Appendix B

Verification of plastic solutions

We compare the approximate solutions from SNAC with the analytic solutions to two simple problems including plasticity: an oedometer test and a problem of a thick cylinder with a pressurized inner wall. Readers are referred to standard textbooks on plasticity for more details of these problems (e.g., Hill, 1998; Davis and Selvadurai, 2002).

B.1 Oedometer test

This simple problem tests if SNAC can properly handle the angular geometry of the Mohr-Coulomb yield surface.

B.1.1 Problem Setup

A cube of Mohr-Coulomb material is pressed on one surface while all the other surfaces are confined such that they have free-slip boundary conditions (Fig.B.1).

Special care is needed for the angular yield envelope of the Mohr-Coulomb model. This oedometer test provides a direct test in this regard because the post-yielding stress state resides on one of the edges of the Mohr-Coulomb yield surface.

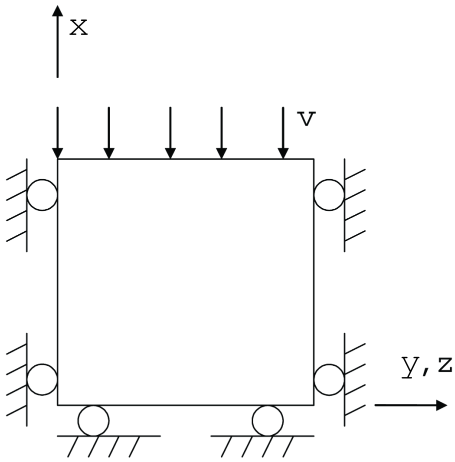

Figure B.1: Schematic diagram depicting the oedometer test.

B.1.2 Analytic Solutions

B.1.2.1 Elastic solution

$$\Delta\epsilon_x = \frac{v\Delta t}{L - vt} \quad \text{(B.1)}$$

$$\Delta\epsilon_y = \Delta\varepsilon_z = 0. \quad \text{(B.2)}$$

$$\Delta\sigma_x = (\lambda + 2\mu)\Delta\epsilon_x \quad \text{(B.3)}$$

$$\Delta\sigma_y = \lambda\Delta\epsilon_x \quad \text{(B.4)}$$

$$\Delta\sigma_z = \sigma_y. \quad \text{(B.5)}$$

B.1.2.2 Plastic solution

Yield criteria for Mohr-Coulomb plasticity are defined as

$$F^1 = \sigma_x - \sigma_y N_\phi + 2c\sqrt{N_\phi} \quad \text{(B.6)}$$

$$F^2 = \sigma_x - \sigma_z N_\phi + 2c\sqrt{N_\phi}. \quad \text{(B.7)}$$

During plastic flow, the strain increments are composed of elastic and plastic parts and we have

$$\Delta\epsilon_x = \Delta\epsilon_x^e + \Delta\epsilon_x^p \tag{B.8}$$

$$\Delta\epsilon_y = \Delta\epsilon_y^e + \Delta\epsilon_y^p \tag{B.9}$$

$$\Delta\epsilon_z = \Delta\epsilon_z^e + \Delta\epsilon_z^p. \tag{B.10}$$

Using the boundary conditions, we may write

$$\Delta\epsilon_x^e = \frac{v\Delta t}{L - vt} - \Delta\epsilon_x^p \tag{B.11}$$

$$\Delta\epsilon_y^e = -\Delta\epsilon_y^p \tag{B.12}$$

$$\Delta\epsilon_z^e = -\Delta\epsilon_z^p. \tag{B.13}$$

The flow rule for plastic flow along the edge of the Mohr-Coulomb criterion corresponding to $\sigma_y = \sigma_z$ has the form

$$\Delta\epsilon_x^p = \lambda_1 \frac{\partial G^1}{\partial \sigma_x} + \lambda_2 \frac{\partial G^2}{\partial \sigma_x} \tag{B.14}$$

$$\Delta\epsilon_y^p = \lambda_1 \frac{\partial G^1}{\partial \sigma_y} + \lambda_2 \frac{\partial G^2}{\partial \sigma_y} \tag{B.15}$$

$$\Delta\epsilon_z^p = \lambda_1 \frac{\partial G^1}{\partial \sigma_z} + \lambda_2 \frac{\partial G^2}{\partial \sigma_z}, \tag{B.16}$$

where G^1 and G^2 are the potential functions corresponding to F^1 and F^2:

$$G^1 = \sigma_x - \sigma_y N_\psi \tag{B.17}$$

$$G^2 = \sigma_x - \sigma_z N_\psi. \tag{B.18}$$

After substitution,

$$\Delta\epsilon_x^p = \lambda_1 + \lambda_2 \tag{B.19}$$
$$\Delta\epsilon_y^p = -\lambda_1 N_\psi \tag{B.20}$$
$$\Delta\epsilon_z^p = -\lambda_2 N_\psi. \tag{B.21}$$

By symmetry, we know $\lambda_1 = \lambda_2$:

$$\Delta\epsilon_x^p = 2\lambda_1 \tag{B.22}$$
$$\Delta\epsilon_y^p = -\lambda_1 N_\psi \tag{B.23}$$
$$\Delta\epsilon_z^p = -\lambda_1 N_\psi. \tag{B.24}$$

The stress increments are given as

$$\Delta\sigma_x = (\lambda + 2\mu)\Delta\epsilon_x^e + 2\lambda\Delta\epsilon_y^e \tag{B.25}$$
$$\Delta\sigma_y = (\lambda + 2\mu)\Delta\epsilon_y^e + \lambda(\Delta\epsilon_x^e + \Delta\epsilon_y^e) \tag{B.26}$$
$$\Delta\sigma_z = \Delta\sigma_y, \tag{B.27}$$

$$\Delta\sigma_x = (\lambda + 2\mu)\left(\frac{v\Delta t}{L - vt} - 2\lambda_1\right) + 2\lambda\lambda_1 N_\psi \tag{B.28}$$
$$\Delta\sigma_y = (\lambda + 2\mu)\lambda_1 N_\psi + \lambda\left(\frac{v\Delta t}{L - vt} - 2\lambda_1 + \lambda_1 N_\psi\right) \tag{B.29}$$
$$\Delta\sigma_z = \Delta\sigma_y. \tag{B.30}$$

During plastic flow, the consistency condition that $\Delta F^1 = 0$ should be satisfied, which takes the form

$$\Delta\sigma_x - \Delta\sigma_y N_\phi = 0. \tag{B.31}$$

Solving for λ_1, we get

$$\lambda_1 = \frac{(\lambda + 2\mu - \lambda N_\phi)\frac{v\Delta t}{L - vt}}{2(\lambda + 2\mu) - 2\lambda(N_\phi + N_\psi) + 2(\lambda + \mu)N_\phi N_\psi}. \tag{B.32}$$

B.1.3 Results

The following parameters are used:

- Bulk modulus: 200 MPa.
- Shear modulus: 200 MPa.
- Cohesion: 1 MPa.
- Friction angle: 10°.
- Dilation angle: 10°.
- Tension cut-off: 5.67 MPa.
- Boundary Conditions: v_x = -1.0×10^{-5} m/sec.
- $\Delta t = 1$ sec.
- Mesh size: 1×1×1 m, and 5×5×5 nodes.

The stress (σ_{xx}) is plotted against the strain (ϵ_{xx}) in Fig.B.2 for the solution of SNAC and the analytic solution. Those two solutions show a good agreement.

B.2 Thick cylinder with a pressurized inner wall

B.2.1 Problem Setup

We compute the equilibrium solution for a thick cylinder with a pressure applied on its inner and outer wall. The cylinder is assumed to be long along its axis such that the problem becomes a plane-strain one.

The problem is constructed by the following mathematical statements:

- Momentum balance:

$$\frac{d\sigma_{rr}}{dr} + \frac{\sigma_{rr} - \sigma_{\theta\theta}}{r} = 0. \tag{B.33}$$

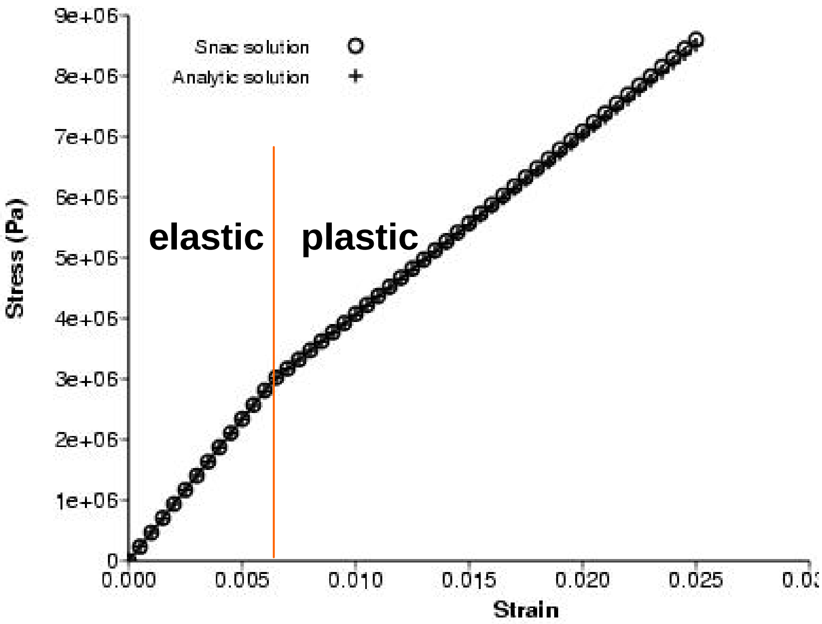

Figure B.2: Plots of stress vs. strain for analytic solutions and those from SNAC.

- Boundary conditions:

$$\sigma_{rr}(a) = -P_i, \tag{B.34}$$

$$\sigma_{rr}(b) = -P_o. \tag{B.35}$$

***Note**: The sign is negative because compressional.*

- Plane strain:

$$\epsilon_{zz} = 0, \tag{B.36}$$

$$\sigma_{zz} = \nu(\sigma_{rr} + \sigma_{\theta\theta}). \tag{B.37}$$

- Constitutive Relations:

$$\epsilon_{rr} = \frac{du_r}{dr}, \tag{B.38}$$

$$\epsilon_{\theta\theta} = \frac{u_r}{r}, \tag{B.39}$$

$$\sigma_{rr} = \lambda e + 2G\epsilon_{rr}, \tag{B.40}$$

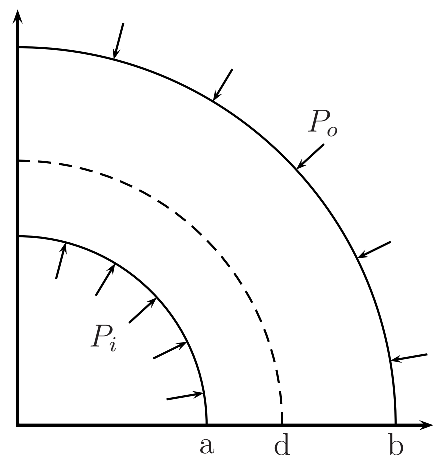

Figure B.3: Schematic diagram for the problem of the thick cylinder with a pressurized inner wall

$$\sigma_{\theta\theta} = \lambda e + 2G\epsilon_{\theta\theta}, \tag{B.41}$$

where e is the volumetric strain.

- Yield and flow functions:

  Since $\sigma_{\theta\theta}(>0) > \sigma_{zz} > \sigma_{rr}(<0)$,

$$f = \sigma_{\theta\theta} - N\sigma_{rr} - 2c\sqrt{N} \leq 0, \;\; N = \frac{1+\sin\phi}{1-\sin\phi}, \tag{B.42}$$

$$g = \sigma_{\theta\theta} - M\sigma_{rr}, \;\; M = \frac{1+\sin\psi}{1-\sin\psi}. \tag{B.43}$$

B.2.2 Analytic Solutions

By combining (B.33) and (B.38)-(B.41), we get the momentum balance equation in terms of the non-trivial radial component of displacement, u_r:

$$\frac{d^2u_r}{dr^2} + \frac{1}{r}\frac{du_r}{dr} - \frac{u_r}{r^2} = 0. \tag{B.44}$$

A general solution has the form

$$u_r = C_1 r + \frac{C_2}{r}. \tag{B.45}$$

Then, strain components become

$$\epsilon_{rr} = C_1 - \frac{C_2}{r^2}, \tag{B.46}$$

$$\epsilon_{\theta\theta} = C_1 + \frac{C_2}{r^2}. \tag{B.47}$$

B.2.2.1 Elastic solution

$$\begin{aligned} \sigma_{rr}(a) = -P_i &= \lambda(2C_1) + 2G\left(C_1 - \frac{C_2}{a^2}\right) \\ &= 2(\lambda + G)C_1 - \frac{2G}{a^2}C_2, \end{aligned} \tag{B.48}$$

$$\begin{aligned} \sigma_{rr}(b) = -P_o &= \lambda(2C_1) + 2G\left(C_1 - \frac{C_2}{b^2}\right) \\ &= 2(\lambda + G)C_1 - \frac{2G}{b^2}C_2. \end{aligned} \tag{B.49}$$

From (B.48) and (B.49),

$$C_1 = \frac{a^2 P_i - b^2 P_o}{2(\lambda + G)(b^2 - a^2)}, \tag{B.50}$$

$$C_2 = \frac{P_i - P_o}{2G}\frac{a^2 b^2}{b^2 - a^2}. \tag{B.51}$$

The full solution for the radial component of displacements is

$$u_r = \frac{a^2 P_i - b^2 P_o}{2(\lambda + G)(b^2 - a^2)} r + \frac{P_i - P_o}{2G}\frac{a^2 b^2}{b^2 - a^2}\frac{1}{r}. \tag{B.52}$$

Components of strains are given as

$$\epsilon_{rr} = \frac{a^2 P_i - b^2 P_o}{2(\lambda + G)(b^2 - a^2)} - \frac{P_i - P_o}{2G}\frac{a^2 b^2}{b^2 - a^2}\frac{1}{r^2}, \tag{B.53}$$

$$\epsilon_{\theta\theta} = \frac{a^2P_i - b^2P_o}{2(\lambda + G)(b^2 - a^2)} + \frac{P_i - P_o}{2G}\frac{a^2b^2}{b^2 - a^2}\frac{1}{r^2}. \tag{B.54}$$

Finally, stress components are

$$\sigma_{rr} = \frac{a^2P_i - b^2P_o}{b^2 - a^2} - \frac{(P_i - P_o)a^2b^2}{b^2 - a^2}\frac{1}{r^2}, \tag{B.55}$$

$$\sigma_{\theta\theta} = \frac{a^2P_i - b^2P_o}{b^2 - a^2} + \frac{(P_i - P_o)a^2b^2}{b^2 - a^2}\frac{1}{r^2}. \tag{B.56}$$

B.2.2.2 Plastic solution

In the elastic region : Let us assume that region of $r \le d$ yielded, where d is the outer radius of the yielded region and $a < d < b$. Also, let σ_d be the stress at $r = d$. Then, solution for the elastic region ($r > d$) are acquired by simply substituting d and $\sigma_d - P_o$ for a and $P_i - P_o$ in (B.52), (B.53), (B.54), (B.55), and (B.56). Specifically, stress components are given as:

$$\sigma_{rr} = \frac{d^2\sigma_d - b^2P_o}{b^2 - d^2} - \frac{(\sigma_d - P_o)d^2b^2}{b^2 - d^2}\frac{1}{r^2}, \tag{B.57}$$

$$\sigma_{\theta\theta} = \frac{d^2\sigma_d - b^2P_o}{b^2 - d^2} + \frac{(\sigma_d - P_o)d^2b^2}{b^2 - d^2}\frac{1}{r^2}. \tag{B.58}$$

In the plastic region : The yield function (B.42) should be 0 in the plastic region. So, we insert

$$\sigma_{\theta\theta} = N\sigma_{rr} + 2c\sqrt{N} \tag{B.59}$$

into (B.33):

$$\frac{d\sigma_{rr}}{dr} + \frac{\sigma_{rr}}{r} - \frac{N\sigma_{rr} + 2c\sqrt{N}}{r} = 0$$

. Then, we get an ordinary differential equation for σ_{rr}

$$\frac{d\sigma_{rr}}{dr} + (1 - N)\frac{\sigma_{rr}}{r} - \frac{2c\sqrt{N}}{r} = 0. \tag{B.60}$$

The solution to the ODE

$$y' = a\frac{y}{x} + \frac{b}{x}$$

is

$$y = -\frac{b}{a} + C_1 x^a \,,$$

where C_1 needs to be determined using a boundary condition. The corresponding coefficients in (B.60) are

$$a = -(1-N), \;\; b = 2c\sqrt{N} \,.$$

The solution for σ_{rr} is

$$\sigma_{rr} = -\frac{2c\sqrt{N}}{N-1} + C_1 r^{N-1} \,. \tag{B.61}$$

The value of C_1 is determined by the stress continuity at $r = d$, *i.e.*, $\sigma_{rr}(d) = -\sigma_d$:

$$C_1 = \left(-\sigma_d + \frac{2c\sqrt{N}}{N-1}\right) d^{1-N}. \tag{B.62}$$

The complete stress solution in the plastic regions is

$$\sigma_{rr}(r) = -\frac{2c\sqrt{N}}{N-1} + \left(-\sigma_d + \frac{2c\sqrt{N}}{N-1}\right)\left(\frac{r}{d}\right)^{N-1}. \tag{B.63}$$

$$\begin{aligned} \sigma_{\theta\theta}(r) &= N\sigma_{rr} + 2c\sqrt{N} \\ &= -\frac{2c\sqrt{N}}{N-1} + N\left(-\sigma_d + \frac{2c\sqrt{N}}{N-1}\right)\left(\frac{r}{d}\right)^{N-1}. \end{aligned} \tag{B.64}$$

In the above formulae, σ_d is still unknown. The elastic stress solutions (B.57 and B.58) should make the yield function (B.42) zero when yieldeing occurs initially at

$r = a$.

$$\begin{aligned}
&\frac{a^2P_i - b^2P_o}{b^2 - a^2} + \frac{(P_i - P_o)a^2}{b^2 - d^2}\frac{b^2}{a^2} \\
&-N\left[\frac{a^2P_i - b^2P_o}{b^2 - a^2} - \frac{(P_i - P_o)a^2}{b^2 - a^2}\frac{b^2}{a^2}\right] - 2c\sqrt{N} = 0 \\
&\frac{(a^2 + b^2)P_i - 2b^2P_o}{b^2 - a^2} + NP_i - 2c\sqrt{N} = 0 \\
&\therefore P_{i0}(a) = \frac{2c\sqrt{N} + (2b^2/(b^2 - a^2))P_o}{N + (b^2 + a^2)/(b^2 - a^2)}.
\end{aligned}$$

Since the same should hold for any end condition.

$$\sigma_d(d) = \frac{2c\sqrt{N} + (2b^2/(b^2 - d^2))P_o}{N + (b^2 + d^2)/(b^2 - d^2)}. \tag{B.65}$$

Finally, d is numerically determined by finding r at which the yield function becomes zero.

To benchmark SNAC's solution, we use (B.57), (B.58), (B.63), and (B.64) with numerically computed d.

B.2.3 Results

We present the results for the following parameters:

- Bulk modulus: 200 MPa.
- Shear modulus: 200 MPa.
- Cohesion: 1 MPa.
- Friction angle: 10 °.
- Dilation angle: 10 °.
- Tension cut-off: 567 MPa.
- Grid size: 31×3×31.

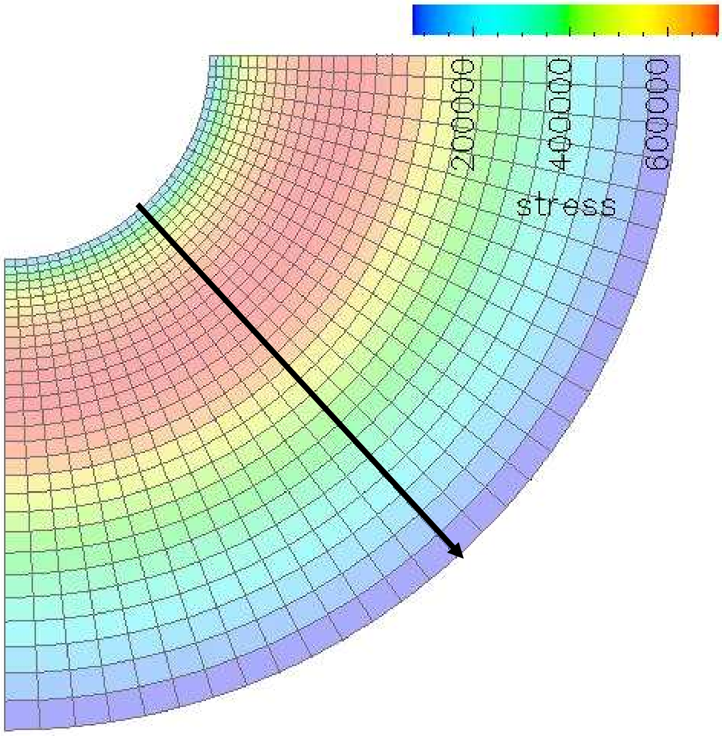

Figure B.4: The square root of the second invariant of stress from SNAC. Profiles shown in Fig.B.5 are extracted along the radial direction (the black arrow)

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- Geometry of cylinder: a (inner radius) = 3.0 m, b (outer radius) = 10.0 m.
- Boundary Conditions: $P_i = 20.0$ MPa, and $P_o = 0.0$ MPa. Two surfaces normal to the axis are free-slip.
- $dt = 1$ sec and results are compared after SNAC proceeds 5000 steps.

The second invariant of stress (II_σ) is chosen as a representative value and Fig.B.4 shows the spatial distribution of the squre root of II_σ.

The radial profile of $\sqrt{II_\sigma}$ for the SNAC's solution is shown in Fig.B.5 together with the analytic and purely analytic solutions. The SNAC's solution shows a good agreement with the analytic solution.

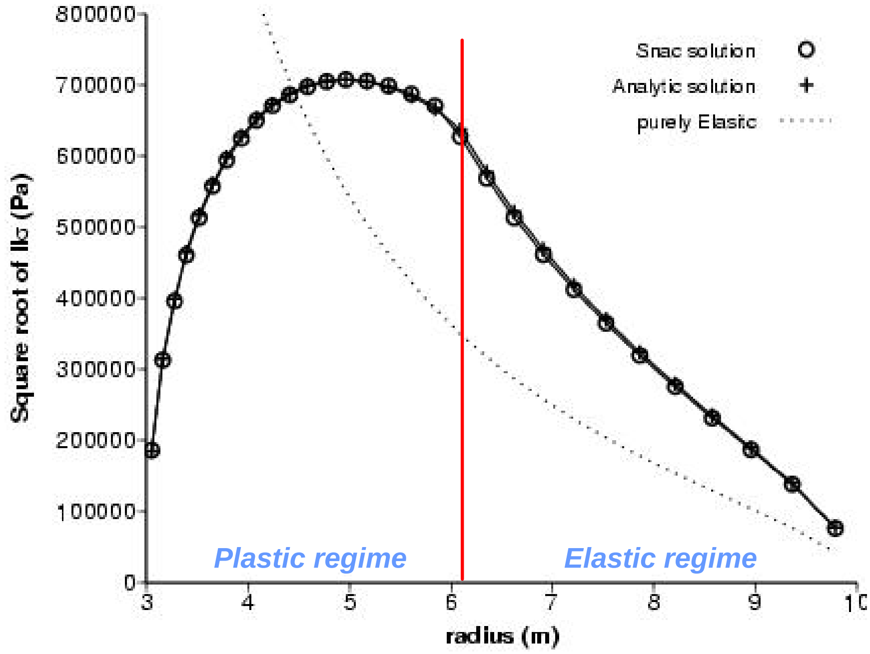

Figure B.5: Radial profiles of the square root of the second invariant of stressi (II_σ): Circles for SNAC's solution, crosses for analytic solution, and a dashed line for the analytic solution for the purely elastic case.

References

Davis, R. O., Selvadurai, A. P. S., 2002. Plasticity and Geomechanics. Cambridge University Press.

Hill, R., 1998. The Mathematical Theory of Plasticity. Oxford University Press, U.S.A.