RADIATIVE CORRECTIONS
TO MUON AND NEUTRON DECAY

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ABSTRACT

The corrections to muon decay due to electromagnetic interactions have been recalculated. Our results differ from those of Behrends, Finkelstein and Sirlin, because those authors used an inconsistent method for handling the infra-red divergences which arise separately in the real and virtual processes. The disagreement is especially significant near the end of the electron (positron) spectrum where our results indicate that the radiative correction to the Michel ρ parameter is approximately 1% larger than previously supposed, a result in the direction of increasing agreement between experiment and theory. With the radiative corrections to muon decay given here the predicted value of the muon lifetime using the universal theory is $(2.27 \pm 0.04) \times 10^{-6}$ sec. As a preliminary to studying the decay of particles with structure the β-decay of the neutron is examined. This leads to an increase in the coulomb F factor independent of the nuclear charge and of amount approximately 2.6%. As a result the universal coupling constant obtained from the decay of O^{14} is decreased to $G = (1.37 \pm 0.02) \times 10^{-49}$ erg cm.^{3} and increases the value of the muon lifetime to $(2.33 \pm 0.05) \times 10^{-6}$ sec.

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I. INTRODUCTION

Recently a universal theory for the decay of unstable elementary particles has been proposed by Feynman and Gell-Mann.[1] The lifetimes, energy spectra, and angular correlations determined by their theory will all require small corrections due to the electromagnetic interaction of the various particles involved in the decay process. These electromagnetic interactions manifest themselves in two ways. First by the emission of real photons as part of the decay process (inner bremsstrahlung) and second by the effect of virtual photon exchange between the particles concerned in the decay. Since the rate for emission of real photons must be positive the net probability for unperturbed decay and inner bremsstrahlung will always exceed the unperturbed decay rate. However the virtual processes produce a damping effect which is always in a direction as to decrease the rate from the unperturbed case. The two electromagnetic effects do not quite cancel but leave a small correction to the lifetime etc. of the order of the fine structure constant $e^2/\hbar c$. For decays which involve an electron in the final state of mass m the order of magnitude of the effect may be established in terms of the per cent change in the rate ΔR as

$$\Delta R/R \approx (1/137)\log^2(\Delta E/m)$$

where ΔE is the maximum energy release in the decay. For the decay of a nucleus there is the additional effect of the static coulomb potential which is taken into account by the usual coulomb factor F.[2]

Of the particles which are unstable perhaps the simplest is the muon. This particle has approximately the Dirac magnetic moment and appears not to participate in any of the 'strong' interactions. Therefore the radiative corrections to its decay can be calculated without regard to couplings with other fields. Although these corrections are small, of the order of a few per cent, they are larger than the errors in the most recent and accurate experiments. It is then necessary, in order to compare the theory and experiment with as much precision as possible, to calculate those corrections to the energy spectrum which are a consequence of electromagnetic interactions.

If we consider the decay of any other particle we find that all the remaining unstable elementary particles exhibit 'strong' interactions and the correct calculation of their decay rate would have to include these strong couplings.

When the muon decay is described by a linear combination of all five types of interaction invariants, the decay electron spectrum (in the limit of zero mass) can be expressed as a one parameter family (Michels[3] ρ) of curves. The ρ parameter is then a certain ratio of the various coupling constants. When the radiative corrections are included it is still possible to express the electron spectrum resulting from a general muon decay as a one parameter family of curves to the extent that the radiative corrections for all types of coupling are approximately the same.[4] The theory of Feynman and Gell-Mann predicts the value of the ρ parameter to be 3/4. When the radiative corrections are included the shape of the electromagnetically unperturbed spectrum is slightly changed but the value obtained for the ρ parameter is still 3/4.

Since $e^2 = 1/137$ is small compared to unity, perturbation theory can be used to calculate the radiative corrections. The problem then divides itself naturally into two parts, the virtual processes which are treated in detail in Section IIA and real photon emission accompanying the decay, treated in Section IIB.

Among the integrals which arise in calculating the matrix element for virtual photons some exhibit a logarithmic divergence for small photon momenta. The covariant method for handling this infra-red divergence is given by Feynman[5] and consists of giving the virtual photon a small rest mass λ_m. The resulting change in the photon propagator is sufficient to prevent the integrals from having a divergence for small photon momenta. However, the expression for the matrix element for virtual photons is now logarithmically dependent on λ_m. This dependence on λ_m disappears when the emission of real photons (inner bremsstrahlung) is taken into account. To this end one integrates the differential cross-section for inner bremsstrahlung over all the space components of the photon momentum compatible with conservation of energy and momentum in the decay process. This integration also contains an infra-red divergence and to be consistent with the method used for the virtual processes the photon must be given a small mass λ_m. When the two parts are combined the dependence on λ_m disappears.

The cancellation of the parts dependent on λ_m may also be accomplished, however erroneously, by treating the emitted photon as having zero rest mass and cutting off the integral over the inner bremsstrahlung cross-section at some lower limit arbitrarily defined as λ_m. This last procedure is clearly not compatible with the method used for the virtual processes.

The radiative corrections to decay processes were first calculated by Behrends, Finkelstein and Sirlin (BFS). For the case of vector and axial vector interaction the results given here are in agreement with those of BFS for the virtual processes, however, the contribution to the energy spectrum from inner bremsstrahlung is different from those authors because they used the inconsistent procedure described above for handling the infra-red divergences.

As a consequence of the radiative corrections to the decay of the muon there is an increase in the lifetime of amount 0.44%. The correction to the muon lifetime determined by BFS was a decrease of 3.5%. From the results shown here the value of the muon lifetime predicted from the universal theory using the value $G = (1.41 \pm 0.01) \times 10^{-49}$ erg cm.^{3} given by beta decay of O^{14} is $(2.27 \pm 0.04) \times 10^{-6}$ sec., while the experimental value is $(2.22 \pm 0.02) \times 10^{-6}$ sec.

To determine the change in the experiment ρ value occasioned by the radiative corrections here the energy spectrum given in Section II must be integrated over the experimental resolution for a particular experiment. This has been done by Dudziak[11] and he finds that the change in the ρ value given by the new radiative corrections over those of BFS is 1%. In this connection we mention that the meaning of the ρ value, as originally proposed by Michel, is altered when the radiative corrections to the electron spectrum are included. It would be an improvement in understanding the relationship between theory and experiment if the idea of a ρ value were abandoned and instead a comparison made of the experimental and theoretical spectra over the complete range of electron energies. This may be best accomplished by comparing the experimental results with equation (29) of Section II which expresses the theoretical electron spectrum including the first order radiative corrections.

In order that a precise value be assigned to the universal coupling constant G some consideration should be given to those effects of electromagnetic origin which could modify the lifetime of O^{14}. As a simpler example than entertaining the problem of the nucleus as a whole the calculation of the radiative corrections to the β-decay of the neutron have been considered. Ordinarily these corrections would be expected to be quite small of order e^2/π (1/5%), however, the result is found that large coefficients of e^2/π arise which are of order $\log (M/m)$, where M and m refer to the mass of the nucleon and electron respectively. Also investigated is the question as to whether the including of anomalous moments can give surprisingly large coefficients such as $(\mu_P - \mu_N)^2$ where μ_P and μ_N are the anomalous moments of the proton and neutron respectively. It is found that this latter type of coefficient appears divided by large numerical factors that depress its effect. These problems are considered in detail in Section III and numerical examples are given.

II. DECAY OF THE MUON

A. The Virtual Processes

We begin by mentioning certain conventions of notation to be used. The choice of units is for $\hbar = C = 1$ and unrationalized electromagnetic units with $e^2 = 1/137$. We use the metric of reference 5 so that the dot product of two four-vectors, a, b is expressed as

$$a \cdot b = a_t b_t - a_x b_x - a_y b_y - a_z b_z = a_\mu b_\mu$$

Three-vectors will be denoted by an overhead arrow $\vec{a}$. Dirac spinors will be normalized so that $\bar{u}\gamma_t u = 2E$ where $\bar{u} = u^* \gamma_t$. In order to simplify certain expressions in Section II the quantity η will be defined by

$$\eta = 2E_2/m_1$$

where E_2 is the energy of an electron and m_1 is the mass of the muon. $\eta = 1$ corresponds to the maximum energy of the electron which is about 55 mev. The ultra-violet and infra-red cutoff factors will be denoted by Λ and λ_m respectively and slashed four vectors $\not{a}$ will denote $a_\mu \gamma_\mu$. The symbol log will always denote the logarithm to the base e.

Muons are observed to decay by the reaction

$$\mu \rightarrow e + \nu + \bar{\nu}$$

Since both the muon and electron are charged their electromagnetic fields will affect the decay process.

In charge retention order the decay of the muon (1) into an electron

(2) and two neutrinos (3) and (4) with electromagnetic coupling is described by the interaction Lagrangian

$$\mathcal{L}_{int} = \sqrt{8}\, G(\bar{\psi}_1\gamma_\mu a\psi_2)(\bar{\psi}_3\gamma_\mu a\psi_4) + e(\bar{\psi}_1\gamma_\mu j_\mu\psi_1) + e(\bar{\psi}_2\gamma_\mu j_\mu\psi_2) + \text{herm. conj.} \tag{1}$$

where

$$\gamma_\mu a = \gamma_\mu(1 + i\gamma_5)/2$$

is the form of the universal beta interaction introduced by Feynman and Gell-Mann[2] and j_μ is the usual four vector electromagnetic current. Applying perturbation theory, the decay process with electromagnetic interactions may be described to order e^2G^2 by the six diagrams of figure 1.

The corrections to the spectrum of the emitted electron arise partly from the virtual processes described by diagrams II thru IV and partly by the process of inner bremsstrahlung described by diagrams V and VI of figure 1.

Using the form of the beta interaction given in equation (1) the probability that the electron has energy η in $d\eta$ will be expressed as

$$P(\eta)d\eta = \frac{m_1^5}{(2\pi)^3(12)}\, G^2\{\eta^2(3 - 2\eta)[1 + \frac{e^2}{2\pi}(\alpha + \beta)]\}\, d\eta \tag{2}$$

In equation (2) α and β correspond to the contributions from the virtual and real processes respectively. In this section we will discuss the calculation of α.

Since most of the observable part of the spectrum lies in an energy region where the rest mass of the electron is negligible compared to its

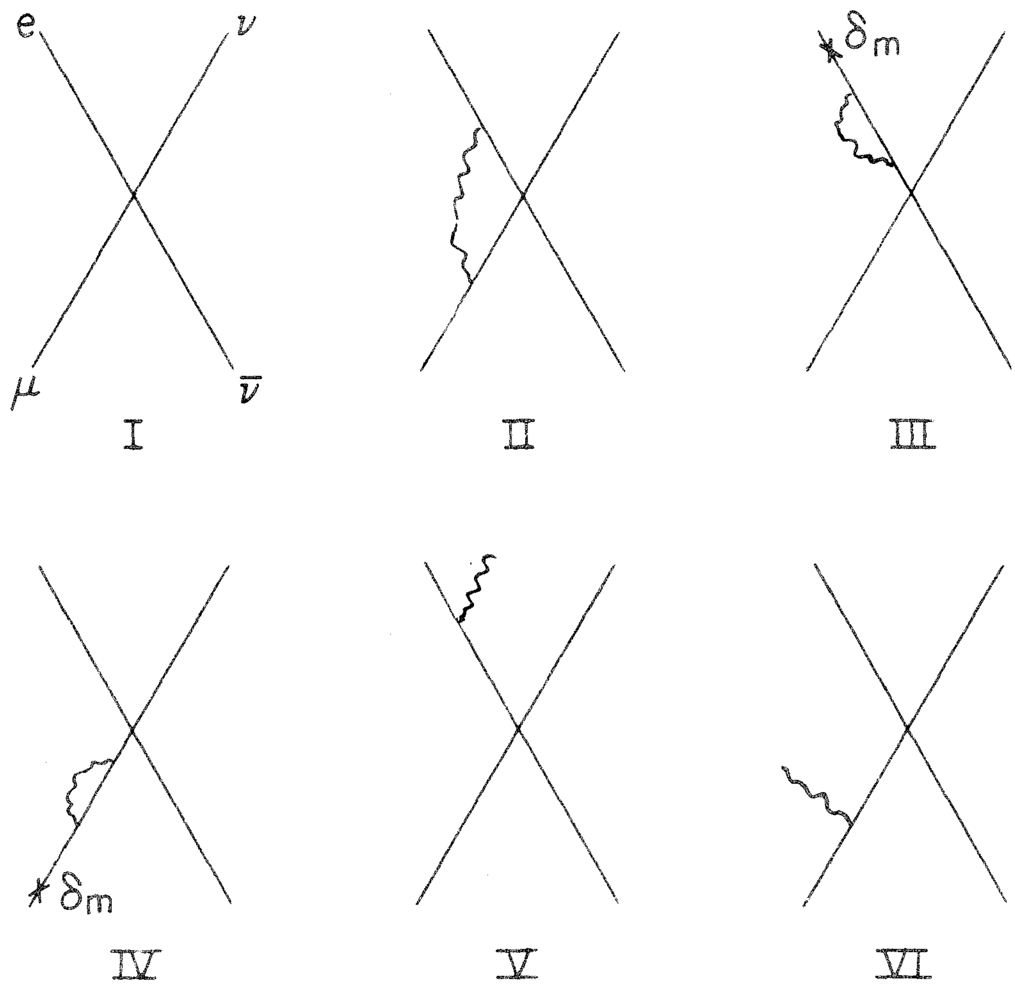

FIGURE I. FEYNMAN DIAGRAMS FOR MUON DECAY.

kinetic energy we will make the approximation that $m_2 = 0$ wherever possible, i.e., unless this introduces a spurious divergence. In this case the radiative corrections for vector and axial vector interactions are equal.

Diagrams III and IV of figure 1 constitute the mass and wave function renormalization. Since we associate a self energy effect, as in diagram III, figure 1, with both the decay vertex and whatever interaction may follow, the mass and wave function renormalization R, are given by one half the sum of the contributions from diagrams III and IV of figure 1. The result is

$$R\gamma_\mu a = e^2/2\pi[\log(\Lambda/m_1) - 2\log(m_2/\lambda_m) + \tfrac{1}{2}\log(m_2/m_1) + \tfrac{9}{4}]\gamma_\mu a \qquad (3)$$

where we have supressed the multiplicative factor $\sqrt{8}\,G$.

Using the rules of quantum electrodynamics we can write the matrix element for the charged leptons $< e|M|\mu >$, arising from diagram II of figure 1 as

$$M = -4\pi e^2 i \int \frac{d^4k}{(2\pi)^4} \frac{C(k^2)}{k^2} [\gamma_\nu \frac{1}{\not p_2 - \not k - m_2} \gamma_\mu a \frac{1}{\not p_1 - \not k - m_1} \gamma_\nu] \qquad (4)$$

where we have again supressed the factor $\sqrt{8}\,G$. Upon making use of the fact that M is to be evaluated between free particle states we have

$$M = -4\pi e^2 i \int \frac{d^4k}{(2\pi)^4} \frac{C(k^2)}{k^2} [\frac{(2p_{2\nu} - \gamma_\nu \not k)\, \gamma_\mu a\, (2p_{1\nu} - \not k \gamma_\nu)}{(k^2 - 2p_1\cdot k)(k^2 - 2p_2\cdot k)}] \qquad (5)$$

or

$$M = -\frac{e^2 i}{\pi} [4p_1\cdot p_2\gamma_\mu a I_1 - (2\not p_1\gamma_\alpha\gamma_\mu a + \gamma_\mu a\gamma_\alpha \not p_2) I_{2\alpha}$$

$$+ 2(g_{\alpha\beta}\gamma_{\mu a} - 2\gamma_\alpha g_{\mu\beta}) I_{3\alpha\beta}] \qquad (5a)$$

where I_1, I_2, I_3 are defined by equations (7), (9) and (12a) below. The factor $C(k^2)$ is the usual convergence factor and as the integrals appearing in equation (5) have at most logarithmic divergences we take

$$C(k^2) = - \int_{\lambda_m^2}^{\Lambda^2} (k^2 - L)^{-2} \, dL \tag{6}$$

The three types of integrals which appear in equation (5a) are type I_1 with no k's in the numerator, type I_2 with one k in the numerator, type I_3 with two k's in the numerator. More explicitly

$$I_1 = \int \frac{d^4k}{(2\pi)^2} \frac{1}{(k^2 - 2p_1 \cdot k)(k^2 - 2p_2 \cdot k)} \frac{C(k^2)}{k^2} \tag{7a}$$

$$= (8i)^{-1} \int dy \, p_y^{-2} \log(p_y^2 / \lambda_m^2) \tag{7b}$$

where we have used reference 5 and where $p_y = yp_1 + (1 - y)p_2$. Since the evaluation of the integral in equation (7b) involves special difficulties we discuss this in detail in Appendix I. It is shown there that in the rest system of particle one and for small mass m_2 equation (7b) may be expressed as

$$I_1 = (8i)^{-1}[2(\log 2E_2/m_2)(\log m_2/\Lambda_m) + \log^2 2E_2/m_2$$

$$- L(1 - 2E_2/m_1)] \tag{8}$$

and where $L(x)$ is the Spence function defined in Appendix I.

The second type of integral arising is

$$I_{2\alpha} = \int \frac{d^4k}{(2\pi)^2} \frac{C(k^2)}{k^2} \frac{k_\nu}{(k^2 - 2p_1 \cdot k)(k^2 - 2p_2 \cdot k)} \tag{9}$$

and using reference 5

$$I_{2\alpha} = (4i)^{-1} \int_0^1 dy\ p_y^{-2}\ p_{y\alpha} \tag{10}$$

The integral (10) is readily done for small m_2, hence

$$I_{2\alpha} = (4i)^{-1}[q_\nu/q^2 \log(1 + q^2/2m_1E_2)$$

$$- p_{2\nu}/m_1E_2 \log 2E_2/m_2] \tag{11}$$

where the four vector $q = p_2 - p_1$.

The third type of integral which arises is

$$I_{3\alpha\beta} = \int \frac{d^4k}{(2\pi)^2} \frac{C(k^2)}{k^2} \frac{k_\mu k_\nu}{(k^2 - 2p_1\cdot k)(k^2 - 2p_2\cdot k)} \tag{12a}$$

and using reference 5 equation (25a)

$$I_{3\alpha\beta} = (8i)^{-1}[\int_0^1 p_y^{-2} p_{y\mu}p_{y\nu}dy + \frac{1}{2} g_{\mu\nu} \int_0^1 dy \log p_y^2/\Lambda^2$$

$$+ \frac{1}{4} g_{\mu\nu}] \tag{12b}$$

From equation (5b) we note that $I_{3\mu\nu}$ arises in the matrix element in the two forms

$$g_{\mu\nu}I_{3\mu\nu} = -e^2/8\pi[-2 + 4 \log m_1/r - 4m_1E_2/q^2 \log 2E_2/m_1]$$

and

$$\gamma_\mu I_{3\mu\nu}a = q^{-4}m_1(p_{1\nu}-p_{2\nu})[q^2-2m_1E_2\log(1 + q^2/2m_1E_2)]\bar{a}$$

$$+ q^{-2}m_1p_{2\nu}\log(1 + q^2/2m_1E_2)\bar{a}$$

$$- [3/4 + \log \Lambda/m_1 - m_1E_2q^{-2}\log 2E_2/m_1]\gamma_\nu a$$

If the coefficient of $e^2\gamma_\mu a$ including R is labeled A and the coefficient of $e^2\bar{a}$ labeled B_μ then the matrix element for the decay

T is

$$T = \sqrt{8}\, G\, \bar{u}_2[(1 + e^2 A)\gamma_\mu a + e^2 B_\mu \bar{a}] u_1 \bar{u}_4 \gamma_\mu a u_3$$

The part of the energy spectrum from the unperturbed case coupled with the virtual processes in the case of unpolarized electrons and unpolarized muons is

$$\frac{m_1^5 G^2}{(12)(2\pi)^3}\{\eta^2(3-2\eta)[1+e^2\alpha/2\pi]\} = \frac{2\pi^2}{(2\pi)^6 m_1 E_2}\int d^4s \int d^4s\, [\tfrac{1}{2} \sum_{\text{spins}}{}' |T|^2$$

$$p_2^2\, dp_2\, \delta(s^2)\delta(t^2)\delta^4(p_1 - p_2 - s - t)] \qquad (13)$$

where s and t are the neutrino four vectors. Since only corrections to order e^2 are desired those terms in $|T|^2$ proportional to AB will not be retained.

The part of $\sum_{\text{spins}}{}' |T|^2$ coming from the neutrinos and depending on s is

$$\sum_{\text{spins}} |\bar{u}_4 \gamma_\mu u_3|^2 = 4[s_\mu t_\nu + s_\nu t_\mu - g_{\mu\nu} s\cdot t] = N_{\mu\nu} \qquad (14)$$

Using the method shown in Appendix II the integral of the above equation becomes

$$4\int d^4s\, \delta(s^2)[s_\mu t_\nu + s_\nu t_\mu - g_{\mu\nu} s\cdot t] = \frac{2\pi}{3}[G_\mu G_\nu - g_{\mu\nu} G^2] \qquad (14a)$$

where $G = p_1 - p_2 = s + t$.

If the indicated summations remaining in equation (13) are performed then we find for the quantity

$$\alpha = 2[-2\log(m_2/\lambda_m)(\log\eta + \omega - 1) - (\log\eta + \omega)^2 + L(1-\eta)$$
$$- (1-\eta)^{-1}\log\eta + 5\omega/2 + 2\log\eta - 2 + \eta/2\log\eta(1-\eta)^{-1}$$
$$+ \eta/2\log\eta(1-\eta)^{-1}(3-2\eta)^{-1}] \qquad (15)$$

where $\omega = \log(m_1/m_2)$.

We see that dependence of α on the ultra-violet cutoff Λ has disappeared just as in the case of pure electrodynamics. Since the interaction used for the beta decay is $V + A$ and since in the limit as $m_2 \to 0$ V and A yield the same spectrum, the vertex operator is just as in electrodynamics and we expect the same renormalization to work here. If any of the other interactions (S,P,T) for the beta vertex is employed the resultant radiative corrections will depend upon the ultra-violet cutoff Λ.[4] Furthermore since the interaction $V + A$ is invariant under the interchange of the final particles ($u_2 \leftrightarrow u_4$) both charge retention and charge exchange order have radiative corrections which are independent of Λ.

That changing the photon propagator from k^{-2} to $[k^2 - \lambda_m^2]^{-1}$ is equivalent to giving the photon a mass λ_m may be seen by considering the field equation for a vector field A_μ, with mass λ_m. This is given in reference 5 footnote 27 as

$$- \partial_\nu(\partial_\nu A_\mu - \partial_\mu A_\nu) - \lambda_m^2 A_\mu = - 4\pi s_\mu \tag{16}$$

where s_μ is the source.

If the source is conserved, i.e., $\partial_\mu s_\mu = 0$ then we have $\partial_\mu A_\mu = 0$ or in momentum space

$$k_\mu a_\mu = 0 \tag{17}$$

where a_μ is the Fourier transform of A_μ. Using this result the Fourier transform of equation (16) becomes $(k^2 - \lambda_m^2)a_\mu(k) = -4\pi s_\mu(k)$. Hence the complete propagator for a spin one particle with divergence free source is $[k^2 - \lambda_m^2]^{-1}$ and it is seen that the method of handling the infra-red divergence in the virtual processes is equivalent to treating the photon as a "neutral vector muon" with mass λ_m. The calculation of the inner bremsstrahlung in the next section will be carried out with the same considerations for the emitted photon.

B. Inner Bremsstrahlung (IB)

The differential transition probability for IB in muon decay has been determined independently by Lennard[7] for $m_2 = 0$ and for $m_2 \neq 0$ by BFS[4]. If the IB cross-section is calculated as part of the radiative corrections to a given interaction to be consistent with the virtual processes we must also treat the emitted photon as having a small mass λ_m. We shall now discuss this aspect of the IB cross-section in detail.

The matrix T_μ to be evaluated between muon and electron spinors for the process of IB is

$$T_\mu = -(4\pi e^2)^{1/2}\left[\gamma_\mu a \frac{1}{\not p_1 - \not k - m_1} \not e + \not e \frac{1}{\not p_2 + \not k - m_2} \gamma_\mu a\right] \tag{18}$$

or after a slight simplification

$$T_\mu = -(4\pi e^2)^{1/2}\Big[2\Big(\frac{p_2 \cdot e}{2p_2 \cdot k + \lambda_m^2} - \frac{p_1 \cdot e}{2p_1 \cdot k + \lambda_m^2}\Big)\gamma_\mu a$$

$$+ \Big(\frac{\not e \not k \gamma_\mu a}{2p_2 \cdot k + \lambda_m^2} - \frac{\gamma_\mu a \not e \not k}{2p_1 \cdot k + \lambda_m^2}\Big)\Big] \tag{19a}$$

For the differential cross-section for IB it is necessary to determine the expression

$$\sum_{\text{pol.}} \sum_{\text{spins}} \left| \bar\psi_2 T_\mu \psi_1 \bar\psi_4 \gamma_\mu a \psi_3 \right|^2 \tag{19b}$$

where 3 and 4 refer to the neutrinos.

From equation (16) we see that since the emitted photon has a mass λ_m there is no gauge condition on the electromagnetic field operator and the massive photon has really three independent directions of polarization. These must be taken so as to satisfy the condition

$$k \cdot e = 0 \tag{20}$$

which follows immediately from equation (17).

If we take the direction of propagation of the momentum vector in the 3 direction, i.e. $k = (\omega, 0, 0, k)$, then the three polarization vectors which are mutually perpendicular and satisfy equation (20) are readily determined as

$$e_1 = (0,1,0,0)$$

$$e_2 = (0,0,1,0)$$

$$e_3 = (\omega,0,0,k)\lambda_m^{-1}$$

in which case a sum of the form $\sum (T \cdot e)^2$, where the sum is over 3 directions of polarization, may be expressed as

$$\sum_{3\ \mathrm{pol.}} (T \cdot e)^2 = (T \cdot e_1)^2 + (T \cdot e_2)^2 + (T \cdot e_3)^2$$

$$= - T^2 + \lambda_m^{-2} (T \cdot k)^2 \tag{21}$$

If $T \cdot e$ is the matrix element for IB then we see from equation (19) that $T \cdot k$ is of order λ_m^2 and that

$$\lim_{\lambda_m \to 0} \sum_{3\ \mathrm{pol.}} (T \cdot e)^2 = -T^2 \tag{22}$$

This is, of course, the same result as one obtains for real massless photons. We conclude that in the limit of $\lambda_m \to 0$ there are <u>no additional</u> terms in the IB differential cross-section as a result of summing over three instead of two directions of polarization. However there is still an infra-red divergence which arises when the differential cross-section for IB is integrated over photon momenta. We examine this point below.

Summing over three directions of polarization final spins and averaging over the initial muon spin as indicated in expression (19a) we find for the differential cross-section for IB in the limit of $m_2 \to 0$

$$\frac{4\pi^2 e^2}{m_1 E_2 \omega} G^2 \frac{d^3p_2 d^3k}{(2\pi)^9} \int \delta(s^2)\delta(G^2 - 2s\cdot G) B_{\mu\nu} N_{\mu\nu} d^4s \qquad (23)$$

where

$$G = p_1 - p_2 - k$$
$$= s + t$$

$N_{\mu\nu}$ is defined by equation (14)

$$\begin{aligned} B_{\mu\nu} = &-4\Omega^2[2p_{1\mu}p_{2\nu} - g_{\mu\nu}p_1\cdot p_2 - 2k_\mu p_{2\nu} + p_2\cdot k g_{\mu\nu} + 2p_{1\nu}k_\mu - (p_1\cdot k)g_{\mu\nu}] \\ &+4\Omega_\sigma[2p_{1\mu}g_{\mu\sigma} - g_{\mu\nu}p_{1\sigma} + 2p_{2\nu}g_{\mu\sigma} - p_{2\sigma}g_{\mu\nu}] \\ &-4(\Omega\cdot p_1)(p_1\cdot k)^{-1}[2p_{1\mu}k_\nu - p_1\cdot k g_{\mu\nu}] \\ &-4(\Omega\cdot p_2)(p_2\cdot k)^{-1}[2p_{2\nu}k_\mu - p_2\cdot k g_{\mu\nu}] \\ &+ 4(p_2\cdot k)^{-1}[2p_{1\mu}k_\nu - g_\mu\, p_1\cdot k] + 4(p_1\cdot k)^{-1}[2p_{2\mu}k_\nu - g_{\mu\nu}p_2\cdot k] \end{aligned}$$

and

$$\Omega_\sigma = p_{2\sigma}(p_2\cdot k)^{-1} - p_{1\sigma}(p_1\cdot k)^{-1}$$

Using the technique of Appendix II the integral of $N_{\mu\nu}$ is

$$\int d^4s\, N_{\mu\nu} = 2\pi/3[G_\mu G_\nu - g_{\mu\nu}G]$$

The differential cross-section for IB can be written, after using the above expansion in equation (23) and in the limit of $m_2 \to 0$, as

$$d^6R_B = \frac{e^2G^2d^3p_2d^3k}{3m_1E_2\omega(2\pi)^6}[\{G^4 - \frac{m_1^2}{2}G^2 - \frac{1}{4}m_1^4\}\Omega^2 + 4G^2 + \frac{k\cdot G}{p_1\cdot k p_2\cdot k}(2G^2 + m_1^2)]$$

(24)

Equation (24) is in agreement with both Lennard and BFS.

For the spectrum of the decay electron equation (24) must be integrated over all photon momentum compatible with conservation of energy and momentum in the decay. This is different from the radiative corrections to scattering processes where the integral over photon momenta is done over a very small range without regard to the conservation laws. The reason for this difference is that in a decay where there is a spectrum of energies, an electron found with low energy could have been made along with a photon of considerable energy whereas in a scattering experiment the maximum energy of the emitted photon is determined by the energy width of the experimental apparatus. The decay spectrum will be sensitive to the experimental resolution only at the upper limit of the energy. In this region the electron could be made along with only a very low energy photon whose maximum energy depends on the resolution. Thus we expect that the usual logarithmic dependence on the experimental resolution which occurs in a perturbation theory calculation of the radiative corrections will only be significant near the high energy end of the spectrum.

From conservation of energy and momentum the maximum photon momentum k_o at a given photon angle θ with respect to the electron in the rest system of the muon is

$$k_o = G^2/2(m_1 - E_2 + p_2 x)^{-1} \qquad (25)$$

where

$$x = \cos\theta$$

The spectrum is determined by integrating equation (24) first from $k = 0$ to $k = k_o$, then over all angles. If we attempt this with $\omega = |\vec{k}|$

then since $\Omega^2 \sim (p\cdot k)^{-2}$ the integral becomes logarithmically divergent for small k. The elimination of this infra-red divergence must be done in a manner consistent with the virtual processes. When the photon is treated as having a mass λ_m the integral of equation (24) is no longer divergent but depends logarithmically on λ_m.

There are three non-trivial integrals arising in the integration of equation (24) which when expressed in the rest system of the muon and in the limit of $m_2 \to 0$ are

$$\int_{-1}^{1} dx \int_0^{k_o} \frac{k^2 dk}{\omega(p_2\cdot k)^2} = \frac{2}{E_2 m_1} \log\left(\frac{2E_2}{m_2}\right) + \frac{1}{E_2 m_1} \log\left(\frac{m_1}{m_1 - 2E_2}\right)$$

$$+ \frac{1}{2E_2^2} \log 2 - \frac{2}{m_2^2} \log\left[\frac{2E_2\lambda_m}{(m_1 - 2E_2)m_2}\right] \tag{26a}$$

$$\int_{-1}^{1} dx \int_0^{k_o} \frac{k^2 dk}{\omega(p_1\cdot k)^2} = \frac{1}{m_1 E_2} \log \frac{\lambda_m}{m_1 - 2E_2} \tag{26b}$$

$$\int_{-1}^{1} dx \int_0^{k_o} \frac{k^2 dk}{\omega(p_1\cdot k)(p_2\cdot k)} = \frac{2}{E_2 m_1} \log\left(\frac{\lambda_m}{m_1}\right)\log\left(\frac{m_1}{2E_2}\right)$$

$$+ \frac{2}{E_2 m_1} \log\left(\frac{m_1}{m_1 - 2E_2}\right) \log\left(\frac{m_2}{2E_2}\right)$$

$$+ \frac{1}{m_1 E_2}\left[\log^2\left(\frac{m_1}{m_2}\right) + \sum_{k=1}^{\infty} \frac{1}{k^2}\left(\frac{m_1}{2E_2}\right)^k - \pi^2/6\right] \tag{26c}$$

Using the above expressions in the integral of equation (24) the probability that the electron has energy η in a range $d\eta$ is

$$p(\eta)d\eta = \frac{m_1^5 G^2 e^2}{(2\pi)^3(12)(2\pi)} \eta^2(3 - 2\eta)\beta \, d\eta \tag{27}$$

where

$$\beta = 2X + \frac{1}{3(3-2\eta)} (\log \eta + \omega - 1)(1-\eta)(5/\eta + 17 - 34\eta)$$

$$+ \frac{5(1-\eta)^2}{3(3-2\eta)\eta} \tag{28}$$

and

$$\omega = \log (m_1/m_2)$$

$$X = (\log \eta + \omega - 1)[2 \log(1-\eta) - \log \eta + \omega - 2\log(\frac{\lambda_m}{m_2})]$$

$$+ \sum_{k=1}^{\infty} \eta^k/k - \pi^2/6 - \frac{1-\eta}{\eta} \log (1-\eta) .$$

The difference between the result given here for β and that of BFS is contained in X and the expression V equation (25d) of BFS. We obtain the result of BFS, that X be replaced by V, if instead in the integrals of equation (26) we set $\omega = |\vec{k}|$ and integrate $|\vec{k}|$ with a lower limit λ_m. If this is done the integrals of equation (26) become in the limit of $m_2 \to 0$,

$$\int_{-1}^{1} dx \int_{\lambda_m}^{k_o} \frac{k^2 dk}{\omega(p_2 \cdot k)^2} = -\frac{2}{m_2^2} \log\left(\frac{2\lambda_m}{m_1 - 2E_2}\right)$$

$$\int_{-1}^{1} dx \int_{\lambda_m}^{k_o} \frac{k^2 dk}{\omega(p_1 \cdot k)^2} = \frac{1}{m_1^2}\left[2 - 2\log\left(\frac{\lambda_m}{m_1}\right) - \frac{m_1}{E_2} \log\left(\frac{m_1}{m_1 - 2E_2}\right)\right]$$

$$\int_{-1}^{1} dx \int \frac{k^2 dk}{\omega(p_1 \cdot k)(p_2 \cdot k)} = \frac{2}{E_2}\left[\log\left(\frac{2\lambda_m}{m_1}\right)\log\left(\frac{m_1}{2E_1}\right)\right.$$

$$\left. + \log\left(\frac{m_1 - 2E_2}{m_1}\right)\log\left(\frac{2E_2}{m_2}\right) + \frac{1}{2}\sum_{k=0}^{\infty} \frac{\eta^k}{k}\right]$$

When the values of these integrals are used we find that β is given by equation (28) with

$$X \to V = \sum_{k=1}^{\infty} \frac{\eta^k}{k^2} - 1 + 2[\omega - \log(\frac{\lambda_m}{m_2}) - \log 2](\log \eta + \omega - 1)$$

$$+ (2 \log \eta + 2\omega - 1 - \frac{1}{\eta}) \log(1 - \eta)$$

The complete spectrum with radiative corrections may be expressed as

$$p(\eta)d\eta = \frac{m_1^5 G^2}{(2\pi)^3 12} [\eta^2 (3 - 2\eta)(1 + h(\eta))]d\eta \tag{29}$$

where

$$h(\eta) = \frac{e^2}{2\pi} (\alpha + \beta)$$

Numerical values are shown in table I for 100 $h(\eta)$ given here as well as 100 $h(\eta)$ given by BFS. Figure 2 is a graph of the two h functions. It is seen that the radiative corrections given here tend to decrease the energy spectrum near the high energy end more than those of BFS.

In the expression for V we notice an apparent divergence as $\eta \to 1$. This divergence is of the same type that arises in the radiative corrections to scattering and is removed when we consider the dependence of the spectrum on the experimental resolution. Near the upper end of the spectrum the maximum energy of the emitted photon is not determined by conservation laws but rather by the energy resolution of the apparatus. If δ is the energy resolution and Δ is an energy interval such that

$$\Delta \leq \delta$$

then since the experiment is incapable of discerning the spectrum in a width Δ we can replace $p(\eta)$ in a region Δ about $\eta = 1$ by its average value in that interval. From the expressions for α eq. (15) and β eq. (28) we see that

$$\lim_{\eta \to 1} p(\eta) = E \log (1 - \eta) + F$$

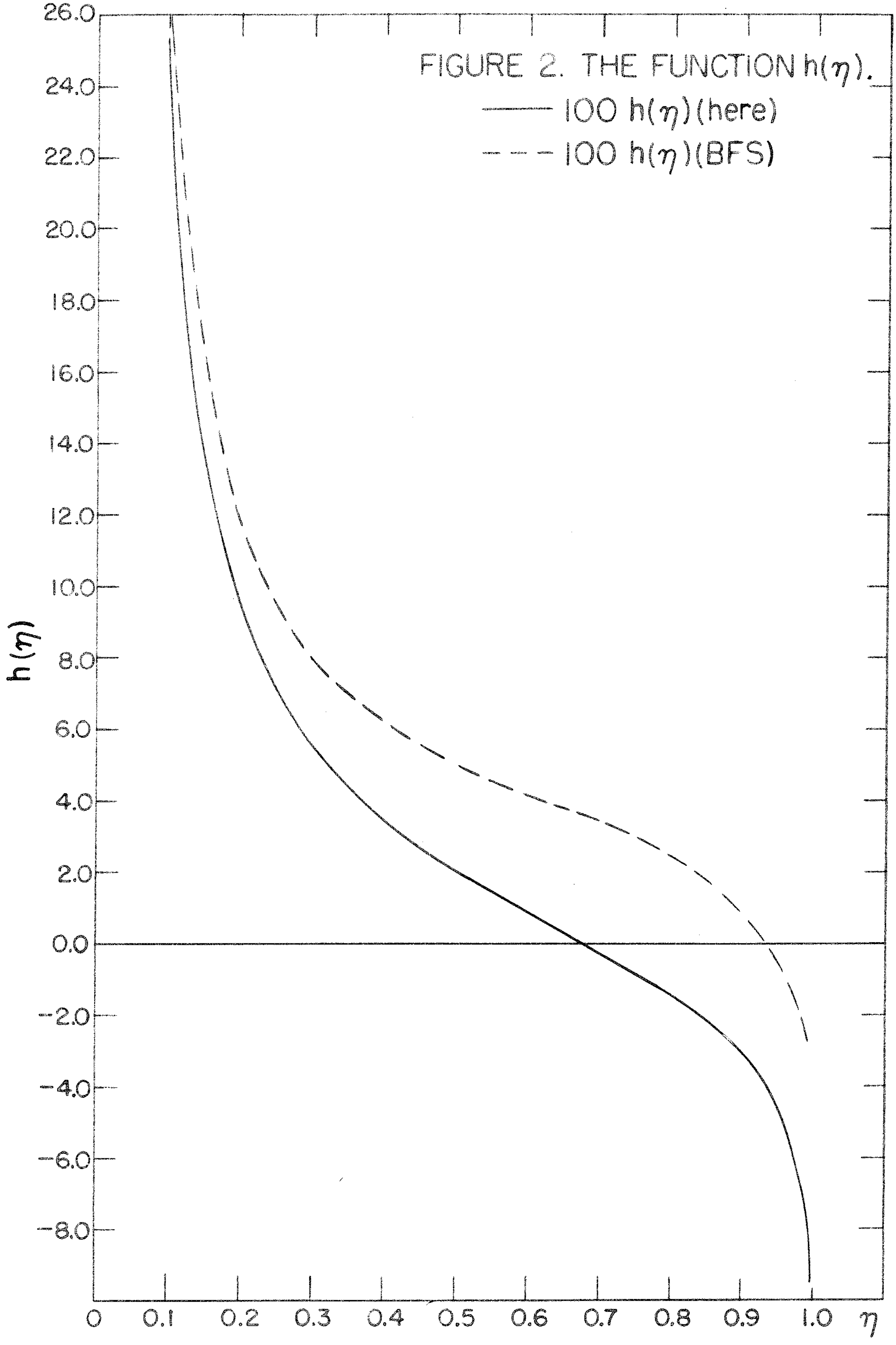
FIGURE 2. THE FUNCTION h(η).
100 h(η)(here)
100 h(η)(BFS)
h(η)
η
26.0
24.0
22.0
20.0
18.0
16.0
14.0
12.0
10.0
8.0
6.0
4.0
2.0
0.0
-2.0
-4.0
-6.0
-8.0
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1.0

where E and F are constants which are finite and do not depend on η in this limit. The average value of $p(\eta)$ is then determined by the average value of $\log(1 - \eta)$ in the interval Δ about $\eta = 1$, which is $\log(\Delta/e)$. By allowing

$$\log(1 - \eta) \to \log(1 - \eta + \Delta/e)$$

over the whole interval we have a function which, as long as Δ is small is $\log(1 - \eta)$ when η is not near 1, and is $\log(\Delta/e)$ near $\eta = 1$.

For the correction to the lifetime of the muon $p(\eta)$ is integrated from $\eta = 0$ to $\eta = 1$ with $\Delta = 0$. The radiative corrections then lead to a 0.44% increase in the lifetime compared to a 3.5% decrease found by BFS. With this correction the predicted value from the universal theory using the value of $G = (1.41 \pm 0.01) \times 10^{-49}$ erg cm.^{3} given by the beta decay of O^{14} is $(2.27 \pm 0.04) \times 10^{-6}$ sec. while the experimental value is $(2.22 \pm 0.02) \times 10^{-6}$ sec.

TABLE 1

The Function $h(\eta)$

η	100 $h(\eta)$ given here	100 $h(\eta)$ given by BFS
0.1	24.68	26.5
0.2	9.69	11.6
0.3	5.54	7.9
0.4	3.43	6.2
0.5	2.01	5.1
0.6	0.85	4.2
0.7	-0.23	3.4
0.8	-1.42	2.3
0.9	-3.06	0.86
0.94	-4.16	-0.16
0.98	-6.45	

III. THE BETA DECAY OF THE NEUTRON

A. The Virtual Processes

Since the muon has apparently no strong couplings we can assume, with confidence, that effects on the electron spectrum due to its interaction with other fields will be much smaller than the radiative corrections calculated in Section II. However, in the case of neutron β-decay it would certainly be incorrect to ignore its couplings with the meson field. Nevertheless, we have studied the electromagnetic corrections to the neutron β-decay in the approximation of neglecting mesons for the following reason.

We take the point of view of Feynman and Gell-Mann(1) that the vector coupling of β-decay remains unrenormalized just as it is in electrodynamics. Then, since it is known that the β-decay of O^{14} proceeds only through the vector interaction, the fact that the nucleons are coupled with mesons will produce no change in the effective O^{14} β-decay interaction, unless higher order forbiddeness is considered. However, electromagnetic interaction of the charged particles involved in the decay, especially between the nucleus and the decay electron, will be responsible for small corrections to the electron spectrum and ft value in O^{14} decay. These corrections are in part due to the point particle nature of the nucleus and in part caused by the fact that the nucleus has a structure. In this section the radiative corrections to neutron decay are calculated treating the neutron as a bare point particle without anomalous moments. These corrections are also relevant for O^{14} decay since they arise from regions of low virtual photon momenta where the form factor is unimportant. However, these corrections are not complete since the effect of structure on either O^{14} or the neutron has not been considered. Below we examine briefly how the inclusion of structure will contribute additional electromagnetic corrections.

For O^{14} there are, in addition to the type of virtual processes described in Section II for the muon, contributions to order e^2 from virtually excited states of the nucleus. In second order perturbation theory we could have the nucleus β-decay to a virtual excited state and return to the daughter nucleus by emission of a photon which is, in turn, absorbed by the electron. These excited states may be close enough together so that there could be a significant contribution from this virtual process to the radiative corrections of O^{14} decay. Similarly we can expect effects of structure of the neutron to affect the corrections to its β-decay. However, since the first excited state would be expected to be approximately 140 Mev. above the ground state the effect of structure may be less important than for O^{14}.

In Section IIIC we have estimated some aspects of the meson-nucleon coupling by including the effect of the anomalous moments on the radiative corrections to neutron decay.

The radiative corrections to the neutron decay are different from the muon decay even if we neglect any electromagnetic interaction via the anomalous moments. This difference is due in part to kinematical factors and in part to the alteration in the β interaction brought about by a rearrangement of the field operators. It will be shown below that the leading terms in the limit of large neutron or muon mass are the same for the radiative corrections to neutron or muon decay. The second difference mentioned above comes about when we use a Fierz[(9)] transformation to arrange the field operators from the usual order in the beta decay of the neutron $(\overline{P}N)(\overline{e}\nu)$ to the order where the charged particles are coupled together $(\bar{P}e)(\bar{N}\nu)$; the original interaction

$$(\bar{P}\gamma_\mu aN)(\bar{e}\gamma_\mu a\nu)$$

goes over to

$$-2(\bar{P}\,\bar{a}\,e)(\bar{\nu}\,\bar{a}\,N)$$

and the interaction $V + A$ has become $2(S - P)$.

We may describe the decay of the neutron with radiative corrections by adding to the above Lagrangian the electromagnetic interaction

$$L_{int} = e\bar{\psi}_P\gamma_\mu A_\mu\psi_P + e\bar{\psi}_e\gamma_\mu A_\mu\psi_e$$

In that case the decay of the neutron with radiative corrections has diagrams A, B, C of figure.3. The matrix T to be evaluated between photon and electron states for diagram A is

$$T = 4\pi ie^2 \int \frac{d^4k}{(2\pi)^4} \left[\gamma_\mu \frac{1}{\not{P}-\not{K}-M}\, 2\bar{a}\, \frac{1}{-\not{t}-\not{K}-m}\, \gamma_\mu C(k^2)\right] \tag{30}$$

where $a = (1 + i\gamma_5)/2$. P_μ and t_μ are the proton and electrom momentum four vectors and $C(k^2)$ is given by equation (16). The minus sign on $\not{t}$ in equation (30) follows from the usual rules of quantum electrodynamics for electrons propagating backwards in time.

If we rationalize equation (30) and make use of the fact that T is to be taken between free proton and electron states then

$$T = 2ie^2/\pi[-4P\cdot tI_1 + [4(t_\mu - P_\mu) + 2M\gamma_\mu]I_{2\mu} + 4g_{\mu\nu}I_{3\mu\nu}]\bar{a}$$

$$+ ie^2/\pi[4m\gamma_\mu aI_{2\mu} + 8MmaI_1] \tag{31}$$

where M and m are the proton and electron mass respectively

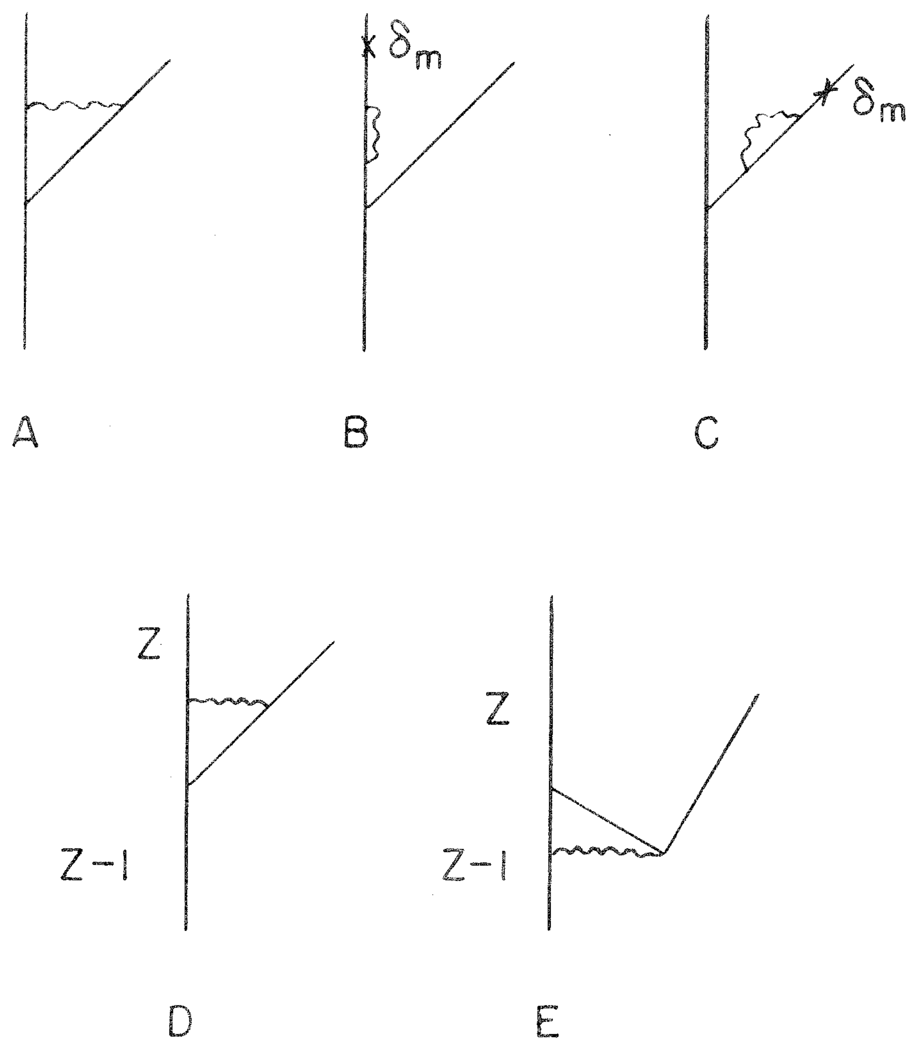

FIGURE 3. FEYNMAN DIAGRAMS FOR NEUTRON AND NUCLEAR β-DECAY.

$$I_1 = \int \frac{d^4k}{(2\pi)^2} \left[\frac{1}{(k^2-2P\cdot k)(k^2+2t\cdot k)} \frac{C(k^2)}{k^2} \right]$$

$$= (8i)^{-1} \int_0^1 dy\, P_y^{-2} \log (P_y^2/\Lambda_m^2) \qquad (\not{P}_y = y\not{P}-(1-y)\not{t}) \qquad (32a)$$

$$I_{2\mu} = \int \frac{d^4k}{(2\pi)^2} \left[\frac{k_\mu}{(k^2-2P\cdot k)(k^2+2t\cdot k)} \frac{C(k^2)}{k^2} \right]$$

$$= (4i)^{-1} \int_0^1 P_y^{-2} P_{y\mu}\, dy \qquad (32b)$$

$$g_{\mu\nu} I_{3\mu\nu} = \int \frac{d^2k}{(2\pi)^2} \left[\frac{k_\mu k_\nu}{(k^2-2P\cdot k)(k^2+2t\cdot k)} \frac{C(k^2)}{k^2} \right] g_{\mu\nu}$$

$$= (-8i)^{-1} \left[\int_0^1 dy \int_0^1 dx \int_0^{(1-x)\Lambda^2} dV\, 2x(2V+x^2P_y^2)(x^2P_y^2+V)^{-2} \right. \qquad (32c)$$

It is interesting to note that if it were the proton undergoing β-decay the matrix to be evaluated between proton and electron spinors for virtual photon exchange is given by equation (30) with $P \to -P$. The effect of this is to change the sign of t in equations (32) with the result, shown below, that the corrections to the electron spectrum in neutron and proton decay differ by just the coulomb effect.

The last integral equation (32c) is logarithmically divergent for large k. Since we expect the cutoff Λ to be of order of the proton mass we give the result for the above integral as a function of Λ. Following the method of Appendix I we have

$$g_{\mu\nu} I_{3\mu\nu} = -i \int [a^2\alpha/b + a/2 \log (4a)]dy \qquad a > 1 \qquad (33a)$$

$$= -i \int_0^1 [a^2\beta/b + a/2 \log 4a]dy \qquad a < 1 \qquad (33b)$$

where $\qquad a = \Lambda^2/4P_y^2$

$$b^2 = a(a - 1) \qquad\qquad a > 1$$

$$ = a(1 - a) \qquad\qquad a < 1$$

$$\alpha = - \coth^{-1} a/b \qquad\qquad \beta = - (\tan^{-1} a/b + \pi/2)$$

If we take Λ large compared to the proton mass then equation (33a) reduces to the form given in reference 5

$$\lim_{\substack{\Lambda \to \infty \\ \text{or } p_y^2 \to 0}} g_{\mu\nu} I_{3\mu\nu} = (4i)^{-1}\left[1 - \int_0^1 \log \frac{\Lambda^2}{p_y^2}\, dy\right] \tag{34}$$

where we have used

$$\lim_{\Lambda \to \infty} \alpha = 1/2 \log (4a - 2)$$

Figure 4 shows the exact value, equation (33), and the approximate value, equation (34), for $1/4 < a < 10$. When a is less than 1, which is the case over most of the y integration, we see that the approximate value is quite poor.

As in the case of the muon decay the effects of wave function renormalization for the charged particles must be taken into account. To this end we have for the electron and proton

$$R\gamma_\mu = ie^2/\pi \int \frac{d^4k}{(2\pi)^2} \frac{C(k^2)}{k^2} \left[\gamma_\nu \frac{1}{\binom{\not P}{\not\chi} - \not k - \binom{M}{m}} \gamma_\mu \frac{1}{\binom{\not P}{\not\chi} - \not k - \binom{M}{m}} \gamma_\nu\right] \tag{35}$$

If we determine the R for electron and proton by assuming Λ larger than both masses then R is given by equation (3) section I. The net effect of R and virtual photons is not independent of Λ as in the case of muon decay since the β-interaction is now $S - P$ instead of $V + A$ for the same order of wave functions as in the muon decay.

If we do not take Λ much larger than the nucleon mass then for the

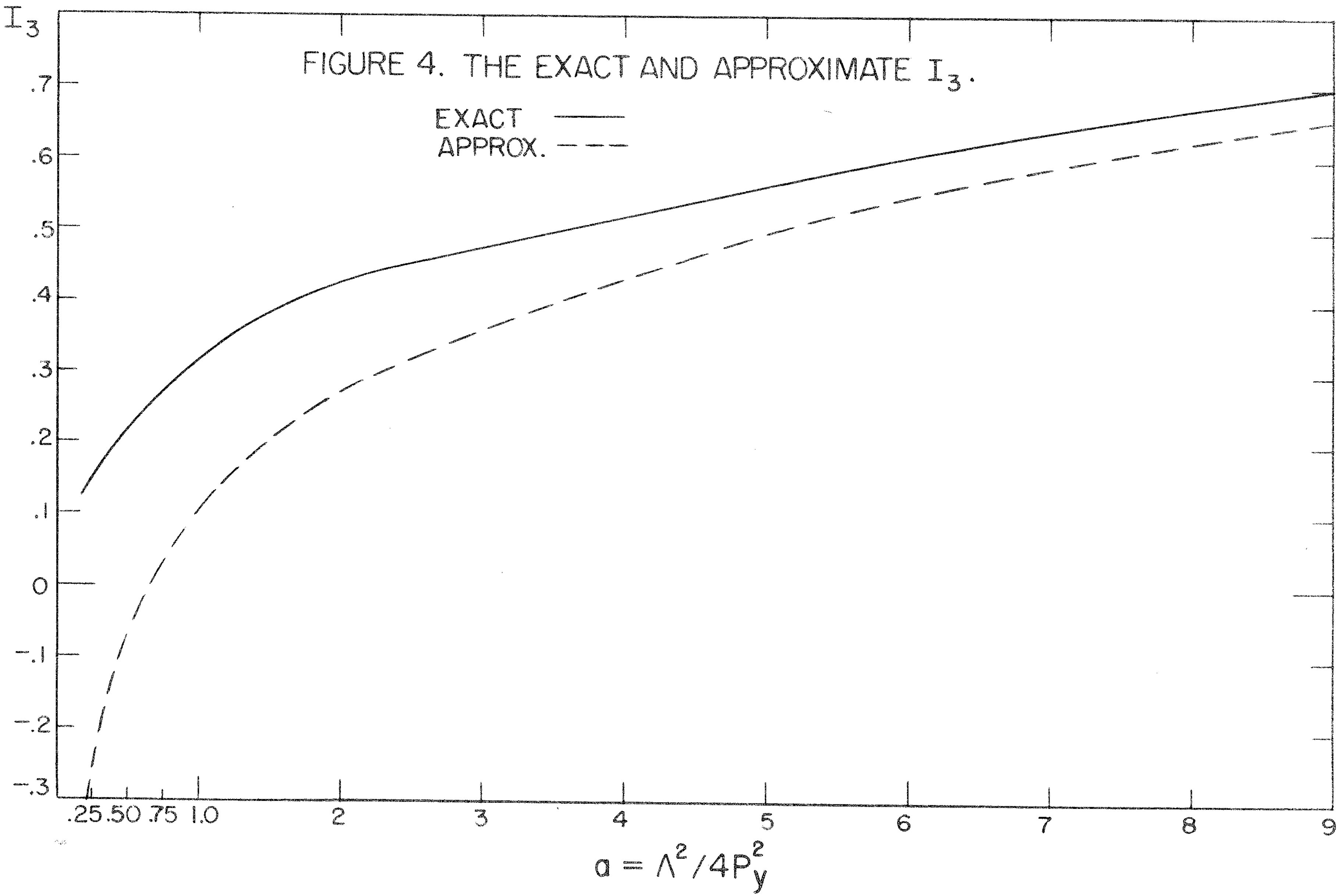

FIGURE 4. THE EXACT AND APPROXIMATE I_3.

proton

$$R_p = e^2/2\pi[-2\log(M/\lambda_m) + 2 + 2\{12a^2 \log 4a + (\frac{24a^3}{b} - \frac{20a^2}{b})\delta - 6a\}] \quad (36)$$

where

$$a = \Lambda^2/4M^2$$

and

$$\left.\begin{aligned} b^2 &= a(a-1) \\ \delta &= -\coth a/b \end{aligned}\right\} \quad a > 1$$

$$\left.\begin{aligned} b^2 &= a(1-a) \\ \delta &= -(\tan^{-1} a/b + \pi/2) \end{aligned}\right\} \quad a < 1$$

For $\Lambda = M$

$$R_p = e^2/2\pi[-2\log(M/\lambda_m) + 2 + 1.24]$$

which may be compared with R for $\Lambda \gg M$

$$R_p = e^2/2\pi[-2\log(M/\lambda_m) + 2 + 1/4 + \log(\Lambda/M)]$$

We will now consider two problems. First the evaluation of T as given by equation (31) in the limit of large proton mass. At first glance it would appear that this would yield the coulomb F factor to order e^2. Instead we find in addition to the usual coulomb factor a large contribution of the form $\log M/m$ where M and m refer to the proton and electron mass respectively. Second we will consider the energy dependent corrections to the spectrum when M is large. In this case it will be necessary to also determine the energy dependent parts arising from IB.

The coulomb F factor is the function which multiplies the unperturbed electron spectrum from β-decay when the static coulomb interaction between electron and the infinitely heavy daughter nucleus is taken into account. This function has been calculated by several authors and for not too small velocities is given approximately by[9]

$$F(z,\varepsilon) = \frac{4}{[\Gamma(3)]^2}\,(2pR)^{2s-2}e^{\pi\gamma/V}\left(\frac{2\pi\, y\,(1-2x)}{e^{\pi y}-e^{-\pi y}}\right)$$

where $s = (1 - e^2z^2)^{1/2}$; $\gamma = e^2z^2$ $x = s - 1$; $y = \gamma/V$, ε and V are the energy and velocity respectively of the electron and z is the charge of the daughter nucleus. The quantity R in the above equation is taken as the radius of the daughter nucleus. Since s is less than one, R cannot be zero; this is the usual infinity of the radial Dirac wave function in the presence of a coulomb field. To compare $F(z,\varepsilon)$ with a perturbation calculation we need the coulomb factor to order e^2. Expanding $F(z,\varepsilon)$ we find

$$F(z,\varepsilon) = 1 + \pi z e^2/V + O(e^4) \tag{37}$$

The dependence on R is contained in the e^4 and higher order terms.

In order to examine to what extent the virtual processes in the limit of large M agree with equation (37) we must evaluate the integrals in equation (32) for the matrix element T, equation (31).

We have

$$I_1 = (8i)^{-1}\int_0^1 dy\; P_y^{-2}\,\log\,(P_y^2/\Lambda_m^2)$$

where

$$P_y^2 = y^2M^2 - 2y(1-y)M\varepsilon + (1-y)^2m^2$$

Since $M \gg m$ we can approximate P_y^2 by

$$P_y^2 \approx y^2M^2 - 2yM\varepsilon + m^2$$

The m^2 cannot be neglected for then I_1 would be divergent at $y = 0$. Furthermore since P_y^2 does not behave as M^2 for very small y we expect that $\lim_{M \to \infty} MI_1$ will not be zero. Using the technique of Appendix I we find*

$$-I_1 = (8iMp)^{-1}[\log(p^2/\lambda_m^2)(\alpha_2+\alpha_1) + \alpha_2 \log \operatorname{csch}^2 \alpha_2 - \pi^2$$

$$+ \alpha_1 \log \operatorname{csch}^2 \alpha_1 + 2\int_{-\alpha_1}^{\alpha_2} \alpha \coth \alpha \, d\alpha] \qquad (38a)$$

where

$$\alpha_1 = \log [(\varepsilon + p)/m]$$

$$\alpha_2 = \coth^{-1} (M - \varepsilon)/p$$

and p is the electron momentum.

We see from equation (31) that for the matrix element the desired quantity is $4M\varepsilon I_1$ and using $\lim_{M \to \infty} \alpha_2 \to 0$ we have

$$\lim_{M \to \infty} i4M\varepsilon I_1 = \varepsilon/2p[\log p^2/\lambda_m^2)\alpha_1 + 2\alpha_1 \log \varepsilon/m - 5\alpha_1^2$$

$$+ \pi^2/6 + \alpha_1 \log[2p(\varepsilon+p)/m^2] + L(m^2/\varepsilon+p)^2]$$

where $L(x)$ is the Spence function[6] of x.

The integrals $I_{2\mu}$ is readily done and yields

$$I_{2\mu} = (4i)^{-1}\int_0^1 P_y^{-2} P_{y\mu} \, dy$$

$$= (4i)^{-1}[(P_\mu+t_\mu)/M^2 \log (M-2\varepsilon)/2\varepsilon - t_\mu/Mp(\alpha_2+\alpha_1) \qquad (38b)$$

---

*The integrand of I_1 has poles at $y = (\varepsilon \pm p)/M$. These poles contribute an additional real part of I_1 represented by the factor π^2 in equation (38a) and give rise to the coulomb correction term. For proton decay I_1 has no poles and we can conclude that the only difference between the neutron and proton decay corrections is the coulomb effect.

and for the matrix element

$$\lim_{M \to \infty} i[4(t_\mu - P_\mu) + 2(M \mp m)\gamma_\mu] I_{2\mu} = -\frac{1}{2}\log\left(\frac{M}{2\varepsilon}\right) - \frac{\varepsilon}{p}\alpha_1$$

$$= -\frac{1}{2}\log\left(\frac{M}{m}\right) + \frac{1}{2}\log\left(\frac{2\varepsilon}{m}\right) - \frac{\varepsilon}{p}\log\left(\frac{\varepsilon+p}{m}\right)$$

where we have made use of the fact that the matrix element is evaluated between free proton and electron states, and where the term proportional to $\bar{a}$ has been dropped since $a\,\bar{a} = 0$ in the matrix element squared. The factor $\log(M/m)$ represents a surprisingly large correction which we shall discuss in more detail below.

We note that for $4P_y^2 > \Lambda^2$ the integral $g_{\mu\nu} I_{3\mu\nu}$ is given by equation (33b). By examing this integral for the range of y such that $4P_y^2 > \Lambda^2$ we see that the integrand vanishes in the limit of large M. When y is very small so that $4P_y^2 < \Lambda^2$ then we may use the asymtotic form of the integral given by equation (34). In that case for the matrix element

$$i4g_{\mu\nu} I_{3\mu\nu} = [1 - \int_0^1 \log\left(\frac{\Lambda^2}{P_y^2}\right) dy]$$

$$= [(1 - \frac{2\varepsilon}{M})\log(1 - \frac{2\varepsilon}{M}) - 1 - \frac{2\varepsilon}{M}\log(\frac{2\varepsilon}{M}) - \log(\frac{\Lambda^2}{M^2})]$$

$$\to -1 - 2\log\left(\frac{\Lambda}{M}\right) \tag{38c}$$

We see that there are no energy dependent terms from $I_{3\mu\nu}$ in the limit of large M. Combining the various integrals, the matrix element T of equation (31) may be expressed as

$$T = -\frac{e^2}{\pi}\left[\frac{\varepsilon}{2p}\left\{\alpha_1 \log\left(\frac{p^2}{\lambda_m^2}\right) + 2\alpha_1 \log\left(\frac{\varepsilon}{m}\right) - 4\alpha_1^2 + \frac{\pi^2}{6} + \alpha_1 \log\left(\frac{2p}{m}\right)\right.\right.$$

$$\left.\left. + L\left(\frac{m^2}{(\varepsilon+p)^2}\right) - \frac{1}{2}\log\left(\frac{M}{2\varepsilon}\right) - 2\log\frac{\Lambda}{M} - 1\right\}\right] 2\bar{a} \qquad (39)$$

For the complete matrix element for the virtual processes we need the contribution of wave function renormalization for the electron and proton

$$R = R_p + R_e = \frac{e^2}{2\pi}\left[-2\log\left(\frac{m}{\lambda_m}\right) - \frac{1}{2}\log\frac{M}{m} + \log\left(\frac{\Lambda}{M}\right)\right.$$

$$\left. + \left\{\frac{17}{8} + 12a^2 \log 4a + \left(\frac{24a^3}{b} - \frac{20a^2}{b}\right)\delta - 6a\right\}\right] \qquad (40)$$

where a, b, and δ are given by equation (36). For $\Lambda \approx M$ the term in braces in equation (40) becomes 11/4. The complete matrix element for the virtual processes is given by $T - R2\bar{a}$.

The main contribution to $T - R2\bar{a}$ is contained in the factor

$$(e^2/2\pi)(3/2)\log(M/m)2\bar{a} \approx (T - R2\bar{a})$$

This term is much larger than any energy dependent terms in T and in the IB which will be calculated in the next section. The effect of this logarithm is to give the β-coupling constant an electromagnetic renormalization of + 2.6% with about 1/2% uncertainty due to neglecting energy dependent parts. Of course none of the effect of mesons is thus far included.

Examination of the coefficient of $I_{2\mu}$ in equation (5a) section IIA in the limit of large p_1 shows that the leading term $3/4 \log m_1/m_2$ relative to the electromagnetically unperturbed interaction is the same for the muon decay as for the neutron decay. This is to be expected since the decay rate for

$$P \rightarrow N + e^{+} + \nu$$

$$\mu^{+} \rightarrow e^{+} + \nu + \bar{\nu}$$

should be the same for large muon mass except for kinematical factors.

The reason why the virtual processes as calculated using the relativistic methods of quantum electrodynamics does not agree with the coulomb F factor to order e^2 may be explained as follows.

As is well known the electromagnetic interaction may be divided into the instantaneous coulomb potential and the transverse wave part. For an infinitely heavy nucleon the transverse wave parts, in diagrams of type D and E of figure 3, do not contribute and the whole effect must come from the coulomb interaction. If we use second order perturbation theory for a nucleus of charge $z - 1$ undergoing a β-decay to a nucleus of charge z, the matrix T to be evaluated between electron and neutrino states is determined in Appendix III and may be expressed as

$$T = 4\pi e^2 \int \frac{d^3k}{(2\pi)^3} \Big[z \, \frac{(\varepsilon_k + \vec{\alpha}\cdot\vec{k} + \beta m)}{2(\varepsilon - \varepsilon_k)\varepsilon_k} + (z-1) \frac{(\varepsilon_k - \vec{\alpha}\cdot\vec{k} - \beta m)}{2(\varepsilon + \varepsilon_k)\varepsilon_k} \Big] \frac{1}{(\vec{p} - \vec{k})^2} \tag{41}$$

where ε and ε_k are the energy of the electron in the final and intermediate state respectively and where the nucleus is treated as infinitely heavy.

The two terms of T are equally logarithmically divergent but of opposite sign, hence if it were not for the fact that $(z - 1)$ instead of z multiplies the negative energy contribution, T would be finite and lead to the F function of order ze^2. But we see that the usual

F function neglects the fact that it is a charge $(z - 1)$ that contributes to diagram E and a charge z that contributes to diagram D. Therefore when $< e|T|v >$ is calculated there appears a logarithmic divergence independent of the nuclear charge z. In addition there will be a logarithmically divergent term from the affect of transverse waves in the electron self energy, diagram C, as mentioned above.

As a check on the amount of $\log M/m$ we have considered the β-decay of a spin zero nucleus to a spin zero nucleus with the emission of an electron and an anti-neutrino. Since the limit of large nuclear mass is taken there should be no difference between the spin 1/2 case and the spin zero case. The spin zero case has been calculated in complete detail in Appendix IV and the result agrees with the $3e^2/4\pi \log M/m$ determined for the spin 1/2 decay. In the next section we compute the portion of the energy spectrum which arises from the process of inner bremsstrahlung in the limit of very large proton mass.

B. Inner Bremsstrahlung (IB) in Neutron Decay

The calculation of IB for neutron decay proceeds in an analogous manner to the case of muon decay. In the representation in which the charged particles are grouped together the matrix element for IB accompanying neutron decay to be evaluated between proton and electron spinors is

$$T = i(4\pi e^2)^{1/2}[2\bar{a} \frac{1}{\not{t}-\not{K}-m} \not{e} + \not{e} \frac{1}{\not{P}_2+\not{K}-M} 2\bar{a}] \tag{42}$$

where $\not{t}u_e = -mu_e$ and $a = (1 + i\gamma_5)/2$ and P_2 refers to the proton. Simplifying equation (42) by rationalizing the denominators and using $k \cdot k = 0$ we have

$$T = i(4\pi e^2)^{1/2}[\frac{2P_2 \cdot e + \not{e}\not{K}}{2P_2 \cdot k} - \frac{2t \cdot e + \not{K}\not{e}}{2t \cdot k}]2\bar{a} \tag{43}$$

where we have replaced t by $-t$ so that in equation (43) $\not{t}u_e = mu_e$.

Summing over spins and polarizations for $|T|^2$ yields

$$\sum_{\text{spins}} \sum_{\text{pol.}} |T|^2 = 32\pi e^2[-\Omega^2 P_2 \cdot t - BP_2 \cdot \Omega k \cdot t + B\Omega \cdot tP_2 \cdot k + B^2 P_2 \cdot kt \cdot k]$$

where $\Omega_\sigma = P_{2\sigma}(P_2 \cdot k)^{-1} - t_\sigma(t \cdot k)^{-1}$ and $B = (P_2 \cdot k)^{-1} + (t \cdot k)^{-1}$.

For the differential transition probability for IB we must multiply $|T|^2$ by the average over initial spins and sum over final spins for the neutron and neutrino. This yields a factor $P_1 \cdot s$ multiplying T^2 where P_1 and s refer to the neutron and neutrino respectively. The differential transition probability for IB is then

$$d^9R = \frac{\pi}{16E_1E_2\omega s\epsilon} 4P_1 \cdot s \sum_{\text{spins}} \sum_{\text{pol.}} |T|^2 \frac{d^3s}{(2\pi)^3} \frac{d^3t}{(2\pi)^3} \frac{d^3k}{(2\pi)^3} \delta(E_o - s - \epsilon - \omega) \tag{44}$$

where E_o is the energy release in the decay.

In the limit of large M_1 and M_2 $(M_1 \approx M_2 = M)$

$$(P_1 \cdot s) \sum_{\text{spins}} \sum_{\text{pol.}} |T|^2 = 128\, e^2 \pi M^2 s\left[-\frac{\varepsilon}{\omega^2} - \frac{\varepsilon m^2}{(t\cdot k)^2} + \frac{2\varepsilon^2}{\omega t\cdot k} - \frac{1}{\omega} + \frac{\varepsilon}{t\cdot k} + \frac{\omega}{t\cdot k}\left(\frac{\varepsilon}{\omega} - \frac{m^2}{t\cdot k}\right) + \frac{\omega}{t\cdot k}\right] \qquad (45)$$

The contribution to the energy spectrum of the decay electron from IB is determined by integrating equation (44) over all photon angles and energies up to a maximum energy $k_o = E_o - m$. The neutron decay in the limit of large nucleon mass is different from the muon decay in this respect for the conservation of momentum need not be taken into account. If we treat the infra-red divergences as in the muon decay then the contribution to the energy spectrum of the electron from IB is

$$\begin{aligned} dR_{IB} = \frac{G^2 e^2}{\pi^4} \Big[&\Big\{ 2I_1 - 2I_2 - 2\varepsilon I_3 - \frac{2k_o}{\varepsilon} + \left(\frac{4}{p} + \frac{k_o}{p\varepsilon}\right)\frac{k_o \alpha_1}{p} - \frac{2k_o}{\varepsilon}\Big\}(E_o - \varepsilon)^2 \\ &- \Big\{-4 - \frac{2k_o}{\varepsilon} + \left(4\varepsilon + 2k_o + \frac{2k_o^2}{3\varepsilon}\right)\frac{\alpha_1}{p}\Big\} 2(E_o - \varepsilon)k_o \\ &+ \Big\{-2 - \frac{4k_o}{3\varepsilon} + \left(2\varepsilon + \frac{4}{3}k_o + \frac{k_o^2}{2\varepsilon}\right)\frac{\alpha_1}{p}\Big\} k_o^2 \Big] p^2 dp \end{aligned} \qquad (46)$$

where

$$I_1 = 1 + \log \frac{\lambda_m}{2k_o} \; ; \quad I_2 = \log \frac{\lambda_m}{2k_o} + \frac{\varepsilon}{p}\alpha_1$$

$$I_3 = -\frac{2\alpha_1}{p} \log \frac{\lambda_m}{k_o} + \frac{\alpha_1}{p} \log 2 + \frac{1}{p}[L(V) - L(-V)] - \frac{1}{2p}[L(2+2V) - L(2-2V)] + \frac{1}{4p}[\log^2(1+V) - \log^2(1-V)]$$

where L is the Spence function and the velocity of the electron $V = p/\varepsilon$.

Using equations (39) and (40) the contribution to the energy spectrum from the virtual processes is

$$dR_V = \frac{2G^2}{\pi^3}(E_o-\varepsilon)^2 p^2 dp - \frac{4G^2e^2}{\pi^4}\Big[\frac{\varepsilon}{p}\Big\{ \log\frac{p}{\lambda_m} + \alpha_1 \log\frac{\varepsilon}{m} - 2\alpha_1^2 - \pi^2/2$$

$$+ \frac{\pi^2}{12} + \frac{\alpha_1}{2}\log\frac{2p}{m} + \frac{1}{2}L\Big(\frac{m^2}{(\varepsilon+P)^2}\Big)\Big\} - \frac{1}{2}\log\frac{m}{2\varepsilon} - \frac{\varepsilon}{p}\alpha_1$$

$$+ 1 - \log\frac{m}{\lambda_m} - \frac{3}{4}\log\frac{M}{m} - \Big\{ \frac{3}{2}\log\frac{\Lambda}{M} + \frac{17}{8} + 12a^2\log 4a$$

$$+ \Big(\frac{24a^3}{b} - \frac{20a^2}{b}\Big)\delta - 6a\Big\}\Big](E_o - \varepsilon)^2 p^2 dp \qquad (47)$$

where a, b, δ are given by equation (36). For $\Lambda = M$ the expression in the last brace becomes $11/4$.

C. The Effect of Anomalous Moments

The second possibility of large coefficients comes from including the anomalous moments of the proton and neutron. If we attempt to compute the radiative corrections by treating the nucleons as bare point particles with anomalous moments we find that the electromagnetic effects introduce quadratic as well as logarithmic divergences. It is necessary, in order that finite results be obtained, to cut off the integrals over virtual photon moments by including the nucleon form factor. There are two problems that arise in this connection.

First, even with form factors the complete end result would not be finite as there remains the logarithmic divergence from the wave function renormalization of the electron (diagram C of figure 3). (However, if only the energy dependent correction is desired this is of no significance since the result is just to provide an electromagnetic renormalization of the β-coupling constant.) Second, the only form factor that is known is for the interaction between electron and nucleon, but the complete electromagnetic corrections would require the form factor for the interaction of the neutron moment with the proton charge and moment. This latter form factor is, of course, unknown. However a reasonable calculation procedure would be to include the effect of the moments by using only the Stanford form factor. However, since the virtual photon interacts with non-free nucleons and the Stanford experiments are done with free nucleons this procedure may not be relevant.

For the purpose of estimating the kind of corrections introduced by the moments as well as the order of magnitude we have considered the problem without form factors and using a cutoff Λ, large compared to

M. The moments give rise to four additional diagrams, two from the interaction of the electron with the neutron and proton moments, and two from the interaction of the neutron with the proton charge and moment. The last type, the neutron proton moment interaction, is quadratically divergent, whereas the other three types are logarithmically divergent.

Let us consider the problem in some detail. The four new diagrams III thru VI of figure 5 are listed below where the circle stands for anomalous moment interaction. The matrix element to be evaluated between proton and electron states for diagram III of figure 5 is

$$T_{III} = \frac{-i4\pi e^2 \mu_P}{4M} \int \frac{d^4k}{(2\pi)^4} [(\gamma_\mu \not{k} - \not{k}\gamma_\mu) \frac{1}{\not{P}-\not{k}-M} 2\bar{a} \frac{1}{\not{t}-\not{k}-m} \gamma_\mu] \frac{C(k^2)}{k^2} \quad (48)$$

where $\not{t}u_e = -mu_e$ refers to the positron and P refers to the proton and $C(k^2)$ is given by equation (6). Employing the rules of gamma algebra T_{III} can be simplified to

$$T_{III} = -\frac{ie^2\mu_P}{M\pi} \int \frac{d^4k}{(2\pi)^2} [\frac{2P\cdot k\not{k} - 2Mk^2 - 3/2\not{k}k^2)2\bar{a}}{(k^2-2P\cdot k)(k^2+2t\cdot k)}] \frac{C(k^2)}{k^2}$$

$$+ O(t_\mu/M) \quad (49)$$

where $t_4 > 0$ and terms of order $1/M$ are dropped in the limit of large M.

There is one new integral in equation (49) not appearing in reference 5 which is readily evaluated by Feynman techniques. We have

$$12i \int \frac{d^4k}{(2\pi)^2} \frac{C(k^2)}{k^2} [\frac{k_\mu k_\nu k_\sigma}{(k^2-2P\cdot k)(k^2+2t\cdot k)}] = \int_0^1 dy \frac{P_{y\mu}P_{y\nu}P_{y\sigma}}{P_y^2}$$

$$- \frac{1}{2}(g_{\mu\nu}P_{y\sigma}+g_{\mu\sigma}P_{y\nu}+g_{\nu\sigma}P_{y\mu})\log\frac{\Lambda^2}{P_y^2} + \frac{7}{12}(g_{\mu\nu}P_{y\sigma}+g_{\mu\sigma}P_{y\nu}+g_{\nu\sigma}P_{y\mu})]$$

$$(50)$$

where

$$P_y = yP - (1-y)t$$

Since we are interested in the above integral for large M we can use $P_y^2 \approx M^2y^2$ which makes the integral on y quite simple. Combining terms and using the integral equation (34) we have

$$T_{III} = -\frac{e^2\mu_N}{8\pi}\left[9 \log\frac{\Lambda}{M} + \frac{15}{4}\right] 2\bar{a} \tag{51}$$

We note that in the standard representation $(\bar{P}N)(\bar{e}\nu)$ the $2\bar{a}$ of equation (51) goes over to the interaction $\gamma_\mu a$.

For diagram IV of figure 5 we use the Furz transformation which changes the order of operators from $(\bar{P}N)(\bar{e}\nu)$ to $(\bar{P}\nu)(\bar{e}N)$ and thus leaves the interaction $\gamma_\mu a$ invariant. Hence the matrix element to be evaluated between electron and neutron states is

$$T_{IV} = -\frac{ie^2\mu_N}{4M\pi}\int\frac{d^4k}{(2\pi)^2}\left[\gamma_\mu\frac{1}{\not{t}-\not{K}-m}\,\gamma_\nu a\,\frac{1}{\not{P}-\not{K}-M}\,(\gamma_\mu\not{K}-\not{K}\gamma_\mu)\right]\frac{C(k^2)}{k^2} \tag{52}$$

where $C(k^2)$ is given by equation (6) and where $\not{t}u_e = mu_e$ and since the limit of large M is to be taken we do not distinguish between proton and neutron momenta and refer to the latter also by P. Collecting those terms in T_{IV} which are non zero in the limit of large M we have

$$T_{IV} = -\frac{ie^2\mu_N}{4M\pi}\int\frac{d^4k}{(2\pi)^2}\left[\frac{8P_\nu k^2\bar{a}-8k_\nu k^2\bar{a}+4mk_\nu\not{K}a+2\gamma_\nu a\not{K}k^2}{(k^2-2t\cdot k)(k^2-2P\cdot k)}\right]\frac{C(k^2)}{k^2}$$

$$+\,O(1/M)$$

With the aid of equation (50) T_{IV} becomes in the limit of $M\to\infty$

$$T_{IV} = \frac{e^2\mu_N}{4\pi}\left[\left(\log\left(\frac{\Lambda}{M}\right)+\frac{1}{8}\right)\gamma_\nu a + \left(\log\left(\frac{\Lambda}{M}\right)+1\right)\frac{2P_\nu}{M}\bar{a}\right] \tag{53}$$

Using the standard representation $(\bar{P}N)(\bar{e}\nu)$ the matrix element to be evaluated between proton and neutron states in diagram V of figure 5 is

$$T_V = \frac{ie^2}{4M\pi}\int \frac{d^4k}{(2\pi)^2}\,\frac{C(k^2)}{k^2}\,[(\gamma_\mu K - K\gamma_\mu)\,\frac{1}{\not P - K - M}\,\gamma_\nu a\,\frac{1}{\not P - K - M}\,\gamma_\mu\,] \qquad (54)$$

where in anticipation of the limit of large M we have labeled both neutron and proton by P and where $C(k^2)$ is again given by equation (6). Simplifying T_V yields

$$T_V = \frac{ie^2\mu_N}{M\pi}\int \frac{d^4k}{(2\pi)^2}\,[-2P_o k^2 a - \frac{1}{2}Kk^2\gamma_\nu a + 2k_\nu k^2 a - Mk^2\gamma_\nu a$$

$$+ 2MKk_\nu a + P\cdot kK\gamma_\nu a]\,\frac{C(k^2)}{k^2(k^2-2P\cdot k)^2}$$

Since we have made no distinction between proton and neutron momentum the integral with three k's in the numerator is given by equation (50) with $P_y \to P$ and $P_y^2 = M^2$. Using the same substitution in the integrals of reference 5 we have after some simplification in the limit of large M

$$T_V = \frac{e^2\mu_N}{8\pi}\,[3\,\log\,(\,\frac{\Lambda}{M}\,) + \frac{1}{4}\,)\gamma_\nu + 2\gamma_\nu\bar{a}\,] \qquad (55)$$

In the standard representation the matrix element to be evaluated between proton and neutron states in diagram VI of figure 5 is

$$T_{VI} = \frac{ie\pi\mu_N\mu_P}{4M^2}\int \frac{d^4k}{(2\pi)^4}\,\frac{C(k^2)}{k^2}[\gamma_\mu K - K\gamma_\mu\,\frac{1}{\not P - K - M}\,\gamma_\nu a\,\frac{1}{\not P - K - M}\,(\gamma_\mu K - K\gamma_\mu)] \quad (56)$$

where both neutron and proton are again labeled by P. The matrix element T_{VI} is quadratically divergent so that we cannot use $C(k^2)$ given by equation (6). Instead the stronger convergence factor

$$C'(k^2) = \Lambda^4 (k^2 - \Lambda^2)^{-2} \tag{57}$$

is used. In order to perform the integrals in equation (56) we introduce auxilliary integrations over x and y so that

$$k^{-2}(k^2-2P\cdot k)^{-2}(k^2-\Lambda^2)^{-2} = \int_0^1 2x\,dx \int_0^1 12y(1-y)^2(k^2-2P\cdot ky-\Delta y)^{-5}dy$$

where

$$\Delta y = (1-y)x\Lambda^2$$

After carrying through a considerable amount of gamma-algebra T_{VI} can be written as

$$T_{VI} = \frac{ie^2\mu_N\mu_P}{2M\pi} \int \frac{d^4k}{(2\pi)^2} [(4P_\mu P_\nu \gamma_\alpha \bar{a} - 4MP_\mu g_{\alpha\nu}\bar{a} + 4M^2\gamma_\mu g_{\alpha\nu} a$$

$$- 2P_\alpha M(a + \bar{a})g_{\mu\nu} - 2MP_\mu \gamma_\alpha \gamma_\nu (a - \bar{a}))I_{\mu\nu}$$

$$+ (-4P_\mu g_{\nu\sigma}\gamma_\alpha \bar{a} - Mg_{\mu\nu}\gamma_\sigma\gamma_\gamma (a - \bar{a}) + 2Mg_{\alpha\mu}g_{\nu\sigma}(a + \bar{a}))I_{\mu\nu\sigma}$$

$$+ (a\gamma_\nu g_{\alpha\mu}g_{\sigma\tau})I_{\nu\sigma\tau}] \tag{58}$$

where

$$I_{\mu\nu} = \int \frac{d^4k}{(2\pi)^2} \frac{k_\mu k_\nu}{(k^2-2P\cdot k)^2} \frac{C(k^2)}{k^2}$$

$$= (8i)^{-1} [\frac{2P_\mu P_\nu}{M^2} - 2\, g_{\mu\nu} \log \frac{\Lambda^2}{M^2}]$$

$$I_{\mu\nu\sigma} = \int \frac{d^4k}{(2\pi)^2} \frac{C(k^2)}{k^2} \frac{k_\mu k_\nu k_\sigma}{(k^2-2P\cdot k)^2}$$

$$= (8i)^{-1}[P_\mu P_\nu P_\nu M^{-2} - \frac{1}{2}(P_\mu g_{\nu\sigma} + P_\nu g_{\mu\sigma} + P_\sigma g_{\mu\nu})(\log(\frac{\Lambda}{M}) - \frac{5}{4})$$

$$I_{\mu\nu\sigma\tau} = \int \frac{d^4k}{(2\pi)^2} \frac{k_\mu k_\nu k_\sigma k_\tau}{(k^2-2P\cdot k)^2} \frac{C'(k^2)}{k^2}$$

$$= (-8i)^{-1}[A_{\mu\nu\sigma\tau}(\frac{1}{3} \log\frac{\Lambda^2}{M^2} - \frac{13}{18}) - \frac{2}{3M^2} B_{\mu\nu\sigma\tau} - C_{\mu\nu\sigma\tau}(\frac{\Lambda^2}{4} - \frac{M^2}{3} \log\frac{\Lambda}{M} + \frac{1}{36})$$

with

$$g_{\sigma\tau}A_{\mu\,\sigma\tau} = 8P_\mu P_\nu + M^2 g_{\mu\nu}$$

$$g_{\sigma\tau}B_{\mu\nu\sigma\tau} = M^2 P_\mu P_\nu$$

$$g_{\sigma\tau}C_{\mu\nu\sigma\tau} = 6g_{\mu\nu}$$

Carrying through the indicated algebra in equation (58) yields

$$T_{VI} = \frac{e^2\mu_N\mu_P}{16}\Big[\Big(\frac{3}{2}\frac{\Lambda^2}{M^2} + \frac{23}{3}\log\frac{\Lambda}{M} + \frac{241}{36}\Big)\gamma_\mu\bar{a}$$

$$+ \Big(\frac{255}{9} + \frac{127}{3}\log\frac{\Lambda}{M}\Big)\gamma_\mu a \qquad (59)$$

Since we have assumed Λ is large compared to M then the main effect of diagrams III through VI of figure 5 is contained in equation (59) and is that the unperturbed β-interaction $\gamma_\mu a$ evaluated between proton and neutron states becomes

$$\gamma_\mu a + \frac{3}{32}\,\frac{e^2}{\pi}\,\mu_N\mu_P\,\frac{\Lambda^2}{M^2}\,\gamma_\mu\bar{a}$$

The added term to the interaction $\gamma_\mu a$ leads to an interference term in the energy spectrum and anisotropy in the decay of polarized neutrons. We can easily calculate this for neutrons at rest and polarized in the z-direction by using the additional projection operator $(\sigma_z + 1)/2$ on the initial free neutron state. The correction to the energy spectrum and anisotropy occasioned by the additional interaction is then given by

$$-\frac{3}{32}\,\frac{e^2G^2}{\pi}\,\frac{\Lambda^2}{M^2}\,\mu_N\mu_P\Big(\frac{1}{2\pi^3}\Big)[\varepsilon q - \vec{t}\cdot\vec{q} + q_z\varepsilon - qt_z][q\varepsilon d\varepsilon\, d(\cos\theta_q)\, d(\cos\theta_t)]$$

where q, $\vec{q}$ and ε, $\vec{t}$ are the energy and momentum of the neutrino and electron respectively.

By examination of the four matrix elements T_{III} thru T_{VI} we see that the only other new operator which results from the inclusion of the anomalous moments is in equation (53), i.e., the operator $P_\mu \bar{\bar{a}}$ taken between electron and neutron states. The energy spectrum and anisotropy from polarized neutrons calculated with the interaction are equal, in the limit of large nucleon mass M, to those of the interaction $\gamma_\mu \bar{\bar{a}}$ between proton and neutron. Hence we can say that in the limit of large M the interaction $\gamma_\mu \bar{\bar{a}}$ is the only new type of operator which arises when the anomalous moments are included.

Since we have calculated the corrections assuming Λ large compared to M it would certainly not be legitimate to use the results of the Stanford experiments which indicate that Λ is of order M. However we have taken $\Lambda \sim M$ in order that a rough estimate be given for the magnitude of the moment corrections. In that case the effect is approximately a 1/5% change in the lifetime, energy spectrum and isotropy.

It is worthwhile to mention that before assuming the corrections given here are the complete electromagnetic corrections it is necessary to examine how nuclear structure will affect these corrections. In the treatment of the neutron decay we have not included those intermediate states which involve the clothed nucleon. The effect of these states is however, even more important for a nucleus where the energy differences between the ground state and excited states would be small compared to the mass of the pion, whereas for an individual nucleon the first excited state would be expected to be approximately 140 Mev. above the ground state.

If we take the point of view that it is an individual nucleon in the nucleus that undergoes a β-decay and hence has the same electromagnetic renormalization as the neutron, then we can apply the log M/m correction to the β-decay of a nucleus. However, in view of the aforementioned facts on the effects of nuclear structure it is unknown as to whether the inclusion of these effects will contribute terms as large as the log M/m term. With this thought in mind we will apply the results of the neutron problem without the inclusion of structure effects to nuclear β-decay with the following conclusions.

There is an increase in the coulomb F factor independent of the nuclear charge Z and electron energy of amount 2.6% with approximately 1/2% uncertainty due to the energy dependent parts and contributions from the anomalous moment interactions. The result decreases the universal coupling constant obtained from O^{14} to $G = (1.37 \pm 0.02) \times 10^{-49}$ erg cm.^{3} and increases the preducted value of the muon lifetime from the value given above to $(2.33 \pm 0.05) \times 10^{-6}$ sec. while the experimental value is $(2.22 \pm 0.02) \times 10^{-6}$ sec. The disagreement between experiment and theory appears to be outside the limit of experimental error and might be regarded as an indication of the lack of universality even by the strangeness conserving part of the vector interaction. However, it is very difficult to understand the mechanism for such a slight deviation from universality, that is, if universality is to be broken at all why should it be by such a small amount? There are two possibilities that may account for this condition. First, the overlap in the O^{14} matrix element might be as low as 95 per cent. However, MacDonald(10) has examined this problem theoretically and he finds that the deviation from perfect overlap should be less than one per cent. Second, the inclusion of nuclear structure in the electromagnetic corrections could be in the direction as to offset the effect of the log M/m term and thus erase the apparent deviation from universality.

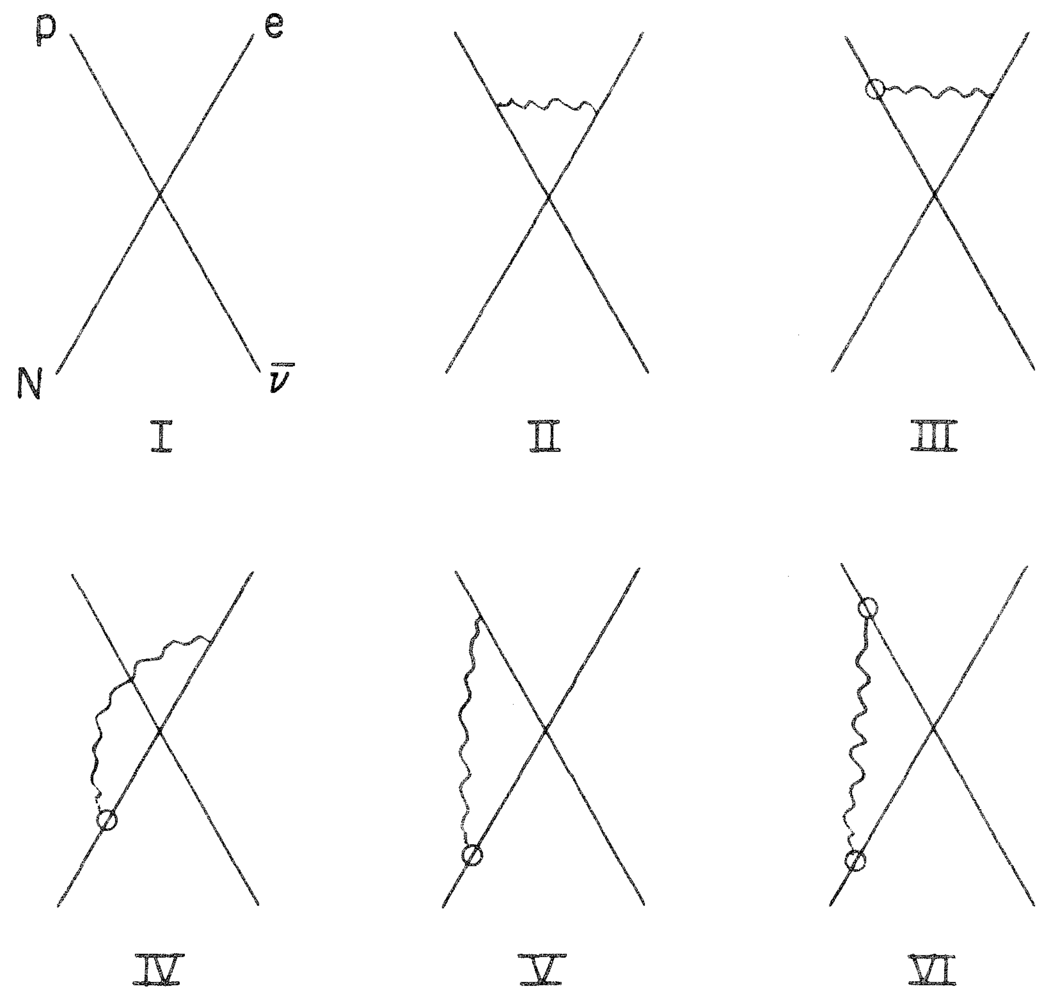

FIGURE 5. FEYNMAN DIAGRAMS FOR NEUTRON DECAY.

APPENDIX I

Evaluation of the Integral I

The integral I_1 appearing in Section IIA is

$$8iI_1 = \int_0^1 P_y^{-2} \log \left(\frac{P_y^2}{\lambda_m^2} \right) dy \tag{A1}$$

where

$$P_y^2 = (p_2 - qy)^2$$

In the rest system of particle one

$$P_y^2 = m_2^2 + 2(m_1 E_2 - m_2^2)y + q^2 y^2$$

With the substitution

$$y = b \coth \alpha - a$$

where

$$a = (m_1 E_2 - m_2^2)/q^2 \text{ and } b^2 = a^2 - m_2^2/q^2$$

I_1 may be expressed as

$$8iI_1 = -1/6q^2 \left[\int_{\alpha_1}^{\alpha_2} d\alpha \log \frac{b^2 q^2}{\lambda_m^2} - 2 \int_{\alpha_1}^{\alpha_2} \log \sinh \alpha \, d\alpha \right] \tag{A2}$$

where

$$2\alpha_1 = 2\coth^{-1}(a/b) = \log \left(\frac{a+b}{a-b} \right) = \log \left[\frac{m_1(E_2+p_2)-m_2^2}{m_1(E_2-p_2)-m_2^2} \right]$$

and

$$\alpha_2 = \coth^{-1}\left(\frac{1+a}{b} \right) = \frac{1}{2} \log \left[\frac{m_1^2-m_1(E_2-p_2)}{m_1^2-m_1(E_2+p_2)} \right]$$

The second integral of (A2) can be readily integrated by using the exponential form of $\sinh \alpha$, in which case

$$2\int_{\alpha_1}^{\alpha_2} d\alpha \log \sinh\alpha = (\alpha_2-\alpha_1)^2 + 2(\alpha_1 - \log 2)(\alpha_2-\alpha_1) + L(e^{-2\alpha}) - L(e^{-2\alpha_2}) \tag{A3}$$

where L is the Spence function defined by

$$L(x) = \int_0^x \frac{dz}{z} \log (1 - z)$$

Using the identity

$$b^2 q^2 = b^2 m_2^2 (a + b)^{-1} (a - b)^{-1}$$

and the relation

$$bq^2 = m_1 P_2$$

I_1 may be expressed as

$$8iI_1 = -\frac{1}{m_1 p_2} [(\alpha_2 - \alpha_1) \log \frac{m_2^2}{\lambda_m^2} + (\alpha_2 - \alpha_1) \log [\frac{b^2}{(a+b)^2}] + L(e^{-2\alpha_2})$$

$$-L(e^{-2\alpha_1}) - 2(\alpha_2 - \alpha_1)^2 + 2(\alpha_2 - \alpha_1) \log 2] \qquad \text{(A4)}$$

Employing the relation

$$2(\alpha_2 - \alpha_1) = \log \left(\frac{E_2 - p_2}{E_2 + p_2} \right)$$

and passing to the limit of $m_2 \to 0$ we obtain the result

$$8iI_1 = \frac{1}{m_1 p_2} [\log \left(\frac{2E_2}{m_2} \right) \log \left(\frac{m_2^2}{\lambda_m^2} \right) + \log^2 \left(\frac{2E_2}{m_2} \right) - L(1 - \frac{2E_2}{m_1})] \qquad \text{(A5)}$$

where we have used

$$\lim_{m_2 \to 0} e^{-2\alpha_1} = 0$$

$$\lim_{m_2 \to 0} \left(\frac{a+b}{b} \right)^2 = 4$$

APPENDIX II

Covariant Method for Integrating Over Neutrino Momentum

Consider an expression of the form

$$I = \int \delta(s^2)\delta(G^2-2s\cdot G)A\cdot s\ B\cdot s\ d^4s \qquad \text{(A6)}$$

$$= \int \delta(s^2)\delta(G^2-2s\cdot G)(A_4s_4-\vec{A}\cdot\vec{s})(B_4s_4-\vec{B}\cdot\vec{s})\ d^4s$$

We evaluate the integral in a coordinate system where G has only a time component and express the result in a covariant form. In this coordinate system the δ functions have no angular dependence so that integrating over the angles of s gives

$$I = 4\pi\int (A_4B_4 - \tfrac{1}{3}\vec{A}\cdot\vec{B})\delta(s_4^2-\vec{s}^2)\delta(G_4^2-2s_4G_4)ds_4s^2ds$$

Performing the integrations over s_4 and $|\vec{s}|$ use up the two δ functions and give

$$I = (\pi/24)G_4^2(A\cdot B + 2A_4B_4)$$

or rewriting in covariant form

$$I = \pi/24[(A\cdot B)G^2 + 2(A\cdot G)(B\cdot G)] \qquad \text{(A7)}$$

In a similar manner we determine

$$\int (A\cdot s)\delta(s^2)\delta(t^2)d^4s = (A\cdot G)\pi/4 \qquad \text{(A8)}$$

Equations (A7) and (A8) yield the right side of Equation (14) Section IIA.

APPENDIX III

Derivation of the Static Coulomb Correction, Equation (41) and the Coulomb F Function to Order e^2

The transition matrix element T_{Fo} for β-decay with coulomb interaction is given in second order perturbation theory by

$$T_{Fo} = \sum_n H_{Fn} H_{no} (E_o - \vec{\alpha} \cdot \vec{k} - \beta m - s)^{-1} \tag{A9}$$

where the β-decay has

$$H_{on} = (\text{Nuc})\ \bar{u}_e\ O\ u_\nu$$

and the coulomb interaction has γ_0 (?)

$$H_{Fn} = (\text{Nuc}) 4\pi z e^2 (\vec{p} - \vec{k})^{-2}\ \bar{u}(P) u(k)$$

and the energy release for a neutrino of energy s is

$$E_o = s + \varepsilon_k$$

$$\varepsilon^2 = p^2 + m^2, \quad \varepsilon_k^2 = k^2 + m^2$$

Since the detailed structure of the nuclear matrix elements is not relevant for the derivation of equation (41) we symbolize their appearance in H_{Fn} and H_{no} by the expression (Nuc). The O appearing in the β-decay matrix element is the operator describing this interaction. Both diagrams D and E of figure 3 are included in T_{Fo} providing we sum over both positive and negative values of the intermediate energy of the electron. From the expression for H_{Fn} and H_{no} we see that summing over spins of the intermediate states gives for the positive and negative energy contributions

$$\sum_{\text{spins}} [u_e^*(P)\ (p_4 - \vec{\alpha}\cdot\vec{k} - \beta m)^{-1} u_e(k) u_e^*(k) u_\nu(s)]$$

$$= \frac{\varepsilon_k + \vec{\alpha}\cdot\vec{k} - \beta m}{2\varepsilon_k(\varepsilon - \vec{\alpha}\cdot\vec{k} - \beta m)} \qquad \text{for } \varepsilon_k > 0 \qquad \text{(A10a)}$$

$$= \frac{\varepsilon_k - \vec{\alpha}\cdot\vec{k} - \beta m}{2\,\varepsilon_k\,(\varepsilon + \vec{\alpha}\cdot\vec{k} - \beta m)} \qquad \text{for } \varepsilon_k < 0 \qquad \text{(A10b)}$$

For $\varepsilon_k > 0$ the interaction is pictured by diagram D of figure 3 and corresponds to an electron scattered by a coulomb potential of strength ze^2. For $\varepsilon_k < 0$ the interaction is given by diagram E of figure 3 and corresponds to the coulomb potential taking an electron from the sea to a positive energy state. However we see that the strength of the coulomb potential is $(z - 1)e^2$. Making use of $(\vec{\alpha}\cdot\vec{k} + \beta m)u_e(k) = \varepsilon_k u_e(k)$ leads to equation (41).

If we disregard the fact that we should have strength z and strength $(z - 1)$ for diagrams D and E respectively then we can write T_{Fo} as

$$T_{Fo} = (\text{Nuc})4\pi ze^2 \int \frac{d^3q}{(2\pi)^3}\, u_e^*(P)\left[\frac{2\varepsilon - \vec{\alpha}\cdot\vec{q}}{2\vec{p}\cdot\vec{q} - \vec{q}^2}\right]\frac{O}{\vec{q}^2}\, u_\nu(s)$$

with integrals

$$\int \frac{d^3q}{(2\pi)^3}\ \frac{1}{(2\vec{p}\cdot\vec{q} - \vec{q}^2)\vec{q}^2} = (16\pi|\vec{p}|)^{-1}$$

$$\int \frac{d^3q}{(2\pi)^3}\ \frac{q_i}{(2\vec{p}\cdot\vec{q} - \vec{q}^2)\vec{q}^2} = 0 \quad (?)$$

Hence

$$2T_{Fo} = u_e^*\, O\, u_\nu \pi ze^2 N$$

whereas for the unperturbed case

$$T'_{Fo} = u_e^* O\, u_\nu$$

If we average and sum over initial and final spins of electron and neutrino and insert the density of states and normalization factors we have the coulomb F factor to order e^2.

APPENDIX IV

Spin Zero Model

In this appendix we calculate the radiative corrections to a β-decay process in which the nucleons are assumed to be spin zero particles while the electron and neutrino behave as before. The analogous spin zero interaction corresponding to γ_μ is given by Feynman[(5)] and hence the corresponding β-interaction Lagrangian is

$$L^o_{int} = \phi_2(p_{1\mu} + p_{2\mu})\phi_1 \bar{\Psi}_e \gamma_\mu a \Psi_{\bar{v}} \qquad \text{(A11)}$$

where p_1 and p_2 refer to the initial and final nucleus. It is impossible to construct an axial vector like $\gamma_\mu a$ for spin zero particles so that the interaction has only $(p_1 + p_2)_\mu$.

Consider a β-decay of a nucleus of charge z to a nucleus of charge $z + 1$ with the emission of an electron and antineutrino, i.e.,

$$z \to (z + 1) + e + \bar{v}$$

The electromagnetic corrections in first order in e^2 to the above decay are described by diagrams I through VI of figure 6. In addition to the diagrams encountered in the neutron decay there are three new types of diagrams IV, V and VI of figure 6. These result as a consequence of gauge invariance which requires the substitution

$$p \to (p - zeA)$$

in equation (A11) and hence we have five field vertices as in diagrams IV, V, VI.

The corresponding matrix elements are

$$T_I = -4\pi e^2 iz(z+1)\int \frac{d^4k}{(2\pi)^4}[(2P_{2\mu}-k_\mu)\frac{1}{(P_2-k)^2-M^2}(P_{2\nu}+P_{1\nu}-2k_\nu)\frac{1}{(P_1-k)^2-M^2}$$

$$(2P_{1\mu}-k_\mu)\ \frac{C(k^2)}{k^2}]\bar{\psi}_t\gamma_\nu a\psi_s \tag{A12}$$

where P_1 and P_2 are the momenta of the nuclei with charge z and $(z+1)$ respectively and t and s are the electron and neutrino momenta respectively.

$$T_{II} = 4\pi e^2 i(z+1)\int \frac{d^4k}{(2\pi)^4}[(2P_{2\mu}-k_\mu)(P_{2\nu}+P_{1\nu}-k_\nu)\frac{1}{(P_2-k)^2-M^2}\ \frac{C(k^2)}{k^2}$$

$$\bar{\psi}_t\gamma_\mu \frac{1}{\not{t}-\not{k}-m}\gamma_\nu a\psi_s] \tag{A13}$$

$T_{III} = T_{II}$ with $P_2 \leftrightarrow - P_1$ and $(z + 1) \to -z$

$$T_{IV} = (z+1)(2z+1)4\pi e^2 i\int \frac{d^4k}{(2\pi)^4}[(2P_{2\mu}-k_\mu)\frac{1}{(P_2-k)^2-M^2}\ \frac{C(k^2)}{k^2}$$

$$\bar{\psi}_t\gamma_\mu a\psi_s] \tag{A14}$$

$T_V = T_{IV}$ with $P_2 \leftrightarrow P_1$ and $(z + 1) \to z$

$$T_{VI} = 4\pi e^2 i(2z+1)\int \frac{d^4k}{(2\pi)^4}[\bar{\psi}_t\gamma_\mu \frac{1}{\not{t}-\not{k}-m}\ \gamma_\mu a\psi_s\ \frac{C(k^2)}{k^2}\ ] \tag{A15}$$

Since we are only interested in determining the corrections in the limit of large nuclear mass we will see here as in the neutron problem that terms of the type log M/m do not arise from integrals which have ultra-violet divergences. Hence for the purposes of evaluation the use of equation (6) for $C(k^2)$ in all the matrix elements will not affect the log M/m terms in the limit of large nuclear mass.

The last matrix element T_{VI} is just a multiple, $(2z' + 1)$, of the electron self energy. Hence

$$T_{VI} = \frac{e^2 m}{2\pi}\left[3 \log\frac{\Lambda}{m} + \frac{3}{4}\right](2z + 1)\bar{\Psi}_t a \Psi_s \tag{A16}$$

Since the unperturbed interaction is proportional to the nuclear momenta and T_{VI} is independent of the nuclear momentum it will not contribute in the limit of large nuclear mass.

T_I can be readily evaluated using the integrals of reference 5 and is especially simple in the limit of $M \to \infty$ in which case $P_1 = P_2$. In the limit we have for T_I

$$T_I = \frac{e^2}{2\pi} z(z+1)\ 2P_{2\nu}\left[- \log\frac{M^2}{\lambda_m^2} - \log\frac{\Lambda}{M} + \frac{7}{4}\right]\bar{\Psi}_t \gamma_\nu a \Psi_s \tag{A17}$$

From equation (A13) we see that the same integrals occurring in equations (38) appear in T_{II} with the additional integral given by equation (50). Of these only the integral of equation (38b) behaves as log (M/m) for large M.

Expressing T_{II} as

$$T_{II} = \frac{(z+1)ie^2}{\pi}\int\frac{d^4k}{(2\pi)^2}\Big[\{2P_{2\mu}(P_{2\nu}+P_{1\nu}) - 2P_{2\mu}k_\nu - k_\mu(P_{1\nu}+P_{2\nu})$$

$$+ k_\mu k_\nu\}\frac{1}{(k^2-2P_2\cdot k)(k^2+2t\cdot k)}\ \bar{\Psi}_t(2t_\mu + \gamma_\mu \not{k})\gamma_\nu a\Psi_s\Big]$$

we see that the part of T_{II} which behaves as $(P_1 + P_2)_\nu \log (M/m)$ in the limit of large M is

$$T_{II} \to (z + 1)\frac{ie^2}{\pi}\ 2P_{2\mu}(P_1+P_2)_\nu I_{2\sigma}\bar{\Psi}_t\gamma_\mu\gamma_\sigma\gamma_\nu a\Psi_s$$

where $I_{2\sigma}$ is given by equation (38b). Using this fact in the limit of large M there is only one component of $P_{1\mu}$ and $P_{2\mu}$ namely the time component we have

$$T_{II} \rightarrow (z+1)\frac{e^2}{2\pi}\log\frac{M}{m}(P_1+P_2)_\nu\bar{\Psi}_t\gamma_\nu a\Psi_s \tag{A18a}$$

and

$$T_{III} \rightarrow -\frac{z^2}{2\pi}\log\frac{M}{m}(P_1+P_2)_\nu\bar{\Psi}_t\gamma_\nu a\Psi_s \tag{A18b}$$

Using the integrals of reference 5 T_{IV} is readily determined as

$$T_{IV} = (z+1)(2z+1)\frac{e^2}{4\pi}\left[3\log\frac{\Lambda}{M}+\frac{9}{4}\right]P_{2\mu}\bar{\Psi}_t\gamma_\mu a\Psi_s$$

and

$$T_V = z(2z+1)\frac{e^2}{4\pi}\left[3\log\frac{\Lambda}{M}+\frac{9}{4}\right]P_{1\mu}\bar{\Psi}_t\gamma_\mu a\Psi_s$$

Both T_{IV} and T_V have no $\log M/m$ behavior in the limit of large M.

To complete the calculation we must now determine the contributions from wave function renormalization. We determine this by using the Ward identity[12]

$$\frac{\partial\Sigma}{\partial p_\mu} = i\Lambda_\mu(P,P) \tag{A19}$$

which relates the derivative of the self energy Σ to the vertex operator for zero momentum photons. For spin 1/2 particles

$$\Lambda_\mu(P,P) = 2\gamma_\mu R \tag{A20}$$

where in both the spin 1/2 and spin zero cases R represents the wave function renormalization. From equations (A19) and (A20) we see that R for the spin zero case is determined by

$$R = \frac{1}{2iM^2} P_\mu \frac{\partial\Sigma}{\partial p_\mu} = \frac{1}{2iM^2} P_\mu \frac{\partial \Delta M^2}{\partial p_\mu} \tag{A21}$$

Using the mass change ΔM for a spin zero particle as given by Feynman[5]

$$\sum = \frac{e^2}{\pi}\int \frac{d^2k}{(2\pi)^2}[(2P-k)^2 \frac{1}{(P-k)^2-M^2} - 4] \frac{C'(k^2)}{k^2} \tag{A22}$$

From equation (A21)

$$R = \frac{e^2}{\pi i M^2}\int \frac{d^4k}{(2\pi)^2}[\frac{2(2M^2-P\cdot k)}{k^2-2P\cdot k} - (\frac{M^2-P\cdot k}{k^2-2P\cdot k})(\frac{4M^2-2P\cdot k}{k^2-2P\cdot k} + 1)]\frac{C(k^2)}{k^2} \tag{A23}$$

where $C(k^2)$ is given by equation (6).

Executing the indicated integrals yields

$$R = \frac{e^2}{\pi}[\frac{3}{2}\log(\frac{\Lambda}{M}) - \log(\frac{M}{\lambda_m}) + 2] \tag{A24}$$

Wave function renormalization is seen to contribute some log M through the second term in R.

Collecting all the log (M/m) terms we have from T_I

$$-(e^2/\pi)z(z+1)\log(M/m)$$

from T_{II} and T_{III}

$$e^2/2\pi \log(M/m)$$

from the wave function renormalization of the nuclei of charge z and z+1

$$(z^2/2)(e^2/\pi)\log(M/m)\ ;\ (z+1)^2(e^2/2\pi)\log(M/m)$$

and from the wave function renormalization of the electron

$$-(e^2/4\pi)\log(M/m)$$

If we add these together we have

$$(3e^2/4\pi)\log(M/m)$$

which is precisely what the spin 1/2 case yielded in the limit of large nucleon mass.

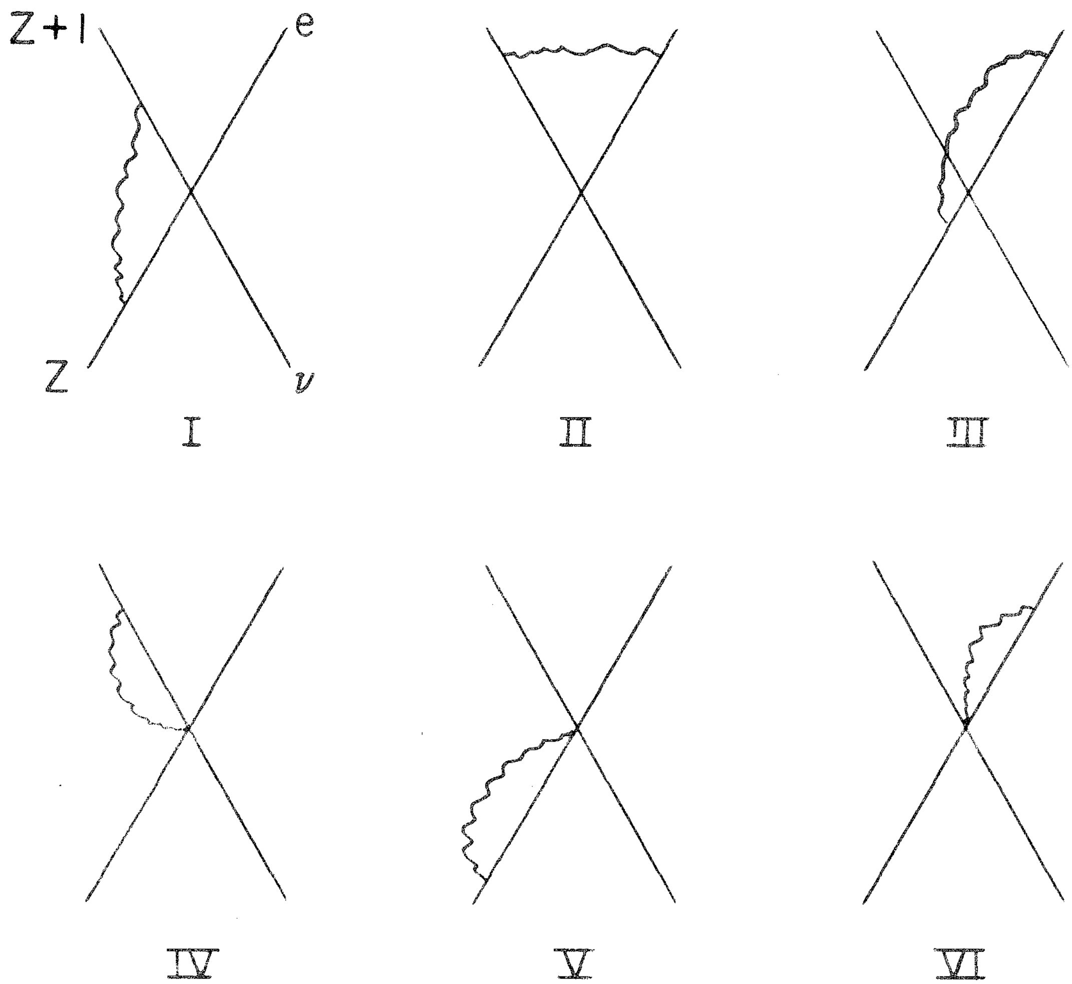

FIGURE 6. FEYNMAN DIAGRAMS FOR THE DECAY OF A SPIN ZERO NUCLEUS.

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