RADIATION OF A POINT CHARGE
MOVING UNIFORMLY OVER AN INFINITE ARRAY
OF CONDUCTING HALF-PLANES

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ABSTRACT

The problem of the excitation of an infinite array of parallel, semi-infinite metallic plates by a uniformly moving point charge is studied by the Wiener-Hopf method. It is treated as a boundary value problem for the potentials of the diffracted electromagnetic fields. The formulation of this problem makes use of the well-known conditions on the electromagnetic fields at a metallic boundary. A method is used to translate these boundary conditions on the fields into boundary conditions on the potentials. In this way the problem is formulated in terms of a set of dual integral equations for the current densities induced on the plates by the point charge. These integral equations are exactly soluble by the Wiener-Hopf technique. The solutions are found to satisfy the famous edge conditions for diffraction problems, and are therefore unique. From these solutions exact expressions for the diffracted fields are derived in the form of Fourier integrals. It is seen that these fields represent a radiation of electromagnetic energy. The method of steepest descent is then used to obtain expressions for the radiation fields, the Poynting vector, the frequency spectrum and the radiation pattern. The radiation shows that the array of plates behaves both like a diffraction grating and a series of parallel-plate waveguides.

I. INTRODUCTION

1.1 Historical Development

The phenomenon of diffraction occurs when light passes close by the edges of an opaque obstacle. The shadow of the latter is not found to have a sharp boundary as predicted by geometrical optics or the corpuscular theory. Instead there appears a series of alternately dark and bright bands extending into the geometrical shadow region. These fringe structures were probably first observed and recorded by Grimaldi (1618-1663) who gave the phenomenon the name "diffraction". Of all the diffraction problems, the simplest and at the same time most fundamental are those involving semi-infinite plane screens. The history of such problems is practically identical with the history of diffraction itself. As is true with other branches of physics, the development of the theory of diffraction can be divided into two periods. In the first period the true nature of the phenomenon is sought; while in the second period, one is concerned with the construction of a deductive, mathematical formulation of the theory--preferably in the form of one embracing principle--and its application to particular problems.

During the century and a half after Grimaldi's discovery, numerous observations on diffraction were made, but no interpretations were offered until the beginning of the nineteenth century. In the meantime two physical theories of light were created; Huygens (1629-1695) formulated a geometrical wave theory and Newton (1642-1727) a corpuscular theory. Huygens thought of light as a longitudinal wave in the aether. Every point of a wave front could be considered to be a center of a secondary disturbance which gave rise to spherical wavelets. The wave front at any later instant might be regarded as the envelope of these wavelets and could be constructed geometrically. Newton suggested that light consisted of a stream of particles emitted rectilinearly from the source. In view of his great reputation his theory dominated in the field of optics for almost a century. Huygens was apparently unaware of Grimaldi's discovery, and so his theory did not deal with diffraction, although it could have covered at least a crude theory of this phenomenon.

Between the years 1801 and 1804 Young (1773-1829) published a series of papers in optics in which he put forth his principle of interference. With the help of this principle he gave a qualitative explanation of diffraction in terms of the wave theory. However, his theory was not widely circulated. It was a period when the corpuscular theory was in vogue. Diffraction, being a deviation from rectilinear propagation, was considered a thing apart--an appendage to the subject of optics. In fact, the Paris Academy of Sciences was so dominated by supporters of the corpuscular theory that it offered in 1818 a prize for an essay on diffraction. Undoubtedly most academicians had in mind the prospects of receiving a brilliant treatise in the spirit of the corpuscular theory, thereby dealing a fatal blow to the wave theory. Such prospects were blighted when the prize was awarded to Fresnel (1788-1827). In his Mémoire couronné (1) Fresnel extended Huygens' geometrical theory by adding periodicity in space and time to Huygens' wave fronts. Thereby the waves were allowed to interfere. In this way, by supplementing Huygens' principle with Young's principle of interference, Fresnel correctly explained diffraction phenomena as being due to the mutual interference of the secondary waves emitted by those portions of the original wave front which were not obstructed by the diffracting obstacle. He was then able to account quantitatively for various diffraction patterns. In particular, he calculated the diffraction pattern of a half-plane. His results were expressed in terms of the definite integrals which now bear his name.

Now that the nature of diffraction was understood, a period of mathematical investigations followed. The first attempt was to give Huygens' principle an analytical expression. Following Fresnel's idea of regarding light as a periodic wave motion in space and time, Helmholtz (1821-1894) considered the wave field as a solution of the partial differential equation $(\nabla^2 + k^2)\, u = 0$. With the help of Green's theorem he succeeded in expressing a wave field at an observation point in the form of a surface integral over the field and its normal derivative on a closed surface surrounding the point. This formulation of Huygens' principle was adopted by Kirchhoff (1824-1887) in his own theory of diffraction. In his work Kirchhoff had to make certain rather arbitrary assumptions about the values of the field and its normal derivative on the enclosing surface. More specifically, these boundary values were assumed to be not appreciably different from the values obtained in the absence of the diffracting screens. In addition, the boundary values on the screens were taken to be zero. This method is therefore only approximate. Poincaré (1854-1912) later showed that Kirchhoff's boundary conditions were not in general self-consistent. Nevertheless his calculations showed good agreement with observation in a great variety of problems where the characteristic dimension was much larger than the wavelength.

Hitherto light was taken to be representable by a scalar wave field. With the appearance of Maxwell's (1831-1879) electromagnetic theory of light, it became evident that an exact treatment of a diffraction problem had to take into account the polarization of light as well. A problem in diffraction now consisted in finding a solution to Maxwell's equations satisfying appropriate boundary conditions. But the transition from a scalar to a vector field added much to the mathematical difficulty of the problem. It is no great surprise that the first exact solution of a diffraction problem did not appear until the last decade of the nineteenth century.

In 1896 Sommerfeld (1868-1951) published the solution of his diffraction problem (2). He treated the case of a monochromatic plane wave incident on an infinitely thin, perfectly conducting half-plane. He took the incident wave to be polarized parallel to the edge of the half-plane, thus reducing the essentially vector problem to a scalar one. Here the boundary conditions were unambiguously provided by the electromagnetic theory. Assumptions like those made by Kirchhoff were not necessary. Sommerfeld had the ingenuity to perceive the similarity of this problem to the electrostatic problem of an infinite conducting plane, and proceeded to synthesize a solution by the method of images. Because he was dealing with a half-plane, the imaging was done on a two-sheet Riemann surface of a double-valued function. In this fashion he constructed an exact solution to the problem, namely, one which satisfied both Maxwell's equations and the boundary conditions. This solution provided not only a check on the accuracy of previous approximate methods but also an insight into the behavior of the electromagnetic fields in the neighborhood of the edge of the conducting half-plane where previous methods failed.

From this point on, one had to wait almost half a century before the next significant advance in diffraction theory took place. In the meantime several variants of Sommerfeld's problem were solved. The technique of solving partial differential equations by separation of variables was applied to obtain series solutions of several plane-wave diffraction problems, for example, that of a finitely conducting sphere, a slit, or a circular aperture in a conducting plane. The utility of such solutions depends on the ease with which computation of the relevant functions can be carried out and the rapidity with which the series converge. For a long time the Sommerfeld-type problems remained the only ones whose solutions could be obtained in closed forms.

In the early 1940's diffraction problems were studied under the approach of integral equation formulations, as first suggested by Rayleigh and Poincaré (3). A number of workers, notably Schwinger (4), Magnus (5), and Copson (6), found that certain problems involving semi-infinite metallic structures yielded integral equations which might be exactly soluble by the Wiener-Hopf method. In particular, Sommerfeld's half-plane problem was re-solved under this new approach.

It is of interest to note that whereas the approaches of Sommerfeld and Schwinger are very different in technique, their foundations are traceable to the analytic continuation of analytic functions of a single complex variable. In Sommerfeld's work the mathematical tool is the continuation of a double-valued function from one sheet of a Riemann surface to the other. In the method of Wiener and Hopf, one finds another version of analytic continuation. Here one encounters two analytic functions that are analytic in two different half-planes and these two half-planes have a common strip of analyticity. These analytic functions do not arise from representations of solutions of the wave equation, as is the case in Sommerfeld's approach, but rather from the Fourier transforms of known and unknown functions in the formulated integral equations.

Since the introduction of the Wiener-Hopf method to diffraction problems, great interest in this field was aroused. The impetus came both from the mathematical challenge and from the possibility of applying the results to radio microwave technology. In general the development proceeded along two directions. On the one hand, one studied the single-plane problem for more sophisticated sources of excitation (7),(8). On the other hand, one attempted multi-plane problems (9),(10). In the latter cases the authors have hitherto considered only two-dimensional cases, namely, the sources of excitation were two-dimensional and so oriented with respect to the half-planes that the problems remained essentially two-dimensional in nature. The advantage of this choice was that the vector problems could be scalarized.

In this work one solves a multi-plane diffraction problem when the source of excitation is a moving point charge. This is the first three-dimensional problem of its kind. As will be seen later, this three-dimensionality of the source compels one to work in terms of the electromagnetic potentials instead of the fields. Part of the difficulty of the problem also consists in the interlocking of the two-dimensionality of the half-planes with the three-dimensionality of the point charge. This manifests itself in the fact that whereas the half-planes are naturally described by a rectangular coordinate system, the moving point charge is more conveniently described in terms of cylindrical coordinates.

1.2 Statement of the Problem

In the present problem we investigate the excitation of an infinite array of parallel metallic plates by a uniformly moving point charge. The plates are taken to be infinitely thin as well as infinitely conducting. They are equally spaced and semi-infinite in extent with their edges lying in a plane which is perpendicular to the plates. The trajectory of the point charge lies at a constant distance above the edges and is perpendicular to them.

We assume that the point charge moves with a uniform velocity. This either implies that the uniform velocity of the point charge is maintained by an external agent, or that the interaction between the plates and the point charge does not alter the motion of the latter appreciably. The second statement is clearly not true, for since the plates are infinite in number, any effect of the force exerted by the plates on the point charge, however small, will multiply indefinitely as the charge traverses the plates in succession. Nevertheless, in the laboratory we can only construct a finite array of plates to which the infinite array in our calculations is a convenient approximation. So in practice when the transition time of the point charge across the finite array of plates is short, the velocity of the point charge may well be considered uniform.

We set up a right-handed rectangular coordinate system as shown in Fig. 1. The positive direction of the z-axis is taken to point out from the plane of the paper. The array of plates lies in the lower half-space $y < 0$. The plates are separated by a distance d from one another. Thus they are located at $x = nd$, $n = 0, \pm 1, \pm 2, \cdots$. Their edges all lie in the z-x-plane. The point charge carrying a charge e moves in the positive x-direction with velocity v . Its trajectory lies at a constant distance a above the x-axis.

We can foresee roughly what happens during the motion of the point charge. The moving point charge generates time-varying electromagnetic fields which act on the free charges on the plates to give rise to induced current densities whose existence is required to satisfy the boundary conditions of the electromagnetic fields on metallic surfaces. The central point of the problem consists in calculating these induced current densities from which the induced electromagnetic fields can be derived. Unlike the fields of a uniformly moving point charge, these induced fields represent an outflow of radiation. This situation is similar to that of Cerenkov radiation where the point charge is taken to move uniformly; the

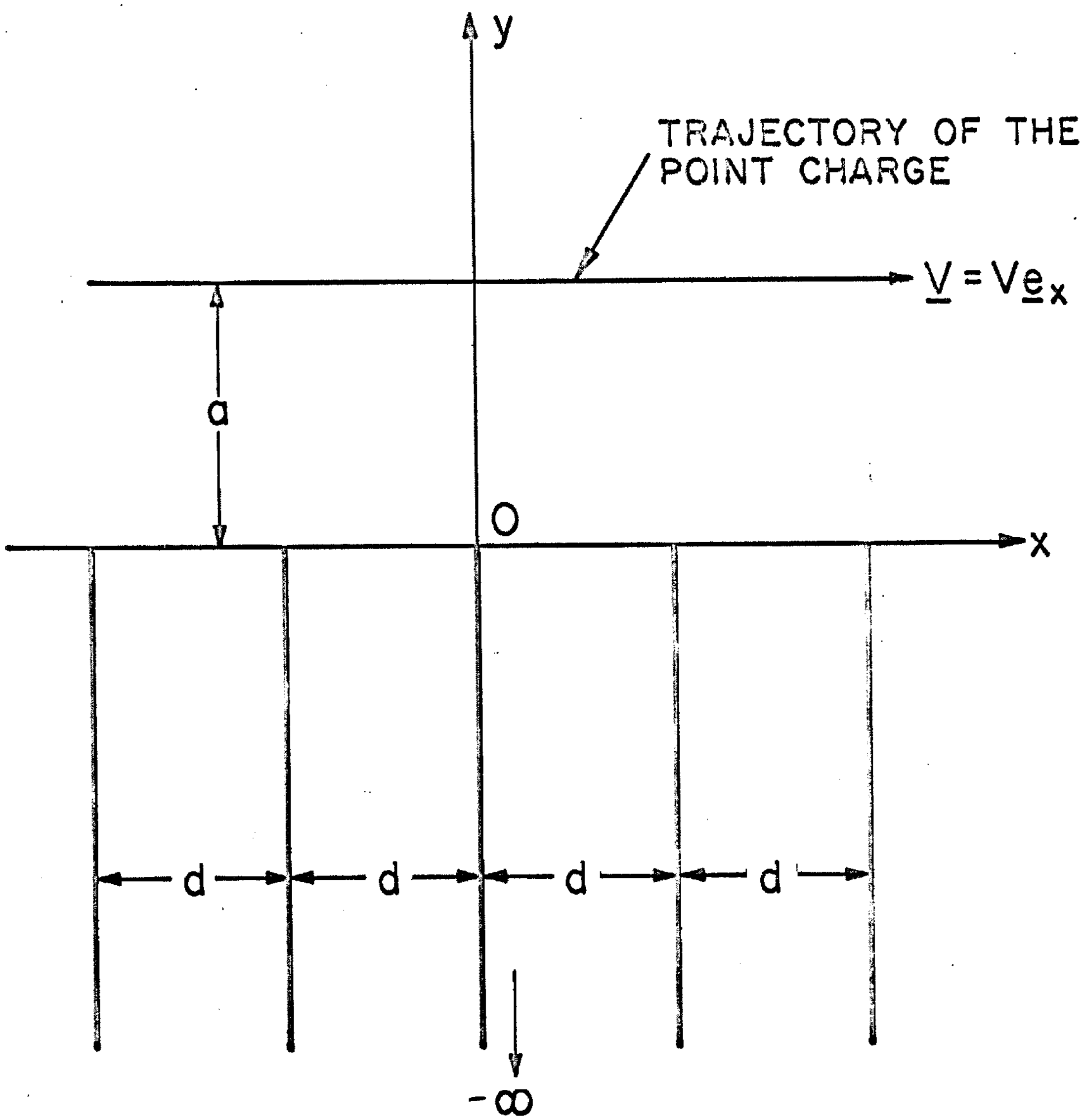

Fig. 1. Geometry of the problem. The array of plates extends to infinity in both directions of the x-axis.

radiation is totally attributed to the excited medium traversed by the charge.

1.3 Notations and Conventions

Throughout this work we will be working mostly with the relativistic four-potential A_μ , $\mu = 1,2,3,4$. Written out more explicitly

$$A_\mu = (\underline{A}, i \frac{\phi}{c}) = (A_x, A_y, A_z, i \frac{\phi}{c})$$

where $\underline{A}$ is the electromagnetic vector potential and ϕ the scalar potential. It is noticed that we use the Minkowski metric with metric tensor $g_{\mu\nu} = \delta_{\mu\nu}$, so that it is not necessary to differentiate between covariant and contravariant indices.

From the principle of superposition the total potential A_μ^{total} at a point in space can be expressed as the sum of the potential A_μ^o due to the point charge and the potential A_μ due to the currents induced on the plates. Thus

$$A_\mu^{total} = A_\mu^o + A_\mu$$

From the geometry of the problem we easily see that

$$A_\mu^o = (A_x^o, 0, 0, i \frac{\phi^o}{c})$$

$$A_\mu = (0, A_y, A_z, i \frac{\phi}{c})$$

The latter equation follows from the fact that the current density j_μ induced on the plates has no component in the x-direction, that is, no current flows out of the plates:

$$j_\mu = (0,\ j_y,\ j_z,\ ic\rho)$$

In the subsequent chapters the technique of Fourier transformation will be used extensively. The convention we will adopt in the transformation from space-time coordinate space to energy-momentum space or vice versa will be as follows:

$$f(\underline{r},t) = \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(\underline{k},\omega)\, e^{i\underline{k}\cdot\underline{r} - i\omega t}\, dk_x dk_y dk_z d\omega$$

$$g(\underline{k},\omega) = \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\underline{r},t)\, e^{-i\underline{k}\cdot\underline{r} + i\omega t}\, dx\, dy\, dz\, dt$$

where $\underline{r} = (x,y,z)$ and $\underline{k} = (k_x,k_y,k_z)$.

A collection of other symbols, defined at various stages of the problem, will be found in Appendix A.

II. FORMULATION OF THE PROBLEM

In this chapter we shall derive integral equations for the current densities induced on the metal plates. Use is made of the periodicity of our system to relate the current on any plate to those on one particular plate, for example, the one at $x = 0$. The integral equations are established by imposing the boundary conditions on the total 4-potential. These conditions are deduced directly from the well-known conditions on the total electromagnetic fields at a metallic boundary. The reason we choose to work in terms of the boundary conditions on the potentials, rather than those on the fields, is that in the former case simpler integral equations result. The price we have to pay is the introduction of unknown constants into the equations. These will be determined in the future after the solution of the equations by the boundary conditions on the induced currents.

2.1 Treatment of the Source

Consider Maxwell's equations:

$$\nabla \cdot \underline{D} = \rho$$

$$\nabla \cdot \underline{B} = 0$$

$$\nabla \times \underline{E} = -\frac{\partial \underline{B}}{\partial t}$$

$$\nabla \times \underline{H} = \underline{j} + \frac{\partial \underline{D}}{\partial t} \qquad (2.1)$$

These fields satisfy in addition the relations

$$\underline{D} = \varepsilon_o \underline{E} \quad , \qquad \underline{B} = \mu_o \underline{H}$$

and

$$\varepsilon_o \mu_o = \frac{1}{c^2}$$

Introduce the electromagnetic potentials through the relations

$$\underline{E} = - \nabla \phi - \frac{\partial \underline{A}}{\partial t}$$

$$\underline{B} = \nabla \times \underline{A} \tag{2.2}$$

Then in terms of the potentials Maxwell's equations become

$$\nabla^2 \underline{A} - \frac{1}{c^2} \frac{\partial^2 \underline{A}}{\partial t^2} - \nabla(\nabla \cdot \underline{A} + \frac{1}{c^2} \frac{\partial \phi}{\partial t}) = - \mu_o \underline{j}$$

$$\nabla^2 \phi - \frac{1}{c^2} \frac{\partial^2 \phi}{\partial t^2} + \frac{\partial}{\partial t} (\nabla \cdot \underline{A} + \frac{1}{c^2} \frac{\partial \phi}{\partial t}) = - \frac{\rho}{\varepsilon_o}$$

If we prescribe between the potentials the Lorentz condition

$$\nabla \cdot \underline{A} + \frac{1}{c^2} \frac{\partial \phi}{\partial t} = 0 \quad , \tag{2.3}$$

the potentials then appear as the solutions of an inhomogeneous wave equation with the current densities as sources:

$$(\nabla^2 - \frac{1}{c^2} \frac{\partial^2}{\partial t^2}) A_\mu = - \mu_o j_\mu \tag{2.4}$$

Let us perform Fourier transformations on the z,t coordinates:

$$A_\mu(x,y,z,t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} A_\mu(x,y,k_z,\omega)\, e^{ik_z z - i\omega t} dk_z\, d\omega$$

$$j_\mu(x,y,z,t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} j_\mu(x,y,k_z,\omega)\, e^{ik_z z - i\omega t}\, dk_z\, d\omega \qquad (2.5)$$

These transformations are necessary in the present approach to the problem in order to obtain simple integral equations that are soluble by the Wiener-Hopf method. If the transformations are not performed our formulation will result in equations involving multiple integrals. Substituting (2.5) into (2.4) we obtain

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + p^2\right) A_\mu(x,y,k_z,\omega) = - \mu_o j_\mu(x,y,k_z,\omega) \qquad (2.6)$$

where we have put

$$p^2 = k^2 - k_z^2 \qquad (2.7)$$

$$k = \omega/c \quad .$$

For the case $k^2 > k_z^2$ we define

$$p = \sqrt{k^2 - k_z^2}$$

The case $k^2 < k_z^2$ will be examined shortly. Let us for the time being return to equation (2.6). The particular solution of the equation can be found with the help of the Green's function $G(x,y,x',y')$ of the two-dimensional Helmholtz equation:

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + p^2\right) G(x,y,x',y') = - 4\pi\ \delta(x-x')\ \delta(y-y')$$

$$G(x,y,x',y') = i\pi H_o^{(1)}[p \sqrt{(x-x')^2 + (y-y')^2}] \qquad (2.8)$$

Here $H_o^{(1)}$ is a Hankel function of the first kind whose asymptotic behavior is

$$H_o^{(1)}(z) \sim (\frac{2}{\pi z})^{1/2} e^{i(z - \frac{1}{4}\pi)}$$

when $|z|$ is large. Thus if p is real and positive $G(x,y,x',y')$ satisfies the outgoing wave condition at infinity. On the other hand if p is purely imaginary, we must take its imaginary part to be positive so that $G(x,y,x'y')$ might not behave singularly at infinity. So for $k^2 < k_z^2$ we are led to set

$$p = i \sqrt{k_z^2 - k^2} \quad .$$

Summarizing the two cases we define

$$p = \sqrt{k^2 - k_z^2}\ , \quad \text{Im } p > 0 \ . \qquad (2.9)$$

Finally, the particular solution of (2.6) assumes the form

$$A_\mu(x,y,k_z,\omega) = \frac{\mu_o}{4\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} i\pi H_o^{(1)} [p \sqrt{(x-x')^2 + (y-y')^2}]$$

$$\times \ j_\mu(x',y',k_z,\omega) \, dx'dy' \qquad (2.10)$$

For our point charge e moving uniformly in the x direction with velocity $\underline{v} = v \, \underline{e}_x$ and at a distance a above the x-axis, the current density is given by

$$j^o_\mu(x,y,z,t) = ec\ \delta(x-c\ t)\ \delta(y-a)\ \delta(z)\ \ (\beta,0,0,i) \qquad (2.11)$$

where $\beta = v/c$. As in Section 1.3 the superscript here indicates quantities associated with the source. Without loss of generality it is assumed that at $t = 0$ the charge is at the point $x = 0$, $y = a$, $z = 0$. Substituting equation 2.11 into the inversion formula

$$j^o_\mu(x,y,k_z,\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} j^o_\mu(x,y,z,t)\ e^{-ik_z z + i\omega t}\ dz\ dt$$

we get

$$j^o_\mu(x,y,k_z,\omega) = \frac{e}{2\pi\beta}\ e^{i\frac{\omega}{v}x}\ \delta(y-a)\ \ (\beta,0,0,i) \qquad (2.12)$$

The potential A^o_μ due to j^o_μ will be the particular solution given by equation 2.10. The complementary solution is not needed. Putting 2.12 into 2.10 we obtain

$$A^o_\mu(x,y,k_z,\omega) = \frac{\mu_o}{4\pi}\frac{e}{2\pi\beta} \int_{-\infty}^{\infty} i\pi H^{(1)}_o\ [p\sqrt{(x-x')^2 + (y-a)^2}\ ] \times e^{i\frac{\omega}{v}x'}\ dx'\ \ (\beta,0,0,i)\ .$$

Now we have the relation

$$\int_{\infty}^{\infty} i\pi H_o^{(1)}[p \sqrt{(x-x')^2 + (y-a)^2}]\, e^{i\frac{\omega}{v}x'}\, dx'$$

$$= 2\pi\, e^{i\frac{\omega}{v}x} \frac{e^{-|y-a|\sqrt{-p^2+(\frac{\omega}{v})^2}}}{\sqrt{-p^2 + (\frac{\omega}{v})^2}} \qquad (2.13)$$

In arriving at the above result we made use of the formula

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} i\pi H_o^{(1)}[p\sqrt{z^2 + a^2}]\, e^{itz} dz = \frac{e^{-|a|\sqrt{-p^2+t^2}}}{\sqrt{-p^2 + t^2}} \qquad (2.14)$$

which we will derive in Appendix B. Thus if we define

$$q = \sqrt{-p^2 + (\frac{\omega}{v})^2} = \sqrt{\alpha^2 k^2 + k_z^2}$$

where $$\alpha = \frac{\sqrt{1 - \beta^2}}{\beta} \qquad (2.15)$$

we obtain

$$A_\mu^o(x,y,k_z,\omega) = \frac{\mu_o}{4\pi} \frac{e}{q\beta}\, e^{i\frac{\omega}{v}x - q|y-a|} (\beta,0,0,i) \qquad (2.16)$$

As a check we see that $A_\mu^o(x,y,k_z,\omega)$ satisfies the Lorentz condition 2.3. Expression 2.16 can be derived alternatively by Fourier transforming the Liénard-Wiechert potentials of a uniformly moving point charge.

The electromagnetic fields associated with the point charge can now be computed from equation 2.16 according to the relations 2.2:

$$E_x^o(x,y,k_z,\omega) = -\frac{\mu_o}{4\pi}\frac{ie\alpha^2\omega}{q}\, e^{i\frac{\omega}{v}x - q|y-a|}$$

$$E_y^o(x,y,k_z,\omega) = \mp\frac{\mu_o}{4\pi}\frac{ec}{\beta}\, e^{i\frac{\omega}{v}x - q|y-a|}$$

$$E_z^o(x,y,k_z,\omega) = -\frac{\mu_o}{4\pi}\frac{iec\ k_z}{q\beta}\, e^{i\frac{\omega}{v}x - q|y-a|}$$

$$B_x^o(x,y,k_z,\omega) = 0$$

$$B_y^o(x,y,k_z,\omega) = \frac{\mu_o}{4\pi}\frac{iec\ k_z}{q}\, e^{i\frac{\omega}{v}x - q|y-a|}$$

$$B_z^o(x,y,k_z,\omega) = \mp\frac{\mu_o}{4\pi}\, e\, e^{i\frac{\omega}{v}x - q|y-a|} \qquad (2.17)$$

In the above the upper and lower signs refer to the half-spaces $y < a$ and $y > a$ respectively. It is easy to see that $\underline{E}^o \cdot \underline{B}^o = 0$.

2.2 Treatment of the Plates

We now turn to expressing the induced 4-potential A_μ in terms of the induced current density j_μ . Unlike the case with the point charge, the form of j_μ is not known. But much information about it can be derived from the geometry of our system. As in the last section, let us perform Fourier transformations on the z,t coordinates:

$$j_\mu(x,y,z,t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} j_\mu(x,y,k_z,\omega)\, e^{ik_z z - i\omega t}\, dk_z d\omega \qquad (2.18)$$

With the help of Dirac's delta-function we can express $j_\mu(x,y,k_z,\omega)$ in the form of an infinite sum of surface current densities induced on the plates:

$$j_\mu(x,y,k_z,\omega) = \sum_{n=-\infty}^{\infty} j_{\mu n}(y,k_z,\omega)\ \delta(x-nd) \qquad (2.19)$$

where $j_{\mu n}(y,k_z,\omega)$ is the surface current density on the plate at $x = nd$.

We now make use of the periodicity of our system of plates to derive a relation between the surface current densities induced on two different plates. Suppose we have a relevant physical quantity $Q_n(y,z,t)$ defined on the plate at $x = nd$. It is easily shown that

$$Q_n(y,z,\frac{nd}{v}) = Q_o(y,z,0) \qquad (2.20)$$

This equality asserts that the value of the quantity Q measured on the plate at $x = 0$ at $t = 0$ is the same as that measured on the plate at $x = nd$ at a later time $t = nd/v$. For at $t = 0$ the charged particle is above the edge of the plate at $x = 0$. After a time $t = nd/v$ it is above the edge of the plate at $x = nd$. Thus at $t = nd/v$ the relation of the charged particle to the plate at $x = nd$ is identical to its relation to the plate at $x = 0$ at $t = 0$. Since any relevant physical quantity on the plates is produced by the passage of the charged particle, equation 2.20 follows. In particular

$$j_{\mu n}(y,z,\frac{nd}{v}) = j_{\mu 0}(y,z,0) \tag{2.21}$$

Writing

$$j_{\mu n}(y,z,t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} j_{\mu n}(y,k_z,\omega)\, e^{ik_z z - i\omega t}\, dk_z d\omega \tag{2.22}$$

we obtain from 2.21 the relation

$$j_{\mu n}(y,k_z,\omega) = e^{i\frac{\omega}{v}nd}\, j_{\mu 0}(y,k_z,\omega) \tag{2.23}$$

Thus all the surface currents differ from that on the plate at $x = 0$ by a phase factor only.

Turning to the induced potential we first write

$$A_\mu(x,y,z,t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} A_\mu(x,y,k_z,\omega)\, e^{ik_z z - i\omega t}\, dk_z d\omega \tag{2.24}$$

In analogy to 2.6 $A_\mu(x,y,k_z,\omega)$ and $j_\mu(x,y,k_z\omega)$ are connected by the equation

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + p^2\right) A_\mu(x,y,k_z\omega) = - \mu_o j_\mu(x,y,k_z,\omega) \tag{2.25}$$

The particular solution is

$$A_\mu(x,y,k_z,\omega) = \frac{\mu_o}{4\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} i\pi H_o^{(1)}[p\sqrt{(x-x')^2(y-y')^2}]$$

$$\cdot\, j_\mu(x',y',k_z,\omega)\, dx'dy' \tag{2.26}$$

Again the complementary solution is not needed. Substituting 2.23 into 2.19 and carrying out the x'-integration in 2.26, we easily get

$$A_\mu(x,y,k_z,\omega) = \frac{\mu_o}{4\pi} \sum_{n=-\infty}^{\infty} e^{i\frac{\omega}{v}nd} \int_{-\infty}^{\infty} i\pi H_o^{(1)}[p\sqrt{(x-nd)^2+(y-y')^2}] \cdot j_{\mu 0}(y',k_z,\omega)\, dy' \tag{2.27}$$

In the present form the summation cannot be performed since n occurs in the argument of the Hankel function. For the purpose of bringing n out we use the following device. First let

$$j_{\mu 0}(y,k_z,\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} j_{\mu 0}(k_y,k_z,\omega)\, e^{ik_y y}\, dk_y \tag{2.28}$$

Then

$$A_\mu(x,y,k_z,\omega) = \frac{\mu_o}{4\pi} \sum_{n=-\infty}^{\infty} e^{i\frac{\omega}{v}nd} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} j_{\mu 0}(k_y,k_z,\omega) \cdot \left\{ \int_{-\infty}^{\infty} i\pi H_o^{(1)}[p\sqrt{(x-nd)^2+(y-y')^2}]\, e^{ik_y y'}\, dy' \right\} dk_y \tag{2.29}$$

In analogy to 2.13 we get

$$\int_{-\infty}^{\infty} i\pi H_o^{(1)}[p\sqrt{(x-nd)^2+(y-y')^2}]\, e^{ik_y y'}\, dy' = 2\pi\, e^{ik_y y} \frac{e^{iw|x-nd|}}{-iw} \tag{2.30}$$

where we have defined

$$w = \sqrt{p^2 - k_y^2} \quad , \qquad \text{Im } w > 0 \tag{2.31}$$

Thus 2.29 becomes

$$A_\mu(x,y,k_z,\omega) = \frac{\mu_0}{4\pi}\sqrt{2\pi}\; i \int_{-\infty}^{\infty} j_{\mu 0}(k_y,k_z,\omega)$$

$$\cdot \sum_{n=-\infty}^{\infty} e^{i\frac{\omega}{v}nd + iw|x-nd|} \frac{e^{ik_y y}}{w}\, dk_y \tag{2.32}$$

The summation can now be carried out since n occurs only in the exponential factor.

Consider the infinite sum

$$S = \sum_{n=-\infty}^{\infty} e^{i\frac{\omega}{v}nd + iw|x-nd|}$$

Assume $md < x < (m+1)d$ for some integer m. Then

(i) for $n \leq m$, $|x - nd| = x - nd$;

(ii) for $n \geq m+1$, $|x - nd| = nd - x$.

We therefore split S into two partial sums:

$$S = \sum_{n=-\infty}^{m} e^{iwx + i(\frac{\omega}{v}d - wd)n}$$

$$+ \sum_{n=m+1}^{\infty} e^{-iwx + i(\frac{\omega}{v}d + wd)n}$$

Now we can easily show that

$$\sum_{n=-\infty}^{m} e^{iwx + i(\frac{\omega}{v}d - wd)n} = \frac{e^{i\frac{\omega}{v}md} \; e^{iw[x-(m+1)d]}}{e^{-iwd} - e^{-i\frac{\omega}{v}d}}$$

$$\sum_{n=m+1}^{\infty} e^{-iwx + i(\frac{\omega}{v}d + wd)n} = \frac{e^{i\frac{\omega}{v}(m+1)d} \; e^{-iw[x-md]}}{e^{-iwd} - e^{i\frac{\omega}{v}d}}$$

Combining the two partial sums we get

$$S = i\, e^{i\frac{\omega}{v}md} \; \frac{\sin w[x - (m+1)d] - e^{i\frac{\omega}{v}d} \sin w[x-md]}{\cos wd - \cos \frac{\omega}{v} d}$$

2.32 finally reduces to

$$A_\mu(x,y,k_z,\omega) = - \frac{\mu_o}{4\pi} \sqrt{2\pi}\, e^{i\frac{\omega}{v}md} \int_{-\infty}^{\infty} \frac{j_{\mu 0}(k_y,k_z,\omega)}{w}$$

$$\frac{\sin w[x - (m+1)d] - e^{i\frac{\omega}{v}d} \sin w[x-md]}{\cos wd - \cos \frac{\omega}{v} d} \, e^{ik_y y} \, dk_y \qquad (2.33)$$

We recall this expression holds for $md < x < (m+1)d$. We notice that, like the induced current density in 2.23, $A_\mu(x,y,k_z,\omega)$ has the phase factor $e^{i\frac{\omega}{v}md}$. In this final form the induced potential depends entirely on the Fourier transform of the surface current density on one single plate at $x = 0$.

2.3 Boundary Values of the Induced Potentials

In the previous two sections we derived expressions for the Fourier transforms of the electromagnetic potentials of the moving point charge and those of the current densities induced on the plates. We recall that in arriving at 2.16 we used mainly the outgoing wave condition at infinity, and in arriving at 2.33 we employed the periodic property of our system. In the present section the main tool will be the conditions on the total electromagnetic fields at the boundary of a conductor.

In our case the boundary conditions on the electromagnetic fields assume the following form

$$E_y^{total} = E_y^o + E_y = 0$$

$$E_z^{total} = E_z^o + E_z = 0$$

$$H_x^{total} = H_x^o + H_x = 0 \qquad (2.34)$$

at $x = md$, $m = 0, \pm 1, \pm 2, \cdots$. Here $\underline{E}^o$, $\underline{H}^o$ are the fields of the charged particle and $\underline{E}, \underline{H}$ those of the induced currents. The first two equations express the vanishing of the tangential component of the total electric field on the surface of a metallic plate, while the third equation expresses the vanishing of the normal component of the total magnetic field. We must keep in mind that 2.34 holds only for $y < 0$. For $y > 0$ another set of boundary conditions must be employed. This is supplied by the fact that no induced currents exist outside the plates. Hence for $y > 0$,

$$j_{\mu m}(y,k_z,\omega) = 0 \qquad (2.35)$$

where $m = 0, \pm 1, \pm 2, \cdots$. By 2.23, equation 2.35 is equivalent to

$$j_{\mu o}(y,k_z,\omega) = 0 \qquad (2.36)$$

Equations 2.34 and 2.36 then constitute a set of integral equations for the unknown components of $j_{\mu o}(k_y,k_z,\omega)$. This can be seen as follows: the fields $\underline{E}^o,\underline{H}^o$ are known through 2.17; the induced fields $\underline{E},\underline{H}$ are derivable from 2.33 according to the relations in 2.2; substitution of the fields into 2.34 yields a set of inhomogeneous integral equations for $j_{\mu o}(k_y,k_z,\omega)$ valid for $y < 0$. For $y > 0$ the combination of 2.28 and 2.36 results in a set of homogeneous integral equations for $j_{\mu o}(k_y,k_z,\omega)$.

However, a difficulty arises in this formulation. Since each field component depends on two components of the 4-potential, the inhomogeneous integral equations derived from 2.34 will involve two unknown components of $j_{\mu o}(k_y,k_z,\omega)$ in one equation. This makes the system of integral equations difficult to solve. One would like to find a method to separate the unknowns so that each equation would contain one unknown quantity only. This suggests that we work directly with the 4-potential, since by 2.33 each component of the potential depends on one current density component only. The first step in our new approach will be to translate the boundary conditions on the fields, as given by 2.34, into boundary conditions on the potentials. As will be seen in the following, this translation is possible due to the fact that we have plane boundaries.

Let us start out with the Lorentz condition and the relation between the electric field and the potentials:

$$\underline{E} = - \nabla \phi - \frac{\partial \underline{A}}{\partial t}$$

$$\nabla \cdot \underline{A} + \frac{1}{c^2} \frac{\partial \phi}{\partial t} = 0$$

These relations hold separately for both the induced potentials and the potentials of the source. Let us take the case of the induced 4-potential and recall that its x-component is zero. Then the above relations become:

$$E_y(x,y,k_z,\omega) = - \frac{\partial}{\partial y} \phi(x,y,k_z,\omega) + i\omega A_y(x,y,k_z,\omega)$$

$$E_z(x,y,k_z,\omega) = - i k_z \phi(x,y,k_z,\omega) + i\omega A_z(x,y,k_z,\omega) \qquad (2.37)$$

$$i \frac{\omega}{c^2} \phi(x,y,k_z,\omega) = \frac{\partial}{\partial y} A_y(x,y,k_z,\omega) + i k_z A_z(x,y,k_z,\omega)$$

In the following the analogous expression for $E_x(x,y,k_z,\omega)$ is not needed and is therefore suppressed. From these we easily obtain

$$\frac{\partial}{\partial y} E_y(x,y,k_z,\omega) + i k_z E_z(x,y,k_z,\omega)$$

$$= - \left(\frac{\partial^2}{\partial y^2} - k_z^2\right) \phi(x,y,k_z,\omega)$$

$$+ i\omega \left[\frac{\partial}{\partial y} A_y(x,y,k_z,\omega) + i k_z A_z(x,y,k_z,\omega) \right]$$

The expression inside the bracket on the right-hand side is just the right-hand side of the Lorentz condition in 2.37. Hence

$$\frac{\partial}{\partial y} E_y(x,y,k_z,\omega) + i\, k_z\, E_z(x,y,k_z,\omega)$$

$$= - \left(\frac{\partial^2}{\partial y^2} + p^2\right) \emptyset(x,y,k_z,\omega) \qquad (2.38)$$

where, as in 2.9,

$$p = \sqrt{k^2 - k_z^2} \quad , \quad \text{Im } p > 0 \quad .$$

Equation 2.38 holds for all points in space. Suppose now we choose a point on one of the plates, say the one at $x = md$. On this plate the electric fields $\underline{E}^o$ and $\underline{E}$ are related by the boundary conditions 2.34

$$E_y^o + E_y = 0$$

$$E_z^o + E_z = 0$$

These conditions provide a means to express the boundary values of A_μ in terms of those of A_μ^o through the application of 2.38. From 2.34

$$\frac{\partial}{\partial y} E_y(md,y,k_z,\omega) + i\, k_z\, E_z(md,y,k_z,\omega)$$

$$= - \frac{\partial}{\partial y} E_y^o(md,y,k_z,\omega) - i\, k_z\, E_z^o(md,y,k_z,\omega)$$

Using 2.38 we obtain

$$\left(\frac{\partial^2}{\partial y^2} + p^2\right) \emptyset(md,y,k_z,\omega)$$

$$= \frac{\partial}{\partial y} E_y^o(md,y,k_z,\omega) + i\, k_z\, E_z^o(md,y,k_z,\omega)$$

Or expressing the field $\underline{E}^o$ in terms of the potentials, we get

$$(\frac{\partial^2}{\partial y^2} + p^2)\ \phi(md,y,k_z,\omega) = (-\frac{\partial^2}{\partial y^2} + k_z^2)\ \phi^o(md,y,k_z,\omega) \tag{2.39}$$

where use has been made of the fact that $A_y^o = A_z^o = 0$. We therefore see that on the plates the induced scalar potential satisfies a second-order differential equation. The right-hand side of the equation can be easily evaluated with the help of 2.16, keeping in mind that we consider the case $y < 0$, $x = md$. We finally obtain

$$(\frac{\partial^2}{\partial y^2} + p^2)\ \phi(md,y,k_z,\omega) = -\frac{\mu_o}{4\pi}\frac{ec\ \alpha^2 k^2}{q\beta}\ e^{i\frac{\omega}{v} md + q(y - a)} \tag{2.40}$$

The solution of this equation is

$$\phi(md,y,k_z,\omega) = c_{m1}\ e^{ipy} + c_{m2}\ e^{-ipy} - \frac{\mu_o}{4\pi}\frac{ec}{q\beta}\ \frac{\alpha^2 k^2}{p^2 + q^2}\ e^{i\frac{\omega}{v} md + q(y - a)} \tag{2.41}$$

Here c_{m1}, c_{m2} are constants of integration which may depend on md, k_z and ω.

Furthermore we note that since $\phi(md,y,k_z,\omega)$ is a quantity defined on the mth plate, it must satisfy a periodicity condition similar to 2.23. This infers that the constants of integration must be of the form

$$c_{m1} = c_1 \, e^{i\frac{\omega}{v} md}$$

$$c_{m2} = c_2 \, e^{i\frac{\omega}{v} md}$$

where c_1, c_2 may depend on k_z and ω . We can then write

$$\phi(md,y,k_z,\omega) = e^{i\frac{\omega}{v} md} \Big[c_1 e^{ipy} + c_2 e^{-ipy} - \frac{\mu_o}{4\pi} \frac{ec\,\alpha^2\beta}{q} e^{q(y-a)} \Big] \qquad (2.42)$$

in analogy to 2.23. Here use has been made of the identity

$$p^2 + q^2 = (\frac{\omega}{v})^2 = (\frac{k}{\beta})^2$$

With the derivation of 2.42 we have succeeded in obtaining the boundary value of the induced scalar potential for $x = md$, $y < 0$.

We now turn to obtaining the corresponding boundary values of the other non-zero components of the induced 4-potential. These can be calculated with the help of 2.42 and the boundary condition in the form of 2.34. From 2.34 and 2.37 we have

$$- \frac{\partial}{\partial y} \phi(md,y,k_z,\omega) + i\,\omega\, A_y(md,y,k_z,\omega) = E_y(md,y,k_z,\omega)$$

$$= - E_y^o(md,y,k_z,\omega)$$

$$- ik_z\phi(md,y,k_z,\omega) + i\,\omega\, A_z(md,y,k_z,\omega) = E_z(md,y,k_z,\omega)$$

$$= - E_z^c(md,y,k_z,\omega) \qquad .$$

We immediately obtain expressions for $A_y(md,y,k_z,\omega)$ and $A_z(md,y,k_z,\omega)$ since the other quantities in the above equations are already known. We here summarize the boundary conditions in this section as follows:

$$A_y(md,y,k_z,\omega) = e^{i\frac{\omega}{v}md}\Big[c_1 \frac{p}{\omega} e^{ipy} - c_2 \frac{p}{\omega} e^{-ipy} - \frac{\mu_o}{4\pi}\frac{iec\beta}{\omega} e^{q(y-a)}\Big]$$

$$A_z(md,y,k_z,\omega) = e^{i\frac{\omega}{v}md}\Big[c_1 \frac{k_z}{\omega} e^{ipy} + c_2 \frac{k_z}{\omega} e^{-ipy} + \frac{\mu_o}{4\pi}\frac{ec\beta\, k_z}{q\omega} e^{q(y-a)}\Big]$$

$$\emptyset(md,y,k_z,\omega) = e^{i\frac{\omega}{v}md}\Big[c_1 e^{ipy} + c_2 e^{-ipy} - \frac{\mu_o}{4\pi}\frac{ec\,\alpha^2\beta}{q} e^{q(y-a)}\Big] \qquad (2.43)$$

These expressions hold for $y < 0$. We also recall that A_x is identically zero. As a check we see that 2.43 satisfies the Lorentz condition.

At this stage we have brought in two constants of integration c_1 and c_2 which are so far unknown. In the next chapter they will be determined by imposing the outgoing wave condition and the requirement that the y-component of the induced current density at the edges of the plates be zero.

2.4 Derivation of the Integral Equations

We are now in the position to combine the results in the previous three sections to derive integral equations for our problem. With the help of equations 2.33, 2.36 and 2.43 we can at once write down the integral equations we look for. Let us first set $x = md$ in 2.33. Then

$$A_\mu(md,y,k_z,\omega) = \frac{\mu_o}{4\pi} \sqrt{2\pi}\, e^{i\frac{\omega}{v} md} \int_{-\infty}^{\infty} \frac{\sin wd}{w} \frac{1}{\cos wd - \cos \frac{\omega}{v} d} \cdot j_{\mu o}(k_y,k_z,\omega)\, e^{i k_y y}\, dk_y \tag{2.44}$$

Equating the expressions for $A_\mu(md,y,k_z,\omega)$ in 2.43 and 2.44 we obtain

$$\int_{-\infty}^{\infty} K(k_y)\, j_{yo}(k_y,k_z,\omega)\, e^{ik_y y} dk_y$$

$$= c_1' \frac{p}{\omega} e^{ipy} - c_2' \frac{p}{\omega} e^{-ipy} - \frac{ie\, c\beta}{\omega \sqrt{2\pi}} e^{q(y - a)}$$

$$\int_{-\infty}^{\infty} K(k_y)\, j_{zo}(k_y,k_z,\omega)\, e^{ik_y y}\, dk_y$$

$$= c_1' \frac{k_z}{\omega} e^{ipy} + c_2' \frac{k_z}{\omega} e^{-ipy} + \frac{ec\beta\, k_z}{q\omega \sqrt{2\pi}} e^{q(y - a)}$$

$$\int_{-\infty}^{\infty} K(k_y)\, c^2 \rho_o(k_y,k_z,\omega)\, e^{ik_y y}\, dk_y$$

$$= c_1'\, e^{ipy} + c_2'\, e^{-ipy} - \frac{ec\ \alpha^2\beta}{q\ \sqrt{2\pi}}\, e^{q(y - a)} \qquad (2.45)$$

where the kernel $K(k_y)$ is defined by

$$K(k_y) = \frac{\sin wd}{w}\ \frac{1}{\cos wd - \cos \frac{\omega}{v} d} \qquad (2.46)$$

For brevity we have also defined

$$c_1 = c_1' \frac{\mu_o}{4\pi} \sqrt{2\pi}$$

$$c_2 = c_2' \frac{\mu_o}{4\pi} \sqrt{2\pi}$$

The equations 2.45 hold only for $y < 0$. They are not sufficient to determine the unknown quantities $j_{\mu o}(k_y,k_z,\omega)$. We must supplement them by other conditions holding for $y > 0$. These conditions are given by 2.36 which says that there are no induced currents in the upper half-space $y > 0$; that is

$$j_{\mu o}(y,k_z,\omega) = 0 \quad , \quad y > 0 .$$

Or, expressed in terms of $j_{\mu o}(k_y,k_z,\omega)$

$$\int_{-\infty}^{\infty} j_{\mu o}(k_y,k_z,\omega)\, e^{ik_y y}\, dk_y = 0 \quad , \ y > 0 \qquad (2.47)$$

Equations 2.45 and 2.47 constitute three independent pairs of integral equations in the three unknowns $j_{\mu o}(k_y,k_z,\omega)$, $\mu = 2,3,4$, each pair containing one unknown only. Each pair has the following form

$$\int_{-\infty}^{\infty} K(k_y)\, j_{\mu o}(k_y,k_z,\omega)\, e^{ik_y y}\, dk_y = g_\mu(y) \;, \quad y < 0$$

$$\int_{-\infty}^{\infty} j_{\mu o}(k_y,k_z,\omega)\, e^{ik_y y}\, dk_y = 0 \qquad\qquad , \quad y > 0$$

Such a pair is called a dual integral equation in the sense that the unknown satisfies one equation for one range of value of the parameter y , and another equation for another range (11).

To conclude this chapter we remark that by utilizing the fact that we have plane boundaries, we have succeeded in obtaining the boundary values of the induced potentials on a plate. This enables us to formulate our problem in terms of three relatively simple dual integral equations. In the next chapter these equations will be solved by the Wiener-Hopf method.

III. SOLUTION OF THE INTEGRAL EQUATIONS

In this chapter we are concerned with the solution of the dual integral equations given by 2.45 and 2.47. This is done through the application of the Wiener-Hopf method (12). In this method the first step is to convert the dual integral equations into algebraic equations, holding within a horizontal strip in the complex plane. By examining the analytic properties of each term of the algebraic equations we then analytically continue the terms into the upper and lower half planes. The solutions of the algebraic equations are obtained through the imposition of definite asymptotic behaviors on the unknowns.

3.1 Derivation of the Wiener-Hopf Equation

The dual integral equations we derived in the last chapter are of the general form

$$\int_{-\infty}^{\infty} K(k_y)\, f(k_y)\, e^{ik_y y}\, dk_y = g(y) \,, \quad y < 0$$

$$\int_{-\infty}^{\infty} f(k_y)\, e^{ik_y y}\, dk_y = 0 \,, \qquad y > 0 \tag{3.1}$$

Here $f(k_y)$ represents any component of the induced current density $j_{\mu o}(k_y, k_z, \omega)$ and $g(y)$ is a known function of y. The kernel $K(k_y)$ is given by 2.46.

The first equation of 3.1 can be easily converted into an algebraic equation in k_y. Let us write it in the following form

$$\int_{-\infty}^{\infty} K(k_y)\, f(k_y)\, e^{ik_y y}\, dk_y = \begin{cases} g(y) & , \quad y < 0 \\ h(y) & , \quad y > 0 \end{cases} \tag{3.2}$$

In this equation the function $h(y)$ is unknown; but $g(y)$ is given explicitly by 2.45 and has the general form

$$g(y) = A\, e^{ipy} + B\, e^{-ipy} + C\, e^{qy}$$

where A, B, C are independent of y .

Taking the Fourier inversion of 3.2 we obtain

$$K(k_y)\, f(k_y) = \frac{1}{2\pi} \int_{0}^{\infty} h(y)\, e^{-ik_y y}\, dy + \frac{1}{2\pi} \int_{-\infty}^{0} g(y)\, e^{-ik_y y}\, dy \tag{3.3}$$

Since the form of $h(y)$ is not known, let us simply define

$$h(k_y) = \frac{1}{2\pi} \int_{0}^{\infty} h(y)\, e^{-ik_y y}\, dy \tag{3.4}$$

Turning now to the second term on the right-hand side of 3.3, we find

$$\frac{1}{2\pi} \int_{-\infty}^{0} g(y)\, e^{-ik_y y}\, dy$$

$$= \frac{1}{2\pi} \int_{-\infty}^{0} \left[A\, e^{ipy} + B\, e^{-ipy} + C\, e^{qy} \right] e^{-ik_y y}\, dy$$

$$= \frac{1}{2\pi} \left[\frac{-i\,A}{p - k_y}\, e^{i(p-k_y)y} + \frac{i\,B}{p + k_y}\, e^{-i(p+k_y)y} \right]_{-\infty}^{0} + \frac{1}{2\pi}\, \frac{C}{q - ik_y} \tag{3.5}$$

For real values of p the expression inside the bracket is undefined at the lower limit $y = -\infty$. In a situation like this the common practice is to give p a small imaginary part. This imaginary part is allowed to tend to zero after the evaluation of the expression.

We now wish to give a physical argument for the origin of this imaginary part and to show that its sign is positive. Hitherto we have considered our system to be immersed in vacuum. Suppose that we replace the vacuum by a slightly dissipative medium. Then instead of equation 2.4, the 4-potential satisfies a wave equation with damping:

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \frac{\varepsilon}{c^2}\frac{\partial}{\partial t}\right) A_\mu(x,y,z,t) = -\mu_o j_\mu(x,y,z,t) \qquad (3.6)$$

where ε is the damping coefficient greater than zero. The Lorentz condition 2.3 is now modified to

$$\nabla \cdot \underline{A} + \frac{1}{c^2}\frac{\partial \phi}{\partial t} + \frac{\varepsilon}{c^2}\phi = 0 \qquad (3.7)$$

An example of a damping medium is a slightly conducting material. The damping coefficient is then given by

$$\varepsilon = c^2 \mu\sigma \qquad (3.8)$$

where σ is the electrical conductivity of the material. If we write

$$A_\mu(x,y,z,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} A_\mu(x,y,z,\omega)\, e^{-i\omega t}\, d\omega \quad ,$$

$$j_\mu(x,y,z,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} j_\mu(x,y,z,\omega)\, e^{-i\omega t}\, d\omega \quad ,$$

the equation for $A_\mu(x,y,z,\omega)$ then reads

$$(\nabla^2 + \frac{\omega^2 + i\omega\varepsilon}{c^2})\, A_\mu(x,y,z,\omega) = -\,\mu_o j_\mu(x,y,z,\omega) \qquad (3.9)$$

This is formally the same as the undamped equation except that the frequency ω is given a small positive imaginary part; that is,

$$\omega \rightarrow \omega + i\varepsilon \quad , \quad \varepsilon > 0 \qquad (3.10)$$

This imaginary part is passed on to p according to 2.9 :

$$p \rightarrow p + i\varepsilon \quad , \quad \varepsilon > 0 \qquad (3.11)$$

It is understood that after the calculations ε is allowed to tend to zero. In this way we recover the undamped vacuum case. In what follows we will see that the introduction of a nonvanishing positive ε greatly simplifies the mathematical analysis of the problem.

Now let us return to 3.5. Replacing p by $p + i\varepsilon$ we find that the first term inside the bracket diverges at the lower limit $y = -\infty$. To remedy this we must require that $A \equiv 0$. This is equivalent to putting the constant of integration c_1 in 2.42 equal to zero. Hence

$$c_1 = 0 \qquad (3.12)$$

What this means physically is that there is to be no disturbance propagating from $y = -\infty$ toward the edges of the plates. The evaluation of 3.5 is now straightforward, namely

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} g(y)\, e^{-ik_y y}\, dy = \frac{1}{2\pi}\,\frac{iB}{p + k_y + i\varepsilon} + \frac{1}{2\pi}\,\frac{C}{q - ik_y} \qquad (3.13)$$

Combining 3.3, 3.4 and 3.13 we get

$$K(k_y)\, f(k_y) = h(k_y) + \frac{1}{2\pi}\,\frac{iB}{p + k_y + i\varepsilon} + \frac{1}{2\pi}\,\frac{C}{q - ik_y} \qquad (3.14)$$

Equation 3.14 is the Wiener-Hopf equation of the problem. It is the algebraic counterpart of the integral equation 3.2. It contains two unknown functions $f(k_y)$ and $h(k_y)$. To solve the equation we need information on the analyticity of these functions. This information is supplied by the second half of the dual integral equation 3.1.

3.2 Analytic Properties of the Wiener-Hopf Equation

In the derivation of 3.14 we considered only real values of k_y . Now we want to check the validity of the equation for complex values of k_y . In other words, we want to continue 3.14 analytically beyond the real axis. For this purpose we need to examine the analytic properties of each term of the equation closely.

Let us go back to the second equation of 3.1

$$\int_{-\infty}^{\infty} f(k_y)\, e^{ik_y y}\, dk_y = 0 , \quad y > 0 \qquad (3.15)$$

An immediate consequence of this equation is that $f(k_y)$ is analytic in the upper half of the complex k_y- plane. In anticipation of this fact we write

$$f(k_y) = f_+(k_y) \tag{3.16}$$

where the plus sign indicates analyticity in the upper k_y plane. This can be seen as follows. Equation 3.15 holds for $y > 0$. Suppose for $y < 0$ the right-hand side of the equation is equal to a function of $y, F(y)$:

$$\int_{-\infty}^{\infty} f_+(k_y)\, e^{ik_y y}\, dk_y = F(y) \quad , \quad y < 0 \tag{3.17}$$

Upon inverting the above equation, we get

$$f_+(k_y) = \frac{1}{2\pi} \int_{-\infty}^{0} F(y)\, e^{-ik_y y}\, dy \tag{3.18}$$

Now let us assume that

$$|F(y)| \sim A\, e^{\tau_1 y} \quad , \quad y \to -\infty$$

where A and τ_1 are real constants with $\tau_1 > 0$. Then from a well-known theorem on the Fourier transform in the complex plane (13) $f_+(k_y)$ is analytic everywhere inside the half plane

$$-\tau_1 < \mathrm{Im}\, k_y < \infty$$

That is, in addition to being analytic in the upper k_y plane, $f_+(k_y)$ is also analytic in a horizontal strip of width τ_1 extending from the real axis into the lower k_y plane. The situation is depicted in Fig. 2. The assumption we made about the asymptotic behavior of

$F(y)$ will be justified later on after the solution of the Wiener-Hopf equation.

In a similar manner we see from 3.4 that since the range of integration is $y > 0$, $h(k_y)$ is finite for $\text{Im } k_y < 0$. It then follows that $h(k_y)$ is analytic in the lower k_y plane. Let us therefore define, in analogy to 3.16,

$$h(k_y) = h_-(k_y) \tag{3.19}$$

where the minus sign indicates analyticity in the lower k_y plane. If, furthermore, we assume that

$$|h(y)| \sim B\, e^{-\tau_2 g} , \quad y \to \infty$$

where B and τ_2 are real constants with $\tau_2 > 0$, we deduce that $h_-(k_y)$ is analytic for

$$-\infty < \text{Im } k_y < \tau_2$$

The situation is depicted in Fig. 2. Again the assumption we made about the asymptotic behavior of $h(y)$ will be justified later on after the solution of the Wiener-Hopf equation.

The analytic properties of the last two terms in 3.14 are obvious. These terms have simple poles off the real axis at $k_y = -p - i\varepsilon$ and $k_y = -iq$. These poles are shown in Fig. 2. As for the kernel we will show in the next section that its only singularities are isolated poles, but that it is analytic within the strip

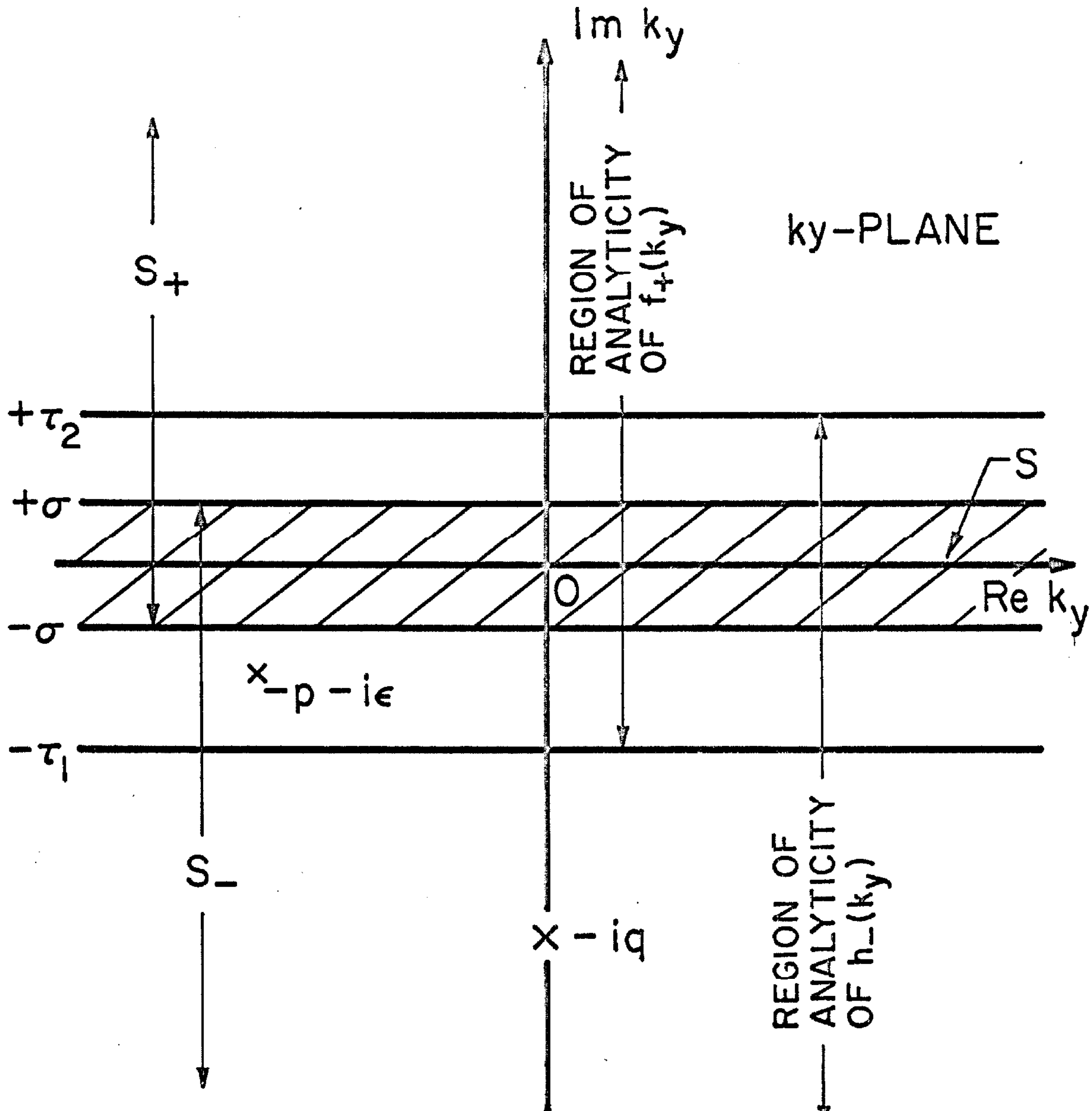

Fig. 2. Singularities and region of analyticity of the Wiener-Hopf equation. Poles of the kernel $K(k_y)$ are not shown.

$$-\varepsilon < \mathrm{Im}\, k_y < \varepsilon$$

where ε is the same small imaginary part of p in 3.11.

The conclusion from all the above discussions is that there exists a horizontal strip S defined by

$$-\sigma < \mathrm{Im}\, k_y < \sigma \quad , \quad 0 < \sigma < \varepsilon \ , \ \tau_1, \ \tau_2$$

within which all the terms in 3.14 are analytic, as shown in Fig. 2. Thus the Wiener-Hopf equation 3.14 holds not only along the real k_y axis, but also within a horizontal strip containing the real axis.

3.3 Factorization of the Kernel

In the last section we have treated one particular feature of the Wiener-Hopf method, namely, the establishment of a common region of analyticity for the terms in the Wiener-Hopf equation. In the present section we are going to deal with another particular feature of the method. This turns out to be the separation of the kernel $K(k_y)$ into two factors, one analytic in the upper half k_y plane and the other in the lower half k_y plane. To be more specific, we want to write $K(k_y)$ in the following form:

$$K(k_y) = \frac{2}{d} \frac{K_+(k_y)\ K_-(k_y)}{k_y^2 + q^2} \tag{3.20}$$

where $K_+(k_y)$ is analytic and has no zeros in the upper k_y plane and $K_-(k_y)$ is analytic and has no zeros in the lower k_y plane.

We recall the definition of the kernel in 2.46

$$K(k_y) = \frac{\sin wd}{w} \frac{1}{\cos wd - \cos \frac{\omega}{v} d}$$

with

$$w = \sqrt{p^2 - k_y^2} \quad , \quad \text{Im } w > 0$$

We first rewrite $K(k_y)$ as follows:

$$K(k_y) = \frac{2}{d} \frac{1}{k_y^2 + q^2} \frac{\sin wd}{wd} \frac{\frac{d}{2}(w - \frac{\omega}{v}) \quad \frac{d}{2}(w + \frac{\omega}{v})}{\sin \frac{d}{2} (w - \frac{\omega}{v}) \quad \sin \frac{d}{2}(w + \frac{\omega}{v})} \tag{3.21}$$

In this form the singularities of the kernel are most easily exhibited. It will be seen that the only singularities are isolated poles.

We may wish to object at this stage that, since in 2.46 $K(k_y)$ contains the radical $w = \sqrt{p^2 - k_y^2}$, we should expect the existence of branch cuts as well as poles in the k_y plane. However, $K(k_y)$ is an even function of w . A change in the sign of w does not produce any change in $K(k_y)$. Thus the branch behavior is only apparent. To be more specific, if we expand $K(k_y)$ in a power series near the points $k_y = \pm p$ ($w = 0$), the series will contain only even powers of w which are single valued.

We notice that in 3.21 $K(k_y)$ contains factors of the form $\sin z/z$ which is an entire function. Its zeros are those of $\sin z$. In analytic function theory we know that such an entire function can be written as an infinite product:

$$\frac{\sin z}{z} = \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{n^2 \pi^2}\right)$$

$$= \prod_{n=-\infty}^{\infty}{}' \left(1 - \frac{z}{n\pi}\right) e^{\frac{z}{n\pi}} \qquad (3.22)$$

where the prime over the product sign indicates that the factor with $n = 0$ is to be excluded. In the second line of 3.22 the purpose of the introduction of the exponential factor is to insure the convergence of the product. It can be shown that a necessary and sufficient condition for the absolute convergence of the infinite product

$$\prod_n (1 + a_n)$$

is the absolute convergence of the series (14)

$$\sum_n a_n$$

In 3.22 for n large

$$\left(1 - \frac{z}{n\pi}\right) e^{\frac{z}{n\pi}} \sim 1 - \frac{z^2}{2n^2 \pi^2} \quad , \qquad n \to \infty$$

and the series

$$\sum_n \frac{1}{n^2}$$

converges. Hence the product in 3.22 converges. On the other hand, if the exponential term is absent the product becomes divergent since the series

$$\sum_n \frac{1}{n}$$

diverges.

We first examine the factor $\sin wd/wd$ in 3.21.

$$\frac{\sin wd}{wd} = \prod_{n=1}^{\infty} \left[1 - \left(\frac{wd}{n\pi}\right)^2 \right]$$

$$= \prod_{n=1}^{\infty} \left[1 - \left(\frac{pd}{n\pi}\right)^2 + \left(\frac{k_y d}{n\pi}\right)^2 \right]$$

$$= F(k_y)\, F(-k_y) \tag{3.23}$$

where

$$F(k_y) = \prod_{n=1}^{\infty} \left[\sqrt{1 - \left(\frac{pd}{n\pi}\right)^2} - i\,\frac{k_y d}{n\pi} \right] e^{i\frac{k_y d}{n\pi}} \tag{3.24}$$

Again the exponential factor is inserted to insure convergence. It is easy to see that $F(k_y)$ is analytic and has no zeros in the upper k_y plane or, more precisely, inside the strip

$$-\sigma < \mathrm{Im}\, k_y < \infty$$

provided that the sign of the sqare root is so chosen that

$$\mathrm{Im}\, \sqrt{1 - \left(\frac{pd}{n\pi}\right)^2} < 0 \tag{3.25}$$

Let us first call this strip S_+ (see Fig. 2). The fact that $F(k_y)$ is analytic inside S_+ is obvious. And if $n\pi > pd$ for all n, no factor in 3.24 vanishes inside S_+. On the other hand, if $n\pi < pd$ for some n, then by our convention 3.25

$$\sqrt{1 - (\frac{pd}{n\pi})^2} = -i \sqrt{(\frac{pd}{n\pi})^2 - 1} + \frac{d}{n\pi} \varepsilon$$

Here we recall that $p \to p + i\varepsilon$. Substituting the square root into 3.24 we have

$$F(k_y) = \prod_{n=1}^{\infty} \left[-i \sqrt{(\frac{pd}{n\pi})^2 - 1} - i \frac{d}{n\pi} (k_y + i\varepsilon) \right] e^{i\frac{k_y d}{n\pi}}$$

which again does not vanish inside S_+ since $\varepsilon > \sigma$. In exactly the same manner we can show that $F(-k_y)$ is analytic and has no zeros inside the strip

$$-\infty < \text{Im } k_y < \sigma$$

which we will call S_- (see Fig. 2.)

Turning to the other factors in 3.21 and proceeding in the same way as above we get

$$\frac{\sin \frac{d}{2}(w - \frac{\omega}{v})}{\frac{d}{2}(w - \frac{\omega}{v})} \cdot \frac{\sin \frac{d}{2}(w + \frac{\omega}{v})}{\frac{d}{2}(w + \frac{\omega}{v})}$$

$$= \prod_{n=1}^{\infty} \left[(1 + \frac{\omega d}{2n\pi v})^2 - (\frac{wd}{2n\pi})^2 \right] \left[(1 - \frac{\omega d}{2n\pi v})^2 - (\frac{wd}{2n\pi})^2 \right]$$

$$= \prod_{n=1}^{\infty} \left[\sqrt{(1 + \frac{\omega d}{2n\pi v})^2 - (\frac{pd}{2n\pi})^2} - i \frac{k_y d}{2n\pi} \right] \cdot$$

$$\cdot \left[\sqrt{(1 + \frac{\omega d}{2n\pi v})^2 - (\frac{pd}{2n\pi})^2} + i \frac{k_y d}{2n\pi} \right] \cdot$$

$$\left[\sqrt{(1-\frac{\omega d}{2n\pi v})^2-(\frac{pd}{2n\pi})^2}-i\,\frac{k_y d}{2n\pi}\right]$$

$$\left[\sqrt{(1-\frac{\omega d}{2n\pi v})^2-(\frac{pd}{2n\pi})^2}+i\,\frac{k_y d}{2n\pi}\right]$$

$$= G(k_y)\;G(-k_y) \qquad (3.26)$$

where

$$G(k_y) = \prod_{n=1}^{\infty}\left[\sqrt{(1+\frac{\omega d}{2n\pi v})^2-(\frac{pd}{2n\pi})^2}-i\,\frac{k_y d}{2n\pi}\right]e^{-\frac{d}{2n\pi}(\frac{\omega}{v}-ik_y)}$$

$$\prod_{n=1}^{\infty}\left[\sqrt{(1-\frac{\omega d}{2n\pi v})^2-(\frac{pd}{2n\pi})^2}-i\,\frac{k_y d}{2n\pi}\right]e^{\frac{d}{2n\pi}(\frac{\omega}{v}+ik_y)} \qquad (3.27)$$

If in analogy to 3.25 we adopt the following sign conventions for the square roots

$$\mathrm{Im}\sqrt{(1+\frac{\omega d}{2n\pi v})^2-(\frac{pd}{2n\pi})^2} < 0$$

$$\mathrm{Im}\sqrt{(1-\frac{\omega d}{2n\pi v})^2-(\frac{pd}{2n\pi})^2} < 0 \qquad (3.28)$$

we readily see that $G(k_y)$ is analytic and has no zeros inside S_+ while $G(-k_y)$ possesses the same properties in S_-.

Summarizing 3.21, 3.23 and 3.26, we can write

$$K(k_y) = \frac{2}{d}\,\frac{1}{k_y^2+q^2}\,\frac{F(k_y)}{G(k_y)}\,\frac{F(-k_y)}{G(-k_y)} \qquad (3.29)$$

In this form we may tentatively make the following identifications:

$$K_+(k_y) = \frac{F(k_y)}{G(k_y)}$$

$$K_-(k_y) = \frac{F(-k_y)}{G(-k_y)}$$

Then $K_+(k_y)$ and $K_-(k_y)$ will have the analytic properties quoted at the beginning of this section, namely, $K_+(k_y)$ is analytic and has no zeros in S_+ and $K_-(k_y)$ is analytic and has no zeros in S_-. However, such an identification is not unique. For if $J(k_y)$ is an entire function which does not vanish in the entire k_y plane, the following definitions of $K_+(k_y)$ and $K_-(k_y)$ will still possess the aforementioned analytic properties:

$$K_+(k_y) = \frac{J(k_y)\ F(k_y)}{G(k_y)}$$

$$K_-(k_y) = \frac{F(-k_y)}{J(k_y)\ G(-k_y)} \tag{3.30}$$

It is therefore seen that the mere specification of the analytic properties of the functions $K_+(k_y)$ and $K_-(k_y)$ is not sufficient to fix the forms of these two functions. To determine the unknown factor $J(k_y)$ in 3.30, additional conditions must be applied. These conditions appear in the form of specified asymptotic behaviors of $K_+(k_y)$ and $K_-(k_y)$. We require that $K_+(k_y)$ and $K_-(k_y)$ should behave like some finite powers of k_y as $|k_y| \to \infty$, that is, $K_+(k_y)$ and $K_-(k_y)$ should have algebraic growth as compared with exponential growth. We shall see in the next section that such specified asymptotic behavior is important in the solution of the Wiener-Hopf equation. We now show that the function $J(k_y)$ in 3.30 can be so chosen that this asymptotic behavior is achieved.

Consider the limit $|k_y| \to \infty$. By 3.24

$$F(k_y) \sim \prod_{n=1}^{\infty} \left[1 - i \frac{k_y d}{n\pi} \right] e^{i \frac{k_y d}{n\pi}}$$

$$= \frac{e^{i\gamma \frac{k_y d}{\pi}}}{-i \frac{k_y d}{\pi} \Gamma(-i \frac{k_y d}{\pi})} \qquad (3.31)$$

where $\gamma = 0.5772 \cdots$ is Euler's constant, and we have used the infinite product representation of the gamma function:

$$\frac{1}{\Gamma(z)} = z\, e^{\gamma z} \prod_{n=1}^{\infty} (1 + \frac{z}{n})\, e^{-z/n} \qquad (3.32)$$

Now the asymptotic behavior of the gamma function is given by Stirling's formula

$$\Gamma(z) \sim \sqrt{\frac{2\pi}{z}}\; e^{-z}\, z^z \quad , \qquad |z| \to \infty \qquad (3.33)$$

From 3.31 and 3.33 we obtain

$$F(k_y) \sim \frac{1}{\sqrt{2\pi}} e^{-i(1-\gamma)\frac{k_y d}{\pi}} \left(-i \frac{k_y d}{\pi}\right)^{-\left(-i \frac{k_y d}{\pi} + \frac{1}{2}\right)} \tag{3.34}$$

Proceeding in the same way we get

$$G(k_y) \sim \left\{ \prod_{n=1}^{\infty} \left[1 - i \frac{k_y d}{2n\pi}\right] e^{i \frac{k_y d}{2n\pi}} \right\}^2$$

$$= \left[\frac{e^{i\gamma \frac{k_y d}{2\pi}}}{-i \frac{k_y d}{2\pi} \Gamma\left(-i \frac{k_y d}{2\pi}\right)} \right]^2 \tag{3.35}$$

Or, using Stirling's formula 3.33,

$$G(k_y) \sim \frac{1}{2\pi} e^{-i(1-\gamma)\frac{k_y d}{\pi}} \left(-i \frac{k_y d}{2\pi}\right)^{-\left(-i \frac{k_y d}{\pi} + 1\right)} \tag{3.36}$$

Therefore combining 3.34 and 3.36 we get

$$\frac{F(k_y)}{G(k_y)} \sim \text{constant} \times k_y^{1/2} \; e^{i \frac{k_y d}{\pi} \ln 2} \tag{3.37}$$

Comparing this with 3.30 we see that if we choose

$$J(k_y) = e^{-i \frac{k_y d}{\pi} \ln 2} \tag{3.38}$$

which is a nonvanishing entire function as we required, we can make $K_+(k_y)$ behave like $k_y^{1/2}$ as $|k_y| \to \infty$, that is to say, $K_+(k_y)$ has algebraic growth. Noticing that

$$\frac{1}{J(k_y)} = J(-k_y)$$

we can show in like manner that $K_-(k_y)$ behaves like $k_y^{1/2}$ as $|k_y| \to \infty$.

To sum up the results of this section we have

$$K(k_y) = \frac{2}{d} \frac{K_+(k_y)\, K_-(k_y)}{k_y^2 + q^2}$$

$$K_+(k_y) = \frac{e^{-i \frac{k_y d}{\pi} \ln 2} \prod_{n=1}^{\infty} \left[\sqrt{1 - (\frac{pd}{n\pi})^2} - i \frac{k_y d}{n\pi} \right] e^{i \frac{k_y d}{n\pi}}}{\prod_{n=1}^{\infty} \left[\sqrt{(1 + \frac{\omega d}{2n\pi v})^2} - (\frac{pd}{2n\pi})^2 - i \frac{k_y d}{2n\pi} \right] e^{-\frac{d}{2n\pi}(\frac{\omega}{v} - i k_y)} \cdot \prod_{n=1}^{\infty} \left[\sqrt{(1 - \frac{\omega d}{2n\pi v})^2 - (\frac{pd}{2n\pi})^2} - i \frac{k_y d}{2n\pi} \right] e^{\frac{d}{2n\pi}(\frac{\omega}{v} + i k_y)}}$$

$$K_-(k_y) = K_+(-k_y)$$

$$K_\pm(k_y) \sim \text{constant} \times k_y^{1/2} , \quad |k_y| \to \infty \tag{3.39}$$

3.4 Solution of the Wiener-Hopf Equation

With the establishment of the results in Sections 3.1, 3.2 and 3.3, we are now in the position to solve the Wiener-Hopf equation 3.14:

$$K(k_y)\ f_+(k_y) = h_-(k_y) + \frac{1}{2\pi}\frac{i\,B}{p + k_y + i\varepsilon} + \frac{1}{2\pi}\frac{C}{q - ik_y}$$

Let us first rewrite this equation in the following form:

$$\frac{2}{d}\ \frac{K_+(k_y)\ f_+(k_y)}{k_y + iq} - \frac{(k_y - iq)\ h_-(k_y)}{K-(k_y)}$$

$$= \frac{i}{2\pi}\ \frac{k_y - iq}{K_-(k_y)}\left[\frac{B}{k_y + p + i\varepsilon} + \frac{C}{k_y + iq}\right] \tag{3.40}$$

We notice that on the left-hand side of 3.40 the first term is analytic in S_+ ; the second term is analytic in S_- . As for the right-hand side, we only know that it is analytic in S (see Fig. 2).

The next task we are facing is to separate the right-hand side into two terms: one analytic in S_+ and the other in S_- . For brevity we define

$$\psi(k_y) = \frac{i}{2\pi}\ \frac{k_y - iq}{K_-(k_y)}\left[\frac{B}{k_y + p + i\varepsilon} + \frac{C}{k_y + iq}\right] \tag{3.41}$$

and write

$$\psi(k_y) = \psi_+(k_y) + \psi_-(k_y) \tag{3.42}$$

where $\psi_+(k_y)$ is analytic in S_+ and $\psi_-(k_y)$ is analytic in S_- . To calculate $\psi_+(k_y)$ and $\psi_-(k_y)$ we consider a closed rectangular contour C bounding the region S as shown in Fig. 3. Since $\psi(k_y)$ is analytic within S we can represent it as a contour

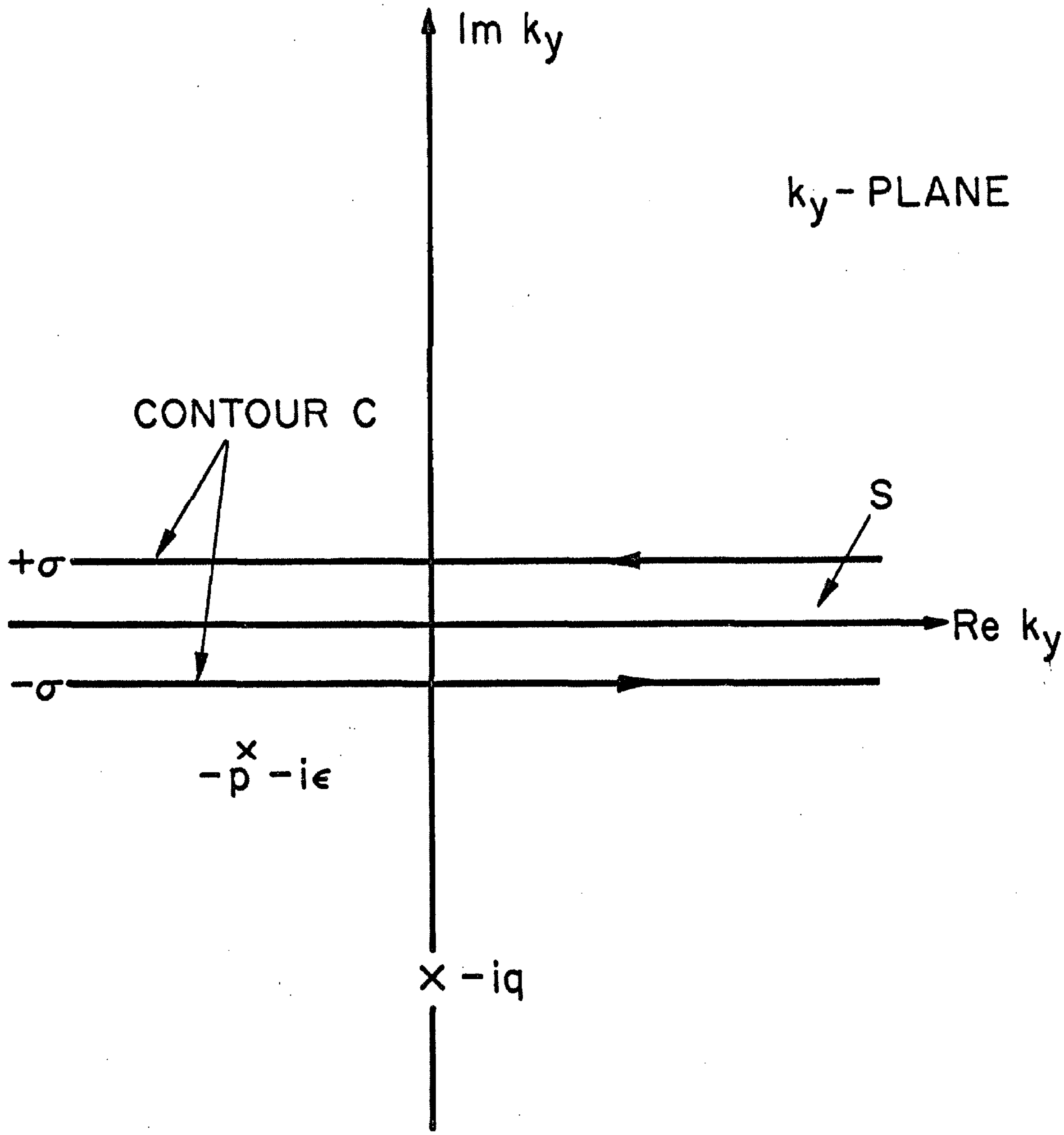

Fig. 3. Contour C used for the separation of the analytic function $\psi(k_y)$.

integral around C . Thus

$$\psi(k_y) = \frac{1}{2\pi i} \oint_c \frac{\psi(k'_y)}{k'_y - k_y} \, dk'_y \tag{3.43}$$

From the asymptotic behavior of $K-(k_y)$ in 3.39 we deduce that $\psi(k_y)$ tends to zero like $k_y^{-1/2}$ as $|k_y| \to \infty$. Hence we can neglect the end sections of the contour C at infinity and 3.43 becomes

$$\psi(k_y) = \frac{1}{2\pi i} \int_{-\infty-i\sigma}^{\infty-i\sigma} \frac{\psi(k'_y)}{k'_y - k_y} \, dk'_y$$

$$+ \frac{-1}{2\pi i} \int_{-\infty+i\sigma}^{\infty+i\sigma} \frac{\psi(k'_y)}{k'_y - k_y} \, dk'_y \tag{3.44}$$

We recall that in the above expression k_y is a point within the contour C , that is, inside the strip S . However, we can actually analytically continue the terms in 3.44 beyond S . Let us examine the first term on the right-hand side of 3.44,

$$\frac{1}{2\pi i} \int_{-\infty-i\sigma}^{\infty-i\sigma} \frac{\psi(k'_y)}{k'_y - k_y} \, dk'_y$$

In the integral $\text{Im } k'_y = -\sigma$; so the integral is well defined if $\text{Im } k_y > -\sigma$. We can therefore continue the integral analytically beyond S into the upper k_y plane. Thus the integral actually defines a function that is analytic in S_+ . In a similar way we can show that the second integral in 3.44 defines a function that is analytic in S_- . Returning to 3.42 we see that we may identify

$\psi_+(k_y)$, $\psi_-(k_y)$ as follows:

$$\psi_+(k_y) = \frac{1}{2\pi i} \int_{-\infty-i\sigma}^{\infty-i\sigma} \frac{\psi(k_y')}{k_y' - k_y} dk_y' \qquad (3.45)$$

$$\psi_-(k_y) = \frac{-1}{2\pi i} \int_{-\infty+i\sigma}^{\infty+i\sigma} \frac{\psi(k_y')}{k_y' - k_y} dk_y'$$

From 3.41 and 3.45 it is straightforward to evaluate $\psi_+(k_y)$. We complete the contour by a large semi circle in the lower k_y plane. The contribution from this semicircle to the integral is zero, since

$$\psi(k_y) \sim A\, k_y^{-1/2} \quad , \qquad |k_y| \to \infty$$

where A is a constant. Our contour will include poles of $\psi(k_y)$ at $k_y = -p - i\varepsilon$ and $k_y = -iq$. The value of $\psi_+(k_y)$ is therefore just the sum of the residues at these poles. Thus

$$\psi_+(k_y) = - \operatorname{Res}\left[\frac{\psi(k_y')}{k_y' - k_y}\right]_{k_y' = -p-i\varepsilon} - \operatorname{Res}\left[\frac{\psi(k_y')}{k_y' - k_y}\right]_{k_y' = -iq}$$

The two minus signs come from the clockwise sense of the contour. Evaluating the residue we get

$$\psi_+(k_y) = \frac{-i}{2\pi}\left[\frac{B}{K - (-p)} \frac{p + iq}{k_y + p + i\varepsilon} + \frac{2iC}{K - (-iq)} \frac{q}{k_y + iq}\right] \quad (3.46)$$

In this final form the analyticity of $\psi_+(k_y)$ in S_+ is explicitly confirmed.

Let us rewrite 3.40 in the following form:

$$\frac{2}{d}\frac{K_+(k_y)\ f_+(k_y)}{k_y + iq} - \psi_+(k_y) = \frac{(k_y - iq)\ h_-(k_y)}{K_-(k_y)} + \psi_-(k_y)$$

Since each term in the equation is analytic in the strip S, each side defines a function $I(k_y)$ that is analytic in S:

$$I(k_y) = \frac{2}{d}\frac{K_+(k_y)\ f_+(k_y)}{k_y + iq} - \psi_+(k_y)$$

$$= \frac{(k_y - iq)\ h_-(k_y)}{K_-(k_y)} + \psi_-(k_y) \qquad (3.47)$$

So far we confine the region of definition of $I(k_y)$ to S. Let us examine the analyticity of $K(k_y)$ in the entire k_y plane. As we have seen, the first line in 3.47 is analytic in S_+. We can regard it as the analytically continued value of $I(k_y)$ in the upper k_y plane. In the same way the second line of 3.47 is analytic in S_-. We can regard it as the analytically continued value of $I(k_y)$ in the lower k_y plane. Thus we see that $I(k_y)$ is analytic over the entire k_y plane. Consequently it is an entire function.

To evaluate $I(k_y)$ we consider the limit $|k_y| \to \infty$. $I(k_y)$ is analytic over the entire k_y plane with the possible exception of the point at infinity which can be a singularity. Let us therefore examine the behavior of $I(k_y)$ at the point at infinity. Now $f_+(k_y)$ is the Fourier transform of a component of the surface current density $j_\mu(y)$. It must approach zero for $|k_y| \to \infty$ if

$j_\mu(y)$ is to be integrable, as is required by physical considerations. The asymptotic behavior of $K_+(k_y)$ and $\psi_+(k_y)$ being already known, the first line of 3.47 shows that $I(k_y)$ is asymptotic to zero in the upper k_y plane. Similarly the second line shows that $I(k_y)$ is asymptotic to zero in the lower k_y plane. Thus $I(k_y)$ is asymptotic to zero in all directions. Since $I(k_y)$ is an entire function and, by the maximum-modulus theorem, the modulus of an entire function attains its maximum at the point at infinity, $I(k_y)$ must be identically zero. Therefore

$$I(k_y) \equiv 0 \tag{3.48}$$

From this we immediately obtain the solution of the Wiener-Hopf equation 3.14

$$f_+(k_y) = \frac{d}{2}\frac{k_y + iq}{K_+(k_y)}\psi_+(k_y)$$

$$= \frac{-id}{4\pi}\frac{1}{K_+(k_y)}\left[\frac{B(p + iq)}{K_-(-p)}\;\frac{k_y + iq}{k_y + p + i\varepsilon} + \frac{2i\;Cq}{K_-(-iq)}\right] \tag{3.49}$$

We see that $f_+(k_y)$ tends to zero like $k_y^{-1/2}$ as $|k_y| \to \infty$, in agreement with our previous assumption.

At this stage $f_+(k_y)$ still contains an unknown constant of integration c_2' which is absorbed in B . To determine what c_2' is we must insert further conditions on our solution. Let us therefore consider the current density component $j_{yo}(k_y,k_z,\omega)$. Comparing 2.45

with 3.1 we see that $j_{yo}(k_y,k_z,\omega)$ can be obtained from 3.49 by setting

$$B = -\, c_2' \, \frac{p}{\omega}$$

$$C = -\, \frac{i\, e\, c\, \beta}{\omega \sqrt{2\pi}} \, e^{-qa}$$

Therefore

$$j_{yo}(k_y,k_z,\omega) = \frac{id}{4\pi\omega} \frac{1}{K_+(k_y)} \left[\frac{c_2' \, p(p + iq)}{K_-(-p)} \; \frac{k_y + iq}{k_y + p + i\varepsilon} - \frac{2e\, c\, \beta\, q}{\sqrt{2\pi}\, K_-(-iq)} \; e^{-qa} \right] \tag{3.50}$$

and

$$j_{yo}(y,k_z,\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} j_{yo}(k_y,k_z,\omega) \; e^{ik_y y} \, dk_y \tag{3.51}$$

The additional condition we are going to use for the determination of the constant of integration c_2' is that no induced current should flow out of the plate. Mathematically this is equivalent to putting

$$j_{yo}(y,k_z,\omega) = 0 \quad , \quad y = 0 \tag{3.52}$$

Or by 3.50 and 3.51

$$\int_{-\infty}^{\infty} \frac{id}{4\pi\omega} \frac{1}{K_+(k_y)} \left[\frac{c_2' \, p(p + iq)}{K_-(-p)} \; \frac{k_y + iq}{k_y + p + i\varepsilon} - \frac{2\, e\, c\, \beta q}{\sqrt{2\pi}\, K_-(-iq)} \; e^{-qa} \right] dk_y = 0 \tag{3.53}$$

From the asymptotic behavior of $K_+(k_y)$ as $|k_y| \to \infty$ we see that the integral diverges unless the expression inside the bracket tends to zero sufficiently rapidly as $|k_y| \to \infty$. This is the case if we adjust c_2' so that

$$c_2' \frac{p(p+iq)}{K_-(-p)} = \frac{2 e c\beta q}{\sqrt{2\pi}\, K_-(-iq)} e^{-qa}$$

With this condition the left-hand side of 3.53 becomes

$$\int_{-\infty}^{\infty} \frac{-iec\beta d}{(2\pi)^{3/2}\omega} \frac{q}{K_-(-iq)} \frac{(p-iq)\, e^{-qa}}{K_+(k_y)\,(k_y+p+i\varepsilon)}\, dk_y$$

which is convergent. The value of the integral is in fact zero in agreement with 3.53. For since $K_+(k_y)$ is analytic and has no zeros in the upper k_y plane, the integrand is analytic in the entire upper k_y plane. Upon completing the contour in the upper k_y plane we readily see that the integral has value zero. Hence we have

$$j_{yo}(k_y,k_z,\omega) = \frac{-iec\beta d}{(2\pi)^{3/2}\omega} \frac{q}{K_-(-iq)} \frac{(p-iq)\, e^{-qa}}{K_+(k_y)\,(k_y+p+i\varepsilon)}$$

To get the other components of the induced currents all we need to do is to take the corresponding values of B and C from 2.45 and substitute them in 3.49, using the calculated value of c_2'. To summarize the results of this section we give the explicit forms of the solutions of the set of integral equations given by 2.45 and 2.47:

$$j_{yo}(k_y,k_z,\omega) = \frac{-i\,e\,c\,\beta\,d}{(2\pi)^{3/2}} \frac{q}{\omega K_-(-iq)} \frac{(p-iq)\,e^{-qa}}{K_+(k_y)\,(k_y+p+i\varepsilon)}$$

$$j_{zo}(k_y,k_z,\omega) = \frac{-\,i\,e\,c\,\beta\,d}{(2\pi)^{3/2}} \frac{k_z}{\omega K_-(-iq)} \frac{e^{-qa}}{K_+(k_y)} \left[\frac{q}{p}\frac{k_y+iq}{k_y+p+i\varepsilon} + i\right]$$

$$c\rho_o(k_y,k_z,\omega) = \frac{-\,i\,e\,c\,\beta\,d}{(2\pi)^{3/2}} \frac{1}{cK_-(-iq)} \frac{e^{-qa}}{K_+(k_y)} \left[\frac{q}{p}\frac{k_y+iq}{k_y+p+i\varepsilon} - i\alpha^2\right] \tag{3.54}$$

As a check we find that these current densities satisfy the equation of continuity. Incidentally we notice in 3.54 that whereas $j_{yo}(y,k_z,\omega)$ vanishes at the edge $y = 0$, $j_{zo}(y,k_z,\omega)$ and $c\rho_o(y,k_z,\omega)$ diverge there. This singular behavior of the current densities at the edge is also present in Sommerfeld's half-plane diffraction problem. We will discuss this point further in the next section.

3.5 Properties of the Solution

In this section we are going to examine the behavior of the solutions in 3.54 in the two limiting cases:

(i) $y \to -\infty$ (asymptotic behavior)

and (ii) $y \to -0$ (edge behavior).

The asymptotic behavior appeared as an assumption in Section 3.2, namely,

$$\left| \int_{-\infty}^{\infty} f_+(k_y)\, e^{ik_y y}\, dk_y \right| \sim A\, e^{\tau_1 y} \quad , \quad y \to -\infty \tag{3.55}$$

where A and τ_1 are real constants with $\tau_1 > \sigma > 0$. Now we wish to justify such an assumption. In the latter case the edge behavior is intimately connected with the uniqueness of the solutions.

In 3.55 $f_+(k_y)$ represents any one of the current density components $j_{yo}(k_y,k_z,\omega)$, $j_{zo}(k_y,k_z,\omega)$ or $c\rho_o(k_y,k_z,\omega)$. From the general form of the current densities in 3.54 we see that the singularities of $f_+(k_y)$ in the k_y plane are poles. In the lower k_y plane it has a pole at $k_y = -p - i\varepsilon$. The other poles are those of $1/K_+(k_y)$. These are located at

$$\text{(i)} \quad k_y = -\sqrt{p^2 - (\tfrac{n\pi}{d})^2} - i\varepsilon \quad , \quad p^2 > (\tfrac{n\pi}{d})^2$$

$$\text{and} \quad \text{(ii)} \quad k_y = -i\sqrt{(\tfrac{n\pi}{d})^2 - p^2} \quad , \quad p^2 < (\tfrac{n\pi}{d})^2$$

where $n = 1,2,3 \cdots$. Thus if we close the contour in 3.55 by a semicircle in the lower k_y plane, the value of the integral will be given by the sum of the residues of the integrand at the poles of $f_+(k_y)$ inside the contour, the contribution of the semicircle to the value of the integral being zero by Jordan's lemma. The y-dependence of the integral is exclusively given by the factor $e^{ik_y y}$. At the poles at

$$k_y = -p - i\varepsilon \quad , \quad -\sqrt{p^2 - (\tfrac{n\pi}{d})^2} - i\varepsilon \quad , \quad -i\sqrt{(\tfrac{n\pi}{d})^2 - p^2}$$

this factor gives respectively contributions proportional to $e^{\varepsilon y}$, $e^{\varepsilon y}$, $e^{\sqrt{(\frac{n\pi}{d})^2 - p^2}\,y}$. These factors immediately confirm the asymptotic behavior in 3.55, since $\varepsilon > \sigma > 0$.

The edge behavior of the induced current density can easily be derived with the help of a well-known result concerning the asymptotic relations between functions and their Fourier transforms. Suppose we consider the component $j_{yo}(y, k_z, \omega)$

$$j_{yo}(y, k_z, \omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} j_{yo}(k_y, k_z, \omega)\, e^{ik_y y}\, dk_y$$

The corresponding inversion formula is

$$j_{yo}(k_y, k_z, \omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{0} j_{yo}(y, k_z, \omega)\, e^{-ik_y y}\, dy \qquad (3.56)$$

The range of integration in 3.56 is from $y = -\infty$ to $y = 0$ since $j_{yo}(y,k_z,\omega) = 0$ for $y > 0$. It can be shown that if

$$j_{yo}(k_y, k_z, \omega) \sim A\, k_y^{-\nu} \quad , \qquad k_y \to \infty$$

where $\nu > 0$, then (15)

$$j_{yo}(y, k_z, \omega) \sim B y^{\nu-1} \quad , \qquad y \to -0$$

From 3.54 we see that $j_{yo}(k_y,k_z,\omega)$, $j_{zo}(k_y,k_z,\omega)$ and $c\rho_o(k_y,k_z,\omega)$ behave respectively like $k_y^{-3/2}$, $k_y^{-1/2}$ and $k_y^{-1/2}$ as $k_y \to \infty$. Hence, according to the above relation $j_{yo}(y,k_z,\omega)$, $j_{zo}(y,k_z,\omega)$ and $c\rho_o(y,k_z,\omega)$ behave like $y^{1/2}$, $y^{-1/2}$ and $y^{-1/2}$ in the neighborhood of the edge $(y = 0)$. Therefore the component of the induced current density that is normal to the edge vanishes at the edge, while the component parallel to the edge diverges like the square root of the inverse distance from the edge. The induced charge density has a similar divergent behavior.

In certain approaches to diffraction problems involving obstacles with sharp edges, the above edge behavior of the induced current density is imposed on the solution as the so-called edge conditions to ensure the uniqueness of the solution. How the edge conditions affect the uniqueness of the solution is a very complicated question. Without going into great mathematical detail, we can describe the situation as follows. The idealization of an infinitely sharp edge results in the appearance of a singularity for the charge density and the parallel component of the current density at the edge. It is precisely this edge singularity which prevents the establishment of a uniqueness proof for the solution of the problem. As a result it is possible to produce more than one solution to the problem every one of which satisfies all the boundary conditions, except that they have singularities of different orders at the edge. Among these solutions the one involving the singularity of lowest possible order is to be taken as representing the solution to the physical problem. This, in fact, rules out any singularities of order greater than $y^{-1/2}$ as $y \to 0$ at the diffracting edge. The physical implication of this chosen order of singularity is that the edge neither radiates nor absorbs energy. We may regard the edge conditions as additional energy conditions which suffice to make the solution of a physical problem unique (16).

The possibility of a multitude of solutions having been seen, we may be inclined to ask why the Wiener-Hopf method apparently yields a unique solution which is, moreover, the correct one judged by the above-mentioned criteria. The answer is that we assume from the outset the possibility of representing the components of the induced current density as convergent Fourier integrals. This precludes them from having singularities of too high an order.

Finally we remark that the achievement of the correct edge behavior for our solution is also a direct consequence of our imposition of algebraic growth for the function $K_+(k_y)$.

IV. INDUCED POTENTIALS AND FIELDS

4.1 Induced Potentials

The central point of the present problem is to solve the set of dual integral equations for the induced current density formulated in Chapter II. In Chapter III these equations were solved and the solutions were given in 3.54. From the induced current density the induced potentials and fields can be obtained by well-known procedures. However, since the results in 3.54 appear as Fourier transforms, additional steps must be taken to perform the inversion transformations as far as possible. In this section we will first perform the k_y inversion to obtain expressions for the induced potentials as functions of x, y, k_z and ω.

Our starting point is 2.33:

$$A_\mu(x,y,k_z,\omega) = -\frac{\mu_c}{4\pi}\sqrt{2\pi}\, e^{i\frac{\omega}{v}md} \int_{-\infty}^{\infty} \frac{j_{\mu o}(k_y,k_z,\omega)}{w}$$

$$\frac{\sin w[x-(m+1)d] - e^{i\frac{\omega}{v}d} \sin w[x-md]}{\cos wd - \cos \frac{\omega}{v} d}\, e^{ik_y y}\, dk_y \,,$$

$$md < x < (m+1)d \,.$$

Using 2.33 and the identity

$$\frac{1}{wK_+(k_y)(\cos wd - \cos \frac{\omega}{v} d)} = \frac{2}{d \sin wd} \; \frac{K_-(k_y)}{k_y^2 + q^2}$$

we write out explicitly

$$A_y(x,y,k_z,\omega) = \frac{\mu_o}{4\pi}\frac{iec\beta}{\pi}\frac{q\ e^{-qa}}{\omega K_-(-iq)}\ e^{i\frac{\omega}{v}md}\ .$$

$$\cdot\int_{-\infty}^{\infty}\frac{p-iq}{k_y+p+i\varepsilon}\ \frac{\sin w[x-(m+1)d] - e^{i\frac{\omega}{v}d}\sin w[x-md]}{\sin wd}\ \frac{K_-(k_y)}{k_y^2+q^2}\ e^{ik_y y}\ dk_y$$

$$A_z(x,y,k_z,\omega) = \frac{\mu_o}{4\pi}\frac{iec\beta}{\pi}\ \frac{k_z\ e^{-qa}}{\omega K_-(-iq)}\ e^{i\frac{\omega}{v}md}$$

$$\cdot\int_{-\infty}^{\infty}\left[\frac{q}{p}\ \frac{k_y+iq}{k_y+p+i\varepsilon}+i\right]\ \frac{\sin w[x-(m+1)d] - e^{i\frac{\omega}{v}d}\sin w[x-md]}{\sin wd}\ \frac{K_-(k_y)}{k_y^2+q^2}$$

$$\times\ e^{ik_y y}\ dk_y$$

$$\phi(x,y,k_z,\omega) = \frac{\mu_o}{4\pi}\frac{iec\beta}{\pi}\frac{e^{-qa}}{K_-(-iq)}\ e^{i\frac{\omega}{v}md}$$

$$\cdot\int_{-\infty}^{\infty}\left[\frac{q}{p}\ \frac{k_y+iq}{k_y+p+i\varepsilon}\ -i\alpha^2\right]\frac{\sin w[x-(m+1)d] - e^{i\frac{\omega}{v}d}\ \sin w[x-md]}{\sin wd}\ \frac{K_-(k_y)}{k_y^2+q^2}$$

$$\times\ e^{ik_y y}\ dk_y$$

$$md < x < (m+1)d \qquad (4.1)$$

To evaluate these integrals we have to treat two cases separately: one for $y < 0$ and the other for $y > 0$. This is clear from physical considerations, since the half-space $y < 0$ is occupied by the metallic plates while the half-space $y > 0$ is, except for the point charge, empty. We would expect to find the behavior of the fields in one half-space quite different from those in the other.

Let us first consider the case $y < 0$; and let us further focus our attention on the space between two particular adjacent plates, that is, we consider the restriction on the x-coordinate

$$md < x < (m+1)\, d$$

Then the contour of the integrals in 4.1 may be closed by a semicircle in the lower k_y plane. By familiar arguments in contour integration the semicircle makes no contribution to the value of the integral. The integral is then determined by the singularities of the integrand inside the contour. These singularities will be seen to be simple poles. Thus the value of the integral is simply the sum of the residues at the poles.

The poles of the integrals in 4.1 arise from three sources, namely, from the factors $k_y^2 + q^2$, $k_y + p + i\varepsilon$ and $\sin wd$ in the denominator of the integrand. From these we find there are poles at

(i) $k_y = -iq$

(ii) $k_y = -p - i\varepsilon$

(iii) $k_y = -\sqrt{p^2 - (\frac{n\pi}{d})^2} - i\varepsilon \ , \qquad n = 1,2,3,\cdots$

How we come to conclusion (iii) can be seen as follows. Recalling that $w = \sqrt{p^2 - k_y^2}$ we see that $\sin wd = 0$ implies that

$$k_y = \pm \sqrt{p^2 - (\frac{n\pi}{d})^2} \ , \qquad n = 1,2,3,\cdots$$

The case $n = 0$ can be discarded here since it is already included in (ii). Giving p a positive imaginary part and taking the minus sign we obtain (iii). The radical $w = \sqrt{p^2 - k_y^2}$ does not give rise to branch cuts since the integrand is an even function of w.

Let us consider these three sets of poles one by one and calculate their separate contributions to the induced potentials. The results are summarized below.

(i) Contribution from pole at $k_y = -iq$:

$$A_y(x,y,k_z,\omega) = \frac{u_o}{4\pi} \frac{ec\beta}{q} \left[\frac{-iq}{\omega}\right] e^{i\frac{\omega}{v} x + q(y-a)}$$

$$A_z(x,y,k_z,\omega) = \frac{\mu_o}{4\pi} \frac{ec\beta}{q} \left[\frac{k_z}{\omega}\right] e^{i\frac{\omega}{v} x + q(y-a)}$$

$$\phi(x,y,k_z,\omega) = \frac{\mu_o}{4\pi} \frac{ec\beta}{q}[-\alpha^2] \, e^{i\frac{\omega}{v} z + q(y-a)} \tag{4.2}$$

(ii) Contribution from pole at $k_y = -p - i\varepsilon$:

$$A_y(x,y,k_z,\omega) = \frac{\mu_o}{4\pi} \frac{2\, ec\beta\, e^{-qa}}{K_-(-iq)} \left[\frac{q}{\omega}(p - iq)\right] e^{i\frac{\omega}{v} md}$$

$$\cdot \frac{[x-(m+1)d] - e^{i\frac{\omega}{v} d}[x-md]}{d} \; \frac{K_-(-p)}{p^2+q^2} \, e^{-ipy}$$

$$A_z(x,y,k_z,\omega) = \frac{\mu_o}{4\pi} \frac{2\, ec\,\beta\, e^{-qa}}{K_-(-iq)} \left[\frac{k_z}{\omega} \frac{q}{p} (-p + iq)\right] e^{i\frac{\omega}{v} md}$$

$$\cdot \frac{[x-(m+1)d] - e^{i\frac{\omega}{v} d} [x-md]}{d} \frac{K_-(-p)}{p^2 + q^2} e^{-ipy}$$

$$\phi(x,y,k_z,\omega) = \frac{\mu_o}{4\pi} \frac{2\, ec\,\beta\, e^{-qa}}{K_-(-iq)} \left[\frac{q}{p} (-p + iq)\right] e^{i\frac{\omega}{v} md}$$

$$\cdot \frac{[x-(m+1)d] - e^{i\frac{\omega}{v} d} [x-md]}{d} \frac{K_-(-p)}{p^2 + q^2} e^{-ipy} \tag{4.3}$$

(iii) Contribution from pole at $k_y = -\sqrt{p^2 - (\frac{n\pi}{d})^2} - i\varepsilon$:

$$A_y(x,y,k_z,\omega) = \frac{\mu_o}{4\pi} \frac{2ec\,\beta\, e^{-qa}}{K_-(-iq)} \frac{n\pi}{d^2\, k_{yn}} \left[\frac{q}{\omega} \frac{p - iq}{k_{yn} + p}\right] e^{i\frac{\omega}{v} md}$$

$$\cdot (-1)^{n(m+1)} \left[(-1)^{n-1} + e^{i\frac{\omega}{v} d}\right] \frac{K_-(k_{yn})}{k_{yn}^2 + q^2} e^{ik_{yn}y} \sin \frac{n\pi}{d} x$$

$$A_z(x,y,k_z,\omega) = \frac{\mu_o}{4\pi} \frac{2ec\,\beta\, e^{-qa}}{K_-(-iq)} \frac{n\pi}{d^2 k_{yn}} \frac{k_z}{\omega} \left[\frac{q}{p} \frac{k_{yn} + iq}{k_{yn} + p} + i\right] e^{i\frac{\omega}{v} md}$$

$$\cdot (-1)^{n(m+1)} \left[(-1)^{n-1} + e^{i\frac{\omega}{v} d} \right] \frac{K_-(k_{yn})}{k_{yn}^2 + q^2} e^{ik_{yn}y} \sin \frac{n\pi}{d} x$$

$$\phi(x,y,k_z,\omega) = \frac{\mu_o}{4\pi} \frac{2\,ec\,\beta\,e^{-qa}}{K_-(-iq)} \frac{n\pi}{d^2\,k_{yn}} \left[\frac{q}{p} \frac{k_{yn}+iq}{k_{yn}+p} - i\alpha^2 \right] e^{i\frac{\omega}{v}md}$$

$$\cdot\,(-1)^{n(m+1)} \left[(-1)^{n-1} + e^{i\frac{\omega}{v}d} \right] \frac{K_-(k_{yn})}{k_{yn}^2 + q^2}\, e^{ik_{yn}y} \sin\frac{n\pi}{d}x \qquad (4.4)$$

where we have used the notation

$$k_{yn} = -\sqrt{p^2 - (\frac{n\pi}{d})^2} - i\varepsilon \qquad (4.5)$$

and the relation

$$\mathrm{Res}\left.\frac{1}{\sin wd}\right|_{k_y=k_{yn}} = \left[\frac{d}{dk_y} \sin\sqrt{p^2 - k_y^2}\,d \right]^{-1}_{k_y=k_{yn}}$$

$$= \frac{n\pi}{d^2 k_{yn}} (-1)^{n-1}$$

As a check we find that in the above three cases the potentials separately satisfy the Lorentz condition.

We now proceed to evaluate the integrals in 4.1 for the case $y > 0$. Now the contour of the integrals may be closed by a semicircle in the upper k_y plane. The contribution from the semicircle to the values of the contour integrals can again be shown to be zero. The only contributions to the integrals are from poles of the integrands in the upper k_y plane. There are two classes of such poles located at

(i) $k_y = iq$

(ii) $k_y = \sqrt{p^2 - (\frac{2n\pi}{d} - \frac{\omega}{v})^2} + i\varepsilon \quad , \quad n = \pm 1, \pm 2, \pm 3, \cdots$

Poles of the second class are those of $K_-(k_y)$ in the upper k_y plane as can be seen from 3.39. It is to be noted that no contributions come from the zeros of the factor sin wd in the denominator, since on examining 3.39 we find that these are exactly cancelled by the zeros of $K_-(k_y)$ in the numerator.

We again evaluate the integrals in 4.1 by the method of residues and summarize our results as follows:

(i) Contribution from pole at $k_y = iq$:

$$A_y(x,y,k_z,\omega) = \frac{\mu_o}{4\pi} \frac{ec\beta}{iq} \frac{K_-(iq)}{K_-(-iq)} \left[\frac{q}{\omega} \frac{p - iq}{p + iq} \right] e^{i\frac{\omega}{v} x - q(y+a)}$$

$$A_z(x,y,k_z,\omega) = \frac{\mu_o}{4\pi} \frac{ec\beta}{iq} \frac{K_-(iq)}{K_-(-iq)} \frac{k_z}{\omega} \left[\frac{q}{p} \frac{2iq}{p + iq} + i \right] e^{i\frac{\omega}{v} x - q(y+a)}$$

$$\phi(x,y,k_z,\omega) = \frac{\mu_o}{4\pi} \frac{ec\beta}{iq} \frac{K_-(iq)}{K_-(-iq)} \left[\frac{q}{p} \frac{2iq}{p + iq} - i\alpha^2 \right] e^{i\frac{\omega}{v} x - q(y+a)} \tag{4.6}$$

(ii) Contribution from pole at $k_y = \sqrt{p^2 - (\frac{2n\pi}{d} - \frac{\omega}{v})^2} + i\varepsilon$, $n = \pm 1 , \pm 2 , \pm 3 \cdots$:

$$A_y(x,y,k_z,\omega) = \frac{\mu_o}{4\pi} \frac{2ec\beta \; e^{-qa}}{K_-(-iq)} \frac{\text{Res } K_-(k'_{yn})}{k'^2_{yn} + q^2}$$

$$\left[\frac{q}{\omega} \; \frac{p - iq}{k'_{yn} + p} \right] e^{i(\frac{\omega}{v} - \frac{2n\pi}{d}) x} \; e^{ik'_{yn} y}$$

$$A_z(x,y,k_z,\omega) = \frac{\mu_o}{4\pi} \frac{2ec\beta\, e^{-qa}}{K_-(-iq)} \frac{\text{Res } K_-(k'_{yn})}{k'^2_{yn} + q^2}$$

$$\frac{k_z}{\omega}\left[\frac{q}{p}\frac{k'_{yn} + iq}{k'_{yn} + p} + i\right] e^{i(\frac{\omega}{v} - \frac{2n\pi}{d})\,x}\; e^{ik'_{yn}\,y}$$

$$\emptyset(x,y,k_z,\omega) = \frac{\mu_o}{4\pi} \frac{2ec\beta e^{-qa}}{K_-(-iq)} \frac{\text{Res } K_-(k'_{yn})}{k'^2_{yn} + q^2}$$

$$\left[\frac{q}{p}\frac{k'_{yn} + iq}{k'_{yn} + p} - i\alpha^2\right] e^{i(\frac{\omega}{v} - (\frac{2n\pi}{d})x}\; e^{ik'_{yn}y} \tag{4.7}$$

where we have used the notation

$$k'_{yn} = \sqrt{p^2 - (\frac{2n\pi}{d} - \frac{\omega}{v})^2} + i\varepsilon \tag{4.8}$$

and

$$\text{Res } K_-(k'_{yn}) = \frac{e^{i\frac{k'_{yn}d}{\pi}\ell n\, 2}\; \prod_{m=1}^{\infty}\left[\sqrt{1 - (\frac{pd}{m\pi})^2} + i\,\frac{k'_{yn}d}{m\pi}\right] e^{-i\frac{k'_{yn}d}{m\pi}}}{\frac{id}{2n\pi}\, e^{\frac{d}{2n\pi}(\frac{\omega}{v} - ik'_{yn})} \prod_{m=1}^{\infty}\left[\sqrt{(1+\frac{\omega d}{2m\pi v})^2 - (\frac{pd}{2m\pi})^2} + i\,\frac{k'_{yn}d}{2m\pi}\right]\cdot}$$

$$\cdot\, e^{-\frac{d}{2m\pi}(\frac{\omega}{v} + ik'_{yn})} \prod_{\substack{m=1\\ m\neq n}}^{\infty}\left[\sqrt{(1-\frac{\omega d}{2m\pi v})^2 - (\frac{pd}{2m\pi})^2} + i\,\frac{k'_{yn}d}{2m\pi}\right]$$

$$\cdot\, e^{\frac{d}{2m\pi}(\frac{\omega}{v} - ik'_{yn})} \tag{4.9}$$

is the residue of $K_-(k_y)$ at the pole $k_y = k'_{yn}$.

It is to be noted that for $y > 0$ the solutions are independent of the integer m , so that the restriction $md > x > (m+1)d$ can be removed. This is evident since the upper half space is not divided up by the plates. As a check we find that the two classes of potentials separately satisfy the Lorentz condition.

To find the total induced potential we need only add up the contributions from all the poles. We should furthermore perform the Fourier inversion for the z coordinate. However, the k_z integration turns out to be far more difficult than the k_y integration. Whereas in the k_y plane the integrand has only isolated poles, it has branch cuts in the k_z plane. In the next chapter we will evaluate the k_z integrals only approximately.

However, in the present stage, even without performing the k_z inversion, the nature of the solutions is already evident. This we will discuss in the next section after we have calculated the induced fields from the potentials.

4.2 Induced Fields

In the last section the induced potentials were explicitly calculated. It was found that they had different forms in the two separate half spaces $y < 0$ and $y > 0$. Furthermore in each of the half spaces the potentials might be expressed as the sum of different classes of terms. In this section we will calculate the induced fields corresponding to each class of induced potentials in accordance with the relations

$$E_x(x,y,k_z,\omega) = -\frac{\partial}{\partial x}\,\phi(x,y,k_z,\omega)$$

$$E_y(x,y,k_z,\omega) = -\frac{\partial}{\partial y}\,\phi(x,y,k_z,\omega) + i\,\omega\,A_y(x,y,k_z,\omega)$$

$$E_z(x,y,k_z,\omega) = -i\, k_z\, \emptyset(x,y,k_z,\omega) + i\,\omega\, A_z(x,y,k_z,\omega)$$

$$B_x(x,y,k_z,\omega) = \frac{\partial}{\partial y} A_z(x,y,k_z,\omega) \quad -ik_z\, A_y(x,y,k_z,\omega)$$

$$B_y(x,y,k_z,\omega) = -\frac{\partial}{\partial x} A_z(x,y,k_z,\omega)$$

$$B_z(x,y,k_z,\omega) = \frac{\partial}{\partial x} A_y(x,y,k_z,\omega) \qquad (4.10)$$

Let us first consider $y < 0$. The results are summarized as follows:

(i) Contribution from pole at $k_y = -iq$

$$E_x(x,y,k_z,\omega) = \frac{\mu_o}{4\pi}\,\frac{ec\beta}{q}\,[i\alpha^2\,\frac{\omega}{v}]\; e^{i\frac{\omega}{v}x + q(y-a)}$$

$$E_y(x,y,k_z,\omega) = \frac{\mu_o}{4\pi}\,\frac{ec\beta}{q}\,[\frac{q}{\beta^2}]\; e^{i\frac{\omega}{v}x + q(y-a)}$$

$$E_z(x,y,k_z,\omega) = \frac{\mu_o}{4\pi}\,\frac{ec\beta}{q}\,[i\,\frac{k_z}{\beta^2}]\; e^{i\frac{\omega}{v}x + q(y-a)}$$

$$B_x(x,y,k_z,\omega) = 0$$

$$B_y(x,y,k_z,\omega) = \frac{\mu_o}{4\pi}\,\frac{ec\,\beta}{q}\,[\frac{-ik_z}{v}]\; e^{i\frac{\omega}{v}x + q(y-a)}$$

$$B_z(x,y,k_z,\omega) = \frac{\mu_o}{4\pi}\,\frac{ec\beta}{q}\,[\frac{q}{v}]\; e^{i\frac{\omega}{v}x + q(y-a)} \qquad (4.11)$$

Upon comparing 4.11 with 2.17 we see that these parts of the induced fields exactly cancel the fields of the moving point charge in the lower half space. Such a cancellation is obviously necessary to satisfy the boundary conditions for the total fields.

(ii) Contribution from pole at $k_y = - p - i\varepsilon$:

$$E_x(x,y,k_z,\omega) = \frac{\mu_o}{4\pi} \frac{2ec\,\beta\, e^{-qa}}{p + iq} \frac{K_-(-p)}{K_-(-iq)} e^{i\frac{\omega}{v} md} \frac{1 - e^{i\frac{\omega}{v} d}}{d} [\frac{q}{p}] e^{-ipy}$$

$$E_y(x,y,k_z,\omega) = 0$$

$$E_z(x,y,k_z,\omega) = 0$$

$$B_x(x,y,k_z,\omega) = 0 \tag{4.12}$$

$$B_y(x,y,k_z,\omega) = \frac{\mu_o}{4\pi} \frac{2ec\,\beta\, e^{-qa}}{p + iq} \frac{K_-(-p)}{K_-(-iq)} e^{i\frac{\omega}{v} md} \frac{1 - e^{i\frac{\omega}{v} d}}{d} \left[\frac{k_z q}{\omega p}\right] e^{-ipy}$$

$$B_z(x,y,k_z,\omega) = \frac{\mu_o}{4\pi} \frac{2ec\,\beta\, e^{-qa}}{p + iq} \frac{K_-(-p)}{K_-(-iq)} e^{i\frac{\omega}{v} md} \frac{1 - e^{i\frac{\omega}{v} d}}{d} [\frac{q}{\omega}] e^{-ipy}$$

These fields have the following properties:

$$\underline{E} \cdot \underline{B} = 0 \quad , \quad E = cB \; .$$

The interpretation is that they represent a wave propagating between the plates at $x = md$ and $x = (m+1)d$, with propagation vector

$$\underline{k} = (0, -p, k_z)$$

From the fact that $\underline{E} \cdot \underline{k} = \underline{B} \cdot \underline{k} = 0$, it is clear that this wave corresponds to a TEM mode of propagation.

(iii) Contribution from pole at $k_y = -\sqrt{p^2 - (\frac{n\pi}{d})^2} - i\varepsilon$, $n = 1,2,3,\cdots$:

$$E_x(x,y,k_z,\omega) = \frac{\mu_o}{4\pi} \frac{2ec\beta\, e^{-qa}}{K_-(-iq)} \frac{n\pi}{d^2 k_{yn}} \left[-\frac{n\pi}{d} \left(\frac{q}{p} \frac{k_{yn} + iq}{k_{yn} + p} - i\alpha^2\right)\right]$$

$$\cdot\, e^{i\frac{\omega}{v} md} (-1)^{n(m+1)} \left[(-1)^{n-1} + e^{i\frac{\omega}{v} d}\right] \frac{K_-(k_{yn})}{k_{yn}^2 + q^2}\, e^{ik_{yn}y} \cos \frac{n\pi}{d} x$$

$$E_y(x,y,k_z,\omega) = \frac{\mu_o}{4\pi} \frac{2ec\beta\, e^{-qa}}{K_-(-iq)} \frac{n\pi}{d^2 k_{yn}} \left[-ik_{yn} \left(\frac{q}{p} \frac{k_{yn} + iq}{k_{yn} + p} - i\alpha^2\right) + iq\, \frac{p - iq}{k_{yn} + p}\right]$$

$$\cdot\, e^{i\frac{\omega}{v} md} (-1)^{n(m+1)} \left[(-1)^{n-1} + e^{i\frac{\omega}{v} d}\right] \frac{K_-(k_{yn})}{k_{yn}^2 + q^2}\, e^{ik_{yn}y} \sin \frac{n\pi}{d} x$$

$$E_z(x,y,k_z,\omega) = \frac{\mu_o}{4\pi} \frac{2ec\beta\, e^{-qa}}{K_-(-iq)} \frac{n}{d^2 k_{yn}} \left[-\frac{k_z}{\beta^2}\right]$$

$$\cdot\, e^{i\frac{\omega}{v} md} (-1)^{n(m+1)} \left[(-1)^{n-1} + e^{i\frac{\omega}{v} d}\right] \frac{K_-(k_{yn})}{k_{yn}^2 + q^2}\, e^{ik_{yn}y} \sin \frac{n\pi}{d} x$$

$$B_x(x,y,k_z,\omega) = \frac{\mu_o}{4\pi} \frac{2ec\,\beta\, e^{-qa}}{K_-(-iq)} \frac{n\pi}{d^2 k_{yn}} \left[ik_{yn} \frac{k_z}{\omega} \left(\frac{q}{p} \frac{k_{yn} + iq}{k_{yn} + p} + i\right) - \right.$$

$$\left. -ik_z \frac{q}{\omega} \frac{p - iq}{k_{yn} + p} \right] e^{i\frac{\omega}{v} md} (-1)^{n(m+1)} \left[(-1)^{n-1} + e^{i\frac{\omega}{v} d} \right] \frac{K_-(k_{yn})}{k_{yn}^2 + q^2} e^{ik_{yn} y}$$

$$\cdot \sin \frac{n\pi}{d} x$$

$$B_y(x,y,k_z,\omega) = \frac{\mu_o}{4\pi} \frac{2ec\,\beta\, e^{-qa}}{K_-(-iq)} \frac{n\pi}{d^2 k_{yn}} \left[-\frac{n\pi}{d} \frac{k_z}{\omega} \left(\frac{q}{p} \frac{k_{yn} + iq}{k_{yn} + p} + i\right) \right]$$

$$\cdot\, e^{i\frac{\omega}{v} md} (-1)^{n(m+1)} \left[(-1)^{n-1} + e^{i\frac{\omega}{v} d} \right] \frac{K_-(k_{yn})}{k_{yn}^2 + q^2} e^{ik_{yn} y} \cos \frac{n\pi}{d} x$$

$$B_z(x,y,k_z,\omega) = \frac{\mu_o}{4\pi} \frac{2ec\,\beta\, e^{-qa}}{K_-(-iq)} \frac{n\pi}{d^2 k_{yn}} \left[\frac{n\pi}{d} \frac{q}{\omega} \frac{p - iq}{k_{yn} + p} \right]$$

$$\cdot\, e^{i\frac{\omega}{v} md} (-1)^{n(m+1)} \left[(-1)^{n-1} + e^{i\frac{\omega}{v} d} \right] \frac{K_-(k_{yn})}{k_{yn}^2 + q^2} e^{ik_{yn} y} \cos \frac{n\pi}{d} x \quad (4.13)$$

These fields satisfy the conditions

$$\underline{E} \cdot \underline{B} = 0 \quad , \quad E = cB$$

If $p^2 > (\frac{n\pi}{d})^2$, $k_{yn} = -\sqrt{p^2 - (\frac{n\pi}{d})^2}$ is real; and the fields represent a wave travelling down the space between two adjacent plates, at the same time being reflected back and forth by them. They correspond to higher excited modes than the TEM mode considered in (ii) above. If $p^2 < (\frac{n\pi}{d})^2$, $k_{yn} = -i\sqrt{(\frac{n\pi}{d})^2 - p^2}$; and the fields are exponentially damped in the negative y direction.

In the half space $y > 0$ the fields take on the following forms:

(i) Contribution from pole at $k_y = iq$:

$$E_x(x,y,k_z,\omega) = \frac{\mu_o}{4\pi} \frac{ec\beta}{iq} \frac{K_-(iq)}{K_-(-iq)} \left[\frac{\omega}{v} \left(\frac{q}{p} \frac{2q}{p+iq} - \alpha^2\right) \right] e^{i\frac{\omega}{v} x - q(y+a)}$$

$$E_y(x,y,k_z,\omega) = \frac{\mu_o}{4\pi} \frac{ec\beta}{iq} \frac{K_-(iq)}{K_-(-iq)} \left[iq\left(\frac{q}{p} \frac{2q}{p+iq} + \frac{p-iq}{p+iq} - \alpha^2\right) \right] \cdot e^{i\frac{\omega}{v} x - q(y+a)}$$

$$E_z(x,y,k_z,\omega) = \frac{\mu_o}{4\pi} \frac{ec\beta}{iq} \frac{K_-(iq)}{K_-(-iq)} \left[-\frac{k_z}{\beta^2} \right] e^{i\frac{\omega}{v} x - q(y+a)}$$

$$B_x(x,y,k_z,\omega) = \frac{\mu_o}{4\pi} \frac{ec\beta}{iq} \frac{K_-(iq)}{K_-(-iq)} \left[-ik_z \frac{q}{\omega} \cdot \left(\frac{q}{p} \frac{2q}{p+iq} + 1 + \frac{p-iq}{p+iq}\right) \right] \cdot e^{i\frac{\omega}{v} x - q(y+a)}$$

$$B_y(x,y,k_z,\omega) = \frac{\mu_o}{4\pi} \frac{ec\beta}{iq} \frac{K_-(iq)}{K_-(-iq)} \left[\frac{k_z}{v} \left(\frac{q}{p} \frac{2q}{p+iq} + 1\right) \right] e^{i\frac{\omega}{v} x - q(y+a)}$$

$$B_z(x,y,k_z,\omega) = \frac{\mu_o}{4\pi} \frac{ec\beta}{iq} \frac{K_-(iq)}{K_-(-iq)} \left[i \frac{q}{v} \frac{p-iq}{p+iq} \right] e^{i\frac{\omega}{v} x - q(y+a)} \qquad (4.14)$$

These fields satisfy the relation

$$\underline{E} \cdot \underline{B} = 0$$

They are attenuated exponentially in the positive y direction. Hence they do not represent radiation fields. From the factor $e^{\frac{i\omega}{v}x}$ we find that

$$k_x = \frac{\omega}{v}$$

From this we can calculate the group velocity of the wave in the x direction:

$$v_{gx} = \frac{\partial\omega}{\partial k_x} = v$$

which is identically the same as the velocity of the point charge. Thus these fields are simply dragged along by the moving point charge in 2.17.

(ii) Contribution from pole at $k_y = \sqrt{p^2 - (\frac{2n\pi}{d} - \frac{\omega}{v})^2} + i\varepsilon$, $n = \pm 1, \pm 2, \pm 3, \cdots$:

$$E_x(x,y,k_z,\omega) = \frac{\mu_o}{4\pi} \frac{2ec\beta\, e^{-qa}}{K_-(-iq)} \frac{\text{Res } K_-(k'_{yn})}{k'^2_{yn} + q^2}$$

$$\cdot \left[-i(\frac{\omega}{v} - \frac{2n\pi}{d})(\frac{q}{p} \frac{k'_{yn} + iq}{k'_{yn} + p} - i\alpha^2) \right] e^{i(\frac{\omega}{v} - \frac{2n\pi}{d})x}\, e^{ik'_{yn}y}$$

$$E_y(x,y,k_z,\omega) = \frac{\mu_o}{4\pi} \frac{2ec\beta\, e^{-qa}}{K_-(-iq)} \frac{\text{Res } K_-(k'_{yn})}{k'^2_{yn} + q^2}$$

$$\cdot \left[-ik'_{yn}(\frac{q}{p} \frac{k'_{yn} + iq}{k'_{yn} + p} - i\alpha^2) + iq\, \frac{p - iq}{k'_{yn} + p} \right] e^{i(\frac{\omega}{v} - \frac{2n\pi}{d})x}\, e^{ik'_{yn}y}$$

$$E_z(x,y,k_z,\omega) = \frac{\mu_o}{4\pi}\frac{2ec\beta\, e^{-qa}}{K_-(-iq)}\;\frac{\text{Res } K_-(k'_{yn})}{k'^2_{yn} + q^2}$$

$$\cdot \left[-\frac{k_z}{\beta^2}\right] e^{i(\frac{\omega}{v} - \frac{2n\pi}{d})x}\; e^{ik'_{yn}y}$$

$$B_x(x,y,k_z,\omega) = \frac{\mu_o}{4\pi}\frac{2ec\beta\, e^{-qa}}{K_-(-iq)}\;\frac{\text{Res } K_-(k'_{yn})}{k'^2_{yn} + q^2}$$

$$\cdot \left[ik'_{yn}\,\frac{k_z}{\omega}\left(\frac{q}{p}\,\frac{k'_{yn} + iq}{k'_{yn} + p} + i\right) - ik_z\,\frac{q}{\omega}\,\frac{p - iq}{k'_{yn} + p}\right]$$

$$\cdot\, e^{i(\frac{\omega}{v} - \frac{2n\pi}{d})x}\; e^{ik'_{yn}y}$$

$$B_y(x,y,k_z,\omega) = \frac{\mu_o}{4\pi}\frac{2ec\beta\, e^{-qa}}{K_-(-iq)}\;\frac{\text{Res } K_-(k'_{yn})}{k'^2_{yn} + q^2}$$

$$\cdot \left[-i\left(\frac{\omega}{v} - \frac{2n\pi}{d}\right)\frac{k_z}{\omega}\left(\frac{q}{p}\,\frac{k'_{yn} + iq}{k'_{yn} + p} + i\right)\right] e^{i(\frac{\omega}{v} - \frac{2n\pi}{d})x}\; e^{ik'_{yn}y}$$

$$B_z(x,y,k_z,\omega) = \frac{\mu_o}{4\pi}\frac{2ec\beta\, e^{-qa}}{K_-(-iq)}\;\frac{\text{Res } K_-(k'_{yn})}{k'^2_{yn} + q^2}$$

$$\cdot \left[i\left(\frac{\omega}{v} - \frac{2n\pi}{d}\right)\frac{q}{\omega}\,\frac{p - iq}{k'_{yn} + p}\right] e^{i(\frac{\omega}{v} - \frac{2n\pi}{d})x}\; e^{ik'_{yn}y} \qquad (4.15)$$

Again these fields satisfy the relations

$$\underline{E} \cdot \underline{B} = 0 \quad , \qquad E = cB$$

For $p^2 > (\frac{2n\pi}{d} - \frac{\omega}{v})^2$, they represent a wave with propagation vector

$$\underline{k} = (\frac{\omega}{v} - \frac{2n\pi}{d}, \; k'_{yn} \; , \; k_z)$$

For $p^2 < (\frac{2n\pi}{d} - \frac{\omega}{v})^2$, the fields are exponentially damped in the positive y direction.

Summarizing the results we see that in the evaluation of the integrals in 4.1 each pole of the integrand gives rise to one characteristic solution of the wave equation. The solutions are of three types. In the first type the waves propagate at equal pace with the point charge, and thus do not represent radiation. In the second type the waves propagate freely into both half spaces $y < 0$ and $y > 0$. In the third type the waves are damped exponentially in the y direction; they thus cling closely to the z-x plane.

V. ASYMPTOTIC PROPERTIES OF SOLUTIONS

Up to the last chapter the results we obtained are exact. The objection we can raise against them is that they are expressed as functions of x, y, k_z and ω. An exact performance of the Fourier inversion for k_z is impossible due to the complex structure of the integrand which contains infinite products. In this chapter we will use the method of steepest descent (17) to derive an expression for the far fields. From the fields the form of Poynting's vector can be computed. In this way we obtain information on the radiation emitted by the excited conducting plates.

5.1 Method of Steepest Descent

Let u be a complex variable and let us consider a contour integral of the type

$$I = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(u)\, e^{i\sqrt{\lambda^2 - u^2}\, y + iuz}\, du \tag{5.1}$$

where λ is a positive real constant. However, for convenience in the analysis, we give λ a small positive imaginary part, that is, $\lambda \to \lambda + i\varepsilon$. Assume that the only singularities of the function $f(u)$ are branch points lying exclusively inside the first and third quadrants in the complex u plane. Furthermore, assume that the branch points in the first quadrant either lie immediately above the real axis or immediately to the right of the imaginary axis. Similarly in the third quadrant the branch points all lie immediately below the real axis or immediately to the left of the imaginary axis. We note at this point that the amplitudes of the induced fields calculated in Chapter IV possess just such a property. The branch cuts can therefore be drawn entirely inside the first and third quadrants. The situation is depicted in Fig. 4.

We introduce polar coordinates through the relations:

$$y = \rho \cos \phi$$

$$z = \rho \sin \phi \quad , \quad -\frac{\pi}{2} < \phi < \frac{\pi}{2} \qquad (5.2)$$

Then 5.1 becomes

$$I = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(u)\, e^{i\rho g(u)}\, du \qquad (5.3)$$

where

$$g(u) = \sqrt{\lambda^2 - u^2} \cos \phi + u \sin \phi \qquad (5.4)$$

We want to evaluate the integral approximately for very large values of the parameter ρ .

Consider the following conformal transformation:

$$u = \lambda \sin \theta \ , \qquad -\frac{\pi}{2} < \text{Re } \theta < \frac{\pi}{2} \qquad (5.5)$$

Under this transformation the entire u plane is mapped into a vertical strip $-\frac{\pi}{2} \leq \text{Re } \theta \leq \frac{\pi}{2}$ of the θ plane as shown in Fig. 5. Comparing with Fig. 4 we see that the real axis of the u plane has been mapped into the broken line ABCG. We can now write 5.3 in the form

$$I = \frac{1}{\sqrt{2\pi}} \int_{ABCG} f(\lambda \sin \theta)\, e^{i\lambda\rho \cos(\theta - \phi)}\, \lambda \cos \theta \, d\theta \qquad (5.6)$$

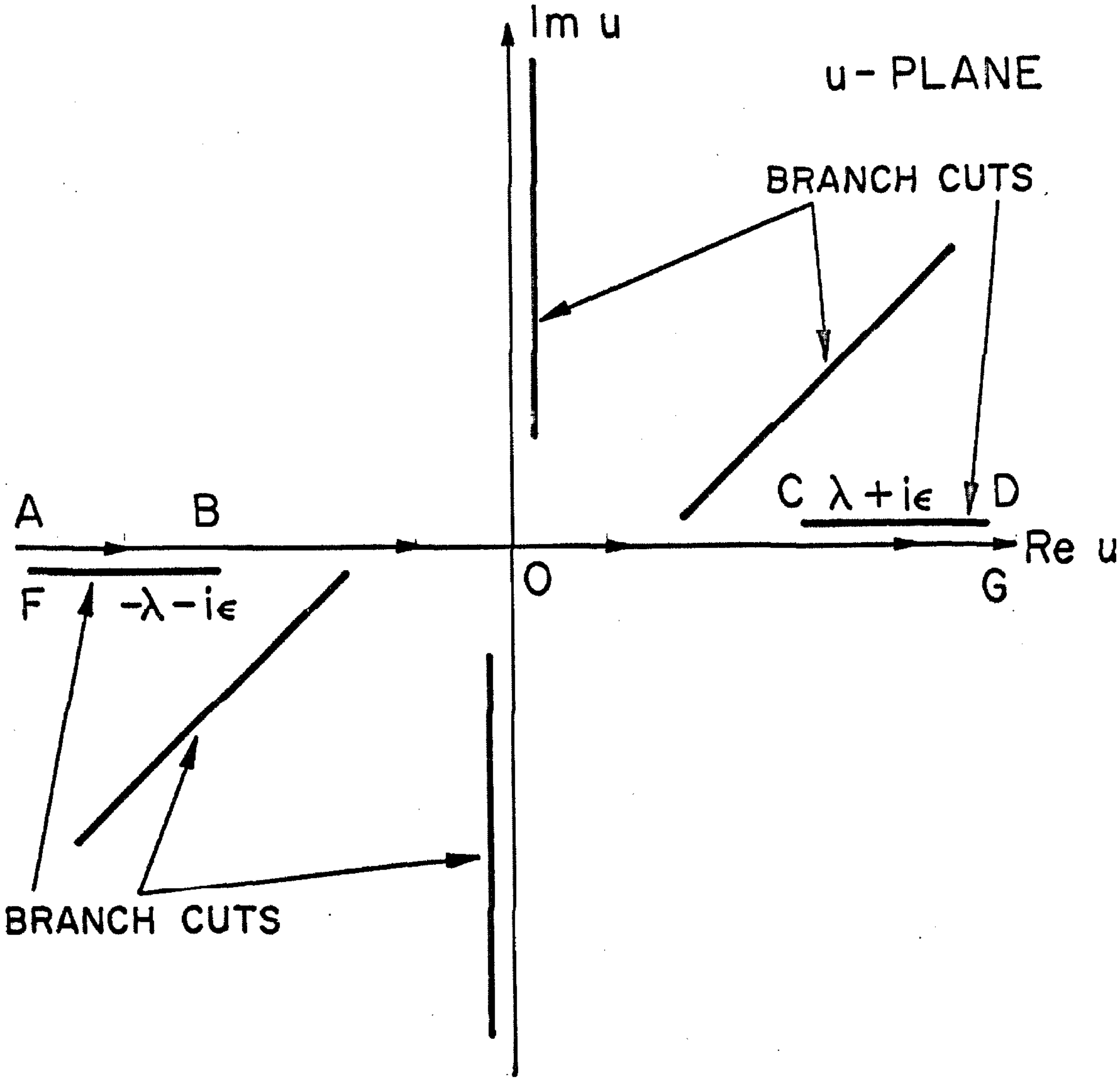

Fig. 4. Singularities in the complex u-plane

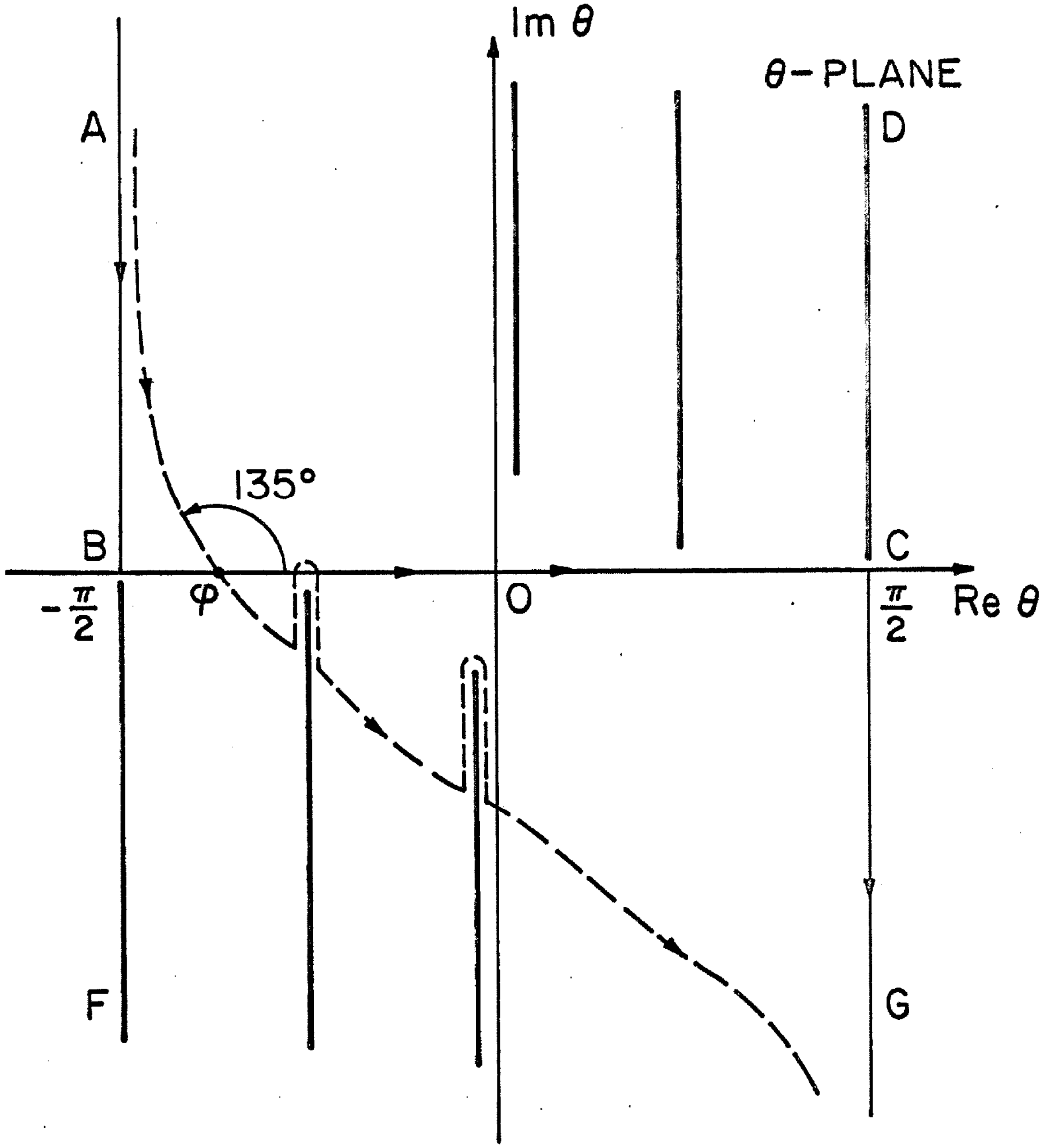

Fig. 5. Image of the u-plane under the conformal transformation

$$u = \lambda \sin \theta \ , \quad -\frac{\pi}{2} \leq \mathrm{Re}\ \theta \leq \frac{\pi}{2}$$

The dotted line is the path of steepest descent.

Here the integral is to be taken along the contour ABCG in the Θ plane.

If λ is finite and ρ is very large, it is clear that any change in $\cos(\Theta - \emptyset)$ will produce a large fluctuation in the exponential factor. If $f(\lambda \sin \Theta)$ is a slowly varying function, such a large fluctuation will tend to produce an average value of zero for the integral. The main contribution to the integral comes from the vicinity of points on the contour at which $\cos(\Theta - \emptyset)$ has the least tendency to change, that is, at which

$$\frac{d}{d\Theta} \cos(\Theta - \emptyset) = 0$$

This equation has the solution $\Theta = \emptyset$. It can be shown that such a point is not an absolute maximum of $\cos(\Theta - \emptyset)$, rather it is a saddle point.

The idea of the present approximate method for the evaluation of the integral 5.6 is to deform the contour ABCG continuously into one which passes through the saddle point along a path of "steepest descent". Through a saddle point there are two characteristic paths: one of steepest descent, the other of steepest ascent. Suppose we put

$$\Theta = \xi + i\eta$$

$$\cos(\Theta - \emptyset) = \psi + i\chi$$

The two characteristic paths are given by the equation

$$\psi(\xi,\eta) = \text{constant.}$$

We note that along such a path the exponential term in 5.6 does not fluctuate. Calculating $\psi(\xi,\eta)$ directly we find the equation to be

$$\cos(\xi - \emptyset) \cosh \eta = 1$$

The constant is taken to be unity so that the curve passes through the saddle point $\xi = \emptyset$, $\eta = 0$. At the saddle point we find that

$$\frac{d\eta}{d\xi} = \pm 1$$

An examination of $\cos(\Theta - \emptyset)$ in the neighborhood of the saddle point shows that the minus sign corresponds to the path of steepest descent.

In Fig. 5 we indicate that the contour ABCG is deformed into the dotted contour which passes through the saddle point at an angle of 135° with the real axis. When it is necessary the branch cuts are circumvented.

It can be shown that along the path of steepest descent the exponential factor in 5.6 decreases monotonically and rapidly on either side of the saddle point. Making use of this fact we can replace $f(\lambda \sin \Theta)\lambda \cos \Theta$ in 5.6 by its constant value at the saddle point. Thus

$$I \simeq \frac{1}{\sqrt{2\pi}} f(\lambda \sin \emptyset) \lambda \cos \emptyset \int e^{i\lambda\rho \cos(\Theta - \emptyset)} d\Theta \qquad (5.7)$$

Now we can deform the dotted contour back into the original contour ABCG. The integral in 5.7 becomes identical to an integral representation of a Hankel function of the first kind: (18)

$$H_o^{(1)}(\lambda\rho) = \frac{1}{\pi} \int_{ABCG} e^{i\lambda\rho \cos(\theta - \phi)} \, d\theta$$

Therefore we have

$$I \simeq \frac{1}{\sqrt{2\pi}} \pi \, f(\lambda \sin \phi) \, \lambda \cos \phi \, H_o^{(1)}(\lambda\rho)$$

Since ρ is large we can further use the asymptotic formula for $H_o^{(1)}(\lambda\rho)$:

$$H_o^{(1)}(\lambda\rho) \sim \sqrt{\frac{2}{\pi\lambda\rho}} \, e^{-i\pi/4} \, e^{i\lambda\rho} \qquad , \qquad \lambda\rho \to \infty$$

Finally we obtain our approximate formula for the integral 5.1:

$$I \simeq \sqrt{\lambda} \; e^{-i\pi/4} f(\lambda \sin \phi) \cos \phi \, \frac{1}{\sqrt{\rho}} \, e^{i\lambda\rho} \tag{5.8}$$

This has the form of a cylindrical wave.

From 5.2 we see that in the above derivation we have assumed ϕ to be in the range $(-\frac{\pi}{2}, \frac{\pi}{2})$. Hence we were only considering the case $y > 0$. For $y < 0$, 5.2 becomes

$$y = -\rho \cos \phi$$

$$z = \rho \sin \phi \qquad , \qquad -\frac{\pi}{2} < \phi < \frac{\pi}{2} \tag{5.9}$$

that is, ϕ is now measured from the negative direction of the y-axis. A glance at the results of Chapter IV shows that when $y < 0$ we will be considering integrals of the type 5.1, but with $\sqrt{\lambda^2 - u^2}$ replaced by $-\sqrt{\lambda^2 - u^2}$. Hence the function $g(u)$ in 5.4 remains the same. And we arrive at the same asymptotic expression as in 5.8.

5.2 Far Fields and Poynting's Vector

In Chapter IV the induced fields were calculated and the results were expressed as functions of x,y,k_z and ω . In order to investigate the radiation properties of our system, it is necessary to express the fields in terms of x,y,z and ω . Results of this form can be obtained by performing the inverse Fourier k_z transformation:

$$\underline{E}(x,y,z,\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \underline{E}(x,y,k_z,\omega)\, e^{ik_z z}\, dk_z \qquad (5.10)$$

Such an integral is of the form 5.1 and hence, at very large distances ($\rho \to \infty$), the integral for the far fields in 5.10 can be approximated by 5.8. In the following we will calculate the asymptotic expressions for the far fields as well as the Poynting vector.

It is clear from 5.10 that the far fields we obtain in this way are only the frequency components of the actual fields. Therefore the Poynting vector corresponding to them will be a function of the frequency ω . We would like to give a physical interpretation for the Poynting vector obtained in this manner. Let us consider the definition of the Poynting vector

$$\underline{s}(\underline{r},t) = \underline{E}(\underline{r},t) \times \underline{H}(\underline{r},t)$$

and let us resolve the electromagnetic fields into their frequency components:

$$\underline{E}(\underline{r},t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \underline{E}(\underline{r},\omega)\, e^{-i\omega t}\, d\omega$$

$$\underline{H}(\underline{r},t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \underline{H}(\underline{r},\omega)\, e^{-i\omega t}\, d\omega$$

These fields are real; their frequency components must satisfy the conditions

$$\underline{E}(\underline{r},-\omega) = \underline{E}^{*}(\underline{r},\omega)$$

$$\underline{H}(\underline{r},-\omega) = \underline{H}^{*}(\underline{r},\omega)$$

Then

$$\underline{s}(\underline{r},t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \underline{E}(\underline{r},\omega) \times \underline{H}^{*}(\underline{r},\omega')\, e^{-i(\omega-\omega')t}\, d\omega\, d\omega'$$

From this we get

$$\int_{-\infty}^{\infty} \underline{s}(\underline{r},t)\, dt = \int_{-\infty}^{\infty} \underline{s}(\underline{r},\omega)\, d\omega \tag{5.11}$$

where we have defined

$$\underline{s}(\underline{r},\omega) = \underline{E}(\underline{r},\omega) \times \underline{H}^{*}(\underline{r},\omega) \tag{5.12}$$

From 5.11 we see that $\underline{s}(\underline{r},\omega)$ represents the total electromagnetic energy per unit frequency interval at ω passing through one unit area at the point $\underline{r}$ during the entire time interval $(-\infty,\infty)$.

We now return to the task of calculating the asymptotic fields and Poynting's vector. Let us begin with the half space $y < 0$, and take $md < x < (m+1)d$. Recalling 5.9 we define

$$y = -\rho \cos \phi$$

$$z = \rho \sin \phi \qquad , \quad -\frac{\pi}{2} < \phi < \frac{\pi}{2}$$

In our calculations it is only necessary to consider the unattenuated part of the solution. The results are summarized as follows:

(i) Contribution from pole at $k_y = -p - i\varepsilon$:

$$\underline{E}(\underline{r}, \omega) = \frac{\mu_o}{4\pi} \frac{2ec\beta}{d} e^{i\frac{\omega}{v} md} (1 - e^{i\frac{\omega}{v} d}) \sqrt{k} \; e^{-i\pi/4} \cos \phi$$

$$\cdot \left[\frac{K_-(-p)}{K_-(-iq)} \frac{q}{p} \frac{e^{-qa}}{p + iq} \right]_{k_z = k \sin \phi} \frac{1}{\sqrt{\rho}} e^{ik\rho} \underline{e}_x$$

$$\underline{B}(\underline{r},\omega) = \frac{\underline{k}}{\omega} \times \underline{E}(\underline{r},\omega) \tag{5.13}$$

$$\underline{s}(\underline{r},\omega) = \frac{|\underline{E}(\underline{r},\omega)|^2}{\omega\mu_o} \underline{k}$$

$$= \frac{\mu_o}{4\pi} \frac{2e^2 v^4}{\pi d^2 \; \rho\omega c^2} (1 - \cos \frac{\omega}{v} d) (\alpha^2 + \sin^2\phi)$$

$$\cdot \left| \frac{K_-(-k \cos \phi)}{K_-(-ik\sqrt{\alpha^2 + \sin^2\phi})} \right|^2 e^{-2|k|a \sqrt{\alpha^2 + \sin^2\phi}} \underline{e}_\rho \tag{5.14}$$

where

$$\underline{k} = k(-\cos \phi \; \underline{e}_y + \sin \phi \; \underline{e}_z) = k \; \underline{e}_\rho \tag{5.15}$$

(ii) Contribution from pole at $k_y = -\sqrt{p^2 - (\frac{n\pi}{d})^2} - i\varepsilon$,

$n - 1,2,3,\cdots,\ p^2 > (\frac{n\pi}{d})^2$:

$$E_x(\underline{r},\omega) = \left\{ f(k_{yn}) \left[-\frac{n\pi}{d} \frac{\underline{q}\ k_{yn} + i[(\frac{k_z}{\beta})^2 - \alpha^2\ \underline{p}k_{yn}]}{p(k_{yn}+p)} \right] \right\}_{k_z = \sqrt{k^2-(\frac{n\pi}{d})^2}\sin\phi}$$

$$\cdot \frac{1}{\sqrt{\rho}} e^{i\sqrt{k^2-(\frac{n\pi}{d})^2}\rho} \cos\frac{n\pi}{d}x$$

$$E_y(\underline{r},\omega) = \left\{ f(k_{yn}) \frac{\alpha^2 p(\frac{n\pi}{d})^2 + (\frac{k_z}{\beta})^2\ (k_{yn}+p) + iq(\frac{n\pi}{d})^2}{p(k_{yn}+p)} \right\}_{k_z = \sqrt{k^2-(\frac{n\pi}{d})^2}\sin\phi}$$

$$\cdot \frac{1}{\sqrt{\rho}} e^{i\sqrt{k^2-(\frac{n\pi}{d})^2}\rho} \sin\frac{n\pi}{d}x$$

$$E_z(\underline{r},\omega) = \left\{ f(k_{yn}) \frac{-k_z}{\beta^2} \right\}_{k_z = \sqrt{k^2-(\frac{n\pi}{d})^2}\sin\phi}$$

$$\cdot \frac{1}{\sqrt{\rho}} e^{i\sqrt{k^2-(\frac{n\pi}{d})^2}\rho} \sin\frac{n\pi}{d}x$$

$$B_x(\underline{r},\omega) = \left\{ f(k_{yn}) \frac{k_z}{\omega} \frac{p(\frac{n\pi}{d})^2 - (\frac{\omega}{v})^2(k_{yn}+p) - iq(\frac{n\pi}{d})^2}{p(k_{yn}+p)} \right\}_{k_z = \sqrt{k^2-(\frac{n\pi}{d})^2}\sin\phi}$$

$$\cdot \frac{1}{\sqrt{\rho}} e^{i\sqrt{k^2-(\frac{n\pi}{d})^2}\rho} \sin\frac{n\pi}{d}x$$

$$B_y(\underline{r},\omega) = \left\{ f(k_{yn}) \left[-\frac{n\pi}{d}\frac{k_z}{\omega}\frac{qk_{yn} + i[(\frac{\omega}{v})^2 + pk_{yn}]}{p(k_{yn}+p)} \right] \right\}_{k_z = \sqrt{k^2-(\frac{n\pi}{d})^2}\sin\phi}$$

$$\cdot \frac{1}{\sqrt{\rho}} e^{i\sqrt{k^2-(\frac{n\pi}{d})^2}\,\rho} \cos\frac{n\pi}{d} x$$

$$B_z(\underline{r},\omega) = \left\{ f(k_{yn}) \frac{n\pi}{d}\frac{q}{\omega}\frac{p - iq}{k_{yn} + p} \right\}_{k_z = \sqrt{k^2-(\frac{n\pi}{d})^2}\sin\phi}$$

$$\cdot \frac{1}{\sqrt{\rho}} e^{i\sqrt{k^2-(\frac{n\pi}{d})^2}\,\rho} \cos\frac{n\pi}{d} x \qquad (5.16)$$

Here we have used the abbreviation

$$f(k_{yn}) = \frac{\mu_o}{4\pi}\frac{2ec\beta}{(\frac{\omega}{v})^2 - (\frac{n\pi}{d})^2} e^{i\frac{\omega}{v}md} \frac{n\pi}{d^2 k_{yn}} (-1)^{n(m+1)}[(-1)^{n-1} + e^{i\frac{\omega}{v}d}]$$

$$\cdot \frac{K_-(k_{yn})}{K_-(-iq)} e^{-qa} [k^2 - (\frac{n\pi}{d})^2]^{1/4} \cos\phi\, e^{-i\pi/4} \qquad (5.17)$$

From these fields we obtain the following expression for the Poynting vector:

$$s_x(\underline{r},\omega) = \left\{ |f(k_{yn})|^2 \frac{n\pi}{d}\frac{1}{\mu_o\omega p^2(k_{yn}+p)^2} i(\frac{\omega}{v})^2 p \left[p\alpha^2(\frac{n\pi}{d})^2 \right.\right.$$

$$\left.\left. + 2(\frac{k_z}{\beta})^2 (k_{yn}+p)\right] \right\}_{k_z = \sqrt{k^2-(\frac{n\pi}{d})^2}\sin\phi} \frac{1}{\rho}\sin\frac{n\pi}{d}x\cos\frac{n\pi}{d}x$$

$$s_y(\underline{r},\omega) = \left\{ |f(k_{yn})|^2 \frac{-1}{\mu_o \omega p(k_{yn}+p)} \left(\frac{k_z}{\beta}\right)^2 \left[p\left(\frac{n\pi}{d}\right)^2 - \left(\frac{\omega}{v}\right)^2 (k_{yn}+p) + iq\left(\frac{n\pi}{d}\right)^2 \right] \right\}_{k_z=\sqrt{k^2-(\frac{n\pi}{d})^2}\sin\phi}$$

$$\cdot \frac{1}{\rho} \sin^2 \frac{n\pi}{d} x + \left\{ |f(k_{yn})|^2 \left(\frac{n\pi}{d}\right)^2 \frac{qp(p+iq)}{\mu_o \omega p^2 (k_{yn}+p)^2} \left[qk_{yn} + i\left(\left(\frac{k_z}{\beta}\right)^2 - p\alpha^2 k_{yn}\right)\right] \right\}_{k_z=\sqrt{k^2-(\frac{n\pi}{d})^2}\sin\phi} \frac{1}{\rho}\cos^2 \frac{n\pi}{d} x$$

$$s_z(\underline{r},\omega) = \left\{ |f(k_{yn})|^2 \left(\frac{n\pi}{d}\right)^2 \frac{k_z}{\mu_o \omega p^2 (k_{yn}+p)^2} \left[q\, k_{yn} + i\left(\left(\frac{k_z}{\beta}\right)^2 - p\alpha^2 k_{yn}\right)\right] \cdot \left[qk_{yn} - i\left(\left(\frac{\omega}{v}\right)^2 + pk_{yn}\right)\right] \right\}_{k_z=\sqrt{k^2-(\frac{n\pi}{d})^2}\sin\phi}$$

$$\cdot \frac{1}{\rho} \cos^2 \frac{n\pi}{d} x - \left\{ |f(k_{yn})|^2 \frac{k_z}{\mu_o \omega p^2 (k_{yn}+p)^2} \cdot \left[p\alpha^2 \left(\frac{n\pi}{d}\right)^2 + \left(\frac{k_z}{\beta}\right)^2 (k_{yn}+p) + iq\left(\frac{n\pi}{d}\right)^2 \right] \cdot \left[p\left(\frac{n\pi}{d}\right)^2 - - \left(\frac{\omega}{v}\right)^2 (k_{yn}+p) + iq\left(\frac{n\pi}{d}\right)^2 \right] \right\}_{k_z=\sqrt{k^2-(\frac{n\pi}{d})^2}\sin\phi} \frac{1}{\rho}\sin^2 \frac{n\pi}{d} x \tag{5.18}$$

In the above form the significance of the Poynting vector is not apparent. However, one thing is to be noticed: its x component is purely imaginary which indicates zero energy flow in the x direction. This is evident from the fact that the metallic plates block the transportation of electromagnetic energy. In order to gain more insight into the property of the Poynting vector $\underline{s}(\underline{r},\omega)$, let us integrate it over the x coordinate. We first define

$$\underline{\Pi}(y,z,\omega) = \int_{md}^{(m+1)d} \underline{s}(\underline{r},\omega)\, dx \tag{5.19}$$

Then we find

$$\underline{\Pi}(y,z,\omega) = \frac{\mu_o}{4\pi}\frac{e^2\omega}{\pi d\rho}\frac{(\frac{n\pi}{d})^2}{[(\frac{\omega}{v})^2-(\frac{n\pi}{d})^2]^2}\left[1-(-1)^n\cos\frac{\omega}{v}d\right]\left\{\left|\frac{K_-(k_{yn})}{K_-(-iq)}\right|^2\cdot\right.$$

$$\left.\cdot\ \frac{e^{-2qa}}{p(k_{yn}+p)^2}\left[p\alpha^2(\frac{n\pi}{d})^2+2(\frac{k_z}{\beta})^2(k_{yn}+p)\right]\right\}_{k_z=\sqrt{k^2-(\frac{n\pi}{d})^2}\sin\phi}\underline{e}_\rho \tag{5.20}$$

It is to be noticed that in the above cases the Poynting vector is independent of the integer m . This is clear from the definition of $\underline{s}(\underline{r},\omega)$ in 5.11. For, since the point charge is moving uniformly, the total energy radiated down the space between two adjacent plates at $x = md$ and $x = (m+1)d$ from $t = -\infty$ to $t = \infty$ must be the same for all values of *m* .

Proceeding in the same way, we can calculate the forms of the fields and the Poynting vector in the half space $y > 0$.

(i) Contribution from pole at $k_y = \sqrt{p^2 - (\frac{2n\pi}{d} - \frac{\omega}{v})^2} + i\varepsilon$,
$n = \pm 1, \pm 2, \pm 3, \cdots$, $p^2 > (\frac{2n\pi}{d} - \frac{\omega}{v})^2$:

$$E_x(\underline{r},\omega) = \left\{ g(k'_{yn}) \left[-i(\frac{\omega}{v} - \frac{2n\pi}{d})(\frac{q}{p} \frac{k'_{yn} + iq}{k'_{yn} + p} - i\alpha^2) \right] \right\}_{k_z = \sqrt{k^2 - (\frac{2n\pi}{d} - \frac{\omega}{v})^2} \sin \phi}$$

$$\cdot \frac{1}{\sqrt{\rho}} e^{i\sqrt{k^2 - (\frac{2n\pi}{d} - \frac{\omega}{v})^2}\rho} \; e^{i(\frac{\omega}{v} - \frac{2n\pi}{d})x}$$

$$E_y(\underline{r},\omega) = \left\{ g(k'_{yn}) \left[-ik'_{yn}(\frac{q}{p} \frac{k'_{yn} + iq}{k'_{yn} + p} - i\alpha^2) + iq \frac{p - iq}{k'_{yn} + p} \right] \right\}_{k_z = \sqrt{k^2 - (\frac{2n\pi}{d} - \frac{\omega}{v})^2} \times \sin \phi}$$

$$\cdot \frac{1}{\sqrt{\rho}} e^{i\sqrt{k^2 - (\frac{2n\pi}{d} - \frac{\omega}{v})^2}\rho} \; e^{i(\frac{\omega}{v} - \frac{2n\pi}{d})x}$$

$$E_z(\underline{r},\omega) = \left\{ g(k'_{yn}) \frac{-k_z}{\beta^2} \right\}_{k_z = \sqrt{k^2 - (\frac{2n\pi}{d} - \frac{\omega}{v})^2} \sin \phi}$$

$$\cdot \frac{1}{\sqrt{\rho}} e^{i\sqrt{k^2 - (\frac{2n\pi}{d} - \frac{\omega}{v})^2}\rho} \; e^{i(\frac{\omega}{v} - \frac{2n\pi}{d})x}$$

$$\underline{B}(\underline{r},\omega) = \frac{\underline{k}}{\omega} \times \underline{E}(\underline{r},\omega) \qquad (5.21)$$

$$\underline{s}(\underline{r},\omega) = \frac{\mu_o}{4\pi}\frac{e^2\omega}{\pi\rho}\frac{k^2-(\frac{2n\pi}{d}-\frac{\omega}{v})^2}{[(\frac{\omega}{v})^2-(\frac{2n\pi}{d}-\frac{\omega}{v})^2]^2}\left\{\left|\frac{\text{Res } K_-(k'_{yn})}{K_-(-iq)}\right|^2\right.$$

$$\cdot\frac{1}{p(k'_{yn}+p)^2}[p\alpha^2(\frac{2n\pi}{d}-\frac{\omega}{v})^2+2(\frac{k_z}{\beta})^2(k'_{yn}+p)]$$

$$\left.\cdot\, e^{-2qa}\right\}_{k_z=\sqrt{k^2-(\frac{2n\pi}{d}-\frac{\omega}{v})^2}\sin\phi}\frac{\cos^2\phi}{\sqrt{k^2-(\frac{2n\pi}{d}-\frac{\omega}{v})^2}}\,\underline{k} \qquad (5.22)$$

where we have defined

$$g(k'_{yn}) = \frac{\mu_o}{4\pi}\frac{2ec\beta\, e^{-qa}}{k'^2_{yn}+q^2}\frac{\text{Res } K_-(k'_{yn})}{K_-(-iq)}$$

$$\cdot\left[k^2-(\frac{2n\pi}{d}-\frac{\omega}{v})^2\right]^{1/4}\cos\phi\, e^{-i\pi/4} \qquad (5.23)$$

and

$$\underline{k} = (\frac{\omega}{v}-\frac{2n\pi}{d})\,\underline{e}_x + \sqrt{k^2-(\frac{2n\pi}{d}-\frac{\omega}{v})^2}\,\underline{e}_\rho \qquad (5.24)$$

5.3 Properties of the Radiation

We are now in the position to examine the properties of the radiation emitted by the excited plates. In the last section the Poynting vectors are given explicitly in 5.14, 5.20 and 5.22. From them we get an expression for the profile of the radiation pattern in a plane parallel to the y-z plane:

$$I(\phi) = \rho\,\underline{s}(\underline{r},\omega)\cdot\underline{e}_\rho \qquad (5.25)$$

where $\rho^2 = y^2 + z^2$. As we recall in the half space $y < 0$, ϕ is measured from the negative direction of the y axis toward the positive direction of the z axis; and in the half space $y > 0$, it is measured from the positive direction of the y axis toward the positive direction of the z axis. The dependence of $I(\phi)$ on ϕ is very complicated due to the fact that $\underline{s}(\underline{r},\omega)$ contains infinite products. Techniques of computation and limited tables for this type of infinite products can occasionally be found in the literature (19),(20). In any case, only a few factors in the product sequences in 3.39 need be computed to obtain satisfactory accuracy, since the factors approach unity quite rapidly as n increases. More discussions on the radiation pattern will be found in the next section.

In the lower half space $y < 0$ at fixed frequency ω , the radiation consists of a superposition of modes of electromagnetic waves that can exist inside a parallel-plane waveguide. These waves appear to be generated from a fictitious source situated along the x axis $(\rho = 0)$. We notice that the contribution to the Poynting vector from each mode is proportional to

$$1 - (-1)^n \cos \frac{\omega}{v} d \quad , \qquad n = 0,1,2,\cdots$$

Thus for some value of the frequency such that

$$\frac{\omega}{v} d = m\pi \tag{5.26}$$

where m is an integer, the factor becomes

$$1 - (-1)^{n+m}$$

which vanishes if m and n are both even or both odd. We therefore find that at certain frequencies some waves are not excited at all between the plates. The explanation of this phenomenon is as follows. From 2.17 we see that the source of excitation is proportional to $e^{i\frac{\omega}{v}x}$. Let us expand this factor as a series in terms of the eigenfunctions $\cos \frac{n\pi}{d} x$ and $\sin \frac{n\pi}{d} x$ of the region between two plates:

$$e^{i\frac{\omega}{v} x} = \frac{A_o}{2} + \sum_{n=1}^{\infty} (A_n \cos \frac{n\pi}{d} x + B_n \sin \frac{n\pi}{d} x)$$

The coefficients in the series are easily found to be

$$A_o = \frac{2}{d} \frac{-1}{i \frac{\omega}{v}} \left[1 - e^{i\frac{\omega}{v} d}\right]$$

$$A_n = \frac{2}{d} \frac{i \frac{\omega}{v}}{(\frac{\omega}{v})^2 - (\frac{n\pi}{d})^2} \left[1 - (-1)^n e^{i\frac{\omega}{v} d}\right]$$

$$B_n = \frac{2}{d} \frac{- \frac{n\pi}{d}}{(\frac{\omega}{v})^2 - (\frac{n\pi}{d})^2} \left[1 - (-1)^n e^{i\frac{\omega}{v} d}\right]$$

The amplitudes of the fields excited in the lower half space must be proportional to these coefficients. From 5.13 and 5.16 we see that this is actually the case. In 5.14 and 5.20 the Poynting vectors are clearly proportional to the absolute square of these coefficients. Thus the phenomenon that at frequencies satisfying 5.26 certain waves are not excited in the lower half space is due to the fact that at these frequencies the source does not contain these waves.

Let us proceed to discuss the radiation in the upper half space $y > 0$. In this region, at fixed frequency ω , the electromagnetic waves also appear to be generated from a source along the x axis ($\rho = 0$), but with propagation vector

$$\underline{k} = k_x \underline{e}_x + k_\rho \underline{e}_\rho$$

where

$$k_x = \frac{\omega}{v} - \frac{2n\pi}{d}$$

$$k_\rho = \sqrt{k^2 - (\frac{\omega}{v} - \frac{2n\pi}{d})^2} \qquad (5.27)$$

and $n = \pm 1 , \pm 2 , \pm 3 , \cdots$ such that $\underline{k}$ is real. If we examine the profile of the radiation pattern in the x-y plane, we see that the radiation exists only in a finite number of discrete directions, the angles these directions make with the y axis being given by

$$\tan^{-1} \frac{k_x}{k_\rho} = \tan^{-1} \frac{\frac{\omega}{v} - \frac{2n\pi}{d}}{\sqrt{k^2 - (\frac{\omega}{v} - \frac{2n\pi}{d})^2}}$$

This phenomenon is easily explained if we regard the edges of the plates as forming a one-dimensional crystal lattice in the x direction. In crystal diffraction theory we know that the intensity of the electromagnetic waves diffracted from a crystal lattice is appreciable only in those directions which satisfy the von Laue condition

$$\underline{k}_i - \underline{k}_d = 2n\pi\underline{p}$$

where $\underline{k}_i$ is the propagation vector of the incident wave, $\underline{k}_d$ that of the diffracted wave, and $\underline{p}$ is a period of the reciprocal lattice. In our case the system is periodic only in the x direction, and we have

$$\underline{k}_i = \frac{\omega}{v}\,\underline{e}_x$$

$$\underline{p} = \frac{1}{d}\,\underline{e}_x$$

The von Laue equation therefore gives

$$\underline{k}_d = \left(\frac{\omega}{v} - \frac{2n\pi}{d}\right)\,\underline{e}_x$$

which is just what we have in 5.27.

Suppose now we let the frequency ω vary and observe the radiation in a fixed direction, making an angle Θ with the x axis. Then

$$k_x = k\cos\Theta = \frac{\omega}{v} - \frac{2n\pi}{d}$$

Since $k = \frac{\omega}{c}$ we get from above

$$\omega = \frac{\frac{2n\pi v}{d}}{1 - \frac{v}{c}\cos\Theta}\,, \qquad n = \pm 1,\ \pm 2,\ \pm 3 \cdots \tag{5.28}$$

Therefore the frequency spectrum of the radiation observed at a fixed angle Θ consists of a sequence of evenly-spaced lines. We can identify in 5.28 a fundamental angular frequency

$$\omega_o = \frac{2\pi v}{d}$$

This is just 2π times the number of plates traversed by the point charge in one second. Thus from 5.28 we find that the frequency spectrum of a point charge moving at a uniform speed over a linear periodic structure is equivalent to that of a uniformly moving harmonic oscillator, each frequency component being shifted by the Doppler factor $1 - \frac{v}{c} \cos \theta$ in the denominator.

5.4 Reduction to the Single-Plate Case

It is possible to get further information on the properties of the radiation by considering a limiting case in which the separation of two neighboring plates is large; or more specifically, $kd >> 1$. Consider equation 2.33 which relates the induced potential to the induced current density on the plate at $x = 0$:

$$A_\mu(x,y,k_z,\omega) = -\frac{\mu_o}{4\pi}\sqrt{2\pi}\, e^{i\frac{\omega}{v}md} \int_{-\infty}^{\infty} \frac{j_{\mu o}(k_y,k_z,\omega)}{w} \frac{\sin w[x-(m+1)d] - e^{i\frac{\omega}{v}d} \sin w[x - md]}{\cos wd - \cos \frac{\omega}{v} d} e^{ik_y y}\, dk_y \qquad (2.33)$$

where $md < x < (m+1)d$. Suppose we take an observation point close to the zeroth plate $0 < |x| < d$ $(m = 0 \text{ or } -1)$. In the limit $kd \to \infty$ the above equation reduces to

$$A_\mu(x,y,k_z,\omega) = \frac{\mu_o}{4\pi}\sqrt{2\pi}\, i \int_{-\infty}^{\infty} \frac{e^{iw|x|}}{w} j_{\mu o}(k_y,k_z,\omega)\, e^{ik_y y}\, dk_y \qquad (5.29)$$

which is the equation for the diffraction problem of one conducting half plane (8). The reason is that when kd is large, many waves can be fitted into the space between two adjacent plates, and the mutual interaction of the plates is small compared to that between the point charge and the plate closest to it. In arriving at 5.29 the fact that $w = \sqrt{p^2 - k_y^2}$ has a small positive imaginary part has been used.

If we supplement equation 5.29 with equation 2.47

$$\int_{-\infty}^{\infty} j_{\mu o}(k_y,k_z,\omega)\, e^{ik_y y}\, dk_y = 0 \quad , \quad y > 0 \tag{2.47}$$

we get a set of dual integral equations which can be solved by the procedures of Chapter III. The solutions are simpler than those of the multiplane case in that they do not contain infinite products:

$$j_{yo}(k_y,k_z,\omega) = \frac{-iec\,\beta\, e^{-qa}}{(2\pi)^{3/2}\,\omega} \sqrt{p + iq}\, \frac{p - iq}{\sqrt{k_y + p}\,(k_y + iq)}$$

$$j_{zo}(k_y,k_z,\omega) = \frac{-iec\,\beta\, e^{-qa}}{(2\pi)^{3/2}\,\omega} \sqrt{p + iq}\, \frac{k_z}{q}\left[\frac{q}{p}\frac{k_y + iq}{k_y + p} + i\right] \frac{\sqrt{k_y + p}}{k_y + iq}$$

$$c\rho_o(k_y,k_z,\omega) = \frac{-iec\,\beta\, e^{-qa}}{(2\pi)^{3/2}\,\omega} \sqrt{p + iq}\, \frac{\omega}{qc}\left[\frac{q}{p}\,\frac{k_y + iq}{k_y + p} - i\alpha^2\right] \frac{\sqrt{k_y + p}}{k_y + iq}$$

(5.30)

As in the multiplane case these current densities satisfy the edge conditions: $j_{yo}(y,k_z,\omega)$ vanishes at the edge $y = 0$ while $j_{z0}(y,k_z,\omega)$ and $c\rho_o(y,k_z,\omega)$ become infinite.

At this stage we may substitute 5.30 into 5.29 to get an exact expression for the induced potential. But since we are interested in the radiation properties, an asymptotic expression will be adequate for the purpose. Let us begin with the particular solution of the wave equation 2.4:

$$A_\mu(x,y,z,\omega) = \frac{\mu_o}{4\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{e^{ikr'}}{r'} j_{\mu o}(x',y',z',\omega)\, dx'dy'dz' \qquad (5.31)$$

where

$$r' = \sqrt{(x-x')^2 + (y-y')^2 + (z-z')^2}$$

Introduce spherical polar coordinates through the transformations:

$$x = r \sin \theta \cos \phi$$

$$y = r \sin \theta \sin \phi$$

$$z = r \cos \theta$$

Then in the radiation zone

$$r' \simeq r - \sin \theta \cos \phi\, x' - \sin \theta \sin \phi\, y' - \cos \theta\, z'$$

Writing

$$j_{\mu o}(x',y',z',\omega) = \delta(x')\, j'_{\mu o}(y',z',\omega) \qquad (5.32)$$

we get

$$A_\mu(x,y,z,\omega) \simeq \frac{\mu_o}{4\pi} \frac{e^{ikr}}{r} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-ik(\sin \theta \sin \phi y' + \cos \theta z')} \cdot j_{\mu o}(y',z',\omega)\, dy'\, dz'$$

$$= \frac{\mu_o}{4\pi} \frac{e^{ikr}}{r} 2\pi\, j_{\mu o}(k \sin \theta \sin \phi,\, k \cos \theta, \omega) \qquad (5.33)$$

which is in the form of a spherical wave. The induced potential is therefore directly proportional to the Fourier transform of the induced current density.

From the potential in 5.33 we can easily compute the electric field

$$\underline{E} = - \nabla \phi - \frac{\partial \underline{A}}{\partial t}$$

Keeping only terms proportional to r^{-1}, we obtain the results:

$$E_x(\underline{r},\omega) = \frac{\mu_o}{4\pi} \frac{-2\pi i\, \omega\, e^{ikr}}{r} [c\rho_o \sin\theta \cos\phi]$$

$$E_y(\underline{r},\omega) = \frac{\mu_o}{4\pi} \frac{-2\pi i\, \omega\, e^{ikr}}{r} [c\rho_o \sin\theta \sin\phi - j'_{yo}]$$

$$E_z(\underline{r},\omega) = \frac{\mu_o}{4\pi} \frac{-2\pi i\, \omega\, e^{ikr}}{r} \frac{\sin\theta}{\cos\theta} [-c\rho_o \sin\theta + j_{yo} \sin\phi] \qquad (5.34)$$

where the dependence of $c\rho_o$, j_{yo} on k_y, k_z, ω is suppressed and we have used the continuity equation

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \underline{j} = 0$$

to eliminate the z component j_{zo}. The magnetic field is found to be given by

$$\underline{B}(\underline{r},\omega) = \frac{1}{c} \underline{e}_r \times \underline{E}(\underline{r},\omega)$$

From these fields the Poynting vector is obtained:

$$\underline{s}(\underline{r},\omega) = s(\underline{r},\omega)\, \underline{e}_r = \frac{1}{c\mu_o} |\underline{E}(\underline{r},\omega)|^2\, \underline{e}_r \quad .$$

After some lengthy calculations we get

$$s(\underline{r},\omega) = \frac{\mu_o}{4\pi}\frac{e^2 v}{8\pi^2 r^2}\;\frac{\cos^2\theta(1+\sin\phi) + (1-\beta^2\sin^2\theta)(1-\sin\phi)}{\sin\theta(1-\beta^2\sin^2\theta)(1-\beta^2\sin^2\theta\cos^2\phi)}$$

$$\cdot\;\exp\left[-\frac{2a|\omega|}{v}\sqrt{1-\beta^2\sin^2\theta}\;\right] \tag{5.35}$$

A plot of $s(\underline{r},\omega)$ as a function of θ and ϕ gives the radiation pattern at frequency ω. Traces of the pattern on the y-z, x-y, and z-x planes are sketched in Figs. 6, 7 and 8 respectively. In general the radiation is concentrated along the y-axis at high frequencies, along the z axis at low frequencies, and along the x axis at high velocities ($\beta \simeq 1$). The infinite behavior of the Poynting vector along the z axis is due to the singularities in the current densities at the edge.

If we integrate $r^2 s(\underline{r},\omega)$ over the frequency ω we get the angular distribution

$$\frac{dW}{d\Omega} = \frac{\mu_o}{4\pi}\frac{e^2 v^2}{8\pi^2 a}\;\frac{\cos^2\theta(1+\sin\phi) + (1-\beta^2\sin^2\theta)(1-\sin\phi)}{\sin\theta(1-\beta^2\sin^2\theta)^{3/2}\,(1-\beta^2\sin^2\theta\cos^2\phi)} \tag{5.36}$$

where W is the radiated energy. Finally if we integrate 5.36 over the solid angle Ω, we obtain the total energy radiated by a single plate

$$W = \frac{\mu_o}{4\pi}\frac{3e^2 v^2}{8a\sqrt{1-\beta^2}} \tag{5.37}$$

Suppose now we have N plates evenly spaced at $x = nd$, $n = 0,1,2,\cdots,$ $N-1$. In the approximation of no interaction among the plates the

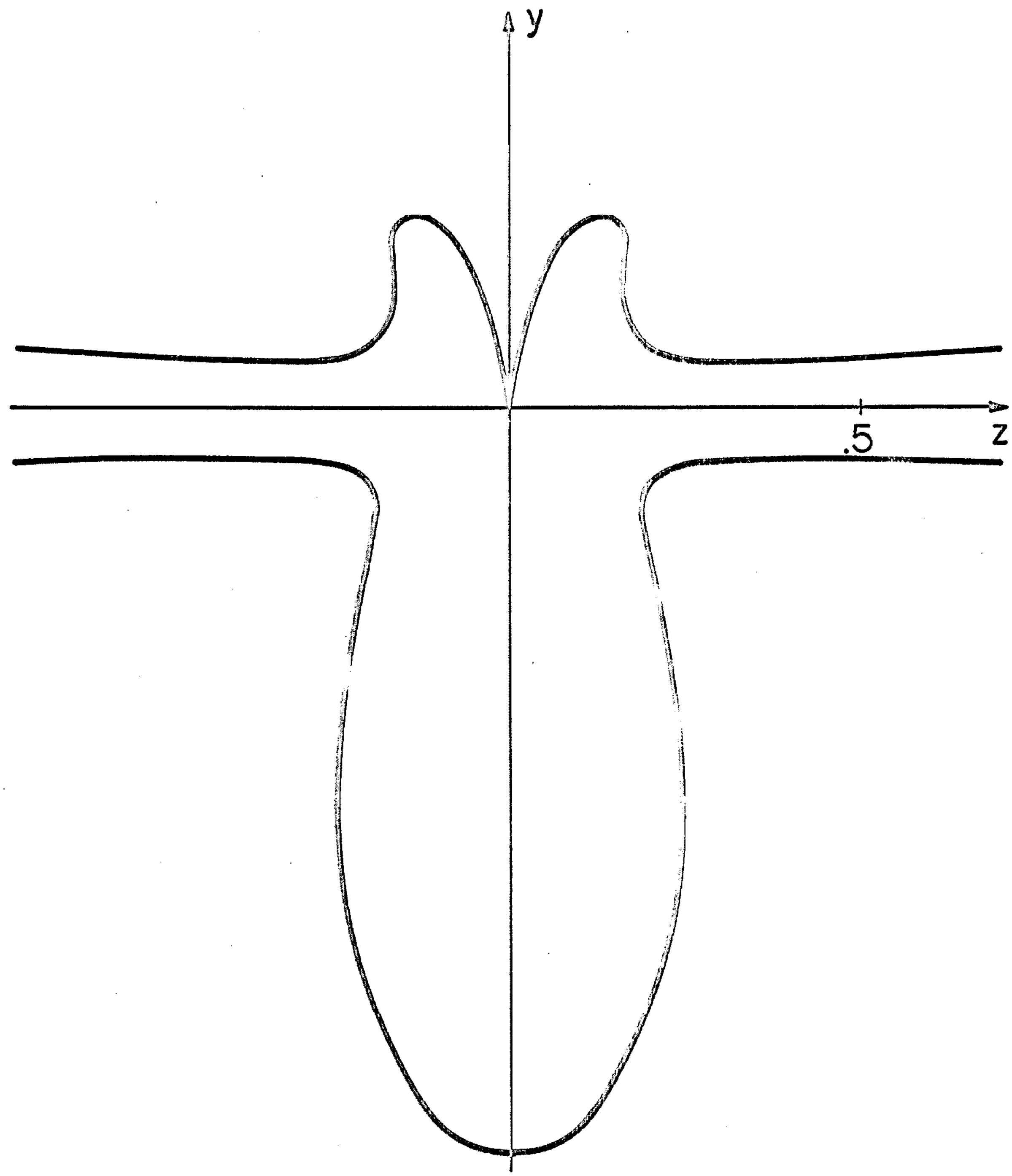

Fig. 6. Radiation pattern for the single-plate case for $\beta^2 = .8$ and $\frac{2a\omega}{v} = 5$.

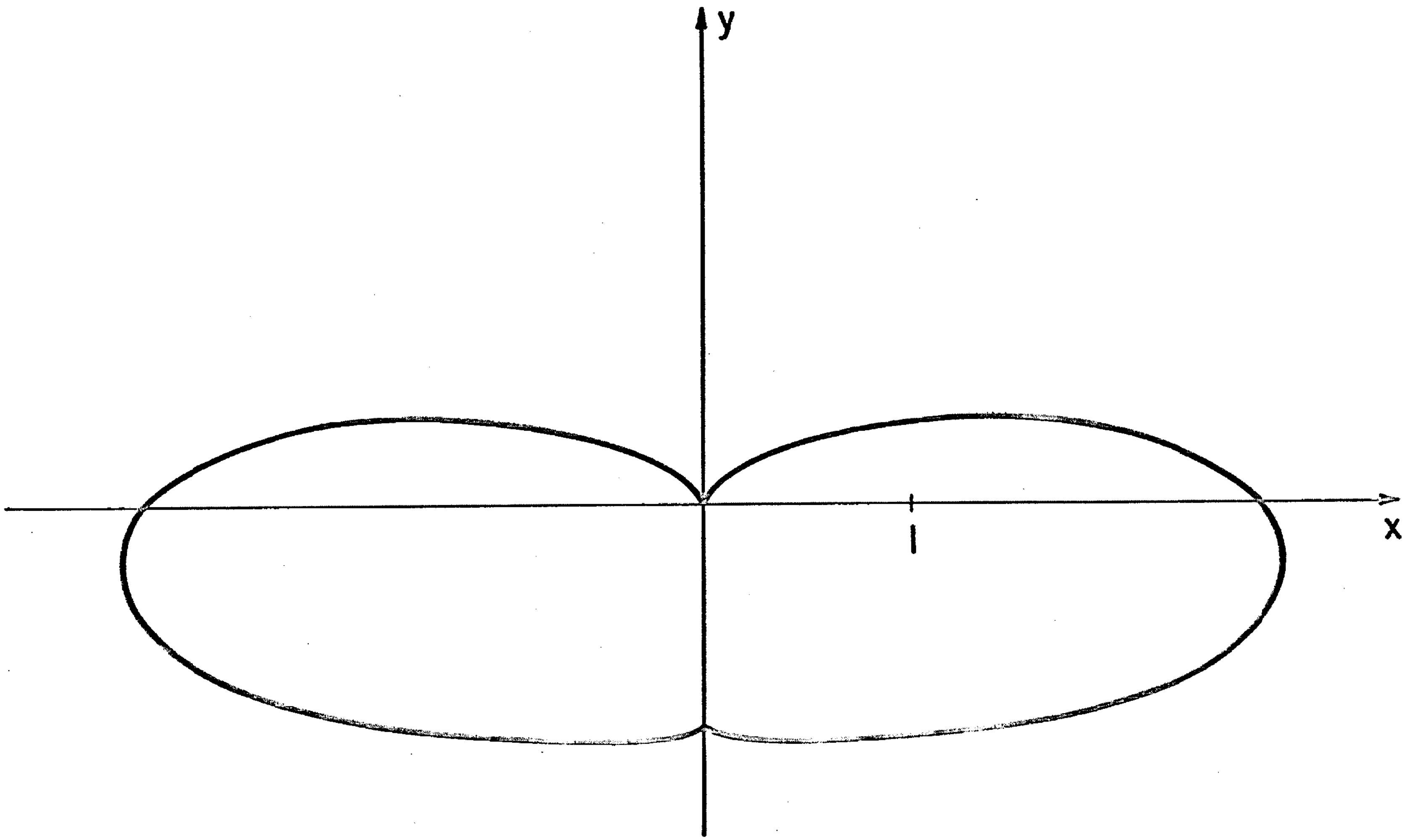

Fig. 7. Radiation pattern for the single-plate case for $\beta^2 = .8$ and $\frac{2a\omega}{v} = 5$.

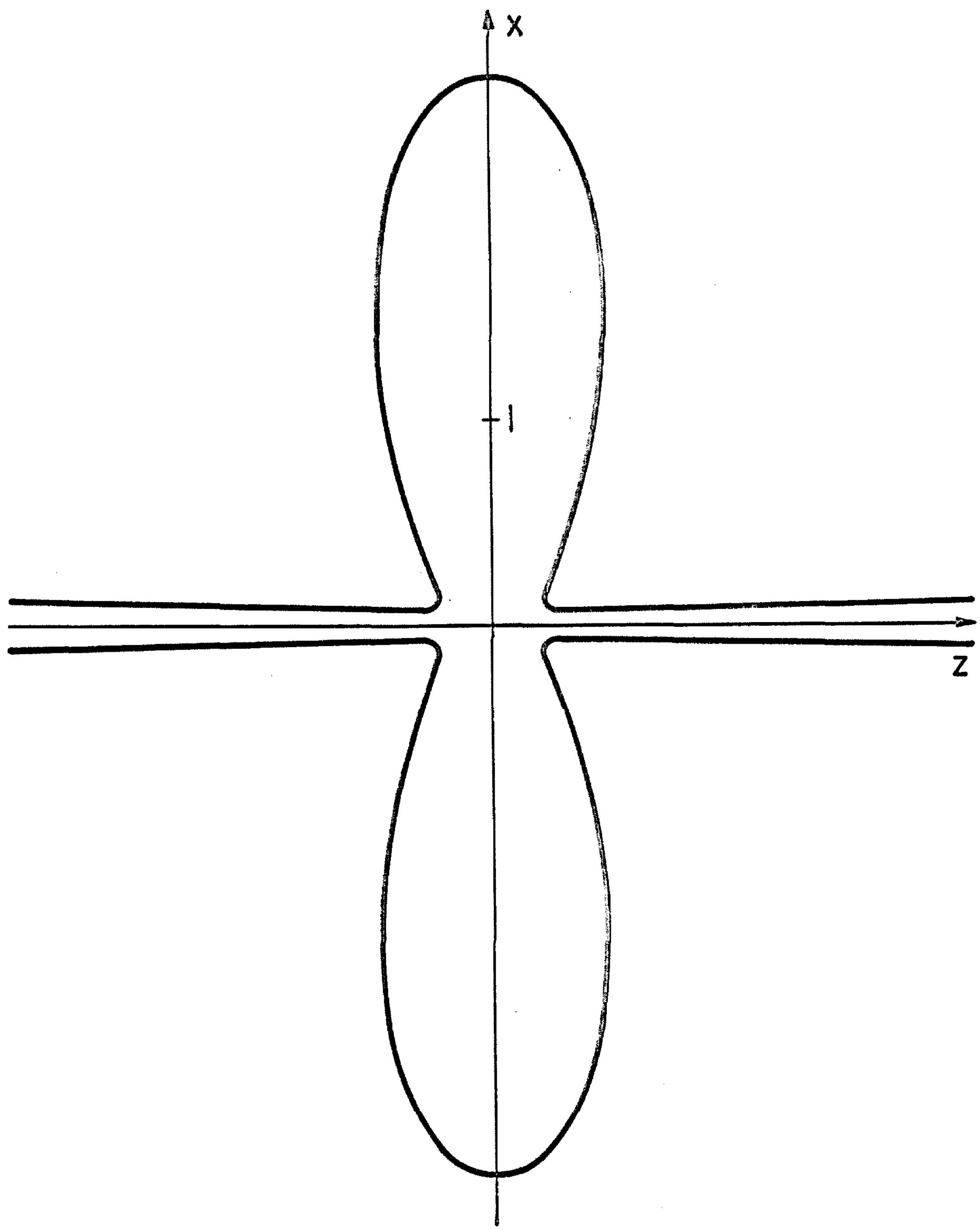

Fig. 8. Radiation pattern for the single-plate case for $\beta^2 = .8$ and $\frac{2a\omega}{v} = 5$.

radiation fields due to these plates can be synthesized from those of a single plate. Instead of $j_{\mu o}(x',y',z',\omega)$ in 5.32, the induced current density is now a sum of N terms:

$$j_\mu(x',y',z',\omega) = \sum_{n=0}^{N-1} \delta(x'-nd)\, j_{\mu n}(y',z',\omega)$$

Or using the periodic condition 2.23, we get

$$j_\mu(x',y',z',\omega) = \sum_{n=0}^{N-1} \delta(x'-nd)\, e^{i\frac{\omega}{v}nd}\, j_{\mu o}(y',z',\omega) \tag{5.38}$$

Substituting this current density into 5.31 we obtain the induced potential

$$A_\mu(x,y,z,\omega) = \frac{\mu_o}{4\pi}\frac{e^{ikr}}{r} A_N(\theta,\phi)\, 2\pi j_{\mu o}(k\sin\theta\sin\phi,\, k\cos\theta,\omega) \tag{5.39}$$

This potential differs from that of a single plate in 5.33 in the "array factor"

$$A_N(\theta,\phi) = \sum_{n=0}^{N-1} \exp[-in(kd\sin\theta\cos\phi - \frac{\omega}{v}d)]$$

$$= \frac{\sin[\frac{N}{2}(kd\sin\theta\cos\phi - \frac{\omega}{v}d)]}{\sin[\frac{1}{2}(kd\sin\theta\cos\phi - \frac{\omega}{v}d)]}$$

$$\cdot \exp[-i\frac{N-1}{2}(kd\sin\theta\cos\phi - \frac{\omega}{v}d)] \tag{5.40}$$

Thus the Poynting vector in 5.35 is modified by a factor $|A_N(\theta,\phi)|^2$.

Various patterns of the function

$$K(\theta,\phi) = \frac{1}{N} \left|A_N(\theta,\phi)\right|$$

for $\theta = \pi/2$ and different values of kd are given in Reference (21). The function is sharply peaked in directions satisfying

$$kd \cos \phi - \frac{\omega}{v} d = - 2n\pi$$

where n is a non-zero integer. Taking $k_x = k \cos \phi$, we recover the von Laue condition in 5.27.

We can also obtain the rate of energy loss of the point charge from equation 5.37. Suppose there are N plates extending over a distance of $(N-1)d$ along the x axis. For N large the loss of energy per unit distance is on the average equal to

$$- \frac{dW}{dx} = \frac{\mu_o}{4\pi} \frac{3e^2v^2}{8ad\sqrt{1 - \beta^2}} \qquad (5.41)$$

Finally we can establish a criterion for the uniformity of the motion of the point charge. Suppose in the laboratory we set up a stack of N plates. From 5.37 the total energy radiated during the passage of the point charge is

$$\frac{\mu_o}{4\pi} \frac{3N\ e^2v^2}{8a\sqrt{1 - \beta^2}}$$

If the motion of the point charge is to remain uniform, this loss of energy must be small compared to the kinetic energy of the point charge. We must therefore require

$$\frac{\mu_o}{4\pi}\frac{3N\ e^2 v^2}{8a\sqrt{1-\beta^2}} << mc^2\left(\frac{1}{\sqrt{1-\beta^2}} - 1\right)$$

or

$$\frac{3N}{8}\frac{r_o}{a} << \frac{1-\sqrt{1-\beta^2}}{\beta}$$

where $r_o = 2.82 \times 10^{-15}m$ is the classical electron radius.

VI. SUMMARY AND CONCLUSIONS

In the present work the problem of the diffraction of the fields of a moving point charge by an infinite array of parallel conducting half-planes is solved by the Wiener-Hopf method.

The problem is treated as a boundary value problem for the potentials of the diffracted electromagnetic fields. A special feature of this approach is the calculation of the values of these potentials on the conducting half-planes. By virtue of the periodicity of the system of half-planes the potentials are found to depend only on the current densities induced on one single half-plane. The problem is then formulated in the form of integral equations for these current densities. The integral equations so obtained are dual integral equations.

The integral equations are first converted into algebraic equations through Fourier transformation. These algebraic equations are solved by the Wiener-Hopf method. The solutions appear as exact expressions for the induced current densities, which are shown to satisfy the edge conditions required in diffraction problems involving sharp edges. Exact formulas for the potentials and the diffracted electromagnetic fields are derived in the form of Fourier integrals. The asymptotic values of these integrals are calculated by using the method of steepest descent. Expressions for the Poynting vector are thereby obtained.

The radiation in the region of the half-planes is found to consist of a superposition of modes of electromagnetic waves that can be excited in a parallel-plane waveguide. The radiation outside the half-plane region displays characteristics that are expected of a system with a linear periodic structure. The diffracting effect of the edges of the half-planes is comparable to that of a one-dimensional crystal lattice. The frequency spectrum of the radiation at a fixed angle of observation is found to consist of a sequence of discrete lines. This latter property makes it possible to use the parallel-plane system as a charged particle detector and velocity measuring device.

APPENDIX A

SUMMARY OF SYMBOLS

For reference purposes we give a list of the symbols most frequently used in the main text.

$$\beta = v/c$$

$$\alpha = \frac{\sqrt{1 - \beta^2}}{\beta}$$

$$k = \omega/c$$

$$p = \sqrt{k^2 - k_z^2} = i\sqrt{k_z^2 - k^2}$$

$$q = \sqrt{-p^2 + (\frac{\omega}{v})^2} = \sqrt{\alpha^2 k^2 + k_z^2}$$

$$w = \sqrt{p^2 - k_y^2} = i\sqrt{k_y^2 - p^2}$$

$$k_{yn} = -\sqrt{p^2 - (\frac{n\pi}{d})^2} = -i\sqrt{(\frac{n\pi}{d})^2 - p^2}\ , \quad n = 1,2,3,\cdots$$

$$k'_{yn} = \sqrt{p^2 - (\frac{2n\pi}{d} - \frac{\omega}{v})^2} = i\sqrt{(\frac{2n\pi}{d} - \frac{\omega}{v})^2 - p^2}, \ n = 0,\pm 1,\pm 2,\cdots$$

APPENDIX B

DERIVATION OF A FOURIER TRANSFORM FORMULA

In this appendix we derive a Fourier transform formula used in Chapter II, namely

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} i\pi H_o^{(1)}[p\sqrt{z^2+a^2}]\, e^{itz}\, dz = \frac{e^{-|a|\sqrt{-p^2+t^2}}}{\sqrt{-p^2+t^2}}$$

The branch of the radical is fixed by the definition

$$\sqrt{-p^2+t^2} = -i\sqrt{p^2-t^2}$$

when $p^2 > t^2$. The above integral formula can be rewritten in the equivalent form

$$\int_{-\infty}^{\infty} \frac{e^{-|a|\sqrt{-p^2+t^2}}}{\sqrt{-p^2+t^2}}\, e^{-itz}\, dt = i\pi H_o^{(1)}[p\sqrt{z^2+a^2}] \qquad (A.1)$$

The latter form turns out to be more convenient to prove.

Let us give p a small imaginary part (see Chapter III):

$$p \to p + i\varepsilon \qquad (A.2)$$

Then on the complex t-plane the integrand in A.1 has branch points at $t = p + i\varepsilon$ and $t = -p - i\varepsilon$. The branch cuts can be drawn from these branch points to infinity parallel to the real axis, as shown in Fig. 9. These cuts are the only singularities of the integrand on the complex t plane.

Let us first transform to polar coordinates through the equations

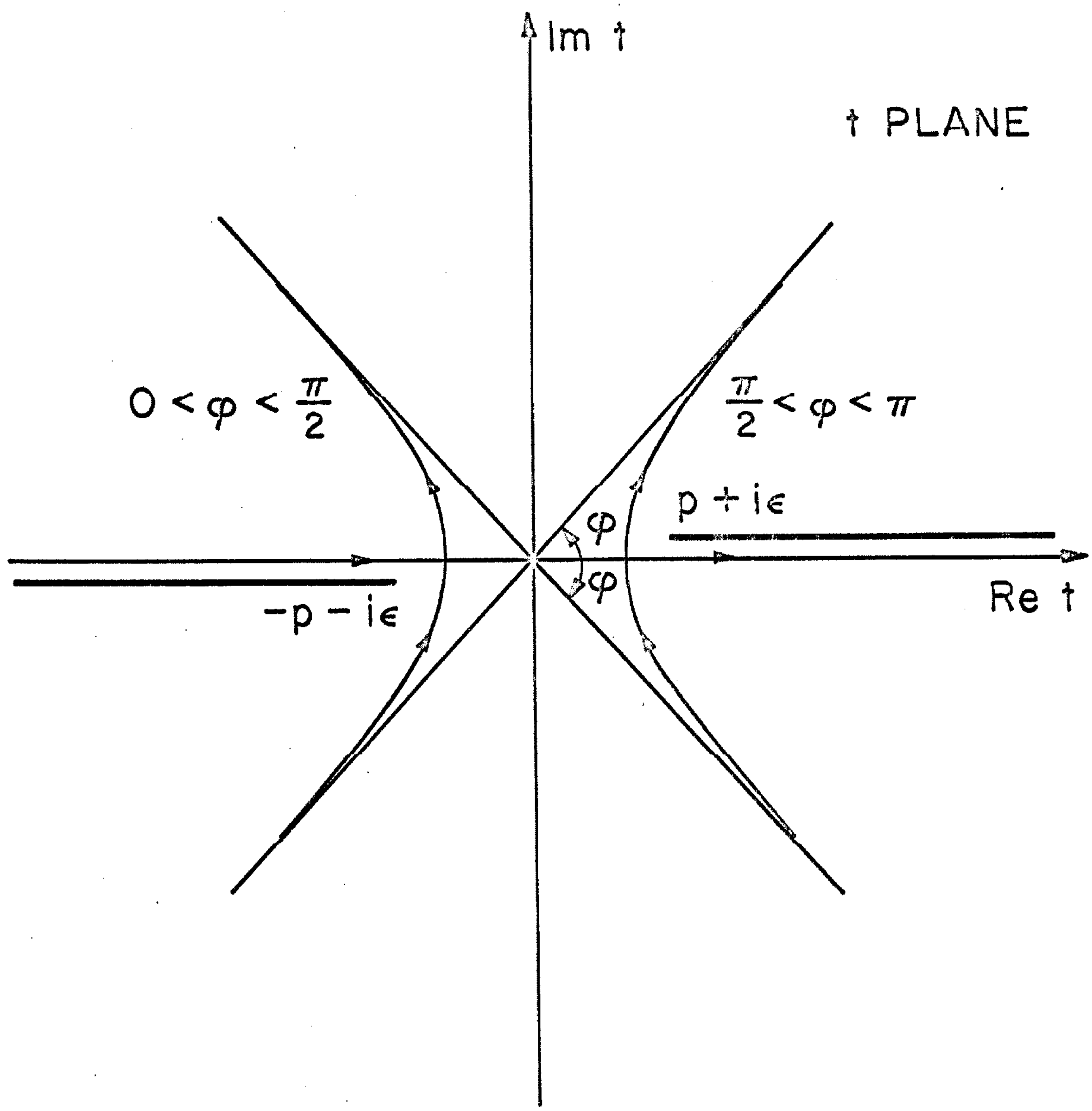

Fig. 9. Singularities and contours in the complex t-plane.

$$z = \rho \cos \phi$$

$$|a| = \rho \sin \phi \quad , \quad 0 < \phi < \pi \qquad (A.3)$$

Consider the following change of the variable of integration

$$t = - \rho \cos(\phi + iu) \quad , \qquad -\infty < u < \infty \qquad (A.4)$$

Equation A.4 represents one branch of a hyperbola as shown in Fig. 9. For $0 < \phi < \pi/2$ we have the branch on the left; for $\pi/2 < \phi < \pi$ we have the branch on the right. The direction of algebraic increase of u is indicated by the arrow. The change of the variable of integration from t to u through A.4 is equivalent to a shift in the contour from the real t-axis to a branch of the hyperbola. Since the real t-axis can be deformed continuously into the appropriate branch of the hyperbola without crossing a singularity, the two corresponding contour integrals are equal.

Now we have

$$\sqrt{-p^2 + t^2} = - i\rho \sin(\phi + iu)$$

$$- itz - |a| \sqrt{-p^2 + t^2} = i p\rho \cosh u$$

$$dt = i\rho \sin(\phi + iu)\, du$$

Therefore

$$\int_{-\infty}^{\infty} \frac{e^{-|a| \sqrt{-p^2 + t^2}}}{\sqrt{-p^2 + t^2}} e^{-itz}\, dt = \int_{-\infty}^{\infty} e^{ip\rho \cosh u}\, du \qquad (A.5)$$

By a well-known integral representation of the Hankel function (22)

$$i\pi H_0^{(1)}(p\rho) = \int_{-\infty}^{\infty} e^{ip\rho \cosh u}\, du$$

A.5 reduces to the formula we set out to prove, since

$$\rho = \sqrt{z^2 + a^2}\ .$$

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