THE 2-LENGTH OF A FINITE SOLVABLE GROUP

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ABSTRACT

In this paper the relationship between the 2-length $\ell_2(G)$ and the 2-exponent $e_2(G)$ of a finite solvable group G is studied. It is shown that $\ell_2(G) \leq 2e_2(G) - 1$ provided that $\ell_2(G) \geq 1$.

The special case of groups satisfying $e_2(G) = 2$, i.e., groups whose Sylow 2-groups are of exponent 4, is investigated to determine whether $\ell_2 \leq e_2$ in this case. This question is not answered but it is shown that a certain normal subgroup (which may be the whole group) satisfies $\ell_2 \leq e_2$. In addition if all the elements of order 4 are contained in this subgroup, then $\ell_2 \leq e_2$ for the whole group as well. As an application of this last result, it is proved that $\ell_2 \leq e_2$ in a group of exponent 12.

I. Introduction

The primary objective of this thesis is to obtain an improved bound for the 2-length of a finite solvable group G in terms of its 2-exponent. In addition the special case where the Sylow 2-subgroup of G is of exponent 4 is studied.

The first results on the p-length of a group were obtained by Hall and Higman in [3]. Following their paper we make the following definitions:

(1) A finite group G is called a p'-group, where p is a prime, provided p does not divide the order of G.

(2) G is p-solvable if each of its composition factor groups is either a p-group or a p'-group.

(3) The upper p-series

$$1 = P_0 \leq N_0 < P_1 < \dots < P_\ell \leq N_\ell = G$$

of a p-solvable group G is defined inductively by setting N_k/P_k to be the greatest normal p'-subgroup of G/P_k and P_{k+1}/N_k the greatest normal p-subgroup of G/N_k.

(4) The least integer ℓ such that $N_\ell = G$ is called the p-length of G and is denoted by $\ell_p(G)$, or, if the group G is understood, simply by ℓ_p.

(5) $c_p(G)$ for a prime p and a finite group G is defined to be the class of a Sylow p-subgroup of G.

(6) For a finite group G and a prime p the p-exponent $e_p(G)$ of G is defined by the rule that $p^{e_p(G)}$ is the exponent of a Sylow p-subgroup of G, i.e., the greatest order of any p-element.

It is shown in [3] that $\ell_p \leq c_p$ in any p-solvable group. An application of this which we use repeatedly is that if G is solvable and $e_2(G) = 1$ then $\ell_2(G) = 1$. This follows since a group of exponent 2 is necessarily abelian.

For an odd prime p, Hall and Higman showed that $\ell_p \leq e_p$ if p is not a Fermat prime and $\ell_p \leq 2e_p$ if p is a Fermat prime. They further showed that these inequalities are best-possible. No results, however, were obtained about the relationship between ℓ_p and e_p when p = 2. This gap was filled by A. H. M. Hoare in [5] who showed that $\ell_2 \leq 3e_2 - 2$ in any 2-solvable group where $\ell_2 \geq 1$. In the present paper it will be shown that under the same conditions $\ell_2 \leq 2e_2 - 1$.

It is not known whether or not this is best-possible and a better result may very well be true. Indeed, the author knows of no solvable group in which ℓ_2 exceeds e_2.

The special case $e_2(G) = 2$ is studied in more detail and one result is that if G is of exponent 12, i.e., $x^{12} = 1$ for all x in G, then $\ell_2(G) \leq e_2(G)$.

II. Statement and Proof of the Main Theorem

For the rest of this paper we adopt the convention that all groups referred to are assumed finite, and if G is such a group then $|G|$ is its order. If H is a proper subgroup of G this is written $H < G$ and if, in addition, H is normal in G we write $H \lhd G$. $GF(p^n)$ denotes the finite field of p^n elements.

Before proceeding to our main result we first state some basic properties of the upper p-series

$$1 = P_0 \trianglelefteq N_0 \lhd P_1 \lhd N_1 \lhd \ldots \lhd P_\ell \trianglelefteq N_\ell = G.$$

Lemma 1. For $i \geq 1$, P_i/N_{i-1} contains its centralizer in G/N_{i-1} and N_i/P_i contains its centralizer in G/P_i.

Lemma 2. If F_i/N_{i-1} is the Frattini subgroup of P_i/N_{i-1} for $i \geq 1$, then P_i/F_i is its own centralizer in G/F_i.

Corollary. For $i \geq 1$, G/P_i is faithfully represented as a group of automorphisms of the elementary abelian p-group P_i/F_i.

Proofs of these results are to be found in [3]. Now an elementary abelian p-group can be considered as a vector space over $GF(p)$. In particular from the corollary we have that G/P_1 is faithfully represented as a linear group operating on the vector space P_1/F_1. G/P_1 has no normal p-group except for the identity and $\ell_p(G/P_1) = \ell_p(G) - 1$.

Now if g is an element of order p^m in G/P_1 then the minimal equation of g on P_1/F_1 is $(x - 1)^r = 0$ where r is some integer $\leq p^m$. If $r < p^m$, i.e., $(g - 1)^{p^m-1} = 0$, then g is said to be exceptional. The following is proved in [3]:

Lemma 3. If g is of order p^m and not exceptional in G/P_1, then $e_p(G) \geq m+1$.

We are now prepared to state our main result:

Theorem 1. If G is a finite solvable group and $\ell_2(G) \geq 1$, then $\ell_2(G) \leq 2e_2(G) - 1$.

I first remark that, since Feit and Thompson have proved the long-standing conjecture that groups of odd order are solvable, a 2-solvable group is in fact solvable. Solvability is quite important in the proof of theorem 1.

Now if $\ell_2 = 1$ the conclusion is trivial and, since $e_2 = 1$ implies $\ell_2 = 1$, we see that $\ell_2 = 2$ implies that $e_2 \geq 2$ so the result again follows. Now if $\ell_2 > 2$, then $\ell_2(G/P_2) = \ell_2(G) - 2 \geq 1$, so that if we could prove that $e_2(G/P_2) \leq e_2(G) - 1$ then theorem 1 would follow by induction on the order of G. For this purpose we only need to concern ourselves with the exceptional elements in G/P_1 because of lemma 3.

It will be shown that if g is of order 2^m and exceptional in G/P_1 then $g^{2^{m-1}}$ is in P_2/P_1. This will immediately prove that $e_2(G/P_2) \leq e_2(G) - 1$ and theorem 1 will be proved.

Theorem 2. Let $\overline{G}$ be a solvable linear group on a field F of characteristic 2 and assume $\overline{G}$ has no normal 2-group other than the identity. Then if N is the largest normal $2'$-subgroup of $\overline{G}$ and if g is an exceptional element of order 2^m in $\overline{G}$, it follows that $g^{2^{m-1}}$ is in the largest normal 2-subgroup of $\overline{G}/N$.

Proof. G/P_1 satisfies the hypothesis of this theorem so that, by our previous discussion, theorem 1 follows from theorem 2. The rest of this section is devoted to the proof of theorem 2.

It should be pointed out that this theorem is a more general result than the special case needed to prove theorem 1. Although in the statement of theorem 2 it is assumed that $\overline{G}$ is finite, neither F nor the dimension of the space on which $\overline{G}$ operates need be finite.

Now neither the hypothesis nor the conclusion of the theorem is affected by an extension of the field F. Thus, without loss of generality, we assume that F is algebraically closed.

Since an element of order 2 cannot be exceptional (since otherwise it would have to be the identity), m must be greater than 1. Let $h = g^{2^{m-2}}$ and then $h^2 = g^{2^{m-1}}$.

In proving theorem 2 we define subgroups H and H_1; $\overline{G} \trianglerighteq H \trianglerighteq H_1$, $h^2 \in H_1$, and g normalizes H_1 (H_1 need not be normal in G). It is then shown that if x is any element in the largest normal 2-subgroup of $H_1/H_1 \cap N$, then

$(h^2, x) = (h, x)^2$. From this it will follow that h^2 is in the largest normal 2-subgroup of $H_1/H_1 \cap N$, and, finally, from this the desired result.

Our first step is to prove two lemmas which are of use later and which also motivate the definition of H. Here, and elsewhere, we denote the space on which $\overline{G}$ operates by V.

Lemma 4. If Q is any $2'$-subgroup of $\overline{G}$ which is normalized by g, then h^2 fixes every minimal characteristic F-Q submodule of V.

Proof. A minimal characteristic F-Q submodule is simply the join of all those F-Q submodules operator isomorphic to a given irreducible F-Q submodule. Now since Q is a $2'$-group V can be written as the direct sum of the minimal characteristic F-Q submodules.

$$V = V_1 \oplus V_2 \oplus \cdots$$

Since g normalizes Q, g must permute the V_i among themselves. Now if h^2 does not fix every V_i then g, as a permutation of the V_i, has a cycle of length 2^m. But $(g - 1)^{2^m-1} = g^{2^m-1} + g^{2^m-2} + \cdots + g + 1$ since F is of characteristic 2. Thus $(g - 1)^{2^m-1}$ could not be zero in this case, contrary to assumption.

Lemma 5. If Q is any abelian $2'$-subgroup of $\overline{G}$ and x is any element of $\overline{G}$ normalizing Q and fixing every minimal characteristic F-Q submodule of V, then x centralizes Q.

Proof. $V = V_1 \oplus V_2 \oplus \cdots$ where the V_i are the minimal characteristic F-Q submodules. Suppose $(x, Q) \neq 1$. Thus there exists a V_i, V_1 say, such that (x, Q) is not the identity on V_i. Now Q is abelian and F is algebraically closed so that Q operates on V_1 as a scalar multiplication, i.e., if $y \in Q$ and $v \in V_1$ then $yv = \chi(y)v$ where $\chi(y)$ is a scalar. Thus we have (this computation is taken from [4])

$$\chi(x^{-1}yx)v = (x^{-1}yx)v = x^{-1}y(xv) = x^{-1}\chi(y)xv = \chi(y)v.$$

Therefore $(y, x)v = v$ for all $y \in Q$, $v \in V_1$ contrary to (Q, x) not being the identity on V_1. Thus the lemma is proved.

Now let H be the following set of elements of $\bar{G}$: $x \in H$ if, and only if, for every normal $2'$-subgroup Q of $\bar{G}$, x fixes every minimal characteristic F-Q submodule of V.

It is easy to see that H is a normal subgroup of $\bar{G}$ and $h^2 \in H$ by lemma 4. Since the largest normal 2-subgroup and the largest normal $2'$-subgroup of H are normal in $\bar{G}$ we have at once that H has no normal 2-subgroup other than the identity and the largest normal $2'$-subgroup of H is $H \cap N$.

We now proceed to construct a characteristic $2'$-subgroup K of H such that K is nilpotent of class 2, no 2-element of H except for the identity centralizes K, and if $K = K_1 \times K_2 \times \dots$ is the representation of K as the direct product of its Sylow subgroups, K_i the Sylow q_i-subgroup of K for some prime q_i, then each K_i is of exponent q_i. The importance of K is due to the fact that a Sylow 2-subgroup of H can be represented faithfully as a group of automorphisms of K, and the restricted nature of K then restricts the structure of the Sylow 2-subgroup.

To construct K, first let $\overline{Q}$ be the largest normal nilpotent subgroup of H. Clearly $\overline{Q} \leq H \cap N$. Furthermore, since H is solvable (this is where solvability is crucial), $\overline{Q}$ must contain its centralizer in H. (This is proved in [1].) Since $\overline{Q}$ is nilpotent we can write

$$\overline{Q} = \overline{Q}_1 \times \overline{Q}_2 \times \dots$$

where $\overline{Q}_i$ is a q_i-group for an odd prime q_i and $q_i \neq q_j$ for $i \neq j$.

<u>Lemma 6</u>. $c(\overline{Q}) = 2$, i.e., $\overline{Q}$ is nilpotent of class 2.

<u>Proof</u>. Since $h^2 \in H$, h^2 does not centralize $\overline{Q}$. Thus by lemmas 4 and 5, $\overline{Q}$ is not abelian. Thus $c(\overline{Q}) \geq 2$. Now suppose $c = c(\overline{Q}) \geq 3$ and let

$$\overline{Q} = \Gamma_1(\overline{Q}) \triangleright \Gamma_2(\overline{Q}) \triangleright \dots \triangleright \Gamma_{c+1}(\overline{Q}) = 1$$

be the lower central series of $\overline{Q}$ and let n be the first integer $\geq (c+1)/2$. Clearly $n \leq c-1$ since $c-1 \geq (c+1)/2$ for $c \geq 3$. But from [2, p. 150] we have

$$(\Gamma_n(\overline{Q}), \overline{Q}) = \Gamma_{n+1}(\overline{Q}) \neq 1, \text{ and from [2, p. 156]}$$

$$(\Gamma_n(\overline{Q}), \Gamma_n(\overline{Q})) \leq \Gamma_{2n}(\overline{Q}) = 1 \text{ since } 2n \geq c+1.$$

Thus we have that $\Gamma_n(\overline{Q})$ is abelian and, of course, normal in $\overline{G}$ but not centralized by $\overline{Q}$. But from the definition of H and lemma 5 we see that this is impossible. Hence $c = 2$.

This naturally implies that each $\overline{Q}_i$ is of class not exceeding 2. Now q_i is an odd prime and so greater than 2. Thus $\overline{Q}_i$ is a regular q_i-group. (For the definition and properties of regular q-groups see [2, pp. 183-186].) Then the elements of order at most ${q_i}^a$ form a characteristic subgroup of $\overline{Q}_i$ which will be denoted $C^a(\overline{Q}_i)$.

Set $K_i = C^1(\overline{Q}_i)$ and $K = K_1 \times K_2 \times \ldots$

We now prove some elementary results which imply that no 2-element of H except for the identity centralizes K.

<u>Lemma 7</u>. If Q is a group, x a non-trivial automorphism of order prime to $|Q|$, and if M is a normal subgroup of Q admitting x, then x cannot centralize both M and Q/M.

<u>Proof</u>. Suppose that x does centralize both M and Q/M. Since x is non-trivial we have that there must be a $y \in Q$ such that $y^x \neq y$. Thus we must have $y^x = yz$ where z is in M and $z \neq 1$. Now $y^{x^2} = yzz = yz^2$ since x centralizes M. By induction we have $y^{x^n} = yz^n$ for all n. This is a contradiction since the orders of x and z are relatively prime.

<u>Lemma 8</u>. If P is a regular p-group with $e_p(P) = n > 1$ and x is a non-trivial automorphism of P of order prime to p then x does not centralize $C^{n-1}(P)$, the subgroup consisting of all elements of P of order dividing p^{n-1}.

<u>Proof</u>. Suppose x does centralize $C^{n-1}(P)$. Then by the previous lemma x cannot centralize $P/C^{n-1}(P)$. But for any $y \in P$, y^p must belong to $C^{n-1}(P)$ and thus

must be centralized by x. Thus $y^p = (y^x)^p$. But since P is regular this implies that $(y^{-1}y^x)^p = 1$. Since $n > 1$ we see that $y^{-1}y^x = (y, x)$ is always in $C^{n-1}(P)$. Hence x centralizes both $C^{n-1}(P)$ and $P/C^{n-1}(P)$ contrary to assumption. Therefore the lemma is proved.

Corollary. If P is a regular p-group, x a non-trivial automorphism of order prime to p then x does not centralize $C^1(P)$.

Proof. If $e_p(P) = 1$, then $C^1(P) = P$. If $e_p(P) > 1$, then the result follows from the lemma and from

$$C^{n-a-1}\left[C^{n-a}(P)\right] = C^{n-a-1}(P).$$

As a result of this we see that no 2-element (except for the identity) of H centralizes K. K and each K_i is a characteristic subgroup of H and thus normal in $\overline{G}$. Since h^2 does not centralize K (since h^2 is a non-idenity 2-element of H), K cannot be abelian. Thus K is of class 2.

We are now prepared to define the subgroup H_1. For this purpose decompose V for each K_i into the sum

$$V = V_{i1} \oplus V_{i2} \oplus \cdots$$

where the V_{ij} are the minimal characteristic F-K_i submodules. Let $C_{ij} = \left\{ x \mid x \in H \text{ and } (K_i, x) \text{ is } 1 \text{ on } V_{ij} \right\}$. Clearly each C_{ij} is a normal subgroup of H although it is not necessarily normal in $\overline{G}$.

Now let H_1 be the intersection of all the C_{ij} which contain h^2. If h^2 is not in any C_{ij} then set H_1 equal to H. In any event $H_1 \trianglelefteq H$ and H_1 is normalized by g.

As was the case with H, H_1 cannot have any normal 2-subgroup other than the identity and the largest normal $2'$-subgroup is $(H \cap N) \cap H_1 = H_1 \cap N$. It will be shown that h^2 is in the greatest normal 2-subgroup of $H_1/H_1 \cap N$. This will imply that h^2 is in the greatest normal 2-subgroup of $H/H \cap N$ which, in turn, will imply that h^2 is in the greatest normal 2-subgroup of $\bar{G}/N$.

Let P be a 2-subgroup of H_1 such that $P(H_1 \cap N)/H_1 \cap N$ is the largest normal 2-subgroup of $H_1/H_1 \cap N$. Since, modulo N, P is normalized by g we can take P and g to belong to the same Sylow 2-subgroup of $\bar{G}$.

Now $h^2 \in H_1$ so that if $h^2 \notin P$ then by lemma 2, h^2 does not centralize $P/\Phi(P)$ where $\Phi(P)$ is the Frattini subgroup of P.

Lemma 9. If $x \in P$, then $(h^2, x) = (h, x)^2$.

Proof. h may or may not belong to H_1 but h normalizes P (since h and P generate a 2-subgroup and h normalizes P modulo a $2'$-group) so that $(h, x) \in P$ and thus $(h, x)^2 \in \Phi(P)$. Therefore once the lemma is proved we have at once that h^2 centralizes $P/\Phi(P)$ which implies that h^2 must be in P which will finish the proof of theorem 2.

Let $k = (h^2, x)(h, x)^{-2}$ and suppose $k \neq 1$. But k is a 2-element of H_1 and thus cannot centralize K. Hence $(K_i, k) \neq 1$ for some i. Choose V_{ij} such that (K_i, k) is not the identity when restricted to V_{ij}. Now $k \in H_1$

so that by the definition of H_1 we must have (K_i, h^2) also not the identity on V_{ij}. (This last statement is the motivation for the definition of H_1.)

In what follows let $V' = V_{ij}$, Q the image of K_i when restricted to V', $q = q_i$, and x_1 the image of x when restricted to V'. Let $g^{2^{m-m_1}}$ be the first power of g fixing V' and let g_1 be the image of this element when restricted to V'. From [3, p. 13] it follows that since g is exceptional, g_1 must be exceptional, i.e.,

$$(g_1 - 1)^{2^{m_1}-1} = 0.$$

Now h^2 is not the identity on V' and so h^2 is not exceptional on V'. Thus $m_1 \geqslant 2$ and so h fixes V'. We define $h_1 = g_1^{2^{m_1-2}}$. Then both (Q, h_1^2) and (Q, k_1), where $k_1 = (h_1^2, x_1)(h_1, x_1)^{-2}$, are not equal to the identity.

Since g_1 is exceptional and $(Q, h_1^2) \neq 1$, Q cannot be abelian. But $c(K) = 2$ so that Q must be of class 2. Since, in addition, Q is of exponent q, we have

$$Z(Q) \geqslant Q' = \Phi(Q) \quad (Z(Q) \text{ is the center of } Q).$$

Now V' is the sum of absolutely irreducible F-Q submodules all of which are operator isomorphic to each other. On an absolutely irreducible F-Q module $Z(Q)$ has to be cyclic and generated by a scalar matrix. The same must hold true for the representation of Q on V'. Thus $Z(Q)$ is cyclic of order q generated by a scalar matrix. Now $Q' \neq 1$ since Q is not abelian. Therefore we must have $Z(Q) = Q'$ which means that Q is an extra-special q-group. (See [3, p. 15].) Note also that if S is the 2-group generated by x_1 and g_1 then $(Z(Q), S) = 1$ since $Z(Q)$ is generated by a scalar matrix.

Now let V'' be an irreducible F-QS submodule of V'. By [3, p. 14] we have that V'' is an irreducible F-Q module. V' is the sum of F-Q modules operator isomorphic to V''. Thus $(Q, h_1^2) \neq 1$ on V'' and of course g_1 must be exceptional on V''. Then by [3, p. 21] we have the following: (1) $2^{m_1}-1$ is a power of q, and (2) if g_1 is faithfully and irreducibly represented on Q_1/Q' (such a Q_1 can always be found since h_1^2 is not the identity on Q/Q'), then Q can be written as the central product of Q_1 and a group Q_2 and g_1 transforms Q_2/Q' trivially. We now need the following result:

<u>Lemma 10</u>. $2^{m_1} - 1 = q$ and $|Q_1/Q'| = q^2$.

<u>Proof</u>. The following argument is essentially due to [5].

First we have $2^{m_1} - 1 = q^n$ for some n. But $m_1 \geq 2$ so that $q^n \equiv 3 \pmod 4$ and n must be odd. Thus $q^n + 1$ is divisible by $q + 1$. Thus we must have $q + 1 = 2^{s-1}$ for some $s \geq 3$. $q - 1 = 2r$ where r is odd. Hence we have $q^2 = 2^s r + 1$ and $q^{2n} - 1 = r^n 2^{sn} + \dots + \binom{n}{2} r^2 2^{2s} + nr2^s$. Since r and n are odd we see that 2^s is the highest power of 2 dividing $q^{2n} - 1$. But $q^n = 2^{m_1} - 1$ so $q^{2n} - 1 = 2^{m_1+1}(2^{m_1-1} - 1)$. It now follows that $s = m_1 + 1$ and then that $q = 2^{m_1} - 1$.

Since 2^{m_1} divides $q^2 - 1$, the polynomial $(t^{2^{m_1}} - 1)$ divides $(t^{q^2} - t)$. Thus the irreducible factors modulo q of $t^{2^{m_1}} - 1$ are all of degree less than or equal to 2. However no element of order 4 can have a faithful 1-dimensional representation over GF(q) since $q \equiv 3 \pmod 4$ so that -1 is not a quadratic residue of q. Thus since g_1 is a 2-element of order at least 4 and g_1 is faithfully and irreducibly represented on Q_1/Q' it follows that $|Q_1/Q'| = q^2$.

Now the representation of Q on V'' is isomorphic to the representation of Q on V' so that $(g_1, Q_2) = 1$ on V'' implies that $(g_1, Q_2) = 1$.

Thus from the preceding lemma we see that the centralizer of g_1 in the space Q/Q' has co-dimension 2 over GF(q). It easily follows that this is also true for all powers of g_1 (except for the identity, of course). Now if g_1 is of order 4 then the equation of g_1 on Q_1/Q' must be $t^2 + 1 = 0$ so that ${g_1}^2$ must have the representation

$$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

on Q_1/Q'. If g_1 is not of order 4 then ${g_1}^2$ can have no faithful 1-dimensional representation over GF(q) so that Q_1/Q' must give an irreducible representation of ${g_1}^2$. These results can be summed up by stating that in the completely reduced representation of ${g_1}^2$ on Q/Q', there is only one non-trivial block unless g_1 is of order 4 in which case there are two non-trivial blocks.

Now Q/Q' can be considered as a vector space M over GF(q). M can be given the structure of a symplectic space as follows [3, p. 17]:

Let c be a generator of Q'. Then for any elements a, b of Q the commutator (a, b) must be a power of c. Define $\rho(a, b)$ by the equation

$$(a, b) = c^{\rho(a, b)}.$$

$\rho(a, b)$ is uniquely defined on Q/Q' and is bilinear and skew symmetric. Since $Q' = Z(Q)$ it follows that ρ is of maximum rank on M.

Now since ρ is of maximum rank the dimension of M must be even, 2r say. Since $(S, Q') = 1$, S preserves the symplectic structure of M. Thus the homomorphic image $\bar{S}$ of S obtained by the representation of S on Q/Q' may be considered as a subgroup of a Sylow 2-subgroup of the symplectic group S(2r, q). (S(2r, q) is the symplectic group on a space of dimension 2r over GF(q).) Let $\bar{x}_1$, $\bar{g}_1$, and $\bar{h}_1$ be the images of x_1, g_1, and h_1, respectively, in the homomorphism $S \to \bar{S}$. From $(Q, h_1^2) \neq 1$ and $(Q, k_1) \neq 1$ it follows that $\bar{h}_1^2 \neq 1$ and $(\bar{h}_1^2, \bar{x}_1) \neq (\bar{h}_1, \bar{x}_1)^2$.

We now describe a Sylow 2-subgroup of S(2r, q). (This is based upon [3, pp. 23-24].)

M is of dimension 2r over GF(q) so that M can be provided with the structure of a space of dimension r over $GF(q^2)$. For an element $\alpha \in GF(q^2)$ define α'

to be α^q. Then if $u_1, u_2, \ldots, u_r$ are a basis for M over $GF(q^2)$, the expression

$$\rho(\Sigma\alpha_i u_i, \Sigma\beta_i u_i) = \Sigma(\alpha_i\beta_i' - \alpha_i'\beta_i)/\gamma,$$

where γ is a primitive fourth root of unity, is a skew symmetric bilinear form on M of rank 2r with values in GF(q). Since all such forms are equivalent we can assume ρ is the fundamental form. $q^2 - 1$ is divisible by 2^{m_1+1} so that $GF(q^2)$ contains a primitive 2^{m_1+1}-th root of unity θ. Then $\theta\theta' = \theta^{q+1} = \theta^{2^{m_1}} = -1$. Let t_1 be the transformation of $GF(q^2)$ $\alpha \to \theta^2\alpha$ and let t_2 be the transformation $\alpha \to \theta\alpha'$.

t_1 is of order 2^{m_1}, t_2 is of order 4 and together they generate a generalized quaternion group of order 2^{m_1+1}. Call this group T and let P_o be a Sylow 2-subgroup of the symmetric group on the numbers 1, 2, ..., r. All transformations $\bar{y}$ of M of the form

$$\bar{y}(\Sigma\alpha_i u_i) = \Sigma(T_i\alpha_i)u_{\sigma(i)},$$

where the T_i are taken from T and σ is a permutation from P_o, form a Sylow 2-subgroup of S(2r, q).

Thus we may suppose $\bar{S}$ to be a subgroup of the group just described. We first need a lemma giving additional information about $\bar{g}_1$ and then we will be ready to finish the proof of lemma 9.

Lemma 11. The permutation σ associated with $\bar{g}_1$ is the identical permutation.

Proof. It is shown in [3, p. 23] that σ is of order smaller than the order of g_1. First suppose σ is of order > 2. Then the order of g_1 is greater than 4. Thus $\bar{g}_1{}^2$ when completely reduced has only one non-trivial block. But the permutation associated with $\bar{g}_1{}^2$ is σ^2 which has at least 2 disjoint non-trivial cycles. Clearly this is a contradiction. Thus $\sigma^2 = 1$.

Now suppose $\sigma \neq 1$. Assume, say, $\sigma(1) = 2$, $\sigma(2) = 1$. In the complete reduction of $\bar{g}_1$ on M there is only one non-trivial block. Thus $\bar{g}_1$ must be the identity on $\sum_{i \neq 1,2} \alpha_i u_i$. Now $\bar{g}_1(\alpha_1 u_1 + \alpha_2 u_2) = T_1 \alpha_1 u_2 + T_2 \alpha_2 u_1$.
Thus $\bar{g}_1{}^2(\alpha_1 u_1 + \alpha_2 u_2) = T_2 T_1 \alpha_1 u_1 + T_1 T_2 \alpha_2 u_2$.
One of $T_2 T_1$ or $T_1 T_2$ must not be the identity of T. But if either one is the identity then the other is also. Thus neither one is the identity and so $\bar{g}_1{}^2$ has two non-trivial blocks which can happen only if $\bar{g}_1$ is of order 4 which implies that $T_1 T_2$ is of order 2. Thus both $T_1 T_2$ and $T_2 T_1$ must be the transformation

$$\alpha \rightarrow -\alpha.$$

(This is the only element of order 2 in T.) Thus the centralizer of $\bar{g}_1{}^2$ in M has co-dimension 4 over $GF(q)$ whereas it should be 2. This proves that $\sigma = 1$.

Therefore $\bar{g}_1$ fixes each u_i and must act trivially on α_i for all but one value of i, i = 1, say. Thus

$$\bar{g}_1(\sum \alpha_i u_i) = A \alpha_1 u_1 + \sum_{i \neq 1} \alpha_i u_i$$

where A is an element of order 2^{m_1} in T. Then we have

$$\bar{h}_1(\sum \alpha_i u_i) = A^{2^{m_1 - 2}} \alpha_1 u_1 + \sum_{i \neq 1} \alpha_i u_i \text{ and}$$

$$\bar{h}_1^{\,2}(\sum \alpha_i u_i) = -\alpha_1 u_1 + \sum_{i \neq 1} \alpha_i u_i .$$

Assume $\bar{x}_1(\sum \alpha_i u_i) = \sum T_i \alpha_i u_{\pi(i)}$.

<u>Case I: $\pi(1) \neq 1$</u>. Assume, say, that $\pi^{-1}(1) = 2$.

A straight forward calculation yields

$$(\bar{h}_1, \bar{x}_1)(\sum \alpha_i u_i) = A^{-2^{m_1-2}} \alpha_1 u_1 + (T_2^{-1} A^{2^{m_1-2}} T_2)\alpha_2 u_2 + \sum_{i \neq 1,2} \alpha_i u_i .$$

But $(A^{-2^{m_1-2}})^2 = -I$ ($-I$ is the transformation $\alpha \to -\alpha$) and $(T_2^{-1} A^{2^{m_1-2}} T_2)^2 = T_2^{-1}(-I)T_2 = -I$. Thus

$$(\bar{h}_1, \bar{x}_1)^2(\sum \alpha_i u_i) = -\alpha_1 u_1 - \alpha_2 u_2 + \sum_{i \neq 1,2} \alpha_i u_i .$$

It is easy to check that this is the same as $(\bar{h}_1^{\,2}, \bar{x}_1)$.

<u>Case II: $\pi(1) = 1$</u>. It is easily verified that in this case $(\bar{h}_1^{\,2}, \bar{x}_1)$ is the identity while

$$(\bar{h}_1, \bar{x}_1)^2(\sum \alpha_i u_i) = (A^{2^{m_1-2}}, T_1)^2 \alpha_1 u_1 + \sum_{i \neq 1} \alpha_i u_i .$$

Now A is of order 2^{m_1} in a generalized quaternion group of order 2^{m_1+1} so that the only conjugates of A in T are A and A^{-1}. Thus

$$(A^{2^{m_1-2}}, T_1)^2 = (A^{-2^{m_1-2}} A^{\pm 2^{m_1-2}})^2 = (-I)(-I) = I.$$

Thus $(\bar{h}_1, \bar{x}_1)^2$ is also the identity.

Therefore it has been shown that

$$(\bar{h}_1, \bar{x}_1)^2 = (\bar{h}_1^{\,2}, \bar{x}_1)$$

in all cases. This finishes the proof of lemma 9. With this, theorems 1 and 2 are also proved.

III. Groups with $e_2 = 2$

Since, if a solvable group satisfies $e_2 \leq 1$, then it also must satisfy $\ell_2 \leq e_2$, it is perhaps a natural question to ask whether this is also true if $e_2 = 2$. This question is even more to the point when it is realized that (to the author's knowledge, at least) there have not been found any examples of groups in which ℓ_2 exceeds e_2.

In an argument similar to that used in the preceding section we show that if $e_2(G) = 2$ then a normal subgroup (which may be the whole group) satisfies $\ell_2 \leq 2$. If all elements of order 4 are contained in this subgroup then it will follow that $\ell_2(G) \leq 2$. As an application we show that this must happen if G is of exponent 12, i.e., if $x^{12} = 1$ for all x in G.

Unfortunately, in the situation analogous to the hypothesis of theorem 2, we need an additional assumption to prove the desired result for groups of exponent 12. This condition is that $\overline{G}$ should be irreducibly represented on the space V. Therefore before proceeding further we prove a reduction theorem which allows us to assume this extra condition.

A proposition is said to be of type A if it has the following form:

If G is a finite p-solvable group satisfying condition B, then $\ell_p(G) \leq f(e_p(G))$,

where f is a monotonically increasing function defined for non-negative integral arguments, $f(0) = 0$, and condition B is either vacuous or states that

$$e_{p_i}(G) \leq a_i \qquad \text{for some set, possibly}$$

infinite, of primes p_i and non-negative integers a_i.

Note that the proposition that if G is of exponent 12 then $\ell_2(G) \leq e_2(G)$ is certainly of type A for a group of exponent 12 is of order $2^a 3^b$ and thus solvable by a well-known theorem of Burnside and the condition that G is of exponent 12 is equivalent to stating that $e_2(G) \leq 2$, $e_3(G) \leq 1$, and $e_p(G) \leq 0$ for all primes $p \neq 2, 3$. We now state and prove our reduction theorem.

Theorem 3. In proving a proposition $\mathfrak{T}$ of type A it suffices to prove it for the following special case:

(1) G is the normal product of V by $\bar{G}$ where V is a vector space over F, a specified finite field of characteristic p, and $\bar{G}$ is a p-solvable linear group on V having no normal p-subgroup other than the identity.

(2) All groups of order at most $|G|$ satisfy $\mathfrak{T}$.

(3) V is an irreducible F-$\bar{G}$ module.

Proof. First it should be explained that F may be arbitrarily picked from among the finite fields of characteristic p, but once it is chosen it is to remain fixed for the rest of the argument.

In proving theorem 3 we assume the proposition $\mathfrak{T}$ is valid for the special case and then prove it is valid for the general case.

Now suppose G is the group of smallest order satisfying the hypothesis of T but not the conclusion, and let

$$1 = P_0 \trianglelefteq N_0 \triangleleft P_1 \triangleleft \cdots \triangleleft P_\ell \trianglelefteq N_\ell = G$$

be the upper p-series of G. Since f(0) = 0 we must have $\ell_p(G) > 0$. Now if F_1/N_0 is the Frattini subgroup of P_1/N_0, then, as is shown in [3], $\ell_p(G/F_1) = \ell_p(G)$ so that if $F_1 \neq 1$ then we would have a proper factor group of G satisfying the hypothesis but not the conclusion of the proposition T.

Therefore assume $F_1 = 1$. Thus P_1 is an elementary abelian p-group which we identify with a vector space V_1 over GF(p). G/P_1 is faithfully represented as a linear group $\bar{G}$ on V_1.

Now by [3, p. 4] we find that we may assume that G has only one minimal normal subgroup. This subgroup must be contained in V_1 and we denote it with M. If $M = V_1$ then $\bar{G}$ is irreducibly represented on V_1. Now if $M \neq V_1$ and if $\bar{G}$ is faithfully represented on V_1/M then we have that $\ell_p(G/M) = \ell_p(G)$ so that we would have a contradiction to the definition of G.

Finally, suppose that $V_1 \neq M$ and $\bar{G}$ is not faithfully represented on V_1/M. Then the elements of $\bar{G}$ centralizing V_1/M form a normal subgroup of $\bar{G}$ greater than the identity. Let Q be a minimal normal subgroup of $\bar{G}$ centralizing V_1/M. Clearly Q must be a p'-group so that V as a Q-module is completely reducible. Thus there exists a Q-module M_1 such that $V_1 = M \oplus M_1$. Clearly Q is the identity on M_1 but is not the identity on M since Q is faithfully represented on V_1. Now let M' be the centralizer of Q in V_1. We have M' normal in G since Q is normal in $\bar{G}$, and $M' \geq M_1$ but $M' \ngeq M$. All this contradicts the fact that M_1 is the unique minimal normal subgroup of G.

Thus we see from the above that we can assume that $\bar{G}$ is irreducibly represented over GF(p). One consequence of this is that if H is any normal subgroup greater than the identity in $\bar{G}$ then H can have no non-zero fixed vector in V_1. For if H did have a non-zero fixed vector then all the vectors fixed by H would form a non-trivial submodule of V_1.

Now F is some finite field of characteristic p so F must be a finite extension of GF(p). Let $1 = \theta_0, \theta_1, \ldots, \theta_r$ be a basis for F over GF(p) and let $v_1, v_2, \ldots, v_s$ be a basis for V_1 over GF(p). Finally let V be the vector space over F with basis $v_1, \ldots, v_s$. Any vector of V is of the form

$$\sum_{j=1}^{s} \sum_{i=0}^{r} a_{ij} \theta_i v_j$$

where $a_{ij} \in GF(p)$. $\bar{G}$ acts on V in the obvious way.

Consider the group $G^* = \bar{G}V$, i.e., the normal product of V by $\bar{G}$. Suppose that g^* is of order p^m in G^*. Then

g^* is the ordered pair $(\bar{g}, \mathbf{v})$ for some $\bar{g} \in \bar{G}$ and $\mathbf{v} \in V$. Now if $\bar{g}$ is not of order p^m then $\bar{g}$ must be of order p^{m-1} and cannot be exceptional on V. Thus there must be a $\mathbf{v}_i$ such that $(\bar{g} - 1)^{p^{m-1}-1}\mathbf{v}_i \neq 0$. Therefore $\bar{g}$ is not exceptional on V_1 and so we have that $e_p(G) \geq (m-1) + 1 = m$.

Thus in any event $e_p(G) \geq e_p(G^*)$. Since for $q \neq p$, $e_q(G^*) = e_q(G)$ we have that G^* satisfies condition B. Furthermore $\ell_p(G) = \ell_p(G^*)$ so that if G^* satisfies T so does G.

Now suppose H is any normal p'-subgroup other than the identity in $\bar{G}$ and suppose

$$\mathbf{v} = \sum_{j=1}^{s} \sum_{i=0}^{r} c_{ij}\, \theta_i \mathbf{v}_j$$

is a non-zero vector fixed by H. Since $\mathbf{v} \neq 0$ the coefficient of $\mathbf{v}_j$ is not zero for some j, j = 1 say. F is a field so there exists $\alpha \in F$ such that

$$\alpha\left(\sum_{i=0}^{r} c_{i1}\, \theta_1\right) = 1.$$

Now H must also fix $\alpha\mathbf{v}$ which can be written in the form $\alpha\mathbf{v} = \mathbf{v}' + \mathbf{v}''$ where

$$\mathbf{v}' = \mathbf{v}_1 + \sum_{j=2}^{s} c'_{0j}\mathbf{v}_j, \quad \mathbf{v}'' = \sum_{j=2}^{s} \sum_{i=1}^{r} c'_{ij}\, \theta_i \mathbf{v}_j$$

and the c'_{ij} are the coefficients for $\alpha\mathbf{v}$. It is clear that for H to fix $\alpha\mathbf{v}$ it must also fix $\mathbf{v}'$. But $\mathbf{v}'$ is a non-zero vector of V_1. Thus H can have no fixed non-zero vector in V.

Now if V is an irreducible F-$\bar{G}$ module then we are already at the special case of the theorem. Therefore assume U is a proper submodule.

If $\bar{G}$ is not faithfully represented on V/U then let Q be a minimal normal subgroup of $\bar{G}$ centralizing V/U. Q must be a p'-group so that V is completely reducible as an F-Q module. Thus there exists an F-Q module U' such that $V = U \oplus U'$. U' contains non-zero vectors since U is a proper submodule, and Q is the identity on U'. We have seen that Q cannot have any non-zero fixed vectors so this is a contradiction.

Therefore $\bar{G}$ is faithfully represented on V/U. Thus $\ell_p(G^*) = \ell_p(G^*/U)$ and of course $e_p(G^*) \geq e_p(G^*/U)$ so that if G^*/U satisfies proposition T so does G^* and then so does G.

Now we still have that any normal non-identity p'-subgroup H of $\bar{G}$ cannot have any fixed non-zero vectors in V/U since V is completely reducible as an F-H module. Thus if $\bar{G}$ is not irreducibly represented on V/U then the same argument as before yields that $\bar{G}$ is faithfully represented on a non-trivial factor module of V/U. Continuing in this way we finally arrive at the case where $\bar{G}$ is faithfully and irreducibly represented on some vector space over the field F. This is just the special case mentioned in the theorem and so theorem 3 is proved.

As was stated previously, one of the results of this section is that if G is of exponent 12, then $\ell_2(G) \leq e_2(G)$. Before proceeding further it might be well to justify this work. For in a group of order $2^a 3^b$ the 2-length and the 3-length can vary at most by one. Thus if it were true that the 3-length of a group of exponent 12 was 1, then it would be quite trivial to state that the 2-length was at most 2. However, in [4, p. 5] is found a group of exponent 12 but with 3-length 2. The group obtained by taking the direct product of this group with the symmetric group on 4 letters is a group of exponent 12 with both the 2-length and the 3-length equal to 2. Thus the stated result about groups of exponent 12 cannot be obtained in a trivial way by arguing on the 3-length.

For the rest of this section we make the following standing assumptions:

(1) $G = \bar{G}V$, the normal product of V by $\bar{G}$, where V is a vector space over a finite field F of characteristic 2.

(2) $\bar{G}$ is faithfully and irreducibly represented as a linear group over V.

(3) $\bar{G}$ is finite, solvable, and has no normal 2-group other than the identity.

(4) $e_2(G) \leq 2$.

Now we are interested in seeing under what conditions can $\ell_2(G)$ exceed $e_2(G)$. But if $e_2(\overline{G}) = 0$ then both $e_2(G)$ and $\ell_2(G)$ are 1 and if $e_2(\overline{G}) = 1$ then $\ell_2(\overline{G}) = 1$ and we have $\ell_2(G) = e_2(G) = 2$. Thus we may as well assume

(5) $e_2(\overline{G}) = 2$.

So far we haven't specified F. Our choice is given by

(6) If Q is any normal nilpotent $2'$-subgroup of class ≤ 2 in $\overline{G}$ then any irreducible representation of Q over F is in fact absolutely irreducible.

Since there are only finitely many subgroups of $\overline{G}$, it follows that by taking F to be a large enough extension of GF(2) we can assume (6) holds. Any finite field of characteristic 2 satisfying (6) is satisfactory for what follows.

Later we shall add to these assumptions the further one that G is of exponent 12. Actually it should be pointed out that until we restrict ourselves to groups of exponent 12, we will make no use of the fact that $\overline{G}$ is irreducibly represented on V.

The approach used to investigate the structure of G is similar to that used in the proof of theorem 2. We will show that if N is the largest normal $2'$-subgroup of $\overline{G}$, then a certain 2-subgroup, to be described later, must be contained in the greatest normal 2-subgroup of $\overline{G}/N$. In particular if $\ell_2(G) > 2$ (which is the same as saying that $\ell_2(\overline{G}) > 1$), we will see that there must exist an element of order 4 of a special type in $\overline{G}$.

First let H be the following normal subgroup of $\bar{G}$: $x \in H$ if, and only if, for every normal nilpotent subgroup Q of class at most 2 in $\bar{G}$, x fixes every minimal characteristic F-Q submodule of V. A normal nilpotent subgroup of $\bar{G}$ must be a $2'$-group since otherwise $\bar{G}$ would have a non-trivial normal 2-subgroup. Thus if Q is such a group then V splits into the sum of minimal characteristic F-Q modules, so the definition of H is intelligible.

Now from (5) there are elements of order 4 in $\bar{G}$, and from (4) all such elements must be exceptional. Thus if g is of order 4 in $\bar{G}$, then g^2 must be in H by lemma 4. Hence H is greater than the identity. H has no normal 2-group except for the identity and the largest normal $2'$-group in H is $H \cap N$ (N being the greatest normal $2'$-group in $\bar{G}$).

Let $\bar{Q}$ be the greatest normal nilpotent subgroup of H. $\bar{Q} = \bar{Q}_1 \times \bar{Q}_2 \times \dots$ where $\bar{Q}_i$ is a q_i-group and the q_i are distinct odd primes for distinct i. Now it is still true that H centralizes any normal abelian subgroup of $\bar{G}$. Thus the proof of lemma 6 is applicable and we obtain $c(\bar{Q}) = 2$. Now let $K_i = C^1(\bar{Q}_i)$ and let $K = K_1 \times K_2 \times \dots$ As in the proof of theorem 2 we find that no non-identity 2-element of H centralizes K.

Now let H_1 be the subgroup of $\bar{G}$ consisting of all elements which fix every minimal characteristic F-K_i module for all i. Since $c(K_i) \leq 2$, $\bar{G} \trianglerighteq H_1 \trianglerighteq H$. H_1 has no normal 2-subgroup except for the identity and its greatest normal $2'$-subgroup is $H_1 \cap N$.

Let P be a Sylow 2-subgroup of H_1. $P \neq 1$ since $e_2(\bar{G}) = 2$ and if g is of order 4 then $g^2 \in H$. Now the square of any element of P must be in H. Thus $P/P \cap H$ is of exponent 2 and thus abelian. Therefore $P' < H$. We now prove two lemmas which will then enable us to show directly that PN/N is normal in $\bar{G}/N$.

Lemma 12. Suppose that g and h are two elements of P and V' is a minimal characteristic F-K_i submodule of V. Let Q, g_1, and h_1 be the restrictions of K_i, g, and h, respectively, to V'. Then, if $(Q, h_1^2) = 1$, it follows that $(Q, (g_1, h_1)) = 1$.

Proof. Assume $(Q, (g_1, h_1)) \neq 1$. Therefore neither g_1 nor h_1 centralizes Q. Now if $(Q, g_1^2) = 1$ then

$$(Q, (g_1, h_1)) = (Q, (g_1h_1)^2) \quad \text{and}$$

$$(Q, (g_1h_1, h_1)) = (Q, h_1^{-1}g_1^{-1}h_1g_1h_1^2) = (Q, (g_1, h_1)^{-1})$$

In this case simply replace g_1 by g_1h_1. Therefore, without loss of generality, we may assume that $(Q, g_1^2) \neq 1$ along with $(Q, h_1^2) = 1$ and $(Q, (g_1, h_1)) \neq 1$.

Now exactly as in the proof of lemma 9 we obtain that Q is an extra-special q-group ($q = 3$ since g_1 is of order 4 and thus exceptional so that $4 - 1$ must be a power of q), Q/Q' is a symplectic space, g_1 and h_1 preserve the symplectic structure of Q/Q', and we may assume that g_1 and h_1 operate on Q/Q' as follows

$$\bar{g}_1(\sum \alpha_i u_i) = A\alpha_1 u_1 + \sum_{i \neq 1} \alpha_i u_i$$

$$\bar{h}_1(\sum \alpha_i u_i) = \sum T_i \alpha_i u_{\sigma(i)}$$

where σ is a permutation of order ≤ 2 (since $(Q, h_1^2) = 1$), and A and the T_i are chosen from a group isomorphic to the quaternion group of order 8 (since $q = 3$). In addition A must be of order 4 since $(Q, g_1^2) \neq 1$.

Now if σ does not fix 1 then $(\bar{g}_1, \bar{h}_1)$ would be of order 4 but its centralizer in Q/Q' would have co-dimension 4 over GF(3). Thus (g_1, h_1) would be of order 4 but not exceptional which is impossible.

Thus σ fixes 1 and since $(Q, h_1^2) = 1$ we must have

$$\bar{h}_1(\sum \alpha_i u_i) = \pm \alpha_1 u_1 + \sum_{i \neq 1} T_i \alpha_i u_{\sigma(i)}.$$

It is now an easy matter to verify that $(\bar{g}_1, \bar{h}_1) = 1$ and the lemma is proved.

<u>Corollary</u>. If $g, h \in P$ and $h^2 = 1$, then $(g, h) = 1$.

<u>Proof</u>. (g, h) is in P' and thus in H. So if (g, h) is not 1 then $(K_i, (g, h)) \neq 1$ for some K_i. The lemma states that this cannot happen.

<u>Lemma 13</u>. If $g, h \in P$, then $(g, h)^2 = 1$.

<u>Proof</u>. Suppose that $(g, h)^2 \neq 1$. Then for some K_i $(K_i, (g, h)^2) \neq 1$. Choose V' to be a minimal characteristic F-K_i submodule of V such that $(K_i, (g, h)^2)$ is not the identity on V'. Define Q, g_1, and h_1 as in the previous lemma. Then if either (Q, g_1^2) or (Q, h_1^2) is the identity, then $(g_1, h_1) = 1$. Therefore assume neither g_1^2 nor h_1^2 centralize Q. Thus g_1 and h_1 are exceptional and of order 4. Hence Q is a 3-group and g_1 and h_1 operate on Q/Q' as follows:

$$\bar{g}_1(\sum \alpha_i u_i) = A\alpha_1 u_1 + \sum_{i \neq 1} \alpha_i u_i$$

$$\bar{h}_1(\sum \alpha_i u_i) = B\alpha_j u_j + \sum_{i \neq j} \alpha_i u_i.$$

Now if $j \neq 1$ then $(\bar{g}_1, \bar{h}_1) = 1$ and if $j = 1$ then we have

$$(\bar{g}_1, \bar{h}_1)^2(\sum \alpha_i u_i) = (A, B)^2\alpha_1 u_1 + \sum_{i \neq 1} \alpha_i u_i.$$

But A and B are elements of a quaternion group so that $(A, B)^2 = 1$ and the lemma is proved.

<u>Theorem 4</u>. $PN/N \trianglelefteq \bar{G}/N$.

<u>Proof</u>. I shall prove that $P(H_1 \cap N)/H_1 \cap N \trianglelefteq H_1/H_1 \cap N$ which is equivalent to the theorem since $H_1 \trianglelefteq \bar{G}$.

Let P_1 be the subgroup of P such that $P_1(H_1 \cap N)/H_1 \cap N$ is the largest normal 2-subgroup of $H_1/H_1 \cap N$. $P_1 \trianglelefteq P$ and $P_1 \trianglerighteq Z(P)$ (since P_1 must contain its centralizer in P). Thus by the corollary to lemma 12, P_1 contains all elements of order 2 in P. Let $P_2 = \{x \mid x^2 = 1, x \in P_1\}$. P_2 is an elementary abelian group which is normal, modulo $H_1 \cap N$, in H_1. Let C consist of those elements of $H_1/H_1 \cap N$ which centralize both P_2 and P_1/P_2. Clearly C is a normal subgroup of $H_1/H_1 \cap N$. But any $2'$-element in C would then have to centralize P_1 contrary to P_1 containing its centralizer in $H_1/H_1 \cap N$. Thus C is a normal 2-subgroup of $H_1/H_1 \cap N$. But from the corollary to lemma 12 **and** from lemma 13, $P(H_1 \cap N)/H_1 \cap N \leq C$. Thus $P = P_1$ and the theorem is proved.

<u>Corollary</u>. $\ell_2(H_1) = 1$.

Now let S be a Sylow 2-subgroup of $\bar{G}$ which contains P. From the theorem it follows that P is normal in S.

Lemma 14. If P contains all elements of order 4 in S, then $\mathcal{l}_2(\bar{G}) = 1$.

Proof. If S = P we are done. Therefore assume $S \neq P$. Then if $x \in S-P$ we must have $x^2 = 1$. Also $x \in S-P$, $y \in P$ implies that $xy \in S-P$ so that $(xy)^2 = 1$ which implies that $x^{-1}yx = y^{-1}$. Thus x induces the autormorphism $y \rightarrow y^{-1}$ of P. This can be an automorphism only if P is abelian. If both x_1 and x_2 are in S-P, then x_1x_2 centralizes P. But $e_2(\bar{G}) = 2$ so that P does contain elements of order 4. Thus x_1x_2 cannot be in S-P.

Therefore $|S/P| = 2$ and P is abelian. Now if $x \in S-P$, $y \in P$, then $(x, y) = x^{-1}y^{-1}xy = y^2 \in \Phi(P)$ and so x centralizes $P/\Phi(P)$. Therefore PN/N cannot be the largest normal 2-subgroup of $\bar{G}$/N. But P is maximal in S. Then SN/N is the largest normal 2-subgroup of $\bar{G}$/N. Thus $\mathcal{l}_2(\bar{G}) = 1$.

To our original assumptions (1)--(6) we now add

(7) G is of exponent 12.

This implies that K must be a 3-group. We prove that $\mathcal{l}_2(\bar{G}) = 1$ in this case by showing that the hypothesis of lemma 14 is satisfied.

Now suppose there exists an element of order 4 in S-P. g^2 is in H so $(K, g^2) \neq 1$. Let $V = V_1 \oplus V_2 \oplus \dots$ be the decomposition of V into minimal characteristic F-K modules. Since $g \in S-P$, g does not fix some V_i. g^2 does fix each V_i and if g^2 is not the identity on a V_i then g must fix that

V_i for otherwise g could not be exceptional. (It is shown in [3] that if g is exceptional then the first power of g fixing a V_i must be exceptional on that V_i.)

Before proceeding further, we first need the following result:

<u>Lemma 15</u>. There exist x and y in K such that $((x, g^2), (y, g^2)) \neq 1$.

<u>Proof</u>. Let $C = \{x \mid x \in K, (x, g^2) \in Z(K)\}$. Clearly $C \geqslant Z(K)$ but $C \neq K$ since then g^2 would centralize Z(K) and K/Z(K) contrary to $(K, g^2) \neq 1$. (g^2 centralizes Z(K) by lemmas 4 and 5.) K/Z(K) is an elementary abelian 3-group so that there must be a GF(3)-g module of K/Z(K) complementary to C/Z(K). Thus $K/Z(K) = L/Z(K) \oplus C/Z(K)$ and g normalizes L. Now for all $x \in L - Z(K)$, $(x, g^2) \notin Z(K)$.

Now suppose $x, y \in L - Z(K)$ and $(x, g^2)(y, g^2)^{-1} \in Z(K)$. But $(xy^{-1}, g^2) = (x, g^2)^{y^{-1}}(y^{-1}, g^2)$. But since K/Z(K) is abelian $(xy^{-1}, g^2) \equiv (x, g^2)(y^{-1}, g^2) \pmod{Z(K)}$, and, similarly, $1 = (yy^{-1}, g^2) \equiv (y, g^2)(y^{-1}, g^2)$. Thus $(xy^{-1}, g^2) \equiv (x, g^2)(y, g^2)^{-1} \equiv 1 \pmod{Z(K)}$. Thus $xy^{-1} \in Z(K)$. Therefore $(x, g^2) \equiv (y, g^2) \pmod{Z(K)}$ if, and only if, $x \equiv y \pmod{Z(K)}$ for $x, y \in L$.

It immediately follows from this that for any $x \in L$, there exists a y such that $x \equiv (y, g^2) \pmod{Z(K)}$. Now L cannot be abelian since g normalizes L and g^2 does not centralize it. From all this we see that there exist $x, y \in L$ such that $((x, g^2), (y, g^2)) \neq 1$.

Now, taking x and y to satisfy the lemma, we may assume without any loss of generality that $((x, g^2), (y, g^2))$ is not the identity on V_1. Clearly this implies that g^2 is not the identity on V_1 so g must fix V_1.

Since g does not fix every V_i, assume g does not fix V_2. Therefore g^2 is the identity on V_2 which then also must be the case for (x, g^2) and (y, g^2).

Now V is an irreducible F-$\overline{G}$ module so that there must be an element taking V_1 into V_2. (This is the only place where irreducibility is made use of.) Such an element must be of the form zh where $h \in S$ and z is from a Sylow 3-subgroup of $\overline{G}$ which necessarily must contain K. Our ultimate contradiction will be that z and K generate elements of order 9 which is impossible in a group of exponent 12.

If $hV_1 = V_m$ then $zV_m = V_2$. Set $g_1 = hgh^{-1}$. Then we have that

$$((x^{h^{-1}}, g_1{}^2), (y^{h^{-1}}, g_1{}^2))$$

is not the identity on V_m. It is further claimed that $g_1V_2 \neq V_2$. For suppose $g_1V_2 = V_2$. Then $gh^{-1}V_2 = h^{-1}V_2$ and, since $gV_2 \neq V_2$, this implies that $h^{-1}V_2 = V_j$, $j \neq 2$. Then we must have that $gV_j = V_j$. But $gh^{-1} \in S$ so that $(gh^{-1})^2 \in H$. Thus $(gh^{-1})^2$ fixes V_2. Therefore $gh^{-1}V_j = V_2$. Also $(h^{-1})^2$ must fix V_2 so we have $h^{-1}V_j = V_2$. We finally conclude that $V_2 = gh^{-1}V_j = gV_2$ which is a contradiction. Hence $g_1V_2 \neq V_2$.

If we now replace V_1, g, x, and y by V_m, g_1, $x^{h^{-1}}$, and $y^{h^{-1}}$, respectively, we may assume that $zV_1 = V_2$, $((x, g^2), (y, g^2))$ is not the identity on V_1, and $gV_2 \neq V_2$. Let $x_1 = (x, g^2)$ and $y_1 = (y, g^2)$. x_1 and y_1 must be the identity on V_2 since g^2 is. Now since G is of exponent 12, z must be of order 3. Thus $zV_1 = V_2$, $zV_2 = V_n$ $(n \neq 1, 2)$, and $zV_n = V_1$.

Now let $V' = V_1 \oplus V_2 \oplus V_n$. V' is fixed by z and the restrictions of x_1, y_1, and z to V' are

$$z = \begin{pmatrix} O & O & A \\ B & O & O \\ O & C & O \end{pmatrix}, \quad x_1 = \begin{pmatrix} M & O & O \\ O & I & O \\ O & O & M_1 \end{pmatrix}, \quad y_1 = \begin{pmatrix} N & O & O \\ O & I & O \\ O & O & N_1 \end{pmatrix},$$

where I is the identity and O the zero matrix. Now (x_1, y_1) is not the identity on V_1 but $(x_1, y_1) \in Z(K)$ and $Z(K)$ is represented on V_1 as a cyclic group generated by a scalar matrix. Thus $(M, N) = \omega I$ where ω is a primitive third root of unity. From $z^3 = 1$ we get $C = A^{-1}B^{-1}$.

Now z, x_1, and y_1 all belong to the same Sylow 3-subgroup of $\overline{G}$ which must be of exponent 3. $(zx_1)^3 = z^3 {x_1}^{z^2} {x_1}^{z} x_1$ so that we must have ${x_1}^{z^2} {x_1}^{z} x_1 = 1$. Direct computation yields that this implies that $M_1 = A^{-1}M^{-1}A$. Similarly $N_1 = A^{-1}N^{-1}A$.

Thus $(M_1, N_1) = A^{-1}(M^{-1}, N^{-1})A$. But M and N generate a group of exponent 3 and class 2. It follows easily that $(M^{-1}, N^{-1}) = (M, N) = \omega I$.

Thus $(x_1, y_1) = \begin{pmatrix} \omega I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & \omega I \end{pmatrix}$.

It is now a simple matter to verify that

$$(x_1, y_1)^{z^2}(x_1, y_1)^{z}(x_1, y_1) \neq 1.$$

Thus $z(x_1, y_1)$ is a 3-element of **order** greater than 3. This contradiction proves that the hypothesis of lemma 14 is satisfied and thus:

Theorem 5. If G is a finite group of exponent 12, then $\ell_2(G) \leq e_2(G)$.

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