ELECTRIC DIPOLE RADIATION IN

ISOTROPIC AND UNIAXIAL PLASMAS

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Electric Dipole Radiation in Isotropic and Uniaxial Plasmas

ABSTRACT

This paper describes an investigation of radiation from an electric point dipole situated in a cold, collisionless, homogeneous, electronic plasma medium. Two limiting cases of a gyroelectric medium are studied. The magnetostatic biasing field $\underset{\sim}{B}_o$ is first taken to be equal to zero, making the medium isotropic, and then it is taken to be infinite, causing a uniaxial anisotropy. The retarded electromagnetic fields and the instantaneous and averaged values of irreversibly radiated power P^{irr} are calculated.

In each medium, the partial differential equations resulting from the two-sided Laplace transformation of Maxwell's equations with an oscillating electric dipole source and the constitutive equations (derived from the appropriate form of the Lorentz force equation) are solved. A particular path deformation of the Laplace inversion integral reveals that the electromagnetic fields and P^{irr} are exactly expressible in terms of circular, cylindrical, and two-variabled Lommel functions. Asymptotic expressions and graphical results of numerical calculations of these quantities are presented.

For the isotropic case, it is shown that the retarded fields are well behaved for all space and time (excluding the origin, of course). P^{irr} eventually settles down to the result derived from the conventional time-harmonic analysis when the dipole oscillation

frequency ω_o is greater than the plasma frequency ω_p . When the value of ω_o is less than that of ω_p , P^{irr} eventually oscillates at a frequency $2\omega_o$ with zero average value.

When the medium is uniaxial, the fields are finite everywhere except at the dipole. The amplitude of the fields does, however, increase with increasing time. This is quite different from the ordinary time-harmonic solution which ignores all time variations different from $e^{-i\omega_o t}$ and which is singular on a conical surface defined by $\theta = \arcsin \omega_o/\omega_p$ for $\omega_o < \omega_p$. The value of P^{irr} in a uniaxial medium is found to be equal to the value of P^{irr} of a dipole in vacuum. It is also shown that the so-called conventional expression for time-averaged radiated power will not give a sensible result since it contains the retarded electric field which never settles down to a steady-state variation with time. The quantity P^{irr}, on the other hand, does not increase with time, oscillates only at the source frequency, and has a well-defined time average.

TABLE OF CONTENTS

1. INTRODUCTION

In recent years radio communication with artificial earth satellites and missiles in the ionosphere has led to an interest in radiation from sources in gyroelectric media. This has led several investigators to study this phenomenon.

Using the dyadic Green function method, Bunkin (1) found formulas for the radiation field of a time-harmonic current distribution in an arbitrary electrically anisotropic medium. Restricting the anisotropy to that of a gyroelectric medium, Bunkin obtained explicit expressions for the Hertz vector in the far zone, expanded the fields in multipoles, and finally found the far-zone electric field of an electric dipole. Although his work represents a great achievement in the development of the theory of radiation from sources in anisotropic media, Bunkin failed to recognize in his far-zone calculations the possibility of multiple stationary-point contributions.

H. Kogelnik (2,3) was the first to investigate the radiation resistance of a point electric dipole in a gyroelectric medium. He used the expression

$$\langle P \rangle = -\frac{1}{2}\,\mathrm{Re}\int_{V_o} \underline{E}^{out}_{\omega}(\underline{r}) \cdot \underline{J}^{*}_{\omega}(\underline{r})\, dV \tag{1.1}$$

to calculate the time-averaged radiated power $\langle P \rangle$. $\underline{E}^{out}$ is the electric field which satisfies the field equations with source current $\underline{J}$ and which represents an outgoing wave. The asterisk on $\underline{J}$ denotes complex conjugation; V_o is the volume enclosing the current $\underline{J}$. The radiation resistance is then defined by the relation

$$R = \frac{2\langle P \rangle}{|I|^2} \tag{1.2}$$

To relate this resistance to an input resistance of the dipole, a current distribution must be chosen; I is then the input current. This method which he used will be referred to as the conventional method of calculating radiation resistance.

Meecham (4) investigated the VLF properties of the radiation from sources in gyroelectric media by following the Green function method. He found the electric field of a magnetic dipole in such a medium when the source frequency is much less than the plasma and gyrofrequencies of the electrons but is greater than the ionic plasma and gyrofrequencies.

Kuehl (5,6) studied the radiation fields and resistance of an electric dipole in a cold, anisotropic plasma. His methods were similar to those of Bunkin, but he corrected some errors made by Bunkin. Kuehl found expressions for the far-zone electric field of an electric dipole in directions both parallel to and perpendicular to the static magnetic biasing field $\underline{B}_o$ for an arbitrary gyroelectric medium. In the low-frequency limit he gave expressions for the far-zone electric fields produced by point electric dipoles parallel and perpendicular to $\underline{B}_o$ and by a linear current in the direction of $\underline{B}_o$. In the uniaxial case; i.e., for an infinite gyrofrequency, the electric and magnetic fields and the Poynting vector were found for electric dipoles parallel and perpendicular to $\underline{B}_o$. On the basis of an infinite value of the Poynting vector on a conical surface, Kuehl concluded that the radiated power is infinite when the source frequency is less than the plasma frequency. This conclusion was based on the unjustified assumptions that the fields are time-harmonic and that the divergence theorem relationship applies.

At the Symposium on Electromagnetic Theory and Antennas held in Copenhagen, Denmark in 1962, several important papers on the subject under discussion were presented. Arbel and Felsen (7,8) reported on their analysis of radiation from sources in anisotropic media via a modal procedure. They refined earlier asymptotic evaluations of the far-zone fields by employing the steepest descent method rather than the stationary phase method. They also studied the problem of a dipole radiating in a gyroelectric medium, with particular attention to the properties of the dispersion curves and the singular field behavior near the boundary between regions of propagation and nonpropagation. Like Kuehl, they concluded that a point electric-dipole source will radiate infinite power if such boundaries exist. They noted the similarity of the field behavior in the neighborhood of one of these boundaries in a gyroelectric medium to the field behavior at the boundary between propagating and nonpropagating regions in a uniaxial medium ($\underline{B}_o = \infty$). Arbel and Felsen also gave the steady-state solution for the fields of a dipole in a uniaxial medium at an arbitrary distance from the source.

Clemmow (9) reported his findings regarding radiation in uniaxial media. By a scaling procedure, he obtained the fields of a dipole and of a uniformly moving point charge (parallel to the infinite $\underline{B}_o$).

Kogelnik and Motz (10) extended Kogelnik's previous work (11) to include radiation from magnetic as well as electric currents. Using a method developed by Lighthill (12), they derived formulas for the far-zone electric field and showed that the time-averaged Poynting vector is parallel to the group velocity.

Mittra and Deschamps (13) showed that the exact evaluation of the fields of a point dipole in a gyroelectric medium could be reduced to elementary functions and a double integral having a finite range.

Walsh and Weil (14) obtained expressions for radiation resistance that are based on Kogelnik's work. Balmain (15) calculated the impedance of a short dipole in a gyroelectric medium by employing quasi-static techniques. Staras (16) evaluated the radiation resistance of an electric dipole with finite dimensions in such a medium. In all of the investigations just cited, it was assumed that after a long time the fields eventually settle down to a steady state given by the time-harmonic solution and that Eqs. 1.1 and 1.2 should be used in calculating radiation resistance.

A study of the transient behavior of electromagnetic waves in plasmas was simultaneously under way. The responses to plane waves incident upon half spaces of isotropic (17,18) and anisotropic (19) plasmas were obtained as functions of time. One of the motives for this study was to use the transient "ringing" of the medium as a diagnostic technique. Deering and Fejer (20) were interested in resonance effects in a warm gyroelectric medium. They calculated the fields due to a pulsed electric dipole in the asymptotic limit of long times.

Recently S. Lee and Mittra (21) reported their solution for the fields generated by a switched-on, sinusoidally oscillating electric dipole in a uniaxial medium. Their solution is both erroneous and misleading. They impose a short rise time τ on the oscillation of the source, perform a Laplace transformation, and then take as their source term a two-term asymptotic expansion in τ at $\tau \to 0$. This asymptotic expansion is not uniform with respect to the transform variable and after inverse transformation it bears little resemblance to the original time variation of the source. They solve and interpret the problem in terms of potentials. The asymptotic behavior of these potentials is very different from that of the fields. The processes of obtaining an asymptotic expression and of deriving fields from potentials are not commutative. Whereas the potential is well behaved with increasing time, the amplitudes of the fields increase with time.

Papas and K. Lee (22,23) introduced to the engineering community a new theory for calculating radiation resistance based on the time-irreversible power radiated by a current distribution P^{irr}. In terms of time-harmonic quantities they use, instead of Eq. 1.1,

$$\langle P^{irr} \rangle = -\frac{1}{2} \mathrm{Re} \int_{V_o} \frac{1}{2} (\underline{E}_\omega^{out} - \underline{E}_\omega^{in}) \cdot \underline{J}_\omega^* \, dV \tag{1.3}$$

and then they would use

$$R = \frac{2\langle P^{irr} \rangle}{|I_\omega|^2} \tag{1.4}$$

instead of Eq. 1.2.

According to this theory, radiation resistance depends on $\langle P^{irr} \rangle$ and not on $\langle P \rangle$. $\langle P^{irr} \rangle$ depends both on the outgoing and the incoming electric fields. These authors demonstrate the invariance of $\langle P^{irr} \rangle$ under a time-reversal transformation, thus the superscript irr for irreversible.

The literature published to date has not questioned the validity of the use of time harmonic analysis in studying radiation in gyroelectric media. Furthermore, a unanimous decision regarding the relative merits of the conventional and the Papas-Lee theory has not been reached. In this paper the fields in isotropic and uniaxial media are evaluated as functions of time and thus the relationship between the exact and the time-harmonic fields can be established. From these fields $\langle P^{irr} \rangle$ is calculated. The simplicity of these calculations is contrasted with the complexities which would arise in the calculation of $\langle P \rangle$. It is shown that in a uniaxial medium $\langle P \rangle$ would always be infinite for a point dipole, whereas $\langle P^{irr} \rangle$ is always finite.

Statement of the Problems

The two problems solved in the following discussion are concerned with the radiation due to point electric dipoles in isotropic and uniaxial media consisting of cold, collisionless, homogeneous, electronic plasmas in which a z-directed static magnetic biasing field takes on the values $\underline{B}_o = 0\underline{e}_z$ (isotropic) and $\underline{B}_o = \infty\underline{e}_z$ (uniaxial). The origin of a rectangular coordinate system is located at the dipole. The directions of the dipole, biasing magnetic field, and positive z-axis are the same; see Figure 1.1. Cylindrical and spherical coordinate systems are also shown.

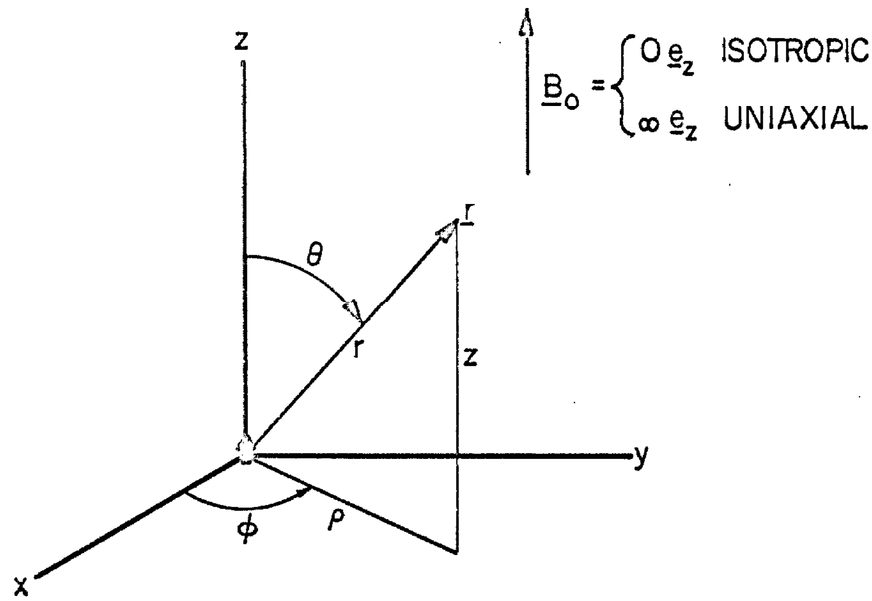


Figure 1.1 Configuration and coordinate systems for the two radiation problems

In each problem the retarded electric and magnetic fields are evaluated, and the amount of power irreversibly radiated by the dipole is calculated.

The dipole is taken to be parallel to $\underline{B}_o$ so that considerations of how current can flow perpendicular to an infinite magnetic field do not enter into the problem. It is believed that the uniaxial medium retains some important characteristics of a gyroelectric medium. From an examination of the existing time-harmonic solutions in both media, it can be seen that there can be directions in which propagation can take place and others in which it cannot. On the boundaries between such regions the time-harmonic fields become infinite. From a study of radiation in a uniaxial medium this effect should become better understood in a gyroelectric medium. Of course, the gyration or Faraday rotation effects are lost in passing to the limit of $\underline{B}_0$ going to infinity.

2. IRREVERSIBLE POWER AND RADIATION RESISTANCE

In this chapter the connection between irreversible power and radiation resistance will be established. Also the computational advantages of using this method will be discussed.

The time-averaged power radiated by a current distribution can be calculated either by using the Poynting vector method, according to which one integrates the normal component of the Poynting vector over a surface enclosing the source, or by means of the induced emf method in which one integrates the work of the current distribution on the fields throughout the source region (24). The Poynting vector method often takes advantage of the simplifications which result from performing far-zone calculations. In dispersive or dispersive anisotropic media the expressions for the far-zone fields may be extremely complicated. In addition, one may have difficulty with nonuniformity of the limits $R \to \infty$ (to be in the far zone) and $t \to \infty$ (to establish a steady-state). For these reasons it seems preferable in the present investigation to use the emf method.

The conventional formulas for time-averaged radiated power that are used in the induced emf method are

$$\langle P \rangle = -\left\langle \int_{V_o} \underline{E}(\underline{r},t) \cdot \underline{J}(\underline{r},t)\, dV \right\rangle \tag{2.1}$$

in time domain calculations, and

$$\langle P \rangle = -\frac{1}{2} \mathrm{Re} \int_{V_o} \underline{E}_\omega(\underline{r},t) \cdot \underline{J}^*_\omega(\underline{r},t)\, dV \tag{2.2}$$

in time-harmonic analysis. The symbol $\langle\ \rangle$ denotes time-averaging over a period of the source frequency. V_o is the region of space occupied by the source. All quantities having a subscript ω are phasors. Most investigators have used the retarded and outgoing fields in Eqs. 2.1 and 2.2, respectively. Such choices for the fields used in the computations can lead to difficulties, even when the current distribution is situated in vacuum. If the spatial dependence of the source is described by a Dirac δ function, the integrals in Eqs. 2.1 and 2.2 diverge. However, if the source is periodic in time and if time averaging is performed before the volume integration in Eq. 2.1, the resulting time-averaged power is finite. Equivalently, in time-harmonic analysis, it is the imaginary part of the integral in Eq. 2.2 which diverges.

This difficulty can be overcome by modifying Eqs. 2.1 and 2.2 to

$$\langle P \rangle = - \int_{V_o} \left\langle \underline{E}^{ret}(\underline{r},t) \cdot \underline{J}(\underline{r},t) \right\rangle dV \tag{2.3}$$

and

$$\langle P \rangle = - \frac{1}{2} \int_{V_o} \mathrm{Re}\{\underline{E}_\omega^{out}(\underline{r}) \cdot \underline{J}_\omega^*(\underline{r})\}\, dV \tag{2.4}$$

respectively. The result of any calculation of time-averaged power by means of these formulas for periodic sources of finite or infinitesimal size in a vacuum is the same as the result obtained by the Poynting vector method. In a vacuum taking the real part of the integrand performs two operations at once, i.e. averaging, and the elimination of the self-reaction of the current upon itself. It appears that this second operation is not performed in some anisotropic media.

As a clue to finding a formula which eliminates self-reaction, we look into the calculation of the power radiated from an arbitrarily moving point-like charged particle in vacuum. Averaging loses meaning for such a source current, and the radiated power based on retarded fields alone (not averaged) as a function of time diverges. This question was studied and resolved by P. A. M. Dirac (25) by using his radiation field $\underline{E}^{rad} \equiv 1/2\ (\underline{E}^{ret} - \underline{E}^{adv})$ = one half of the difference between the retarded and advanced electric fields instead of only the retarded electric field when calculating the power actually radiated by the particle. Dirac's scheme of using the radiation field results in a finite radiation reaction for an accelerated point charge in free space. The power radiated by a current distribution based on Dirac's method is invariant under a time-reversal transformation and is denoted by $P^{irr}(t)$.

$$P^{irr}(t) = - \int_{V_o} \frac{1}{2} (\underline{E}^{ret}(\underline{r},t) - \underline{E}^{adv}(\underline{r},t)) \cdot \underline{J}(\underline{r},t)\ dV \qquad (2.5)$$

This invariance property means that if one observes this process in a frame of reference K' whose time coordinate is reversed with respect to that of frame $K(t' = -t)$, he then sees that the radiated power in the K' frame $P^{irr'}(t')$ equals $P^{irr}(t)$. Corresponding to Eq. 2.5, we have, in terms of time-harmonic quantities

$$\langle P^{irr} \rangle = \frac{1}{2} \mathrm{Re} - \int_{V_o} \frac{1}{2} (\underline{E}_\omega^{out}(\underline{r}) - \underline{E}_\omega^{in}(\underline{r})) \cdot \underline{J}_\omega^*(\underline{r})\ dV \qquad (2.6)$$

The relationships among $\langle P^{irr} \rangle$, $\langle P \rangle$ and radiation resistance must now be found. The radiation resistance of an antenna is that value of ohmic resistance which would dissipate an amount of power equal to that radiated by the antenna, when both have the same current applied. Because we think of radiation resistance in terms of an equivalent resistance, it must possess the same time symmetry as a resistance, i.e. it must be invariant under time reversal. The conventional expression for radiation resistance is

$$R_{rad} = \frac{2\langle P \rangle}{|I|^2} \tag{2.7}$$

The expressions for $\langle P \rangle$, Eqs. 2.3 and 2.4, possess no special time symmetry, although for some simple media and vacuum the value of $\langle P \rangle$ is invariant under time reversal. In these exceptional cases Eq. 2.7 defines a meaningful radiation resistance. We can, however, write

$$\langle P \rangle = \langle P^{irr} \rangle + \langle P^{rev} \rangle \tag{2.8}$$

$\langle P \rangle$ and $\langle P^{irr} \rangle$ are defined by Eqs. 2.4 and 2.6, respectively. These imply that

$$\langle P^{rev} \rangle = -\frac{1}{2}\,\mathrm{Re}\int_{V_o} \frac{1}{2}\left(\underline{E}^{out}_{\omega}(\underline{r}) + \underline{E}^{in}_{\omega}(\underline{r})\right)\cdot \underline{J}^{*}_{\omega}(\underline{r})\, dV \tag{2.9}$$

Similar expressions can be written for time-dependent fields and currents. $\langle P^{rev} \rangle$ often equals zero, but in a gyroelectric or uniaxial medium it may not be. In a lossless gyroelectric medium it has been shown that $\langle P^{irr} \rangle$ is invariant under time reversal and $\langle P^{rev} \rangle$

reverses its sign under that transformation (26). A radiation resistance based on $\langle P \rangle$ when $\langle P^{rev} \rangle \neq 0$ is unacceptable according to the considerations above. Therefore only $\langle P^{irr} \rangle$ is used in the expression for radiation resistance,

$$R_{rad} = \frac{2\langle P^{irr} \rangle}{|I|^2} \tag{2.10}$$

It will be seen later that the volume integration in the calculation of $P^{irr}(t)$, which will be performed in both isotropic and uniaxial media, is quite simple, and the subsequent time-averaging is not too complicated. The conventional method would require that time-averaging of the product of the current and the electric field be performed before the volume integration, Eq. 2.3. This would be a formidable task in view of the complicated time variation of the fields, as will be obvious once those fields are found. Thus this method, in addition to its theoretical foundation, is also easier to use.

In summary, it has been noted that in some complicated media the conventional method for the calculation of radiation resistance can result in an answer which is not invariant under time reversal. The method based on P^{irr}, on the other hand, will always yield a result which is invariant under time reversal, at least for a gyroelectric medium and its limiting cases, i.e. uniaxial and isotropic plasmas, simple media and vacuum. The computations involved in this method are also easier to carry out than those of the conventional method.

3. RADIATION IN AN ISOTROPIC PLASMA

We first consider the radiation by an oscillating point dipole in an isotropic, cold, lossless, homogeneous, linear, electronic plasma. The mathematical methods used in solving for the fields in this medium and for those in a uniaxial medium are the same, but in this case the expressions are far less complex.

The field quantities obey Maxwell's equations, i.e.

$$\nabla \times \underline{E} = -\dot{\underline{B}} \tag{3.1}$$

$$\nabla \times \underline{H} = \underline{J} + \dot{\underline{D}} \tag{3.2}$$

$$\nabla \cdot \underline{B} = 0 \tag{3.3}$$

$$\nabla \cdot \underline{D} = \rho \tag{3.4}$$

A dot above a quantity denotes partial differentiation with respect to time. The field quantities are further interrelated by constitutive relations

$$\underline{B} = \mu_o \underline{H} \tag{3.5}$$

$$\underline{D} = \varepsilon_o \underline{E} + \underline{P} \tag{3.6}$$

The electric polarization $\underline{P}$ has been introduced to account for the interaction of the field with the medium. $\underline{P}$ can be related to the electric field.

The plasma is composed of neutral molecules, free electrons and positive ions. It is assumed that the neutral molecules contribute nothing to the polarization. Also, the ions are assumed to be so massive with respect to the electrons that their contribution to the total polarization will be negligible compared with that of the electrons. Therefore the polarization will be

$$\underline{P}(\underline{r},t) = Nq\,\underline{\xi}(\underline{r},t) \tag{3.7}$$

where N is the free electron density, q is the electronic charge, and $\underline{\xi}(\underline{r},t)$ is the displacement of an electron at time t from its mean position $\underline{r}$. The equation of motion for $\underline{\xi}$ now follows from Newton's second law and the Lorentz force equation.

$$Nm\ddot{\underline{\xi}} = Nq(\underline{E} + \dot{\underline{\xi}} \times \underline{B}) \tag{3.8}$$

where m is the electron mass. Linearizing Eq. 3.8 and making use of 3.7, one obtains the desired relation between $\underline{P}$ and $\underline{E}$.

$$\ddot{\underline{P}} = \varepsilon_o\,\omega_p^2\,\underline{E} \tag{3.9}$$

where $\omega_p \equiv \sqrt{Nq^2/m\varepsilon_o}$ is the plasma frequency.

Taking the curl of 3.1 and using Eqs. 3.2, 3.5, 3.6 and 3.9 yield

$$\nabla \times (\nabla \times \underline{E}) + \frac{1}{c^2}\ddot{\underline{E}} + \frac{\omega_p^2}{c^2}\underline{E} = -\mu_o\,\dot{\underline{J}} \tag{3.10}$$

where $c^2 = 1/\mu_o\varepsilon_o$. One more equation is required for the scalarization of this vector equation. Taking the second time derivatives of Eqs. 3.4 and 3.6, and then substituting Eq. 3.9 give

$$\nabla \cdot \ddot{\underline{E}} + \omega_p^2\,\nabla \cdot \underline{E} = \ddot{\rho}/\varepsilon_o \tag{3.11}$$

Equations for $\underline{B}$ have not been sought, since $\underline{B}$ can be found from Eq. 3.1.

The source, an electric point dipole, has a moment which is a turned-on sinusoid of amplitude p and is z-directed. It is represented mathematically by

$$\underline{M}(\underline{r},t) = \underline{e}_z \, p \sin \omega_o t \; H(t) \; \delta(\underline{r}) \tag{3.12}$$

Here $\delta(\underline{r})$ is the three-dimensional Dirac δ function and $H(t)$ is the Heaviside unit step function. The source current $\underline{J}$ and the charge density ρ are related to $\underline{M}$ by

$$\underline{J}(\underline{r},t) = \dot{\underline{M}}(\underline{r},t) = \underline{e}_z \, p\,\omega_o \cos \omega_o t \; H(t) \; \delta(\underline{r}) \tag{3.13}$$

$$\rho(\underline{r},t) = -\nabla \cdot \underline{M}(\underline{r},t) = -p \sin \omega_o t \; H(t) \frac{\partial}{\partial z} \delta(\underline{r}) \tag{3.14}$$

Two-sided Laplace transformation with respect to time is now performed, assuming that, at a fixed position in space, the fields are identically zero before some fixed time. If $A(t)$ is the function of time under consideration, $\hat{A}(s)$, defined as

$$\hat{A}(s) = \int_{-\infty}^{\infty} A(t) \, e^{-st} \, dt \tag{3.15}$$

is its two-sided Laplace transform. Subject to a few rather minor restrictions, the original function $A(t)$ is recoverable by means of the inversion formula

$$A(t) = \frac{1}{2\pi i} \int_{\Gamma} \hat{A}(s) \, e^{st} \, ds \tag{3.16}$$

Γ is the straight-line path $\sigma - i\infty$ to $\sigma + i\infty$, and the path lies in the strip of convergence which, in the problems under study, is the entire right-hand s-plane (27), i.e. $\sigma > 0$.

The results of this transformation of equations 3.10, 3.11, 3.13 and 3.14 are

$$\nabla \times (\nabla \times \hat{\underline{E}}(\underline{r},s)) + \frac{s^2 + \omega_p^2}{c^2} \hat{\underline{E}}(\underline{r},s) = -\mu_o s \hat{\underline{J}}(\underline{r},s) \tag{3.17}$$

$$\nabla \cdot \hat{\underline{E}}(\underline{r},s) = \frac{s^2}{\varepsilon_o(s^2 + \omega_p^2)} \hat{\rho}(\underline{r},s) \tag{3.18}$$

$$\hat{\underline{J}}(\underline{r},s) = \underline{e}_z \frac{p\ \omega_o s}{s^2 + \omega_o^2} \delta(\underline{r}) \tag{3.19}$$

and

$$\hat{\rho}(\underline{r},s) = \frac{-p\omega_o}{s^2 + \omega_o^2} \frac{\partial}{\partial z} \delta(\underline{r}) \tag{3.20}$$

respectively. By using the vector identity $\nabla \times (\nabla \times \hat{\underline{E}}) = -\nabla^2 \hat{\underline{E}} + \nabla(\nabla \cdot \hat{\underline{E}})$ these equations give, suppressing the arguments $\underline{r}$ and s ,

$$\nabla^2 \hat{\underline{E}} - \frac{s^2 + \omega_p^2}{c^2} \hat{\underline{E}} =$$

$$\underline{e}_z \frac{\mu_o p \omega_o s^2}{s^2 + \omega_o^2} \delta(\underline{r}) - \frac{p\omega_o\ s^2}{\varepsilon_o(s^2 + \omega_o^2)(s^2 + \omega_p^2)} \nabla \frac{\partial}{\partial z} \delta(\underline{r}) \tag{3.21}$$

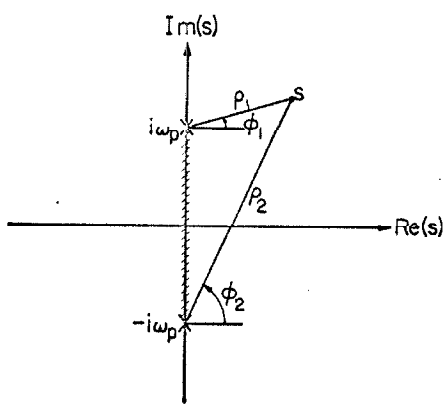
This vector equation can be considered as three separate partial differential equations, one for each of the three cartesian components of $\underline{E}$. The scalar equations are of the form

$$\nabla^2 \hat{f} - K^2 \hat{f} = -\hat{g}(\underline{r}) \tag{3.22}$$

The solutions of 3.22 are

$$\hat{f}^{\pm}(\underline{r},s) = \int_{V_o} \frac{\hat{g}(\underline{r}',s)\, e^{\pm K|\underline{r}-\underline{r}'|}}{4\pi|\underline{r}-\underline{r}'|} dV' \tag{3.23}$$

It has been assumed that the source term $\hat{g}(\underline{r})$ is nonzero only within a finite volume V_o and that the observation point lies outside V_o. A comparison of Eqs. 3.21 and 3.22 shows that $K = \frac{1}{c}\sqrt{s^2+\omega_p^2}$. The square root is defined to be positive for real, positive s. For the purposes of the exact inversion technique the definition of the square root will be continued into the left half plane by Eqs. 3.24 and Figure 3.1.



$$\left.\begin{aligned} \sqrt{s^2+\omega_p^2} &= \sqrt{\rho_1\rho_2}\, e^{\frac{i}{2}(\phi_1+\phi_2)} \\ -\frac{\pi}{2} &\le \phi_{1,2} < \frac{3\pi}{2} \end{aligned}\right\} \tag{3.24}$$

Figure 3.1 A definition of $\sqrt{s^2+\omega_p^2}$

Inversion of Eq. 3.23 by means of Eq. 3.16 gives

$$f^{\pm}(\underline{r},t) = \frac{1}{2\pi i}\int_{\Gamma}\int_{V_o} \frac{\hat{g}(\underline{r}',s)\, e^{st \pm \frac{|\underline{r}-\underline{r}'|}{c}\sqrt{s^2+\omega_p^2}}}{4\pi|\underline{r}-\underline{r}'|} dV'\, ds \tag{3.25}$$

By reversing the order in which the two integrations in Eq. 3.25 are performed and by closing the contour Γ with an arc of infinite radius in the right half s-plane, one finds that f^+ is zero for $t < -\frac{1}{c}$ times the maximum of $|\underline{r} - \underline{r}'|$, $\underline{r}'$ within V_o . Similarly f^- is zero for $t < \frac{1}{c}$ times the minimum of $|\underline{r} - \underline{r}'|$, $\underline{r}'$ within V_o . f^+ and f^- are, respectively, solutions converging onto and diverging from the source. They are therefore the advanced and retarded solutions of the second order partial differential equation from which Eq. 3.22 is obtained. If the source is point-like, one solution can be obtained from the other by a replacement of c by $-c$.

We are now in a position to find the retarded solution of Eq. 3.21 $\hat{\underline{E}}^{ret}(\underline{r},s)$. It is

$$\hat{\underline{E}}^{ret}(\underline{r},s) = -\frac{\mu_o \omega_o p}{4\pi} \frac{s^2}{s^2 + \omega_o^2} \int_{V_o} \frac{e^{-\frac{\sqrt{s^2+\omega_p^2}}{c}|\underline{r} - \underline{r}'|}}{|\underline{r} - \underline{r}'|}$$

$$\times \left[\underline{e}_z \, \delta(\underline{r}') - \frac{c^2}{s^2+\omega_p^2} \nabla' \frac{\partial}{\partial z'} \delta(\underline{r}') \right] dV' \qquad (3.26)$$

Carrying out the indicated integration throughout the volume V_o results in

$$\hat{\underline{E}}^{ret}(\underline{r},s) = -\frac{\mu_o \omega_o p}{4\pi} \frac{s^2}{s^2 + \omega_o^2} \left[\underline{e}_z \frac{e^{-\frac{r}{c}\sqrt{s^2+\omega_p^2}}}{r} \right.$$

$$\left. - \frac{c^2}{s^2+\omega_p^2} \nabla \frac{\partial}{\partial z} \frac{e^{-\frac{r}{c}\sqrt{s^2+\omega_p^2}}}{r} \right] \qquad (3.27)$$

After performing the differentiations and expressing the components of $\hat{\underline{E}}^{ret}$ in spherical coordinates, we have

$$\hat{E}_r^{ret} = \frac{\mu_o \omega_o p}{4\pi} \frac{s^2}{s^2 + \omega_o^2} 2 \cos \theta \left(\frac{c}{r^2 \sqrt{s^2 + \omega_p^2}} + \frac{c^2}{r^3 (s^2 + \omega_p^2)} \right) e^{-\frac{r}{c}\sqrt{s^2 + \omega_p^2}} \tag{3.28}$$

$$\hat{E}_\theta^{ret} = \frac{\mu_o \omega_o p}{4\pi} \frac{s^2}{s^2 + \omega_o^2} \sin \theta \left(\frac{1}{r} + \frac{c}{r^2 \sqrt{s^2 + \omega_p^2}} + \frac{c^2}{r^3 (s^2 + \omega_p^2)} \right) e^{-\frac{r}{c}\sqrt{s^2 + \omega_p^2}} \tag{3.29}$$

$$\hat{E}_\emptyset^{ret} = 0 \tag{3.30}$$

To obtain an expression for $\underline{B}^{ret}$, we Laplace transform Eq. 3.1.

$$\nabla \times \hat{\underline{E}}^{ret}(\underline{r}, s) = - s\hat{\underline{B}}^{ret}(\underline{r}, s) \tag{3.31}$$

Since $\hat{E}_\emptyset^{ret}$ and all $\emptyset$ derivatives are equal to zero,

$$\hat{\underline{B}}^{ret} = - \frac{1}{s} \underline{e}_\emptyset \frac{1}{r} \left[\frac{\partial}{\partial r}(r \hat{E}_\theta^{ret}) - \frac{\partial \hat{E}_r^{ret}}{\partial \theta} \right] \tag{3.32}$$

Thus we find that

$$\hat{B}_r^{ret} = 0 \tag{3.33}$$

$$\hat{B}_\theta^{ret} = 0 \tag{3.34}$$

$$\hat{B}_\emptyset^{ret} = \frac{\mu_o \omega_o p}{4\pi} \frac{s}{s^2 + \omega_o^2} \sin \theta \left(\frac{\sqrt{s^2 + \omega_p^2}}{rc} + \frac{1}{r^2} \right) e^{-\frac{r}{c}\sqrt{s^2 + \omega_p^2}} . \tag{3.35}$$

The radial electric field in the time domain is

$$E_r^{ret}(\underline{r},t) = \frac{1}{2\pi i}\int_\Gamma \hat{E}_r^{ret}(\underline{r},s)\, e^{st}\, ds \tag{3.36}$$

$$= \frac{1}{2\pi i}\frac{\mu_o\omega_o p}{4\pi}\, 2\cos\theta\, \frac{c}{r^2}\int_\Gamma \frac{s^2}{s^2+\omega_o^2}\left(\frac{1}{\sqrt{s^2+\omega_p^2}}+\frac{c}{r(s^2+\omega_p^2)}\right)$$

$$\times\; e^{st-\frac{r}{c}\sqrt{s^2+\omega_p^2}}\, ds \tag{3.37}$$

Performing the transformations of contour and the substitutions indicated in Appendix B for integrals of this type, one obtains

$$E_r^{ret} = \frac{\mu_o\omega_o pc\cos\theta}{4\pi^2 r^2}\int_0^{2\pi}\frac{(1+\xi^2)^2}{(\xi^2-\xi_o^2)(\xi^2-\xi_0^{-2})}\left(1+\frac{2ci}{r\omega_p}\frac{\xi}{\xi^2-1}\right)$$

$$\times\; e^{iq\cos\psi}\, d\psi\, H(t-\frac{r}{c}) \tag{3.38}$$

In this chapter $\beta \equiv r/ct$, $q \equiv \omega_p t\sqrt{1-\beta^2}$, $\gamma \equiv \sqrt{(1-\beta)/(1+\beta)}$, $\xi = \gamma e^{i\psi}$ and $\xi_o \equiv (\omega_o + \sqrt{\omega_o^2 - \omega_p^2})/\omega_p$ when $\omega_o > \omega_p$ or $\xi_o \equiv (\omega_o + i\sqrt{\omega_p^2-\omega_o^2})/\omega_p$ when $\omega < \omega_p$. Partial fractioning of the integrand yields

$$E_r^{ret} = \frac{\mu_o \omega_o p\ c \cos\theta}{4\pi^2 r^2} \int_0^{2\pi} \left[1 + \frac{\omega_o}{\sqrt{\omega_o^2 - \omega_p^2}} \left(\frac{\xi_o^2}{\xi^2 - \xi_o^2} - \frac{\xi_o^{-2}}{\xi^2 - \xi_o^{-2}} \right) \right.$$

$$\left. + \frac{ci}{r(\omega_o^2 - \omega_p^2)} \left(\frac{\xi_o \omega_o \xi}{\xi^2 - \xi_o^2} + \frac{\xi_o^{-1} \omega_o \xi}{\xi^2 - \xi_o^{-2}} - \frac{2\omega_p \xi}{\xi^2 - 1} \right) \right] e^{iq\cos\psi}\, d\psi\ H(t - \frac{r}{c}) \tag{3.39}$$

Using the integration formulas in Appendix C, we find that

$$E_r^{ret} = \frac{\mu_o \omega_o pc \cos\theta}{2\pi\ r^2} \left\{ J_o(q) + \frac{\omega_o}{\sqrt{\omega_o^2 - \omega_p^2}} \left(U_o(\gamma\xi_o q, q) - U_o(\gamma\xi_o^{-1} q, q) \right) \right.$$

$$\left. + \frac{c}{r(\omega_o^2 - \omega_p^2)} \left[\omega_o \left(U_1(\gamma\xi_o q, q) + U_1(\gamma\xi_o^{-1} q, q) \right) - 2\omega_p U_1(\gamma q, q) \right] \right\} H(t - \frac{r}{c}) \tag{3.40}$$

See Appendix A for some mathematical properties of Lommel functions of two variables $U_n(w,z)$.

By noticing that

$$E_\theta^{ret}(\underline{r},t) = \frac{\mu_o \omega_o p \sin\theta}{4\pi r} F + \frac{1}{2} \tan\theta\ E_r \tag{3.41}$$

in which

$$F = \frac{1}{2\pi i} \int_\Gamma \frac{s^2}{s^2 + \omega_o^2}\, e^{st - \frac{r}{c}\sqrt{s^2 + \omega_p^2}}\, ds \tag{3.42}$$

some duplication of calculations can be avoided. The behavior of the integrand of Eq. 3.42 as $s \to \infty$ indicates the presence of a δ

function in E_θ. Treating this behavior separately by using the fact that the inverse transform of $e^{-b\sqrt{s^2+a^2}}$ is

$$\delta(t-b) - \frac{a^2 b}{a\sqrt{t^2-b^2}} J_1(a\sqrt{t^2-b^2})\ H(t-b)$$

we find that

$$F = \delta(t-\frac{r}{c}) - \frac{r\omega_p^2}{c}\frac{J_1(q)}{q} H(t-\frac{r}{c})$$
$$- \frac{\omega_o^2}{2\pi i}\int_\Gamma \frac{1}{s^2+\omega_o^2} e^{st - \frac{r}{c}\sqrt{s^2+\omega_p^2}}\, ds \qquad (3.43)$$

The integration technique described in Appendix B gives

$$F = \delta(t-\frac{r}{c}) - \frac{r\omega_p^2}{c}\frac{J_1(q)}{q} H(t-\frac{r}{c})$$
$$- \frac{\omega_o^2}{\omega_p \pi i}\int_0^{2\pi} \frac{\xi(1-\xi^2)\, e^{iq\cos\psi}}{(\xi^2-\xi_o^2)(\xi^2-\xi_o^{-2})}\, d\psi\ H(t-\frac{r}{c}) \qquad (3.44)$$

Partial fractioning followed by the use of integration formulas in Appendix C yields

$$F = \delta(t-\frac{r}{c}) - \frac{r\omega_p^2}{c}\frac{J_1(q)}{q} H(t-\frac{r}{c})$$
$$- \omega_o \left[U_1(\gamma\xi_o q, q) + U_1(\gamma\xi_o^{-1} q, q)\right] H(t-\frac{r}{c}) \qquad (3.45)$$

With this result and Eqs. 3.40 and 3.41, the value of E_θ is found to be

$$E_\theta^{ret} = \frac{\mu_o \omega_o p \sin\theta}{4\pi r} \Bigg\{ 2\delta(t - \frac{r}{c}) - \frac{\omega_p^2 r}{c} \frac{J_1(q)}{q} - \omega_o \Big[U_1(\gamma\xi_o q, q) + $$

$$+ U_1(\gamma\xi_o^{-1} q, q) \Big] + \frac{c}{r} \Big[J_o(q) + \frac{\omega_o}{\sqrt{\omega_o^2 - \omega_p^2}} \Big(U_o(\gamma\xi_o q, q) - U_o(\gamma\xi_o^{-1} q, q) \Big) \Big]$$

$$+ \frac{c^2}{r^2(\omega_o^2 - \omega_p^2)} \Big[\omega_o \Big(U_1(\gamma\xi_o q, q) + U_1(\gamma\xi_o^{-1} q, q) \Big) - 2\omega_p U_o(\gamma q, q) \Big] \Bigg\} H(t - \frac{r}{c})$$

$$(3.46)$$

The generalized function identity $2\delta(x)\, H(x) = \delta(x)$ has been used.

The magnetic induction field $B_\emptyset$ also contains a δ function. This behavior can be treated separately, i.e.

$$B_\emptyset^{ret}(\underline{r}, t) = \frac{\mu_o \omega_o p \sin\theta}{4\pi rc} \Big[\delta(t - \frac{r}{c}) - \frac{r\omega_p^2}{c} \frac{J_1(q)}{q} H(t - \frac{r}{c}) \Big]$$

$$+ \frac{\mu_o \omega_o p \sin\theta}{8\pi^2 i\, rc} \int_\Gamma \Big[\frac{s\sqrt{s^2 + \omega_p^2}}{s^2 + \omega_o^2} - 1 + \frac{cs}{r(s^2 + \omega_o^2)} \Big] e^{st - \frac{r}{c}\sqrt{s^2 + \omega_p^2}}\, ds$$

$$(3.47)$$

Using the integration technique of Appendix B, we obtain

$$B_\emptyset^{ret} = \frac{\mu_o \omega_o p \sin\theta}{4\pi rc} \Big[\delta(t - \frac{r}{c}) - \frac{r\omega_p^2}{c} \frac{J_1(q)}{q} H(t - \frac{r}{c}) \Big] - \frac{\mu_o \omega_o p \sin\theta}{8\pi^2 irc}$$

$$\times \int_0^{2\pi} \Big[\frac{\omega_p}{2} \frac{-2\xi^2 + \xi_o^2 + \xi_o^{-2}}{(\xi^2 - \xi_o^2)(\xi^2 - \xi_o^{-2})} \xi(1 - \xi^2) + \frac{c}{ri} \frac{1 - \xi^4}{(\xi^2 - \xi_o^2)(\xi^2 - \xi_o^{-2})} \Big] e^{iq\cos\psi}\, d\psi$$

$$\times \quad H(t - \frac{r}{c}) \qquad (3.48)$$

When this expression undergoes partial fractioning and when the formulas in Appendix C are applied, we obtain

$$B_{\phi}^{ret} = \frac{\mu_o\omega_o p \sin\theta}{4\pi rc}\Bigg\{2\delta(t - \frac{r}{c}) - \frac{r\omega_p^2}{c}\frac{J_1(q)}{q} - \omega_p\gamma\, J_1(q)$$

$$- \sqrt{\omega_o^2 - \omega_p^2}\left[U_1(\gamma\xi_o q,q) - U_1(\gamma\xi_o^{-1}q,q)\right] + \frac{c}{r}\Big[- J_o(q) + U_o(\gamma\xi_o q,q) +$$

$$+ U_o(\gamma\xi_o^{-1}q,q)\Big]\Bigg\}\ H(t - \frac{r}{c}) \qquad (3.49)$$

We shall now obtain asymptotic formulas for the fields as $\gamma \to 0$, corresponding to times just after the arrival of the first disturbance. For small γ, Eqs. 3.40, 3.46 and 3.49 give, respectively,

$$E_r^{ret}(\underline{r},t) = \frac{\mu_o\omega_o p \cos\theta c}{2\pi r^2}\left\{J_o(q) + \frac{c}{r}\frac{2\gamma}{\omega_p}J_1(q)\right\}H(t - \frac{r}{c}) + O(\gamma^2) \qquad (3.50)$$

$$E_\theta^{ret}(\underline{r},t) = \frac{\mu_o\omega_o p \sin\theta}{4\pi r}\Bigg\{2\delta(t - \frac{r}{c}) - \frac{r}{c}\omega_p^2\frac{J_1(q)}{q} - \frac{2\omega_o^2}{\omega_p}\gamma J_1(q)$$

$$+ \frac{c}{r}J_o(q) + \frac{c^2 2\gamma}{r^2\omega_p}J_1(q)\Bigg\}\ H(t - \frac{r}{c}) + O(\gamma^2) \qquad (3.51)$$

$$B_\phi^{ret}(\underline{r}_1 t) = \frac{\mu_o\omega_o p \sin\theta}{4\pi\ rc}\Bigg\{2\delta(t - \frac{r}{c}) - \frac{r}{c}\omega_p^2\frac{J_1(q)}{q} + \frac{\omega_p^2 - 2\omega_o^2}{\omega_p}\gamma J_1(q) +$$

$$+ \frac{c}{r}J_o(q)\Bigg\}\ H(t - \frac{r}{c}) + O(\gamma^2) \qquad (3.52)$$

These equations, which contain Bessel functions with arguments $q = \omega_p\sqrt{t^2 - r^2/c^2}$, represent oscillations having a frequency that is initially high but that continually decreases to ω_p . This behavior and in fact the method of obtaining these formulas which is described in Appendices B and C closely resemble the oscillatory behavior and the method of analysis in calculating the first forerunner found by Sommerfeld (28) in his work on the propagation of light in dispersive media. It should also be noted that these expressions are finite when $\omega_o = \omega_p$, whereas Eqs. 3.40 and 3.46 appear to be infinite under that condition.

The asymptotic behavior of the fields at a point $\underline{r}$ as $t \to \infty$ can be obtained through the use of asymptotic formulas D.37 to D.45 in Appendix D. We find for $\omega_o > \omega_p$

$$E_r^{ret} \sim \frac{\mu_o \omega_o p \cos\theta c}{2\pi r^2} \left\{ \frac{\omega_o \cos(\omega_o t - \frac{r}{c}\sqrt{\omega_o^2 - \omega_p^2})}{\sqrt{\omega_o^2 - \omega_p^2}} \right.$$

$$\left. + \frac{c}{r(\omega_o^2 - \omega_p^2)} \left[\omega_o \sin(\omega_o t - \frac{r}{c}\sqrt{\omega_o^2 - \omega_p^2}) - \omega_p \sin \omega_p t \right] \right\} \quad (3.53)$$

$$E_\theta^{ret} \sim \frac{\mu_o \omega_o p \sin\theta}{4\pi r} \left\{ -\omega_o \sin(\omega_o t - \frac{r}{c}\sqrt{\omega_o^2 - \omega_p^2}) + \right.$$

$$+ \frac{c}{r} \frac{\omega_c \cos(\omega_o t - \frac{r}{c}\sqrt{\omega_o^2 - \omega_p^2})}{\sqrt{\omega_o^2 - \omega_p^2}} + \frac{c^2}{r^2(\omega_o^2 - \omega_p^2)} \left[\omega_o \sin(\omega_o t - \right.$$

$$\left. \left. - \frac{r}{c}\sqrt{\omega_o^2 - \omega_p^2}) - \omega_p \sin \omega_p t \right] \right\} \quad (3.54)$$

$$B_\phi^{ret} \sim \frac{\mu_o \omega_o p \sin\theta}{4\pi rc} \left\{ -\sqrt{\omega_o^2 - \omega_p^2}\, \sin\left(\omega_o t - \frac{r}{c}\sqrt{\omega_o^2 - \omega_p^2}\right) + \frac{c}{r} \cos\left(\omega_o t - \frac{r}{c}\sqrt{\omega_o^2 - \omega_p^2}\right) \right\} \tag{3.55}$$

and for $\omega_o < \omega_p$

$$E_r^{ret} \sim \frac{\mu_o \omega_o p \cos\theta c}{2\pi r^2} \left\{ -\frac{\omega_o}{\sqrt{\omega_p^2 - \omega_o^2}} e^{-\frac{r}{c}\sqrt{\omega_p^2 - \omega_o^2}} \sin \omega_o t + \frac{c}{r(\omega_o^2 - \omega_p^2)} \left[\omega_o e^{-\frac{r}{c}\sqrt{\omega_p^2 - \omega_o^2}} \sin\omega_o t - \omega_p \sin\omega_p t \right] \right\} \tag{3.56}$$

$$E_\theta^{ret} \sim \frac{\mu_o \omega_o p \sin\theta}{4\pi r} \left\{ -\omega_o\, e^{-\frac{r}{c}\sqrt{\omega_p^2 - \omega_o^2}} \sin \omega_o t - \frac{c}{r} \frac{\omega_o}{\sqrt{\omega_p^2 - \omega_o^2}} e^{-\frac{r}{c}\sqrt{\omega_p^2 - \omega_o^2}} \sin \omega_o t + \frac{c^2}{r^2(\omega_o^2 - \omega_p^2)} \left[\omega_o\, e^{-\frac{r}{c}\sqrt{\omega_p^2 - \omega_o^2}} \sin \omega_o t - \omega_p \sin \omega_p t \right] \right\} \tag{3.57}$$

$$B_\phi^{ret} \sim \frac{\mu_o \omega_o p \sin\theta}{4\pi rc} \left\{ -\sqrt{\omega_o^2 - \omega_p^2} + \frac{c}{r} \right\} e^{-\frac{r}{c}\sqrt{\omega_p^2 - \omega_o^2}} \cos \omega_o t \tag{3.58}$$

The neglected terms have amplitudes vanishing faster than $t^{-1/2}$ as $t \to \infty$. It is readily seen that the magnetic field quickly assumes its steady-state value, whereas the electric field has a residual oscillation at the plasma frequency. This oscillation is restricted to the near zone, and hence it does not contribute to radiation. The slightest loss in the medium would damp out such an oscillation at the plasma frequency. Loss, expressed in terms of a collision frequency ω_{eff}, would displace the branch cut shown in Fig. 3.1 into the left half plane a distance $\omega_{eff}/2$. Additional branch points would appear at $s = 0 , - \omega_{eff}$. If the ion plasma frequency had also been considered in the problem formulation, other branch points would occur near the origin of the complex s-plane. A limitation on this technique is for the elliptical path to lie well away from these additional singularities so that the principal contributions to a more inclusive formulation of the problem would arise from the poles of the source at $s = \pm i\omega_o$ and the branch points due to the plasma frequency of the medium at $s = \pm i\omega_p$. If the first unaccounted for effect is due to collisions, then by Eq. B.7, the condition

$$\frac{\omega_{eff}}{2} \ll \frac{\omega_p \beta}{\sqrt{1 - \beta^2}} \tag{3.59}$$

must be fulfilled for the path to be well away from the influence of the collision frequency. Since the elliptical integration path approaches the branch cut as time increases, an upper limit on time for the neglect of collisions can be established

$$t \ll \frac{2\omega_p}{\omega_{eff}} \frac{r}{c} \sqrt{1 + (\frac{\omega_{eff}}{2\omega_p})^2} \tag{3.60}$$

Numerical calculations of the field quantities were performed. Since the medium is isotropic, the fields were only computed in the equitorial plane ($z = 0$). In this plane only the fields E_θ and B_ϕ are nonzero. Figures 3.2 through 3.5 show these fields as a function of normalized time $\tau = \omega_o t$. The time-harmonic electric field's amplitude is denoted by horizontal dashed lines in each figure. Figure 3.2 corresponds to a condition of low dispersion and long distance from the source with $\omega_p = 0.1\ \omega_o$ and $r = 1000\ c/\omega_o$. In this figure only E_θ is shown because the magnitude of cB_ϕ is only about .5% smaller than E_θ . Figure 3.3 shows the fields for $\omega_p = 0.5\ \omega_o$ and $r = 60c/\omega_o$. Figure 3.4 shows the fields for $\omega_p = 0.9\ \omega_o$ and $r = 10c/\omega_o$, corresponding to the observation of the fields near the source in a dense plasma. Figure 3.5 shows the overdense case in which $\omega_p = 1.2\omega_o$ and $r = 5c/\omega_o$. We examine the near field because only there does the portion of the solution which is exponentially damped in space contribute significantly to the total solution. (See Eqs. 3.57 and 3.58.) For larger distances from the source, the oscillation at frequency ω_p would dominate the solution for $\omega_o < \omega_p$. In Figure 3.5 the magnetic field is only shown during the initial transient period because its amplitude becomes too small to plot for later times. The beating of oscillations at two frequencies (ω_o and ω_p) can be seen in E_θ in Figures 3.3 through 3.5.

The irreversibly radiated power will now be calculated by using Eq. 2.5. As was mentioned while solving the equations for the fields, the advanced fields can be obtained from the retarded ones through a substitution of $-c$ for c . See Eq. 3.25 and subsequent discussion. The simplest way to evaluate the radiation field parallel to and at the

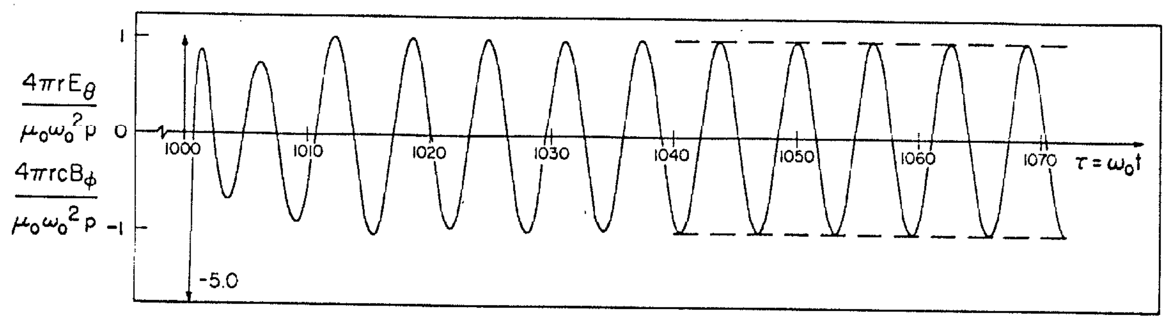


Figure 3.2. E_θ for $\omega_p = 0.1\ \omega_o$, $r = 1000c/\omega_o$, $\theta = 90^o$. Horizontal dashed lines indicate the time harmonic amplitude of E_θ . A δ-function is denoted by an arrow.

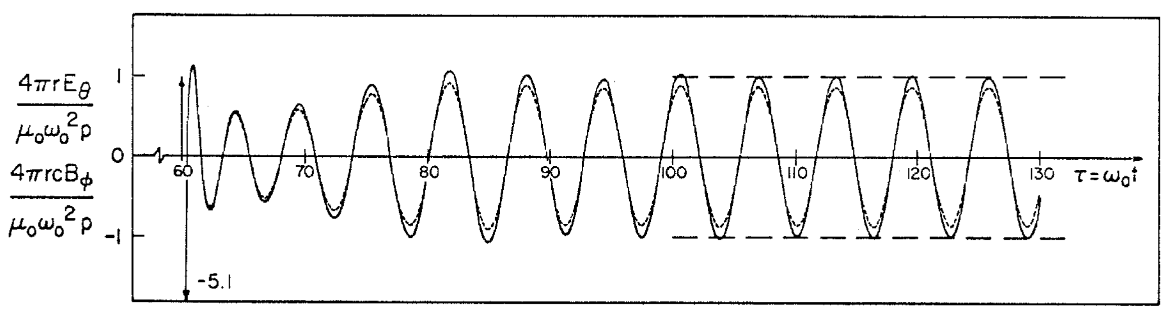


Figure 3.3. E_θ(____) and B_ϕ(----) for $\omega_p = 0.5\ \omega_o$, $r = 60c/\omega_o$, $\theta = 90^o$. Horizontal dashed lines indicate the time harmonic amplitude of E_θ . A δ-function is denoted by an arrow.

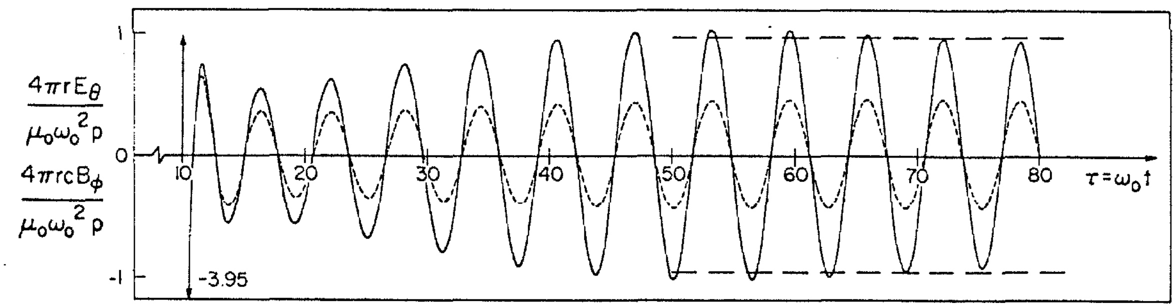


Figure 3.4. E_θ(____) and B_ϕ(----) for $\omega_p = 0.9\omega_o$, $r = 10c/\omega_o$, $\theta = 90^o$. Horizontal dashed lines indicate the time harmonic amplitude of E_θ. A δ-function is denoted by an arrow.

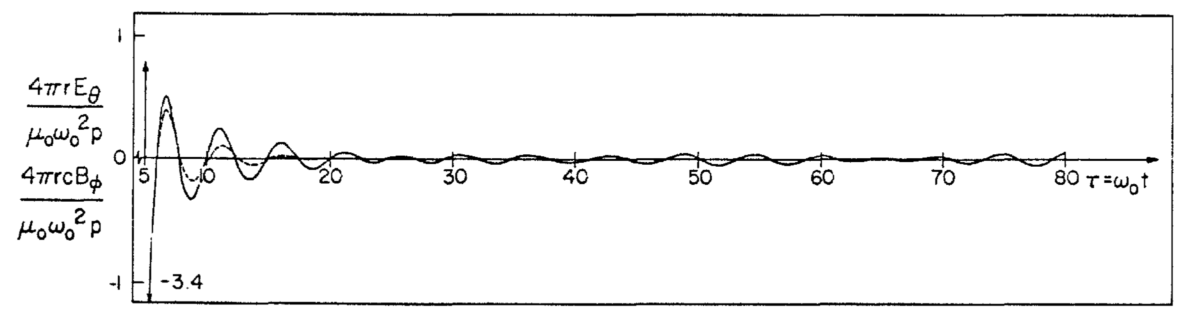


Figure 3.5. E_θ(____) and B_ϕ(----) for $\omega_p = 1.2\omega_o$, $r = 5c/\omega_o$, $\theta = 90^o$. A δ-function is denoted by an arrow.

dipole is to begin with Eq. 3.27.

$$\hat{E}_z^{rad}(\underline{r},s) \equiv \frac{1}{2}\,[\hat{E}_z^{ret}(\underline{r},s) - \hat{E}_z^{adv}(\underline{r},s)] = -\frac{\mu_o\omega_o p}{8\pi}\,\frac{s^2}{s^2+\omega_o^2}\left[\frac{e^{-\frac{r}{c}\sqrt{s^2+\omega_p^2}}}{r} - \right.$$

$$\left. - \frac{e^{\frac{r}{c}\sqrt{s^2+\omega_p^2}}}{r} - \frac{c^2}{s^2+\omega_p^2}\,\frac{\partial^2}{\partial z^2}\left(\frac{e^{-\frac{r}{c}\sqrt{s^2+\omega_p^2}}}{r} - \frac{e^{\frac{r}{c}\sqrt{s^2+\omega_p^2}}}{r}\right)\right] \tag{3.61}$$

Expressing the exponentials as hyperbolic functions, we obtain

$$\hat{E}_z^{rad}(\underline{r},s) = \frac{\mu_o\omega_o p}{4\pi}\,\frac{s^2}{s^2+\omega_o^2}\left[\frac{\sinh\frac{r}{c}\sqrt{s^2+\omega_p^2}}{r} - \frac{c^2}{s^2+\omega_p^2}\,\frac{\partial^2}{\partial z^2}\left(\frac{\sinh\frac{r}{c}\sqrt{s^2+\omega_p^2}}{r}\right)\right] \tag{3.62}$$

Carrying out the differentiation we find, as $r \to 0$

$$\hat{E}_z^{rad}(\underline{r},s) = \frac{1}{6\pi c}\;\frac{\mu_o\omega_o p s^2}{s^2+\omega_o^2}\;\sqrt{s^2+\omega_p^2} + O(r) \tag{3.63}$$

This transform can now be inverted by the same technique as was used in finding the retarded fields. As $r \to 0$, we obtain for $t > |\underline{r}|/c$

$$E_z^{rad}(\underline{r},t) = \frac{\mu_o\omega_o p}{6\pi c}\left\{\delta'(t) - \frac{1}{2\pi}\int_0^{2\pi} \frac{1-\xi^4\,(\xi^2-\xi_o^2)(\xi^2-\xi_o^{-2})}{(\xi^2-\xi_o^2)(\xi^2-\xi_o^{-2})}\left(\frac{\omega_p}{2\xi}\right)^2 \right.$$

$$\left. \times\;(1-\xi^4)\,e^{i\omega_p t\cos\psi}\,d\psi\right\} + O(|\underline{r}|) \tag{3.64}$$

$$E_z^{rad}(\underline{r},t) = \frac{\mu_o\omega_o p}{6\pi c}\left\{\delta'(t) - \frac{1}{2\pi}\int_0^{2\pi}\left[\frac{\alpha^2}{2}\xi^2 + \frac{\alpha^2}{2}\,\frac{\xi_o^2+\xi_o^{-2}}{2}\right.\right.$$

$$\left.\left. + \frac{\alpha^2}{4}\,\frac{\xi_o^4-1}{\xi^2-\xi_o^2} + \frac{\alpha^2}{4}\,\frac{\xi_o^{-4}-1}{\xi^2-\xi_o^{-2}}\right] e^{i\omega_p t\cos\psi}\, d\psi\right\} + O(|\underline{r}|) \quad (3.65)$$

and finally

$$E_z^{rad}(\underline{r},t) = \frac{\mu_o\omega_o p}{6\pi c}\left\{\delta'(t) + \omega_p^2\,\frac{J_1(\omega_p t)}{\omega_p t} - \omega_o^2 J_o(\omega_p t)\right.$$

$$\left. - \omega_o\sqrt{\omega_o^2-\omega_p^2}\,[U_o(\xi_o\omega_p t,\omega_p t) - U_o(\xi_o^{-1}\omega_p t,\omega_p t)]\right\} + O(|\underline{r}|) \quad (3.66)$$

The same result can be obtained by taking the general expressions for the electric radiation field in the z-direction (derivable from Eqs. 3.40 and 3.46) and expanding it in a Laurent series in r . The coefficients of all inverse powers of r cancel and the $O(1)$ term in r is identical to Eq. 3.66.

The irreversible power is therefore

$$P^{irr}(t) = -\frac{\mu_o\omega_o^2 p^2}{6\pi c}\cos\omega_o t\left\{\omega_p^2\,\frac{J_1(\omega_p t)}{\omega_p t} + \delta'(t) - \omega_o^2 J_o(\omega_p t)\right.$$

$$\left. - \omega_o\sqrt{\omega_o^2-\omega_p^2}\,[U_o(\xi_o\omega_p t,\omega_p t) - U_o(\xi_o^{-1}\omega_p t,\omega_p t)]\right\} \quad (3.67)$$

When the source frequency is greater than the plasma frequency, ξ_o is greater than 1 . We can then use the asymptotic formulas D.11 and D.14 to obtain the long time limit of $P^{irr}(t)$. The result is

$$P^{irr}(t) \underset{t\to\infty}{\sim} \frac{\mu_o\omega_o^2 p^2 \cos\omega_o t}{6\pi c}\left\{\omega_o\sqrt{\omega_o^2-\omega_p^2}\cos\omega_o t\right.$$

$$\left. + \frac{\omega_p^4}{\omega_o^2-\omega_p^2}\frac{1}{\omega_p t}\sqrt{\frac{2}{\pi\omega_p t}}\sin(\omega_p t-\frac{\pi}{4})\right\} \qquad (3.68)$$

Thus, as $t \to \infty$

$$\left\langle P^{irr}(t)\right\rangle \to \frac{\mu_o\omega_o^3 p^2}{12\pi c}\sqrt{\omega_o^2-\omega_p^2} \qquad (3.69)$$

When $\omega_o = \omega_p$

$$P^{irr}(t) = -\frac{\mu_o\omega_o^4 p^2}{6\pi c}\cos\omega_o t\left\{\frac{J_1(\omega_o t)}{\omega_o t}+\frac{\delta'(t)}{\omega_o^2}-J_o(\omega_o t)\right\} \qquad (3.70)$$

and, after a long time,

$$\left\langle P^{irr}(t)\right\rangle \to 0 \qquad (3.71)$$

For $\omega_o < \omega_p$ the asymptotic formula D.30 must be used. Thus,

$$P^{irr}(t) \underset{t\to\infty}{\sim} \frac{\mu_o\omega_o^2 p^2}{6\pi c}\cos\omega_o t\left[\omega_o\sqrt{\omega_p^2-\omega_o^2}\sin\omega_o t\right.$$

$$\left. + \frac{4}{\pi}\frac{\omega_o^2}{\sqrt{1-(\frac{\omega_o}{\omega_p})^2}}\frac{\sin\omega_p t}{\omega_p t}-\frac{\omega_p^2}{\omega_p t}J_1(\omega_p t)\right] \qquad (3.72)$$

whence

$$\left\langle P^{irr}\right\rangle \to 0 \qquad (3.73)$$

Since the limits $t \to \infty$ and $r \to 0$ are commutative, the preceding results for $\langle P^{irr} \rangle$ could have been obtained by using the formulas 3.53, 3.54, 3.56 and 3.57, which describe the behavior of the electric field after a long time. If $\langle P \rangle$ is calculated from these expressions, the oscillation of the electric field at the frequency ω_p multiplied by the time variation of $\underline{J}$ does not average to zero over one period of the source frequency. For $\omega_o > \omega_p$, we find that

$$\langle P \rangle = \frac{\mu_o \omega_o^3 p^2}{12\pi c} \sqrt{\omega_o^2 - \omega_p^2}$$

$$+ \frac{\mu_o \omega_o^2 p^2 c^2 \omega_p}{4\pi(\omega_o^2 - \omega_p^2)} \int_{V_o} \left\langle \frac{(2-3\sin^2\theta)}{r^3} \sin \omega_p t \cos \omega_o t \; \delta(\underline{r}) \, dV \right\rangle \tag{3.74}$$

and for $\omega_o < \omega_p$

$$\langle P \rangle = 0 + \frac{\mu_o \omega_o^2 p^2 c^2 \omega_p}{4\pi(\omega_o^2 - \omega_p^2)} \int_{V_o} \left\langle \frac{(2-3\sin^2\theta)}{r^3} \sin \omega_p t \cos \omega_o t \; \delta(\underline{r}) \, dV \right\rangle \tag{3.75}$$

To keep $\langle P \rangle$ finite, we must use an average which gives zero for $\langle \sin \omega_p t \cos \omega_o t \rangle$, e.g.

$$\langle f \rangle' = \lim_{T \to \infty} \frac{1}{T} \int_0^T f(t) \, dt \tag{3.76}$$

Using this definition for average, we find that

$$\langle P \rangle' = \langle P^{irr} \rangle = \langle P^{irr} \rangle' \tag{3.77}$$

for an isotropic plasma.

4. RADIATION IN A UNIAXIAL PLASMA

The medium in which we now consider electric dipole radiation consists of a macroscopically neutral ionized gas to which a magnetostatic biasing field $\underline{B}_o$ of infinite magnitude is applied. $\underline{B}_o$ allows the charged particles to move only in directions parallel to $\underline{B}_o$. Thus the induced polarization is also parallel to $\underline{B}_o$. Both $\underline{B}_o$ and an oscillating electric dipole located at the origin of the coordinate system are oriented parallel to the z-axis; see Figure 1.1.

Again, as the foundation for the derivation of the differential equations satisfied by the fields, we begin with Maxwell's equations and the constitutive equations, 3.1 to 3.6. The equation relating the electric polarization $\underline{P}$ to the electric field $\underline{E}$ must now be found. As before, we assume that the only contribution to $\underline{P}$ is due to the electrons. $\underline{P}(\underline{r},t)$ is related to the displacement $\underline{\xi}(\underline{r},t)$ of a typical electron at time t from its mean position $\underline{r}$ by

$$\underline{P}(\underline{r},t) = Nq\, \underline{\xi}(\underline{r},t) \tag{4.1}$$

where N is the free electron density and q is the electronic charge.

The equation of motion for $\underline{\xi}(\underline{r},t)$ can be derived from Newton's second law and the Lorentz force equation

$$Nm\, \ddot{\underline{\xi}} = Nq[\underline{E} + \dot{\underline{\xi}} \times (\underline{B} + \underline{B}_o)] \tag{4.2}$$

Linearizing Eq. 4.2 and expressing it in terms of $\underline{P}$ via Eq. 4.1

gives

$$\ddot{\underline{P}} = \varepsilon_o \omega_p^2 \underline{E} + \dot{\underline{P}} \times \underline{\omega}_g \tag{4.3}$$

where $\omega_p \equiv \sqrt{Nq^2/m\varepsilon_o}$ and $\underline{\omega}_g \equiv q\underline{B}_o/m$ are the electron plasma frequency and gyrofrequency, respectively. In this problem $\underline{B}_o$ and hence $\underline{\omega}_g$ take on an infinite magnitude. To see the influence of taking this limit, we take two derivatives with respect to time on both sides of Eq. 4.3, resubstitute Eq. 4.3 twice and obtain (29)

$$\ddot{\ddot{\underline{P}}} + \underline{\omega}_g \cdot \underline{\omega}_g \ddot{\underline{P}} = \varepsilon_o \omega_p^2 \left\{ \ddot{\underline{E}} + \dot{\underline{E}} \times \underline{\omega}_g + (\underline{E} \cdot \underline{\omega}_g) \underline{\omega}_g \right\} \tag{4.4}$$

Assuming that the time derivatives of all quantities in Eq. 4.4 remain bounded as $\underline{\omega}_g \to \infty$, we obtain by taking that limit

$$\ddot{\underline{P}} = \varepsilon_o \omega_p^2 \frac{(\underline{E} \cdot \underline{\omega}_g) \underline{\omega}_g}{\underline{\omega}_g \cdot \underline{\omega}_g} \tag{4.5}$$

Since $\underline{\omega}_g$ is parallel to the z axis, we have

$$\ddot{\underline{P}} = \varepsilon_o \omega_p^2 E_z \underline{e}_z \tag{4.6}$$

We note that if the fields in Eqs. 4.6 and 3.6 are assumed to have harmonic time dependence, $e^{-i\omega t}$, then they imply that electric displacement $\underline{D}_\omega$ is related to the electric field $\underline{E}_\omega$ through the dyadic relation

$$\underline{D}_\omega = \underline{\underline{\varepsilon}} \cdot \underline{E}_\omega \tag{4.7}$$

where $\underline{\underline{\varepsilon}}$ is the familiar dielectric tensor of a uniaxial plasma

$$\underline{\underline{\varepsilon}} = \varepsilon_o \underline{e}_x \underline{e}_x + \varepsilon_o \underline{e}_y \underline{e}_y + \varepsilon_o \left(1 - \frac{\omega_p^2}{\omega^2}\right) \underline{e}_z \underline{e}_z \tag{4.8}$$

$\underline{B}$, $\underline{H}$, and $\underline{D}$ can now be eliminated in deriving a differential equation for $\underline{E}$ from Maxwell's equations, 3.5, 3.6 and 4.6.

$$\nabla \times (\nabla \times \underline{E}) + \frac{1}{c^2} \ddot{\underline{E}} + \frac{\omega_p^2}{c^2} E_z \underline{e}_z = - \mu_o \dot{\underline{J}} \tag{4.9}$$

In a similar way an equation for $\underline{H}$ can be derived

$$\nabla^2 \underline{H} - \frac{1}{c^2} \ddot{\underline{H}} = - \nabla \times \underline{J} - \nabla \times \dot{\underline{P}} \tag{4.10}$$

Since both $\underline{J}$ and $\underline{P}$ are z-directed, the z components of their curls are zero. Therefore, $H_z = 0$. From this fact and Eq. 3.1 it follows that

$$\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} = 0 \tag{4.11}$$

Taking two time derivatives of Eq. 4.9, expanding the double curl operator, and using the divergence of Eq. 4.9

$$\nabla^2 \ddot{\underline{E}} + \omega_p^2 \nabla \frac{\partial E_z}{\partial z} - \frac{1}{c^2} \ddddot{\underline{E}} - \frac{\omega_p^2}{c^2} \ddot{E}_z \underline{e}_z = \mu_o \dddot{\underline{J}} + \frac{1}{\varepsilon_o} \nabla \ddot{\rho} \tag{4.12}$$

As it stands, only the z component of this vector equation is uncoupled. However, the x and y components of Eq. 4.9 and the relation 4.11 can be used to uncouple the other two cartesian components of the vector equation 4.12. The three uncoupled equations can now be written as one vector partial differential equation, each cartesian component of which is the equation for the electric field in that direction

$$\nabla^2 \ddot{\underline{E}} + \omega_p^2 \frac{\partial^2 \underline{E}}{\partial z^2} - \frac{1}{c^2} \ddot{\ddot{\underline{E}}} - \frac{\omega_p^2}{c^2} \ddot{\underline{E}} = \mu_o \dddot{\underline{J}} + \frac{1}{\varepsilon_o} \nabla \ddot{\rho} \tag{4.13}$$

Taking the curl of Eq. 4.13, using Eq. 3.1, and integrating the result once with respect to time give an equation for $\underline{B}$

$$\nabla^2 \ddot{\underline{B}} + \omega_p^2 \frac{\partial^2 \underline{B}}{\partial z^2} - \frac{1}{c^2} \ddot{\ddot{\underline{B}}} - \frac{\omega_p^2}{c^2} \ddot{\underline{B}} = - \mu_o \nabla \times \ddot{\underline{J}} \tag{4.14}$$

Each cartesian component of Eq. 4.14 gives an uncoupled scalar equation for the corresponding cartesian component of $\underline{B}$.

Two-sided Laplace transformation of Eqs. 4.13 and 4.14 is performed to yield

$$\nabla^2 \hat{\underline{E}} + \frac{\omega_p^2}{s^2} \frac{\partial^2}{\partial z^2} \hat{\underline{E}} - \frac{s^2 + \omega_p^2}{c^2} \hat{\underline{E}} = \mu_o s \, \hat{\underline{J}} + \frac{1}{\varepsilon_o} \nabla \hat{\rho} \tag{4.15}$$

and

$$\nabla^2 \hat{\underline{B}} + \frac{\omega_p^2}{s^2} \frac{\partial^2}{\partial z^2} \hat{\underline{B}} - \frac{s^2 + \omega_p^2}{c^2} \hat{\underline{B}} = - \mu_o \nabla \times \hat{\underline{J}} \tag{4.16}$$

respectively. Since the retarded fields are identically zero for $t < 0$, their transforms are analytic in the right half s-plane We shall solve Eqs. 4.15 and 4.16 subject to the retardation condition by taking s to be real and positive. Those solutions, considered as functions of complex s will be analytic in the right half s-plane. Therefore, by the identity theorem for analytic functions (30) those solutions will be the Laplace-transformed retarded fields.

For s real and positive, we can perform the coordinate transformation $\bar{x} = x$, $\bar{y} = y$, $\bar{z} = \frac{s}{\sqrt{s^2 + \omega_p^2}} z$. Doing this and using the expressions for the current and charge densities, i.e., using Eqs. 3.19 and 3.20, Eqs. 4.15 and 4.16 become

$$\nabla^2_{\underline{r}} \hat{\underline{E}} - \frac{s^2 + \omega_p^2}{c^2} \hat{\underline{E}} = \frac{\mu_o p \ \omega_o s^3}{(s^2+\omega_o^2)\sqrt{s^2+\omega_p^2}} \delta(\bar{\underline{r}}) - \frac{1}{\varepsilon_o}\left(\underline{e}_x \frac{\partial}{\partial \bar{x}} + \right.$$

$$\left. + \underline{e}_y \frac{\partial}{\partial \bar{y}} + \underline{e}_z \frac{s}{\sqrt{s^2+\omega_p^2}} \frac{\partial}{\partial \bar{z}}\right) \frac{p\omega_o s^2}{(s^2+\omega_o^2)(s^2+\omega_p^2)} \frac{\partial}{\partial \bar{z}} \delta(\bar{\underline{r}}) \qquad (4.17)$$

and

$$\nabla^2_{\underline{r}} \hat{\underline{B}} - \frac{s^2 + \omega_p^2}{c^2} \hat{\underline{B}} = - \nabla_{\underline{r}} \times \underline{e}_z \frac{\mu_o p \omega_o s^2 \ \delta(\bar{\underline{r}})}{(s^2+\omega_o^2)\sqrt{s^2+\omega_p^2}} \qquad (4.18)$$

respectively.

In obtaining these equations the relation $\delta(z) = \left|\frac{d\bar{z}}{dz}\right| \delta(\bar{z})$ has been used. Now each cartesian component of Eqs. 4.17 and 4.18 is an equation like Eq. 3.22, the retarded solutions of which are given by Eq. 3.23 with a minus sign in the exponent. Therefore the retarded solutions of Eqs. 4.17 and 4.18 are

$$\hat{\underline{E}}^{ret}(\bar{\underline{r}},s) = - \frac{\mu_o p \omega_o s^3 \underline{e}_z}{4\pi\sqrt{s^2+\omega_p^2}(s^2+\omega_o^2)} \int_{\bar{V}_o} \frac{\delta(\bar{\underline{r}}')}{|\bar{\underline{r}} - \bar{\underline{r}}'|} e^{-\frac{|\bar{\underline{r}} - \bar{\underline{r}}'|}{c}\sqrt{s^2+\omega_p^2}} d\bar{V}'$$

$$+ \frac{p\omega_o s^2}{4\pi\varepsilon_o(s^2+\omega_p^2)(s^2+\omega_o^2)} \int_{\bar{V}_o} \left[\left(\underline{e}_x \frac{\partial}{\partial \bar{x}'} + \underline{e}_y \frac{\partial}{\partial \bar{y}'} + \underline{e}_z \frac{s}{\sqrt{s^2+\omega_p^2}} \frac{\partial}{\partial \bar{z}'}\right) \frac{\partial}{\partial \bar{z}'} \delta(\bar{\underline{r}}')\right] \times$$

$$\times \quad \frac{e^{-\frac{|\underline{\bar{r}} - \underline{\bar{r}}'|}{c}\sqrt{s^2+\omega_p^2}}}{|\underline{\bar{r}} - \underline{\bar{r}}'|} \; d\bar{V}' \tag{4.19}$$

and

$$\hat{\underline{B}}^{ret}(\underline{\bar{r}},s) = \frac{p\mu_o\omega_o s^2}{4\pi(s^2+\omega_o^2)\sqrt{s^2+\omega_p^2}} \int_{\bar{V}_o} \left[\nabla_{\underline{\bar{r}}'} \times \underline{e}_z \; \delta(\underline{\bar{r}}')\right] \frac{e^{-\frac{|\underline{\bar{r}}-\underline{\bar{r}}'|}{c}\sqrt{s^2+\omega_p^2}}}{|\underline{\bar{r}} - \underline{\bar{r}}'|} \, d\bar{V}' \tag{4.20}$$

respectively.

After carrying out these volume integrations and returning to physical x, y, z space, we obtain

$$\hat{\underline{E}}^{ret}(\underline{r},s) = \frac{\mu_o p\omega_o s}{s^2+\omega_o^2}\left[-s^2 \; \underline{e}_z + c^2 \; \nabla\frac{\partial}{\partial z}\right] \frac{e^{-\frac{r}{c}\sqrt{s^2+\omega_p^2\sin^2\theta}}}{4\pi r\sqrt{s^2+\omega_p^2\sin^2\theta}} \tag{4.21}$$

and

$$\hat{\underline{B}}^{ret}(\underline{r},s) = -\frac{\mu_o p\omega_o s^2}{s^2+\omega_o^2} \, \underline{e}_\phi \, \frac{\partial}{\partial\rho} \; \frac{e^{-\frac{r}{c}\sqrt{s^2+\omega_p^2\sin^2\theta}}}{4\pi r\sqrt{s^2+\omega_p^2\sin^2\theta}} \tag{4.22}$$

The other solutions to Eqs. 4.17 and 4.18, i.e., those with a + sign in the exponential, are easily shown to be advanced fields, and they differ from the retarded fields only by the substitution of $-c$ for c in the formulas for the retarded fields, Eqs. 4.21 and 4.22. In cylindrical coordinates, the electric field has only ρ and z components and the magnetic field has only a ϕ component. Carrying out the differentiations indicated by Eqs. 4.21 and 4.22 we obtain for $\hat{E}_\rho^{ret}$

$$\hat{E}_\rho^{ret} = \frac{\mu_o p\omega_o s^3 \cos\theta \sin\theta}{4\pi r} \; \frac{s^2+\omega_p^2}{s^2+\omega_o^2}\left[\frac{1}{(s^2+\alpha^2)^{3/2}} + \frac{3c}{r(s^2+\alpha^2)^2}\right.$$

$$\left. + \frac{3c^2}{r^2(s^2+\alpha^2)^{5/2}}\right] e^{-\frac{r}{c}\sqrt{s^2+\alpha^2}} \tag{4.23}$$

and for $\hat{E}_z^{ret}$

$$\hat{E}_z^{ret} = -\frac{\mu_o p\omega_o s^3}{4\pi r(s^2+\omega_o^2)}\left[\frac{1}{(s^2+\alpha^2)^{1/2}} + \frac{c}{r(s^2+\alpha^2)} + \frac{c^2}{r^2(s^2+\alpha^2)^{3/2}}\right.$$

$$\left. - \frac{s^2\cos^2\theta}{s^2+\alpha^2}\left(\frac{1}{(s^2+\alpha^2)^{1/2}} + \frac{3c}{r(s^2+\alpha^2)} + \frac{3c^2}{r^2(s^2+\alpha^2)^{3/2}}\right)\right] e^{-\frac{r}{c}\sqrt{s^2+\alpha^2}} \tag{4.24}$$

and for $\hat{B}_\phi^{ret}$

$$\hat{B}_\phi^{ret} = \frac{\mu_o p\omega_o s^2 \sin\theta}{4\pi rc} \; \frac{s^2+\omega_p^2}{s^2+\omega_o^2}\left[\frac{1}{s^2+\alpha^2} + \frac{c}{r(s^2+\alpha^2)^{3/2}}\right] e^{-\frac{r}{c}\sqrt{s^2+\alpha^2}} \tag{4.25}$$

where $\alpha \equiv \omega_p \sin\theta$.

The fields as functions of time can now be obtained by using the Laplace inversion formula, Eq. 3.16. We treat first the inversion of $E_\rho^{ret}(\underline{r},t)$

$$E_\rho^{ret}(\underline{r},t) = \frac{\mu_o p\omega_o \cos\theta \sin\theta}{8\pi^2 i r} \int_\Gamma \frac{s^3(s^2+\omega_p^2)}{s^2+\omega_o^2}\left[\frac{1}{(s^2+\alpha^2)^{3/2}} + \frac{3c}{r(s^2+\alpha^2)^2} + \right.$$

$$\left. + \frac{3c^2}{r^2(s^2+\alpha^2)^{5/2}}\right] e^{st - \frac{r}{c}\sqrt{s^2+\alpha^2}} \, ds \tag{4.26}$$

This integral is of the same form as the one in the preceding problem dealing with radiation in an isotropic plasma. It can be solved by using the method described in Appendix B . After subtracting away the δ function behavior and performing the integration contour transformations and substitutions described in Appendix B, we arrive at

$$E_\rho^{ret}(\underline{r},t) = \frac{\mu_c p\omega_o \cos\,\theta\,\sin\,\theta}{4\pi r}\left[\delta(t - \tfrac{r}{c}) - \frac{r\alpha^2}{c}\frac{J_1(q)}{q}\,H(t - \tfrac{r}{c})\right]$$

$$+ \frac{\mu_o p\omega_o \cos\,\theta\,\sin\,\theta}{8\pi^2 r}\int_0^{2\pi}\left\{\frac{\alpha i}{2\xi}\left[\frac{(\xi^2-\mathrm{ctn}^2\,\frac{\theta}{2})(\xi^2-\tan^2\,\frac{\theta}{2})(1+\xi^2)^3}{(\xi^2-\xi_o^2)(\xi^2-\xi_o^{-2})(1-\xi^2)^2} - 1 + \xi^2\right]\right.$$

$$\left.+ \frac{3(1+\xi^2)^3\,(\xi^2-\mathrm{ctn}^2\,\frac{\theta}{2})(\xi^2-\tan^2\,\frac{\theta}{2})}{(\xi^2-\xi_o^2)(\xi^2-\xi_o^{-2})(1-\xi^2)^3}\left(\frac{c}{r} + \frac{2c^2}{\alpha i r^2(1-\xi^2)}\right)\right\}e^{iq\cos\psi}\,d\psi\,H(t-\tfrac{r}{c}) \qquad (4.27)$$

where $\beta \equiv r/ct$, $q \equiv \alpha t\sqrt{1-\beta^2}$, $\gamma \equiv \sqrt{(1-\beta)/(1+\beta)}$, $\xi = \gamma e^{i\psi}$, and $\xi_o \equiv (\omega_o + \sqrt{\omega_o^2 - \alpha^2})/\alpha$ when $\omega_o > \alpha$ or $\xi_o \equiv (\omega_o + i\sqrt{\alpha^2 - \omega_o^2})/\alpha$ when $\omega_o < \alpha$.

Partial fractioning of the integrand yields

$$E_\rho^{ret}(\underline{r},t) = \frac{\mu_o p\omega_o \cos\,\theta\,\sin\,\theta}{4\pi r}\left[\delta(t - \tfrac{r}{c}) - \frac{r\alpha^2}{c}\frac{J_1(q)}{q}\;H(t - \tfrac{r}{c})\right]$$

$$+ \frac{\mu_o p\omega_o \cos\,\theta\,\sin\,\theta}{8\pi^2 r}\int_0^{2\pi}\left\{-\frac{c}{r}\,\alpha i\,\left[\xi + \frac{\omega_o^2(\omega_o^2 - \omega_p^2)}{\alpha(\omega_o^2 - \alpha^2)^{3/2}}\left(\frac{\xi_o\xi}{\xi^2-\xi_o^2} - \frac{\xi_o^{-1}\,\xi}{\xi^2 - \xi_o^{-2}}\right)\right.\right.$$

$$+ \frac{2\omega_p^2 \cos^2\theta}{\omega_o^2 - \alpha^2}\left(\frac{2\xi}{(\xi^2-1)^2} + \frac{\xi}{\xi^2-1}\right)\Bigg] - \frac{3c}{r}\Bigg[1 + \frac{\omega_o^2(\omega_o^2-\omega_p^2)}{(\omega_o^2-\alpha^2)^2}\left(\frac{\xi_o^2}{\xi^2-\xi_o^2} + \frac{\xi_o^{-2}}{\xi^2-\xi_o^{-2}}\right)$$

$$+ \frac{8\omega_p^2\cos^2\theta}{\omega_o^2-\alpha^2}\frac{1}{(\xi^2-1)^3} + \frac{12\omega_p^2\cos^2\theta}{\omega_o^2-\alpha^2}\frac{1}{(\xi^2-1)^2} + \frac{2\omega_p^2\cos^2\theta}{\omega_o^2-\alpha^2}\left(4-\sec^2\theta+\frac{\alpha^2}{\omega_o^2-\alpha^2}\right)\frac{1}{\xi^2-1}\Bigg]$$

$$+ \frac{6c^2}{(\omega_o^2-\alpha^2)\alpha i r^2}\Bigg[\frac{\alpha\omega_o^2(\omega_o^2-\omega_p^2)}{2(\omega_o^2-\alpha^2)^{3/2}}\left(\frac{\xi_o\xi}{\xi^2-\xi_o^2} - \frac{\xi_o^{-1}\xi}{\xi^2-\xi_o^{-2}}\right) + 4\omega_p^2\cos^2\theta\left(\frac{2\xi}{(\xi^2-1)^4}+\frac{3\xi}{(\xi^2-1)^3}\right)$$

$$+ 2\omega_p^2\cos^2\theta\left(4-\sec^2\theta+\frac{\alpha^2}{\omega_o^2-\alpha^2}\right)\frac{\xi}{(\xi^2-1)^2} + \omega_p^2\cos^2\theta\Bigg(2-\sec^2\theta \;+$$

$$+ \frac{\alpha^2}{\omega_o^2-\alpha^2}\Bigg)\frac{\xi}{\xi^2-1}\Bigg]\Bigg\} e^{iq\cos\psi}\, d\psi\, \Pi(t-\tfrac{r}{c}) \qquad (4.28)$$

When we carry out these integrations by using the integration formulas in Appendix C, we obtain

$$E_\rho^{ret}(\underline{r},t) = \frac{\mu_o p\omega_o \cos\theta\sin\theta}{4\pi r}\Bigg\{2\delta(t-\tfrac{r}{c}) - \frac{r\alpha^2}{c}\frac{J_1(q)}{q} - \alpha\gamma J_1(q)$$

$$- \frac{\omega_o^2(\omega_o^2-\omega_p^2)}{(\omega_o^2-\alpha^2)^{3/2}}\left(U_1(\gamma\xi_o q,q) - U_1(\gamma\xi_o^{-1}q,q)\right) - \frac{\alpha\omega_p^2\cos^2\theta}{\omega_o^2-\alpha^2}\, q(\gamma U_o + \gamma^{-1}U_2)$$

$$- \frac{3c}{r}\Bigg[J_o(q) - \frac{\omega_o^2(\omega_o^2-\omega_p^2)}{(\omega_o^2-\alpha^2)^2}\left(U_o(\gamma\xi_o q,q) + U_o(\gamma\xi_o^{-1}q,q)\right)$$

$$- \frac{1}{4}\frac{\omega_p^2\cos^2\theta}{\omega_o^2-\alpha^2}\left(q^2(\gamma^2U_{-2} + 2U_o + \gamma^{-2}U_2) + 2q(\gamma U_{-1} - \gamma^{-1}U_1)\right)$$

$$- \frac{2\omega_p^2 \cos^2\theta}{\omega_o^2 - \alpha^2}\left(\frac{\omega_o^2}{\omega_o^2 - \alpha^2} - \tan^2\theta\right) U_o \Bigg]$$

$$+ \frac{3c^2}{\alpha(\omega_o^2 - \alpha^2)r^2}\Bigg[\frac{\alpha\omega_o^2(\omega_o^2 - \omega_p^2)}{(\omega_o^2 - \alpha^2)^{3/2}}\left(U_1(\gamma\xi_o q,q) - U_1(\gamma\xi_o^{-1}q,q)\right)$$

$$+ \omega_p^2 \cos^2\theta\left(\frac{\omega_o^2}{\omega_o^2 - \alpha^2} - \tan^2\theta\right) q(\gamma U_o + \gamma^{-1}U_2)$$

$$+ \frac{\omega_p^2 \cos^2\theta}{24}\left\{q^3(\gamma^3 U_{-2} + 3\gamma U_o + 3\gamma^{-1}U_2 + \gamma^{-3}U_4) + 6q^2(\gamma^2 U_{-1} - \gamma^{-2}U_3)\right\}\Bigg]\Bigg\} H(t - \frac{r}{c}) \tag{4.29}$$

where for brevity we write U_n for $U_n(\gamma q,q)$.

By performing the same sequence of operations on Eqs. 4.24 and 4.25 we obtain for E_z^{ret} and B_ϕ^{ret}

$$E_z^{ret}(\underline{r},t) = \frac{\omega_o \mu_o p}{4\pi r}\Bigg\{-2\sin^2\theta\ \delta(t - \frac{r}{c}) + \sin^2\theta\ \frac{r\alpha^2}{c}\frac{J_1(q)}{q} + \alpha \sin^2\theta\, \gamma J_1(q)$$

$$+ \frac{\omega_o^2(\omega_o^2 - \omega_p^2)\sin^2\theta}{(\omega_o^2 - \alpha^2)^{3/2}}\left(U_1(\gamma\xi_o q,q) - U_1(\gamma\xi_o^{-1}q,q)\right) + \frac{\alpha^3\cos^2\theta}{\omega_o^2 - \alpha^2} q(\gamma U_o + \gamma^{-1}U_2)$$

$$+ \frac{c}{r}\Bigg[(1 - 3\cos^2\theta)J_o(q) - \frac{\omega_o^2(\omega_o^2 - \alpha^2 - 3\omega_o^2\cos^2\theta)}{(\omega_o^2 - \alpha^2)^2}\left\{U_o(\gamma\xi_o q,q) + U_o(\gamma\xi_o^{-1}q,q)\right\}$$

$$- \frac{3}{4}\frac{\alpha^2\cos^2\theta}{\omega_o^2 - \alpha^2}\left\{q^2(\gamma^2 U_{-2} + 2U_o + \gamma^{-2}U_2) + 2q(\gamma U_{-1} - \gamma^{-1}U_1)\right\}$$

$$-12\frac{\alpha^2\cos^2\theta}{\omega_o^2 - \alpha^2}\left(1 + \frac{1}{2}\frac{\alpha^2}{\omega_o^2 - \alpha^2} - \frac{1}{6}\sec^2\theta\right) U_o\Bigg]$$

$$- \frac{c^2}{r^2}\left[\frac{\omega_o^2(\omega_o^2 - \alpha^2 - 3\omega_o^2\cos^2\theta)}{(\omega_o^2 - \alpha^2)^{5/2}}\Big(U_1(\gamma\xi_o q,q) - U_1(\gamma\xi_o^{-1}q,q)\Big)\right.$$

$$+ \frac{1}{8}\frac{\alpha\cos^2\theta}{\omega_o^2 - \alpha^2}\Big(q^3(\gamma^3 U_{-2} + 3\gamma U_o + 3\gamma^{-1}U_2 + \gamma^{-3}U_4) + 6q^2(\gamma^2 U_{-1} - \gamma^{-2}U_3)\Big)$$

$$\left. + \frac{6\alpha\cos^2\theta}{\omega_o^2 - \alpha^2}\Big(1 + \frac{1}{2}\frac{\alpha^2}{\omega_o^2 - \alpha^2} - \frac{1}{6}\sec^2\theta\Big)\, q(\gamma U_o + \gamma^{-1}U_2)\right]\Bigg\} H(t - \frac{r}{c}) \qquad (4.30)$$

and

$$B_\phi^{ret}(\underline{r},t) = \frac{\mu_o\omega_o p\sin\theta}{4\pi rc}\Bigg\{2\delta(t - \frac{r}{c}) - \frac{r}{c}\ \frac{\alpha^2 J_1(q)}{q} + \frac{\omega_o(\omega_p^2 - \omega_o^2)}{\omega_o^2 - \alpha^2}\Big(U_1(\gamma\xi_o q,q)$$

$$+ U_1(\gamma\xi_o^{-1}q,q)\Big) - \frac{2\omega_p^2\cos^2\theta\alpha U_1}{\omega_o^2 - \alpha^2} + \frac{c}{r}\Big[J_o(q) + \frac{\omega_o(\omega_o^2 - \omega_p^2)}{(\omega_o^2 - \alpha^2)^{3/2}}\Big(U_o(\gamma\xi_o q,q) -$$

$$- U_o(\gamma\xi_o^{-1}q,q)\Big) + \frac{\omega_p^2\cos^2\theta}{\omega_o^2 - \alpha^2}\, q(\gamma^{-1}U_1 + \gamma U_{-1})\Big]\Bigg\} H(t - \frac{r}{c}) \qquad (4.31)$$

Near the wavefront, the fields behave like

$$E_\rho^{ret} = \frac{\omega_o\mu_o p\cos\theta\sin\theta}{4\pi r}\Bigg\{\Bigg[2\delta(t - \frac{r}{c}) - \frac{r}{c}\,\frac{\alpha^2 J_1(q)}{q} + \frac{(2\omega_p^2 - 2\omega_o^2 - 3\alpha^2)}{\alpha}\gamma J_1(q)$$

$$+ \frac{3c}{r}J_o(q) + \frac{6c^2}{\alpha r^2}\gamma\, J_1(q)\Bigg] + O(\gamma^2)\Bigg\} H(t - \frac{r}{c}) \qquad (4.32)$$

$$E_z^{ret} = -\frac{\omega_o\mu_o p}{4\pi r}\Bigg\{\Bigg[\sin^2\theta\Big(2\delta(t - \frac{r}{c}) - \frac{r}{c}\,\frac{\alpha^2 J_1(q)}{q} + \frac{(2\omega_p^2 - 2\omega_o^2 - 3\alpha^2)}{\alpha}\gamma\, J_1(q)\Big)$$

$$+ \frac{c}{r}(1 - 3\cos^2\theta)J_o(q) + \frac{c^2}{r^2}\ \frac{1 - 3\cos^2\theta}{\alpha}\,2\gamma J_1(q)\Bigg] + O(\gamma^2)\Bigg\} H(t - \frac{r}{c}) \qquad (4.33)$$

and

$$B_{\phi}^{ret} = \frac{\omega_o \mu_o p \sin\theta}{4\pi rc}\left\{\left[2\delta(t-\frac{r}{c}) - \frac{r}{c}\,\frac{\alpha^2 J_1(q)}{q} + \frac{\omega_p^2-\omega_o^2-\alpha^2}{\alpha}\,2\gamma J_1(q) + \frac{c}{r}\,J_o(q)\right] + O(\gamma^2)\right\} H(t-\frac{r}{c}) \tag{4.34}$$

This initial behavior is quite similar to that found in the isotropic medium (see Eqs. 3.50 to 3.52) with $\alpha = \omega_p \sin\theta$ substituted for ω_p. To zeroth order in γ, a uniaxial medium behaves like an isotropic plasma having a plasma frequency $\omega_p \sin\theta$.

The behavior of the fields after a long time can be obtained by using the recursion relation A.5 in Appendix A and the asymptotic expression D.38 through D.45 in Appendix D. Discarding terms which vanish as $t \to \infty$, we obtain for $\omega_o > \omega_p \sin\theta$

$$E_\rho^{ret} \sim \frac{\omega_o \mu_o p \cos\theta \sin\theta}{4\pi r}\left\{-\frac{\omega_o^2(\omega_o^2-\omega_p^2)}{(\omega_o^2-\alpha^2)^{3/2}}\sin(\omega_o t - \frac{r}{c}\sqrt{\omega_o^2-\alpha^2}) + \frac{3c}{r}\,\frac{\omega_o^2(\omega_o^2-\omega_p^2)}{(\omega_o^2-\alpha^2)^2}\cos(\omega_o t - \frac{r}{c}\sqrt{\omega_o^2-\alpha^2}) + \frac{3c^2}{r^2(\omega_o^2-\alpha^2)}\left[\frac{\omega_o^2(\omega_o^2-\omega_p^2)}{(\omega_o^2-\alpha^2)^{3/2}}\sin(\omega_o t - \frac{r}{c}\sqrt{\omega_o^2-\alpha^2}) + \omega_p^2\cos^2\theta\left(\frac{\omega_o^2}{\omega_o^2-\alpha^2} - \tan^2\theta\right) t\, J_o(q) - \frac{\omega_p^2\cos^2\theta\;\alpha t^2}{3} J_1(q)\right]\right\} \tag{4.35}$$

$$E_z^{ret} \sim \frac{\omega_o \mu_o p}{4\pi r} \left\{ \frac{\omega_o^2(\omega_o^2 - \omega_p^2)\sin^2\theta}{(\omega_o^2 - \alpha^2)^{3/2}} \sin(\omega_o t - \frac{r}{c}\sqrt{\omega_o^2 - \alpha^2}) \right.$$

$$- \frac{c}{r} \frac{\omega_o^2}{(\omega_o^2 - \alpha^2)^2} (\omega_o^2 - \alpha^2 - 3\omega_o^2 \cos^2\theta) \cos(\omega_o t - \frac{r}{c}\sqrt{\omega_o^2 - \alpha^2})$$

$$- \frac{c^2}{r^2} \left[\frac{\omega_o^2(\omega_o^2 - \alpha^2 - 3\omega_o^2 \cos^2\theta)}{(\omega_o^2 - \alpha^2)^{5/2}} \sin(\omega_o t - \frac{r}{c}\sqrt{\omega_o^2 - \alpha^2}) - \frac{\alpha^3\cos^2\theta}{\omega_o^2 - \alpha^2} t^2 J_1(q) \right.$$

$$\left. \left. + \frac{6\alpha^2\cos^2\theta}{\omega_o^2 - \alpha^2} \left(1 + \frac{1}{2}\frac{\alpha^2}{\omega_o^2 - \alpha^2} - \frac{1}{6}\sec^2\theta \right) t\, J_o(q) \right] \right\} \tag{4.36}$$

and

$$B_\phi^{ret} \sim \frac{\omega_o \mu_o p \sin\theta}{4\pi r c} \left\{ \frac{\omega_o(\omega_p^2 - \omega_o^2)}{\omega_o^2 - \alpha^2} \sin(\omega_o t - \frac{r}{c}\sqrt{\omega_o^2 - \alpha^2}) \right.$$

$$\left. + \frac{c}{r} \left[\frac{\omega_o(\omega_o^2 - \omega_p^2)}{(\omega_o^2 - \alpha^2)^{3/2}} \cos(\omega_o t - \frac{r}{c}\sqrt{\omega_o^2 - \alpha^2}) - \frac{\alpha\omega_p^2 \cos^2\theta}{\omega_o^2 - \alpha^2} t\, J_1(q) \right] \right\} \tag{4.37}$$

When $\omega_o < \omega_p \sin\theta$ we obtain

$$E_\rho^{ret} \sim \frac{\omega_o^3 \mu_o p \cos\theta \sin\theta}{4\pi r} \frac{(\omega_o^2 - \omega_p^2)}{(\alpha^2 - \omega_o^2)^{3/2}} \left[1 + \frac{3c}{r\sqrt{\alpha^2 - \omega_o^2}} \right.$$

$$\left. + \frac{3c^2}{r^2(\alpha^2 - \omega_o^2)} \right] e^{-\frac{r}{c}\sqrt{\alpha^2 - \omega_o^2}} \cos\omega_o t + \frac{3\omega_o \omega_p^2 \mu_o p c^2 \cos^3\theta \sin\theta}{4\pi r^3(\alpha^2 - \omega_o^2)}$$

$$\times \left[\left(\frac{\omega_o^2}{\alpha^2 - \omega_o^2} + \tan^2\theta \right) t\, J_o(q) + \frac{\alpha t^2}{3} J_1(q) \right] \tag{4.38}$$

$$E_z^{ret} \sim \frac{\omega_o^3 \mu_o p}{4\pi r(\alpha^2-\omega_o^2)^{3/2}} \left[(\omega_p^2 - \omega_o^2)\sin^2\theta + \frac{c}{r} \frac{(\alpha^2-\omega_o^2+3\omega_o^2 \cos^2\theta)}{\alpha^2-\omega_o^2} \right.$$

$$\left. + \frac{c^2}{r^2}\left(\frac{\alpha^2-\omega_o^2+3\omega_o^2\cos^2\theta}{\alpha^2-\omega_o^2}\right)\right] e^{-\frac{r}{c}\sqrt{\alpha^2-\omega_o^2}} \cos \omega_o t$$

$$- \frac{\alpha^2 \omega_o \cos^2\theta \; \mu_o p c^2}{4\pi r^3(\alpha^2-\omega_o^2)} \left[\alpha t^2 J_1(q) + (\sec^2\theta - 6 + \frac{3\alpha^2}{\alpha^2-\omega_o^2})\; t \; J_o(q)\right] \qquad (4.39)$$

and

$$B_\rho^{ret} \sim \frac{\omega_o^2 \mu_o p \sin\theta}{4\pi r c}\left(\frac{\omega_o^2-\omega_p^2}{\alpha^2-\omega_o^2}\right)\left(1+\frac{c}{r\sqrt{\alpha^2-\omega_o^2}}\right) e^{-\frac{r}{c}\sqrt{\alpha^2-\omega_o^2}} \sin \omega_o t$$

$$+ \frac{\alpha\omega_o\omega_p^2 \; \mu_o p \sin\theta \cos^2\theta}{4\pi r^2(\alpha^2-\omega_o^2)}\; t \; J_1(q) \qquad (4.40)$$

In these formulas for the long time asymptotic behaviors of the fields, we see a definite increase in amplitudes as the polar angle θ approaches $\sin^{-1} \omega_o/\omega_p$. These asymptotics are not uniformly valid as $\theta \to \sin^{-1} \omega_o/\omega_p$. The limit $\theta \to \sin^{-1} \omega_o/\omega_p$ can be taken in Eqs. 4.29, 4.30 and 4.31. The result would show that there is no singular behavior of the fields at that angle. However, at that angle the rate at which the amplitudes of the fields grow is increased by one more power of t . Thus instead of the terms oscillating at frequency $\omega_p \sin\theta$ increasing in amplitude like $t^{1/2}$ (as in $\underline{B}$) or $t^{3/2}$ (as in $\underline{E}$), they will at this angle increase like $t^{3/2}$ or $t^{5/2}$ respectively. The two oscillations become inextricably combined when $\omega_o = \omega_p \sin\theta$. These comments will be borne out in the results of some numerical calculations which follow. Also, as could be anticipated by the singularities of the Laplace transformed fields near $s = \pm i\alpha$, we see that the field quantities contain oscillations at a frequency $\alpha = \omega_p \sin\theta$ and that these oscillations increase in amplitude with time. As noted in the discussion of the fields in an isotropic medium, there is also an upper time limit on the validity of the solutions derived here if additional effects, e.g. losses, are taken into account. The parts of the asymptotic solutions which oscillate at the source frequency ω_o are the conventional time-harmonic solutions to this problem (31).

From the long time asymptotic behaviors of the fields, we can readily recognize that the region of space for which $0 < \sin\theta < \omega_o/\omega_p$ is a region of propagation and the region described by $\sin\theta > \omega_o/\omega_p$ is a region of nonpropagation.

For $\omega_o > \omega_p \sin\theta$ the Lommel functions which eventually contribute to the oscillation of frequency ω_o are $U_o(\gamma\xi_o q,q)$ and $U_1(\gamma\xi_o q,q)$. By the series definition of Lommel functions of two variables A.1, these functions are equal to

$$U_n(\gamma\xi_o q,q) = \sum_{m=0}^{\infty} (-)^m \, (\gamma\xi_o)^{n+2m} \, J_{n+2m}(q) \tag{4.41}$$

in which n takes on the values 0 and 1. For $\gamma\xi_o < 1$, the convergence of this series is governed by the coefficients of the Bessel functions, but when $\gamma\xi_o$ is greater than unity, the convergence of the series is due to the Bessel functions. In the latter case it is more convenient to consider the equivalent representation for

$U_n(\gamma\xi_o q,q)$

$$U_n(\gamma\xi_o q,q) = \cos(\omega_o t - \frac{r}{c}\sqrt{\omega_o^2 - \alpha^2} - \frac{n\pi}{2}) + (-)^n \sum_{m=0}^{\infty} (-)^m (\gamma\xi_o)^{n-2-2m}$$

$$\times \; J_{2m+2-n}(q) \qquad (4.42)$$

in which the convergence of the series is again established by the coefficients of the Bessel functions $\gamma\xi_o$. That series is, according to Eqs. D.5 and D.15 $O(t^{-1/2})$ as $t \to \infty$. The critical condition $\gamma\xi_o = 1$ corresponds to two significant events. Mathematically, $\gamma\xi_o = 1$ when the elliptical integration path of the integration technique described in Appendix B crosses the poles of the integrand due to the source at $s = \pm i\omega_o$. Physically $\gamma\xi_o = 1$ at time $T = r/v_{group}$ where r is the distance from the source to the observation point, and $v_{group} = c\sqrt{1 - (\omega_p^2 \sin^2\theta/\omega_o^2)}$ is the group velocity, calculable from the dispersion relation associated with the differential operator acting on the fields in Eqs. 4.13 and 4.14. $\gamma\xi_o = 1$ just at the time when the main signal of frequency ω_o arrives at the observation point from the source.

For $\omega_o > \omega_p \sin\theta$ a saddle point technique asymptotic analysis of the inversion integral gives the same results as Eqs. 4.35 through 4.37. In a saddle point integration the saddle point moves with time and crosses the poles of the integrand at $s = \pm i\omega_o$ at the time $T = \frac{r}{c}\sqrt{1 - (\omega_p^2 \sin^2\theta/\omega_o^2)}$. At this time the oscillation at frequency ω_o begins to contribute to the signal and we note that this is also the time when $\gamma\xi_o = 1$ if the method followed in this paper is adopted.

Numerical calculations of E_θ and B_ϕ were performed. Plasma frequencies of $0.9\omega_o$ and $\sqrt{2}\ \omega_o$ were chosen to represent under- and over-dense plasmas, respectively. The fields were evaluated at several polar angles, so that the effects of the anisotropy could be illustrated. Figures 4.1 through 4.8 show E_θ and B_ϕ as a function of normalized time $\tau = \omega_o t$. Figures 4.1 and 4.2 show the fields at polar angles 45^o and 90^o for $\omega_p = 0.9\omega_o$ and $r = 10c/\omega_o$. In both figures the beating of the oscillations having frequencies ω_o and $\omega_p \sin\theta$ is quite evident. This is an extreme case, in the sense of there being a very strong oscillation at frequency $\omega_p \sin\theta$, because the medium is quite dense and because the distance from the source to the observation point is relatively small. From the asymptotic formulas 4.35 through 4.37 it can be seen that eventually the oscillation at frequency $\alpha = \omega_p \sin\theta$ will dominate the solution. The horizontal dashed lines show the amplitude of the oscillation at frequency ω_o , i.e., the amplitude of the conventional steady-state solution.

In the over-dense case ($\omega_p = \sqrt{2}\ \omega_o$) the observation point is taken to be $r = 50c/\omega_o$ (meters) from the source. The fields E_θ and B_ϕ are shown for polar angles 15^o, 30^o, 45^o, 60^o and 90^o in Figures 4.3, 4.4, 4.5, 4.6, 4.7 and 4.8, respectively. The polar angle at which the steady state solutions become infinite is equal to 45^o for this choice of ω_p . In the region of propagation, at polar angles 15^o and 30^o, there is strong beating between the oscillations at frequencies ω_o and $\omega_p \sin\theta$. It was not possible to calculate the fields right at 45^o with the formulas derived here. The fields were

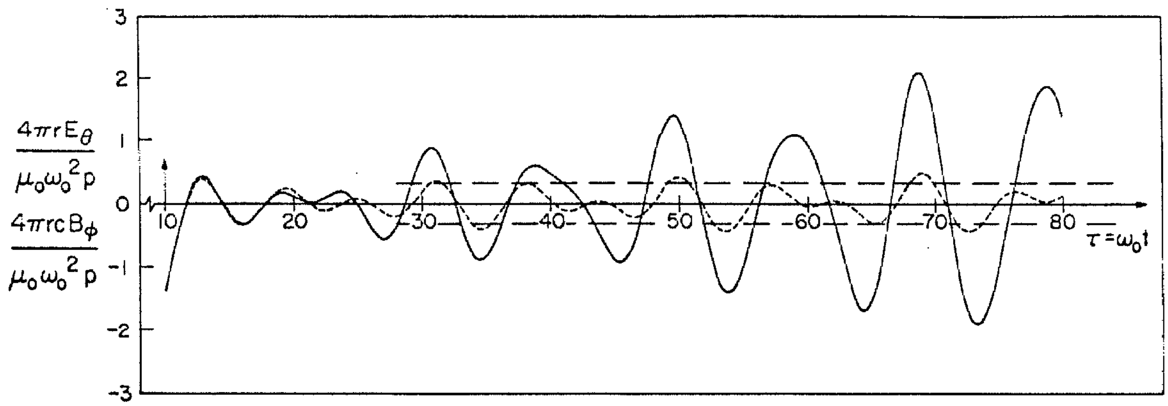


Figure 4.1. E_θ(____) and B_ϕ(----) for $\omega_p = 0.9\omega_o$, $r = 10c/\omega_o$, $\theta = 45^\circ$. Horizontal dashed lines indicate the time harmonic amplitude of E_θ . A δ-function is denoted by an arrow.

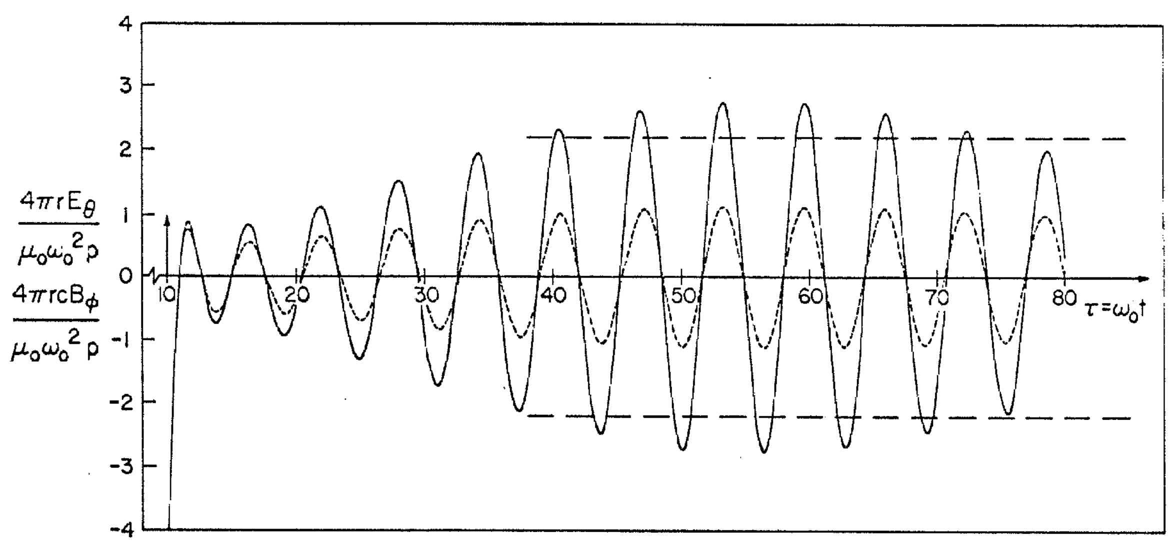


Figure 4.2. E_θ(____) and B_ϕ(----) for $\omega_p = 0.9\omega_o$, $r = 10c/\omega_o$, $\theta = 90^\circ$. Horizontal dashed lines indicate the time harmonic amplitude of E_θ . A δ-function is denoted by an arrow.

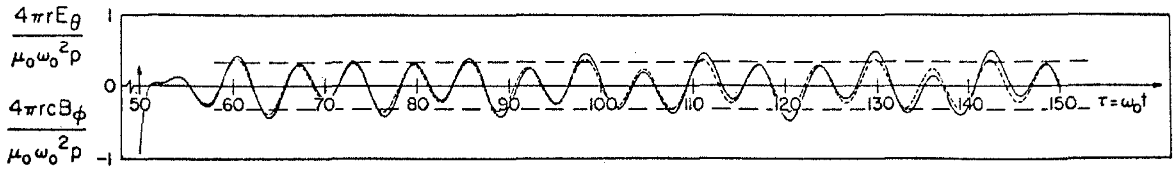


Figure 4.3. E_θ (____) and B_ϕ(----) for $\omega_p = \sqrt{2}\,\omega_o$, $r = 50c/\omega_o$, $\theta = 15^\circ$. Horizontal dashed lines indicate the time harmonic amplitude of E_θ . A δ-function is denoted by an arrow.

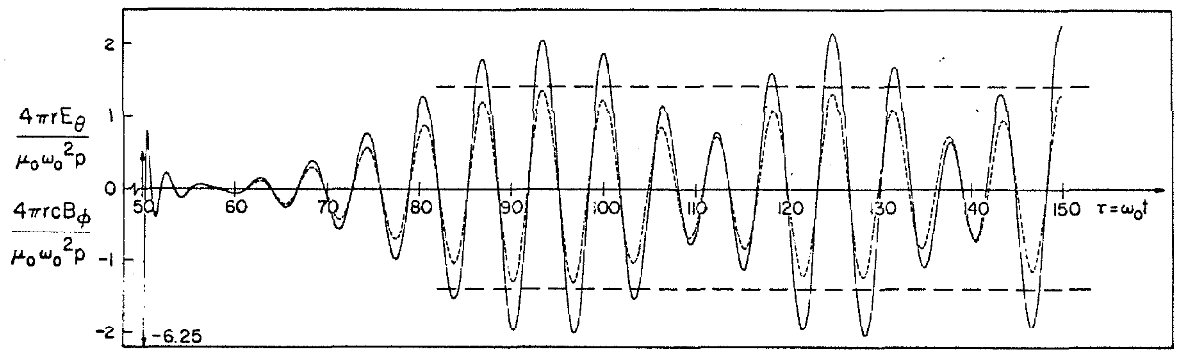


Figure 4.4. E_θ (____) and B_ϕ (----) for $\omega_p = \sqrt{2}\,\omega_o$, $r = 50c/\omega_o$, $\theta = 30^\circ$. Horizontal dashed lines indicate the time harmonic amplitude of E_θ . A δ-function is denoted by an arrow.

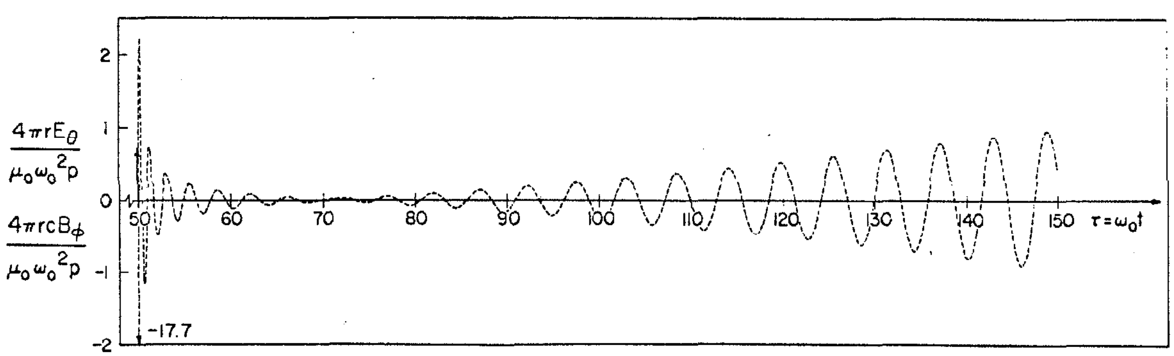


Figure 4.5. B_ϕ (----) for $\omega_p = \sqrt{2}\,\omega_o$, $r = 50c/\omega_o$, $\theta = 45^\circ$. A δ-function is denoted by an arrow.

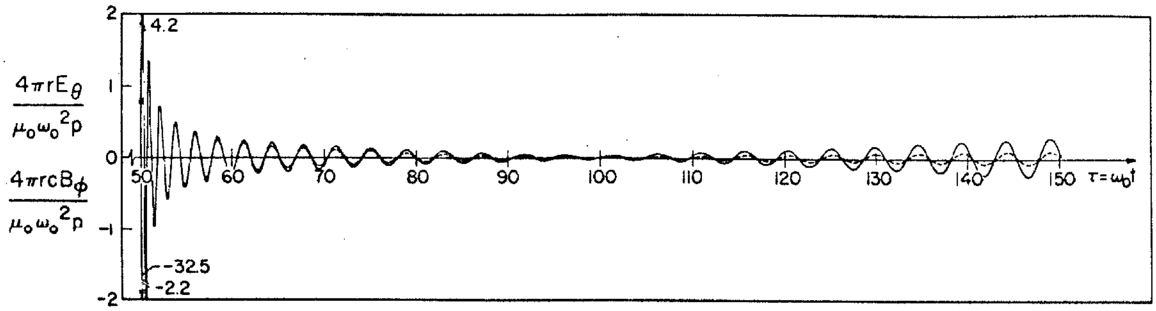


Figure 4.6. E_θ (____) and B_ϕ (----) for $\omega_p = \sqrt{2}\ \omega_o$, $r = 50c/\omega_o$, $\theta = 60^o$. A δ-function is denoted by an arrow.

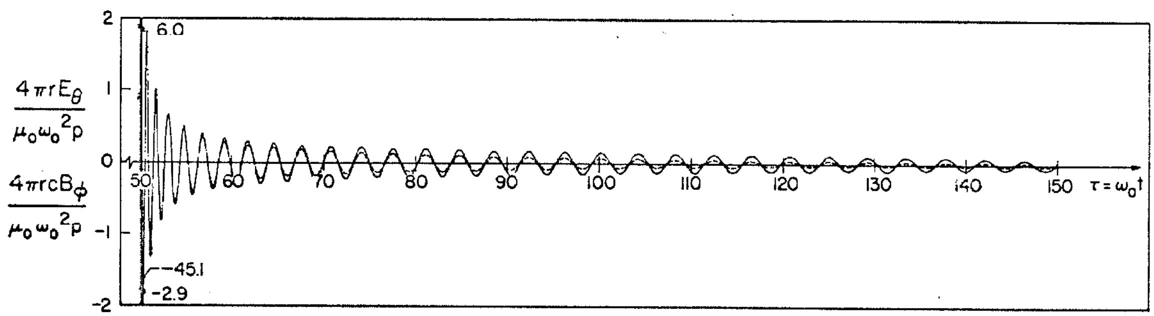


Figure 4.7. E_θ (____) and B_ϕ(----) for $\omega_p = \sqrt{2}\ \omega_o$, $r = 50c/\omega_o$, $\theta = 75^o$. A δ-function is denoted by an arrow.

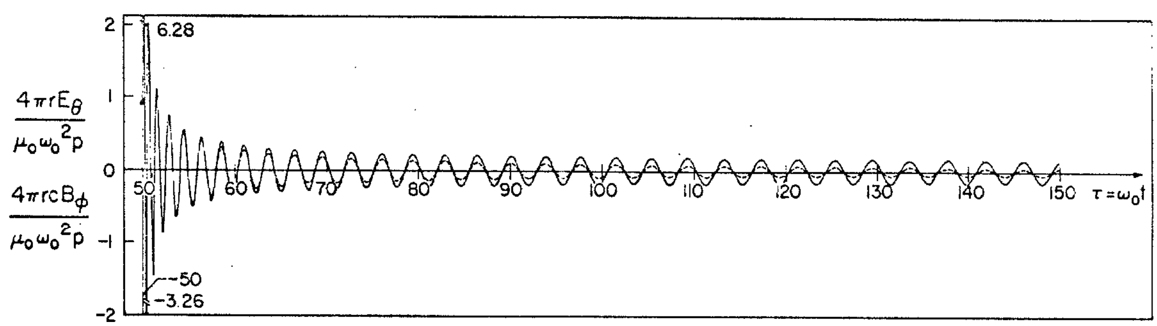


Figure 4.8. E_θ(____) and B_ϕ (----) for $\omega_p = \sqrt{2}\ \omega_o$, $r = 50c/\omega_o$, $\theta = 90^o$. A δ-function is denoted by an arrow.

computed at an angle slightly larger than 45°, and then again at an angle slightly less than 45°. The magnetic fields so calculated were in very close agreement and so they are taken to be the values at 45°. Numerical difficulties precluded the use of this method for calculating the electric field. Only the magnetic field is shown in Fig. 4.5, and it increases in amplitude very rapidly with time. In the region of nonpropagation, the fields are shown at polar angles 60°, 75° and 90°. After a large initial transient, the fields decrease in amplitude temporarily until the terms of ever-increasing amplitude (see Eqs. 4.38 to 4.40) begin to dominate the solution. The conventional steady-state solutions corresponding to those shown for $\theta = 60^{\circ}, 75^{\circ}$ and 90° are several orders of magnitude too small to be shown with the ordinate scale used here.

To calculate P^{irr} we must first find $E_z^{rad} = \frac{1}{2}(E_z^{ret} - E_z^{adv})$. As has been mentioned before, the advanced electric field can be obtained by the substitution of $-c$ for c in the expression for $\underline{E}^{ret}$. Obtaining E_z^{adv} from Eq. 4.30 in this manner, we find after many simplifications that

$$E_z^{rad}(\underline{r},t) = \frac{\mu_o p \omega_o^3 \cos \omega_o t}{4\pi r(\omega_o^2 - \alpha^2)^{3/2}} \Bigg[(\alpha^2 - \omega_o^2 \sin^2\theta) \sin(\tfrac{r}{c}\sqrt{\omega_o^2 - \alpha^2})$$

$$- \frac{c}{r} \frac{(\omega_o^2 - \alpha^2 - 3\omega_o^2 \cos^2\theta)}{\sqrt{\omega_o^2 - \alpha^2}} \cos(\tfrac{r}{c}\sqrt{\omega_o^2 - \alpha^2})$$

$$+ \frac{c^2}{r^2} \frac{(\omega_o^2 - \alpha^2 - 3\omega_o^2 \cos^3\theta)}{\omega_o^2 - \alpha^2} \sin(\tfrac{r}{c}\sqrt{\omega_o^2 - \alpha^2}) \Bigg] \qquad (4.42)$$

for $t > \frac{r}{c}$. The simplicity of Eq. 4.42 as compared to Eq. 4.30 is amazing. As $r \to 0$, we find that

$$E_z^{rad}(\underline{r},t) = -\frac{2}{3}\frac{\mu_o\omega_o^3 \, p \cos \omega_o t}{4\pi c} + O(|\underline{r}|) \tag{4.43}$$

Therefore by Eq. 2.5, we obtain

$$\langle P^{irr}(t)\rangle = \frac{\mu_o\omega_o^4 p^2}{12\pi c} \tag{4.44}$$

whether ω_o is greater than or less than the plasma frequency. $\langle P^{irr}(t)\rangle$ is equal to the time-averaged power radiated by a dipole in free space. This is the same result which Papas and Lee obtained using time-harmonic analysis (32).

The computation of $\langle P \rangle$ by Eq. 2.3 would entail a time averaging of $E_z^{ret}(\underline{r},t)$ multiplied by $\cos \omega_o t$, the time dependence of the current distribution. A straightforward, exact evaluation of $\langle P \rangle$ would be a formidable task in view of the complicated time dependence of $E_z^{ret}(\underline{r},t)$ as given by Eq. 4.30. The long-time asymptotic behavior of E_z^{ret} might be used, but the terms containing $tJ_o(q)$ and $t^2J_1(q)$ would never average to zero except for special polar angles θ. The quantity $\langle \underline{E}^{ret} \cdot \underline{J} \rangle$ would depend on θ and thus the evaluation of this quantity at $|\underline{r}| = 0$ would depend critically on how the origin is approached. In general, one would obtain $\langle P \rangle = \infty$, in contradistinction to the result of steady-state analysis in which the result

$$\langle P \rangle = \begin{cases} \dfrac{\mu_c \omega_o^4 p^2}{12\pi c} & \omega_o > \omega_p \\ \\ \infty & \omega_o < \omega_p \end{cases} \tag{4.45}$$

has been obtained (33). However, in the calculation of $\langle P \rangle$ via Eq. 2.3, if one performs the x and y integrations before the time averaging, one obtains

$$\langle P \rangle = \langle P^{irr} \rangle \tag{4.46}$$

where $\langle P^{irr} \rangle$ is given by Eq. 4.44. Thus the concept of P^{irr} provides an analytically convenient way to obtain the physically meaningful result for time averaged radiated power, given by Eq. 4.44.

5. CONCLUSIONS

In this paper exact formulas have been derived for the electric and magnetic fields caused by a turned-on oscillating point electric dipole in isotropic and uniaxial media. From these formulas the long-time asymptotic behaviors of the fields were then derived. It was discovered that in addition to the oscillations of the fields at the source frequency there are other oscillatory terms which do not vanish as $t \to \infty$. In an isotropic plasma there is a term in the expression for the electric field which oscillates at the plasma frequency. In a uniaxial plasma, the expressions for the electric and magnetic fields contain oscillations at the frequency $\omega_p \sin\theta$, and the amplitudes of these terms increase with time. Thus in a lossless uniaxial medium, the retarded fields do not reach a steady-state condition.

The terms which do not oscillate at the source frequency cause difficulties in the calculation of the average power radiated by the source $\langle P \rangle$. However, $\langle P^{irr} \rangle$ is free of these difficulties because as $t \to \infty$, $\frac{1}{2}(\underline{E}^{ret} - \underline{E}^{adv})$ oscillates only at the source frequency. In calculating $\langle P^{irr} \rangle$, the order in which spatial and time-averaging integrations are performed is not critical. Furthermore, the results of this computation do not depend on the method of averaging. The calculation of $\langle P \rangle$ does not enjoy either of these two convenient mathematical advantages. Therefore, in addition to the theoretical foundation for doing so, the calculation of radiation resistance via $\langle P^{irr} \rangle$ is also easier to carry out analytically than the calculation via $\langle P \rangle$.

Since radiation resistance is proportional to $\langle P^{irr} \rangle$, we conclude that in an isotropic plasma

$$R_{rad} = \begin{cases} 0 & \omega_o < \omega_p \\ \sqrt{1 - \dfrac{\omega_p^2}{\omega_o^2}}\; R_o & \omega_o > \omega_p \end{cases} \tag{5.1}$$

and in a uniaxial medium

$$R_{rad} = R_o \tag{5.2}$$

R_o is the radiation resistance of the dipole in vacuum. If a constant current flows along the length ℓ of the dipole, then R_o is given by

$$R_o = \frac{\mu_o \ell^2 \omega_o^2}{6\pi c} . \tag{5.3}$$

APPENDIX A

PROPERTIES OF LOMMEL FUNCTIONS OF TWO VARIABLES

Since it is quite possible that the reader may not be familiar with Lommel functions of two variables, some of their properties are summarized in this appendix (34).

The Lommel functions of two variables for integral orders are defined by the series

$$U_n(w,z) = \sum_{m=0}^{\infty} (-1)^m \left(\frac{w}{z}\right)^{n+2m} J_{n+2m}(z) \tag{A.1}$$

$$V_n(w,z) = \sum_{m=0}^{\infty} (-1)^m \left(\frac{w}{z}\right)^{-n-2m} J_{-n-2m}(z) \tag{A.2}$$

The two types of Lommel functions are interrelated in the following ways:

$$U_n(w,z) - V_{-n+2}(w,z) = \cos\left(\frac{w}{2} + \frac{z^2}{2w} - \frac{n\pi}{2}\right) \tag{A.3}$$

$$V_n(w,z) = (-1)^n U_n(z^2/w,z) \tag{A.4}$$

It can be easily seen from the defining series that these functions satisfy the recursion relations

$$U_n(w,z) + U_{n+2}(w,z) = \left(\frac{w}{z}\right)^n J_n(z) \tag{A.5}$$

$$V_n(w,z) + V_{n+2}(w,z) = \left(\frac{w}{z}\right)^{-n} J_{-n}(z) \tag{A.6}$$

APPENDIX B

AN INTEGRATION TECHNIQUE(35)

The inversion integrals used in this paper are all of the form

$$A(t) = \frac{1}{2\pi i} \int_{\Gamma} \frac{A_1(s, \sqrt{s^2+\alpha^2})}{A_2(s, \sqrt{s^2+\alpha^2})} e^{st - \frac{r}{c}\sqrt{s^2+\alpha^2}} ds \qquad \text{(B.1)}$$

A_1 and A_2 are polynomials in their arguments. As s goes to infinity, $A_1/A_2 \sim a_0 + a_1/s + a_2/s^2$ in the instances encountered here. If $a_0 \neq 0$, then there is a Dirac δ function in $A(t)$. This can be subtracted away and inverted separately, in view of the fact that the inverse of $e^{-\frac{r}{c}\sqrt{s^2+\alpha^2}}$ is $\delta(t-\frac{r}{c}) \frac{r}{c} a^2 \frac{J_1(q)}{q} H(t - \frac{r}{c})$ where $q = \alpha\sqrt{t^2 - \frac{r^2}{c^2}}$. Then $A_1/A_2 - a_0 = \tilde{A}_1/A_2$ which is $O(s^{-1})$ as s goes to infinity. It should be noted that by the very standard technique of closing the contour with a very large arc in the right-hand plane, $A(t)$ can be shown to be zero for $t < r/c$.

The singularities in the integrand occur only at $s = \pm i\omega_o$ and $s = \pm i\alpha$. They are indicated by crosses in Figure B.1 for the two cases $\omega_o < \alpha$ and $\omega_o > \alpha$. For the purpose of this method, the complex s plane is cut along the imaginary axis between $\pm i\alpha$.

Two successive deformations of the integration contour are now made. First the integral along Γ is shown to be equal to an integral along a path C around the branch cut plus contributions from the resudies. Figure B.2 indicates the method of proof.

In Case I, in the limit $L \to \infty$, path C becomes Γ_1; the integrals along C_4 and C_{14}, C_6 and C_{10} and C_{12} cancel one another,

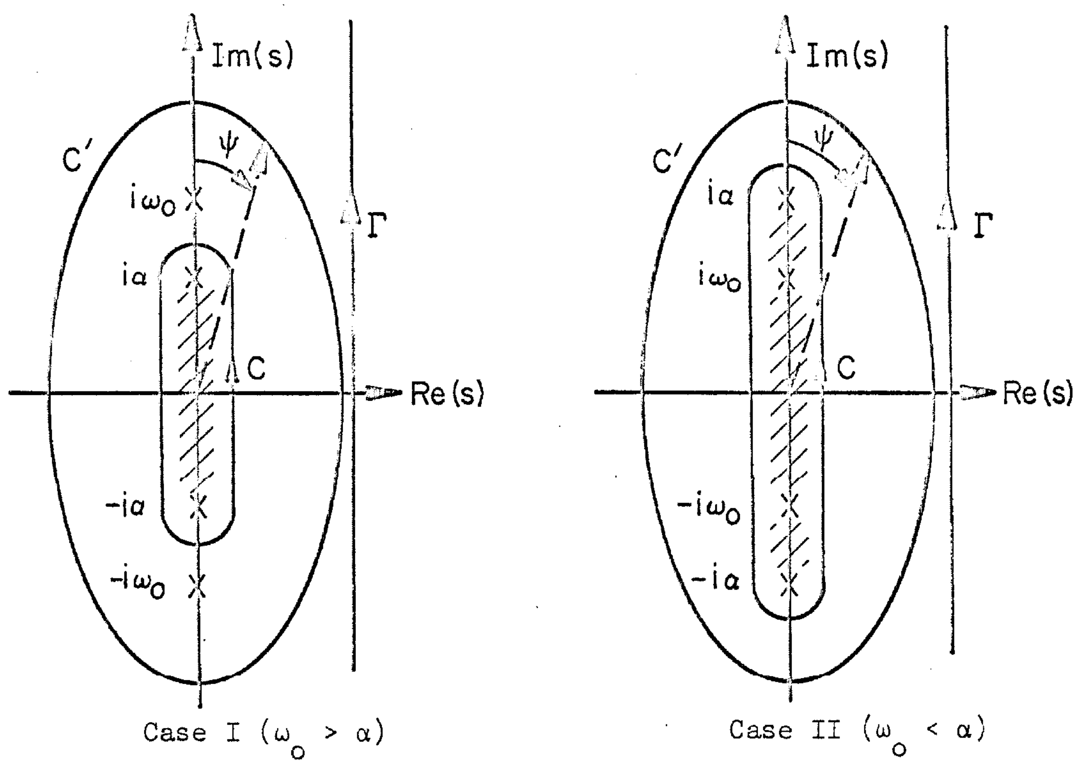


Figure B.1. Diagrams showing integration contours for Cases I and II

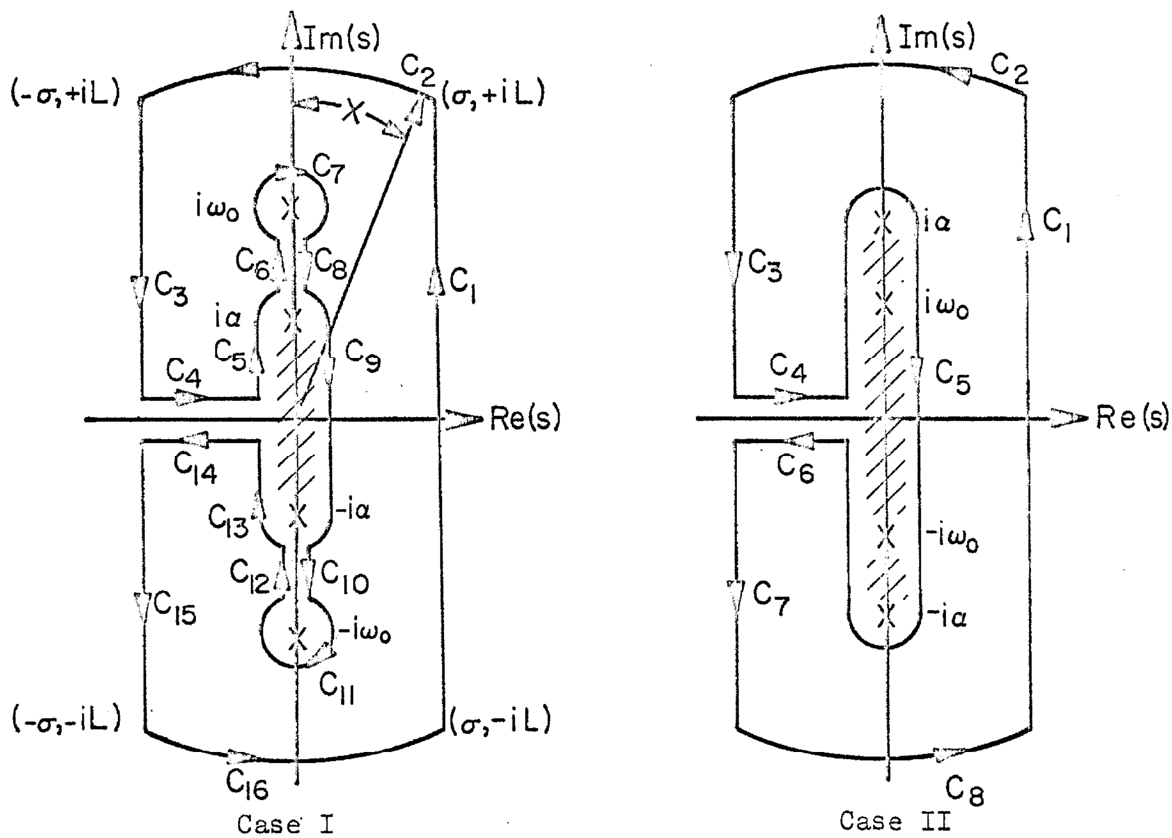


Figure B.2. Diagrams showing integration paths C_j

and the integrals C_7 and C_{11} have values equal to $-2\pi i$ times the residues of the integrand at $\pm i\omega_o$, respectively. The value of the sum of the integrals along C_3 and C_{15} is zero when $L \to \infty$ for $t > r/c$ since there are no singularities to the left of Re $= -\sigma$. That the integral along C_2 goes to zero as $L \to \infty$ can be shown in the following way.

$$\lim_{L \to \infty} \left| \frac{1}{2\pi i} \int_{C_2} \frac{\tilde{A}_1}{A_2} e^{st - \frac{r}{c}\sqrt{s^2 + \alpha^2}} ds \right| \leq$$

$$\leq \lim_{L \to \infty} \frac{1}{2\pi} \int_{= \frac{\pi}{2} - x}^{\frac{\pi}{2} + x} M e^{\sqrt{\sigma^2 + L^2}\,(t - \frac{r}{c}) \cos \phi} d\phi \tag{B.2}$$

where $x = \tan^{-1} \sigma/t$ and M is a constant which arises from the asymptotic property of $\tilde{A}_1/A_2$ as $s \to \infty$. Changing variables in the integral on the right-hand side one obtains for that integral

$$\frac{M}{2\pi} \int_{-x}^{x} e^{(t - \frac{r}{c})\sqrt{\sigma^2 + L^2} \sin \phi'} d\phi'$$

Using $\frac{2}{\pi}\phi' \leq \sin \phi' \leq \phi'$ for $0 \leq \phi' \leq \frac{\pi}{2}$ it can be shown that this integral is less than

$$\frac{M[1 - e^{-\frac{2}{\pi}(t - \frac{r}{c})\tan^{-1}\frac{\sigma}{L}}]}{4\,(t - \frac{r}{c})\,\sqrt{\sigma^2 + L^2}} + \frac{M\left\{e^{(t - \frac{r}{c})\sqrt{\sigma^2 + L^2}\tan^{-1}\frac{\sigma}{L}} - 1\right\}}{2\pi(t - \frac{r}{c})\,\sqrt{\sigma^2 + L^2}}$$

which vanishes as $L \to \infty$. Therefore

$$\lim_{L \to \infty} \frac{1}{2\pi i} \int_{C_2} \tilde{A}_1/A_2 \; e^{st - \frac{r}{c}\sqrt{s^2+\alpha^2}} \, ds = 0 \tag{B.3}$$

Similarly the contribution from the integral on path C_{16} can be shown to go to zero as $L \to \infty$. It follows from Cauchy's integral formula that

$$\sum_{n=1}^{16} \frac{1}{2\pi i} \int_{C_n} \tilde{A}_1/A_2 \; e^{st - \frac{r}{c}\sqrt{s^2+\alpha^2}} \, ds = 0 \tag{B.4}$$

and

$$A(t) + \frac{1}{2\pi i} \int_{C=C_5+C_9+C_{13}} \tilde{A}_1/A_2 \; e^{st - \frac{r}{c}\sqrt{s^2+\alpha^2}} \, ds - \sum_{\pm i\omega_o} \text{Res} \; \tilde{A}_1/A_2 \; e^{st - \frac{r}{c}\sqrt{s^2+\alpha^2}} = 0 \tag{B.5}$$

In exactly the same way, except for the absence of residue contributions, it can be shown in Case II that

$$A(t) + \frac{1}{2\pi i} \int_{C} \tilde{A}_1/A_2 \; e^{st - \frac{r}{c}\sqrt{s^2+\alpha^2}} \, ds = 0 \tag{B.6}$$

The path of integration about the branch cut is now made to conform to the ellipse described by

$$s = \frac{\alpha}{\sqrt{1-\beta^2}} [\beta \sin \psi + i \cos \psi] \; , \qquad \beta \equiv r/ct > 1$$
$$0 \leq \psi \leq 2\pi \tag{B.7}$$

This path is denoted by C' in Figure B.1. In Case I if the path C' lies outside $\pm i\omega_o$, i.e. $\alpha/\sqrt{1-\beta^2} > \omega_o$, then

$$A(t) - \frac{1}{2\pi i}\int_{C'} \frac{\tilde{A}_1}{A_2} e^{st - \frac{r}{c}\sqrt{s^2+\alpha^2}}\, d = 0 \tag{B.8}$$

If $\alpha/\sqrt{1-\beta^2} < \omega_o$, then

$$A(t) - \frac{1}{2\pi i}\int_{C'} \frac{\tilde{A}_1}{A_2} e^{st - \frac{r}{c}\sqrt{s^2+\alpha^2}}\, ds - \sum_{\pm i\omega_o} \text{Res}\, \frac{\tilde{A}_1}{A_2} e^{st - \frac{r}{c}\sqrt{s^2+\alpha^2}} = 0 \tag{B.9}$$

On C'

$$\sqrt{s^2+\alpha^2} = \frac{\alpha}{\sqrt{1-\beta^2}}\,[\sin\psi + i\beta\cos\psi] \tag{B.10}$$

and

$$st - \frac{r}{c}\sqrt{s^2+\alpha^2} = i\alpha t\sqrt{1-\beta^2}\cos\psi = iq\cos\psi \tag{B.11}$$

If we let $\xi = \gamma e^{i\psi}$ where $\gamma = \sqrt{(1-\beta)/(1+\beta)}$,

$$s = \frac{\alpha i}{2\xi}(1+\xi^2) \tag{B.12}$$

$$\sqrt{s^2+\alpha^2} = \frac{\alpha i}{2\xi}(1-\xi^2) \tag{B.13}$$

$$ds = \frac{\alpha}{2\xi}(1-\xi^2)d\psi \tag{B.14}$$

Therefore the integral in Eq. B.9 takes the form

$$\int_0^{2\pi} \frac{P(e^{i\phi})}{Q(e^{i\phi})} e^{iq\cos\psi}\, d\psi \tag{B.15}$$

P and Q are polynomials in $e^{i\phi}$. This ratio of polynomials can be expanded into partial fractions

$$\frac{P}{Q} = \sum_{n=0}^{N} a_n e^{in\phi} + \sum_{m=1}^{M} \sum_{\ell=1}^{L} b_{m,\ell} \frac{1}{(C_{m,\ell} + e^{i\phi})^{\ell}} \tag{B.16}$$

The first sum can be integrated term by term by using the integral representation for the Bessel function in order n :

$$2\pi i^n J_n(q) = \int_0^{2\pi} e^{\pm in\psi} e^{iq \cos \psi} d\psi \tag{B.17}$$

Each term in the second sum can be expanded into a Taylor or Laurent series (geometric series) in $e^{i\phi}$, depending on the magnitude of $C_{m,\ell}$. If $|C_{m,\ell}| \geq 1 + \delta$, $\delta > 0$, it is expandable into a uniformly convergent Taylor series or if $|C_{m,\ell}| \leq 1 - \delta$, $\delta > 0$, it is expandable into a uniformly convergent Laurent series. Thus in either instance the order of summation and integration can be interchanged. Using the integral representation for the Bessel function above, one obtains an infinite series of Bessel functions (a Neumann series) for the value of a typical term in the second sum of Eq. B.16. This series can be expressed in terms of Lommel functions of two variables.

APPENDIX C

EVALUATION OF AN INTEGRAL

In this appendix the following family of integrals is evaluated

$$I_{\ell,m}(q,K) = \int_0^{2\pi} \frac{\xi^\ell e^{iq\cos\psi}}{(\xi^2 - K^2)^m} d\psi \qquad \text{(C.1)}$$

for $\ell = 0,1$; $m = 1$ to 4 when $K = 1$; and $m = 1$ when $K = \xi_o > 1$, $K = \xi_o < 1$ or $|K| = |\xi_o| = |\xi_o^{-1}| = 1$.

First the integrals $K = 1, \xi_o$ with $\ell = 0,1$ and $m = 1$ are considered. For $\ell = 0$, $m = 1$ and K left as a variable parameter $\geqq 1$,

$$I_{o,1}(q,K) = \int_0^{2\pi} \frac{e^{iq\cos\psi}}{\xi^2 - K^2} d\psi = -\frac{1}{K^2}\int_0^{2\pi} \frac{e^{iq\cos\psi}}{1 - \frac{\gamma^2}{K^2} e^{i2\psi}} d\psi \qquad \text{(C.2)}$$

For $\beta = r/ct > 0$ and $\gamma/K < 1$, the denominator can be expanded into a geometric series

$$I_{o,1}(q,K) = -\frac{1}{K^2}\int_0^{2\pi} \sum_{k=0}^{\infty} \left[\frac{\gamma}{K} e^{i\psi}\right]^{2k} e^{iq\cos\psi}\, d\psi \qquad \text{(C.3)}$$

When the order of integration and summation is reversed and the integral representation for the Bessel function is used, we obtain

$$I_{o,1}(q,K) = -\frac{2\pi}{K^2}\sum_{k=0}^{\infty} (-1)^k \left(\frac{\gamma}{K}\right)^{2k} J_{2k}(q) \qquad \text{(C.4)}$$

In terms of Lommel functions of two variables

$$I_{o,1}(q,K) = -\frac{2\pi}{K^2} U_o(\frac{\gamma q}{K},q) \tag{C.5}$$

$$I_{o,1}(q,\xi_o) = -2\pi\xi_o^{-2}U_o(\gamma\xi_o^{-1}q,q) \tag{C.6}$$

$$I_{o,1}(q,1) = -2\pi U_o(\gamma q,q) \tag{C.7}$$

The integrals $K = 1$, $\ell = 0$, and $m = 2,3,4$ can be obtained from C.5 by successive differentiations with respect to K. Doing this and then setting $K = 1$, we obtain the formulas

$$I_{o,2}(q,1) = 2\pi U_o + \frac{\pi}{2}\, q(\gamma U_{-1} + \gamma^{-1}U_1) \tag{C.8}$$

$$I_{o,3}(q,1) = -2\pi U_o - \frac{\pi q}{8}(7\gamma U_{-1} + 5\gamma^{-1}U_1) - \frac{\pi q^2}{16}(\gamma^2 U_{-2} + 2U_o + \gamma^{-2}U_2) \tag{C.9}$$

$$I_{o,4}(q,1) = 2\pi U_o + \frac{\pi q}{16}(19\gamma U_{-1} + 11\gamma^{-1}U_1) + \frac{\pi q^2}{32}(5\gamma^2 U_{-2} + 8U_o + 3\gamma^{-2}U_2) + \frac{\pi q^3}{192}(\gamma^3 U_{-3} + 3\gamma U_{-1} + 3\gamma^{-1}U_1 + \gamma^{-3}U_3) \tag{C.10}$$

For the sake of notational simplicity, the arguments $(\gamma q,q)$ of the Lommel functions have been suppressed in Eqs. C.8, C.9 and C.10, i.e. $U_n = U_n(\gamma q,q)$. The instances $|\xi_o\gamma| < 1$ and $|\xi_o\gamma| > 1$ must be considered separately in evaluating $I_{o,1}(q,\xi_o^{-1})$. For $|\xi_o\gamma|$ we have as before (from Eq. C.4)

$$I_{o,1}(a,\xi_o^{-1}) = - 2\pi\xi_o^2 \, U_o(\gamma\xi_o q,q) \tag{C.11}$$

On the other hand, when $|\xi_o\gamma| > 1$

$$I_{o,1}(q,\xi_o^{-1}) = \int_0^{2\pi} \frac{e^{iq\cos\psi}}{\xi^2 - \xi_o^{-2}} \, d\psi =$$

$$= \int_0^{2\pi} \gamma^{-2} e^{-i2\psi} \sum_{k=0}^{\infty} (\gamma^{-1}\xi_o^{-1} e^{-i\psi})^{2k} e^{iq\cos\psi} \, d\psi$$

$$= \sum_{k=0}^{\infty} \gamma^{-2-2k} \xi_o^{-2k} \, 2\pi(-1)^{k+1} J_{2k+2}(q)$$

$$= - 2\pi\xi_o^2 \, U_2(\gamma^{-1}\xi_o^{-1} q,q) \tag{C.12}$$

By means of Eqs. A.3 and A.4 we obtain

$$U_2(\gamma^{-1}\xi_o^{-1},q,q) = U_o(\gamma\xi_o q,q) - \cos(\omega_o t - \frac{r}{c}\sqrt{\omega_o^2 - \alpha^2}) \tag{C.13}$$

After combining Eqs. C.11 and C.12 we obtain

$$I_{o,1}(q,\xi_o^{-1}) = -2\pi\xi_o^2 \, U_o(\gamma\xi_o q,q) + 2\pi\xi_o^2 \cos(\omega_o t - \frac{r}{c}\sqrt{\omega_o^2 - \alpha^2}) \, H\left(t - \frac{r}{c}\sqrt{1-(\frac{\alpha}{\omega_o})^2}\right) \tag{C.14}$$

In the Laplace inversion there is also a contribution due to poles in the integrand. See Eq. B.9. This term appears only for $t > \frac{r}{c}\sqrt{1 - (\frac{\alpha}{\omega_o})^2}$, and it exactly cancels the second term on the right hand side of Eq. C.14. Therefore for the value of $I_{o,1}(q,\xi_o^{-1})$ we shall formally use Eq. C.11 and the residue contribution will automatically be taken care of.

Similarly, it can be shown that

$$I_{1,1}(q,K) = -\frac{2\pi i}{K} U_1(qK^{-1}\gamma,q) \tag{C.15}$$

$$I_{1,1}(q,\xi_o) = -2\pi i\ \xi_o^{-1}\ U_1(\gamma\xi_o^{-1}q,q) \tag{C.16}$$

$$I_{1,1}(q,1) = -2\pi i\ U_1(\gamma q,q) \tag{C.17}$$

$$I_{1,2}(q,1) = \pi i U_1 + \frac{\pi i}{2} q(\gamma U_o + \gamma^{-1}U_2) \tag{C.18}$$

$$\begin{aligned} I_{1,3}(q,1) &= -\frac{3\pi i}{4} U_1 - \frac{\pi i}{8} q(5\gamma U_o + 3\gamma^{-1}U_2) \\ &\quad - \frac{\pi i}{16} q^2(\gamma^2 U_{-1} + 2U_1 + \gamma^{-2}U_3) \end{aligned} \tag{C.19}$$

$$\begin{aligned} I_{1,4}(q,1) &= \frac{5\pi i}{8} U_1 + \frac{\pi i}{16} q(11\gamma U_o + 5\gamma^{-1}U_2) \\ &\quad + \frac{\pi i}{16} q^2(2\gamma^2 U_{-1} + 3U_1 + \gamma^{-2}U_3) \\ &\quad + \frac{\pi i q^3}{192}(\gamma^3 U_{-2} + 3\gamma U_o + 3\gamma^{-1}U_2 + \gamma^{-3}U_4) \end{aligned} \tag{C.20}$$

$$I_{1,1}(q,\xi_o) = -2\pi i\ \xi_o^{-1}\ U_1(\gamma\xi_o^{-1}q,q) \tag{C.21}$$

$$I_{1,1}(q,\xi_o^{-1}) = \begin{cases} -2\pi i\ \xi_o\ U_1(\gamma\xi_o q,q) & |\xi_o\gamma| < 1 \quad \text{(C.22)} \\ 2\pi i\ \xi_o\ U_1(\gamma^{-1}\xi_o^{-1}q,q) & |\xi_o\gamma| > 1 \quad \text{(C.23)} \end{cases}$$

We have denoted $U_n(\gamma q,q)$ by U_n in Eqs. C.18, C.19, and C.20. Again, using Eqs. A.3 and A.4, we obtain

$$U_1(\gamma^{-1}\xi_o^{-1}q,q) = - U_1(\gamma\xi_o q,q) + \sin(\omega_o t - \frac{r}{c}\sqrt{\omega_o^2 - \alpha^2}) \quad (C.24)$$

Combining Eqs. C.22 through C.24 results in

$$I_{1,1}(q,\xi_o^{-1}) = - 2\pi i \ \xi_o U_1(\gamma\xi_o q,q) + 2\pi i \ \xi_o \sin\left(\omega_o t - \frac{r}{c}\sqrt{\omega_o^2 - \alpha^2}\right) H\left(t - \frac{r}{c}\sqrt{1 - (\frac{\alpha}{\omega_o})^2}\right) \quad (C.25)$$

Our comments following Eq. C.14 concerning $I_{o,1}(q,\xi_o^{-1})$ apply to this integral as well, and formally we shall use

$$I_{1,1}(q,\xi_o^{-1}) = - 2\pi i \ \xi_o U_1(\gamma\xi_o q,q) \quad (C.26)$$

for the value of $I_{1,1}(q,\xi_o^{-1})$.

APPENDIX D

EVALUATION OF SOME ASYMPTOTIC FORMULAS

In this appendix, some required asymptotic expressions for Lommel functions of two variables are derived. In Chapter 3 we need to know the behavior of $U_o(cx,x)$ as $x \to \infty$, when c and x are real.

If $c < 1$, a one-term formula can be obtained from the results given by Watson (36). However, we need a second term for the calculations in Chapter 3. The method used by Watson is rather cumbersome, especially if one attempts to obtain the second term in the asymptotic expansion. A technique using the method of stationary phase is used here.

By definition A.1 we have

$$U_o(cx,x) = \sum_{k=0}^{\infty} (-)^k \; c^{2k} \; J_{2k}(x) \tag{D.1}$$

Using an integral representation for the Bessel function, a slight modification of Eq. B.17

$$J_n(x) = \frac{1}{2\pi} \int_0^{2\pi} e^{in\theta} \; e^{-ix \sin \theta} \; d\theta \tag{D.2}$$

we obtain, after the interchange of integration and summation orders,

$$U_o(cx,x) = \frac{1}{2\pi} \int_0^{2\pi} \sum_{k=0}^{\infty} (-)^k (ce^{i\theta})^{2k} e^{-ix \sin \theta} \; d\theta \tag{D.3}$$

Recognizing that the summation is a geometric series, we can now write

$$U_o(cx,x) = \frac{1}{2\pi}\int_0^{2\pi} \frac{e^{-ix\sin\theta}}{1+c^2e^{i2\theta}}\, d\theta \tag{D.4}$$

By the method of stationary phase the first term in the asymptotic expansion of $U_o(cx,x), x \to \infty$, $c < 1$ is

$$U_o(cx,x) \sim \frac{1}{1-c^2}\sqrt{\frac{2}{\pi x}}\cos(x-\frac{\pi}{4}) \sim \frac{1}{1-c^2} J_o(x) \tag{D.5}$$

This is the result obtained by Watson. To obtain the next term in the expansion, we add and subtract $J_o(x)/(1-c^2)$ in Eq. D.5 using Eq. D.2

$$U_o(cx,x) = \frac{1}{1-c^2} J_o(x) + \frac{1}{2\pi}\int_0^{2\pi}\left[\frac{1}{1+c^2e^{i2\theta}} - \frac{1}{1-c^2}\right] e^{-ix\sin\theta}\, d\theta \tag{D.6}$$

Integration by parts gives

$$U_o(cx,x) = \frac{1}{1-c^2} J_o(x) + \frac{1}{2\pi\, ix}\int_0^{2\pi}\left\{\frac{\sin\theta}{\cos^2\theta}\left[\frac{1}{1+c^2e^{i2\theta}} - \frac{1}{1-c^2}\right] + \frac{1}{\cos\theta}\left[\frac{-2ic^2e^{i2\theta}}{(1+c^2e^{i2\theta})^2}\right]\right\} e^{-ix\sin\theta}\, d\theta \tag{D.7}$$

$$= \frac{1}{1-c^2} J_o(x) - \frac{c^2}{1-c^2}\,\frac{1}{\pi x}\int_0^{2\pi}\frac{[1-c^2e^{i2\theta}]e^{i\theta}}{[1+c^2e^{i2\theta}]^2}\, e^{-ix\sin\theta}\, d\theta \tag{D.8}$$

Again, applying the method of stationary phase

$$U_o(cx,x) \sim \frac{1}{1-c^2} J_o(x) - \frac{c^2(1+c^2)}{(1-c^2)^3} \frac{2}{x} \sqrt{\frac{2}{\pi x}} \sin(x - \tfrac{\pi}{4}) \qquad (D.9)$$

Using Hankel's (37) asymptotic expansion of $J_o(x)$ to two terms

$$J_o(x) \sim \sqrt{\frac{2}{\pi x}} \left[\cos(x - \tfrac{\pi}{4}) + \frac{1}{8x} \sin(x - \tfrac{\pi}{4})\right] \qquad (D.10)$$

we obtain

$$U_o(cx,x) \sim \frac{1}{1-c^2} \sqrt{\frac{2}{\pi x}} \cos(x - \tfrac{\pi}{4}) + \left[1 - \frac{16c^2(1+c^2)}{(1-c^2)^2}\right] \frac{1}{1-c^2} \sqrt{\frac{2}{\pi x}} \frac{1}{8x} \sin(x - \tfrac{\pi}{4}) \qquad (D-11)$$

If $c = 1$, then according to Eqs. A.3, A.4 and A.5, we have

$$U_o(x,x) = \tfrac{1}{2}(J_o(x) + \cos x) \qquad (D.12)$$

An asymptotic formula for $U_o(x,x)$ can then be found, to any desired number of terms, from Hankel's asymptotic expansion for Bessel functions of large argument.

If $c > 1$, we can obtain the asymptotic expansion via the interrelationships between Lommel functions given in Appendix A, viz. Eqs. A.3, A.4, and A.5, along with the result just obtained for $c < 1$.

$$\begin{aligned} U_o(cx,x) &= \cos[\tfrac{x}{2}(c+c^{-1})] + J_o(x) - U_o(c^{-1}x,x) \\ &\sim \cos[\tfrac{x}{2}(c+c^{-1})] + \frac{1}{1-c^2}\sqrt{\frac{2}{\pi x}} \cos(x - \tfrac{\pi}{4}) + \left(1 - \frac{16c^2(1+c^2)}{(1-c^2)^2}\right) \\ &\quad \times \frac{1}{1-c^2} \frac{1}{8x} \sqrt{\frac{2}{\pi x}} \sin(x - \tfrac{\pi}{4}) \end{aligned} \qquad (D.14)$$

We also need $U_1(cx,x)$ for $c < 1$ to order $x^{-1/2}$. The result of computations similar to those resulting in Eq. D.5 is

$$U_0(cx,x) \sim \frac{c}{1-c^2} J_1(x) \tag{D.15}$$

Another asymptotic formula we must derive is one for

$$U_0(e^{i\delta}x,x) - U_0(e^{-i\delta}x,x) \ , \quad \text{as} \quad x \to \infty$$

where x and δ are real. To do this we will convert these Lommel functions back into a Laplace transform inversion integral.

From the definition of the Lommel functions, Eq. A.1, and the representation of exponentials as sinusoids, we arrive at

$$U_0(e^{i\delta}x,x) - U_0(e^{-i\delta}x,x) = -2i \sum_0^{\infty} (-)^k \sin[2\delta(k+1)]\, J_{2k+2}(x) \tag{D.16}$$

The following convolution integral is expressible as a sum of Bessel functions

$$\sin\delta \int_0^z e^{i(z-t)} J_\nu(t)dt = 2\sum_{k=0}^{\infty} (-)^k \sin(2k+1)\delta\, J_{2k+1}(z) + 2i\sum_{k=0}^{\infty} (-)^k \sin 2\delta(k+1)\, J_{2k+z+\nu}(z) \tag{D.17}$$

Using the imaginary part of Eq. D.17 with $\nu = 0$, we see that Eq. D.16 becomes

$$U_o(e^{i\delta}x,x) - U_o(e^{-i\delta}x,x) = -i \sin\delta \int_0^x \sin[(x-t)\cos\delta] J_o(t)dt \tag{D.18}$$

When that convolution is expressed as a Laplace inversion, we have

$$U_o(e^{i\delta}x,x) - U_o(e^{-i\delta}x,x) = -i \sin\delta \frac{1}{2\pi i} \int_\Gamma \frac{\cos\delta}{s^2 + \cos^2\delta} \frac{e^{sx}}{\sqrt{s^2+1}} ds \tag{D.19}$$

Figure D.1 shows the contour Γ and the path composed of C_j, $j = 1$ to 12, to which Γ can be transformed via an argument identical to that in Appendix B.

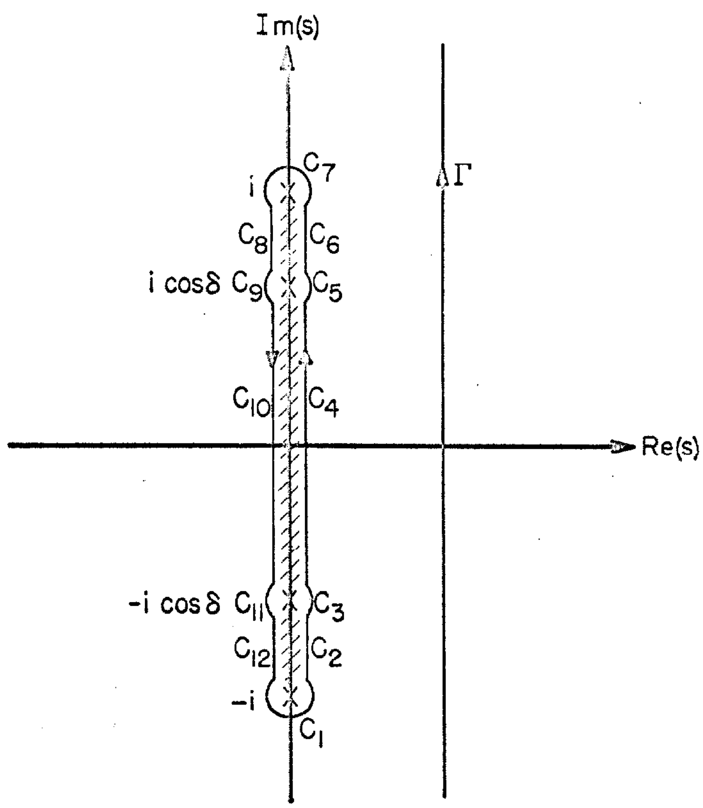


Figure D.1. Diagram showing equivalent paths Γ and one composed of C_j, $j = 1$ to 12.

Equation D.19 can now be written

$$U_o(e^{i\delta}x,x) - U_o(e^{-i\delta}x,x) = -\frac{i \sin \delta \cos \delta}{2\pi i} \sum_{j=1}^{12} \int_{c_j} \frac{e^{sx}}{s^2+\cos^2\delta} \frac{ds}{\sqrt{s^2+1}} \tag{D.20}$$

$$= -\frac{\sin 2\delta}{4\pi} \sum_{j=1}^{12} I_j \tag{D.21}$$

Analysis of the integrals I_j along the paths C_j as the complete path approaches the branch cut shows that

$$\begin{aligned} &I_1 \to 0 \qquad I_7 \to 0 \\ &I_3 = I_5^* \doteq -\frac{\pi}{\sin 2\delta} e^{-ix\cos\delta} \\ &I_9 = I_{11}^* = \frac{\pi}{\sin 2\delta} e^{ix\cos\delta} \end{aligned} \tag{D.22}$$

By the definition of Cauchy principal value, which we denote by P.V., we can write

$$I_2 + I_4 + I_6 = \text{P.V.} \int_{-1}^{1} \frac{e^{ixt}\, i dt}{(\cos^2\delta - t^2)\sqrt{1-t^2}}$$

$$= 2i \text{ P.V.} \int_0^1 \frac{\cos xt \, dt}{\sqrt{1-t^2}\,(\cos^2\delta - t^2)} \tag{D.23}$$

and

$$I_8 + I_{10} + I_{12} = \text{P.V.} \int_1^{-1} \frac{e^{ixt}\, i dt}{-\sqrt{1-t^2}\,(\cos^2\delta - t^2)}$$

$$= 2i\ \text{P.V.} \int_0^1 \frac{\cos xt\, dt}{\sqrt{1-t^2}\,(\cos^2\delta - t^2)} \qquad \text{(D.24)}$$

Combining these results, we see that

$$U_o(e^{i\delta}x,x) - U_o(e^{-i\delta}x,x) = -\frac{i}{\pi}\sin 2\delta\ \text{P.V.} \int_0^1 \frac{\cos xt\, dt}{\sqrt{1-t^2}\,(\cos^2\delta - t^2)} \qquad \text{(D.25)}$$

Consider the integral

$$\text{P.V.} \int_0^1 \frac{\cos xt}{\cos^2\delta - t^2}\, dt = \text{P.V.} \int_0^\infty \frac{\cos xt\, dt}{\cos^2\delta - t^2} - \int_1^\infty \frac{\cos xt}{\cos^2\delta - t^2}\, dt \qquad \text{(D.26)}$$

The first integral is tabulated and repeated integration by parts will yield an asymptotic formula for the second integral.

$$\text{P.V.} \int_0^1 \frac{\cos xt}{\cos^2\delta - t^2}\, dt = \frac{\pi}{2\cos\delta}\sin(x\cos\delta) + \frac{1}{x}\frac{\sin x}{\sin^2\delta} + 0(x^{-2}) \qquad \text{(D-27)}$$

It is also well known that

$$\int_0^1 \frac{\cos xt}{\sqrt{1-t^2}}\, dt = \frac{\pi}{2} J_o(x) \qquad \text{(D.28)}$$

Using Eqs. D.27 and D.28 as aids in the asymptotic evaluation of Eq. D.25 we find that

$$\text{P.V.} \int_0^1 \frac{\cos xt\, dt}{\sqrt{1-t^2}\,(\cos^2\delta - t^2)} = \text{P.V.} \int_0^1 \left[\frac{1}{\sqrt{1-t^2}\,(\cos^2\delta - t^2)} \right.$$

$$\left. - \frac{1}{\sin\delta} \frac{1}{\cos^2\delta - t^2} + \frac{1}{\sin^2\delta} \frac{1}{\sqrt{1-t^2}} \right] \cos xt\, dt$$

$$+ \frac{1}{\sin\delta} \left[\frac{\pi}{2\cos\delta} \sin(x\cos\delta) + \frac{1}{x} \frac{\sin x}{\sin^2\delta} + 0(x^{-2}) \right] - \frac{\pi}{2} \frac{1}{\sin^2\delta} J_o(x) \tag{D.29}$$

Examination of the integrand shows that its singularities have been removed. The P.V. symbol on the integral on the right-hand side can now be dropped. This integral can be integrated by parts to reach, as a final result,

$$U_o(e^{i\delta}x,x) - U_o(e^{-i\delta}x,x) = -i\left[\sin(x\cos\delta) - \frac{\cos\delta}{\sin\delta} J_o(x) \right.$$
$$\left. + \frac{4}{\pi} \frac{\cos\delta}{\sin^2\delta} \frac{\sin x}{x} \right] + 0(x^{-2}) \tag{D.30}$$

To obtain asymptotic expressions for Lommel functions as $t \to \infty$ when the function arguments depend on t through $\gamma = \sqrt{(1 - \frac{r}{ct})/(1 + \frac{r}{ct})}$ and $q = \alpha t\sqrt{1 - (\frac{r}{ct})^2}$, it seems best to convert the Lommel functions back into inverse Laplace transform integrations, redefine the branch cuts, and then evaluate the residue and asymptotic branch cut contributions to the integral. We illustrate this approach in detail for $U_o(\gamma q,q)$ and present only results for the other cases.

From Eqs. C.7 and C.1 it follows that

$$U_o(\gamma q,q) = -\frac{1}{2\pi}\int_0^{2\pi} \frac{e^{iq\cos\psi}}{\xi^2 - 1}\, d\psi \tag{D.31}$$

Using Eqs. B.11 to B.14 we obtain

$$U_o(\gamma q,q) = \frac{1}{2\pi i}\int \frac{1}{2}\,\frac{s + \sqrt{s^2+\alpha^2}}{s^2+\alpha^2}\, e^{st - \frac{r}{c}\sqrt{s^2+\alpha^2}}\, ds \tag{D.32}$$

Tabulated Laplace inversion formulas lead to

$$U_o(\gamma q,q) = \frac{1}{2} J_o(q) + \frac{1}{2}\cos\alpha t + \frac{1}{2\pi i}\int_\Gamma \frac{1}{2}\,\frac{s}{s^2+\alpha^2}\left[e^{-\frac{r}{c}\sqrt{s^2+\alpha^2}} - 1\right] e^{st}\, ds \tag{D.33}$$

Instead of the branch cut definition previously used, we now define the square root by

$$\sqrt{s^2+\alpha^2} = \sqrt{\rho_1\rho_2}\; e^{\frac{i}{2}(\phi_1+\phi_2)}, \quad -\pi \le \phi_{1,2} < \pi \tag{D.34}$$

See Figure D.2. It can easily be shown that the integral along Γ is equal to the sum of the integrals along the branch cuts in the directions shown in Fig. D.2. The contributions from arcs closing the contour at infinity and at the branch points are equal to zero.

Writing down these branch cut integrals we obtain

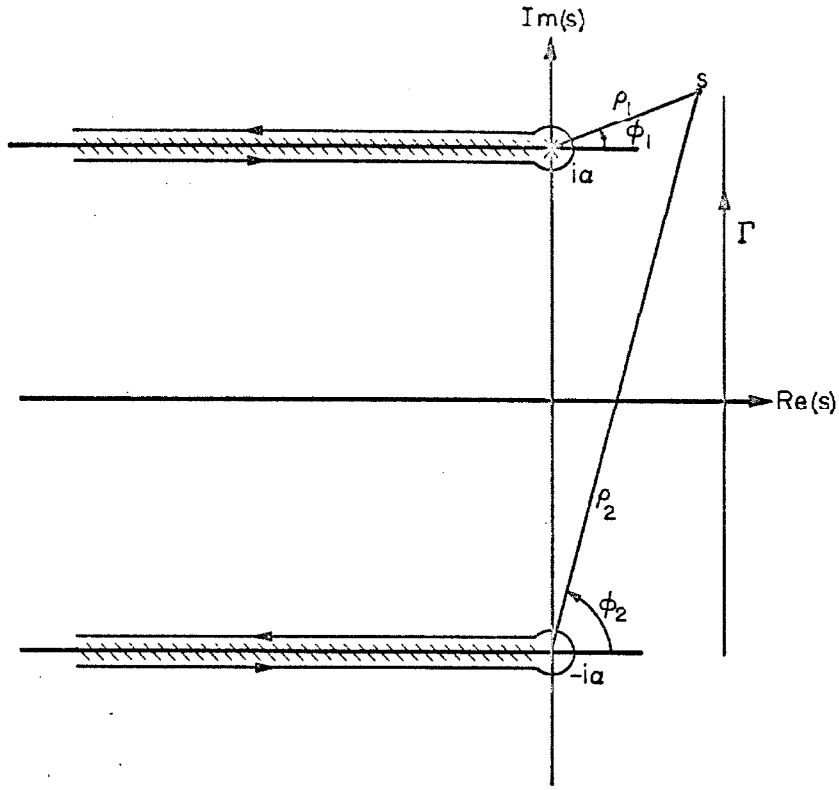


Fig. D.2. Definitions of branch cuts and directions of branch cut integrals for asymptotic evaluation of some integrals.

$$U_o(\gamma q, q) = \frac{1}{2} J_o(q) + \frac{1}{2} \cos \alpha t - \frac{e^{i\alpha t}}{2\pi i} \int_0^\infty \frac{i\alpha - x}{-2i\alpha x + x^2}$$

$$\times \ \sinh\left(\frac{r}{c} \sqrt{-2i\alpha x + x^2}\right) e^{-xt} \, dx + \frac{e^{-i\alpha t}}{2\pi i} \int_0^\infty \frac{-i\alpha - x}{2i\alpha x + x^2}$$

$$\times \ \sinh\left(\frac{r}{c} \sqrt{-2i\alpha x + x^2}\right) e^{-xt} \, dx \qquad \text{(D.35)}$$

The two integrals in Eq. D.35 are of the Laplace type and their asymptotic behavior to any desired number of terms as $t \to \infty$ may be found using the following theorem (38).

Let $0 < \lambda_1 < \lambda_2 \cdots$. If $f(t) = \lim_{X \to \infty} \int_0^X e^{-tx} \phi(x)\, dx$ exists for some $t = t_o$ and if $\phi \sim \sum a_n x^{\lambda_n - 1}$ to N terms as $x \to 0$, then $f \sim \sum \Gamma(\lambda_n) a_n t^{-\lambda_n}$ to N terms, uniformly in arg t as $t \to \infty$ in the sector $|\arg t| < \frac{\pi}{2} - \Delta$, $\Delta > 0$.

Retaining only the first term in the asymptotic evaluation of Eq. D.35, according to this theorem, we obtain

$$U_o(\gamma q, q) \sim \frac{1}{2} J_o(q) + \frac{1}{2} \cos \alpha t + \frac{1}{2} \frac{r\alpha}{c} \sqrt{\frac{2}{\pi\alpha t}} \cos(\alpha t - \frac{3\pi}{4}) \qquad (D.36)$$

In a similar manner we obtain for $U_1(\gamma q, q)$

$$U_1(\gamma q, q) = \frac{1}{2} \sin \alpha t - \frac{1}{2} \frac{r\alpha}{c} \sqrt{\frac{2}{\pi\alpha t}} \cos(\alpha t - \frac{\pi}{4}) \qquad (D.37)$$

An additional complication arises when a Lommel function contains a ξ_o or ξ_o^{-1} in its first argument. The integrand of the Laplace inverse transform then has poles at $s = \pm i\omega_o$. Thus, in addition to the branch cut contributions, we must also include the contributions due to these poles. The results for certain combinations of Lommel functions appearing in this paper are

$$\frac{U_1(\gamma\xi_o q,q) - U_1(\gamma\xi_o^{-1}q,q)}{\sqrt{\omega_o^2 - \alpha^2}}$$

$$\sim \begin{cases} \dfrac{e^{-\frac{r}{c}\sqrt{\alpha^2-\omega_o^2}}}{\sqrt{\alpha^2-\omega_o^2}} \cos\omega_o t - \dfrac{\alpha}{\omega_o^2-\alpha^2}\sqrt{\dfrac{2}{\pi\alpha t}}\cos(\alpha t - \tfrac{3\pi}{4}), \quad \omega_o < \alpha & \text{(D.38)} \\ \dfrac{1}{\sqrt{\omega_o^2-\alpha^2}}\sin\left(\omega_o t - \dfrac{r}{c}\sqrt{\omega_o^2-\alpha^2}\right) - \dfrac{\alpha}{\omega_o^2-\alpha^2}\sqrt{\dfrac{2}{\pi\alpha t}}\cos(\alpha t - \tfrac{3\pi}{4}), \quad \omega_o > \alpha & \text{(D.39)} \end{cases}$$

$$U_o(\gamma\xi_o q,q) + U_o(\gamma\xi_o^{-1}q,q)$$

$$\sim \begin{cases} e^{-\frac{r}{c}\sqrt{\alpha^2-\omega_o^2}}\cos\omega_o t + \sqrt{\dfrac{2}{\pi\alpha t}}\cos(\alpha t - \tfrac{\pi}{4}), \quad \omega_o < \alpha & \text{(D.40)} \\ \cos\left(\omega_o t - \dfrac{r}{c}\sqrt{\omega_o^2-\alpha^2}\right) + \sqrt{\dfrac{2}{\pi\alpha t}}\cos(\alpha t - \tfrac{\pi}{4}), \quad \omega_o > \alpha & \text{(D.41)} \end{cases}$$

$$\frac{U_o(\gamma\xi_o q,q) - U_o(\gamma\xi_o^{-1}q,q)}{\sqrt{\omega_o^2 - \alpha^2}}$$

$$\sim \begin{cases} -\dfrac{e^{-\frac{r}{c}\sqrt{\alpha^2-\omega_o^2}}}{\sqrt{\omega^2-\alpha_o^2}}\sin\omega_o t - \dfrac{\omega_o}{\omega_o^2-\alpha^2}\sqrt{\dfrac{2}{\pi\alpha t}}\cos(\alpha t - \tfrac{\pi}{4}), \quad \omega_o < \alpha & \text{(D.42)} \\ \dfrac{\cos\left(\omega_o t - \frac{r}{c}\sqrt{\omega_o^2-\alpha^2}\right)}{\sqrt{\omega_o^2-\alpha^2}} - \dfrac{\omega_o}{\omega_o^2-\alpha^2}\sqrt{\dfrac{2}{\pi\alpha t}}\cos(\alpha t - \tfrac{\pi}{4}), \quad \omega_o > \alpha & \text{(D.43)} \end{cases}$$

$$U_1(\gamma\xi_o q,q) + U_1(\gamma\xi_o^{-1} q,q)$$

$$\sim \begin{cases} e^{-\frac{r}{c}\sqrt{\alpha^2-\omega_o^2}} \sin \omega_o t \quad , & \omega_o < \alpha \qquad \text{(D.44)} \\ \sin\left(\omega_o t - \frac{r}{c}\sqrt{\omega_o^2 - \alpha^2}\right) \quad , & \omega_o > \alpha \qquad \text{(D.45)} \end{cases}$$

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