VARIATIONAL PRINCIPLES
AND APPLICATIONS
IN FINITE ELASTIC STRAIN THEORY

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ABSTRACT

The variational principles of finite elastostatic strain theory are presented in a unified manner for both compressible and incompressible bodies. Whereas the principle of stationary potential energy, a restricted case of the general principle of Hu and Washizu, is valid for any elastic deformation, it is found that the principle of stationary complementary energy is valid only for infinitesimal elastic strains. Consequently, Reissner's Theorem is the appropriate stationary principle to use in finite elastic strain theory when the complementary strain energy density is to be the argument function.

The potential energy principle is applied to several problems dealing with the finite straining of a neo-Hookean material. All but one of these problems are concerned with plane strain deformations; the one other problem, in a spherical geometry, involves an unusual stability question. Approximate solutions are obtained for some mixed boundary value problems which are not amenable to the semi-inverse methods of solution frequently used in finite elastic strain theory.

Another plane strain problem, requiring more detailed stress information than can be obtained from the potential energy principle, is studied approximately by means of Reissner's Theorem.

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CHAPTER I. VARIATIONAL PRINCIPLES FOR THE FINITE DEFORMATION OF A PERFECTLY ELASTIC SOLID

1. Introduction

In general, the field theories of physics may be presented in two alternative formulations. These are a) the appropriate field (differential) equations together with the associated boundary and initial conditions; b) a variational principle which is equivalent to the first formulation. Equivalence means that the Euler equations, boundary conditions, and initial conditions (if any) of the variational principle correspond to the field equations, boundary conditions, and initial conditions of the physical theory. [1] *

In some cases certain of the field equations, boundary conditions, and initial conditions of the theory are used as admissibility conditions. Then one constructs a restricted variational principle which will have the remaining field equations and conditions as its Euler equations and conditions.

Frequently the mathematical problem of solving the field equations, subject to given conditions, may be intractable. In that event the variational formulation of the theory may provide a means for obtaining an approximation to the desired solution. The Rayleigh-Ritz method is the best known of the so-called direct methods of the calculus of variations which are used to obtain such approximations. [2] In this method an approximating function(s) of known form, but containing free parameters, which satisfies the admissibility

*Numbers in brackets refer to references listed at the end of this thesis.

conditions is substituted into the functional of the variational principle and the functional is made stationary with respect to the free parameters. The problem is thereby reduced from a problem in the Calculus of Variations to a problem in the calculus of a function of a finite number of variables. The functional is thus given a value as close to its true value for the problem under consideration as is consistent with the approximating function(s). In this sense the variational principle provides one with a "best approximation."

The Rayleigh-Ritz method has proved to be of great value in the approximate solution of difficult problems in the infinitesimal theory of elasticity.* The literature does not seem to contain applications of the Rayleigh-Ritz method to problems of the theory of large elastic strains, i.e. problems involving physical nonlinearities. A correct statement of the variational principles governing the finite straining of a perfectly elastic solid is the first step which must be taken if such problems are to be treated by a Rayleigh-Ritz procedure. The approach taken is similar to that of Johnson [6] who formulated variational principles for a non-Newtonian fluid.

Reissner [7] made the first contribution toward this end with his theorem. There is a possible objection, however, in starting with a theorem which assumes that the complementary strain energy density may be written down explicitly. This possibility exists only if the constitutive equations for an elastic material can be inverted. As Truesdell and Toupin [8] have noted, it is not to be expected

* The principles involved and the methods used are discussed in great detail in the books by Sokolnikoff, [3] Timoshenko and Goodier, [4] and Hoff. [5]

that this inversion will be possible for an arbitrary, non-quadratic, strain energy density simply because of algebraic difficulties.

In view of this last point, it is decided to take as fundamental a theorem, due independently to Hu [9], Washizu [10], and the author, which is stated in terms of the strain energy density. This theorem can be transformed into Reissner's Theorem by means of a Legendre (contact) transformation when the constitutive equations are invertible. The potential energy theorems governing finite straining of both compressible and incompressible perfectly elastic media will be derived from appropriate forms of the fundamental theorem. Similar restrictions on Reissner's Theorem lead to the corresponding complementary energy theorems provided that only infinitesimal strains are considered; for finite strains the meaning of the functional obtained by restricting Reissner's Theorem is not clear. However, since the question is studied in some detail it is hoped that the difficulties alluded to by other workers in the field [11], [12], [13] will be clarified to the extent that further investigation of this question becomes possible.

2. The Fundamental Variational Principle for a Compressible Elastic Solid

In this section the very general variational principle of Hu and Washizu governing the static behavior of a perfectly elastic solid is presented. As in Reissner's Theorem, stresses, displacements, and strains are all allowed to vary independently. Only the symmetry of the stress tensor and sufficient regularity conditions to permit the required analytic operations are assumed. The variational principle then provides all the field equations and boundary conditions of the theory of finite elastic deformations.

Convected coordinates are used throughout this paper and, in general, the notation of Green and Zerna is used. [14]

Consider the functional

$$J = \iiint_{\tau} \Big\{ \frac{W(\gamma_{ij})}{\sqrt{III_M}} + \tau^{ij} \big[\tfrac{1}{2} (\overline{G}_i \cdot \overline{v}_{,j} + \overline{G}_j \cdot \overline{v}_{,i} - \overline{v}_{,j} \cdot \overline{v}_{,i}) - \gamma_{ij} \big] - \rho \Phi (\overline{v}) \Big\} \, d\tau - \iint_{S_t} \overline{v} \cdot \underset{\sim}{\overline{t}}_i \, dS_i - \iint_{S_v} (\overline{v} - \underset{\sim}{\overline{v}}) \cdot \overline{t}_i \, dS_i \tag{I.1}$$

where W, the strain energy density per unit volume of the undeformed body, is taken to be a function of the finite strain components, γ_{ij}, through the invariants $\{I_M, II_M, III_M\}$ of the mixed deformation tensor. $\sqrt{III_M} = d\tau/d\tau_o$ is the ratio of a volume element in the deformed body to that of a volume element in the undeformed body, and is the square root of the third invariant of the mixed deformation tensor

$$g^{ij} G_{jk} = M^{i}_{\cdot k}; \; III_M = \det M^{i}_{\cdot k} \, . \tag{I.2}$$

$$I_M = M^{i}_{\cdot i} = \operatorname{tr} M^{i}_{\cdot k} \tag{I.3}$$ is the first invariant of the mixed deformation tensor.

$$II_M = III_M \operatorname{tr} (M^{-1})^{i}_{\cdot k} \tag{I.4}$$ is the second invariant of the mixed deformation tensor.

τ^{ij} is the Cauchy-Green true stress tensor.

$\overline{v}$ is the displacement vector and has the prescribed value $\underset{\sim}{\overline{v}}$ on the part of the bounding surface S_v.

$\{\overline{G}_i\}$ are the base vectors in the deformed body and are related to the base vectors in the undeformed body by

$$\overline{G}_i = \overline{g}_i + \overline{v}_{,i} \tag{I.5}$$

G_{ij} is the metric tensor for the deformed body and is related to the metric tensor for the undeformed body by

$$G_{ij} = g_{ij} + \overline{g}_i \cdot \overline{v}_{,j} + \overline{g}_j \cdot \overline{v}_{,i} + \overline{v}_{,i} \cdot \overline{v}_{,j} \tag{I.6}$$

$\Phi(\overline{v})$ is the body force potential

$\overline{t}_i$ is the stress vector in the deformed body and is related to the stress tensor by

$$\overline{t}_i \sqrt{G^{ii}} = \tau^{i\ell}\, G_\ell \tag{I.7}$$

$\overline{t}_i$ has the value $\overline{\underset{\sim}{t}}_i$ on the part of the bounding surface S_t.

$S_t + S_v = S$ is the total bounding surface of the deformed body B.

In order to avoid any confusion as to what is meant by variational operations on (I.1) it is desirable to define the surface and volume elements in the undeformed body B_o and transform (I.1) into a functional defined with respect to B_o. If this is done (I.1) becomes

$$J = \iiint_{\tau_o} \{W(\gamma_{ij}) + S^{ij}[\tfrac{1}{2}(\overline{g}_i \cdot \overline{v}_{,j} + \overline{g}_j \cdot \overline{v}_{,i} + \overline{v}_{,i} \cdot \overline{v}_{,j}) - \gamma_{ij}]$$
$$- \rho_o \Phi(\overline{v})\} d\tau_o - \iint_{S_{o_t}} \overline{v} \cdot \overline{\underset{\sim}{t}}_{o_i} dS_{o_i} - \iint_{S_{o_v}} (\overline{v} - \underset{\sim}{\overline{v}}) \cdot \overline{t}_{o_i} dS_{o_i} \tag{I.8}$$

where $S^{ij} = \sqrt{III_M}\, \tau^{ij}$ (I.9)

is the engineering stress with components along the basis $\{\overline{G}_i\}$; $\overline{t}_{o_i}$ is the stress vector in the undeformed body and has the value $\overline{\underset{\sim}{t}}_{o_i}$ on the portion of the bounding surface of

B_o denoted by S_{a_t}.

$\overline{t}_{o_i}$ is related to $\overline{t}_i$ by

$$\overline{t}_{o_i} = \overline{t}_i \, dS_i/dS_{o_i} = \overline{t}_i \sqrt{III_M} \sqrt{G^{ii}/g^{ii}} \qquad (I.10)$$

Now form

$$\begin{aligned}\delta J = \iiint_{\tau_o} \{ & \frac{\partial W}{\partial \gamma_{ij}} \delta\gamma_{ij} + \delta S^{ij} [\tfrac{1}{2}(\overline{g}_i \cdot \overline{v}_{,j} + \overline{g}_j \cdot \overline{v}_{,i} + \overline{v}_{,i} \cdot \overline{v}_{,j}) - \gamma_{ij}] \\ & + S^{ij} [\tfrac{1}{2}(\overline{g}_i \cdot \delta\overline{v}_{,j} + \overline{g}_j \cdot \delta\overline{v}_{,i} + \overline{v}_{,i} \cdot \delta\overline{v}_{,j} + \overline{v}_{,j} \cdot \delta\overline{v}_{,i}) - \delta\gamma_{ij}] \\ & - \rho_o \overline{F} \cdot \delta\overline{v} \} \, d\tau_o - \iint_{S_{o_t}} \delta\overline{v} \cdot \overline{t}_{o_i} \, dS_{o_i} \\ & - \iint_{S_{o_v}} [\delta\overline{v} \cdot \overline{t}_{o_i} + (\overline{v} - \underline{v}) \cdot \delta t_{o_i}] \, dS_{o_i}\end{aligned} \qquad (I.11)$$

where

$$\delta\Phi = \overline{F} \cdot \delta\overline{v} \text{ ; } \overline{F} \text{ is the body force per unit mass of the body} \qquad (I.12)$$

Consider the following term in δJ

$$\begin{aligned}& \iiint_{\tau_o} \sqrt{g} \, S^{ij} \, (\overline{g}_j + \overline{v}_{,j}) \cdot \delta\overline{v}_{,i} \frac{d\tau_o}{\sqrt{g}} \\ & = \iiint_{\tau_o} [\sqrt{g} \, S^{ij} (\overline{g}_j + \overline{v}_{,j}) \cdot \delta\overline{v}]_{,i} \frac{d\tau_o}{\sqrt{g}} \\ & - \iiint_{\tau_o} [\sqrt{g} \, S^{ij} \, (\overline{g}_j + \overline{v}_{,j})]_{,i} \cdot \delta\overline{v} \frac{d\tau_o}{\sqrt{g}}\end{aligned} \qquad (I.13)$$

which becomes by Green's theorem, and by (I.5) and (I.10),

$$= \iint_{S_o} S^{ij}\, \overline{G}_j \cdot \delta\overline{v}\, \frac{dS_{o_i}}{\sqrt{g^{ii}}} - \iiint_{\tau_o} (\sqrt{G}\, \tau^{ij} G_j)_{,i} \cdot \delta\overline{v}\, \frac{d\tau_o}{\sqrt{g}} \qquad (I.14)$$

and by use of (I.7) and (I.9)

$$= \iint_S \overline{t}_i \cdot \delta\overline{v}\, dS_i - \iiint_\tau \frac{1}{\sqrt{G}} (\sqrt{G}\, \tau^{ij}\, \overline{G}_j)_{,i} \cdot \delta\overline{v} d\tau \qquad (I.15)$$

so that

$$\begin{aligned} \delta J = & \iiint_{\tau_o} \{(\frac{\partial W}{\partial \gamma_{ij}} - S^{ij})\delta\gamma_{ij} + \delta S^{ij}[\tfrac{1}{2}(\overline{g}_i \cdot \overline{v}_{,j} + \overline{g}_j \cdot \overline{v}_{,i} + \overline{v}_{,i} \cdot \overline{v}_{,j}) - \gamma_{ij}]\}\, d\tau_o \\ & - \iiint_\tau [\frac{1}{\sqrt{G}} (\sqrt{G}\, \tau^{ij}\, \overline{G}_j)_{,i} + \rho\, \overline{F}] \cdot \delta\overline{v}\, d\tau \\ & + \iint_{S=S_t+S_v} \overline{t}_i \cdot \delta\overline{v}\, dS_i - \iint_{S_t} \delta\overline{v} \cdot \underset{\sim}{\overline{t}_i}\, dS_i \\ & - \iint_{S_v} [\delta\overline{v} \cdot \overline{t}_i + (\overline{v} - \underset{\sim}{\overline{v}}) \cdot \delta\overline{t}_i]\, dS_i \end{aligned} \qquad (I.16)$$

In order for δJ to vanish it is necessary that

$$S^{ij} = \frac{\partial W}{\partial \gamma_{ij}} \qquad (I.17)$$

$$\gamma_{ij} = \tfrac{1}{2}(\overline{g}_i \cdot \overline{v}_{,j} + \overline{g}_j \cdot \overline{v}_{,i} + \overline{v}_{,i} \cdot \overline{v}_{,j})$$

or $\qquad (I.18)$

$$\gamma_{ij} = \tfrac{1}{2}(\overline{G}_i \cdot \overline{v}_{,j} + \overline{G}_j \cdot \overline{v}_{,i} - \overline{v}_{,i} \cdot \overline{v}_{,j})$$

$$\frac{1}{\sqrt{G}} (\sqrt{G}\ \tau^{ij}\ \overline{G}_j)_{,i} + \rho \overline{F} = 0$$

or

$$\frac{1}{\sqrt{g}} [\sqrt{g}\ S^{ij}\ (\overline{g}_j + \overline{v}_{,j})]_{,i} + \rho_o\ \overline{F} = 0 \qquad \text{(I.19)}$$

or

$$\frac{1}{\sqrt{g}} (\sqrt{g}\ t^{ij}\ \overline{g}_j)_{,i} + \rho_o\ \overline{F} = 0$$

where $$t^{ij} = S^{ir}\ [\delta^j_r - v^j|_r] \qquad \text{(I.20)}$$

is an unsymmetrical stress tensor referred to the $\{\overline{g}_i\}$ basis in B_o [15], [16].

In addition it is necessary that

on $$S_t : \overline{t}_i = \underset{\sim}{\overline{t}}_i \quad \text{or on} \quad S_{o_t} : \overline{t}_{o_i} = \underset{\sim}{\overline{t}}_{o_i} \qquad \text{(I.21)}$$

and on S_v or S_{o_v} : $\overline{v} = \underset{\sim}{\overline{v}}$ (I.22)

depending upon which basis $\overline{v}$ is referred to.

(I.17) is the constitutive equation for the deformation of a perfectly elastic solid. (I.18) correctly defines the strain tensor γ_{ij}. (I.19) are various representations of the vector equilibrium equation. (I.21) and (I.22) are the boundary conditions over the appropriate portions of the bounding surface of B_o or B. [15][16] Consequently the vanishing of δJ implies the complete set of relations governing the finite deformation of a perfectly elastic solid and, conversely, the equations of finite elasticity imply that $\delta J = 0$. This is what shall be called fundamental variational principle of finite elastostatics.

At this point it is noted that when the constitutive equation, (I.17), can be inverted, i.e. if γ_{ij} can be expressed as $\gamma_{ij}(S^{k\ell})$, then there exists a function $W_c(S^{ij})$ such that

$$W_c = S^{ij}\ \gamma_{ij}(S^{k\ell}) - W[\gamma_{ij}(S^{k\ell})] \qquad \text{(I.23)}$$

and $$\gamma_{ij}\ (S^{k\ell}) = \frac{\partial W_c}{\partial S^{ij}} \qquad \text{(I.24)}$$

(I.23), which defines the complementary strain energy density, is a Legendre (contact) transformation on W.[17]

3. Reissner's Theorem for a Compressible Solid

In this section it is assumed that (I.23) holds so that (I.8) may be written as

$$J = \iiint_{\tau_o} \{\tfrac{1}{2} S^{ij}(\overline{g}_i \cdot \overline{v}_{,j} + \overline{g}_j \cdot \overline{v}_{,i} + \overline{v}_{,i} \cdot \overline{v}_{,j}) - W_c(S^{ij}) - \rho_o \Phi(\overline{v})\}\ d\tau_o - \iint_{S_{o_t}} \overline{v} \cdot \overline{\underset{\sim}{t}}_{o_i}\ dS_{o_i} - \iint_{S_{o_v}} (\overline{v} - \underset{\sim}{\overline{v}}) \cdot \overline{t}_{o_i}\ dS_{o_i} \qquad \text{(I.25)}$$

In addition to the weak admissibility conditions of Section II it is also assumed that (I.18) holds. Then

$$J = \iiint_{\tau_o} \{S^{ij}\ \gamma_{ij} - W_c - \rho_o \Phi\}\ d\tau_o - \iint_{S_{o_t}} \overline{v} \cdot \overline{\underset{\sim}{t}}_{o_i}\ dS_{o_i} - \iint_{S_{o_v}} (\overline{v} - \underset{\sim}{\overline{v}}) \cdot \overline{t}_{o_i}\ dS_{o_i} \qquad \text{(I.26)}$$

In this case

$$\delta J = \iiint_{\tau_o} (\gamma_{ij} - \frac{\partial W_c}{\partial S^{ij}}) \delta S^{ij} \, d\tau_o$$

$$- \iiint_{\tau} [\frac{1}{\sqrt{G}} (\sqrt{G} \, \tau^{ij} \, \overline{G}_j)_{,i} + \rho \overline{F}] \cdot \delta \overline{v} d\tau \qquad (I.27)$$

$$- \iint_{S_t} \delta \overline{v} \cdot (\underset{\sim}{t_i} - \overline{t}_i) dS_i - \iint_{S_v} (\overline{v} - \underset{\sim}{v}) \cdot \delta \overline{t}_i dS_i$$

since

$$S^{ij} \delta \gamma_{ij} = S^{ij} (\overline{g}_j + \overline{v}_{,j}) \cdot \delta \overline{v}_{,i} \qquad (I.28)$$

The analysis is quite the same as that required to obtain (I.16).

The vanishing of δJ requires that (I.19), (I.21), (I.22), and (I.24) hold. Since (I.18) was assumed to hold, $\delta J = 0$ provides all the relations governing the finite deformations of an elastic solid in the case when $W_c(S^{ij})$ may be written explicitly. Since the converse is also true $\delta J = 0$ is a valid variational principle for finite elastic deformations. When J is written as in (I.26) the variational principle is called Reissner's Theorem. [7]

Reissner's Theorem is completely equivalent to the fundamental variational principle presented in Section 2 when the constitutive equation can be inverted explicitly; otherwise it is without meaning.

4. The Principles of Stationary Potential Energy and Stationary Complementary Energy for Compressible Solids

The theorems given in Sections 2 and 3 have the theoretical advantages of generality and elegance; this very generality may prove burdensome, however, in the actual solution of particular problems by one of the direct methods of the Calculus of Variations. This is because one would be required to choose approximations to all of the field simultaneously and thereby one might be pushed beyond the bounds of intuition and reasonable computational effort.

For this reason it is desirable to have principles which require that only one field, either the displacement field or the stress field, need be chosen in some trial form. The potential energy and complementary energy principles of infinitesimal elasticity have this feature.

The potential energy theorem has been established, in the past, by means of the principle of virtual work; [18] now it will be shown to be a restricted case of the fundamental theorem presented in Section 2.

Again consider the functional J with the argument function W. In addition to the requirements that τ^{ij} be symmetric and that all functions be sufficiently regular now it is required also that (I.17), (I.18), and (I.22) hold.

Then

$$J = \iiint_{\tau_o} \{W - \rho_o \Phi\} \, d\tau_o - \iint_{S_{o_t}} \overline{v} \cdot \overline{\underset{\sim}{t}}_{o_i} \, dS_{o_i} \tag{I.29}$$

so that the functional J can be computed if $\overline{v}$ be given.

$$\delta J = \iiint_{\tau_o} \{\frac{\partial W}{\partial \gamma_{ij}} \delta\gamma_{ij} - \rho_o \overline{F} \cdot \delta\overline{v}\} \, d\tau_o - \iint_{S_{o_t}} \overline{\underset{\sim}{t}}_{o_i} \cdot \delta\overline{v} \, dS_{o_i} \qquad (I.30)$$

After invoking (I.18) and performing the same analysis as in Section 2

$$\delta J = -\iiint_{\tau} [\frac{1}{\sqrt{G}} (\sqrt{G}\, \tau^{ij}\, \overline{G}_j)_{,i} + \rho \overline{F}] \cdot \delta\overline{v} d\tau - \iint_{S_t} (\overline{\underset{\sim}{t}}_i - \overline{t}_i) \cdot \delta\overline{v} dS \qquad (I.31)$$

The vanishing of δJ implies the relations of elasticity not taken as admissibility conditions and, conversely, if these relations hold $\delta J = 0$. This is the principle of stationary potential energy.

If (I.17) is called to mind it is clear that (I.30) is a statement of the principle of virtual work.

In order to apply the principle of stationary potential energy it is necessary only to know the form of the appropriate strain energy function in terms of a convenient deformation measure and to choose a trial deformation field which is continuous and satisfies (I.22), i.e. satisfies the displacement boundary conditions.

Clearly it would be desirable to have a variational principle in which only a trial stress field, satisfying (I.19) and (I.21), need be chosen.

Consider (I.26) and assume that only the stress tensor S^{ij} (or τ^{ij}) is varied. [8] Then

$$J = \iiint_{\tau_o} \{S^{ij}\gamma_{ij} - W_c\}\, d\tau_o - \iint_{S_{o_v}} (\overline{v} - \underline{v})\cdot \overline{t}_{o_i} dS_{o_i} \qquad (I.32)$$

since when δJ is formed the omitted terms contribute nothing.

It is not clear how (I. 32) is to be calculated since there are admissibility conditions only on the stress tensor; $\overline{v}$ is required to be continuous only. The following analysis must be considered, therefore, as purely formal.

Form

$$\delta J = \iiint_{\tau_o} \{\gamma_{ij} - \frac{\partial W_c}{\partial S^{ij}}\}\, \delta S^{ij} d\tau_o - \iint_{S_{o_v}} (\overline{v}-\underline{v})\cdot \delta\overline{t}_{o_i} d S_{o_i} \qquad (I.33)$$

The vanishing of δJ implies those relations of elasticity, (I. 22) and (I. 24), which have not been used as admissibility conditions. Conversely the complete set of relations of elasticity require that $\delta J = 0$. This establishes a principle of stationary complementary energy <u>in a purely formal sense only since, as has been indicated, there is no rule on how to choose $\overline{v}$</u>. This is the result indicated by Truesdell and Toupin.[8]

If (I. 32) is rewritten so as to take (I. 18) into account and an apparently extraneous term is included then

$$J = \iiint_{\tau_o} \{S^{ij}(\tfrac{1}{2})(\overline{g}_i\cdot\overline{v}_{,j} + \overline{g}_j\cdot\overline{v}_{,j} + \overline{v}_{,i}\cdot\overline{v}_{,j}) - W_c\} d\tau_o$$
$$- \iint_{S_{o_v}} (\overline{v}-\underline{v})\cdot \overline{t}_{o_i} dS_{o_i} - \iint_{S_{o_t}} \overline{v}\cdot \underline{t}_{o_i}\, dS_{o_i} \qquad (I.34)$$

Note that

$$\iiint_{\tau_o} S^{ij}(\tfrac{1}{2})(\overline{g}_i\cdot\overline{v}_{,j}+\overline{g}_j\cdot\overline{v}_{,i})d\tau_o = \iiint_{\tau_o} S^{ij}v_j|_i\, d\tau_o \tag{I.35}$$

because of the symmetry of the stress tensor. The right hand side of (I.35) is a divergence term and if Green's theorem is used it is found that (I.34) becomes

$$J = \iiint_{\tau_o}\{\tfrac{1}{2}S^{ij}\overline{v}_{,i}\cdot\overline{v}_{,j}-W_c\}d\tau_o+\iint_{S_{o_v}}\overline{\underset{\sim}{v}}\cdot\overline{t}_{o_i}dS_{o_i}-\iint_{S_{o_t}}v\cdot(\underset{\sim}{\overline{t}}_{o_i}-\overline{t}_{o_i})dS_{o_i} \tag{I.36}$$

The integral over S_{o_t} vanishes because (I.21) is assumed to hold, i.e.

$$J = \iiint_{\tau_o}\{\tfrac{1}{2}S^{ij}\,\overline{v}_{,i}\cdot\overline{v}_{,j}-W_c\}\,d\tau_o+\iint_{S_{o_v}}\overline{\underset{\sim}{v}}\cdot t_{o_i}\,dS_{o_i} \tag{I.36a}$$

It is no clearer how to calculate (I.36a) than it is to calculate (I.32) unless one makes the assumption that quadratic terms in displacement gradients are negligible, i.e. one passes over to the infinitesimal theory of elasticity. Then

$$J = -\iiint_{\tau_o}W_c\,d\tau_o+\iint_{S_{o_v}}\overline{\underset{\sim}{v}}\cdot\overline{t}_{o_i}dS_{o_i} \tag{I.37}$$

and the functional J now may be computed once an admissible stress field is chosen. (I.37) is, except for a change in sign, the functional considered in the classical principle of minimum complementary energy of infinitesimal elasticity.

(I.37), or even (I.32), may be obtained as a Legendre transformation on the principle of stationary potential energy as well as by restrictions on Reissner's Theorem. Since, for stable equilibrium, the potential energy principle is a minimum principle it is clear that (I. 37) leads to a maximum principle [17] and by changing signs, as is usually done, a minimum principle is obtained.

If (I.19) is appended to (I. 37) by means of a vector Lagrange multiplier, $\overline{v}$, then the appropriate Euler equations are obtained. [12]

It is now possible to understand the constant references in the literature, e.g. Hoff, [5] concerning the application of the complementary energy theorem to nonlinear elastic and even inelastic deformations. Obviously these comments apply to physical nonlinearities and not to geometrical nonlinearities. The use of a complementary energy principle may be justified, then, in the case of limited plastic deformation of metals whereas the application of such a principle to the large elastic deformations of thin bodies, i.e. plates, shells, beams, etc., is questionable, since such problems always involve geometrical nonlinearities.

5. The Fundamental Principle for an Incompressible Solid

As is well known, the incompressible elastic solid must be treated as a special case in the theory of finite elastic strain. [14, 19] This special case is, however, of great theoretical and practical interest. Most of the exact solutions of finite elastic problems given in the literature are limited to the incompressible solid and, of greater importance to the engineer, the only engineering materials capable of sustaining truly large strains are the almost incompressible rubberlike materials, natural and synthetic. In this section the fundamental variational principle for an incompressible elastic solid, corresponding to the principle for compressible solids of Section 2, is derived.

Consider, now, the functional

$$I = \iiint_{\tau} \{\frac{W(\gamma_{ij})}{\sqrt{III_M}} - \tau_d^{ij}[\tfrac{1}{2}(\overline{G}_i\cdot\overline{v}_{,j}+\overline{G}_j\cdot\overline{v}_{,i}-\overline{v}_{,i}\cdot\overline{v}_{,j})-\gamma_{ij}]$$

$$-\rho\Phi + \frac{k}{\sqrt{III_M}} \ \ell n \sqrt{III_M}\}d\tau - \iint_{S_t} \overline{v}\cdot\underset{\sim}{\overline{t}}_i \, dS_i \qquad \text{(I. 38)}$$

$$-\iint_{S_v} (\overline{v} - \underset{\sim}{\overline{v}})\cdot t_i \, dS_i$$

where τ_d^{ij} is essentially the stress deviator tensor.

k is a scalar invariant function of position.

$$\overline{t}_i\sqrt{G^{ii}} = \tau^{i\ell}\,\overline{G}_\ell = (\tau_d^{ij} + \frac{kG^{ij}}{\sqrt{III_M}})\,\overline{G}_j$$

so that $\frac{kG^{ij}}{\sqrt{III_M}}$ has the interpretation of being the hydrostatic portion of the stress tensor. k is called by some authors the hydrostatic pressure. [14]

All other symbols have been defined previously.

If the integrations are referred to the undeformed body B_o, then

$$I = \iiint_{\tau_o} \{W(\gamma_{ij}) + S_d^{ij}[\tfrac{1}{2}(\overline{g}_i\cdot\overline{v}_{,j}+\overline{g}_j\cdot\overline{v}_{,i}+\overline{v}_{,i}\cdot\overline{v}_{,j})-\gamma_{ij}]$$

$$-\rho_o\Phi + k\,\ell n\sqrt{III_M}\}d\tau_o - \iint_{S_{o_t}} \overline{v}\cdot\underset{\sim}{\overline{t}}_{o_i} dS_{o_i} \qquad \text{(I. 39)}$$

$$-\iint_{S_{o_v}} (\overline{v}-\underset{\sim}{\overline{v}})\cdot t_{o_i} dS_{o_i}$$

where S_d^{ij} is the deviator of the S^{ij} tensor.

Now form

$$\delta I = \iiint_{\tau_o} \{ \frac{\partial W}{\partial \gamma_{ij}} \delta\gamma_{ij} + \delta S_d^{ij} [\tfrac{1}{2}(\overline{g}_i \cdot \overline{v}_{,j} - \overline{g}_j \cdot \overline{v}_{,i} + \overline{v}_{,i} \cdot \overline{v}_{,j}) - \gamma_{ij}]$$

$$+ S_d^{ij} [\tfrac{1}{2}(\overline{g}_i \cdot \delta\overline{v}_{,j} + \overline{g}_j \cdot \delta\overline{v}_{,i} + \overline{v}_{,i} \cdot \delta\overline{v}_{,j} + \overline{v}_{,j} \cdot \delta\overline{v}_{,i}) - \delta\gamma_{ij}]$$

$$- \rho_o \overline{F} \cdot \delta\overline{v} + \delta k \ln\sqrt{III_M} + k\delta \ln\sqrt{III_M} \} d\tau_o \qquad \text{(I. 40)}$$

$$- \iint_{S_{o_t}} \delta\overline{v} \cdot \underset{\sim}{t}_{o_i} dS_{o_i} - \iint_{S_{o_v}} \delta\overline{v} \cdot \overline{t}_{o_i} dS_{o_i}$$

$$- \iint_{S_{o_v}} (\overline{v} - \underset{\sim}{v}) \cdot \delta t_{o_i} dS_{o_i}$$

The term $\delta \ln\sqrt{III_M}$ will be considered now.

$$\delta \ln\sqrt{III_m} = \tfrac{1}{2}\delta \ln III_M = \tfrac{1}{2} \frac{\delta III_M}{III_M} = \frac{\delta G}{2G} = \frac{1}{2G} \frac{\partial G}{\partial G_{ij}} \delta G_{ij} \qquad \text{(I. 41)}$$

but

$$\frac{\partial G}{\partial G_{ij}} = D^{ij} = GG^{ij} \quad [20] \qquad \text{(I. 42)}$$

where G is $\det(G_{ij})$

D^{ij} is the cofactor of G_{ij} in G

G^{ij} is the associated metric tensor in B

Therefore

$$\delta \ln\sqrt{III_M} = \tfrac{1}{2} G^{ij} \delta G_{ij} \qquad \text{(I. 43)}$$

or

$$\delta \ln\sqrt{III_M} = G^{ij} (\overline{g}_j + \overline{v}_{,j}) \delta\overline{v}_{,i} \qquad \text{(I. 44)}$$

due to the symmetry of G^{ij}. (I. 44) follows immediately from

Eq. (2.6.2) in Green and Zerna [14] and (I.5).

If (I.44) is substituted into (I.40), then it follows that

$$\delta I = \iiint_{\tau_o} \{ (\frac{\partial W}{\partial \gamma_{ij}} - S_d^{ij})\delta\gamma_{ij} - \delta S_d^{ij}[\tfrac{1}{2}(\overline{g}_i\cdot\overline{v}_{,j} + \overline{g}_j\cdot\overline{v}_{,i} + \overline{v}_{,i}\cdot\overline{v}_{,j}) - \gamma_{ij}]$$
$$+ (S_d^{ij} + kG^{ij})(\overline{g}_j + \overline{v}_{,j})\delta\overline{v}_{,i} - \rho_o\overline{F}\cdot\delta\overline{v} + \ell n\sqrt{III_M}\delta k\} d\tau_o$$
$$- \iint_{S_{o_t}} \delta\overline{v}\cdot\overline{\underset{\sim}{t}}_{o_i} dS_{o_i} - \iint_{S_{o_v}} \delta\overline{v}\cdot\overline{t}_{o_i} dS_{o_i} \tag{I.45}$$
$$- \iint_{S_{o_v}} (\overline{v} - \underset{\sim}{\overline{v}})\cdot\delta\overline{t}_{o_i} dS_{o_i}$$

After the usual manipulations it is found that

$$\delta I = \iiint_{\tau_o} \{ (\frac{\partial W}{\partial \gamma_{ij}} - S_d^{ij})\delta\gamma_{ij} + \delta S_d^{ij}[\tfrac{1}{2}(\overline{g}_i\cdot\overline{v}_{,j} + \overline{g}_j\cdot\overline{v}_{,i} + \overline{v}_{,i}\cdot\overline{v}_{,j}) - \gamma_{ij}]$$
$$- (\frac{[\sqrt{g}(S_d^{ij} + kG^{ij})(g_j + v_{,j})]_{,i}}{\sqrt{g}} + \rho_o\overline{F})\cdot\delta\overline{v} + \ell n\sqrt{III_M}\delta k\} d\tau_o \tag{I.46}$$
$$- \iint_{S_{o_t}} (\underset{\sim}{\overline{t}}_{o_i} - \overline{t}_{o_i})\cdot\delta\overline{v} dS_{o_i} - \iint_{S_{o_v}} (\overline{v} - \underset{\sim}{\overline{v}})\cdot\delta\overline{t}_{o_i} dS_{o_i}$$

The vanishing of δI implies, since all variations are taken independently, that

$$S_d^{ij} = \frac{\partial W(\gamma_{ij})}{\partial \gamma_{ij}} \tag{I.47}$$

$$\gamma_{ij} = \tfrac{1}{2}(\overline{g}_i\cdot\overline{v}_{,j} + \overline{g}_j\cdot\overline{v}_{,i} + \overline{v}_{,i}\cdot\overline{v}_{,j}) \tag{I.18}$$

$$\frac{1}{\sqrt{g}}[\sqrt{g}(S_d^{ij} + kG^{ij})(\overline{g}_j + \overline{v}_{,j})]_{,i} + \rho_o\overline{F} = 0 \tag{I.48}$$

but since $S_d^{ij} + kG^{ij} = S^{ij}$ (I.49) this is merely

$$\frac{1}{\sqrt{g}}\left[\sqrt{g}\, S^{ij}\, (\overline{g}_j + \overline{v}_{,j})\right]_{,i} + \rho_o \overline{F} = 0 \tag{I.19}$$

and may be written as either of the other two forms of (I.19) that were given previously.

$$\ell n \sqrt{III_M} = 0 \quad \text{or} \quad III_m = 1 \tag{I.50}$$

which is the incompressibility condition.

In addition it is necessary that the boundary conditions (I.21) and (I.22) hold.

$$\text{on } S_t : \overline{t}_i = \underset{\sim}{\overline{t}}_i \quad \text{or on } S_{o_t} : \overline{t}_{o_i} = \underset{\sim}{\overline{t}}_{o_i} \tag{I.21}$$

$$\text{on } S_v \quad \text{or } S_{o_v} : \overline{v} = \underset{\sim}{\overline{v}} \tag{I.22}$$

Note that for an incompressible material, since $III_M = 1$, one may write τ^{ij} or S^{ij} (or their deviators) interchangeably.

$$S^{ij} = \tau^{ij} = \frac{\partial W}{\partial \gamma_{ij}} + k\, G^{ij} \tag{I.51}$$

Thus all the equations governing the finite straining of an incompressible, perfectly elastic solid are implied by $\delta I = 0$ and, conversely, these equations imply $\delta I = 0$.

If (I.47) can be inverted to give $\gamma_{ij} = \gamma_{ij}(S_d^{ij})$ then, as in the case of a compressible solid, a complementary strain energy density, W_c, may be constructed by means of a Legendre transformation.

$$W_c(S_d^{ij}) = S_d^{ij}\, \gamma_{ij}(S_d^{ij}) - W[\gamma_{ij}(S_d^{ij})] \tag{I.52}$$

and $$\gamma_{ij} = \frac{\partial W_c(S_d^{ij})}{\partial S_d^{ij}} \tag{I.53}$$

6. Reissner's Theorem for an Incompressible Solid

Assume that (I. 52) holds so that

$$\begin{aligned} I = \iiint_{\tau_o} \{ & \frac{S_d^{ij}}{2} (\overline{g}_i \cdot \overline{v}_{,j} + \overline{g}_j \cdot \overline{v}_{,i} + \overline{v}_{,i} \cdot \overline{v}_{,j}) \\ & - W_c(S_d^{ij}) - \rho_o \Phi + k \ln\sqrt{III_M} \} d\tau_o \\ & - \iint_{S_{o_t}} \overline{v} \cdot \underset{\sim}{\overline{t}}_{o_i} dS_{o_i} - \iint_{S_{o_v}} (\overline{v} - \underset{\sim}{\overline{v}}) \cdot \overline{t}_{o_i} dS_{o_i} \end{aligned} \tag{I.54}$$

and if (I. 18) is taken as an admissibility condition

$$\begin{aligned} I = \iiint_{\tau_o} & \{ S_d^{ij} \gamma_{ij} - W_c - \rho \Phi + k \ln\sqrt{III_M} \} d\tau_o \\ & - \iint_{S_{o_t}} \overline{v} \cdot \underset{\sim}{\overline{t}}_{o_i} dS_{o_i} - \iint_{S_{o_v}} (\overline{v} - \underset{\sim}{\overline{v}}) \cdot \overline{t}_{o_i} dS_{o_i} \end{aligned} \tag{I.55}$$

Then

$$\begin{aligned} \delta I = \iiint_{\tau_o} & \{ (\gamma_{ij} - \frac{\partial W_c}{\partial S_d^{ij}}) \delta S_d^{ij} + S_d^{ij} \delta \gamma_{ij} - \rho_o \overline{F} \cdot \delta \overline{v} \\ & + \ln\sqrt{III_M} \delta k + k \delta \ln\sqrt{III_M} \} d\tau_o \\ & - \iint_{S_{o_t}} \delta \overline{v} \cdot \underset{\sim}{\overline{t}}_{o_i} dS_{o_i} - \iint_{S_{o_v}} (\overline{v} - \underset{\sim}{\overline{v}}) \cdot \delta \overline{t}_{o_i} dS_{o_i} \\ & - \iint_{S_{o_v}} \delta \overline{v} \cdot \overline{t}_{o_i} dS_{o_i} \end{aligned} \tag{I.56}$$

The now familiar operations lead to

$$\delta I = \iiint_{\tau_o} \{(\gamma_{ij} - \frac{\partial W_c}{\partial S_d^{ij}})\,\delta S_d^{ij} - (\frac{1}{\sqrt{g}}\,[\sqrt{g}\,(S_d^{ij}+kG^{ij})(\overline{g}_j+\overline{v}_{,j})]_{,i}$$

$$+ \rho_o\overline{F})\cdot\delta\overline{v} + \ln\sqrt{III_M}\,\delta k\}\,d\tau_o \qquad (I.57)$$

$$-\iint_{S_{o_t}} (\overline{t}_{o_i} - \overline{t}_{o_i})\cdot\delta\overline{v}\,dS_{o_i} - \iint_{S_{o_v}} (\overline{v}-\underline{v})\cdot\delta\overline{t}_{o_i}\,dS_{o_i}$$

The vanishing of δI implies

$$\gamma_{ij} = \frac{\partial W_c}{\partial S_d^{ij}} \qquad (I.58)$$

and (I.19), (I.21), (I.22), and (I.50). Thus, since (I.18) was taken as an admissibility condition, all the equations of elasticity are given when δI vanishes and all the equations of elasticity imply $\delta I = 0$. This is Reissner's Theorem for an incompressible, perfectly elastic solid.

Again, as in the case of a compressible solid, it is noted that Reissner's Theorem is completely equivalent to the fundamental theorem when the constitutive equation, (I.47), can be inverted.

7. The Principles of Stationary Potential Energy and Stationary Complementary Energy for Incompressible Solids

Consider, once more, the functional I with the argument function $W(\gamma_{ij})$. The admissibility conditions are now extended to include (I.18), (I.22), and (I.47) as well as $S_d^{ij} = S_d^{ji}$ and the required regularity conditions.

Then

$$I = \iiint_{\tau_o} \{W - \rho_o \Phi + k\, \ell n \sqrt{III_M}\} d\tau_o - \iint_{S_{o_t}} \overline{v} \cdot \overline{\underset{\sim}{t}}_{o_i} dS_{o_i} \tag{I.59}$$

If δI is formed and operated upon as usual it is found that

$$\delta I = - \iiint_{\tau_o} \{ (\frac{1}{\sqrt{g}} [\sqrt{g}\, (S_d^{ij} + kG^{ij})(\overline{g}_j + \overline{v}_{,j})]_{,i} + \rho_o \overline{F}) \\ - \ell n \sqrt{III_M}\, \delta k \} d\tau_o - \iint_{S_{o_t}} (\overline{\underset{\sim}{t}}_{o_i} - \overline{t}_{o_i}) \cdot \delta \overline{v} dS_{o_i} \tag{I.60}$$

The vanishing of δI implies (I.19), (I.21), and (I.50) which are the remaining relations of the theory of finite strain of an incompressible elastic solid. Conversely, (I.19), (I.21), and (I.50) imply $\delta I = 0$. This is the principle of stationary potential energy for an incompressible elastic solid.

Since a k field must be chosen to compute (I.59) a somewhat different formulation of the potential energy principle for incompressible solids is desirable. If only incompressible displacement fields are allowed as trial fields, i.e. $\sqrt{III_M} \equiv 1$, then the principle may be written as

$$\delta I = \delta [\iiint_{\tau_o} \{W - \rho_o \Phi\} d\tau_o - \iint_{S_{o_t}} v \cdot \overline{\underset{\sim}{t}}_{o_i} dS_{o_i}] = 0 \tag{I.61}$$

Although (I.61) appears to be formally the same as (I.29) it must be remembered that for the incompressible case (I. 47) and not (I.17) holds.

In the formal sense of Section 4, it is possible to establish a complementary energy principle for an incompressible elastic solid.

Consider (I. 54) and assume that (I. 18) and (I. 19) hold. Furthermore only S_d^{ij} and k are to be varied. Then

$$I = \iiint_{\tau_o} \{S_d^{ij}\gamma_{ij} - W_c + k \ln\sqrt{III_M}\} d\tau_o - \iint_{S_{o_v}} (\overline{v}-\underline{\overline{v}})\cdot \overline{t}_{o_i} dS_{o_i} \quad \text{(I. 62)}$$

and when the formal variation, indicated above, is performed it is found that

$$\delta I = \iiint_{\tau_o} \{(\gamma_{ij} - \frac{\partial W}{\partial S_d^{ij}})\delta S_d^{ij} + \delta k \ \ln\sqrt{III_M}\} \ d\tau_o - \iint_{S_{o_v}} (\overline{v}-\underline{\overline{v}})\cdot \delta\overline{t}_{o_i} dS_{o_i} \quad \text{(I. 63)}$$

$\delta I = 0$ if and only if (I. 22), (I. 47), and (I. 50) hold. Since these are the remaining relations of elasticity the formal principle of stationary complementary energy is established.

An analysis similar to that of Section 4 shows that the functional (I. 62) may be written as

$$I = \iiint_{\tau_o} \{\tfrac{1}{2} S_d^{ij} \overline{v}_{,i}\cdot\overline{v}_{,j} - W_c + k \ \ln\sqrt{III_M}\} d\tau_o + \iint_{S_{o_v}} \underline{\overline{v}}\cdot\overline{t}_{o_i} dS_{o_i} \quad \text{(I. 64)}$$

If it is possible to express

$$S_d^{ij} = S_d^{ij}(S^{k\ell}) \ , \quad \text{(I. 65)}^*$$

---
*Reissner has shown this to be possible for a neo-Hookean material. [7]

i.e. k can be found as a function of $S^{k\ell}$, and if (I.65) implies that $\ell n\sqrt{III_M} \equiv 0$, then it is possible to write

$$I = \iiint_{\tau_o} \{\tfrac{1}{2} S_d^{ij}(S^{k\ell}) \bar{v}_{,i} \cdot \bar{v}_{,j} - W_c [S_d^{ij}(S^{k\ell})]\} d\tau_o + \iint_{S_{o_v}} \bar{\underset{\sim}{v}} \cdot \bar{t}_{o_i} \, dS_{o_i} \tag{I.66}$$

(I.66) can be viewed in the same manner as (I.36a) or, in other words, the meaning of (I.66) is only clear for the case of infinitesimal strains.

8. Concluding Remarks and Summary

A very general variational principle, called the fundamental principle, for finite strain of both compressible and incompressible elastic solids has been presented. Aside from regularity requirements the principle has as its only admissibility condition that the true stress tensor be symmetric, i.e. moment equilibrium is enforced. Stress, displacement, and strain fields are varied independently. From this principle one obtains all the relations governing the finite deformation of a perfectly elastic solid. The functional used in the principle has the strain energy density, W, as an argument function and hence the principle is valid for all perfectly elastic solids.

Reissner's Theorem, involving the complementary strain energy density, W_c, is obtained from the fundamental variational principle with the aid of a Legendre (contact) transformation. Since there is no a priori reason to believe that this transformation can be performed for an arbitrarily given strain energy density, Reissner's Theorem may possibly be of more limited applicability than the fundamental principle.

By imposing suitable restrictions one obtains the principle of stationary potential energy from the fundamental principle. If, in a similar manner, one attempts to obtain the principle of stationary complementary energy from Reissner's Theorem it is discovered that the desired result is obtained only if the strains are taken to be infinitesimal; for finite strains the resulting functional apparently cannot be computed in a precisely defined manner. This question warrants further study.

For convenience the various principles are grouped below.

Compressible Solid

Fundamental Principle

$$\delta\Big[\iiint_{\tau_o}\{W + S^{ij}[\tfrac{1}{2}(\bar{g}_i\cdot\bar{v}_{,j}+\bar{g}_j\cdot\bar{v}_{,i}+\bar{v}_{,i}\cdot\bar{v}_{,j})-\gamma_{ij}] - \rho_o\Phi\}\,d\tau_o - \iint_{S_{o_t}}\bar{v}\cdot\bar{\underset{\sim}{t}}_{o_i}\,dS_{o_i} - \iint_{S_{o_v}}(\bar{v}-\underset{\sim}{\bar{v}})\cdot\bar{t}_{o_i}\,dS_{o_i}\Big] = 0 \qquad (I.8)^*$$

Reissner's Theorem

$$\delta\Big[\iiint_{\tau_o}\{S^{ij}\gamma_{ij}-W_c-\rho_o\Phi\}\,d\tau_o - \iint_{S_{o_t}}\bar{v}\cdot\bar{\underset{\sim}{t}}_{o_i}\,dS_{o_i} - \iint_{S_{o_v}}(\bar{v}-\underset{\sim}{\bar{v}})\cdot\bar{t}_{o_i}\,dS_{o_i}\Big] = 0 \qquad (I.26)$$

* Equation numbers correspond to equations defining the appropriate functionals in text.

Principle of Stationary Potential Energy

$$\delta \left[\iiint_{\tau_o} \{W - \rho_o \Phi\} \, d\tau_o - \iint_{S_{o_t}} \bar{v} \cdot \bar{\underset{\sim}{t}}_{o_i} dS_{o_i} \right] = 0 \tag{I. 29}$$

Principle of Stationary Complementary Energy (infinitesimal strain)

$$\delta \left[-\iiint_{\tau_o} W_c \, d\tau_o + \iint_{S_{o_v}} \bar{\underset{\sim}{v}} \cdot \bar{t}_{o_i} dS_{o_i} \right] = 0 \tag{I. 37}$$

Incompressible Solid

Fundamental Principle

$$\delta \left[\iiint_{\tau_o} \{W + S_d^{ij} [\tfrac{1}{2}(\bar{g}_i \cdot \bar{v}_{,j} + \bar{g}_j \cdot \bar{v}_{,i} + \bar{v}_{,i} \cdot \bar{v}_{,j}) - \gamma_{ij}] \right.$$

$$- \rho_o \Phi + k \ln \sqrt{III}_M \} d\tau_o - \iint_{S_{o_t}} \bar{v} \cdot \bar{\underset{\sim}{t}}_{o_i} dS_{o_i} \tag{I. 39}$$

$$\left. - \iint_{S_{o_v}} (\bar{v} - \bar{\underset{\sim}{v}}) \cdot \bar{t}_{o_i} dS_{o_i} \right] = 0$$

Reissner's Theorem

$$\delta \left[\iiint_{\tau_o} \{S_d^{ij} \gamma_{ij} - W_c - \rho \Phi + k \ln \sqrt{III}_M\} \, d\tau_o \right.$$

$$\left. - \iint_{S_{o_t}} \bar{v} \cdot \bar{\underset{\sim}{t}}_{o_i} dS_{o_i} - \iint_{S_{o_v}} (\bar{v} - \bar{\underset{\sim}{v}}) \cdot \bar{t}_{o_i} dS_{o_i} \right] = 0 \tag{I. 55}$$

Principle of Stationary Potential Energy

$$\delta \left[\iiint_{\tau_o} \{W - \rho_o \Phi\} \, d\tau - \iint_{S_{o_t}} \bar{v} \cdot \bar{\underset{\sim}{t}}_{o_i} dS_{o_i} \right] = 0 \tag{I. 61}$$

provided that only incompressible displacement fields are admitted as trial displacement fields

Principle of Stationary Complementary Energy (infinitesimal strain)

(I.37) may be used for a Hookean solid provided that Poisson's ratio is given the value 1/2. If a nonlinear stress-strain law is assumed then

$$\delta\left[-\iiint_{\tau_o} W_c[S^{ij}_d(S^{k\ell})]\, d\tau_o + \iint_{S_{o_V}} \bar{\underset{\sim}{v}}\cdot \bar{t}_{o_i}\, dS_{o_i}\right] = 0 \qquad \text{(I.67)}$$

CHAPTER II. PLANE STRAIN OF A MOONEY-RIVLIN MATERIAL

1. Introduction

The purpose of this chapter is to introduce the ideas and formulas that will be useful for the variational solution, exact or approximate, of plane strain problems of a Mooney-Rivlin material. It will be shown that in plane strain there is no distinction between a Mooney-Rivlin material and a neo-Hookean material.

The question of choosing incompressible deformation fields is also given some attention in this chapter.

2. Plane Strain of a Mooney-Rivlin Solid

In this section it is assumed that all deformations occur in a plane and that the direction normal to the plane may be considered as a principal direction of the deformation. A rectangular Cartesian coordinate system, (x_i), is associated with material points in the undeformed body B_o such that the deformations occur in the $x_1 - x_2$ plane. When convenient, Greek indices taking only the values 1 and 2 will be used.

As the body undergoes deformation the point (x_1, x_2) moves to (X_1, X_2) in the deformed body B and $x_3 = X_3$. It is possible to write, in a common rectangular frame, that

$$X_\alpha = x_\alpha + v_\alpha \qquad \text{(II.1)}$$

where v_α are components of the displacement vector $\bar{v}$.

The metric and associated metric tensors in the undeformed body are given by

$$g_{ij} = g^{ij} = \delta_{ij} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \text{(II. 2)}$$

In the present case it is convenient to write

$$x_1 = x; \quad x_2 = y$$
$$X_1 = X; \; X_2 = Y \qquad \text{(II. 3)}$$

and to denote partial differentiation by subscripts. If a convected coordinate system is associated with the rectangular coordinate system in the undeformed body, and the body is considered to be incompressible, it is possible to write the metric and associated metric tensors in the deformed body, B, as

$$G_{ij} = \begin{pmatrix} X_x^2 + Y_x^2 & X_xX_y + Y_xX_y & 0 \\ X_xX_y + Y_xY_y & X_y^2 + Y_y^2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \text{(II. 4)}$$

$$G^{ij} = \begin{pmatrix} X_y^2 + Y_y^2 & -(X_xX_y + Y_xY_y) & 0 \\ -(X_xX_y + Y_xY_y) & X_x^2 + Y_x^2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \text{(II. 5)}$$

since $G = g = 1$ for an incompressible body in the coordinate systems considered. In general, it should be noted that the notation of Green and Zerna [12] will be used.

The Mooney-Rivlin material is characterized by the strain energy density

$$W = \frac{\mu}{2}\left[(1-C)(I_M-3) + C(II_M-3)\right]; \; III_M = 1 \qquad \text{(II.6)}$$

where μ is the classical shear modulus.

C is a material constant.

If λ_1, λ_2, λ_3 are the principle extension ratios of a deformation then

$$\begin{aligned} I_M &= \lambda_1^2 + \lambda_2^2 + \lambda_3^2 \\ II_M &= \lambda_1^2\lambda_2^2 + \lambda_1^2\lambda_3^2 + \lambda_2^2\lambda_3^2 \\ III_M &= \lambda_1^2\lambda_2^2\lambda_3^2 \end{aligned} \qquad \text{(II.7)}$$

In the case of incompressible plane strain

$$\lambda_3^2 = 1 \; ; \quad \lambda_2^2 = 1/\lambda_1^2 \qquad \text{(II.8)}$$

so that

$$\begin{aligned} I_M &= \lambda_1^2 - \lambda_2^2 + 1 \\ II_M &= 1 + \lambda_1^2 + \lambda_2^2 = I_M \\ III_M &= 1 \end{aligned} \qquad \text{(II.9)}$$

and the Mooney-Rivlin strain energy density becomes

$$W = \frac{\mu}{2}(I_M-3); \; III_M = 1 \qquad \text{(II.10)}$$

which is identical in form to the neo-Hookean strain energy density, [19] i.e., in plane strain there is no distinction between Mooney-Rivlin and neo-Hookean materials.

Rivlin [19] has shown that for the neo-Hookean material it is possible to write

$$W = \mu g^{rs} \gamma_{rs} = \mu \gamma_{ii} \tag{II.11}$$

in rectangular Cartesian coordinates.

$$\gamma_{ij} = \tfrac{1}{2}(G_{ij} - g_{ij}) \tag{II.12}$$

is the strain tensor.

It is found, then, that

$$W = \frac{\mu}{2}(X_x^2 + X_y^2 + Y_x^2 + Y_y^2 - 2); \; (X_x Y_y - X_y Y_x) = 1 \tag{II.13}$$

Introduce the unsymmetrical, nominal stress tensor t^{ij} which is referred to the base vectors in the undeformed body. [15] If no body forces are acting the stress equilibrium equations are

$$t^{ij}\big|_i = 0 \tag{II.14}$$

where $t^{ij} = \sqrt{III_M}\, \tau^{ir}(\delta^j_r + v^j|_r)$ and τ^{ir} is the Cauchy-Green true stress tensor.

For an incompressible material

$$t^{ij} = t^{ij}_d + kG^{ir}\,(\delta^j_r + v^j|_r) \tag{II.15}$$

where t^{ij}_d is the deviator of t^{ij} and

k is a hydrostatic pressure term [14]

In rectangular Cartesian coordinates

$$(\delta^j_r + v^j|_r) = \partial X_j/\partial x_r \tag{II.16}$$

$$t^{ij}_d = \frac{\partial W}{\partial(v_j|_i)}[13] = \frac{\partial W}{\partial(X_{j,i})}\frac{\partial(X_{j,i})}{\partial(v_{j,i})} = \frac{\partial W}{\partial(X_{j,i})} \tag{II.17}$$

so that (II.14) may be written as

$$\left[\frac{\partial W}{\partial(X_{\beta,\alpha})} + kG^{\alpha\gamma}\frac{\partial X^\beta}{\partial x_\gamma}\right]_{,\alpha} = 0 \tag{II.18}$$

for the case of plane strain.

In particular, for the Mooney-Rivlin material and $\beta = 1$, (II.18) may be written as

$$\frac{\partial}{\partial x}\left[\frac{\partial W}{\partial(X_x)}+k(G^{11}X_x+G^{12}X_y)\right] + \frac{\partial}{\partial y}\left[\frac{\partial W}{\partial(X_y)}+k(G^{21}X_x+G^{22}X_y)\right]=0 \tag{II.19}$$

From (II.13) it is found that

$$\frac{\partial W}{\partial(X_x)} = \mu X_x\ ;\ \frac{\partial W}{\partial(X_y)} = \mu X_y \tag{II.20}$$

With the use of (II.5) and (II.20) it is possible to write (II.19) as

$$\frac{\partial}{\partial x}[\mu X_x + k\{(X_y^2 + Y_y^2)X_x - (X_xX_y + Y_xY_y)X_y\}]$$
$$+ \frac{\partial}{\partial y}[\mu X_y + k\{(X_x^2 + Y_x^2)X_y - (X_xX_y+Y_xY_y)X_x\}] = 0 \tag{II.21}$$

If the incompressibility condition (II.13.2) is used (II.21) becomes

$$\frac{\partial}{\partial x}[\mu X_x + kY_y] + \frac{\partial}{\partial y}[\mu X_y - kY_x] = 0 \tag{II.22}$$

and finally

$$\nabla^2 X = 1/\mu [k_y Y_x - k_x Y_y] \tag{II.23}$$

In a similar manner it is found that

$$\nabla^2 Y = 1/\mu \ [k_x X_y - k_y X_x] \tag{II.24}$$

when $\beta = 2$.

Call the boundary of B_o in the x-y plane S_o and denote the unit normal vector to S_o by $\overline{n}_o$. Then if n_{o_α} are the components (direction cosines) of $\overline{n}_o$ in the coordinate directions

$$t^{\alpha\beta} \ n_{o_\alpha} = [\frac{\partial W}{\partial(X_{\beta,\alpha})} + kG^{\alpha\gamma} \ \frac{\partial X_\beta}{\partial x_\gamma}] \ n_{o_\alpha} = \overline{t}_{o_\beta} \tag{II.25}$$

represents the surface traction in the β direction. For the Mooney-Rivlin material (II.25) becomes in the x direction

$$\overline{t}_{o_x} = [(\mu X_x + k \ Y_y)n_{o_x} + (\mu X_y - kY_x)n_{o_y}] \tag{II.26}$$

and in the y direction

$$\overline{t}_{o_y} = [(\mu Y_x - kX_y)n_{o_x} + (\mu Y_y + kX_x)n_{o_y}] \tag{II.27}$$

where it is understood that the Greek subscripts on the n_{o_α} refer to directions rather than to partial derivatives.

The boundary conditions on S_o are that at each point of S_o either $\overline{t}_o$, the stress vector with components $\overline{t}_{o_\alpha}$, or the displacement vector $\overline{v}$ is given. If $\overline{v}$ is given then equivalently, of course, X_α is given.

It will now be shown that (II.23), (II.24), (II.25), and (II.26) are a consequence of the principle of stationary potential energy. It is again assumed that no body forces act.

For the purpose at hand the principle may be stated as

$$\delta\left[\iint_{\tau_o} W^{*} dxdy - \int_{S_{o_t}} \overline{t}_o \cdot \overline{v}\, dS_o\right] = 0 \tag{II.28}$$

where

$$W^{*} = W + k\,(X_x Y_y - X_y Y_x - 1) \tag{II.29}$$

is the strain energy density modified to include the incompressibility condition (II.13.2) through the use of the Lagrange multiplier k.

S_{o_t} is the portion of S_o on which the stress is prescribed.

(II.28), when written out, becomes

$$\begin{aligned}\delta\Big[\iint_{\tau_o} &\Big[\frac{\mu}{2}(X_x^2+X_y^2+Y_y^2+Y_x^2-2) + k(X_xY_y-X_yY_x-1)\Big]\,dxdy \\ &-\int_{S_{o_t}} [\overline{t}_{o_x}(X-x) + \overline{t}_{o_y}(Y-y)]\,dS_o\Big] = 0\end{aligned} \tag{II.30}$$

When the indicated variations are performed it is found that

$$\begin{aligned}\iint_{\tau_o} &[\mu(X_x\delta X_x+X_y\delta X_y+Y_y\delta Y_y+Y_x\delta Y_x) \\ &+k(x,y)(Y_y\delta X_x+X_x\delta Y_y-Y_x\delta X_y-X_y\delta Y_x)]\,dxdy \\ &-\int_{S_{o_t}} [\overline{t}_{o_x}\delta X + t_{o_y}\delta Y]\,dS_o = 0\end{aligned} \tag{II.31}$$

Assume that X and Y, i.e., $\bar{v}$, are sufficiently regular so that

$$X_{yx} = X_{xy} \; ; \quad Y_{yx} = Y_{xy} \tag{II. 32}$$

Then after performing the indicated integrations by parts (or really by use of Green's Theorem) it is found that

$$\begin{aligned}
-\iint_{\tau_o} \{ &[\mu\nabla^2 X + k_x Y_y - k_y Y_x]\,\delta X + [\mu\nabla^2 Y + k_y X_x \\
&- k_x X_y]\,\delta Y\}\, dxdy + \int_{S_{o_t}} \{[(\mu X_x + kY_y) n_{o_x} \\
&+ (\mu X_y - kY_x) n_{o_y} - \bar{\underset{\sim}{t}}_{o_x}]\,\delta X + [(\mu Y_x - kX_y) n_{o_x} \\
&+ (\mu Y_y + kX_x) n_{o_y} - \bar{\underset{\sim}{t}}_{o_y}]\,\delta Y\}\, dS_o + \int_{S_{o_v}} \{[(\mu X_x + kY_y) n_{o_x} \\
&+ (\mu X_y - kY_x) n_{o_y}]\,\delta X + [(\mu Y_x - kX_y) n_{o_x} \\
&+ (\mu_{Y_y} + kX_x) n_{o_y}]\,\delta Y\}\, dS_o = 0
\end{aligned} \tag{II. 33}$$

where S_{o_v} is the portion of S_o on which the displacement $\bar{v}$ is prescribed.

(II. 33) implies the equilibrium equations (II. 23) and (II. 24) as well as the natural boundary conditions, making use of (II. 26) and (II. 27), that

$$\text{on} \quad S_{o_t} : \bar{t}_{o_x} = \bar{\underset{\sim}{t}}_{o_x} \; ; \; \bar{t}_{o_y} = \bar{\underset{\sim}{t}}_{o_y} \tag{II. 34}$$

$$\text{on} \quad S_{o_v} : \bar{v} = \bar{\underset{\sim}{v}} \; ; \; \text{i.e.} \;\; \delta X = \delta Y = 0 \tag{II. 35}$$

Now the Lagrange multiplier k will be eliminated from the equilibrium equations to give one equation in X and Y which together with (II.13.2) will constitute the field equations for the plane strain of a Mooney-Rivlin material. These have been obtained previously in a different manner by Adkins. [21]

Multiply (II.23) by X_x and (II.24) by Y_x. Then add the resulting equations to obtain

$$X_x \nabla^2 X + Y_x \nabla^2 Y = -\frac{1}{\mu} k_x \qquad \text{(II.36)}$$

If the same operations are performed with X_y and Y_y

$$X_y \nabla^2 X + Y_y \nabla^2 Y = -\frac{1}{\mu} k_y \qquad \text{(II.37)}$$

is obtained.

Differentiate (II.36) with respect to y and (II.37) with respect to x and subtract one from the other to obtain

$$\frac{\partial}{\partial y}(X_x \nabla^2 X + Y_x \nabla^2 Y) - \frac{\partial}{\partial x}(X_y \nabla^2 X + Y_y \nabla^2 Y) = 0 \qquad \text{(II.38)}$$

which is identical to Adkins' equation

$$\frac{\partial(X, \nabla^2 X)}{\partial(x,y)} + \frac{\partial(Y, \nabla^2 Y)}{\partial(x,y)} = 0 \qquad \text{(II.39)}$$

(II.38) and (II.13.2) are the required, coupled field equations in rectangular Cartesian coordinates.

It is also possible to eliminate k from the stress boundary conditions and obtain a natural boundary condition for (II.38) and (II.13.2). From (II.26)

$$k = \frac{\bar{t}_{o_x} - \mu(X_x n_{o_x} + X_y n_{o_y})}{(Y_y n_{o_x} - Y_x n_{o_y})} \qquad \text{(II.40)}$$

and from (II.27)

$$k = \frac{\bar{t}_{o_y} - \mu(Y_x n_{o_x} + Y_y n_{o_y})}{(X_x n_{o_y} - X_y n_{o_x})} \tag{II.41}$$

so that the stress boundary condition on S_{o_t} may be written as

$$\begin{gathered}\bar{t}_{o_X}(X_y n_{o_x} - X_x n_{o_y}) + \bar{t}_{o_y}(Y_y n_{o_x} - Y_x n_{o_y}) = \\ \mu\{(X_x^2 - X_y^2 + Y_x^2 - Y_y^2)(n_{o_x} n_{o_y}) + (X_x X_y + Y_x Y_y)(n_{o_y}^2 - n_{o_x}^2)\}\end{gathered} \tag{II.42}$$

It is not clear that (II.13.2), (II.38), and (II.35) and/or (II.42) define a well posed mathematical problem.

There may be cases, especially in curvilinear coordinates, where certain of the earlier results of this section will be useful if expressed in terms of the displacement vector rather than the deformation mapping.

$$I_M = g^{rs}G_{rs} = g^{rs}[g_{rs} + \bar{g}_r \cdot \bar{v}_{,s} + \bar{g}_s \cdot \bar{v}_{,r} + \bar{v}_{,r} \cdot \bar{v}_{,s}] \tag{II.43}$$

so that

$$(I_M - 3) = g^{rs}[v_{r|s} + v_{s|r} + v_{m|r} v^m|_s] \tag{II.44}$$

and for the plane strain of a Mooney-Rivlin material

$$W = \frac{\mu}{2} g^{\alpha\beta}[v_{\alpha|\beta} + v_{\beta|\alpha} + v_{\gamma|\alpha} v^{\gamma}|_{\beta}] \tag{II.45}$$

(II.45) is clearly the generalized tensor expression for (II.11).

In rectangular Cartesian coordinates (II.45) becomes

$$W = \frac{\mu}{2} [u_x^2 + u_y^2 + v_x^2 + v_y^2 + 2(u_x + v_y)] \qquad \text{(II.46)}$$

where u and v are the components of the displacement vector in the x and y directions respectively.

The incompressibility condition, (II.13.2), may be written as

$$u_x + v_y + u_x v_y - u_y v_x = 0 \qquad \text{(II.47)}$$

(II.47) may be used to simplify (II.46).

$$W = \frac{\mu}{2} [(u_x - v_y)^2 + (u_y - v_x)^2] \qquad \text{(II.46a)}$$

It is noted that results analogous to those presented in this section will be obtained if the convected coordinate system is associated with a Cartesian system in the deformed body B.

3. On Obtaining Incompressible Deformation Fields

When the Rayleigh-Ritz method is used to obtain approximate solutions of problems in the infinitesimal theory of elasticity the only restrictions placed on admissible deformation fields are those of continuity and that the geometrical boundary conditions be satisfied. [22] The incompressibility condition, if imposed, is satisfied by letting Poisson's ratio assume the value of one-half. For finite strain problems there are two ways to deal with incompressibility. The first is to require that only incompressible deformation fields be admissible. The second, and more complicated way, is to choose both a deformation field and a hydrostatic k field as indicated in Chapter I.

It is believed that in most cases the first way of dealing with the incompressibility condition is the preferable one. With this view in mind the case of plane strain is studied. If the incompressibility condition for this case, (II. 13. 2), is examined it is seen to be a first order, bilinear partial differential equation for X and Y. If it is possible, on the basis of physical intuition or experience, to choose an appropriate form for either X or Y then (II. 13. 2) becomes a first order, linear partial differential equation for the remaining function and it may be solved by the method of characteristics. [23]

Assume that $Y = Y(x, y)$ is given. Then (II. 13. 2) becomes

$$a(x,y)\, X_x + b(x,y)\, X_y = 1 \tag{II. 48}$$

where

$$a = Y_y \; ; \quad b = -Y_x \tag{II. 49}$$

The family of characteristics of (II. 48) is defined by

$$\frac{dx}{a} = \frac{dy}{b} = \frac{dX}{1} \tag{II. 50}$$

The first equation (II. 50)

$$bdx - ady = 0$$

or (II. 51)

$$Y_x\, dx + Y_y\, dy = 0$$

is an exact differential equation whose solution is

$$Y(x,y) = C_1 \tag{II. 52}$$

If it is possible to solve (II. 52) for x or y in terms of the other variable then from (II. 50) it is possible to obtain

$$X - \int \frac{dy}{b} = C_2$$

or

$$X - \int \frac{dx}{a} = C_3 \qquad \text{(II. 53)}$$

The general solution of (II. 48) is [21]

$$\Phi \left([Y(x,y)] \, , \, [X - \int \frac{dx}{a}] \right) = 0$$

or

$$\Phi \left([Y(x,y)] \, , \, [X - \int \frac{dy}{b}] \right) = 0 \qquad \text{(II. 54)}$$

where Φ is an arbitrary function of its arguments.

A simpler, restricted form of (II. 54) is

$$X = \int \frac{dx}{a} + \Psi(Y)$$

or

$$X = \int \frac{dy}{b} + \Psi(Y) \qquad \text{(II. 55)}$$

where Ψ is an arbitrary function of its argument.

Similar results for Y are obtained if X is assumed to be known.

The method presented in this section is a systematic way of constructing incompressible deformation fields for plane strain problems. The extension to three dimensions is straightforward. In that case two of the three deformation components are to be chosen.

CHAPTER III. ILLUSTRATIVE APPLICATIONS OF THE PRINCIPLE OF STATIONARY POTENTIAL ENERGY

1. Introduction

The present chapter deals with some illustrative applications of the principle of stationary potential energy to problems of the finite deformation of a neo-Hookean material.* Two of the problems are solved exactly and one of these involves an unusual sort of stability question. An approximate solution is presented for a problem which is not amenable to the usual semi-inverse methods of solution used in solving finite elastic strain problems. The approximate solution is obtained by making the sort of assumption that is more easily associated with a "Strength of Materials" approach rather than with a "Theory of Elasticity" approach.

2. A Two-dimensional Slump Problem

The application of the Rayleigh-Ritz method to plane strain problems of a neo-Hookean material is illustrated by the following "slump" problem. An infinite slab of thickness t, shown in Figure III. 1, is bonded to rigid walls which always remain a distance t apart. The system is subjected to an upward acceleration of Ng so that the total force acting on an element of unit depth (into the plane of the paper) is in the y direction and of magnitude

$$F_y = \rho (N) g \qquad \text{(III. 1)}$$

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* Henceforth, wherever they are equivalent, the term neo-Hookean material will be used instead of the term Mooney-Rivlin material.

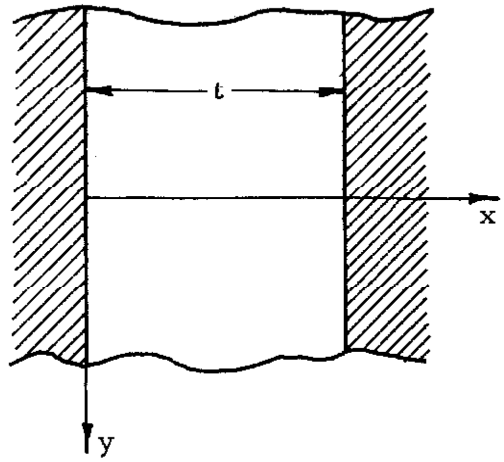

FIGURE III. 1. Infinite Slab.

where ρ is the mass density of the material and

g is the gravitational constant

From conditions of symmetry and the infinite extent of the slab it is reasonable to assume that vertical fibers in the undeformed body remain vertical after deformation and that their spacing remains unchanged, i.e., it is assumed that

$$X = x \tag{III. 2}$$

and therefore, from (II. 13. 2)

$$Y_y = 1$$

or

$$Y = y + f(x); \quad v = f(x) \tag{III. 3}$$

Since the following boundary conditions must be satisfied

$$v(0) = v(t) = 0 \quad \text{and} \quad \frac{dv}{dx}(t/2) = 0 \tag{III. 4}$$

v is assumed to be of the form

$$v = \sum_{n=1,3,5,\ldots}^{\infty} A_n \sin\left(\frac{n\pi x}{t}\right) \tag{III. 5}$$

The principle of stationary potential energy, with body forces and no surface tractions, states in view of (II. 46a) that

$$\delta\left[\int_{y_1}^{y_2}\int_0^t \left\{\frac{\mu}{2}(v_x)^2 - \rho(N)gv\right\}dxdy\right] = 0 \tag{III. 6}$$

or

$$\delta\left[\int_0^t \left\{\frac{\mu}{2}(v_x)^2 - \rho(N)\,gv\right\}dx\right] = 0 \tag{III. 6a}$$

since the problem has no y dependence.

Performing the indicated integrations it is found that

$$\delta\left[\frac{\mu\pi^2}{4t}\sum_{n=1,3,\ldots}^{\infty} A_n^2 n^2 - 2\rho\left(\frac{N}{\pi}\right) g \sum_{n=1,3,\ldots}^{\infty} \frac{A_n}{n}\right] = 0 \tag{III. 7}$$

Since the variational operation is equivalent to partial differentiation with respect to each A_n

$$\frac{\mu(n\pi)^2 A_n}{2t} - \frac{2t\rho(N)g}{\pi n} = 0 \tag{III. 8}$$

from which

$$A_n = \frac{4t^2}{(\pi n)^3}\,\frac{\rho(N)g}{\mu} \tag{III. 9}$$

and

$$v(x) = \frac{4t^2\rho(N)g}{\pi^3\mu}\left[\sin\left(\frac{\pi x}{t}\right) + \frac{1}{27}\sin\left(\frac{3\pi x}{t}\right) + \ldots\right] \tag{III. 10}$$

The sine series (III. 10) may be written in the following closed form.

$$v(x) = \frac{t^2 \rho(N) g}{2\mu} \left(\frac{x}{t} - \frac{x^2}{t^2}\right) \tag{III. 11}$$

and then

$$Y = y + \frac{t^2 \rho(N) g}{2\mu} \left(\frac{x}{t} - \frac{x^2}{t^2}\right) \tag{III. 12}$$

The nominal stress tensor t^{ij} will now be calculated. It will be recalled that one may write

$$t^{ij} = \mu(g^{ir} + \overline{k} G^{ir})(\delta^j_r + v^j\big|_r) \tag{III. 13}$$

which is merely another form of (II. 15).

From (II. 5) it is found that

$$G^{ij} = \begin{pmatrix} 1 & -v_x & 0 \\ -v_x & 1+v_x^2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{III. 14}$$

so that

$$t^{ij} = \mu \begin{pmatrix} 1+\overline{k} & -v_x\overline{k} & 0 \\ -v_x\overline{k} & 1+\overline{k}(1+v_x^2) & 0 \\ 0 & 0 & 1+\overline{k} \end{pmatrix} + \mu \begin{pmatrix} 0 & v_x(1+\overline{k}) & 0 \\ 0 & -\overline{k}v_x^{\;2} & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$t^{ij} = \mu \begin{pmatrix} 1+\overline{k} & v_x & 0 \\ -v_x\overline{k} & 1+\overline{k} & 0 \\ 0 & 0 & 1+\overline{k} \end{pmatrix} \tag{III. 15}$$

where

$$\overline{k} = k/\mu \tag{III.16}$$

The equations of equilibrium are*

$$t^{ij}{}_{,i} + F_j = 0 \tag{III.17}$$

When written explicitly for the present problem, noting that

$$\frac{\partial(\;)}{\partial y} = \frac{\partial(\;)}{\partial z} = 0 \;, \tag{III.18}$$

it is found that

$$\begin{aligned} &\overline{k}_x = 0 \\ &\mu v_{xx} + \rho N g = 0 \end{aligned} \tag{III.19}$$

(III.19.2) is identically satisfied and, if $\overline{k}$ is a constant, (III.19.1) is satisfied so that the exact solution has been obtained.

Since there are no stress boundary conditions to be satisfied $\overline{k}$ may be chosen arbitrarily. If $\overline{k} = -1$ is chosen

$$t^{ij} = \mu \begin{pmatrix} 0 & v_x & 0 \\ v_x & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & \frac{t\rho N g}{2}(1-\frac{2x}{t}) & 0 \\ \frac{t\rho N g}{2}(1-\frac{2x}{t}) & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{III.20}$$

Note that t^{ij} is symmetric due to the choice of k (or $\overline{k}$) but t^{ij} in general is not symmetric, although τ^{ij} is always symmetric.

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* (II.23) and (II.24) could be used as well.

It is of interest to examine the results obtained by using only the first term in the expansion of v.

$$v_{approx.} = \frac{4t^2 \rho N g}{\pi^3 \mu} \sin\left(\frac{\pi x}{t}\right) \qquad \text{(III.21)}$$

and

$$t^{ij}_{approx.} = \begin{pmatrix} 0 & \frac{4t\rho Ng}{\pi^2}\cos\left(\frac{\pi x}{t}\right) & 0 \\ \frac{4t\rho Ng}{\pi^2}\cos\left(\frac{\pi x}{t}\right) & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \text{(III.22)}$$

(III.22) does not satisfy the equilibrium equations (III.17).

Comparisons of the approximate and exact solutions to the "slump" problem are given below in Table III.1.

$\frac{X}{t}$	$\frac{v}{t^2\rho Ng/\mu}$	$\frac{v_{approx.}}{t^2\rho Ng/\mu}$	$\frac{t^{12}}{t\rho Ng}$	$\frac{t^{12}_{approx.}}{t\rho Ng}$
0	0	0	0.500	0.406
0.125	0.0515	0.0494	0.375	0.375
0.250	0.0938	0.0912	0.250	0.287
0.375	0.1171	0.1192	0.125	0.155
0.500	0.1250	0.1290	0	0

Table III.1

As is to be expected when the principle of stationary potential energy is used to obtain an approximate solution, the displacements are approximated with much greater accuracy than are the stresses. In the case under consideration the maximum error in the displacement is only 3.2% whereas the maximum stress is underestimated by 18.8%. Still, this error is within allowable limits for most engineering stress analysis.

The slump problem considered proved to be a linear one but it should not be expected that this will be the usual situation when a Rayleigh-Ritz solution is attempted. For example, the problem of the pure shear of a neo-Hookean rectangular block, shown in Figure III.2, can be solved by assuming

$$\begin{matrix} X = ax + by \\ Y = cx + dy \end{matrix} \quad ; \qquad ad - bc = 1 \tag{III.23}$$

with "a," "b," "c," and "d" as parameters.

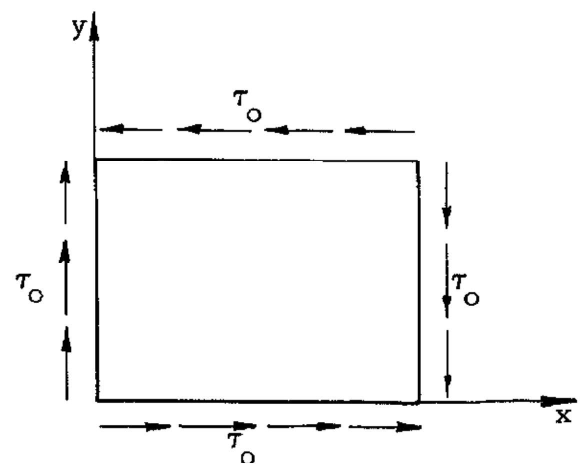

FIGURE III.2 Pure Shear

Substitution of (III.23) into the variational principle clearly leads to non-linear algebra. In this problem the exact solution will be obtained.

3. <u>An Unsymmetrical Plane Strain Problem</u>

In this section an unsymmetrical, mixed boundary value problem is studied and an approximate solution is obtained. The method of approach is different than the one used to study the slump problem in that a certain geometric assumption is made which leads, through the variational principle, to an easily integrated ordinary differential equation.

Consider the following unsymmetric problem. The right, isosceles triangular block shown in Figure III. 3 is bonded to a rigid base (indicated by shading) and is loaded by a linearly varying horizontal load on the face x = a.

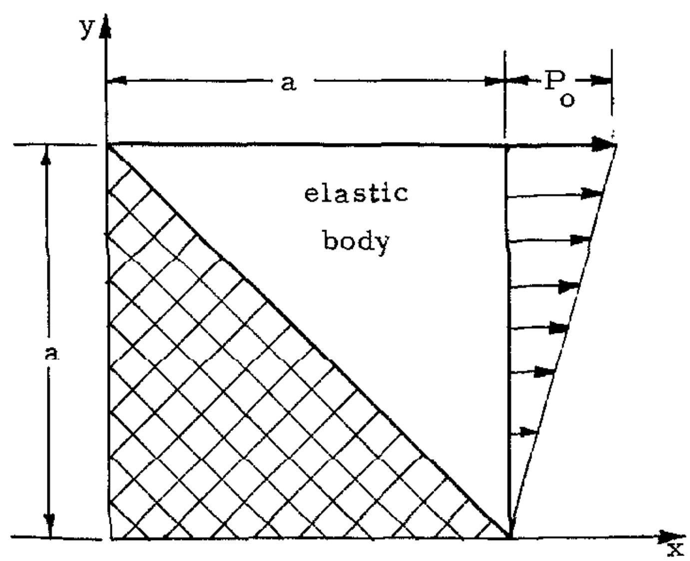

FIGURE III. 3. Unsymmetrical Problem

The problem will be considered to be one of plane strain.

It is assumed that the loading is a dead loading, i.e., the loading always acts horizontally and furthermore each "load element" is of constant magnitude and always acts on the same surface element.

The geometrical boundary condition that must be satisfied is that along the bond line, x + y = a, the displacements vanish. This suggests that the following assumption be made.

$$u = g(x + y)$$

and (III.24)

$$v = -g(x + y)$$

with $g(a) = 0$ (III.25)

as the boundary condition.

(III.24) satisfies the incompressibility condition (II.47). This is the reason that v was chosen to be $-g(x + y)$ rather than $h(x + y)$.

The strain energy density, given by (II.46a), becomes

$$W = 2\mu \left(\frac{dg}{d(x + y)}\right)^2 \tag{III.26}$$

and the variational principle is written as

$$\delta \left[2\mu \int_0^a \int_{a-x}^a \left[\frac{dg}{d(x+y)}\right]^2 dy dx - \frac{p_o}{a} \int_0^a y g(a+y) dy \right] = 0 \tag{III.27}$$

Now introduce new variables

$$\eta = x + y, \ \xi = x \tag{III.28}$$

for which the Jacobean of the transformation is unity. Therefore

$$\int_0^a \int_{a-x}^a \left[\frac{dg}{d(x+y)}\right]^2 dy dx = \int_a^{2a} \int_{\eta-a}^a \left(\frac{dg}{d\eta}\right)^2 d\xi d\eta \tag{III.29}$$

The choice of limits is easily seen by examining Figure III.4 below.

Similarly, letting $(a + y) = \eta$,

$$\int_0^a y g(a + y) dy = \int_a^{2a} (\eta - a) g(\eta) d\eta \tag{III.30}$$

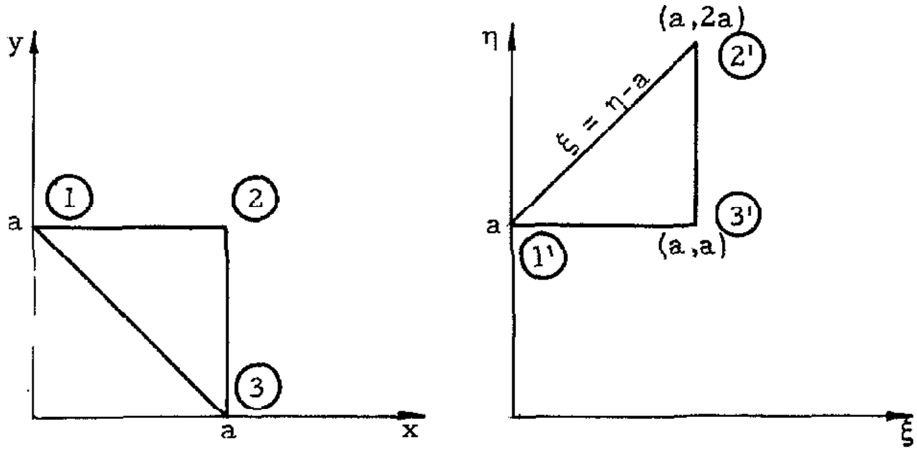

FIGURE III. 4. Transformation from x-y to ξ-η coordinates.

so that (III. 27) becomes

$$\delta\left[2\mu \int_a^{2a}\int_{\eta-a}^{a} \left(\frac{dg}{d\eta}\right)^2 d\xi d\eta - \frac{P_o}{a}\int_a^{2a} (\eta-a)g(\eta)d\eta\right] = 0 \qquad \text{(III. 31)}$$

After the variational operation is performed it is found that

$$4\mu(2a-\eta)\frac{dg}{d\eta}\delta g \Bigg|_a^{2a} - \int_a^{2a} \left\{4\mu \frac{d}{d\eta}\left[(2a-\eta)\frac{dg}{d\eta}\right] + \frac{P_o}{a}(\eta-a)\right\}\delta g d\eta = 0 \qquad \text{(III. 32)}$$

so that the Euler equation for the problem is

$$\frac{d}{d\eta}\left[(2a-\eta)\frac{dg}{d\eta}\right] = \frac{P_o}{4\mu a}(a-\eta) \qquad \text{(III. 33)}$$

and the natural boundary conditions are that at

$$\begin{aligned} \eta = a &: \frac{dg}{d\eta} = 0 \quad \text{or } g \text{ is specified} \\ \eta = 2a &: \frac{dg}{d\eta} \neq \infty \quad \text{or } g \text{ is specified} \end{aligned} \qquad \text{(III. 34)}$$

For the problem under consideration at

$$\eta = a : g = 0$$

$$\eta = 2a : \frac{dg}{d\eta} \text{ is finite} \qquad \text{(III. 34a)}$$

The function satisfying (III. 33) and (III. 34a) is

$$g = \frac{P_o a}{16\mu} \left(\frac{\eta^2}{a^2} - 1\right) \qquad \text{(III. 35)}$$

or

$$u = -v = \frac{P_o a}{16\mu} \left(\frac{(x+y)^2}{a^2} - 1\right) \qquad \text{(III. 36)}$$

For the case of the unit square, with $P_o/\mu = 1$, the deformation is as given in the following table and Figure III. 5.

TABLE III. 2

η	$u = -v$
1	0
5/4	0.035
3/2	0.078
7/4	0.129
2	0.188

If the equilibrium equations, (II. 24) and (II. 25), and the stress boundary conditions, (II. 26) and (II. 27), are considered, then it will become apparent that the solution of (III. 33) is only an approximation to the solution of the elasticity problem posed and studied in this section.

Since

$$\frac{\partial(\ )}{\partial x} = \frac{\partial(\ )}{\partial \eta}\frac{\partial \eta}{\partial x} = \frac{\partial(\ )}{\partial \eta} = \frac{\partial(\ )}{\partial y} \tag{III.37}$$

(II.23) and (II.24) reduce to a single equation

$$\frac{\partial k}{\partial \eta} = \frac{dk}{d\eta} = \frac{P_o}{4a} \tag{III.38}$$

so that

$$k = \frac{P_o \eta}{4a} + \mu \text{Const.} \tag{III.39}$$

and it is clear from (II.26) and (II.27) that the boundary conditions on the faces x = a, y = a will not be satisfied. However, since the deformation field satisfies the equilibrium equations without body forces an exact solution of some problem(s) has been found and it would be instructive to search out such a problem. The problem will be determined by how the arbitrary constant in (III.39) is determined. In view of the problem studied it would seem reasonable to require on the face y = a that

$$\bar{t}^{22} = \frac{1}{a}\int_0^a t^{22}\, dx = 0 \tag{III.40}$$

or, using (II.27),

$$\bar{t}^{22} = \frac{1}{a}\int_0^a \{\mu[1-\frac{P_o}{8\mu}(\frac{x}{a}+1)] + [\frac{P_o}{4}(\frac{x}{a}+1)+\mu\text{Const}][1+\frac{P_o}{8\mu}(\frac{x}{a}+1)]\}\, dx=0 \tag{III.40a}$$

It is found that

$$\text{Const.} = -(1 + \frac{7(\frac{P_o}{\mu})^2}{96 + 18\frac{P_o}{\mu}}) \tag{III.41}$$

so that

$$k = \frac{P_o(x+y)}{4a} - \mu\left(1 + \frac{7\left(\frac{P_o}{\mu}\right)^2}{96 + 18\frac{P_o}{\mu}}\right) \qquad \text{(III. 42)}$$

Using (II. 26) and (II. 27) the following numerical results are found for the case $P_o/\mu = 1$.

TABLE III. 3

On x = a

y/a	t^{11}/μ
0	0.415
0.5	0.630
1.0	0.829

y/a	t^{12}/μ
0	0.039
0.5	0.109
1.0	0.140

On y = a

x/a	t^{22}/μ
0	-0.038
0.5	-0.003
1.0	0.048

x/a	t^{21}/μ
0	-0.039
0.5	-0.109
1.0	-0.140

It is seen from the results presented in Table III. 3 that the shears and t^{22}, which vanish in the problem posed, are small compared to t^{11}. Unfortunately the distribution of t^{11} along the side x = a is not a good approximation to the assumed form

$$\frac{t^{11}}{\mu}(a, y) = y/a \qquad \text{(III. 43)}$$

since the result given in Table III. 3 can be represented approximately by

$$\frac{t^{11}(a,y)}{\mu} = 0.415\,(1 + \frac{y}{a}) \qquad \text{(III. 44)}$$

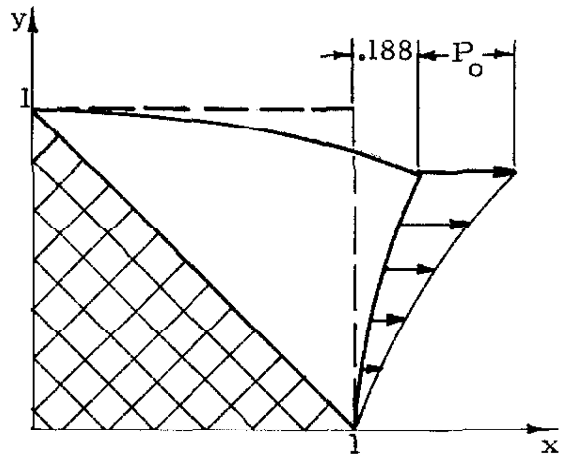

FIGURE III. 5. Deformed Triangle.

It is likely that the approximate solution represents a fair estimate of the displacements but certainly it will give a poor estimate of the stresses, particularly t^{11} (or τ^{11}). Considering the complexity of the problem and the simplicity (or simplemindedness) of the method of approximation little more was to be expected. The approach was essentially a "strength of materials" one such as might be used if a quick estimate was required for design purposes.

If an attempt is made to use assumption (III. 24) as the basis of a semi-inverse type solution it will be found that the displacements are quadratic functions of η; however there will be no rational basis for choosing any particular parabolas. Consequently a semi-inverse approach cannot be considered appropriate for the problem just considered.

3. Internally Pressurized Neo-Hookean Sphere

The principle of stationary potential energy will be applied now to the problem of a hollow sphere which is internally pressurized; the sphere is composed of a neo-Hookean material and has inner radius "a" and outer radius "b" in its undeformed state.

$$W = \frac{\mu}{2}(I_M - 3) \; ; \quad III_M = 1 \tag{III. 45}$$

where

$$\begin{aligned} I_M &= \lambda_1^2 + \lambda_2^2 + \lambda_3^2 \\ III_M &= \lambda_1^2 \, \lambda_2^2 \, \lambda_3^2 \end{aligned} \tag{III. 46}$$

and the λ_i's are the principal extension ratios. If a spherical coordinate system, (r, θ, φ), is used to parametrize the sphere in its undeformed state then

$$\lambda_1 = \lambda_r, \; \lambda_2 = \lambda_\theta, \; \lambda_3 = \lambda_\varphi = \lambda_2 \tag{III. 47}$$

due to the symmetry of both the body and the loading.

Let $\lambda_2 = \lambda(r)$; then from (III. 45. 2) $\lambda_1 = (\frac{1}{\lambda(r)})^2$ and

$$I_M = 2\lambda^2 + \frac{1}{\lambda^4} \tag{III. 48}$$

The incompressibility condition may be used to find

$$\lambda(r) = \frac{a}{r}\left[(\lambda_a^3 - 1) + \frac{r^3}{a^3}\right]^{1/3} \tag{III. 49}$$

where λ_a denotes the value of λ at the inner radius "a" of the sphere.

Since $W = W(\lambda_a)$ and $\overline{v} = \overline{v}(\lambda_a)$ in this problem, the variational operation becomes merely the total differentiation of a function of one variable

$$W = \frac{\mu}{2}\left(\frac{1}{\lambda^4} + 2\lambda^2 - 3\right) \; ; \; \delta W = 2\mu\left(\lambda - \frac{1}{\lambda^5}\right)\delta\lambda \qquad \text{(III. 50)}$$

and from (III. 49) it is found that

$$\delta\lambda = \frac{a}{r}\,\lambda_a^2\,\left[(\lambda_a^3 - 1) + \frac{r^3}{a^3}\right]^{-2/3}\delta\lambda_a = \frac{a^3}{r^3}\,\frac{\lambda a^2}{\lambda^2}\,\delta\lambda_a \qquad \text{(III. 51)}$$

The expressions for W and δW are integrated over the volume of the undeformed sphere noting that

$$\frac{dr}{r} = \frac{\lambda^2 d\lambda}{1 - \lambda^3} \qquad \text{(III. 52)}$$

Then

$$\iiint_{\tau_o} W d\tau_o = 2\pi\mu a^3(\lambda_a^3 - 1)\left[\frac{2\lambda^2 + 2\lambda + 1}{\lambda^3 + \lambda^2 + \lambda}\right]_{\lambda_a}^{\lambda_b} \qquad \text{(III. 53)}$$

and

$$\iiint_{\tau_o} \delta W d\tau_o = 8\pi\mu a^3\lambda_a^2\left[\frac{1}{\lambda} + \frac{1}{4\lambda^4}\right]_{\lambda_a}^{\lambda_b}\delta\lambda_a \qquad \text{(III. 54)}$$

By integrating over the surface of the deformed body it is found that the potential energy of the external forces is

$$-\iint_S \underset{\sim}{\overline{t}} \cdot \overline{v} dS = -\iint_S P(\lambda_a - 1) a dS = -\frac{4}{3} P\pi (a\lambda_a)^3 \qquad \text{(III. 55)}$$

and

$$-\delta\iint_S \underset{\sim}{\overline{t}} \cdot \overline{v}dS = -4P\pi(a\lambda_a)^2 a\delta\lambda_a \tag{III.56}$$

where P is the internal pressure.

Note that there are additive constants in (III.53) and (III.55) which may be disregarded or chosen so as to normalize the total potential energy in some desired fashion.

The principle of stationary potential energy requires that the sum of (III.54) and (III.56) vanish

$$\delta\lambda_a[8\pi\mu a^3\lambda_a^2\{(\frac{1}{\lambda_b} - \frac{1}{\lambda_a}) + (\frac{1}{4\lambda_b^4} - \frac{1}{4\lambda_a^4})\} - 4\pi a^3\lambda_a^2 P] = 0 \tag{III.57}$$

so that for arbitrary $\delta\lambda_a$

$$\frac{P}{\mu} = [\tfrac{1}{2}(\frac{1}{\lambda_b^4} - \frac{1}{\lambda_a^4}) + 2(\frac{1}{\lambda_b} - \frac{1}{\lambda_a})] \tag{III.58}$$

which is the result obtained by a conventional semi-inverse analysis. [24]

It is possible to make a complete characterization of the behavior of (III.58). After some manipulation it is found that

$$\frac{d}{d\lambda_a}(\frac{P}{\mu}) = \frac{2}{\lambda_a^2}[\frac{1}{\lambda_a^3}\{1 - \frac{b^4}{a^4}(1 - \frac{b^3-a^3}{b^3+(\lambda_a^3-1)a^3})^{\frac{7}{3}}\} + \{1 - \frac{b}{a}(1 - \frac{b^3-a^3}{b^3+(\lambda_a^3-1)a^3})^{\frac{4}{3}}\}] \tag{III.59}$$

Observe that

$$\begin{aligned} &a)\quad \frac{d}{d\lambda_a}(\frac{P}{\mu})\Big|_{\lambda_a=1} = 4(1 - \frac{a^3}{b^3}) > 0 \\ &b)\quad \frac{d}{d\lambda_a}(\frac{P}{\mu})\Big|_{\lambda_a\to\infty} = 0 \\ &c)\quad \frac{d}{d\lambda_a}(\frac{P}{\mu}) < 0 \qquad \text{for } \lambda_a \text{ sufficiently large but finite} \end{aligned} \tag{III.60}$$

From a) and c) it is concluded, since $d(P/\mu)/d\lambda_a$ is a continuous function, that there is a zero of $d(P/\mu)/d\lambda_a$ for at least one finite λ_a. It will be shown that there is only one such zero.

Assume λ_a is the value at which the first zero of $(P/\mu)/d\lambda_a$ occurs, i.e., $d(P/\mu)/d\lambda_a > 0$ for $\lambda_a < \lambda_{a_o}$. It is clear that the sign of the right hand side of (III. 59) is determined by the terms in the square bracket. Since the terms in the square bracket are monotonically decreasing with increasing λ_a and vanish at $\lambda_a = \lambda_{a_o}$ (by assumption) it follows that $d(P/\mu)/d\lambda_a$ is negative for all $\lambda_a > \lambda_{a_o}$. Therefore $d(P/\mu)/d\lambda_a$ has only one finite zero, namely at $\lambda_a = \lambda_{a_o}$.

It is found by differentiating (III. 59) that

$$\frac{d^2(\frac{P}{\mu})}{d\lambda_a^2} = \frac{2}{\lambda_a^3}\left[\frac{1}{\lambda_a^3}\left\{\frac{b^4}{a^4}\left[1 - \frac{b^3-a^3}{b^3+(\lambda_a^3-1)a^3}\right]^{\frac{7}{3}} \times \right.\right.$$

$$\times \left[7\frac{b}{a}\left(1 - \frac{b^3-a^3}{b^3+(\lambda_a^3-1)a^3}\right)^{\frac{1}{3}} - 2\right] - 5\Big\} + \tag{III. 61}$$

$$+ \frac{b}{a}\left[1 - \frac{b^3-a^3}{b^3+(\lambda_a^3-1)a^3}\right]^{\frac{4}{3}}\left[4\frac{b}{a}\left(1 - \frac{b^3-a^3}{b^3+(\lambda_a^3-1)a^3}\right)^{\frac{1}{3}} - 2\right] - 2\Big]$$

Observe that

a) $\left.\frac{d^2(\frac{P}{\mu})}{d\lambda_a^2}\right|_{\lambda_a = 1} = -14\left(1 - \frac{a^3}{b^3} < 0\right)$

b) $\left.\frac{d^2(\frac{P}{\mu})}{d\lambda_a^2}\right|_{\lambda_a \to \infty} = 0$ (III. 62)

c) $\frac{d^2(\frac{P}{\mu})}{d\lambda_a^2} > 0$ for λ_a sufficiently large but finite

An analysis paralleling the one made for $d(P/\mu)/d\lambda_a$ leads to the conclusion that there is one, and only one, finite inflection point in the curve of P/μ vs. λ_a.

Figure III.6 below illustrates this behavior

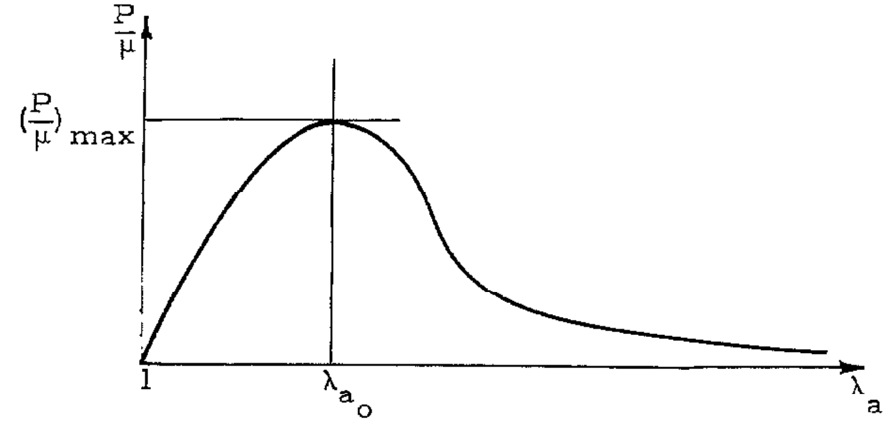

FIGURE III.6. Qualitative Behavior of a Neo-Hookean Sphere.

It is noted that for each $P/\mu < (P/\mu)_{max}$ there are two equilibrium values of λ_a. It shall be made clear in the following analysis that the portion of the curve to the left of the maximum represents stable equilibrium and the portion to the right unstable equilibrium.

The static condition that the equilibrium be stable is that the second variation of the potential energy be positive in the equilibrium configuration where the first variation vanished. In the present case this is merely that the second derivative of the potential energy with respect to λ_a be positive, i.e.,

$$\frac{d}{d\lambda_a}\{\lambda_a^2[2(\frac{1}{\lambda_b}-\frac{1}{\lambda_a})+\tfrac{1}{2}(\frac{1}{\lambda_b^4}-\frac{1}{\lambda_a^4})-\frac{P}{\mu}]\}>0 \qquad \text{(III.63)}$$

or

$$2\lambda_a[2(\frac{1}{\lambda_b}-\frac{1}{\lambda_a})+\tfrac{1}{2}(\frac{1}{\lambda_b^4}-\frac{1}{\lambda_a^4})-\frac{P}{\mu}] + \lambda_a^2[\frac{d}{d\lambda_a}\{2(\frac{1}{\lambda_b}-\frac{1}{\lambda_a})-\tfrac{1}{2}(\frac{1}{\lambda_b^4}-\frac{1}{\lambda_a^4})\}]>0 \qquad \text{(III.63a)}$$

The first term is identically zero for equilibrium positions from (III.58) and so the stability criterion is really

$$\frac{d}{d\lambda_a}(\frac{P}{\mu})_{equil.}>0 \qquad \text{(III.64)}$$

This is the analytic confirmation of the intuitive belief that the equilibrium is stable when P/μ increases with increasing λ_a and unstable when P/μ decreases with increasing λ_a. λ_a corresponding to $(P/\mu)_{max}$ is the transition point and must be considered as a point of unstable equilibrium.

To illustrate these ideas with a numerical example the case $b/a = 2$ is considered. It is found from (III.58) and (III.59) that at $\lambda_a = 1.82$, $(P/\mu)_{max} = 0.83$. The potential energy of the system for this value of P/μ is plotted against λ_a in Figure III.7* and it is seen that this curve has no maximum or minimum although it does have a stationary value at $\lambda_a = 1.82$. This means that for $(P/\mu)_{max}$ there is really an instability. To show the behavior of the potential energy for any $(P/\mu) < (P/\mu)_{max}$ the case $(P/\mu) = 0.55$ is considered.

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* Full page figures providing quantitative information are grouped at the end of the thesis.

This is found to correspond to $\lambda_a = 1.25$ and $\lambda_a = 3.60$. It is found that at $\lambda_a = 1.25$ the potential energy has a minimum and at $\lambda_a = 3.60$ it has a maximum. These are the stable and unstable equilibrium positions respectively for $(P/\mu) = 0.55$. This, too, is plotted in Figure III.7.

It is to be remembered that the potential energy is determined only to within an additive constant. For $(P/\mu)_{max}$ the constant was chosen so that the potential energy vanished in the (unstable) equilibrium position while in the case $(P/\mu) = 0.55$ the constant was chosen so that the potential energy vanished at the stable equilibrium position. These choices were purely arbitrary.

The variational solution of this problem has the advantage that with little additional effect the static stability criterion is found or, viewed in another way, in the course of finding the stability criterion one automatically finds the equilibrium relation.

CHAPTER IV. THE FINITE PLANE STRAIN DEFORMATION OF A THIN PAD

1. Introduction

The finite elastic deformation of a thin neo-Hookean pad bonded to the faces of a rigid testing machine is studied in this chapter under the assumption of plane strain. Similar studies, under the assumptions of infinitesimal elasticity, have been made by Gent and Lindley [23] and Lindsey, et al. [24] The latter work considers both compressible and incompressible materials.

The appropriate differential equation governing the problem, together with the associated boundary conditions, is derived by applying the principle of stationary potential energy to a suitably restricted deformation field. For thin pads an approximation to this boundary value problem can be solved in closed form. It is then possible to show that for some cases of practical interest the solution of the approximate differential equation is an excellent approximation to the solution of the full differential equation derived from the variational principle.

2. Formulation of the Thin Pad Problem

It is assumed that in the deformed state the thin pad, shown in Figure IV.1 below, has a rectangular cross-section in the plane of the paper.* It is also assumed that the pad has undergone no

* If the cross-section of the undeformed pad was rectangular the formulation of the problem would be similar except for the fact that undeformed coordinates would be the independent variables. The computation of true stresses would be more complicated.

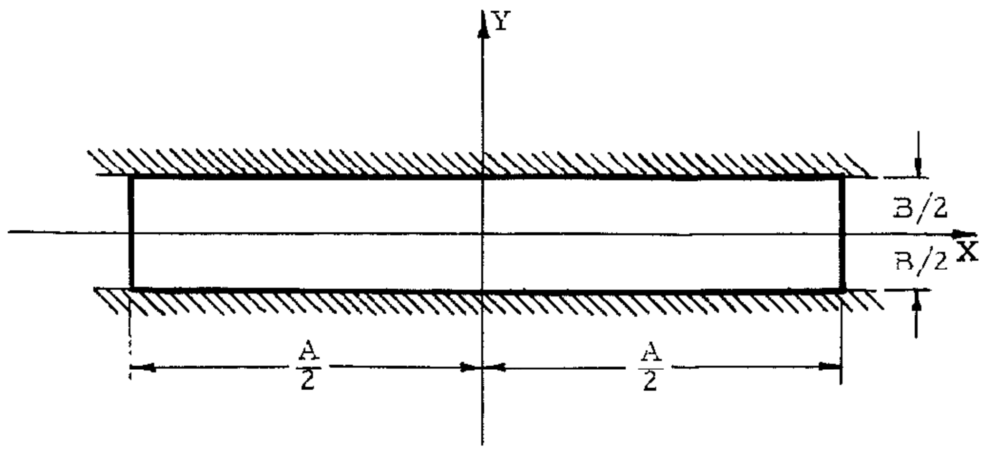

FIGURE IV.1. Thin Pad in Deformed State.

deformation normal to the plane of the paper, i.e. a plane strain deformation is considered.

The pad is bonded at the faces $Y = \pm B/2$ and is stress free on the faces $X = \pm A/2$. The deformation has consisted of a uniform stretching in the Y direction such that the original thickness of the pad is given by B/λ. A point (X, Y) in the deformed body was originally at (x, y) in the undeformed body.

The problem, as posed above, requires the solution of the coupled Adkins and incompressibility equations, taking into account the appropriate boundary conditions. [21] This is a formidable task well beyond the bounds of present mathematical knowledge. In order to simplify the problem it is assumed that

$$y = y(Y) \tag{IV.1}$$

provided that $A \gg B$. This assumption has been made by those who have studied the problem within the context of infinitesimal elasticity and should be reasonable for the finite deformation of a thin pad except in a region of non-uniform depth close to the free edges. Since, from a stress point of view, the region of greatest interest is at the origin, i.e. center of the pad, it seems justifiable to proceed on assumption (1).

The incompressibility condition

$$x_X y_Y - x_Y y_X = 1 \qquad \text{(IV.2)}$$

where subscripts indicate partial differentiation, becomes in the present case

$$x_X = 1/y_Y \qquad \text{(IV.2a)}$$

and the mapping x is given by

$$x = X/y_Y \qquad \text{(IV.3)}$$

since $x(0, Y)$ must vanish.

Previously the strain energy function for a neo-Hookean body has been shown to be

$$W = \frac{\mu}{2}[x_X^2 + x_Y^2 + y_X^2 + y_Y^2 - 2] \qquad \text{(IV.4)}$$

for the case of plane strain. Therefore the statement of the principle of stationary potential energy for the problem under consideration becomes

$$\delta\int_0^{A/2}\int_0^{B/2}[\frac{1}{y_Y^2} + \frac{X^2 y_{YY}^2}{y_Y^4} + y_Y^2 - 2]\, dYdX = 0 \qquad \text{(IV.5)}$$

If (IV.5) is integrated with respect to X, and

$$\overline{Y} = \frac{Y}{B/2}\,;\quad \overline{y} = \frac{y}{B/2} \tag{IV.6}$$

are introduced, it is found that

$$\delta \int_0^1 \left[\frac{\overline{y}^2_{\overline{Y}\,\overline{Y}}}{\overline{y}^4_{\overline{Y}}} + \frac{3B^2}{A^2} \left\{ \frac{\overline{y}^2_{\overline{Y}} - 1}{\overline{y}_{\overline{Y}}} \right\} \right] d\overline{Y} = 0 \tag{IV.7}$$

At this point the cumbersome barred notation may be dropped and, unless otherwise noted, y and Y will denote the nondimensional quantities defined in (IV.6).

If the variational operation indicated in (IV.7) is performed, the Euler-Lagrange differential equation is found to be

$$\frac{d}{dY}\left\{\frac{d}{dY}\left(\frac{d^2y}{dY^2} \Big/ \left(\frac{dy}{dY}\right)^4\right) + 2\left(\frac{d^2y}{dY^2}\right)^2 \Big/ \left(\frac{dy}{dY}\right)^5 - \frac{3B^2}{A^2}\left(\frac{dy}{dY} - 1\Big/\left(\frac{dy}{dY}\right)^3\right)\right\} = 0 \tag{IV.8}$$

together with the boundary conditions

$$\frac{d^2y}{dY^2}\,\delta\left(\frac{dy}{dY}\right)\Big|_0^1 = 0$$

$$\left[\frac{3B^2}{A^2}\,\frac{\left[\left(\frac{dy}{dY}\right)^4 - 1\right]}{\left(\frac{dy}{dY}\right)^3} + \frac{2\left(\frac{d^2y}{dY^2}\right)^2}{\left(\frac{dy}{dY}\right)^5} - \frac{d}{dY}\left(\frac{d^3y}{dY^3} \Big/ \left(\frac{dy}{dY}\right)^4\right)\right]\delta y\,\Big|_0^1 = 0 \tag{IV.9}$$

The correct choice of boundary conditions for the problem being studied is as follows

$$\delta\,\frac{dy(1)}{dY} = 0\,;\quad \frac{dy(1)}{dY} = 1 \tag{IV.10}$$

since $x(X,1) = X$,

$$\frac{d^2 y(0)}{dY^2} = 0 \tag{IV.11}$$

since $\frac{\partial x}{\partial Y}(X,0) = 0$,

$$\delta y(1) = 0\ ;\quad y(1) = \frac{1}{\lambda}\ ,\quad \text{and} \tag{IV.12}$$

$$\delta y(0) = 0\ ;\quad y(0) = 0 \tag{IV.13}$$

Thus, under assumption (1), the bonded thin pad problem for a neo-Hookean material is reduced to the study of the fourth order, non-linear, one-dimensional boundary value problem defined by (IV.8), (IV.10), (IV.11), (IV.12), and (IV.13).

3. Partial Integration of the Full Differential Equation

If one integration of (IV.8) is performed and the substitution

$$w = \frac{dy}{dY} \tag{IV.14}$$

is made then (IV.8) becomes

$$\frac{d}{dY}\left(\frac{dw}{dY}/w^4\right) + 2\left(\frac{dw}{dY}\right)^2/w^5 - \frac{3B^2}{A^2}\left(w - \frac{1}{w^3}\right) = C \tag{IV.15}$$

where C is an arbitrary constant of integration.

Note that

$$\frac{d}{dY} = \frac{dy}{dY}\frac{d}{dy} = w\frac{d}{dy} \tag{IV.16}$$

so that after some manipulation (IV.15) may be written as

$$w\frac{d^2w}{dy^2} - \left(\frac{dw}{dy}\right)^2 - 3\frac{B^2}{A^2}(w^4 - 1) = Cw^3 \tag{IV.17}$$

The substitution

$$\sqrt{z} = \frac{dw}{dy} \; ; \quad \frac{d}{dy} = \sqrt{z}\, \frac{d}{dw} \tag{IV.18}$$

leads to

$$\frac{w}{2}\, \frac{dz}{dw} - z - \frac{3B^2}{A^2}\,(w^4 - 1) = Cw^3 \tag{IV.19}$$

which may be further simplified by the substitution

$$t = \frac{z}{w^2} \; ; \quad \frac{dz}{dw} = w^2\, \frac{dt}{dw} + 2wt \tag{IV.20}$$

The resulting equation in terms of t is

$$\frac{dt}{dw} = 2C + \frac{3B^2}{A^2}\left(w - \frac{1}{w^3}\right) \tag{IV.21}$$

which may be immediately integrated to yield

$$t = D + 2Cw + \frac{3}{2}\,\frac{B^2}{A^2}\left(w^2 + \frac{1}{w^2}\right) \tag{IV.22}$$

where D is another arbitrary constant of integration.

By inverting the order of the various substitutions it is found that

$$dy = \frac{dw}{\sqrt{(3B^2/2A^2)\, w^4 + 2Cw^3 + Dw^2 + 3B^2/2A^2}}$$

$$dY = \frac{dw}{w\sqrt{(3B^2/2A^2)\, w^4 + 2Cw^3 + Dw^2 + 3B^2/2A^2}} \tag{IV.23}$$

The problem thus has been reduced to the consideration of two elliptic integrals. Unfortunately, due to the nonlinearity of the problem, the arbitrary constants of integration C and D occur under the radical and hence a straightforward evaluation of the integrals is not possible. It is not even clear that one can find a workable numerical procedure to solve the problem for given values of the parameters B/A and λ.

4. Solution of the Asymptotic Differential Equation

Assumption (1) was made on the basis that $A >> B$. If this is so, there is also justification for the investigation of the approximation to the boundary value problem posed in Section 2 which is defined by

$$\frac{d}{dY}\left\{\frac{d}{dY}\left(\frac{d^2y}{dY^2}/(\frac{dy}{dY})^4\right) + 2(\frac{d^2y}{dY^2})^2/(\frac{dy}{dY})^5\right\} = 0 \qquad (IV.24)$$

and (IV.10), (IV.11), (IV.12), and (IV.13). It is difficult to assess the worth of this approximation a priori; however an a posteriori judgement may be made by solving (IV.24) together with the boundary conditions and then determining how closely this solution satisfies (IV.8), point by point, for a given set of parameters A/B and λ.

It is noted at this point that at $Y = y = 0$ a solution of (IV.24) which satisfies the boundary conditions also satisfies (IV.8) because of (IV.11). In addition such a solution of (IV.24) satisfies (IV.15) at $Y = 1$, $y = 1/\lambda$ because of (IV.10).

The solution of (IV.24) is attempted in the same manner as was the solution of (IV.8) and, analogous to (IV.23), it is found that

$$dy = \frac{dw}{w\sqrt{Cw+D}} \qquad (IV.25)$$

$$dY = \frac{dw}{w^2\sqrt{Cw+D}} \tag{IV.26}$$

Fortunately these expressions may be integrated in closed form and the solution of (IV.24) may be expressed simply in terms of elementary functions.

It is necessary to distinguish between the two cases $D > 0$ and $D < 0$. In the course of the analysis it will become clear that $D > 0$ applies to the case of compression ($\lambda < 1$) and $D < 0$ applies to the case of extension ($\lambda > 1$).

1) The Case $D > 0$

From (IV.25) one obtains

$$y-E = -\frac{2}{\sqrt{D}} \tanh^{-1}\left(\sqrt{\frac{C}{D} w + 1}\right) \tag{IV.27}$$

which may be inverted to yield

$$w = \frac{dy}{dY} = -\frac{D}{C}\operatorname{sech}^2\left(\frac{\sqrt{D}}{2}(E-y)\right) \tag{IV.28}$$

From (IV.28) one easily obtains an implicit expression for y in terms of Y

$$Y-F = \frac{C}{2D^{3/2}}\left[\sinh(\sqrt{D}(E-y)) + \sqrt{D}(E-y)\right] \tag{IV.29}$$

Also

$$\frac{d^2y}{dY^2} = \frac{D^{5/2}}{C^2}\operatorname{sech}^4\left(\frac{\sqrt{D}}{2}(E-y)\right)\tanh\left(\frac{\sqrt{D}}{2}(E-y)\right) \tag{IV.30}$$

Boundary condition (IV.11) implies, from (IV.30), that $E = 0$ and this result together with boundary condition (IV.13) implies, from (IV.29), that F = 0.

The remaining two boundary conditions, (IV.10) and (IV.12), require that

$$\frac{D}{C}\,\mathrm{sech}^2\left(\frac{\sqrt{D}}{2\lambda}\right) + 1 = 0 \qquad \text{(IV.31)}$$

and

$$\frac{C}{2D^{3/2}}\left(\sinh\left(\frac{\sqrt{D}}{\lambda}\right) + \frac{\sqrt{D}}{\lambda}\right) - 1 = 0 \qquad \text{(IV.32)}$$

From (IV.31) it is found that

$$C = \frac{-2D}{\cosh\left(\frac{\sqrt{D}}{\lambda}\right) + 1} \qquad \text{(IV.33)}$$

which, when substituted into (IV.32), provides a transcendental equation for D.

$$\cosh\left(\frac{\sqrt{D}}{\lambda}\right) + 1 = \frac{1}{\sqrt{D}}\sinh\left(\frac{\sqrt{D}}{\lambda}\right) + \frac{1}{\lambda} \qquad \text{(IV.34)}$$

It is easy to show that if D is real and positive, as was assumed, then $\lambda < 1$ is implied. Expand both sides of (IV.34) in Taylor series.

$$2 + \frac{D}{2!\,\lambda^2} + \frac{D^2}{4!\,\lambda^4} + \dots = \frac{2}{\lambda} + \frac{D}{3!\,\lambda^3} + \frac{D^2}{5!\,\lambda^5} + \dots \qquad \text{(IV.35)}$$

If $\lambda > 1$ then each term on the right side of (IV.35) is less than the corresponding term on the left side of (IV.35) and the equation cannot hold. Therefore $\lambda \leq 1$. The condition $\lambda = 1$ implies D = 0 since only then is $\cosh\left(\frac{\sqrt{D}}{\lambda}\right) = \frac{1}{\sqrt{D}}\sinh\left(\frac{\sqrt{D}}{\lambda}\right)$. Consequently $D > 0$ implies $\lambda < 1$.

Simple arguments establish that D is uniquely determined for $\lambda > \frac{1}{3}$. For any $\lambda < 1$ the expression on the left side of (IV.35) is smaller than the expression on the right side of (IV.35) provided that D is sufficiently small. If (IV.35) is differentiated with respect to D it is found that

$$\frac{1}{2\lambda^2} + \frac{D}{12\lambda^4} + \frac{D^2}{240\lambda^6} + \dots = \frac{1}{\lambda}\left(\frac{1}{6\lambda^2} + \frac{D}{60\lambda^4} + \frac{D^2}{1680\lambda^6} + \dots\right) \qquad \text{(IV.36)}$$

For $\lambda > \frac{1}{3}$ the expression on the left side of (IV.36) is greater than the expression on the right side for all D. The two arguments presented in this paragraph establish that there is at most one value of D satisfying (IV.35). The existence of D for any given case can be established by computation. The fact that these arguments have established the uniqueness of D (in the case of compression) only for $\frac{1}{3} < \lambda$ is of no practical importance as anyone who has attempted to squeeze a rubber pad (or block) to one third of its original thickness can testify.

The stress field in the pad can be found from the following formulas.

$$\begin{aligned}
\tau^{11}/\mu &= \bar{k} + x_Y^2 + y_Y^2 \\
\tau^{22}/\mu &= \bar{k} + x_X^2 + y_X^2 \\
\tau^{33}/\mu &= \bar{k} + 1 \\
\tau^{12}/\mu &= -\left(x_X x_Y - y_X y_Y\right)
\end{aligned} \qquad \text{(IV.37)}$$

where $\bar{k}$ is a nondimensional hydrostatic stress term to be determined from the equilibrium equations and boundary conditions.

Using the results previously found one has, recalling that X is a coordinate with length dimensions and that $\frac{dy}{dY} = \frac{d\bar{y}}{d\bar{Y}}$ but that $\frac{dx}{dY} = \frac{2}{B}\frac{dx}{d\bar{Y}}$,

$$\tau^{11}/\mu = \bar{k} + 4\frac{X^2}{B^2}D\frac{(\cosh(\sqrt{D}\,y)-1)}{(\cosh(\sqrt{D}\,y)+1)} + \frac{D^2}{C^2}\operatorname{sech}^4(\frac{\sqrt{D}}{2}\,y)$$

$$\tau^{22}/\mu = \bar{k} + \frac{C^2}{D^2}\cosh^4(\frac{\sqrt{D}}{2}\,y) \tag{IV.38}$$

$$\tau^{33}/\mu = \bar{k} + 1$$

$$\tau^{12}/\mu = \frac{C}{\sqrt{D}}\,\frac{X}{B}\sinh(\sqrt{D}\,y)$$

Since the displacement field is based on assumption (1) it is not to be expected that the stresses, (IV.38), will satisfy the equilibrium equations

$$\tau^{ij}{}_{,i} = 0 \tag{IV.39}$$

exactly.

The equilibrium equation in the X direction becomes, in terms of (IV.38),

$$\bar{k}_X + \frac{4XD}{B^2}\,\frac{(\cosh(\sqrt{D}\,y)-2)}{(\cosh(\sqrt{D}\,y)+1} = 0 \tag{IV.40}$$

and that in the Y direction is

$$\bar{k}_Y - \frac{1}{B}\,\frac{C}{\sqrt{D}}\sinh(\sqrt{D}\,y) = 0 \tag{IV.41}$$

In obtaining (IV.40) and (IV.41) one must recall that the stresses are explicit functions of y and the differentiation is to be performed with respect to Y.

Since both (IV.40) and (IV.41) cannot be satisfied simultaneously it is necessary to choose a way to compute $\bar{k}$. In view of the stress free boundary condition at $X = \pm A/2$ it will be assumed that (IV.40) holds and $\bar{k}$ will be completely determined by enforcing the condition

$$\tau^{11}(\pm A/2, Y) = 0 \qquad \text{(IV.42)}$$

For (IV.40) to hold it is necessary that

$$\bar{k} = \frac{2X^2D}{B^2}\,\frac{(2-\cosh(\sqrt{D}\,y))}{(\cosh(\sqrt{D}\,y)+1)} + f(Y) \qquad \text{(IV.43)}$$

Substitution of (IV.38.1) and (IV.43) into the boundary condition (IV.42) leads to

$$f(Y) = -\frac{D^2}{C^2}\,\mathrm{sech}^4\left(\frac{\sqrt{D}}{2}\,y\right) - \frac{D}{2}\,\frac{A^2}{B^2}\,\frac{\cosh(\sqrt{D}\,y)}{(\cosh(\sqrt{D}\,y)+1)} \qquad \text{(IV.44)}$$

so that the stresses are given by

$$\tau^{11}/\mu = \frac{2D\cosh(\sqrt{D}\,y)}{(1+\cosh(\sqrt{D}\,y))}\left(\frac{X^2}{B^2} - \frac{A^2}{4B^2}\right)$$

$$\tau^{22}/\mu = \frac{2D\cosh(\sqrt{D}\,y)}{(1+\cosh(\sqrt{D}\,y))}\left\{(2\,\mathrm{sech}(\sqrt{D}\,y) - 1)\frac{X^2}{B^2} - \frac{A^2}{4B^2}\right\}$$

$$+\frac{C^2}{D^2}\cosh^4\left(\frac{\sqrt{D}}{2}\,y\right) - \frac{D^2}{C^2}\,\mathrm{sech}^4\left(\frac{\sqrt{D}}{2}\,y\right) \qquad \text{(IV.38a)}$$

$$\tau^{33}/\mu = \frac{2D\cosh(\sqrt{D}\,y)}{(1+\cosh(\sqrt{D}\,y))}\ \{(2\text{sech}\ (\sqrt{D}\,y)-1)\frac{X^2}{B^2} - \frac{A^2}{4B^2}\}$$

$$+ 1 - \frac{D^2}{C^2}\text{sech}^4\left(\frac{\sqrt{D}}{2}y\right) \qquad \text{(IV.38a)}$$

$$\tau^{12}/\mu = \frac{C}{\sqrt{D}}\ \frac{X}{B}\sinh\sqrt{D}\,y$$

2) <u>The Case D < 0</u>

The difference between the case $D > 0$ and the case $D < 0$ depends on the fact that for $D < 0$

$$y - E = \frac{2}{\sqrt{|D|}}\tan^{-1}\left(\sqrt{\frac{C}{|D|}w - 1}\right) \qquad \text{(IV.45)}$$

It is found that in the present case

$$Y = \frac{C}{2|D|^{3/2}}\ (\sin(\sqrt{|D|}\ y) + \sqrt{|D|}\ y) \qquad \text{(IV.46)}$$

$$\frac{dy}{dY} = w = \frac{|D|}{C}\sec^2\left(\frac{\sqrt{|D|}}{2}y\right) \qquad \text{(IV.47)}$$

$$\frac{d^2y}{dY^2} = \frac{|D|^{5/2}}{C^2}\sec^4\left(\frac{\sqrt{|D|}}{2}y\right)\tan\left(\frac{\sqrt{|D|}}{2}y\right) \qquad \text{(IV.48)}$$

and that

$$C = \frac{2|D|}{\cos(\sqrt{|D|}/\lambda) + 1} \qquad \text{(IV.49)}$$

$$\cos\left(\frac{\sqrt{|D|}}{\lambda}\right) + 1 = \frac{1}{\sqrt{|D|}}\sin\left(\frac{\sqrt{|D|}}{\lambda}\right) + \frac{1}{\lambda} \qquad \text{(IV.50)}$$

In this case, $D < 0$, it is desired to show that $\lambda > 1$ is implied. Expand (IV.50) in Taylor series.

$$2 - \frac{|D|}{2!\lambda^2} + \frac{|D|^2}{4!\lambda^4} + \ldots = \frac{1}{\lambda}\left(2 - \frac{|D|}{3!\lambda^2} + \frac{|D|^2}{5!\lambda^4} + \ldots\right) \qquad \text{(IV.51)}$$

The series on the right side will be smaller than the series on the left side for $|D|/\lambda^2$ sufficiently small so that in this case $\lambda > 1$ is implied if (IV.51) is to hold. If D is a continuous function of λ, and since $\lambda = 1 \Longleftrightarrow D = 0$, then for all $D < 0$ $\lambda > 1$.

Since the results of this subsection are, in a sense, analytic continuations of the results for $\lambda < 1$ as λ passes through unity, i.e. as $\lambda \to 1+$ and $\lambda \to 1-$ corresponding expressions coalesce, it becomes reasonable to choose those values of D which make D a continuous function of λ across $\lambda = 1$ and so the question of the uniqueness of D for the case of extension is bypassed.

By means of computations similar to those made in the case of $\lambda < 1$ it is found that

$$\tau_{11}/\mu = \frac{2|D|\cos(\sqrt{|D|}y)}{(1+\cos(\sqrt{|D|}\,y))}\left(\frac{A^2}{4B^2} - \frac{X^2}{B^2}\right)$$

$$\tau_{22}/\mu = \frac{2|D|\cos(\sqrt{|D|}\,y)}{(1+\cos(\sqrt{|D|}\,y))}\left\{\frac{A^2}{4B^2} - (2\sec(\sqrt{|D|}\,y)-1)\frac{X^2}{B^2}\right\}$$
$$+ \frac{C^2}{D^2}\cos^4\left(\frac{\sqrt{|D|}}{2}y\right) - \frac{D^2}{C^2}\sec^4\left(\frac{\sqrt{|D|}}{2}y\right)$$

$$\tau_{33}/\mu = \frac{2|D|\cos(\sqrt{|D|}\,y)}{1+\cos(\sqrt{|D|}\,y)}\left\{\frac{A^2}{4B^2} - (2\sec(\sqrt{|D|}\,y)-1)\frac{X^2}{B^2}\right\}$$
$$+ 1 - \frac{D^2}{C^2}\sec^4\left(\frac{\sqrt{|D|}}{2}\,y\right) \qquad \text{(IV.52)}$$

$$\tau_{12}/\mu = \frac{C}{\sqrt{|D|}}\ \frac{X}{B} \sin\left(\sqrt{|D|}\ y\right) \tag{IV.52}$$

5. The Linearized Problem

Before any numerical results based on Section 4 are presented it is desired to make an analysis of the thin, incompressible pad under the assumptions of infinitesimal elasticity. The analysis based on assumption (1) and pursued by means of the principle of stationary potential energy seems more straightforward than those found in the literature [23, 24] although, of course, Lindsey, et al. were only incidentally concerned with the special case of an incompressible material. The gross results of the two prior analyses and the present one are in agreement; however the details of the present analysis seem to be cleaner.

For the linear, i.e. infinitesimal, case (IV.24) reduces to

$$\frac{d^4 y}{dY^4} = 0 \tag{IV.53}$$

Since it is customary, in infinitesimal elasticity, to work with the displacement vector introduce now

$$v = (Y - y) \tag{IV.54}$$

so that the governing differential equation may be written as

$$\frac{d^4 v}{dY^4} = 0 \quad \text{or} \quad \frac{d^4 v}{dy^4} = 0 \tag{IV.55}$$

since the mapping is the identity mapping to within a first order infinitesimal mapping. The second form of (IV.55) which is consistent with the usual formulation of infinitesimal elasticity will be used.

The appropriate boundary conditions, corresponding to (IV.10), (IV.11), (IV.12), and (IV.13) are

$$\frac{dv}{dy}(1) = 0 \tag{IV.56}$$

$$\frac{d^2v}{dy^2}(0) = 0 \tag{IV.57}$$

$$v(1) = \Delta \ ; \quad \Delta = \lambda - 1 \tag{IV.58}$$

$$v(0) = 0 \tag{IV.59}$$

The solution to the boundary value problem is

$$v = \frac{\Delta}{2}(3y - y^3) \tag{IV.60}$$

Introduce, now, the displacement component

$$u = X - x = -x\frac{dv}{dy} = -\frac{3}{2}\Delta x\,(1-y^2) \tag{IV.61}$$

where the final form follows directly from the linearized incompressibility condition for the case of plane strain.

$$\begin{aligned} \varepsilon_x &= \frac{\partial u}{\partial x} = -3/2\,\Delta\,(1-y^2) \\ \varepsilon_y &= \frac{\partial v}{\partial y} = 3/2\,\Delta\,(1-y^2) \\ 2\gamma_{xy} &= \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = 6\Delta\frac{xy}{B} \end{aligned} \tag{IV.62}$$

are the components of the infinitesimal strain tensor. The factor 2/B enters into γ_{xy} because x was never nondimensionalized. Note that dv/dy remains the same whether or not v and y are nondimensionalized.

The nominal stresses are given by the following expressions for the infinitesimal plane strain of an incompressible material.

$$
\begin{aligned}
t^{11}/\mu &= 2\varepsilon_x + \bar{p} \\
t^{22}/\mu &= 2\varepsilon_y + \bar{p} \qquad \text{(IV.63)} \\
t^{12}/\mu &= 2\gamma_{xy}
\end{aligned}
$$

where $\bar{p}$ is a nondimensional hydrostatic term.

For the remainder of this section dimensional x and y will be used.

When the expressions for stresses are placed in terms of displacement components and the stresses are then substituted into the (nominal) equilibrium equations it is found that

$$
\begin{aligned}
\bar{p}_x + 3\Delta\, x/(\tfrac{B}{2})^2 &= 0 \\
&\qquad \text{(IV.64)} \\
\bar{p}_y - 3\Delta\, y/(\tfrac{B}{2})^2 &= 0
\end{aligned}
$$

Consequently

$$\bar{p} = \frac{3\Delta}{2}\left[y^2/(\tfrac{B}{2})^2 - x^2/(\tfrac{B}{2})^2\right] + \text{const.} \qquad \text{(IV.65)}$$

where the constant will be determined from a mean stress free condition at $x = A/2$, i.e.

$$\int_0^{B/2} t^{11}\big|_{x=A/2}\, dy = 0 \qquad \text{(IV.66)}$$

Thus

$$\bar{p} = \frac{3\Delta}{2}\left[(y^2/(\tfrac{B}{2})^2 + 1) + (\frac{A^2}{B^2} - x^2/(\tfrac{B}{2})^2) \right] \qquad \text{(IV.67)}$$

and finally

$$t^{11}/\mu = \frac{3\Delta}{2}\left[(\frac{A^2}{B^2} - \frac{4x^2}{B^2}) + (\frac{12y^2}{B^2} \quad 1) \right]$$

$$t^{22}/\mu = \frac{3\Delta}{2}\left[(\frac{A^2}{B^2} - \frac{4x^2}{B^2}) + (3 - \frac{4y^2}{B^2}) \right] \qquad \text{(IV.63a)}$$

$$t^{12} = 12\Delta \frac{xy}{B^2}$$

These results are virtually the same as those of references [23] and [24] but the details of the present analysis are cleaner. As a matter of fact, the "extraneous" stresses acting on the free edges are small compared to the stresses at the center of the pad for $B \ll A$ so that if St. Venant's principle is invoked it may be claimed that the analysis is virtually exact near the center of the pad. Note that in the present infinitesimal elasticity analysis it was possible to satisfy both equilibrium equations whereas this was not possible for the finite strain analysis.

6. Numerical Examples

In this section the results of Sections 4 and 5 are used to compute displacement and stress fields for particular numerical cases. The ratio A/B is taken to be 20 in all cases as was done in reference [23], and the values 1.1, 0.9, and 0.5 are chosen for the gross stretch ratio λ.

1a) <u>The Case $\lambda = 1.1$ (non-linear analysis)</u>

Newton's method is used to find D from (IV.50) for $\lambda = 1.1$ and then C is computed from (IV.49). The results are

$$\begin{aligned} D &= -0.669 \\ C &= 0.771 \end{aligned} \tag{IV.68}$$

so that the displacement field is found to be, from (IV.46), (IV.47), and (IV.3),

$$\begin{aligned} Y &= 0.704\,(\sin(0.818y) + 0.818y) \\ x &= 1.152\,X\cos^2(0.409y) \end{aligned} \tag{IV.69}$$

The stress field, (IV.52), is given by

$$\tau^{11}/\mu = \frac{1.338\cos(0.818y)}{(1+\cos(0.818y)}\,(100 - \frac{X^2}{B^2})$$

$$\begin{aligned} \tau^{22}/\mu &= \frac{1.338\cos(0.818y)}{(1+\cos(0.818y)}\,[100-(2\sec(0.818y)-1)\,\frac{X^2}{B^2}] \\ &\quad + 1.372\cos^4(0.409y) - 0.754\sec^4(0.409y) \end{aligned} \tag{IV.70}$$

$$\begin{aligned} \tau^{33}/\mu &= \frac{1.338\cos(0.818y)}{(1+\cos(0.818y))}\,[100 - (2\sec(0.818y)-1)\frac{X^2}{B^2}] \\ &\quad + 1-0.754\sec^4(0.409y) \end{aligned}$$

$$\tau^{12}/\mu = 0.942\,\frac{X}{B}\sin(0.818y)$$

At the center of the pad it is found that

$$\tau^{11}/\mu = 66.9$$

$$\tau^{22}/\mu = 67.5 \qquad \text{(IV.70a)}$$

$$\tau^{33}/\mu = 67.1$$

so that a state of virtually hydrostatic tension exists there.

(IV.69.2) states that the undeformed pad bulged out 15.2% of A/2 at its center as shown in Figure IV.2 below.

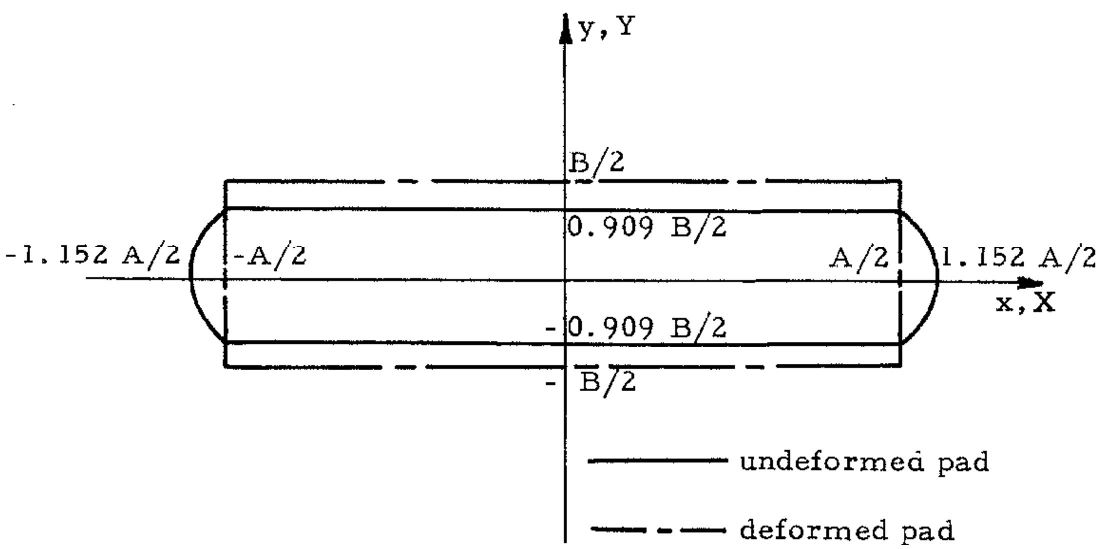

FIGURE IV.2. Deformed and Undeformed Pad for $\lambda = 1.1$
(not to scale)

It is noted that the analysis gives, on the edges $X = \pm A/2$, an "extraneous" shear stress

$$\tau^{12}\big|_{X=\pm A/2} = 9.42 \sin(0.818y) \qquad \text{(IV.71)}$$

whose maximum value at $y = 0.909$ (or $Y = 1$) is 6.37. Since the total shearing force on each of the edges is equipollent to zero and the maximum value of the shearing stress on each of the edges is an order of magnitude smaller than the normal stresses acting at the center of the pad it is reasonable to conclude from St. Venant's principle (assumed here to be valid for finite elasticity) that these "extraneous" shear stresses have little effect on the normal stresses at the center of the pad.

At this point it is pertinent to inquire about the relation between the solution of the asymptotic equation (IV.24), and the full equation, (IV.8), i.e. how well does the solution of (IV.24) satisfy (IV.8)? For the present case, $A/B = 20$ and $\lambda = 1.1$, direct calculation gives the following results.

a) at $Y = 0$ the solution of (IV.24) satisfies (IV.8) exactly. This is a general result noted previously.

b) at $Y = 0.516$ ($y = 1/2\lambda = 0.4545$) the neglected terms are about 2% of the magnitude of each of the terms which are retained.

c) at $Y = 1$ the neglected terms are about 3% of the magnitude of each of the terms which are retained.

Because of the smooth behavior of the functions involved the above information seems sufficient to conclude that the solution of (IV.24) is an excellent approximation to the solution of (IV.8) for the particular case considered.

1b) The Case $\lambda = 1.1$ (linear analysis)

The case $\lambda = 1.1$ corresponds to $\Delta = 0.1$ when the terminology of Section 5 is used.

Recalling that

$$Y = y + v \tag{IV.54}$$

it is found that

$$Y = y + 0.05\,(3y - y^3) \tag{IV.72}$$

A tabular comparison of (IV.64.1) and (IV.72) is made below.

y	$Y_{nonlinear}$	Y_{linear}
0	0	0
0.227	0.261	0.261
0.455	0.517	0.518
0.682	0.765	0.768
0.909	1	1

Table IV.1. $Y_{nonlinear}$ vs. Y_{linear} for $\lambda = 1.1$

The agreement between the nonlinear and linear results for the moderate strains considered is very close indeed. It must be borne in mind, however, that somewhat different physical problems are being compared. The linear results are for a body of originally rectangular form which is deformed to a concave section whereas the nonlinear analysis is for an originally convex section which is deformed to a rectangle (cf. Fig. IV.2).

The stresses given by (IV.63) are nominal stresses. At the center of the pad these are

$$t^{11}/\mu = 59.9$$

$$t^{22}/\mu = 60.5 \qquad \text{(IV.73)}$$

$$t^{33}/\mu = \tfrac{1}{2}\left(\frac{t^{11}+t^{22}}{\mu}\right) = 60.2$$

It is seen that if the undeformed body is of rectangular cross-section the state of nominal stress at the center of the pad is approximately hydrostatic tension.

The relation between the true stresses and nominal stresses is given by

$$t^{ij} = \tau^{ir}\left(\delta^j_r + \frac{\partial v^j}{\partial x^r}\right) \qquad \text{(IV.74)}$$

for a rectangular coordinate system so that at the center of the pad the true stresses are

$$\tau^{11}/\mu = 69.8$$

$$\tau^{22}/\mu = 52.6 \qquad \text{(IV.75)}$$

$$\tau^{33}/\mu = 60.2$$

This is a state of triaxial tension only roughly approximating hydrostatic tension.

It may be concluded, then, that if it is desired to reach an almost hydrostatic tensile state in a specimen which fails at more than a couple of percent extension, that the original cross-section should be barrel shaped. Once the stretch ratio at failure is known approximately it is possible to design a suitable specimen. These comments apply as well to the common "poker chip" test specimen.

Examination of the nominal stress expressions, (IV.63a), shows that at the free edges the "extraneous" stresses are small compared to the stresses at the center of the pad. t^{11}/μ varies parabolically from -0.15 to 0.3 as y goes from 0 to 0.909 and τ^{12} varies linearly from 0 to 6 in the same domain. Since in the linear case the equilibrium equations are both satisfied it is possible to have great confidence in the results obtained at the center of the pad when appeal is made to St. Venant's principle.

2a) <u>The Case $\lambda = 0.9$ (nonlinear analysis)</u>

From (IV.33) and (IV.34) it is found that

$$C = -0.457 \qquad D = 0.537 \tag{IV.76}$$

so that from (IV.28), (IV.29), and (IV.3) the displacement field is found to be

$$Y = 0.593\,(\sinh(0.734y) + 0.734y) \qquad x = 0.851\,X\cosh^2(0.367y) \tag{IV.77}$$

The stress field is given by

$$\tau^{11}/\mu = \frac{1.074 \cosh(0.734y)}{(1+\cosh(0.734y))}\left(\frac{X^2}{B^2} - 100\right)$$

$$\tau^{22}/\mu = \frac{1.074 \cosh(0.734y)}{(1+\cosh(0.734y))}\left\{(2\operatorname{sech}(0.734y)-1)\frac{X^2}{B^2} - 100\right\}$$

$$+ 0.723 \cosh^4(0.367y) - 1.384 \operatorname{sech}^4(0.367y)$$

$$\tau^{33}/\mu = \frac{1.074 \cosh(0.734y)}{(1+\cosh(0.734y))}\left\{(2\operatorname{sech}(0.734y)-1)\frac{X^2}{B^2} - 100\right\} \qquad \text{(IV.78)}$$

$$+ 1 - 1.384 \operatorname{sech}^4(0.367y)$$

$$\tau^{12}/\mu = -0.623 \frac{X}{B} \sinh(0.734y)$$

At the center of the pad

$$\tau^{11}/\mu = -53.7$$

$$\tau^{22}/\mu = -54.4 \qquad \text{(IV.78a)}$$

$$\tau^{33}/\mu = -54.1$$

Note the significantly different absolute values of the almost hydrostatic stresses for the cases of 10% extension ($\lambda = 1.1$) and 10% compression ($\lambda = 0.9$). This is not really surprising when it is observed that in the compression case the undeformed pad was concave whereas in the extension case it was convex.

The absolute value of the "extraneous" shear stress at the bonded corners is found to be 5.66 in this case.

Computation again indicates that for the present case the solution of (IV.24) is an excellent approximation to the solution of (IV.8). In fact, the degree of approximation is about the same as for the case $\lambda = 1.1$.

2b) The Case $\lambda = 0.9$ (linear analysis)

The case $\lambda = 0.9$ corresponds to $\Delta = -0.1$ so that the displacement and nominal stress fields are the same, except for a change in sign, as in the case of $\lambda = 1.1$. The true stresses at the center of the pad are

$$\tau^{11}/\mu = -52.6$$
$$\tau^{22}/\mu = -69.8 \qquad \text{(IV.80)}$$
$$\tau^{33}/\mu = -60.2$$

which again only roughly approximates a state of hydrostatic stress.

3) The Case $\lambda = 0.5$ (nonlinear analysis)

In this case the values of C and D from (IV.33) and (IV.34) are

$$C = -0.440$$
$$D = 1.438 \qquad \text{(IV.81)}$$

so that the displacement field is found to be

$$Y = 0.127\ (\sinh(1.199y) + 1.199y)$$
$$x = 0.306\ X \cosh^2(0.600y) \qquad \text{(IV.82)}$$

The stress field is given by

$$\tau^{11}/\mu = \frac{2.876\cosh(1.199y)}{(1+\cosh(1.199y))}\left[\frac{X^2}{B^2} - 100\right]$$

$$\tau^{22}/\mu = \frac{2.876\cosh(1.199y)}{(1+\cosh(1.199y))}\left[(2\ \mathrm{sech}(1.199y)-1)\frac{X^2}{B^2} - 100\right]$$
$$+ 0.094\cosh^4(0.600y) - 10.68\ \mathrm{sech}^4(0.600y)$$

(IV.83)

$$\tau^{33}/\mu = \frac{2.876\cosh(1.199y)}{(1+\cosh(1.199y))}\left[(2\mathrm{sech}(1.199y)-1)\frac{X^2}{B^2} - 100\right]$$
$$+ 1 - 10.68\mathrm{sech}^4(0.600y)$$

$$\tau^{12}/\mu = -0.367\frac{X}{B}\sinh(1.199y)$$

At the center of the pad

$$\tau^{11}/\mu = -143.8$$

$$\tau^{22}/\mu = -154.4 \qquad \text{(IV.83a)}$$

$$\tau^{33}/\mu = -153.5$$

The magnitude of the shear stress, τ^{12}/μ, is found to be 19.9 at the corners.

The results just given indicate that for very large compression the state of stress at the center of the pad does not approximate hydrostatic compression as well as it does for moderate compression.

One obvious reason that the stress at the center of the pad does not increase as rapidly as the amount of compression is that the undeformed body taken for increased compression is softer since its concavity is greater.

The deformation field for $\lambda = 0.5$ is calculated now so that the shape of the undeformed body may be compared with the final shape.

y	Y	$x\vert_{X=A/2}$
0	0	0.306 A/2
0.5	0.157	0.335 A/2
1.0	0.344	0.429 A/2
1.5	0.602	0.629 A/2
2.0	1.0	A/2

Table IV.2. Deformation Field for $\lambda = 0.5$.

The results tabulated above are sketched in Figure IV.3 in order to give a graphic demonstration of the magnitude of the deformation.

Finally, an estimate will be given of how well the full differential equation, (IV.8), is satisfied.

a) at Y = 0 (IV.8) is satisfied exactly.

b) at Y = 0.602 (y = 1.5) the neglected terms of (IV.8) are about 2% of the magnitude of each of the terms in (IV.24).

c) at Y = 1.0 (y = 2.0) the neglected terms of (IV.8) are almost 4% of the magnitude of each of the terms retained in (IV.24).

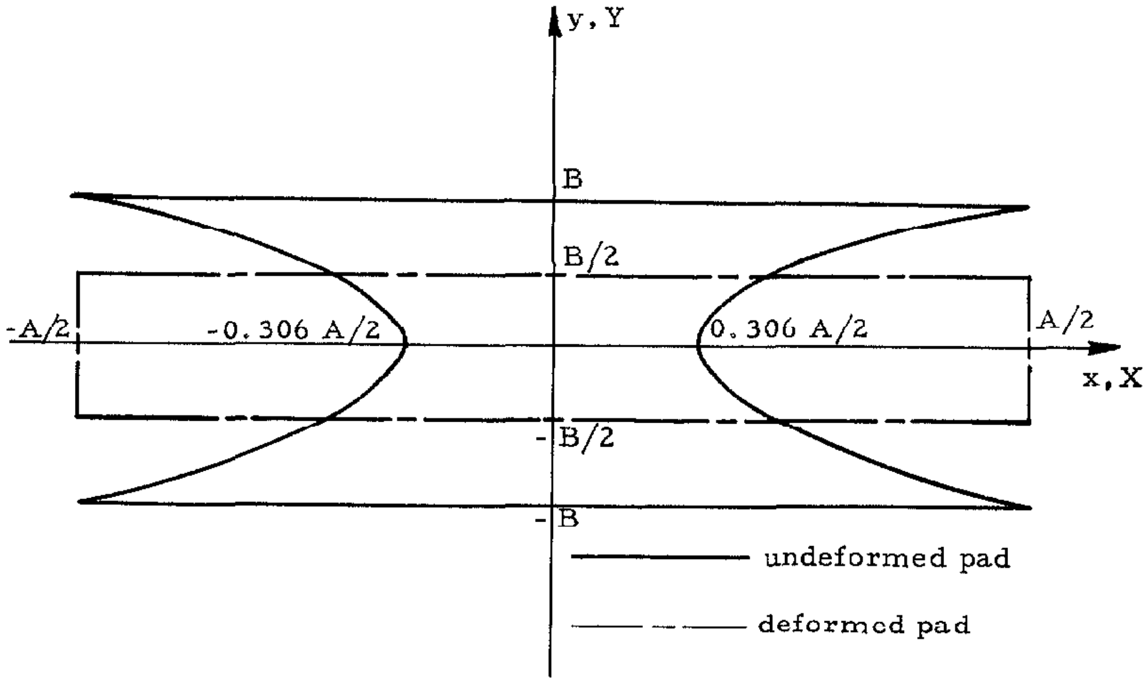

FIGURE IV.3. Deformed and Undeformed Pad for $\lambda = 0.5$
(y, Y scales stretched 3 times)

Comparing the above results with the cases $\lambda = 1.1$ and $\lambda = 0.9$ shows that the error introduced by neglecting the part of (IV.8) multiplied by B^2/A^2 is essentially independent of λ and depends almost wholly on B^2/A^2 so that for B^2/A^2 sufficiently small the use of (IV.24) instead of (IV.8) is justified and introduces errors into the analysis which are probably of less consequence than those introduced into the problem by assumption (1).

7. A Related Elasticity Problem

The work already presented in this chapter is based on the application of the principle of stationary potential energy to a partially restricted deformation field. It is assumed, when the governing Euler-Lagrange differential equation is derived, that the stress boundary conditions are satisfied. Therefore, within the restriction of assumption (1), the solution of the Euler-Lagrange equation provides a deformation field which renders the total potential energy of the system a minimum (if the equilibrium is stable). Since assumption (1) is not exactly true for the physical problem considered a penalty must be paid in that the deformation field leads to a stress field which does not satisfy the equilibrium equations. Because it was assumed that the elastic medium was incompressible it was possible to satisfy one equilibrium equation exactly (and also the normal stress boundary condition). The overall solution is, in an energy sense, the best to be had once assumption (1) is made. Indeed, by comparing the solution of the nonlinear problem, for moderate strains, with the results of the linearized problem which satisfies the equilibrium equations it seems likely that an excellent estimate of the almost hydrostatic stress at the center of the pad is obtained.

In this section a solution of the Adkins equation

$$x_X(\nabla^2 x)_Y - x_Y(\nabla^2 x)_X + y_X(\nabla^2 y)_X - y_Y(\nabla^2 y)_X = 0 \qquad \text{(IV.84)}$$

based on assumption (1) will be presented. This solution will be the exact solution to some problem in the theory of plane strain of a neo-Hookean solid. However, since no attempt is made in the process to effect a reconciliation between the contradictory assumptions that the surface at $X = A/2$ is stress free and that (1) holds, it will be found that the hydrostatic stress predicted as occurring at the center of the pad will differ significantly from the hydrostatic stress predicted by the variational analysis. In addition, the "extraneous" stresses acting on the free boundary will be of the same order of magnitude as the hydrostatic stress at the center of the pad. On the basis of the energy criterion, and what has just been stated, the variational solution would appear to be the more reliable approximation to the solution of the physical problem which was proposed originally.

Assumption (1), when substituted into (IV.84), leads after some manipulation to

$$\frac{d}{dY}\left(\frac{\frac{d^3y}{dY^3}}{\frac{dy}{dY}}\right) - \frac{2\left(\frac{d^2y}{dY^2}\right)^2}{\left(\frac{dy}{dY}\right)^2} = 0 \tag{IV.85}$$

where y and Y are to be interpreted as nondimensional quantities for the present.

If substitutions of the sort used in Section 3 are made the solution of (IV.85) is found to depend on the integration of

$$dy = \frac{dw}{\sqrt{Dw^2 - C}} \tag{IV.86}$$

$$dY = \frac{dw}{w\sqrt{Dw^2 - C}} \tag{IV.87}$$

together with the boundary conditions (IV.10), (IV.11), (IV.12), (IV.13). Again $w = dy/dY$.

The analytic form of the solution of (IV.85) depends on the sign of D. The case $D < 0$ is solved by

$$y = \frac{1}{\sqrt{|D|}} \cos^{-1}\left(\operatorname{sech}\left(\sqrt{|C|}\, Y\right)\right) \tag{IV.88}$$

or

$$Y = \frac{1}{\sqrt{|C|}} \operatorname{sech}^{-1}\left(\cos\sqrt{|D|}\, y\right) \tag{IV.88a}$$

This also may be written as [25]

$$Y = \frac{1}{\sqrt{|C|}} \ln\left[\sec\left(\sqrt{|D|}\, y\right) + \tan\left(\sqrt{|D|}\, y\right)\right] \tag{IV.88b}$$

The relation between C and D is given by

$$D = C \operatorname{sech}^2\left(\sqrt{|C|}\right) \tag{IV.89}$$

and the transcendental equation to be solved is

$$\cos\left(\frac{\sqrt{|C|}}{\lambda} \operatorname{sech}\sqrt{|C|}\right) = \operatorname{sech}\sqrt{|C|} \tag{IV.90}$$

As is shown in Figure IV.4, where $|C|$ is given as a function of λ, $|C| \to 0$ as $\lambda \to 1$ and $|C| \to \infty$ as $\lambda \to 0$.

The stress field, associated with the above displacement field, which satisfies the normal stress boundary condition in the mean, i.e.

$$\int_0^1 \frac{\tau^{11}}{\mu}\Big|_{X=A/2} dY = 0 \ , \tag{IV.91}$$

is

$$\tau^{11}/\mu = \frac{|D|\sinh(2\sqrt{|C|})}{4\sqrt{|C|}}\left(\frac{1}{2|C|} - \frac{A^2}{B^2}\right) + |D|\cosh(2\sqrt{|C|}Y)\left(\frac{2X^2}{B^2} - \frac{1}{4|C|}\right)$$

$$- \frac{\sqrt{|C|}}{|D|}\tanh\sqrt{|C|} + \frac{C}{D}\operatorname{sech}^2(\sqrt{|C|}\ Y$$

$$\tau^{22}/\mu = \frac{|D|\sinh(2\sqrt{|C|})}{4\sqrt{|C|}}\left(\frac{1}{2|C|} - \frac{A^2}{B^2}\right) + 2|D|\frac{X^2}{B^2}$$

$$- \frac{\sqrt{|C|}}{|D|}\tanh\sqrt{|C|} + \frac{D}{2C}\left(1 + \tfrac{1}{2}\cosh\left(2\sqrt{|C|}\ Y\right)\right) \qquad \text{(IV.92)}$$

$$\tau^{33}/\mu = \frac{|D|\sinh(2\sqrt{|C|})}{4\sqrt{|C|}}\left(\frac{1}{2|C|} - \frac{A^2}{B^2}\right) + 2|D|\frac{X^2}{B^2}$$

$$- \frac{D}{4C}\cosh\left(2\sqrt{|C|}\ Y\right) - \frac{\sqrt{|C|}}{|D|}\tanh\sqrt{|C|} + 1$$

$$\tau^{12}/\mu = \frac{D}{\sqrt{|C|}}\ \frac{X}{B}\sinh\left(2\sqrt{|C|}\ Y\right)$$

For $\lambda = 0.9$ it is found that

$$C = -0.349$$

$$D = -0.251 \qquad \text{(IV.93)}$$

so that

$$y = 1.995\cos^{-1}(\operatorname{sech}(0.591Y)) \qquad \text{(IV.88a)}$$

$$\frac{dy}{dY} = w = 1.182\operatorname{sech}(0.591\ Y) \qquad \text{(IV.94)}$$

$$x = 0.846\ X \cosh (0.591\ Y) \tag{IV.95}$$

and, for A/B = 20,

$$\tau^{11}/\mu = -63.8 + \cosh (1.182Y)(0.502 \frac{X^2}{B^2} - 0.178) + 1.39 \operatorname{sech}^2 (0.591Y)$$

$$\tau^{22}/\mu = -63.4 + 0.502 \frac{X^2}{B^2} + 0.360 \cosh (1.182Y) \tag{IV.92a}$$

$$\tau^{33}/\mu = -62.8 + 0.502 \frac{X^2}{B^2} - 0.178 \cosh (1.182Y)$$

$$\tau^{12}/\mu = -0.425 \frac{X}{B} \sinh (1.182Y)$$

At the center of the pad (IV.92a) shows that

$$\tau_{11}/\mu = -62.6$$

$$\tau_{22}/\mu = -63.0 \tag{IV.96}$$

$$\tau_{33}/\mu = -63.0$$

The virtually hydrostatic stress at the center of the pad is, in this case, about 16% greater in magnitude than that obtained from the previously presented variational solution. In addition, the present elasticity solution requires, at the free edges, an "extraneous" normal stress τ^{11}/μ whose magnitude varies from -12.4 at Y = 0 to 26.5 at Y = 1. While it is true that the present calculations deal with an exact elasticity solution to some problem it is believed, as previously indicated, that the variational solution is a better approximation to the physical problem which was originally posed.

Figures IV.5 through IV.8 provide a comparison of the stresses given by the two solutions on the lines $X = 0$, $X = A/2$, $Y = 0$, and $Y = 1$.

The following table compares the deformation fields given by the two solutions. It is computed from (IV.28), (IV.77.1), (IV.88a), and (IV.94). dy/dY is given in the table because $x = \frac{X}{\frac{dy}{dY}}$.

Y	$y_{var.}$	$y_{elas.}$	$\frac{dy}{dY}$ var.	$\frac{dy}{dY}$ elas.
1	1.111	1.111	1.000	1.000
0.802	0.888	0.891	1.057	1.062
0.592	0.666	0.675	1.105	1.111
0.391	0.444	0.449	1.143	1.153
0.194	0.222	0.227	1.170	1.174
0	0	0	1.175	1.182

Table IV.3. Deformation fields for the case $A/B = 20$, $\lambda = 0.9$

The above table shows that the deformation fields computed in the two different manners are very similar. These results also are shown in Figure IV.9.

The case $D > 0$ has the solution

$$y = \frac{1}{\sqrt{D}} \cosh^{-1} (\sec(\sqrt{C}\, Y)) \qquad \text{(IV.97)}$$

or

$$y = \frac{1}{\sqrt{D}} \ln [\sec(\sqrt{C}\, Y) + \tan (\sqrt{C}\, Y)] \qquad \text{(IV.97a)}$$

$$D = C \sec^2 (\sqrt{C}) \qquad \text{(IV.98)}$$

and $$\cosh\left(\frac{\sqrt{C}}{\lambda}\sec(\sqrt{C})\right) = \sec(\sqrt{C}) \tag{IV.99}$$

8. Summary

In this chapter an approximate solution to a non-trivial, mixed boundary value, plane strain problem of a neo-Hookean body was found. The problem, that of a thin pad bonded on its faces to a rigid testing machine and then subjected to extension or compression, was solved by making a reasonable geometrical assumption that led to a highly nonlinear fourth order ordinary boundary value problem. An asymptotic approximation to the differential equation involved was solved exactly and this solution was shown to be an excellent approximation to the solution of the complete differential equation provided that the pad was sufficiently thin. The errors introduced by the geometrical assumption were not assessed since to be able to do so would imply a knowledge of the exact solution to the problem that is not available. However on the basis of a linear analysis and St. Venant's principle (assumed valid for large strain theory) it seems likely that the significant stresses, at the center of the pad, were determined reasonably well. Comparison with a related large strain elasticity problem indicates that the deformation field is approximated very closely by the variational solution.

Of possible future practical interest is the observation that if a "poker chip" specimen fails at more than a couple of percent extension it is worthwhile to consider making it barrel shaped rather than cylindrical in order that a more nearly hydrostatic state of tension will exist at its center.

CHAPTER V. THE FINITE PLANE STRETCHING OF A STRIP BONDED AT ITS ENDS

1. Introduction

The finite elastic deformation of a neo-Hookean strip bonded at its ends to rigid supports and stretched lengthwise is studied in this chapter under the assumption of plane strain. The appropriate differential equation, derived by means of the principle of stationary potential energy, is essentially the same as that arising in the study of the thin pad. However, different techniques are used to obtain approximate solutions for the strip problem because in the present case a small parameter multiplies the higher order terms of the differential equation rather than the lower order terms and, consequently, the solution of the strip problem behaves in a markedly different manner than does the solution of the thin pad problem. The techniques used are

a) a direct minimization of the energy integral

b) a singular perturbation study of the Euler-Lagrange equation associated with the problem.

It is found that for the particular numerical example considered both approximate solutions provide similar information about the pertinent aspects of the problem.

2. Formulation of the Strip Problem

In the undeformed state the strip, shown in Figure V.1 below, is bonded to rigid supports at $y = \pm B/2$ and is free on its sides $x = \pm A/2$. The deformation consists of a gross stretching of

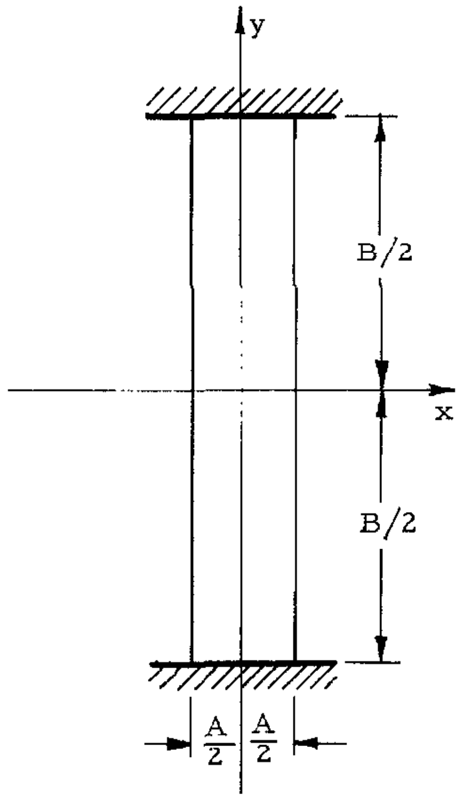

FIGURE V.1. The undeformed strip.

the strip in the y direction so that the final length of the strip is λB. It is assumed that there is no deformation normal to the x-y plane, i.e. a plane strain deformation is considered. This situation can be approximated, experimentally, near the center of a strip whose dimension in the direction normal to the x-y plane is considerably greater than B.

In the deformed strip a material point which was originally at (x, y) moves to the position (X, Y). For A/B sufficiently small it is reasonable to assume that

$$Y = Y(y) \tag{V.1}$$

This assumption is based on the experimental evidence that, except near the bonded edges, the strip is in a state of uniaxial tension (in the x-y plane). [26] It is this assumption that makes the strip problem similar, in formulation, to the thin pad problem.

The incompressibility condition

$$X_x Y_y - X_y Y_x = 1 \qquad (V.2)$$

is reduced, on the basis of (V.1), to

$$X_x = \frac{1}{Y_y} \qquad (V.2a)$$

so that

$$X = \frac{x}{Y_y} \qquad (V.3)$$

since X(0,y) must vanish.

If the notation

$$\bar{y} = \frac{y}{\frac{B}{2}} \; ; \; \bar{Y} = \frac{Y}{\frac{B}{2}} \qquad (V.4)$$

is introduced, the statement of the principle of stationary potential energy becomes, after integration with respect to x,

$$\delta\int_0^1 [\frac{A^2}{3B^2} \frac{\bar{Y}_{\bar{y}\bar{y}}}{\bar{Y}_{\bar{y}}^4} + (\frac{\bar{Y}_{\bar{y}}^2 - 1}{\bar{Y}_{\bar{y}}})^2] \, d\bar{y} = 0 \qquad (V.5)$$

At this point the cumbersome barred notation will be dropped and, unless otherwise noted, y and Y will denote the nondimensional quantities defined in (V.4).

The Euler-Lagrange differential equation found by performing the variation indicated in (V.5) is

$$\frac{d}{dy}\{\epsilon^2[\frac{d}{dy}(\frac{\frac{d^2Y}{dy^2}}{(\frac{dY}{dy})^4}) + \frac{2(\frac{d^2Y}{dy^2})^2}{(\frac{dY}{dy})^5}] - (\frac{dY}{dy} - \frac{1}{(\frac{dY}{dy})^3})\} = 0 \qquad (V.6)$$

where

$$\epsilon = A/\sqrt{3}\ B \qquad (V.7)$$

Using arguments similar to those used in the case of the thin pad the appropriate boundary conditions on (V.6) are found to be

$$Y(0) = 0 \qquad (V.8)$$

$$\frac{d^2Y(0)}{dy^2} = 0 \qquad (V.9)$$

$$Y(1) = \lambda \qquad (V.10)$$

$$\frac{dY(1)}{dy} = 1 \qquad (V.11)$$

Equations (V.6) through (V.11) define the approximate boundary value problem for the thin strip. The striking difference between the strip and thin pad problems is that in the present case the small parameter ϵ multiplies the leading terms of (V.6) instead of the lower order terms so that entirely different approaches to the approximate solution of the mathematical problem must be used.

3. A Direct Minimization of the Energy Integral

Examination of (V.6) suggests that, except near the boundary points, Y behaves like a linear function of y. A simple function having such behavior in the interval $-1 \leq y \leq 1$ is

$$Y = ay + by^n \qquad (V.12)$$

provided that n is large compared to unity. For (V.12) to be an admissible function for the variational principle it is necessary that it satisfy the boundary conditions (V.8), (V.9), (V.10), and (V.11); (V.8) and (V.9) are satisfied automatically and (V.10) and (V.11) will be satisfied if the constants a and b are related to the constant n as follows.

$$a = (\lambda + \frac{\lambda-1}{n-1})$$
$$b = -(\frac{\lambda-1}{n-1}) \qquad (V.13)$$

Therefore an admissible function for the strip problem is

$$Y = (\lambda + \frac{\lambda-1}{n-1})\, y - (\frac{\lambda-1}{n-1})\, y^n \qquad (V.12a)$$

which contains n as the only parameter to be determined by the variational principle.

If n is truly large compared to unity when λ is a moderate multiple of unity it is seen that, except near the bonded edges, the deformation closely approximates that of a uniaxial specimen stretched to λ times its original length.

When (V.12a) is substituted into (V.5) it is found that

$$\frac{d}{dn}\int_0^1 \left\{ \epsilon^2 \frac{n^2(\lambda-1)^2 y^{2(n-2)}}{[(\lambda + \frac{\lambda-1}{n-1}) - \frac{n(\lambda-1)}{(n-1)} y^{(n-1)}]^4} + \left(\frac{[(\lambda + \frac{\lambda-1}{n-1}) - \frac{n(\lambda-1)}{(n-1)} y^{(n-1)}]^2 - 1}{[(\lambda + \frac{\lambda-1}{n-1}) - \frac{n(\lambda-1)}{(n-1)} y^{(n-1)}} \right)^2 \right\} dy = 0 \qquad (V.14)$$

The simplest way to perform the minimization, given ϵ and λ, is to evaluate the integral in (V. 14) for various values of n and then to determine, from a curve, the value of n satisfying (V. 14).

4. A Singular Perturbation Approach to the Euler-Lagrange Equation

Let $$w = \frac{dY}{dy}; \quad w' = \frac{dw}{dy} \qquad (V.15)$$

so that (V. 6) may be written as

$$\frac{d}{dy}\left\{\epsilon^2 \left(\frac{d}{dy}\left(\frac{w'}{w^4}\right) + \frac{2(w')^2}{w^5}\right) - \left(w - \frac{1}{w^3}\right)\right\} = 0 \qquad (V.6a)$$

As $\epsilon \to 0$ the order of the differential equation changes and a solution of the reduced second order equation

$$\frac{d}{dy}\left(w - \frac{1}{w^3}\right) = 0 \qquad (V.16)$$

cannot satisfy all of the boundary conditions associated with (V. 6a). (V. 16) has for its only real valued solution w = constant or that Y is a linear function of y. A solution satisfying the boundary conditions at the origin is that

$$Y_o = \overline{\lambda}\, y \tag{V.17}$$

where $\overline{\lambda}$ is a constant which is not yet specified.

(V.6a) will be studied using some of the ideas arising in the theory of singular perturbation problems. [29] Because of the rather complicated nonlinear nature of (V.6a) an ad hoc approach to the problem will be adopted.

Assume

$$Y = Y_o + \epsilon^{\beta} Y_1(y) = \overline{\lambda} y + \epsilon^{\beta}\, Y_1(y) \tag{V.18}$$

and let

$$\xi = \frac{1-y}{\epsilon^{\alpha}} \; . \tag{V.19}$$

α and β are constants.

When (V.18) and (V.19) are substituted into (V.6) it is found that

$$\frac{d}{d\xi}\Big\{-\epsilon^{(2+\beta-3\alpha)}\frac{d}{d\xi}\Big(\frac{\frac{d^2Y_1}{d\xi^2}}{(\overline{\lambda}-\epsilon^{(\beta-\alpha)}\frac{dY_1}{d\xi})^4}\Big) + \frac{2\epsilon^{2(1+\beta-2\alpha)}(\frac{d^2Y_1}{d\xi^2})^2}{(\overline{\lambda}-\epsilon^{(\beta-\alpha)}\frac{dY_1}{d\xi})^5}$$

$$-\Big[(\overline{\lambda}-\epsilon^{(\beta-\alpha)}\frac{dY_1}{d\xi}) - \frac{1}{(\overline{\lambda}-\epsilon^{(\beta-\alpha)}\frac{dY_1}{d\xi})^3}\Big]\Big\} = 0 \tag{V.20}$$

If $\alpha = \beta = 1$ the leading terms of (V.20) will be retained, independent of ϵ, and if

$$W = \bar{\lambda} - \frac{dY_1}{d\xi} \tag{V.21}$$

it is possible to write (V.20) as

$$\frac{d}{d\xi}\left[\frac{d}{d\xi}\left(\frac{\frac{dW}{d\xi}}{W^4}\right) + \frac{2\left(\frac{dW}{d\xi}\right)^2}{W^5} - \left(W - \frac{1}{W^3}\right)\right] = 0 \tag{V.22}$$

The boundary conditions (V.8), (V.9), (V.10), and (V.11) become, in terms of Y_1 and ξ,

$$Y_1(\infty) = 0 \tag{V.23}$$

$$\frac{d^2Y_1(\infty)}{d\xi^2} = 0 \tag{V.24}$$

$$Y_1(0) = \frac{\lambda - \bar{\lambda}}{\epsilon} \tag{V.25}$$

$$\frac{dY_1(0)}{d\xi} = (\bar{\lambda} - 1) \tag{V.26}$$

(V.23) and (V.24) are expressions of the basic assumption underlying a singular perturbation analysis, i.e. that the perturbation term Y_1 decays exponentially in the "stretched" coordinate ξ. Consequently it is assumed that at $y = 0$ $\xi \to \infty$ for all practical purposes.

From the analysis of the thin pad problem it is known that (V.22) may be integrated twice to yield

$$\frac{dW}{d\xi} = W\sqrt{W^4 + 2CW^3 + DW + 1} \tag{V.27}$$

where C and D are the first two constants of integration, respectively, found in integrating (V.22).

If, furthermore, the basic assumption about Y_1 is again invoked it may be assumed that for all $n \geq 0$

$$\frac{d^{(n)}Y_1(\infty)}{d\xi^{(n)}} = 0 \qquad \text{(V.28)}$$

Assumption (V.28) permits the evaluation of C and D in terms of $\bar{\lambda}$ as follows. From (V.22)

$$C = \left(\frac{1}{W^3} - W\right)\Big|_{\xi=\infty} = \frac{1}{\bar{\lambda}^3} - \bar{\lambda} \qquad \text{(V.29)}$$

Furthermore, from (V.27) it is found that

$$D = \bar{\lambda}^2 - 3/\bar{\lambda}^2 \qquad \text{(V.30)}$$

It is possible to write, then, that

$$\frac{dW}{d\xi} = \pm W(W-\bar{\lambda})\sqrt{W^2 + \frac{2}{\bar{\lambda}^3}W - \frac{1}{\bar{\lambda}^2}} \qquad \text{(V.27a)}$$

To aid in determining the appropriate sign to use in (V.27a) it is observed that from gross incompressibility considerations $\bar{\lambda} > \lambda$. Consequently $Y_1(0) < 0$ and $dY_1(0)/d\xi > 0$. If the plus sign in (V.27a) were to hold at $\xi = 0$ it is seen that $dW/d\xi(0) < 0$ (or $d^2Y_1(0)/d\xi^2 > 0$). Then for Y_1 to behave as assumed in (V.28) for $\xi \rightarrow \infty$ it is necessary that there be an inflection point in the Y_1 vs. ξ curve for some finite ξ, i.e. $dW/d\xi = 0$ for some finite ξ. There are two possible ways for this to happen. Either $W = 0$ or $(W-\bar{\lambda}) = 0$. The second possibility implies that $dY_1/d\xi$ and $d^2Y_1/d\xi^2$ vanish simultaneously for some finite ξ. But for Y_1, a smooth function of ξ, this is impossible unless $d^2Y_1/d\xi^2 = 0$ for some smaller value of ξ also. This could happen only if $W = 0$ so that one is led to consider the first possibility which would make $dW/d\xi$ vanish. This possibility is examined most easily by integrating (V.27a).

$$(\xi - E) = \ln\left\{\left(\frac{2\left[(\bar{\lambda}^2 + \frac{3}{\bar{\lambda}^2}) - (\bar{\lambda} + \frac{1}{\bar{\lambda}^3})(W - \bar{\lambda}) - \sqrt{(\bar{\lambda}^2 + \frac{3}{\bar{\lambda}^2})(W^2 + \frac{2W}{\bar{\lambda}^3} + \frac{1}{\bar{\lambda}^2})}\right]}{W - \bar{\lambda}}\right)^{\frac{1}{\sqrt{\bar{\lambda}^4 + 3}}} \times \left(\frac{\sqrt{\bar{\lambda}^4 W^2 + 2\bar{\lambda} W + \bar{\lambda}^2} + W\bar{\lambda}^2}{W\bar{\lambda}^2}\right)\right\} \tag{V.31}$$

if the plus sign holds. E is the constant obtained by evaluating the right side of (V.31) for $W = 1$. Observe that $W = 0$ implies $\xi \to \infty$. It is possible to conclude, then, that if (V.28) is a valid assumption the plus sign cannot hold in (V.27a). Therefore

$$\frac{dW}{d\xi} = W(\bar{\lambda} - W)\sqrt{W^2 + 2W/\bar{\lambda}^3 + 1/\bar{\lambda}^2} \tag{V.27b}$$

and

$$\xi = \int_1^W \frac{dW'}{W'(\bar{\lambda} - W')\sqrt{W'^2 + 2W'/\bar{\lambda}^3 + 1/\bar{\lambda}^2}} \tag{V.32}$$

It can be shown that $W \to \bar{\lambda}$ implies $\xi \to \infty$.

Since $W = \bar{\lambda} - dY_1/d\xi$

$$Y_1 = Y_1(0) + \int_1^W \frac{dW'}{W'\sqrt{W'^2 + 2W'/\bar{\lambda}^3 + 1/\bar{\lambda}^2}} \qquad (V.33)$$

and, in particular, since $Y_1(\infty) = 0$

$$\frac{\lambda - \bar{\lambda}}{\epsilon} + \int_1^{\bar{\lambda}} \frac{dW'}{\sqrt{W'^2 + 2W'/\bar{\lambda}^3 + 1/\bar{\lambda}^2}} = 0 \qquad (V.34)$$

This is a transcendental equation for $\bar{\lambda}$ which may be written as

$$\left(\frac{\lambda}{\bar{\lambda}} - 1\right) = \epsilon \, \ell n \left\{ \frac{\sqrt{\bar{\lambda}^4 + 3} + 2}{\sqrt{\bar{\lambda}^4 + 2\bar{\lambda} + \bar{\lambda}^2} + \bar{\lambda} + 1} \right\} \qquad (V.35)$$

Clearly any root $\bar{\lambda}$ greater than unity must also be greater than λ (which is consistent with a prior observation). Since the left side of (V.35) is monotone decreasing in $\bar{\lambda}$ for $\bar{\lambda}$ greater than unity (starting at a positive value for $\bar{\lambda} = 1$) and the right side of (V.35) vanishes at both $\bar{\lambda} = 1$ and $\bar{\lambda} \to \infty$ (taking on negative values for $\bar{\lambda}$ positive) it is likely that there is only one root $\bar{\lambda}$ in the interval $1 < \bar{\lambda} < \infty$ due to the smooth nature of the functions appearing in (V.35).

It is now possible to compute $\bar{\lambda}$ for any particular numerical case and thus evaluate the boundary conditions (V.25) and (V.26). Then one may compute $Y_1(\xi)$ in tabular form from the parametric equations (V.32) and (V.33).

The worth of this singular perturbation analysis depends on how rapidly $dY_1/d\xi$ and Y_1 decay. For large ξ (V.27b) may be approximated by

$$\frac{dW}{d\xi} \doteq \bar{\lambda}\sqrt{\bar{\lambda}^2 + 3/\bar{\lambda}^2}\ (\bar{\lambda} - W) \tag{V.36}$$

so that

$$\frac{dY_1}{d\xi} \sim e^{-\bar{\lambda}\sqrt{\bar{\lambda}^2 + 3/\bar{\lambda}^2}\,\xi} \tag{V.37}$$

This indicates that for $\bar{\lambda} > 1$, as it must be, $dY_1/d\xi$ and Y_1 do indeed decay rapidly.

5. A Numerical Example: $\lambda = 2$, $A/B = 1/8$

a. Direct Minimization of the Energy Integral

For the numerical values $\lambda = 2$ and $A/B = 1/8$ the integral in (V.14) becomes

$$\int_0^1 \left\{\frac{1}{192}\,\frac{n^2 y^{2(n-2)}}{\left[\left(2 + \frac{1}{(n-1)}\right) - \frac{n}{(n-1)}\,y^{(n-1)}\right]^4} + \left(\frac{\left[\left(2 + \frac{1}{(n-1)}\right) - \frac{n}{(n-1)}\,y^{(n-1)}\right]^2 - 1}{\left(2 + \frac{1}{(n-1)}\right) - \frac{1}{(n-1)}y^{(n-1)}}\right)^2\right\} dy \tag{V.38}$$

Integrating (V.38) numerically for various values of n shows that (V.38) is minimized by $n \doteq 15$. On the basis of the numerical work performed it does not seem justifiable to specify n to more than 2 digits.

From (V.12a) and (V.3) it is found that in the present case

$$Y = 2.071y - 0.071y^{15}$$
$$X = x/(2.071 - 1.071y^{14}) \qquad (V.39)$$

The deformations of the free edges are given in some detail in Table V.1 below.

y	Y	$\frac{X}{A/2}$
0.0	0	0.483
0.2	0.414	0.483
0.4	0.828	0.483
0.6	1.243	0.483
0.7	1.445	0.485
0.8	1.655	0.495
0.9	1.849	0.548
0.95	1.934	0.647
1.0	2.0	1

Table V.1. Deformed Geometry of Free Edge

b. Singular Perturbation Analysis

From (V.35) it is found that if $\lambda = 2$ and $\epsilon = 0.0722$ the appropriate value of $\overline{\lambda}$ is $\overline{\lambda} = 2.032$.

In this case (V.32) can be written as

$$\xi = 0.216 +$$
$$+ \ln\left\{\left(\frac{W-2.032}{9.7112+4.3024(W-2.032)-\sqrt{19.422(W^2+0.2384W+0.2422)}}\right)^{0.22331} \times \right. \qquad (V.32a)$$
$$\left. \times \left(\frac{4.129W}{\sqrt{17.049W^2+4.064W+4.129}+4.129W}\right)\right\}$$

To show how rapidly $dY_1/d\xi$ and Y_1 decay (V.32a) is evaluated at W = 2. It is found that $\xi|_{W=2} = 0.82$ so that $dY_1/d\xi\,(0.82) = 0.032$ as compared to $dY_1/d\xi(0) = 1.032$. Furthermore, in the present case (V.33) may be written as

$$Y_1(\xi) = -0.443 + \int_1^W \frac{dW'}{W'\sqrt{W'^2 + 0.2384W' + 0.2422}} \qquad \text{(V.33a)}$$

When (V.33a) is evaluated for W = 2 it is found that $Y_1(0.82) = -0.008$. Clearly $Y_1(\xi)$ and $dY_1/d\xi$ do decay rapidly. As a matter of fact the decay is even more rapid than the decay of the asymptotic forms of Y_1 given by (V.37), since the function

$$Y_1^{(a)} = -0.443\, e^{-4.47\xi} = -0.443\, e^{-62.0(1-y)} \qquad \text{(V.40)}$$

evaluated at $\xi = 0.82$ has the value $Y_1^{(a)}(0.82) = -0.012$.

At $\xi = 0.82$, $y = 0.941$ and $X(A/2, 0.941) = 0.493\ A/2$ as compared with $X(A/2, 0) = 0.492\ A/2$. Consequently the portion of the strip originally in the interval $-0.94 < y < 0.94$ can certainly be considered to be undergoing uniaxial tension (in the x-y plane). Note that the true stress in the central portion of the strip differs by less than 2% from the true stress in a similar strip undergoing a completely homogeneous deformation.

The singular perturbation analysis is believed to be the more accurate of the two approximate analyses which have been presented for the simplified strip problem. This is because the basic assumption, (V.28), concerning the exponentially decaying behavior of Y_1 has been confirmed for a reasonable numerical example. The rest of the singular perturbation analysis is exact (except for numerical error).

The reasonableness of assumption (1) is also confirmed by the singular perturbation analysis. The portion of the deformed strip adjacent to the rigid support which undergoes a nonhomogeneous deformation will be referred to as the "boundary layer." The mean "boundary layer" width (X-dimension) is about twice the "boundary layer" depth if $\xi = 0.82$ is taken as the extent of the "boundary layer" (this is a reasonably conservative definition). It is observed that in this case assumption (1) is likely to hold even over a good portion of the "boundary layer." Of course the state of stress at the bonded surface is in no way given by the present analysis.

CHAPTER VI. THE APPLICATION OF REISSNER'S THEOREM TO THE SLUMP OF A BLOCK OF FINITE LENGTH

1. Introduction

The deformation of a neo-Hookean block of finite length, bonded to rigid walls on two parallel faces and accelerated in the direction parallel to the walls, is studied in this chapter under the assumption of plane strain. The analysis is based on suitable approximate stress and displacement fields which are characterized by a single parameter. The appropriate value of the parameter is determined by using Reissner's Theorem.

2. The Neo-Hookean Complementary Strain Energy Function in the Case of Plane Strain

As a preliminary to the study of the slump problem it is necessary to construct the neo-Hookean complementary strain energy function, W_c, for the case of plane strain.

For principal coordinates in the deformed body it is possible to write the constitutive equation for a neo-Hookean material as

$$\tau_i = \mu \lambda_i^2 + k; \quad \prod_{i=1}^{3} \lambda_i^2 = 1 \tag{VI. 1}$$

where τ_i and λ_i are the i^{th} principal true stress and the i^{th} principal stretch ratio respectively; k is the hydrostatic portion of the stress. (VI. 1) may be used to express the first deformation invariant in terms of the first stress invariant.

$$I_M = \frac{I_\tau - 3k}{\mu} \tag{VI. 2}$$

Consequently the neo-Hookean strain energy function may be written as

$$W = \tfrac{1}{2}\,[I_\tau - 3(k + \mu)] \qquad \text{(VI. 3)}$$

Consider the case of plane strain, i.e. $\lambda_3 = 1$,

$$\tau_1 = \mu\lambda_1^2 + k$$

$$\tau_2 = \mu\lambda_2^2 + k \qquad ; \qquad \lambda_1^2\lambda_2^2 = 1 \qquad \text{(VI. 4)}$$

$$\tau_3 = \mu + k$$

Since $\lambda_1^2 = \frac{\tau_1 - k}{\mu}$ and $\lambda_2^2 = \frac{\tau_2 - k}{\mu}$ it is seen that

$$k^2 - (\tau_1 + \tau_2)\,k + (\tau_1\tau_2 - \mu^2) = 0 \qquad \text{(VI. 5)}$$

or, using the definitions of the invariants of the two dimensional stress tensor,

$$k^2 - I_\tau\, k + (II_\tau - \mu^2) = 0 \qquad \text{(VI. 6)}$$

(VI. 6) may be solved for k to give

$$k = \frac{I_\tau}{2} \pm \mu\sqrt{I + 1} \qquad \text{(VI. 7)}$$

where

$$I = \left(\frac{I_\tau}{2\mu}\right)^2 - \frac{II_\tau}{\mu^2} = \left(\frac{\tau_1 - \tau_2}{2\mu}\right)^2 \qquad \text{(VI. 8)}$$

In other than principal coordinates

$$I = \left(\frac{\tau^{11} - \tau^{22}}{2\mu}\right)^2 + \left(\frac{\tau^{12}}{\mu}\right)^2 \qquad \text{(VI. 8a)}$$

The proper sign to choose in (VI. 7) is the minus sign since Rivlin [17] has shown that the hydrostatic term, k, must take on the algebraically smallest value possible, i.e.

$$k = \frac{I_\tau}{2} - \mu\sqrt{I + 1} \qquad \text{(VI. 7a)}$$

Substitution of (VI. 7a) into (VI. 3), specialized for plane strain, leads to

$$W = \mu[\sqrt{I + 1} - 1] \qquad \text{(VI. 9)}$$

when it is recalled that for plane strain $\tau_3 = k + \mu$. If the radical in (VI. 9) is expanded in a power series in I it is found that

$$W = \mu\left[\frac{I}{2} - \frac{I^2}{8} + \frac{I^3}{16} - \ldots\right] \;;\; I < 1 \qquad \text{(VI. 10)}$$

The first term of (VI. 10) can be shown to correspond to the strain energy function (expressed in terms of stresses) of an incompressible Hookean solid.

It is now possible to construct the neo-Hookean complementary strain energy function for the case of plane strain. Using the present terminology in the definition of the complementary strain energy function given in Chapter I it is found that

$$W_c = \frac{1}{2}\sum_{i=1}^{2} [\tau_i - (k + \mu)]\,[\lambda_i^2 - 1] - W \qquad \text{(VI. 11)}$$

Since $(\lambda_i^2 - 1) = \dfrac{\tau_i - (k + \mu)}{\mu}$, (VI. 11) may be written as

$$W_c = \frac{1}{2\mu} \sum_{i=1}^{2} [\tau_i - (k + \mu)]^2 - \mu[\sqrt{I+1} - 1] \qquad \text{(VI. 11a)}$$

Using (VI. 7) and (VI. 8) it is possible to write (VI. 11a) as

$$W_c = \mu[2I + 3(1 - \sqrt{I+1})] \qquad \text{(VI. 11b)}$$

If the radical in (VI. 11b) is expanded in terms of a power series in I then

$$W_c = \mu[I/2 + 3/8\, I^2 - 3/16\, I^3 - \ldots] \; ; \; I < 1 \qquad \text{(VI. 12)}$$

The leading term of (VI. 12) is, as it should be, the same as the leading term of (VI. 10).

3. Formulation of the Slump Problem

It is assumed that in the deformed state the block, shown in Figure VI. 1 below, has a rectangular cross-section in the plane of

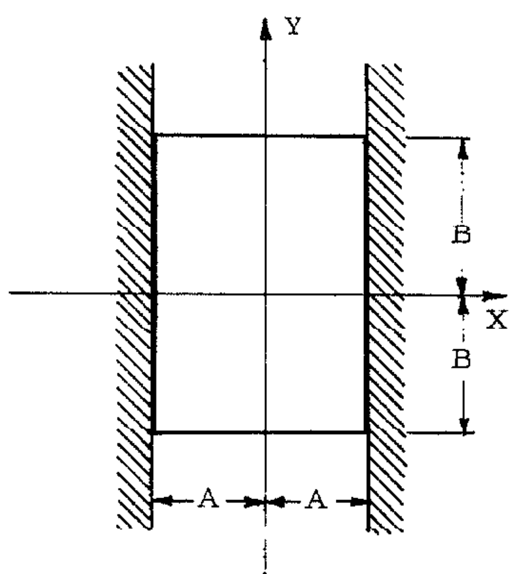

FIGURE VI. 1. Deformed Block

the paper. The deformation of the block is presumed to be due to a constant upward acceleration of magnitude Ng where g is the acceleration due to gravitational forces.

If the block were of infinite extent in the Y direction, i.e. $B \to \infty$, then the exact solution of the slump problem would be given by

$$x = X$$
$$y = Y + \frac{\rho N g}{2\mu} A^2 \left(1 - \frac{X^2}{A^2}\right) \qquad \text{(VI. 13)}$$

and

$$\frac{\tau^{11}}{\mu} = 0$$
$$\frac{\tau^{22}}{\mu} = \left(\frac{\rho N g}{\mu} X\right)^2 \qquad \text{(VI. 14)}$$
$$\frac{\tau^{12}}{\mu} = \frac{\rho N g}{\mu} X$$

where ρ is the mass density of the material.

(VI. 13) and (VI. 14) aid one in choosing suitable approximate deformation and stress fields for the case of a block of finite length 2B. The choices made will be relatively simple. For the deformation field choose

$$x = X$$
$$y = Y + \alpha B \left(1 - \frac{X^2}{A^2}\right) \qquad \text{(VI. 15)}$$

where α is an undetermined parameter. (VI. 15) satisfies the incompressibility condition identically. The stress field given below is determined by assuming τ^{12}/μ and then using the stress equilibrium equations to generate τ^{11}/μ and τ^{22}/μ. τ^{22}/μ is determined uniquely in this manner whereas τ^{11}/μ is determined only to within an arbitrary function of Y. This arbitrary function is taken to be zero since the resulting expression for τ^{11}/μ behaves as would be expected intuitively.

$$\frac{\tau^{11}}{\mu} = \alpha n\left(\frac{X^2}{A^2}\right)e^{-n(1-Y/B)}$$

$$\frac{\tau^{22}}{\mu} = \left(\frac{B\rho Ng}{\mu} - 2\alpha\frac{B^2}{A^2}\right)\left(\frac{Y}{B} - 1\right) - \frac{2\alpha B^2}{nA^2}\left[1-e^{-n(1-Y/B)}\right] \qquad \text{(VI. 16)}^*$$

$$\frac{\tau^{12}}{\mu} = 2\alpha\frac{B}{A}\left(\frac{X}{A}\right)\left[1-e^{-n(1-Y/B)}\right]$$

where n is an undetermined parameter. Note that the hydrostatic portion of the stress, $(k + \mu)$, may be considered as vanishing in this case so that $\tau^{ij} = \tau_d^{ij}$. It is possible to relate the parameters α and n by a purely static argument. The total shear force acting along each block-wall bond surface must balance one half of the total body force acting on the block.

$$\int_0^B \tau^{12}\big|_{X=A}\, dY = AB\,\rho Ng \qquad \text{(VI. 17)}$$

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* (VI. 16) is valid for $0 \leq Y \leq B$. An obvious modification is required for $-B \leq Y \leq 0$.

Consequently

$$\alpha = \frac{1}{2}\left(\frac{A^2}{B^2}\right)\left(\frac{B\rho Ng}{\mu}\right)\left[\frac{n}{n - (1 - e^{-n})}\right] \qquad \text{(VI. 18)}$$

For the present problem Reissner's Theorem may be written as

$$\delta\int_{\tau}\left(\frac{\tau^{\alpha\beta}}{\mu}\gamma_{\alpha\beta} - \frac{W_c}{\mu} - \frac{\rho\overline{F}\cdot\overline{v}}{\mu}\right)d\tau = 0 \qquad \text{(VI. 19)}^{*}$$

where $\gamma_{\alpha\beta}$ is the strain tensor and $\rho\overline{F}\cdot\overline{v}$ is the scalar product of the body force per unit volume and the displacement vector.

From (VI. 15) and the definition of the strain tensor it is found that

$$\gamma_{11} = -2\alpha^2\left(\frac{B^2}{A^2}\right)\left(\frac{X^2}{A^2}\right)$$

$$\gamma_{22} = 0 \qquad \text{(VI. 20)}$$

$$\gamma_{12} = \gamma_{21} = \alpha\left(\frac{B}{A}\right)\left(\frac{X}{A}\right)$$

(VI. 15) also provides

$$\frac{\rho\overline{F}\cdot\overline{v}}{\mu} = \frac{\alpha B\rho Ng}{\mu}\left(1 - \frac{X^2}{A^2}\right) \qquad \text{(VI. 21)}$$

Using (VI. 16) and (VI. 20) it is found that

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* (VI. 19) holds because the boundary conditions and the incompressibility condition have been satisfied identically by the assumed trial fields.

$$\frac{\tau^{\alpha\beta}}{\mu}\gamma_{\alpha\beta} = -2\alpha^3 n(\frac{B^2}{A^2})(\frac{X^4}{A^4})\, e^{-n(1-Y/B)} + 4\alpha^2(\frac{B^2}{A^2})(\frac{X^2}{A^2})[1-e^{-n(1-Y/B)}] \tag{VI. 22}$$

Recall that

$$I = (\frac{\tau^{11}-\tau^{22}}{2\mu})^2 + (\frac{\tau^{12}}{\mu})^2 \tag{VI. 8a}$$

so that in the present case

$$I = [\frac{\alpha n}{2}(\frac{X^2}{A^2})e^{-n(1-Y/B)} - (\frac{B\rho Ng}{2\mu} - \alpha\frac{B^2}{A^2})(\frac{Y}{B}-1) + \frac{\alpha}{n}\frac{B^2}{A^2}(1-e^{-n(1-Y/B)})]^2 + 4\alpha^2(\frac{B^2}{A^2})(\frac{X^2}{A^2})[1-e^{-n(1-Y/B)}]^2 \tag{VI. 23}$$

The substitution of (VI. 11b), (VI. 18), (VI. 21), (VI. 22), and (VI. 23) into (VI. 19) provides the basis for determining the single parameter n. Except for the term under the radical in W_c the integrations may be performed explicitly. However, as things stand the simplest way of determining n for a given case is to evaluate the functional of Reissner's Theorem, (VI. 19), numerically for various values of n and thus determine the value of n which makes the functional stationary.

4. Numerical Examples

The following cases are considered.

$$\rho g = 0.06 \text{ lb./in.}^3$$
$$N = 100$$
$$\mu = 100 \text{ psi} \qquad \text{(VI. 24)}$$
$$A = 10 \text{ in.}$$
$$\frac{B}{A} = 3; \frac{B}{A} = 10$$

<u>Case a)</u> B/A = 3

In this case

$$\frac{B\rho N g}{\mu} = 1.8 \qquad \text{(VI. 25)}$$

and

$$a = \frac{n}{10[n-(1-e^{-n})]} \qquad \text{(VI. 26)}$$

Consequently, from (VI. 22), (VI. 23), and (VI. 21),

$$\frac{\tau^{\alpha\beta}\gamma_{\alpha\beta}}{\mu} = \frac{-0.018n^4}{[n-(1-e^{-n})]^3} \left(\frac{X^4}{A^4}\right) e^{-n(1-Y/B)} + \frac{0.36n^2}{[n-(1-e^{-n})]^2} \left(\frac{X^2}{A^2}\right)[1 - e^{-n(1-Y/B)}] , \qquad \text{(VI. 27)}$$

$$I = \frac{n^2}{[n-(1-e^{n})]^2} \left\{ \left[0.05ne^{-n(1-Y/B)}\left(\frac{X^2}{A^2}\right) + \frac{0.9}{n}\left([1-e^{-n}][\frac{Y}{B} - 1] + [1-e^{-n(1-Y/B)}]\right)\right]^2 + 0.36\left(\frac{X^2}{A^2}\right)[1-e^{-n(1-Y/B)}]^2 \right\} , \qquad \text{(VI. 28)}$$

and

$$\frac{\rho\overline{F}\cdot\overline{v}}{\mu} = \frac{0.18n}{[n - (1 - e^{-n})]} \left(1 - \frac{X^2}{A^2}\right) \tag{VI.29}$$

The integral to be evaluated for various values of n is

$$J(n) = \int_0^1 \int_0^1 \left\{\frac{\tau^{\alpha\beta}\gamma_{\alpha\beta}}{\mu} - [2I + 3(1 - \sqrt{I+1})] - \frac{\rho\overline{F}\cdot\overline{v}}{\mu}\right\} d\left(\frac{X}{A}\right) d\left(\frac{Y}{B}\right) \tag{VI.30}$$

When (VI.27), (VI.28), and (VI.29) are substituted into (VI.30), and a numerical integration is performed for various n by means of Simpson's rule for double integration, [28] it is found that (VI.30) is <u>maximized</u> by

$$n = 10.8 \tag{VI.31}$$

Although the numerical integrations have been performed with a rather coarse 6 by 6 net the result, (VI.31), is probably fairly accurate since it depends on the comparison of (VI.30) evaluated for different values of n rather than on the absolute value of (VI.30) calculated for each n. It does not seem reasonable that the relative magnitude of J(n) will be affected by the numerical scheme which has been employed if n varies over a relatively small domain.

For n = 10.8 it is found that

$$a = 0.110 \tag{VI.32}$$

so that the deformation and stress fields are given by

$$\frac{x}{A} = \frac{X}{A}$$

$$\frac{y}{B} = \frac{Y}{B} + 0.110\ (1 - \frac{X^2}{A^2}) \tag{VI.33}$$

and

$$\frac{\tau^{11}}{\mu} = 1.19\ (\frac{X^2}{A^2})e^{-10.8(1-Y/B)}$$

$$\frac{\tau^{22}}{\mu} = 0.186\ (1 - \frac{Y}{B}) - 0.183\ [1-e^{-10.8(1-Y/B)}] \tag{VI.34}$$

$$\frac{\tau^{12}}{\mu} = 0.662\ (\frac{X}{A})[1-e^{-10.8(1-Y/B)}]$$

The stresses acting at the wall-block bond surface are shown in Figure VI.2.

Case b) B/A = 10

In this case

$$\frac{B\rho Ng}{\mu} = 6.0 \tag{VI.35}$$

and

$$\alpha = 0.03\ \frac{n}{[n - (1-e^{-n})]} \tag{VI.36}$$

Consequently

$$\frac{\tau^{\alpha\beta}\gamma_{\alpha\beta}}{\mu} = \frac{-0.0054\ n^4}{[n-(1-e^{-n})]^3}\ (\frac{X^4}{A^4})e^{-n(1-Y/B)} + \frac{0.36n^2}{[n-(1-e^{-n})]^2}\ (\frac{X^2}{A^2})\ [1-e^{-n(1-Y/B)}]\ , \tag{VI.37}$$

$$I = \frac{n^2}{[n-(1-e^n)]^2} \left\{ [0.015ne^{-n(1-Y/B)}(\frac{X^2}{A^2}) + \right.$$

$$+ \frac{3}{n}([1-e^{-n}][Y/B-1] + [1-e^{-n(1-Y/B)}])]^2 + \qquad \text{(VI. 38)}$$

$$\left. + 0.36 (\frac{X^2}{A^2}) [1-e^{-n(1-Y/B)}]^2 \right\} ,$$

and

$$\frac{\rho \overline{F} \cdot \overline{v}}{\mu} = \frac{0.6n}{[n-(1 - e^{-n})]} (1 - \frac{X^2}{A^2}) \qquad \text{(VI. 39)}$$

Again, (VI. 30) is to be made stationary with respect to n; in this case (VI. 37), (VI. 38), and (VI. 39) are to be substituted into (VI. 30).

A comparison of (VI. 27), (VI. 28), and (VI. 29) with (VI. 37), (VI. 38), and (VI. 39) suggests that n may be approximately proportional to B/A. This observation is confirmed when (VI. 30) is computed for various values of n and it is found that

$$n = 32 \qquad \text{(VI. 40)}$$

maximizes the functional.*

The physical interpretation of the above suggested relation between n and B/A is that the free surface, i.e. finite length, effect is essentially independent of B/A provided that B/A is large enough

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*The computational net used in this case consisted of four equal divisions in the X/A direction while in the Y/B direction the mesh points were 0, 0.45, 0.90, 0.925, 0.950, 0.975, and 1.

for the effect of the free surface to become negligibly small as the center of the block, $Y/B = 0$, is approached.

For $n = 32$ it is found that

$$\alpha = 0.031 \tag{VI.41}$$

so that the deformation and stress fields are given by

$$\frac{x}{A} = \frac{X}{A}$$

$$\frac{y}{B} = \frac{Y}{B} + 0.031 \left(1 - \frac{X^2}{A^2}\right) \tag{VI.42}$$

and

$$\frac{\tau^{11}}{\mu} = 0.992 \left(\frac{X^2}{A^2}\right) e^{-32(1-Y/B)}$$

$$\frac{\tau^{22}}{\mu} = 0.200 \,(1-Y/B) - 0.194 \left[1-e^{-32(1-Y/B)}\right] \tag{VI.43}$$

$$\frac{\tau^{12}}{\mu} = 0.620 \frac{X}{A} \left[1-e^{-32(1-X/B)}\right]$$

The stresses acting at the wall-block bond surface are shown in Figure VI.2.

The present analysis indicates that a block of finite length sustains greater displacements and maximum shear stresses than does a similarly accelerated body of infinite length; the shorter the block, the greater are these differences.

Note the complete change in the character of the normal stresses when the block of finite length is compared to the infinitely long body. Of particular interest is the rapid increase in τ^{11}/μ as a corner is approached. This suggests a possible singularity in τ^{11}/μ at the corners. Unfortunately, simple energy approximations "wash out" stress singularities which are not allowed for explicitly.

The approximate solution of the slump problem for a block of finite length which has been presented indicates the usefulness of Reissner's Theorem for problems in which more information about the stress field is required than can be provided by the principle of stationary potential energy.

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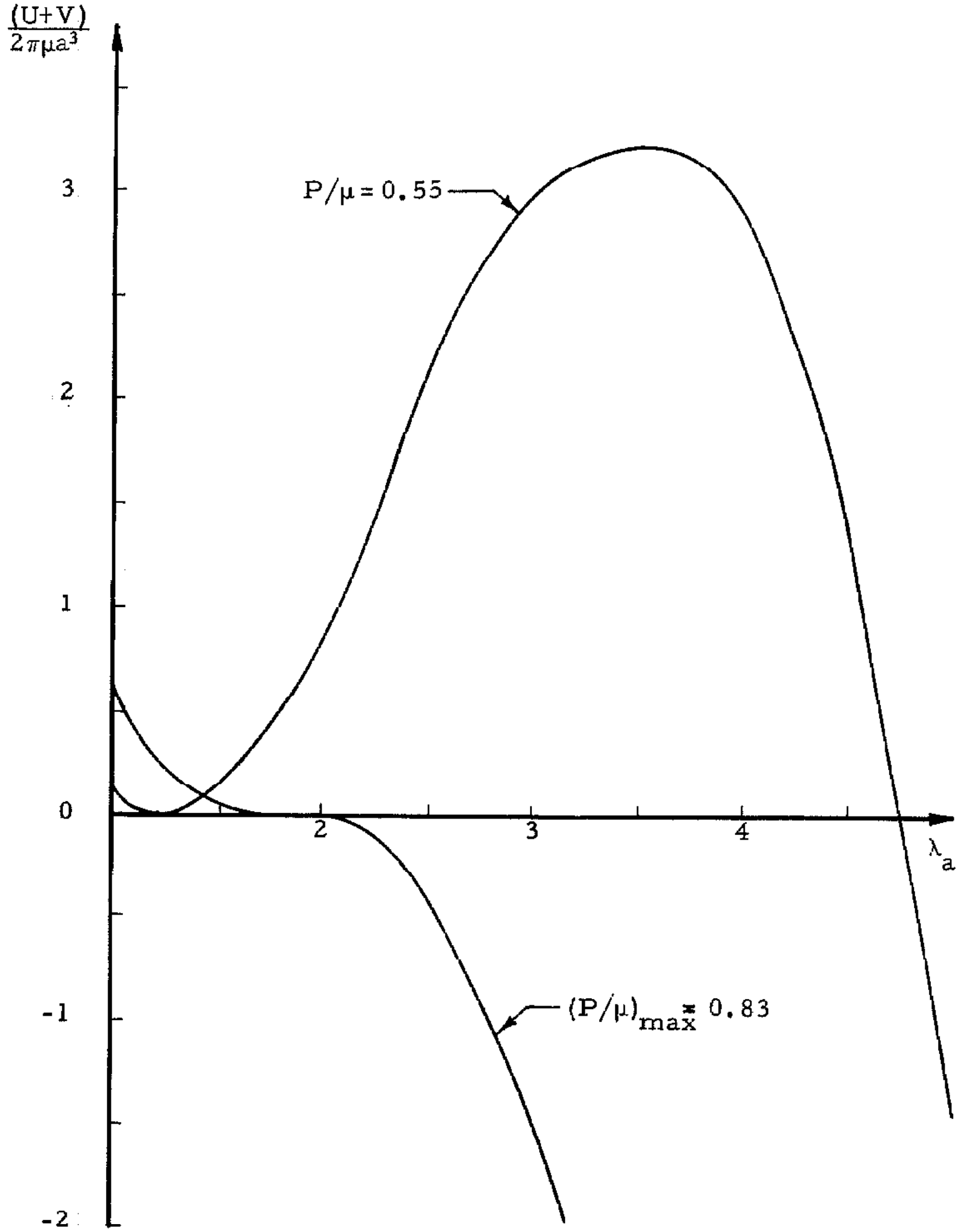

FIGURE III. 7. Total Potential of Neo-Hookean Sphere for $P/\mu = 0.55$ and $(P/\mu)_{max} = 0.83$; $b/a = 2$.

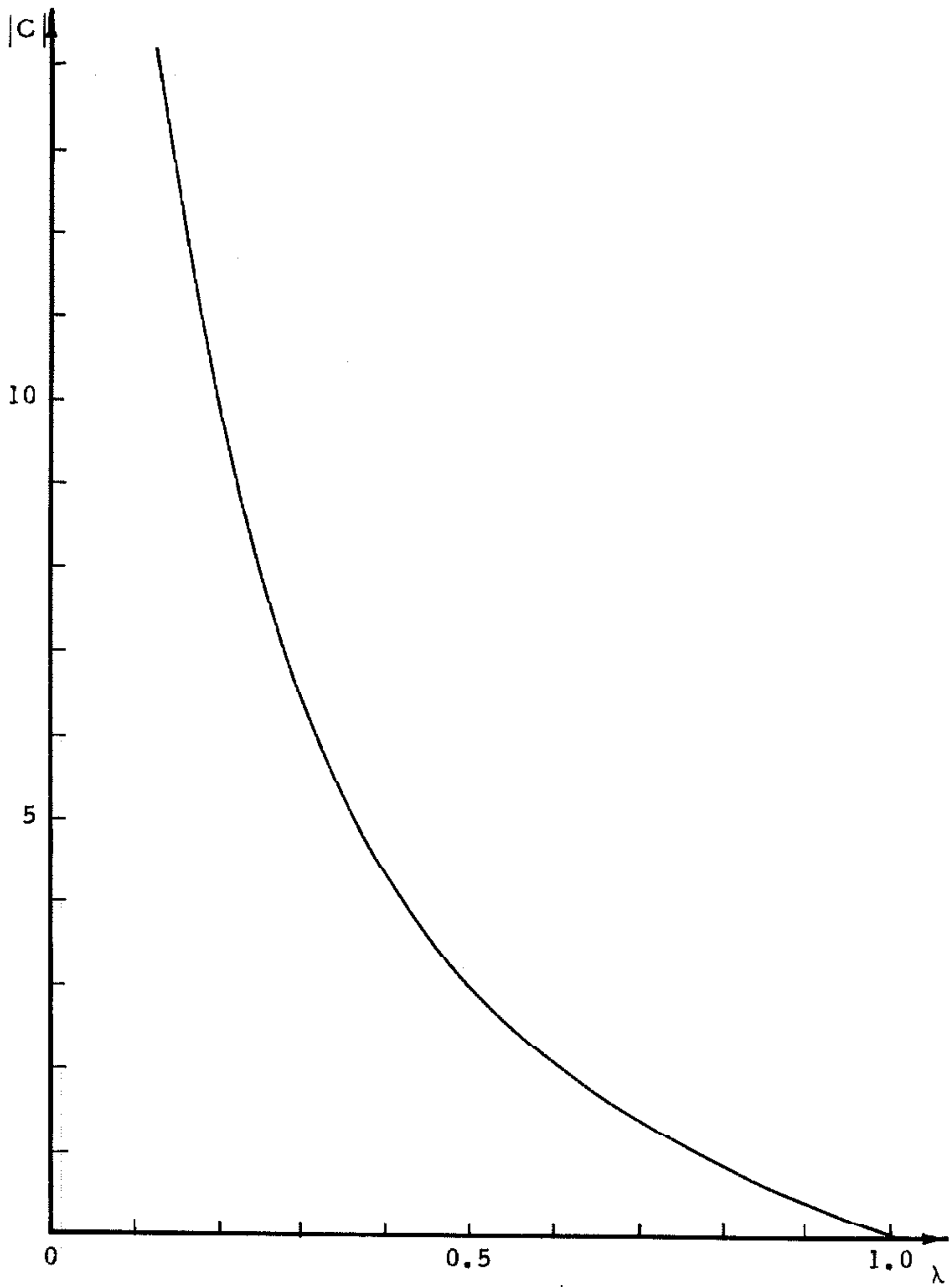

FIGURE IV.4. Arbitrary Constant |C| as Function of Stretch Ratio λ for Compression Case of the Related Elasticity Problem.

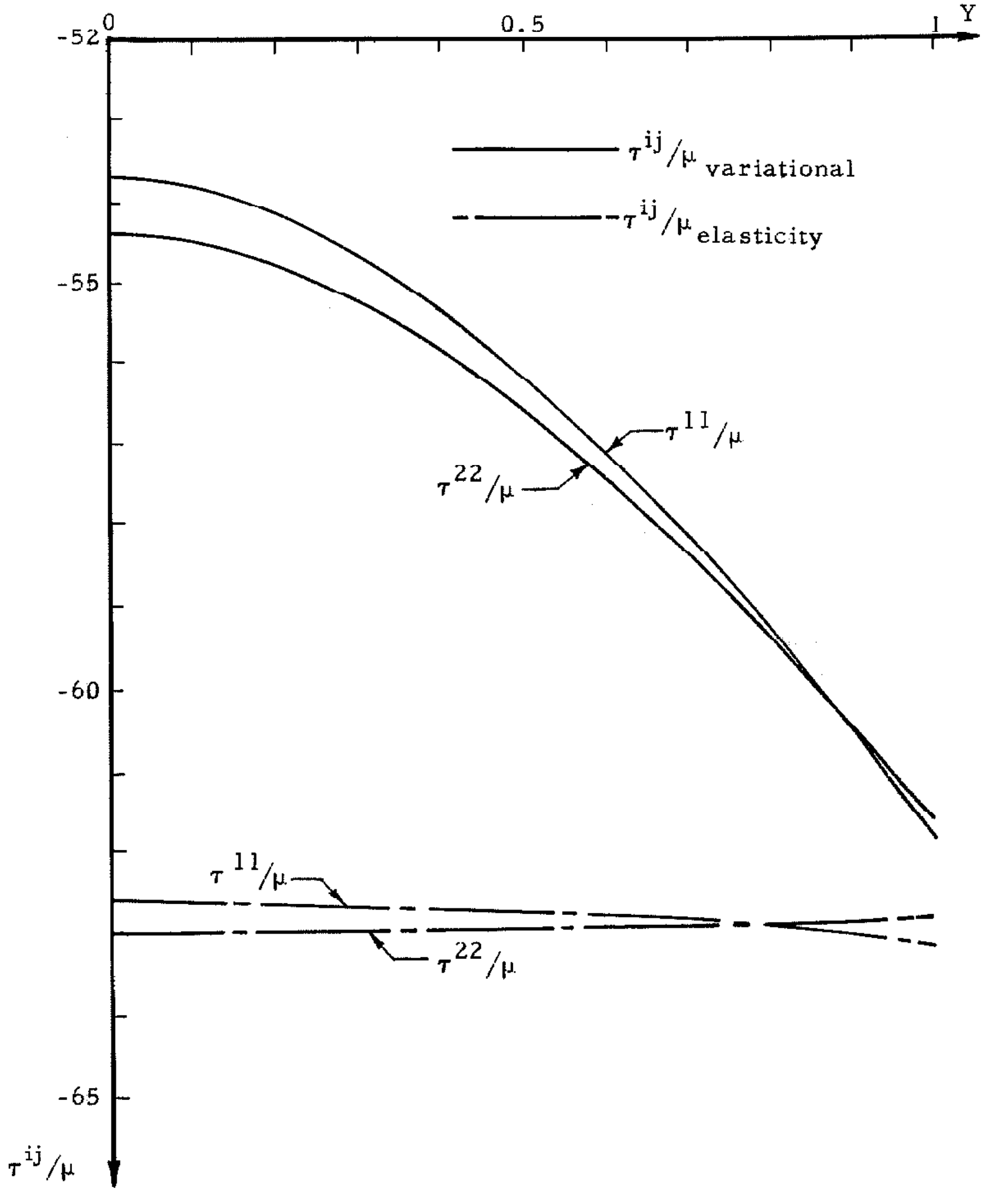

FIGURE IV.5. Stresses on X = 0 from Variational and Related Elasticity Solutions for the Case $\lambda = 0.9$; $A/B = 20$.

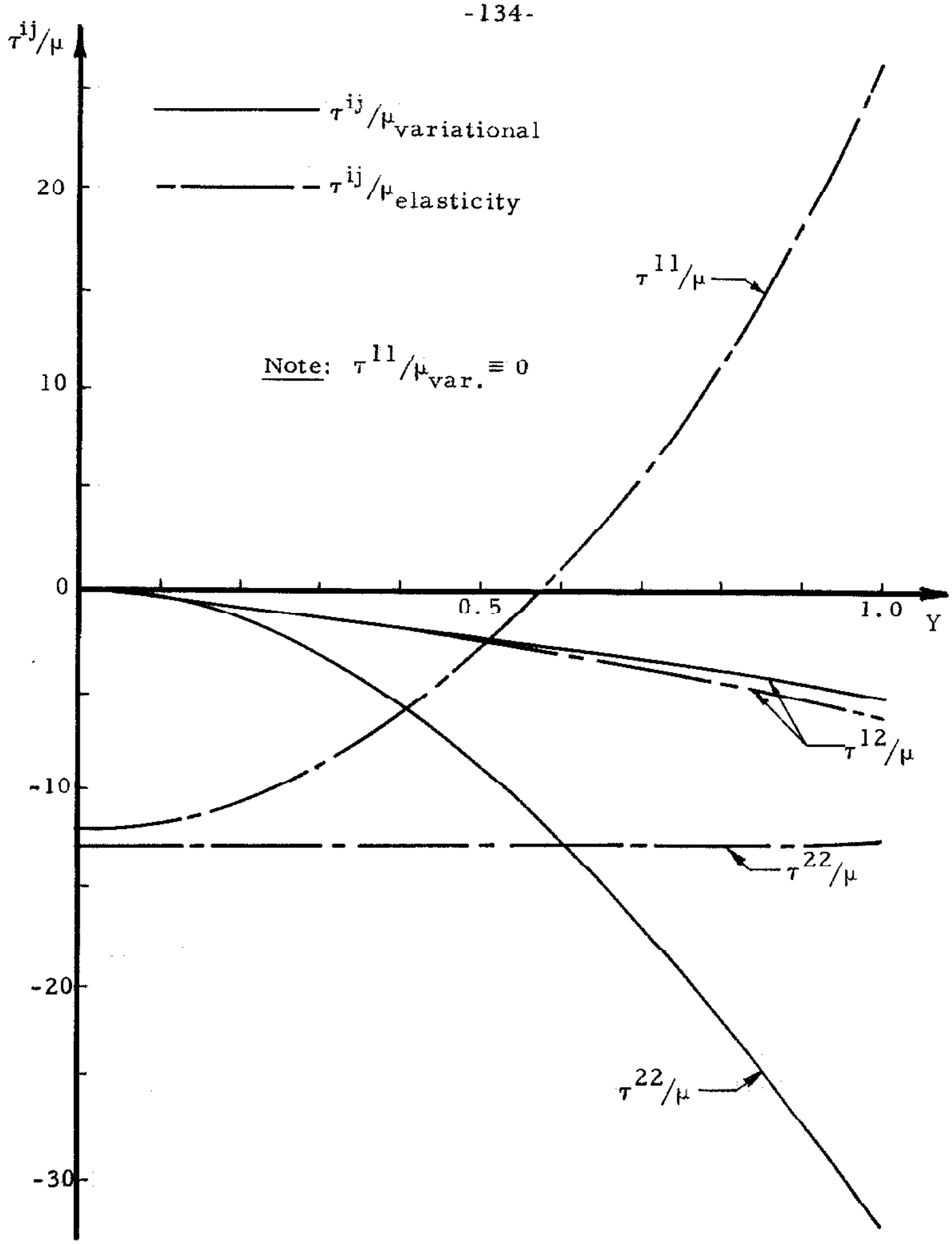

FIGURE IV.6. Stresses on X = A/2 from Variational and Related Elasticity Solutions for the Case λ = 0.9; A/B = 20.

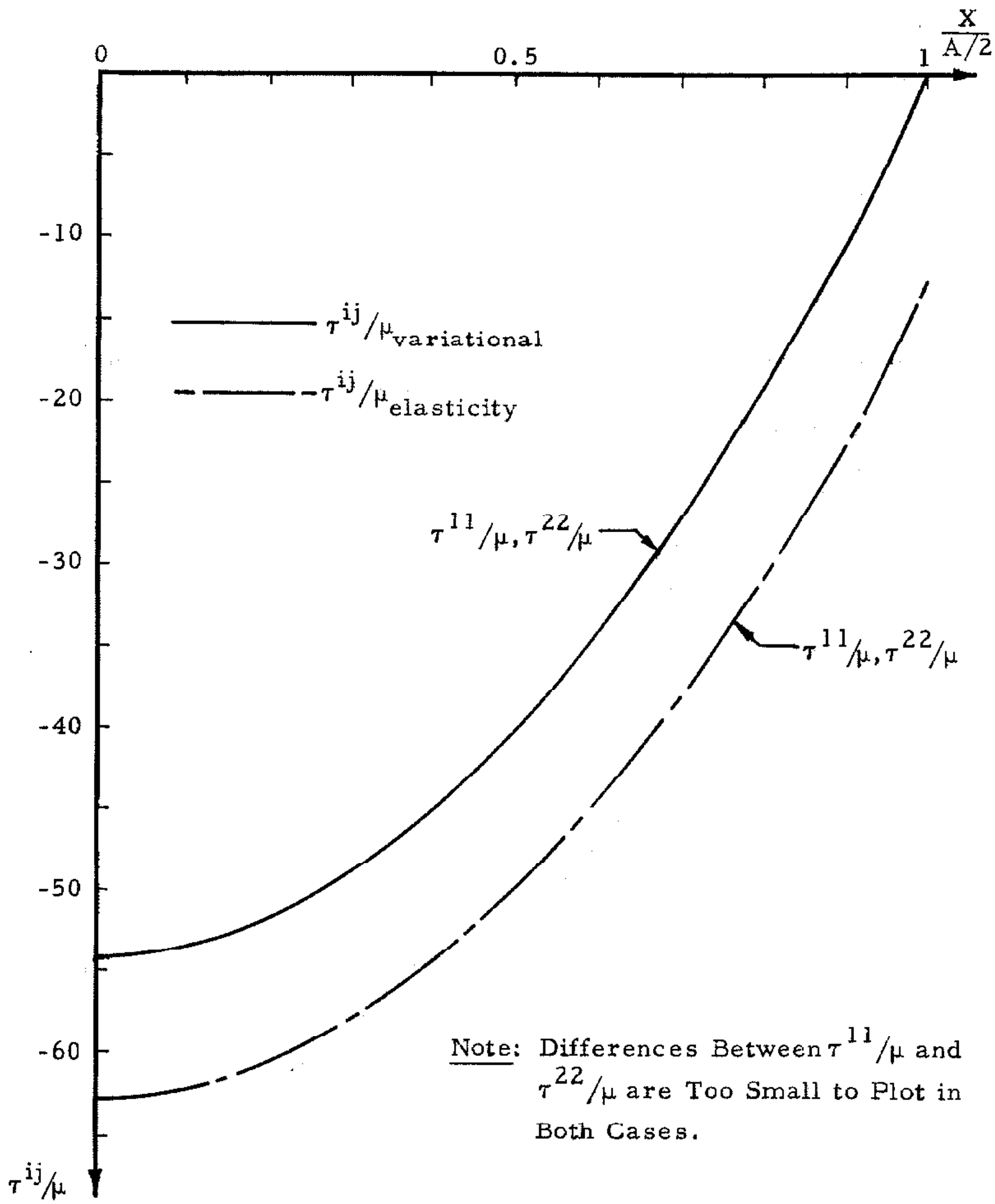

FIGURE IV.7. Stresses on Y = 0 from Variational and Related Elasticity Solutions for the Case $\lambda = 0.9$; $A/B = 20$.

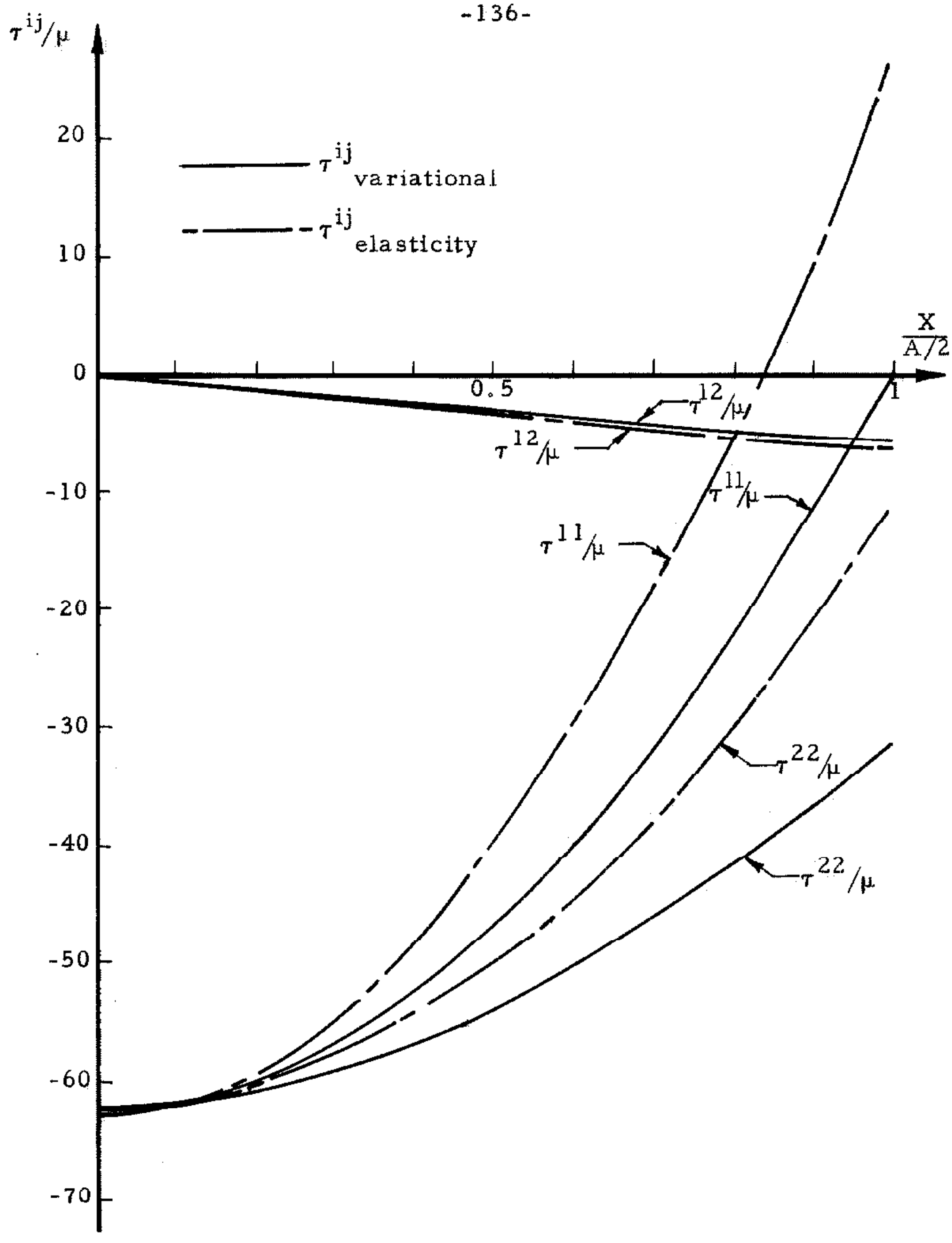

FIGURE IV.8. Stresses on Y = 1 from Variational and Related Elasticity Solutions for the Case $\lambda = 0.9$; $A/B = 20$.

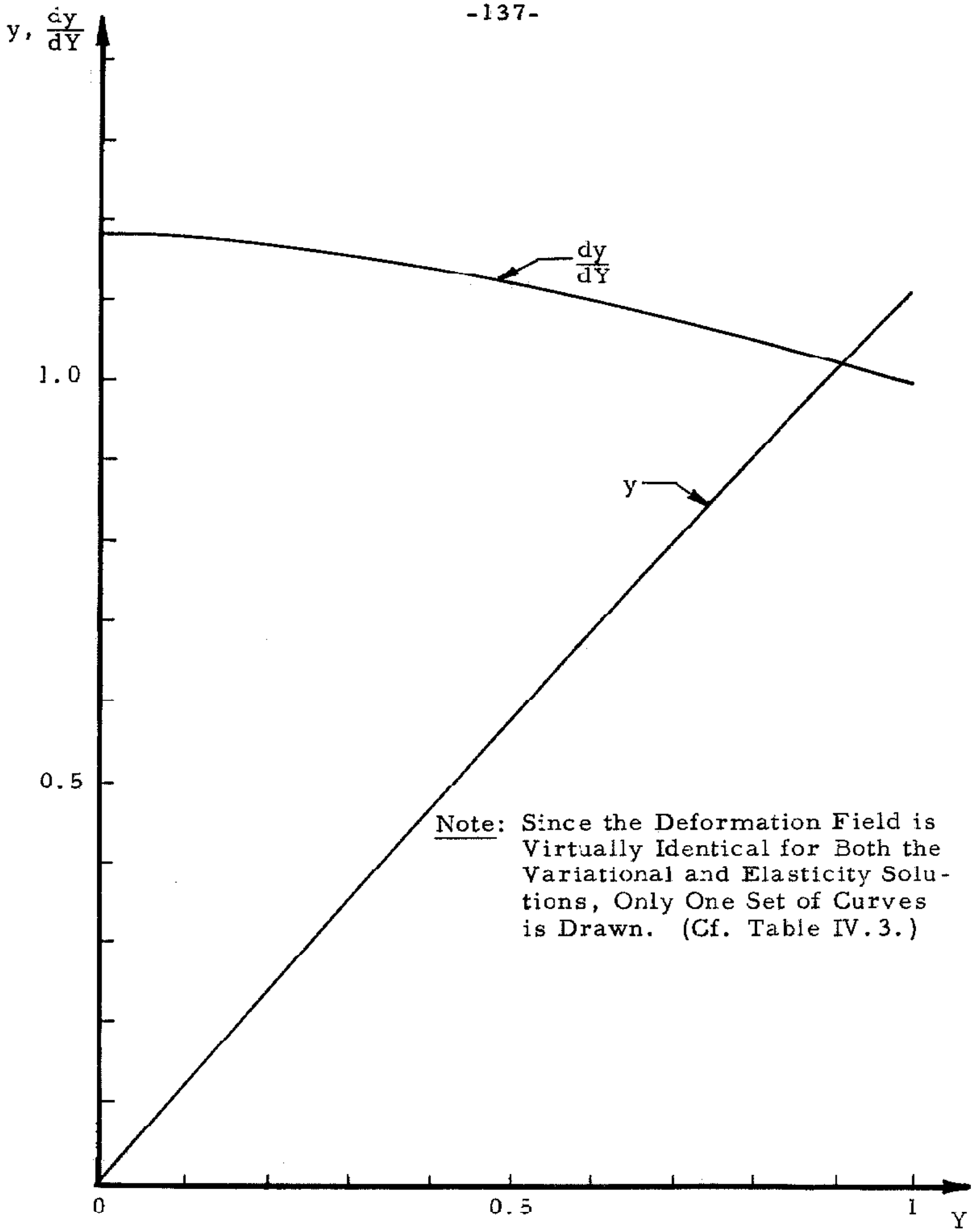

FIGURE IV.9. Deformation Field for A/B = 20, $\lambda = 0.9$.

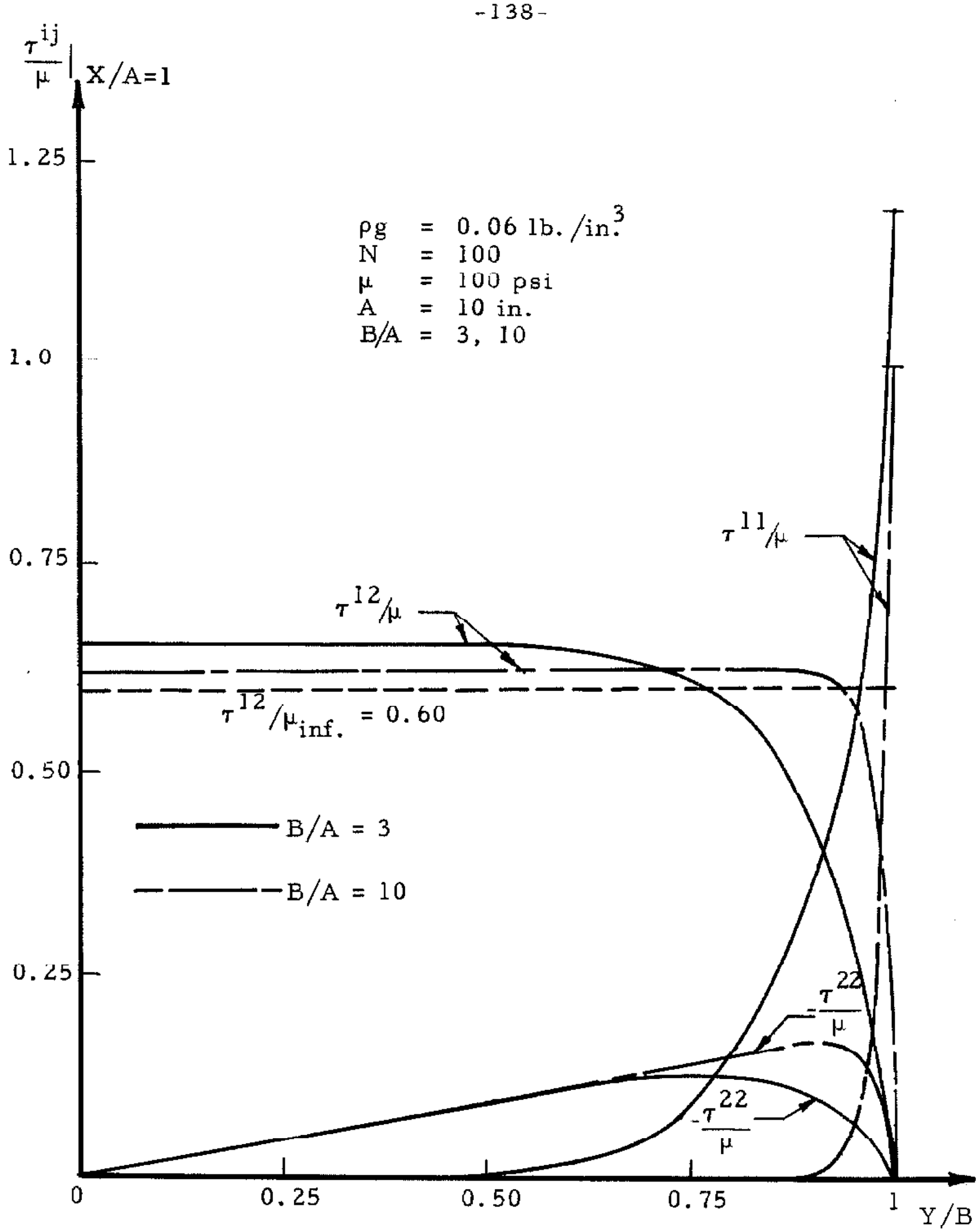

FIGURE VI.2. Stresses at Wall-Block Bond Surface.