Changes of Variables and the Renormalization Group

Thesis by

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In Partial Fulfillment of the Requirements

for the Degree of

Doctor of Philosophy

California Institute of Technology

Pasadena, California

1985

(Submitted May 21, 1985)

to my father

Acknowledgements

I am sincerely grateful to F.Zachariasen, my advisor, for his guidance and support and in particular, for the freedom he allowed me in pursuing my research.

I would like to acknowledge my debt to my brother, friend and colleague, Nestor Caticha, with whom I have had the pleasure to collaborate and to indulge in lengthy and most stimulating discussions.

I am thankful to R.Gomez for sharing with me his enthusiasm for the art of teaching, for his advice and for his concern.

Those whose friendship I have enjoyed over these past few years are too numerous to list. They have contributed an important part to my education, both scientific and otherwise. I must, however, recognize the special friendship of M.Gomez who has been an unfailing source of encouragement.

Finally, I must express my gratitude to both my parents for their advice and very specially for the high example they have set before me. To them and to my two brothers, for their love and support I am most deeply indebted.

Abstract

A class of exact infinitesimal renormalization group (RG) transformations is studied. These transformations are pure changes of variables (i.e., no integration or elimination of some degrees of freedom is required) such that a saddle point approximation is more accurate, becoming, in some cases, asymptotically exact as the transformations are iterated. The formalism provides a simplified and unified approach to several known renormalization groups. The RG equations for a scalar field theory are obtained and solved both by expanding in $\varepsilon=d-4$ and also by expanding in a single coupling constant. This calculation yields results in agreement with conventional methods.

Next we study the application of this kind of RG to Yang-Mills theories. A simple exact gauge covariant RG transformation is constructed; the corresponding RG equations are obtained and solved in the weak coupling regime. This calculation shows that only certain initial conditions (i.e., bare actions) are compatible with the constraint that all the RG evolution be described by a single coupling constant. It also shows that at the tree level the β function for the $SU(N)$ gauge theory is $-\frac{21}{6}\frac{N}{(4\pi)^2}g^3$. A one-loop calculation yields the usual result $(-\frac{22}{6}\frac{N}{(4\pi)^2}g^3)$. Unlike the scalar theory case the iteration of the RG transformation does not lead to an asymptotic situation in which the saddle-point approximation is exact. A lattice gauge theory is proposed for which the application of this RG formalism is straightforward.

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Chapter 1. Introduction

1.1 Historical Background

The Renormalization Group (RG) is the common name given to several concepts and calculational techniques which have been found useful in dealing with some difficult problems in physics. The origin of the difficulty is that in order to describe the behavior of some systems one must take into account an extremely large number of degrees of freedom which are coupled in nontrivial ways.

Problems of this kind include critical phenomena, the problem of magnetic impurities in nonmagnetic metals (the Kondo problem) and many others. At first sight, these might seem to be rather esoteric problems; after all, materials are not normally found near their critical point, which is just one point in the whole phase diagram. But this is precisely the situation for quantum field theories. In this case the correlation lengths (or the inverse masses of particles) are much longer than any cutoff distance describing the microscopic structure of spacetime (for example, the Planck length). Thus, in this sense, quantum field theories are arbitrarily near to or very precisely on their critical points and RG concepts should be relevant. It is then not unexpected to find that RG ideas lie at the very heart of theories such as quantum chromodynamics and grand unified theories. In fact, historically the development of the RG originated in quantum electrodynamics and only later was the relevance to critical phenomena realized.

The beginnings of the understanding of critical phenomena from a microscopic point of view can be traced back to van der Waals [1] over a century ago. He produced the first theoretical description of a critical point, the liquid-gas critical point, in an attempt to explain the phenomenon of critical opalescence which had only recently been discovered by Andrews. Later, early this century, Weiss [2] reached an analogous understanding of magnetic critical phenomena following experimental work of P.Curie. These two theories are particular instances of mean-field theories which were formulated in a unified way by Landau [3] in 1937.

The mean-field theories provide rather accurate descriptions of the phase diagram except in the immediate vicinity of the critical point. The basic idea is to focus one's attention on that particular configuration (the mean-field) that gives the dominant contribution to the partition function and ignore all fluctuations. In field theory this is known as the classical, or the tree approximation.

With the improvement of experimental techniques and especially with the exact solution of the two-dimensional Ising model by Onsager [4] in 1944, discrepancies with the mean-field predictions could be unambiguously established. This stimulated a larger effort leading on the one side to extensive numerical calculations and on the other side to the so-called scaling hypotheses developed by Widom, Fisher, Kadanoff and others (these matters are reviewed in e.g. [5-6]). The contribution of Kadanoff [7] was particularly important. It provided an intuitive understanding of scaling based on the block-spin idea, a primitive form of RG. From all this, and also from extensive parallel experimental work, it became clear that the critical behavior of very different systems is surprisingly similar, exhibiting a striking universality.

This universality is expressed through the critical exponents, which describe the critical behavior of thermodynamic quantities and turn out to be independent of the detailed microscopic characteristics of the system. At the level of mean-field theory, universality is already present (for example in the Law of Corresponding States) but in a rather trivial manner; the mean-field behavior is the same for absolutely all critical transitions. In contrast, evidence from either experiments or numerical calculations showed that the critical exponents exhibit a marked dependence on the nature (i.e. the symmetry) of the degrees of freedom involved and on the number of dimensions of space.

The independence from the precise nature of microscopic interactions is important for two reasons. First, it was identified as the basic qualitative feature of critical phenomena that needed to be explained. And second, it suggests a connection with the concept of renormalizability in quantum field theory, which emerged from a completely independent line of development.

In the late forties and early fifties a theory of renormalization was developed in the works of Bethe, Feynman, Schwinger, Tomonaga, Dyson and others (see e.g. [8]) which allows one to by-pass the annoying problem of the divergent results for the radiative corrections in quantum electrodynamics. It was found that finite results could be obtained if the electron and electromagnetic fields were rescaled in an appropriate way and the results were expressed not in terms of the bare electron mass and charge parameters appearing in the lagrangean but in terms of some renormalized parameters.

Stueckelberg and Peterman [9] were the first to realize that there are many different equivalent ways to choose the renormalized parameters and that physical quantities would be invariant under the group of transformations from one set of parameters to another. They named this group the renormalization group. In 1954 Gell-Mann and Low [10] independently discovered the RG and considered the ultraviolet asymptotics of electrodynamic Green's functions. Later, Bogoliubov and Shirkov (see e.g. [11]) formulated the RG invariance in terms of the normalization momenta and used it to study also the infrared asymptotics of Green's functions in electrodynamics.

At high energies where particle masses can be neglected, one could naively expect quantum field theories to be scale invariant. Gell-Mann and Low pointed out that this was not necessarily so; the scale invariance would be broken by the large cutoff introduced during the renormalization process. Still, this cutoff could be taken into account, and useful information could be obtained, provided an appropriate choice of the renormalized charge was made. Furthermore, while physical quantities should be invariant under the RG, perturbative expansions to any finite order need not be so. One may then impose the desired invariance and construct improved perturbation expansions.

The connection between the field-theoretical RG of Gell-Mann and Low and critical phenomena was achieved in the early seventies mainly through the work of K.Wilson (this is reviewed in [12]) although many other workers (see, e.g., [13]) independently realized that such a connection should exist. Wilson [14] conceived renormalization as a process in which high energy degrees of freedom should be gradually eliminated in a sequence of steps before tackling the experimentally accessible lower energies. This point of view turned out to be very convenient. By lifting the previous restriction to a fixed number of coupling constants it allowed an important generalization. Also, it provided a more intuitive understanding of renormalization, which, together with Wilson's study of the implications of fixed points for scaling behavior [15] led to the first RG calculation of critical exponents [16]. Soon after, it became clear that practical calculations could more easily be carried out using Feynman diagram techniques in a space of dimensionality close to four (Wilson and Fisher [17]). This powerful calculational technique together with the intuitive understanding of critical behavior has produced a vast wealth of applications (for references, see, e.g., [18]).

In the early seventies interest in the RG was independently awakened by work of an altogether different nature. In 1968 the MIT-SLAC experiment on deep inelastic electron proton scattering showed a very simple scaling behavior, Bjorken scaling (see, e.g., [19]). This stimulated new studies on the scaling symmetry and its breaking of field theories at high energies, the conformal anomaly was discovered by Coleman and Jackiw [20] and a new version of the RG appeared in works of Callan and Symanzik [21]. Then the work of Politzer, Gross and Wilczek and also 't Hooft [22] showed that the observed scaling could be explained if strong interactions were described by a non-abelian gauge theory (Yang and Mills [23]), in this case quantum chromodynamics (Gell-Mann et al., Weinberg [24]). This theory exhibits the remarkable property, known as asymptotic freedom, that at high momenta the interactions become weak. Foremost among its implications is the fact that strong interactions at high energies become accessible to perturbative methods. Also, the circumstance that at low momenta the forces become stronger is widely recognized to be intimately connected to the phenomenon of quark confinement. Furthermore, asymptotic freedom permits one to overcome what is perhaps the largest obstacle to the unification of all interactions, namely the fact that at low energies they have very different strengths (Georgi et al. [25]).

Many important parallel developments have been omitted from the account above (for example lattice gauge theories, Monte Carlo RG's, turbulence, etc.) but those mentioned are certainly sufficient to indicate the extreme degree to which RG concepts and methods have become important in modern physics.

1.2 An Overview of this Work

This work does not deal with the physics of any particular system undergoing a critical transition. Neither does it deal with the physics of quarks and gluons nor other particles and fields. It is not a work about any particular physical system but rather about the renormalization group (RG) itself as a technique that has been useful in approaching many problems in physics. A study of this kind might offer new insights into why the RG techniques are useful. This, in turn, may allow one to modify and simplify those techniques preserving only the very essential features which make them work. It may also suggest new ways, new problems to which the modified RG's might profitably be applied.

For quite some time now it has been apparent that the success of the RG methods in dealing with problems involving many degrees of freedom is connected to an appropriate choice of variables. That is, the various RG's provide systematic ways to focus one's attention on the degrees of freedom which are most important in the problem under consideration.

For example, in Wilson's approach to the problem of critical phenomena [12] it is recognized that short wavelength degrees of freedom are not interesting in themselves, but only indirectly through the effective interactions they induce between the experimentally accessible long wavelength degrees of freedom. The strategy then is to eliminate the short wavelengths in a sequence of steps. We start by integrating out the shortest ones first, then slightly longer ones, and so on, gradually working our way towards an effective lagrangean which contains only the relevant degrees of freedom.

In RG's such as the original Gell-Mann and Low RG [10] the appropriate choice of variables was achieved in quite a different way. The point is that while all wavelengths contribute to a loop integration in a given Feynman diagram, the actual relative contribution of the short versus the long wavelengths depends on the renormalization scale chosen. The freedom to change the renormalization scale thus allows us to emphasize some degrees of freedom over others and therefore to improve the perturbative calculation.

The two approaches above are sufficiently different that in spite of yielding the same results when applied to a given problem the precise connection between the two has been a matter of some confusion.

The Wilson RG transformation involves not only an elimination of some degrees of freedom but also an explicit change of variables. Some consequences of this fact appeared in works of Jona-Lasinio [26] and Wegner [27]. These authors were concerned with the possibility of choosing more general RG transformations and showing that physically significant quantities such as critical exponents are independent of such a choice. Thus, Jona-Lasinio defines generalized renormalization transformations as all those that leave the effective action Γ invariant in value. It seems unlikely that one can be more general than that, but this evades the important issue of which transformations are useful. Wegner goes further. He recognizes that transformations can be made in a rather general way and makes the essential remark that the elimination of degrees of freedom is not a necessary step since some changes of variable effectively accomplish such an elimination. He then goes on to exhibit explicitly the transformation which generates Wilson's incomplete-integration RG [12] and to conjecture that useful RG transformations would involve some kind of nonlinearity perhaps through some unspecified dependence on the hamiltonian.

In Chapter 2 we study a class of exact infinitesimal RG transformations for field theories in continuum space which are pure changes of variables, i.e., no additional elimination or integration of certain degrees of freedom is required. To isolate the minimal structure a change of variables needs to include in order to actually accomplish this, we consider in Section 2.2 three exact RG's in differential form. First we review the sharp-cutoff RG of Wegner and Houghton as simplified by Weinberg [28] and then formulate simplified versions of the incomplete-integration RG of Wilson [12] and of the hard-soft splitting RG (Wilson [29], Lowenstein and Mitter [30], Mitter and Valent [31], Shalloway [32]).

In Section 2.3 the required change of variables is obtained as well as the RG equations both in functional form and as an infinite set of integro-differential equations. The reason why this class of RG's is useful is immediately apparent: the changes of variables are such that a classical or saddle-point approximation in the new variables is more accurate. No mention is made of the question of long versus short wavelengths; this is important. On iterating the RG transformations (i.e., on solving the equations for the RG evolution of the action or of the hamiltonian) the classical approximation becomes better and better approaching the exact result. Since these RG equations are much simpler than other sets of equations which need to be tackled in order to solve quantum field theories (e.g. the Schwinger-Dyson equations), we feel this is a promising way (as is the case with some other RG's) to leap beyond the limitations of perturbation theory. As we will see a further fortunate feature is related to the possibility of applying these RG's to gauge theories.

The calculation of Green's functions is considered in Section 2.4.

As with most manipulations with path-integrals, the level of mathematical rigor is fairly low. Changes of variables occasionally produce surprises in that the new lagrangean differs from the one that would be naively obtained. In some situations the additional terms can be cast into the form of an extra potential of order $\hbar^2$; in other situations they can be traced to non-trivial jacobian factors and they generate anomalies (see e.g. [33-34] and references therein). Two explicit solutions of the RG equations in Section 2.5 serve as a check that in our case no such surprises occur. The first solution is an expansion in $\varepsilon = d-4$ (Wilson and Kogut [12], Wilson and Fisher [17], Shukla and Green [35]) the second is an expansion in a single coupling constant. Both give the same results, but they represent differing viewpoints. The former emphasizes analyticity, the latter is somewhat closer to the Gell-Mann and Low spirit.

Some of the details of the calculations and a pedagogical example, a scalar field theory in zero dimensions (a single integral), are discussed in the appendices to Chapter 2.

As discussed in the previous section, Quantum Chromodynamics (QCD) (some popular textbooks are listed in [36]) is quite universally believed to be the correct theory of the strong interactions. Renormalization group methods have been instrumental in bringing about this state of things: QCD is the only candidate field theory which is renormalizable, consistent with the known symmetries of the strong interactions and which exhibits the remarkable phenomenon of asymptotic freedom (Gross and Wilczek, Politzer [22]).

The challenging problem is to understand how, as the interactions become strong, quarks and gluons are permanently confined inside hadrons. Even in high energy experiments where couplings are small and perturbative calculations may be performed, remaining vestiges of strong coupling effects obscure the comparison with experimental results yielding only qualitative agreement.

Perhaps RG methods which have been so useful at short distances, may also be used to describe phenomena at long distances in the strong coupling regime. This might be achieved by eliminating the uninteresting short wavelength degrees of freedom and obtaining an effective action or hamiltonian which describes only the interesting long wavelengths. This idea is old and has been tried before. In the context of QCD, it was first suggested by Wilson [29] and then developed by Shalloway [32] who used it in the small coupling perturbative regime. The RG transformation consisted of splitting the propagators into hard and soft parts and integrating out the hard part. The procedure had a serious problem: the gauge invariance was not manifest. Also, although in principle the transformation could be exact, in practice it was only approximate. On iterating this approximate transformation, errors could accumulate which would violate the gauge symmetry. For small coupling this problem was not fatal because the errors could be bounded and their accumulation controlled, but the strong coupling regime lay completely beyond the reach of this RG. At the expense of introducing an additional ghost field, Mitter and Valent [31] developed a manifestly gauge invariant way to split the propagator. This would perhaps allow one to overcome the limitations of Shalloway's work [32] but because of the ghost field the formalism is rather complicated. A simple gauge covariant RG transformation is required.

Once the structure of changes of variables which are also RG transformations is identified, the actual construction of gauge covariant transformations is simple. An exact gauge covariant RG transformation would allow one to develop approximation schemes compatible with the gauge symmetry even for strong coupling. For example, one possible such scheme could be as follows. The infinite set of differential equations for the vertex functions appearing in the RG-evolved action can be truncated in a manner compatible with the gauge symmetry and one might perhaps obtain nonperturbative solutions. This is quite analogous to the procedure developed by Baker, Ball and Zachariasen (see [37] and references therein) to truncate the Schwinger-Dyson equations and obtain nonperturbative information about the gluon propagator in the axial gauge at long distances. Considering that the RG equations are much simpler than the Schwinger-Dyson equations and that obtaining solutions to the Ward identities is rather simple, (see Section 3.4 below) we are confident that implementing this scheme should be quite feasible.

A word of caution is, however, necessary. It may happen that the most convenient variables to describe a given problem are topologically different from the original variables one has chosen. They cannot therefore be reached through the continuous sequence of RG transformations considered in Chapter 2. It is quite likely that this is actually the case with QCD at long distances. If the QCD vacuum behaves as a chromomagnetic superconductor ('t Hooft, Mandelstam [38]) then it is possible that the appropriate variables at long distances are the electric vector potentials C_μ which are dual to the usual magnetic vector potentials A_μ (Mandelstam [39]). Baker et al. [40] have developed this idea further. Once the theory is formulated in terms of the C_μ potentials the RG of Chapter 2 becomes useful again because it can be used to improve further on the choice of the C_μ variables.

In Chapter 3 we formulate an exact gauge covariant RG and obtain solutions for the RG evolution of the pure Yang-Mills action in the weak coupling regime. Matter fields are not considered.

In Section 3.2 a simple exact gauge covariant RG transformation is given as well as the corresponding RG equations. It is interesting that even though the path integral for the Green's generating functional $Z(j)$ requires that a particular gauge be chosen and a Fadeev-Popov determinant introduced (for popular textbooks on this subject, see, e.g., [36]) these complications do not affect the RG equation. This is shown in Section 3.3. In fact, since one never needs to invert the quadratic vertex function to obtain a propagator, the solutions to the RG equation are not affected either.

Although it is not at all required, it is convenient to take advantage of the gauge symmetry when solving the RG equations. In Section 3.4 the n-point vertex functions are separated into two components, one which is transverse to the momenta entering through each of its n legs, and another which is (partially) longitudinal and is completely determined by the Ward identities in terms of the vertex function with $n-1$ legs. A procedure is given to calculate the longitudinal components and some solutions are given. In ref.[41], Kim and Baker studied the constraints of gauge invariance on the cubic gluon vertex in the axial and covariant gauges. Our method differs considerably from theirs, being simpler partially due to the fact that there are neither ghosts nor an axial gauge direction n_μ to worry about.

The perturbative solution to the RG equations is carried out in Section 3.5. This calculation yields two interesting results. A common feature of all RG-evolved actions is that they include an infinite number of complicated interaction terms, in fact all terms consistent with the symmetries of the system. Such actions are obviously not renormalizable in the usual sense. In Section 3.5 we impose on the solution of the RG equations the condition that all the evolution be described in terms of a single running coupling constant, the same that appears in the covariant derivative. It is found that solutions of this kind are possible but only if the right initial conditions (i.e., the right bare action) are chosen. The usual QCD bare action is precisely such a right initial condition. It is quite plausible that this strong constraint on the bare actions is equivalent to the requirement of renormalizability in quantum field theories, either in its usual form or in the generalized version known as "asymptotic safety" proposed by Weinberg [42].

The second interesting result is that in four spacetime dimensions the β function describing the RG evolution of the action of the $SU(N)$ gauge theory turns out to be

$$\beta(g) = -\frac{21}{6}\frac{N}{(4\pi)^2}g^3 \ .$$

This result, which is the tree level approximation to the β' function which describes the evolution of the effective action Γ (the generating functional of one-particle-irreducible Green's functions), is quite close to the usual one-loop result

$$\beta'(g) = -\frac{22}{6}\frac{N}{(4\pi)^2}g^3 \ ,$$

obtained using the conventional Feynman graph methods.

In Section 3.6 we show that on computing the effective action Γ to order one loop an additional term with a coefficient of $\frac{-1}{6}$ is obtained. Our final result therefore coincides with the conventional one, as it should. The existence of a large contribution to β' already at the tree level is in accord with the basic idea behind this kind of RG, namely a perturbative calculation performed in the new variables is more accurate. Unfortunately, the fact that β and β' differ indicates that unlike the scalar field theory case, the classical approximation is improved when the RG transformation is iterated but never becomes exact.

The details of some calculations appear in the appendices to Chapter 3. Appendix A contains an interesting result concerning the RG evolution of the Fadeev-Popov determinant.

The conclusions and some final comments appear in Chapter 4. A number of applications of this formalism can be envisaged. In the remaining sections of Chapter 4 we consider in a very preliminary form how one could make a start towards implementing them.

We mentioned it should be possible to obtain non-perturbative solutions to the QCD RG equations by following a procedure analogous to that of Baker, Ball and Zachariasen [37], namely, truncating the system of equations in a manner which does not violate the gauge symmetry. To do this we need trial forms for the vertex functions which satisfy the Ward identities. The longitudinal components of such trial solutions were obtained in Section 3.4, the necessary transverse components are obtained in Section 4.2.

A very convenient feature of these exact RG transformations is that they do not require the elimination of degrees of freedom and can therefore be applied to field theories defined on a lattice.The lattice can be finite and even small, and thus the RG evolution might be studied numerically. The RG equations appropriate to a scalar field lattice theory appear in Section 4.3.

Finally, in Section 4.4 we propose a lattice gauge theory formulated in terms of the field variables A_μ instead of the usual rotation matrix variables U_μ (for a detailed account of the latter formalism see, e.g., the review by Kogut [43]). The Ward identities and RG equations for this lattice theory are obtained and shown to be remarkably similar to the corresponding equations for the continuum theory.

1.3 Basic Concepts: Wilson's Renormalization Group

To introduce the basic ideas behind Wilson's RG approach to critical phenomena [6,12], consider a system described by a real scalar field φ. This field φ may represent for example the magnetization of a uniaxial ferromagnet. The partition function for such a system is

$$Z = \int \left[\prod_{q<\Lambda} d\varphi(q)\right] \exp -H_0(\varphi(q)) \quad , \tag{3.1}$$

where H_0/β is the hamiltonian and Λ is a momentum cutoff.

The features about critical phenomena that we wish to study are the independence from the microscopic details of H_0 and the scaling behavior. This suggests we should describe the behavior of long wavelengths by an effective hamiltonian from which the irrelevant short wavelengths have been removed and the remaining long wavelengths have been appropriately scaled. Thus, a typical RG transformation

$$R_b H_0 = H_1 \quad , \tag{3.2}$$

is performed in two steps:

1. the field components with wavevectors q between Λ/b and Λ with $b>1$ are integrated out, and

2. the remaining fields with $q<\Lambda/b$ are rescaled, the wavevectors are dilated by a factor b to restore the cutoff to Λ, and the fields are multiplied by a factor ζ_b.

The precise form of the factor ζ_b will be determined later. However, since we want $R_b R_{b'} = R_{bb'}$ or $\zeta_b \zeta_{b'} = \zeta_{bb'}$, it must be that $\zeta_b = b^y$ with y a constant independent of b.

The RG transformation R_b is then defined by

$$\exp -R_b H_0 = \int \left[\prod_{\Lambda/b<q<\Lambda} d\varphi(q)\right] \exp -H_0 \Bigg|_{\varphi(q)\to\zeta_b\varphi(bq)} , \tag{3.3}$$

so that (3.1) becomes

$$Z = \int \left[\prod_{bq<\Lambda} \zeta_b d\varphi(bq)\right] \exp -R_b H_0(\varphi(q)) . \tag{3.4}$$

and quantities computed from (3.4) are simply related to those computed from (3.1). For example, the two-spin correlation function is

$$<\varphi(q)\varphi(-q)>_{H_0} \equiv \Gamma^{(2)}(q,H_0) = \zeta_b^2\Gamma^{(2)}(bq,R_b H_0) , \tag{3.5}$$

or equivalently (in d space dimensions)

$$\Gamma^{(2)}(r,H_0) = b^{2y-d}\Gamma^{(2)}(\frac{r}{b},R_b H_0) . \tag{3.6}$$

This shows explicitly that distances are scaled by b^{-1}, in particular the correlation length ξ is scaled to

$$\xi(R_b H_0) = b^{-1}\xi(H_0) . \tag{3.7}$$

On iterating the RG transformation one generates a sequence of effective hamiltonians

$$H_{n+1} = R_b H_n = (R_b)^n H_0 . \tag{3.8}$$

The new effective hamiltonians we have generated in this way are much more complicated than the original one; they contain all kinds of complicated interactions. It is certainly not clear so far that any progress has been achieved.

Consider, however, the special situation in which ζ_b is chosen so that as the RG transformation is iterated a fixed point is approached,

$$\lim_{n\to\infty} H_n = H^* \text{ so that } R_b H^* = H^* \tag{3.9}$$

In fact, (3.9) may be considered as an equation for both H^* and ζ_b or y. At the fixed point (3.6) and (3.7) become

$$\Gamma^{(2)}(q,H^*) = b^{2y}\Gamma^{(2)}(bq,H^*) \text{ or } \Gamma^{(2)}(r,H^*) = b^{2y-d}\Gamma^{(2)}(\frac{r}{b},H^*) \ . \tag{3.10}$$

and

$$\xi(H^*) = b^{-1}\xi(H^*) \ . \tag{3.11}$$

Equation (3.11) has two solutions $\xi=0$ and $\xi=\infty$, the second of which describes a critical point, furthermore (3.10) exhibits the scaling behavior we set out to uncover; for large r Equation (3.10) implies (choose e.g. $b=\Lambda r$)

$$\Gamma^{(2)}(r,H^*) \sim \frac{1}{r^{d-2y}} \equiv \frac{1}{r^{d-2+\eta}} \ . \tag{3.12}$$

The last equality defines the critical exponent η which is given by

$$\eta = 2-2y \ . \tag{3.13}$$

The simple ideas above have very remarkable implications. First, systems are at their critical points when the iterated RG transformation leads to the appropriate fixed point. Second, the mere existence of the fixed point already gives us scaling behavior and third, the calculation of H^* gives a critical exponent as a by-product. Furthermore, fourth, critical behavior is described by hamiltonians in the vicinity of H^*, the details of the original H_0 are irrelevant in the sense that many different H_0's could lead to the same H^*. Universality has been explained (more on this later).

Some other characteristics of critical behavior (i.e.,the remaining critical exponents) we would wish to calculate refer to systems that are close but not quite at the fixed point. This study is accomplished by considering the effect of RG transformations near the fixed point. Let

$$R_b(H^*+\delta H) = H^* + L_b\delta H + O(\delta H^2) = H^* + \delta H' + O(\delta H^2) \ . \qquad (3.14)$$

A convenient way to analyze this problem has been proposed by Wegner [44]. Suppose we can express δH as a linear combination of a complete set of operators Ω_i which are also eigen-operators of the linearized transformation L_b (it is at present not yet clear whether this can be done in a mathematically sound way),

$$\delta H = \sum_i \mu_i \Omega_i \ , \qquad (3.15)$$

$$L_b \Omega_i = \lambda_i(b)\Omega_i \ . \qquad (3.16)$$

As before, the fact that we want

$$L_b L_{b'} = L_{bb'} \quad \text{or} \quad \lambda_i(b)\lambda_i(b') = \lambda_i(bb') \ ,$$

restricts the form of the eigenvalues to

$$\lambda_i(b)=b^{y_i} \ ,$$

with y_i independent of b, so that

$$\delta H' = \sum_i \mu'_i \Omega_i \ , \ \mu'_i = \mu_i b^{y_i} \ . \qquad (3.17)$$

After solving (3.16), the RG evolution of H near H^* can be easily studied. Relevant operators (those with $y_i>0$) will tend to grow exponentially as L_b is iterated driving H_n away from the fixed point H^*. Conversely, irrelevant operators (those with $y_i<0$) will vanish exponentially and only slightly affect critical behavior. Marginal operators (those with $y_i=0$) require a more careful study; in general, going beyond the linear approximation of (3.14) is sufficient to determine whether they are truly marginal or not.

Usually a hamiltonian describes a system at its critical point if all μ_i's for the relevant operators, of which there are normally just a small number, have been carefully tuned to zero. For example, in magnetic transitions we have to tune just two parameters, the temperature and the magnetic field, to reach the critical point. On the other hand, the number of irrelevant operators is large. In fact, most of the microscopic details contained in the original H_0 are represented by irrelevant operators. This is the microscopic cause of universality.

We suggested above that in the simple situation we are studying here there are only two relevant operators which must be tuned to zero to reach the critical point, thus

$$\delta H = \mu_t b^{y_t} \Omega_t + \mu_h b^{y_h} \Omega_h + (\text{irrelevant } \Omega\text{'s}) \ ,$$

where

$$\mu_t \sim t \quad \text{and} \quad \mu_h \sim h \ .$$

h is the magnetic field and $t = \dfrac{T - T_c}{T_c}$ (in fact, this defines T_c).

As an example of how critical exponents are determined once (3.15) is solved, consider a situation in which $h = 0$ and $t > 0$. Rewrite (3.6) as

$$\Gamma^{(2)}(r, \mu_t) = b^{-(d-2+\eta)} \, \Gamma^{(2)}\left(\frac{r}{b}, \mu_t b^{y_t}\right) \ .$$

But b is arbitrary, so we may choose $b = \Lambda r$ and then

$$\Gamma^{(2)}(r, t) = \frac{1}{(\Lambda r)^{d-2+\eta}} \Gamma^{(2)}\left(\frac{1}{\Lambda}, t\,(\Lambda r)^{y_t}\right) \equiv \frac{1}{r^{d-2+\eta}} D(\Lambda r t^{1/y_t}) \ ,$$

for some function D. On the other hand

$$\Gamma^{(2)}(r,t) = \frac{1}{r^{d-2+\eta}} D(\frac{\Lambda r}{\xi}) \ ,$$

defines the correlation length ξ, and therefore

$$\xi(t) \sim t^{-\nu} \ , \ \nu = \frac{1}{y_t} \ .$$

The first equality is the conventional definition of the exponent ν and the second shows how solving (3.16) allows one to obtain critical exponents. As a final example, consider the magnetization $m = <\varphi>$ which satisfies an equation analogous to Equations (3.5,6), namely

$$m(\mu_t) = b^{-(d-2+\eta)/2} m(\mu_t b^{y_t}) \ .$$

Now choose $b = t^{-\nu}$, then

$$m(t) = t^{\beta} m(const)$$

is the desired scaling law with the critical exponent β given by

$$\beta = \frac{\nu}{2}(d-2+\eta) \ .$$

Other critical exponents may be similarly obtained.

1.4 Basic Concepts: The Field-Theoretical Renormalization Group

Renormalizability is the property of a quantum field theory which allows the Green's functions to be rendered free of ultraviolet infinities provided the fields, coupling constants and masses are rescaled in an appropriate way. To be specific consider, for example, a scalar field theory in euclidean space-time. This theory is described by the lagrangean density

$$L = \frac{1}{2}(\partial\varphi_0)^2 + \frac{1}{2}m_0\varphi_0^2 + \frac{g_0}{4!}\varphi_0^4 \ . \qquad (4.1)$$

The theory may be regularized by calculating in $d=4-\varepsilon$ dimensions. Then the appropriate renormalization constants are of the form

$$\varphi_0 = Z_\varphi^{1/2}(g,\frac{m}{\mu})\varphi \ , \ m_0^2 = Z_m(g,\frac{m}{\mu})m^2 \ , \ g_0 = Z_g(g,\frac{m}{\mu})\mu^{\varepsilon/2}g \ , \qquad (4.2)$$

where g is the dimensionless renormalized coupling constant, m and φ are the renormalized mass and field respectively and μ is an arbitrary mass scale. It is this freedom in the choice of μ that generates a renormalization group.

Renormalizability is thus expressed by the following equality between the bare and the renormalized one-particle-irreducible Green's functions

$$\Gamma_0^{(n)}(q_i;g_0,m_0) = Z_\varphi^{-n/2}\,\Gamma^{(n)}(q_i;g,m,\mu) \ ,$$

where $\Gamma^{(n)}$ is finite as $\varepsilon\to 0$ and $\Gamma_0^{(n)}$ is independent of μ. Differentiating with respect to μ produces a RG equation

$$\left[\mu\frac{\partial}{\partial\mu} + \beta\frac{\partial}{\partial g} + m\beta_m\frac{\partial}{\partial m} - n\gamma_\varphi\right]\Gamma^{(n)}(q_i;g,m,\mu) = 0 \ , \qquad (4.3)$$

where

$$\beta(g,\frac{m}{\mu}) = \mu\frac{\partial g}{\partial\mu} \ , \qquad (4.4a)$$

$$\beta_m(g,\frac{m}{\mu}) = \frac{\mu}{m}\frac{\partial m}{\partial \mu} \quad , \tag{4.4b}$$

and

$$\gamma_\varphi(g,\frac{m}{\mu}) = \frac{\mu}{2}\frac{\partial \log Z_\varphi}{\partial \mu} \quad . \tag{4.4c}$$

These quantities are dimensionless and analytic as $\varepsilon\to 0$, they can be explicitly computed using perturbation theory but are so far arbitrary because we have not yet chosen a definite renormalization prescription.

To study the behavior of $\Gamma^{(n)}$ when the momenta are scaled, $q\to\lambda q$, Equation (4.3) can be rewritten in a more convenient form. The dimension (in the sense of dimensional analysis) of $\Gamma^{(n)}$ is $d-n(\frac{d}{2}-1)$, therefore

$$\Gamma^{(n)}(\lambda q_i;g,m,\mu) = \mu^{d-n(\frac{d}{2}-1)}\Gamma^{(n)}(\frac{\lambda q_i}{\mu},g,\frac{m}{\mu}) \quad ,$$

or

$$\left[\mu\frac{\partial}{\partial\mu} + \lambda\frac{\partial}{\partial\lambda} + m\frac{\partial}{\partial m} - d + n(\frac{d}{2}-1)\right]\Gamma^{(n)}(\lambda q_i;g,m,\mu) = 0 \quad .$$

Subtracting from (4.3) we obtain the desired equation,

$$\left[\lambda\frac{\partial}{\partial\lambda} - \beta\frac{\partial}{\partial g} + (1-\beta_m)m\frac{\partial}{\partial m} + n\gamma_\varphi + n(\frac{d}{2}-1) - d\right]\Gamma^{(n)}(\lambda q_i;g,m,\mu) = 0 \quad . \tag{4.5}$$

The difficulty in solving (4.3) is that β,β_m and γ_φ depend on two variables, g and m/μ. A most elegant way out of this problem results from noticing that the reason the renormalization constants of Equations (4.2) were introduced was to absorb infinities due to integrations over high momenta. But at high momenta it should be possible to neglect m. Therefore a renormalization prescription must exist for which the renormalization constants and also β,β_m and γ_φ are mass independent. Such an idea is implemented by choosing the finite parts of the counterterms to be identically zero. This "mass-independent" or "minimal-subtraction" prescription was invented independently by 't Hooft and Weinberg [45].

The solution to Equation (4.5) in this case becomes

$$\Gamma^{(n)}(\lambda q_i;g,m,\mu) = \lambda^{d-n(d/2-1)} \exp\left[-n\int_1^\lambda \gamma_\varphi(g(\lambda'))\frac{d\lambda'}{\lambda'}\right] \Gamma^{(n)}(q_i;g(\lambda),m(\lambda),\mu) \ . \tag{4.6}$$

where $g(\lambda)$ and $m(\lambda)$ are the solutions of

$$\lambda\frac{\partial g(\lambda)}{\partial\lambda} = \beta(g(\lambda)) \ , \ g(\lambda=1) = g \ , \tag{4.7a}$$

and

$$\lambda\frac{\partial m(\lambda)}{\partial\lambda} = m(\lambda)\left[\beta_m(g(\lambda))-1\right] \ , \ m(\lambda=1) = m \ . \tag{4.7b}$$

The interpretation of Equation (4.6) is that a Green's function evaluated at the scaled momenta equals the Green's function evaluated at non-scaled momenta multiplied by the usual factor $\lambda^{d-n(d/2-1)}$ given by dimensional analysis, provided that the latter Green's function be evaluated using the effective coupling and mass of (4.7) and that there be an additional anomalous dimension factor.

To illustrate the use of the formalism above we will briefly consider the critical point example of the previous section. As we saw there, the hamiltonian describing such a system may contain arbitrarily complicated microscopic interactions but most of these are irrelevant and may be omitted. The system may then be represented by a hamiltonian of the same form as the lagrangean of (4.1),

$$H = \frac{1}{2}(\partial\varphi_0)^2 + \frac{1}{2}m_0\varphi_0^2 + \frac{g_0}{4!}\varphi^4 \ .$$

It is convenient to renormalize at the critical point, that is, choose the renormalized mass to be zero (*). Suppose now that at a certain value g^* of the coupling $\beta(g^*)=0$. Equation (4.7a) shows this is a fixed point, while Equation (4.6) becomes

$$\Gamma^{(n)}(\lambda q_i; g^*, \mu) = \lambda^{d-n(d/2-1)} \lambda^{-n\gamma_\varphi(g^*)}\, \Gamma^{(n)}(q_i; g^*, \mu) \ ,$$

which is a simple scaling behavior. In particular, for $n=2$

$$\Gamma^{(2)}(\lambda q; g^*, \mu) = \lambda^{2-2\gamma_\varphi(g^*)}\, \Gamma^{(2)}(\lambda q; g^*, \mu) \ ,$$

which, comparing with (3.10) and (3.13) yields $\eta=2\gamma_\varphi(g^*)$. Furthermore suppose that in the vicinity of the fixed point, $\beta(g) \approx \beta_n\,(g-g^*)^n + \ldots$, where n is an odd integer. Using (4.7a) it is not difficult to see that if $\beta_n<0$ the fixed point is attractive in the ultraviolet (i.e. $\lim\limits_{\lambda\to\infty} g(\lambda)=g^*$) while if $\beta_n>0$ it is attractive in the infrared (i.e. $\lim\limits_{\lambda\to 0} g(\lambda)=g^*$). If n is an even integer and $\beta_n>0$ then the fixed point is ultraviolet and infrared stable for $g<g^*$ and $g>g^*$ respectively (and vice-versa for $\beta_n<0$).

In this scalar field example a direct computation leads to

$$\beta(g) = -\varepsilon g + \frac{3}{(4\pi)^2} g^2 + O(g^2\varepsilon, g^3) \ , \tag{4.8a}$$

$$\gamma_\varphi(g) = \frac{1}{12(4\pi)^4} g^2 + O(g^2\varepsilon, g^3) \ . \tag{4.8b}$$

This shows that $g^*=0$ is an ultraviolet stable fixed point, while $g^* = \frac{(4\pi)^2}{3}\varepsilon$ is the infrared stable fixed point which describes the critical behavior. The corresponding anomalous dimension is $\gamma_\varphi(g^*) = \frac{\varepsilon^2}{108}$.

(*) The perturbative expansion of a massless φ^4 is strongly infrared divergent in $d<4$ space dimensions even for non-vanishing external momenta. This forces upon us an (asymptotic) expansion in $\varepsilon=4-d$; the theory is then well defined to any finite order in $g^n\varepsilon^l$.

1.5 Basic Concepts: Yang-Mills Theory in the Background Field Gauge

The Euclidean spacetime Green's functions of a Yang-Mills theory [23] without fermions are generated by the functional (some introductory texts are listed in [36])

$$Z(j) = \int D(Q\theta\bar{\theta}) \exp -\left[S(Q) + S_{GF} + S_G - \int j.Q \right] , \tag{5.1}$$

where Q_μ^a is the gauge field. The action S is given in terms of the field strength

$$F_{\mu\nu}^a = \partial_\mu Q_\nu^a - \partial_\nu Q_\mu^a - g f^{abc} Q_\mu^b Q_\nu^c ,$$

by

$$S = \int d^d x \ \frac{1}{4} F_{\mu\nu}^a F_{\mu\nu}^a , \tag{5.2}$$

and is invariant under the non-Abelian gauge transformation

$$\delta Q_\mu^a(x) = D_\mu^{ab}(Q)\Omega^b(x) = \partial_\mu \Omega^a(x) - g f^{abc} Q_\mu^b(x)\Omega^c(x) . \tag{5.3}$$

$D_\mu^{ab}(Q)$ is the covariant derivative and f^{abc} are the structure constants of the gauge group. The gauge fixing term corresponding to the gauge condition $G(Q)=0$ is

$$S_{GF} = \frac{1}{2\alpha} \int d^d x \ G^a(Q) G^a(Q) . \tag{5.4}$$

Finally, the Fadeev-Popov ghost action is

$$S_G = -i \int d^d x \, d^d y \, \bar{\theta}^a(x) \frac{\delta G^a(x)}{\delta \Omega^b(y)} \theta^b(y) \bigg|_{G=0} . \tag{5.5}$$

It has been found that calculations are simplified if one takes advantage of the gauge symmetry. This is most conveniently done using the background field method originally proposed by De Witt [46] and further developed by many others (see, e.g., Abbott [47] and references therein).

The generating functional in the presence of a background field A^a_μ is

$$Z(j,A) = \int D(Q\theta\bar{\theta}) \exp -\left[S(Q+A) + S_{GF} + S_G - \int j.Q \right] . \tag{5.6}$$

The following choice of gauge,

$$G^a(Q) = \partial_\mu Q^a_\mu - g f^{abc} A^b_\mu Q^c_\mu = D^{ab}_\mu (A) Q^b_\mu = 0 \ , \tag{5.7}$$

is convenient because $Z(j,A)$ is then invariant under the transformations

$$\delta A^a_\mu = \partial_\mu \Omega^a - g f^{abc} A^b_\mu \Omega^c \ , \tag{5.8a}$$

and

$$\delta j^a_\mu = -g f^{abc} j^b_\mu \Omega^c \ . \tag{5.8b}$$

This is easily shown by making a change of integration variables in (5.6) of the form

$$\delta Q^a_\mu = -g f^{abc} Q^b_\mu \Omega^c \ . \tag{5.8c}$$

Equations (5.8a) and (5.8c) are just a gauge transformation of the total field $A+Q$, and therefore $S(Q+A)$ is invariant. Furthermore, since Equations (5.8b) and (5.8c) are adjoint rotations, the terms $\int j.Q$ and S_{GF} are clearly invariant. Similarly, making appropriate changes of the integration variables θ and $\bar{\theta}$ one may show that the ghost action,

$$S_G = -i \int d^d x \, \bar{\theta}^a D^{ab}_\mu (A) D^{bc}_\mu (A+Q) \theta^c \ ,$$

is invariant (*).

(*) Notice that an axial gauge $n_\mu Q^a_\mu = 0$ is also an appropriate gauge choice for the background field method.

It follows that the effective action (the generating functional of one-particle-irreducible Green's functions) $\Gamma(\bar{Q},A)$ is invariant under (5.8a) and

$$\delta \bar{Q}^a_\mu = -g f^{abc} \bar{Q}^b_\mu \Omega^c \quad . \tag{5.8d}$$

Finally, it is not difficult to show (Abbott [47]) that the arguments $\bar{Q}$ and A only enter in Γ in the combination $\bar{Q}+A$,

$$\Gamma(\bar{Q},A) = \Gamma(\bar{Q}+A) \quad . \tag{5.9}$$

Therefore, adjusting the sources j so that $\bar{Q}=0$ we obtain an effective action $\Gamma(0,A)=\Gamma(A)$ which is gauge invariant and which is sufficient (because of (5.9)) to compute all Green's functions.

The advantage of preserving explicit gauge invariance is that the coupling and field renormalization constants defined by (*)

$$g_0 = Z_g \mu^{\varepsilon/2} g \quad , \quad \text{and} \quad A_0 = Z_A^{1/2} \mu^{-\varepsilon/2} A \quad , \tag{5.10}$$

are related and one only needs the vacuum polarization to compute them. The radiative corrections to the ghost propagator and to the cubic vertices are not needed.

The infinities appearing in Γ must be in the gauge invariant form of a divergent constant times $(F^a_{\mu\nu})^2$. But $F^a_{\mu\nu}$ is renormalized by

$$(F^a_{\mu\nu})_0 = Z_A^{1/2} \mu^{-\varepsilon/2} \left[\partial_\mu A^a_\nu - \partial_\nu A^a_\mu - g Z_g Z_A^{1/2} f^{abc} A^b_\mu A^c_\nu \right] \quad ,$$

and will be gauge covariant only if it is proportional to $F^a_{\mu\nu}$. Therefore

(*) This is an unconventional field renormalization which is convenient because it leads to (5.11). Abbott [47] chooses $g_0 = Z_g g$ and $A_0 = Z_A^{1/2} A$ which also preserves (5.11) but involves a dimensionful coupling.

$$Z_g = Z_A^{-1/2} \quad . \tag{5.11}$$

To compute the β function one notes that since g_0 is independent of μ,

$$\mu \frac{\partial g_0}{\partial \mu} = 0 = Z_g \mu^{\varepsilon/2} \left[\frac{\varepsilon}{2} g + \mu \frac{\partial g}{\partial \mu} + g\, \mu \frac{\partial \log Z_g}{\partial \mu} \right] .$$

And, using (5.11),

$$\beta(g) = -\frac{\varepsilon}{2} g + \frac{g\mu}{2} \frac{\partial \log Z_A}{\partial \mu} = -\frac{\varepsilon}{2} g + \frac{g}{2} \beta(g) \frac{\partial \log Z_A}{\partial g} \quad . \tag{5.12}$$

In the minimal subtraction scheme Z_A is a series of poles in ε,

$$Z_A = 1 + \sum_{n=0}^{\infty} \frac{Z_n(g)}{\varepsilon^n} \quad .$$

which, when substituted into (5.12), must all cancel away so that β turns out to be finite. The result is

$$\beta(g) = -\frac{\varepsilon}{2} g - \frac{g^2}{4} \frac{\partial Z_1}{\partial g} \quad . \tag{5.13}$$

A direct calculation of the vacuum polarization in $d=4$ dimensions for $SU(N)$ Yang-Mills theory leads to

$$\beta(g) = -\frac{11}{3} \frac{g^3 N}{(4\pi)^2} + \cdots \tag{5.14}$$

This shows that $g=0$ is an ultraviolet stable fixed point and thus, that the pure Yang-Mills theory exhibits asymptotic freedom.

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Chapter 2. Changes of Variables and the Renormalization Group

2.1 Summary

A class of exact infinitesimal renormalization group (RG) transformations is proposed and studied. The form of the transformations suggests itself rather naturally after we formulate several known exact differential RG's. First we review the sharp-cutoff RG of Wegner and Houghton [1], then we formulate a simplified version of the incomplete-integration RG of Wilson [2] and finally we consider the hard-soft splitting RG developed by Wilson, Lowenstein et al. and others [3-6]. The transformations are pure changes of variables (i.e., no integration or elimination of some degrees of freedom is required) such that a saddle point approximation is more accurate, becoming, in some cases, asymptotically exact as the transformations are iterated. The formalism provides a simplified and unified approach to several known renormalization groups. The calculation of Green's functions is considered and solutions for a scalar field theory are obtained both as an expansion in $\varepsilon=d-4$ and as an expansion in a single coupling constant. Some of the details of the calculations and a pedagogical example, a scalar field theory in zero dimensions (a single integral), are discussed in the appendices.

2.2.1 Sharp-Cutoff RG

Here we review the sharp-cutoff RG derived by Wegner and Houghton as simplified by Weinberg [1].

Consider the Green's functions generating functional in Euclidean space-time,

$$Z = \int D\varphi \exp -S_\tau(\varphi) \ . \tag{2.1}$$

For the moment we are not concerned with coupling the field φ to external sources. This problem will be addressed in Section 4.

Suppose field components with momenta larger than a certain cutoff $\Lambda_\tau = \Lambda e^{-\tau}$ have been integrated out, i.e.

$$\varphi(q) = 0 \quad \text{for} \quad q > \Lambda_\tau \ .$$

This implies that the effective action S_τ consists not just of the simple interactions contained in the bare action $S = S_{-\infty}$ but rather of an infinite number of arbitrarily complicated interactions. Our problem is to study how the action S_τ evolves when the cutoff Λ_τ is slightly decreased to $\Lambda_{\tau+\delta\tau}$. Suppose we separate out the field components with momenta in the thin shell between Λ_τ and $\Lambda_{\tau+\delta\tau}$:

$$\varphi(x) \to \varphi(x) + \sigma(x) \ ,$$

where now $\varphi(q) = 0$ for $q > \Lambda_{\tau+\delta\tau}$ and $\sigma(q) = 0$ for q outside the thin shell of thickness $\Lambda_\tau \delta\tau$.

On integrating out the fields σ, the new action will be given by

$$\exp -S_{\tau+\delta\tau}(\varphi) = \int D\sigma \ \ \exp -S_\tau(\varphi+\sigma) \ . \tag{2.2}$$

The integration is performed perturbatively in three steps: first expand $S_\tau(\varphi+\sigma)$ in a power series in σ, second, isolate a convenient σ-field propagator, and third, treat the remaining σ^p vertices ($p\geq 1$) as a perturbation. This procedure, carried out in detail in Appendix 2.A shows that to first order in the shell thickness only diagrams with one internal σ-field line contribute. The result is

$$S_{\tau+\delta\tau}(\varphi) - S_\tau(\varphi) =$$

$$= \int d^d q\,(2\pi)^d \Delta_\tau(q)\left[\frac{\delta^2 S_\tau}{\delta\varphi(q)\delta\varphi(-q)} - \frac{\delta S_\tau}{\delta\varphi(q)}\,\frac{\delta S_\tau}{\delta\varphi(-q)}\right] . \tag{2.3a}$$

where $\Delta_\tau(q)$ is a convenient σ propagator which vanishes for q outside the shell. Alternatively

$$S_{\tau+\delta\tau}(\varphi) - S_\tau(\varphi) =$$

$$= (2\pi)^d \Lambda_\tau^{d-2}\delta\tau \int d\,\Omega_d\left[\frac{\delta^2 S_\tau}{\delta\varphi(q)\delta\varphi(-q)} - \frac{\delta S_\tau}{\delta\varphi(q)}\,\frac{\delta S_\tau}{\delta\varphi(-q)}\right] . \tag{2.3b}$$

where now $q^2=\Lambda_\tau^2$.

To obtain RG equations an additional dilatation change of variables is required. This is a rather trivial step which we will address later in Section 2.3.

The basic improvement of eqs.(2.3) over those of Wegner and Houghton's is that their equations include all diagrams with one internal σ-field loop (i.e. many internal σ propagators) while these include the much smaller set of diagrams with only one σ-field propagator.

2.2.2 Incomplete-Integration RG

In order to avoid the unphysical difficulties introduced by the discontinuous cutoff considered in the last section, Wilson [2] introduced the concept of incomplete-integration designed to achieve a smooth interpolation between those degrees of freedom which have been integrated out and those that have not.

This idea is implemented through the introduction of an auxiliary functional $\delta_\alpha(\varphi)$. In the case of an ordinary single integral $\delta_\alpha(x)$ is a function such that as α goes from 0 to ∞, the function $z_\alpha(x)$,

$$z_\alpha(x) = \int dy\, z_0(y)\, \delta_\alpha(y-x) \quad ,$$

smoothly interpolates between the integrand $z_0(x)$ and the integral $z_\infty(x)$. All that is required is that

$$\delta_\alpha(x) \to \begin{cases} \delta(x) & \text{for } \alpha\to 0 \\ 1 & \text{for } \alpha\to\infty \end{cases} \; .$$

Wilson's choice for δ_α was the Green's function of a certain differential equation. It is perhaps simpler to use a Gaussian:

$$\delta_\alpha(x) = \left[\frac{1}{4\pi\alpha} + 1\right]^{\frac{1}{2}} \exp-\frac{x^2}{4\alpha} \; .$$

This single integral case is pursued further in an appendix, now we return to the functional integral problem. We let $\alpha=\alpha(q,\tau)$ and introduce

$$\text{const} = \int D\varphi\, \delta_\alpha(\varphi-\Phi)$$

into the generating functional Z,

$$Z = \int D\Phi\, exp -S(\Phi) = \int D\varphi\, exp -S_\tau(\varphi)$$

where

$$\exp -S_\tau(\varphi) = \int D\Phi \; \delta_\alpha(\varphi-\Phi) \; \exp -S(\Phi)$$

$$= \int D\Phi \exp -\left[S(\Phi) + \int d\tilde{q} \frac{|\varphi(q)-\Phi(q)|^2}{4\alpha(q,\tau)} \right] \quad . \tag{2.4}$$

neglecting an unimportant overall factor. In eq.(2.4) and in the following we adopt the notation $d\tilde{q}=d^d q/(2\pi)^d$ and also $\tilde{\delta}(q)=(2\pi)^d \delta^d(q)$.

The conventional usage is to choose $\alpha(q,\tau)$ such that $S_\tau(\varphi)$ describes modes with an effective cutoff about $\Lambda_\tau=\Lambda e^{-\tau}$ which means that $\alpha(q,\tau)$ is very large for $q>\Lambda_\tau$ and very small for $q<\Lambda_\tau$. A convenient, but by no means obligatory choice is one in which the mode $\varphi(qe^{-\delta\tau})$ in $S_{\tau+\delta\tau}$ is integrated out to the same extent as $\varphi(q)$ in S_τ, i.e.

$$\alpha(q,\tau) = \alpha(q\, e^{-\delta\tau}, \tau+\delta\tau)$$

This implies that α depends on q and τ only through the combination qe^τ:

$$\alpha = \alpha(q/\Lambda_\tau) \quad .$$

The functional integral (2.4) can be transformed into the functional differential equation

$$\frac{dS_\tau}{d\tau} = \int d^d q \, (2\pi)^d \dot{\alpha}(q,\tau) \left[\frac{\delta^2 S_\tau}{\delta\varphi(q)\delta\varphi(-q)} - \frac{\delta S_\tau}{\delta\varphi(q)} \frac{\delta S_\tau}{\delta\varphi(-q)} \right] \quad , \tag{2.5}$$

where $\dot{\alpha}=d\alpha/d\tau$. This equation is obtained noticing that differentiation of eq.(2.4) with respect to τ brings down factors of $(\varphi-\Phi)$ on the right hand side which may also be brought down through functional differentiation with respect to φ. Comparison of eq.(2.5) with the remarkably similar eq.(2.3a) shows that here $\dot{\alpha}d\tau$ is playing the role of a propagator.

Again, the full RG equations require an additional dilatation which we postpone until Section 2.3.

2.2.3 Hard-Soft Splitting RG

Another method which allows elimination of the short wavelength degrees of freedom was suggested by Wilson [3]. It consists of splitting the propagator of a massless particle into two pieces, one of which contributes dominantly for high momenta while the other does so for low momenta:

$$\frac{1}{p^2} = \frac{1}{p^2+\mu^2} + \frac{\mu^2}{p^2(p^2+\mu^2)} \quad . \tag{2.6}$$

The idea was to take advantage of the UV asymptotic freedom of Yang-Mills theories to integrate out the hard components in renormalized perturbation theory, and generate an effective action for the soft components which could be treated using other techniques better suited to the strong coupling regime.

The method was further developed by Lowenstein and Mitter [4], and by Mitter and Valent [5], and applied to the weak coupling regime of quantum chromodynamics by Shalloway [6]. In this section we formulate it in a simple way which allows immediate comparison with the RG's described in the preceding sections.

The actual splitting of hard and soft components is accomplished by introducing

$$\text{const} = \int D\chi \exp - \int dx \frac{1}{2}\mu^2\chi^2$$

into the path integral for Z,

$$Z = \int D\Phi \, exp - S(\Phi) \quad \text{where} \quad S(\Phi) = \int dx \left[\frac{1}{2}\partial\Phi\partial\Phi + V(\Phi)\right] \quad ,$$

and then making the change of variables $\Phi = \varphi + \varphi_h$ and $\chi = \varphi_h + \partial^2\varphi^2/\mu^2$. The result is

$$Z = \int D\varphi D\varphi_h \exp\int dx\left[\frac{1}{2}\varphi\partial^2(1-\frac{\partial^2}{\mu^2})\varphi + \frac{1}{2}\varphi_h(\partial^2-\mu^2)\varphi_h - V(\varphi+\varphi_h)\right],$$

where the separation of (2.6) is explicit. Integrating over φ_h leads once more to

$$Z = \int D\varphi\, exp -S_\tau(\varphi) \;,$$

where

$$exp -S_\tau(\varphi) = \int D\varphi_h\, exp -\left[S(\varphi+\varphi_h) + \int dx \frac{1}{2}\mu^2(\frac{\partial^2}{\mu^2}\varphi+\varphi_h)^2\right] .$$

We wish to study the evolution of S_τ under changes of τ,(we are taking $\mu=\mu(\tau)$). It is convenient to shift variables back to Φ :

$$exp -S_\tau(\varphi) = \int D\Phi\, exp -\left[S(\Phi) + \int d\tilde{q}\, \frac{1}{2}\mu^2|\rho\varphi(q)-\Phi(q)|^2\right] .$$

where $\rho=1+q^2/\mu^2$. This equation, which is very similar to eq.(2.4) can also be transformed into a functional differential equation:

$$\frac{dS_\tau}{d\tau} = \int \frac{d^dq}{2\rho^2}\left[(2\pi)^d\frac{d\mu^{-2}}{d\tau}\left(\frac{\delta^2 S_\tau}{\delta\varphi(q)\delta\varphi(-q)} - \frac{\delta S_\tau}{\varphi(q)}\frac{\delta S_\tau}{\delta\varphi(-q)}\right) + \right.$$
$$\left. + \frac{d\rho^2}{d\tau}\varphi(q)\frac{\delta S_\tau}{\delta\varphi(q)}\right] . \qquad (2.7)$$

This differs from eq.(2.3) and eq.(2.4) only in the last term, which by now one might suspect is not essential.

Incidentally, one could consider situations where μ and ρ have momentum dependences other than those assumed above, in particular one could choose

$$(2\pi)^d\mu^{-2} = \rho^2+c$$

where c is independent of τ. Then eq.(2.7) becomes identical with the original incomplete-integration equation of Wilson (eq.11.8 of ref.[2]).

The important conclusion to be drawn is that the various examples of exact RG's considered above are characterized by a certain common feature, which we might guess is what makes them useful RG's in the first place. The variations are presumably inessential in principle, although in practice they may be important. For example, the sharp-cutoff RG is definitely more inconvenient to calculate with.

2.3. The RG as a Change of Variables

Functional integrals are a particularly convenient way to formulate quantum field theories not just because they readily allow for perturbative expansions but also because the implications of the invariances or of the changes induced in those theories by transformations of the dynamical variables can be easily studied. This feature has been found particularly useful in the case of non-Abelian gauge transformations. In this section we consider infinitesimal variable changes which reproduce the exact RG transformations described previously.

Let us go back to eq.(2.1) and investigate the changes in the action S_τ induced by the variable transformation

$$\varphi(q) \to \varphi(q) + \delta\tau\, \eta(\varphi,q) \quad . \tag{3.1}$$

where $\eta(\varphi,q)$ is some sufficiently well behaved functional of φ. Taking into account the jacobian of this transformation, eq.(2.1) becomes

$$Z = \int D\varphi \left[1+\delta\tau \int d^d q \frac{\delta\eta(\varphi,q)}{\delta\varphi(q)}\right] exp - \left\{S_\tau + \delta\tau \int d^d q \frac{\delta S_\tau}{\delta\varphi(q)} \eta(\varphi,q)\right\}$$
$$\equiv \int D\varphi \; exp - S_{\tau+\delta\tau}(\varphi) \quad ,$$

where

$$S_{\tau+\delta\tau}(\varphi) = S_\tau(\varphi) + \delta\tau \int d^d q \left[\frac{\delta S_\tau}{\delta\varphi(q)} \eta(\varphi,q) - \frac{\delta\eta(\varphi,q)}{\delta\varphi(q)}\right] \quad . \tag{3.2}$$

Suppose one chooses

$$\eta(\varphi,q) = -(2\pi)^d \dot{\alpha}(q,\tau) \frac{\delta S_\tau(\varphi)}{\delta\varphi(-q)} \quad . \tag{3.3}$$

Then eq.(3.2) becomes identical to the Gaussian incomplete-integration RG. More generally, if one also includes an inessential rescaling of the field,

$$\eta(\varphi,q) = -(2\pi)^d \dot{\alpha}(q,\tau) \frac{\delta S_\tau(\varphi)}{\delta\varphi(-q)} + \zeta(q,\tau)\varphi(q) \ ,$$

one obtains an equation of the form of eq.(2.7). From now on we will deal only with the simpler transformation of eq.(3.3).

The conclusion is that transformations of the form of eq.(3.3) are RG transformations.

Furthermore, one can see why they are useful transformations. A field configuration φ which is a solution of the classical equation of motion $\delta S_\tau / \delta\varphi = 0$ will not be affected by (3.3). Any other configuration will flow with τ until it becomes a classical solution (i.e., a stationary point), then it ceases to flow. As τ goes to infinity a situation is approached in which all field configurations are classical solutions, i.e. $\delta S_\tau / \delta\varphi = 0$ for all φ. The action approaches a constant.

In the Appendix 2.B an example in zero spacetime dimensions, an ordinary integral, is worked out. It allows one very clearly to see what is happening. The changes of variables are such that a "classical" approximation, i.e., a steepest descent approximation becomes better and better as τ increases, approaching the exact result as $\tau \to \infty$. The reason the approximation is improved is not that the integrand becomes steeper as one might at first guess, but rather that it approaches a Gaussian for which the steepest descent method is exact. The fact that this Gaussian is increasingly flatter (the action approaches a constant) is not a serious obstacle. It merely requires us to calculate the integral before the limit $\tau \to \infty$ is taken.

Traditionally RG techniques have been applied to problems which exhibit some kind of symmetry under changes of scale. In these cases it is convenient to perform an additional dilatation change of variables.

Consider a situation in which the effective cutoff is Λ. Under the change of variables

$$\varphi(q) \to \varphi(q) + \delta\tau\, \eta(\varphi, q, \tau{=}0) \ ,$$

this effective cutoff is changed to $\Lambda e^{-\delta\tau}$. The scaling transformation $q \to q e^{-\delta\tau}$ then guarantees that the new momenta will span the same range $(0 \to \Lambda)$ as before. Thus one takes

$$\delta_{dil}\varphi(q) = \delta\tau\, (d - d_\varphi + q.\frac{\partial}{\partial q})\varphi(q) \ ,$$

where the field dimension

$$d_\varphi = \frac{d}{2} - 1 + \gamma_\varphi \ ,$$

includes an anomalous dimension term.

The full RG transformation is

$$\varphi(q) \to \varphi(q) + \delta\tau\, \eta(\varphi, q, \tau{=}0) + \delta\tau \left[d - d_\varphi + q.\frac{\partial}{\partial q}\right]\varphi(q) \ , \tag{3.4}$$

and the full RG equation is

$$\frac{dS_\tau}{d\tau} = \int d^d q \left\{ (2\pi)^d \dot{\alpha}(q) \left[\frac{\delta^2 S_\tau}{\delta\varphi(q)\delta\varphi(-q)} - \frac{\delta S_\tau}{\delta\varphi(q)} \frac{\delta S_\tau}{\delta\varphi(-q)} \right] + \right.$$
$$\left. + \frac{\delta S_\tau}{\delta\varphi(q)} (d - d_\varphi + q.\frac{\partial}{\partial q})\varphi(q) \right\} \ , \tag{3.5}$$

where now $\dot{\alpha} = d\alpha / d\tau$ at $\tau = 0$. This functional differential equation can be transformed into an infinite set of integro-differential equations. Substituting an action of the general form

$$S_\tau(\varphi) = \sum_{n \ even}^{\infty} \frac{1}{n!} \int d\tilde{q}_1 \ldots d\tilde{q}_n \tilde{\delta}(\sum_{j=1}^{n} q_j) u_n(q_1, \ldots, q_n, \tau) \varphi(q_1) \ldots \varphi(q_n) \tag{3.6}$$

into eq.(3.5) and equating the coefficients of terms of the same degree in φ

one obtains (omitting the τ dependence)

$$\frac{\partial u_n(q_1 \ldots q_n)}{\partial \tau} = \left[d - n d_\varphi - \sum_{j=1}^{n} q_j \cdot \frac{\partial}{\partial q_j} \right] u_n(q_1 \ldots q_n) + \int d\tilde{k}\, \dot{\alpha}(k) u_{n+2}(q_1 \ldots q_n, k, -k) +$$
$$- \sum_{m \geq 2}^{n} \binom{n}{m-1} \frac{1}{n!} \sum_{\{q_j\}} \dot{\alpha}(k_m) u_m(q_1 \cdots q_{m-1}, k_m) u_{n-m+2}(q_m \cdots q_n, -k_m) \quad (3.7)$$

where $k_m = -\sum_{j=1}^{m-1} q_j = \sum_{j=m}^{n} q_j$ and where $\sum_{\{q_j\}}$ denotes a sum over all permutations of the q_j's.

The equations (3.7) have an interesting graphical representation (fig. 2.1.) in which the internal broken lines have factors of $\dot{\alpha}$ acting as propagators. When a sharp-cutoff is employed, as discussed in Section 2.2.2, they actually do correspond to propagators in the conventional sense.

2.4. Green's Functions

The calculation of Green's functions or of correlation functions brings us to the problem of coupling the field φ to an external source. The study of how Green's functions calculated from the bare action S are related to those calculated from S_τ can be done in a number of ways (see, e.g., Wilson and Kogut [2]). We would like to address this question in the spirit of the previous section, regarding the RG as an infinitesimal change of variables.

Consider the generating functional

$$Z_\tau(j) = \int D\varphi\, exp\left[-S_\tau(\varphi)+\int j\varphi\right] .$$

Performing a change of variables of the form of equations (3.1) and (3.3), (for simplicity we do not include the dilatation change of variables) we obtain

$$Z_\tau(j) = \int D\varphi\left[1-\delta\tau\int d^dq\, j(-q)\dot{\alpha}(q,\tau)\frac{\delta S_\tau}{\delta\varphi(-q)}\right]\exp\left(-S_\tau+\int j\varphi\right) .$$

But,

$$0 = \int D\varphi\frac{\delta}{\delta\varphi(-q)}\, e^{-S_\tau+\int j\varphi} = \int D\varphi\left[\frac{j(q)}{(2\pi)^d} - \frac{\delta S_\tau}{\delta\varphi(-q)}\right] e^{-S_\tau+\int j\varphi} ,$$

and therefore

$$\frac{dZ_\tau}{d\tau} = \left[\int d\tilde{q}\, j(-q)\dot{\alpha}(q,\tau)j(q)\right] Z_\tau(j) .$$

Integrating in τ with the initial condition $S_{-\infty}=S$, i.e.

$$Z_{-\infty}(j) = Z(j) = \int D\varphi\, exp\left(-S(\varphi)+\int j\varphi\right) ,$$

leads to

$$Z(j) = \exp\left[-\int d\tilde{q}\, j(-q)\alpha(q,\tau)j(q)\right] Z_\tau(j) ,$$

which exhibits the desired relation in a particularly simple form.

The generating functionals for connected Green's functions, $W=-\log Z$ and the corresponding W_τ are related by

$$W(j) = W_\tau(j) + \int d\tilde{q}\, j(-q)\alpha(q,\tau)j(q) \ .$$

This shows that the connected n-point functions computed with S_τ are identical to those computed with S for $n \geq 3$ while for $n=2$ they differ in a rather trivial way. In this formulation it is then particularly clear that physically significant quantities such as critical exponents or S-matrix elements can be computed with either Z or Z_τ and that they are independent of the choice of α, that is, independent of the choice of the RG, which is a well known fact.

2.5. Solutions

Obtaining solutions to the RG evolution equations (3.7) is a challenging problem. In this section two conventional approximations are discussed, an expansion in $\varepsilon=4-d$ and an expansion in a single coupling constant. One motivation is to give us confidence that the expressions in the previous sections are correct in spite of the lack of mathematical rigor employed in their deduction. Another motivation is to compare the two approximation schemes which, although leading to differential equations of similar structure, represent different viewpoints. Finally, yet a third motivation is to stress the larger freedom in the choice of the RG. This is important, not only because it allows one to construct RG's in which the usual restriction of integrating only over short wavelength fluctuations is lifted, but also because it will allow us to construct gauge covariant RG's.

A standard approach to solving eq.(3.7) consists of finding a fixed point and studying the evolution of small perturbations about this fixed point. One looks for a fixed point solution S^* for which the vertex functions $u_n{}^*$ do not depend on τ as an expansion in ε :

$$u_2{}^* = V_{20} + V_{21}\varepsilon + V_{22}\varepsilon^2 + \ldots,$$

$$u_4{}^* = V_{41}\varepsilon + V_{42}\varepsilon^2 + \ldots \ , \ u_6{}^* = V_{62}\varepsilon^2 + \ldots \tag{5.1a}$$

with the anomalous dimension given by

$$\gamma_\varphi = \gamma_1\varepsilon + \gamma_2\varepsilon^2 + \ldots \tag{5.1b}$$

The crucial extra condition imposed on the solution and on the RG transformation (3.4) (i.e., on the anomalous dimension γ_φ) is that the solution be analytic in the momenta. The need for this condition can be vaguely argued as follows. Non-analyticity in momentum space translates into long range or nonlocal interactions in position space for which some features of critical behavior, like universality, are known not to hold.

Details of these calculations, which are similar to those obtained by Shukla and Green [7] for Wilson's incomplete-integration RG, can be found in Appendix 2.C.

An alternative perturbative approach consists in expanding in a single coupling constant $g(\tau)$ without referring to any fixed point. One looks for a solution of the form

$$u_2 = U_{20} + gU_{21} + g^2U_{22} + \ldots \ ,$$

$$u_4 = \Lambda^{\varepsilon}\,(gU_{41} + g^2U_{42} + \ldots) \ ,$$

$$u_6 = \Lambda^{2\varepsilon}\,(g^2U_{62} + \ldots) \ . \tag{5.2a}$$

with an anomalous dimension given by

$$\gamma_{\varphi} = \gamma_1 g + \gamma_2 g^2 + \ldots \tag{5.2b}$$

and $g(\tau)$ flowing according to

$$\frac{dg}{d\tau} = -\beta(g) = b_1 g + b_2 g^2 + \ldots \tag{5.2c}$$

Factors of Λ^{ε} have been made explicit so that the various U's have the same dimensions they would have at $d=4$.

The crucial extra condition imposed on the solution and on the RG transformation in this perturbative approach is that all τ dependence be contained in the single function $g(\tau)$. This brings us somewhat closer to the spirit of the Gell-Mann and Low RG. The other functions are required not to depend on τ but could be non-analytic. These calculations are carried out in Appendix 2.D.

While none of the results obtained in those calculations are new, the freedom in the choice of the "cutoff" function $\dot{\alpha}(q)$ is explicit. In particular, one is not required to integrate only short wavelengths, that is $\dot{\alpha}(0)=0$, as for example in the usual $\dot{\alpha}(q)=\frac{q^2}{\Lambda^4}$. One can integrate the long wavelengths as well, for example $\dot{\alpha}(q)=\frac{1}{\Lambda^2}\exp\frac{q^2}{\Lambda^2}$ or even integrate all wavelengths simultaneously to the same extent by taking $\dot{\alpha}=\frac{1}{\Lambda^2}=const$.

Although physically significant quantities such as γ_φ or $\beta(g)$ are independent of $\dot{\alpha}$ the same is not true of the vertex functions u_n. In particular one should be careful with the otherwise very convenient choice $\dot{\alpha}=const$. For this choice of $\dot{\alpha}$ the vertex functions contain parts which are divergent as $d\to4$. This is annoying but nothing more. The way around this problem is the usual one, to think of the u_n's as separated into two parts $u_n=u_n^R+u_n^C$ one of which is finite while the other is a divergent counterterm. The RG equations (3.7) keep track of the evolution of both the finite and the divergent parts. The presence of these divergences is a manifestation of the fact that while the RG was historically connected to renormalization theory such a connection, although sometimes convenient, is not at all necessary.

Appendix 2.A. The Sharp-Cutoff RG

In this appendix we integrate out the σ fields in the thin momentum shell and deduce equations (2.3). As discussed in Section 2.1 this process involves three steps.

step 1: Expand $S_\tau(\varphi+\sigma)$ in a power series about $\sigma=0$.

$$S_\tau(\varphi+\sigma) = \sum_{p=0}^{\infty} \frac{1}{p!}\int dx_1 \cdots dx_p S_\tau^{(p)}(\varphi, x_1 \cdots x_p)\sigma(x_1)\ldots\sigma(x_p) \ , \qquad (A.1)$$

where

$$S_\tau^{(p)}(\varphi, x_1 \ldots x_p) = \frac{\delta^p S_\tau(\varphi)}{\delta\varphi(x_1)\ldots\delta\varphi(x_p)} \ .$$

step 2: Identify a convenient σ-field propagator.

Rewrite the quadratic term in eq.(A.1) as

$$\frac{1}{2}\int dx_1 dx_2 S_\tau^{(2)}(\varphi, x_1, x_2)\sigma(x_1)\sigma(x_2) =$$

$$= \frac{1}{2}\int dx\, \sigma(x)\partial^2\sigma(x) + \frac{1}{2}\int dx_1 dx_2 \bar{S}_\tau^{(2)}(\varphi, x_1, x_2) \ ,$$

and let $S_\tau^{(p)} = \bar{S}_\tau^{(p)}$ for $p \neq 2$, so that a convenient propagator is

$$\Delta_\tau(x-y) = \int d\tilde{q}\, \Delta_\tau(q) e^{-iq(x-y)} = \frac{\delta\tau\Lambda_\tau^{d-2}}{(2\pi)^d}\int d\Omega_d e^{-iq(x-y)} \ . \qquad (A.2)$$

step 3: Treat the σ^p vertices perturbatively.

Rewrite eq.(2.2) in the form

$$\exp -S_{\tau+\delta\tau}(\varphi) = \exp -\left[\sum_{p=0}^{\infty}\frac{1}{p!}\int dx_1 \cdots dx_p \bar{S}_\tau^{(p)} \frac{\delta}{\delta j(x_1)} \cdots \frac{\delta}{\delta j(x_p)}\right]$$

$$\int D\varphi \exp -\int dx \left(\frac{1}{2}\sigma\Delta_\tau^{-1}\sigma - j\sigma\right)\Big|_{j=0} \ .$$

Since each σ propagator contributes a factor $\delta\tau$, (eq.(A.2)), while each vertex $\bar{S}_\tau^{(p)}$ contributes a factor $(\delta\tau)^0$ it follows that to first order in $\delta\tau$ only diagrams with one internal σ line need be included. Therefore

$$S_{\tau+\delta\tau}(\varphi) = S_\tau(\varphi) - \int dx_1 dx_2 \frac{1}{2}\Delta_\tau(x_1-x_2)\Big[S_\tau^{(2)}(x_1,x_2) - S_\tau^{(1)}(x_1)S_\tau^{(1)}(x_2)\Big] +$$

$$+ \; O(\delta\tau^2) \; ,$$

where at the expense of a φ independent constant, which may for our purposes be ignored, we have dropped the bars. Rewriting this expression in momentum space leads to eq.(2.3) as desired.

Appendix 2.B. A Field Theory in Zero Dimensions

In this appendix we consider in more detail the application of the RG formalism described previously to a field theory in zero dimensions for which the partition function is an ordinary integral. This study serves to clarify the concepts in a much simpler setting exhibiting the essence of the RG transformation as a change to variables better suited for a semiclassical approximation, and also to illustrate the fact that these RG's are not restricted to systems with an infinite number of degrees of freedom. First we deduce the incomplete-integration RG equation interpreting it as a change of variables and then show that a steepest descent approximation becomes asymptotically exact for the RG evolved action. Finally, as a practical example we perform a RG improved perturbative calculation.

As discussed in Section 2.2.2 "incomplete integration" is achieved through the introduction of a constant,

$$1 = N_\alpha^{-1}\int_{-\infty}^{\infty} dx \, \delta_\alpha(y-x) \;\; , \;\; \text{where} \;\; N_\alpha = (1+4\pi\alpha)^{\frac{1}{2}} \;\; ,$$

into the "partition function" z, so that

$$z = N_\alpha^{-1}\int_{-\infty}^{\infty} dx \exp -S_\alpha(x) \;\; , \tag{B.1}$$

with

$$\exp -S_\alpha(x) = \int_{-\infty}^{\infty} dy \;\delta_\alpha(y-x) \exp -S(y) \quad . \tag{B.2}$$

Using

$$\frac{d\delta_\alpha(x)}{d\alpha} = \frac{2\pi}{1+4\pi\alpha}\delta_\alpha(x) + \frac{d^2\delta_\alpha(x)}{dx^2} \quad ,$$

equation (B.2) can be turned into a RG differential equation

$$\frac{dS_\alpha(x)}{d\alpha} = \frac{d^2S_\alpha(x)}{dx^2} - \left[\frac{dS_\alpha(x)}{dx}\right]^2 - \frac{2\pi}{1+4\pi\alpha} \quad . \tag{B.3}$$

This evolution can be interpreted as a sequence of changes of variables. Changing x to $x+\eta d\alpha$ in (B.1) leads to

$$z = N_{\alpha+d\alpha}^{-1} \int_{-\infty}^{\infty} dx \; exp -S_{\alpha+d\alpha}(x) \quad ,$$

where

$$S_{\alpha+d\alpha}(x) = S_\alpha(x) + \frac{dS_\alpha(x)}{dx}\eta d\alpha - \frac{d\eta}{dx}d\alpha - \frac{2\pi d\alpha}{1+4\pi\alpha} \quad .$$

If one chooses $\eta = -\frac{dS_\alpha}{dx}$ this is precisely the RG equation (B.3).

Equation (B.3) can be transformed into a system of differential equations for the evolution of the "vertex functions". Substituting

$$S_\alpha(x) = \sum_{n=0,even}^{\infty} \frac{1}{n!} u_n(\alpha) x^n \quad ,$$

into (B.3) one obtains

$$\frac{du_0}{d\alpha} = u_2 - \frac{2\pi}{1+4\pi\alpha} \quad , \tag{B.4a}$$

and for $n \neq 0$

$$\frac{du_n}{d\alpha} = u_{n+2} - \sum_{m=2}^{n} \binom{n}{m-1} u_m u_{n-m+2} \quad . \tag{B.4b}$$

Now we come to the question of why it is useful to go through the trouble of solving (B.3). Consider a steepest descent approximation to (B.1) :

$$z = N_\alpha^{-1}\left[\frac{2\pi}{S_\alpha^{(2)}(\bar{x}_\alpha)}\right]^{\frac{1}{2}} exp - S_\alpha(\bar{x}_\alpha) + \ldots$$

where $S_\alpha^{(n)}$ is the n-th derivative of S_α and $\bar{x}_\alpha$ is the saddle point, $S_\alpha^{(1)}(\bar{x}_\alpha)=0$.

The incomplete integration was designed so that as $\alpha\to\infty$ the exponential factor on the right hand side, e^{-S_α} tends to the desired exact value z. If the leading steepest descent approximation is to become exact it should be true that

$$\lim_{\alpha\to\infty} N_\alpha^{-1}\left[\frac{2\pi}{S_\alpha^{(2)}(\bar{x}_\alpha)}\right]^{\frac{1}{2}} = 1 \ . \tag{B.5}$$

It is easy to see that this is so by referring back to eq.(B.3). As $\alpha\to\infty$ the left hand side vanishes. Evaluating at the saddle point $\bar{x}_\alpha$, the second term on the right also vanishes and one gets

$$S_\alpha^{(2)}(\bar{x}_\alpha) \approx \frac{2\pi}{1+4\pi\alpha} \quad , \tag{B.6}$$

which implies (B.5) as desired.

It is interesting to see what is happening from another point of view. Consider evaluating

$$z = \int_{-\infty}^{\infty} dx \ exp - (\frac{1}{2}wx^2 + \frac{1}{4!}\lambda x^4) \ .$$

If λ is small enough one could try a perturbative expansion

$$z \approx (\frac{2\pi}{w})^{\frac{1}{2}}(1 - \frac{\lambda}{8w^2} + O(\lambda^2)) \tag{B.8}$$

But one could refer to eq.(B.1) and try a RG improved expansion :

$$z \approx N_\alpha^{-1}\, e^{-u_0(\alpha)}(\frac{2\pi}{u_2(\alpha)})^{\frac{1}{2}}(1 - \frac{1}{8}\frac{u_4(\alpha)}{u_2^2(\alpha)} + \ldots) \ , \tag{B.9}$$

where $u_0(\alpha)$, $u_2(\alpha)$ and $u_4(\alpha)$ are solutions of (B.4) with the initial conditions $u_0(0)=0$, $u_2(0)=w$ and $u_4(0)=\lambda$. To order λ these solutions are

$$u_0(\alpha) = \frac{1}{2}\log\frac{1+2w\alpha}{1+4\pi\alpha} + \frac{\lambda\alpha^2}{2(1+2w\alpha)^2} + O(\lambda^2) \ , \tag{B.10a}$$

$$u_2(\alpha) = \frac{w}{1+2w\alpha} + \frac{\lambda\alpha}{(1+2w\alpha)^3} + O(\lambda^2) \ , \tag{B.10b}$$

$$u_4(\alpha) = \frac{\lambda}{(1+2w\alpha)^4} + O(\lambda^2) \ . \tag{B.10c}$$

According to (B.10b) $u_2(\alpha)$ tends to vanish as α increases. This could mean trouble since as the integrand becomes flatter and flatter (this is apparent in eq.(B.6) also), the reliability of the steepest descent approximation would become increasingly doubtful. Fortunately we are saved by (B.10c) which shows that the "interaction" u_4 vanishes much faster and the integrand approaches a Gaussian, a rather flat one but still a Gaussian. The perturbative correction u_4/u_2^2 in eq.(B.9) asymptotically vanishes.

Given the vertices (B.10) correct to order λ, the best approximation is obtained by letting $\alpha\to\infty$:

$$z \approx \left(\frac{2\pi}{w}\right)^{\frac{1}{2}} exp - \frac{\lambda}{8w^2} \ , \tag{B.11}$$

which has the typical exponential of RG improved calculations.

It is quite remarkable that one can study the powerful RG techniques at work in such a simple example. It is perhaps even more remarkable that in this simple study even their limitations become apparent. For the simple case (B.7) the exact result is known and for strong coupling (large λ) the correct dependence on λ is the power law

$$z \approx \frac{1}{2}\Gamma\left(\frac{1}{4}\right)\left(\frac{6}{\lambda}\right)^{\frac{1}{4}} \ ,$$

and not the exponential dependence of (B.11). This illustrates the known fact that while RG perturbation expansions are an improvement over plain perturbation expansions, they remain nevertheless restricted to the small coupling regime. Needless to say, this is not a limitation of the RG itself (eqs.(B.3-4) are exact) but of the perturbative solution (B.10) to the RG equation (B.4).

Appendix 2.C. The ε-Expansion

The RG system of equations (3.7) is greatly simplified if one realizes that the solutions of interest are not the most general solutions of the first order partial differential equations in which the momenta q_j are independent variables, but rather those special solutions with interesting scaling properties when all q_j are scaled together. The unwanted solutions can be discarded by evaluating (3.7) at momenta λq_j with $\lambda = e^{\tau}$ instead of q_j. This eliminates the partial derivatives and (3.7) becomes

$$\left\{\lambda\frac{d}{d\lambda}+nd_{\varphi}-d\right\}u_n(\lambda q_1\cdots\lambda q_n,\lambda) = \int d\tilde{q}\;\dot{\alpha}(k)u_{n+2}(\lambda q_1\cdots\lambda q_n,k,-k,\lambda) -$$
$$\sum_{m\geq 2}^{n}\binom{n}{m-1}\frac{1}{n!}\sum_{\{q_j\}}\dot{\alpha}(\lambda k_m)u_m(\lambda q_1\cdots\lambda q_{m-1},\lambda k_m,\lambda)u_{n-m+2}(\lambda q_m\cdots\lambda q_n,-\lambda k_m,\lambda). \tag{C.1}$$

Substituting eq.(5.1) into (C.1) leads to a set of first order ordinary differential equations for the V's :

$$(\lambda\frac{d}{d\lambda}-2)V_{20} = -2\dot{\alpha}V_{20}^2 \quad , \tag{C.2}$$

$$(\lambda\frac{d}{d\lambda}-2)V_{21} + 2\gamma_1 V_{20} = \int d\tilde{k}\,\dot{\alpha}\,V_{41} - 4\dot{\alpha}V_{20}V_{21} \quad , \tag{C.3}$$

$$(\lambda\frac{d}{d\lambda}-2)V_{22} + 2\gamma_1 V_{21} + 2\gamma_2 V_{20} = \int d\tilde{k}\,\dot{\alpha}\,V_{42} + \frac{d}{d\varepsilon}\int d\tilde{k}\,\dot{\alpha}\,V_{41}\Big|_{\varepsilon=0}$$
$$- 2\dot{\alpha}V_{21}^2 - 4\dot{\alpha}V_{20}V_{21} \quad , \tag{C.4}$$

$$\lambda \frac{d}{d\lambda} V_{41} = -2(\sum_j^4 \dot{\alpha} V_{20}) V_{41} \quad . \tag{C.5}$$

$$\lambda \frac{d}{d\lambda} V_{42} + (4\gamma_1 - 1) V_{41} = \int d\tilde{k}\, \dot{\alpha} V_{62} - 2(\sum_j^4 \dot{\alpha} V_{20}) V_{42} - 2(\sum_j^4 \dot{\alpha} V_{21}) V_{41} \quad , \tag{C.6}$$

$$(\lambda \frac{d}{d\lambda} + 2) V_{62} = -2(\sum_j^6 \dot{\alpha} V_{20}) V_{62} - 2(\dot{\alpha} V_{41} V_{41} + 9 \text{ perm}) \quad . \tag{C.7}$$

The arguments of the V's can be easily obtained by referring back to eq.(C.1). In eq.(C.7) the ten permutations refer to the inequivalent ways of grouping six momenta into two sets of three.

Equation (C.2) is of the Bernoulli type. The solution behaving as $V_{20} \approx q^2$ for small q is

$$V_{20}(q) = q^2 f(q) \quad ,$$

where

$$f(q) = \frac{1}{1 + q^2 \int_0^1 d\lambda \lambda^2 \dot{\alpha}(\lambda q)} = exp -2 \int_0^1 \frac{d\lambda}{\lambda} \dot{\alpha}(\lambda q) V_{20}(\lambda q) \quad . \tag{C.8}$$

The second equality is very useful because it will allow us to construct integrating factors for all the remaining equations (C.3-7) which are linear. The solution to (C.5) is

$$V_{41}(q_1 \ldots q_4) = A \prod_{j=1}^4 f(q_j) \quad .$$

Next we solve (C.3). The fixed point ($\partial V_{21} / \partial \lambda = 0$) and the analyticity requirements force us to choose $\gamma_1 = 0$, so that

$$V_{21}(q) = (C q^2 - \frac{1}{2} AB) f^2(q) \quad ,$$

where C is a constant and

$$B = \int d\tilde{k}\, \dot{\alpha}(k) f^2(k) \quad . \tag{C.9}$$

This completes the solution to order ε.

The solution for (C.7) is straightforward,

$$V_{62}(q_1 \ldots q_6) = -A^2 \left[\prod_{j=1}^{6} f(q_j)\right] \left[D(q_1+q_2+q_3) + 9 \text{ perm}\right] , \qquad \text{(C.10)}$$

where

$$D(k) = \frac{1}{k^2}[1-f(k)] .$$

The solution for V_{42} is messier, the only important point being that in order to eliminate a divergence at $\lambda=0$ (or $q=0$) the constant A in V_{41} must be chosen to be $A=(4\pi)^2/3$.

Finally, we turn to V_{22}. Again its explicit form is not very illuminating but the requirement that it be analytic at $q=0$ determines both $V_{42}(0 \ldots 0)$ which is not in itself very interesting and also γ_2 :

$$\gamma_2 = -\frac{(4\pi)^4}{18} \int d\tilde{k}\, d\tilde{k}' \frac{1}{k^2} f(k) f'(k') f''(Q) , \qquad \text{(C.11)}$$

where $\vec{Q}=\vec{k}+\vec{k}'$ and $f'=\frac{d}{dk^2}f(k)$. The integral in eq.(C.11) is complicated, it can be done analytically for $\dot{\alpha}=\frac{1}{\Lambda^2}e^{q^2/\Lambda^2}$ or else numerically for $\dot{\alpha}=\frac{1}{\Lambda^2}$ or for $\dot{\alpha}=\frac{q^2}{\Lambda^4}$. The result is $\gamma_2=\frac{1}{108}$ so that $\gamma_\varphi=\frac{\varepsilon^2}{108}$ as it should be. This completes the solution to order ε^2.

Appendix 2.D. The Perturbative Solution

Substituting eqs.(5.2) into (C.1) leads to the following set of equations for the U's :

$$(\lambda \frac{d}{d\lambda} - 2) U_{20} = -2\dot{\alpha} U_{20}^2 , \qquad \text{(D.1)}$$

$$(\lambda \frac{d}{d\lambda} - 2 + b_1) U_{21} + 2\gamma_1 U_{20} = \Lambda^\varepsilon \int d\tilde{k}\, \dot{\alpha} U_{41} - 4\dot{\alpha} U_{20} U_{21} . \qquad \text{(D.2)}$$

$$(\lambda\frac{d}{d\lambda}-2+2b_1)U_{22} + (b_2+2\gamma_1)U_{21} + 2\gamma_2 U_{20} = \Lambda^\varepsilon\int d\bar{k}\,\dot{\alpha}\,U_{42} +$$

$$- 2\dot{\alpha}\,U_{21}^2 - 4\dot{\alpha}\,U_{20}U_{22} \;. \tag{D.3}$$

$$(\lambda\frac{d}{d\lambda}+b_1-\varepsilon)U_{41} = -2(\sum_j^4\dot{\alpha}\,U_{20})U_{41} \;. \tag{D.4}$$

$$(\lambda\frac{d}{d\lambda}+2b_1-\varepsilon)U_{42} + (b_2+4\gamma_1)U_{41} = \Lambda^\varepsilon\int d\bar{k}\,\dot{\alpha}\,U_{62} +$$

$$- 2(\sum_j^4\dot{\alpha}\,U_{20})U_{42} - 2(\sum_j^4\dot{\alpha}\,U_{21})U_{41} \;. \tag{D.5}$$

$$(\lambda\frac{d}{d\lambda}+2b_1+2-2\varepsilon)U_{62} = -2(\sum_j^6\dot{\alpha}\,U_{20})U_{62} - 2(\dot{\alpha}\,U_{41}U_{41} + 9\text{ perm}) \;. \tag{D.6}$$

These equations are naturally very similar to those of Appendix 2.C and their solution proceeds exactly as before except that now, as discussed in Section 2.5, no requirement of analyticity is made.

The solution of (D.1) behaving as $U_{20}\approx q^2$ for small q is

$$U_{20}(q) = q^2 f(q)$$

with $f(q)$ given by (C.8) as before. The integration of (D.4) is straightforward. We want U_{41} independent of λ so that all the evolution of gU_1 is attributed to the evolution of g. This forces us to choose $b_1=\varepsilon$. Further we normalize $U_{41}(0...0)=1$ which is the conventional normalization for g. Then

$$U_{41}(q_1...q_4) = \prod_{j=1}^4 f(q_j)$$

Next consider (D.2). The requirement that U_{21} is independent of λ implies $\gamma_1=0$, and one obtains the in general non-analytic expression

$$U_{21}(q) = \Lambda^\varepsilon(C\,q^{2-\varepsilon}-\frac{B}{2-\varepsilon})f^2(q) \;,$$

where B is given by (C.9).

The solution for U_{62} is uneventful, one obtains the expression in eq.(C.10) with $A=1$.

Finally, the solution U_{42} to eq.(D.5) does not depend on λ provided one chooses $b_2=-3/(4\pi)^2 + O(\varepsilon)$, that is

$$-\frac{dg}{d\tau} = \beta(g) = -\varepsilon g + \left(\frac{3}{(4\pi)^2} + O(\varepsilon) \right) g^2 + \ldots$$

as it should be. We stop here since the solution for U_{22} and γ_2 proceeds in just the same way.

One final comment concerning the choice of the "cutoff" function $\dot{\alpha}$. If one chooses a constant $\dot{\alpha}=1/\Lambda^2$ the vertex functions develop divergences as $d\to 4$. This is evident when one computes the constant B given by (C.9). As discussed further in Section 2.5. this is not really a problem, particularly since physically significant quantities such as γ_φ and $\beta(g)$ are perfectly finite and independent of $\dot{\alpha}$.

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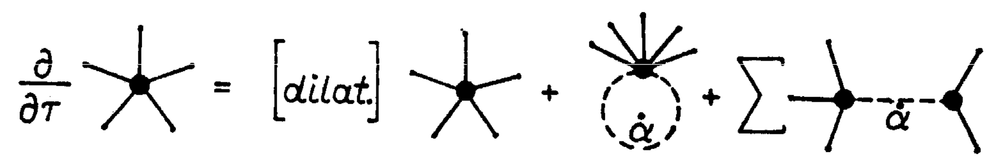

Fig.1.- Graphical representation of the renormalization group equations (3.7).

Chapter 3. A Gauge Covariant Renormalization Group

3.1 Summary

As an application of the ideas of the previous chapter we propose an exact gauge covariant renormalization group transformation which is a pure change of variables. The corresponding RG equations are obtained. By taking the subtleties of gauge fixing into account it is shown that the gauge invariant RG-evolved action is equivalent to the usual action as far as the computation of gauge invariant quantities is concerned.

Although the gauge symmetry is manifestly preserved by the RG evolution it is convenient to study the constraints which are imposed on the vertex functions by gauge invariance. A method to do this is devised which is simpler than a previously existing one (Kim and Baker [1]).

The perturbative solution of the RG equations shows that only certain initial conditions (i.e., bare actions) are compatible with the constraint that all the RG evolution be described by a single coupling constant. The β function for this single coupling for the $SU(N)$ gauge theory is calculated to be $-\frac{21}{6}\frac{N}{(4\pi)^2}g^3$ at the tree level while a one-loop calculation yields the usual result, $-\frac{22}{6}\frac{N}{(4\pi)^2}g^3$. Unlike the scalar theory case the iteration of the RG transformation does not lead to an asymptotic situation in which the saddle-point approximation is exact. The details of some calculations and the demonstration of an interesting result concerning the RG evolution of the Fadeev-Popov determinant appear in the appendices.

3.2. The Renormalization Group Equations

The Green's functions generating functional $Z(j)$ for the pure $SU(N)$ Yang-Mills theory in Euclidean spacetime is given by (see e.g.[2])

$$Z(j) = \int DA \, \Delta_G(A) \, \delta(G(A)) \exp -\left[S(A) - \int dx \, jA\right] , \tag{2.1}$$

where the action $S(A)$ is gauge invariant, $G(A)=0$ is the gauge condition and $\Delta_G(A)$ is the Fadeev-Popov determinant.

As discussed in Chapter 2, exact RG transformations may be constructed which are infinitesimal changes of variables consisting of two parts

$$A_\mu^a \to A_\mu^a + \delta_{dil} A_\mu^a + \delta_{int} A_\mu^a .$$

The first piece, the dilatation $\delta_{dil} A$, is rather trivial. It is conventionally included, although this is not necessary, because it simplifies the analysis of problems in which scaling phenomena are the object of investigation. The second piece $\delta_{int} A$, is the important one. It is such that a saddle point approximation performed in the new variables is more accurate than in the old variables. In special limiting cases it accomplishes the elimination of certain degrees of freedom, a characteristic feature of many RG transformations.

We saw in Chapter 2 that the change of variables $\delta_{int} A$ includes a term linear in $\frac{\delta S_\tau}{\delta A}$, where S_τ is the RG-evolved action and τ is the parameter which describes that evolution. In order that the transformation be gauge covariant, one chooses for example

$$\delta_{int} A(x) = -\delta\tau \, \dot{\alpha} \, \frac{\delta S_\tau}{\delta A(x)} ,$$

where $\dot{\alpha} = \dot{\alpha}(D,\tau)$ is some differential operator and D is the covariant derivative. The simplest choice, $\dot{\alpha}=const.$, dependent only on τ, leads to

$$\delta_{int} A_\mu^a(x) = -\frac{\delta\tau}{\Lambda_\tau^2}\frac{\delta S_\tau}{\delta A_\mu^a(x)} \quad , \tag{2.2a}$$

where $\Lambda_\tau = \Lambda e^{-\tau}$, or equivalently, in momentum space

$$\delta_{int} A_\mu^a(q) = -\frac{\delta\tau(2\pi)^d}{\Lambda_\tau^2}\frac{\delta S_\tau}{\delta A_\mu^a(-q)} \quad . \tag{2.2b}$$

This gauge covariant transformation, which is the one we will employ in what follows, differs from the transformations usually employed in studies of critical phenomena in that it tends to integrate out not only the short wavelengths but alsc, and to the same extent, the long wavelengths. This strange feature seems to be necessary if gauge covariance is to be achieved.

For the moment we will neglect the complications due to gauge fixing to which we will return in the next section. Following the same steps that led to (2;3.2), the evolution of the action S_τ under the transformation (2.2) is given by the RG equation

$$\frac{dS_\tau(A)}{d\tau} = \frac{(2\pi)^d}{\Lambda_\tau^2}\int d^dq\left[\frac{\delta^2 S_\tau}{\delta A_\mu^a(q)\delta A_\mu^a(-q)} - \frac{\delta S_\tau}{\delta A_\mu^a(q)}\,\frac{\delta S_\tau}{\delta A_\mu^a(-q)}\right] \; . \tag{2.3}$$

By construction the evolution is such that if the bare Yang-Mills action $S(A)=S_{-\infty}(A)$ is gauge invariant then the evolved action $S_\tau(A)$ will also be so.

As mentioned above one may want to perform a dilatation also

$$\delta_{dil} A_\mu^a(x) = -\delta\tau\,(d_A + x.\frac{\partial}{\partial x})A_\mu^a(x) \quad , \tag{2.4}$$

where the field dimension

$$d_A = \frac{d}{2} - 1 + \gamma_A$$

includes an anomalous dimension term γ_A. Equation (2.4) may be rewritten as

$$\delta_{du} A_\mu^a(x) = -\delta\tau\left[x_\nu F_{\nu\mu}^a + D_\mu^{ab}(x_\nu A_\nu^b) + (d_A-1)A_\mu^a\right] .$$

While the first two terms are gauge covariant (in fact, the second is just a gauge transformation which could be omitted if desired), the last term is not. Fortunately, this noncovariant term is not a problem because it is just an overall multiplication of the field by a constant and this may be reabsorbed into a running coupling constant $g(\tau)$. Thus the RG transformation will take an action S_τ invariant under the gauge transformation

$$\delta A_\mu^a(x) = \partial_\mu \Omega^a(x) - g(\tau) f^{abc} A_\mu^b(x)\Omega^c(x) , \tag{2.5}$$

into another action $S_{\tau+\delta\tau}$ which is also gauge invariant but now under a different gauge transformation

$$\delta A_\mu^a(x) = \partial_\mu \Omega^a(x) - g(\tau+\delta\tau) f^{abc} A_\mu^b(x)\Omega^c(x) .$$

The evolution of $g(\tau)$ is such that under $A \to e^{-\delta\tau(d_A-1)}A$,

$$g(\tau)A \to g(\tau)e^{-\delta\tau(d_A-1)}A \equiv g(\tau+\delta\tau)A .$$

Therefore

$$\beta(g) \equiv -\frac{dg}{d\tau} = \frac{d-4}{2}g + g\gamma_A . \tag{2.6}$$

This relation between γ_A and β is the same that holds in the ghost free axial gauges and in the manifestly gauge invariant background field gauge .

In analogy to Equations (2;3.4) and (2;3.5), the full RG transformation is

$$A_\mu^a(q) \to A_\mu^a(q) - \delta\tau \frac{(2\pi)^d}{\Lambda^2} \frac{\delta S_\tau}{\delta A_\mu^a(-q)} + \delta\tau \left[d - d_A + q \cdot \frac{\partial}{\partial q} \right] A_\mu^a(q) \quad . \tag{2.7}$$

(note that here $\delta_{int}A$ is evaluated at $\tau=0$) and the corresponding RG equation is

$$\frac{dS_\tau(A)}{d\tau} = \int d^d q \left\{ \frac{(2\pi)^d}{\Lambda^2} \left[\frac{\delta^2 S_\tau}{\delta A_\mu^a(q) \delta A_\mu^a(-q)} - \frac{\delta S_\tau}{\delta A_\mu^a(q)} \frac{\delta S_\tau}{\delta A_\mu^a(-q)} \right] + \right.$$

$$\left. + \frac{\delta S_\tau}{\delta A_\mu^a(q)} (d - d_A + q \cdot \frac{\partial}{\partial q}) A_\mu^a(q) \right\} \quad . \tag{2.8}$$

The RG functional Equations (2.3) and (2.8) may be transformed into an infinite set of integro-differential equations. Consider an action of the general form

$$S_\tau(A) = \sum_{n=1}^{\infty} \frac{1}{n!} \int d\tilde{q}_1 \cdots d\tilde{q}_n \tilde{\delta}(\sum_{j=1}^{n} q_j) w_n \begin{bmatrix} a_1 & \cdots & a_n \\ q_1 & \cdots & q_n \\ \mu_1 & \cdots & \mu_n \end{bmatrix} A_{\mu_1}{}^{a_1}(q_1) \cdots A_{\mu_n}{}^{a_n}(q_n) \tag{2.9}$$

where the τ dependence of the vertex functions w_n is not written explicitly. As before, we use the notation $d\tilde{q} = d^d q / (2\pi)^d$ and $\tilde{\delta}(q) = (2\pi)^d \delta^d(q)$. Substituting into (2.3) and equating the coefficients of terms of the same degree in A one obtains

$$\frac{\partial}{\partial\tau} w_n \begin{bmatrix} a_1 & \cdots & a_n \\ q_1 & \cdots & q_n \\ \mu_1 & \cdots & \mu_n \end{bmatrix} = \frac{1}{\Lambda_\tau^2} \int d\tilde{k}\, w_{n+2} \begin{bmatrix} a_1 & \cdots & a_n & a & a \\ q_1 & \cdots & q_n & k & -k \\ \mu_1 & \cdots & \mu_n & \mu & \mu \end{bmatrix} -$$

$$\frac{1}{\Lambda_\tau^2} \sum_{m=2}^{n} \binom{n}{m-1} \frac{1}{n!} \sum_{\{q_j\}} w_m \begin{bmatrix} a_1 & \cdots & a_{m-1} & a \\ q_1 & \cdots & q_{m-1} & k_m \\ \mu_1 & \cdots & \mu_{m-1} & \mu \end{bmatrix} w_{n-m+2} \begin{bmatrix} a_m & \cdots & a_n & a \\ q_m & \cdots & q_n & -k_m \\ \mu_m & \cdots & \mu_n & \mu \end{bmatrix} \tag{2.10}$$

where $k_m = -\sum_{j=1}^{m-1} q_j = \sum_{j=m}^{n} q_j$ and where $\sum_{\{q_j\}}$ denotes a sum over all permutations of the q_j's.

The same procedure may be carried out for the RG equation with dilatations, Equation (2.8). As discussed in Appendix 2.C, it is convenient to evaluate at momenta λq_j with $\lambda = e^{\tau}$. The resulting system of RG equations is

$$(\lambda \frac{d}{d\lambda} + n d_A - d)\, w_n \begin{bmatrix} a_1 \cdots & a_n \\ \lambda q_1 \cdots & \lambda q_n \\ \mu_1 \cdots & \mu_n \end{bmatrix} = \frac{1}{\Lambda^2} \int d\tilde{k}\; w_{n+2} \begin{bmatrix} a_1 \cdots & a_n & a & a \\ \lambda q_1 \cdots & \lambda q_n & k & -k \\ \mu_1 \cdots & \mu_n & \mu & \mu \end{bmatrix} +$$

$$- \frac{1}{\Lambda^2} \sum_{m=2}^{n} \binom{n}{m-1} \frac{1}{n!} \sum_{\{q_j\}} w_m \begin{bmatrix} a_1 \cdots & a_{m-1} & a \\ \lambda q_1 \cdots & \lambda q_{m-1} & \lambda k_m \\ \mu_1 \cdots & \mu_{m-1} & \mu \end{bmatrix} w_{n-m+2} \begin{bmatrix} a_m \cdots & a_n & a \\ \lambda q_m \cdots & \lambda q_n & -\lambda k_m \\ \mu_m \cdots & \mu_n & \mu \end{bmatrix}. \tag{2.11}$$

Below we will obtain perturbative solutions to both (2.10) and (2.11), but first we must show the RG equations above are legitimate when the gauge fixing term and the Fadeev-Popov determinant in (2.1) are taken into account.

3.3. Gauge Fixing, Expectation Values and the RG Transformation

In this section we wish to study in more detail how the transformation (2.2),

$$A \to \Phi(A) = A + \delta_{int} A \quad , \tag{3.1}$$

is carried out. For simplicity we will not consider the full transformation involving also a dilatation.

We will take advantage of the fact that obtaining the gauge dependent Green's functions is not an end in itself but merely a useful intermediate calculational step and thus, of the freedom to choose different gauge conditions. For example, the generating functional

$$Z_\tau(j) = \int DA\ \Delta_G(A)\ \delta(G(A))\ \exp -(S_\tau(A) - \int dx\ jA) \quad , \tag{3.2}$$

could equally well have been expressed in a different gauge condition $G'(A)=0$,

$$Z_\tau(j) = \int DA\ \Delta_{G'}(A)\ \delta(G'(A))\ \exp -(S_\tau(A) - \int dx\ jA_\Omega) \quad . \tag{3.3}$$

Here, $\Omega(A)$ is the gauge transformation that takes a field configuration A in the G' gauge ($G'(A)=0$) to the corresponding field configuration A_Ω in the G gauge ($G(A_\Omega)=0$). Consider now the functional $Z'_\tau(j)$ obtained by dropping the subscript Ω in (3.3),

$$Z'_\tau(j) = \int DA\ \Delta_{G'}(A)\ \delta(G'(A))\ \exp -(S_\tau(A) - \int dx\ jA) \quad .$$

Green's functions obtained with $Z'_\tau(j)$ clearly differ from those obtained with $Z_\tau(j)$. This is a reflection of the fact that Green's functions are gauge dependent quantities. However, it is quite obvious that Z_τ and Z'_τ are completely equivalent as far as the computation of gauge invariant quantities is concerned. We will denote this equivalence by the symbol "$\approx$", for example: $Z_T \approx Z'_T$.

Let us then change gauge from $G(A)=0$ to $G'(A)=G(\Phi^{-1}(A))=0$ and make the RG change of variables (3.1). We obtain

$$Z_T(j) \approx \int DA \, \det\frac{\delta\Phi}{\delta A} \, \Delta_{G(\Phi^{-1})}(\Phi(A)) \, \delta(G(A)) \exp-(S_T(\Phi(A))-\int dx \, j\,\Phi(A)) \quad .(3.4)$$

In the Appendix 3.A it is shown that the Fadeev-Popov determinant for any linear gauge G is invariant under these successive changes of gauge and of variables,

$$\Delta_{G(\Phi^{-1})}(\Phi(A)) = \Delta_G(A) \quad . \qquad (3.5)$$

This is a non-trivial statement dependent on the particular form of Φ given by (3.1) and (2.2). Thus, (3.4) may be rewritten as

$$Z_T(j) \approx \int DA \, \Delta_G(A) \, \delta(G(A)) \exp-(S_{T+\delta T}(A)-\int dx \, j\,\Phi(A)) \quad , \qquad (3.6)$$

where

$$S_{T+\delta T}(A) = S_T(\Phi(A)) - \log\det\frac{\delta\Phi}{\delta A} \quad . \qquad (3.7)$$

Using (2.2) one sees that (3.7) is the RG Equation (2.3) we wished to obtain.

Finally, we would like to study how expectation values obtained using the RG-evolved actions in (3.2) and (3.6) are related to those calculated using the bare action in (2.1).

It is convenient to choose an axial gauge (which as discussed above implies no loss of generality). This simplifies dealing with $\Delta_G(A)$ which is in this case field independent and can now be taken outside the integral sign. Notice that from our point of view this property (the field independence of Δ_G

in the axial gauge)is useful only if it is preserved by the RG evolution. This is guaranteed by (3.5).

Then, using (2.2), (3.1) and also

$$0 = \int DA \left[j(x) + \frac{\delta S_\tau}{\delta A(x)} \right] \delta(G(A)) \exp -(S_\tau - \int dx\, jA) \ ,$$

Equation (3.6) may be rewritten as

$$Z_\tau(j) \approx Z_{\tau+\delta\tau}(j) - \frac{\delta\tau}{\Lambda_\tau^2} \left[\int dx\, j^2(x) \right] Z_\tau(j) \ ,$$

or equivalently as

$$Z_\tau(j) \approx \exp \left[\frac{\delta\tau}{\Lambda_\tau^2} \int dx\, j^2(x) \right] Z_{\tau-\delta\tau}(j) \ .$$

Iterating this expression leads to

$$Z_\tau(j) \approx \exp \left[\int_{\tau_0}^{\tau} \frac{d\tau}{\Lambda_\tau^2} \int dx\, j^2(x) \right] Z_{\tau_0}(j) \ .$$

Taking $S_{-\infty}=S$ or $Z_{-\infty}=Z$ as the initial condition one obtains

$$Z_\tau(j) \approx \exp \left[\frac{1}{2\Lambda_\tau^2} \int dx\, j^2(x) \right] Z(j) \ , \tag{3.8}$$

just as for the scalar theory (Section (2.4)).

The conclusion is that Equation (2.3) is a legitimate RG equation provided that $Z_\tau(j)$ be used to compute gauge invariant quantities only. This is not a serious limitation.

3.4. Constraints on the Vertex Functions Due to Gauge Invariance

The RG evolution expressed by (2.3) or (2.8) is such that provided we choose a gauge invariant initial condition (i.e., the bare action S) the invariance of the RG-evolved action S_τ automatically follows and we need not worry about it. However, it is useful, and as we shall see in the next section it may provide further insights, to inquire what information about the structure of the vertex functions can be obtained from the gauge symmetry alone.

The invariance of the action S_τ under the gauge transformation (2.5) is expressed by

$$D_\mu^{ab} \frac{\delta S_\tau}{\delta A_\mu^b(x)} = 0 \ , \tag{4.1}$$

or, in momentum space

$$iq_\mu \frac{\delta S_\tau}{\delta A_\mu^a(q)} = g(\tau) f^{abc} \int d\tilde{k}\, A_\mu^b(k-q) \frac{\delta S_\tau}{\delta A_\mu^c(k)} \ . \tag{4.2}$$

This may be transformed into a set of identities, a form of Ward identities, between the vertex functions. Substituting (2.9) into (4.2) and equating the coefficients of terms of the same degree in A, one obtains the well known identities which determine the longitudinal projections of an $(n+1)$-legged vertex in terms of the vertices with n legs,

$$iq_\mu w_{n+1}\begin{bmatrix} a & a_1 & \cdots & a_n \\ q & q_1 & \cdots & q_n \\ \mu & \mu_1 & \cdots & \mu_n \end{bmatrix} = g \sum_{j=1}^{n} f^{aa_jb}\, w_n \begin{bmatrix} a_1 & \cdots & a_{j-1} & b & \cdots & a_n \\ q_1 & \cdots & q_{j-1} & q_j+q & \cdots & q_n \\ \mu_1 & \cdots & \mu_{j-1} & \mu_j & \cdots & \mu_n \end{bmatrix} \ , \tag{4.3}$$

where $q + \sum_{j=1}^{n} q_j = 0$.

For $n=1$, (4.3) becomes

$$q_\mu w_2 \begin{bmatrix} a & a_1 \\ q & q_1 \\ \mu & \mu_1 \end{bmatrix} = 0 \ , \quad q + q_1 = 0 \ ,$$

and the tensor structure of w_2 is therefore completely determined

$$w_2\begin{bmatrix} a & b \\ q & -q \\ \mu & \nu \end{bmatrix} = \delta^{ab}\, P_{\mu\nu}(q)\, w_2(q) \quad , \tag{4.4}$$

where $P_{\mu\nu}(q) \equiv \delta_{\mu\nu} - \frac{q_\mu q_\nu}{q^2}$.

The statements made so far in this section are well known and have been included mainly for completeness and to establish the notation. Now we study other consequences of (4.3). All vertex functions can be decomposed into two parts

$$w_n = w_n^T + w_n^L \quad ,$$

the first of which, w_n^T, is transverse in all n legs, that is

$$w_n^T\begin{bmatrix} a_1 & \cdots & a_n \\ q_1 & \cdots & q_n \\ \mu_1 & \cdots & \mu_n \end{bmatrix} = P_{\mu_1\nu_1}(q_1) \cdots P_{\mu_n\nu_n}(q_n)\, w_n^T\begin{bmatrix} a_1 & \cdots & a_n \\ q_1 & \cdots & q_n \\ \nu_1 & \cdots & \nu_n \end{bmatrix} \quad . \tag{4.6}$$

This completely transverse component obviously makes no contribution to the left hand side of the Ward identities. The second component w_n^L contains terms which are longitudinal in at least one of the momenta entering through its legs. This is expressed by

$$P_{\mu_1\nu_1}(q_1) \cdots P_{\mu_n\nu_n}(q_n)\, w_n^L\begin{bmatrix} a_1 & \cdots & a_n \\ q_1 & \cdots & q_n \\ \nu_1 & \cdots & \nu_n \end{bmatrix} = 0 \quad . \tag{4.7}$$

Equation (4.7) is very useful because together with (4.3) it completely determines the (partially) longitudinal component w_n^L. It is easy to see how this works. On multiplying out the projection operators on the left one finds there is one term with zero momenta (just a product of Kronecker δ's), n terms with two momenta, then terms with four momenta and so on,

$$w_n^L\begin{bmatrix} a_1 & \cdots & a_n \\ q_1 & \cdots & q_n \\ \mu_1 & \cdots & \mu_n \end{bmatrix} = \left\{ \frac{(q_1)_{\mu_1}(q_1)_{\nu_1}}{q_1^2}\, w_n^L\begin{bmatrix} a_1 & \cdots & a_n \\ q_1 & \cdots & q_n \\ \nu_1 & \cdots & \mu_n \end{bmatrix} + \cdots \right\} +$$

$$+ \left\{ \text{terms with 4 q's} \right\} + \cdots \tag{4.8}$$

Now one eliminates all w_n's on the right using the Ward identities (4.3) and the desired unique expression for w_n^L in terms of $w_{n-1}, w_{n-2}, \ldots$ results.

Carrying out this procedure for $n=3$ (see Appendix 3.B) one obtains

$$w_3^L\begin{bmatrix} a & b & c \\ q & p & k \\ \mu & \nu & \sigma \end{bmatrix} = igf^{abc}\left\{\frac{q_\mu}{q^2}(w_2(p)P_{\nu\sigma}(p) - w_2(k)P_{\nu\sigma}(k)) + \right.$$

$$\left. + \frac{p_\nu k_\sigma}{p^2 k^2} w_2(q)\, k_\lambda P_{\lambda\mu}(q)\right\} + \left\{2 \text{ cyclic perm.}\right\} \tag{4.9}$$

Notice that w_3^L contains singularities when any of the momenta vanish. This is quite natural since the definition (4.7) of the w_n^L's contains projection operators and the property of being transverse to a momentum q becomes ill-defined as $q \to 0$. Therefore the singularities must necessarily occur.

In the past (see, e.g., Baker et al. [3] and references therein) it has been found useful to employ a trial expression for a vertex function which satisfies the Ward identities and is free of the kinematic singularities described above. Such trial vertex functions may be constructed by adding to the w_n^L calculated as above a completely transverse tensor carefully tailored to cancel the singularities (see Section 4.2 below). This transverse tensor is clearly not unique.

Following a method very different from the one we have described Kim and Baker [1] have obtained a trial form for the cubic gluon one-particle-irreducible vertex in both the axial and the covariant gauges.

The calculation of explicit expressions for the w_n^L's for $n>3$ is straightforward although quite laborious. It is quite fortunate that when solving the RG Equations (2.11) we will not need the full quartic vertex w_4 but only a highly contracted form

$$\widetilde{w}_4(q,p) \equiv w_4\begin{bmatrix} a & a & b & b \\ q & -q & p & -p \\ \mu & \mu & \nu & \nu \end{bmatrix} . \tag{4.10}$$

In the Appendix 3.B the longitudinal component of this contracted vertex is calculated to be

$$\widetilde{w}_4^L(q,p) = g^2N^2\Bigg\{\frac{(d-1)}{q^2}\Big[w_2(p+q)-2w_2(p)+w_2(p-q)\Big] +$$

$$+\frac{(d-1)}{p^2}\Big[w_2(p+q)-2w_2(q)+w_2(p-q)\Big] - \left[1-\frac{(p.q)^2}{p^2q^2}\right]\left[\frac{w_2(p+q)}{(p+q)^2}+\frac{w_2(p-q)}{(p-q)^2}\right]\Bigg\}.$$

(4.11)

It is interesting to note that while the Ward identities constrain w_4^L to be completely determined in terms of w_3 and w_2, the constraint on the contracted form $\widetilde{w}_4^L$ is far greater: $\widetilde{w}_4^L$ is completely determined by w_2 alone.

The gauge constraints studied above may be used to simplify the RG equations. For example, (4.4) shows that the quadratic vertex contains just one unknown scalar function. Then the RG equation (2.11) for $n=2$ may be considerably simplified without any loss of information by taking its trace, in which case one obtains

$$(\lambda\frac{d}{d\lambda}+2d_A-d)\, w_2(\lambda q) = \frac{1}{N(d-1)\Lambda^2}\int d\tilde{k}\, \widetilde{w}_4(\lambda q,k) - \frac{2}{\Lambda^2}w_2^2(\lambda q) \qquad (4.12)$$

Furthermore, for $n>2$ one needs to study only the RG evolution of the transverse components w_n^T (the longitudinal ones are determined by gauge invariance alone). Multiplying (2.11) by a string of projection operators $P_1,\ldots,P_n$, and rearranging one obtains (for $n\neq 2$)

$$(\lambda\frac{d}{d\lambda}+nd_A-d)w_n^T = \frac{1}{\Lambda^2}\int d\tilde{k}\left\{w_{n+2}^T + P_1\ldots P_n\frac{k_\mu k_\nu}{k^2}w_{n+2}\begin{bmatrix}\cdots & a & a\\ \cdots & k & -k\\ \cdots & \mu & \nu\end{bmatrix}\right\} +$$

$$\frac{1}{\Lambda^2}\sum_{m=2}^{n}\binom{n}{m-1}\frac{-1}{n!}\sum_{\{q_j\}}\left\{w_m^T w_{n-m+2}^T + P_1\ldots P_n\frac{k_{m\mu}k_{m\nu}}{k^2}w_m\begin{bmatrix}\cdots & a\\ \cdots & \lambda k_m\\ \cdots & \mu\end{bmatrix}w_{n-m+2}\begin{bmatrix}\cdots & a\\ \cdots & -\lambda k_m\\ \cdots & \nu\end{bmatrix}\right\}.$$

(4.13)

The second and fourth terms on the right may be rewritten further using the Ward identities and the multiplication by $P_1,...,P_n$ eliminates all longitudinal components. This means that the RG equation for w_n^T contains on the right hand side only the transverse components of other vertex functions.

Clearly, analogous manipulations may be carried out for the RG equations without dilatations (2.10), they are very similar and not worth repeating.

One final comment before closing this section. As far as the gauge invariance is concerned one may take all $w_n^T=0$ for $n>2$ (putting $w_2=0$ is certainly not interesting). Is this zero value preserved by the RG evolution ? In other words is $w_n^T(\tau)=0, n>2$ a solution of the RG equations (4.12)? The answer is no; for $n=4, m=3$ the last term in (4.12) is proportional to $w_2 w_2 \neq 0$ so that even if one takes $w_n^T=0$ as an initial condition, non-zero values will be generated by the RG evolution. Thus, the transverse components, undetermined by gauge invariance, are unavoidable.

3.5. Perturbative Solution of the RG Equations

We will look for solutions of the RG Equations (2.11) written in the form (4.12) and (4.13) which are expansions in the single coupling constant $g(\tau)$. Later we will consider the similar solutions to the RG equations without dilatations, (2.10).

In $d=4-\varepsilon$ spacetime dimensions, let

$$w_2 = W_{20} + g^2 W_{22} + \dots \ , \tag{5.1a}$$

$$w_3 = \Lambda^{\varepsilon/2}(g W_{31} + \cdots) \ , \tag{5.1b}$$

$$w_4 = \Lambda^{\varepsilon}(g^2 W_{42} + \cdots) \ , \tag{5.1c}$$

with the anomalous dimension and β function given by

$$\gamma_A = \gamma g^2 + \cdots \quad \text{and} \quad \beta(g) = -\lambda\frac{dg}{d\lambda} = -\frac{\varepsilon}{2}g + \gamma g^3 + \cdots \tag{5.2}$$

We have rescaled $g \to \Lambda^{\varepsilon/2} g$ and made explicit factors of $\Lambda^{\varepsilon/2}$ in (5.1) in order that g be dimensionless and that the W's may have the same dimensions they have at $d=4$.

The anomalous dimension γ_A and, as we shall see, the solution itself is determined by the requirement that all $\lambda=e^{\tau}$ dependence be contained in the single function $g(\lambda)$, i.e. that the W's be explicitly independent of λ.

Substituting (5.1) and (5.2) into (4.12) and (4.13) one obtains

$$(\lambda\frac{d}{d\lambda}-2)W_{20}(\lambda q) = -\frac{2}{\Lambda^2}W_{20}^2(\lambda q) \ , \tag{5.3}$$

$$(\lambda\frac{d}{d\lambda}-2+\varepsilon+\frac{4}{\Lambda^2}W_{20}(\lambda q))W_{22}(\lambda q) = -2\gamma W_{20}(\lambda q) + \frac{\int d\tilde{k}\, W_{42}(\lambda q, k)}{(d-1)N\Lambda^{2-\varepsilon}} \ , \tag{5.4}$$

$$(\lambda\frac{d}{d\lambda}-1+\frac{2}{\Lambda^2}\sum_{j=1}^{3}W_{20}(\lambda q_j))\,W_{31}^{T}\begin{bmatrix} a & b & c\\ \lambda q_1 & \lambda q_2 & \lambda q_3\\ \mu & \nu & \sigma\end{bmatrix}=0\ , \tag{5.5}$$

$$\left[\lambda\frac{d}{d\lambda}+\frac{4}{\Lambda^2}(W_{20}(\lambda q)+W_{20}(\lambda k))\right]\tilde{W}_{42}^{T}(\lambda q,\lambda k)=\frac{2}{\Lambda^2}\left\{W_{31}^{T}\left[\begin{matrix} a & b & c\\ -\lambda q & -\lambda k & \lambda(q+k)\\ \mu & \nu & \sigma\end{matrix}\right]\right\}^2+$$

$$-\frac{2N^2}{\Lambda^2\lambda^2(q+k)^2}\left[W_{20}(\lambda q)-W_{20}(\lambda k)\right]^2P(q).P(k)+(\text{same with } k\to -k)\ . \tag{5.6}$$

To solve these equations initial conditions are required. As $\lambda\to 0$ (i.e., $\tau\to-\infty$) we would like the action S_τ to reproduce the usual QCD bare action, therefore the initial conditions should be (remember that the q's are not the physical momenta but have been dilated)

$$\lim_{\lambda\to 0}\frac{1}{\lambda^2}W_{20}(\lambda q)=q^2\ , \tag{5.7}$$

$$\lim_{\lambda\to 0}\frac{1}{\lambda}W_{31}\begin{bmatrix} a & b & c\\ \lambda q & \lambda p & \lambda k\\ \mu & \nu & \sigma\end{bmatrix}=v_3\begin{bmatrix} a & b & c\\ q & p & k\\ \mu & \nu & \sigma\end{bmatrix}\equiv \tag{5.8}$$

$$\equiv i f^{abc}\left\{\delta_{\mu\nu}(q-p)_\sigma+\delta_{\nu\sigma}(p-k)_\mu+\delta_{\sigma\mu}(k-q)_\nu\right\}\ ,$$

while for W_{42} we only need the contracted form

$$\lim_{\lambda\to 0}\tilde{W}_{42}(\lambda q,\lambda k)=\tilde{v}_4(q,k)\equiv 2d(d-1)N^2\ . \tag{5.9}$$

It follows from the discussion of the previous section that once condition (5.7) is chosen the longitudinal components of (5.8-9) are automatically satisfied but one would be free to choose different transverse components. Instead of (5.8-9) let us then consider a slightly more general situation in which

$$\lim_{\lambda\to 0}\frac{1}{\lambda}W_{31}^{T}\begin{bmatrix} a & b & c\\ \lambda q & \lambda p & \lambda k\\ \mu & \nu & \sigma\end{bmatrix}=\xi v_3^{T}\begin{bmatrix} a & b & c\\ q & p & k\\ \mu & \nu & \sigma\end{bmatrix}\ , \tag{5.10}$$

$$\lim_{\lambda\to 0} \hat{W}^T_{42}(\lambda q, \lambda k) = \eta \tilde{v}^T_4(q, k) \tag{5.11}$$

which includes the usual QCD action as the particular case $\xi=\eta=1$.

Equation (5.3) is of the Bernoulli type. The solution satisfying (5.7) is

$$W_{20}(q) = q^2 f(q) \ , \tag{5.12}$$

where

$$f(q) = \frac{1}{1+q^2/\Lambda^2} = \exp -\frac{2}{\Lambda^2}\int_0^1 \frac{d\lambda}{\lambda} W_{20}(\lambda q) \ . \tag{5.13}$$

Equation (5.13) is very useful since it provides a way of constructing integrating factors for the linear Equations (5.4-6).

Solving (5.5) and (5.6) is a straightforward though somewhat laborious exercise. We obtain

$$W^T_{31}\begin{bmatrix} a & b & c \\ q & p & k \\ \mu & \nu & \sigma \end{bmatrix} = f(q)f(p)f(k)\xi v^T_3\begin{bmatrix} a & b & c \\ q & p & k \\ \mu & \nu & \sigma \end{bmatrix} \ . \tag{5.14}$$

and

$$\hat{W}^T_{42}(q,k) = f^2(q)f^2(k)\left[\eta\tilde{v}^T_4(q,k) + \frac{N^2}{\Lambda^2}\left\{f(q+k)\left[v^T_3\begin{bmatrix} q & k & -q-k \\ \mu & \nu & \sigma \end{bmatrix}\right]^2 + \right.\right.$$
$$\left.\left. -\frac{1}{(q+k)^2}(q^2-k^2)^2 P(q).P(k) + (k\to -k)\right\}\right] \ . \tag{5.15}$$

Notice that both W_{31} and W_{42} are explicitly independent of λ; so far all λ dependence in the action is contained in the single coupling $g(\lambda)$.

The solution of (5.4) is more interesting. Integrating from λ_0 to λ and rescaling the momenta back $\lambda q \to q$, the solution can be put in the form

$$W_{22}(q) = f^2(q)\left\{\left(\frac{\lambda}{\lambda_0}\right)^{2-\varepsilon} W_{22}\left(\frac{\lambda_0 q}{\lambda}\right) f^{-2}\left(\frac{\lambda_0 q}{\lambda}\right) + \right.$$

$$+\int_{\frac{\lambda_0}{\lambda}}^{1}\frac{d\mu}{\mu^{1-\varepsilon}}\left[\frac{-2\gamma q^2}{f(\mu q)}+\frac{I(0)+I'(0)\mu^2 q^2}{(d-1)N\mu^2 f^2(\mu q)}\right]+\int_{\frac{\lambda_0}{\lambda}}^{1}\frac{d\mu}{\mu^{1-\varepsilon}}\frac{I(\mu q)-I(0)-I'(0)\mu^2 q^2}{(d-1)N\mu^2 f^2(\mu q)}\Bigg\} \quad (5.16)$$

where

$$I(q)=\Lambda^{\varepsilon-2}\int d\tilde{k}\, W_{42}(q,k)=I(0)+I'(0)q^2+\cdots \quad (5.17)$$

Equation (5.16) shows an explicit dependence on λ which must be eliminated by taking the limit $\lambda_0\to 0$. The third term in (5.16) is not singular and offers no danger, but the first two diverge. In order that the limit of the first term exist we are required to choose

$$W_{22}\left(\frac{\lambda_0 q}{\lambda}\right)\approx C\Lambda^{\varepsilon}\left[\frac{\lambda_0 q}{\lambda}\right]^{2-\varepsilon}+\cdots \quad (5.18)$$

for small λ_0. The second term is more problematic, the integrand contains terms diverging as $\mu^{\varepsilon-3}$ and as $\mu^{\varepsilon-1}$ as $\mu\to 0$ and their cofficients must be made to vanish independently.

Consider first the $\mu^{\varepsilon-3}$ divergence. The calculation of $I(0)$ from (5.17) and (5.15) yields

$$I(0)=\frac{4N^2\Lambda^2\Gamma(3-d/2)}{(4\pi)^{d/2}}\frac{(d-1)^2}{d(d-2)(4-d)}\left[d^2(\eta-\xi^2)+d(2\xi^2-3\eta+1)+2(\eta-1)\right] \quad (5.19)$$

Choosing the parameters η and ξ appropriately $I(0)$ might vanish for any dimension d; we have to solve an overdetermined system of three equations in two unknowns. It is fortunate, first, that a solution exists,

$$\xi^2=\eta=1\ , \quad (5.20)$$

and second, that this solution includes the usual QCD bare action.

Apparently the $\mu^{\varepsilon-1}$ divergence is simple to deal with. One just has to choose an appropriate value for γ,

$$\gamma = \frac{I'(0)}{2(d-1)N} \tag{5.21}$$

There is however a potential problem. As Equation (5.19) already shows, $I(q)$ contains divergences as $d \to 4$. A finite value of γ can thus be chosen to cancel away the finite part of $I'(0)$, but then we have no more free parameters to eliminate the diverging $\frac{1}{\varepsilon}$ terms. These must vanish on their own. Fortunately (5.17) yields

$$I'(0) = \frac{N^2\Gamma(1+\varepsilon/2)}{(4\pi)^{2-\varepsilon/2}}\left[\frac{-18}{\varepsilon}(1+3\eta-4\xi^2) + \frac{1}{2}(-2+81\eta-121\xi^2) + O(\varepsilon)\right] \tag{5.22}$$

so that the same choice (5.20) solves this problem too. Equations (5.21) and (5.22) then imply

$$\gamma = -\frac{21}{6}\frac{N}{(4\pi)^2} + O(\varepsilon) \ , \tag{5.23}$$

or, using (2.6)

$$\beta(g) = -\frac{21}{6}\frac{N}{(4\pi)^2}g^3 + O(\varepsilon) \ . \tag{5.24}$$

This perturbative solution of the RG equations has thus yielded two surprises. One is the result (5.24). It differs from the one obtained employing conventional Feynman diagram methods which contains a factor $\frac{22}{6}$ instead of our $\frac{21}{6}$. We will return to this apparent discrepancy later.

The other surprise is that in the process of solution of the RG equations we have learnt not only how the action evolves but also what the possible initial conditions or bare actions are. In other words, the condition that all the RG evolution be contained in just one function $g(\tau)$ (this function being the same that appears in the gauge transformation (2.5)) is a very strong constraint on the possible bare actions. Perhaps the condition above is equivalent to the requirement that the theory be either renormalizable in the usual sense or that it be "asymptotically safe" (Weinberg [4]).

The initial condition (5.18) contains a free parameter C, and (5.20) allows two choices $\xi=\pm 1$. It is very likely that this remaining arbitrariness will be restricted further or altogether eliminated when pursuing this solution to higher orders (say computing W_{33}).

Before turning our attention to the problem of the missing 1/6 it is convenient to consider the perturbative solution of the RG equations without dilatations, Equations (2.10). Except for some details the algebra involved is essentially a repetition of the previous solution.

The natural expansion parameter is the coupling constant appearing in the covariant derivative. As shown in Section 2, in the absence of dilatations this coupling constant is not altered by the RG transformation and therefore stays firmly fixed at its initial value, g_0. Let

$$w_2 = U_{20} + g_0^2 U_{22} + \ldots, \quad w_3 = g_0 U_{31} + \ldots, \quad \text{and} \quad w_4 = g_0^2 U_{42} + \cdots \quad . \tag{5.25}$$

The U's now depend on $\lambda = e^{\tau}$ and satisfy equations analogous to (5.3-6) for the W's, namely

$$\lambda \frac{d}{d\lambda} U_{20}(q) = -2 \frac{\lambda^2}{\Lambda^2} U_{20}^2(q) \ , \tag{5.26a}$$

$$(\lambda \frac{d}{d\lambda} + 4 \frac{\lambda^2}{\Lambda^2} U_{20}(q)) U_{22}(q) = \frac{\lambda^2}{(d-1)N\Lambda^2} \int d\tilde{k}\ \tilde{U}_{42}(q,k) \ , \tag{5.26b}$$

$$(\lambda \frac{d}{d\lambda} + 2 \frac{\lambda^2}{\Lambda^2} \sum_{j=1}^{3} U_{20}(q_j))\, U_{31}^T \begin{bmatrix} a & b & c \\ q_1 & q_2 & q_3 \\ \mu & \nu & \sigma \end{bmatrix} = 0 \ , \tag{5.26c}$$

$$\left[\lambda \frac{d}{d\lambda} + 4 \frac{\lambda^2}{\Lambda^2} (U_{20}(q) + U_{20}(k))\right] \tilde{U}_{42}^T(q,k) = 2 \frac{\lambda^2}{\Lambda^2} \left\{ U_{31}^T \begin{bmatrix} a & b & c \\ -q & -k & (q+k) \\ \mu & \nu & \sigma \end{bmatrix} \right\}^2 +$$

$$- \frac{2N^2\lambda^2}{\Lambda^2(q+k)^2}\Big(U_{20}(q)-U_{20}(k)\Big)^2 P(q).P(k) + (\text{same with } k \to -k) \qquad (5.26d)$$

We will take the usual QCD bare action as the initial condition, $U_{20}(\lambda=0)=q^2$, $U_{31}(\lambda=0)=v_3$, and $\tilde{U}_{42}(\lambda=0)=\tilde{v}_4$. Integrating (5.26a,c and d) is straightforward and we obtain

$$U_{20}(q) = q^2 f_\lambda(q) \ . \qquad (5.27a)$$

$$U^T_{31}\begin{bmatrix} a & b & c \\ q & p & k \\ \mu & \nu & \sigma \end{bmatrix} = f_\lambda(q) f_\lambda(p) f_\lambda(k) v^T_3 \begin{bmatrix} a & b & c \\ q & p & k \\ \mu & \nu & \sigma \end{bmatrix} \ . \qquad (5.27b)$$

and

$$\tilde{U}^T_{42}(q,k) = f^2_\lambda(q) f^2_\lambda(k)\Bigg[\tilde{v}^T_4(q,k) + \frac{N^2\lambda^2}{\Lambda^2}\Bigg\{\Big[f_\lambda(q+k)\Big[v^T_3\begin{bmatrix} q & k & -q-k \\ \mu & \nu & \sigma \end{bmatrix}\Big]\Big]^2 +$$

$$- \frac{1}{(q+k)^2}(q^2-k^2)^2 P(q).P(k) + (k \to -k)\Bigg\}\Bigg] \ . \qquad (5.27c)$$

where now

$$f_\lambda(q) = \frac{1}{1+q^2\lambda^2/\Lambda^2} \ . \qquad (5.27d)$$

As before, solving (5.26b) is sightly more interesting. Let

$$I_\lambda(q) = \frac{\lambda^2}{\Lambda^2}\int d\tilde{k}\ \tilde{U}_{42}(q,k,\lambda) = I_\lambda(0) + I'_\lambda(0) q^2 + \ldots \qquad (5.28a)$$

The first two terms on the right can be shown to be precisely (5.19) and (5.22),

$$I_\lambda(0) = 0 \ , \ \text{and} \ \ I'_\lambda(0) = -\frac{21N^2}{(4\pi)^2} + O(\varepsilon) \ . \qquad (5.28b)$$

Integrating (5.26b) from λ_0 to λ and rearranging gives

$$U_{22}(q) = f^2_\lambda(q)\Bigg\{\Bigg[\frac{U_{22}(q,\lambda_0)}{f^2_\lambda(q,\lambda_0)} - \frac{I'_\lambda(0)q^2}{(d-1)N}\log\lambda_0\Bigg] +$$

$$+\left[\frac{I'_\lambda(0)q^2}{(d-1)N}\log\lambda+\frac{1}{(d-1)N}\int_{\lambda_0}^{\lambda}\frac{d\mu}{\mu}\left[\frac{I_\mu(q)}{f_\mu^2(q)}-I'_\lambda(0)q^2\right]\right]\Bigg\}$$

The term in the second square brackets is perfectly well behaved as $\lambda_0 \to 0$, and therefore so must the first term be. This first term may consistently be chosen to vanish. Thus,

$$U_{22}(q)=\frac{f_\lambda^2(q)}{(d-1)N}\left\{I'_\lambda(0)q^2\log\lambda+\int_0^\lambda\frac{d\mu}{\mu}\left[\frac{I_\mu(q)}{f_\mu^2(q)}-I'_\lambda(0)q^2\right]\right\}. \qquad (5.29)$$

Naturally, this solution yields the same anomalous dimension as before. If the quadratic vertex function is of the form

$$w_2(q,\lambda)=m^2(\lambda)+q^2z^{-1}(\lambda)+\ldots,$$

then the anomalous dimension is defined by

$$\gamma_A=\frac{\lambda}{2}\frac{d}{d\lambda}\log z^{-1}(\lambda)\ ,$$

and therefore (using (5.27a),(5.28b) and (5.29))

$$\gamma_A=-\frac{21}{6}\frac{N}{(4\pi)^2}g_0^2+O(\varepsilon)\ .$$

as expected.

3.6. The One-Loop Correction to the Effective Action

The solution obtained in the last section for the RG evolution of the coupling constant differs from the result evaluated following more conventional Feynman graph methods. We wish to show that there is no inconsistency. The β function we have computed describes the RG flow of the coupling constant appearing in the action S_τ while the corresponding β' function which is usually calculated describes the flow of the coupling constant appearing in the effective action Γ_τ. For scalar theories these two β functions coincide, but for non-Abelian gauge theories they do not.

To calculate the β' function it is most convenient to employ the background field method (see, e.g., Abbott [5] and references therein) which manifestly preserves the gauge invariance. One needs to compute only the vacuum polarization. The radiative corrections to the ghost propagator or to the cubic vertices are not needed. Let

$$Z_\tau(j,A) = \int D(Q\bar{\theta}\theta)\exp -\left[S_\tau(A+Q) + S_{GF} + S_G - \int jQ\right] \; . \tag{6.1}$$

where A and Q are the background and quantum fields respectively. The gauge-fixing term,

$$S_{GF} = \frac{1}{2a}\int d^d x \left[D_\mu^{ab}(A)Q_\mu^b\right]^2 \; . \tag{6.2}$$

involves the derivative $D_\mu(A)$ which is covariant with respect to the background field A. The ghost action is

$$S_G = -i\int d^d x\, \bar{\theta}^a D_\mu^{ab}(A) D_\mu^{bc}(A+Q)\theta^c \; . \tag{6.3}$$

The action S_τ is the RG-evolved action (2.9) with the vertex functions w_n given by Equations (5.25,27 and 29). The relevant Feynman rules and some details of the rather messy calculation of the vacuum polarization are left to the Appendix 3.C.

To calculate the field renormalization Z_A we only need the part of the vacuum polarization quadratic in the momenta which is

$$\Pi^{ab}_{\mu\nu}(q) = \frac{\Lambda_\tau^\varepsilon g^2 N}{(4\pi)^2}\,\delta^{ab}\,P_{\mu\nu}(q)\,q^2\left[\frac{21}{3}\log\lambda + \frac{1}{3}\frac{1}{\varepsilon}\right] + \ldots \tag{6.4}$$

where g is the dimensionless coupling constant ($g_0 = Z_g \Lambda_\tau^{\varepsilon/2} g$). Then

$$Z_A = 1 + \frac{g^2 N}{(4\pi)^2}\left[\frac{21}{3}\log\lambda + \frac{1}{3}\frac{1}{\varepsilon}\right] + \ldots \tag{6.5}$$

This is of the form

$$Z_A = 1 + \sum_{n=0}^{\infty} \frac{z_n(g,\lambda)}{\varepsilon^n} \quad . \tag{6.6}$$

One should note the presence of an $n=0$ term and also that the z_n's show an explicit λ dependence in addition to the λ dependence that is implicit in g. In this case, as shown in the Appendix 3.D, the β' function is given by

$$\beta'(g) = -\lambda\frac{dg}{d\lambda} = -\frac{\varepsilon}{2}g - \frac{\varepsilon}{4}g\frac{\partial z_0}{\partial g} - \frac{1}{4}g^2\frac{\partial z_1}{\partial g} - \frac{1}{2}g\lambda\frac{\partial z_0}{\partial \lambda} \,. \tag{6.7}$$

The first and third terms are the conventional ones ('t Hooft [6]), the second and fourth are new.

The use of Equation (6.7) for the theory described by S_τ in Equation (6.1) might be considered questionable since S_τ does not look renormalizable. One should remember, however, that S_τ is completely equivalent to the usual renormalizable QCD action (since it is obtained by mere changes of variables) and therefore must also be renormalizable, although certainly not manifestly so.

The Equations (6.5-7) then imply (as $\varepsilon \to 0$) the desired result

$$\beta'(g) = -\frac{22}{6}\frac{g^3 N}{(4\pi)^2} + \ldots$$

The result (6.4) is interesting in two ways. First it shows that λ acts as regulator which eliminates only some of the divergences; this is the reason why β and β' differ. Second, the first term in (6.4) comes from the quadratic vertex U_{22}, it is a tree graph and not a loop correction. It is the second term which is the correction due to loop graphs containing ghosts, cubic and quartic vertices. In the conventional QCD variables the loop correction is much larger (it has a factor $\frac{22}{3}$ instead of just $\frac{1}{3}$). This is in agreement with the idea behind this kind of RG, namely a saddle point calculation performed in the new variables is more accurate.

One last comment is in order. In the scalar theory of Chapter 2 the sequence of RG changes of variables was such that a saddle point approximation would become asymptotically exact (i.e., the loop corrections vanish as $\tau \to \infty$). This is not the case for the non-Abelian gauge theory although there is a vast improvement.

Appendix 3.A. A Theorem Concerning the RG Evolution of $\Delta_G(A)$

We wish to prove that the Fadeev-Popov determinant is invariant under the RG transformation (2.2) and the change of gauge described in Section 3.3. Recall that (see, e.g., Fadeev and Slavnov [2])

$$\Delta_G(A) = \Big[\det M\Big]_{G=0} \ ,$$

where M is the derivative of the gauge condition with respect to a gauge transformation, that is $\dfrac{\delta G(A_\Omega)}{\delta\Omega}$. Using the chain rule this is

$$M^{ab}(x,y) = G_\mu^{ac}(x)\, D_\mu^{cb}(A,x)\, \delta(x-y) \ ,$$

where

$$G_\mu^{ac}(x) = \frac{\partial G^a(A(x))}{\partial A_\mu^c(x)}$$

is independent of A for linear gauge conditions.

The determinant we wish to evaluate is

$$\Delta_{G(\Phi^{-1})}(\Phi(A)) = \Big[\det M'\Big]_{G(\Phi^{-1})=0} \ , \quad M' = \frac{\delta}{\delta\Omega} G(\Phi^{-1}(\Phi_\Omega(A))) \ .$$

Let us first consider

$$\frac{\delta}{\delta\Omega} G(\Phi^{-1}(A_\Omega)) = -\, G_\nu^{ac}(x) D_\mu^{bd}(A,y) \frac{\delta(\Phi^{-1}(x))_\nu^c}{\delta A_\mu^d(y)} \ ,$$

which for Φ given by (3.1) and (2.2) becomes

$$\frac{\delta}{\delta\Omega} G(\Phi^{-1}(A_\Omega)) = G_\mu^{ac}(x)\, D_\mu^{cb}(A,x)\, \delta(x-y) - \frac{\delta\tau}{\Lambda_\tau^2}\, G_\nu^{ac}(x)\, D_\mu^{bd}(A,y) \frac{\delta^2 S_\tau}{\delta A_\mu^d(y) \delta A_\nu^c(x)} \ .$$

M' is obtained by evaluating this expression at $\Phi(A)$ instead of A:

$$M'^{ab}(x,y) = M^{ab}(x,y) + G_\mu^{ac}(x) \frac{\delta}{\delta A_\mu^c(x)} \left[D_\nu^{bd}(A,y) \frac{\delta S_\tau}{\delta A_\nu^d(y)} \right] \ .$$

But gauge invariance, expressed by (4.1), implies that the second term vanishes, therefore

$$\Delta_{G(\Phi^{-1})}(\Phi(A)) = \Delta_G(A) \quad ,$$

as desired. Notice that the particular form of Φ was essential; the Fadeev-Popov determinant is not invariant under transformations with arbitrary Φ.

Appendix 3.B. The Longitudinal Cubic and Contracted Quartic Vertex Functions

Let $$(L_1)_{\mu_1\nu_1} = \frac{(q_1)_{\mu_1}(q_1)_{\nu_1}}{q_1^2} \quad .$$

Then for $n=3$, Equation (4.8) reads

$$w_3^L[\,123\,] = \left\{(L_1+L_2+L_3) - (L_1L_2+L_1L_3+L_2L_3) + L_1L_2L_3\right\} w_3[\,123\,] \quad , \qquad \text{(B.1)}$$

where we have dropped the superscript L on the right.

The Ward identity (4.3) for the cubic vertex

$$q_\mu w_3\begin{bmatrix} a & b & c \\ q & p & k \\ \mu & \nu & \sigma \end{bmatrix} = i\,g\,f^{abc}\,(w_2(p)\,P_{\nu\sigma}(p) - w_2(k)\,P_{\nu\sigma}(k)) \quad , \qquad \text{(B.2a)}$$

implies

$$q_\mu p_\nu w_3\begin{bmatrix} a & b & c \\ q & p & k \\ \mu & \nu & \sigma \end{bmatrix} = -i\,g\,f^{abc}\,w_2(k)\,p_\nu P_{\nu\sigma}(k) \quad , \qquad \text{(B.2b)}$$

and

$$q_\mu p_\nu k_\sigma w_3\begin{bmatrix} a & b & c \\ q & p & k \\ \mu & \nu & \sigma \end{bmatrix} = 0 \quad . \qquad \text{(B.2c)}$$

Substituting (B.2) into (B.1) the expression (4.9) for w_3^L follows immediately.

The same procedure may be repeated for w_4^L, but we will need only the contracted form $\widetilde{w}_4^L$ which is much simpler to obtain. The contracted form of (4.8) for $n=4$ is

$$\widetilde{w}_4^L(q,p) = \left\{ \frac{q_\mu q_\nu}{q^2}\delta_{\sigma\rho} + \frac{p_\sigma p_\rho}{p^2}\delta_{\mu\nu} + \frac{q_\mu q_\nu p_\sigma p_\rho}{q^2 p^2} \right\} w_4 \begin{bmatrix} a & a & b & b \\ q & -q & p & -p \\ \mu & \nu & \sigma & \rho \end{bmatrix} . \tag{B.3}$$

The Ward identity (4.3) for the quartic vertex is

$$iq_\mu w_4 \begin{bmatrix} a & b & c & d \\ q & k & p & r \\ \mu & \nu & \sigma & \rho \end{bmatrix} = gf^{abe}\, w_3 \begin{bmatrix} e & c & d \\ k+q & p & r \\ \nu & \sigma & \rho \end{bmatrix} +$$

$$+ gf^{ace}\, w_3 \begin{bmatrix} b & e & d \\ k & p+q & r \\ \nu & \sigma & \rho \end{bmatrix} + gf^{ade}\, w_3 \begin{bmatrix} b & c & e \\ k & p & r+q \\ \nu & \sigma & \rho \end{bmatrix} , \tag{B.4a}$$

which implies

$$q_\mu q_\nu w_4 \begin{bmatrix} a & a & b & b \\ q & -q & p & -p \\ \mu & \nu & \sigma & \rho \end{bmatrix} =$$

$$= g^2 N^2 \left\{ w_2(p+q)\, P_{\sigma\rho}(p+q) - 2w_2(p)\, P_{\sigma\rho}(p) + w_2(p-q)\, P_{\sigma\rho}(p-q) \right\} , \tag{B.4b}$$

and

$$q_\mu q_\nu p_\sigma p_\rho w_4 \begin{bmatrix} a & a & b & b \\ q & -q & p & -p \\ \mu & \nu & \sigma & \rho \end{bmatrix} = g^2 N^2 (q^2 p^2 - (p.q)^2) \left[\frac{w_2(p+q)}{(p+q)^2} + \frac{w_2(p-q)}{(p-q)^2} \right] . \tag{B.4c}$$

Notice that w_3, while present in (B.4a), does not appear in (B.4b,c).

Substituting (B.4b) and (B.4c) into (B.3) Equation (4.11) for $\widetilde{w}_4^L$ follows.

Appendix 3.C. The One-Loop Vacuum Polarization

The Euclidean spacetime Feynman rules we will need for the calculation of the field renormalization Z_A are as follows. The ghost propagator is $\frac{-i\delta^{ab}}{k^2}$ and the cubic ghost-ghost-field vertex corresponding to the term

$$A_\mu^a(q)\bar{\theta}^b(k)\theta^c(p)\delta(q+p-k) \quad \text{is} \quad f^{abc}(p_\mu + k_\mu) .$$

The gluon $Q-Q$ propagator is

$$\frac{\delta^{ab} P_{\mu\nu}(q)}{U_{20}(q)} + \frac{\delta^{ab} q_\mu q_\nu}{a\, q^4} \qquad \text{(C.1)}$$

We will need the quadratic $A-A$ vertex

$$-g^2 \Lambda_\tau^\varepsilon\, U_{22}(q) \delta^{ab} P_{\mu\nu}(q) \ , \qquad \text{(C.2)}$$

the cubic $A^a_\mu(q) Q^b_\nu(p) Q^c_\sigma(k)$ vertex,

$$-g \Lambda_\tau^{\varepsilon/2}\, U_{31} \begin{bmatrix} a & b & c \\ q & p & k \\ \mu & \nu & \sigma \end{bmatrix} - \frac{i}{a} g \Lambda_\tau^{\varepsilon/2} f^{abc} \left(k_\sigma \delta_{\mu\nu} - p_\nu \delta_{\mu\sigma} \right) \ , \qquad \text{(C.3)}$$

and the quartic $A^a_\mu(q) A^b_\nu(k) Q^c_\rho(p) Q^d_\sigma(r)$ vertex,

$$-g^2 \Lambda_\tau^\varepsilon\, U_{42} \begin{bmatrix} a & b & c & d \\ q & k & p & r \\ \mu & \nu & \rho & \sigma \end{bmatrix} - \frac{g^2 \Lambda_\tau^\varepsilon}{a} \left[f^{acx} f^{bdx} \delta_{\mu\sigma} \delta_{\nu\rho} + f^{adx} f^{bcx} \delta_{\mu\rho} \delta_{\nu\sigma} \right] \ . \qquad \text{(C.4)}$$

It is convenient to calculate in a Landau-like gauge $a=0$. We need to compute just three one-loop diagrams, the usual graph with one ghost loop, a tadpole graph containing the quartic vertex, and the loop graph with two cubic vertices. The ghost loop diagram is precisely the same that occurs in QCD,

$$\Pi_G = \frac{g^2 \Lambda_\tau^\varepsilon N}{(4\pi)^2} \delta^{ab}\, P_{\mu\nu}(q)\, q^2 \frac{2}{3\varepsilon} + \ldots \qquad \text{(C.5)}$$

The tadpole diagram containing the quartic vertex does not, as is usually the case, vanish in dimensional regularization because of the structure of the gluon propagator (C.1). The vertex U_{42} in (C.4) is not completely known, we only know its contracted form $\bar{U}_{42}$. Fortunately that is all that is needed. The rules (C.3) and (C.4) are complicated. To simplify the algebra, and since this is enough to obtain the field renormalization Z_A, we compute just the term quadratic in the momenta q. The longitudinal components of the tadpoles and the cubic vertex contribution cancel out giving

$$\Pi_3 + \Pi_4 = \frac{g^2 \Lambda_T^\varepsilon N}{(4\pi)^2} \delta^{ab} P_{\mu\nu}(q) q^2 \left[\frac{-1}{3\varepsilon}\right] + \ldots \tag{C.6}$$

Finally, the rule (C.2) with (5.29) gives a tree graph contribution

$$\Pi_2 = \frac{g^2 \Lambda_T^\varepsilon N}{(4\pi)^2} \delta^{ab} P_{\mu\nu}(q) q^2 \left[-I'_\lambda(0) \log \lambda\right] \tag{C.7}$$

Summing (C.5,6 and 7) gives (6.4).

Appendix 3.D. Calculation of the β' function

In the background field gauge, the coupling renormalization constant Z_g,

$$g_0 = Z_g \Lambda_T^{\varepsilon/2} g \quad ,$$

is related to the background field renormalization constant Z_A, ($A_0 = Z_A^{1/2} \Lambda_T^{-\varepsilon/2} A$) by (*)

$$Z_g = Z_A^{-1/2} \quad . \tag{D.1}$$

Let

$$\beta'(g) = \Lambda_T \frac{d}{d\Lambda_T} g = -\lambda \frac{d}{d\lambda} g \quad , \quad \Lambda_T = \frac{\Lambda}{\lambda} \quad .$$

Then since g_0 does not depend on Λ_T,

$$\Lambda_T \frac{d}{d\Lambda_T} g_0 = 0 = Z_g \Lambda_T^{\varepsilon/2} \left\{ \frac{\varepsilon}{2} g + \beta'(g) + g \frac{\Lambda_T}{Z_g} \frac{dZ_g}{d\Lambda_T} \right\} \quad ,$$

and therefore

$$\beta'(g) = -\frac{\varepsilon}{2} g - \frac{g}{2} \lambda \frac{d}{d\lambda} \log Z_A \quad .$$

(*) This is an unconventional field renormalization which is convenient because it leads to (D.1). Abbott [5] chooses $g_0 = Z_g g$ and $A_0 = Z_A^{1/2} A$ which also leads to (D.1) but involves a dimensionful coupling.

We can use the chain rule $\lambda\frac{d}{d\lambda}=\lambda\frac{\partial}{\partial\lambda}-\beta'(g)\frac{\partial}{\partial g}$ so that

$$\beta'(g) = -\frac{\varepsilon}{2}g - \frac{g}{2}\lambda\frac{\partial}{\partial\lambda}\log Z_A + \frac{g}{2}\beta'(g)\frac{\partial}{\partial g}\log Z_A \ .$$

We have followed closely the standard treatment ('t Hooft [6]), except that now, due to the explicit λ dependence in Z_A there is the additional second term. Solving for β'

$$\beta'(g) = \left[-\frac{\varepsilon}{2}g - \frac{g}{2}\lambda\frac{\partial}{\partial\lambda}\log Z_A\right]\left[1 + \frac{g}{2}\frac{\partial}{\partial g}\log Z_A + \ldots\right] \ .$$

and substituting for Z_A using (6.6), Equation (6.7) follows. As in the standard formalism the coefficients of the poles in ε should identically vanish furnishing a set of constraints on the z_n's.

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Chapter 4. Conclusions and Further Developments

4.1. Conclusions

In this work an approach to the Renormalization Group has been developed in which the RG transformations are convenient changes of variables. The main conclusions of our work are enumerated below.

(1) A class of exact infinitesimal RG transformations has been proposed. The form of the transformations is suggested quite naturally after several known exact RG's are formulated in a conveniently simplified way. Conversely, those exact RG's can be treated as special cases of a more general formalism.

(2) The transformations are pure changes of variables (i.e., no explicit integration or elimination of some degrees of freedom is required) such that a saddle point approximation is more accurate, becoming, in some cases, asymptotically exact as the transformations are iterated.

(3) Solutions of the RG equations for a scalar field theory were obtained both as an expansion in $\varepsilon=d-4$ and as an expansion in a single coupling constant. Physically significant results agree with those obtained following conventional methods. The well-known fact that physical quantities (such as critical exponents) are independent of the particular RG employed emerges quite clearly.

The consideration of RG's from this generalized point of view has a number of attractive features which immediately suggest many possible applications. We concentrated our attention on the application of this kind of RG to Yang-Mills theories.

(4) An exact gauge covariant RG transformation has been proposed. The corresponding RG equations have been obtained and solved in the weak coupling regime. By taking the subtleties of gauge fixing into account it was shown that the gauge-invariant RG-evolved action is equivalent to the usual action as far as the computation of gauge invariant quantities is concerned.

(5) The requirement that the RG evolution of the action be described by a single coupling constant restricts the choice of the actions to be used as possible initial conditions. It is plausible that this restriction is equivalent to the usual requirement of renormalizability. The conventional QCD action is included among such possible initial conditions.

(6) The β and the β' functions describing the evolution of the coupling constants in the action S_τ and in the effective action Γ are almost equal but quite definitely different; β is the tree approximation to β'.

(7) The small difference between β and β' is evidence that for the Yang-Mills theory the iteration of the RG transformation improves substantially a saddle point approximation but that the approximation never becomes exact.

(8) Although somewhat outside the line of argument of this work, it is convenient as a technical aid in solving the RG equations to study the constraints imposed by gauge invariance on the vertex functions. A new method to do this was devised.

A number of other applications of this RG formalism can be envisaged. For example, the method could be extended to any problem where a saddle point approximation is used, one could perhaps obtain improved large N expansions (Halpern [1]).

The role of dilatations is deemphasized and one might profitably attack problems where the issue is not the symmetry under scale transformations or its breaking. It should be possible to study the phenomena of dynamical symmetry breaking or of dynamical symmetry restoration. The localization of the minima of the classical action S and of the RG-evolved action S_τ need not coincide and it is the latter that will give more reliable information about the true minima. Another related possible application could be in the study of the so-called anomalies. Again, the true symmetry of a quantum theory could be more reliably established by classically studying the RG-evolved action S_τ, which includes some quantum effects, than by classically studying the action S.

In the remaining sections of this chapter we discuss in a rather preliminary form how one could begin the study of some other applications which we feel might be interesting.

We mentioned in Chapter 1 the possibility of obtaining non-perturbative solutions to the RG equations of the Yang-Mills theory by truncating them in a manner compatible with the gauge symmetry. The longitudinal components of the vertex functions were obtained in Section 3.4, in addition we need trial forms for the transverse components. These are obtained in Section 4.2.

The RG's discussed do not require the successive elimination of degrees of freedom and can therefore be applied to systems with a small number or even just one degree of freedom (see Appendix 2.B). In Section 4.3 the exact RG transformations are applied to a scalar field theory defined on a lattice (which can be finite) and the corresponding RG equations are obtained.

Finally, in Section 4.4 a lattice gauge theory is proposed for which the RG formalism described in this work can be applied in a very straightforward manner. The theory is formulated in terms of the field variables A_μ instead of the more usual rotation matrices U_μ (for an account of the latter formalism see, e.g., Kogut [2] and references therein). The Ward identities and RG equations for this lattice theory are very similar to those corresponding to the continuum Yang-Mills theory. We feel the exact RG transformations of this work might represent an improvement over those approximate ones proposed by Migdal [3] and Kadanoff [4] especially for small lattices.

4.2 Trial Forms for the Transverse Vertex Functions

As we have mentioned previously, it should be possible to obtain non-perturbative solutions of the QCD RG equations by truncating them in a form which is compatible with the gauge symmetry. This requires that we use trial forms for the vertex functions which satisfy the Ward identities.

For example, consider the RG equation for w_2, Equation (3;4.12). The right hand side of this equation contains the contracted quartic vertex $\tilde{w}_4$, the longitudinal component of which, $\tilde{w}_4^L$, is completely determined by the gauge symmetry (Equation (3;4.12)). Were the transverse component, $\tilde{w}_4^T$, also known, one would be able to solve exactly for w_2. Since $\tilde{w}_4^T$ is not known an alternative procedure is to guess reasonable trial forms for $\tilde{w}_4^T$ and study their implications for w_2.

As far as the gauge symmetry is concerned the transverse vertex functions are completely arbitrary,

$$w_n^T\begin{bmatrix} a_1 \cdots & a_n \\ q_1 \cdots & q_n \\ \mu_1 \cdots & \mu_n \end{bmatrix} = P_{\mu_1\nu_1}(q_1) \cdots P_{\mu_n\nu_n}(q_n)\ V_n\begin{bmatrix} a_1 \cdots & a_n \\ q_1 \cdots & q_n \\ \mu_1 \cdots & \mu_n \end{bmatrix} ,$$

where V_n is some unknown function with the appropriate Bose symmetry. There are, however, some constraints which can reasonably be imposed on w_n^T. First, one would like to study RG trajectories that lead as $\lambda \to 0$ to the conventional QCD action. In fact, in Section 3.5 we saw that other RG trajectories do not necessarily yield consistent results.

A second constraint is the requirement that w_n be free from kinematic singularities. As discussed in Section 3.4, the property of being transverse to a momentum q necessarily introduces singularities as $q \to 0$. It is these singularities which we wish to avoid for the following reasons: (a) the singularities are absent in the conventional QCD action (the initial condition for the RG evolution), (b) the RG evolution along a trajectory is smooth (singularities are absent from the action or hamiltonian and they appear only on integrating to obtain the partition function), and (c) for mathematical convenience.

Let us first consider the transverse cubic vertex. It is not difficult to see that the kinematic singularities are avoided if we require w_3 to satisfy the "low energy theorem"

$$w_3\begin{bmatrix} a & b & c \\ 0 & p & -p \\ \mu & \nu & \sigma \end{bmatrix} = -igf^{abc}\frac{\partial}{\partial p_\mu}\left[w_2(p)P_{\nu\sigma}(p)\right] ,$$

obtained directly from the Ward identity for w_3. Extracting the transverse part leads to

$$V_3\begin{bmatrix} a & b & c \\ 0 & p & -p \\ \mu & \nu & \sigma \end{bmatrix} = -igf^{abc}\frac{\partial w_2(p)}{\partial p_\mu}\delta_{\nu\sigma} + (\text{longitudinal terms}) .$$

The other constraint on V_3 is that when w_2 tends to q^2, V_3 must reproduce the conventional QCD cubic vertex. A trial form which satisfies these requirements is

$$V_3\begin{bmatrix} a & b & c \\ q & p & k \\ \mu & \nu & \sigma \end{bmatrix} = -igf^{abc}\left\{\delta_{\mu\nu}(q_\sigma-p_\sigma)\frac{w_2(k)}{k^2}\frac{w_2(q)-w_2(p)}{q^2-p^2} + (\text{cyclic permutations})\right\} .$$

In a similar way a trial form for the transverse quartic vertex may be obtained. The appropriate "low energy theorem" to be satisfied is

$$w_4\begin{bmatrix} a & a & b & a \\ 0 & 0 & p & -p \\ \mu & \nu & \sigma & \rho \end{bmatrix} = N^2g^2\frac{\partial^2}{\partial p_\mu \partial p_\nu}\left[w_2(p)P_{\sigma\rho}(p)\right] .$$

One possible trial expression which reproduces the usual quartic vertex (in the contracted form used in Chapter 3) as $w_2(q)\to q^2$ is

$$\tilde{w}_4(q,p) = N^2g^2P_{\mu\nu}(q)P_{\sigma\rho}(p)\left\{2\delta_{\mu\nu}\delta_{\sigma\rho}\frac{w_2(p+q)-w_2(p-q)}{(p+q)^2-(p-q)^2} + 4p_\mu p_\nu\delta_{\sigma\rho}\frac{\partial^2 w_2(p)}{\partial(p^2)^2} + \right.$$

$$+ \, 4q_\sigma q_\rho \delta_{\mu\nu} \frac{\partial^2 w_2(q)}{\partial (q^2)^2} - \frac{1}{2}\left[\frac{w_2(p+q)}{(p+q)^2} + \frac{w_2(p-q)}{(p-q)^2}\right](\delta_{\mu\rho}\delta_{\nu\sigma} + \delta_{\mu\sigma}\delta_{\nu\rho})\Bigg\}$$

It is important to emphasize that these trial forms are by no means unique and that perhaps more convenient ones can be found.

4.3. RG Equations for a Scalar Field Theory on a Lattice

The RG formalism described in this thesis can be applied in a straightforward manner to field theories defined on a lattice. The only difficulty is to develop an appropriate notation. This is done below.

At each site R of an infinite Bravais lattice let there be a field variable $\varphi(R)$ the dynamics of which is described by a hamiltonian (into which a factor of β has been absorbed) of the general form

$$H_T = \sum_{n=1}^{\infty} \frac{1}{n!} \sum_{R_1 \ldots R_n} u_n(R_1 \ldots R_n) \varphi(R_1) \ldots \varphi(R_n) \ . \tag{3.1}$$

The formalism is closest to the continuum field theory if one works in momentum space. Let

$$\varphi(R) = \int_{1BZ} d\tilde{q}\, \varphi(q) \exp -iq.R \ , \ d\tilde{q} \equiv \frac{v d^d q}{(2\pi)^d} \ . \tag{3.2}$$

The integral is over the first Brillouin zone (1BZ) and v is the volume of the d-dimensional unit cell. Outside the 1BZ the field is defined to be periodic, $\varphi(q)=\varphi(q+Q)$, where Q is any reciprocal lattice vector ($e^{iQ.R}=1$ for all R and Q). Then the hamiltonian can be written as

$$H_T = \sum_{n=1}^{\infty} \frac{1}{n!} \int_{1BZ} d\tilde{q}_1 \ldots d\tilde{q}_n\, u_n(q_1 \ldots q_n) \varphi(q_1) \ldots \varphi(q_n) \ , \tag{3.3}$$

where

$$u_n(q_1 \cdots q_n) = \sum_{R_1 \ldots R_n} u_n(R_1 \ldots R_N) \exp -i \sum_{j=1}^{n} q_j R_j \ . \tag{3.4}$$

It is easy to see that the symmetry of the $u_n(R_1 \ldots R_n)$ under translations by a lattice vector R implies

$$u_n(q_1 \cdots q_n) = \sum_Q u_{n,Q}(q_1 \cdots q_n) \tilde{\delta}(\sum_{j=1}^{n} q_j + Q) \ , \ \tilde{\delta}(q) \equiv \frac{(2\pi)^d}{v} \delta^d(q) \ , \tag{3.5a}$$

$$u_{n,Q+Q'}(q_1 \cdots q_n) = u_{n,Q}(q_1 \cdots q_j + Q' \ldots q_n) \ . \tag{3.5b}$$

By analogy to Equation (2;3.3) the RG transformation will be

$$\varphi(q) \to \varphi(q) - \delta\tau \frac{(2\pi)^d}{v} \dot{\alpha}(q,\tau) \frac{\delta H_\tau(\varphi)}{\delta\varphi(-q)} . \qquad (3.6)$$

If we want to preserve the periodicity $\varphi(q)=\varphi(q+Q)$ under RG transformations we have to integrate out $\varphi(q)$ and $\varphi(q+Q)$ to the same extent and therefore the cutoff function $\dot{\alpha}$ must also be periodic, $\dot{\alpha}(q)=\dot{\alpha}(q+Q)$.

The transformation (3.6) leads to the RG equation

$$\frac{dH_\tau}{d\tau} = \int_{1BZ} d^d q \frac{(2\pi)^d}{v} \dot{\alpha}(q,\tau) \left[\frac{\delta^2 H_\tau}{\delta\varphi(q)\delta\varphi(-q)} - \frac{\delta H_\tau}{\delta\varphi(q)} \frac{\delta H_\tau}{\delta\varphi(-q)} \right] , \qquad (3.7)$$

or, equivalently, to the following system of equations for the $u_{n,Q}$'s :

$$\frac{d}{d\tau} u_{n,Q}(q_1 \ldots q_n) = \int_{1BZ} d\tilde{k}\, \dot{\alpha}(k) u_{n+2,Q}(q_1 \ldots q_n, k, -k) +$$

$$- \sum_{m\geq 2}^{n} \frac{1}{(m-1)!(n-m+1)!} \sum_{\{q_j\}} \dot{\alpha}(k_m) u_{m,0}(q_1 \ldots q_{m-1}, k_m) u_{n-m+2,0}(q_m \ldots q_n, k'_m) \qquad (3.8)$$

where

$$k_m = -\sum_{j=1}^{m-1} q_j \ , \ k'_m = -\sum_{j=m}^{n} q_j \quad \text{and} \quad k_m + k'_m = Q \ .$$

As before, $\sum_{\{q_j\}}$ denotes a sum over all permutations of the q_j's.

Notice that the last term of (3.8) involves only u's corresponding to Q=0. This implies that even if we set $u_{n,Q}$=0 for $Q\neq 0$ as an initial condition, terms with $Q\neq 0$ will be generated by the RG evolution.

The RG equations for a finite lattice can be developed in an analogous way. These equations are obtained from those above by restricting all momenta q to only those discrete values which are compatible with periodic boundary conditions and by replacing all integrals over the 1BZ by the appropriate sums.

4.4. RG Equations for a Lattice Gauge Theory

While it might be possible to study the RG of a lattice gauge theory formulated in terms of the usual rotation matrix variables U_μ (see, e.g., Kogut [2]), in order to apply the RG techniques described above it is more convenient to employ the field variables A_μ. Below we propose a lattice gauge theory of this kind.

At each site R of a Bravais lattice let there be variables $A_\mu^a(R)$ which have a Fourier expansion of the form

$$A_\mu^a(R) = \int_{1BZ} d\tilde{q}\, A_\mu^a(q) \exp -iqR \quad . \tag{4.1}$$

The integral is over the first Brillouin zone (1BZ), and $A_\mu^a(q)$ is defined to be periodic, i.e. $A_\mu^a(q+Q)=A_\mu^a(q)$ where Q is any reciprocal lattice vector. The notation $d\tilde{q}$ means $v d^d q/(2\pi)^d$, where v is the volume of the unit cell and similarly $\tilde{\delta}(q)=(2\pi)^d \delta^d(q)/v$.

We will consider actions of the general form

$$S_\tau(A) = \sum_{n=1}^{\infty} \frac{1}{n!} \sum_Q \int_{1BZ} d\tilde{q}_1 \ldots d\tilde{q}_n \tilde{\delta}(Q+\sum_{j=1}^{n} q_j) w_{n,Q} \begin{bmatrix} a_1 & \cdots & a_n \\ q_1 & \cdots & q_n \\ \mu_1 & \cdots & \mu_n \end{bmatrix} A_{\mu_1}{}^{a_1}(q_1) \ldots A_{\mu_n}{}^{a_n}(q_n). \tag{4.2}$$

The translational symmetry of the Bravais lattice implies

$$w_{n,Q}\left[q_1, \ldots, q_n + Q'\right] = w_{n,Q+Q'}\left[q_1, \ldots, q_n\right] \quad . \tag{4.3}$$

To replace the continuum spacetime gauge transformation (3;2.5) we need an analogue transformation involving a function $\Omega^a(R)$ defined only at the lattice sites,

$$\Omega^a(R) = \int_{1BZ} d\tilde{q}\, \Omega^a(q) \exp -iqR \quad , \quad \Omega^a(q+Q) = \Omega^a(q) \quad .$$

An appropriate infinitesimal lattice gauge transformation is easily written in momentum space,

$$\delta A^a_\mu(q) = -iq_\mu \Omega^a(q) - g f^{abc} \int_{1BZ} d\tilde{k}\, A^b_\mu(q-k)\Omega^c(k) \ . \tag{4.4}$$

The invariance of the action of (4.2) under the transformation (4.4) is expressed by the analogue of (3;4.2),

$$iq_\mu \frac{\delta S_\tau}{\delta A^a_\mu(q)} = g f^{abc} \int_{1BZ} d\tilde{k}\, A^b_\mu(k-q)\, \frac{\delta S_\tau}{\delta A^c_\mu(k)} \ , \tag{4.5}$$

or equivalently by a set of Ward identities between the $w_{n,Q}$'s ,

$$iq_\mu w_{n+1,Q}\begin{bmatrix} a_1 & \cdots a_n & a \\ q_1 & \cdots q_n & q \\ \mu_1 & \cdots \mu_n & \mu \end{bmatrix} = g\sum_{j=1}^{n} f^{aa_jb}\, w_{n,Q}\begin{bmatrix} a_1 & \cdots & a_{j-1} & b & \cdots & a_n \\ q_1 & \cdots & q_{j-1} & q_j+q & \cdots & q_n \\ \mu_1 & \cdots & \mu_{j-1} & \mu & \cdots & \mu_n \end{bmatrix} , \tag{4.6}$$

where $Q+q+\sum_{j=1}^{n} q_j = 0$.

By analogy to Equation (3;2.2), the RG transformation will be taken to be

$$A^a_\mu(q) \to A^a_\mu(q) - \frac{\delta\tau}{\Lambda^2_\tau}\frac{(2\pi)^d}{v}\frac{\delta S_\tau}{\delta A^a_\mu(-q)} \ , \tag{4.7}$$

which leads to the RG equation

$$\frac{dS_\tau}{d\tau} = \frac{(2\pi)^d}{v\Lambda^2_\tau}\int_{1BZ} d^dq \left[\frac{\delta^2 S_\tau}{\delta A^a_\mu(q)\delta A^a_\mu(-q)} - \frac{\delta S_\tau}{\delta A^a_\mu(q)}\frac{\delta S_\tau}{\delta A^a_\mu(-q)}\right] , \tag{4.8}$$

or equivalently

$$\frac{d}{d\tau} w_{n,Q}\begin{bmatrix} a_1 & \cdots a_n \\ q_1 & \cdots q_n \\ \mu_1 & \cdots \mu_n \end{bmatrix} = \frac{1}{\Lambda^2_\tau}\int_{1BZ} d\tilde{k}\, w_{n+2,Q}\begin{bmatrix} a_1 & \cdots a_n & a & a \\ q_1 & \cdots q_n & k & -k \\ \mu_1 & \cdots \mu_n & \mu & \mu \end{bmatrix} +$$

$$- \frac{1}{\Lambda^2_\tau}\sum_{m=2}^{n}\binom{n}{m-1}\frac{1}{n!}\sum_{\{q_j\}} w_{m,0}\begin{bmatrix} a_1 & \cdots a_{m-1} & a \\ q_1 & \cdots q_{m-1} & k_m \\ \mu_1 & \cdots \mu_{m-1} & \mu \end{bmatrix} w_{n-m+2,0}\begin{bmatrix} a_m & \cdots a_n & a \\ q_m & \cdots q_n & k'_m \\ \mu_m & \cdots \mu_n & \mu \end{bmatrix} , \tag{4.9}$$

where

$$k_m = -\sum_{j=1}^{m-1} q_j \ , \quad k'_m = -\sum_{j=m}^{n} q_j \quad \text{and} \quad k_m + k'_m = Q \ .$$

It is very plausible that a perturbative solution of (4.9) will yield the same result of Section 5, namely $\beta=-\frac{21}{6}\frac{N}{(4\pi)^2}g^3$.

Finally, everything that was said here about an infinite lattice applies also to a theory formulated on a finite lattice. The equations appropriate to this case are obtained by replacing all integrals over the 1BZ by sums over those wavevectors within the 1BZ which are consistent with periodic boundary conditions.

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