MEASURES IN TOPOLOGICAL SPACES

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ABSTRACT

Let X be a completely-regular topological space and let $C^*(X)$ denote the space of all bounded, real-valued continuous functions on X. For a positive linear functional φ on $C^*(X)$, consider the following two continuity conditions. φ is said to be a B-integral if whenever $\{u_n\} \subseteq C^*(X)$ and $u_n(x) \downarrow 0$ for all $x \in X$, then $\varphi(u_n) \downarrow 0$. φ is said to be B-normal if whenever $\{u_\tau\} \subseteq C^*(X)$ is a directed system with $u_\tau(x) \downarrow 0$ for all $x \in X$, then $\varphi(u_\tau) \downarrow 0$. It is obvious that a B-normal functional is always a B-integral. The main concern of this paper is what can be said in the converse direction.

Methods are developed for discussing this question. Of particular importance is the representation of $C^*(X)$ as a space $\mathcal{M}(X)$ of finitely-additive set functions on a certain algebra of subsets of X. This result was first announced by A. D. Alexandrov, but his proof was obscure. Since there seem to be no proofs readily available in the literature, a complete proof is given here. Supports of functionals are discussed and a relatively simple proof is given of the fact that every B-integral is B-normal if and only if every B-integral has a support.

The space X is said to be B-compact if every B-integral is B-normal. It is shown that B-compactness is a topological invariant and various topological properties of B-compact spaces are investigated. For instance, it is shown that B-compactness is permanent on the closed sets and the co-zero sets of a B-compact space. In the case that the spaces involved are locally-compact, it is shown that countable

products and finite intersections of B-compact spaces are B-compact.

Also B-compactness is studied with reference to the classical compactness conditions. For instance, it is shown that if X is B-compact, then X is realcompact. Or that if X is paracompact and if the continuum hypothesis holds, then X is B-compact if and only if X is realcompact.

Finally, the methods and results developed in the paper are applied to formulate and prove a very general version of the classical Kolmogorov consistency theorem of probability theory. The result is as follows. If X is a locally-compact, B-compact space and if S is an abstract set, then a necessary and sufficient condition that a finitely-additive set function defined on the Baire (or the Borel) cylinder sets of X^S be a measure is that its projection on each of the finite coordinate spaces be Baire (or regular Borel) measures.

TABLE OF CONTENTS

INTRODUCTION

The main objective of this paper is to study the relation between certain continuity conditions for linear functionals on spaces of continuous functions. More specifically, let X be a completely-regular topological space and let $C^*(X)$ denote the bounded, real-valued continuous functions on X. A non-negative linear functional φ on $C^*(X)$ is said to be a B-integral if for every decreasing sequence $\{u_n\} \subseteq C^*(X)$ with $u_n(x) \downarrow 0$ for all $x \in X$, then $\varphi(u_n) \downarrow 0$. φ is said to be B-normal if for every decreasing net $\{u_\tau\} \subseteq C^*(X)$ with $u_\tau(x) \downarrow 0$ for all $x \in X$, then $\varphi(u_\tau) \downarrow 0$. It is clear that every B-normal functional is a B-integral. The main concern in the paper will be what can be said in the converse direction.

The paper is divided into four parts and ten chapters. The first part is a preliminary section which catalogues the background information required in the sequel. In two chapters, the basic properties of Riesz spaces and completely regular topological spaces are listed.

The second part is devoted to a study of the dual space of $C^*(X)$ and contains chapters three through six. In chapter three, the dual space of $C^*(X)$ is represented as a space of finitely-additive set functions. In chapters five and six, this representation is used to show that B-integrals and B-normal functionals correspond respectively to so-called σ-additive and net-additive set-functions. In chapter six, the concept of support is introduced and studied. Part two culminates with a proof of the very useful fact that all B-integrals are B-normal if and only if there are no B-integrals which are entirely without support.

In part three, the methods developed previously are employed to study the problem posed; namely, when are B-integrals also B-normal. A completely regular space is said to be B-compact if all B-integrals on C*(X) are B-normal. In chapter seven, the topological implications of B-compactness are studied. In chapter eight, the considerations are restricted to locally-compact, B-compact spaces. It is shown that countable products and finite intersections of locally compact, B-compact spaces are B-compact. In chapter nine, B-compactness is studied in relation to other compactness conditions of point-set topology. For instance, it is shown that if the continuum hypothesis holds a paracompact space is B-compact if and only if it is realcompact.

In part four, an application of the methods and results developed is made to probability theory. Here two very general versions of the classical Kolmogorov consistency theorem are formulated and proved.

PART I

PRELIMINARY INFORMATION

This section contains the background information required in the paper. The results here will be stated without proof. As a reference for chapter 1, the reader is referred to [14] and [15]. For the results in chapter 2, the reader should consult [2].

CHAPTER I

RIESZ SPACES

Definition: Let L be a linear space over the field R of real numbers, and let L have a partial ordering $\leq$. L is said to be an ordered linear space if it satisfies the two conditions:

(i) $0 \leq u, v \in L$ implies that $0 \leq u + v$.

(ii) $0 \leq u \in L$ and $0 \leq \alpha \in R$ implies that $0 \leq \alpha u$.

Definition: If L is an ordered linear space and if L is a lattice with respect to its partial ordering, then L is said to be a Riesz space.

Let L be a Riesz space. For $u \in L$, let $u^+ = \sup(u, 0)$, $u^- = -\inf(u, 0)$, and $|u| = \sup(u, -u)$. Then $u = u^+ - u^-$, $|u| = u^+ + u^-$, $u^+ = \frac{1}{2}(u + |u|)$, and $u^- = \frac{1}{2}(|u| - u)$. Hence an ordered linear space is a Riesz space if and only if for each $u \in L$, one of u^+, u^-, or $|u|$ is in L.

Definition: A Riesz space L is said to be Dedekind complete if for any set $A \subseteq L$ such that there exists $v \in L$ with $u \leq v$ for all $u \in A$, then $\sup A \in L$.

Theorem 1.1. Let L be a Riesz space. L is Dedekind complete if and only if every set of positive elements in L which is bounded above has a least upper bound.

The proof of this theorem can be found in [14], Note VI.

Definition: Let L be a Riesz space and K be a Riesz subspace of L. K is said to be an ideal if $u \in L$ and $|u| \leq v$ for some $v \in K$ implies that $u \in K$.

Definition: Let L be a Riesz space and K an ideal in L. K is said to be a band if whenever $A \subseteq K$ with $\sup A \in L$, then $\sup A \in K$.

The intersection of an arbitrary collection of bands in a Riesz space is again a band. If L is a Riesz space and $A \subseteq L$, then the intersection of all the bands which contain A is said to be the band generated by A.

Theorem 1.2. Let L be a Dedekind complete Riesz space and let K be a band in L. Then $L = K \oplus K^p$, where $K^p = \{u \in L: \inf(|u|, |v|) = 0 \text{ for all } v \in K\}$.

The decomposition above is called a Riesz decomposition.

Definition: Let L be a Riesz space. A linear functional φ on L is said to be order bounded if for all $0 \leq u \in L$, $\sup \{\varphi(v): 0 \leq v \leq u, v \in L\} < +\infty$. The collection of all order bounded linear functionals on L is called the order dual of L and is denoted by $L^{\sim}$.

Theorem 1.3. *If L is a Riesz space, then $L^{\sim}$ is a Dedekind complete Riesz space.*

Theorem 1.4. *If L is a Riesz space and if $\varphi \in L^{\sim}$, then for* $0 \leq u \in L$,

$$\varphi^{+}(u) = \sup \{\varphi(v) : 0 \leq v \leq u,\ v \in L\},$$

$$\varphi^{-}(u) = -\inf \{\varphi(v) : 0 \leq v \leq u,\ v \in L\},$$

$$|\varphi|(u) = \sup \{|\varphi(v)| : 0 \leq v \leq u,\ v \in L\}.$$

CHAPTER II

COMPLETELY-REGULAR SPACES

Definition. A topological space X is said to be completely-regular if X is Hausdorff and if for each $x \in X$ and for each closed set $G \subseteq X$ with $x \notin G$, there is a continuous function which vanishes on G and is 1 on x.

In what follows, C(X) will denote the space of all real-valued continuous functions on X and C*(X) will denote the subspace of C(X) consisting of bounded functions. C(X) and C*(X) are both Riesz spaces with respect to the usual pointwise ordering. Furthermore, C*(X) is a Banach lattice with respect to the topology of uniform convergence on X. The Banach dual and the order dual of C*(X) are identical.

Theorem 2.1. Let X be a topological space. X is completely-regular if and only if the topology on X coincides with the weak topology induced on X by C*(X).

Definition: Let X be a topological space. If $u \in C^*(X)$, the set $Z(u) = \{x \in X : u(x) = 0\}$ is called a zero set. The collection of all zero sets on X is denoted by $\mathfrak{Z}(x)$. Likewise, the set $U(u) = \{x \in X : u(x) \neq 0\}$ is called a u-set (or a co-zero set). The collection of all u-sets of X is denoted by $\mathcal{U}(x)$.

Theorem 2.2. Let X be a topological space. X is completely-regular if and only if $\mathcal{U}(x)$ ($\mathfrak{Z}(x)$) is a basis for the open (closed) sets in X.

A finite union or a countable intersection of zero sets is again a zero set. Dually, a finite intersection or a countable union of u-sets is again a u-set.

Theorem 2.3. Let X be completely-regular. Then there is a compactification βX of X such that each element of $C^*(X)$ has a unique extension to an element of $C(\beta X)$. Furthermore, βX is unique up to homeomorphism.

The space βX is called the Stone-Cêch compactification of X. Note that each $f \in C(X)$ also has a unique extension to βX although the extended function may be $+\infty$ or $-\infty$ at some points of βX.

Definition: Let X be completely-regular. X is said to be realcompact if for each point $x \in \beta X - X$, there is a $0 \leq f \in C(X)$ such that the extension $\bar{f}$ of f to βX has $\bar{f}(x) = +\infty$.

Theorem 2.4. Let X be completely-regular. Then there is a realcompact space υX containing X as a dense subspace such that each element of $C(X)$ has a unique extension to an element of $C(\upsilon X)$. Furthermore, υX is unique up to homeomorphism.

Theorem 2.5. Let X be completely-regular. Then X is realcompact if and only if it is homeomorphic to a closed subspace of $R^{C(X)}$.

Theorem 2.6. Let X be completely-regular. Then υX is the intersection of a family of locally-compact, σ-compact subsets of βX.

For a point $x \in \beta X$, let $\mathcal{Z}_x$ denote the family of zero sets Z in X such that the closure of Z in βX contains x. Then $\mathcal{Z}_x$ is a filter and it is principal if and only if $x \in X$. If the intersection of a countable number of elements in $\mathcal{Z}_x$ is again in $\mathcal{Z}_x$, then $\mathcal{Z}_x$ is said to be closed under countable intersection.

Theorem 2.7. Let X be completely-regular. Then $\upsilon X \subseteq \beta X$. Furthermore, $x \in \upsilon X$ if and only if $\mathcal{Z}_x$ is closed under countable intersection.

Theorem 2.8. Let X be completely-regular and let $\{u_n\} \subseteq C^*(X)$ be a monotone decreasing sequence with $u_n(x) \downarrow 0$ for all $x \in X$. If $\overline{u}_n$ denotes the extension of u_n to βX, then $\overline{u}_n(t) \downarrow 0$ for all $t \in \upsilon X$.

Proof: Let $t \in \upsilon X$ and assume that $\overline{u}_n(t) \downarrow \alpha > 0$. Set $Z_n = \{x : u_n(x) \geq \alpha\}$. Then $\{Z_n\} \subseteq \mathcal{Z}_t$ and $\mathcal{Z}_t$ is closed under countable intersection by Theorem 2.7. Thus $Z = \cap\{Z_n : n \in N\} \in \mathcal{Z}_t$ and so in particular, $Z \neq \varphi$. But if $x_0 \in Z$, $u_n(x_0) \geq \alpha$ for all n. This contradicts the fact that $u_n(x) \downarrow 0$ for all $x \in X$.

PART II

FUNCTIONAL ANALYTIC PRELIMINARIES

CHAPTER III

A REPRESENTATION THEOREM

In this chapter, a very useful representation of the dual space of C*(X) will be given (see Theorem 3.13). This representation was first announced by A. D. Alexandrov, [5]; but the proof is rather inaccessible in his work. Since it seems not to be readily available in the literature, a complete proof will be given here. Since the proof is long, it will be broken into three parts. In the first part a space $\mathcal{M}(X)$ of set functions will be studied; in the second part, $\mathcal{M}(X)$ will be imbedded in $[C^*(X)]^\sim$; in the third part, it will be shown that $\mathcal{M}(X)$ provides a very suitable representation of $[C^*(X)]^\sim$

A. The Space $\mathcal{M}(X)$

In the following X will always denote a completely-regular topological space. As above, $\mathcal{Z}(X)$ and $\mathcal{U}(X)$ denote the zero sets and the u-sets of X respectively. Let $\mathfrak{F}(X)$ denote the algebra of subsets of X generated by $\mathcal{Z}(X)$. (This is, of course, the same as the algebra generated by $\mathcal{U}(X)$.)

Lemma 3.1. $A \in \mathfrak{F}(X)$ if and only if there is a positive integer n and zero sets Z_i, Z_i' for $i = 1, \ldots, n$ with

$$A = \bigcup_{i=1}^{n} Z_i \cap \complement Z_i' .$$

<u>Proof</u>: Let $\mathcal{B}$ denote the family of all sets of the form

$$A = \bigcup_{i=1}^{n} Z_i \cap \complement Z_i'$$

as in the Lemma. It is clear that $\mathcal{B} \subseteq \mathfrak{F}(X)$. Furthermore, $\mathfrak{Z}(X) \subseteq \mathcal{B}$ and $\mathcal{B}$ is closed under finite unions. If it can be shown that $\mathcal{B}$ is closed under complementation, then $\mathcal{B}$ will be an algebra. It will then follow that $\mathfrak{F}(X) \subseteq \mathcal{B}$ and the proof will be complete.

Let

$$A = \bigcup_{i=1}^{n} Z_i \cap \complement Z_i' \in \mathcal{B} .$$

Then
$$\complement A = \bigcap_{i=1}^{n} \complement Z_i \cup Z_i' .$$

Let $\mathcal{P}$ be the power set of $\{1, \ldots, n\}$. For $S \in \mathcal{P}$, let $S' = \{1, \ldots, n\} - S$. For each $S \in \mathcal{P}$, set $Z_S = \bigcap_{i \in S} Z_i'$ and $Z_S' = \bigcup_{i \in S'} Z_i$. Then for each $S \in \mathcal{P}$, $Z_S, Z_S' \in \mathfrak{Z}(X)$. Furthermore, $\complement A = \bigcup_{S \in \mathcal{P}} Z_S \cap \complement Z_S'$. Hence it is clear that $\complement A \in \mathcal{B}$.

<u>Definition</u>: A finitely-additive, real-valued set function m defined on $\mathfrak{F}$ will be called <u>regular</u> if for each $A \in \mathfrak{F}$ and each $\epsilon > 0$, there is a $U \in \mathcal{U}(X)$ with $A \subseteq U$ such that $|m(B)| < \epsilon$ for each $B \in \mathfrak{F}$ for which $B \subseteq U - A$.

<u>Lemma 3. 2.</u> <u>Let</u> $0 \leq m$ <u>be a finitely-additive set function on</u> $\mathfrak{F}(X)$. <u>Then the following are equivalent.</u>

(i) m <u>is regular.</u>

(ii) <u>For all</u> $A \in \mathfrak{F}(X)$, $m(A) = \inf\{m(U) : A \subseteq U,\ U \in \mathcal{U}(X)\}$.

(iii) <u>For all</u> $A \in \mathfrak{F}(X)$, $m(A) = \sup\{m(Z) : Z \subseteq A,\ Z \in \mathfrak{Z}(X)\}$.

Definition: $\mathcal{M}(X)$ will denote the space of all regular real-valued, finitely-additive set functions m defined on $\mathfrak{F}(X)$ and such that

$$\sup\{|m(A)| : A \in \mathfrak{F}\} < \infty .$$

For $m_1, m_2 \in \mathcal{M}(X)$ and $\alpha, \beta \in R$, define $(\alpha m_1 + \beta m_2)(A) = \alpha m_1(A) + \beta m_2(A)$ for all $A \in \mathfrak{F}(X)$. It is clear that $\alpha m_1 + \beta m_2$ is again in $\mathcal{M}(X)$. Furthermore, define $m_1 \leq m_2$ if $m_1(A) \leq m_2(A)$ for all $A \in \mathfrak{F}$. This relation is a partial ordering compatible with the linear structure of $\mathcal{M}(X)$. Hence $\mathcal{M}(X)$ is an ordered linear space.

Definition: Let $m \in \mathcal{M}(X)$. For $A \in \mathfrak{F}$, define

$$m^+(A) = \sup\{m(B) : B \subseteq A,\ B \in \mathfrak{F}\},$$

$$m^-(A) = -\inf\{m(B) : B \subseteq A,\ B \in \mathfrak{F}\},$$

$$|m|(A) = \sup\{m(B) - m(C) : B \cup C \subseteq A,\ B \cap C = \emptyset,\ B, C \in \mathfrak{F}\}.$$

Lemma 3.3. Let $m \in \mathcal{M}(X)$. Then $m^+, m^-, |m| \in \mathcal{M}(X)$.

Proof: It is easy to check that $m^- = m^+ - m$ and that $|m| = m^+ + m^-$. Since $\mathcal{M}(X)$ is a linear space, it will thus be sufficient to show that $m^+ \in \mathcal{M}(X)$. It is clear that m^+ is a non-negative, real-valued function on $\mathfrak{F}$ and that $\sup\{m^+(A) : A \in \mathfrak{F}(X)\} < +\infty$.

a) m^+ is finitely-additive on $\mathfrak{F}(X)$.

Let $A, B \in \mathfrak{F}$ and $\varepsilon > 0$. There exists $C \in \mathfrak{F}(X)$ with $C \subseteq A \cup B$ and $m^+(A \cup B) < m(C) + \varepsilon$. From the finite-additivity of m and the definition of m^+,

$$m^+(A \cup B) - \epsilon < m(C) = m(C \cap A) + m(C-A) \leq m^+(A) + m^+(B) .$$

Since $\epsilon > 0$ was arbitrary, m^+ is subadditive on $\mathfrak{F}(X)$.

Now assume that $A \cap B = \phi$. There exist $A_1 \subseteq A$ and $B_1 \subseteq B$ for which $m^+(A) < \epsilon + m(A_1)$ and $m^+(B) < \epsilon + m(B_1)$. Since $A_1 \cap B_1 = \phi$ and m is additive,

$$m^+(A) + m^+(B) - 2\epsilon < m(A_1) + m(B_1) = m(A_1 \cup B_1) \leq m^+(A \cup B) .$$

Since $\epsilon > 0$ was arbitrary, the result follows from the last equation and the subadditivity of m^+.

b) <u>m^+ is regular on $\mathfrak{F}(X)$.</u>

Let $A \in \mathfrak{F}$ and $\epsilon > 0$. Since m is regular, there exist $U \in \mathcal{U}(X)$ with $A \subseteq U$ such that for every $B \in \mathfrak{F}$ with $B \subseteq U-A$, $|m(B)| < \epsilon$. Hence for such a B,

$$0 \leq m^+(B) = \sup\{m(C) : C \subseteq B \text{ and } C \in \mathfrak{F}\} \leq \epsilon .$$

This proves the regularity of m^+.

The next result is analogous to the Hahn decomposition theorem of measure theory.

<u>Theorem 3.4.</u> <u>Let</u> $m \in \mathcal{M}(X)$. <u>For every</u> $\epsilon > 0$, <u>there exist sets</u> $Z_1, Z_2 \in \mathfrak{Z}(X)$ <u>with</u> $Z_1 \cap Z_2 = \phi$ <u>such that</u> $m^+(Z_2) < \epsilon$ <u>and</u> $m^-(Z_1) < \epsilon$. <u>Furthermore</u>, $m^+(Z_1) > m^+(X) - \epsilon$ <u>and</u> $m^-(Z_2) > m^-(X) - \epsilon$.

<u>Proof:</u> There exists $Z_1 \in \mathfrak{Z}(X)$ for which $m^+(X) < m(Z_1) + \epsilon/2$ by the regularity of m. Hence,

$$m^-(Z_1) = m^+(Z_1) - m(Z_1) < m^+(Z_1) - m^+(X) + \frac{\varepsilon}{2} < \varepsilon \ .$$

Also,

$$(*) \quad m^-(X-Z_1) = m^+(X-Z_1) - m(X-Z_1)$$
$$\geq m^-(X) + m(Z_1) - m^+(Z_1)$$
$$\geq m^-(X) - \frac{\varepsilon}{2} \ .$$

Choose $Z_2 \subseteq X - Z_1$, $Z_2 \in \mathfrak{Z}(X)$ such that $m^-(Z_2) > m^-(X-Z_1) - \varepsilon/2$. Then $m^-(Z_2) > m^-(X) - \varepsilon$ by (*). Furthermore, $m^+(Z_2) \leq m^+(X - Z_1) < \varepsilon/2 < \varepsilon$. Finally, it is obvious that $Z_1 \cap Z_2 = \phi$.

Definition: For $m \in \mathcal{M}(X)$, define $||m|| = |m|\,(X)$.

It is clear that $||\cdot||$ is a norm on $\mathcal{M}(X)$. Furthermore, $|m_1| \leq |m_2|$ implies that $||m_1|| \leq ||m_2||$.

Lemma 3.5. *Let* $0 \leq m_n \in \mathcal{M}(X)$ *for* $n = 1, 2, \ldots$. *If*

$$\sum_{n=1}^{\infty} ||m_n|| < \infty \ ,$$

then there exists $m \in \mathcal{M}(X)$ *such that*

$$m = \sum_{n=1}^{\infty} m_n \ .$$

Proof: For each $A \in \mathfrak{F}(X)$,

$$\left\{ \sum_{k=1}^{n} m_k(A) \right\}_{m \in N}$$

is a positive increasing sequence of real numbers. Furthermore,

$$\sum_{k=1}^{n} m_k(A) \le \sum_{k=1}^{n} m_k(X) \le \sum_{n=1}^{\infty} ||m_n|| \ .$$

Thus define

$$m(A) = \sum_{k=1}^{\infty} m_k(A) \ .$$

It is clear that m is a non-negative, finitely-additive set function on $\mathfrak{F}(X)$ and that $m(X) < \infty$. The proof will be complete if m is shown to be regular. Thus let $A \in \mathfrak{F}(X)$ and $\epsilon > 0$. Choose n large enough so that

$$m(X) < \sum_{k=1}^{n} m_k(X) + \epsilon \ .$$

Since $\sum_{k=1}^{n} m_k$ is regular, there exists $U \in \mathcal{U}(X)$ with $A \subseteq U$ and

$$\sum_{k=1}^{n} m_k(U-A) < \epsilon \ .$$

Thus,

$$\begin{aligned} m(U-A) &= m(X) - m(X - (U-A)) \\ &< \sum_{k=1}^{n} m_k(X) + \epsilon - \sum_{k=1}^{n} m_k(X - (U-A)) \\ &< \sum_{k=1}^{n} m_k(U-A) + \epsilon \\ &< 2\epsilon \ . \end{aligned}$$

Hence m is regular.

Theorem 3.6. 1. $\mathcal{M}(X)$ is a Dedekind complete Riesz space.

2. $\mathcal{M}(X)$ is a Banach lattice with respect to the norm $||m|| = |m|(X)$.

Since the norm in condition (2) has the property that, for $0 \leq m_1, m_2 \in \mathcal{M}(X)$, $||m_1 + m_2|| = ||m_1|| + ||m_2||$, $\mathcal{M}(X)$ is an L-space in the sense of Kakutani.

Proof: To show that $\mathcal{M}(X)$ is a Riesz space, it is enough to show that for each $m \in \mathcal{M}(X)$, $m^+ = \sup(m, 0)$. It will then follow that $m^- = \sup(-m, 0)$ and $|m| = \sup(m, -m)$.

It is clear that $0 \leq m^+$ and that $m \leq m^+$. Let $m' \in \mathcal{M}(X)$ with $0 \leq m'$ and $m \leq m'$. Then for all $A \in \mathfrak{F}(X)$,

$$\begin{aligned} m^+(A) &= \sup\{m(B) : B \subseteq A,\ B \in \mathfrak{F}(X)\} \\ &\leq \sup\{m'(B) : B \subseteq A,\ B \in \mathfrak{F}(X)\} \\ &\leq m'(A) \ , \end{aligned}$$

since $0 \leq m'$ implies that m' is monotone. Thus $m^+ \leq m'$, and so $m^+ = \sup(m, 0)$.

To see that $\mathcal{M}(X)$ is Dedekind complete, let $\Omega \subseteq \mathcal{M}(X)$ with $m \leq m_0$ for all $m \in \Omega$ where $m_0 \in \mathcal{M}(X)$. By Theorem 1.1, assume that $0 \leq m$ for all $m \in \Omega$. Since $\mathcal{M}(X)$ is a Riesz space, assume without loss of generality that Ω is directed upward. (That is, assume that Ω is closed with respect to taking the supremum over a finite number of elements in Ω.)

For $A \in \mathfrak{F}(X)$, define $m'(A) = \sup\{m(A) : m \in \Omega\}$. If it can be shown that $m' \in \mathcal{M}(X)$, then $m' = \sup \Omega$ and the proof will be done. But m' is clearly finitely-additive and $0 \leq \sup\{m'(A) : A \in \mathfrak{F}(X)\} \leq m_0(X) < +\infty$. Hence all that need be shown is that m' is regular.

Let $A \in \mathfrak{F}(X)$ and $\varepsilon > 0$. Let $m \in \Omega$ be such that $m'(X) < m(X) + \varepsilon/2$. Since m is regular, there exists $U \in \mathcal{U}(X)$ with $A \subseteq U$ and $0 \leq m(U-A) < \varepsilon/2$. Hence,

$$\begin{aligned} 0 \leq m'(U-A) &= m'(X) - m'(X - (U-A)) \\ &< m(X) + \frac{\varepsilon}{2} - m(X - (U-A)) \\ &< m(U-A) + \frac{\varepsilon}{2} < \varepsilon \ . \end{aligned}$$

Thus m' is regular.

(2) If $|m_1| \leq |m_2|$, then $||m_1|| \leq ||m_2||$. Thus all that need by proven is the completeness under the given norm. Thus let $||m_n - m_k|| \to 0$ as $n, k \to \infty$. Without loss of generality, assume that

$$\sum_{n=1}^{\infty} ||m_{n+1} - m_n|| < \infty \ .$$

By Lemma 3.5,

$$m = \sum_{n=1}^{\infty} |m_{n+1} - m_n| \in \mathcal{M}(X) \ .$$

Define

$$m_n' = m_n - m_1 + \sum_{k=1}^{n-1} |m_{k+1} - m_k|$$

for $n = 1, 2, \ldots$. Since $m_{n+1}' - m_n' = (m_{n+1} - m_n) + |m_{n+1} - m_n| \geq 0$ and since $m_{n+1}' - m_n' \leq 2|m_{n+1} - m_n|$, it follows from Lemma 3.5 that

$$m' = \sum_{k=1}^{\infty} (m_{k+1}' - m_k') \in \mathcal{M}(X) .$$

Set $m'' = m' - m + m_1$.

The claim is that $||m'' - m_n|| \to 0$ as $n \to \infty$ which will complete the proof. Indeed,

$$\begin{aligned} m'' - m_n &= m' - m + m_1 - m_n \\ &= m' - m + \sum_{k=1}^{n-1} |m_{k+1} - m_k| - m_n' \\ &= \sum_{k=n}^{\infty} (m_{k+1}' - m_k') - \sum_{k=n}^{\infty} (m_{k+1} - m_k) . \end{aligned}$$

Hence,

$$|m'' - m_n| \le \sum_{k=n}^{\infty} (m_{k+1}' - m_k') + \sum_{k=n}^{\infty} |m_{k+1} - m_k| .$$

From the last equation, it is clear that $||m'' - m_n|| \to 0$ as $n \to \infty$.

The proof given of the completeness of $\mathcal{M}(X)$ is due to Luxemburg and Zaanen (see [14], Note VIII).

B. The Imbedding of $\mathcal{M}(X)$ in $[C^*(X)]^\sim$

The description of the structure of $\mathcal{M}(X)$ is now complete. The next step is to show that it can be imbedded suitably in the space $[C^*(X)]^\sim$.

By L(X), denote the Riesz space of step functions on $\mathfrak{F}$. That is, L(X) consists of all functions of the form

$$u = \sum_{k=1}^{n} \alpha_k \chi_{A_k} ,$$

where α_k is real, $A_k \in \mathfrak{F}(X)$, and χ_{A_k} is the characteristic function of A_k for $k = 1, \ldots, n$. Since $\mathfrak{F}$ is a ring, each element of $L(X)$ has a canonical representation such that $A_k \cap A_\ell = \phi$ for $k \neq \ell$.

Let $0 \leq m \in \mathfrak{M}(X)$. For

$$u = \sum_{k=1}^{n} \alpha_k \chi_{A_k}$$

in canonical form, define

$$\varphi_m(u) = \sum_{k=1}^{n} \alpha_k m(A_k) \ .$$

Then φ_m is a positive linear functional on $L(X)$. By the usual Riemann process, φ_m may be extended to a positive linear functional Φ_m on the space $\mathcal{R}_m$ of all Riemann m-integrable functions. The following theorem is recalled without proof.

Theorem 3.7. The following statements are equivalent.

(i) $u \in \mathcal{R}_m$.

(ii) There exist $u_1, u_2 \in L(X)$ such that $u_1 \leq u \leq u_2$ and $\varphi_m(u_2 - u_1) < \epsilon$.

Since every $f \in C^*(X)$ can be uniformly approximated by functions in $L(X)$, the above theorem guarantees that $C^*(X) \subseteq \mathcal{R}_m$. Let $\hat{m}$ denote the restriction of Φ_m to $C^*(X)$. Then $0 \leq \hat{m} \in [C^*(X)]^{\sim}$.

Thus far only non-negative elements in $\mathfrak{M}(X)$ have been considered. For an arbitrary $m \in \mathfrak{M}(X)$, define $\hat{m} = (m^+)^{\wedge} - (m^-)^{\wedge}$. Then $\hat{m} \in [C^*(X)]^{\sim}$. The next theorem shows that the mapping $\wedge$ has the right properties.

Theorem 3.8. The mapping $\wedge : \mathcal{M}(X) \to [C^*(X)]^\sim$ has the following properties.

(i) $(\alpha m_1 + \beta m_2)^\wedge = \alpha \hat{m}_1 + \beta \hat{m}_2$.

(ii) $m_1 \le m_2 \Rightarrow \hat{m}_1 \le \hat{m}_2$.

(iii) $(m^+)^\wedge = (\hat{m})^+$; $(m^-)^\wedge = (\hat{m})^-$; $(|m|)^\wedge = |\hat{m}|$.

(iv) $\hat{m}_1 = \hat{m}_2 \Rightarrow m_1 = m_2$.

(v) $||m|| = ||\hat{m}||$.

Proof: (i) Let $u \in C^*(X)$. Take a sequence $\{u_n\} \subseteq L(X)$ such that $u_n \to u$ uniformly on x. If

$$u_n = \sum_{k=1}^{K(n)} \alpha_{n,k} \chi_{A_{n,k}}$$

then

$$(\alpha m_1 + \beta m_2)^\wedge(u) = \lim_{n\to\infty} \sum_{k=1}^{K(n)} \alpha_{n,k} (\alpha m_1 + \beta m_2)(A_{n,k})$$

$$= \alpha \cdot \lim_{n\to\infty} \sum_{k=1}^{K(n)} \alpha_{n,k} m_1(A_{n,k}) + \beta \cdot \lim_{n\to\infty} \sum_{k=1}^{K(n)} \alpha_{n,k} m_2(A_{n,k})$$

$$= \alpha \hat{m}_1(u) + \beta \hat{m}_2(u) .$$

(ii) If $0 \le m \in \mathcal{M}(X)$, then the definition of $\hat{m}$ guarantees that $0 \le \hat{m}$. Let $m_1 \le m_2$. Then by (i),

$$0 \le (m_2 - m_1)^\wedge = \hat{m}_2 - \hat{m}_1 .$$

Thus $\hat{m}_1 \le \hat{m}_2$.

(iii) Let $m \in \mathcal{M}(X)$. By (ii) $0 \le (m^+)^\wedge$; and, by definition, $\hat{m} = (m^+)^\wedge - (m^-)^\wedge \le (m^+)^\wedge$. Thus $(\hat{m})^+ \le (m^+)^\wedge$.

By Theorem 3.4, there exist $Z_1, Z_2 \in \mathcal{Z}(X)$ with $Z_1 \cap Z_2 = \phi$ and such that $m^+(X-Z_1) < \varepsilon$ and $m^-(X-Z_2) < \varepsilon$. Let $w \in C^*(X)$ be such that $0 \le w \le 1$, $w = 1$ on Z_1, and $w = 0$ on Z_2.

For $0 \le u \in C^*(X)$ with $0 \le u \le 1$,

$$(m^+)^\wedge(u - w\cdot u) + (m^-)^\wedge(w\cdot u) \le m^+(X-Z_1) + m^-(X-Z_2) < 2\varepsilon .$$

Thus,

$$\begin{aligned}(m^+)^\wedge(u) &< (m^+)^\wedge(w\cdot u) - (m^-)^\wedge(w\cdot u) + 2\varepsilon = \hat{m}(w\cdot u) + 2\varepsilon \\ &< \sup\{\hat{m}(v) : 0 \le v \le u,\ v \in C^*(X)\} + 2\varepsilon \\ &< (\hat{m})^+(u) + 2\varepsilon .\end{aligned}$$

Since $\varepsilon > 0$ was arbitrary, $(m^+)^\wedge(u) \le (\hat{m})^+(u)$ for all $0 \le u \le 1$, $u \in C^*(X)$. Thus $(m^+)^\wedge \le (\hat{m})^+$. Hence from above, it follows that $(m^+)^\wedge = (\hat{m})^+$.

Finally, $(m^-)^\wedge = (m^+ - m)^\wedge = (m^+)^\wedge - \hat{m} = (\hat{m})^+ - \hat{m} = (\hat{m})^-$. Also $(|m|)^\wedge = (m^+ + m^-)^\wedge = (m^+)^\wedge + (m^-)^\wedge = (\hat{m})^+ + (\hat{m})^- = |\hat{m}|$.

(iv) Assume that $\hat{m}_1 = \hat{m}_2$. Since $(\hat{m}_1)^+ = (\hat{m}_2)^+$ and $(\hat{m}_1)^- = (\hat{m}_2)^-$, it may be assumed by (iii) that $0 \le m_1$ and $0 \le m_2$. Furthermore, by the regularity of m_1 and m_2, it is sufficient to show that $m_1(U) = m_2(U)$ for every $U \in \mathcal{U}(X)$.

Thus let $U \in \mathcal{U}(X)$. By the regularity of m_1 and m_2, there is a $Z \in \mathcal{Z}(X)$ with $Z \subseteq U$ and such that $m_1(U-Z) < \varepsilon$, $m_2(U-Z) < \varepsilon$. Take $w \in C^*(X)$ with $0 \le w \le 1$ and $\chi_Z \le w \le \chi_U$. Then it follows that,

$$m_1(U) - m_2(U) < \varepsilon + m_1(Z) - m_2(U)$$
$$< \varepsilon + \hat{m}_1(w) - \hat{m}_2(w) = \varepsilon .$$

Similarly, $m_2(U) - m_1(U) < \varepsilon$; and hence $|m_2(U) - m_1(U)| < \varepsilon$. Since $\varepsilon > 0$ was arbitrary, $m_1(U) = m_2(U)$. It follows that $m_1 = m_2$.

(v) $|\hat{m}(1)| \le |\hat{m}|(1) = (|m|)^{\wedge}(1) = |m|(X) = ||m||$. Thus $||\hat{m}|| \le ||m||$. But also,

$$||\hat{m}|| = \sup \{|\hat{m}(u)| : u \in C^*(X),\ ||u||_\infty \le 1\}$$
$$\ge \sup \{|\hat{m}(u)| : u \in C^*(X),\ 0 \le u \le 1\}$$
$$\ge |\hat{m}|(1) = (|m|)^{\wedge}(1) = |m|(X) = ||m|| .$$

Thus $||\hat{m}|| = ||m||$ and the proof is complete.

C. The Representation of $C^*(X)$

In Theorem 3. 8, it has been shown that $\mathcal{M}(X)$ is isometric and isomorphic to a closed subspace of $[C^*(X)]^\sim$. The representation will be complete once it is shown that the mapping $\wedge$ defined above is onto.

Definition: For $0 \le \varphi \in [C^*(X)]^\sim$ define, for each $U \in \mathcal{U}(X)$, $\lambda(U) = \sup \{\varphi(v) : 0 \le v \in C^*(X) \text{ and } v \le \chi_U\}$.

Lemma 3. 9.

(i) λ is non-negative, monotone, and subadditive on $\mathcal{U}(X)$.

(ii) λ is additive on $\mathcal{U}(X)$.

(iii) If $U \in \mathcal{U}(X)$ and if $\varepsilon > 0$, then there exists $W \in \mathcal{U}(X)$ with $\overline{W} \subseteq U$ such that $\lambda(U) < \lambda(W) + \varepsilon$.

Proof: (i) λ is obviously non-negative and monotone. Let $U_1, U_2 \in \mathcal{U}(X)$ and let $\epsilon > 0$. Choose $0 \le v \in C^*(X)$ such that $v \le \chi_{U_1 \cup U_2}$ and $\varphi(v) + \epsilon > \lambda(U_1 \cup U_2)$. Next set $Z = \{x : v(x) \ge \epsilon\} \cap \complement U_2$. Let $u \in C^*(X)$ be such that $0 \le u \le 1$, $u = 1$ on Z, and $u = 0$ on $\complement U_1$. Then $0 \le v \cdot u \le \chi_{U_1}$ and $0 \le (v - v \cdot u - \epsilon)^+ \le \chi_{U_2}$. Hence,

$$\lambda(U_1 \cup U_2) < \varphi(v) + \epsilon \le \varphi(v \cdot u) + \varphi((v - v \cdot u - \epsilon)^+) + 2\epsilon$$

$$\le \lambda(U_1) + \lambda(U_2) + 2\epsilon .$$

Since $\epsilon > 0$ was arbitrary, the result follows.

(ii) Let $U_1, U_2 \in \mathcal{U}(X)$ with $U_1 \cap U_2 = \phi$. Let $0 \le v_1$, $v_2 \in C^*(X)$ be such that $v_1 \le \chi_{U_1}, v_2 \le \chi_{U_2}$ with $\lambda(U_1) < \varphi(v_1) + \epsilon$, $\lambda(U_2) < \varphi(v_2) + \epsilon$. Since $U_1 \cap U_2 = \phi$, $v_1 + v_2 = \sup(v_1, v_2)$. Hence, $0 \le v_1 + v_2 \le \chi_{U_1 \cup U_2}$ and

$$\lambda(U_1) + \lambda(U_2) < \varphi(v_1 + v_2) + 2\epsilon \le \lambda(U_1 \cup U_2) + 2\epsilon .$$

Since $\epsilon > 0$ was arbitrary, $\lambda(U_1 \cup U_2) \ge \lambda(U_1) + \lambda(U_2)$. This fact together with (i) yields the additivity.

(iii) Let $U \in \mathcal{U}(X)$ and $\epsilon > 0$ be given. Choose $0 \le u \in C^*(X)$ such that $u \le \chi_U$ and $\lambda(U) < \varphi(u) + \epsilon$. Set $W = \{x : u(x) > \epsilon\}$. Then observe that $\overline{W} \subseteq U$. Set $w = (u - \epsilon)^+$. Then $0 \le w \in C^*(X)$ and $w \le \chi_W$. Hence,

$$\lambda(W) \ge \varphi(w) = \varphi((u - \epsilon)^+) \ge \lambda(U) - 2\epsilon .$$

Thus the result is proved.

Definition: For $A \subseteq X$, define $m^*(A) = \inf\{\lambda(U) : U \in \mathcal{U}(X)$ and $A \subseteq U\}$.

Lemma 3.10.

(i) m^* is an outer measure.

(ii) If $U \in \mathcal{U}(X)$, then $m^*(U) = \lambda(U)$.

(iii) If $U \in \mathcal{U}(X)$, then U is m^*- measurable in the sense of Caratheodory.

Proof: (i) That m^* is an outer measure follows immediately from Lemma 3.7 and the definition of m^*.

(ii) This is clear from the definition of m^*.

(iii) Let $A \subseteq X$ and $U \in \mathcal{U}(X)$. It must be shown that $m^*(A) \geq m^*(A \cap U) + m^*(A-U)$. Let $\epsilon > 0$ be arbitrary. Choose $V \in \mathcal{U}(X)$ such that $A \subseteq V$ and $m^*(A) > \lambda(V) - \epsilon$. By (iii) of Lemma 3.7, choose $W \in \mathcal{U}(X)$ with $\overline{W} \subseteq U$ and $\lambda(W) > \lambda(U) - \epsilon$. Then,

$$\begin{aligned} \epsilon + m^*(A) &> \lambda(V) = \lambda[(V \cap W) \cup (V-W)] \\ &\geq \lambda(V \cap W) + \lambda(V-\overline{W}) \\ &\geq \lambda(V \cap W) + m^*(A-U) \;. \end{aligned}$$

Thus,

$$(*) \qquad \epsilon + m^*(A) \geq \lambda(V \cap W) + m^*(A-U) \;.$$

But

$$\begin{aligned} m^*(V \cap U) &\leq m^*(V \cap W) + m^*(V \cap (U-W)) \\ &\leq \lambda(V \cap W) + m^*(U-W) \;. \end{aligned}$$

Thus,

$$(**) \qquad m^*(V \cap U) \leq m^*(V \cap W) + m^*(U-W) \;.$$

Now choose $T \in \mathcal{U}(X)$ such that $\overline{T} \subseteq W$ and $\lambda(W) < \lambda(T) + \varepsilon$. Then

$$\begin{aligned} m^*(U-W) &\leq m^*(U-\overline{T}) = \lambda(U-\overline{T}) \\ &\leq \lambda(U) - \lambda(T) \\ &\leq \lambda(U) - \lambda(W) + \varepsilon < 2\varepsilon , \end{aligned}$$

by Lemma 3.9. Hence $m^*(U-W) < 2\varepsilon$. Combining this fact with (*) and (**) above, it follows that,

$$\begin{aligned} \varepsilon + m^*(A) &\geq \lambda(V \cap U) + m^*(A-U) - 2\varepsilon \\ &\geq m^*(A \cap U) + m^*(A-U) - 2\varepsilon . \end{aligned}$$

Since $\varepsilon > 0$ was arbitrary, the result follows.

From Lemmas 3.9 and 3.10, the following result is immediate.

Corollary 3.11. Let $0 \leq \varphi \in [C^*(X)]^{\sim}$ and let m^* be the outer measure associated with φ. If $Z \in \mathcal{Z}(X)$, then $m^*(Z) = \inf\{\varphi(u) : 0 \leq u \in C^*(X),\ u \geq \chi_Z\}$.

It is known from measure theory that the family of Caratheodory m^*-measurable subsets of X form an algebra Ω_{m^*} and that m^* is a finitely-additive, monotone set function on Ω_{m^*}. By (iii) of Lemma 3.10, $\mathfrak{F}(X) \subseteq \Omega_{m^*}$. Let m_φ denote the restriction of m^* to $\mathfrak{F}(X)$.

Lemma 3.12. For $0 \leq \varphi \in [C^*(X)]^{\sim}$, there is a finitely-additive set function m_φ on $\mathfrak{F}(X)$ such that,

(i) $m_\varphi \in \mathcal{M}(X)$.

(ii) $\hat{m}_\varphi = \varphi$.

Proof: (i) m_φ is certainly non-negative and finitely-additive on $\mathfrak{F}(X)$. The regularity is an immediate consequence of the definition of m^*.

(ii) To prove (ii), it is clearly sufficient to show that, for all $u \in C^*(X)$ with $0 \le u < 1$, $\varphi(u) = \hat{m}_\varphi(u)$. ($u < 1$ here means $u(x) < 1$ for all $x \in X$.) But to show this it is enough to show that $0 \le u < 1$ implies $\hat{m}_\varphi(u) \ge \varphi(u)$. Indeed an application of this inequality to $1-u$ then yields the desired result.

Fix $u \in C^*(X)$ with $0 \le u < 1$. Define $Z_k = \{x : u(x) \ge k/n\}$ for $k = 0, 1, \ldots, n$. Then

$$\hat{m}_\varphi(u) \ge \sum_{k=0}^{n-1} \frac{k}{n} \cdot m_\varphi(Z_k - Z_{k+1})$$

$$\ge \sum_{k=0}^{n-1} \frac{k}{n} \cdot \{m_\varphi(Z_k) - m_\varphi(Z_{k+1})\}$$

$$\ge \frac{1}{n} \cdot \sum_{k=1}^{n} m_\varphi(Z_k) \;,$$

since $u < 1$ implies that $m_\varphi(F_n) = 0$.

But by Corollary 3.11, there are $u_k \in C^*(X)$ with $\chi_{Z_k} \le u_k$ and $m_\varphi(Z_k) \ge \varphi(u_k) - 1/n$, for $k = 1, 2, \ldots, n$. Thus

$$\frac{1}{n} \sum_{k=1}^{n} m_\varphi(Z_k) \ge \varphi\left(\frac{1}{n} \sum_{k=1}^{n} u_k\right) - \frac{1}{n} \cdot \varphi(1) \;.$$

It is clear that $u \le \frac{1}{n} \cdot \sum_{k=1}^{n} u_k + \frac{1}{n}$. Hence,

$$\varphi(u) \le \varphi\left(\frac{1}{n}\sum_{k=1}^{n} u_k\right) + \frac{1}{n}\varphi(1) .$$

Thus it follows that

$$\varphi(u) \le \hat{m}_{\varphi}(u) + \frac{2}{n}\varphi(1) .$$

Letting n tend to ∞, it follows that $\varphi(u) \le \hat{m}_{\varphi}(u)$ and the proof is complete.

Theorem 3.13. *Let X be a completely-regular topological space. If $C^*(X)$ is the space of all bounded, continuous, real-valued functions on X, then $[C^*(X)]^{\sim}$ is isomorphic as a Banach lattice to $\mathcal{M}(X)$. Moreover, the isomorphism is given by $m \to \hat{m}$ where $\hat{m}$ denotes the Riemann integral of m on $C^*(X)$.*

Proof: By Theorem 3.8, $\mathcal{M}(X)$ is isometrically isomorphic to a subspace of $C^*(X)$ under the mapping $\wedge$. The proof will be complete if $\wedge$ is shown to be onto. Let $\varphi \in [C^*(X)]^{\sim}$. By Lemma 3.12, there exists $m_{\varphi^+}, m_{\varphi^-} \in \mathcal{M}(X)$ such that $\hat{m}_{\varphi^+} = \varphi^+$ and $\hat{m}_{\varphi^-} = \varphi^-$. Set $m = m_{\varphi^+} - m_{\varphi^-} \in \mathcal{M}(X)$. By the linearity of $\wedge$, $\hat{m} = \varphi$ and the result follows.

CHAPTER IV

B-NORMAL FUNCTIONALS AND B-INTEGRALS

Definition: Let $\{u_\tau\}_{\tau \in T} \subseteq C^*(X)$. $\{u_\tau\}$ is said to be a downward (upward) directed system if for every pair u_{τ_1}, u_{τ_2}, there exists u_{τ_3} such that $u_{\tau_3} \le \inf(u_{\tau_1}, u_{\tau_2})$ $(u_{\tau_3} \ge \sup(u_{\tau_1}, u_{\tau_2}))$.

Definition: Let $\{u_\tau\}_{\tau \in T} \subseteq C^*(X)$ be a downward (upward) directed system. An element $u \in C^*(X)$ is said to be the B-limit of $\{u_\tau\}$ if for each $x \in X$, $u(x) = \inf\{u_\tau(x) : \tau \in T\}$ $(u(x) = \sup\{u_\tau(x) : \tau \in T\})$. This is written $u_\tau \downarrow^B u$ $(u_\tau \uparrow_B u)$.

Concerning the above definition, it should be noted that the B-limit of a directed system $\{u_\tau\}$ is not in general the same as the order limit of the system $\{u_\tau\}$. (An element $u \in C^*(X)$ is said to be the order limit of the downward (upward) directed system $\{u_\tau\}$ if $u \le u_\tau$ $(u \ge u_\tau)$ for all τ, and if $v \in C^*(X)$ and $v \le u_\tau$ $(v \ge u_\tau)$ for all τ imply that $v \le u$ $(v \ge u)$.) It is clear that the existence of the B-limit implies the existence of the order limit and the equality of the two. However, the order limit may exist without the B-limit existing.

If $C^*(X)$ is considered as a Riesz subspace of the space $B(X)$ of all bounded, real-valued functions on X, then the B-limit of a directed system $\{u_\tau\}$ in $C^*(X)$ is the order limit of the system $\{u_\tau\}$ in $B(X)$.

The notion of a B-limit above gives a meaning 'a fortiori' to a B-limit of a monotone sequence in $C^*(X)$. Because of its importance, however, this definition is given separately.

Definition: Let $\{u_n\}_{n \in N} \subseteq C^*(X)$ be a decreasing (increasing) sequence. An element $u \in C^*(X)$ is called the B-limit of $\{u_n\}$ if for all $x \in X$, $u_n(x) \downarrow u(x)$ $(u_n(x) \uparrow u(x))$. This is written $u_n \downarrow^B u$ $(u_n \uparrow_B u)$.

The above concepts of convergence can be used to define continuity properties for elements in $[C^*(X)]^\sim$. These are of central importance in this paper.

Definition: Let $\varphi \in [C^*(X)]^\sim$. Then φ is said to be B-normal if for every downward directed system $\{u_\tau\} \subseteq C^*(X)$ with $u_\tau \downarrow^B 0$, it follows that $|\varphi|(u_\tau) \downarrow 0$. The set of all B-normal functionals in $[C^*(X)]^\sim$ is denoted by $(C^*)^\sim_n$.

Definition: Let $\varphi \in [C^*(X)]^\sim$. Then φ is said to be a B-integral if for every decreasing sequence $\{u_n\} \subseteq C^*(X)$ with $u_n \downarrow^B 0$, it follows that $|\varphi|(u_n) \downarrow 0$. The set of all B-integrals in $[C^*(X)]^\sim$ is denoted by $(C^*)^\sim_c$.

Note that $\varphi \in [C^*(X)]^\sim$ is B-normal if and only if $u_\tau \downarrow^B u$ $(u_\tau \uparrow u)$ implies that $|\varphi|(u_\tau) \downarrow |\varphi|(u)$ $(|\varphi|(u_\tau) \uparrow |\varphi|(u))$. Also φ is a B-integral if and only if $u_n \downarrow^B u$ $(u_n \uparrow_B u)$ implies that $|\varphi|(u_n) \downarrow |\varphi|(u)$ $(|\varphi|(u_n) \uparrow |\varphi|(u))$.

It is clear that $(C^*)^\sim_n \subseteq (C^*)^\sim_c$. The converse relation is not in general true. Consider the following example.

Example 4.1. Set $X = \Omega$, where Ω denotes the space of ordinals less than the first uncountable ordinal with the order topology. Then Ω is a completely-regular, locally-compact space (see [2], p. 72). Furthermore, Ω is pseudocompact so that every element of $[C^*(X)]^\sim$ is a B-integral (see [6]).

The Stone-Ceĉh compactification is given by the one-point compactification $\beta\Omega = \Omega \cup \{\Omega\}$. Thus for each $f \in C^*(\Omega)$, define $\varphi(f) = \bar{f}(\Omega)$, where $\bar{f}$ denotes the unique extension of f to $\beta\Omega$. Since Ω is pseudocompact, φ is a B-integral.

For $\alpha \in \Omega$, define

$$f_\alpha(t) = \begin{cases} 0, & t \leq \alpha \\ 1, & t > \alpha . \end{cases}$$

Then $\{f_\alpha\}_{\alpha \in \Omega} \subseteq C^*(\Omega)$ is a downward directed system with $f_\alpha \downarrow^B 0$. But $\varphi(f_\alpha) = 1$ for all $\alpha \in \Omega$. Hence φ is not B-normal.

CHAPTER V

NET-ADDITIVE AND σ-ADDITIVE MEASURES

Definition: Let $\{A_\tau\}_{\tau \in T}$ be a family of subsets of X. $\{A_\tau\}$ is said to be downward (upward) directed if for each pair A_{τ_1}, A_{τ_2}, there is an A_{τ_3} with $A_{\tau_2} \subseteq A_{\tau_1} \cap A_{\tau_3}$ $\left(A_{\tau_3} \supseteq A_{\tau_1} \cup A_{\tau_2}\right)$. In this case, $A_\tau \downarrow A$ ($A_\tau \uparrow A$) where $A = \cap\{A_\tau : \tau \in T\}$ ($A = \cup\{A_\tau : \tau \in T\}$).

Definition: Let $m \in \mathcal{M}(X)$. Then m is said to be net-additive if for every downward directed family $\{Z_\tau\} \subseteq \mathcal{Z}(X)$ with $Z_\tau \downarrow \varphi$, it follows that $|m|(Z_\tau) \downarrow \phi$. The set of all net-additive elements of $\mathcal{M}(X)$ is denoted by $\mathcal{M}_n$.

Definition: Let $m \in \mathcal{M}(X)$. Then m is said to be σ-additive if for every decreasing sequence $\{Z_n\} \subseteq \mathcal{Z}(X)$ with $Z_n \downarrow \phi$, it follows that $|m|(Z_n) \downarrow 0$. The set of all σ-additive elements of $\mathcal{M}(X)$ is denoted by $\mathcal{M}_c$.

It is clear that $\mathcal{M}_n \subseteq \mathcal{M}_c$. However, the converse is not true in general. (See Theorem 5.11 and Example 4.1).

Lemma 5.1. Let $m \in \mathcal{M}(X)$. Then the following are equivalent.

(i) m is net-additive.

(ii) $\{Z_\tau\} \subseteq \mathcal{Z}(X)$, $Z \in \mathcal{Z}(X)$, and $Z_\tau \downarrow Z$ imply that $|m|(Z_\tau) \downarrow |m|(Z)$.

(iii) $\{U_\tau\} \subseteq \mathcal{U}(X)$, $U \in \mathcal{U}(X)$, and $U_\tau \uparrow U$ imply that $|m|(U_\tau) \uparrow |m|(U)$.

(iv) $\{U_\tau\} \subseteq \mathcal{U}(X)$ and $U_\tau \uparrow X$ implies that $|m|(U_\tau) \uparrow |m|(X)$.

Proof: It is obvious that (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv) $\Rightarrow$ (i). Only (i) $\Rightarrow$ (ii) need be proved. By the regularity, for $\epsilon > 0$, there is a $U \in \mathcal{U}(X)$ with $Z \subseteq U$ and $m(U) < m(Z) + \epsilon$. Thus $Z_\tau - U \downarrow \phi$ and hence $|m|(Z_\tau - U) \downarrow 0$. But

$$|m|(Z_\tau - U) = |m|(Z_\tau) - |m|(Z) - |m|(Z_\tau \cap (U-Z))$$

$$\geq |m|(Z_\tau) - |m|(Z) - \epsilon .$$

Hence, $\epsilon \geq \limsup |m|(Z_\tau) - |m|(Z) \geq 0$. Since $\epsilon > 0$ was arbitrary, the result follows.

Lemma 5.2. Let $0 \leq m \in \mathcal{M}(X)$. Then the following are equivalent.

(i) m is σ-additive.

(ii) If $\{A_n\} \subseteq \mathfrak{Z}(X)$ and $A_n \downarrow \varphi$, then $m(A_n) \downarrow 0$.

(iii) If $\{A_n\} \subseteq \mathfrak{Z}(X)$,

$$A = \bigcup_{n=1}^{\infty} A_n \in \mathfrak{Z}(X),$$

and $A_n \cap A_m = \phi$ for $n \neq m$, then

$$m(A) = \sum_{n=1}^{\infty} m(A_n) .$$

Proof: (i) (ii). Let $\{A_n\} \subseteq \mathfrak{Z}(X)$ with $A_n \downarrow \phi$. Assume that $m(A_n) \downarrow \alpha > 0$. By the regularity of m, for each $n \in N$, there is a

$Z_n \in \mathcal{Z}(X)$ with $Z_n \subseteq A_n$ and $m(A_n - Z_n) < \alpha/2^{n+1}$. Define

$$Z'_n = \bigcap_{k=1}^{n} Z_n \in \mathcal{Z}(X) .$$

Then $Z'_n \downarrow \emptyset$ and hence $m(Z'_n) \downarrow 0$.

However, for each $n \in N$,

$$m(A_n - Z'_n) \leq \sum_{k=1}^{n} m(A_k - Z_k) < \frac{\alpha}{2} , \quad \text{and}$$

$$m(A_n - Z'_n) = m(A_n) - m(Z'_n) \geq \alpha - m(Z'_n) .$$

Thus $m(Z'_n) \geq \alpha/2 > 0$ for all n. This is a contradiction and the result follows.

(ii) $\Rightarrow$ (iii). If $\{A_n\} \subseteq \mathfrak{Z}$ is as in (iii) above, define

$$B_n = A - \bigcup_{k=1}^{n} A_k .$$

Then $\{B_n\} \subseteq \mathfrak{Z}(X)$ and $B_n \downarrow \emptyset$. That is

$$m\left(\bigcup_{k=1}^{n} A_k\right) = \sum^{n} m(A_k) \uparrow m(A) .$$

(iii) $\Rightarrow$ (i). Let $\{Z_n\} \subseteq \mathcal{Z}(X)$ and $Z_n \downarrow \emptyset$. Then

$$Z_n = \bigcup_{k=n}^{\infty} (Z_k - Z_{k+1}) .$$

Hence by assumption,

$$m(Z_n) = \sum_{k=n}^{\infty} m(Z_k - Z_{k+1}) .$$

Since $m(Z_n)$ is the remainder after n terms of a convergent sequence, $m(Z_n) \to 0$.

Definition: Let $Ba(X)$ denote the σ-algebra generated by $\mathcal{U}(X)$. $Ba(X)$ is called the Baire sets of X.

Definition: Let $Bo(X)$ denote the σ-algebra generated by the open sets in X. $Bo(X)$ is called the Borel sets of X.

Note that $Ba(X)$ is the σ-algebra generated by $\mathcal{Z}(X)$ or by $\mathfrak{F}(X)$. Likewise, $Bo(X)$ is the σ-algebra generated by the closed sets of X. Furthermore, $Ba(X) \subseteq Bo(X)$. However, the equality $Ba(X) = Bo(X)$ is not in general valid.

Definition: Let $0 \leq m$ be a measure on $Ba(X)$. Then m is said to be a regular Baire measure if for all $A \in Ba(X)$, $m(A) = \inf\{m(U) : U \in \mathcal{U}(X) \text{ and } A \subseteq U\}$. (Equivalently, $m(A) = \sup\{m(Z) : Z \in \mathcal{Z}(X) \text{ and } Z \subseteq A\}$.)

Definition: Let $0 \leq m$ be a measure on $Bo(X)$. Then m is said to be a regular Borel measure if for all $A \in Bo(X)$, $m(A) = \inf\{m(O) : O \text{ is open and } A \subseteq O\}$. (Equivalently, $m(A) = \sup\{m(G) : G$ is closed and $G \subseteq A\}$.)

Theorem 5.3. Let $0 \leq m \in \mathcal{M}_c$. Then there is a unique extension of m to a regular Baire measure.

Proof: By Lemma 5.2, m is a measure on $\mathfrak{F}(X)$. Hence m may be extended by the usual Caratheodory process to a regular Baire measure.

The uniqueness of the extension is an immediate consequence of the regularity.

Definition: Let $0 \leq m$ be a Borel measure. Then m is said to be net-additive if for every downward directed family $\{G_\tau\}$ of closed sets with $G_\tau \downarrow \phi$, it follows that $m(G_\tau) \downarrow 0$.

Note the m is a net-additive Borel measure if and only if for every upward directed family $\{O_\tau\}$ of open sets with $O_\tau \uparrow X$, it follows that $m(O_\tau) \uparrow m(X)$.

Theorem 5.4. Let $0 \leq m \in \mathcal{M}_n$. Then there is a unique extension of m to a regular, net-additive Borel measure.

Before a proof of this theorem can be given a series of lemmas must be established. The process for extending a measure considered here is sometimes called the Bourbaki extension procedure.

Lemma 5.6. Let $0 \leq m \in \mathcal{M}_n$. Assume that $O \subseteq X$ is open and $\{U_\tau\}$, $\{V_\sigma\} \subseteq \mathcal{U}(X)$ are upward directed with $U_\tau \uparrow O$ and $V_\sigma \uparrow O$. Then $\lim_\tau m(U_\tau) = \lim_\sigma m(V_\sigma)$.

Proof: Let $m(U_\tau) \uparrow \alpha$. For $\epsilon > 0$, there is U_{τ_0} such that $m(U_{\tau_0}) > \alpha - \epsilon$. Since $V_\sigma \cap U_{\tau_0} \uparrow U_{\tau_0}$, the net-additivity of m implies that

$$\lim_\sigma m(V_\sigma) \geq m(U_{\tau_0}) > \alpha - \epsilon .$$

Thus $\lim_\sigma m(V_\sigma) \geq \alpha = \lim_\tau m(U_\tau)$. The result then follows.

Definition: For $0 \le m \in \mathcal{M}_n$, define $\lambda(O) = \sup\{m(U) : U \in \mathcal{U}(X)$ and $U \subseteq O\}$ for each open set O in X.

Lemma 5. 7.

1. λ is monotone and non-negative.

2. If O_1, O_2 are open, then $\lambda(O_1 \cup O_2) \le \lambda(O_1) + \lambda(O_2)$.

3. For $U \in \mathcal{U}(X)$, $\lambda(U) = m(U)$.

Proof:

1. This is obvious.

2. Let $\{U_\tau\}, \{V_\sigma\} \subseteq \mathcal{U}(X)$ with $U_\tau \uparrow O_1$ and $V_\sigma \uparrow O_2$. By Lemma 5. 6, $m(U_\tau \cup V_\sigma) \uparrow \lambda(O_1 \cup O_2)$. Thus,

$$\lambda(O_1 \cup O_2) = \lim_{\tau,\sigma} m(U_\tau \cup V_\sigma) \le \lim_{\tau,\sigma} (m(U_\tau) + m(V_\sigma))$$
$$\le \lambda(O_1) + \lambda(O_2) .$$

3. This is obvious.

Definition: Let $0 \le m \in \mathcal{M}_n$. For $A \subseteq X$, define $m^*(A) = \inf\{\lambda(O) : O$ is open and $A \subseteq O\}$.

Lemma 5. 8.

1. m^* is an outer measure.

2. If $O \subseteq X$ is open, then O is m^*-measurable in the sense of Caratheodory.

Proof:

1. This follows from Lemma 5. 7.

2. Let $A \subseteq X$. Take $O \subseteq X$ open. By 1, it is sufficient to

show that $m^*(A) \geq m^*(A \cap O) + m^*(A-O)$. Hence let $\varepsilon > 0$. Choose an open set $O_1 \subseteq X$ with $A \subseteq O_1$ and $\varepsilon + m^*(A) > \lambda(O_1)$. Let $\{U_\tau\} \subseteq \mathcal{U}(X)$ with $U_\tau \uparrow O_1$. Finally, take $V \in \mathcal{U}(X)$ subject only to $V \subseteq O$. By the regularity of m, there is a $Z \in \mathcal{Z}(X)$ with $Z \subseteq V$ and $m(Z) + \varepsilon > m(V)$. Thus,

$$\begin{aligned}
\varepsilon + m^*(A) > \lambda(O_1) &= \lim_\tau m(U_\tau) \\
&> \lim_\tau [m(U_\tau \cap V) + m(U_\tau - V)] \\
&> \lambda(O_1 \cap V) + \lim_\tau m(U_\tau - Z) - \varepsilon \\
&> \lambda(O_1 \cap V) + \lambda(O_1 - Z) - \varepsilon \\
&> \lambda(O_1 \cap V) + m^*(O_1 - O) - \varepsilon \\
&> \lambda(O_1 \cap V) + m^*(A-O) - \varepsilon .
\end{aligned}$$

Since $V \in \mathcal{U}(X)$, $V \subseteq O$ was arbitrary, it follows that

$$\begin{aligned}
2\varepsilon + m^*(A) &\geq \lambda(O_1 \cap O) + m^*(A-O) \\
&\geq m^*(A \cap O) + m^*(A-O) .
\end{aligned}$$

Since $\varepsilon > 0$ was arbitrary, the result follows.

If $\mathcal{B}$ denotes the algebra of subsets generated by the open subsets of X, the above lemma implies that m^* is a finitely-additive set function on $\mathcal{B}$. Let μ denote the restriction of m^* to $\mathcal{B}$.

Lemma 5.9.

1. If $\{G_\tau\}$ is a downward directed family of closed sets with $G_\tau \downarrow \phi$, then $\mu(G_\tau) \downarrow 0$.

2. If $\{A_n\} \subseteq \mathcal{B}$, $A = \bigcup_{n=1}^{\infty} A_n \in \mathcal{B}$, and $A_n \cap A_m = \phi$,

for $n \neq m$, then

$$\mu(A) = \sum_{n=1}^{\infty} \mu(A_n) .$$

Proof: 1. Since $\mathcal{Z}(X)$ is a basis for the closed sets in X, $G_\tau = \cap\{Z : Z \in \mathcal{Z}(X) \text{ and } G_\tau \subseteq Z\}$. Set $\mathcal{S} = \{Z : Z \in \mathcal{Z}(X) \text{ and } Z \supseteq G_\tau$ for some $\tau\}$. Since $G_\tau \downarrow \phi$, $\mathcal{S} \downarrow \phi$. Since m is net-additive and since $m(Z) = \mu(Z)$ for all $Z \in \mathcal{Z}(X)$, it follows that $0 = \inf\{\mu(Z) : Z \in \mathcal{S}\}$. Hence by the monotonicity of μ, $\mu(G_\tau) \downarrow 0$.

2. Set

$$B_n = A - \bigcup_{k=1}^{n} A_k \in \mathcal{Y} .$$

Assume that $\mu(B_n) \downarrow \alpha > 0$. By the definition of m^*, there exists a closed $G_n \subseteq B_n$ such that $m(B_n) < m(G_n) + \alpha/2^{n+1}$. Set

$$G'_n = \bigcap_{k=1}^{n} G_k .$$

Then $G'_n \downarrow \phi$; and hence by (1), $\mu(G'_n) \downarrow 0$. However,

$$\mu(B_n - G'_n) \geq \sum_{k=1}^{n} \mu(B_k - G_k) \geq \frac{\alpha}{2},$$

and

$$\mu(B_n - G'_n) = \mu(B_n) - \mu(G'_n) \geq \alpha - \mu(G'_n) .$$

Thus $\inf_n \mu(G'_n) \geq \alpha/2 > 0$. This is a contradiction and the result follows.

Proof of Theorem 5.4: By Lemma 5.9, μ is an extension of m to a measure on $\mathcal{Y}$. Then by the usual Caratheodory process, μ can be extended to a regular Borel measure which again is denoted by μ. Furthermore, μ is net-additive by Lemma 5.9.

The uniqueness of the extension follows from the regularity. Indeed if λ is an extension of m satisfying the conditions in Theorem 5.4, then for any open set O in X, $\lambda(O) = \sup\{m(U) : U \in \mathcal{U}(X)$ and $U \subseteq O\} = \mu(O)$. The regularity then implies that $\lambda = \mu$.

Corollary 5.10. Let $0 \leq m \in \mathcal{M}_n$. If μ and λ are two regular Borel extensions of m, then $\mu = \lambda$.

Proof: Let μ be the extension given by Theorem 5.4. For a closed set $G \subseteq X$,

$$\lambda(G) \leq \inf\{\lambda(Z) : Z \in \mathfrak{Z}(X),\ Z \supseteq G\} = \mu(G),$$

since μ is net-additive.

Hence if $\{G_\tau\}$ is a downward directed family of closed sets with $G_\tau \downarrow 0$,

$$0 \leq \inf_\tau \lambda(G_\tau) \leq \inf_\tau \mu(G_\tau) = 0.$$

Thus λ is net-additive and hence $\lambda = \mu$ by Theorem 5.4.

In order to extend a measure from the Baire sets to the Borel sets, net-additivity is in general indispensable. It will be seen later that there are in general σ-additive elements of $\mathcal{M}(X)$ which have no extensions to Borel measures even if the requirement of regularity is dropped.

There is an intimate relation between net-additive set functions and B-normal functionals. This will be the subject of the next theorem. Let $\varphi \in [C^*(X)]^\sim$ and let m_φ denote the corresponding element of $\mathcal{M}(X)$ according to Theorem 3.13. That is, φ is the Riemann integral on $C^*(X)$ with respect to the set function m_φ.

Theorem 5.11. Let $\varphi \in [C^*(X)]^\sim$ and let $m_\varphi \in \mathcal{M}(X)$ correspond to φ.

1. φ is a B-integral if and only if m_φ is σ-additive.
2. φ is B-normal if and only if m_φ is net-additive.

Proof: Without loss of generality, assume $0 \le \varphi$. Since 1. is proved in exactly the same manner as 2., only the proof of 2. will be given.

Hence assume that φ is B-normal. Let $\{Z_\tau\}_{\tau \in T} \subseteq \mathcal{Z}(X)$ with $Z_\tau \downarrow \phi$. For each $\tau \in T$, take $v_\tau \subseteq C^*(X)$ with $0 \le v_\tau \le 1$ and $Z_\tau = \{x : v_\tau(x) = 1\}$. Let S denote the family of all finite subsets of T. For each $\sigma \in S$ and $n \in N$, define

$$u_{(\sigma, n)} = [\inf\{v_\tau : \tau \in \sigma\}]^n .$$

It is clear that $\{u_{(\sigma, n)}\} \subseteq C^*(X)$ is a downward directed system with $u_{(\sigma, n)} \downarrow^B 0$. Hence $\varphi(u_{(\sigma, n)}) \downarrow 0$. Also,

$$(*) \quad \varphi(u_{(\sigma, n)}) \ge m_\varphi\,(\cap\{Z_\tau : \tau \in \sigma\}) .$$

Since $\{Z_\tau\}$ is downward directed, (*) implies that $m_\varphi(Z_\tau) \downarrow 0$. Hence m_φ is net-additive.

On the other hand, assume that m_φ is net-additive. Let $\{u_\tau\} \subseteq C^*(X)$ be a downward directed system with $u_\tau \downarrow^B 0$. Furthermore, assume that $u_\tau \le 1$ for all τ. For $\epsilon > 0$, set $Z_\tau = \{x : u_\tau(x) \ge \epsilon\}$. Then $\{Z_\tau\} \subseteq \mathcal{Z}(X)$ and $Z_\tau \downarrow \phi$. Hence $m_\varphi(Z_\tau) \downarrow 0$ and,

$$\varphi(u_\tau) = \int_{Z_\tau} u_\tau \, dm_\varphi + \int_{X-Z_\tau} u_\tau \, dm_\varphi$$

$$\leq 1 \circ m_\varphi(Z_\tau) + \varepsilon \cdot m_\varphi(X) \; .$$

Hence $0 \leq \limsup_{\tau} \varphi(u_\tau) \leq \varepsilon \circ m_\varphi(X)$. Since $\varepsilon > 0$ was arbitrary, $\varphi(u_\tau) \downarrow 0$ and the proof is complete.

CHAPTER VI

SUPPORTS

Definition: Let $\varphi \in [C^*(X)]^\sim$. Set $\mathcal{U}_\varphi = \{u : u \in C^*(X), 0 \leq u \leq 1, \text{ and } |\varphi|(u) = |\varphi|(1)\}$. Define $S_\varphi = \cap\{W(u) : u \in \mathcal{U}_\varphi\}$, where $W(u) = \{x : u(x) = 1\}$. S_φ is said to be the support of φ. If $S_\varphi = \phi$, then ϕ is said to be entirely without support.

Definition: Let $m \in \mathcal{M}(X)$. Set $T_m = \cap\{Z : Z \in \mathcal{Z}(X) \text{ and } |m|(Z) = |m|(X)\}$. T_m is said to be the support of m. If $T_m = \phi$, then m is said to be entirely without support.

It is obvious that S_φ and T_m are always closed.

Theorem 6.1. Let $m \in \mathcal{M}(X)$. Then $x \in T_m$ if and only if whenever $U \in \mathcal{U}(X)$ and $x \in U$, then $|m|(U) > 0$.

Proof: Clearly if $x \notin T_m$, then there is a $U \in \mathcal{U}(X)$ with $x \in U$ and $|m|(U) = 0$. On the other hand, assume $x \in T_m$ and $U \in \mathcal{U}(X)$ with $x \in U$. If $|m|(U) = 0$, then $T_m \subseteq X - U$ which gives a contradiction.

Corollary 6.2. Let $m \in \mathcal{M}(X)$. Then m is entirely without support if and only if for each $x \in X$, there exists $U \in \mathcal{U}(X)$ such that $x \in U$ and $|m|(U) = 0$.

Theorem 6.3. Let $\varphi \in [C^*(X)]^\sim$ and let m_φ be the corresponding element in $\mathcal{M}(X)$. Then $S_\varphi = T_{m_\varphi}$.

Proof: Assume that $0 \le \varphi$. Let $Z \in \mathfrak{Z}(X)$ with $m_\varphi(X) = m_\varphi(Z)$. Take $u \in C^*(X)$ with $0 \le u \le 1$ and $Z = \{x : u(x) = 1\}$. Then,

$$\varphi(1) = m_\varphi(X) = m_\varphi(Z) \le \varphi(u) \le \varphi(1) .$$

Hence $u \in \mathcal{U}_\varphi$ and so $S_\varphi \subseteq Z$. It then follows that $S_\varphi \subseteq T_{m_\varphi}$.

If $T_{m_\varphi} = \phi$, the theorem is proved. Thus assume that $x_0 \in T_{m_\varphi} - S_\varphi$. Then there is a $u \in \mathcal{U}_\varphi$ such that $u(x_0) = \alpha < 1$. Set $v = (u-\alpha)^+/1-\alpha$. It is clear that $0 \le v \le 1$. Furthermore,

$$\varphi(1) \ge \varphi(v) \ge \frac{\varphi(u) - \alpha\varphi(1)}{1-\alpha} = \frac{\varphi(1)(1-\alpha)}{1-\alpha} = \varphi(1) .$$

Hence $v \in \mathcal{U}_\varphi$ and $v(x_0) = 0$.

Set $W = \{x : v(x) < 1/2\}$. Then $x_0 \in W$ and $W \in \mathcal{U}(X)$. Hence by Theorem 6.1, $m_\varphi(W) > 0$. But,

$$0 = \varphi(1-v) \ge \int_W (1-v)\, dm_\varphi \ge \frac{1}{2} \cdot m_\varphi(W) > 0 .$$

This is a contradiction and the theorem is proved.

Corollary 6.4. Let $\varphi \in [C^*(X)]^\sim$ and let m_φ be the corresponding element in $\mathfrak{M}(X)$. Then φ is entirely without support if and only if m_φ is entirely without support.

Theorem 6.5.

1. Let $0 \ne m \in \mathfrak{M}(X)$. If m is net-additive, then $T_m \ne \phi$.
2. Let $0 \ne \varphi \in [C^*(X)]^\sim$. If φ is B-normal, then $S_\varphi \ne \phi$.

Proof: 1. Let $\mathcal{S} = \{Z : |m|(X) = |m|(Z),\ Z \in \mathfrak{Z}(X)\}$. Then $\mathcal{S}$ is a downward directed family with $\mathcal{S} \downarrow T_m$. If $T_m = \phi$, the net-additivity of m implies that $\inf\{|m|(Z) : Z \in \mathcal{S}\} = 0$. Since $m \neq 0$, this is a contradiction. Hence $T_m \neq \phi$.

2. If φ is B-normal, m_φ is net-additive by Theorem 5.11. Thus $T_{m_\varphi} \neq \phi$ by (1). Finally $T_{m_\varphi} = S_\varphi \neq \phi$ by Theorem 6.3.

<u>Theorem 6.6.</u>

1. <u>If</u> $0 \neq m \in \mathcal{M}(X)$ <u>is not net-additive, then there exists</u> $0 < m' \in \mathcal{M}(X)$ <u>with</u> $m' \leq |m|$ <u>and</u> m' <u>entirely without support.</u>

2. <u>If</u> $0 \neq \varphi \in [C^*(X)]^\sim$ <u>is not B-normal, then there exists</u> $0 < \varphi' \in [C^*(X)]^\sim$ <u>with</u> $\varphi' < |\varphi|$ <u>and</u> φ' <u>entirely without support.</u>

<u>Proof:</u> 1. Since m is not net-additive, there exists a downward directed family $\{Z_\tau\}_{\tau \in T} \subseteq \mathcal{Z}(X)$ with $Z_\tau \downarrow \phi$ and $|m|(Z_\tau) \downarrow \alpha > 0$. For $A \in \mathfrak{F}(X)$ define $m'(A) = \inf\{|m|(A \cap Z_\tau) : \tau \in T\}$. It is clear that $0 \leq m' \in \mathcal{M}(X)$. Also $m'(X) = \alpha > 0$ implies $m' > 0$. Furthermore, $m' \leq m$. Finally, $m'(Z_\tau) = m'(X)$ for all $\tau \in T$ implies that $T_{m'} \subseteq \cap\{Z_\tau : \tau \in T\} = \phi$. That is, m' is entirely without support.

2. Since φ is not B-normal, m_φ is not net-additive by Theorem 5.11. Thus there exists $0 < m' < |m_\varphi|$ with m' entirely without support by (1). Let φ' correspond to m' according to Theorem 3.13. Then φ' is entirely without support by Corollary 6.4. Since $0 < \varphi' \leq |\varphi|$, the result is proved.

This theorem is given in a different setting in [15], Note XV. It is also proved in [17], but the proof given here is essentially different and somewhat simpler.

Let $\mathcal{M}_0$ and $(C^*)_0^\sim$ denote the elements of $\mathcal{M}(X)$ and $[C^*(X)]^\sim$

respectively, which are entirely without support. It is easy to show that $\mathcal{M}_0$ and $(C^*)^{\sim}_0$ are ideals in $\mathcal{M}(X)$ and $[C^*(X)]^{\sim}$. Let $\{\mathcal{M}_0\}$ and $\{(C^*)^{\sim}_0\}$ denote the bands generated by $\mathcal{M}_0$ and $(C^*)^{\sim}_0$ respectively.

Theorem 6.7.

1. $\mathcal{M}^{\sim}_n$ is a band in $\mathcal{M}(X)$ and $\mathcal{M}(X) = \mathcal{M}_n \oplus \{\mathcal{M}_0\}$ is a Riesz decomposition.

2. $C^{\sim}_n$ is a band in $[C^*(X)]^{\sim}$ and $[C^*(X)]^{\sim} = (C^*)^{\sim}_n \oplus \{(C^*)^{\sim}_0\}$ is a Riesz decomposition.

Proof: 1. It is easy to see that $\mathcal{M}^{\sim}_n$ is an ideal in $\mathcal{M}(X)$. Furthermore, $\mathcal{M}_n = (\mathcal{M}_0)^p$ by Lemma 6.6 and hence $\mathcal{M}_n$ is a band. The decomposition follows from Theorem 1.2.

2. This follows immediately from Theorems 3.13 and 5.11.

Theorem 6.8. All B-integrals are B-normal if and only if there are no B-integrals which are entirely without support.

Proof: This follows immediately from Theorem 6.7 above.

Corollary 6.9. All B-integrals are B-normal if and only if there are no σ-additive elements of $\mathcal{M}(X)$ which are entirely without support.

PART III

B-COMPACT SPACES

CHAPTER VII

TOPOLOGICAL PROPERTIES OF B-COMPACT SPACES

Definition: A completely regular topological space X is said to be B-compact if every B-integral on X is B-normal.

Theorem 7.1. Let X and Y be homeomorphic completely regular spaces. Then X is B-compact if and only if Y is B-compact.

Proof: Let $\tau : X \to Y$ be a homeomorphism. Then the map $\sigma : C^*(X) \to C^*(Y)$ defined by $\sigma(u) = u \circ \tau^{-1}$ is a Riesz space isomorphism which preserves B-limits. Hence the mapping $\upsilon : [C^*(X)]^\sim \to [C^*(Y)]^\sim$ defined by $\upsilon(\varphi) = \varphi \circ \sigma^{-1}$ is a Riesz space isomorphism which preserves B-integrals and B-normal functionals. The result is then obvious.

The above theorem shows that B-compactness is a 'bona fide' topological condition. The following technical lemma will be of considerable usefulness in studying this condition.

Lemma 7.2. Let X be a completely regular space and let Y be a subspace of X. Then there is a canonical Riesz space homomorphism of $\mathfrak{M}(Y)$ into $\mathfrak{M}(X)$ which preserves σ-additive and net-additive measures. Furthermore if $m \in \mathfrak{M}(Y)$ is σ-additive and if $\overline{m}$ denotes its image in $\mathfrak{M}(X)$, then for all $A \in \mathfrak{F}(X)$, $\overline{m}(A) = m(A \cap Y)$.

Proof: The restrictions of elements in $C^*(X)$ to Y form a Riesz subspace L of $C^*(Y)$. If $m \in \mathfrak{M}(Y)$ and φ is the corresponding functional on $C^*(Y)$, it is clear that φ may be extended to $\overline{\varphi} \in [C^*(X)]^\sim$. Let $\overline{m} \in \mathcal{M}(X)$ correspond to $\overline{\varphi}$. Then $m \to \overline{m}$ is a Riesz space homomorphism of $\mathcal{M}(Y)$ into $\mathcal{M}(X)$ which clearly preserves σ-additivity and net-additivity.

Now assume $0 \le m \in \mathcal{M}(Y)$ is σ-additive. To show that $\overline{m}(A) = m(A \cap Y)$, it is sufficient by Lemma 3.1 to show that $\overline{m}(Z) = m(Z \cap Y)$ for all $Z \in \mathcal{Z}(X)$. But if $0 \le u \in C^*(X)$ with $\chi_Z \le u$ and $\overline{m}(Z) > \overline{\varphi}(u) - \epsilon$, then

$$\overline{m}(Z) > \overline{\varphi}(u) - \epsilon = \varphi(u|Y) - \epsilon \ge m(Z \cap Y) - \epsilon .$$

Thus $\overline{m}(Z) \ge m(Z \cap Y)$.

On the other hand, let $u \in C^*(X)$ with $0 \le u \le 1$ and $Z = \{x : u(x) = 1\}$. Then $u^n(x) \downarrow \chi_Z$ as $n \to \infty$. By the monotone convergence theorem, if m is σ-additive, then $\varphi(u^n|Y) \downarrow m(Z \cap Y)$. Thus,

$$\overline{m}(Z) \le \overline{\varphi}(u^n) = \varphi(u^n|Y) ,$$

and hence $\overline{m}(Z) \le \lim_{n\to\infty} \varphi(u^n|Y) = m(Z \cap Y)$. This completes the proof.

<u>Theorem 7.3.</u> <u>Let</u> X <u>be B-compact.</u> <u>If</u> $Y \subseteq X$ <u>is closed</u>, <u>then</u> Y <u>is B-compact.</u>

<u>Proof:</u> Let $0 \le m \in \mathcal{M}(Y)$ be σ-additive, and take $\overline{m} \in \mathcal{M}(X)$ according to Lemma 7.2. Since X is B-compact, $\overline{m}$ is net-additive.

Let $\{Z_\tau\}_{\tau \in T} \subseteq \mathcal{Z}(Y)$ be downward directed with $Z_\tau \downarrow \phi$. Since Y is closed in X, Z_τ is closed in X for each $\tau \in T$. Define $\mathcal{S}_\tau = \{Z \in \mathcal{Z}(X) : Z \supseteq Z_\tau\}$ for each $\tau \in T$. Since $\mathcal{Z}(X)$ is a basis for the closed sets of X, $\mathcal{S}_\tau$ is directed downward with $\mathcal{S}_\tau \downarrow Z_\tau$. Hence

$\mathcal{S} = \bigcup\{\mathcal{S}_\tau : \tau \in T\} \subseteq \mathcal{Z}(X)$ and $\mathcal{S}$ is directed downward with $\mathcal{S}\downarrow 0$. By the net-additivity of $\overline{m}$, $\inf\{\overline{m}(Z) : Z \in \mathcal{S}\} = 0$.

But for each $Z \in \mathcal{S}$, there exists $Z_\tau \subseteq Z$ and so $m(Z_\tau) \leq m(Z \cap Y) = \overline{m}(Z)$. Hence $m(Z_\tau)\downarrow 0$ and m is net-additive.

Theorem 7.4. Let X be B-compact. If $U \in \mathcal{U}(X)$, then U is B-compact.

Proof: Let $0 < m \in \mathcal{M}(U)$ be σ-additive. Since $U \in \mathcal{U}(X)$, there exists a sequence $\{Z_n\} \subseteq \mathcal{Z}(X)$ with $Z_n \uparrow U$. Then $m(Z_n)\uparrow m(U)$. Take Z_{n_0} with $m(Z_{n_0}) > 0$ and define $m'(A) = m(A \cap Z_{n_0})$ for $A \in \mathfrak{F}(U)$. Then $0 < m' \in \mathcal{M}(U)$ is σ-additive. Let $\overline{m}' \in \mathcal{M}(X)$ according to Lemma 7.2.

Since X is B-compact, $\overline{m}'$ has a non-empty support T. But,

$$\begin{aligned} \overline{m}'(Z_{n_0}) &= m'(Z_{n_0} \cap U) \\ &= m'(U) = \overline{m}'(X) . \end{aligned}$$

Hence $T \subseteq Z_{n_0} \subseteq U$.

Let $x \in T$ and $W \in \mathcal{U}(U)$ with $x \in W$. Then there is $V \in \mathcal{U}(X)$ such that $x \in V \cap U \subseteq W$. By Theorem 6.1, $0 < \overline{m}'(V) = m'(V \cap U) \leq m'(W)$ and hence x is in the support of m. Thus m is not entirely without support and the result follows from Corollary 6.9.

Theorem 7.5. Let X_1 be B-compact and let X_2 be compact. Then the Cartesian product $X_1 \times X_2$ is B-compact.

Proof: Let $0 \leq m \in \mathcal{M}(X_1 \times X_2)$ be σ-additive. Assume that m is entirely without support. For $A \in \mathfrak{F}(X_1)$, $\pi_1^{-1}[A] \in \mathfrak{F}(X_1 \times X_2)$

where π_1 denotes the projection of $X_1 \times X_2$ onto X_1. Thus for $A \in \mathfrak{Z}(X_1)$, define $m_1(A) = m(\pi_1^{-1}[A]) = m(A \times X_2)$. It is clear that $m_1 \in \mathcal{M}(X_1)$ is σ-additive. Since X_1 is B-compact, the support of m_1 is not empty. Let $x \in X_1$ be in the support of m_1.

Set $K = \{x\} \times X_2$. Then K is compact. Since m is entirely without support, there exists by Theorem 6.1 a cover $\{U_\alpha : \alpha \in A\}$ of K by u-sets such that $m(U_\alpha) = 0$ for each $\alpha \in A$. Let $\{U_1, \ldots, U_n\}$ be a finite subcover. Furthermore, assume without loss of generality that for $i = 1, \ldots, n$, $U_i = U_i^{(1)} \times U_i^{(2)}$, where $U_i^{(j)} \in \mathcal{U}(X_j)$ for $j = 1, 2$. Set

$$U = \bigcap_{i=1}^{n} U_i^{(1)} \in U(X_1) .$$

Then

$$\pi_1^{-1}[U] = U \times X_2 \subseteq \bigcup_{i=1}^{n} U_i^{(1)} \times U_i^{(2)} .$$

Hence

$$m_1(U) = m(U \times X_2) \leq \sum_{i=1}^{n} m(U_i) = 0 .$$

But $x \in U$ and x is in the support of m_1 implies that $m_1(U) > 0$. This is a contradiction and the result follows.

There is little more to say about the topological properties of a general B-compact space. As will be seen in Chapter IX, many of the interesting topological conditions generally fail for B-compact spaces. For instance, an arbitrary product or an arbitrary intersection of B-compact spaces need not be B-compact. It is an open question whether a finite product or a finite intersection of B-compact spaces is again B-compact. However, a fairly complete answer to these questions can be given in the case that the spaces involved are locally-compact. This will be the topic of the next chapter.

CHAPTER VIII

LOCALLY-COMPACT, B-COMPACT SPACES

The considerations in this section will concern locally-compact, completely regular spaces. The following lemma is very important. It is precisely this lemma which allows more to be said for locally-compact, B-compact spaces than can be said in general.

Lemma 8.1. Let X be locally-compact and let $0 < m \in \mathcal{M}(X)$ be net-additive. Then there is a sequence $\{G_n\}$ of compact zero-sets of X such that $m(G_n) \uparrow m(X)$.

Proof: For each $x \in X$, let G_x be a compact zero-set neighborhood of x. Let $\mathcal{G}$ denote finite unions of the sets G_x. Then $\mathcal{G}$ is directed upward with $\mathcal{G} \uparrow X$. Since m is net-additive, for each $k \in N$ there is $F_k \in \mathcal{G}$ such that $m(X - F_k) < 1/k$. Set

$$G_n = \bigcup_{k=1}^{n} F_k .$$

Then $\{G_n\}$ fulfills the conditions of the theorem.

Theorem 8.2. Let X be completely-regular. If X_1 and X_2 are locally-compact, B-compact subspaces of X, then $X_1 \cap X_2$ is B-compact.

Proof: Let $0 < m \in \mathcal{M}(X_1 \cap X_2)$ be σ-additive. For $A \in \mathfrak{Z}(X_1 \cup X_2)$, define $\overline{m}(A) = m(A \cap X_1 \cap X_2)$. By Lemma 7.2, $\overline{m}$ is

σ-additive. Similarly for i = 1, 2, define $\overline{m}_i(A) = m(A \cap X_1 \cap X_2)$ for $A \in \mathfrak{Z}(X_i)$. Then $0 < \overline{m}_i \in \mathcal{M}(X_i)$ is σ-additive.

The claim is that $\overline{m}$ is net-additive. Indeed, let $\{Z_\tau\} \subseteq \mathfrak{Z}(X_1 \cup X_2)$ be directed downward with $Z_\tau \downarrow \phi$. Then $\{Z_\tau \cap X_1\} \downarrow \phi$. Since X_1 is B-compact, $\overline{m}_1$ is net-additive; and hence,

$$\overline{m}(Z_\tau) = m(Z_\tau \cap X_1 \cap X_2) = \overline{m}_1(Z_\tau \cap X_1) \downarrow 0 .$$

Thus $\overline{m}$ is net-additive as claimed.

Let μ, μ_1 and μ_2 denote the unique regular Borel extensions of $\overline{m}$, $\overline{m}_1$ and $\overline{m}_2$. For $A \in Bo(X_1 \cup X_2)$ define $\lambda_i(A) = \mu_i(A \cap X)$ for i = 1, 2. It is clear that λ_i is a net-additive Borel measure on $X_1 \cup X_2$. Furthermore, for $A \in \mathfrak{Z}(X_1 \cup X_2)$, $A \cap X_i \in \mathfrak{Z}(X_i)$; and,

$$\begin{aligned} \lambda_i(A) &= \mu_i(A \cap X_i) = \overline{m}_i(A \cap X_i) \\ &= m(A \cap X_1 \cap X_2) = \mu(A) . \end{aligned}$$

If it can be shown that λ_i is regular, the uniqueness of the extension μ will imply that $\mu = \lambda_i$ for i = 1, 2.

The claim is that λ_i is regular. Indeed, let $A \in Bo(X_1 \cup X_2)$ and let $\varepsilon > 0$. Since μ_i is regular, there is a closed set $F \subseteq X_1 \cup X_2$ such that $F \cap X_i \subseteq A \cap X_i$ and $\lambda_i(A) < \lambda_i(F) + \varepsilon$. By Lemma 8.1, there are compact zero sets $\{G_n\}$ in X_i with $\mu_i(G_n) \uparrow \mu_i(X_i)$. Thus $\mu_i(G_n \cap F \cap X_i) \uparrow \mu_i(F \cap X_i)$ and so there is a compact set $G \subseteq X_i$ such that $\lambda_i(A\text{-}G) < \varepsilon$. But since G is compact, G is closed in $X_1 \cup X_2$ and the regularity of λ_i is proved.

Let T be the support of $\overline{m}$. Then T is a non-empty closed subset of $X_1 \cup X_2$ since $\overline{m}$ is net-additive. The claim is that

$T \cap X_1 \cap X_2 \neq \emptyset$. Indeed, assume that $T \cap X_1 \cap X_2 = \emptyset$. Then for $i = 1, 2$, $T \cap X_i$ is closed in X_i. Hence there is a compact set $G_i \subseteq T \cap X_i$ with $\mu_i(X_i \cap T) < \mu_i(G_i) + \epsilon$. Thus,

$$\mu(T) = \lambda_i(T) = \mu_i(X_i \cap T) < \mu_i(G_i) + \epsilon$$
$$< \lambda_i(G_i) + \epsilon = \mu(G_i) + \epsilon .$$

Since $G_1 \cap G_2 \subseteq T \cap X_1 \cap X_2 = \emptyset$, it follows that

$$2\mu(T) < \mu(G_1 \cup G_2) + 2\epsilon \leq \mu(T) + 2\epsilon .$$

Thus $\mu(T) = 0$ which is a contradiction since $0 < m$. Thus $T \cap X_1 \cap X_2 \neq \emptyset$.

Finally, let $x \in T \cap X_1 \cap X_2$ and let $W \in \mathcal{U}(X_1 \cap X_2)$ with $x \in W$. There exists $V \in \mathcal{U}(X_1 \cup X_2)$ such that $x \in V \cap X_1 \cap X_2 \subseteq W$; and, hence,

$$0 < \overline{m}(V) = m(V \cap X_1 \cap X_2) \leq m(W) .$$

By Theorem 6.1, x is in the support of m. Thus m is not entirely without support and the proof is complete.

Corollary 8.3. The intersection of a finite number of locally-compact, B-compact spaces is B-compact.

The next theorem concerns products of B-compact spaces. It will be convenient to have a definition of a projection of a measure.

Definition: Let X_1 and X_2 be completely-regular spaces. If $m \in \mathcal{M}(X_1 \times X_2)$ and if π_i is the projection mapping of $X_1 \times X_2$ onto X_i $(i=1,2)$, then define $m_i(A) = m(\pi_i^{-1}[A])$ for $A \in \mathfrak{F}(X_i)$. m_i is called the projection of m on X_i.

Lemma 8.4. *Let X_1 and X_2 be completely-regular. If $0 \le m \in \mathcal{M}(X_1 \times X_2)$ is σ-additive, then the projection $m_i \in \mathcal{M}(X_i)$ and is σ-additive for $i = 1, 2$.*

Proof: It is clear that m_i is a non-negative, finitely-additive set function on $\mathfrak{F}(X_i)$. Furthermore, the σ-additivity of m_i is obvious. Thus only regularity need be shown.

Let $U \in \mathcal{U}(X_i)$ and take an upward directed sequence $\{Z_n\} \subseteq \mathcal{Z}(X_i)$ with $Z_n \uparrow U$. Then $\{\pi_i^{-1}[Z_n]\} \subseteq \mathcal{Z}(X_1 \times X_2)$ and $\pi_i^{-1}[Z_n] \uparrow \pi_i^{-1}[U] \in \mathcal{U}(X_1 \times X_2)$. Since m is σ-additive,

$$m_i(Z_n) = m(\pi_i^{-1}[Z_n]) \uparrow m(\pi_i^{-1}[U]) = m_i(U) .$$

Thus m_i is regular by Lemma 3.1.

Theorem 8.5. *If X_1 and X_2 are locally-compact, B-compact spaces, then $X_1 \times X_2$ is B-compact.*

Proof: Let $0 < m \in \mathcal{M}(X_1 \times X_2)$ be σ-additive. If m is not net-additive, assume that m is entirely without support. Furthermore, without loss of generality, assume that $m(X_1 \times X_2) = 1$. If m_1 is the projection of m on X_1, then $m_1 \in \mathcal{M}(X_1)$ is σ-additive. Since X_1 is B-compact, m_1 is net-additive. Hence by Lemma 8.1, there is a compact zero set $G_1 \subseteq X_1$ such that $m_1(G_1) > 0$. For $A \in \mathfrak{F}(X_1 \times X_2)$, define

$$m'(A) = \frac{m(A \cap G_1 \times X_2)}{m(G_1 \times X_2)} .$$

It is clear that $0 < m' \in \mathcal{M}(X_1 \times X_2)$ is σ-additive, that $m'(X_1 \times X_2) = 1$, and that m' is entirely without support.

The same procedure may now be applied to m'. Hence there is a compact zero set $G_2 \leq X_2$ such that $0 < m'' \in \mathcal{M}(X_1 \times X_2)$ is σ-additive and entirely without support. Furthermore, for $A \in \mathfrak{Z}(X_1 \times X_2)$,

$$m''(A) = \frac{m'(A \cap X_1 \times G_2)}{m'(X_1 \times G_2)} .$$

Set $H = G_1 \times G_2$. Then H is a compact zero-set in $X_1 \times X_2$. Furthermore, since $m'[(X_1 - G_1) \times X_2] = 0$,

$$\begin{aligned} 1 = m''(X_1 \times G_2) &= m'(G_1 \times G_2) + m'[(X_1 - G_1) \times G_2] \\ &= m'(G_1 \times G_2) . \end{aligned}$$

Thus $m''(H) = 1$.

Since m'' is entirely without support, there is a cover of H by u-sets of m''-measure 0. Since H is compact, a finite number of these cover H and hence $m''(H) = 0$. This is a contradiction. Hence m is not entirely without support and the result follows from Corollary 6.9.

Corollary 8.6. The product of a finite number of locally-compact, B-compact spaces is B-compact.

The next theorem shows that the above corollary can be strengthened to countable products. First consider the following definition.

Definition: Let $\{X_\alpha : \alpha \in A\}$ be a family of completely-regular spaces and let $X = \Pi\{X_\alpha : \alpha \in A\}$. A subset $K \subseteq X$ is said to be <u>compact-like</u> if there is a finite subset $F \subseteq A$ and a compact subset K_F of $X_F = \Pi\{X_\alpha : \alpha \in F\}$ with $K = K_F \times \Pi\{X_\alpha : \alpha \in A - F\}$.

Lemma 8.7. *Let $K \subseteq X = \Pi\{X_\alpha : \alpha \in A\}$ be compact-like. Then K is closed in X.*

Proof: This is clear from the definition.

Lemma 8.8. *Let $X = \Pi\{X_\alpha : \alpha \in A\}$. If $\mathcal{B}$ is a filter of compact-like sets, then $\cap\mathcal{B} \neq \emptyset$.*

Proof: Without loss of generality, assume for each $\alpha \in A$, there is $K \subset \mathcal{B}$ with $\pi_\alpha[K]$ compact in X_α. Let $\mathcal{U}$ be an ultrafilter finer than $\mathcal{B}$. For $\alpha \in A$, set $\mathcal{U}_\alpha = \{\pi_\alpha[B] : B \in \mathcal{U}\}$. Then $\mathcal{U}_\alpha$ is an ultrafilter on X_α and $\mathcal{U}_\alpha$ contains a compact subset of X_α. Thus $\mathcal{U}_\alpha$ converges to $t_\alpha \in X_\alpha$. Let $t = \{t_\alpha : \alpha \in A\} \in X$.

Take $B \in \mathcal{U}$. If V_α is open in X_α and $t_\alpha \in V_\alpha$, then $\pi_\alpha^{-1}[V_\alpha] \cap B \neq \emptyset$. Thus $\pi_\alpha^{-1}[V_\alpha] \in \mathcal{U}$. Since a basis for the neighborhood system at the point t consists of sets which are finite intersections of sets of the form $\pi_\alpha^{-1}[V_\alpha]$ where $t_\alpha \in V_\alpha$, it follows that $\mathcal{U}$ is finer than the neighborhood filter at t. Hence t is in the closure of every element of $\mathcal{U}$. But by Lemma 8.7, the members of $\mathcal{B}$ are closed. Hence $t \in \cap\mathcal{B}$.

Theorem 8.9. *Let $\{X_n\}_{n \in N}$ be a sequence of locally-compact, B-compact spaces. Then $X = \Pi\{X_n : n \in N\}$ is B-compact.*

Proof: Let $0 < m \in \mathfrak{M}(X)$ be σ-additive and for convenience assume that $m(X) = 1$. If $Y_n = \Pi\{X_k : 1 \le k \le n\}$, let m_n denote the projection of m on Y_n. By Lemma 8.4, $0 < m_n \in \mathcal{M}(Y_n)$ is σ-additive. Hence by Corollary 8.6, m_n is net-additive. By Lemma 8.1, there exists $G_n \subseteq Y_n$ with G_n a compact zero-set and $1 = m_n(Y_n) < m_n(G_n) + 1/2^n$. Let T_n denote the support of m_n and let μ_n be the unique extension of m_n to a regular Borel measure on Y_n.

Set $K_n = \pi_n^{-1}[G_n \cap T_n]$ where π_n is the projection mapping of X onto Y_n. Then K_n is clearly compact-like. Furthermore,

$$\bigcap_{n=1}^{\ell} K_n \neq \phi$$

for all $\ell \in N$. Indeed,

$$\bigcap_{n=1}^{\ell} K_n = \bigcap\{\pi_{\ell,n}^{-1}[G_n \cap T_n] : 1 \leq n \leq \ell\} \times \Pi\{X_k : k > \ell\} ,$$

where $\pi_{\ell,n}$ denotes the projection mapping of Y_ℓ onto Y_n. Hence it is enough to show that $\bigcap\{\pi_{\ell,n}^{-1}[G_n \cap T_n] : 1 \leq n \leq \ell\} \neq \phi$. But

$$\mu_\ell(\bigcap\{\pi_{\ell,n}^{-1}[G_n \cap T_n] : 1 \leq n \leq \ell\}) =$$

$$= 1 - \mu_\ell(\bigcup\{Y_\ell - \pi_{\ell,n}^{-1}[G_n \cap T_n] : 1 \leq n \leq \ell\})$$

$$\geq 1 - \sum_{n=1}^{\ell} \mu_\ell(Y_\ell - \pi_{\ell,n}^{-1}[G_n \cap T_n])$$

$$\geq 1 - \sum_{n=1}^{\ell} \mu_n(Y_n - (G_n \cap T_n))$$

$$\geq 1 - \sum_{n=1}^{\ell} \mu_n(Y_n - G_n)$$

$$\geq 1 - \sum_{n=1}^{\ell} m_n(Y_n - G_n)$$

$$\geq 1 - \sum_{n=1}^{\ell} 1/2^n > 0 .$$

Thus $\cap\{\pi_{\ell,n}^{-1}[G_n \cap T_n] : 1 \leq n \leq \ell\} \neq \emptyset$.

By Lemma 8. 8,

$$\bigcap_{n=1}^{\infty} K_n = K \neq \emptyset .$$

Let $x \in K$ and $U \in \mathcal{U}(X)$. Then for some n, there is $W \subseteq \mathcal{U}(Y_n)$ with $x \in \pi_n^{-1}[W] \subseteq U$. Thus $\pi_n(x) \in W \cap T_n$. Thus by Theorem 6. 1, $m_n(W) > 0$. Hence $0 < m_n(W) = m(\pi_n^{-1}[W]) \leq m(U)$ and so x is in the support of m. Thus m is not entirely without support and the result follows.

An arbitrary product of B-compact spaces is not in general B-compact as will be seen in the next chapter. (In fact, R^S is not in general B-compact where S is an abstract set of cardinal $> \aleph_0$.)

CHAPTER IX

B-COMPACTNESS AND OTHER TOPOLOGICAL CONDITIONS

Theorem 9.1. If X is a Lindelöf space, then X is B-compact.

Proof: Let $0 < m \in \mathcal{M}(X)$ be σ-additive and assume that m is entirely without support. Then there exists a cover of X by u-sets of m-measure 0. Since X is Lindelöf, there is a countable cover of this type. But the σ-additivity of m then implies that $m(X) = 0$ which is a contradiction. Thus the result follows.

Corollary 9.2. If X is σ-compact, then X is B-compact.

Corollary 9.3. If X is compact, then X is B-compact.

Definition: Let S be an abstract set. S is said to have a measurable cardinal if there is a non-negative, countably additive measure defined for all subsets of S which is zero on the points of S and 1 on S.

It is known that the class of sets whose cardinals are not measurable is a closed class containing $\aleph_0$. Hence if measurable cardinals exist, they must be strongly inaccessible. Furthermore, the class of sets with an accessible cardinal form a model for Bernays-Frankel-Gödel set theory. Hence the assumption that there are no measurable cardinals is consistent with axiomatic set theory. Whether the statement is independent is not known.

Theorem 9.4. Assume X has a non-measurable cardinal. If X is paracompact, then X is B-compact.

Proof: Let $0 < m \in \mathcal{M}(X)$ be σ-additive and entirely without support. Let $\mathcal{U} = \{U_x : x \in X\}$ be a cover of X by u-sets of m-measure 0. Let $\mathcal{P} = \{u_\alpha : \alpha \in A\}$ be a partition of unity subordinate to the cover $\mathcal{U}$. (Such a partition exists since X is paracompact.)

Let $0 < \varphi \in [C^*(X)]^\sim$ correspond to m by Theorem 3.13. For $B \subseteq A$, let $u_B = \sum \{u_\alpha : \alpha \in B\}$. Then $0 \leq u_B \leq 1$ and $u_B \in C^*(X)$. Define $\mu(B) = \varphi(u_B)$. Then μ is a finitely-additive set function defined for all subsets of A. Since φ is a B-integral, μ is countably-additive. Furthermore, for each $\alpha \in A$, $\mu(\{\alpha\}) = \varphi(u_\alpha) = 0$ since u_α vanishes outside some member of $\mathcal{U}$. Finally, $\mu(A) = \varphi(1) > 0$. Hence A has a measurable cardinal.

However, card $A \leq$ card R^X and card R^X is non-measurable since card X is non-measurable. Hence card A is non-measurable which is a contradiction.

Theorem 9.5. If X is B-compact, then X is realcompact.

Proof: Assume that X is not realcompact. Let $x_0 \in \upsilon X - X$. Since $\upsilon X \subseteq \beta X$, for $u \in C^*(X)$, define $\varphi(u) = \overline{u}(x_0)$, where $\overline{u}$ denotes the unique extension of u to βX. Then $0 \leq \varphi \in [C^*(X)]^\sim$. Furthermore, φ is a B-integral. Indeed, assume $\{u_n\} \subseteq C^*(X)$ with $u_n \downarrow^B 0$. Then by Theorem 2.8, $\overline{u}_n(x_0) \downarrow 0$. That is $\varphi(u_n) \downarrow 0$.

However, φ is not B-normal. Indeed, let

$$\mathcal{B} = \{u \in C^*(X) : \overline{u}(x_0) = 1 \quad \text{and} \quad 0 \leq u \leq 1\}.$$

Then $\mathcal{B}$ is downward directed with $\mathcal{B}\downarrow^{B}0$. But $\varphi(u) = 1$ for all $u \in \mathcal{B}$. Hence X is not B-compact and the result follows.

For a long time, it was thought that the above theorem had a converse. That is, that the B-compact spaces coincided with the realcompact spaces. In fact, several incorrect proofs of this conjecture found their way into the literature. (See [7], [11], and [13].) However, the following example is a realcompact space which is not B-compact. This example was sent to the author by J. D. Knowles who had in turn received it from an undisclosed source.

Example 9.6. Let $X = \Gamma$ where Γ is the following space. $\Gamma = \{(x, y) : x, y \in R \text{ with } y \geq 0\}$. Let $D = \{(x, 0) : x \in R\}$. The topology of Γ is an enlargement of the product topology. In addition to the usual neighborhoods for the product topology, for each $r > 0$, the set

$$V_r(x, 0) = \{(x, 0)\} \cup \{(y, z) : (y-x)^2 + (z-r)^2 < r^2\}$$

is a neighborhood of $(x, 0)$.

The following properties of Γ will be required. They are stated without proof and the reader is referred to [2], p. 50 and p. 122.

1. Γ is a completely-regular topological space.
2. Γ is realcompact.
3. D is a discrete zero-set in Γ.
4. $V_r(x, 0)$ is a u-set for each $r > 0$ and $(x, 0) \in D$.

It will be shown that Γ is not B-compact. The following lemma will be needed.

<u>Lemma.</u> <u>Let</u> $Z \in \mathcal{Z}(\Gamma)$ <u>and</u> $Z \subseteq D$. <u>Then</u> Z <u>is measurable with respect to Lebesgue measure on</u> R.

<u>Proof:</u> Let $u \in C^*(\Gamma)$, $0 \leq u \leq 1$, and $Z = \{t \in \Gamma : u(t) = 0\}$. Assume $Z \subseteq D$. For $m, n \in N$, set $A_{n,m} = \{(x,0) \in Z : V_{1/m}(x,0) \subseteq u^{-1}([0, 1/2n])\}$. By the continuity of u,

$$Z \subseteq \bigcup_{m=1}^{\infty} A_{m,n}$$

for each $n \in N$.

Let $\bar{A}_{m,n}$ denote the closure of $A_{m,n}$ in the topology of the real line. If $(x,0) \in \bar{A}_{m,2n}$, then there exists a sequence $\{(x_n, 0)\} \subseteq A_{m,2n}$ and converging to $(x,0)$ in the topology of the real line. Hence there is a sequence $\{t_n\}$ with $t_n \in V_{1/n}(x_n, 0)$ such that $t_n \to (x,0)$ in Γ. This implies that $u(x,0) \leq 1/4n < 1/2n$. That is $A_{m,2n} \subseteq u^{-1}([0, 1/2n])$.

Finally,

$$Z \subseteq \bigcap_{n}^{\infty} \bigcup_{m}^{\infty} A_{m,2n} \subseteq \bigcap_{n}^{\infty} \bigcup_{m}^{\infty} \bar{A}_{m,2n} \subseteq \bigcap_{n}^{\infty} u^{-1}([0, 1/2n]) = Z .$$

Thus

$$Z = \bigcap_{n}^{\infty} \bigcup_{m}^{\infty} \bar{A}_{m,2n}$$

which is obviously Lebesgue measurable and the lemma is proved.

Since D is a zero set, it follows from the above lemma that all of the zero sets of Γ are Lebesgue measurable sets in the plane. Hence $\mathfrak{F}(\Gamma)$ is a ring of Lebesgue measurable sets.

If λ denotes Lebesgue measure restricted to $[0, 1]$, define for each $A \in \mathfrak{F}(\Gamma)$, $m(A) = \lambda(A \cap I)$ where $I = \{(x,0) : x \in [0,1]\}$. It is then clear that $0 < m \in \mathfrak{M}(\Gamma)$ is σ-additive.

The claim is that m is entirely without support. Indeed, if $x \in \Gamma - I$, let B_x be a ball about x such that $B_x \cap I = \phi$. Then $m(B_x) = 0$. If $x \in I$, $V_r(x)$ is a u-set containing x, and $V_r(x) \cap I = \{x\}$. Hence $m(V_r(x)) = 0$. Thus Γ can be covered by u-sets of m-measure 0; and so, by Theorem 6.1, m is entirely without support. Since m is σ-additive and not net-additive, Γ is not B-compact.

In view of the above example, it seems natural to ask what topological conditions in conjunction with realcompactness guarantee B-compactness. If the continuum hypothesis is assumed, one such condition is paracompactness. Indeed, Varadarajan in [4] has shown that if X is paracompact and if every closed, discrete subset of X has a non-measurable cardinal, then X is B-compact. Furthermore, in [12], Katětov has shown that a paracompact space is realcompact if and only if every closed discrete subset X_0 of X has the property that the only two-valued probability measure which is defined for all subsets of X_0 and which vanishes on points is the zero measure. The continuum hypothesis (or even the weaker assumption that $\mathfrak{c}$ is a non-measurable cardinal) implies that a set S has a non-measurable cardinal if and only if the only two-valued probability measure defined on all subsets of S which is zero on points is the zero measure. Combining the above remarks with Theorem 9.5, the following result is obtained.

<u>Theorem 9.7.</u> *Let X be paracompact. If the continuum hypothesis holds, then X is B-compact if and only if X is realcompact.*

Corollary 9.8. Let X be a discrete space. If the continuum hypothesis holds, X is B-compact if and only if X is realcompact.

The results in Chapter VIII suggest that possibly a locally-compact space is B-compact if and only if it is realcompact. However, this is an open question.

Using the fact that there is a realcompact space which is not B-compact, it is possible to show that B-compact spaces do not have some of the nice topological properties.

If X is a topological space and if X_1 and X_2 are subspaces of X with $X = X_1 \cup X_2$, then X is said to be the union of X_1 and X_2.

A. The union of two B-compact spaces need not be B-compact.

Proof: In example 9.6, Γ-D is a Lindelöf space and hence B-compact. D is a discrete space of cardinal $\mathfrak{c}$. If one assumes the continuum hypothesis, $\mathfrak{c}$ is a non-measurable cardinal. Hence if one assumes the continuum hypothesis, D is B-compact. But $\Gamma = \Gamma\text{-D} \cup D$ is not B-compact.

B. An arbitrary intersection of locally-compact, B-compact spaces need not be B-compact.

Proof: Let X be realcompact, but not B-compact. Then X is the intersection of a family of locally-compact, σ-compact subspaces of βX. (See Theorem 2.6.) But a σ-compact space is B-compact by Corollary 9.2. This completes the proof.

C. A product of real lines need not be B-compact.

Proof: Let X be realcompact, but not B-compact. Then X is homeomorphic to a closed subspace of $R^{C(X)}$ by Theorem 2.4. If $R^{C(X)}$ were B-compact, then X would be B-compact by Theorem 7.3. Hence $R^{C(X)}$ is not B-compact.

As a final remark with respect to example 9.6, note that the measure m defined there has no extension to a measure on $Bo(\Gamma)$. For if it did, then the restriction of this extension to the discrete space D would be a measure defined for all subsets of D which would vanish on the points of D and which would be 1 on D. That is, D would have a measurable cardinal. But if one assumes the continuum hypothesis, the cardinal of D is non-measurable.

Before proceeding to the next chapter, a few remarks are in order with respect to the space C(X) of all continuous functions on X. It is clear that the notions of being a B-integral or being B-normal can easily be defined for elements of $[C(X)]^{\sim}$. It is then natural to ask when B-integrals are B-normal in this new setting. It will be seen that a complete answer to this question can be given. (See [9].) The following theorem is given in [15], Note XVa.

Theorem 9.9. If $\varphi \in [C(X)]^{\sim}$, then φ is a B-integral.

Proof: Assume that $0 \le \varphi \in [C(X)]^{\sim}$. Let $\{u_n\} \subseteq C(X)$ with $u_n(x) \downarrow 0$ for all $x \in X$. It must be shown that $\varphi(u_n) \downarrow 0$. Hence assume that $\varphi(u_n) \downarrow \alpha > 0$. Set $v_n = (u_n - \alpha/2)^+$. Then $\{v_n\} \subseteq C(X)$ and $v_n(x) \downarrow 0$ for each $x \in X$. In fact for each $x \in X$, there is an $n(x) \in N$ such that

$v_n(x) = 0$ for $n \geq n(x)$. It is easy to see that

$$v = \sum_{n=1}^{\infty} v_n \in C(X) .$$

However,

$$\varphi(v) \geq \sum_{n=1}^{K} \varphi(v_n) \geq \sum_{n=1}^{K} \frac{\alpha}{2} = K \cdot \frac{\alpha}{2} ,$$

for each $K \in N$. This implies that $\varphi(v) = \infty$ which is a contradiction. Hence φ is a B-integral.

If X is not realcompact, let $x_0 \in \upsilon X - X$. By Theorem 2.4, each $u \in C(X)$ has a unique extension $\overline{u} \in C(\upsilon X)$. Define $\varphi(u) = \overline{u}(x_0)$ for each $u \in C(X)$. Then φ is a B-integral by Theorem 9.19. However, φ is not B-normal since its restriction to $C^*(X)$ is not B-normal. Thus, as in the case of $C^*(X)$, a necessary condition that all B-integrals on $C(X)$ be B-normal is that X be realcompact. This is also sufficient in the present case.

Lemma 9.10. Let $0 < \varphi \in [C(X)]^{\sim}$. For each $0 \leq u \in C(X)$, there is $n(u) \in N$ such that $\varphi(u) = \varphi(\inf(u, n))$ for $n \geq n(u)$.

Proof: Set $u_n = (u - n)^+ \downarrow 0$ and let $\varphi(u_n) = \alpha_n \geq 0$. The claim is that $\alpha_n = 0$ for $n \geq n(u)$. If this is shown, then $u = \inf(u, n) + u_n$ implies that $\varphi(u) = \varphi(\inf(u,n))$ for $n \geq n(u)$.

Hence assume that $\alpha_n > 0$ for all $n \in N$. Let $w_n \in C^*(X)$ with $0 \leq w_n \leq 1/\alpha_n$, $w_n = 0$ on $\{x : u(x) \leq n-1\}$, and $w_n = 1/\alpha_n$ on $\{x : u(x) \geq n\}$. Then

$$w = \sum_{n=1}^{\infty} w_n \cdot u \in C(X) .$$

But $w_n \cdot u \geq u_n/\alpha_n$ and hence,

$$\varphi(w) \geq \sum^{k} \varphi(w_n \cdot u) \geq \sum^{k} \frac{1}{\alpha_n} \varphi(u_n) = k .$$

Thus $\varphi(w) = \infty$ which is a contradiction.

The above lemma is due to Gould and Mahowald, [9].

Theorem 9. 11. Let X be completely regular. The following are equivalent.

(i) X is realcompact.

(ii) All B-integrals in C(X) are B-normal.

(iii) Every functional in $[C(X)]^{\sim}$ is B-normal.

Proof: (ii)⇒(iii). This is a consequence of Theorem 9. 9.

(iii)⇒(i). This has been observed above.

(i)⇒(ii). Let $0 < \varphi \in [C(X)]^{\sim}$ be a B-integral. By Lemma 9. 10, it is enough to show that the restriction ψ to $C^*(X)$ is B-normal. If it is not, then without loss of generality, assume ψ is entirely without support. The extension $\overline{\psi}$ of ψ to $C^*(\beta X)$ has a support which contains a point $x_0 \in \beta X - X$.

Since X is realcompact, there is $0 \leq u \in C(X)$ with $\overline{u}(x_0) = +\infty$. Thus if $W = \{x \in \beta X : \overline{u}(x) > n\}$, $\overline{m}(W) > 0$ by Theorem 6. 1 where $\overline{m}$ is the measure on βX associated with $\overline{\psi}$. Hence $\varphi((u-n)^+) > 0$ for each $n \in N$. That is $\varphi(\inf(u,n)) < \varphi(u)$ for all $n \in N$. This contradicts Lemma 9. 10.

PART IV

APPLICATIONS

CHAPTER X

THE KOLMOGOROV CONSISTENCY THEOREM

In this section a generalized version of the Kolmogorov consistency theorem of probability theory will be proved. First recall the classical theorem due to Kolmogorov.

If S is an abstract set, a <u>cylinder set</u> in R^S is a subset $A \subseteq R^S$ for which there exists a finite set $F \subseteq S$ and a subset $B \subseteq R^F$ with $A = \pi_F^{-1}[B]$. (π_F denotes the projection mapping of R^S onto R^F.) If B is a Borel set in R^F, then A is said to be a Borel cylinder set. Let Σ denote the σ-algebra generated by the Borel cylinder sets in R^S. The following is the theorem due to Kolmogorov.

Theorem 10.1. Let S be an abstract set. For each finite $F \subseteq S$, let m_F be a Borel measure in R^F. Furthermore, for $G \subseteq F$, assume that $m_G(A) = m_F(\pi_{F,G}^{-1}[A])$ for each $A \in Bo(R^G)$. ($\pi_{F,G}$ denotes the projection mapping of R^F onto R^G.) Then there is a unique measure m defined on Σ in R^S such that the projection of m on R^F is m_F for each finite $F \subseteq S$.

This section will be devoted to showing that Theorem 10.1 can be generalized by replacing R by a general locally-compact, B-compact space. However, some care must be taken since in the general case the Baire sets and the Borel sets may not coincide.

Let $\{X_\alpha : \alpha \in S\}$ be a family of completely-regular topological spaces and let $X = \Pi\{X_\alpha : \alpha \in S\}$. If $F \subseteq S$, then π_F will denote the projection mapping of X onto $X_F = \Pi\{X_\alpha : \alpha \in F\}$.

Definition: Let $A \subseteq X$. A is said to be a Borel (Baire) cylinder set if there is a finite set $F \subseteq S$ and a Borel(Baire) set $B \subseteq X_F$ such that $A = \pi_F^{-1}[B]$. BoC (BaC) denotes the family of Borel (Baire) cylinder sets in X.

Definition: Let $A \in$ BoC ($A \in$ BaC). If F is the smallest subset of S for which there is $B \in Bo(X_F)$ ($B \in Ba(X_F)$) with $A = \pi_F^{-1}[B]$, B is said to be the base of A.

It is clear that BoC and BaC are both algebras of subsets of X and that BoC $\subseteq$ Bo(X) and BaC $\subseteq$ Ba(X).

Lemma 10.2. Let $\{X_k : k \in N\}$ be a sequence of locally-compact, B-compact spaces and let $X = \Pi\{X_k : k \in N\}$. Let m be a non-negative, finitely-additive set function on BaC. Then the following are equivalent.

(i) m is countably-additive on BaC.

(ii) For each $n \in N$, m_n is countably-additive, where m_n is the projection of m on the space $Y_n = \Pi\{X_k : 1 \le k \le n\}$.

Proof: (i)$\Rightarrow$(ii). This follows immediately.

(ii)$\Rightarrow$(i). Let $\{A_\ell\} \subseteq$ BaC with $A = \overset{\infty}{\underset{k}{\cup}} A_\ell \in$ BaC and $A_\ell \cap A_k = \phi$ for $k \neq \ell$. Then $B_k = A - \cup A_\ell \in$ BaC and $B_k \downarrow \phi$. It is enough to show that $m(B_k) \downarrow 0$. Hence assume that $m(B_k) \downarrow \alpha > 0$. It will be shown that this implies $\bigcap^{\infty} B_k \neq \emptyset$ and the theorem will be proved.

By Corollary 8. 6, m_n is net-additive. Each B_k has a base $B'_k \subseteq Y_{n(k)}$ for some $n(k) \in N$. Since $Y_{n(k)}$ is locally-compact, by Lemma 8. 1 there is a compact zero-set $G_{n(k)}$ in $Y_{n(k)}$ such that $G_{n(k)} \subseteq B'_k$ and $m_{n(k)}(B'_k) < m_{n(k)}(G_{n(k)}) + \alpha/2^{k+1}$.

Define $K_k = \pi^{-1}_{n(k)}[G_{n(k)}]$. Then it is clear that K_k is a compact-like subset of X and $K_k \subseteq B_k$. Furthermore, $K_k \subseteq BaC$. Hence

$$\begin{aligned} m\left(B_k - \bigcap_{\ell=1}^{k} K_\ell\right) &\leq \sum_{\ell=1}^{k} m(B_k - K_\ell) \\ &\leq \sum_{\ell=1}^{k} m(B_\ell - K_\ell) \\ &\leq \sum_{\ell=1}^{k} m_{n(\ell)}(B'_\ell - G_{n(\ell)}) \\ &< \sum_{\ell=1}^{k} \alpha/2^{\ell+1} < \alpha/2 . \end{aligned}$$

It follows that $m\left(\bigcap^{k} K_\ell\right) > m(B_k) - \alpha/2 > \alpha/2$. Hence $\{K_k : k \in N\}$ is a family of compact-like sets with the finite intersection property. Thus by Lemma 8. 8, $\bigcap^{\infty} K_k \neq \emptyset$. But $\emptyset \neq \bigcap^{\infty} K_k \subseteq \bigcap^{\infty} B_k$ and the proof is complete.

<u>Theorem 10. 3.</u> <u>Let</u> $\{X_\alpha : \alpha \in S\}$ <u>be a family of locally-compact, B-compact spaces and let</u> $X = \Pi\{X_\alpha : \alpha \in S\}$. <u>Let</u> m <u>be a non-negative, finitely-additive set function on</u> BaC. <u>Then the following are equivalent.</u>

(i) m <u>is countably-additive on</u> BaC.

(ii) <u>For each finite set</u> $F \subseteq S$, m_F <u>is countably-additive.</u> (m_F <u>is the projection of</u> m <u>on</u> $X_F = \Pi\{X_\alpha : \alpha \in F\}$.

<u>Proof</u>: (i)$\Rightarrow$(ii). This is obvious.

(ii)$\Rightarrow$(i). Let $\{A_\ell\} \subseteq$ BaC. As above it is enough to show that $A_\ell \downarrow \emptyset$ implies that $m(A_\ell) \downarrow 0$. But each A_ℓ has a base A'_ℓ in $X_{F_\ell} = \Pi\{X_\alpha : \alpha \in F_\ell\}$, where $F_\ell \subseteq S$ is finite. Set $F = \cup\{F_\ell : \ell \in N\}$. If m_F is the projection of m on $X_F = \Pi\{X_\alpha : \alpha \in F\}$, then m_F is a non-negative, finitely-additive set function on $BaC(X_F)$. Furthermore, since the projection m_G of m_F on X_G is the same as the projection of m on X_G, m_G is σ-additive for each finite $G \subseteq F$. Hence by Lemma 10.2, m_F is countably-additive. Thus, if $B_\ell \in BaC(X_F)$ with base A'_ℓ, then $B_\ell \downarrow \emptyset$. Hence $m(A_\ell) = m_F(B_\ell) \downarrow 0$ and the proof is complete.

In the generality considered in this chapter, two consistency theorems may be formulated--one for BaC and the other for BoC. Both types will be considered.

<u>Definition:</u> Let $X = \Pi\{X_\alpha : \alpha \in S\}$. For each finite $F \subseteq S$, let a non-negative, finitely-additive set function m_F be defined on $Bo(X_F)$ $(Ba(X_F))$. The family $\{m_F : F \subseteq S$ is finite$\}$ is said to be <u>Borel consistent</u> (<u>Baire consistent</u>) if for every finite $F \subseteq S$ and every $G \subseteq F$, the projection of m_F on X_G is m_G.

<u>Theorem 10.4.</u> <u>Let</u> $X = \Pi\{X_\alpha : \alpha \in S\}$ <u>where</u> X_α <u>is locally-compact, B-compact for</u> $\alpha \in S$. <u>If</u> $\{m_F : F \subseteq S$ <u>is finite</u>$\}$ <u>is a Baire</u>

consistent family of Baire measures, then there is a unique measure m on BaC such that the projection of m on X_F is m_F for each finite $F \subseteq S$.

Proof: Let $A \in BaC$. Then there is a finite $F \subseteq S$ and $A' \in Ba(X_F)$ such that $A = \pi_F^{-1}[A']$. Define $m(A) = m_F(A')$. By the consistency condition $m(A)$ is independent of the choice of F. Furthermore, m is a finitely-additive set function on BaC. Indeed, let A_1, $A_2 \in BaC$ with $A_1 \cap A_2 = \emptyset$. Choose $F \subseteq S$ finite such that there exists $A_1' \in Ba(X_{F_i})$ with $\pi_F^{-1}[A_1'] = A_i$, $i = 1, 2$. Then $A_1' \cap A_2' = \emptyset$ and hence

$$\begin{aligned} m(A_1 \cup A_2) &= m_F(A_1' \cup A_2') \\ &= m_F(A_1') + m_F(A_2') \\ &= m(A_1) + m(A_2) . \end{aligned}$$

Finally, it is clear that, for $F \subseteq S$ finite, m_F is the projection of m on X_F. Thus by Theorem 10.3, m is countably-additive. The uniqueness is obvious.

Let $X = \Pi\{X_\alpha : \alpha \in S\}$ where X is completely-regular. A finitely-additive set function μ on BoC will be called regular if for each $A \in BoC$, $\mu(A) = \sup\{\mu(G) : G \in BoC$ and G is closed in X$\}$.

Definition: μ will be called net-additive if for every downward directed family $\{G_\tau\} \subseteq BoC$ such that G_τ is closed and $G_\tau \downarrow \emptyset$, then $\mu(G_\tau) \downarrow 0$.

Lemma 10. 5. *Let μ be a regular, net-additive set function on BoC. Then there is a unique, net-additive regular Borel measure λ on X such that μ is the restriction of λ to BoC.*

The proof of this theorem is exactly the same as the proof of Theorem 5. 4. All that was needed there was that the u-sets formed a basis for the topology. Since the open cylinder sets form a basis for the topology of $X = \Pi\{X_\alpha : \alpha \in S\}$, the same proof is valid with the open cylinder sets replacing the u-sets.

Lemma 10. 6. *Let $X = \Pi\{X_k : k \in N\}$ where X_k is a locally-compact, B-compact space. Let μ be a finitely-additive set function on BoC such that the projection μ_n on $Y_n = \Pi\{X_k : 1 \le k \le n\}$ is a net-additive regular Borel measure. Then there is a net-additive regular Borel measure λ on X such that μ is the restriction of λ to BoC.*

Proof: Since μ_n is regular for each $n \in N$, μ is regular. If it can be shown that μ is net-additive, then the result will follow from Lemma 10. 5. Hence let $\{G_\tau\}_{\tau \in T} \subseteq$ BoC be a family of closed sets directed downward with $G_\tau \downarrow \phi$. Without loss of generality, assume that finite intersections of elements in $\{G_\tau\}$ are again in $\{G_\tau\}$.

For each $n \in N$, let $\mathcal{G}_n$ denote the set of bases of those G_τ with base in Y_n. Then $\mathcal{G}_n$ is directed downward to a closed set $H_n \subseteq Y_n$. Furthermore, since μ_n is net-additive, if $G_n = \pi_n^{-1}[H_n]$,

$$\mu(G_n) = \mu_n(H_n) = \inf\{\mu_n(G) : G \in \mathcal{G}_n\}$$

$$= \inf\{\mu(G_\tau) : G_\tau \text{ has a base in } Y_n\} .$$

Hence,

$$\inf\{\mu(G_n) : n \in N\} = \inf\{\mu(G_\tau) : \tau \in T\} .$$

It is thus enough to show that $\mu(G_n) \downarrow 0$.

Assume $\mu(G_n) \downarrow \alpha > 0$. Since Y_n is locally compact, there is a compact set $L_n \subseteq H_n$ with $\mu_n(H_n) < \mu_n(L_n) + \alpha/2^{n+1}$. Let $K_n = \pi_n^{-1}[L_n]$. Then K_n is a compact-like set in X and

$$\mu\left(G_\ell - \bigcap_{n=1}^{\ell} K_n\right) \geq \sum_{n=1}^{\ell} \mu(G_n - K_n) > \alpha/2 .$$

Hence

$$\bigcap_{n=1}^{\ell} K_n \neq \phi .$$

Since $\{K_n : n \in N\}$ is a family of compact-like sets with the finite intersection property, $\bigcap^{\infty} K_n \neq \phi$. But $\bigcap^{\infty} K_n \subseteq \bigcap^{\infty} G_n \subseteq \bigcap\{G_\tau : \tau \in T\} = \phi$. This is a contradiction, and the lemma is proved.

Theorem 10.7. Let $X = \Pi\{X_\alpha : \alpha \in S\}$ where X_α is locally-compact, B-compact for $\alpha \in S$. If $\{\mu_F : F \subseteq S$ is finite$\}$ is a Borel consistent family of regular Borel measures, then there is a unique measure μ on BoC such that the projection of μ on X_F is μ_F for each finite $F \subseteq S$.

Proof: Let $A \in$ BoC. Then there is a finite $F \subseteq S$ and an $A' \in Ba(X_F)$ such that $A = \pi_F^{-1}[A']$. Define $\mu(A) = \mu_F(A')$. By the consistency condition $\mu(A)$ is independent of the choice of F. As in Theorem 10.4, μ is a non-negative, finitely-additive set function on

BoC. Furthermore, it is clear that for $F \subseteq S$ finite, μ_F is the projection of μ on X_F. All that is left is to show that μ is countably-additive.

Let $\{A_\ell\} \subseteq$ BoC with $A_\ell \downarrow \phi$. Each A_ℓ has a base A'_ℓ in $X_{F_\ell} = \Pi\{X_\alpha : \alpha \in F_\ell\}$ where $F_\ell \subseteq S$ is finite. Let $R = \bigcup F_\ell$ and let μ_R be the projection of μ on X_R. For $F \subseteq R$ finite, μ_F is the projection of μ_R on X_F. If it can be shown that μ_F is a net-additive, regular Borel measure on X_F, then by Lemma 10.6 that μ_R is a measure. Hence $\mu(A_\ell) = \mu_R(\pi_R[A_\ell]) \downarrow 0$ and the result will follow.

The claim is that μ_F is a net-additive, regular Borel measure on X_F. Indeed, the restriction m_F of μ_F to $\mathfrak{F}(X_F)$ is σ-additive. Since X_F is B-compact by Corollary 8.6, m_F is net-additive. Hence μ_F is the unique regular Borel extension of m_F by Corollary 5.10. Hence μ_F is net-additive by Theorem 5.4 and the proof is complete.

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