4 + \epsilon_2 h_5 + \cdots \\ \ell &= \ell_o + \epsilon_1 \ell_1 + \epsilon_2 \ell_2 + \cdots \end{aligned} \tag{40}$$

The form for h will be dictated by the matching principle.

Upon substitution of equation (38) and equation (40) into equation (28)

$$z = \int_1^{\zeta} e^{\Omega_o(\zeta)} \left\{ 1 + \sum_1^n \epsilon_k \Omega_k + \frac{1}{2} \left(\sum_1^n \epsilon_k \Omega_k \right)^2 + \cdots \right\} H(\zeta)\, d\zeta \tag{41a}$$

which may be regrouped to yield the form

$$z(\zeta; \beta) = z_o(\zeta) + \epsilon_1 z_1(\zeta) + \epsilon_2 z_2(\zeta) + \cdots . \tag{41b}$$

Inner Solution of Zeroth Order;

It will be found later that $\epsilon_{-1} = \log \beta$. Hence on substitution of the inner expansion of Ω into the inner problem and taking the inner limit, the zeroth order problem becomes

$$1 - e^{-2\tau_o} = 0 \qquad (\eta = 0, \ |\xi| > 1) \tag{42}$$

$$\theta_o = \begin{cases} -\alpha_o & (\eta = 0, \quad c < \xi < 1) \\ \pi - \alpha_o & (\eta = 0, \quad -1 < \xi < c) \end{cases} \tag{43}$$

From equation (42) it follows that

$$\tau_o = 0 \qquad (\eta = 0, \quad |\xi| > 1) \tag{44}$$

Let

$$\Omega_o(\zeta) = -\log w_o(\zeta) \ , \tag{45}$$

then

$$|w_o| = 1 \qquad (\eta = 0, \quad |\xi| > 1) \ , \tag{46}$$

$$\arg w_o = \begin{cases} \alpha_o & (\eta = 0, \quad c < \xi < 1) \\ \alpha_o - \pi & (\eta = 0, \quad -1 < \xi < c) \end{cases} \ , \tag{47}$$

with

$$w_o(c) = 0 \ . \tag{48}$$

This represents a semi-circle in the w_o-plane which can be mapped onto the lower half of the ζ-plane to give as a solution

$$\begin{aligned} w_o(\zeta) &= e^{i\alpha_o} \frac{\zeta - c}{(1 - \zeta c) + \sqrt{1 - \zeta^2}\sqrt{1 - c^2}} \\ &= e^{i\alpha_o} \frac{(1 - \zeta c) - \sqrt{1 - \zeta^2}\sqrt{1 - c^2}}{\zeta - c} \end{aligned} \tag{49}$$

$\sqrt{1 - \zeta^2}$ is defined as the branch of $(1 - \zeta^2)^{\frac{1}{2}}$ with its branch cut on the real ζ-axis between $\zeta = \pm 1$, the arguments of $\zeta + 1$, $\zeta - 1$ lying between $-\pi$ and π. α_o will be found by matching with the outer solution.

8. OUTER EXPANSION

The variables are $\tilde{\zeta}$ and $\tilde{w}(\tilde{\zeta}, \beta)$; the outer limit is to let β tend to zero for fixed $\tilde{\zeta}$. Transformation of equation (29b) to outer variables gives

$$\tilde{w} \; \bar{\tilde{w}} \; (\tilde{w} \; \bar{\tilde{w}})' = 2 \, \mathrm{Im} \, \tilde{w} \, H \, (\tilde{\xi}/\beta) \qquad (\tilde{\eta} = 0) \tag{50}$$

where the prime denotes differentiation with respect to $\tilde{\xi}$. The boundary corresponding to the plate shrinks to the origin in the $\tilde{\zeta}$-plane. Thus $\tilde{w}$ is an analytic function of $\tilde{\zeta}$ with a possible singularity at the origin.

Let the outer expansion be

$$\tilde{w}(\tilde{\zeta};\beta) = \tilde{w}_o(\tilde{\zeta}) + \delta_1(\beta) \, \tilde{w}_1(\tilde{\zeta}) + \delta_2(\beta) \, \tilde{w}_2(\tilde{\zeta}) + \ldots \tag{51}$$

where the $\delta_i(\beta)$ form an asymptotic series such that

$$\lim_{\beta \to 0} \delta_{i+1}/\delta_i = 0, \qquad \delta_o = 1 \tag{52}$$

To find $\tilde{w}_o(\tilde{\zeta})$, consider the outer limit of the flow field as $\beta = g\ell/U^2$ tends to zero. One way of doing this is to let the characteristic length ℓ tend to zero for fixed g and U. The plate shrinks to a point singularity on an otherwise undisturbed uniform stream. In the limit as $\beta \to 0$ it is obvious that the uniform stream is recovered; therefore

$$\tilde{w}_o(\tilde{\zeta}) = 1 \tag{53}$$

This can also be seen by noting that the equation for $\tilde{w}_o(\zeta)$ is

$$(\tilde{w}_o \bar{\tilde{w}}_o)(\tilde{w}_o \bar{\tilde{w}}_o)^{1} = 2\,A\,\mathrm{Im}\,\tilde{w}_o \qquad (\tilde{\eta} = 0) \tag{54}$$

where $\tilde{w}_o$ can have a singularity at $\tilde{\zeta} = 0$. Since $\tilde{w}_o \sim 1$ as $\tilde{\zeta} \longrightarrow \infty$, an expansion of the form

$$\tilde{w}_o(\tilde{\zeta}) = 1 + \sum_{n=1}^{\infty} c_n / \tilde{\zeta}^{\,n}$$

can be substituted into equation (54) and relationships can be found for the constants c_n. By use of the matching process, it turns out that $c_n = 0$ for all n .

9. MATCHING

The inner and outer solutions are the inner and outer expansions of the total exact solution. The resulting correspondence between the two solutions is expressed by the matching principle. The following statement of it is most useful in the present case.

> "The n-th order inner expansion of the n-term outer solution, and the n-th order outer expansion of the n-term inner solution should match when written in the same coordinates."

The original work on this principle is due to Kaplun[11]. A general statement and discussion is given by Van Dyke[14].

Outer Expansion of Zeroth Order Inner Solution

From equation (49)

$$w_o(\zeta) = e^{i\alpha_o} \frac{(1 - \zeta c) - i\sqrt{\zeta^2 - 1}\sqrt{1 - c^2}}{\zeta - c}$$

is the one term inner solution. The outer limit is now applied to it.

$$w_o(\zeta) \sim e^{i\alpha_o} \frac{(\beta - \tilde{\zeta}c) - i\sqrt{\tilde{\zeta}^2 - \beta^2}\sqrt{1 - c^2}}{\tilde{\zeta} - \beta c}$$

$$\sim e^{i\alpha_o} \frac{-\tilde{\zeta}c - i\tilde{\zeta}\sqrt{1 - c^2}}{\tilde{\zeta}} + 0(\beta)$$

$$= -e^{i\alpha_o}(c + i\sqrt{1 - c^2}) + 0(\beta)$$

The one term outer expression of the one term inner solution is thus

$$w_o(\zeta) \sim -e^{i\alpha_o}(c + i\sqrt{1 - c^2}) \tag{55}$$

Note that no terms in $1/\beta$ occur in this one term expansion and hence for the coefficients in $w_o(\zeta)$

$$c_1 = 0$$

$$c_2 = 0$$

Hence

$$\tilde{w}_o(\tilde{\zeta}) = 1 \quad .$$

The one term inner expansion of the one term outer solution given in equation (51) is therefore

$$\tilde{w}(\zeta) \sim 1 \tag{56}$$

Matching equations (55) and (56) gives the first matching condition

$$c = - \cos \alpha_o$$

or (57)

$$\alpha_o = \cos^{-1}(-c)$$

It is more convenient to use α_o rather than c from now on. Hence

$$w_o(\zeta) = e^{i\alpha_o} \frac{(1 + \zeta \cos \alpha_o) - \sqrt{1 - \zeta^2} \sin \alpha_o}{\zeta + \cos \alpha_o}$$

$$= e^{i\alpha_o} \frac{\zeta + \cos \alpha_o}{1 + \zeta \cos \alpha_o + \sqrt{1 - \zeta^2} \sin \alpha_o}$$

Selection of Expansion Parameters:

To select the ϵ_i, δ_i consider the inner equation

$$1 - e^{-2\tau} = 2\beta (y_s - h) \tag{29}$$

Care has to be exercised in the choice of singular perturbation expansions due to the possible occurrence of logarithmic terms (cf., Van Dyke[14], p. 29). This is hinted at here since Green's solution, which is the limiting case of $\beta = 0$, gives a draft which is logarithmically singular. This indicates that ϵ_{-1} in the expansion (40) for h should be $\log \beta$. It turns out that this is indeed required to allow matching with the outer flow.

Then, expanding equation (29), one obtains

$$\epsilon_1 \tau_1 + \epsilon_2 \tau_2 + \dots = \beta y_{os} - \beta \log\beta \, h_1 - \beta h_2 + \dots \tag{58}$$

The choice is thus indicated to be

$$\epsilon_1 = \beta \log\beta, \quad \epsilon_2 = \beta, \quad \dots .$$

The outer solution must be matched with the inner and hence similar terms will appear indicating that

$$\delta_1 = \beta \log\beta, \quad \delta_2 = \beta, \quad \dots .$$

Then, together with $\epsilon_{-1} = \log \beta$, the terms are

$$\epsilon_1 = \beta \log \beta = \delta_1, \quad \epsilon_2 = \beta = \delta_2 \qquad (59)$$

This turns out to be the correct choice, no other terms being required for matching to $0(\beta)$ and for matching to higher orders, the choice of $\epsilon_3, \epsilon_4, \ldots$ becomes straightforward.

Therefore

$$\Omega(\zeta, \beta) = \Omega_o(\zeta) + \beta \log \beta \, \Omega_1(\zeta) + \beta \, \Omega_2(\zeta) + \ldots \qquad (60)$$

$$\tilde{w}(\tilde{\zeta}, \beta) = 1 + \beta \log \beta \, \tilde{w}_1(\tilde{\zeta}) + \beta \, \tilde{w}_2(\tilde{\zeta}) + \ldots \qquad (61)$$

Inner Boundary Conditions:

Noting $\tau_o = 0$, one substitutes equation (60) into equation (29a) and compares terms, giving

$$\left.\begin{aligned} \tau_1 &= -h_{-1} \\ \tau_2 &= y_o(\xi) - h_2 \end{aligned}\right\} \quad (\eta = 0, \ |\xi| > 1) \qquad (62)$$

Substitution into equation (31) gives

$$\theta_k = -\alpha_k \qquad (k = 1, 2, \ldots, \quad \eta = 0, \ |\xi| < 1) \qquad (63)$$

The apparent angle of attack for the free streamline solution is α_o, which gives c as the stagnation point. When the free streamline solution is perturbed to match with the outer solutions the apparent angle of attack changes so as to retain c as the stagnation point (indirect approach).

Outer Boundary Conditions:

Substitution of equation (61) into equation (50) gives

$$\frac{d}{d\tilde{\xi}} \operatorname{Re} \tilde{w}_1 = A \operatorname{Im} \tilde{w}_1 \qquad (\tilde{\eta} = 0)$$

$$\frac{d}{d\tilde{\xi}} \operatorname{Re} \tilde{w}_2 = A \operatorname{Im} \tilde{w}_2 \qquad (\tilde{\eta} = 0)$$

These equations can be written

$$\operatorname{Re} (\tilde{w}_k{}' + i A \tilde{w}_k) = 0 \qquad (\tilde{\eta} = 0 \quad k = 1, 2) \tag{64}$$

where the prime indicates differentiation with respect to $\tilde{\xi}$.

Remark on Inner Problem:

The general form of the boundary value problem for $\Omega_n(\zeta)$ is

$$\tau_n = f_n(\xi) \qquad (\eta = 0, \quad |\xi| > 1)$$

$$\theta_n = -\alpha_n \qquad (\eta = 0, \quad |\xi| < 1)$$

This mixed-type boundary problem can be expressed in terms of a Riemann-Hilbert problem, the solution of which can be readily obtained (cf. Muskhelishvili[15]), and may be written

$$\Omega_n(\zeta) = -i\alpha_n - \frac{\sqrt{\zeta^2 - 1}}{\pi i} \int_L \frac{f_n(t)}{\sqrt{t^2 - 1}\,(t - \zeta)} dt \tag{65}$$

where L is the real axis, $|\xi| > 1$.

This is the form required when Ω_n has no singularities at $\zeta = \pm 1$.

Inner Problem of Order $\beta \log \beta$:

$$\tau_1 = -h_1$$

$$\theta_1 = -\alpha_1$$

Hence from equation (65)

$$\Omega_1 = -(h_1 + i\alpha_1) \tag{66}$$

Inner Problem of Order β:

$$\tau_2 = y_o - h_o \qquad (\eta = 0, \quad |\xi| > 1)$$

$$\theta_2 = -\alpha_2 \qquad (n = 0, \quad |\xi| < 1)$$

Hence

$$\Omega_2(\zeta) = -h_2 - i\alpha_2 - \frac{\sqrt{\zeta^2 - 1}}{\pi i} \int_L \frac{y_o(t)}{\sqrt{t^2 - 1}\,(t - \zeta)}\, dt \tag{67}$$

where

$$y_o(\xi) = A \operatorname{Im} e^{-i\alpha_o} \int_1^{\xi} \frac{1 + \zeta \cos\alpha_o + i\sqrt{\zeta^2 - 1}\sin\alpha_o}{\zeta + b}\, d\zeta \tag{68}$$

from equations (28) and (49). After some manipulation the solution is found to be

$$\Omega_2(\zeta) = -(h_2 + i\alpha_2) + A\Big\{\sin\alpha_o\,(b\cos\alpha_o - 1)\log\frac{\zeta + b}{b + 1}$$

$$- \sin\alpha_o\cos\alpha_o\,(\zeta - \sqrt{\zeta^2 - 1} - 1) + B_1\log(\zeta + \sqrt{\zeta^2 - 1})$$

$$+ B_2 \log\frac{1 + \zeta b + \sqrt{b^2 - 1}\sqrt{\zeta^2 - 1}}{\zeta + b} + \frac{i}{\pi}\sin\alpha_o\cos\alpha_o\int_1^{\zeta}\frac{\sqrt{t^2 - 1}}{t + b}$$

$$\log\frac{t + 1}{t - 1}\, dt\Big\} \tag{69}$$

where

$$B_1 = -b \sin a_o \cos a_o + i\,(b - \cos a_o - \frac{1}{\pi} \sin 2a_o)$$

$$B_2 = \sin a_o \cos a_o \sqrt{b^2 - 1} - i \Big\{ (b \cos a_o - 1) \cos a_o$$

$$+ \sin^2 a_o \sqrt{b^2 - 1} - \frac{1}{\pi} \sin a_o \cos a_o \sqrt{b^2 - 1} \log \frac{b+1}{b-1} \Big\} \qquad (70)$$

It may be noted that from equation (19)

$$w = e^{-\Omega}$$

$$= e^{-\Omega_o} (1 - \beta \log \beta \Omega_1 - \beta\Omega_2 + \ldots).$$

Hence if

$$w = w_o (1 + \beta \log \beta w_1 + \beta w_2 + \ldots) \qquad (71)$$

then

$$w_o = e^{-\Omega_o}$$

$$w_1 = -\Omega_1 \qquad (72)$$

$$w_2 = -\Omega_2$$

The parameters a_1, a_2, h_2 are yet to be found by matching with the outer solution.

10. HIGHER ORDER OUTER SOLUTIONS

From equation (64)

$$\mathrm{Re}\,(\tilde{w}_k{}' + i A \tilde{w}_k) = 0 \quad \text{on} \quad \tilde{\eta} = 0$$

except for $\tilde{\xi} = 0$ which corresponds to the plate in outer variables.

Let

$$\tilde{w}_k' + iA\tilde{w}_k = ig_k(\tilde{\zeta}) \tag{73}$$

Then $g_k(\tilde{\zeta})$ is analytic in the lower half plane and takes on real values on the real axis. Hence by Schwarz's principle of reflection $g_k(\tilde{\zeta})$ can be continued analytically into the upper half $\tilde{\zeta}$-plane by defining a function $g^*(\tilde{\zeta})$ such that

$$g_k^*(\tilde{\zeta}) = g_k(\tilde{\zeta}) \qquad (\tilde{\eta} < 0)$$

$$g_k^*(\tilde{\zeta}) = \bar{g}_k(\bar{\tilde{\zeta}}) \qquad (\tilde{\eta} > 0)$$

Consequently, the function $g_k^*(\tilde{\zeta})$ is analytic and regular everywhere except possibly at the origin. Once $g_k(\tilde{\zeta})$ is determined, $w_k(\tilde{\zeta})$ can be found by integration of equation (73).

$$\tilde{w}_k(\tilde{\zeta}) = i\,e^{-iA\tilde{\zeta}} \int^{\tilde{\zeta}} e^{iAt} g_k(t)\,dt \tag{74}$$

A particular solution for $\tilde{w}_k$ is obtained when $g_k = 0$. Then

$$\tilde{w}_k(\tilde{\zeta}) = \text{a real constant .} \tag{75}$$

From the conditions on $g_k(\tilde{\zeta})$ it can be expanded as

$$g_k(\tilde{\zeta}) = \sum_{n=1}^{\infty} \frac{c_{kn}}{\tilde{\zeta}^n} \qquad (c_{kn} \text{ real}) \tag{76}$$

There are no terms for $n \leq 0$ since g_k is regular at infinity. $n = 0$ gives a real constant already considered in equation (75).

The radiation condition requires that no waves propagate to upstream infinity. This indicates that the lower limit in equation (74) should be $-\infty$. Therefore the general solution for $\tilde{w}_k$ is

$$\tilde{w}_k(\tilde{\zeta}) = ie^{-iA\tilde{\zeta}} \sum_{n=1}^{\infty} c_{kn} \int_{-\infty}^{\tilde{\zeta}} \frac{e^{iAt}}{t^n} dt \tag{77}$$

where the c_{kn} are real constants. The path of integration is taken always to be in the lower half $\tilde{\zeta}$-plane.

Consideration of the integrals in equation (77) shows the following. Each term contains the integral for $n = 1$ together with a polynomial in $1/\tilde{\zeta}$ of degree $n - 1$ having complex coefficients. This is easily seen from integration by parts. The integral gives the wave behavior on the downstream side with no waves upstream. The other terms, as well as parts of the integral, give the local behavior which decreases at least like $1/\tilde{\zeta}$ as $\tilde{\zeta} \rightarrow \infty$.

This general feature of $\tilde{w}_k$ can be seen clearly, for example, from the case $n = 2$,

$$\int_{-\infty}^{\tilde{\zeta}} e^{iAt} \frac{dt}{t^2} = -\frac{1}{\tilde{\zeta}} e^{iA\tilde{\zeta}} + iA \int_{-\infty}^{\tilde{\zeta}} e^{iAt} \frac{dt}{t} ,$$

with

$$\int_{-\infty}^{\tilde{\zeta}} e^{iAt} \frac{dt}{t} \sim -\frac{i}{A\tilde{\zeta}} e^{iA\tilde{\zeta}} + 0(1/\tilde{\zeta}^2) \text{ as } \tilde{\zeta} \rightarrow -\infty ,$$

$$\sim i\pi - \frac{i}{A\tilde{\zeta}} e^{iA\tilde{\zeta}} + 0(1/\tilde{\zeta}^2) \text{ as } \tilde{\zeta} \rightarrow +\infty .$$

Therefore

$$ie^{-iA\tilde{\zeta}} \int_{-\infty}^{\tilde{\zeta}} e^{iAt} \frac{dt}{t^2} \sim 0(1/\tilde{\zeta}^2) \qquad \text{as } \tilde{\zeta} \longrightarrow -\infty ,$$

$$\sim \quad -\pi e^{-iA\tilde{\zeta}} + 0(1/\tilde{\zeta}^2) \qquad \text{as } \tilde{\zeta} \longrightarrow +\infty ,$$

as noted above. This satisfies the radiation condition and gives a wave at downstream infinity.

It may be noted that equation (33) gives

$$e^{-iA\tilde{\zeta}} \sim e^{-i\beta z} \quad \text{at infinity.}$$

The outer solution will be completed to order β when the coefficients c_{kn}, which give the strength of the singularity at the origin, are found from matching with the inner solution.

11. MATCHING TO ORDER β

To carry out the matching to order β, one expands the inner solution up to the β-order in outer variables to order β; and expands the outer solution up to the β-order in inner variables, also to order β. The matching principle says that these two procedures give the same result when written in the same variables.

After some algebra the inner solution gives

$$w(\tilde{\zeta}, \beta) \sim 1 + \beta\log\beta\,(h_1 + i\alpha_1) - A\left\{\sin\alpha_o - i\,(b - \cos\alpha_o)\right\}\beta\log\beta$$
$$+ i\beta\,\frac{\sin\alpha_o}{\tilde{\zeta}} + A\beta\left\{\sin\alpha_o - i\,(b - \cos\alpha_o)\right\}\log\tilde{\zeta}$$
$$+ \beta\left\{h_o + i\alpha_2 - AB_3\right\} + \dots \qquad (78)$$

where

$$B_3 = \sin a_o \cos a_o - \sin a_o (b \cos a_o - 1) \log (b + 1) + B_1 \log 2$$

$$+ B_2 \log (b + \sqrt{b^2 - 1}) + \frac{i}{\pi} \sin a_o \cos a_o M_o(b)$$

$$M_o(b) = \int_o^1 \left\{ \frac{\sqrt{1 - t^2}}{1 + tb} \log \frac{1 + t}{1 - t} - 2t \right\} \frac{dt}{t^2} \tag{79}$$

and B_1, B_2 are given by equation (70) as functions of b and c.

Note that the highest term in $1/\zeta$ is of order β/ζ. The remarks following equation (77) therefore indicate that

$$\tilde{w}_2 = ie^{-iA\tilde{\zeta}} \int_{-\infty}^{\tilde{\zeta}} e^{iAt} \left(\frac{c_{21}}{t} + \frac{c_{22}}{t^2} \right) dt \tag{80}$$

No term in $\beta \log \beta$ is required for matching.

Let

$$\tilde{w} (\tilde{\zeta}, \beta) = 1 + i\beta e^{-iA\tilde{\zeta}} \int_{-\infty}^{\tilde{\zeta}} e^{iAt} \left(\frac{c_{21}}{t} + \frac{c_{22}}{t^2} \right) dt \tag{81}$$

By using the asymptotic expansion

$$\int_{-\infty}^{\tilde{\zeta}} e^{-iAt} \frac{dt}{t} = \int_{\infty}^{iA\beta\zeta} e^{-t} \frac{dt}{t} = Ei(-iA\beta\zeta) \sim \gamma + \log(iA\beta\zeta) + 0(\beta) \tag{82}$$

where γ is Euler's Constant, 0.577215, the outer solution (equation (81)) can be written in inner variables and expanded to order β, giving

$$\tilde{w} \sim 1 - i \frac{c_{22}}{\zeta} - \beta\log\beta \, (ic_{21} - Ac_{22}) +$$

$$+ i\beta(c_{21} + iAc_{22}) \log \zeta + i\beta(c_{21} + iAc_{22})(\gamma + \log A + i\frac{\pi}{2}) .$$

Rewriting in outer variables to match with the outer expansion of $w(\zeta, \beta)$

$$\tilde{w} \sim 1 - i\beta \frac{c_{22}}{\tilde{\zeta}} + i\beta (c_{21} + iAc_{22}) \log \tilde{\zeta} + \beta c$$

where

$$c = i(c_{21} + iAc_{22})(\gamma + \log A + i\frac{\pi}{2}) .$$

Matching of the $1/\tilde{\zeta}$ term gives $c_{22} = -\sin \alpha_o$.

Matching of the $\log \tilde{\zeta}$ term gives $c_{21} = -A(b - \cos \alpha_o)$.

The $\beta \log \beta$ term from the inner solution is

$$h_1 + i\alpha_1 - A \sin \alpha_o + iA(b - \cos \alpha_o) ,$$

while the $\beta \log \beta$ term from the outer solution is c_1. Since only real constants arise from the outer solution, the matching gives

$$\alpha_1 = -A(b - \cos \alpha_o) = c_{21} , \qquad (83)$$

$$h_1 = A \sin \alpha_o \qquad (84)$$

Finally, matching the constant terms gives

$$h_2 + i\alpha_2 = A \{ B_3 - iBQ \} \qquad (85)$$

where B_3 is given by equation (79) and

$$B = b - e^{-i\alpha_o} \qquad (86)$$

$$Q = \gamma + \log A + i\frac{\pi}{2} \qquad (87)$$

Therefore α_o, α_1, α_2, h_1 and h_2 have been found in terms of a, b, and c. The values of ℓ_1, ℓ_2, ℓ_3 can be found by integration of equation (28) along the plate. The values of h_3 and h_4 cannot be found from the order β theory, since being of order $\beta \log \beta$ and β respectively they enter the problem through the terms of order $\beta^2 \log\beta$ and β^2 as can be seen from equations (21) and (28).

The complete solution to order β can now be written.

Inner Solution:

$$w(\zeta, \beta) = \frac{e^{i\alpha_o}}{\zeta + \cos\alpha_o}\left\{ (1 + \zeta \cos\alpha_o) - i \sin\alpha_o \sqrt{\zeta^2 - 1} \right\} .$$

$$. \left\{ 1 + \beta \log\beta \, w_1(\zeta) + \beta \, w_2(\zeta) + \ldots . \right\}$$

$$w_1(\zeta) = h_1 + i\alpha_1 = A \sin\alpha_o - iA (b - \cos\alpha_o) = - iAB$$

$$w_2(\zeta) = h_2 + i\alpha_2 - A \left\{ \sin\alpha_o (b \cos\alpha_o - 1) \log \frac{\zeta + b}{b + 1} \right.$$

$$- \sin\alpha_o \cos\alpha_o (\zeta - \sqrt{\zeta^2 - 1} - 1) + B_1 \log (\zeta + \sqrt{\zeta^2 - 1})$$

$$+ B_2 \log \frac{1 + \zeta b + \sqrt{b^2 - 1} \sqrt{\zeta^2 - 1}}{\zeta + b} + \frac{i}{\pi} \sin\alpha_o \cos\alpha_o$$

$$\left. \int_1^{\zeta} \frac{\sqrt{t^2 - 1}}{t + b} \log \frac{t + 1}{t - 1} \, dt \right\} \qquad (88)$$

where

$$A = \frac{a}{\pi (b + c)}$$

$$\alpha_o = \cos^{-1}(-c)$$

h_2, α_2 given by equation (87) .

Outer Solution:

$$\tilde{w}(\tilde{\zeta}, \beta) = 1 - i\beta A e^{-iA\tilde{\zeta}} \int_{-\infty}^{A\tilde{\zeta}} e^{it} \left\{ \frac{\sin \alpha_o}{t^2} + \frac{(b - \cos \alpha_o)}{t} \right\} dt \qquad (89)$$

This can also be written

$$\tilde{w}(\tilde{\zeta}, \beta) = 1 + i\beta \frac{\sin \alpha_o}{\tilde{\zeta}} - iABe^{-iA\tilde{\zeta}} \, Ei(-iA\beta\zeta) \qquad (90)$$

where

$$Ei(z) = \int_{\infty}^{-z} \frac{e^{-t}}{t} dt \quad .$$

12. LENGTH OF PLATE

In the indirect problem the length of the plate is not a primary parameter but is obtained as an expansion

$$\ell = \ell_o + \ell_1 \beta \log \beta + \ell_2 \beta + \ldots . \qquad (40)$$

As shown below ℓ_i can be found in terms of a, b, c. For a given b, c, the condition that

$$\ell = 1 \qquad (91)$$

(in non-dimensional variables), allows a to be determined.

Let z_L, z_T be the positions of the leading and trailing edges of the plate in the physical plane. From equation (28),

$$z_T - z_L = \int_{-1}^{1} \frac{H(\zeta)}{w(\zeta)} \, d\zeta \ .$$

Directly from the physical plane (Figure 1)

$$z_T - z_L = \ell e^{-i\alpha}$$

Therefore

$$\ell = e^{i\alpha} \int_{-1}^{1} \frac{H(\zeta)}{w(\zeta)} \, d\zeta \tag{92}$$

Expanding

$$\ell = e^{i\alpha} \int_{-1}^{1} \frac{1}{w_o} \left\{ 1 - (\beta \log \beta) w_1 - \beta w_2 + \ldots \right\} \quad H(\zeta) \, d\zeta$$

$$= e^{i\alpha} \int_{-1}^{1} \frac{1}{w_o} \left\{ 1 - (h_1 + i\alpha_1)\beta \log \beta - i\alpha_2 \beta + \beta \tau_2(\zeta) + \ldots \right\} H(\zeta) d\zeta$$

$$= e^{i\alpha_o} \int_{-1}^{1} \frac{H(\zeta)}{w_o(\zeta)} d\zeta - \beta \log \beta \, h_1 e^{i\alpha} \int_{-1}^{1} \frac{H(\zeta)}{w_o(\zeta)} \, d\zeta$$

$$+ \beta e^{i\alpha} \int_{-1}^{1} \frac{H(\zeta)}{w_o(\zeta)} \tau_2(\zeta) \, d\zeta + \ldots$$

The α_1, α_2 terms are absorbed by the $e^{-i\alpha}$ term. Hence

$$\ell_o = A \left\{ (b \cos \alpha_o - 1) \log \frac{b - 1}{b + 1} + 2 \cos \alpha_o \right.$$

$$\left. + \pi \sin \alpha_o \, (b - \sqrt{b^2 - 1}) \right\} \tag{93}$$

This is the result found by Green[5] and given in Section 12.26 of Milne-Thomson[16]

$$\ell_1 = - A \sin \alpha_o \, \ell_o \tag{94}$$

$$\ell_2 = A \int_{-1}^{1} \frac{1 + \zeta \cos \alpha_o + \sin \alpha_o \sqrt{1 - \zeta^2}}{\zeta + b} \tau_2(\zeta) d\zeta \qquad (95)$$

From equation (88) τ_2 can be found for $|\xi| < 1$ and ℓ_2 can be written as a function of a, b, c, which can be tabulated.

The parameter a is found by applying condition (91) using equations (93), (94), and (95) for a given value of β.

13. LIFT AND DRAG ON THE PLATE

Let p(z) be the pressure in the fluid, then from equation (3)

$$p = \frac{1}{2}(1 - |w|^2) - \beta y \qquad (96)$$

The force on the plate is given by

$$N = i \int p \, dz \qquad (97)$$

where the integral is taken along the plate. This force is normal to the plate and can thus be written

$$N = iN^* e^{-i\alpha} \qquad (98)$$

where N^* is real.

Let L, D be the lift and drag respectively, then

$$L = N^* \cos \alpha , \qquad D = N^* \sin \alpha . \qquad (99)$$

Combining equations (96) and (97) gives

$$N = \frac{1}{2} i \int_{-1}^{1} \left\{ 1 - |w|^2 \right\} \frac{dz}{d\zeta} d\zeta - i\beta \int_P y \, dz \qquad (100)$$

Hence

$$N^* = \frac{1}{2} e^{i\alpha} \int_{-1}^{1} \left\{ 1 - |w|^2 \right\} \frac{H(\zeta)}{w(\zeta)} d\zeta - \beta e^{i\alpha} \int_P y \, dz$$

$$= \frac{1}{2} e^{i\alpha} \int_{-1}^{1} \left\{ 1 - |w|^2 \right\} \frac{H(\zeta)}{w(\zeta)} d\zeta - \frac{1}{2} \beta \ell (\ell \sin \alpha - 2h)$$

Expanding in terms of β and using the boundary conditions for $w(\zeta)$

$$N^* = \pi A \sin \alpha_o (b - \sqrt{b^2 - 1}) + \beta (h - \frac{1}{2} \sin \alpha)$$

$$- \beta \log \beta A^2 \sin \alpha_o \left\{ 2 \cos \alpha_o - (b \cos \alpha_o - 1) \log \frac{b+1}{b-1} \right\}$$

$$+ \beta A \int_{-1}^{1} \frac{1 + \zeta \cos \alpha_o}{\zeta + b} \tau_2(\zeta) \, d\zeta \qquad (101)$$

upon using equation (91). The first term agrees with Green's solution.

The final integral can be evaluated and tabulated as a function of b and c .

14. INNER SOLUTION TO ORDER β^2

Continuing the inner expansion as given in equation (60) the inner expansion can be written

$$\Omega)\zeta) = \Omega_o(\zeta) + \beta \log\beta\, \Omega_1(\zeta) + \beta\Omega_2(\zeta) + \beta^2 \log^2\beta\, \Omega_3(\zeta)$$

$$+ \beta^2 \log\beta\, \Omega_4(\zeta) + \beta^2 \Omega_5(\zeta) + \ldots . \qquad (102)$$

$$h = h_1 \log\beta + h_2 + h_3 \beta \log^2\beta + h_4 \beta \log\beta + h_5 \beta + \ldots$$

Substitution of equation (102) into equation (29a) gives the following boundary conditions for Ω_3, Ω_4 and Ω_5

$$\tau_3 = h_1^2 - h_3 \qquad (103)$$

$$(\eta = 0, \ |\xi| > 1)$$

$$\tau_4 = y_1 - h_4 - 2h_1 \tau_2 \qquad (104)$$

$$\tau_5 = y_2 + y_o^2 - 2h_2 y_o + h_2^2 - h_5 \qquad (105)$$

with

$$\theta_k = -\alpha_k \quad (k = 1, 2, \ldots, \qquad \eta = 0, \ |\xi| < 1). \qquad (63)$$

Order $\beta^2 \log^2\beta$;

The particular solution is easily seen to be

$$\Omega_3(\zeta) = -(h_3 + i\alpha_3) + A^2 \sin^2 \alpha_o \qquad (106)$$

Order $\beta^2 \log\beta$;

Let

$$\Omega_4(\zeta) = -(h_4 + i\alpha_4) + \Omega_4^*(\zeta) - 3A \sin\alpha_o \Omega_2(\zeta) - A \sin\alpha_o (h_2 + 3i\alpha_2)$$

Then

$$\tau_4^* = f(\xi) \qquad (\eta = 0, \ |\xi| > 1)$$

$$\theta_4^* = 0 \qquad (\eta = 0, \ |\xi| < 1),$$

where from equations (66) and (41)

$$f(\xi) = -\alpha_1 \operatorname{Re} \int_1^{\xi} \frac{H(\zeta)}{w_o(\zeta)}\, d\zeta \tag{107}$$

From which it can be found that

$$\begin{aligned}\Omega_4(\zeta) = &-(h_4 + i\alpha_4) + A^2 (b - \cos\alpha_o) \Big\{ \zeta \cos^2\alpha_o \\ &+ \sqrt{\zeta^2 - 1}\, \sin^2\alpha_o - \cos\alpha_o (b\cos\alpha_o - 1) \log\frac{\zeta + b}{b + 1} - \cos^2\alpha_o \\ &+ B_4 \log(\zeta + \sqrt{\zeta^2 - 1}) + B_5 \log\Big(\frac{1 + b\zeta + \sqrt{b^2 - 1}\sqrt{\zeta^2 - 1}}{\zeta + b}\Big) \\ &+ \sin^2\alpha_o \frac{i}{\pi} \int_1^{\zeta} \frac{\sqrt{t^2 - 1}}{t + b} \log\frac{t + 1}{t - 1}\, dt \Big\} \\ &- 3A \sin\alpha_o \Omega_2(\zeta) - A \sin\alpha_o (h_2 + 3i\alpha_2)\end{aligned} \tag{108}$$

where

$$\begin{aligned} B_4 &= -b\sin^2\alpha_o - i\sin\alpha_o\Big(\frac{2}{\pi}\sin\alpha_o + 1\Big) \\ B_5 &= \sqrt{b^2 - 1}\,\sin^2\alpha_o + i\sin\alpha_o \Big\{\frac{1}{\pi}\sqrt{b^2 - 1}\,\log\frac{b + 1}{b - 1}\,\sin\alpha_o \\ &\qquad - (b - \sqrt{b^2 - 1})\cos\alpha_o + 1\Big\}\end{aligned} \tag{109}$$

The asymptotic expansion as $\zeta \longrightarrow \infty$ is

$$\begin{aligned}\Omega_4(\zeta) \sim\ & A^2(b - \cos\alpha_o)\zeta - A^2 B (B + 2i\sin\alpha_o)\log\zeta \\ & -(h_4 + i\alpha_4) + A^2(b - \cos\alpha_o) B_6 \\ & - 3iA^2 BQ \sin\alpha_o - A\sin\alpha_o (h_2 + 3\, i\alpha_2) \quad ,\end{aligned} \tag{110}$$

$$B_6 = \cos \alpha_o (b \cos \alpha_o - 1) \log (b + 1) - \cos^2 \alpha_o$$

$$+ \frac{i}{\pi} \sin^2 \alpha_o M_o(b) + B_4 \log 2 + B_5 \log (b + \sqrt{b^2 - 1})$$

Order β^2;

From equation (105), on $\eta = 0$

$$\tau_5 = y_2 + y_o^2 - 2h_2(y_o - h_2) - h_2^2 - h_5 \qquad |\xi| > 1$$

$$\theta_5 = - \alpha_5 \qquad |\xi| < 1$$

Therefore let

$$\Omega_5(\zeta) = - (h_5 + i\,\alpha_5) - h_2(h_2 + 2i\alpha_2) - 2h_2\Omega_2(\zeta) + \Omega_5^*(\zeta) \qquad (111)$$

where, on $\eta = 0$,

$$\tau_5^* = y_2 + y_o^2 \qquad |\xi| > 1$$

$$\theta_5^* = 0 \qquad |\xi| < 1 \qquad (112)$$

$y_o(\xi)$ is given by equation (68) and

$$y_2(\xi) = \mathrm{Im} \int_1^{\xi} \frac{H(\zeta)}{w_o(\zeta)} \Omega_2(\zeta)\, d\zeta \qquad (113)$$

The complexity of $\Omega_2(\zeta)$ makes solution of the boundary value problem for $\Omega_5^*(\zeta)$ difficult, although an integral solution may be written down as in equation (65). However the order β^2 inner and outer expansions can be matched to within an imaginary constant if the asymptotic behavior of $\Omega_5^*(\zeta)$, as ζ tends to infinity, is known to $0(1)$. In particular h_4 and h_5 can be found.

Let $$g(\xi) = y_o^2(\xi) + y_2(\xi)$$

As $\xi \longrightarrow \infty$ it is found that

$$g(\xi) \longrightarrow A^2 \mathrm{Re} \Big\{ B\xi \log \xi + B(Q-1)\xi - B\left(\tfrac{1}{2}B + i \sin \alpha_o\right) \log^2 \xi - (B^2 Q + iD_1 - i\pi |B|^2 - 2iBD_2) \log \xi + D_2^2 \Big\} + C_g + 0\left(\frac{\log \xi}{\xi}\right) \tag{114}$$

where

$$D_1 = \frac{1}{2} \sin \alpha_o \cos \alpha_o - b \sin \alpha_o + i(b^2 - 1) \sin^2 \alpha_o + i\sqrt{b^2 - 1}\,(b \cos \alpha_o - 1) \cos \alpha_o + \frac{i}{\pi} \sin \alpha_o \cos \alpha_o \left\{ 2b - (b^2 - 1) \log \frac{b+1}{b-1} \right\} \tag{115}$$

$$D_2 = \mathrm{Re} B_3$$

C_g is given in Appendix I.

D_1 is the term in $\frac{1}{\zeta}$ in the expansion of $\Omega_2(\zeta)$ for large ζ.

Thus $\Omega_5^*(\zeta)$ can be written, as $\zeta \to \infty$

$$\Omega_5^*(\zeta) \sim A^2 B\zeta \log \zeta + A^2 B(Q - 1)\zeta - A^2 B(\tfrac{1}{2} B + i \sin \alpha_o) \log^2 \zeta$$

$$- A^2(B^2 Q + iD_1 - i\pi |B|^2 - 2iBD_1) \log \zeta + A^2 D_2^2$$

$$+ C_g + iC_f + \omega(\zeta) + 0 \left(\frac{\log \zeta}{\zeta} \right) . \qquad (116)$$

where C_f is an arbitrary real constant.

This result can be combined with equation (111) to give the expansion of $\Omega_5(\zeta)$ for large ζ. This is enough to allow matching to order β^2. Note that only the real constant term is found by this method. The imaginary term $i\alpha_5$ cannot be determined unless the full solution is known.

The Inner Velocity $w(\zeta, \beta)$;

By continuing the procedure used to find the relations given in equation (72) the following results are obtained for the relationship between the Ω_i and the w_i.

$$w_3 = -\Omega_3 + \frac{1}{2} w_1^2 \qquad (117)$$

$$w_4 = -\Omega_4 + w_1 w_2 \qquad (118)$$

$$w_5 = -\Omega_5 + \frac{1}{2} w_2^2 \qquad (119)$$

Therefore w_o through w_4 are known exactly while w_5 is known to order 1 as $\zeta \to \infty$. This allows the outer expansion of $w(\zeta, \beta)$ to order β^2 so as to match the outer solution.

15. OUTER SOLUTION TO ORDER β^2

The outer expansion can be written

$$\tilde{w}(\tilde{\zeta}) = 1 + \beta \log\beta\, \tilde{w}_1(\tilde{\zeta}) + \beta\, \tilde{w}_2(\tilde{\zeta}) + \beta^2 \log^2\beta\, \tilde{w}_3(\tilde{\zeta})$$

$$+ \beta^2 \log\beta\, \tilde{w}_4(\tilde{\zeta}) + \beta^2\, \tilde{w}_5(\tilde{\zeta}) \ldots . \qquad (120)$$

Substitution of equation (120) into equation (50) gives the boundary value problems for $\tilde{w}_3$, $\tilde{w}_4$, $\tilde{w}_5$.

On $\tilde{\eta} = 0$,

$$\tilde{u}_3' - A\tilde{v}_3 = 0 \qquad (121)$$

$$\tilde{u}_4' - A\tilde{v}_4 = -3\tilde{u}_1\tilde{u}_2' = 0 \qquad (122)$$

$$\tilde{u}_5' - A\tilde{v}_5 = -3\tilde{u}_2\tilde{u}_2' - \tilde{v}_2\tilde{v}_2' - A\tilde{v}_2 \frac{b - \cos\alpha_o}{\tilde{\xi}} \qquad (123)$$

except for $\tilde{\xi} = 0$, and where the prime denotes differentiation with respect to $\tilde{\xi}$.

The homogeneous solutions are of the form given in equation (77).

Particular Solution for $\tilde{w}_5(\tilde{\zeta})$;

After rearranging the right hand side of equation (123) the following result can be obtained.

$$\tilde{w}_5(\tilde{\zeta}) = \frac{1}{2}\tilde{w}_2^{\,2} + iA\,(b - \cos\alpha_o)\, e^{-iA\tilde{\zeta}} \int_{-\infty}^{\tilde{\zeta}} \tilde{w}_2(t)\, e^{iAt} \frac{dt}{t}$$

$$+ \frac{3}{2} iAe^{-iA\tilde{\zeta}} \int_{-\infty}^{\tilde{\zeta}} \tilde{w}_2{}^2(t)\, e^{iAt} dt \tag{124}$$

To this must be added the appropriate homogeneous solutions with their as yet undetermined coefficients. These will be determined from matching to order β^2 with the inner solution. To do this the asymptotic expansion of the outer solution is required for small $\tilde{\zeta}$.

Expansion of Particular Outer Solution;

The inner expansion of the outer solution to order β^2 is carried out as in section 11, except that terms up to order β^2 are now required. The outer expansion up to and including $\tilde{w}_5(\tilde{\zeta})$, but excluding the homogeneous terms in $\tilde{w}_3$, $\tilde{w}_4$, $\tilde{w}_5$, gives the following when rewritten in outer variables.

$$\tilde{w}(\tilde{\zeta}) \sim 1 + i\beta \frac{\sin \alpha_o}{\tilde{\zeta}} - i\beta ABQ - i\beta AB \log \tilde{\zeta} - \beta A^2 B\, \tilde{\zeta} \log \tilde{\zeta}$$

$$- \beta A^2 B\, (Q - 1)\, \tilde{\zeta} - \frac{1}{2} \beta^2 \frac{\sin^2 \alpha_o}{\tilde{\zeta}^2}$$

$$+ \beta^2 AB \sin \alpha_o \frac{\log \tilde{\zeta}}{\tilde{\zeta}} + \beta^2 AB\, (Q + 1) \sin \alpha_o \frac{1}{\tilde{\zeta}}$$

$$+ \frac{1}{2} i \beta^2 A \sin^2 \alpha_o \frac{1}{\tilde{\zeta}} + i\beta^2 A^2 B \sin \alpha_o \log^2 \tilde{\zeta}$$

$$+ \beta^2 A^2 \sin \alpha_o \left\{ 2iB\, (Q - 1) + \frac{1}{2} \sin \alpha_o \right\} \log \tilde{\zeta}$$

$$+ \beta^2 A^2 \left\{ B^2 Q^2 + \frac{1}{2} Q \sin^2 \alpha_o - 2B^2 q_1 \right.$$

$$\left. + iB \sin \alpha_o (1 - 2Q + 2 q_1) \right\} + 0(\beta^3 \log^3 \beta) \qquad (125)$$

where q_1 is the constant term in the expansion, for small ζ, of

$$\int_{-\infty}^{A\zeta} \frac{1}{t} \int_{-\infty}^{t} e^{iu} \frac{du}{u} \, dt$$

$$q_1 = \frac{1}{2}\gamma^2 - \frac{5\pi^2}{24} - i\frac{\pi\gamma}{2} \quad . \qquad (126)$$

The Homogeneous Solutions;

Appropriate homogeneous solutions must be added to $\widetilde{w}_3$, $\widetilde{w}_4$ and $\widetilde{w}_5$ to allow matching with the inner solution. These are of the form given in equation (77). It will be found that

$$\widetilde{w}_3(\widetilde{\zeta}) = 0$$

$$\widetilde{w}_4(\widetilde{\zeta}) = 0$$

$$\widetilde{w}_5(\widetilde{\zeta}) = ie^{-iA\widetilde{\zeta}} \int_{-\infty}^{\zeta} \left\{ \frac{c_{53}}{t^3} + \frac{c_{52}}{t^2} + \frac{c_{51}}{t} \right\} e^{iAt}\, dt \qquad (127)$$

where the c_{5j} are real constants.

The inner expansion, to order 1, of $\widetilde{w}_5$ is

$$\widetilde{w}_5 \sim -\frac{i}{2} c_{53} \frac{1}{\widetilde{\zeta}^2} + \left(\frac{1}{2} A c_{53} - i c_{52}\right) \frac{1}{\widetilde{\zeta}}$$

$$+ i\left(c_{51} + iA c_{52} - \frac{1}{2} A^2 c_{53}\right) \log \widetilde{\zeta} + iQ\left(c_{51} + iAc_{52} - \frac{1}{2} A^2 c_{53}\right) \qquad (128)$$

when rewritten in outer variables.

16. MATCHING TO ORDER β^2.

Matching is done as in section 11 except that all expansions are carried out to order β^2. It is found that in addition to the particular solutions already obtained the following homogeneous terms are required.

Inner Solution;

No additional term is required of order $\beta^2 \log^2 \beta$. For order $\beta^2 \log \beta$ a term which behaves as $-iA^2 \sin \alpha_o \zeta$ is needed. This indicates that

$$w_4(\zeta) = -iA^2 \sin \alpha_o \sqrt{\zeta^2 - 1} \qquad (129)$$

is the required homogeneous solution which must be included in w_4. No homogeneous solution of order ≥ 1 is needed for w_5.

Outer Solution;

No homogeneous solutions are required for $\tilde{w}_3$, $\tilde{w}_4$. The coefficients for $\tilde{w}_5$ are

$$c_{53} = \sin \alpha_o \cos \alpha_o \qquad (130)$$

$$c_{52} = A \left\{ \frac{3}{2} \sin^2 \alpha_o + \text{Im} D_1 \right\} \qquad (131)$$

$$c_{53} = A^2 \left\{ 2 \, \text{Im} B^2 Q - \pi |B|^2 \right\} . \qquad (132)$$

The Constant Term;

The above particular and homogeneous solutions allow the inner and outer solutions to be matched to order β^2. In particular the constant terms can be matched as in equation (87). Therefore h_3, h_4, h_5, a_3 and a_4 can be found. As noted in section 14, a_5 cannot be determined if the asymptotic solution of Ω_5^* is used. The expressions for these constants are somewhat lengthy and are given in Appendix I.

17. CALCULATION SCHEME

The flow as shown in Figure 1 requires the Froude number to be large. The method used to solve the problem assumed that the interaction of the jet with the mainstream is small. Thus β must be small and/or b be close to unity, otherwise difficulty with the convergence of solutions will exist. For a deeply submerged plate a separate expansion to account for the large jet would be required. This will be discussed in the concluding section.

Suppose that α, h and β are given. The first step is to assume some values for b and c. From equation (57) $c = -\cos \alpha_o$, and α_o should be close to α for small β. Therefore a range of α_o near α, and a suitable range of b should be chosen. These values, together with that of β, are substituted into

$$1 = \ell(a, b, c, \beta) \quad . \tag{133}$$

The resulting root for a, if it exists, is then substituted into similar equations for h and α. Thus, for example, tables of h and α can be obtained for the values of b and c for given values of β. By cross plotting, b and c can be found for given h and α and can then be substituted into the expression for the thrust. Therefore for example the lift coefficient

$$C_L = 2N^* \cos\alpha \quad , \tag{134}$$

can be found as $C_L(\alpha, h, \beta)$.

This can only be done, of course, if equation (133) possesses a root to the order of β taken. If β is too large, then more terms are required since a root must always exist in the exact solution. The accuracy of a particular solution depends very much on the accuracy of the root for a. Hence the results will be most accurate for β small and b near to unity.

18. CONCLUSIONS

A problem in the nonlinear theory of water waves has been solved using the method of singular perturbation expansions for large Froude number. By using a parametric ζ-plane the problem was reduced to the solution of Laplaces equation in a half plane with non-linear boundary conditions. The inner and outer expansions have been matched to order β^2 and expressions obtained for the length, physical angle of attack and depth of submergence of the plate in terms of the parameters b, c and β. The thrust on the plate has also been calculated and a method outlined for obtaining it as a function of α, h and β.

The inner solution of order unity turns out to be the well known free streamline solution for the impingement of a semi-infinite jet on an inclined flat plate. The force on the plate is, to the first approximation, the same as for this case. The first outer solution is a uniform flow $\tilde{w}_o = 1$. It is found that due to the logarithmic nature of the first inner solution a simple expansion in terms of powers of β will not suffice. The expansion parameters are combinations of powers of β and of $\log \beta$. Due to the occurrence of logarithmic terms in the inner and outer solutions matching must be carried out for specified groups of terms. These are

order (1)

order $(1, \beta \log \beta, \beta)$

order $(1, \beta \log \beta, \beta, \beta^2 \log^2 \beta, \beta^2 \log \beta, \beta^2)$

or in general $1, \beta \log \beta, \beta, \ldots \beta^n \log^n \beta, \ldots \beta^n \log \beta, \beta^n$.

The inner solution up to order β contains a constant term of order $\beta \log \beta$ and a more complicated term of order β. A notable feature of these terms is the presence of the depth of submergence. It has not been possible in the linearized theory to prescribe this depth in formulating the problem. Since the present method is not restricted to small angles of attack and small disturbances the depth of submergence can now be taken as arbitrary, within certain limits to be noted below. The outer solution up to order β^2 does not require terms containing $\log \beta$ and it seems likely that they do not occur in the outer solution at all. When equation (89) is compared with the solution obtained from linearized theory for a moving source and vortex on the surface of an infinitely deep fluid, the outer solution is seen to be that for a sink of strength $\pi A(b - \cos \alpha_o)$ and a vortex of strength $-\pi A \sin \alpha_o$ (Wehausen and Laitone[16], equation (13.43)). From equations (32) and (57), the sink is seen to be of strength a as required. When carried to order β^2 the outer solution gives higher order singularities as well as non-linear terms which are corrections to the sinusoidal waves found in the order β solution. Thus for the case β equal to zero and the case ζ, and hence z, large the theory reproduces known limiting cases.

One of the main results is that the depth of submergence of the trailing edge of the plate can now be taken into account in prescribing the problem. The only restriction is that it not be too large unless β is very small. This is due to neglect of the jet interaction with the main flow. To take this into account a "jet variable"

$$\tilde{\tilde{\zeta}} = \beta(\zeta + b) = \tilde{\zeta} + \beta b$$

should be used in the region containing the jet in the ζ-plane. Thus the validity of the results obtained above depends on βb being small. Subject to this condition the depth of submergence can be prescribed, a result which is not found using linearized theory. Of course h cannot be an arbitrarily large negative number since the energy of the flow would not be enough to support the plate.

The method of singular perturbation has worked well in this case to bridge the gap between the extreme solutions of a free-streamline flow and a moving point singularity. It should be noted that the basic singularity of the solution comes in this case not from the differential equation, as in viscous flow theory, but from the boundary conditions. No fundamental difficulties are hinted at which would prevent extension of the solutions to higher order, the main difficulty being the tedious algebra involved. Since the boundary conditions, beyond the first order, are linear the solutions can be written down in principle. Even for the case of a flat plate the terms of order β^2 are difficult to obtain. The problem for flow past a curved body of arbitrary profile with fixed separation points can be treated in a way similar to the present problem except that the details of the calculation would be somewhat more involved. When the flow is such that it separates at some undetermined point on a smooth curved body, for example a circular cylinder, the problem is much more difficult due to the unknown position of the separation points.

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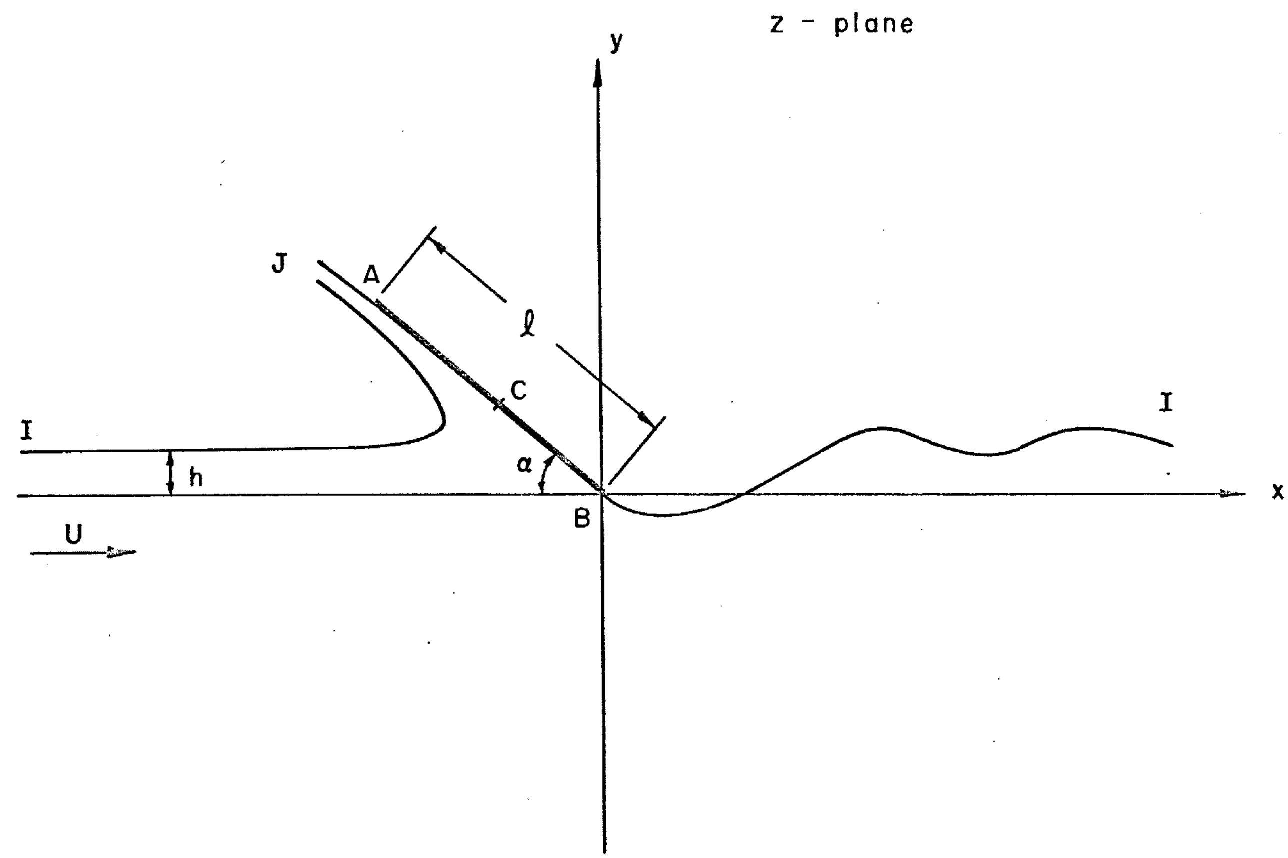

Figure 1

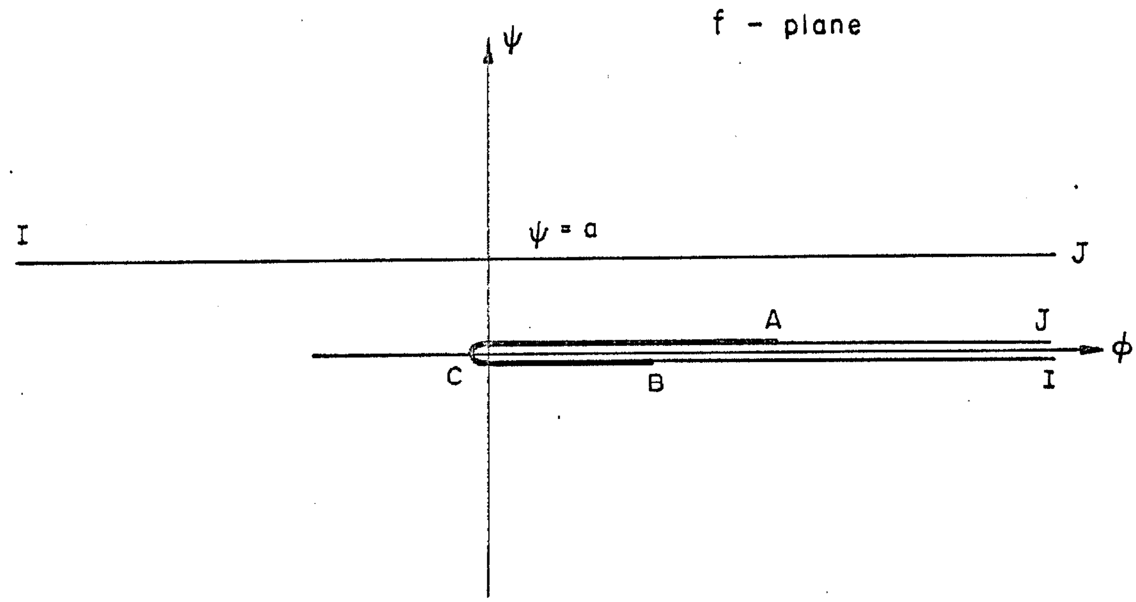

Figure 2

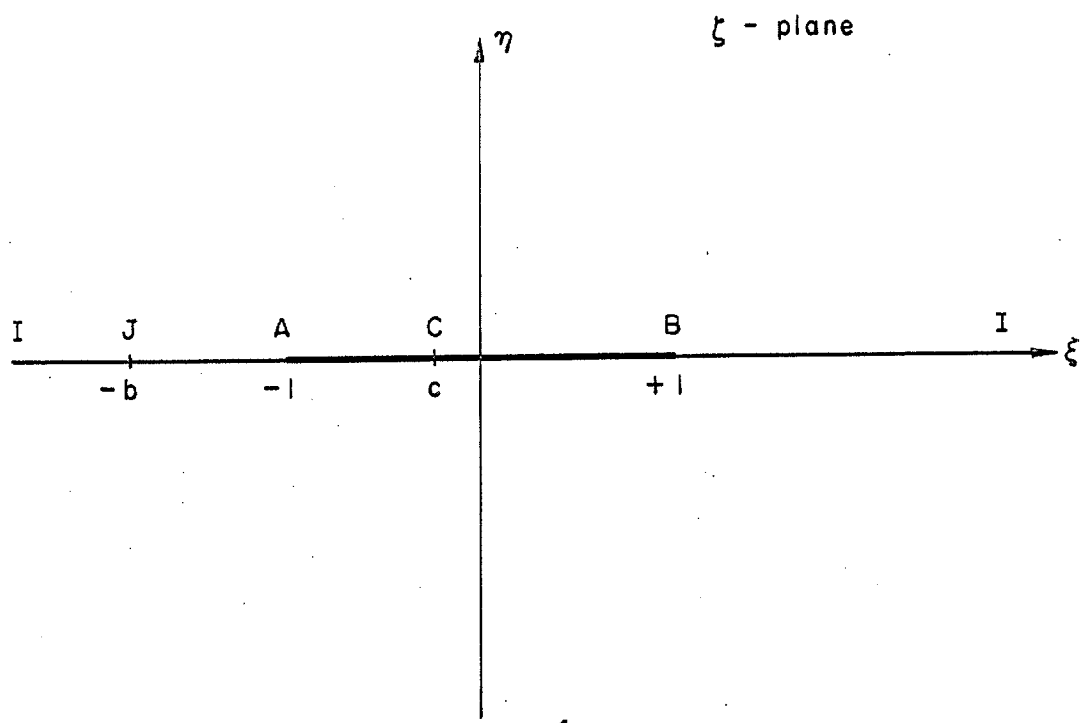

Figure 3

APPENDIX I

MATCHING OF ORDER $\beta^2 \log^2 \beta$;

$$h_3 = \frac{1}{2} A^2 \left\{ (b - \cos \alpha_o)^2 - \sin^2 \alpha_o \right\} \tag{I.1}$$

$$\alpha_3 = 2A^2 \sin \alpha_o (b - \cos \alpha_o) \tag{I.2}$$

MATCHING OF ORDER $\beta^2 \log \beta$;

$$h_4 = A^2 \left\{ (b - \cos \alpha_o)(\mathrm{Re}B_6 - \mathrm{Re}BQ) - 3 \sin \alpha_o \mathrm{Im}BQ + \sin \alpha_o \mathrm{Re}B_3 - \mathrm{Im}D_1 + \sin^2 \alpha_o \right\} \tag{I.3}$$

$$\alpha_4 = A^2 \left\{ (b - \cos \alpha_o)(\mathrm{Im}B_6 + \mathrm{Im}BQ) + 3 \sin \alpha_o \mathrm{Re}BQ - \sin \alpha_o \mathrm{Im}B_3 + \mathrm{Re}D_1 - \pi |B|^2 \right\} \tag{I.4}$$

MATCHING OF ORDER β^2;

$$h_5 = C_g + A^2 \left\{ \sin^2 \alpha_o - \sin \alpha_o \mathrm{Im}BQ + \frac{1}{2} \mathrm{Re}B^2 Q^2 + (\mathrm{Im}BQ)^2 \right\} \tag{I.5}$$

α_5 cannot be found by the asymptotic method

THE CONSTANT C_g ;

(For conciseness, α_o will be written as α in this section.)

$$C_g = T_1 \sin^4\alpha + T_2 \sin^3\alpha \cos\alpha + T_3 \sin^3\alpha + T_4 \sin^2\alpha \cos\alpha$$

$$+ T_5 \sin^2\alpha + T_6 \sin\alpha \cos\alpha + T_7 \sin\alpha + T_8 \cos\alpha + T_9$$

where the T_i are functions of b, given by

$$T_1 = b(b-1) - \frac{1}{2} b^2 \left\{\log 2(b+1)\right\}^2 - b(b - \sqrt{b^2 - 1}) \log (b+1) - bM_1$$

$$+ b(b^2 - 1)M_8 + b^2 M_9 + (b - 2\sqrt{b^2 - 1})(bM_{10} - M_{11} + M_{12}) ,$$

$$\pi T_2 = b \log^2 2 - 3 + b(1 - \log 2)(2 - M_o) + \sqrt{b^2 - 1} \log (b + \sqrt{b^2 - 1}) M_o$$

$$+ M_3 - bM_4 - (b^2 - 1)M_5 - \sqrt{b^2 - 1}\, M_7 - 2(b^2 - 1)M_8 - 2bM_9$$

$$+ \sqrt{b^2 - 1} \log \frac{b+1}{b-1} \left\{ bM_{10} - M_{11} + M_{12} \right\} ,$$

$$\pi T_3 = M_4 + 2M_9 - \sqrt{b^2 - 1} \log \frac{b+1}{b-1} M_{10}$$

$$T_4 = b - 2 - b \log^2 (b+1) - \sqrt{b^2 - 1} \left\{ 1 - \log (b+1) \right\} \log (b + \sqrt{b^2 - 1})$$

$$- (b \log 2 - 1) \log (b+1) + b(1 + \frac{1}{2} \log 2) \log 2$$

$$- M_1 + 2(b - \sqrt{b^2 - 1}) M_{10} - M_{11} + M_{12} ,$$

$$T_5 = \frac{1}{2} + 2b - b^2 + (2b^2 + b + 1)\log 2 + \frac{1}{2}(b^2 + 1)\log^2(b + 1)$$

$$+ (\log 2 - 1)b^2 \log(b + 1) - (b - 1)\log(b + 1)$$

$$+ \left\{(2b + 3)\sqrt{b^2 - 1} - b\right\}\log(b + \sqrt{b^2 - 1}) - b\sqrt{b^2 - 1}\,.$$

$$.\log(b + \sqrt{b^2 - 1}) + bM_1 + M_9 - M_{10} - b(2bM_{10} - 2M_{11} + M_{12})$$

$$+ \sqrt{b^2 - 1}\,(3bM_{10} - 3M_{11} + M_{12}) ,$$

$$\pi T_6 = 2b + 2 - \sqrt{b^2 - 1}\log\frac{b + 1}{b - 1}\log(b + \sqrt{b^2 - 1}) - M_3 + bM_4$$

$$+ 2bM_9 - \sqrt{b^2 - 1}\log\frac{b + 1}{b - 1}\left\{bM_{10} - M_{11}\right\}$$

$$\pi T_7 = - M_4 - 2M_9 + \sqrt{b^2 - 1}\ \log\frac{b + 1}{b - 1} M_{10}$$

$$T_8 = 2bM_9 - 2bM_{10} + M_{11} - \log(b + \sqrt{b^2 - 1})$$

$$T_9 = b\log(b + \sqrt{b^2 - 1}) - (b^2 + 1)M_9 + (b^2 + 1)M_{10} - bM_{11}$$

with the M_i being the constant terms in the expansion for large ζ of the integrals from 1 to ζ of the following functions.

$$M_o; \qquad \frac{\sqrt{t^2 - 1}}{t + b}\log(t + b)$$

M_1; $\quad \frac{1}{t+b} \log (t + \sqrt{t^2 - 1})$

M_3; $\quad \frac{t\sqrt{t^2 - 1}}{t+b} \log \frac{t+1}{t-1}$

M_4; $\quad \frac{\sqrt{t^2 - 1}}{t+b} \log \frac{t+1}{t-1} \log (t+b)$

M_5; $\quad \frac{1}{t+b} \log \frac{t+1}{t-1}$

M_6; $\quad \frac{\sqrt{t^2 - 1}}{t+b} \log \frac{t+1}{t-1} \log (t + \sqrt{t^2 - 1})$

M_7; $\quad \frac{\sqrt{t^2 - 1}}{t+b} \log \frac{t+1}{t-1} \log \left(\frac{1 + tb + \sqrt{b^2 - 1}\sqrt{t^2 - 1}}{t+b}\right)$

M_8; $\quad \frac{\log (t + \sqrt{t^2 - 1})}{t+b} \quad \frac{1}{\sqrt{t^2 - 1}}$

M_9; $\quad \frac{1}{t+b} \log (t + \sqrt{t^2 - 1})$

M_{10}; $\quad \frac{1}{t+b} \log \left(\frac{1 + tb + \sqrt{b^2 - 1}\sqrt{t^2 - 1}}{t+b}\right)$

M_{11}; $\quad \log \left(\frac{1 + tb + \sqrt{b^2 - 1}\sqrt{t^2 - 1}}{t+b}\right)$

M_{12}; $\quad \frac{\sqrt{t^2 - 1}}{t+b} \log \left(\frac{1 + tb + \sqrt{b^2 - 1}\sqrt{t^2 - 1}}{t+b}\right)$

e.g., $\quad M_o = \int_1^{\infty} \left\{ \frac{\sqrt{t^2 - 1}}{t+b} \log \frac{t+1}{t-1} - \frac{2}{t} \right\} dt \, .$