ON THE DOPPLER EFFECT IN A MEDIUM

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ABSTRACT

The problem of calculating the frequency of the wave scattered by a body moving in a medium is formulated from field-theoretic considerations. The Doppler equation for a homogeneous dispersive medium is obtained on the basis of the fact that the frequency and the wave vector of a plane wave form a 4-vector. It is found that the solutions of the Doppler equation can be classified into two kinds. In one kind, the solutions are close to the frequency of the incident wave. In the other kind they appear near the poles of the refractive index of the medium on the ω-axis. In the case of an isotropic plasma, the monochromaticity of the incident wave is shown to be preserved after the wave is scattered by a moving body. However, in the case of a magneto-active plasma, the scattered wave contains more than one frequency for a monochromatic incident wave. The physical interpretations of these frequencies are given. In an inhomogeneous medium the Doppler equation has to be derived from a different starting point. The crucial point of the derivation is to perform spectral decompositions of the transformed fields and then to apply, under the assumption of gradual inhomogeneity, the method of stationary phase to determine the critical points. It is shown how the phase functions of the fields can be obtained by transforming Maxwell's equations into equations of Riccati-type. Approximate solutions of the Doppler equation are obtained for isotropic as well as for gyroelectric stratified media.

I. INTRODUCTION

When a wave impinges on a moving body, the frequency of the scattered wave is known to differ from that of the incident wave. This is called the "Doppler effect". This effect has been calculated by the conventional method of ray optics in which the wave properties of the fields are completely ignored. In using this method most authors (1,2,3) have not, however, taken into account the presence of a medium through which the wave is propagated. In so doing, some interesting phenomena which result from the dispersive character of the medium, disappear. To account for these phenomena it is necessary to extend the conventional method to include the properties of the medium which, in phenomenological description, are expressed by its refractive index. This can be easily done in the ray optics analysis and the extension of the conventional method is outlined below*.

In the treatment of ray optics the concept of instantaneous frequency is used. That is, the frequency of a wave is given by the time derivative of the total phase which is equal to $\omega t - L\omega/c$, where ω is the frequency of the transmitted wave, c the speed of light in vacuum, L the optical path length between the transmitter and the point of observation. The optical path length L between two points in space is defined as c times the minimum time for a light wave traveling from one point to the other. In mathematical language, L is given by

$$L = c \int_{\underline{r}_1}^{\underline{r}_2} \frac{\underline{e} \cdot d\underline{r}}{v} , \qquad (1.1)$$

*One part of the problem, i.e., the Doppler effect from a transmitter moving in a refractive medium, has received extensive study in the literature. See, for example, Ref. (4).

where $\underline{r}_1$ and $\underline{r}_2$ are the position vectors of two spatial points, v the velocity of light in the medium, $\underline{e}$ the unit vector tangent to the ray which is determined by Fermat's principle. By introducing the refractive index n of the medium defined by $n = c/v$, eq. 1.1 can also be written as

$$L = \int_{\underline{r}_1}^{\underline{r}_2} n\, \underline{e} \cdot d\underline{r} \tag{1.2}$$

which is the optical path length between $\underline{r}_1$ and $\underline{r}_2$ in a medium whose properties are characterized by the function $n(\underline{r},\omega)$.

By using the concept of instantaneous frequency, the extension of the conventional method is then easily made. The Doppler shift is just given by the time rate of change of the difference between two optical path lengths measured in wavelengths (See Fig. 1). One of these paths is from the transmitter (which is assumed to be at infinity) to the scatterer and from the scatterer to the receiver, the other being from the transmitter directly to the receiver. With expression 1.2 for the optical path length, one then has

$$\Delta\omega = \omega_s - \omega_i = \frac{\omega_i}{c}\frac{d}{dt}\int_{\infty}^{0} n(\underline{r}',\omega_i)\, \underline{e}_o \cdot d\underline{r}' - \frac{\omega_i}{c}\frac{d}{dt}\int_{\infty}^{\underline{r}(t)} n(\underline{r}',\omega_i)\underline{e}_i \cdot d\underline{r}'$$
$$- \frac{\omega_s}{c}\frac{d}{dt}\int_{\underline{r}(t)}^{0} n(\underline{r}',\omega_s)\, \underline{e}_s \cdot d\underline{r}' \tag{1.3}$$

where ω_s and ω_i are respectively the frequencies of the scattered and the incident waves; $\underline{r}(t)$ is the position of the scatterer; $\underline{e}_o$, $\underline{e}_i$ and $\underline{e}_s$ are defined in Fig. 1.

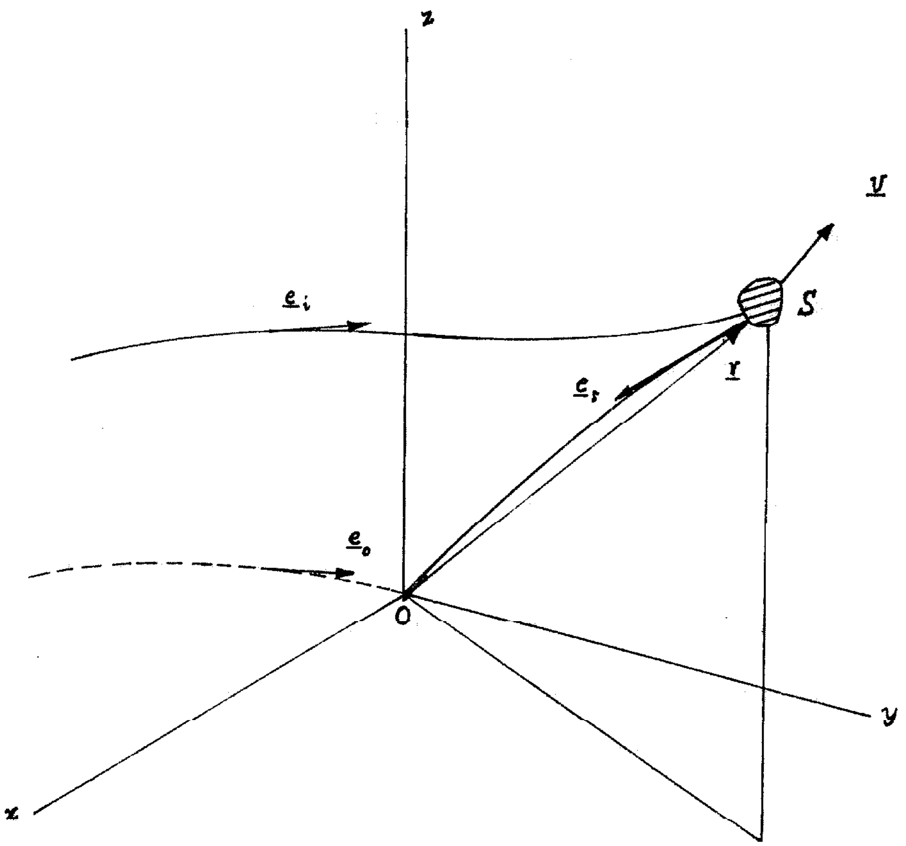

Fig. 1. $\underline{e}_s$, $\underline{e}_i$ and $\underline{e}_o$ are unit vectors tangent to the indicated paths. O is the observer (receiver) situated at the origin. S is the scatterer moving with velocity $\underline{v}$.

Carrying out the differentiations, one obtains

$$\omega_s - \omega_i = -\frac{\omega_i}{c}\, n(\omega_i,\underline{r})\ \underline{e}_i(\underline{r}) \cdot \underline{v}(\underline{r}) + \frac{\omega_s}{c}\, n(\omega_s,\underline{r})\ \underline{e}_s(\underline{r}) \cdot \underline{v}(\underline{r}) , \quad (1.4)$$

where $\underline{v}(\underline{r})$ is the velocity of the scatterer.

In the treatment of those authors mentioned above, n is set equal to unity, and in this case ω_s can be easily obtained by solving eq. 1.4 with $n = 1$. With the medium taken into account, however, eq. 1.4 has as yet to be solved for a given n .

This method of deriving the Doppler equation 1.4 is quite simple, nevertheless it is not at all clear how good it is to use the instantaneous frequency to define the frequency of a wave. Furthermore, it is not obvious how far the approximate equation 1.4 is valid in the microwave range which is of concern to us here*. Thus the treatment of the problem by ray optics is unsatisfactory.

An acceptable approach to the problem is found from field-theoretic considerations. Because of certain inherent difficulties in this approach it has not been recognized in the literature. It is the purpose of this paper to show how the problem can be treated rigorously from the field-theoretic point of view, that is, we shall apply Maxwell's equations and the special theory of relativity to treat the problem as fully as possible.

In past years some related problems have been solved by the field-theoretic approach. Frank (5) [1943] solved the problem of a radiating dipole moving in a homogeneous medium and demonstrated the existence of

*Eq. 1.4 is approximate since in obtaining the equation the phase function is found from the laws of ray optics rather than from Maxwell's equations and since the former can be obtained from the latter as a limiting case, i.e., by letting $\lambda \to 0$.

"complex Doppler modes". He then concluded that the dispersion of the medium splits the frequency. Later Rydbeck (6) solved the same problem for a stratified dispersive medium, first by the treatment of ray optics and then from field-theoretic considerations. In the latter approach he showed the mathematical difficulties involved in the problem. The inverse problem, i.e., the observer is moving and the source of radiation is stationary with respect to the medium, was first attacked by Tischer (7) in 1960. However, he treated only the case of an isotropic stratified medium. Lee and Papas (8) solved the same problem from a different starting point and thus avoided the mathematical difficulties involved in Tischer's treatment. Moreover, they were able to study the more general case where the medium is anisotropic in addition to being inhomogeneous.

The text of this paper is divided essentially into two parts and each part is again subdivided into three chapters.

In the first part, homogeneous dispersive media are considered. The Doppler equation is derived in Chapter II on the basis of the fact that the frequency and the wave vector of a plane wave form a 4-vector. The general method of obtaining the approximate solutions of the Doppler equation is given in Chapter III and is applied to the case of an isotropic plasma where exact solution can be obtained. The approximate solution is shown to be in good agreement with the exact one for $v \ll c$. In Chapter IV gyroelectric media are treated. The resulting four Doppler equations are solved by the method developed in Chapter III. The two kinds of roots thus obtained are discussed.

The second part is devoted to the study of the Doppler effect in inhomogeneous media. The Doppler equation is obtained in Chapter V under the assumption of gradual inhomogeneity. It is shown that the scattered wave consists of a continuous spectrum. The Doppler equation is first solved in Chapter VI for the case of an isotropic stratified medium. By defining the fields in terms of the exponentials of a set of complex quantities, Maxwell's equations are transformed into equations of Riccati-type which are suitable for successive approximations. In Chapter VII the case of a gyroelectric medium is studied. The technique given in Chapter VI is used to obtain a set of coupled nonlinear equations. These equations are rearranged into the forms which can be solved by the method of iteration.

II. THE DOPPLER EQUATION IN HOMOGENEOUS DISPERSIVE MEDIA

2.1 Statement of the Problem

Consider a scatterer of arbitrary shape traveling with constant velocity $\underline{v}$ through a homogeneous dispersive (isotropic or anisotropic) medium of infinite extent. A plane monochromatic electromagnetic wave of frequency ω_i and wave vector $\underline{k}_i$ is incident on the moving scatterer and is subsequently scattered.

At any point $\underline{r}$ in space, the scattered fields can be represented as a sum of plane waves by the following Fourier integrals:

$$\underline{E}_s(\underline{r},t) = \int \underline{E}_s(\omega_s,\underline{k}_s)e^{i(\underline{k}_s \cdot \underline{r} - \omega_s t)} \, d\omega_s \, d^3k_s$$

$$\underline{B}_s(\underline{r},t) = \int \underline{B}_s(\omega_s,\underline{k}_s)e^{i(\underline{k}_i \cdot \underline{r} - \omega_s t)} \, d\omega_s \, d^3k_s \quad . \tag{2.1}$$

Here $\underline{E}_s(\omega_s,\underline{k}_s)$ and $\underline{B}_s(\omega_s,\underline{k}_s)$ are related by one of the Maxwell equations

$$\nabla \times \underline{E}_s(\underline{r},t) = -\frac{\partial}{\partial t}\underline{B}_s(\underline{r},t)$$

and the relationship is given by

$$\underline{B}_s(\omega_s,\underline{k}_s) = \frac{\underline{k}_s \times \underline{E}_s(\omega_s,\underline{k}_s)}{\omega_s}$$

where ω_s and $\underline{k}_s$ denote, respectively, the frequency and the wave vector of the scattered wave.

Physically, equations 2.1 say that at any point in space the field vectors (electric or magnetic) are given by the sum of plane waves coming from all directions. However, if the observation point is far away from the scatterer, the scattered wave there is practically a plane wave coming from the direction θ_s (see Fig. 2). It is required to find the frequency ω_s of this plane wave in terms of ω_i, $\underline{v}$, and the properties of the medium.

2.2 Derivation of the Doppler Equation

In the frame S' where the scatterer is stationary, the frequency of the scattered wave remains equal to that of the incident wave, i.e.,

$$\omega_s' = \omega_i' \tag{2.2}$$

where the primed quantities are measured in S'. Eq. 2.2 can be obtained from the following argument. Since Maxwell's equations are covariant in all Lorentz frames, in S' we simply have (in a source-free region)

$$\begin{aligned} \nabla' \times \underline{E}' &= -\frac{\partial \underline{\underline{B}}'}{\partial t'} \\ \nabla' \times \underline{H}' &= \frac{\partial \underline{\underline{D}}'}{\partial t'} \end{aligned} \tag{2.2'}$$

These equations hold regardless of the state of the medium. Across the surface of the scatterer, the tangential $\underline{E}'$ and the tangential $\underline{H}'$ have to be continuous, as can be easily seen from the Maxwell equations 2.2'. It thus follows that $\omega_s' = \omega_i'$ in order that these boundary conditions be satisfied for all time.

The question then remains as to how this information 2.2 can be carried over to the laboratory frame S where the medium is stationary.

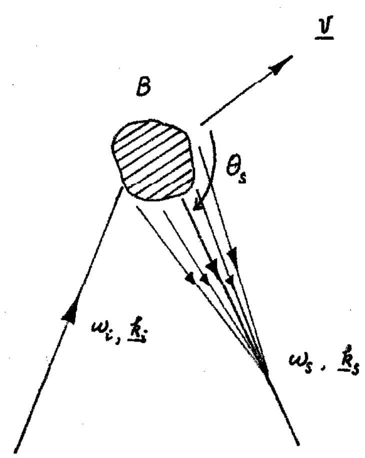

Fig. 2. $(\underline{k}_i, i\frac{\omega_i}{c})$ and $(\underline{k}_s, i\frac{\omega_s}{c})$ are respectively the 4-vectors of the incident and the scattered waves. $\underline{v}$ is the velocity vector of the scatterer B .

Let us recall that the frequency ω and the wave vector $\underline{k}$ of a plane wave form a 4-vector in Minkowski space (see Appendix A). Denote this 4-vector by K_μ :

$$K_\mu = (k_x, k_y, k_z, \frac{i\omega}{c}) = (\underline{k}, \frac{i\omega}{c}) \tag{2.3}$$

where c is the speed of light in vacuum and μ varies from 1 to 4 (in the following all Greek letters used as subscripts are understood to vary from 1 to 4). Let us now construct another 4-vector U_μ for the velocity of the scatterer. As is well known (see Appendix B), U_μ is given by

$$U_\mu = (\gamma \underline{v}, i\gamma c), \tag{2.4}$$

where $\gamma = (1 - \frac{v^2}{c^2})^{-1/2}$, $\underline{v}$ being the ordinary velocity of the scatterer. Since the scalar product of two 4-vectors in Minkowski space is invariant under the Lorentz transformation, $K_\mu U_\mu$ is then an invariant scalar product and eq. 2.2 can be written in the following covariant form

$$K_\mu^{(s)'} U_\mu' = K_\mu^{(i)'} U_\mu' , \tag{2.5}$$

where $K_\mu^{(s)'} = (\underline{k}_s', i\omega_s'/c)$, $K_\mu^{(i)'} = (\underline{k}_i', i\omega_i'/c)$, and $U_\mu' = (0, ic)$. After we have translated eq. 2.2 into a covariant form 2.5, in the laboratory frame S we simply have

$$K_\mu^{(s)} U_\mu = K_\mu^{(i)} U_\mu . \tag{2.6}$$

Writing eq. 2.6 out into components, we obtain

$$\omega_s - \underline{k}_s \cdot \underline{v} = \omega_i - \underline{k}_i \cdot \underline{v} \qquad (2.7)$$

which is the required Doppler equation. Other methods of deriving this equation are given in Appendix C.

2.3 Dispersion Relations

In deriving the Doppler equation 2.7 the medium was assumed only to be homogeneous so that $\underline{k}$ and ω form a 4-vector. We did not, however, specify whether the medium was isotropic or anisotropic. Thus equation 2.7 is applicable both for isotropic and for anisotropic media. Before we are able to solve eq. 2.7 for ω_s, we first have to find the relationship between ω and $\underline{k}$, i.e., the dispersion relation, and then solve eq. 2.7 algebraically. To obtain this dispersion relation we resort to Maxwell's equations which, in a source-free region, take the following forms:

$$\nabla \times \underline{E} = -\frac{\partial \underline{B}}{\partial t} \qquad (2.8)$$

$$\nabla \times \underline{H} = \frac{\partial \underline{D}}{\partial t} \quad . \qquad (2.9)$$

In addition we must have two constitutive equations which describe the electromagnetic properties of the medium. In the following we shall use exclusively a non-magnetic medium whose permittivity tensor is a function of ω. Thus for the constitutive equations we take

$$\underline{B} = \mu_o \underline{H} \qquad (2.10)$$

$$\underline{D} = \int_{-\infty}^{t} \underline{\underline{\epsilon}}(t-\tau) \cdot \underline{E}(\tau, \underline{r}) d\tau \qquad (2.11)$$

where μ_o is the free space permeability and $\underline{\underline{\epsilon}}$ is a tensor of nine components. Taking the curl of eq. 2.8 and by virtue of eqs. 2.9, 2.10 and 2.11, we obtain

$$\nabla \times \nabla \times \underline{E} = -\mu_o \frac{\partial^2}{\partial t^2} \int_{-\infty}^{t} \underline{\underline{\epsilon}}(t-\tau) \cdot \underline{E}(\tau,\underline{r}) d\tau \quad . \tag{2.12}$$

Substitution of

$$\underline{E}(\underline{r},t) = \int \underline{E}(\underline{k},\omega)\, e^{i(\underline{k}\cdot\underline{r} - \omega t)}\, d\omega\, d^3k$$

into 2.12 gives

$$\int \left\{ \underline{k} \times (\underline{k} \times \underline{E}) + \mu_o \omega^2 \underline{\underline{\epsilon}}(\omega) \cdot \underline{E} \right\} e^{i(\underline{k}\cdot\underline{r} - \omega t)}\, d\omega\, d^3k = 0 \tag{2.13}$$

where

$$\underline{\underline{\epsilon}}(\omega) = \int_0^{\infty} \underline{\underline{\epsilon}}(x)\, e^{+i\omega x}\, dx \, . \tag{2.14}$$

From eq. 2.13 we conclude that

$$\underline{k} \times (\underline{k} \times \underline{E}) + \mu_o \omega^2\, \underline{\underline{\epsilon}}(\omega) \cdot \underline{E} = 0 \quad ,$$

or

$$(\underline{k} \cdot \underline{E})\underline{k} + (\mu_o \omega^2\, \underline{\underline{\epsilon}} - k^2 I) \cdot \underline{E} = 0 \, ,$$

or

$$(\underline{k}\,\underline{k} + \mu_o \omega^2 \underline{\underline{\epsilon}} - k^2 I) \cdot \underline{E} = 0 \tag{2.15}$$

where I is the unit dyad.

For $\underline{E} \neq 0$ we must set the following determinant equal to zero, viz.

$$\det \left| k_i k_j + \mu_o \omega^2 \epsilon_{ij} - k^2 \delta_{ij} \right| = 0 \tag{2.16}$$

where δ_{ij} is the Kronecker delta. Eq. 2.16 is the dispersion relation for a homogeneous medium whose permittivity tensor is given by $\epsilon_{ij}(\omega)$.

(i) Isotropic Media

In this case $\underline{\underline{\epsilon}}$ reduces to a scalar ϵ and hence $\underline{k} \cdot \underline{E} = 0$ because $\nabla \cdot \epsilon \underline{E} = \epsilon \nabla \cdot \underline{E} = 0$. We immediately obtain

$$k^2 = \omega^2 \mu_o \epsilon(\omega) \tag{2.17}$$

as the dispersion relation for an isotropic dispersive medium.

(ii) Anisotropic Media

In the following we shall limit ourselves only to the case where the anisotropy of the medium is caused by a magnetostatic field $\underline{B}_o$ as in the ionosphere. In this case $\underline{\underline{\epsilon}}(\omega)$ takes the following form*:

$$\underline{\underline{\epsilon}} = \epsilon_o \begin{pmatrix} \epsilon_1 & ig \cos\theta & -ig \sin\theta \\ -ig\cos\theta & \epsilon_1 \cos^2\theta + \epsilon_2 \sin^2\theta & \frac{\epsilon_2 - \epsilon_1}{2} \sin 2\theta \\ ig \sin\theta & \frac{\epsilon_2 - \epsilon_1}{2} \sin 2\theta & \epsilon_1 \sin^2\theta + \epsilon_2 \cos^2\theta \end{pmatrix} \tag{2.18}$$

where ϵ_o is the free space permittivity. θ is the angle between the direction of propagation and the direction of $\underline{B}_o$. Moreover, in expression 2.18

*See, for example, C. H. Papas, Unpublished Class Notes on Electromagnetic Theory, Calif. Inst. of Tech.

$$\epsilon_1 = 1 - \frac{\omega_p^2}{\omega^2 - \omega_g^2}$$

$$\epsilon_2 = 1 - \frac{\omega_p^2}{\omega^2} \qquad (2.19)$$

$$g = - \frac{\omega_p^2 \omega_g}{\omega(\omega^2 - \omega_g^2)} \quad .$$

Here ω_p is the plasma frequency of the medium and ω_g its gyrofrequency. Substituting eq. 2.18 for ϵ_{ij} into eq. 2.16, and after some manipulations, we obtain, noting that $\underline{k}$ has been assumed to be in the z-direction,

$$k_\pm^2 = \omega^2 \mu_o \epsilon_o$$

$$\times \frac{(\epsilon_1^2 - g^2 - \epsilon_1\epsilon_2)\sin^2\theta + 2\epsilon_1\epsilon_2 \pm \sqrt{(\epsilon_1^2 - g^2 - \epsilon_1\epsilon_2)^2 \sin^4\theta + 4g^2\epsilon_2^2 \cos^2\theta}}{2(\epsilon_1 \sin^2\theta + \epsilon_2 \cos^2\theta)} \qquad (2.20)$$

as the dispersion relations for a gyroelectric medium. For $\theta = 0$ and $\theta = \pi/2$, eqs. 2.20 are reduced to considerably simpler forms:

$$k_\pm^2 = \omega^2 \mu_o \epsilon_o (\epsilon_1 \pm g) \quad , \quad \text{parallel case } (\theta = 0) \qquad (2.21)$$

$$\left.\begin{aligned} k_+^2 &= \omega^2 \mu_o \epsilon_o \frac{\epsilon_1^2 - g^2}{\epsilon_1} \\ k_-^2 &= \omega^2 \mu_o \epsilon_o \epsilon_2 \end{aligned}\right\} \text{perpendicular case } (\theta = \pi/2) \qquad (2.22)$$

By virtue of the dispersion relations just obtained, $\underline{k}_s$ and $\underline{k}_i$ can be eliminated from the Doppler equation 2.7, and the resulting equation will contain only the unknown ω_s . In subsequent chapters we shall solve the resulting Doppler equation for isotropic as well as for gyrolelectric media.

III. SOLUTIONS OF THE DOPPLER EQUATION IN ISOTROPIC, TRANSPARENT AND DISPERSIVE MEDIA

The electromagnetic properties of a medium are usually described by its index of refraction $n(\omega)$ defined by

$$n(\omega) = \frac{c}{v_p(\omega)} = \frac{c}{\omega/k} = \frac{ck}{\omega} \tag{3.1}$$

where $v_p(\omega)$ is the phase velocity of a plane wave of frequency ω. According to the dispersion relation 2.17, $n(\omega)$ is then equal to $\sqrt{\epsilon(\omega)/\epsilon_o}$, where ϵ_o is the permittivity of a vacuum. To determine $n(\omega)$ or $\epsilon(\omega)$ of a given medium, one actually has to consider the interactions of the charged particles of the medium with the self-consistent electric and magnetic fields. However, we shall not go into this, since the determination of $n(\omega)$ of a given medium is well known.

Substitution of eq. 3.1 into the Doppler equation 2.7 yields

$$\omega_s - \beta_s\omega_s n(\omega_s) = \omega_i - \beta_i\omega_i n(\omega_i) \tag{3.2}$$

where

$$\beta_s = \frac{v}{c}\cos\theta_s = \beta\cos\theta_s \quad \text{and} \quad \beta_i = \frac{v}{c}\cos\theta_i = \beta\cos\theta_i$$

θ_i and θ_s being, respectively, the angles of incidence and scattering with respect to the velocity $\underline{v}$ of the scatterer. In general $n(\omega)$ is a complicated function of ω and hence it is not always possible to solve eq. 3.2 analytically for ω_s. However, because of the fact that β is a small parameter, we are able to find approximately all the roots of eq. 3.2 up to the order of β^2, and the method of solution will be given in the next section.

3.1 Method of Solution of the Doppler Equation

For simplicity we shall replace ω_s by ω in the following and rewrite eq. 3.2 as

$$n(\omega) = \frac{\omega - \bar{\omega}_i}{\beta_s \omega} \tag{3.3}$$

where $\bar{\omega}_i = \omega_i - \beta_i \omega_i n(\omega_i)$. In order to get some idea as to how to solve this equation, we shall first examine the properties of the function $(\omega - \bar{\omega}_i)/\beta_s \omega$. Differentiating it with respect to ω, we obtain

$$\frac{d}{d\omega}\left(\frac{\omega - \bar{\omega}_i}{\beta_s \omega}\right) = \frac{\bar{\omega}_i}{\beta_s \omega^2} \quad . \tag{3.4}$$

Thus we see that this function is monotonic increasing or decreasing depending on the sign of β_s. Moreover, at $\bar{\omega}_i$ it starts to rise rapidly for $\beta_s > 0$ and decreases rapidly to zero for $\beta_s < 0$, since its slope given by eq. 3.4 is a large quantity. The properties of the function $n(\omega)$ for transparent media are well known (see Ref. (9)): $n^2(\omega)$ is always a monotonic increasing function and may possess poles and zeros on the ω-axis. These poles correspond to the resonant frequencies of the medium under consideration.

With this information about the functions $n(\omega)$ and $(\omega - \bar{\omega}_i)/\beta_s \omega$ we can draw some conclusions about the roots of eq. 3.3. Here there are two kinds of roots to be distinguished (see Fig. 3).

(i) Roots near $\bar{\omega}_i$

To obtain this kind of root we rewrite eq. 3.3 as

$$\omega = \bar{\omega}_i + \beta_s \omega \, n(\omega) \tag{3.5}$$

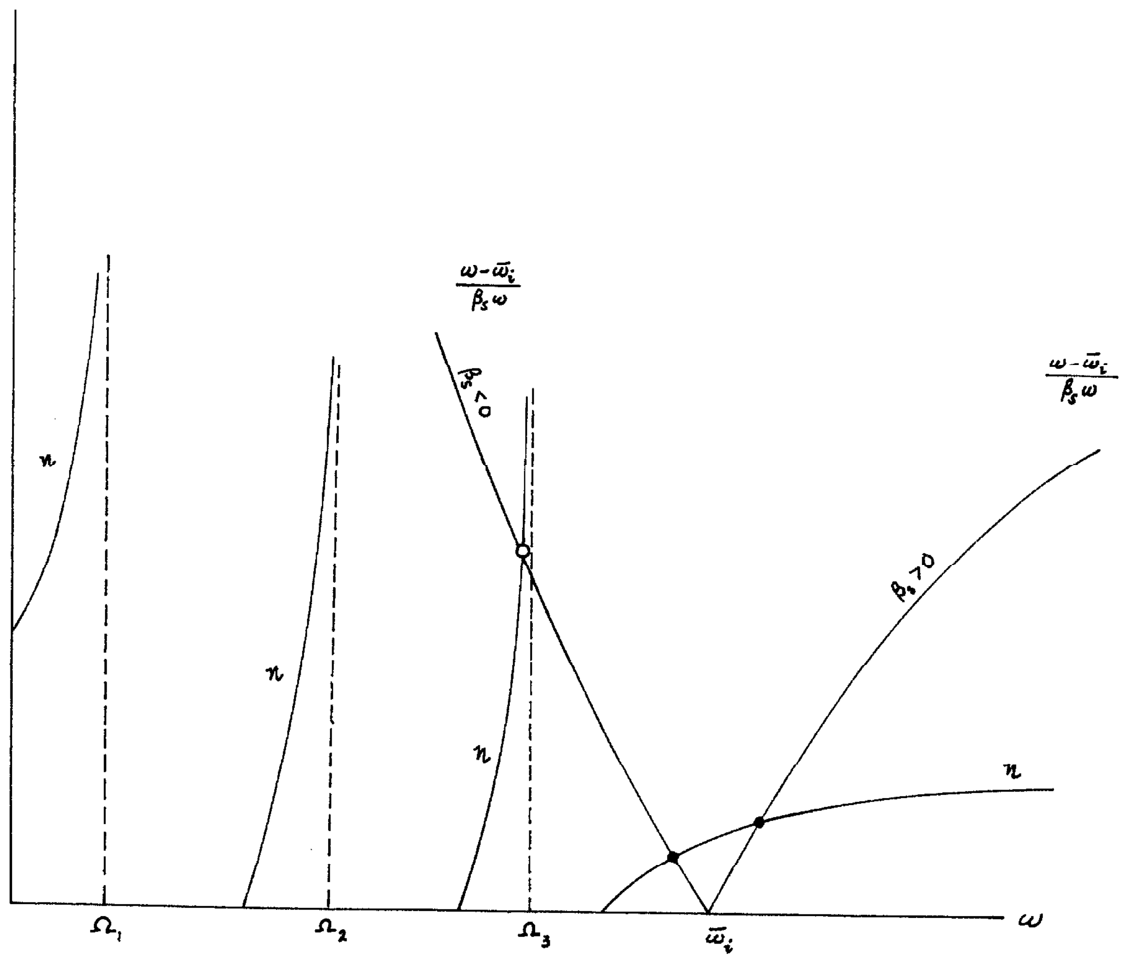

Fig. 3. Graphical solutions of eq. 3.3 for n(ω) having three poles

• ---- solutions of the first kind

o ---- solutions of the second kind, two of which are not shown in the figure where β_s can be positive or negative. The second term on the right hand side of eq.3.5 can be treated as a small perturbation, since it contains the small parameter β_s. By iterating eq. 3.5 one easily gets

$$\omega = \bar{\omega}_i + \beta_s \bar{\omega}_i n(\bar{\omega}_i) + \cdots$$

$$= \omega_i - (\beta_i - \beta_s) \omega_i n(\omega_i) + O(\beta^2) \quad . \qquad (3.6)$$

In the following we shall refer to this kind of solution as the root of the first kind.

(ii) Roots near the Poles of $n(\omega)$

It is always possible to express $n^2(\omega)$ in the following form:

$$n^2(\omega) = \frac{P(\omega)}{\prod_j (\Omega_j^2 - \omega^2)} \qquad (3.7)$$

where $P(\omega)$ is a polynomial of the same degree as $\Pi(\Omega_j^2 - \omega^2)$ so that $n^2 \to 1$ as $\omega \to \infty$. Squaring both sides of eq. 3.3 and using 3.7 for n^2 we have

$$(\omega - \bar{\omega}_i)^2 \prod_j (\Omega_j^2 - \omega^2) = \beta_s^2 \omega^2 P(\omega) \quad . \qquad (3.8)$$

Since the right hand side of this equation is very small, this suggests that we write

$$\omega = \Omega_k(1 - \delta_k) \qquad (3.9)$$

where $\delta_k \ll 1$ and Ω_k is one of those Ω_j's . Substituting 3.9 into 3.8 and after a straightforward manipulation, we find

$$\delta_k \approx \frac{P(\Omega_k)}{2(\Omega_k - \omega_i)^2 \prod_{j \neq k} (\Omega_j^2 - \Omega_k^2)} \quad \beta_s^2 \tag{3.10}$$

which is indeed a very small quantity of the order of β^2. Hence, for this kind of root we simply have

$$\omega_j = \Omega_j - 0_j(\beta^2) \tag{3.11}$$

where $0_j(\beta^2)$ denotes a term similar to expression 3.10. In the following we shall refer to this kind of root as the root of the second kind.

3.2 Exact Solution of the Doppler Equation for an Isotropic Plasma

We take an isotropic plasma to be a completely ionized gas without a biasing magnetostatic field $\underline{B}_o$. Thus the medium considered here exhibits no resonant frequencies and hence the roots of the second kind do not exist. Since $n(\omega)$ in this case is given by the simple form

$$n(\omega) = \sqrt{1 - \frac{\omega_p^2}{\omega^2}} \quad , \tag{3.12}$$

we shall first solve the Doppler equation 3.2 exactly and then compare the exact solutions with those obtained by the short-cut method given in the previous section.

Before proceeding to solve eq. 3.2 analytically, one should note that there is only one solution for each sign of β_s, since $n(\omega)$ given by 3.12 is a monotonic increasing function of ω (Fig. 4). Rewriting eq. 3.3 as

$$\beta_s \omega \sqrt{1 - \frac{\omega_p^2}{\omega^2}} = \omega - \bar{\omega}_i \tag{3.13}$$

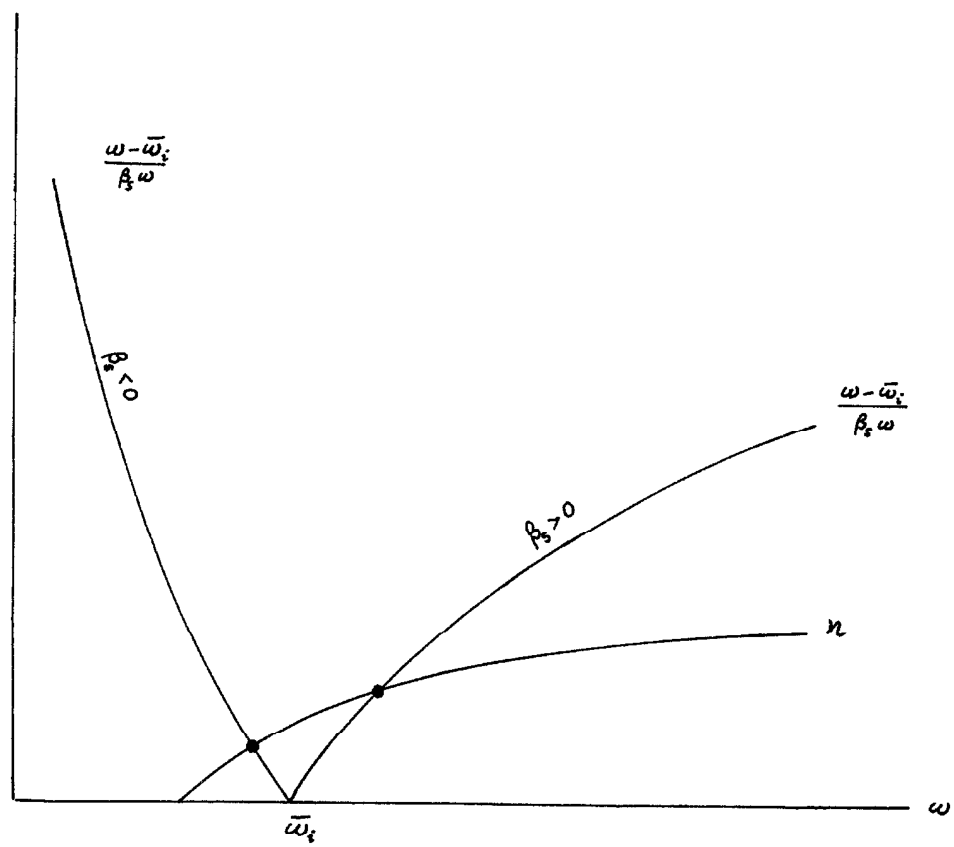

Fig. 4. Graphical solution of the Doppler equation for

$$n(\omega) = \sqrt{1 - \omega_p^2/\omega^2} \quad .$$

and squaring both sides of eq. 3.13 we have

$$(1 - \beta_s^2)\omega^2 - 2\bar{\omega}_i \omega + (\bar{\omega}_i^2 + \beta_s^2 \omega_p^2) = 0 \quad . \tag{3.14}$$

The roots of eq. 3.14 can be easily found to be

$$\omega = \frac{1}{1 - \beta_s^2} \left[\omega_i - \beta_i \sqrt{\omega_i^2 - \omega_p^2} \pm \beta_s \sqrt{(1+\beta_i^2)(\omega_i^2 - \omega_p^2) - 2\omega_i\beta_i \sqrt{\omega_i^2 - \omega_p^2} + \beta_s^2 \omega_p^2} \right] . \tag{3.15}$$

One can easily verify that only the solution with the "+" sign in 3.15 satisfies the original equation 3.13, and thus we have

$$\omega = \frac{1}{1 - \beta_s^2} \left[\omega_i - \beta_i \sqrt{\omega_i^2 - \omega_p^2} + \beta_s \sqrt{(1+\beta_i^2)(\omega_i^2 - \omega_p^2) - 2\omega_i\beta_i \sqrt{\omega_i^2 - \omega_p^2} + \beta_s^2 \omega_p^2} \right] \tag{3.16}$$

as the only root for eq. 3.13. Expanding 3.16 in powers of β we get

$$\omega = \omega_i - (\beta_i - \beta_s) \sqrt{\omega_i^2 - \omega_p^2} - \omega_i(\beta_i - \beta_s)\beta_s + O(\beta^3) \quad . \tag{3.17}$$

Let us now solve eq. 3.13 by the short-cut method, i.e., the method of iteration. To do this, let us rewrite eq. 3.13 as

$$\omega = \omega_i - \beta_i \sqrt{\omega_i^2 - \omega_p^2} + \beta_s \sqrt{\omega^2 - \omega_p^2} \quad . \tag{3.18}$$

In the first approximation we take $\omega = \omega_i - \beta_i \sqrt{\omega_i^2 - \omega_p^2}$. In the next approximation we substitute this value of ω into the third term on the right hand side of 3.18 and obtain

$$\omega = \omega_i - \beta_i\sqrt{\omega_i^2 - \omega_p^2} + \beta_s\sqrt{(\omega_i - \beta_i\sqrt{\omega_i^2 - \omega_p^2})^2 - \omega_p^2}$$

$$= \omega_i - (\beta_i - \beta_s)\sqrt{\omega_i^2 - \omega_p^2} - \omega_i\beta_i\beta_s + 0(\beta^3) \quad . \qquad (3.19)$$

Comparing 3.17 with 3.19 we see that the two expressions agree with each other up to the order of β . In practice $\beta \ll 1$ and we can neglect terms of orders higher than β . It is therefore sufficient to use the short-cut method to obtain a very good approximate solution.

IV. SOLUTIONS OF THE DOPPLER EQUATION IN GYROELECTRIC MEDIA

4.1 The Four Doppler Equations

When an ionized gas is subject to an external magnetostatic field $\underline{B}_o$, it becomes an anisotropic medium such as the ionosphere. As has been shown in section 2.3, a plane wave propagating in such a medium is split into two waves with different indices of refraction. Each of these waves, after being scattered by the moving scatterer, is again split into two. Hence there result four Doppler equations:

$$\omega - \beta_s \omega n_{\pm}(\omega) = \omega_i - \beta_i \omega_i n_{\pm}(\omega_i) \quad , \tag{4.1}$$

where, as before, ω denotes the frequency of the scattered wave. Fig. 5 shows schematically the splitting of the incident and the scattered waves and the coupling between them. In the following section we shall solve these four equations 4.1 by the technique developed in section 3.1.

4.2 Solutions of the Four Doppler Equations

Substituting expressions 2.19 into eqs. 2.20 and using $n_{\pm}^2 = k_{\pm}^2/\omega^2\mu_o\epsilon_o$, we get

$$n_{\pm}^2(\omega) =$$

$$\frac{2\omega(\omega^2-\omega_p^2)(\omega^2-\omega_p^2-\omega_g^2)-\omega\omega_p^2\omega_g^2\sin^2\theta \pm \omega^2\omega_g^2\sqrt{\omega^2\omega_g^2\sin^4\theta+4(\omega^2-\omega_p^2)^2\cos^2\theta}}{2\omega\left[\omega^4-(\omega_p^2+\omega_g^2)\,\omega^2+\omega_p^2\,\omega_g^2\cos^2\theta\right]} \tag{4.2}$$

The denominator of eqs. 4.2 vanishes when $\omega = 0$ and

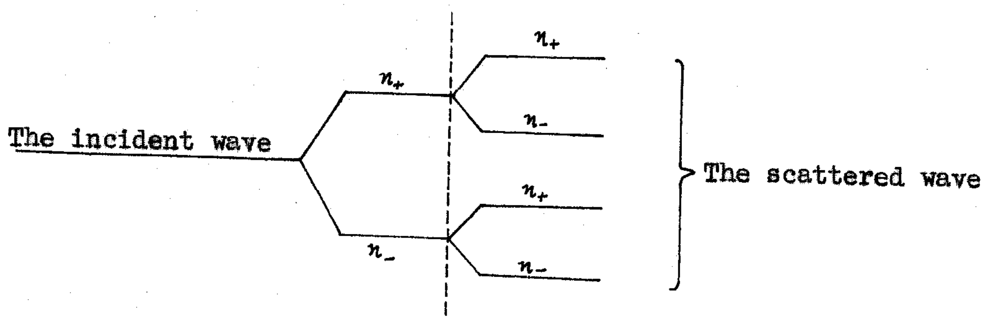

Fig. 5. The splitting of the incident and the scattered waves

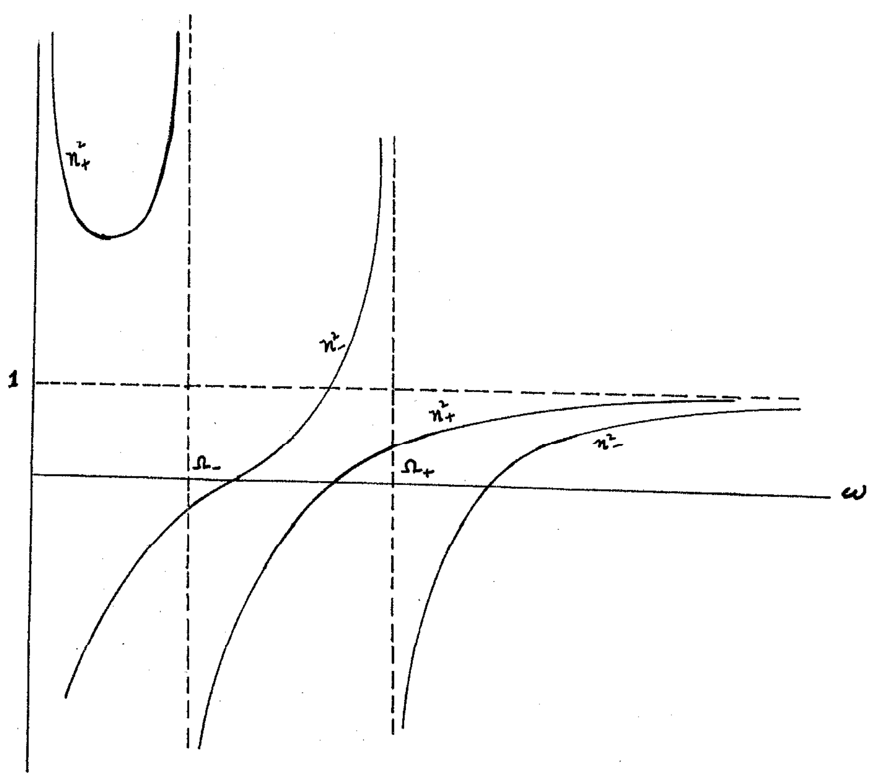

Fig. 6. Curves of $n_{\pm}^2$ vs. ω for arbitrary θ

$$\omega^2 = \frac{\omega_p^2 + \omega_g^2 \pm \sqrt{(\omega_p^2 + \omega_g^2)^2 - 4\omega_p^2 \omega_g^2 \cos^2\theta}}{2} = \Omega_\pm^2 \quad . \tag{4.3}$$

As shown in Appendix D, n_+^2 has simple poles at $\omega = 0$ and $\omega = \Omega_-$, while n_-^2 has simple poles at $\omega = 0$ and $\omega = \Omega_+$. Curves for n_+^2 and n_-^2 versus ω are shown in Fig. 6; and Fig. 7 illustrates the graphical solutions of eqs. 4.1, from which we see that the roots of the first and the second kinds exist for $\beta_s < 0$. For $\beta_s > 0$ only the root of the first kind appears since we have assumed that $\omega_i > \Omega_+$.

By employing the technique developed in section 3.1 one can immediately write the roots of eqs. 4.1

(i) Roots of the First Kind

There are four roots for $\beta_s > 0$ or $\beta_s < 0$, viz.

$$\begin{aligned} \omega &= \omega_i - (\beta_i - \beta_s)\, \omega_i\, n_\pm(\omega_i) + O(\beta^2) \\ \omega &= \omega_i - \left[\beta_i\, n_\pm(\omega_i) - \beta_s\, n_\mp(\omega_i) \right] + O(\beta^2) \end{aligned} \tag{4.4}$$

where $n_\pm$ are given by expressions 4.2.

As $\omega_i \to \infty$, both n_+ and n_- reduce to unity and the medium behaves like a vacuum. In this case the four roots given by 4.4 coincide and become

$$\omega = \omega_i - (\beta_i - \beta_s)\, \omega_i + O(\beta^2) \tag{4.5}$$

as one would expect.

(ii) Roots of the Second Kind

To each pole of $n_\pm(\omega)$ there correspond two roots for $\beta_s < 0$. Since there are three poles at $\omega = 0$ and $\Omega_\pm$, we then have six roots

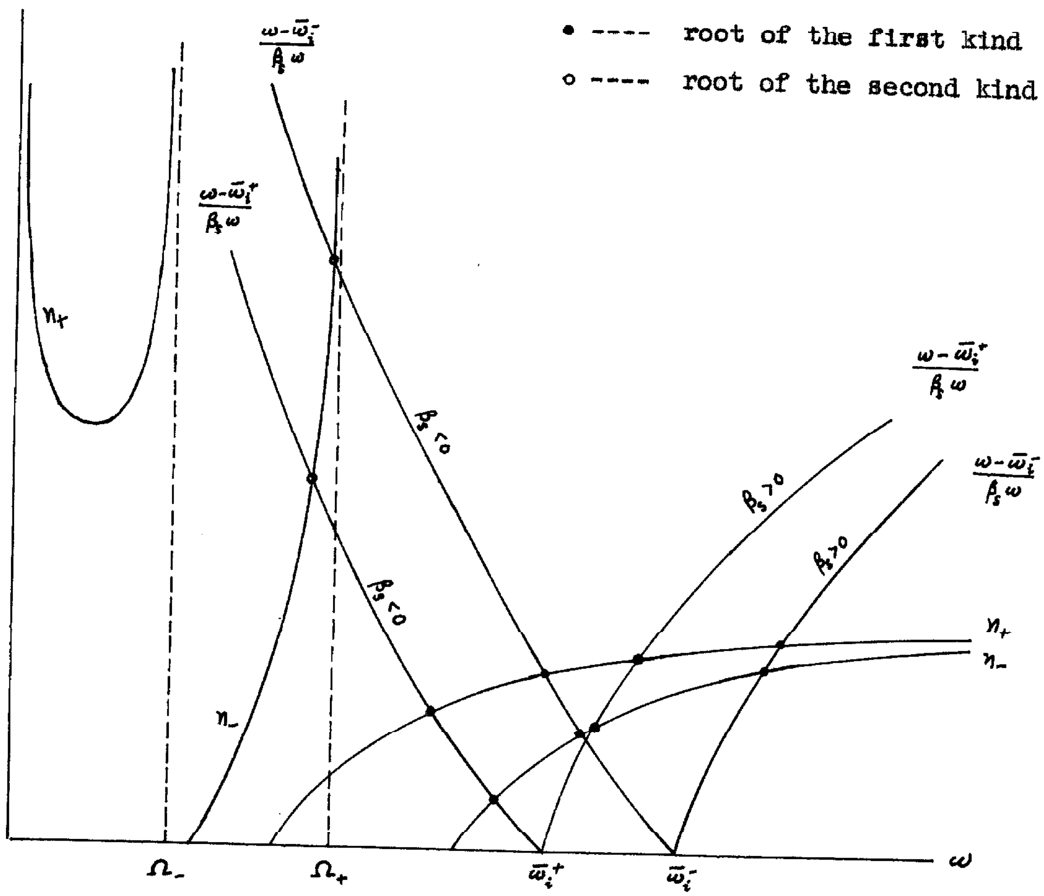

Fig. 7. Graphical solution of eqs. 4.1 with $n_{\pm}$ given by eqs. 4.2. In the figure,

$$\bar{\omega}_i^{\pm} = \omega_i - \beta_i \omega_i n_{\pm}(\omega_i)$$

of the second kind (see Fig. 7). By using the method in Section 3.1 we can write the roots as

$$\left.\begin{aligned} \omega &= \Omega_{-} - O(\beta^2) \\ \omega &= \Omega_{-} - O(\beta^2) \\ \omega &= O(\beta^2) \\ \omega &= O(\beta^2) \end{aligned}\right\} \quad \text{for the extraordinary wave with refractive index } n_+ \qquad (4.6)$$

and

$$\left.\begin{aligned} \omega &= \Omega_{+} - O(\beta^2) \\ \omega &= \Omega_{+} - O(\beta^2) \end{aligned}\right\} \quad \text{for the ordinary wave with refractive index } n_\theta \qquad (4.7)$$

where $\Omega_{\pm}$ are given by expressions 4.3 .

The two special cases, namely $\theta = 0$ and $\theta = \pi/2$, can be easily treated in the same way, and the graphical solutions of eqs. 4.1 are shown respectively in Fig. 8 and Fig. 9. The analytic solutions will not be given here, since they are of no particular interest.

4.3 Discussions of the Roots

In the preceding sections it has been shown that in a magneto-active plasma there result four Doppler equations, the solutions of which can be classified into two kinds. The roots of the first kind are close to the frequency ω_i of the incident wave, and the scattered waves corresponding to these frequencies are propagated freely in space, since the refractive indices at these frequencies are very close to that of a vacuum. The roots of the second kind are shown to lie in the immediate neighborhoods of the resonant frequencies of the medium where absorption

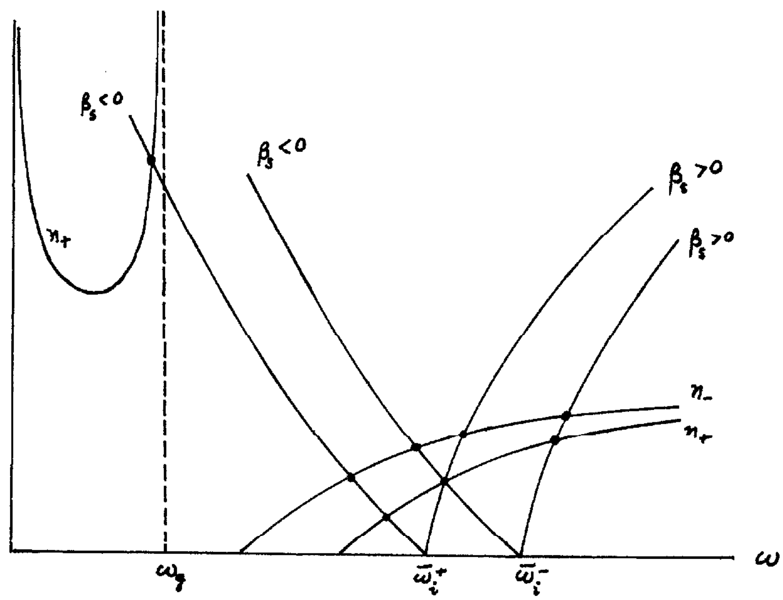

Fig. 8. Graphical solution of eqs. 4.1 for $\theta = 0$

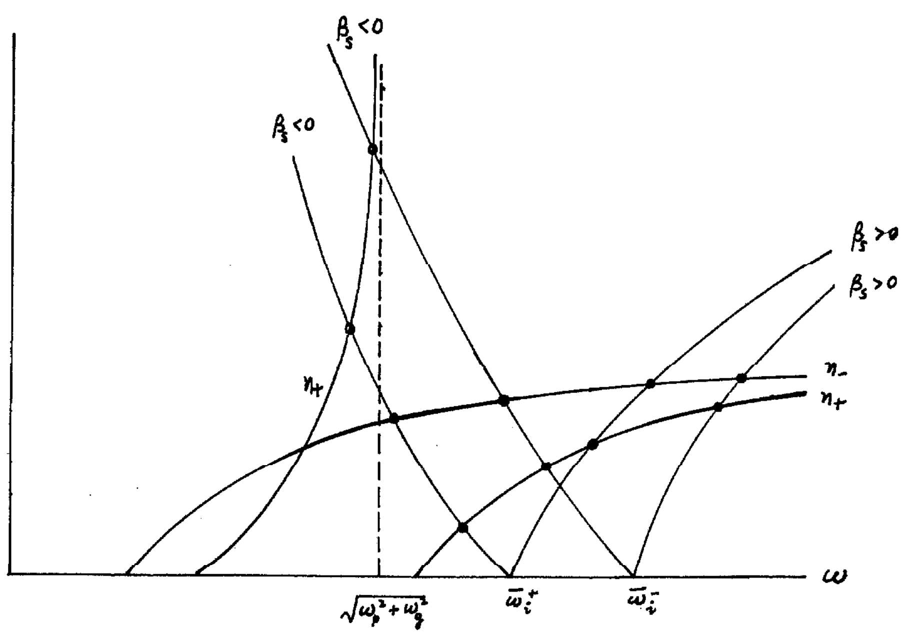

Fig. 9. Graphical solution of eqs. 4.1 for $\theta = \pi/2$

predominates and the intensity of the wave decreases exponentially with distance. Thus, the scattered waves corresponding to these frequencies are attenuated in space. One then may say that part of the scattered energy is lost in "heating up" the medium.

In the case of an isotropic plasma it has been shown in section 3.2 that there is only one root of the first kind for the Doppler equation. This means that the scattered wave is also monochromatic for a plane monochromatic incident wave. For other kinds of isotropic dispersive media such as a rarefied gas, the refractive indices may exhibit poles on the ω-axis. In these cases roots of the first and the second kinds exist at the same time. The above method of analysis can be similarly applied to any specific case under consideration.

V. THE DOPPLER EQUATION IN INHOMOGENEOUS DISPERSIVE MEDIA

5.1 Introduction

In going from a homogeneous medium to an inhomogeneous one we immediately encounter one serious difficulty: plane waves are, in general, no longer solutions of Maxwell's equations. We recall that the derivation of the Doppler equation in a homogeneous medium is based on the fact that the wave vector and the frequency of a plane wave form a 4-vector. We immediately see that this method no longer applies to an inhomogeneous medium. However, the difficulty is resolved if the following assumption is made:

> The free-space wavelength λ of the incident wave is much smaller than the length scale ℓ of the inhomogeneity of the medium*.

An immediate consequence of this assumption is that the distance the scatterer has traveled during one period of the incident wave is small compared to ℓ, i.e., $v\lambda/c < \ell$, where v is the velocity of the scatterer and c the velocity of light in vacuum. Thus the scattering occurs in an essentially homogeneous medium.

With the preceding assumption one may treat a wave as plane wave over a small spatial region whose maximum linear dimension is less than ℓ. This suggests thattthe electric and the magnetic vectors should be represented by the forms $\mathrm{Re}\left\{\underline{E}_o(\underline{r},t)e^{i\psi(\underline{r},t)}\right\}$ and $\mathrm{Re}\left\{\underline{B}_o(\underline{r},t)e^{i\psi(\underline{r},t)}\right\}$ respectively, where $\underline{E}_o(\underline{r},t)$ and $\underline{B}_o(\underline{r},t)$ are slowly varying functions

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*By the length scale ℓ of the inhomogeneity of the medium we mean that the properties of the medium do not change appreciably over a distance smaller than ℓ.

of position and time and $e^{i\psi}$ is a rapidly oscillating term*. With these forms of representation of the fields the Doppler equation can be derived for a slightly non-uniform medium. The essential feature of the derivation is to perform spectral decompositions of the transformed fields in the moving frame. The method of stationary phase is then applied to determine the stationary points from which the required Doppler equation follows.

The resulting Doppler equation contains the function $\psi(\underline{r},t)$ explicitly. Hence, before proceeding to solve this equation for the frequencies of the scattered wave, one must know the explicit form of ψ in the frame where the medium is stationary. Finding this function ψ amounts to solving Maxwell's equations in an inhomogeneous medium. This, as one knows, gives rise to certain mathematical difficulties. We shall overcome these difficulties by defining three complex quantities whose real parts represent the phase functions of the field components and whose imaginary parts correspond to the negative logarithms of the field amplitudes. In so doing we will obtain from Maxwell's equations, equations of Riccati-type which are suitable for successive approximations.

In the following chapters we shall first derive the Doppler equation and then solve it for the frequencies of the wave scattered from a body traveling in an isotropic stratified medium and in a gyroelectric stratified medium such as the ionosphere.

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*A detailed discussion will be given in next section.

5.2 Derivation of the Doppler Equation in an Inhomogeneous Medium

The electric and the magnetic vectors can be represented in the most general forms:

$$\underline{E}(\underline{r},t) = \mathrm{Re}\left[\underline{e}_x E_x e^{i\phi_x} + \underline{e}_y E_y e^{i\phi_y} + \underline{e}_z E_z e^{i\phi_z}\right]$$
$$\underline{B}(\underline{r},t) = \mathrm{Re}\left[\underline{e}_x B_x e^{i\bar{\phi}_x} + \underline{e}_y B_y e^{i\bar{\phi}_y} + \underline{e}_z B_z e^{i\bar{\phi}_z}\right] \qquad (5.1)$$

where E_x, ϕ_x and $B_x \bar{\phi}_x$, etc. are real functions of position and time.

In a slightly non-uniform medium, waves can be considered to be plane in each small spatial region whose maximum linear dimension is less than the length scale of the inhomogeneity. We therefore transform the representation of the fields 5.1 to the forms similar to those of plane waves. To do this let us factor out the rapidly oscillating part $e^{i\psi}$ from eqs. 5.1 and write

$$\underline{E}(\underline{r},t) = \mathrm{Re}\left\{e^{i\psi}\left[\underline{e}_x E_x e^{i\delta_x} + \underline{e}_y E_y e^{i\delta_y} + \underline{e}_z E_z e^{i\delta_z}\right]\right\} = \mathrm{Re}\left\{\underline{E}_o e^{i\psi}\right\}$$
$$\underline{B}(\underline{r},t) = \mathrm{Re}\left\{e^{i\psi}\left[\underline{e}_x B_x e^{i\bar{\delta}_x} + \underline{e}_y B_y e^{i\bar{\delta}_y} + \underline{e}_z B_z e^{i\bar{\delta}_z}\right]\right\} = \mathrm{Re}\left\{\underline{B}_o e^{i\psi}\right\} \qquad (5.2)$$

where $\underline{E}_o$ and $\underline{B}_o$ are complex functions defined respectively by the expressions inside the square brackets in 5.2; δ_x, $\bar{\delta}_x$, etc. are, respectively, equal to $\phi_x - \psi$, $\bar{\phi}_x - \psi$, etc. In the case of a plane wave, $\underline{E}_o$ and $\underline{B}_o$ become complex constants, and $\psi = \underline{k}\cdot\underline{r} - \omega t$, $\underline{k}$ and ω being constant. In the case of a gradually inhomogeneous medium we may say that $\underline{E}_o$ and $\underline{B}_o$ are slowly varying functions of position and time in comparison with the rapidly oscillating term $e^{i\psi}$.

To see the physical implication of the function $\psi(\underline{r},t)$, let us transform the representation 5.2 to the $\underline{k} - \omega$ space by the usual Fourier integral technique:

$$\underline{E}(\underline{k},\omega) = \int e^{i(\omega t - \underline{k}\cdot\underline{r})} \operatorname{Re}\left\{\underline{E}_o(\underline{r},t)e^{i\psi(\underline{r},t)}\right\} d^3x\, dt \qquad (5.3)$$

and a similar expression for $\underline{B}(\underline{k},\omega)$. Since $\underline{E}_o(\underline{r},t)$ is almost constant, the major contribution to the value of the integral 5.3 arises from the vicinity of those points where $\omega t - \underline{k}\cdot\underline{r} + \psi(\underline{r},t)$ is stationary, i.e.

$$\underline{k} = \nabla\psi(\underline{r},t) \qquad \text{and} \qquad \omega = -\frac{\partial\psi(\underline{r},t)}{\partial t} \qquad (5.4)$$

from which the stationary points $(\underline{r},t)$ can be determined in terms of $\underline{k}$ and ω . It is easily seen that in the $\underline{k} - \omega$ space $\underline{E}(\underline{k},\omega)$ and $\underline{B}(\underline{k},\omega)$ have their maximum values at $\underline{k} = \nabla\psi$ and $\omega = -\partial\psi/\partial t$. Within each small region in space it is therefore permissible to treat the wave as a harmonic one moving in the direction $\underline{k} = \nabla\psi$ with frequency $\omega = -\partial\psi/\partial t$. In the following we shall call this wave the main spectral component and restrict ourselves only to the consideration of this component.

Before proceeding to derive the Doppler equation, we should bear in mind the fact that in the frame S' where the scatterer is stationary, the electric and the magnetic fields must satisfy certain boundary conditions on the surface of the scatterer for all time t' . It thus follows that the frequency of the scattered wave must equal that of the incident wave at each point on the surface, i.e., $\omega_s' = \omega_i'$. We shall assume that the length scale of the inhomogeneity of the medium is larger than the maximum linear dimension of the scatterer so that the medium in the immediate neighborhood of the scatterer is essentially homogeneous. Then one may say that

$$\omega_s'(\underline{r}',t') = \omega_i'(\underline{r}',t') \tag{5.5}$$

where $\underline{r}'$ is the position of the scatterer in S'. It should be emphasized that eq. 5.5 holds only at the position of the scatterer, i.e., at the place where the scattering occurs. Moreover, since the moving inhomogeneous medium changes its properties with time at any fixed spatial point in S', ω_i' and ω_s' should in general depend on t' explicitly.

Eq. 5.5 suggests that one should look at the scattering process in the frame S'. In the laboratory frame S, we take the fields to be of the forms 5.2, and the transformation of the incident fields from S to S' is given by eq. B-5 for $v \ll c$, viz.,

$$\underline{E}_i'(\underline{r}',t') = \mathrm{Re}\left\{ e^{i\psi_i(\underline{r},t)} \left[\underline{E}_{oi}(\underline{r},t) + \underline{v} \times \underline{B}_{oi}(\underline{r},t) \right] \right\} , \tag{5.6}$$

and a similar expression for the scattered $\underline{E}_s'(\underline{r}',t')$. We now take the Fourier transform of eq. 5.6 and obtain the spectrum $\underline{E}_i'(\underline{r}',\omega_i')$:

$$\underline{E}_i'(\underline{r}',\omega_i') = \int_{-\infty}^{\infty} dt' \, e^{i\omega_i' t'} \, \mathrm{Re}\left\{ e^{i\psi_i(\underline{r},t)} \left[\underline{E}_{oi}(\underline{r},t) + \underline{v} \times \underline{B}_{oi}(\underline{r},t) \right] \right\} \tag{5.7}$$

and a similar expression for the spectrum $\underline{E}_s'(\underline{r}',\omega_s')$. We now apply the method of stationary phase to the integral 5.7, and find out where the main spectral component of $\underline{E}_i'(\underline{r}',\omega_i')$ is located on the ω_i' - axis. It is easily seen that it is situated at

$$\omega_i' + \frac{\partial\psi_i(\underline{r},t)}{\partial t'} = 0 \quad , \tag{5.8}$$

and $\omega_i' - \frac{\partial\psi_i}{\partial t'} = 0$ is extraneous. Since $\underline{r}$ and t are functions of t' and they are given by the Lorentz transformations B-10 for $v \ll c$ we then have

$$\frac{\partial\psi_i}{\partial t'} = \frac{\partial t}{\partial t'}\frac{\partial\psi_i}{\partial t} + \sum_{j=1}^{3}\frac{\partial x_j}{\partial t'}\frac{\partial\psi_i}{\partial x_j} = \frac{\partial\psi_i}{\partial t} + \underline{v}\cdot\nabla\psi_i \quad . \tag{5.9}$$

However, we are only interested in the portion of the wave which is incident on the scatterer. We therefore replace ∇ by ∇_i in eq. 5.9, where ∇_i is the gradient operator taken along the direction of incidence. The frequency of the main spectral component of the incident wave is then given by

$$\omega_i' = -\frac{\partial\psi_i}{\partial t'} = -\frac{\partial\psi_i}{\partial t} - \underline{v}\cdot\nabla_i\,\psi_i \quad . \tag{5.10}$$

Similarly, the frequency of the main spectral component of the scattered wave is found to be

$$\omega_s' = -\frac{\partial\psi_s}{\partial t} - \underline{v}\cdot\nabla_s\,\psi_s \tag{5.11}$$

where ∇_s is the gradient operator along the direction of scattering.

By virtue of eq. 5.5 we then obtain the required Doppler equation in the laboratory frame S :

$$\frac{\partial \psi_s}{\partial t} + \underline{v} \cdot \nabla_s \psi_s = \frac{\partial \psi_i}{\partial t} + \underline{v} \cdot \nabla_i \psi_i \tag{5.12}$$

which, of course, holds only at the position $\underline{r}$ of the scatterer. In the case of a homogeneous medium, eq. 5.12 reduces rigorously to eq. 2.7.

5.3 The Doppler Equation in the ω-Domain

The Doppler equation given in eq. 5.12 is described in the time domain. It is desirable for later calculations to transform the equation to the frequency domain. To do this we recall from Maxwell's equations that the electric vector $\underline{E}$ satisfies the following equation:

$$\nabla \times \nabla \times \underline{E}(\underline{r},t) = -\mu_o \frac{\partial^2}{\partial t^2} \underline{\underline{\hat{\epsilon}}}(\underline{r}) \cdot \underline{E}(\underline{r},t) \quad , \tag{5.13}$$

where

$$\underline{\underline{\hat{\epsilon}}}(\underline{r}) \cdot \underline{E}(\underline{r},t) = \int_{-\infty}^{t} \underline{\underline{\epsilon}}(\underline{r},t-\tau) \cdot \underline{E}(\underline{r},\tau) d\tau = \underline{D}(\underline{r},t) \tag{5.14}$$

which is the constitutive relation between $\underline{D}$ and $\underline{E}$ in an anisotropic, dispersive, and inhomogeneous medium. If one writes

$$\underline{E}(\underline{r},t) = \int_{-\infty}^{\infty} \underline{E}(\underline{r},\omega)\, e^{i\phi(\underline{r},\omega) - i\omega t} d\omega \, , \tag{5.15}$$

substitution of eq. 5.15 into eq. 5.13 shows that each spectral component $\underline{E}(\underline{r},\omega)e^{i\phi - i\omega t}$ satisfies the following equation:

$$\nabla \times \nabla \times \underline{E}(\underline{r},\omega)e^{i\phi(\underline{r},\omega)} = k^2 \underline{\underline{\epsilon}}(\underline{r},\omega) \cdot \underline{E}(\underline{r},\omega)e^{i\phi(\underline{r},\omega)} \tag{5.16}$$

where $k^2 = \omega^2 \mu_o \epsilon_o$, and

$$\underline{\underline{\epsilon}}(\underline{r},\omega) = \int_0^\infty \underline{\underline{\epsilon}}(\underline{r},t)\ e^{+i\omega t}\ dt \quad .$$

In the case where the incident wave is monochromatic, there is only one spectral component. However, as has been shown above, the scattered wave contains a spectrum and the function $\psi_s(\underline{r},t)$ of the main spectral components satisfy the Doppler equation 5.12. Hence, substituting $\psi_s = \phi_s(\underline{r},\omega_s) - \omega_s t$ and $\psi_i = \phi_i(\underline{r},\omega_i) - \omega_i t$ into the Doppler equation one immediately obtains the Doppler equation in the ω-domain:

$$\omega_s - \underline{v} \cdot \nabla_s\ \phi_s(\underline{r},\omega_s) = \omega_i - \underline{v} \cdot \nabla_i\ \phi_i(\underline{r},\omega_i) \quad . \tag{5.17}$$

The number of the spectral components contained in the scattered wave depends therefore on the number of roots of ω_s in eq. 5.17.

5.4 Calculations of ϕ_s and ϕ_i

It is necessary to know $\nabla_s\phi_s$ and $\nabla_i\phi_i$ as functions of frequency before one is able to solve eq. 5.17 for ω_s. Since the scattered and the incident waves satisfy the same differential equation, namely eq. 5.16, we shall focus our attention on this equation for the solution of $\phi(\underline{r},\omega)$. For convenience, we write, in rectangular coordinates

$$\underline{E}(\underline{r},\omega)e^{i\phi(\underline{r},\omega)} = \underline{e}_x e^{i\eta_x} + \underline{e}_y e^{i\eta_y} + \underline{e}_z e^{i\eta_z} \quad , \tag{5.18}$$

where

$$\begin{aligned} \eta_x &= \phi + \delta_x - i\,\ell n\, E_x \\ \eta_y &= \phi + \delta_y - i\,\ell n\, E_y \\ \eta_z &= \phi + \delta_y - i\,\ell n\, E_z \quad . \end{aligned} \tag{5.19}$$

Here E_x, E_y, E_z are real quantities, and δ_x, δ_y, δ_z account for the phase differences among the components. Substitution of eq. 5.18 into eq. 5.16 yields the following three partial differential equations

$$\left(\frac{\partial\eta_x}{\partial y}\right)^2 + \left(\frac{\partial\eta_x}{\partial z}\right)^2 - i\left(\frac{\partial^2\eta_x}{\partial y^2} + \frac{\partial^2\eta_x}{\partial z^2}\right)$$
$$+ \left(i\,\frac{\partial^2\eta_y}{\partial x\partial y} - \frac{\partial\eta_y}{\partial x}\,\frac{\partial\eta_y}{\partial y}\right) e^{i(\eta_y - \eta_x)}$$
$$+ \left(i\,\frac{\partial^2\eta_z}{\partial x\partial z} - \frac{\partial\eta_z}{\partial x}\,\frac{\partial\eta_z}{\partial z}\right) e^{i(\eta_z - \eta_x)} = k^2\epsilon_1 + ik^2 g\cos\theta\, e^{i(\eta_y - \eta_x)}$$
$$- ik^2 g\sin\theta\, e^{i(\eta_z - \eta_x)} \tag{5.20}$$

$$\left(\frac{\partial\eta_y}{\partial z}\right)^2 + \left(\frac{\partial\eta_y}{\partial x}\right)^2 - i\left(\frac{\partial^2\eta_y}{\partial z^2} + \frac{\partial^2\eta_y}{\partial x^2}\right)$$
$$+ \left(i\,\frac{\partial^2\eta_z}{\partial y\partial z} - \frac{\partial\eta_z}{\partial y}\,\frac{\partial\eta_z}{\partial z}\right) e^{i(\eta_z - \eta_y)}$$
$$+ \left(i\,\frac{\partial^2\eta_x}{\partial y\partial x} - \frac{\partial\eta_x}{\partial y}\,\frac{\partial\eta_x}{\partial x}\right) e^{i(\eta_x - \eta_y)} = k^2\epsilon_1\cos^2\theta + k^2\epsilon_2\sin^2\theta$$
$$- ik^2 g\cos\theta\, e^{i(\eta_x - \eta_y)}$$
$$+ \frac{k^2}{2}(\epsilon_2 - \epsilon_1)\sin 2\theta\, e^{i(\eta_z - \eta_y)} \tag{5.21}$$

$$\left(\frac{\partial \eta_z}{\partial x}\right)^2 + \left(\frac{\partial \eta_z}{\partial z}\right)^2 - i\left(\frac{\partial^2 \eta_z}{\partial x^2} + \frac{\partial^2 \eta_z}{\partial y^2}\right)$$

$$+ \left(i \frac{\partial^2 \eta_x}{\partial y \partial x} - \frac{\partial \eta_x}{\partial z}\frac{\partial \eta_x}{\partial x}\right) e^{i(\eta_x - \eta_z)}$$

$$+ \left(i \frac{\partial^2 \eta_y}{\partial z \partial y} - \frac{\partial \eta_y}{\partial z}\frac{\partial \eta_y}{\partial y}\right) e^{i(\eta_y - \eta_z)} = k^2(\epsilon_1 \sin^2\theta + \epsilon_2 \cos^2\theta)$$

$$+ ik^2 g \sin\theta \, e^{i(\eta_x - \eta_z)}$$

$$+ \frac{k^2}{2}(\epsilon_2 - \epsilon_1)\sin 2\theta \, e^{i(\eta_y - \eta_z)} \tag{5.22}$$

where $\underline{\underline{\epsilon}}(\underline{r},\omega)$ has been assumed to take the form 2.18 with ϵ_1, ϵ_2 and g being functions of position through the plasma frequency ω_p. In general, these equations cannot be easily handled. To make them mathematically tractable we shall assume that the medium is plane stratified i.e., the stratification is perpendicular to one of the coordinate axes. The following chapters will be devoted to solving these equations in isotropic and anisotropic plane stratified media.

VI. SOLUTIONS OF THE DOPPLER EQUATION IN ISOTROPIC, PLANE-STRATIFIED MEDIA

In an isotropic, plane-stratified medium the tensor $\underline{\underline{\epsilon}}$ becomes $\epsilon(z)I$, where I is the unit dyad and the stratification is assumed to be perpendicular to the z-axis. A linearly polarized wave propagated in such a medium must be one of two kinds. In one kind the electric vector is perpendicular to the plane of incidence, while in the other kind the electric vector is parallel to that plane. These two cases of polarization are independent and must be treated separately. The normal incidence is just the special case of these two and, therefore, needs no additional consideration. In the following sections we shall first find the phase function ϕ for these two separate cases and then solve the Doppler equation for the frequency ω_s of the scattered wave.

6.1 The Electric Vector Perpendicular to the Plane of Incidence

Consider a monochromatic wave whose electric vector is linearly polarized parallel to the x-axis propagating in the positive yz-direction (Fig. 10). Since the medium is homogeneous in x and y , η_x will then be a function of z plus a term linear in y . To see this, we start from the equation

$$\nabla \times \nabla \times \underline{E} - k^2\epsilon(z)\underline{E} = 0 . \tag{6.1}$$

In the case considered here $\underline{E} = \underline{e}_x E_x$. Hence, from the divergence condition $\nabla \cdot \left[\epsilon(z)\, \underline{e}_x E_x \right] = 0$, we have $\nabla \cdot \underline{e}_x E_x = 0$, and eq. 6.1 becomes

$$\frac{\partial^2 E_x}{\partial y^2} + \frac{\partial^2 E_x}{\partial z^2} + k^2\epsilon(z)E_x = 0 . \tag{6.2}$$

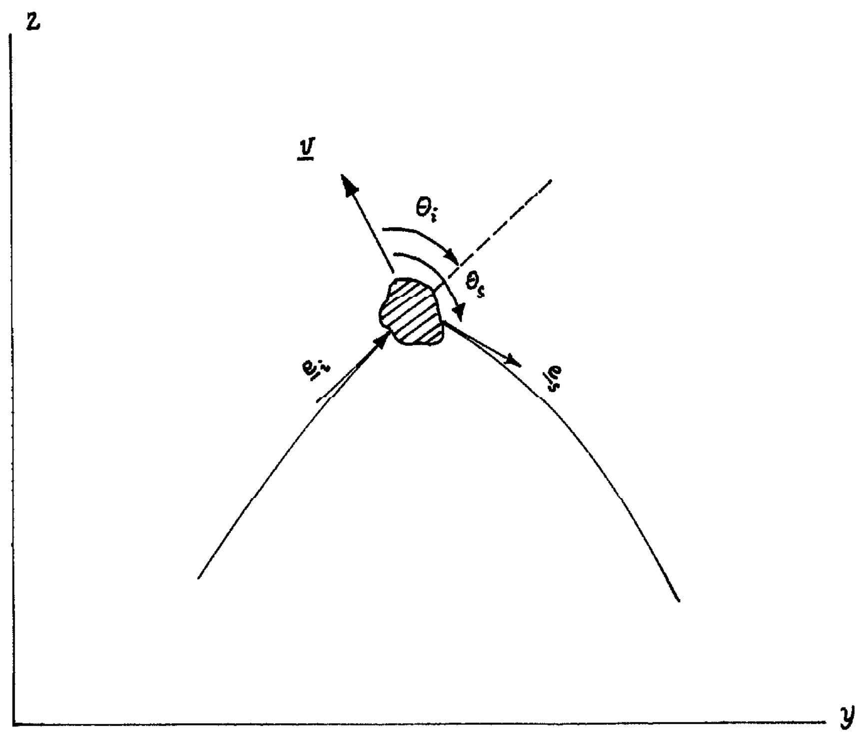

Fig. 10. The scattering diagram

$\underline{e}_s$ and $\underline{e}_i$ are unit vectors in the directions of scattering and incidence. $\underline{v}$ does not necessarily lie in the yz-plane.

Assuming $E_x(y,z) = Y(y)\ Z(z)$, we see that Y is of the form e^{ipy} , where p is a constant which describes the inclination of the direction of propagation with respect to the z-axis, and that $Z(z)$ satisfies the following equation

$$\frac{d^2Z}{dz^2} + \left[k^2\epsilon(z) - p^2 \right] Z = 0 \quad . \tag{6.3}$$

Thus one can write $E_x = e^{ipy + \ell n\, Z} = e^{i\eta_x}$, and this proves our assertion.

From the above consideration we see that eq. 5.20 reduces to

$$\left(\frac{\partial\eta_x}{\partial z}\right)^2 = k^2\epsilon(z) - p^2 + i\,\frac{\partial^2\eta_x}{\partial z^2} \tag{6.4}$$

Since the medium has been assumed to be slowly varying, the term $\partial^2\eta_x/\partial z^2$ is small compared to the others. In the zeroth approximation we neglect $\partial^2\eta_x/\partial z^2$ and obtain

$$\frac{\partial\eta_x^{(0)}}{\partial z} = \sqrt{k^2\epsilon(z) - p^2} \quad . \tag{6.5}$$

In the next approximation we use eq. 6.5 to calculate $\partial^2\eta_x/\partial z^2$ in eq. 6.4 and get

$$\frac{\partial\eta_x^{(1)}}{\partial z} = \kappa(z) + i\,\frac{1}{2\kappa}\,\frac{d\kappa}{dz} \quad , \tag{6.6}$$

where $\kappa(z) = \sqrt{k^2\epsilon(z) - p^2}$. The recursion formula is easily seen to be

$$\frac{\partial\eta_x^{(n)}}{\partial z} = \kappa(z) + i\,\frac{1}{2\kappa}\,\frac{\partial\eta_x^{(n-1)}}{\partial z} \quad , \tag{6.7}$$

from which we obtain

$$\frac{\partial\eta_x}{\partial x} = \sum_{n=0}^{\infty} \left(\frac{i}{2\kappa}\frac{d}{dz}\right)^n \frac{\partial\eta_x^{(0)}}{\partial z} = \sum_{n=0}^{\infty} \left(\frac{i}{2\kappa}\frac{d}{dz}\right)^n \kappa(z) \quad . \tag{6.8}$$

In order that expression 6.8 be valid, the derivatives of $\kappa(z)$ of all orders must exist, and the medium is assumed to be a smooth one such that this is the case. It should also be noted that the infinite series 6.8 is convergent because $\left\{\left(\frac{i}{2\kappa}\frac{d}{dz}\right)^n \kappa\right\}$ is an alternating decreasing sequence and converges to zero as $n \to \infty$.

We shall now sum the series 6.8 in a closed form. To do this, let

$$D^n = \left(\frac{i}{2\kappa}\frac{d}{dz}\right)^n \quad . \tag{6.9}$$

Thus, by the theory of operators we have

$$\sum_{n=0}^{\infty} \left(\frac{i}{2\kappa}\frac{d}{dz}\right)^n \kappa(z) = \left(\sum_{n=0}^{\infty} D^n\right) \kappa(z) = \frac{1}{1-D}\kappa(z) \quad . \tag{6.10}$$

Let us denote

$$F(z) = \frac{1}{1-D}\kappa(z) \quad .$$

We thus have a differential equation for $F(z)$:

$$DF - F = -\kappa(z) \quad ,$$

or

$$\frac{dF}{dz} + 2i\kappa F = 2i\kappa^2 \quad . \tag{6.11}$$

By the theory of elementary differential equations we find $F(z)$ to be

$$F(z) = 2i \exp\left[-2i\int^{z} \kappa(z')dz'\right] x \int^{z} \kappa^{2}(z') \exp\left[2i\int^{z'} \kappa(z'')dz''\right] dz' . \tag{6.12}$$

Hence we obtain

$$\frac{\partial\eta_x}{\partial z} = 2i \exp\left[-2i\int^{z} \kappa(z')dz'\right] x \int^{z} \kappa^{2}(z') \exp\left[2i\int^{z'} \kappa(z'')dz''\right] dz' . \tag{6.13}$$

If the medium were homogeneous, κ would reduce to a constant and the integral in 6.13 could be evaluated in a straightforward manner. Then we would have

$$\frac{\partial\eta_x}{\partial z} = \kappa$$

as one could have expected.

Separating expression 6.13 into its real and imaginary parts we get

$$\mathrm{Re}\,\frac{\partial\eta_x}{\partial z} = 2 \sin 2\bar{\kappa}\int^{z} \kappa^{2}(\lambda) \cos 2\bar{\kappa}(\lambda)\, d\lambda - 2 \cos 2\bar{\kappa} \int^{z} \kappa^{2}(\lambda) \sin 2\bar{\kappa}(\lambda)\, d\lambda \tag{6.14}$$

$$\mathrm{Im}\,\frac{\partial\eta_x}{\partial z} = 2 \sin 2\bar{\kappa}\int^{z} \kappa^{2}(\lambda) \sin 2\bar{\kappa}(\lambda)\, d\lambda + 2 \cos 2\bar{\kappa} \int^{z} \kappa^{2}(\lambda) \cos 2\bar{\kappa}(\lambda)\, d\lambda \tag{6.15}$$

where $\bar{\kappa}(z) = \int^{z} \kappa(\lambda)\, d\lambda$.

Integrating 6.14 and 6.15 by parts and noting that $d\bar{\kappa} = \kappa\, dz$, one can easily obtain

$$\mathrm{Re}\,\frac{\partial\eta_x}{\partial z} = \kappa - \sin 2\bar{\kappa}\int^{z} \kappa' \sin 2\bar{\kappa}\, d\lambda - \cos 2\bar{\kappa}\int^{z} \kappa' \cos 2\bar{\kappa}\, d\lambda \tag{6.16}$$

$$\mathrm{Im}\,\frac{\partial\eta_x}{\partial z} = \frac{\kappa'}{2\kappa} - \frac{1}{2}\sin 2\bar{\kappa}\int^{z}\left(\frac{\kappa'}{\kappa}\right)'\sin 2\bar{\kappa}\,d\lambda - \frac{1}{2}\cos 2\bar{\kappa}\int^{z}\left(\frac{\kappa'}{\kappa}\right)'\cos 2\bar{\kappa}\,d\lambda \quad , \tag{6.17}$$

where the primes denote differentiations.

Since the medium has been assumed to be slowly varying, we then have $\kappa > \kappa' > \kappa'' \cdots$ and consequently $\mathrm{Re}\,\frac{\partial\eta_x}{\partial z} \gg \mathrm{Im}\,\frac{\partial\eta_x}{\partial z}$ from 6.16 and 6.17.

From the definition of η_x given by 5.19 with $\delta_x = 0$, we immediately obtain

$$\phi_\perp = py + \int^{z}\left(\mathrm{Re}\,\frac{\partial\eta_x}{\partial z}\right)dz \tag{6.18}$$

$$E_x = \exp\left[-\int^{z}\left(\mathrm{Im}\,\frac{\partial\eta_x}{\partial z}\right)dz\right] \tag{6.19}$$

where $\mathrm{Re}\,\frac{\partial\eta_x}{\partial z}$ and $\mathrm{Im}\,\frac{\partial\eta_x}{\partial z}$ are given by 6.16 and 6.17, respectively, and $\phi_\perp$ denotes the phase function of the wave whose electric vector is perpendicular to the plane of incidence. We see from 6.17 and 6.19 that E_x is indeed a slowly varying function of position.

One can also obtain the magnetic intensity vector $\underline{H}$ from the equation $i\omega\,\mu_o\underline{H} = \nabla \times \underline{E}$. Its components are simply given by

$$H_y = \frac{1}{\omega\mu_o}\,e^{i\eta_x}\,\frac{\partial\eta_x}{\partial z}$$

$$H_z = -\frac{1}{\omega\mu_o}\,p\,e^{i\eta_x}$$

$$H_x = 0 \quad .$$

We see that H_z has a phase function equal to $\mathrm{Re}\,\eta_x$, i.e., $\phi_\perp$. But H_y has a different phase function which is given by $\phi_\perp + \Delta\phi$, where

$$\Delta\phi = \tan^{-1} \left(\mathrm{Im}\,\frac{\partial\eta_x}{\partial z}\right) \Big/ \left(\mathrm{Re}\,\frac{\partial\eta_x}{\partial z}\right) \quad .$$

However, as shown in eqs. 6.16 and 6.17, $\Delta\phi$ is very small in comparison with $\phi_\perp$ and may be neglected. In fact $\Delta\phi$ reduces to zero in a homogeneous medium. Thus one can say that all the field components have approximately the same phase function, namely $\phi_\perp$. This agrees with the assumption we have made in deriving the Doppler equation for a slightly non-uniform medium.

6.2 The Electric Vector Parallel to the Plane of Incidence

In this case it is easier to deal with the magnetic intensity vector $\underline{H}$ than with the electric field vector $\underline{E}$. From Maxwell's equations we have

$$\nabla \times \left(\frac{1}{\epsilon}\nabla \times \underline{H}\right) = k^2 \underline{H} \quad . \tag{6.20}$$

Consider now a monochromatic wave with $\underline{H}$ perpendicular to the plane of incidence, i.e., the yz-plane, traveling in an isotropic medium stratified in the z-direction. For convenience, let us write

$$\underline{H} = \underline{e}_x \, e^{ih_x} \tag{6.21}$$

where $h_x = \phi_{||} - i \ln H_x$, $\phi_{||}$ being the phase function of the wave whose electric vector is parallel to the plane of incidence.

Substitution of 6.21 into 6.20 yields

$$\left(\frac{\partial h_x}{\partial y}\right)^2 + \left(\frac{\partial h_x}{\partial z}\right)^2 = k^2\epsilon(z) + i\left(\frac{\partial^2 h_x}{\partial y^2} + \frac{\partial^2 h_x}{\partial z^2} - \frac{\epsilon'}{\epsilon}\frac{\partial h_x}{\partial z}\right) , \tag{6.22}$$

where $\epsilon' = d\epsilon/dz$. Since the medium is homogeneous in the y-direction, h_x then depends linearly on y , and eq. 6.22 reduces to

$$\left(\frac{\partial h_x}{\partial z}\right)^2 = \kappa^2 + i\left(\frac{\partial^2 h_x}{\partial z^2} - \frac{\epsilon'}{\epsilon}\frac{\partial h_x}{\partial z}\right) . \tag{6.23}$$

Here, as before, $\kappa^2 = k^2\epsilon(z) - p^2$. Eq. 6.23 can be solved by iterations as in the previous case, and the solution is simply given by

$$\frac{\partial h_x}{\partial z} = \sum_{n=0}^{\infty}\left[\frac{i}{2\kappa}\left(\frac{d}{dz} - \frac{\epsilon'}{\epsilon}\right)\right]^n \kappa$$

$$= 2i \exp\left[-\int^z \left(2i\kappa - \frac{\epsilon'}{\epsilon}\right)dz'\right] \int^z \kappa^2(z') \exp\left[\int^z \left(2i\kappa - \frac{\epsilon'}{\epsilon}\right)dz''\right]dz' . \tag{6.24}$$

Separating 6.24 into real and imaginary parts and then integrating by parts, one can easily obtain the following:

$$\text{Re}\,\frac{\partial h_x}{\partial z} = \kappa - \epsilon \sin 2\bar{\kappa} \int^z \left(\frac{\kappa}{\epsilon}\right)' \sin 2\bar{\kappa}(\lambda)\, d\lambda$$

$$- \epsilon \cos 2\bar{\kappa} \int^z \left(\frac{\kappa}{\epsilon}\right)' \cos 2\bar{\kappa}(\lambda)\, d\lambda \tag{6.25}$$

$$\text{Im}\,\frac{\partial h_x}{\partial z} = \frac{1}{2\kappa}\left(\frac{\kappa}{\epsilon}\right)' - \frac{1}{2}\epsilon \sin 2\bar{\kappa}\int^z \left(\frac{1}{\kappa}\left(\frac{\kappa}{\epsilon}\right)'\right)' \sin 2\bar{\kappa}(\lambda)\, d\lambda$$

$$- \frac{1}{2}\epsilon \cos 2\bar{\kappa}\int^z \left(\frac{1}{\kappa}\left(\frac{\kappa}{\epsilon}\right)'\right)' \cos 2\bar{\kappa}(\lambda)\, d\lambda \tag{6.26}$$

where $\bar{\kappa}(\lambda) = \int^\lambda \kappa(z)dz$ and the primes denote differentiations. From the definition of h_x we get

$$\phi_{||} = py + \int^{z} (\mathrm{Re}\, \frac{\partial h_x}{\partial z}) dz \tag{6.27}$$

$$H_x = \exp\left[- \int^{z} (\mathrm{Im}\, \frac{\partial h_x}{\partial z}) dz \right] \tag{6.28}$$

where $\mathrm{Re}\, \frac{\partial h_x}{\partial z}$ and $\mathrm{Im}\, \frac{\partial h_x}{\partial z}$ are given respectively by 6.25 and 6.26.

The electric vector $\underline{E}$ is related to $\underline{H}$ by the equation $-i\omega\epsilon \underline{E} =$ [redacted] $\nabla \times \underline{H}$, and its components are simply

$$E_y = -\frac{1}{\omega\epsilon} e^{ih_x} \frac{\partial h_x}{\partial z}$$

$$E_z = \frac{1}{\omega\epsilon} p\, e^{ih_x}$$

$$E_x = 0 \quad .$$

Again we can see that $\underline{E}$ and $\underline{H}$ have approximately the same phase function $\phi_{||}$ in a slightly non-uniform medium.

6.3 Solutions of the Doppler Equation

In the previous two sections we have obtained the phase functions $\phi_{\perp}$ and $\phi_{||}$ for the two cases of polarization. These phase functions are functions of ω, $\underline{r}$ and p , and become equal when p is equal to zero, i.e., when the wave is propagated perpendicular to the plane of stratification. In this section we shall obtain ω_s from the Doppler equation

$$\omega_s - \underline{v} \cdot \nabla_s \phi_s(\omega_s) = \omega_i - \underline{v} \cdot \nabla_i \phi_i(\omega_i) \quad . \tag{5.17}$$

Introducing θ_s and θ_i as angles of scattering and incidence with respect to the velocity of the scatterer and noting that ϕ_s and ϕ_i can be either $\phi_\perp$ or $\phi_{||}$, we write eq. 5.26 as

$$\omega_s - v \cos \theta_s \left|\nabla \phi_\mu(p_s,\omega_s)\right| = \omega_i - v \cos \theta_i \left|\nabla \phi_\nu(p_i,\omega_i)\right| \quad ,$$

$$\text{where} \quad \mu,\nu = \perp, || \quad . \tag{6.29}$$

Here p_s and p_i are constants which describe, respectively, the directions of scattering and incidence with respect to the z-axis.

Eqs. 6.29 can be solved by iteration since the second term on the left side is of the order of β . The solutions are given by

$$\omega_s = \omega_i - v \cos \theta_i \left|\nabla \phi_\nu(p_i,\omega_i)\right| + v \cos \theta_s \left|\nabla\phi_\mu(p_s,\omega_i)\right| + O(\beta^2) \quad , \tag{6.30}$$

where

$$\left|\nabla\phi_{\mu,\nu}\right| = \sqrt{\left(\frac{\partial\phi_{\mu,\nu}}{\partial y}\right)^2 + \left(\frac{\partial\phi_{\mu,\nu}}{\partial z}\right)^2} \quad ,$$

and $\phi_{\mu,\nu}$ are given either by expression 6.18 or by expression 6.27.

It should be noted that the right hand side of 6.30 is evaluated at the position $\underline{r}$ of the scatterer, and the value of ω_s thus obtained remains constant when the scattered wave is propagated away from the scatterer. This is due to the fact that the medium considered here is a stationary one, i.e., its properties at each point in space do not change with time. To show this we see from eq. 5.4 that $\underline{k}$ and ω satisfy the following equation:

$$\frac{\partial k_i}{\partial t} + \frac{\partial \omega}{\partial x_i} = 0 \quad , \qquad i = 1,2,3 \quad . \tag{6.31}$$

Since $\omega = \omega(\underline{k}(\underline{r}))$, i.e., ω is treated to depend on $\underline{r}$ implicitly, we get $\frac{\partial \omega}{\partial x_i} = \frac{\partial \omega}{\partial k_j} \frac{\partial k_j}{\partial x_i}$, where the summation convention is used. Moreover, because of the irrotational property of $\underline{k}$, i.e., $\nabla \times \underline{k} = 0$, we have

$$\frac{\partial k_j}{\partial x_i} = \frac{\partial k_i}{\partial x_j}$$

and

$$\frac{\partial \omega}{\partial x_i} = \frac{\partial \omega}{\partial k_j} \frac{\partial k_i}{\partial x_j} \quad .$$

On substituting into eq. 6.31 we get

$$\frac{\partial k_i}{\partial t} + \frac{\partial \omega}{\partial k_j} \frac{\partial k_i}{\partial x_j} = 0 \quad , \tag{6.32}$$

from which we obtain

$$\frac{dx_j}{dt} = \frac{\partial \omega}{\partial k_j} \quad . \tag{6.33}$$

The total time derivative of the frequency is given by

$$\frac{d\omega}{dt} = \frac{\partial \omega}{\partial t} + \frac{\partial \omega}{\partial x_i} \frac{dx_i}{dt} + \frac{\partial \omega}{\partial k_i} \frac{\partial k_i}{\partial t} \quad . \tag{6.34}$$

By virtue of 6.31 and 6.33 we see that the last two terms of 6.34 cancel. Since the medium is stationary, ω does not depend on time explicitly and hence $\partial\omega/\partial t = 0$. Thus we have $d\omega/dt = 0$.

There is another important point which should also be mentioned. The angles θ_s and θ_i in 6.30 are measured at $\underline{r}$. But, in practice, the scattering and the incidence angles are measured at other points which are usually far away from the scatterer. Since the medium considered here is inhomogeneous, the scattering or the incidence angle changes from point to point along a given ray. The connection between the measurements of these angles at two different points on the same ray can be easily obtained from the Snell law. However, we will not go into this consideration.

Finally, it should be remarked that we have given above only solutions of the first kind for the Doppler equation. Solutions of the second kind can be easily written down according to eq. 3.11 if the functions $\nabla \phi_{\nu,\mu}(\omega)$ exhibit poles on the ω-axis.

VII. SOLUTIONS OF THE DOPPLER EQUATION IN GYROELECTRIC PLANE-STRATIFIED MEDIA

In a gyroelectric inhomogeneous medium η_x, η_y and η_z are coupled in a very complicated fashion as shown in eqs. 5.20, 5.21 and 5.22. In general, these equations are not solvable. To make them mathematically tractable we shall assume that the direction of propagation is always perpendicular to the plane of stratification. Hence, only the case of backward scattering from a moving body will be considered. We shall treat in detail the following three separate cases: the direction of propagation is oriented (1) perpendicular, (2) parallel, and (3) arbitrarily with respect to the biasing magnetostatic field $\underline{B}_o$.

7.1 Perpendicular Case

In this case (Fig. 11) $\Theta = \pi/2$ and $\frac{\partial}{\partial x} = \frac{\partial}{\partial y} = 0$, since the medium is homogeneous in x and y . Eqs. 5.20, 5.21 and 5.22 then become, respectively,

$$\left(\frac{d\eta_x}{dz}\right)^2 - i\,\frac{d^2\eta_x}{\partial z^2} = k^2\epsilon_1(z) - ik^2 g(z)\,e^{i(\eta_z - \eta_x)} \tag{7.1}$$

$$\left(\frac{d\eta_y}{dz}\right)^2 - i\,\frac{d^2\eta_y}{dz^2} = k^2\epsilon_2(z) \tag{7.2}$$

$$\epsilon_1(z) + ig(z)\,e^{i(\eta_x - \eta_z)} = 0 \tag{7.3}$$

Eliminating η_z from eq. 7.1 by means of eq. 7.3 we obtain the following equation for η_x :

$$\left(\frac{d\eta_x}{dz}\right)^2 - i\,\frac{d^2\eta_x}{dz^2} = k^2\,\frac{\epsilon_1^2 - g^2}{\epsilon_1} \quad . \tag{7.4}$$

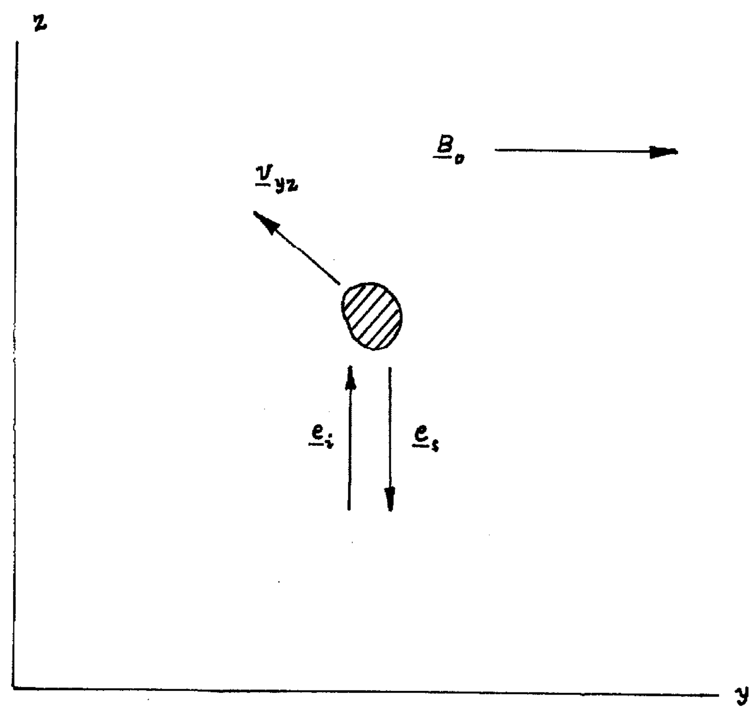

Fig. 11. Perpendicular case

$\underline{e}_i$ and $\underline{e}_s$ are the unit vectors in the directions of incidence and scattering. $\underline{v}_{yz}$ is the velocity of the scatterer projected on the yz-plane. The $\underline{\underline{\epsilon}}(z)$ tensor is a function of z only.

Thus we see that in the perpendicular case η_x, η_y and η_z are uncoupled.

Eqs. 7.2 and 7.4 are of the same form except that the right hand sides are given by two different known functions. Again we can solve them by the method of iteration. Following section 6.1, we obtain for eqs. 7.2 and 7.4

$$\frac{d\eta_y}{dz} = \sum_{n=0}^{\infty} \left(\frac{i}{2\kappa_1} \frac{d}{dz}\right)^n \kappa_1(z)$$

$$= 2i\; e^{-2i\bar{\kappa}_1(z)} \int^{z} \kappa_1^2(\lambda)\; e^{2i\bar{\kappa}_1(\lambda)}\, d\lambda \qquad (7.5)$$

$$\frac{d\eta_x}{dz} = \sum_{n=0}^{\infty} \left(\frac{i}{2\kappa_2} \frac{d}{dz}\right)^n \kappa_2(z)$$

$$= 2i\; e^{-2i\,\bar{\kappa}_2(z)} \int^{z} \kappa_2^2(\lambda)\; e^{2i\bar{\kappa}_2(\lambda)}\, d\lambda \qquad (7.6)$$

where

$$\bar{\kappa}_{1,2}(z) = \int^{z} \kappa_{1,2}(\lambda)d\lambda$$

$$\kappa_1(z) = k\sqrt{\epsilon_2(z)}$$

$$\kappa_2(z) = k\sqrt{\frac{\epsilon_1^2(z) - g^2(z)}{\epsilon_1(z)}} \quad .$$

From eq. 7.3 we can express η_z in terms of η_x as follows:

$$\eta_z = \eta_x - i\left(\ln \frac{g}{\epsilon_1} - \frac{\pi}{2}\right) \quad . \qquad (7.7)$$

Since g and ϵ_1 are real functions, we then see from 7.7 that

Re η_z = Re η_x , i.e., E_x and E_z have the same phase function. Thus, in the perpendicular case, there are two distinct phase functions, one associated with E_x and E_z and the other with E_y .

Having found $d\eta_x/dz$ and $d\eta_y/dz$, we now proceed to obtain the frequency of the scattered wave from the Doppler equation. Let θ denote the angle between the direction of incidence and the velocity of the scatterer. The Doppler equation 5.17 then becomes *

$$\omega_s + v \cos\theta \left|\nabla\phi\ (\omega_s)\right| = \omega_i - v \cos\theta \left|\nabla\phi\ (\omega_i)\right| \tag{7.8}$$

where ϕ has been used to replace ϕ_i and ϕ_s since they have the same functional dependence on frequency and position.

By the method of iteration we solve eq. 7.8 up to the order of β and obtain

$$\omega_s = \omega_i - 2v \cos\theta \left|\nabla\phi\ (\omega_i)\right| + O(\beta^2) \ . \tag{7.9}$$

Using the real parts of 7.5 and 7.6 for $|\nabla\phi|$ we finally have

$$\begin{aligned}\omega_s = \omega_i - 4v \cos\theta \Big(&\sin 2\overline{\kappa}_1 \int^z \kappa_1^2(\lambda) \cos 2\overline{\kappa}_1(\lambda)d\lambda \\ &- \cos 2\overline{\kappa}_1 \int^z \kappa_1^2(\lambda) \sin 2\overline{\kappa}_1(\lambda)d\lambda \Big) + O(\beta^2)\end{aligned} \tag{7.10}$$

for one wave, and

$$\begin{aligned}\omega_s = \omega_i - 4v \cos\theta \Big(&\sin 2\overline{\kappa}_2 \int^z \kappa_2^2(\lambda) \cos 2\overline{\kappa}_2(\lambda)d\lambda \\ &- \cos 2\overline{\kappa}_2 \int^z \kappa_2^2(\lambda) \sin 2\overline{\kappa}_2(\lambda)d\lambda \Big) + O(\beta^2)\end{aligned} \tag{7.11}$$

for the other wave. Here κ_1 and κ_2 are known functions of ω_i ;

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* See the footnote on p. 67.

z which occurs in the limits of the integrals depends on the position of the scatterer which is, of course, a linear function of time.

As in the case of homogeneous gyroelectric media, because of the presence of a pole in the phase function ϕ on the ω-axis*, there are in the Doppler equation other solutions of ω_s different from those given by expressions 7.10 and 7.11. However, as has been shown before, these solutions are very close to the resonant frequencies of the medium and hence they are insignificant in this sense. In the following, we shall therefore ignore completely this kind of solution and calculate only the meaningful ones such as those given by expressions 7.10 and 7.11.

7.2 Parallel Case

We now go on to a consideration of the case where the biasing field $\underline{B}_o$ and the direction of propagation are both perpendicular to the plane of stratification (see Fig. 12). In this case $\theta = 0$ and eqs. 5.20 and 5.21 become, respectively,

$$\left(\frac{d\eta_x}{dz}\right)^2 - i \frac{d^2\eta_x}{dz^2} = k^2\epsilon_1 + ik^2 g\, e^{i(\eta_y - \eta_x)} \qquad (7.12)$$

$$\left(\frac{d\eta_y}{dz}\right)^2 - i \frac{d^2\eta_y}{dz^2} = k^2\epsilon_1 - ik^2 g\, e^{i(\eta_x - \eta_y)} \quad . \qquad (7.13)$$

To make eq. 5.22 consistent, one has to set $\mathrm{Im}\ \eta_z = 0$, and this implies that $E_z = 0$. Thus the wave is completely transverse.

There is a symmetry between eq. 7.12 and eq. 7.13. If one replaces η_x by $\eta_y \pm \frac{\pi}{2}$, or vice versa, the two equations become identical. In

*Fig. 8 shows that the extraordinary wave possesses a finite non-zero pole on the ω-axis. The other pole at $\omega = 0$ is insignificant.

other words, the electric fields in x and y components are in phase quadrature. This consideration suggests the following change of the dependent variables:

$$e^{i\eta_{\pm}} = e^{i\eta_x} + e^{i(\eta_y \pm \frac{\pi}{2})} = e^{i\eta_x} \pm i\, e^{i\eta_y} \quad . \tag{7.14}$$

From eqs. 7.12 and 7.3 one can easily see that η_+ and η_- satisfy the following equations:

$$\left(\frac{d\eta_+}{dz}\right)^2 - i\,\frac{d^2\eta_+}{dz^2} = k^2(\epsilon_1 + g) \tag{7.15}$$

$$\left(\frac{d\eta_-}{dz}\right)^2 - i\,\frac{d^2\eta_-}{dz^2} = k^2(\epsilon_1 - g) \quad . \tag{7.16}$$

By the method of iteration the solutions of these equations are given by

$$\begin{aligned}\frac{d\eta_{\pm}}{dz} &= \sum_{n=0}^{\infty} \left(\frac{i}{2\kappa_{\pm}}\,\frac{d}{d\,z}\right)^n \kappa_{\pm}(z) \\ &= 2i\, e^{-2i\bar{\kappa}_{\pm}} \int^{z} \kappa_{\pm}^2(\lambda)\, e^{2i\bar{\kappa}_{\pm}(\lambda)}\, d\lambda\end{aligned} \tag{7.17}$$

where

$$\begin{aligned}\bar{\kappa}_{\pm}(z) &= \int^{z} \kappa_{\pm}(\lambda)\, d\lambda \\ \kappa_{\pm}(z) &= k\sqrt{\epsilon_1(z) \pm g(z)} \quad .\end{aligned} \tag{7.18}$$

We can now solve the Doppler equation for ω_s and obtain*

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* See the footnote on p. 67.

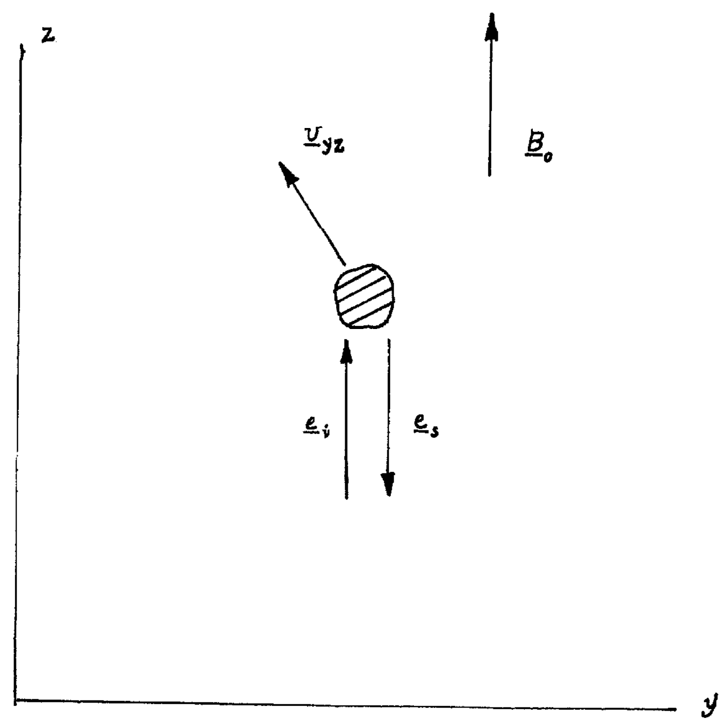

Fig. 12. Parallel case. The notations are the same as in Fig. 11.

$$\omega_s^{\pm} = \omega_i - 4v \cos\theta \left(\sin 2\bar{\kappa}_{\pm} \int^z \kappa_{\pm}^2(\lambda) \cos 2\bar{\kappa}_{\pm}(\lambda) d\lambda - \cos 2\bar{\kappa}_{\pm} \int^z \kappa_{\pm}^2(\lambda) \sin 2\bar{\kappa}_{\pm}(\lambda) d\lambda \right) + O(\beta^2) \quad . \qquad (7.19)$$

7.3 General Case

In the preceding two cases we were able to uncouple eqs. 5.20, 5.21 and 5.22 into equations of Riccati-type. This is due to the fact that in the perpendicular case the component of the electric field along the direction of $\underline{B}_o$ can be separated out from the start, and in the parallel case the polarization defined by $e^{i(\eta_y - \eta_x)}$ is constant and equal to $\pm i$. However, this is no longer the case when the direction of propagation is oriented arbitrarily with respect to $\underline{B}_o$ (see Fig. 13). Here eqs. 5.20, 5.21 and 5.22 become

$$\left(\frac{d\eta_x}{dz}\right)^2 - i\,\frac{d^2\eta_x}{dz^2} = k^2\epsilon_1 + ik^2 g \cos\theta\, e^{i(\eta_y - \eta_x)} - ik^2 g \sin\theta\, e^{i(\eta_z - \eta_x)} \qquad (7.20)$$

$$\left(\frac{d\eta_y}{dz}\right)^2 - i\,\frac{d^2\eta_y}{dz^2} = k^2\epsilon_1\cos^2\theta + k^2\epsilon_2\sin^2\theta - ik^2 g \cos\theta\, e^{i(\eta_x - \eta_y)} + \frac{k^2}{2}(\epsilon_2 - \epsilon_1)\sin 2\theta\, e^{i(\eta_z - \eta_y)} \qquad (7.21)$$

$$ig \sin\theta\, e^{i(\eta_x - \eta_z)} + \frac{\epsilon_2 - \epsilon_1}{2} \sin 2\theta\, e^{i(\eta_y - \eta_z)} + \epsilon_1\sin^2\theta + \epsilon_2\cos^2\theta = 0. \qquad (7.22)$$

Solving 7.22 for $e^{i\eta_z}$ we have

$$e^{i\eta_z} = -\frac{1}{\epsilon_1\sin^2\theta + \epsilon_2\cos^2\theta}\left(ig \sin\theta\, e^{i\eta_x} + \frac{\epsilon_2 - \epsilon_1}{2} \sin 2\theta\, e^{i\eta_y}\right) . \qquad (7.23)$$

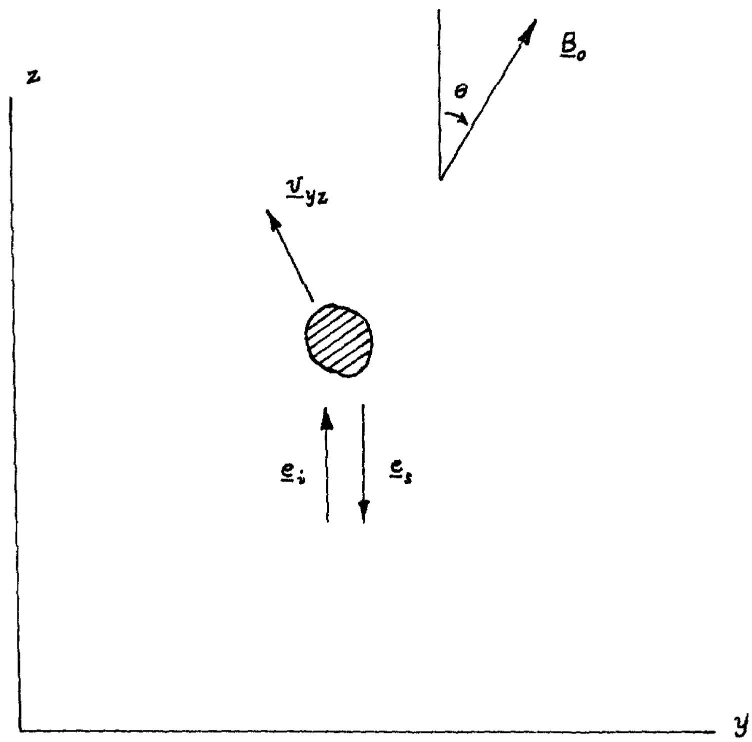

Fig. 13. General case. The notations are the same as in Fig. 11.

Eliminating $e^{i\eta_z}$ in 7.20 and 7.21 by means of 7.23, we obtain

$$\left(\frac{d\eta_x}{dz}\right)^2 - i\,\frac{d^2\eta_x}{dz^2} = \frac{k^2(\epsilon_1^2 - g^2)\sin^2\theta + k^2\epsilon_1\epsilon_2\cos^2\theta}{\epsilon_1\sin^2\theta + \epsilon_2\cos^2\theta} + i\,\frac{k^2 g\,\epsilon_2\cos\theta}{\epsilon_1\sin^2\theta + \epsilon_2\cos^2\theta}\,e^{i(\eta_y - \eta_x)} \tag{7.24}$$

$$\left(\frac{d\eta_y}{dz}\right)^2 - i\,\frac{d^2\eta_y}{dz^2} = \frac{k^2\epsilon_1\epsilon_2}{\epsilon_1\sin^2\theta + \epsilon_2\cos^2\theta} - i\,\frac{k^2 g\,\epsilon_2\cos\theta}{\epsilon_1\sin^2\theta + \epsilon_2\cos^2\theta}\,e^{i(\eta_x - \eta_y)} \tag{7.25}$$

These equations can be rewritten in the following forms:

$$-\frac{d^2}{dz^2}\,e^{i\eta_x} = A(z)\,e^{i\eta_x} + iB(z)\,e^{i\eta_y} \tag{7.26}$$

$$-\frac{d^2}{dz^2}\,e^{i\eta_y} = C(z)\,e^{i\eta_y} - iB(z)\,e^{i\eta_x} \tag{7.27}$$

where

$$A(z) = \frac{k^2(\epsilon_1^2 - g^2)\sin^2\theta + k^2\epsilon_1\epsilon_2\cos^2\theta}{\epsilon_1\sin^2\theta + \epsilon_2\cos^2\theta} \tag{7.28}$$

$$B(z) = \frac{k^2 g\,\epsilon_2\cos\theta}{\epsilon_1\sin^2\theta + \epsilon_2\cos^2\theta} \tag{7.29}$$

$$C(z) = \frac{k^2\epsilon_1\epsilon_2}{\epsilon_1\sin^2\theta + \epsilon_2\cos^2\theta}\ . \tag{7.30}$$

Unlike the case of parallel propagation, there is no symmetry between eqs. 7.26 and 7.27 and hence one cannot uncouple them. However, we can put them into the forms suitable for successive approximations by a proper change of the dependent variables. To this end let us assume momentarily that the medium is homogeneous. In this case, plane waves would be solutions of eqs. 7.26 and 7.27. Denoting by N the index of refraction, we would obtain

$$k^2N^2 e^{i\eta_x} = A e^{i\eta_x} + iB e^{i\eta_y} \tag{7.31}$$

$$k^2N^2 e^{i\eta_y} = C e^{i\eta_y} - iB e^{i\eta_x} \quad . \tag{7.32}$$

For nontrivial solutions we must set

$$\begin{vmatrix} k^2N^2 - A & -iB \\ iB & k^2N^2 - C \end{vmatrix} = 0$$

from which

$$k^2N_{\pm}^2 = \frac{A + C \pm \sqrt{(A-C)^2 + 4B^2}}{2} \tag{7.33}$$

where A, B and C were constants.

Defining in the usual way the polarization factor P by

$$P = e^{i(\eta_y - \eta_x)} \quad ,$$

we get from eq. 7.31 or eq. 7.32

$$P_{\pm} = \frac{k^2N_{\pm}^2 - A}{iB} \quad \text{or} \quad \frac{iB}{C - k^2N_{\pm}^2} \quad . \tag{7.34}$$

On substituting 7.33 into 7.34, we have

$$P_{\pm} = \frac{C - A \pm \sqrt{(A-C)^2 + 4B^2}}{2i\ B} \quad . \tag{7.35}$$

Thus a wave propagating in a homogeneous medium will, in general, be split into two elliptically polarized waves rotating in opposite senses.

Let us now go back to the gyroelectric medium stratified in the z-direction. We see that the homogeneity in x and y is preserved and hence $P_{\pm}$ are again given by eq. 7.35, but now are functions of z. Let

$$e^{i\eta_{\pm}} = e^{i\eta_x} - P_{\pm}(z)\, e^{i\eta_y} \quad . \tag{7.36}$$

Multiplying eq. 7.27 by $P_{\pm}$ and subtracting the resulting equation from eq. 7.26, we get

$$- \frac{d^2}{dz^2} e^{i\eta_x} + P_{\pm} \frac{d^2}{dz^2} e^{i\eta_y} = (A + iBP_{\pm})\, e^{i\eta_{\pm}} + \left[A - C + iB(1 + P_{\pm}^2) \right] e^{i\eta_y} \quad . \tag{7.37}$$

Using eq. 7.35 and noting that $P_+P_- = 1$ one can easily show that

$$A - C + iB(1 + P_{\pm}^2) = 0 \quad .$$

Moreover, from eq. 7.36, $e^{i\eta_y} = \dfrac{e^{i\eta_+} - e^{i\eta_-}}{P_- - P_+}$.

Eq. 7.37 then reduces to

$$-\frac{d^2}{dz^2} e^{i\eta_\pm} - (A + iBP_\pm)e^{i\eta_\pm} = 2P'_\pm \frac{d}{dz} \frac{e^{i\eta_+} - e^{i\eta_-}}{P_- - P_+}$$

$$+ P''_\pm \frac{e^{i\eta_+} - e^{i\eta_-}}{P_- - P_+} \tag{7.38}$$

where the primes denote differentiations with respect to z. Since from eq. 7.34 $A + iBP_\pm = k^2N_\pm^2$, eqs. 7.38 can also be written as

$$\left(\frac{d^2}{dz^2} + k^2N_+^2\right)e^{i\eta_+} = L_+(e^{i\eta_+} - e^{i\eta_-}) \tag{7.39}$$

$$\left(\frac{d^2}{dz^2} + k^2N_-^2\right)e^{i\eta_-} = L_-(e^{i\eta_-} - e^{i\eta_+}) \tag{7.40}$$

where

$$L_\pm = \frac{2P'_\pm}{P_\pm - P_\mp}\frac{d}{dz} + \left(\frac{P''_\pm}{P_\pm - P_\mp} - \frac{2P'_\pm(P'_\pm - P'_\mp)}{(P_\pm - P_\mp)^2}\right) . \tag{7.41}$$

In the case of parallel propagation the polarization factors $P_\pm$ reduce to $\pm i$ and $N_\pm$ reduce to $\sqrt{\epsilon_1 \pm g}$ as can be shown respectively from eq. 7.35 and eq. 7.33. Hence, $L_\pm$ vanish identically and eqs. 7.39 and 7.40 become eqs. 7.15 and 7.16. It can also be easily seen that these two equations are respectively reduced to eqs. 7.4 and 7.2 in the case of perpendicular propagation.

To put eqs. 7.39 and 7.40 in the forms for successive approximations, we rewrite them as

$$\left(\frac{d\eta_+}{dz}\right)^2 - k^2N_+^2 = F_+(\eta_+,\eta_-) \tag{7.42}$$

$$\left(\frac{d\eta_-}{dz}\right)^2 - k^2N_-^2 = F_-(\eta_+,\eta_-) \tag{7.43}$$

where

$$F_{\pm}(\eta_+,\eta_-) = i\,\frac{d^2\eta_{\pm}}{dz^2} - e^{-i\eta_{\pm}}\,L_{\pm}(e^{i\eta_{\pm}} - e^{i\eta_{\mp}}) \ . \qquad (7.44)$$

Now we use the assumption that the medium is slowly varying. Thus $F_{\pm}$ are small compared to the terms on the left hand sides of eqs. 7.42 and 7.43. In the zeroth approximation we neglect $F_{\pm}$ and obtain

$$\frac{d\eta_+^{(o)}}{dz} = k\,N_+(z) \qquad (7.45)$$

$$\frac{d\eta_-^{(o)}}{dz} = k\,N_-(z) \quad . \qquad (7.46)$$

Next approximation simply yields

$$\frac{d\eta_+^{(1)}}{dz} = kN_+ + \frac{1}{2kN_+}\,F_+(\eta_+^{(0)},\eta_-^{(0)}) \qquad (7.47)$$

$$\frac{d\eta_-^{(1)}}{dz} = kN_- + \frac{1}{2kN_-}\,F_-(\eta_+^{(0)},\eta_-^{(0)}) \quad . \qquad (7.48)$$

This process can be continued as many times as one pleases. But, as one can see, the computations become rather tedious as the order of approximation is increased. We shall not therefore carry out the calculations higher than the zeroth order.

Substituting eqs. 7.28, 7.29 and 7.30 into eqs. 7.33 we obtain, after a simple manipulation

$$\frac{d}{dz}\eta_{\pm} =$$

$$\frac{k\left\{(\epsilon_1^2 - g^2 - \epsilon_1\epsilon_2)\sin^2\theta + 2\epsilon_1\epsilon_2 \pm \sqrt{(\epsilon_1^2 - g^2 - \epsilon_1\epsilon_2)^2\sin^4\theta + 4g^2\epsilon_2^2\cos^2\theta}\right\}^{\frac{1}{2}}}{\sqrt{2(\epsilon_1\sin^2\theta + \epsilon_2\cos^2\theta)}}$$

$$+ \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \qquad (7.49)$$

where ϵ_1, ϵ_2 and g are functions of z .

We now go on to solve the Doppler equation 5.17. As before we solve the equation by the method of iteration. A straightforward calculation yields*

$$\omega_s = \omega_i - 2v\cos\Theta \;\mathrm{Re}\frac{d}{dz}\eta_{\pm}(\omega_i, z) + \cdots\cdots \qquad (7.50)$$

where $\mathrm{Re}\frac{d}{dz}\eta_{\pm}$ are given by eqs. 7.49, and Θ is the angle between the direction of incidence and the velocity of the scatterer.

*Here, for simplicity, we only give two solutions of the Doppler equation. The other two solutions which arise from the conversion of the ordinary incident wave to the extraordinary scattered wave, and vice versa, can be easily written down by using the appropriate ϕ on each side of eq. 5.17.

VIII. SUMMARY AND CONCLUSIONS

The problem of calculating the frequency of the scattered wave from a body moving in a medium is considered from field-theoretic viewpoint.

It is shown that the Doppler equation for a homogeneous dispersive medium can be derived by three different methods. The essential feature of each derivation is that in the frame where the scatterer is stationary the frequency of the scattered wave remains equal to that of the incident wave and that the wave vector and the frequency of a plane wave form a 4-vector. In general, the Doppler equation does not lend itself to direct manipulation for exact solutions. A general method is therefore given as to how the approximate solutions of the Doppler equation can be obtained. It is found that, for a gyroelectric medium such as the ionosphere, the scattered wave contains more than one frequency for a monochromatic incident wave. Some frequencies lie very close to the resonant frequencies of the medium and waves corresponding to these frequencies are damped exponentially in space. Other frequencies are just the normal Doppler frequencies modified by the presence of the medium.

The Doppler equation for an inhomogeneous dispersive medium is obtained from a different approach, since plane waves do not in general exist in such a medium. Under the assumptions of gradual inhomogeneity and slow velocity of the scatterer, the Doppler equation is derived by performing spectral decompositions of the transformed fields in the frame where the scatterer is stationary. It is found that the resulting Doppler equation contains explicitly the functions ψ_i and ψ_s of the incident and the scattered waves. The method of obtaining these functions is first to transform Maxwell's equations into a set of coupled equations of Riccati-type and then to rearrange them in the forms suitable for successive approximations. The case of an isotropic stratified medium is discussed and approximate solutions of the Doppler equation are obtained. In the case of a gyroelectric stratified medium, three separate cases for the orientation of the biasing magnetostatic field with respect to the direction of propagation are studied in detail. It is found that, in the perpendicular and the parallel cases, the polarization factor does not depend on position, while in the general case it is a function of space. Thus, the former two cases are easily handled, since the transformed Maxwell equations can be uncoupled rigorously. In the latter case, however, they remain coupled and therefore the method of solution is considerably involved. Nevertheless, it is shown how the approximate solutions can be obtained formally.

APPENDIX A

ON THE 4-VECTOR K_μ

In this appendix we shall prove that the frequency ω and the wave vector $\underline{k}$ of a plane wave form a 4-vector. To do this we write

$$F_{\mu\nu} = a_{\mu\nu} e^{iK_\lambda X_\lambda} = a_{\mu\nu} e^{i(\underline{k}\cdot\underline{r} - \omega t)} \tag{A.1}$$

where $K_\lambda = (\underline{k}, i\frac{\omega}{c})$, $X_\lambda = (\underline{r}, ict)$, and $F_{\mu\nu}$ is the field tensor of second rank and takes the form:

$$F_{\mu\nu} = \begin{pmatrix} 0 & B_z & -B_y & -i\frac{E_x}{c} \\ -B_z & 0 & B_x & -i\frac{E_y}{c} \\ B_y & -B_x & 0 & -i\frac{E_z}{c} \\ i\frac{E_x}{c} & i\frac{E_y}{c} & i\frac{E_z}{c} & 0 \end{pmatrix} .$$

In eq. A.1 , $a_{\mu\nu}$ is constant in the case of a plane wave and represents the amplitudes of the field components. From one of the Maxwell equations $\nabla \times \underline{E} = -\frac{\partial}{\partial t}\underline{B}$, we have

$$\frac{\partial F_{\mu\nu}}{\partial X_\sigma} + \frac{\partial F_{\nu\sigma}}{\partial X_\mu} + \frac{\partial F_{\sigma\mu}}{\partial X_\nu} = 0 \tag{A.2}$$

which is a tensor equation of third rank. Substitution of A.1 into A.2 gives

$$F_{\mu\nu} K_\lambda \frac{\partial X_\lambda}{\partial X_\sigma} + F_{\nu\sigma} K_\lambda \frac{\partial X_\lambda}{\partial X_\mu} + F_{\sigma\mu} K_\lambda \frac{\partial X_\lambda}{\partial X_\nu} = 0 \ . \tag{A.3}$$

But $\frac{\partial X_\lambda}{\partial X_\sigma} = \delta_{\lambda\sigma}$ (Kronecker delta). Eq. A.3 then takes the following form

$$K_\sigma F_{\mu\nu} + K_\mu F_{\nu\sigma} + K_\nu F_{\sigma\mu} = 0 \quad . \tag{A.4}$$

Since eq. A.4 is of third rank, it follows that K_μ must be a 4-vector and transforms like the 4-vector X_μ . However, if $a_{\mu\nu}$ were not a constant, K_μ would no longer be a 4-vector.

APPENDIX B

ON CERTAIN TRANSFORMATION LAWS

Here we give some useful transformation laws for the electric vector $\underline{E}$, the magnetic vector $\underline{B}$, the 4-vector X_μ , and the 4-vector K_μ . The transformation rules can be put into generalized form without specifying a coordinate system. Let S and S' be the two frames of reference moving with a constant relative velocity $\underline{v}$. Then, by the Lorentz transformation we have

$$\underline{E}'_{\parallel} = \underline{E}_{\parallel} \quad ,$$

$$\underline{E}'_{\perp} = \gamma(\underline{E}_{\perp} + \underline{v} \times \underline{B}) \tag{B.1}$$

where $\parallel$ denotes components parallel and $\perp$ components perpendicular to $\underline{v}$. The first of eqs. B.1 can also be rewritten as

$$\frac{(\underline{E}' \cdot \underline{v})\underline{v}}{v^2} = \frac{(\underline{E} \cdot \underline{v})\underline{v}}{v^2} \quad . \tag{B.2}$$

Since $\underline{E}'_{\perp} = \underline{E}' - \underline{E}'_{\parallel}$ and $\underline{E}_{\perp} = \underline{E} - \underline{E}_{\parallel}$, by means of eq. B.2 we can write the second of eqs. B.1 in the following generalized form:

$$\underline{E}' - \frac{(\underline{E}' \cdot \underline{v})\underline{v}}{v^2} = \gamma\left[\underline{E} - \frac{(\underline{E} \cdot \underline{v})\underline{v}}{v^2} + \underline{v} \times \underline{B}\right] \quad ,$$

or

$$\underline{E}' = \gamma(\underline{E} + \underline{v} \times \underline{B}) + (1 - \gamma)\frac{(\underline{E} \cdot \underline{v})\underline{v}}{v^2} \quad . \tag{B.3}$$

Similarly,

$$\underline{B}' = \gamma(\underline{B} - \frac{\underline{v} \times \underline{E}}{c^2}) + (1 - \gamma) \frac{(\underline{B} \cdot \underline{v})\underline{v}}{v^2} \quad . \tag{B.4}$$

At low velocity, i.e., $v \ll c$, eqs. B.3 and B.4 reduce respectively to

$$\begin{aligned} \underline{E}' &= \underline{E} + \underline{v} \times \underline{B} \\ \underline{B}' &= \underline{B} \quad . \end{aligned} \tag{B.5}$$

We shall now find a similar transformation rule for $X_\mu = (\underline{r}, ict)$. It is known that

$$\underline{r}'_{||} = \gamma(\underline{r}_{||} - \underline{v}t) \ , \quad \text{or} \quad \underline{v} \frac{(\underline{r}' \cdot \underline{v})}{v^2} = \gamma(\frac{\underline{r} \cdot \underline{v}}{v^2} \underline{v} - \underline{v}t) \tag{B.6}$$

and

$$\underline{r}'_{\perp} = \underline{r}_{\perp} \ , \quad \text{or} \quad \underline{r}' - \frac{(\underline{r}' \cdot \underline{v})\underline{v}}{v^2} = \underline{r} - \frac{(\underline{r} \cdot \underline{v})\underline{v}}{v^2} \quad . \tag{B.7}$$

Substituting B.6 into B.7 we have

$$\underline{r}' = \underline{r} - \gamma \underline{v}t + (\gamma - 1) \frac{(\underline{r} \cdot \underline{v})\underline{v}}{v^2} \quad . \tag{B.8}$$

Moreover, we have

$$t' = \gamma(t - \frac{\underline{r} \cdot \underline{v}}{c^2}) \quad . \tag{B.9}$$

At low velocity B.8 and B.9 reduce respectively to

$$\underline{r}' = \underline{r} - \underline{v}t \quad \text{and} \quad t' = t \quad . \tag{B.10}$$

The transformation rule for $K_\mu = (\underline{k}, i\frac{\omega}{c})$ follows immediately from that for $X_\mu = (\underline{r}, ict)$. Replacing $\underline{r}$ by $\underline{k}$ and t by ω/c^2 in B.8

and B.9, we obtain

$$\underline{k}' = \underline{k} - \gamma \frac{\omega}{c^2} \underline{v} + (\gamma - 1) \frac{(\underline{k} \cdot \underline{v})\underline{v}}{v^2} \quad , \tag{B.11}$$

and

$$\omega' = \gamma(\omega - \underline{v} \cdot \underline{k}) \quad . \tag{B.12}$$

At low velocity these expressions reduce to

$$\underline{k}' = \underline{k} \quad , \quad \text{and} \quad \omega' = \omega - \underline{v} \cdot \underline{k} \quad . \tag{B.13}$$

Let us now demonstrate that U_μ defined by 2.4 is a 4-vector. We have

$$U_\mu = (\gamma\underline{v}, i\gamma c) \qquad \text{in} \qquad S$$

$$U'_\mu = (0, ic) \qquad \text{in} \qquad S' \quad .$$

Replacing in eq. B.8 $\underline{r}$ by $\gamma\underline{v}$, t by γ , and $\underline{r}'$ by 0 , we see that the right hand side of B.8 vanishes identically and is therefore equal to its left hand side. Thus U_μ is 4-vector.

APPENDIX C

OTHER DERIVATIONS OF THE DOPPLER EQUATION

In Section 2.2 the Doppler equation was derived by transforming the equation

$$\omega_s' = \omega_i' \tag{C.1}$$

in S' to S by means of a scalar invariant formed from the 4-velocity U_μ of the scatterer and the 4-wave vector K_μ of a plane wave. In this appendix we shall give two other approaches to arrive at the Doppler equation, and each approach has its own domain of interest.

C.1 The Indirect Method

This method of attacking the problem is shown schematically in Fig. C.1. We start with the information in S and carry it over to S' by Lorentz transformations. Then we compare the transformed frequency of the scattered wave with that of the incident wave and obtain the Doppler equation.

In the mathematical language we have, from eq. B.12,

$$\omega_i' = \gamma(\omega_i - \underline{v} \cdot \underline{k}_i) \tag{C.2}$$

$$\omega_s' = \gamma(\omega_s - \underline{v} \cdot \underline{k}_s) \ . \tag{C.3}$$

Since $\omega_s' = \omega_i'$, by equating the right hand sides of eqs. C.2 and C.3 we immediately arrive at the Doppler equation:

$$\omega_s - \underline{v} \cdot \underline{k}_s = \omega_i - \underline{v} \cdot \underline{k}_i \tag{C.4}$$

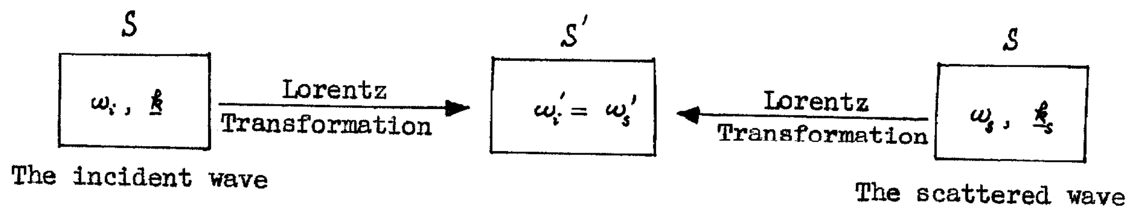

Fig. C-1. Schematic diagram of the indirect method

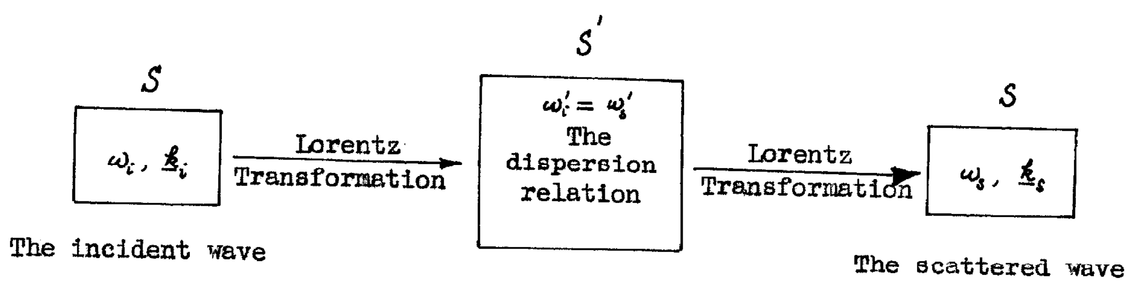

Fig. C-2. Schematic diagram of the direct method which is just the same as eq. 2.7. Although this method is very simple, it is quite indirect, because the very physical fact, i.e., $\omega_s' = \omega_i'$, is exploited only at the final step of the derivation.

C.2 The Direct Method

The more natural approach to the problem is perhaps to start with the incident wave in S and transform it to S'. In S' we then have a scattering process with an incident wave of known frequency and wave vector. Since we are interested only in the frequency of the wave, we need not solve this scattering process as a boundary-value problem. However, the dispersion relation has to be found in S' before one can proceed from S' back to S. This method is shown schematically in Fig. C.2.

Let us now go on to the derivation of the Doppler equation according to the procedure outlined above. We have, from eq. B.12

$$\omega_i' = \gamma(\omega_i - \underline{v} \cdot \underline{k}_i) \qquad (C.5)$$

$$\omega_s = \gamma(\omega_s' + \underline{v} \cdot \underline{k}_s') \quad . \qquad (C.6)$$

Since $\omega_s' = \omega_i'$, substitution of C.5 into C.6 gives

$$\omega_s = \gamma^2(\omega_i - \underline{v} \cdot \underline{k}_i)(1 + \frac{\underline{v} \cdot \underline{k}_s'}{\omega_s'}) \quad . \qquad (C.7)$$

In eq. C.7 it remains to find k_s'/ω_s' in S' where the medium is moving. To do this we have to solve Maxwell's equations in a moving homogeneous medium. In the following, tensor notations will be used, since we wish to write Maxwell's equations and the constitutive equations in covariant forms which hold in all Lorentz frames.

Maxwell's equations are

$$\partial_\lambda F_{\mu\nu} + \partial_\mu F_{\nu\lambda} + \partial_\nu F_{\lambda\mu} = 0 \tag{C.8}$$

$$\partial_\nu G_{\mu\nu} = 0 \tag{C.9}$$

where

$$\partial_\mu = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}, \frac{1}{ic}\frac{\partial}{\partial t}\right) = \left(\nabla, \frac{1}{ic}\frac{\partial}{\partial t}\right) ,$$

$$F_{\mu\nu} = \begin{pmatrix} 0 & B_z & -B_y & -i\frac{E_x}{c} \\ -B_z & 0 & B_x & -i\frac{E_y}{c} \\ B_y & -B_x & 0 & -i\frac{E_z}{c} \\ i\frac{E_x}{c} & i\frac{E_y}{c} & i\frac{E_z}{c} & 0 \end{pmatrix}$$

and

$$G_{\mu\nu} = \begin{pmatrix} 0 & H_z & -H_y & -i\,cD_x \\ -H_z & 0 & H_x & -i\,cD_y \\ H_y & -H_x & 0 & -i\,cD_z \\ icD_x & icD_y & icD_z & 0 \end{pmatrix} .$$

The constitutive equations are

$$G_{\mu\nu} U_\nu = c^2 \epsilon F_{\mu\nu} U_\nu \quad , \tag{C.10}$$

$$F_{\mu\nu} U_\sigma + F_{\nu\sigma} U_\mu + F_{\sigma\mu} U_\nu = \mu(G_{\mu\nu} U_\sigma + G_{\nu\sigma} U_\mu + G_{\sigma\mu} U_\nu) \qquad (C.11)$$

which reduce to $\underline{D} = \epsilon \underline{E}$ and $\underline{B} = \mu \underline{H}$, respectively when the medium is at rest. Here one should not confuse the permeability μ with the subscript μ. Multiplying C.11 with U_σ and noting that $U_\sigma U_\sigma = -c^2$ we have

$$-c^2 F_{\mu\nu} + F_{\nu\sigma} U_\mu U_\sigma + F_{\sigma\mu} U_\nu U_\sigma = - c^2 \mu \, G_{\mu\nu} + \mu(G_{\nu\sigma} U_\mu U_\sigma + G_{\sigma\mu} U_\nu U_\sigma) \qquad (C.12)$$

Substituting C.10 into C.12 and making use of the antisymmetric properties of $F_{\mu\nu}$ and $G_{\mu\nu}$, we get

$$- c^2 F_{\mu\nu} + F_{\nu\sigma} U_\sigma U_\mu - F_{\mu\sigma} U_\sigma U_\nu = - c^2 \mu \, G_{\mu\nu} + c^2 \mu \, (F_{\nu\sigma} U_\sigma U_\mu - F_{\mu\sigma} U_\sigma U_\nu) \quad .$$

Solving this equation for $G_{\mu\nu}$ we obtain

$$G_{\mu\nu} = \frac{F_{\mu\nu}}{\mu} + \frac{(c^2 \epsilon\mu - 1)}{\mu c^2} (F_{\nu\sigma} U_\mu U_\sigma - F_{\mu\sigma} U_\sigma U_\nu) \quad . \qquad (C.13)$$

Substitution of C.13 into C.9 yields

$$\partial_\nu F_{\mu\nu} + \kappa \left[\partial_\nu (F_{\nu\sigma} U_\sigma U_\mu) - \partial_\nu (F_{\mu\sigma} U_\sigma U_\nu) \right] = 0 \qquad (C.14)$$

where $\kappa = \dfrac{c^2 \epsilon \, \mu - 1}{c^2} = \dfrac{n^2 - 1}{c^2}$.

In eq. C.14 the second term vanishes identically, i.e., $\partial_\nu (F_{\nu\sigma} U_\sigma) U_\mu = 0$. To see this, we note that

$$\partial_\nu (G_{\nu\sigma} U_\sigma) = \gamma \, \underline{v} \cdot \left(\frac{\partial D}{\partial t} - \nabla \times \underline{H}\right) + \gamma \, c^2 \, \nabla \cdot \underline{D} \quad . \tag{C.15}$$

Since each term on the right hand side of C.15 vanishes identically in the source-free region, we then have

$$\partial_\nu (G_{\nu\sigma} U_\sigma) = 0$$

and by virtue of C.10 we conclude that

$$\partial_\nu (F_{\nu\sigma} U_\sigma) = 0 \quad .$$

Thus C.14 reduces to

$$\partial_\nu F_{\mu\nu} - \kappa \, \partial_\nu (F_{\mu\sigma} U_\sigma U_\nu) = 0 \quad . \tag{C.16}$$

The field tensor $F_{\mu\nu}$ is also defined in terms of the 4-potential $A_\mu = (\underline{A} \, , \, i\frac{\phi}{c})$ by the following equation:

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \tag{C.17}$$

which automatically satisfies eq. C.8.

Substituting C.17 into C.16 we get

$$(\partial_\nu^2 - \kappa \, U_\nu \partial_\nu U_\sigma \partial_\sigma) A_\mu = \partial_\mu (\partial_\nu A_\nu - \kappa \, U_\nu \partial_\nu U_\sigma A_\sigma) \quad . \tag{C.18}$$

Now we impose on A_μ the subsidiary condition

$$\partial_\nu A_\nu - \kappa \, U_\nu \partial_\nu U_\sigma A_\sigma = 0 \tag{C.19}$$

which takes the following form in the ordinary space:

$$\nabla \cdot \underline{A} + \frac{1}{c^2}\frac{\partial \phi}{\partial t} - \kappa\,\gamma^2(\underline{v} \cdot \nabla + \frac{\partial}{\partial t})(\underline{v} \cdot \underline{A} - \phi) = 0 \quad . \qquad (C.20)$$

When $\underline{v} = 0$, C.20 reduces to

$$\nabla \cdot \underline{A} + \epsilon\mu \frac{\partial \phi}{\partial t} = 0$$

which is just the Lorentz gauge in the frame where the medium is at rest. When $\kappa = 0$, i.e., in vacuum, C.20 becomes

$$\nabla \cdot \underline{A} + \frac{1}{c^2}\frac{\partial \phi}{\partial t} = 0$$

as it should. Thus C.19 is the covariant Lorentz gauge in a medium, and eq. C.18 becomes

$$(\partial_\nu^2 - \kappa\, U_\nu \partial_\nu U_\sigma \partial_\sigma)A_\mu = 0 \quad . \qquad (C.21)$$

Writing C.21 out in components, we have

$$\left[\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \kappa\,\gamma^2(\underline{v} \cdot \nabla + \frac{\partial}{\partial t})^2\right]\underline{A} = 0 \qquad (C.22)$$

$$\left[\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \kappa\,\gamma^2(\underline{v} \cdot \nabla + \frac{\partial}{\partial t})^2\right]\phi = 0 \quad . \qquad (C.23)$$

We note that these equations reduce to the ordinary wave equations when either $\epsilon\mu = 1/c^2$ or $\underline{v} = 0$.

To find the dispersion relation we substitute

$$\underline{A}(\underline{r},t) = \mathrm{Re}\left\{\underline{A}_o\, e^{i(\underline{k} \cdot \underline{r} - \omega t)}\right\}$$

into eq. C.22 and obtain

$$-(ck)^2 + \omega^2 + \gamma^2(n^2 - 1)(\underline{k} \cdot \underline{v} - \omega)^2 = 0 \quad . \tag{C.24}$$

Solving this equation for ck/ω , we get

$$\frac{ck}{\omega} = \frac{\gamma^2(n^2 - 1)\beta \cos\theta \pm \sqrt{1 + \gamma^2(n^2 - 1)(1 - \beta^2\cos^2\theta)}}{\gamma^2(n^2 - 1)\beta^2\cos^2\theta - 1} \tag{C.25}$$

where θ is the angle between $\underline{v}$ and $\underline{k}$.

When $\underline{v} = 0$, ck/ω should reduce to n and hence we have to take the "-" sign in expressions C.25, i,e.,

$$\frac{ck}{\omega} = \frac{\sqrt{1 + \gamma^2(n^2 - 1)(1 - \beta^2\cos^2\theta)} - \gamma^2(n^2 - 1)\beta \cos\theta}{1 - \gamma^2(n^2 - 1)\beta^2\cos^2\theta} \quad . \tag{C.26}$$

It should be noted that n is the refractive index of the medium when the medium is at rest. When the medium is in motion, the refractive index is likewise defined by expression C.26 which, however, depends on the direction of propagation. At low velocity, i.e., $v \ll c$, the refractive index takes the approximate form

$$\frac{ck}{\omega} \approx n - (n^2 - 1)\beta \cos\theta \quad . \tag{C.27}$$

In the case of a dispersive medium, n is a function of $\gamma(\omega - \underline{v} \cdot \underline{k})$ and becomes a function of frequency only when $\underline{v} = 0$. Thus the right hand side of C.26 depends implicitly on ω as well as on k . To each given ω there correspond, in principle, several values of k , and in this sense the motion of a dispersive medium splits the wave.

Replacing θ by $-\theta'$ in C.26 and then substituting the resulting equation into C.7, we obtain

$$\omega_s = \gamma^2(\omega_i - \underline{v}\cdot\underline{k}_i)\left\{\frac{1+\beta\cos\theta'\sqrt{1+\gamma^2(n^2-1)(1-\beta^2\cos^2\theta')}}{1-\gamma^2(n^2-1)\beta^2\cos^2\theta'}\right\} \tag{C.28}$$

Now it remains to find how θ' is transformed to θ, which is the angle between $\underline{v}$ and $\underline{k}$ in the S frame. To do this we have, from eq. B.11,

$$k'_{\|} = \gamma(k_{\|} - \frac{\omega\beta}{c}) \tag{C.29}$$

$$k'_{\perp} = k_{\perp} \quad . \tag{C.30}$$

Eqs. C.29 and C.30 can also be written as

$$k'\cos\theta' = \gamma(k\cos\theta - \frac{\omega\beta}{c}) \tag{C.31}$$

$$k'\sin\theta' = k\sin\theta \tag{C.32}$$

from which we immediately get

$$\tan\theta' = \frac{n\sin\theta}{\gamma(n\cos\theta-\beta)} \quad ,$$

and consequently

$$\cos\theta' = \frac{\gamma(n\cos\theta-\beta)}{\sqrt{n^2\sin^2\theta+\gamma^2(n\cos\theta-\beta)^2}} \,. \tag{C.33}$$

Substitution of C.33 into C.28 yields, after some manipulation,

$$\omega_s = \gamma^2(\omega_i - \underline{v} \cdot \underline{k}_i) \frac{1}{\gamma^2(1 - n\beta \cos \Theta)} ,$$

or

$$\omega_s - \underline{v} \cdot \underline{k}_s = \omega_i - \underline{v} \cdot \underline{k}_i \tag{C.34}$$

which is the same Doppler equation as eq. 2.7.

In this derivation three interesting points have been brought out.

(1) In a moving medium the vector potential $\underline{A}$ and the scalar potential ϕ no longer satisfy the ordinary wave equation, but rather they satisfy the more complicated equations, namely eqs. C.22 and C.23.

(2) The refractive index of a moving medium depends not only on the velocity of the medium, but also on the direction of propagation.

(3) The motion of a dispersive medium splits a wave of given frequency into several waves of the same frequency. It can also be stated that to each fixed $\underline{k}$ there correspond several values of ω in a moving dispersive medium.

APPENDIX D

ON THE POLES OF n_+^2 AND n_-^2

Here we shall examine the singularities of n_+^2 and n_-^2 defined by eqs. 4.2 .

For convenience, let

$$p = 2\omega\,(\omega^2 - \omega_p^2)(\omega^2 - \omega_p^2 - \omega_g^2) - \omega\,\omega_p^2\,\omega_g^2 \sin^2\theta$$

$$q = \omega_p^2\,\omega_g \sqrt{\omega^2\,\omega_g^2 \sin^4\theta + 4(\omega^2 - \omega_p^2)^2 \cos^2\theta} \tag{D.1}$$

Eqs. 4.2 can then be written as

$$n_\pm^2(\omega) = \frac{p(\omega) \pm q(\omega)}{2\omega\left[\omega^4 - (\omega_p^2 + \omega_g^2)\omega^2 + \omega_p^2\,\omega_g^2 \cos^2\theta\right]} \; . \tag{D.2}$$

Since

$$(p+q)(p-q) = \left[\omega^4 - (\omega_p^2 + \omega_g^2)\omega^2 + \omega_p^2\,\omega_g^2 \cos^2\theta\right]\left[(\omega^2 - \omega_p^2 - \omega_g^2)(\omega^2 - \omega_p^2) - \omega_p^2\,\omega_g^2\right] \tag{D.3}$$

we then have

$$(p+q)(p-q) = 0 \qquad \text{at} \qquad \omega = \Omega_\pm \; ,$$

where

$$\Omega_\pm^2 = \frac{\omega_p^2 + \omega_g^2 \pm \sqrt{(\omega_p^2 + \omega_g^2)^2 - 4\omega_p^2\,\omega_g^2 \cos^2\theta}}{2} \; .$$

In what follows we shall prove that $p + q \neq 0$ at $\omega = \Omega_-$, and

$p - q \neq 0$ at $\omega = \Omega_+$.

To prove that $p + q \neq 0$ at $\omega = \Omega_-$, it suffices to show that $p \geqq 0$ at $\omega = \Omega_-$ because q is always positive. Evaluating p at $\omega = \Omega_-$ we have

$$\frac{p}{\Omega_-} = \omega_p^2 \left[\omega_p^2 + \sqrt{(\omega_p^2 + \omega_g^2)^2 - 4\omega_p^2 \omega_g^2 \cos^2\theta} - \omega_g^2 \cos^2\theta \right]$$

$$> \omega_p^2 \left[\omega_p^2 \pm (\omega_p^2 - \omega_g^2) - \omega_g^2 \right] \geqq 0 \quad , \quad \text{for } \theta \neq 0$$

where the "+" and "-" signs are for $\omega_p > \omega_g$ and $\omega_p < \omega_g$, respectively. Moreover, Ω_- is always positive and it follows that $p \geqq 0$. Thus, $p + q \neq 0$ at $\omega = \Omega_-$. Similarly, $p - q \neq 0$ at $\omega = \Omega_+$. But

$$(p + q)(p - q) = 0 \quad \text{at} \quad \omega = \Omega_\pm \quad .$$

This implies that

$$p - q = 0 \qquad \text{at} \qquad \omega = \Omega_- \ ,$$

and

$$p + q = 0 \qquad \text{at} \qquad \omega = \Omega_+ \ .$$

In view of D.2 and D.3 we can also write

$$n_+^2 = \frac{(\omega^2 - \omega_p^2 - \omega_g^2)(\omega^2 - \omega_p^2) - \omega_p^2 \omega_g^2}{2\omega\,(p - g)}$$

and

$$n_-^2 = \frac{(\omega^2 - \omega_p^2 - \omega_g^2)(\omega^2 - \omega_p^2) - \omega_p^2 \omega_g^2}{2\omega(p + q)} \qquad .$$

Hence n_+^2 has simple poles at $\omega = 0$ and $\omega = \Omega_-$, while n_-^2 has simple poles at $\omega = 0$ and $\omega = \Omega_+$.

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