STUDIES ON THE PROPAGATION OF ELASTIC WAVES IN SOLID MEDIA

Thesis by

Charles Kenneth Grimes

In Partial Fulfillment of the Requirements

For the Degree of

Doctor of Philosophy

California Institute of Technology

Pasadena, California

1964

Acknowledgments

It is a pleasure to acknowledge the many valuable suggestions, criticisms and inestimable assistance given by my research chairman, Professor Julian D. Cole, during all stages of my research program. The sponsorship of the United States Air Force, without which my graduate study program would have been impossible, is also gratefully acknowledged. In addition, the assistance of Mrs. Jennavie H. Fiebiger, Air Force Institute of Technology, in preparation of the final draft of the manuscript is deeply appreciated.

Abstract

Several aspects of three basic problems concerned with the propagation of elastic waves in solid media are explored.

Stress and displacement correction terms resulting from application of a subsonically moving point load to the free surface of the infinite half-space are obtained using Fourier transform techniques (the load moves subsonically with respect to the longitudinal and transverse wave speeds). It is shown, for the supersonically travelling point load, that the solution is given, in the limit as the load velocity becomes large, by the well known solution of Sauter for the impulsive point load.

Analytic function theory is used to predict the existence of Rayleigh waves on the free surface of the infinite half-space and Stoneley waves along the welded interface between two dissimilar solid media. A brief analysis shows that free-running waves are also possible on the interior surface of an infinitely long cylindrical cavity. These waves are dispersive, however, because of the introduction of a characteristic length.

The early and long time approximations for the hoop stress generated through scattering of a plane dilatational wave by a cylindrical cavity in an infinite medium are developed. Use is made of Friedlandler's Riemann surface representation (early time) and expansion in Fourier series (long time).

Table of Contents

I. Stresses Produced by a Moving Point Load on the Surface of an Infinite Half-Space

1. Introduction

Numerous investigators have considered the stresses produced by the application of a travelling line load to the surface of an infinite half-space. The earliest work of interest is that of Lamb published in 1916 (1). In this paper, Lamb extended his earlier (2) solution for the stationary line load with periodic time dependent amplitude to the case of a travelling line load by use of an integral technique. Briefly, his method is based on summing up the contributions from an infinite array of stationary line loads to generate a solution for the moving line load. His principal application was to fluids which, of course, cannot transmit transverse waves and, therefore, concern us little here. Lamb also considered the problem of the travelling point source on the surface of an infinite fluid half-space (3).

More recently, Sneddon (4) has considered the problem of the travelling line load, assumed to have been moving with constant velocity for a long time, applied to the surface of the elastic half-space. His velocity is subsonic with respect to the dilatational and equivoluminal wave speeds of the medium. Still later, Cole and Huth (5) have extended

Sneddon's work to the cases of a subsonic, transonic and supersonic line load velocity. Finally, Ang (6) has obtained a transient solution for the subsonically moving line load whose motion is started impulsively from rest at time zero.

Extension of the work referenced above to the three-dimensional case is of interest for comparative purposes. This is easily done, in principle, by assuming a travelling delta function load to be applied to the free surface of an infinite half-space.

The stresses produced by application of the line load in the two-dimensional case and the delta function load in the three-dimensional case are, in effect, Green's functions. Their value is inherent in the fact that stresses caused by arbitrary loading can be obtained from them. This is done formally by applying the principle of superposition and is valid provided the medium being studied remains elastic.

2. Equations of Motion

It is convenient to define coordinates x, y, z as being fixed in the elastic medium. Consider a concentrated load, positive downward, acting on the free surface of the infinite half-space and moving in the negative x-direction with speed U. This configuration is defined in Fig. 1 below:

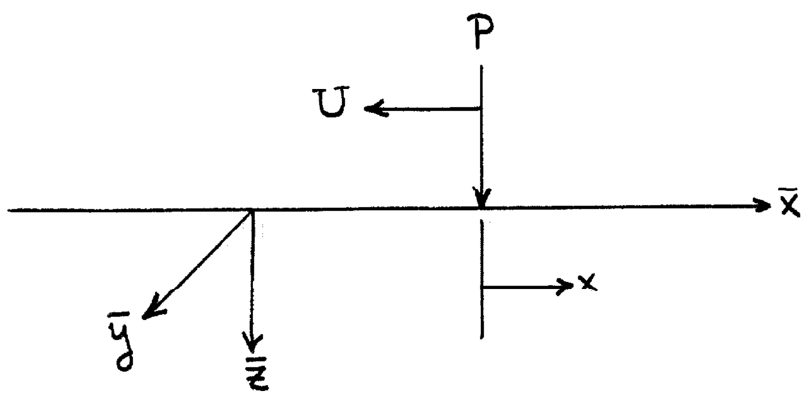


Fig. 1 Coordinates and Boundary Conditions

The location of the load can be specified by the equation:

$$\bar{x} + U\bar{t} = 0 \tag{2.1}$$

We can write the equation of motion for the elastic medium as

$$(\lambda+2\mu)\,\mathrm{grad}\,\mathrm{div}\,\vec{u} - \mu\,\mathrm{curl}\,\mathrm{curl}\,\vec{u} = \rho\vec{u}_{tt} \tag{2.2}$$

where λ and μ are Lame's constants, ρ is the mass density and $\vec{u} = \vec{u}(u,v,w)$ is the vector displacement (7).

Helmholtz vector decomposition theorem can be used to separate the equation of motion into its longitudinal and transverse parts. Write:

$$\vec{u} = \mathrm{grad}\,\phi + \mathrm{curl}\,\vec{\psi}\ ;\ \mathrm{div}\,\vec{\psi} = 0 \tag{2.3}$$

The equation of motion decomposes into:

$$(\lambda+2\mu)\,\mathrm{div}\,\mathrm{grad}\,\phi - \rho\phi_{tt} = 0 \tag{2.4}$$

$$\mu\,\mathrm{curl}\,\mathrm{curl}\,\vec{\psi} + \rho\vec{\psi}_{tt} = 0\ ;\ \mathrm{div}\,\vec{\psi} = 0 \tag{2.5}$$

The vector identity:

$$\mathrm{curl}\,\mathrm{curl}\,\vec{\psi} = \mathrm{grad}\,\mathrm{div}\,\vec{\psi} - \nabla^2\vec{\psi} \tag{2.6}$$

applies in rectangular cartesian coordinates and equations 2.4 and 2.5 become:

$$\nabla^2\phi = \frac{1}{C_L^2}\phi_{tt} \tag{2.7}$$

$$\nabla^2\vec{\psi} = \frac{1}{C_T^2}\vec{\psi}_{tt}\ ;\ \mathrm{div}\,\vec{\psi} = 0 \tag{2.8}$$

where:

$$C_L^2 = \frac{\lambda+2\mu}{\rho} \tag{2.9}$$

$$C_T^2 = \frac{\mu}{\rho} \tag{2.10}$$

and C_L^2, C_T^2 are the longitudinal and transverse wave speeds, respectively. It is obvious that solutions for the potentials ϕ and $\vec{\psi}$ yield a solution for the displacement $\vec{u}$ by application of equation 2.3.

It is now assumed that the load has been moving for a long time and that a steady stress pattern has been established. A Galilean transformation

$$x = \bar{x} + U\bar{t} \tag{2.11}$$

where $t = \bar{t}$, $y = \bar{y}$ and $z = \bar{z}$ has the effect of placing the observer on the load, i.e., he sees a steady stress pattern. Define:

$$M_L^2 = \frac{U^2}{C_L^2} \tag{2.12}$$

$$M_T^2 = \frac{U^2}{C_T^2} \tag{2.13}$$

and the potential equations become:

$$(1 - M_L^2)\phi_{xx} + \phi_{yy} + \phi_{zz} = 0 \tag{2.14}$$

$$(1 - M_T^2)\vec{\psi}_{xx} + \vec{\psi}_{yy} + \vec{\psi}_{zz} = 0 \; ; \; \operatorname{div} \vec{\psi} = 0 \tag{2.15}$$

3. Stresses and Boundary Conditions

The stresses may be obtained, after calculation of $\vec{u}$, from the stress tensor:

$$\bar{\bar{\tau}} = \lambda(\mathrm{div}\,\vec{u})\,\bar{\bar{I}} + \mu\{\mathrm{grad}\,\vec{u} + (\mathrm{grad}\,\vec{u})^{*}\} \tag{3.1}$$

where $\bar{\bar{I}}$ is the unit tensor and * represents the transpose.

At $z=0$, the boundary conditions are:

$$\tau_{zz} = \lambda\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) + 2\mu\frac{\partial w}{\partial z} = -\delta(x)\delta(y)P \tag{3.2}$$

$$\tau_{zx} = \tau_{xz} = \mu\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right) = 0 \tag{3.3}$$

$$\tau_{yz} = \tau_{zy} = \mu\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right) = 0 \tag{3.4}$$

and the definition for $\vec{u}$ can be used to obtain:

$$u = \frac{\partial\phi}{\partial x} + \left(\frac{\partial\psi^{(3)}}{\partial y} - \frac{\partial\psi^{(2)}}{\partial z}\right) \tag{3.5}$$

$$v = \frac{\partial\phi}{\partial y} + \left(\frac{\partial\psi^{(1)}}{\partial z} - \frac{\partial\psi^{(3)}}{\partial x}\right) \tag{3.6}$$

$$w = \frac{\partial\phi}{\partial z} + \left(\frac{\partial\psi^{(2)}}{\partial x} - \frac{\partial\psi^{(1)}}{\partial y}\right) \tag{3.7}$$

where the superscripts refer to the components of the vector potential $\vec{\Psi}$.

Use of equations 3.2 through 3.7 and the potential equations for ϕ and $\vec{\Psi}$ allows the boundary conditions to be rewritten after some manipulation as:

$$\tau_{zz} = \mu\phi_{xx}(M_T^2-2) + 2\mu(-\phi_{yy} + \psi^{(2)}_{xz} - \psi^{(1)}_{yz}) = -\delta(x)\delta(y)P \tag{3.8}$$

$$\tau_{zx} = \mu\left\{2\phi_{zx} - (M_T^2-2)\psi^{(2)}_{xx} - \psi^{(1)}_{yx} + \psi^{(3)}_{yz} + \psi^{(2)}_{yy}\right\} = 0 \qquad (3.9)$$

$$\tau_{yz} = \mu\left\{2\phi_{yz} + (M_T^2-1)\psi^{(1)}_{xx} - 2\psi^{(1)}_{yy} - \psi^{(3)}_{xz} + \psi^{(2)}_{xy}\right\} = 0 \qquad (3.10)$$

The div $\vec{\psi} = 0$ relation may be used to eliminate one of the transverse potential components from the above equations. However, this operation is reserved until later.

4. General Solution Procedure

Two cases must be distinguished depending on whether the moving load is subsonic $(0 < \beta_L^2, \beta_T^2 < 1)$ with respect to the longitudinal and transverse wave speeds of the medium or supersonic $(-\infty < \beta_L^2, \beta_T^2 < 0)$. Exact equations for the potential functions associated with the subsonically moving load will be developed in a subsequent section. Since exact solutions to these equations are not readily obtainable and since the supersonic problem is suspected to be of similar intractability, the limiting cases of low subsonic and high supersonic speeds will be explored in still later sections.

The range of x and y is from $-\infty$ to ∞. This suggests the use of an operational technique based on the double Fourier transform. In this study the double Fourier transform pair is defined by:

$$\bar{\phi}(\omega_1, \omega_2, z) = \iint_{-\infty}^{\infty} \phi(x, y, z)\, e^{-i(\omega_1 x + \omega_2 y)}\, dx\, dy \tag{4.1}$$

$$\phi(x, y, z) = \frac{1}{4\pi^2} \iint_{-\infty}^{\infty} \bar{\phi}(\omega_1, \omega_2, z)\, e^{i(\omega_1 x + \omega_2 y)}\, d\omega_1\, d\omega_2 . \tag{4.2}$$

In evaluating the Fourier transforms of the equations of motion and boundary conditions, the potential functions and their first derivatives with respect to x and y are assumed to vanish at $x, y = \pm\infty$. Also, the functions $\phi_{xx}, \ldots\ldots, \vec{\psi}_{xx}, \ldots$ are assumed to be square integrable over the range $(-\infty, \infty)$.

The transformed equations of motion become:

$$\bar{\phi}_{zz} - (\beta_L^2 \omega_1^2 + \omega_2^2)\bar{\phi} = 0 \; ; \; \beta_L^2 = 1 - M_L^2 \tag{4.3}$$

$$\bar{\vec{\psi}}_{zz} - (\beta_T^2 \omega_1^2 + \omega_2^2)\bar{\vec{\psi}} = 0 \; ; \; \beta_T^2 = 1 - M_T^2 \tag{4.4}$$

and the side condition, div $\vec{\psi} = 0$, transforms to:

$$\bar{\psi}_z^{(3)} = -i\left(\omega_1 \bar{\psi}^{(1)} + \omega_2 \bar{\psi}^{(2)}\right). \tag{4.5}$$

The transformed boundary conditions are:

$$\bar{\tau}_{zz} = -\mu\omega_1^2(M_T^2 - 2)\bar{\phi} + 2\mu(\omega_2^2\bar{\phi} + i\omega_1\bar{\psi}_z^{(2)} - i\omega_2\bar{\psi}_z^{(1)}) = -P \tag{4.6}$$

$$\frac{\bar{\tau}_{zx}}{\mu} = 2i\omega_1\bar{\phi}_z + \omega_1^2(M_T^2 - 2)\bar{\psi}^{(2)} + \omega_1\omega_2\bar{\psi}^{(1)} + i\omega_2\bar{\psi}_z^{(3)} - \omega_2^2\bar{\psi}^{(2)} = 0 \tag{4.7}$$

$$\frac{\bar{\tau}_{yz}}{\mu} = 2i\omega_2\bar{\phi}_z - \omega_2^2(M_T^2 - 1)\bar{\psi}^{(1)} + 2\omega_2^2\bar{\psi}^{(1)} - i\omega_1\bar{\psi}_z^{(3)} - \omega_1\omega_2\bar{\psi}^{(2)} = 0 \tag{4.8}$$

The complete solution to this boundary value problem is defined by the inverse Fourier transform of the solution obtained from equations 4.1 through 4.8. As noted earlier, two cases must be distinguished, i.e., the subsonic and supersonic moving load.

5. Subsonic Case

For this case, $0 < \beta_L^2, \beta_T^2 < 1$, and the potential equations are always elliptic. Solutions to equations 4.3 and 4.4 can be written as:

$$\bar{\phi}(\omega_1, \omega_2, z) = C_1 e^{az} + C_2 e^{-az} \; ; \; a^2 = \beta_L^2 \omega_1^2 + \omega_2^2 \tag{5.1}$$

$$\bar{\Psi}^{(i)}(\omega_1, \omega_2, z) = C_3^{(i)} e^{bz} + C_4^{(i)} e^{-bz} \; ; \; b^2 = \beta_T^2 \omega_1^2 + \omega_2^2 \tag{5.2}$$

where $a, b > 0$ always and i = 1, 2, 3.

The requirement for bounded solutions as z ranges from 0 to ∞ implies that $C_1 = C_3^{(i)} = 0$. Our solutions become:

$$\bar{\phi}(\omega_1, \omega_2, z) = C_2 e^{-az} \tag{5.3}$$

$$\bar{\Psi}^{(i)}(\omega_1, \omega_2, z) = C_4^{(i)} e^{-bz} \tag{5.4}$$

Equations 5.3 and 5.4 can now be substituted into the boundary conditions as defined by equations 4.6 - 4.8. Application of Cramer's rule then allows evaluation of the coefficients C_2 and $C_4^{(i)}$:

$$\mu |D| C_2 = -P\{2\omega_2^2 - \omega_1^2(M_T^2 - 2)\} \tag{5.5}$$

$$\mu |D| C_4^{(1)} = -P\{2ia\omega_2\} \tag{5.6}$$

$$\mu |D| C_4^{(2)} = P\{2ia\omega_1\} \tag{5.7}$$

and from the side condition:

$$C_4^{(3)} = \frac{i\,(\omega_1 C_4^{(1)} + \omega_2 C_4^{(2)})}{b} = 0 \qquad (5.8)$$

where the determinant of coefficients is:

$$|D| = \{2\omega_2^2 - \omega_1^2(M_T^2 - 2)\}^2 - 4ab(\omega_1^2 + \omega_2^2) \qquad (5.9)$$

The exact solutions for $\phi(x,y,z)$ and $\psi^{(i)}(x,y,z)$ can now be written in integral form as:

$$\phi(x,y,z) = \frac{P}{4\pi^2\mu}\iint_{-\infty}^{\infty} \frac{[\omega_1^2(M_T^2-2) - 2\omega_2^2]\, e^{-\sqrt{\omega_1^2\beta_L^2+\omega_2^2}\,z + i(\omega_1 x + \omega_2 y)}}{|D|}\, d\omega_1 d\omega_2 \qquad (5.10)$$

$$\psi^{(1)}(x,y,z) = \frac{-2iP}{4\pi^2\mu}\iint_{-\infty}^{\infty} \frac{\omega_2\sqrt{\omega_1^2\beta_L^2+\omega_2^2}\; e^{-\sqrt{\omega_1^2\beta_T^2+\omega_2^2}\,z + i(\omega_1 x + \omega_2 y)}}{|D|}\, d\omega_1 d\omega_2 \qquad (5.11)$$

$$\psi^{(2)}(x,y,z) = \frac{2iP}{4\pi^2\mu}\iint_{-\infty}^{\infty} \frac{\omega_1\sqrt{\omega_1^2\beta_L^2+\omega_2^2}\; e^{-\sqrt{\omega_1^2\beta_T^2+\omega_2^2}\,z + i(\omega_1 x + \omega_2 y)}}{|D|}\, d\omega_1 d\omega_2 \qquad (5.12)$$

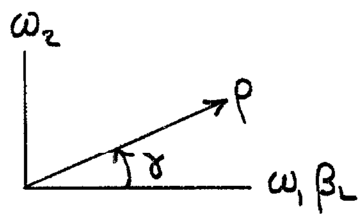
$$\psi^{(3)}(x,y,z) = 0 \qquad (5.13)$$

these improper integrals do not exist near the origin, however, differentiation with respect to x, y, or z eliminates this difficulty. For example:

$$\frac{\partial\phi}{\partial z} = \frac{P}{4\pi^2\mu}\iint_{-\infty}^{\infty} \frac{[2\omega_2^2-\omega_1^2(M_T^2-2)]\sqrt{\beta_L^2\omega_1^2+\omega_2^2}\; e^{-\sqrt{\beta_L^2\omega_1^2+\omega_2^2}\,z+i(\omega_1 x+\omega_2 y)}}{|D|}\, d\omega_1\, d\omega_2 \qquad (5.14)$$

which is now an odd function of ω_1 or ω_2 near the origin, thus, there is a change for the integral to exist.

A transformation of coordinates can be affected according to the following scheme:



Hence:

$$\omega_1\beta_L = \rho\cos\gamma \qquad (5.15)$$

$$\omega_2 = \rho\sin\gamma \qquad (5.16)$$

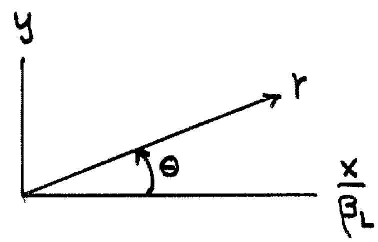
and the Jacobean of the transformation is:

$$|J| = \frac{\partial(\omega_1,\omega_2)}{\partial(\rho,\gamma)} = \frac{\rho}{\beta_L} \qquad (5.17)$$

Equation 5.14 can be rewritten as:

$$\frac{\partial \phi}{\partial z} = \frac{P}{4\pi^2 \mu} \int_0^{\infty} \int_0^{2\pi} \Phi\left(\frac{\rho}{\beta_L}\cos\gamma,\ \rho\sin\gamma\right) e^{-\rho z + i\rho r\cos(\theta-\gamma)} \frac{\rho}{\beta_L}\, d\gamma\, d\rho \qquad (5.18)$$

where:



$$x = r\beta_L \cos\theta \qquad (5.19)$$

$$y = r \sin\theta \qquad (5.20)$$

and:

$$\Phi\left(\frac{\rho}{\beta_L}\cos\gamma,\ \rho\sin\gamma\right)$$

$$= \frac{2\sin^2\gamma - (M_T^2 - 2)\dfrac{\cos^2\gamma}{\beta_L^2}}{\rho\left\{\left[2\sin^2\gamma - (M_T^2-2)\dfrac{\cos^2\gamma}{\beta_L^2}\right]^2 - 4\sqrt{\dfrac{\beta_T^2}{\beta_L^2}\cos^2\gamma + \sin^2\gamma}\ \text{times}\left(\dfrac{\cos^2\gamma}{\beta_L^2} + \sin^2\gamma\right)\right\}} \qquad (5.21)$$

Equation 5.18 may be integrated with respect to ρ and since $z > 0$, the expression reduces to:

$$\frac{\partial \phi}{\partial z} = \frac{P}{4\pi^2 \mu} \int_0^{2\pi} \frac{\beta_L \left[2\beta_L^2 \sin^2 \gamma - (M_T^2 - 2)\cos^2 \gamma \right] \, d\gamma}{\left[z - ir\cos(\theta - \gamma) \right] \left\{ \left[2\beta_L^2 \sin^2 \gamma - (M_T^2 - 2)\cos^2 \gamma \right]^2 - 4\beta_L \sqrt{\beta_T^2 \cos^2 \gamma + \beta_L^2 \sin^2 \gamma} \left(\cos^2 \gamma + \beta_L^2 \sin^2 \gamma \right) \right.} \qquad (5.22)$$

The potential functions, as well as their derivatives, must be real. However, even after rationalization of equation 5.22, it is not apparent that the imaginary part can be deleted. It appears that the complete expression must be retained at this stage in its evaluation.

A standard trick may now be employed in an attempt to evaluate the integral in equation 5.22. Let $\zeta = e^{i\gamma}$ and integrate around the unit circle in the complex ζ - plane. We have:

$$\cos \gamma = \frac{1}{2} \left(\zeta + \frac{1}{\zeta} \right)$$

$$i \sin \gamma = \frac{1}{2} \left(\zeta - \frac{1}{\zeta} \right)$$

$$d\gamma = -i \, \frac{d\zeta}{\zeta}$$

and equation 5.22 becomes:

$$\frac{\partial \phi}{\partial z} = \frac{P}{4\pi^2\mu}\left(\frac{i}{r}\right)\oint_{|\zeta|=1} \frac{\Psi(\zeta)\left(-i\,\frac{d\zeta}{\zeta}\right)}{\left\{ i\,\frac{z\zeta}{r} + \frac{\cos\theta}{2}\left(\zeta^2+1\right) - i\,\frac{\sin\theta}{2}\left(\zeta^2-1\right)\right\}\frac{1}{\zeta}} \tag{5.23}$$

where:

$$\Psi(\zeta) = \frac{\beta_L\left[-\frac{\beta_L^2}{2}\left(\zeta-\frac{1}{\zeta}\right)^2 + \frac{(1+\beta_T^2)}{4}\left(\zeta+\frac{1}{\zeta}\right)^2\right]}{\left[-\frac{\beta_L^2}{2}\left(\zeta-\frac{1}{\zeta}\right)^2 + \frac{(1+\beta_T^2)}{4}\left(\zeta+\frac{1}{\zeta}\right)^2\right]^2 - 4\beta_L\sqrt{\frac{\beta_T^2}{4}\left(\zeta+\frac{1}{\zeta}\right)^2 - \frac{\beta_L^2}{4}\left(\zeta-\frac{1}{\zeta}\right)^2}\left[\frac{1}{4}\left(\zeta+\frac{1}{\zeta}\right)^2 - \frac{\beta_L^2}{4}\left(\zeta-\frac{1}{\zeta}\right)^2\right]} \tag{5.24}$$

The poles implied by the denominator of equation 5.23 are readily found to be:

$$\zeta_{1,2} = \left\{-i\,\frac{z}{r} \pm \frac{i}{r}\sqrt{z^2+r^2}\right\} e^{i\theta} \tag{5.25}$$

and since $\frac{z}{r} > 0$ always, it can be shown that:

$$|\zeta_1| \leq 1$$

$$|\zeta_2| \geq 1$$

always. Thus, ζ_1 , lies inside the unit circle; ζ_2 outside.

Very little progress can be made in analysis of the function $\Psi(\zeta)$ unless we assume special values of Poisson's ratio, ν . For example, assume $\nu = 0$. Then, since:

$$\frac{C_L^2}{C_T^2} = \frac{M_T^2}{M_L^2} = \frac{\lambda + 2\mu}{\mu} = \frac{2(1-\nu)}{1-2\nu} \tag{5.26}$$

this special value of ν yields:

$$M_T^2 = 2M_L^2$$

and equation 5.24 becomes:

$$\Psi(\zeta) = \frac{\beta_L^2}{2\beta_L^3 - 2\sqrt{\beta_L^2 - \frac{M_L^2}{4\zeta^2}(\zeta^2+1)^2}\left[\beta_L^2 + \frac{M_L^2}{4\zeta^2}(\zeta^2+1)^2\right]} \tag{5.27}$$

which is still not susceptible to simple analysis since the denominator is of $O(\zeta^4)$ and the term in the radical implies the presence of four branch points.

Similar difficulties appear in attempts to evaluate the potentials $\psi^{(1)}$ and $\psi^{(2)}$ defined by equations 5.11 and 5.12. However, equations 5.10 - 5.13 are the exact integral expressions for the potentials. These can be evaluated, in principle at least, provided appropriate derivatives are taken to eliminate the difficulty at the origin. Certainly, the equations or integrals can be evaluated provided numerical values are assigned to say M_L^2, M_T^2, as well as ν, even though numerical methods may have to be resorted to in evaluating, for example, either equations 5.10 - 5.13 or in obtaining the roots of the denominator in equation 5.23. In particular, if $M_L^2 = M_T^2 = 0$, ν arbitrary, the solutions to equations 5.10 - 5.13 must yield the same stress field as that obtained by Boussinesq for the stationary point load problem. This result is indeed obtained as will be shown in a subsequent section.

It is of interest, before departing this section, to examine the singularities of the common denominator of equations 5.10 - 5.13. We can equate this denominator to zero and after rationalization write:

$$\left[2\omega_2^2 - \omega_1^2(M_T^2 - 2)\right]^4 - 16\left(\omega_1^2\beta_L^2 + \omega_2^2\right)\left(\omega_1^2\beta_T^2 + \omega_2^2\right)\left(\omega_1^2 + \omega_2^2\right)^2 = 0 \,. \qquad (5.28)$$

The roots of equation 5.28 yield the desired singularities. Now let $\frac{\omega_1^2}{\omega_2^2} = \lambda$ and divide the above expression by $(\omega_2^2)^4$ to obtain:

$$\left[2+\lambda(1+\beta_T^2)\right]^4 - 16(\lambda\beta_L^2+1)(\lambda\beta_T^2+1)(\lambda+1)^2 = 0 \qquad (5.29)$$

which yields on expansion:

$$A\lambda^4 + B\lambda^3 + C\lambda^2 + D\lambda = 0 \qquad (5.30)$$

Therefore, $\lambda = 0$ is always a root. The coefficients are:

$$\begin{aligned}
A &= 1 + 4\beta_T^2(1-4\beta_L^2) + 6\beta_T^4 + 4\beta_T^6 + \beta_T^8 \\
B &= 8 + 8\beta_T^2 + 24\beta_T^4 + 8\beta_T^6 - 16\beta_L^2(1+2\beta_T^2) \\
C &= 8 + 16\beta_T^2 + 24\beta_T^4 - 16\beta_L^2(2+\beta_T^2) \\
D &= 16(\beta_T^2 - \beta_L^2)
\end{aligned} \qquad (5.31)$$

A very simple analysis consisting of plotting the above coefficients for various values of β_T^2, ν fixed and for several values of ν, β_T^2 fixed shows that only coefficient A changes algebraic sign for the subsonic case. We deduce then that the character of the solution changes, dependent on whether coefficient A is positive, negative or zero. Suppose $A = 0$. Substitute

$$\beta_T^2 = 1 - M_T^2 = 1 - \frac{U^2}{C_T^2} \quad ; \quad \beta_L^2 = 1 - M_L^2 = 1 - \frac{U^2}{C_L^2}$$

to obtain

$$A = 0 = \left(\frac{U^2}{C_T^2}\right)^4 - 8\left(\frac{U^2}{C_T^2}\right)^3 + 24\left(\frac{U^2}{C_T^2}\right)^2 - 16\frac{U^2}{C_T^2}\left(1 + \frac{U^2}{C_L^2}\right) + 16\frac{U^2}{C_L^2} \tag{5.32}$$

Note that:

$$\frac{U^2}{C_T^2} = \frac{U^2\rho}{\mu} \tag{5.33}$$

$$\frac{U^2}{C_L^2} = \frac{U^2\rho}{\lambda + 2\mu} \tag{5.34}$$

Equation 5.32 is identical to equation 50, Love, pg. 308, provided (7):

$$\frac{U^2}{C_T^2} = \kappa'^2$$

$$\frac{U^2}{C_L^2} = h'^2$$

Love defines

$$\kappa'^2 = \frac{p^2\rho}{f^2\mu}$$

$$h'^2 = \frac{p^2\rho}{f^2(\lambda + 2\mu)}$$

where $\frac{p}{f}$ is the velocity of propagation of surface waves. Therefore, if we let $U \to \frac{p}{f}$, we conclude that the vanishing of coefficient A occurs when the surface load travels at a speed equal to the Rayleigh wave speed. Of particular significance is the fact that in the three-dimensional case no singularity is generated when the load travels at the Rayleigh wave speed. This is not so in the two-dimensional case. For example, see the subsonic portion of the work of Cole and Huth (5) in which the solutions indicate a singularity exists when the surface load travels at the Rayleigh wave speed.

As a concluding remark of perhaps academic interest, some simplification of the equations results if we assume $A = 0$. Then equation 5.30 becomes a quadratic in λ ($\lambda = 0$ always a root) whose roots are easily determinable. Assumption of specific values of λ and μ simplify the problem even further. For example, one might assume Poisson's condition $\lambda \equiv \mu$. That is to say, the problem is simplified, not simple!

6. Subsonic Approximation

In this instance we assume U to be very small compared to the longitudinal and transverse wave speeds in the medium. Then $\beta_L, \beta_T \to 1$ since $M_L, M_T \to 0$. We will compute the static terms plus the first correction terms resulting from the motion of the load for the stress τ_{zz} and the displacement in the x-direction u. As noted previously, the static portions must be identical with the results of Boussinesq.

We proceed with the computation of τ_{zz} first. The expression for τ_{zz} is obtained from equation 3.1 as:

$$\tau_{zz} = \mu \phi_{xx}(M_T^2 - 2) + 2\mu\left(-\phi_{yy} + \psi^{(2)}_{xz} - \psi^{(1)}_{yz}\right) \qquad (6.1)$$

and the integral expression for τ_{zz} is obtained by substitution of equations 5.10 - 5.12 into equation 6.1. We obtain:

$$\tau_{zz} = \frac{P}{4\pi^2}\iint_{-\infty}^{\infty}\Big\{-\left[\omega_1^2(M_T^2-2) - 2\omega_2^2\right]^2 e^{-\sqrt{\beta_L^2\omega_1^2+\omega_2^2}\,z + i(\omega_1 x + \omega_2 y)}$$
$$+ 4\sqrt{\beta_L^2\omega_1^2+\omega_2^2}\sqrt{\beta_T^2\omega_1^2+\omega_2^2}\,(\omega_1^2+\omega_2^2)\, e^{-\sqrt{\beta_T^2\omega_1^2+\omega_2^2}\,z + i(\omega_1 x + \omega_2 y)}\Big\}\frac{d\omega_1\, d\omega_2}{|D|} \qquad (6.2)$$

where $|D|$ is the determinant of coefficients defined by equation 5.9.

We assume that all appropriate terms in equation 6.2 can be expanded in a Taylor's series and obtain:

$$-\left[\omega_1^2(M_T^2-2)-2\omega_2^2\right]^2 = 4(\omega_1^2+\omega_2^2)^2\left\{1-\frac{\omega_1^2 M_T^2}{2(\omega_1^2+\omega_2^2)}\right\}^2$$

$$= 4(\omega_1^2+\omega_2^2)^2 - 4(\omega_1^2+\omega_2^2)\omega_1^2 M_T^2 + \omega_1^4 M_T^4 \qquad (6.3)$$

and:

$$\sqrt{\beta_L^2\omega_1^2+\omega_2^2} = \sqrt{(1-M_L^2)\omega_1^2+\omega_2^2} = (\omega_1^2+\omega_2^2)^{\frac{1}{2}}\left\{1-\frac{\omega_1^2 M_L^2}{\omega_1^2+\omega_2^2}\right\}^{\frac{1}{2}}$$

$$\cong (\omega_1^2+\omega_2^2)^{\frac{1}{2}}\left\{1-\frac{\omega_1^2 M_L^2}{2(\omega_1^2+\omega_2^2)}-\frac{\omega_1^4 M_L^4}{8(\omega_1^2+\omega_2^2)}+\cdots\right\} \qquad (6.4)$$

plus a similar expansion for $\sqrt{\beta_T^2\omega_1^2+\omega_2^2}$. The exponential terms become:

$$e^{-\sqrt{(1-M_L^2)\omega_1^2+\omega_2^2}\,z} \cong e^{-Az} + \frac{M_L^2\omega_1^2 z}{2A}e^{-Az}$$

$$+\frac{\omega_1^4 z M_L^4}{8A^3}e^{-Az} + \frac{\omega_1^4 z^2 M_L^4}{8A^2}e^{-Az} + \cdots \qquad (6.5)$$

where $A = (\omega_1^2+\omega_2^2)^{\frac{1}{2}}$. A similar expansion is obtained for the remaining exponential term.

The determinant of coefficients $|D|$ can be written after expansion and simplification as:

$$|D| \cong -2\omega_1^2(\omega_1^2+\omega_2^2)(M_T^2-M_L^2) + \omega_1^4 M_T^2(M_T^2-M_L^2)$$

$$+ \frac{\omega_1^4}{2}(M_T^4+M_L^4) + \cdots$$

or equivalently:

$$|D| \cong \frac{\omega_1^2(M_L^2-M_T^2)}{2}\left\{4(\omega_1^2+\omega_2^2) - 2\omega_1^2 M_T^2 + \omega_1^2\left(\frac{M_T^4+M_L^4}{M_L^2-M_T^2}\right) + \cdots\right\}$$

$$= \frac{\omega_1^2(M_L^2-M_T^2)}{2}\left\{4(\omega_1^2+\omega_2^2) - \omega_1^2 k M_L^2 + \cdots\right\}$$

where:

$$-kM_L^2 = -2M_T^2 + \left(\frac{M_T^4+M_L^4}{M_L^2-M_T^2}\right) = M_L^2\,\frac{\left(\frac{3M_T^2}{M_L^2} - 2 + \frac{M_L^2}{M_T^2}\right)}{\frac{M_L^2}{M_T^2}-1}$$

$$= -\left\{\frac{3\lambda^2+10\mu\lambda+9\mu^2}{\mu(\lambda+\mu)}\right\}M_L^2 \qquad (6.6)$$

since , $\dfrac{M_L^2}{M_T^2} = \dfrac{\mu}{\lambda+2\mu}$.

It is convenient to invert $|D|$. We obtain:

$$|D|^{-1} \cong \left\{\frac{2}{\omega_1^2(M_L^2-M_T^2)}\right\}\left\{\frac{1}{4(\omega_1^2+\omega_2^2)}\right\}\left\{1 + \frac{\omega_1^2 k M_L^2}{4(\omega_1^2+\omega_2^2)} + \cdots\right\} \qquad (6.7)$$

Now substitute equations 6.3 - 6.7 into 6.2 and after some simplification obtain:

$$\tau_{zz} \cong -\frac{P}{4\pi^2}\int\!\!\int_{-\infty}^{\infty} \Big\{ 1 + \frac{z}{4(\omega_1^2+\omega_2^2)^{\frac{1}{2}}}\Big[4(\omega_1^2+\omega_2^2)$$

$$+ \omega_1^2(M_L^2 - M_T^2) + k\omega_1^2 M_L^2\Big]$$

$$+ \frac{\omega_1^2 z^2(M_L^2+M_T^2)}{4}\Big\} e^{-(\omega_1^2+\omega_2^2)^{\frac{1}{2}} + i(\omega_1 x+\omega_2 y)} d\omega_1\, d\omega_2 \qquad (6.8)$$

The integrals in equation 6.8 can easily be evaluated if we transform to polar coordinates according to the following scheme:

$$\omega_1 = \rho\cos\gamma \;; \quad \omega_2 = \rho\sin\gamma$$

$$x = r\cos\theta \;; \quad y = r\sin\theta$$

We obtain:

$$\tau_{zz} = -\frac{P}{4\pi^2}\int_0^{\infty}\int_0^{2\pi}\Big\{\rho + \frac{z}{4}\Big[4\rho^2 + \rho^2\cos^2\gamma\left(M_L^2 + kM_L^2 - M_T^2\right)$$

$$+ \frac{z^2}{4}\left(M_L^2 + M_T^2\right)\rho^2\cos^2\gamma\Big\}\, e^{-\rho z + i\rho r\cos(\theta-\gamma)}\, d\gamma\, d\rho \qquad (6.9)$$

The term:

$$\frac{M_L^2(1+k) - M_T^2}{4} = \frac{M_T^2}{2}\left(\frac{\lambda + 2\mu}{\lambda + \mu}\right)$$

after some manipulation and equation 6.9 becomes:

$$\tau_{zz} = -\frac{P}{4\pi^2}\int_0^{2\pi}\int_0^{\infty} e^{-\rho z + i\rho r\cos(\theta-\gamma)}\,\rho\, d\rho\, d\gamma$$

$$-\frac{P}{4\pi^2}\int_0^{2\pi}\int_0^{\infty} z\rho^2 e^{-\rho z + i\rho r\cos(\theta-\gamma)}\, d\rho\, d\gamma$$

$$-\frac{P}{4\pi^2}\int_0^{2\pi}\int_0^{\infty} \frac{M_T^2 z}{2}\left(\frac{\lambda+2\mu}{\lambda+\mu}\right)\rho^2\cos^2\gamma\, e^{-\rho z + i\rho r\cos(\theta-\gamma)}\, d\rho\, d\gamma$$

$$-\frac{P}{4\pi^2}\int_0^{2\pi}\int_0^{\infty} \frac{M_T^2 z^2}{4}\left(\frac{\lambda+3\mu}{\lambda+2\mu}\right)\rho^2\cos^2\gamma\, e^{-\rho z + i\rho r\cos(\theta-\gamma)}\, d\rho\, d\gamma$$

$$(6.10)$$

where:

$$M_L^2 + M_T^2 = M_T^2\left(1 + \frac{M_L^2}{M_T^2}\right) = M_T^2\left(\frac{\lambda+3\mu}{\lambda+2\mu}\right)$$

The first two integrals represent the stress for the stationary point load where as the last two integrals constitute the corrections for the motion of the load. The correction terms include all terms of $O(M_L^2, M_T^2)$.

The integrals in equation 6.10 are readily evaluated. Their evaluation is carried out in Appendix A. See equation A.14. The result is:

$$\begin{aligned}\tau_{zz} = \frac{P}{2\pi}\Bigg\{ -\frac{3z^3}{(z^2+r^2)^{\frac{5}{2}}} + M_T^2\left(\frac{\lambda+2\mu}{\lambda+\mu}\right)\Bigg[\frac{z}{4(z^2+r^2)^{\frac{3}{2}}} \\ -\frac{3z^3}{4(z^2+r^2)^{\frac{5}{2}}} + \frac{3}{4}\,\frac{zr^2\cos 2\theta}{(z^2+r^2)^{\frac{5}{2}}} \\ + M_T^2\left(\frac{\lambda+3\mu}{\lambda+2\mu}\right)\Bigg[\frac{z^2}{8(z^2+r^2)^{\frac{3}{2}}} \\ -\frac{3z^4}{8(z^2+r^2)^{\frac{5}{2}}} + \frac{3}{8}\,\frac{z^2r^2\cos 2\theta}{(z^2+r^2)^{\frac{5}{2}}}\Bigg]\Bigg\}.\end{aligned} \qquad (6.11)$$

Noting that:

$$\cos 2\theta = 2\cos^2\theta - 1 = \frac{x^2-y^2}{x^2+y^2}$$

equation 6.11 can be rewritten in better form as:

$$\tau_{zz} = \frac{P}{2\pi}\left\{-\frac{3z^3}{R^5} + M_T^2\left(\frac{\lambda+2\mu}{\lambda+\mu}\right)\left[\frac{z}{4R^3} - \frac{3z^3}{4R^5} + \frac{3z(x^2-y^2)}{4R^5}\right]\right.$$
$$\left. + M_T^2\left(\frac{\lambda+3\mu}{\lambda+2\mu}\right)\left[\frac{z^2}{8R^3} - \frac{3z^4}{8R^5} + \frac{3z^2(x^2-y^2)}{8R^5}\right]\right\} \qquad (6.12)$$

where:

$$R^2 = x^2+y^2+z^2.$$

Comparison of τ_{zz} above with that of Cole and Huth (5), Figure 2 shows the same behavior near $x=y=0$. That is, the stress becomes more negative with increasing M_T. The first term, which represents the static contribution to the stress, agrees exactly with the Boussinesq result as given in Love (7), pg. 192.

Encouraged by the success of the computation for τ_{zz}, let us proceed to evaluate an approximate expression for the displacement in the x-direction, u.

We have:

$$u = \phi_x + \psi_y^{(3)} - \psi_z^{(2)} \qquad (3.5)$$

Substitution of the integral expressions 5.10, 5.12, and 5.13 into 3.5 above gives:

$$u = \frac{iP}{4\pi^2\mu}\iint_{-\infty}^{\infty} \omega_1\left[\omega_1^2(M_T^2-2)-2\omega_2^2\right] e^{-\sqrt{\beta_L^2\omega_1^2+\omega_2^2}\,z + i(\omega_1 x+\omega_2 y)} \frac{d\omega_1\, d\omega_2}{|D|}$$

$$+ \frac{2iP}{4\pi^2\mu}\iint_{-\infty}^{\infty} \omega_1\sqrt{\beta_L^2\omega_1^2+\omega_2^2}\,\sqrt{\beta_T^2\omega_1^2+\omega_2^2}\; e^{-\sqrt{\beta_T^2\omega_1^2+\omega_2^2}\,z + i(\omega_1 x+\omega_2 y)} \frac{d\omega_1\, d\omega_2}{|D|}$$

(6.13)

We assume again that the appropriate terms in equation 6.13 can be expanded in a Taylor series as $M_L^2, M_T^2 \to 0$. The expansions of equations 6.4 - 6.7 apply here also. Equation 6.7 can be cast into different form by making the substitution:

$$M_T^2 - M_L^2 = M_T^2\left(1 - \frac{M_L^2}{M_T^2}\right) = M_T^2\left(\frac{\lambda+\mu}{\lambda+2\mu}\right) \qquad (6.14)$$

We obtain from equation 6.7 then:

$$|D|^{-1} \cong -\left(\frac{\lambda+2\mu}{\lambda+\mu}\right)\frac{1}{2\omega_1^2 M_T^2(\omega_1^2+\omega_2^2)}\left[1 + \frac{\omega_1^2 k M_L^2}{4(\omega_1^2+\omega_2^2)} + \cdots\right]. \qquad (6.15)$$

Therefore, when expanding the numerators in the integrands of equation 6.13, we must retain all terms of $O(M_L^4, M_T^4)$ in order to obtain the first correction term for the moving load. After expansion and simplification, we obtain for the u-displacement:

$$u = \frac{iP}{4\pi^2\mu}\iint_{-\infty}^{\infty}\left\{-\frac{\omega_1 z}{2A} + \frac{\omega_1^3 z}{8A^3}\left(\frac{M_T^4+M_L^4}{M_T^2-M_L^2}\right) - \frac{\omega_1^3 z^2}{8A^2}\left(M_T^2+M_L^2\right)\right.$$

$$\left. + \frac{\omega_1 M_L^2}{2A^2(M_T^2-M_L^2)} + \frac{\omega_1^3}{8A^4}\left(M_T^2-M_L^2\right)\right\}\left\{1\right.$$

$$\left. + \frac{k\omega_1^2 M_L^2}{4A^2} + \cdots\right\} e^{-Az+i(\omega_1 x+\omega_2 y)}\, d\omega_1\, d\omega_2 \qquad (6.16)$$

where:

$$A = \left(\omega_1^2+\omega_2^2\right)^{\frac{1}{2}}$$

as before and k is defined by equation 6.6.

The quantities $\left(M_T^2 + M_L^2\right)$, etc., can be expressed in terms of M_T^2 and Lame's constants as:

$$\frac{M_T^4 + M_L^4}{M_T^2 - M_L^2} = M_T^2\left[\frac{\lambda^2 + 4\mu\lambda + 5\mu^2}{(\lambda+\mu)(\lambda+2\mu)}\right]$$

$$M_T^2 + M_L^2 = M_T^2\left(\frac{\lambda+3\mu}{\lambda+2\mu}\right)$$

$$M_T^2 - M_L^2 = M_T^2\left(\frac{\lambda+\mu}{\lambda+2\mu}\right) \qquad (6.17)$$

$$\frac{M_L^2}{M_T^2 - M_L^2} = \frac{\mu}{\lambda+\mu}$$

$$M_L^2 = M_T^2\left(\frac{\mu}{\lambda+2\mu}\right)$$

Substitution of equations 6.17 into 6.16 gives

$$u = \frac{iP}{4\pi^2\mu}\iint_{-\infty}^{\infty}\left\{-\frac{\omega_1 z}{2A} + \frac{\omega_1^3 z M_T^2}{8A^3}\frac{(\lambda^2+4\mu\lambda+5\mu^2)}{(\lambda+\mu)(\lambda+2\mu)} - \frac{\omega_1^3 z^2}{8A^2}M_T^2\left(\frac{\lambda+3\mu}{\lambda+2\mu}\right)\right.$$

$$+ \frac{\omega_1}{2A^2}\left(\frac{\mu}{\lambda+\mu}\right) + \frac{\omega_1^3}{8A^4}M_T^2\left(\frac{\lambda+\mu}{\lambda+2\mu}\right) - \frac{k\omega_1^3 z}{8A^3}M_T^2\left(\frac{\mu}{\lambda+2\mu}\right)$$

$$\left. + \frac{k\omega_1^3}{8A^4}M_T^2\frac{\mu^2}{(\lambda+\mu)(\lambda+2\mu)}\right\} e^{-Az+i(\omega_1 x+\omega_2 y)}\, d\omega_1 d\omega_2 \qquad (6.18)$$

which can be combined somewhat to yield:

$$u = \frac{iP}{4\pi^2\mu}\iint_{-\infty}^{\infty}\left\{-\frac{\omega_1 z}{2A} + \frac{\omega_1}{2A^2}\left(\frac{\mu}{\lambda+\mu}\right) + \frac{\omega_1^3 M_T^2}{8A^4}\left[\frac{\lambda^2+4\mu\lambda+5\mu^2}{(\lambda+\mu)^2}\right]\right.$$

$$\left. - \frac{\omega_1^3 z M_T^2}{4A^3} - \frac{\omega_1^3 z^2 M_T^2}{8A^2}\left(\frac{\lambda+3\mu}{\lambda+2\mu}\right)\right\} e^{-Az+i(\omega_1 x+\omega_2 y)}\, d\omega_1\, d\omega_2 \qquad (6.19)$$

We effect the same transformation to polar coordinates as was used in the evaluation of τ_{zz} to obtain:

$$u = \frac{iP}{4\pi^3\mu}\int_0^{\infty}\int_0^{2\pi}\left\{-\frac{\rho z\cos\gamma}{2} + \frac{\cos\gamma}{2}\left(\frac{\mu}{\lambda+\mu}\right) + \frac{\cos^3\gamma\, M_T^2}{8}\left[\frac{\lambda^2+4\mu\lambda+5\mu^2}{(\lambda+\mu)^2}\right]\right.$$

$$\left. - \frac{\rho z\cos^3\gamma\, M_T^2}{4} - \frac{\rho^2 z^2\cos^3\gamma\, M_T^2}{8}\left(\frac{\lambda+3\mu}{\lambda+2\mu}\right)\right\} e^{-\rho z+i\rho r\cos(\theta-\gamma)}\, d\gamma\, d\rho.$$

$$(6.20)$$

This integral is again readily evaluated, although somewhat tediously. We obtain the solution from Appendix A, equation A.36 as:

$$u = \frac{P}{4\pi\mu}\frac{xz}{R^3} - \frac{P}{4\pi(\lambda+\mu)}\frac{x}{R(z+R)}$$

$$-\frac{P}{32\pi\mu}M_T^2 z\left\{\frac{r^3\cos 3\theta}{(z+R)^3}\left[\frac{3}{R^2}+\frac{z}{R^3}\right] - \frac{3x}{z+R}\left[\frac{1}{R^2}+\frac{z}{R^3}\right]\right\}$$

$$+ \frac{PM_T^2}{64\pi\mu}\left[\frac{\lambda^2+4\mu\lambda+5\mu^2}{(\lambda+\mu)^2}\right]\left[\frac{r^3\cos 3\theta}{(z+R)^3 R} - \frac{3x}{R(z+R)}\right]$$

$$- \frac{PM_T^2 z^2}{64\pi\mu}\left(\frac{\lambda+3\mu}{\lambda+2\mu}\right)\left\{\frac{r^3\cos 3\theta}{(z+R)^3}\left[\frac{8}{R^3} + \frac{9z}{R^4} + \frac{3z^2}{R^5}\right]\right.$$

$$\left. - \frac{3x}{z+R}\left[\frac{3z}{R^4} + \frac{3z^2}{R^5}\right]\right\} \; ; \; R^2 = x^2+y^2+z^2. \qquad (6.21)$$

The first two terms in equation 6.21 represent the displacement resulting from the stationary load. These agree with Love (7), pg. 191. The remaining terms represent the first correction for motion of the load.

Before departing this section it is noted that, in principle, additional correction terms may be developed using this same technique. However, it is anticipated that such an attempt would prove to be prohibitively time consuming.

7. Supersonic Approximation

The load travels, in this instance, at a speed exceeding the longitudinal and transverse wave speeds in the medium, *i.e.*,

$$U >> C_L, C_T \tag{7.1}$$

Therefore, we expect to see only backward running waves emanating from the source of the disturbance. Suppose we pass a plane through the origin of coordinates parallel to the x - z plane. We would expect to see backward running rays along which the disturbances originating at the surface propagate with the appropriate wave speeds somewhat as shown in Figure 2 below.

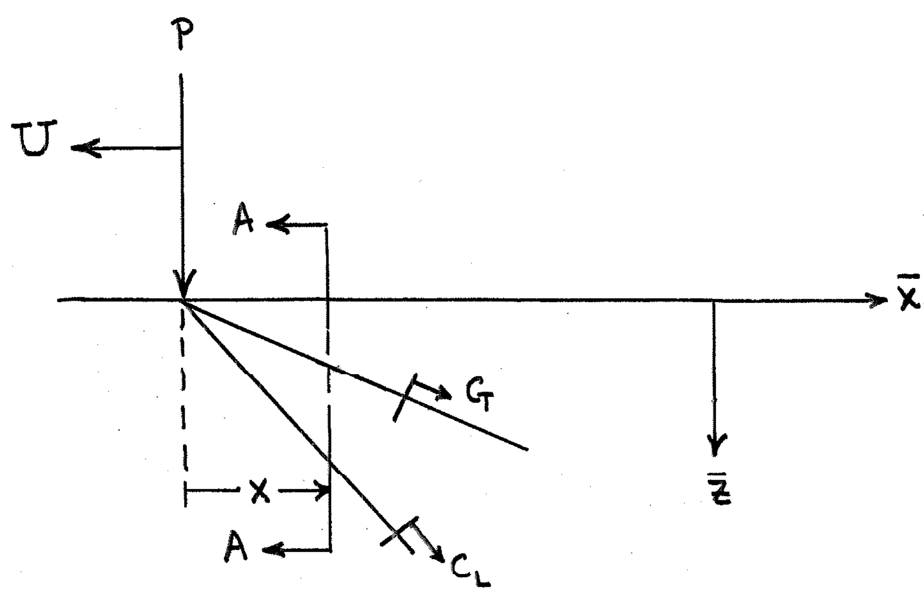


Fig. 2. Supersonic Load

We could use the double Fourier transform technique as in the subsonic approximation. However, we postulate that the solution in the y - z plane may behave as though an impulsive, concentrated load were applied at the origin y = z = 0. Such a result would indeed be fortuitous since Sauter (8) has extensively studied this problem. His results would then carry over to the supersonic problem to be studied here. We shall explore this question further.

Assume that our longitudinal and transverse potentials can be expanded in power series, e.g.,

$$\begin{aligned} \phi(x,y,z) &= \phi_0(\tilde{t},\tilde{x},\tilde{z}) + \frac{1}{M_L^2}\phi_1(\tilde{t},\tilde{x},\tilde{z}) + \cdots \\ \vec{\psi}(x,y,z) &= \vec{\psi}_0(\tilde{t},\tilde{x},\tilde{z}) + \frac{1}{M_T^2}\vec{\psi}_1(\tilde{t},\tilde{x},\tilde{z}) + \cdots \end{aligned} \tag{7.2}$$

where we need consider only the leading terms in equation 7.2 as $M_L^2, M_T^2 \rightarrow$ large.

We note that the distance from the moving load to Section A-A, fixed in the medium, is given by $x = \bar{x} + U\bar{t}$. See Figure 2. Therefore, x grows like Ut and we choose as coordinates:

$$\begin{aligned} x &= U\tilde{t} \\ y &= -\tilde{x} \\ z &= \tilde{z} \end{aligned} \tag{7.3}$$

which are reflected in the potential expansions above. We have chosen $y = -\tilde{y}$ so that the coordinate directions will be consistent with those of Sauter. Note that $t = \bar{t} = \tilde{t}$ throughout.

Equations 2.14 - 2.15 are valid and represent the potential equations after the Galilean transformation has been made. We rewrite them for the supersonic case, since $M_L^2, M_T^2 > 1$ as:

$$(M_L^2 - 1)\phi_{xx} - \phi_{yy} - \phi_{zz} = 0$$
$$(M_T^2 - 1)\vec{\psi}_{xx} - \vec{\psi}_{yy} - \vec{\psi}_{zz} = 0 \;;\; \mathrm{div}\,\vec{\psi} = 0 \qquad (7.4)$$

Using equations 7.2 and letting $U \rightarrow$ large, equations 7.4 become:

$$\frac{1}{C_L^2}\phi_{o\tilde{t}\tilde{t}} = \tilde{\nabla}^2\phi_o$$
$$\frac{1}{C_T^2}\vec{\psi}_{o\tilde{t}\tilde{t}} = \tilde{\nabla}^2\vec{\psi}_o \;;\; \mathrm{div}\,\vec{\psi}_o = 0 \qquad (7.5)$$

where

$$\tilde{\nabla}^2 = \frac{\partial^2}{\partial\tilde{x}^2} + \frac{\partial^2}{\partial\tilde{z}^2}$$

The boundary conditions of equations 3.8 - 3.10 become, after substitution of equations 7.2 - 7.3 and 7.5:

$$\frac{\tau_{zz}}{\mu} = \frac{\tau_{\tilde{z}\tilde{z}}}{\mu} = \frac{(M_T^2-2)}{U^2}\,\phi_{o_{\tilde{t}\tilde{t}}} - 2\phi_{o_{\tilde{x}\tilde{x}}} + 2\psi^{(2)}_{o_{\tilde{t}\tilde{z}}}\left(\frac{1}{U}\right)$$

$$+ 2\,\psi^{(1)}_{o_{\tilde{x}\tilde{z}}} = -\frac{\delta(U\tilde{t})\,\delta(-\tilde{x})\,P}{\mu} = -\frac{\delta(\tilde{t})\,\delta(\tilde{x})\,P}{\mu U} \qquad (7.6)$$

$$\frac{\tau_{yz}}{\mu} = -\frac{\tau_{\tilde{x}\tilde{z}}}{\mu} = -2\phi_{o_{\tilde{x}\tilde{z}}} + \frac{(M_T^2-1)}{U^2}\,\psi^{(1)}_{o_{\tilde{t}\tilde{t}}} - 2\psi^{(1)}_{o_{\tilde{x}\tilde{x}}}$$

$$-\psi^{(3)}_{o_{\tilde{t}\tilde{z}}}\left(\frac{1}{U}\right) - \psi^{(2)}_{o_{\tilde{t}\tilde{x}}}\left(\frac{1}{U}\right) = 0 \qquad (7.7)$$

$$\frac{\tau_{xz}}{\mu} = 2\phi_{o_{\tilde{z}\tilde{t}}}\left(\frac{1}{U}\right) - \frac{(M_T^2-2)}{U^2}\,\psi^{(2)}_{o_{\tilde{t}\tilde{t}}} + \psi^{(1)}_{o_{\tilde{x}\tilde{t}}}\left(\frac{1}{U}\right)$$

$$-\psi^{(3)}_{o_{\tilde{x}\tilde{z}}} + \psi^{(2)}_{o_{\tilde{x}\tilde{x}}} = 0 \qquad (7.8)$$

In the limit, as $U \rightarrow$ large we obtain from the above equations:

$$(C_L^2 - 2C_T^2)\,\phi_{0\tilde{x}\tilde{x}} + C_L^2\,\phi_{0\tilde{z}\tilde{z}} + 2C_T^2\,\psi^{(1)}_{0\tilde{x}\tilde{z}}$$

$$= -\frac{\delta(\tilde{t})\,\delta(\tilde{x})\,C_T^2\,P}{\mu\,U} \qquad (7.9)$$

$$2\phi_{0\tilde{x}\tilde{z}} - \psi^{(1)}_{0\tilde{z}\tilde{z}} + \psi^{(1)}_{0\tilde{x}\tilde{x}} = 0 \qquad (7.10)$$

$$-\frac{1}{C_T^2}\psi^{(2)}_{0\tilde{t}\tilde{t}} - \psi^{(3)}_{0\tilde{x}\tilde{z}} + \psi^{(2)}_{0\tilde{x}\tilde{x}} = 0 \qquad (7.11)$$

The div $\vec{\psi}_0 = 0$ relation can be used to eliminate equation 7.11. It becomes, in the new coordinate system:

$$\frac{\partial\psi_0^{(1)}}{\partial\tilde{t}}\left(\frac{1}{U}\right) - \frac{\partial\psi_0^{(2)}}{\partial\tilde{x}} + \frac{\partial\psi_0^{(3)}}{\partial\tilde{z}} = 0 \qquad (7.12)$$

and as $U \rightarrow$ large:

$$-\frac{\partial\psi_0^{(2)}}{\partial\tilde{x}} + \frac{\partial\psi_0^{(3)}}{\partial\tilde{z}} = 0 \qquad (7.13)$$

Differentiation with respect to x yields:

$$-\psi^{(2)}_{0\tilde{x}\tilde{x}} + \psi^{(3)}_{0\tilde{x}\tilde{z}} = 0$$

thus, equation 7.11 reduces to:

$$-\frac{1}{C_T^2}\psi^{(2)}_{0\tilde{t}\tilde{t}} = 0 \qquad (7.14)$$

We note that in the two-dimensional system $(\tilde{x}, \tilde{z})$ there can only be one transverse potential. Therefore, $\psi_0^{(2)} = 0$ and equation 7.11, which implies 7.14, is automatically satisfied.

We must now show that the boundary conditions of equations 7.9 - 7.10 are identical to those of Sauter(8). Using the two-dimensional form of the stress tensor, equation 3.1, we readily obtain Sauter's boundary conditions, defined in the referenced work by equation 1.5, in the form:

$$\frac{\tau_{zz}}{\mu} = \frac{\rho}{\mu}\left\{C_L^2 \frac{\partial w}{\partial z} + (C_L^2 - 2C_T^2)\frac{\partial u}{\partial x}\right\} = \sigma_1 \frac{\delta(x)\delta(t)}{\mu} \tag{7.15}$$

$$\frac{\tau_{xz}}{\mu} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} = 0 \tag{7.16}$$

where "u" has been substituted for S_x , etc.

If $\vec{u} = \vec{i}u + \vec{k}w$ is substituted into Navier's equation, equation 2.2, we easily obtain Sauter's equations of motion as defined by equation 1.3 in his work. We deduce then that the equations of motion of Sauter and those for our supersonic approximation are identical - without further ado . The combination of Navier's equation (equation 2.2) and Helmholtz vector decomposition theorem (equation 2.3), both in two-dimensional form

(x,z) , imply the existence of the following wave equations for the potentials ϕ and ψ :

$$\nabla^2\phi = \frac{1}{C_L^2}\phi_{tt} \qquad (7.17)$$

$$\nabla^2\vec{\psi} = \frac{1}{C_T^2}\vec{\psi}_{tt} \; ; \quad \operatorname{div}\vec{\psi} = 0$$

where $\vec{\psi} = \vec{j}\,\psi^{(2)}$ has only one component.

From Helmholtz theorem we obtain

$$u = \phi_x - \psi_z^{(2)} \qquad (7.18)$$

$$w = \phi_z + \psi_x^{(2)}$$

and substitution of these results into equation 7.15 - 7.16 gives:

$$(C_L^2 - 2C_T^2)\phi_{xx} + C_L^2\phi_{zz} + 2C_T^2\psi_{xz}^{(2)} = \sigma_1 \frac{\delta(x)\delta(t)C_T^2}{\mu} \qquad (7.19)$$

$$2\phi_{xz} - \psi_{zz}^{(2)} + \psi_{xx}^{(2)} = 0 . \qquad (7.20)$$

Neglecting the right hand sides for the moment, equations 7.19 - 7.20 agree with equations 7.9 - 7.10 if we identify:

$$\begin{aligned} \phi_0 &= \phi \\ \psi_0^{(1)} &= \psi^{(2)} \\ \tilde{x} &= x \\ \tilde{z} &= z \\ \tilde{t} &= t. \end{aligned} \qquad (7.21)$$

The question of algebraic sign of the forcing function is simply a matter of definition. The forcing functions are identical if:

$$-\frac{P}{U} = \sigma_1$$

Dimensionally:

$$\frac{[P]}{[U]} = \left[\frac{ML}{T^2}\cdot\frac{T}{L}\right] = \left[\frac{M}{T}\right]$$

which is impulse per unit length. The forcing function on the right hand side of equation 7.15 must be non-dimensional, therefore:

$$[\sigma_1] = \left[\frac{\mu}{\delta(x)\delta(t)}\right] = \frac{\left[\frac{ML}{T^2}\cdot\frac{1}{L^2}\right]}{\left[\frac{1}{LT}\right]} = \left[\frac{M}{T}\right]$$

and the forcing functions are identical.

We conclude then that our original postulate was correct, *i.e.*, for the supersonic approximation, the results for the impulsive, concentrated at the origin, loading problem do indeed carry over and define the behavior in our y - z plane - at a fixed station. The familiar wave front pattern is expected to be observed in the y - z plane for this approximation and is shown in Figure 3 below.

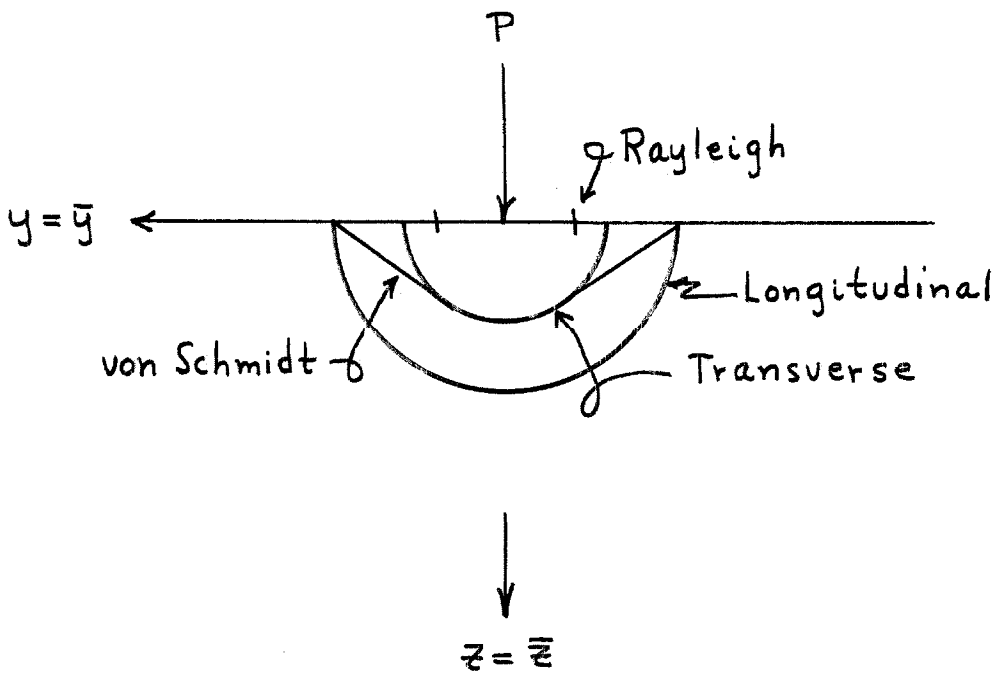


Fig. 3. Supersonic Approximation - Wave Pattern at Section A-A.

II. Surface Waves of Elastic Solids

1. Introduction

The original work of Lord Rayleigh (9), wherein he develops the relation for the speed of plane waves propagating over the surface of an elastic solid, assumes the x , y dependence of the displacements to be simple harmonic. Then the dilatation, θ , decays exponentially with distance, z , into the interior of the medium. Lord Rayleigh's original analysis is three-dimensional but is easily specialized to two dimensions as is done in Love (7), pp. 307-309.

Then again, the frequency or speed equation for waves at the welded contact surface of two semi-infinite media was developed by Stoneley under the assumption of a simple harmonic plane wave which vanishes as z , the distance into the interior of the medium, becomes infinite. The work of Stoneley is discussed in Ewing, Jardetzky and Press (10), pp. 111 - 113.

The purpose of this section is to show, that for the two-dimensional case, e.g., plane strain, the assumption of simple harmonic disturbances is not required to predict the existence and speed of surface waves. In particular, analytic function theory provides a beautiful tool for analysis of the two-dimensional cases discussed above.

A brief analysis on the question of the existence of surface waves which travel parallel to the geometric axis of an infinitely long cylindrical cavity in an infinite medium will show that these surface waves do indeed exist also. However, the introduction of a characteristic length into the problem causes the wave to become dispersive. This problem has been studied previously by Biot (11). The formulation of the problem herein differs slightly from that of Biot, however, in that Fourier transform techniques are used to generate the wave speed equation. Both results are equivalent.

It is well known that circumferential waves also exist on the interior surface of an infinitely long cylindrical cavity. This problem has been studied for the steady harmonic wave propagation case by Viktorov (12) who determined that the waves are also dispersive in this case. These remarks and the analysis referred to above are included only to direct attention to the fact that introduction of a characteristic length results in a dispersive surface wave.

2. Rayleigh Waves

Consider the infinite half-space. We formulate the problem by asking the question:

"Is it possible to have a deformation, travelling along the surface at constant speed, which satisfies the boundary conditions of zero stress?".

The deformation (arbitrary) and coordinate configuration is as shown in Figure 4 below.

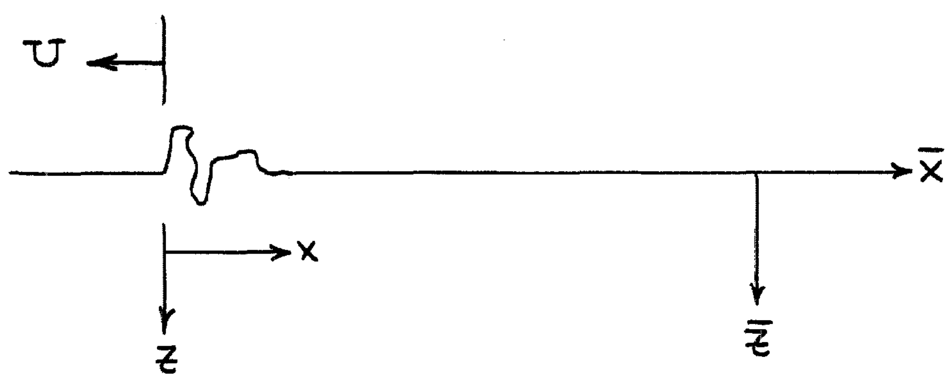


Fig. 4. Surface Deformation Travelling at Constant Speed (Arbitrary Shape)

The medium extends to infinity in the y-direction, thus, the problem is one of plane strain. It is convenient to make the Galilean transformation:

$$x = \bar{x} + U\bar{t} \tag{2.1}$$

and since we are in essence looking for the Rayleigh wave speed, define

$$U \equiv C_R \tag{2.2}$$

Restricting our discussion to two dimensions the results of Part I, Section 2 can be used to obtain the equations for the longitudinal and transverse potentials in the form:

$$\begin{aligned} \alpha_L^2 \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial z^2} &= 0 \\ \alpha_T^2 \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial z^2} &= 0 \end{aligned} \tag{2.3}$$

where:

$$\begin{aligned} \alpha_L^2 &= 1 - \frac{C_R^2}{C_L^2} \\ \alpha_T^2 &= 1 - \frac{C_R^2}{C_T^2} \end{aligned} \tag{2.4}$$

Solutions to the potential equations can be obtained in terms of analytic functions of the complex variables:

$$\begin{aligned} \zeta_L &= x + i\alpha_L z \\ \zeta_T &= x + i\alpha_T z \end{aligned} \tag{2.5}$$

We write:

$$\phi = Re\{\Phi(\zeta_L)\}$$
$$\psi = Re\{\Psi(\zeta_T)\} \tag{2.6}$$

where:

$$\Phi = \phi + i\phi^*$$
$$\Psi = \psi + i\psi^* \tag{2.7}$$

and * denotes the harmonic conjugate. That equations 2.6 are indeed solutions to equations 2.3 is readily verified by direct substitution.

The boundary conditions for the stress free surface are obtained from Part I, Section 3 as:

$$\frac{\tau_{zz}}{\mu} = 0 = \frac{\lambda}{\mu}\phi_{xx} - \left(\frac{\lambda+2\mu}{\mu}\right)\alpha_L^2\phi_{xx} + 2\psi_{xz}$$
$$\frac{\tau_{xz}}{\mu} = 0 = 2\phi_{xz} + \psi_{xx} + \alpha_T^2\psi_{xx} \tag{2.8}$$

The boundary conditions can be integrated with respect to x. This introduces, at most, a constant displacement.

The result is:

$$\left\{\frac{\lambda}{\mu} - \alpha_L^2\left(\frac{\lambda+2\mu}{\mu}\right)\right\}\phi_x + 2\psi_z = 0$$

$$2\phi_z + (1+\alpha_T^2)\psi_x = 0 \tag{2.9}$$

Noting that $\zeta_L = \zeta_T$ at $z = 0$, the boundary conditions can be expressed in terms of the analytic functions Φ and Ψ as:

$$\left\{\frac{\lambda}{\mu} - \alpha_L^2\left(\frac{\lambda+2\mu}{\mu}\right)\right\}\mathrm{Re}\{\Phi'\} + 2\,\mathrm{Re}\{i\alpha_T\Psi'\} = 0$$

$$2\,\mathrm{Re}\{i\alpha_L\Phi'\} + (1+\alpha_T^2)\,\mathrm{Re}\{\Psi'\} = 0 \tag{2.10}$$

or equivalently:

$$\left(\frac{C_R^2}{C_T^2}-2\right)\mathrm{Re}\{\Phi'\} - 2\sqrt{1-\frac{C_R^2}{C_T^2}}\;\mathrm{Im}\{\Psi'\} = 0$$

$$-2\sqrt{1-\frac{C_R^2}{C_L^2}}\;\mathrm{Im}\{\Phi'\} + \left(2-\frac{C_R^2}{C_T^2}\right)\mathrm{Re}\{\Psi'\} = 0 \tag{2.11}$$

since:

$$\mathrm{Re}\{i\Phi'\} = -\,\mathrm{Im}\{\Phi'\}$$

$$\mathrm{Re}\{i\Psi'\} = -\,\mathrm{Im}\{\Psi'\}. \tag{2.12}$$

The first of equations 2.11 is satisfied if:

$$\Phi' = -i \frac{2\sqrt{1-\frac{C_R^2}{C_T^2}}}{\left(\frac{C_R^2}{C_T^2}-2\right)} \Psi' \qquad (2.13)$$

Substitution of the above result into the second of equations 2.11 gives:

$$\left\{-4\sqrt{1-\frac{C_R^2}{C_L^2}}\sqrt{1-\frac{C_R^2}{C_T^2}} + \left(2-\frac{C_R^2}{C_T^2}\right)^2\right\} Re\{\Psi'\} = 0 \qquad (2.14)$$

and since the $Re\{\Psi'\} \neq 0$, we must have:

$$\left(2-\frac{C_R^2}{C_T^2}\right)^2 - 4\sqrt{1-\frac{C_R^2}{C_L^2}}\sqrt{1-\frac{C_R^2}{C_T^2}} = 0 \qquad (2.15)$$

Satisfaction of equation 2.15 implies the existence of a surface deformation, travelling at speed C_R which satisfies the boundary conditions of zero stress.

This result is independent of frequency or any other parameter except the elastic constants λ and μ*. The speed is, thus, non-dispersive. It is easily verified that C_R is indeed the Rayleigh wave speed. We have,

$$\left(2-\frac{C_R^2}{C_T^2}\right)^2 = 4\sqrt{1-\frac{C_R^2}{C_L^2}}\sqrt{1-\frac{C_R^2}{C_T^2}}$$

Squaring both sides and rearranging gives:

$$\left(\frac{C_R^2}{C_T^2}\right)^4 - 8\left(\frac{C_R^2}{C_T^2}\right)^3 + 24\left(\frac{C_R^2}{C_T^2}\right)^2 - 16\frac{C_R^2}{C_T^2}\left(1+\frac{C_R^2}{C_L^2}\right) + 16\frac{C_R^2}{C_L^2} = 0 \qquad (2.16)$$

which has been shown to yield the Rayleigh wave speed in Part I, Section 5. It is only necessary to identify $C_R \equiv U$.

The beauty of this analysis lies in its simplicity. Note that the only restrictions are that Φ and Ψ be analytic functions.

---

* and the density of the medium, ρ.

3. Stoneley Waves

Consider two infinite half-spaces in contact as shown in Figure 5 below.

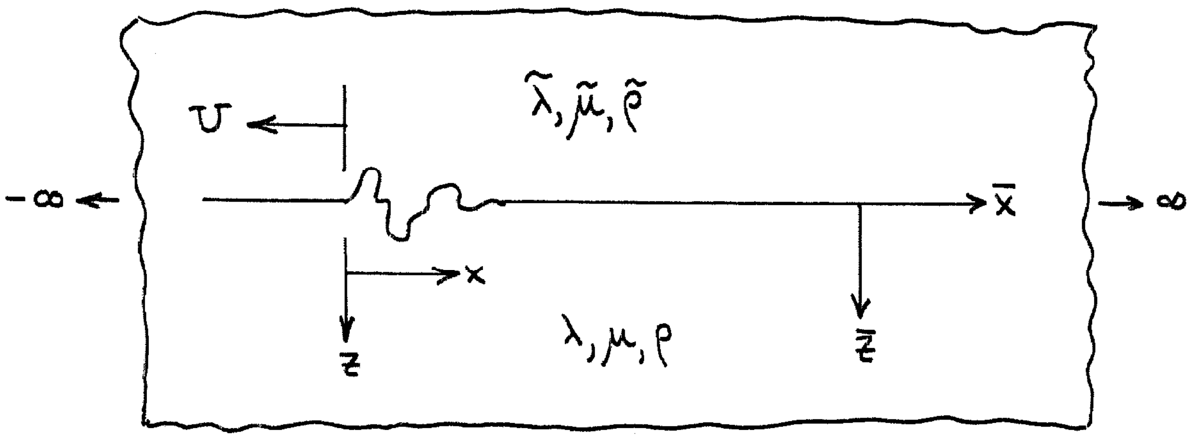


Fig. 5. Interface Deformation Travelling at Constant Speed (Arbitrary Shape)

The problem can be formulated by asking the question, similar to that of the previous section, i.e.:

"Is it possible to have an arbitrary deformation which travels at constant speed along the welded interface and which satisfies the boundary conditions prescribed there?".

The assumption of plane strain can again be made. Making use of the results obtained after the Galilean transformation in the previous section, the equations applicable to the first medium may be written as:

$$\alpha_L^2 \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial z^2} = 0$$
$$\alpha_T^2 \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial z^2} = 0 \qquad (3.1)$$

and to the second medium as:

$$\tilde{\alpha}_L^2 \frac{\partial^2 \tilde{\phi}}{\partial x^2} + \frac{\partial^2 \tilde{\phi}}{\partial z^2} = 0$$
$$\tilde{\alpha}_T^2 \frac{\partial^2 \tilde{\psi}}{\partial x^2} + \frac{\partial^2 \tilde{\psi}}{\partial z^2} = 0 \qquad (3.2)$$

where:

$$\alpha_L^2 = 1 - \frac{C_s^2}{C_L^2} \; ; \; \alpha_T^2 = 1 - \frac{C_s^2}{C_T^2}$$
$$\tilde{\alpha}_L^2 = 1 - \frac{C_s^2}{\tilde{C}_L^2} \; ; \; \tilde{\alpha}_T^2 = 1 - \frac{C_s^2}{\tilde{C}_T^2} . \qquad (3.3)$$

The substitution $U \equiv C_S$ has been made since the Stoneley wave speed is to be sought.

The stresses and displacements must be continuous across the welded interface, therefore, the boundary conditions become:

$$\begin{aligned} \tau_{zz} &= \tilde{\tau}_{zz} \\ \tau_{zx} &= \tilde{\tau}_{zx} \\ u &= \tilde{u} \\ w &= \tilde{w} \end{aligned} \tag{3.4}$$

at $z = 0$.

Solutions of equations 3.1 - 3.2 can be obtained in terms of analytic functions of the complex variables:

$$\begin{aligned} \zeta_L &= x + i\alpha_L z \;;\quad \zeta_T = x + i\alpha_T z \\ \tilde{\zeta}_L &= x + i\tilde{\alpha}_L z \;;\quad \tilde{\zeta}_T = x + i\tilde{\alpha}_T z \end{aligned} \tag{3.5}$$

This approach implies that:

$$\begin{aligned} \phi &= Re\{\Phi(\zeta_L)\} \;;\quad \psi = Re\{\Psi(\zeta_T)\} \\ \tilde{\phi} &= Re\{\tilde{\Phi}(\tilde{\zeta}_L)\} \;;\quad \tilde{\psi} = Re\{\tilde{\Psi}(\tilde{\zeta}_T)\} \end{aligned} \tag{3.6}$$

where:

$$\Phi = \phi + i\phi^{*} \; ; \quad \Psi = \psi + i\psi^{*}$$
$$\tilde{\Phi} = \tilde{\phi} + i\tilde{\phi}^{*} \; ; \quad \tilde{\Psi} = \tilde{\psi} + i\tilde{\psi}^{*} \tag{3.7}$$

and * indicates a harmonic conjugate.

Equations 2.8 - 2.11 of the previous section can be used to obtain the stress boundary conditions:

$$\mu\beta^{2}\,\mathrm{Re}\{\Phi'\} - 2\mu\alpha_{T}\,\mathrm{Im}\{\Psi'\} = \tilde{\mu}\tilde{\beta}^{2}\,\mathrm{Re}\{\tilde{\Phi}'\} - 2\tilde{\mu}\tilde{\alpha}_{T}\,\mathrm{Im}\{\tilde{\Psi}'\}$$
$$-2\mu\alpha_{T}\,\mathrm{Im}\{\Phi'\} - \mu\beta^{2}\,\mathrm{Re}\{\Psi'\} = -2\tilde{\mu}\tilde{\alpha}_{T}\,\mathrm{Im}\{\tilde{\Phi}'\} - \tilde{\mu}\tilde{\beta}^{2}\,\mathrm{Re}\{\tilde{\Psi}'\} \tag{3.8}$$

where:

$$\beta^{2} = \frac{C_s^{2}}{C_T^{2}} - 2 \; ; \quad \tilde{\beta}^{2} = \frac{C_s^{2}}{\tilde{C}_T^{2}} - 2 \tag{3.9}$$

Noting that:

$$u = \frac{\partial\phi}{\partial x} - \frac{\partial\psi}{\partial z} \; ; \quad w = \frac{\partial\phi}{\partial z} + \frac{\partial\psi}{\partial x}$$
$$\tilde{u} = \frac{\partial\tilde{\phi}}{\partial x} - \frac{\partial\tilde{\psi}}{\partial z} \; ; \quad \tilde{w} = \frac{\partial\tilde{\phi}}{\partial z} + \frac{\partial\tilde{\psi}}{\partial x} \tag{3.10}$$

the displacement boundary conditions can be written as (at $z = 0$) :

$$\mathrm{Re}\{\Phi'\} - \mathrm{Re}\{i\alpha_{T}\Psi'\} = \mathrm{Re}\{\tilde{\Phi}'\} - \mathrm{Re}\{i\tilde{\alpha}_{T}\tilde{\Psi}'\}$$
$$\mathrm{Re}\{i\alpha_{L}\Phi'\} + \mathrm{Re}\{\Psi'\} = \mathrm{Re}\{i\tilde{\alpha}_{L}\tilde{\Phi}'\} + \mathrm{Re}\{\tilde{\Psi}'\} \tag{3.11}$$

or equivalently:

$$\begin{aligned} &\mathrm{Re}\{\Phi'\} + \alpha_T \, \mathrm{Im}\{\Psi'\} = \mathrm{Re}\{\tilde{\Phi}'\} + \tilde{\alpha}_T \, \mathrm{Im}\{\tilde{\Psi}'\} \\ &-\alpha_L \, \mathrm{Im}\{\Phi'\} + \mathrm{Re}\{\Psi'\} = -\tilde{\alpha}_L \, \mathrm{Im}\{\tilde{\Phi}'\} + \mathrm{Re}\{\tilde{\Psi}'\} \end{aligned} \qquad (3.12)$$

Equations 3.8 and 3.12 are satisfied if:

$$\begin{aligned} &\mu\beta^2\Phi' - \tilde{\mu}\tilde{\beta}^2\tilde{\Phi}' + 2i\mu\alpha_T\Psi' - 2i\tilde{\mu}\tilde{\alpha}_T\tilde{\Psi}' = 0 \\ &2i\mu\alpha_L\Phi' - 2i\tilde{\mu}\tilde{\alpha}_L\tilde{\Phi}' - \mu\beta^2\Psi' + \tilde{\mu}\tilde{\beta}^2\tilde{\Psi}' = 0 \\ &\Phi' - \tilde{\Phi}' - i\alpha_T\Psi' + i\tilde{\alpha}_T\tilde{\Psi}' = 0 \\ &i\alpha_L\Phi' - i\tilde{\alpha}_L\tilde{\Phi}' + \Psi' - \tilde{\Psi}' = 0 \end{aligned} \qquad (3.13)$$

Equations 3.13 can possess a solution, other than the trivial one, only if the determinant of coefficients vanishes, i.e.:

$$\Delta = \begin{vmatrix} \mu\beta^2 & -\tilde{\mu}\tilde{\beta}^2 & 2i\mu\alpha_T & -2i\tilde{\mu}\tilde{\alpha}_T \\ 2i\mu\alpha_L & -2i\tilde{\mu}\tilde{\alpha}_L & -\mu\beta^2 & \tilde{\mu}\tilde{\beta}^2 \\ 1 & -1 & -i\alpha_T & i\tilde{\alpha}_T \\ i\alpha_L & -i\tilde{\alpha}_L & 1 & -1 \end{vmatrix} = 0 \qquad (3.14)$$

The determinant can be expanded by a well-known method. The result is:

$$
\begin{aligned}
&-2\mu\tilde{\mu}(\alpha_L-\tilde{\alpha}_L)(\alpha_T\tilde{\beta}^2-\tilde{\alpha}_T\beta^2)+\tilde{\mu}^2(1-\alpha_L\alpha_T)(\tilde{\beta}^4-4\tilde{\alpha}_L\tilde{\alpha}_T)\\
&-\mu\tilde{\mu}(1-\alpha_L\tilde{\alpha}_T)(\beta^2\tilde{\beta}^2-4\tilde{\alpha}_L\alpha_T)-\mu\tilde{\mu}(1-\tilde{\alpha}_L\alpha_T)(\beta^2\tilde{\beta}^2-4\alpha_L\tilde{\alpha}_T)\\
&+2\mu\tilde{\mu}(\alpha_T-\tilde{\alpha}_T)(\tilde{\alpha}_L\beta^2-\alpha_L\tilde{\beta}^2)+(1-\tilde{\alpha}_L\tilde{\alpha}_T)\mu^2(\beta^4-4\alpha_L\alpha_T)=0
\end{aligned}
\qquad (3.15)
$$

This result is symmetrical and contains no parameters other than the elastic constants $\lambda, \mu, \tilde{\lambda}, \tilde{\mu}$ and the densities $\rho, \tilde{\rho}$. We conclude that the Stoneley wave speed, which is contained in the terms $\alpha_L, \tilde{\alpha}_L, \beta^2, \tilde{\beta}^2$, etc. is non-dispersive. The existence of the Stoneley wave is, of course, contingent upon the existence of real roots of equation 3.15.

Substitute the following terms into equation 3.15:

$$\mu = \rho C_T^2 \; ; \; \tilde{\mu} = \rho \tilde{C}_T^2$$

$$\beta^2 = \frac{C_s^2}{C_T^2} - 2 \; ; \; \tilde{\beta}^2 = \frac{C_s^2}{\tilde{C}_T^2} - 2$$

$$\alpha_L = \sqrt{1-\frac{C_s^2}{C_L^2}} = A_1 \; ; \; \tilde{\alpha}_L = \sqrt{1-\frac{C_s^2}{\tilde{C}_L^2}} = A_2$$

$$\alpha_T = \sqrt{1-\frac{C_s^2}{C_T^2}} = B_1 \; ; \; \tilde{\alpha}_T = \sqrt{1-\frac{C_s^2}{\tilde{C}_T^2}} = B_2$$

and after some manipulation obtain the result:

$$C_S^4\{(\rho-\tilde{\rho})^2-(\rho A_2-\tilde{\rho}A_1)(\rho B_2-\tilde{\rho}B_1)$$

$$+4C_S^2(\rho C_T^2-\tilde{\rho}\tilde{C}_T^2)(\rho A_2B_2-\tilde{\rho}A_1B_1-\rho+\tilde{\rho}) \tag{3.16}$$

$$+4(\rho C_T^2-\tilde{\rho}\tilde{C}_T^2)^2(1-A_1B_1)(1-A_2B_2) = 0 .$$

This result, which is believed to be correct, agrees with equation 3.139 of Reference (10), except for the algebraic signs of $\tilde{\rho}A_1$ and $\tilde{\rho}B_1$ in the C_S^4 term. The original work of Stoneley was not available at the time of this writing, therefore, equation 3.139 of the cited reference could not be verified.

Suppose the density of the second medium vanishes, i.e., $\tilde{\rho}\rightarrow 0$. The following result is obtained:

$$(1-A_2B_2)\{C_S^4+4C_S^2C_T^2+4C_T^4(1-A_1B_1)\} = 0 \tag{3.17}$$

The coefficient:

$$(1-A_2B_2)$$

$$= 1-\sqrt{1-\frac{C_S^2}{\tilde{C}_L^2}}\sqrt{1-\frac{C_S^2}{\tilde{C}_T^2}} \neq 0$$

and can be cancelled unless $C_S^2 = 0$ or $\tilde{C}_L^2$ and $\tilde{C}_T^2 \rightarrow \infty$.

The equation which remains can be written as:

$$\left(\frac{C_S^2}{C_T^2} - 2\right)^2 - 4\sqrt{1 - \frac{C_S^2}{C_L^2}}\sqrt{1 - \frac{C_S^2}{C_T^2}} = 0 \qquad (3.18)$$

which is the familiar expression for the ordinary Rayleigh wave speed as shown in Part I, Section 5.

In the event that $\rho \to 0$, the Rayleigh wave speed equation applicable to the second medium is obtained. This fact provides additional validity to the conclusion that equation 3.16 is indeed correct.

4. Waves on the Surface of a Cylindrical Cavity in an Infinite Medium.

This analysis will show that the introduction of a characteristic length results in a surface wave which is dispersive in nature.

Consider an infinitely long cylindrical cavity of radius "a" in an infinite medium as depicted in Figure 6 below.

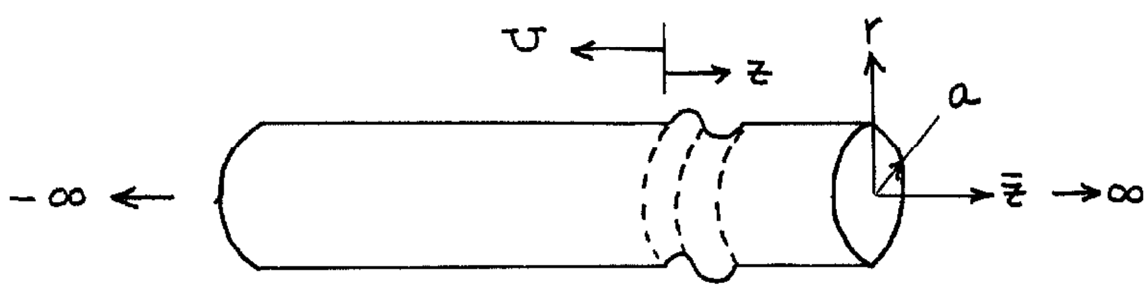


Fig. 6. Coordinate System for a Cylindrical Cavity in an Infinite Medium

It is convenient to use circular cylindrical coordinates. Them, because of axial symmetry, there is no θ-dependence of the variables in the problem.

The problem can again be formulated by inquiring as to the possibility of waves travelling along the surface of the cavity at constant speed which satisfy the boundary conditions of zero stress on the surface.

The equations of motion are easily obtained from:

$$\rho \frac{\partial^2 \vec{u}}{\partial t^2} = \operatorname{div} \bar{\bar{\tau}} \tag{4.1}$$

where $\bar{\bar{\tau}}$ is the general stress tensor:

$$\bar{\bar{\tau}} = \lambda \bar{\bar{I}} \operatorname{div} \vec{u} + \mu \left\{ \operatorname{grad} \vec{u} + (\operatorname{grad} \vec{u})^* \right\} \tag{4.2}$$

and * indicates the transpose. $\bar{\bar{I}}$ is the identity tensor. Recognizing:

$$\operatorname{div}\left(\lambda \bar{\bar{I}} \operatorname{div} \vec{u}\right) = \lambda \operatorname{grad} \operatorname{div} \vec{u} \tag{4.3}$$

equation 4.1 takes the form:

$$\rho \frac{\partial^2 \vec{u}}{\partial t^2} = \lambda \operatorname{grad} \operatorname{div} \vec{u} + \mu \operatorname{div} \operatorname{grad} \vec{u} + \mu \operatorname{div} (\operatorname{grad} \vec{u})^* \tag{4.4}$$

In cartesian tensor notation:

$$\rho \frac{\partial^2 u_i}{\partial t^2} = \lambda \frac{\partial}{\partial x_i}\left(\frac{\partial u_j}{\partial x_j}\right) + \mu \frac{\partial}{\partial x_j}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) \tag{4.5}$$

which can be rearranged into the form:

$$\rho \frac{\partial^2 u_i}{\partial t^2} = \lambda \frac{\partial}{\partial x_i}\left(\frac{\partial u_j}{\partial x_j}\right) + \mu \frac{\partial}{\partial x_j}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} + \frac{\partial u_j}{\partial x_i} - \frac{\partial u_j}{\partial x_i}\right) \tag{4.6}$$

or:

$$\rho \frac{\partial^2 u_i}{\partial t^2} = (\lambda+2\mu)\frac{\partial}{\partial x_i}\left(\frac{\partial u_j}{\partial x_j}\right) + \mu\left\{\frac{\partial}{\partial x_j}\left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\right)\right\} \tag{4.7}$$

which can be written in vector notation as:

$$\rho \frac{\partial^2 \vec{u}}{\partial t^2} = (\lambda+2\mu)\,\text{grad div}\,\vec{u} - \mu\,\text{curl curl}\,\vec{u} \tag{4.8}$$

Application of Helmholtz theorem:

$$\vec{u} = \text{grad}\,\phi + \text{curl}\,\vec{\Psi}\ ;\ \text{div}\,\vec{\Psi} = 0$$

allows equation 4.8 to be separated into two equations for the longitudinal (ϕ) and transverse ($\vec{\Psi}$) potentials, i.e.:

$$\begin{aligned} \nabla^2\phi &= \frac{1}{C_L^2}\phi_{tt} \\ -\text{curl curl}\,\vec{\Psi} &= \frac{1}{C_T^2}\vec{\Psi}_{tt}\ ;\ \text{div}\,\vec{\Psi} = 0 \end{aligned} \tag{4.9}$$

where, for the problem being considered, $\vec{\Psi}$ has only a component associated with the θ-direction.

In circular cylindrical coordinates, equations 4.9 become:

$$\begin{aligned} \frac{\partial^2\phi}{\partial r^2} + \frac{1}{r}\frac{\partial\phi}{\partial r} + \frac{\partial^2\phi}{\partial z^2} &= \frac{1}{C_L^2}\frac{\partial^2\phi}{\partial t^2} \\ \frac{\partial^2\psi}{\partial r^2} + \frac{1}{r}\frac{\partial\psi}{\partial r} + \frac{\partial^2\psi}{\partial z^2} &= \frac{1}{C_T^2}\frac{\partial^2\psi}{\partial t^2} + \frac{\psi}{r^2} \end{aligned} \tag{4.10}$$

where the curl curl $\vec{\Psi}$ components have been obtained from Magnus and Oberhettinger (13).

After the Galilean transformation:

$$z = \bar{z} + U\bar{t} \tag{4.11}$$

equation 4.10 become:

$$\frac{\partial^2\phi}{\partial r^2} + \frac{1}{r}\frac{\partial\phi}{\partial r} + \alpha_L^2 \frac{\partial^2\phi}{\partial \bar{z}^2} = 0$$

$$\frac{\partial^2\psi}{\partial r^2} + \frac{1}{r}\frac{\partial\psi}{\partial r} + \alpha_T^2 \frac{\partial^2\psi}{\partial \bar{z}^2} - \frac{\psi}{r^2} = 0 \tag{4.12}$$

where:

$$\alpha_L^2 = 1 - \frac{C^2}{C_L^2} ; \quad C \equiv U$$

$$\alpha_T^2 = 1 - \frac{C^2}{C_T^2} ; \quad C \equiv U \tag{4.12a}$$

The range of z suggests the application of the Fourier transform. Equations 4.12 become:

$$\frac{\partial^2\tilde{\phi}}{\partial r^2} + \frac{1}{r}\frac{\partial\tilde{\phi}}{\partial r} - \alpha_L^2\omega^2\tilde{\phi} = 0$$

$$\frac{\partial^2\tilde{\psi}}{\partial r^2} + \frac{1}{r}\frac{\partial\tilde{\psi}}{\partial r} - \left(\alpha_T^2\omega^2 + \frac{1}{r^2}\right)\tilde{\psi} = 0 \tag{4.13}$$

Where the Fourier transform pair is defined by:

$$\tilde{\phi}(r,\omega) = \int_{-\infty}^{\infty} \phi(r,z)\, e^{-i\omega z}\, dz$$

$$\phi(r,z) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \tilde{\phi}(r,\omega)\, e^{i\omega z}\, d\omega \tag{4.14}$$

Solutions to equations 4.13 are:

$$\tilde{\phi}(r,\omega) = A(\omega)\, I_0(\alpha_L \omega r) + B(\omega)\, K_0(\alpha_L \omega r)$$

$$\tilde{\psi}(r,\omega) = C(\omega)\, I_1(\alpha_T \omega r) + D(\omega)\, K_1(\alpha_T \omega r) \tag{4.15}$$

The functions $A(\omega)$ and $C(\omega)$ are chosen equal to zero to insure regularity of the solutions as $r \to \infty$. We obtain:

$$\phi(r,z) = \frac{1}{2\pi}\int_{-\infty}^{\infty} B(\omega)\, K_0(\alpha_L \omega r)\, e^{i\omega z}\, d\omega$$

$$\psi(r,z) = \frac{1}{2\pi}\int_{-\infty}^{\infty} D(\omega)\, K_1(\alpha_T \omega r)\, e^{i\omega z}\, d\omega . \tag{4.16}$$

Equations for the radial (τ_{rr}) and tangential (τ_{rz}) stresses may be obtained from Reference (7) as:

$$\tau_{rr} = \lambda \nabla^2 \phi + 2\mu \frac{\partial u_r}{\partial r}$$
$$\tau_{rz} = \mu\left(\frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r}\right) \tag{4.17}$$

where:

$$u_r = \frac{\partial \phi}{\partial r} - \frac{\partial \psi}{\partial z}$$
$$u_z = \frac{\partial \phi}{\partial z} + \frac{1}{r}\frac{\partial}{\partial r}(r\psi) \tag{4.18}$$

Using equations 4.12. 4.17, and 4.18, the stresses can be written as:

$$\tau_{rz} = \mu\left\{2\frac{\partial^2 \phi}{\partial r \partial z} - (1+\alpha_T^2)\frac{\partial^2 \psi}{\partial z^2}\right\} = 0 \quad \text{at } r = a \tag{4.19}$$

We can integrate equation 4.19 with respect to z and take the arbitrary functional equal to zero. This introduces an arbitrary displacement into the problem at most. The result is:

$$\left\{2\frac{\partial \phi}{\partial r} - (1+\alpha_T^2)\frac{\partial \psi}{\partial z}\right\}_{r=a} = 0 . \tag{4.20}$$

The radial stress can be written as:

$$\tau_{rr} = \lambda\left(1-\alpha_L^2\right)\frac{\partial^2\phi}{\partial z^2} + 2\mu\frac{\partial}{\partial r}\left(\frac{\partial\phi}{\partial r} - \frac{\partial\psi}{\partial z}\right) = 0 \quad \text{at } r = a \qquad (4.21)$$

which becomes after substituting for λ :

$$\lambda = \mu\left(\frac{C_L^2}{C_T^2} - 2\right) \qquad (4.22)$$

the following:

$$\tau_{rr} = \left\{\left(-1-\alpha_T^2+2\alpha_L^2\right)\frac{\partial^2\phi}{\partial z^2} + 2\left(\frac{\partial^2\phi}{\partial r^2} - \frac{\partial^2\psi}{\partial r\partial z}\right)\right\}_{r=a} = 0 \qquad (4.23)$$

It is convenient to obtain the Fourier transforms of equations 4.20 and 4.23 as:

$$\left\{2\frac{\partial\tilde{\Phi}}{\partial r} - i\omega\left(1+\alpha_T^2\right)\tilde{\Psi}\right\}_{r=a} = 0 \qquad (4.24)$$

$$\left\{\left(-1-\alpha_T^2+2\alpha_L^2\right)\left(-\omega^2\tilde{\Phi}\right) + 2\left(\frac{\partial^2\tilde{\Phi}}{\partial r^2} - i\omega\frac{\partial\tilde{\Psi}}{\partial r}\right)\right\}_{r=a} = 0 \qquad (4.25)$$

Substitution of equations 4.15 into 4.24 and 4.25 yields:

$$\left(1+\alpha_T^2-2\alpha_L^2\right)\omega^2 B\,K_0(\alpha_L\omega a) + 2\alpha_L^2\omega^2 B\,K_0''(\alpha_L\omega a)$$

$$-\,2i\omega^2\alpha_T\,D\,K_1'(\alpha_T\omega a) = 0 \qquad (4.26)$$

$$2\alpha_L\omega\,B\,K_0'(\alpha_L\omega a) - \left(1+\alpha_T^2\right)i\omega\,D\,K_1(\alpha_T\omega a) = 0. \qquad (4.27)$$

The above relations may be simplified by use of the identities:

$$K_0''(z) = -K_1'(z) \tag{4.28}$$
$$K_0'(z) = -K_1(z)$$

We obtain:

$$(1+\alpha_T^2-2\alpha_L^2)\,B K_0(\alpha_L\omega a) - 2\alpha_L^2 B K_1'(\alpha_L\omega a) - 2i\alpha_T D K_1'(\alpha_T\omega a) = 0 \tag{4.29}$$
$$-2\alpha_L B K_1(\alpha_L\omega a) - (1+\alpha_T^2)\, i\, D K_1(\alpha_T\omega a) = 0$$

A non-trivial solution of equations 4.29 is obtained only if the determinant of the coefficients $B(\omega)$ and $D(\omega)$ vanishes. This condition yields:

$$-(1+\alpha_T^2-2\alpha_L^2)(1+\alpha_T^2)K_0(\alpha_L\omega a)K_1(\alpha_T\omega a)$$
$$+\,2\alpha_L^2(1+\alpha_T^2)K_1'(\alpha_L\omega a)K_1(\alpha_T\omega a)$$
$$-\,4\alpha_L\alpha_T K_1(\alpha_L\omega a)K_1'(\alpha_T\omega a) = 0 \tag{4.30}$$

This is clearly an equation for the speed of the surface wave if one considers the definitions of α_L^2, α_T^2. The above equation for the speed, C, of the free-running surface wave is clearly dispersive because of its dependence on ω, the frequency of the disturbance, as contained in the Bessel functions $K_0(\ )$ and $K_1(\ )$.

Suppose a $\rightarrow$ large, but not infinite, then:

$$K_n(x) \sim \frac{e^{-x}}{\sqrt{\frac{2}{\pi}x}}$$

$$K_n'(x) \sim -\frac{e^{-x}}{\sqrt{\frac{2}{\pi}x}}$$

and equation 4.30 becomes:

$$e^{-\alpha_L \omega a - \alpha_T \omega a}\left\{-\left(1+\alpha_T^2\right)^2 + 4\alpha_L \alpha_T\right\} = 0 \qquad (4.31)$$

Assuming that the exponential term does not vanish, the expression in braces yields the speed of the surface wave. The definition of α_L^2 , α_T^2 allow equation 4.31 to be written as:

$$\left(2-\frac{c^2}{c_T^2}\right)^2 - 4\sqrt{1-\frac{c^2}{c_L^2}}\,\sqrt{1-\frac{c^2}{c_T^2}} = 0 \qquad (4.32)$$

Comparison with equation 2.15 of Part II shows that the true Rayleigh wave, which is non-dispersive, is recovered by this approximation. Clearly, the assumption of $\omega \rightarrow$ large yields the same result.

III. Scattering of a Plane Dilatational Wave by a Cylindrical Cavity in an Infinite Medium

1. Introduction

Consider an infinitely long right circular cylindrical cavity contained in an infinite, isotropic and homogeneous medium. A plane wave of dilatation, which travels at speed C_L in the medium, is assumed to engulf the cavity. The direction of travel of the wave is normal to the geometric axis of the cavity. See Figure 7 below.

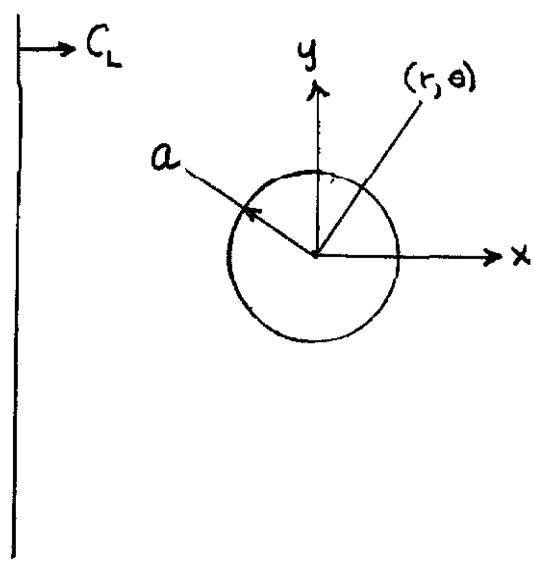


Figure 7. Coordinates and Direction of the Dilatational Wave

If the stress produced by the oncoming wave of dilatation is expressed as a Heaviside step, e.g.:

$$\sigma_x = \sigma H\left(t - \frac{x+a}{C_L}\right) \qquad (1.1)$$

then solutions for disturbances which vary arbitrarily with time may be constructed by application of Duhamel's integral to the solution for the step. This idea then establishes the importance of knowing the solution for the step input plane wave.

Much interest has been devoted to this problem recently. Gilbert (14) has considered scattering of a plane compressional wave by a cylindrical cavity for both normal and oblique incidence. His work considers only the illuminated portion of the cylinder, that is, the portion on which the incident wave impinges, and is applicable for early time. Gilbert and Knopoff (15) have used a method based on a development by Friedlander (16) to explore the effects of scattering of compressional waves, emanating from a line source in the medium, by an embedded rigid cylinder. Early time solutions for the illuminated and shadow (diffracted wave) zones were obtained. Soldate and Hook (17) have also contributed to the cavity scattering problem. They have obtained a low frequency - long time solution.

The most recent work on this problem has been accomplished by Miklowitz (18). He considers the circular cylindrical cavity subject to either an oncoming plane transient wave of dilalation or subject to an impulsively applied line load acting on the interior surface of the cavity.

His representation is based on Freidlander's method and by application of Fourier-Laplace transforms and contour integration, he has generated exact integral solutions to the problem. His work shows that the Rayleigh wave is predominant for long time and further, that the Rayleigh wave is non-decaying in space (Θ) and time. In a related paper, Miklowitz (19) has shown that the introduction of visco-elasticity (Maxwell model) has the effect of attenuating the Rayleigh wave.

The object of the investigation herein is manifold. First, the exact integral representation for the hoop stress at the boundary of the cavity is to be developed. This quantity is expected to be of primary importance as a design parameter for shock resistant structures. The method to be followed is essentially that of Baron and Matthews (20) in which the solution is represented in Fourier series. It is anticipated, and shown later, that the resulting integral reprentations are not amenable to solution in closed form. Thus, certain approximations will be introduced to obtain the early and long time behavior of the hoop stress at the cavity boundary. It appears that the use of Fourier series for long time and Friedlander's method for early time solutions is appropriate. These representations are explored in subsequent portions of this paper. Unfortunately, the work of Miklowitz was not available during this investigation and the validity of the Fourier series representation for long time is in doubt, especially in light of the non-decaying nature of the Rayleigh waves as shown by Miklowitz.

2. Boundary Conditions

Coordinate z, along the geometric axis of the cylinder, extends to infinity in both directions. Thus, the problem is one of plane strain.

Hooke's law can be written as:

$$\epsilon_x = \frac{1}{E}\left[\sigma_x - \nu(\sigma_y + \sigma_z)\right]$$

$$\epsilon_y = \frac{1}{E}\left[\sigma_y - \nu(\sigma_z + \sigma_x)\right] \qquad (2.1)$$

$$\epsilon_z = \frac{1}{E}\left[\sigma_z - \nu(\sigma_x + \sigma_y)\right]$$

where:

E = Young's modulus

ν = Poisson's ratio.

The shearing stresses, $\tau_{xy}, \tau_{yz}, \tau_{xz}$ are identically zero.

If a dilatational potential of the form:

$$\phi = \frac{\sigma}{2\rho} H\left(t - \frac{x+a}{c_L}\right)\left(t - \frac{x+a}{c_L}\right)^2 \qquad (2.2)$$

is chosen, then, it is easily shown that the wave equation (which will be shown to apply in a subsequent section):

$$\nabla^2 \phi = \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = \frac{1}{C_L^2} \frac{\partial^2 \phi}{\partial t^2} \tag{2.3}$$

is satisfied. Also, expressing $\vec{u} = \text{grad}\ \phi$ and using the expression for the stress tensor (Part I, equation 3.1) results in the equation for the stress as given in equation 1.1 above.

A most useful result is obtained from the preceding analysis, namely:

$$\epsilon_y = \frac{\partial v}{\partial y} = \frac{\partial^2 \phi}{\partial y^2} = 0 \tag{2.4}$$

then, equations 2.1 allow the relation between σ_x and σ_y to be written as:

$$\sigma_y = \frac{\nu(1+\nu)}{1-\nu^2} \sigma_x = \frac{\nu}{1-\nu} \sigma_x = -\epsilon \sigma_x \tag{2.5}$$

This result will be used throughout the following analysis.

It is convenient to express the boundary conditions in circular cylindrical coordinates:

$$\begin{aligned} \sigma_r &= 0 \quad \text{at } r = a \\ \tau_{r\theta} &= 0 \quad \text{at } r = a \end{aligned} \tag{2.6}$$

where:

$$\sigma_r = \frac{1}{2}(\sigma_x + \sigma_y) + \frac{1}{2}(\sigma_x - \sigma_y)\cos 2\theta$$

$$\tau_{r\theta} = -\frac{1}{2}(\sigma_x - \sigma_y)\sin 2\theta \qquad (2.7)$$

$$\sigma_\theta = \frac{1}{2}(\sigma_x + \sigma_y) - \frac{1}{2}(\sigma_x - \sigma_y)\cos 2\theta$$

or equivalently, using equation 2.5:

$$\sigma_r = \sigma_x\left(\cos^2\theta - \epsilon\sin^2\theta\right)$$

$$\tau_{r\theta} = -\left(\frac{1+\epsilon}{2}\right)\sigma_x \sin 2\theta \qquad (2.8)$$

$$\sigma_\theta = \sigma_x\left(\sin^2\theta - \epsilon\cos^2\theta\right)$$

which are valid for σ_x equal a tensile stress.

The stress field generated by the incident wave, assuming the cavity to be absent, is given by equations 2.8. This field is defined for every point in the medium. Addition of stresses $-\sigma_r$ and $-\tau_{r\theta}$ on a circle of radius "a" in essence creates a cavity for which the boundary conditions, equations 2.6, are satisfied. Then, when the resulting stress field, which can be considered the scattered field, is added to the incident field, the complete solution is found.

Thus, for the scattered field, the boundary conditions:

$$\sigma_r = -\sigma H\left[t - \frac{a}{c_L}(1+\cos\theta)\right]\left(\cos^2\theta - \epsilon\sin^2\theta\right)$$

$$\tau_{r\theta} = \sigma H\left[t - \frac{a}{c_L}(1+\cos\theta)\right]\left(\frac{1+\epsilon}{2}\right)\sin 2\theta \qquad (2.9)$$

must be satisfied at $r = a$.

3. Equations of Motion and Stress Equations - Scattered Field

The equations of motion can be obtained from Part II, Section 4 as:

$$\nabla^2\phi = \frac{1}{C_L^2}\frac{\partial^2\phi}{\partial t^2}$$

$$-\,\mathrm{curl}\;\mathrm{curl}\;\vec{\psi} = \frac{1}{C_T^2}\frac{\partial^2\vec{\psi}}{\partial t^2}\;;\quad \mathrm{div}\;\vec{\psi} = 0 \tag{3.1}$$

where $\vec{\psi}$ has only one component which is associated with the z-direction. In circular cylindrical coordinates, (r, θ) only, the relation:

$$\mathrm{curl}\;\mathrm{curl}\;\vec{\psi} = \mathrm{grad}\;\mathrm{div}\;\vec{\psi} - \nabla^2\vec{\psi}$$

is valid and the equation for $\vec{\psi}$ becomes:

$$\nabla^2\vec{\psi} = \frac{1}{C_T^2}\frac{\partial^2\vec{\psi}}{\partial t^2} \tag{3.2}$$

where

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\,.$$

The stress equations are obtained from Reference (7) as:

$$\sigma_r = \lambda\Delta + 2\mu\frac{\partial u_r}{\partial r}$$

$$\tau_{r\theta} = \mu\left\{\frac{1}{r}\frac{\partial u_r}{\partial \theta} + r\frac{\partial}{\partial r}\left(\frac{u_\theta}{r}\right)\right\} \tag{3.3}$$

$$\sigma_\theta = \lambda\Delta + 2\mu\left(\frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{u_r}{r}\right)$$

where:

$$\Delta = \operatorname{div} \vec{u} = \nabla^2 \phi. \tag{3.4}$$

Writing:

$$\vec{u} = \operatorname{grad} \phi + \operatorname{curl} \vec{\psi}\ ;\ \operatorname{div} \vec{\psi} = 0^{\,*} \tag{3.5}$$

Using this result in equations 3.3 yields:

$$\sigma_r = \lambda \nabla^2 \phi + 2\mu \left(\frac{\partial^2 \phi}{\partial r^2} - \frac{1}{r^2} \frac{\partial \psi}{\partial \theta} + \frac{1}{r} \frac{\partial^2 \psi}{\partial r \partial \theta} \right)$$

$$\tau_{r\theta} = \mu \left(\frac{2}{r} \frac{\partial^2 \phi}{\partial r \partial \theta} - \frac{2}{r^2} \frac{\partial \phi}{\partial \theta} + \frac{1}{r} \frac{\partial \psi}{\partial r} - \frac{\partial^2 \psi}{\partial r^2} - \frac{1}{r^2} \frac{\partial^2 \psi}{\partial \theta^2} \right) \tag{3.7}$$

$$\sigma_\theta = \lambda \nabla^2 \phi + 2\mu \left(\frac{1}{r^2} \frac{\partial^2 \phi}{\partial \theta^2} - \frac{1}{r} \frac{\partial^2 \psi}{\partial r \partial \theta} + \frac{1}{r} \frac{\partial \phi}{\partial r} + \frac{1}{r^2} \frac{\partial \psi}{\partial \theta} \right).$$

---

* which implies:

$$u_r = \frac{\partial \phi}{\partial r} + \frac{1}{r} \frac{\partial \psi}{\partial \theta}\ ;\ u_\theta = \frac{1}{r} \frac{\partial \phi}{\partial \theta} - \frac{\partial \psi}{\partial r}\ . \tag{3.6}$$

4. Solution Procedure

A Laplace transformation can be effected to eliminate the time derivatives. The equations of motion become:

$$\nabla^2\bar{\Phi} - \frac{p^2}{c_L^2}\bar{\Phi} = 0 \; ; \; p = \text{Laplace operator}$$
$$\nabla^2\bar{\Psi} - \frac{p^2}{c_T^2}\bar{\Psi} = 0 \tag{4.1}$$

where it is recalled that $\vec{\Psi}$ has only one component.

The stresses, σ_r and σ_θ, are even and periodic in θ. However, the shear stress, $\tau_{r\theta}$, is odd and periodic in θ. Inspection of the stress equations 3.7 suggests writing $\bar{\Phi}$ and $\bar{\Psi}$ as Fourier cosine and sine series, respectively. Thus:

$$\bar{\Phi} = \sum_{n=0}^{\infty} \bar{\bar{\Phi}}_n \cos n\theta$$
$$\bar{\Psi} = \sum_{n=1}^{\infty} \bar{\bar{\Psi}}_n \sin n\theta \tag{4.2}$$

Then each harmonic of equations 4.1 must satisfy:

$$\frac{\partial^2\bar{\bar{\Phi}}_n}{\partial r^2} + \frac{1}{r}\frac{\partial\bar{\bar{\Phi}}_n}{\partial r} - \left(\frac{n^2}{r^2} + \frac{p^2}{c_L^2}\right)\bar{\bar{\Phi}}_n = 0$$
$$\frac{\partial^2\bar{\bar{\Psi}}_n}{\partial r^2} + \frac{1}{r}\frac{\partial\bar{\bar{\Psi}}_n}{\partial r} - \left(\frac{n^2}{r^2} + \frac{p^2}{c_T^2}\right)\bar{\bar{\Psi}}_n = 0 . \tag{4.3}$$

Solutions to equations 4.3 may be expressed as modified Bessel functions of the first and second kinds, i.e.:

$$\bar{\bar{\Phi}}_n = A_n I_n\left(\frac{rp}{c_L}\right) + B_n K_n\left(\frac{rp}{c_L}\right)$$
$$\bar{\bar{\Psi}}_n = C_n I_n\left(\frac{rp}{c_T}\right) + D_n K_n\left(\frac{rp}{c_T}\right) \tag{4.4}$$

The constants $A_n = C_n = 0$ to insure outgoing waves as $r \longrightarrow$ large.

If the definition of the Heaviside step function is used:

$$H\left[t - \frac{a}{c_L}(1+\cos\theta)\right] = 1 \; ; \; t \geq \frac{a}{c_L}(1+\cos\theta)$$
$$= 0 \; ; \; \text{otherwise} \tag{4.5}$$

the Laplace transformations of the boundary conditions as expressed by equations 2.9 can be written:

$$\bar{\sigma}_r(a,\theta,p) = -\frac{\sigma}{p}\left(\cos^2\theta - \epsilon\sin^2\theta\right)e^{-\frac{ap}{c_L}(1+\cos\theta)}$$
$$\bar{\tau}_{r\theta}(a,\theta,p) = \frac{\sigma}{p}\left(\frac{1+\epsilon}{2}\right)\sin 2\theta \cdot e^{-\frac{ap}{c_L}(1+\cos\theta)} \quad . \tag{4.6}$$

The boundary conditions can also be written as Fourier cosine and sine series, respectively:

$$\bar{\sigma}_r(a,\theta,p) = \sum_{n=0}^{\infty} a_n \cos n\theta$$

$$\bar{\tau}_{r\theta}(a,\theta,p) = \sum_{n=0}^{\infty} b_n \sin n\theta \tag{4.7}$$

where:

$$a_0 = \frac{1}{\pi}\int_0^{\pi} \bar{\sigma}_r(a,\theta,p)\, d\theta$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} \bar{\sigma}_r(a,\theta,p) \cos n\theta \, d\theta \tag{4.8}$$

$$b_n = \frac{2}{\pi}\int_0^{\pi} \bar{\tau}_{r\theta}(a,\theta,p) \sin n\theta \, d\theta$$

Using the results from equations 3.7, 4.2 and 4.7, the boundary conditions for the scattered field can be written as:

$$a_n = \left\{ \lambda \left(\frac{\partial^2 \bar{\bar{\Phi}}_n}{\partial r^2} + \frac{1}{r}\frac{\partial \bar{\bar{\Phi}}_n}{\partial r} - \frac{n^2}{r^2}\bar{\bar{\Phi}}_n \right) + 2\mu \left(\frac{\partial^2 \bar{\bar{\Phi}}_n}{\partial r^2} - \frac{n}{r^2}\bar{\bar{\Psi}}_n + \frac{n}{r}\frac{\partial \bar{\bar{\Psi}}_n}{\partial r} \right) \right\}_{r=a}$$

$$b_n = \left\{ \mu \left(-\frac{2n}{r}\frac{\partial \bar{\bar{\Phi}}_n}{\partial r} + \frac{2n}{r^2}\bar{\bar{\Phi}}_n + \frac{1}{r}\frac{\partial \bar{\bar{\Psi}}_n}{\partial r} - \frac{\partial^2 \bar{\bar{\Psi}}_n}{\partial r^2} - \frac{n^2}{r^2}\bar{\bar{\Psi}}_n \right) \right\}_{r=a} \qquad (4.9)$$

where each harmonic must satisfy these equations.

The evaluation of coefficients a_n and b_n is carried out in Appendix B. The results are:

$$a_n = -\frac{2\sigma}{\rho} e^{-\frac{ap}{c_L}} \left\{ \left(\frac{1-\epsilon}{2}\right)(-)^n I_n\left(\frac{ap}{c_L}\right) + \left(\frac{1+\epsilon}{4}\right)\left[(-)^{n+2} I_{n+2}\left(\frac{ap}{c_L}\right) + (-)^{n-2} I_{n-2}\left(\frac{ap}{c_L}\right)\right]\right\} \qquad (4.10)$$

$$b_n = \frac{\sigma}{\rho} e^{-\frac{ap}{c_L}} \left(\frac{1+\epsilon}{2}\right)\left\{(-)^{n-2} I_{n-2}\left(\frac{ap}{c_L}\right) - (-)^{n+2} I_{n+2}\left(\frac{ap}{c_L}\right)\right\}$$

Substitution of equations 4.4 and 4.10 into 4.9 allows determination of the constants $\mathcal{B}_n$ and $\mathcal{D}_n$. However, equations 4.9 can be rewritten in a more convenient form by use of equations 4.3 and the relation:

$$\frac{\lambda + 2\mu}{\mu} = \frac{C_L^2}{C_T^2} \tag{4.11}$$

as:

$$\frac{a_n}{\mu} = \left\{ \left[\frac{p^2}{C_T^2} + \frac{2n^2}{r^2} \right] \bar{\bar{\phi}}_n - \frac{2}{r}\frac{d\bar{\bar{\phi}}_n}{dr} + \frac{2n}{r}\left(\frac{d\bar{\bar{\psi}}_n}{dr} - \frac{\bar{\bar{\psi}}_n}{r} \right) \right\}_{r=a}$$

$$\frac{b_n}{\mu} = \left\{ -\frac{2n}{r}\left(\frac{d\bar{\bar{\phi}}_n}{dr} - \frac{\bar{\bar{\phi}}_n}{r} \right) - \left[\frac{p^2}{C_T^2} + \frac{2n^2}{r^2} \right] \bar{\bar{\psi}}_n + \frac{2}{r}\frac{d\bar{\bar{\psi}}_n}{dr} \right\}_{r=a} \tag{4.12}$$

Equations 4.12 become, after the indicated substitutions:

$$-\frac{\sigma}{\mu p} e^{-\frac{ap}{C_L}} (-)^n \left\{ (1-\epsilon) I_n\left(\frac{ap}{C_L}\right) + \left(\frac{1+\epsilon}{2}\right)\left[I_{n+2}\left(\frac{ap}{C_L}\right) + I_{n-2}\left(\frac{ap}{C_L}\right) \right] \right\}$$

$$= B_n \left\{ \left[\frac{p^2}{C_T^2} + \frac{2n^2}{a^2} \right] K_n\left(\frac{ap}{C_L}\right) - \frac{2p}{aC_L} K_n'\left(\frac{ap}{C_L}\right) \right\}$$

$$+ D_n \left\{ \frac{2np}{aC_T} K_n'\left(\frac{ap}{C_T}\right) - \frac{2n}{a^2} K_n\left(\frac{ap}{C_T}\right) \right\} \tag{4.13}$$

and:

$$\frac{\sigma}{\mu p} e^{-\frac{ap}{c_L}} (-)^n \left(\frac{1+\epsilon}{2}\right)\left\{ I_{n-2}\left(\frac{ap}{c_L}\right) - I_{n+2}\left(\frac{ap}{c_L}\right)\right\}$$

$$= B_n \left\{ -\frac{2np}{a c_L} K_n'\left(\frac{ap}{c_L}\right) + \frac{2n}{a^2} K_n\left(\frac{ap}{c_L}\right)\right\}$$

$$+ D_n \left\{ -\left[\frac{2n^2}{a^2} + \frac{p^2}{c_T^2}\right] K_n\left(\frac{ap}{c_T}\right) + \frac{2p}{a c_T} K_n'\left(\frac{ap}{c_T}\right)\right\}. \qquad (4.14)$$

The coefficients may be expressed as:

$$|D| B_n = -\frac{\sigma}{\mu p} e^{-\frac{ap}{c_L}} (-)^n \left\{ (1-\epsilon) I_n\left(\frac{ap}{c_L}\right) + \left(\frac{1+\epsilon}{2}\right)\left[I_{n+2}\left(\frac{ap}{c_L}\right) + I_{n-2}\left(\frac{ap}{c_L}\right)\right]\right\}$$

$$\text{times} \left\{ \frac{2p}{a c_T} K_n'\left(\frac{ap}{c_T}\right) - \left(\frac{2n^2}{a^2} + \frac{p^2}{c_T^2}\right) K_n\left(\frac{ap}{c_T}\right)\right\}$$

$$-\frac{\sigma}{\mu p} e^{-\frac{ap}{c_L}} (-)^n \left(\frac{1+\epsilon}{2}\right)\left\{ I_{n-2}\left(\frac{ap}{c_L}\right) - I_{n+2}\left(\frac{ap}{c_L}\right)\right\}$$

$$\text{times} \left\{ \frac{2np}{a c_T} K_n'\left(\frac{ap}{c_T}\right) - \frac{2n}{a^2} K_n\left(\frac{ap}{c_T}\right)\right\} \qquad (4.15)$$

$$|D|\,D_n = \frac{\sigma}{\mu p}\, e^{-\frac{ap}{c_L}} (-)^n \left\{ \left(\frac{p^2}{c_T^2} + \frac{2n^2}{a^2}\right) K_n\left(\frac{ap}{c_L}\right) - \frac{2p}{ac_L} K_n'\left(\frac{ap}{c_L}\right) \right\}$$

$$\text{times} \left\{ \frac{1+\epsilon}{2} \left[I_{n-2}\left(\frac{ap}{c_L}\right) - I_{n+2}\left(\frac{ap}{c_L}\right) \right] \right\}$$

$$+ \frac{\sigma}{\mu p}\, e^{-\frac{ap}{c_L}} (-)^n \left\{ (1-\epsilon) I_n\left(\frac{ap}{c_L}\right) + \left(\frac{1+\epsilon}{2}\right) \left[I_{n+2}\left(\frac{ap}{c_L}\right) + I_{n-2}\left(\frac{ap}{c_L}\right) \right] \right\}$$

$$\text{times} \left\{ -\frac{2np}{ac_L} K_n'\left(\frac{ap}{c_L}\right) + \frac{2n}{a^2} K_n\left(\frac{ap}{c_L}\right) \right\} \qquad (4.16)$$

where:

$$|D| = \left[\frac{4n^2}{a^4}(1-n^2) - \frac{p^4}{c_T^4} - \frac{4n^2p^2}{a^2c_T^2} \right] K_n\left(\frac{ap}{c_L}\right) K_n\left(\frac{ap}{c_T}\right)$$

$$+ \frac{2p^3}{ac_T^3} K_n\left(\frac{ap}{c_L}\right) K_n'\left(\frac{ap}{c_T}\right) + \frac{2p^3}{ac_Lc_T^2} K_n'\left(\frac{ap}{c_L}\right) K_n\left(\frac{ap}{c_T}\right)$$

$$+ \frac{4p^2}{a^2c_Lc_T}(n^2-1) K_n'\left(\frac{ap}{c_L}\right) K_n'\left(\frac{ap}{c_T}\right). \qquad (4.17)$$

Some simplification in coefficients B_n and D_n may be effected by use of the Wronskian:

$$I_\nu(\alpha) K_\nu'(\alpha) - I_\nu'(\alpha) K_\nu(\alpha) = -\frac{1}{\alpha} \qquad (4.18)$$

and the recurrence relations:

$$I_{\nu-1}(\alpha) - I_{\nu+1}(\alpha) = \frac{2\nu}{\alpha} I_\nu(\alpha)$$

$$\alpha I_\nu'(\alpha) - \nu I_\nu(\alpha) = \alpha I_{\nu+1}(\alpha) \tag{4.19}$$

Using equations 4.19, the result is obtained:

$$I_{n+2}\left(\frac{ap}{c_L}\right) + I_{n-2}\left(\frac{ap}{c_L}\right)$$

$$= \left(2 + \frac{4c_L^2 n^2}{a^2p^2}\right) I_n\left(\frac{ap}{c_L}\right) - \frac{4c_L}{ap} I_n'\left(\frac{ap}{c_L}\right)$$

$$(4.20)$$

$$I_{n+2}\left(\frac{ap}{c_L}\right) - I_{n-2}\left(\frac{ap}{c_L}\right)$$

$$= \frac{4c_L^2 n}{a^2p^2} I_n\left(\frac{ap}{c_L}\right) - \frac{4c_L n}{ap} I_n'\left(\frac{ap}{c_L}\right)$$

Then, substituting equations 4.20 into 4.15 - 4.16 and using equation 4.18 in 4.16, the result is obtained, after some manipulation, that:

$$|D|B_n = \frac{\sigma}{\mu p} e^{-\frac{ap}{c_L}} (-)^{n+1} \left\{ \left[\frac{8c_T^2 n^2}{a^4 p^2}(1-n^2) - \frac{8n^2}{a^2} - \frac{2p^2}{c_T^2} \right] I_n\left(\frac{ap}{c_L}\right) K_n\left(\frac{ap}{c_T}\right) \right.$$

$$+ \frac{4p}{ac_T} I_n\left(\frac{ap}{c_L}\right) K_n'\left(\frac{ap}{c_T}\right) + \frac{4p}{ac_L} I_n'\left(\frac{ap}{c_L}\right) K_n\left(\frac{ap}{c_T}\right)$$

(con't)

$$+ \frac{8C_T}{a^2C_L}(n^2-1)\, I_n'\left(\frac{ap}{C_L}\right) K_n'\left(\frac{ap}{C_T}\right)\Bigg\} \qquad (4.21)$$

$$|D| D_n = \frac{\sigma}{\mu p} e^{-\frac{ap}{C_L}} (-)^n \left\{ \frac{4n}{a^2} + \frac{8C_T^2 n}{a^4 p^2}(n^2-1) \right\} \qquad (4.22)$$

where the results:

$$-\epsilon = \frac{\nu}{1-\nu} = 1 - 2\frac{C_T^2}{C_L^2}$$

$$1-\epsilon = 2 - 2\frac{C_T^2}{C_L^2} \qquad (4.23)$$

and

$$\frac{1+\epsilon}{2} = \frac{C_T^2}{C_L^2}$$

have been used to obtain the expressions for the coefficients as given by equations 4.21 - 4.22.

The complete solutions for the dilatational and transverse potentials are to be obtained by evaluating the inverse Laplace transforms of the harmonics, $\bar{\bar{\Phi}}_n$ and $\bar{\bar{\Psi}}_n$, then summing the series as implied by equations 4.2. Application of the Fourier-Mellin inversion theorem yields the result:

$$\phi(r,\theta,t) = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} B_0 K_0\left(\frac{rp}{C_L}\right) e^{pt}\, dp + \sum_{n=1}^{\infty} \cos n\theta \cdot \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} B_n K_n\left(\frac{rp}{C_L}\right) e^{pt}\, dp \qquad (4.24)$$

where:

$$B_0 = \frac{1}{2}\left\{B_n\right\}_{n=0} \qquad (4.25)$$

and:

$$\psi(r,\theta,t) = \sum_{n=1}^{\infty} \sin n\theta \cdot \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} D_n K_n\left(\frac{rp}{C_T}\right) e^{pt}\, dp. \qquad (4.26)$$

The above potentials are those associated with the scattered field. The displacements and stresses may be computed by application of equations 3.6 - 3.7. The total stress and displacement at any point must, of course, include the effects of the scattered field and the incident wave as discussed in Section 2.

The hoop stress at the cavity boundary is perhaps of primary importance, especially in the design of shock resistant structures of cylindrical shape. Noting that the terms $(-)^n$, $I_n(\ )$, $K_n(\ )$, $I_n'(\ )$, $K_n'(\ )$, B_n and nD_n are even in "n" and using equations 3.7 and 4.24 - 4.26, the hoop stress at $r = a$ caused by the scattered field may be written as:

$$\sigma_\theta(a,\theta,t) = \frac{1}{2}\sum_{n=-\infty}^{\infty} e^{in\theta} \cdot \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} \left\{\left[\frac{\lambda p^2}{C_L^2} - \frac{2\mu n^2}{a^2}\right] K_n\left(\frac{ap}{C_L}\right) + \frac{2\mu p}{aC_L} K_n'\left(\frac{ap}{C_L}\right)\right\} B_n\, e^{pt}\, dp$$

(con't)

$$+ \frac{1}{2} \sum_{n=-\infty}^{\infty} e^{in\theta} \cdot \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} \left\{ -\frac{2\mu n p}{a C_T} K_n'\left(\frac{ap}{C_T}\right) \right.$$

$$\left. + \frac{2\mu n}{a^2} K_n\left(\frac{ap}{C_T}\right) \right\} D_n e^{pt} dp. \qquad (4.27)$$

The Lame constant λ may be expressed as:

$$\lambda = \mu\left(\frac{C_L^2}{C_T^2} - 2\right). \qquad (4.28)$$

Using this result and substituting for the coefficients B_n and D_n from equations 4.21 - 4.22 yields the following expression for the hoop stress:

$$\sigma_\theta(a,\theta,t) = \frac{1}{2} \sum_{n=-\infty}^{\infty} e^{in\theta} \cdot \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} \frac{\sigma}{p} e^{-\frac{ap}{C_L}} (-)^{n+1} \left\{ \left[\frac{p^2}{C_T^2} - \frac{2p^2}{C_L^2} \right.\right.$$

$$\left. - \frac{2n^2}{a^2} \right] K_n\left(\frac{ap}{C_L}\right)$$

$$+ \frac{2p}{aC_L} K_n'\left(\frac{ap}{C_L}\right) \Bigg\} \Bigg\{ \left[\frac{8C_T^2 n^2}{a^4 p^2}(1-n^2) - \frac{8n^2}{a^2} - \frac{2p^2}{C_T^2} \right] I_n\left(\frac{ap}{C_L}\right) K_n\left(\frac{ap}{C_T}\right)$$

$$+ \frac{4p}{aC_T} I_n\left(\frac{ap}{C_L}\right) K_n'\left(\frac{ap}{C_T}\right) + \frac{4p}{aC_L} I_n'\left(\frac{ap}{C_L}\right) K_n\left(\frac{ap}{C_T}\right)$$

(Con't)

$$+ \frac{8C_T}{a^2 C_L}(n^2-1) I_n'\left(\frac{ap}{C_L}\right) K_n'\left(\frac{ap}{C_T}\right)\Big\} \frac{e^{pt}}{|D|}\, dp$$

$$+ \frac{1}{2} \sum_{n=-\infty}^{\infty} e^{in\theta} \cdot \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} \frac{\sigma}{p} e^{-\frac{ap}{C_L}} (-)^n \Big\{ - \frac{2np}{aC_T} K_n'\left(\frac{ap}{C_T}\right)$$

$$+ \frac{2n}{a^2} K_n\left(\frac{ap}{C_T}\right)\Big\}\Big\{\frac{4n}{a^2} + \frac{8C_T^2 n}{a^4 p^2}(n^2-1)\Big\} \frac{e^{pt}}{|D|}\, dp \qquad (4.29)$$

where $|D|$ is defined by equation 4.17.

Expansion of the preceding equation does not result in any appreciable simplification since no combinations of Wronskian type occur. The denominator determinant $|D|$ of equation 4.17 is also of such form as to add additional complexity to attempts to solve the equation for the hoop stress. Perhaps the application of digital machine computation techniques could be effected to obtain a solution to equation 4.29 or for that matter, for the potential equations 4.24 - 4.26. This possibility has not been pursued.

It is considered that some useful results can be generated by consideration of the early and long time behavior of the solution to equation 4.29. This idea is carried out in subsequent sections.

5. Long Time Approximation - Scattering of a Plane Dilatational Wave

It is a well known fact that solutions for large time may be associated with small values of the Laplace operator "p". This fact will be used in this section to generate a solution which is applicable for long time.

The limiting forms $(p \to 0)$ of the modified Bessel functions appearing in equation 4.29 could be utilized to obtain an approximate expression for the hoop stress. However, the modified Bessel function of the second kind (K_n) contains logarithmic terms and the resulting expansion becomes somewhat tedious to handle. Considerable simplification results from making the approximation, $p \to 0$, at a much earlier stage.

Equation 2.8 may be rewritten as:

$$
\begin{aligned}
\sigma_r &= \sigma_x \left(\frac{1-\epsilon}{2} + \frac{1+\epsilon}{2} \cos 2\theta \right) \\
\tau_{r\theta} &= -\sigma_x \left(\frac{1+\epsilon}{2} \right) \sin 2\theta \qquad (5.1) \\
\sigma_\theta &= \sigma_x \left(\frac{1-\epsilon}{2} - \frac{1+\epsilon}{2} \cos 2\theta \right) .
\end{aligned}
$$

If the algebraic signs are reversed, the above expressions represent the forcing functions for the scattered field portion of the solution. Therefore, as $t \rightarrow \infty$, only the $n=0$ and $n=2$ harmonics will influence the solution. With this fact in mind then, only the first three harmonics $(n=0,1,2)$ will be computed.

Assuming $p \rightarrow 0$, the transformed boundary conditions for the scattered field as expressed by equations 4.6 can be written as:

$$\bar{\sigma}_r(a,\theta,p) \cong -\frac{\sigma}{p}\left\{\frac{1-\epsilon}{2} + \frac{1+\epsilon}{2}\cos 2\theta\right\}\left\{1 - \frac{ap}{C_L}(1+\cos\theta)\right.$$

$$\left. + \frac{a^2p^2}{C_L^2}\left(1 + 2\cos\theta + \frac{1}{2} + \frac{1}{2}\cos 2\theta + \cdots\right.\right\}$$

$$\bar{\tau}_{r\theta}(a,\theta,p) \cong \frac{\sigma}{p}\left(\frac{1+\epsilon}{2}\sin 2\theta\right)\left\{1 - \frac{ap}{C_L}\left(1+\cos\theta\right)\right.$$

$$\left. + \frac{a^2p^2}{C_L^2}\left(1 + 2\cos\theta + \frac{1}{2} + \frac{1}{2}\cos 2\theta + \cdots\right.\right\}$$

which can be rewritten up to $O(p^3)$ as:

$$\bar{\sigma}_r(a,\theta,p) \cong -\frac{\sigma}{p}\Bigg\{\frac{1-\epsilon}{2}\left(1-\frac{ap}{C_L}\right)+\frac{7-5\epsilon}{16}\,\frac{a^2p^2}{C_L^2}$$

$$+\frac{3-\epsilon}{4}\left(-\frac{ap}{C_L}+\frac{a^2p^2}{C_L^2}\right)\cos\theta$$

$$+\left[\frac{1+\epsilon}{2}\left(1-\frac{ap}{C_L}\right)+\frac{2+\epsilon}{4}\,\frac{a^2p^2}{C_L^2}\right]\cos 2\theta$$

$$+\frac{1+\epsilon}{4}\left(-\frac{ap}{C_L}+\frac{a^2p^2}{C_L^2}\right)\cos 3\theta$$

$$+\frac{1+\epsilon}{4}\,\frac{a^2p^2}{C_L^2}\,\frac{\cos 4\theta}{4}+O(p^3)\Bigg\} \qquad (5.2)$$

$$\bar{\tau}_{r\theta}(a,\theta,p) \cong \frac{\sigma}{p}\left\{\frac{1+\epsilon}{2}\right\}\Bigg\{\left(-\frac{ap}{2C_L}-\frac{a^2p^2}{2C_L^2}\right)\sin\theta$$

$$+\left(1-\frac{ap}{C_L}+\frac{3a^2p^2}{4C_L^2}\right)\sin 2\theta$$

$$+\left(-\frac{ap}{2C_L}+\frac{a^2p^2}{2C_L^2}\right)\sin 3\theta$$

$$+\frac{a^2p^2}{8C_L^2}\sin 4\theta+O(p^3)\Bigg\}. \qquad (5.3)$$

Thus, the appropriate Fourier coefficients can be selected from the above expansions bearing in mind that:

$$\bar{\sigma}_r(a,\theta,p) = \sum_{n=0}^{\infty} a_n \cos n\theta$$

$$\bar{\tau}_{r\theta}(a,\theta,p) = \sum_{n=0}^{\infty} b_n \sin n\theta \qquad (4.7)$$

The transformed boundary conditions are still expressed by equations 4.12 and solutions for the transformed potentials by equation 4.4.

The expansions for the modified Bessel function of the second kind may be obtained from reference (21) as:

$$K_n(z) = (-)^{n+1} I_n(z) \log\left(\tfrac{z}{2}\right)$$
$$+ \frac{1}{2}\sum_{m=0}^{n-1} (-)^m \left(\tfrac{z}{2}\right)^{2m-n} \frac{(n-m-1)!}{m!}$$
$$+ \frac{1}{2}(-)^n \sum_{m=0}^{\infty} \left(\tfrac{z}{2}\right)^{n+2m} \left\{ \frac{\psi(n+m+1) + \psi(m+1)}{m!\,(n+m)!} \right\} \qquad (5.4)$$

where $n \neq 0$; $n = 1, 2, 3, \ldots$ and:

$$I_n(z) = \sum_{m=0}^{\infty} \frac{\left(\frac{z}{2}\right)^{n+2m}}{m!\,(n+m)!} \tag{5.5}$$

Also:

$$K_0(z) = -I_0(z) \log\left(\frac{z}{2}\right) + \sum_{m=0}^{\infty} \left(\frac{z}{2}\right)^{2m} \frac{\psi(m+1)}{m!^2}$$

where:

$$\psi(m+1) = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{m} - \gamma$$

and:

γ = Euler's constant = 0.5772157 ....

For small $z = \frac{ap}{C_T}$ or $\frac{ap}{C_L}$, the above expansions may be written as:

$$K_n(z) \cong \frac{1}{2}\left(\frac{z}{2}\right)^{-n}(n-1)! = \frac{1}{2}\left(\frac{z}{2}\right)^{-n}\Gamma(n)$$

$$K_n'(z) \cong -\frac{n}{z} K_n(z) \text{ ; leading term only} \tag{5.6}$$

and $K_0(z) \cong -\log\left(\frac{z}{2}\right)$.

5.1 Zeroth Harmonic $(n = 0,\ p \to 0)$

The results of equations 4.4 and 4.12 may be used to obtain:

$$a_0 \cong \mu \frac{p^2}{C_T^2} B_0\left(-\log \frac{ap}{2C_L}\right) - 2\frac{\mu}{a} B_0\left(-\frac{1}{a}\right)$$

$$b_0 \cong -\mu \frac{p^2}{C_T^2} D_0\left(-\log \frac{ap}{2C_T}\right) + 2\frac{\mu}{a} D_0\left(-\frac{1}{a}\right) \qquad (5.7)$$

The approximations of equations 5.6 have been utilized.

The above equations may be solved for B_0 and D_0. Introducing the results for a_0 and b_0 from equations 5.2 - 5.3, these coefficients may be written as:

$$B_0 \cong \frac{a^2}{2\mu} a_0 = -\frac{a^2}{2\mu}\frac{\sigma}{p}\left\{\frac{1-\epsilon}{2}\left(1-\frac{ap}{C_L}\right) + \frac{7-5\epsilon}{16}\frac{a^2p^2}{C_L^2}\right\}$$

$$D_0 \cong -\frac{a^2}{2\mu} b_0 = -\frac{a^2}{2\mu}\frac{\sigma}{p}\left\{0\right\}. \qquad (5.8)$$

Solutions for potentials ϕ_0 and ψ_0 may be expressed, by reference to equations 4.24 - 4.26 as:

$$\phi_0(r,\theta,t) \cong \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} -\frac{a^2\sigma}{2\mu}\left\{\left(\frac{1-\epsilon}{2}\right)\frac{1}{p} - \left(\frac{1-\epsilon}{2}\right)\frac{a}{c_L} + \left(\frac{7-5\epsilon}{16}\right)\frac{a^2 p}{c_L^2}\right\} K_0\left(\frac{rp}{c_L}\right) e^{pt}\, dp$$

which may be integrated to give, using the results of reference (13):

$$\begin{aligned}\phi_0(r,\theta,t) \cong -\frac{a^2\sigma}{2\mu}\Bigg\{ & \frac{1-\epsilon}{2} \ln\left|\frac{c_L t}{r} + \sqrt{\left(\frac{c_L t}{r}\right)^2 - 1}\right| \\ & - \left(\frac{1-\epsilon}{2}\right)\frac{a}{c_L}\cdot\frac{\partial}{\partial t}\ln\left|\frac{c_L t}{r} + \sqrt{\left(\frac{c_L t}{r}\right)^2 - 1}\right| \\ & + \left(\frac{7-5\epsilon}{16}\right)\frac{a^2}{c_L^2}\cdot\frac{\partial^2}{\partial t^2}\ln\left|\frac{c_L t}{r} + \sqrt{\left(\frac{c_L t}{r}\right)^2 - 1}\right| \\ & + \cdots\cdots \end{aligned} \qquad (5.9)$$

The transverse potential, ψ_0, is obviously zero.

Assuming $\frac{c_L t}{r} \rightarrow$ large, the dilatational potential simplifies to:

$$\phi_0(r,\theta,t) \cong -\frac{a^2\sigma}{2\mu}\left\{\frac{1-\epsilon}{2}\ln\left(\frac{2C_Lt}{r}\right)-\left(\frac{1-\epsilon}{2}\right)\frac{a}{C_Lt}\right.$$

$$\left.-\left(\frac{7-5\epsilon}{16}\right)\frac{a^2}{C_L^2t^2}+\cdots\right. \qquad (5.10)$$

5.2 <u>First Harmonic</u> $(n=1,\ p\to 0)$

Substitution of equations 4.4 into 4.12 yields:

$$a_1 \equiv \left(\mu\frac{p^2}{C_T^2}+2\frac{\mu}{a^2}\right)B_1K_1\left(\frac{ap}{C_L}\right)-2\frac{\mu p}{aC_L}B_1K_1'\left(\frac{ap}{C_L}\right)$$

$$+2\frac{\mu}{a}\left[\frac{p}{C_T}D_1K_1'\left(\frac{ap}{C_T}\right)-\frac{1}{a}D_1K_1\left(\frac{ap}{C_T}\right)\right]$$

$$(5.11)$$

$$b_1 \equiv -2\frac{\mu}{a}\left[\frac{p}{C_L}B_1K_1'\left(\frac{ap}{C_L}\right)-\frac{1}{a}B_1K_1\left(\frac{ap}{C_L}\right)\right]$$

$$-\left(\mu\frac{p^2}{C_T^2}+2\frac{\mu}{a^2}\right)D_1K_1\left(\frac{ap}{C_T}\right)+2\frac{\mu p}{aC_T}D_1K_1'\left(\frac{ap}{C_T}\right).$$

The leading terms in $K_1\left(\frac{ap}{C_L}\right), K_1'\left(\frac{ap}{C_L}\right)$ etc., as implied by equations 5.6 may be substituted above to yield:

$$\frac{a_1}{\mu} \cong \left(\frac{p^2}{C_T^2} + \frac{4}{a^2}\right)\frac{C_L}{ap} B_1 - \frac{4C_T}{a^3 p} D_1$$

$$\frac{b_1}{\mu} \cong \frac{4C_L}{a^3 p} B_1 - \left(\frac{p^2}{C_T^2} + \frac{4}{a^2}\right)\frac{C_T}{ap} D_1 \tag{5.12}$$

which can be solved to obtain:

$$\frac{\mu a^3 p}{4C_T} |D| B_1 \cong - a_1 + b_1$$

$$\frac{\mu a^3 p}{4C_L} |D| D_1 \cong - a_1 + b_1 \tag{5.13}$$

where $|D|$ is the denominator determinant and:

$$|D|^{-1} \cong \left(- \frac{8C_L}{a^4 C_T} - \frac{C_L p^2}{a^2 C_T^3}\right)^{-1}$$

$$\cong - \frac{a^4 C_T}{8C_L}\left(1 - \frac{a^2 p^2}{8 C_T^2} + \cdots \right) \tag{5.14}$$

Extracting the Fourier coefficients a_1 and b_1

from equations 5.2 and 5.3, coefficients B_1 and D_1 become:

$$B_1 \cong \frac{4C_T}{\mu a^3 p} \cdot \frac{\sigma}{p} \left\{ \frac{3-\epsilon}{4} \left(\frac{ap}{C_L} - \frac{a^2p^2}{C_L^2} \right) + \frac{1+\epsilon}{4} \left(\frac{ap}{C_L} - \frac{a^2p^2}{C_L^2} \right) \right\}$$

$$\text{times} \left\{ -\frac{a^4 C_T}{8C_L} \left(1 - \frac{a^2p^2}{8C_T^2} + \cdots \right) \right\}$$

$$D_1 \cong \frac{4C_L}{\mu a^3 p} \cdot \frac{\sigma}{p} \left\{ \frac{3-\epsilon}{4} \left(\frac{ap}{C_L} - \frac{a^2p^2}{C_L^2} \right) + \frac{1+\epsilon}{4} \left(\frac{ap}{C_L} - \frac{a^2p^2}{C_L^2} \right) \right\}$$

$$\text{times} \left\{ -\frac{a^4 C_T}{8C_L} \left(1 - \frac{a^2p^2}{8C_T^2} + \cdots \right) \right\}$$

which simplify, for $p \to 0$, to:

$$B_1 \cong \frac{a^2 C_T^2 \sigma}{2\mu C_L^2} \left(\frac{1}{p} - \frac{a}{C_L} - \frac{a^2 p}{8C_T^2} + \cdots \right)$$

$$(5.15)$$

$$D_1 \cong \frac{a^2 C_T \sigma}{2\mu C_L} \left(\frac{1}{p} - \frac{a}{C_L} - \frac{a^2 p}{8C_T^2} + \cdots \right)$$

Equations 4.24 - 4.26 permit the solutions for ϕ_1 and ψ_1 to be written as:

$$\frac{\phi_1}{\cos\theta} = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} B_1 K_1\left(\frac{rp}{C_L}\right)e^{pt}\,dp$$

$$\frac{\psi_1}{\sin\theta} = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} D_1 K_1\left(\frac{rp}{C_T}\right)e^{pt}\,dp \qquad (5.16)$$

It is convenient to use the expansion for $K_1\left(\frac{rp}{C_T}\right)$ in the form:

$$K_1\left(\frac{rp}{C_T}\right) \cong I_1\left(\frac{rp}{C_T}\right)\log\left(\frac{rp}{2C_T}\right) + \frac{C_T}{rp} + \cdots$$

and since $K_0\left(\frac{rp}{C_T}\right) \cong -\log\left(\frac{rp}{2C_T}\right) + \cdots\cdots$:

$$K_1\left(\frac{rp}{C_T}\right) \cong -K_0\left(\frac{rp}{C_T}\right)\left(\frac{rp}{2C_T} + \frac{r^3p^3}{16C_T^3} + \cdots\right)$$

$$+\frac{C_T}{rp} + \cdots \qquad (5.17)$$

A similar result may be obtained for $K_1\left(\frac{rp}{C_L}\right)$.

Incorporation of equation 5.17 into 5.16 then allows the potentials to be written as:

$$\frac{\phi_1}{\cos\theta} \cong \frac{\sigma}{2\mu}\cdot\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}\left\{-K_0\left(\frac{rp}{C_L}\right)\left[\frac{a^2 r C_T^2}{2C_L^3} - \frac{a^3 r C_T^2}{2C_L^4}p - \cdots\right] + \frac{a^2 C_T^2}{rC_L p^2} - \frac{a^3 C_T^2}{rC_L^2 p} + \cdots\right\} e^{pt}\,dp$$

and:

$$\frac{\psi_1}{\sin\theta} \cong \frac{\sigma}{2\mu}\cdot\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}\left\{-K_0\left(\frac{rp}{C_T}\right)\left[\frac{a^2 r}{2C_L} - \frac{a^3 r p}{2C_L^2} - \cdots\right] + \frac{a^2 C_T^2}{rC_L p^2} - \frac{a^3 C_T^2}{rC_L^2 p} - \cdots\right\} e^{pt}\,dp. \qquad (5.18)$$

The integrals for ϕ_1 and ψ_1 may be evaluated by use of reference (13). The result is:

$$\frac{2\mu}{\sigma}\frac{\phi_1}{\cos\theta} \cong -\frac{a^2 r C_T^2}{2C_L^3}\frac{\partial}{\partial t}\ln\left|\frac{C_L t}{r}+\sqrt{\left(\frac{C_L t}{r}\right)^2-1}\right|$$
$$+\frac{a^3 r C_T^2}{2C_L^4}\frac{\partial^2}{\partial t^2}\ln\left|\frac{C_L t}{r}+\sqrt{\left(\frac{C_L t}{r}\right)^2-1}\right|$$
$$+\frac{a^2 C_T^2 t}{r C_L}-\frac{a^3 C_T^2}{r C_L^2}-\cdots \tag{5.19}$$

$$\frac{2\mu}{\sigma}\frac{\psi_1}{\sin\theta} \cong -\frac{a^2 r}{2C_L}\frac{\partial}{\partial t}\ln\left|\frac{C_T t}{r}+\sqrt{\left(\frac{C_T t}{r}\right)^2-1}\right|$$
$$+\frac{a^3 r}{2C_L^2}\frac{\partial^2}{\partial t^2}\ln\left|\frac{C_T t}{r}+\sqrt{\left(\frac{C_T t}{r}\right)^2-1}\right|$$
$$+\frac{a^2 C_T^2 t}{r C_L}-\frac{a^3 C_T^2}{r C_L^2}-\cdots$$

which reduce, for $\frac{C_L t}{r}, \frac{C_T t}{r} >> 1$, to:

$$\frac{2\mu}{\sigma}\frac{\phi_1}{\cos\theta} \cong -\frac{a^2 r C_T^2}{2C_L^3 t}-\frac{a^3 r C_T^2}{2C_L^4 t^2}+\frac{a^2 C_T^2 t}{r C_L}-\frac{a^3 C_T^2}{r C_L^2}-\cdots$$
$$\frac{2\mu}{\sigma}\frac{\psi_1}{\sin\theta} \cong -\frac{a^2 r}{2C_L t}-\frac{a^3 r}{2C_L^2 t^2}+\frac{a^2 C_T^2 t}{r C_L}-\frac{a^3 C_T^2}{r C_L^2}-\cdots \tag{5.20}$$

5.3 Second Harmonic $(n=2,\ p\to 0)$

It is obvious that the higher harmonics require the retention of additional terms in the Bessel function expansions. For example, $K_2\left(\frac{rp}{c_L}\right)\sim \frac{c_L^2}{r^2p^2}$, therefore, all terms up to $O(p^2)$ in the coefficients B_2 and D_2 must be retained to obtain all possible contributions from the inversion integrals.

For the second harmonic, equations 4.12 become:

$$\frac{a_2}{\mu}\equiv\left(\frac{p^2}{c_T^2}+\frac{8}{a^2}\right)B_2K_2\left(\frac{ap}{c_L}\right)-\frac{2p}{ac_L}B_2K_2'\left(\frac{ap}{c_L}\right)$$
$$+\frac{4}{a}\left[\frac{p}{c_T}K_2'\left(\frac{ap}{c_T}\right)-\frac{1}{a}K_2\left(\frac{ap}{c_T}\right)\right]D_2$$

$$(5.21)$$

$$\frac{b_2}{\mu}\equiv-\frac{4}{a}\left[\frac{p}{c_L}K_2'\left(\frac{ap}{c_L}\right)-\frac{1}{a}K_2\left(\frac{ap}{c_L}\right)\right]B_2$$
$$-\left(\frac{p^2}{c_T^2}+\frac{8}{a^2}\right)D_2K_2\left(\frac{ap}{c_T}\right)+\frac{2p}{ac_T}D_2K_2'\left(\frac{ap}{c_T}\right).$$

The Bessel functions $K_2(\ ),\ K_2'(\ )$ may be expanded as follows:

$$K_2(\alpha)\cong\frac{2}{\alpha^2}-\frac{1}{2}-\frac{\alpha^2}{8}\log\left(\frac{\alpha}{2}\right)\cong\frac{2}{\alpha^2}-\frac{1}{2}+\frac{\alpha^2}{8}K_0(\alpha)$$

$$K_2'(\alpha) \equiv -\frac{1}{2}\left[K_3(\alpha) + K_1(\alpha)\right]$$

$$\cong -\frac{4}{\alpha^3} - \frac{\alpha}{16} - \frac{\alpha}{4}\log\left(\frac{\alpha}{2}\right) \cong -\frac{4}{\alpha^3} - \frac{\alpha}{16} + \frac{\alpha}{4}K_0(\alpha),$$

using equations 5.4 - 5.5, the identity for $K'(\alpha)$ and the approximation for small argument:

$$K_0(\alpha) \cong -\log\left(\frac{\alpha}{2}\right).$$

With the above results and noting that $\lim_{\alpha\to 0} \alpha^n \log\left(\frac{\alpha}{2}\right) \to 0;$ $n \geq 1,$ equations 5.21 may be rewritten as:

$$\frac{a_2}{\mu} \cong \left(\frac{24C_L^2}{a^4p^2} + \frac{2C_L^2}{a^2C_T^2} - \frac{4}{a^2}\right)B_2 - \left(\frac{24C_T^2}{a^4p^2} - \frac{2}{a^2}\right)D_2$$

$$(5.22)$$

$$\frac{b_2}{\mu} \cong \left(\frac{24C_L^2}{a^4p^2} - \frac{2}{a^2}\right)B_2 - \left(\frac{24C_T^2}{a^4p^2} - \frac{2}{a^2}\right)D_2.$$

Solutions for the coefficients B_2 and D_2 may be obtained without recourse to determinants. These are:

$$\frac{a_2 - b_2}{\mu} \cong \left(\frac{2C_L^2}{a^2 C_T^2} - \frac{2}{a^2}\right) B_2$$

$$= \frac{2}{a^2}\left(\frac{1-\epsilon}{1+\epsilon}\right) B_2$$

or:

$$B_2 \cong \frac{a^2}{2\mu}\left(\frac{1+\epsilon}{1-\epsilon}\right)(a_2 - b_2) \qquad (5.23)$$

using the results of equations 4.23.

Substitution of B_2 into the second of equations 5.22 gives:

$$\frac{b_2}{\mu} \cong \left(\frac{24C_L^2}{a^4 p^2} - \frac{2}{a^2}\right)\frac{a^2}{2\mu}\left(\frac{1+\epsilon}{1-\epsilon}\right)(a_2 - b_2) - \left(\frac{24C_T^2}{a^4 p^2} - \frac{2}{a^2}\right) D_2$$

or:

$$D_2 \cong \frac{a^2}{1-\epsilon}\left(\frac{a_2 - b_2}{\mu}\right) + \frac{a^4 p^2}{24 C_T^2}\left(\frac{a_2 - 2b_2}{\mu}\right) \qquad (5.24)$$

after manipulation and disregarding terms of greater than $O(p^2)$.

Use of equations 4.24 - 4.26 and the values for B_2 and D_2 above, plus the Fourier coefficients a_2 and b_2 from equations 5.2 - 5.3, allows the solutions for the potentials to be written as:

$$\frac{\phi_2}{\cos 2\theta} \cong -\frac{a^2\sigma}{2\mu}\left(\frac{1+\epsilon}{1-\epsilon}\right)\cdot\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}\left\{(1+\epsilon)\left(\frac{1}{p}-\frac{a}{c_L}\right)\right.$$

$$\left.+\left(\frac{7+5\epsilon}{8}\right)\frac{a^2p}{c_L^2}\right\}K_2\left(\frac{rp}{c_L}\right)e^{pt}\,dp$$

(5.25)

$$\frac{\psi_2}{\sin 2\theta} \cong -\frac{a^2\sigma}{(1-\epsilon)\mu}\cdot\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}\left\{(1+\epsilon)\left(\frac{1}{p}-\frac{a}{c_L}\right)\right.$$

$$\left.+\left(\frac{7+5\epsilon}{8}\right)\frac{a^2p}{c_L^2}\right\}K_2\left(\frac{rp}{c_T}\right)e^{pt}\,dp$$

$$-\frac{a^4\sigma}{24c_T^2\mu}\cdot\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}\left\{3\left(\frac{1+\epsilon}{2}\right)\left(\frac{1}{p}-\frac{a}{c_L}\right)\right.$$

$$\left.+\left(\frac{5+4\epsilon}{4}\right)\frac{a^2p}{c_L^2}\right\}p^2K_2\left(\frac{rp}{c_T}\right)e^{pt}\,dp.$$

It is again convenient to approximate the $K_2(\ )$ functions. Using equations 5.4 - 5.5, the result is obtained for small argument that:

$$K_2(\alpha) \cong \frac{2}{\alpha^2} - \frac{1}{2} + \frac{\alpha^2}{8} K_0(\alpha).$$

Perhaps it is well to remark here that the functions $K_2(\ )$ and $K_1(\ )$ of the previous section have been approximated because the inversion integrals for $K_\nu\left(\frac{p}{a}\right)$ on pg. 125 of reference (13) contain misprints. This fact is readily verified by comparison with Chapter V of reference (22).

The approximation for $K_2(\alpha)$ above results in the following expressions for the potentials:

$$\frac{\phi_2}{\cos 2\theta} \cong -\frac{a^2\sigma}{2\mu}\left(\frac{1+\epsilon}{1-\epsilon}\right)\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}\left\{(1+\epsilon)\left[\frac{2C_L^2}{r^2p^3} - \frac{1}{2p} - \frac{2aC_L}{r^2p^2} + \frac{r^2p}{8C_L^2}K_0\left(\frac{rp}{C_L}\right)\right] + \left(\frac{7+5\epsilon}{8}\right)\frac{2a^2}{r^2p}\right\}e^{pt}\,dp \qquad (5.26a)$$

$$\frac{\psi_2}{\sin 2\theta} \cong -\frac{a^2\sigma}{(1-\epsilon)\mu}\cdot\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}\left\{(1+\epsilon)\left[\frac{2C_T^2}{r^2p^3}-\frac{1}{2p}\right.\right.$$

$$\left.+\frac{r^2p}{8C_T^2}K_0\left(\frac{rp}{C_T}\right)-\frac{2aC_T^2}{r^2C_Lp^2}\right]$$

$$\left.+\left(\frac{7+5\epsilon}{8}\right)2\frac{a^2C_T^2}{r^2C_L^2p}\right\}e^{pt}\,dp$$

$$-\frac{a^4\sigma}{24C_T^2\mu}\cdot\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}3\left(\frac{1+\epsilon}{2}\right)\frac{2C_T^2}{r^2p}\,e^{pt}\,dp. \qquad (5.26b)$$

The above integrals are readily evaluated using reference (13) as:

$$\frac{\phi_2}{\cos 2\theta} \cong -\frac{a^2\sigma}{2\mu}\left(\frac{1+\epsilon}{1-\epsilon}\right)\left\{(1+\epsilon)\left[\frac{C_L^2t^2}{r^2}-\frac{1}{2}-\frac{2aC_Lt}{r^2}\right.\right.$$

$$\left.\left.+\frac{r^2}{8C_L^2}\frac{\partial^2}{\partial t^2}\ln\left|\frac{C_Lt}{r}+\sqrt{\left(\frac{C_Lt}{r}\right)^2-1}\right|\right]+\left(\frac{7+5\epsilon}{4}\right)\frac{a^2}{r^2}\right\}$$

$$(5.27)$$

$$\frac{\psi_2}{\sin 2\theta} \cong -\frac{a^2\sigma}{(1-\epsilon)\mu}\left\{(1+\epsilon)\left[\frac{C_T^2t^2}{r^2}-\frac{1}{2}+\frac{r^2}{8C_T^2}\frac{\partial^2}{\partial t^2}\ln\left|\frac{C_Tt}{r}+\sqrt{\left(\frac{C_Tt}{r}\right)^2-1}\right|\right.\right.$$

$$\left.\left.-\frac{2aC_T^2t}{r^2C_L}\right]+\left(\frac{7+5\epsilon}{4}\right)\frac{a^2C_T^2}{r^2C_L^2}\right\}$$

$$-\frac{a^4\sigma}{8\mu r^2}(1+\epsilon).$$

Assuming that $\frac{C_L t}{r}$, $\frac{C_T t}{r} >> 1$ the potentials can be rewritten as:

$$\frac{\phi_2}{\cos 2\theta} \cong -\frac{a^2\sigma}{2\mu}\left(\frac{1+\epsilon}{1-\epsilon}\right)\left\{(1+\epsilon)\left[\frac{C_L^2 t^2}{r^2} - \frac{2aC_L t}{r^2} - \frac{r^2}{8C_L^2 t^2}\right] + \left(\frac{7+5\epsilon}{4}\right)\frac{a^2}{r^2}\right. \tag{5.28}$$

$$\frac{\psi_2}{\sin 2\theta} \cong -\frac{a^2\sigma}{\mu}\left(\frac{1+\epsilon}{1-\epsilon}\right)\left\{\frac{C_T^2 t^2}{r^2} - \frac{2aC_T^2 t}{r^2 C_L} - \frac{r^2}{8C_T^2 t^2}\right\}$$

$$-\frac{a^4\sigma}{4\mu}\left(\frac{7+5\epsilon}{1-\epsilon}\right)\frac{C_T^2}{r^2 C_L^2} - \frac{a^4\sigma}{\mu}\left(\frac{1+\epsilon}{8r^2}\right).$$

Letting $2\frac{C_T^2}{C_L^2} = 1+\epsilon$ from equation 4.23 allows ψ_2 to be written in better form as:

$$\frac{\psi_2}{\sin 2\theta} \cong -\frac{a^2\sigma}{\mu}\left(\frac{1+\epsilon}{1-\epsilon}\right)\left\{\frac{C_T^2 t^2}{r^2} - \frac{2aC_T^2 t}{r^2 C_L} - \frac{r^2}{8C_T^2 t^2}\right.$$

$$\left. + \frac{a^2}{r^2}\left(\frac{7+5\epsilon}{8}\right) + \frac{a^2}{8r^2}(1-\epsilon)\right\}. \tag{5.29}$$

5.4 Hoop Stress (σ_θ) - Long Time Approximation

The hoop stress is given by equation 3.7 and for the scattered field only, the result for the first three harmonics is:

$$\sigma_{\theta_{sc}}(r,\theta,t) \cong -a^2\sigma\left\{\epsilon\left(\frac{1-\epsilon}{1+\epsilon}\right)\frac{1}{2C_L^2t^2} - \frac{1}{r^2}\left(\frac{1-\epsilon}{2}\right) - \frac{\epsilon r}{2C_L^3t^3}\left(1+\frac{3a}{C_Lt}\right)\cos\theta\right.$$

$$\left.+\left[\frac{3\epsilon}{4}\left(\frac{1+\epsilon}{1-\epsilon}\right)\frac{r^2}{C_L^4t^4} + \frac{(1+\epsilon)^2}{(1-\epsilon)}\frac{1}{4C_L^2t^2} + \left(\frac{1+\epsilon}{1-\epsilon}\right)\frac{1}{2C_T^2t^2} + (1+\epsilon)\frac{3a^2}{2r^4}\right]\cos 2\theta\right\}.$$

(5.30)

The hoop stress resulting from the incident field is given by either equation 2.8 or 5.1 as:

$$\sigma_{\theta_{in}}(r,\theta,t) = \sigma\left(\frac{1-\epsilon}{2} - \frac{1+\epsilon}{2}\cos 2\theta\right)$$

thus, the total hoop stress is given by:

$$\sigma_\theta(r,\theta,t) = \sigma_{\theta_{in}} + \sigma_{\theta_{sc}} = \sigma\left(\frac{1-\epsilon}{2} - \frac{1+\epsilon}{2}\cos 2\theta\right)$$

$$-a^2\sigma\left\{\epsilon\left(\frac{1-\epsilon}{1+\epsilon}\right)\frac{1}{2C_L^2t^2} - \frac{1}{r^2}\left(\frac{1-\epsilon}{2}\right) - \frac{\epsilon r}{2C_L^3t^3}\left(1+\frac{3a}{C_Lt}\right)\cos\theta\right.$$

$$+\left[\frac{3\epsilon}{4}\left(\frac{1+\epsilon}{1-\epsilon}\right)\frac{r^2}{C_L^4t^4} + \frac{(1+\epsilon)^2}{1-\epsilon}\frac{1}{4C_L^2t^2}\right]\cos 2\theta$$

(con't)

$$+\left[\left(\frac{1+\epsilon}{1-\epsilon}\right)\frac{1}{2C_T^2t^2}+(1+\epsilon)\frac{3a^2}{2r^4}\right]\cos 2\theta \tag{5.31}$$

When $\epsilon \rightarrow 0$ ($\nu =$ Poisson's ratio $\rightarrow 0$) and $t \rightarrow \infty$, the above result reduces to:

$$\sigma_\theta(r,\theta,t) \cong \frac{\sigma}{2}(1-\cos 2\theta)$$

$$-a^2\sigma\left(-\frac{1}{2r^2}+\frac{3a^2}{2r^4}\cos 2\theta\right)$$

or:

$$\sigma_\theta = \frac{\sigma}{2}\left(1+\frac{a^2}{r^2}\right)-\frac{\sigma}{2}\left(1+\frac{3a^4}{r^4}\right)\cos 2\theta \tag{5.32}$$

This is the familiar result for the thin plate (plane stress) containing a circular hole of radius "a". At $\theta = \frac{\pi}{2}$ and $r = a$, the hoop stress is three times the applied stress and is a tensile stress if the applied stress is tensile.

6. Early Time Approximation - Scattering of a Plane Dilatational Wave - Fourier Series Representation.

The early time behavior of the solutions for stresses and displacements is expected to be of greater significance than that for long time. This conclusion may be drawn from consideration of the response of a simple dynamic system, which contains damping, to a Heaviside step forcing function. Unfortunately, in the cavity problem being considered here, the early time solutions are most difficult to obtain.

Application of the Fourier series representation, as in the previous section for long time, requires summation of an infinite number of harmonics to obtain the early time solutions. This infinite summation is particularly required when the rate of envelopment of the cavity by the incident wave is relatively slow. Then, it is expected that the response will be too rich in harmonics to be described by a few modes. This result, or expected behavior, is discussed in reference (23) with respect to stress waves produced by pressure loads on a spherical shell.

Early time behavior, in Laplace transform theory, is associated with large values of the operator "p".

Observe that "p" has dimensions of frequency, thus large "p" implies high frequency responses which occur during short time intervals, for example, across a wave front. Note also that in wave motion problems, the time variable must be replaced by $t - t_0(x,y)$ where $t_0(x,y)$ is the time at which the wave reaches the point in question. It follows that a complete description of the early time behavior depends on the following ranges of "p" and "n":

$$p \sim [T]^{-1} \sim \text{large}$$

$$0 \leq n \leq \infty .$$

The range of "p" and "n" requires use of the uniformly valid approximation for the modified Bessel functions $I_n\left(\frac{rp}{C_L}\right)$, $K_n\left(\frac{rp}{C_L}\right)$, etcetera, as found in reference (21), pg. 86, for example. These approximations are:

$$K_n(\alpha) \cong \sqrt{\frac{\pi}{2}}\,(n^2+\alpha^2)^{-\frac{1}{4}} \exp\left[-(n^2+\alpha^2)^{\frac{1}{2}} + n \sinh^{-1}\frac{n}{\alpha}\right]$$

$$I_n(\alpha) \cong \frac{1}{\sqrt{2\pi}}\,(n^2+\alpha^2)^{-\frac{1}{4}} \exp\left[(n^2+\alpha^2)^{\frac{1}{2}} - n \sinh^{-1}\frac{n}{\alpha}\right] \qquad (6.1)$$

where $\alpha = \frac{rp}{c_L}, \frac{rp}{c_T}$ and $n, \alpha > 0$ is required. From these expressions, the results are obtained that:

$$K_n'(\alpha) \cong -\frac{(n^2+\alpha^2)^{\frac{1}{2}}}{\alpha} K_n(\alpha)$$

$$I_n'(\alpha) \cong \frac{(n^2+\alpha^2)^{\frac{1}{2}}}{\alpha} I_n(\alpha) \qquad (6.2)$$

The boundary conditions and solutions for the potentials $\bar{\bar{\phi}}_n$ and $\bar{\bar{\psi}}_n$, after the Laplace transformation, are represented by equations 4.12 and 4.4 respectively. The following expressions for coefficients B_n and D_n are obtained from equation 4.12 after use of equations 6.1 - 6.2 above:

$$B_n \cong \frac{a_n}{\mu\left(\frac{p^2}{c_T^2} + \frac{2n^2}{a^2}\right) K_n\left(\frac{ap}{c_L}\right)}$$

$$D_n \cong \frac{b_n}{\mu\left(\frac{p^2}{c_T^2} + \frac{2n^2}{a^2}\right) K_n\left(\frac{ap}{c_T}\right)} \qquad (6.3)$$

which are valid for p $\longrightarrow$ large and all values of n. The Fourier coefficients a_n and b_n are defined by equations 4.10. Use of equations 4.20 and 6.1 - 6.2 allows these coefficients to be written:

$$a_n \cong -\frac{2\sigma}{p} e^{-\frac{ap}{c_L}} (-)^n \left\{ 1+(1+\epsilon)\left[\frac{c_L^2 n^2}{a^2p^2} - \frac{c_L^2}{a^2p^2}\left(n^2+\frac{a^2p^2}{c_L^2}\right)^{\frac{1}{2}}\right]\right\} I_n\left(\frac{ap}{c_L}\right)$$

$$b_n \cong \frac{2\sigma}{p} e^{-\frac{ap}{c_L}} (-)^n (1+\epsilon)\frac{c_L^2 n}{a^2p^2}\left(n^2+\frac{a^2p^2}{c_L^2}\right)^{\frac{1}{2}} I_n\left(\frac{ap}{c_L}\right) \tag{6.4}$$

which are also valid for p $\longrightarrow$ large and all n.

Solutions for the potentials ϕ and ψ are given, by reference to equations 4.24 - 4.26 and noting that B_n is even in n where as D_n is odd, as:

$$\phi(r,\theta,t) \cong \frac{1}{2} \mathcal{R}e\left\{\sum_{n=-\infty}^{\infty} e^{in\theta} \cdot \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} B_n K_n\left(\frac{rp}{c_L}\right) e^{pt}\, dp\right\}$$

$$\psi(r,\theta,t) \cong \frac{1}{2} \mathcal{I}m\left\{\sum_{n=-\infty}^{\infty} e^{in\theta} \cdot \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} D_n K_n\left(\frac{rp}{c_T}\right) e^{pt}\, dp\right\} \tag{6.5}$$

where the contour lies to the right of all singularities in the complex p-plane. The behavior of B_n and D_n with respect to "n" is best assessed by consideration of equation 4.10 and 4.20 and substitution of the identity:

$$I_n'(\alpha) = \frac{1}{2}\left[I_{n+1}(\alpha) + I_{n-1}(\alpha)\right]$$

which is even in n. See reference (13).

It is again convenient to evaluate the hoop stress since some of the Bessel functions will be eliminated by this artifice. In particular, the hoop stress at $r = a$ is to be evaluated. Equation 4.27 may be applied to give:

$$\sigma_\theta(a,\theta,t) \cong \frac{1}{2}\sum_{n=-\infty}^{\infty} e^{in\theta} \cdot \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} e^{pt}\left\{\frac{\lambda}{C_L^2}p^2 B_n K_n\left(\frac{ap}{C_L}\right)\right.$$

$$-\frac{2\mu n^2}{a^2}B_n K_n\left(\frac{ap}{C_L}\right) + \frac{2\mu p}{aC_L} B_n K_n'\left(\frac{ap}{C_L}\right)$$

$$\left. -\frac{2\mu n p}{aC_T} D_n K_n'\left(\frac{ap}{C_T}\right) + \frac{2\mu n}{a^2} D_n K_n\left(\frac{ap}{C_T}\right)\right\} dp \quad (6.6)$$

Substitution of the results from equation 4.23, 4.28 and 6.2 - 6.4 allows the hoop stress equation to be rewritten as:

$$\sigma_\theta(a,\theta,t) \cong \frac{1}{2}\sum_{n=-\infty}^{\infty} e^{in\theta} \cdot \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}\left\{\frac{e^{p\left(t-\frac{a}{c_L}\right)}}{\left(\frac{p^2}{c_T^2}+\frac{2n^2}{a^2}\right)} \cdot \frac{2\sigma}{p}(-)^n I_n\left(\frac{ap}{c_L}\right)\right\}$$

$$\text{times}\left\{\left[\frac{2\epsilon}{1+\epsilon}\cdot\frac{p^2}{c_L^2}+\frac{2n^2}{a^2}\right]\left[1+(1+\epsilon)\left(\frac{c_L^2 n^2}{a^2p^2}-\frac{c_L^2}{a^2p^2}\sqrt{n^2+\frac{a^2p^2}{c_L^2}}\right)\right]\right.$$

$$\left.-\frac{2n}{a^2}(1+\epsilon)\sqrt{n^2+\frac{a^2p^2}{c_T^2}}\left[\frac{c_L^2 n}{a^2p^2}\sqrt{n^2+\frac{a^2p^2}{c_L^2}}\right]\right\}dp. \qquad (6.7)$$

Equation 6.7 yields the hoop stress for the scattered field and is valid for p $\rightarrow$ large and all values of n.

The above expression can be rewritten once more using equation 6.1 for the I_n function as:

$$\sigma_\theta(a,\theta,t) \cong \frac{1}{2}\sum_{n=-\infty}^{\infty} e^{in\theta} \cdot \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}\frac{e^{p\left(t-\frac{a}{c_L}\right)}}{\left(\frac{p^2}{c_T^2}+\frac{2n^2}{a^2}\right)} \cdot \sqrt{\frac{2}{\pi}} \cdot \sigma\left(n^2+\frac{a^2p^2}{c_L^2}\right)^{-\frac{1}{4}}(-)^n$$

$$\text{times}\left\{\exp\left[\left(n^2+\frac{a^2p^2}{c_L^2}\right)^{\frac{1}{2}} - n\sinh^{-1}\frac{nc_L}{ap}\right]\right\}$$

(con't)

$$\text{times}\left\{\frac{2\epsilon}{1+\epsilon}\cdot\frac{p}{c_L^2}+\frac{2\epsilon}{a^2p}\left(n^2-\sqrt{n^2+\frac{a^2p^2}{c_L^2}}\right)+\frac{2n^2}{a^2p}\right.$$

$$+\frac{2n^2c_L^2}{a^4p^3}(1+\epsilon)\left(n^2-\sqrt{n^2+\frac{a^2p^2}{c_L^2}}\right)$$

$$\left.-\frac{2n^2c_L^2}{a^4p^3}(1+\epsilon)\sqrt{n^2+\frac{a^2p^2}{c_T^2}}\sqrt{n^2+\frac{a^2p^2}{c_L^2}}\right\}dp. \qquad (6.8)$$

Assuming that n is fixed, the complex integration implied by equation 6.8 may, in principle, be carried out. However, there are branch points at:

$$p=\pm i\frac{nc_T}{a},\ \pm i\frac{nc_L}{a}$$

and poles at:

$$p=0,\ \pm i\frac{\sqrt{2}\,nc_T}{a}$$

to be considered and evaluation of the integral in closed form does not appear to be mathematically tractable.

On the other hand, if uniform convergence of the series for the hoop stress is <u>assumed</u>, the integration and summation operations may be interchanged. This implies a summation:

$$\sum_{n=-\infty}^{\infty} f(n)$$

to be carried out. Note that the question of uniform convergence has not been resolved. In fact, it is questionable that the series converges in any sense because of the presence of the exp $\left[\left(n^2+\frac{a^2p^2}{c_L^2}\right)^{\frac{1}{2}}\right]$ term.

However, if the series were uniformly convergent, a beautiful method from the theory of complex variables could be applied to sum it. See, for example, reference (24). The method requires that $f(n)$ be a meromorphic function of the (complex) variable n and that $\lim_{|n|\to\infty} nf(n) \to 0$. Unfortunately, these conditions are not satisfied by $f(n)$ and the method does not apply.

A method is applied in the next section to determine the early time behavior which eliminates the requirement for summing an infinite series.

7. Early Time Approximation - Scattering of a Plane Dilatational Wave - Friedlander's Method.

The method to be applied in this section is based on a development by Friedlander (16). All quantities to be found are periodic in θ . Rather than express the potentials, etcetera, in Fourier series, Friedlander suggests writing the potentials as:

$$\phi'(r,\theta,t) = \sum_{m=-\infty}^{\infty} \phi(r,\theta+2m\pi,t)$$
$$\psi'(r,\theta,t) = \sum_{m=-\infty}^{\infty} \psi(r,\theta+2m\pi,t) \tag{7.1}$$

which are also periodic in θ . This definition implies that the potentials ϕ , ψ , etcetera, are defined on a Riemann surface R having a branch point at the origin with its sheets defined by:

$$(2m-1)\pi \leq \theta \leq (2m+1)\pi$$

where:

$$m = \cdots\cdots, -1, 0, 1, \cdots\cdots$$

Successive sheets are joined along the negative axis as shown in Figure 8 below:

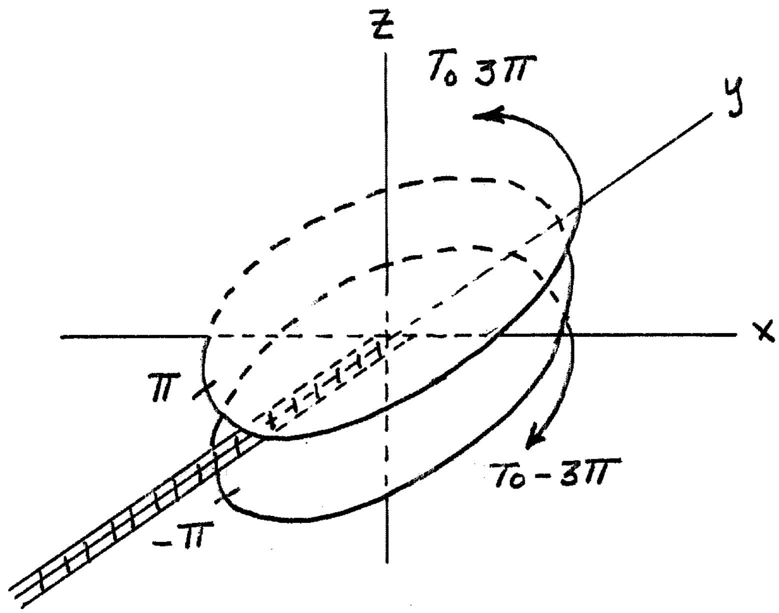


Fig. 8. Riemann Surface R

The $m=0$ sheet extends from $-\pi \leq \theta \leq \pi$, the $m=1$ sheet from $\pi \leq \theta \leq 3\pi$, etcetera.

It is easily verified that ϕ', ψ' are periodic in θ , however, ϕ, ψ are not necessarily periodic. This method is particularly simple since only one solution for ϕ , say $\phi(r, \theta, t)$ need be generated to obtain the complete solution for $\phi'(r, \theta, t)$

The boundary conditions to be satisfied on the plane $m=0$ are:

$$\sigma_r(a,\theta,t) = -\sigma H\left[\frac{a}{c_L}(\cos\theta - 1) + t\right]\left(\frac{1-\epsilon}{2} + \frac{1+\epsilon}{2}\cos 2\theta\right); -\pi \leq \theta \leq \pi \qquad (7.2)$$

$$= 0; \text{ otherwise}$$

$$\tau_{r\theta}(a,\theta,t) = \sigma H\left[\frac{a}{c_L}(\cos\theta - 1) + t\right]\left(\frac{1+\epsilon}{2}\sin 2\theta\right); -\pi \leq \theta \leq \pi \qquad (7.3)$$

$$= 0; \text{ otherwise}$$

A left-running wave has been assumed for convenience. These boundary conditions define the scattered field and were obtained from equations 2.9. The stresses resulting from the scattered field must be added to those from the incident field to obtain the complete solution. Boundary conditions to be satisfied on the other sheets may be phrased similarly.

The early time behavior depends on the solutions from the $m = 0$ sheet, that is, as long as the wave position is defined by $|\theta| < \pi$. When the incident wave reaches the position $|\theta| = \pi$, contributions from the $m=1$ and $m=-1$ sheets arrive simultaneously and must be included. In what follows, attention shall be restricted to the $m=0$ sheet.

The time dependence can be removed by taking the Laplace transform of the wave equations. The Laplace transform pair is defined by:

$$\bar{\Phi}(r,\theta,p) = \int_0^\infty \phi(r,\theta,t)\, e^{-pt}\, dt \tag{7.4}$$

$$\phi(r,\theta,t) = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} \bar{\Phi}(r,\theta,p)\, e^{pt}\, dp$$

Since the range of θ is $-\infty \leq \theta \leq \infty$, a Fourier transform with respect to "θ" may be applied to the wave equations to eliminate the θ -dependence. The Fourier transform pair is defined by:

$$\tilde{\Phi}(r,\omega,p) = \int_{-\infty}^{\infty} \bar{\Phi}(r,\theta,p)\, e^{-i\omega\theta}\, d\theta \tag{7.5}$$

$$\bar{\Phi}(r,\theta,p) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \tilde{\Phi}(r,\omega,p)\, e^{i\omega\theta}\, d\omega .$$

The wave equations may be written by reference to equations 3.1 - 3.2 as:

$$\nabla^2 \phi' = \frac{1}{c_L^2} \phi'_{tt}$$
$$\nabla^2 \psi' = \frac{1}{c_T^2} \psi'_{tt} \qquad (7.6)$$

where:

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}$$

and:

$$\phi'(r,\theta,t) = \sum_{m=-\infty}^{\infty} \phi(r, \theta + 2m\pi, t)$$
$$\psi'(r,\theta,t) = \sum_{m=-\infty}^{\infty} \psi(r, \theta + 2m\pi, t) \qquad (7.7)$$

Each ϕ and ψ must, of course, satisfy the appropriate wave equation. The Fourier-Laplace transforms of the wave equations are ($m = 0$ sheet):

$$\frac{d^2\tilde{\phi}}{dr^2} + \frac{1}{r}\frac{d\tilde{\phi}}{dr} - \left(\frac{\omega^2}{r^2} + \frac{p^2}{c_L^2}\right)\tilde{\phi} = 0$$

$$\frac{d^2\tilde{\psi}}{dr^2} + \frac{1}{r}\frac{d\tilde{\psi}}{dr} - \left(\frac{\omega^2}{r^2} + \frac{p^2}{c_T^2}\right)\tilde{\psi} = 0 \tag{7.8}$$

with solutions:

$$\tilde{\phi} = A(\omega)\, I_{|\omega|}\left(\frac{rp}{c_L}\right) + B(\omega)\, K_{|\omega|}\left(\frac{rp}{c_L}\right)$$

$$\tilde{\psi} = C(\omega)\, I_{|\omega|}\left(\frac{rp}{c_T}\right) + D(\omega)\, K_{|\omega|}\left(\frac{rp}{c_T}\right) \tag{7.9}$$

The functions $A(\omega) = C(\omega) = 0$ to insure outgoing waves as $r \rightarrow \infty$.

The boundary conditions expressed as functions of the potentials ϕ and ψ are given by equations 3.7. Their Fourier-Laplace transforms are:

$$\frac{\tilde{\sigma}_r(a,\omega,p)}{\mu} = \left\{ \left(\frac{p^2}{c_T^2} + \frac{2\omega^2}{r^2} \right) \tilde{\phi} - \frac{2}{r}\frac{d\tilde{\phi}}{dr} + \frac{2i\omega}{r}\left(\frac{d\tilde{\psi}}{dr} - \frac{\tilde{\psi}}{r} \right) \right\}_{r=a}$$

$$\frac{\tilde{\tau}_{r\theta}(a,\omega,p)}{\mu} = \left\{ \frac{2i\omega}{r}\left(\frac{d\tilde{\phi}}{dr} - \frac{\tilde{\phi}}{r} \right) - \left(\frac{p^2}{c_T^2} + \frac{2\omega^2}{r^2} \right)\tilde{\psi} + \frac{2}{r}\frac{d\tilde{\psi}}{dr} \right\}_{r=a} \qquad (7.10)$$

where σ_r and $\tau_{r\theta}$ are defined by equations 7.2 - 7.3.

The range of p and ω again suggest the use of the uniformly valid approximations for the modified Bessel functions. From reference (21), these approximations are:

$$K_{|\omega|}(\alpha) \cong \sqrt{\frac{\pi}{2}}\,(\omega^2+\alpha^2)^{-\frac{1}{4}} \exp\left[-(\omega^2+\alpha^2)^{\frac{1}{2}} + \omega \sinh^{-1}\frac{\omega}{\alpha} \right]$$

$$I_{|\omega|}(\alpha) \cong \frac{1}{\sqrt{2\pi}}\,(\omega^2+\alpha^2)^{-\frac{1}{4}} \exp\left[(\omega^2+\alpha^2)^{\frac{1}{2}} - \omega \sinh^{-1}\frac{\omega}{\alpha} \right] \qquad (7.11)$$

from which:

$$K'_{|\omega|}(\alpha) \cong -\frac{(\omega^2+\alpha^2)^{\frac{1}{2}}}{\alpha} K_{|\omega|}(\alpha)$$

$$I'_{|\omega|}(\alpha) \cong \frac{(\omega^2+\alpha^2)^{\frac{1}{2}}}{\alpha} I_{|\omega|}(\alpha). \qquad (7.12)$$

Substitution of equations 7.9 and 7.11 - 7.12 into 7.10 yields the result, valid for p $\longrightarrow$ large and all ω-values:

$$B(\omega) \cong \frac{\tilde{\sigma}_r(a,\omega,p)}{\mu\left(\frac{p^2}{c_T^2} + \frac{2\omega^2}{a^2}\right) K_\omega\left(\frac{ap}{c_L}\right)}$$

(7.13)

$$D(\omega) \cong -\frac{\tilde{\tau}_{r\theta}(a,\omega,p)}{\mu\left(\frac{p^2}{c_T^2} + \frac{2\omega^2}{a^2}\right) K_\omega\left(\frac{ap}{c_T}\right)}$$

These equations are analogous to equations 6.3 obtained through use of the Fourier series representation. Note that $K_\omega(\alpha) = K_{-\omega}(\alpha)$, therefore the absolute value signs have been removed on the order.

The transformed quantities $\tilde{\sigma}_r(a,\omega,p)$ and $\tilde{\tau}_{r\theta}(a,\omega,p)$ may be obtained from equations 7.2 - 7.3.

$$\tilde{\sigma}_r(a,\omega,p) = -\int_{-\pi}^{\pi} \frac{\sigma}{p} e^{-\frac{ap}{c_L}(1-\cos\theta)} \left(\frac{1-\epsilon}{2} + \frac{1+\epsilon}{2}\cos 2\theta\right) e^{-i\omega\theta}\, d\theta$$

(7.14)

$$\tilde{\tau}_{r\theta}(a,\omega,p) = \int_{-\pi}^{\pi} \frac{\sigma}{p} e^{-\frac{ap}{c_L}(1-\cos\theta)} \left(\frac{1+\epsilon}{2}\sin 2\theta\right) e^{-i\omega\theta}\, d\theta.$$

The limits of integration are established by noting that $\sigma_r(a,\theta,t)$ and $\tau_{r\theta}(a,\theta,t)$ are zero outside the range $-\pi \leq \theta \leq \pi$.

There exist several alternative ways to evaluate the Fourier transforms above, for example:

(1) Apply the method of steepest descent where p is a large positive parameter.

(2) Direct calculation or

(3) Application of Poisson's summation formula.

Poisson's summation formula will be used, not because it is easiest but because this approach will illustrate the relation between the Fourier transforms $\tilde{\sigma}_r$, $\tilde{\tau}_{r\theta}$ and their respective Fourier coefficients a_n , b_n . It is shown in Appendix C that:

$$\frac{1}{2\pi}\tilde{\sigma}_r(a,n,p) \equiv \frac{a_n}{2}$$
$$\frac{1}{2\pi}\tilde{\tau}_{r\theta}(a,n,p) \equiv -i\,\frac{b_n}{2} \tag{7.15}$$

from which it is deduced that:

$$\frac{1}{2\pi}\tilde{\sigma}_r(a,\omega,p) = \frac{a(\omega)}{2}$$
$$\frac{1}{2\pi}\tilde{\tau}_{r\theta}(a,\omega,p) = -i\,\frac{b(\omega)}{2} \tag{7.16}$$

Application of equations 7.6 to equations 4.10 yields the desired result:

$$\tilde{\sigma}_r(a,\omega,p) = -\frac{\sigma\pi}{p}e^{-\frac{ap}{c_L}}\Big\{(1-\epsilon)I_\omega\Big(\frac{ap}{c_L}\Big)+\Big(\frac{1+\epsilon}{2}\Big)\Big[I_{\omega-2}\Big(\frac{ap}{c_L}\Big)$$

$$+I_{\omega+2}\Big(\frac{ap}{c_L}\Big)\Big]\Big\}$$

$$\tilde{\tau}_{r\theta}(a,\omega,p) = i\frac{\sigma\pi}{p}e^{-\frac{ap}{c_L}}\Big(\frac{1+\epsilon}{2}\Big)\Big\{I_{\omega+2}\Big(\frac{ap}{c_L}\Big)-I_{\omega-2}\Big(\frac{ap}{c_L}\Big)\Big\} \tag{7.17}$$

This result is readily verified by direct evaluation of the integrals expressed by equations 7.14. Note that the term $(-)^{\omega}$ is eliminated from equations 4.10 by the assumption of a left-running wave.

The approximations for the $I_\omega(\ )$ and $I'_\omega(\ )$ functions as defined by equations 7.11 - 7.12 plus the recurrence relations given by equations 4.19 allow the boundary stresses of equations 7.17 to be written as:

$$\tilde{\sigma}_r(a,\omega,p) \cong -\frac{\sigma\pi}{p}e^{-\frac{ap}{c_L}}\sqrt{\frac{2}{\pi}}\Big\{\Big[1+(1+\epsilon)\frac{c_L^2\omega^2}{a^2p^2}\Big]\Big(\omega^2+\frac{a^2p^2}{c_L^2}\Big)^{-\frac{1}{4}} \tag{7.18a}$$

$$-(1+\epsilon)\frac{c_L^2}{a^2p^2}\Big(\omega^2+\frac{a^2p^2}{c_L^2}\Big)^{\frac{1}{4}}\Big\}\exp\Big[\Big(\omega^2+\frac{a^2p^2}{c_L^2}\Big)^{\frac{1}{2}}-\omega\sinh^{-1}\frac{\omega c_L}{ap}\Big]$$

$$\tilde{\tau}_{r\theta}(a,\omega,p) \cong i\frac{\sigma\pi}{p}e^{-\frac{ap}{c_L}}(1+\epsilon)\sqrt{\frac{2}{\pi}}\left\{\frac{c_L^2\omega}{a^2p^2}\left(\omega^2+\frac{a^2p^2}{c_L^2}\right)^{-\frac{1}{4}} - \frac{c_L^2\omega}{a^2p^2}\left(\omega^2+\frac{a^2p^2}{c_L^2}\right)^{\frac{1}{4}}\right\}\exp\left[\left(\omega^2+\frac{a^2p^2}{c_L^2}\right)^{\frac{1}{2}} - \omega\sinh^{-1}\frac{\omega c_L}{ap}\right] \tag{7.18b}$$

Equations 7.9, 7.13 and 7.18 imply that the potentials associated with the $m=0$ sheet may be written in integral form as:

$$\phi(r,\theta,t) = \frac{1}{4\pi^2 i}\int_{-\infty}^{\infty}\int_{\gamma-i\infty}^{\gamma+i\infty} B(\omega)K_\omega\left(\frac{rp}{c_L}\right)e^{i\omega\theta+pt}\,dp\,d\omega$$

$$\psi(r,\theta,t) = \frac{1}{4\pi^2 i}\int_{-\infty}^{\infty}\int_{\gamma-i\infty}^{\gamma+i\infty} D(\omega)K_\omega\left(\frac{rp}{c_T}\right)e^{i\omega\theta+pt}\,dp\,d\omega \tag{7.19}$$

It is again convenient to evaluate the hoop stress at $r=a$ to eliminate some of the Bessel functions. As remarked earlier, it is perhaps of greatest importance in the design of shock resistant structures in any event.

The Fourier-Laplace transform of the hoop stress (equation 3.7) is:

$$\frac{\tilde{\sigma}_\theta}{\mu} = \left(-\frac{\epsilon p^2}{c_T^2} - \frac{2\omega^2}{r^2}\right)\tilde{\phi} + \frac{2}{r}\frac{d\tilde{\phi}}{dr} + \frac{2i\omega}{r}\left(\frac{\tilde{\psi}}{r} - \frac{d\tilde{\psi}}{dr}\right) \tag{7.20}$$

where the result:

$$\frac{1}{c_T^2} - \frac{2}{c_L^2} = -\frac{\epsilon}{c_T^2} \tag{7.21}$$

has been substituted from equations 4.23.

Substitution of $\tilde{\phi}$ and $\tilde{\psi}$ from equations 7.9 and $B(\omega)$ and $D(\omega)$ from equations 7.13 gives the following form for the transformed hoop stress:

$$\tilde{\sigma}_\theta(a,\omega,p) \cong \frac{\left(-\frac{\epsilon p^2}{c_T^2} - \frac{2\omega^2}{a^2}\right)}{\left(\frac{p^2}{c_T^2} + \frac{2\omega^2}{a^2}\right)}\tilde{\sigma}_r(a,\omega,p)$$

$$-i\frac{2\omega}{a^2}\frac{\left(\omega^2 + \frac{a^2p^2}{c_T^2}\right)^{\frac{1}{2}}}{\left(\frac{p^2}{c_T^2} + \frac{2\omega^2}{a^2}\right)}\tilde{\tau}_{r\theta}(a,\omega,p) \tag{7.22}$$

where $\tilde{\sigma}_r$ and $\tilde{\tau}_{r\theta}$ are defined by equations 7.18.

The transformed hoop stress written out is:

$$\tilde{\sigma}_\theta(a,\omega,p) \cong \sqrt{2\pi}\,\frac{\sigma}{p}\,e^{-\frac{ap}{c_L}} \cdot \frac{1}{\left(\frac{p^2}{c_T^2}+\frac{2\omega^2}{a^2}\right)}$$

$$\text{times}\left\{\left[\frac{\epsilon p^2}{c_T^2}+\frac{2\omega^2}{a^2}\right]\left[\left(1+(1+\epsilon)\frac{c_L^2\omega^2}{a^2p^2}\right)\left(\omega^2+\frac{a^2p^2}{c_L^2}\right)^{-\frac{1}{4}}-(1+\epsilon)\frac{c_L^2}{a^2p^2}\left(\omega^2+\frac{a^2p^2}{c_L^2}\right)^{\frac{1}{4}}\right]\right.$$

$$\left.+\frac{2\omega}{a^2}\left(\omega^2+\frac{a^2p^2}{c_T^2}\right)^{\frac{1}{2}}(1+\epsilon)\frac{c_L^2\omega}{a^2p^2}\left[\left(\omega^2+\frac{a^2p^2}{c_L^2}\right)^{-\frac{1}{4}}-\left(\omega^2+\frac{a^2p^2}{c_L^2}\right)^{\frac{1}{4}}\right]\right\}$$

$$\text{times}\left\{\exp\left[\left(\omega^2+\frac{a^2p^2}{c_L^2}\right)^{\frac{1}{2}}-\omega\sinh^{-1}\frac{\omega c_L}{ap}\right]\right\}. \qquad (7.23)$$

The inverse Fourier transform of the hoop stress is:

$$\bar{\sigma}_\theta(a,\theta,p) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\tilde{\sigma}_\theta(a,\omega,p)\,e^{i\omega\theta}\,d\omega \qquad (7.24)$$

and the method of steepest descent can be applied to evaluate the integral. The operator "p" is assumed to be large and positive. A saddle point occurs where $f'(\omega)=0$.

We have:

$$f(\omega) = \left(\omega^2 + \frac{a^2 p^2}{C_L^2}\right)^{\frac{1}{2}} - \omega \sinh^{-1} \frac{\omega C_L}{a p} + i\omega\theta$$

$$f'(\omega) = -\sinh^{-1} \frac{\omega C_L}{a p} + i\theta$$

$$f''(\omega) = -\left(\omega^2 + \frac{a^2 p^2}{C_L^2}\right)^{-\frac{1}{2}}$$

thus, the saddle point occurs at:

$$\omega_0 = i\,\frac{ap}{C_L} \sin\theta \tag{7.25}$$

and:

$$f(\omega_0) = \frac{ap}{C_L} \cos\theta$$

$$(7.26)$$

$$f''(\omega_0) = -\frac{1}{\frac{ap}{C_L}\cos\theta}$$

The saddle point is located on the $\mathrm{Im}\{\omega\}$ axis and climbs up the axis as Θ varies between 0 and $\frac{\pi}{2}$. The solution to be obtained here is only valid for $0 \leq |\Theta| \leq \frac{\pi}{2}$ since the integrand implied by equation 7.23 is not defined at $|\Theta| = \frac{\pi}{2}$. Special techniques are available for handling the case of diffraction of a plane acoustic pulse by a cylinder in the range $|\Theta| \sim \frac{\pi}{2}$ and $|\Theta| > \frac{\pi}{2}$. See reference (16). These techniques do not apply here, primarily because of the requirement to satisfy two boundary conditions rather than one. Thus, the solution generated here is applicable only to the reflected wave zone. Additional study is indicated in order to generate a solution applicable to the diffracted wave zone, i.e., $|\Theta| \geq \frac{\pi}{2}$.

These are branch points located at:

$$\omega = \pm i \frac{ap}{c_L}, \pm i \frac{ap}{c_T}$$

and poles at:

$$\omega = \pm i \frac{ap}{\sqrt{2}\, c_T}.$$

The relation between wave speeds in the medium can be written as:

$$\frac{C_L^2}{C_T^2} = \frac{2(1-\nu)}{1-2\nu} \tag{7.27}$$

where ν = Poisson's ratio. Considering only positive values of ν, although negative values are not excluded thermodynamically, the following results are obtained:

$$\begin{aligned} \nu = 0; \quad & \frac{C_L^2}{C_T^2} = 2 \\ & C_L = \sqrt{2}\, C_T \\ \nu = \tfrac{1}{2}; \quad & \frac{C_L^2}{C_T^2} = \infty \\ & C_L \to \infty \end{aligned} \tag{7.28}$$

It can be concluded that $C_L \geq \sqrt{2}\, C_T$ for $\nu \geq 0$, i.e., $0 \leq \nu \leq \frac{1}{2}$. The branch points, poles and contours of integration are then as shown in Figure 9, on page 133.

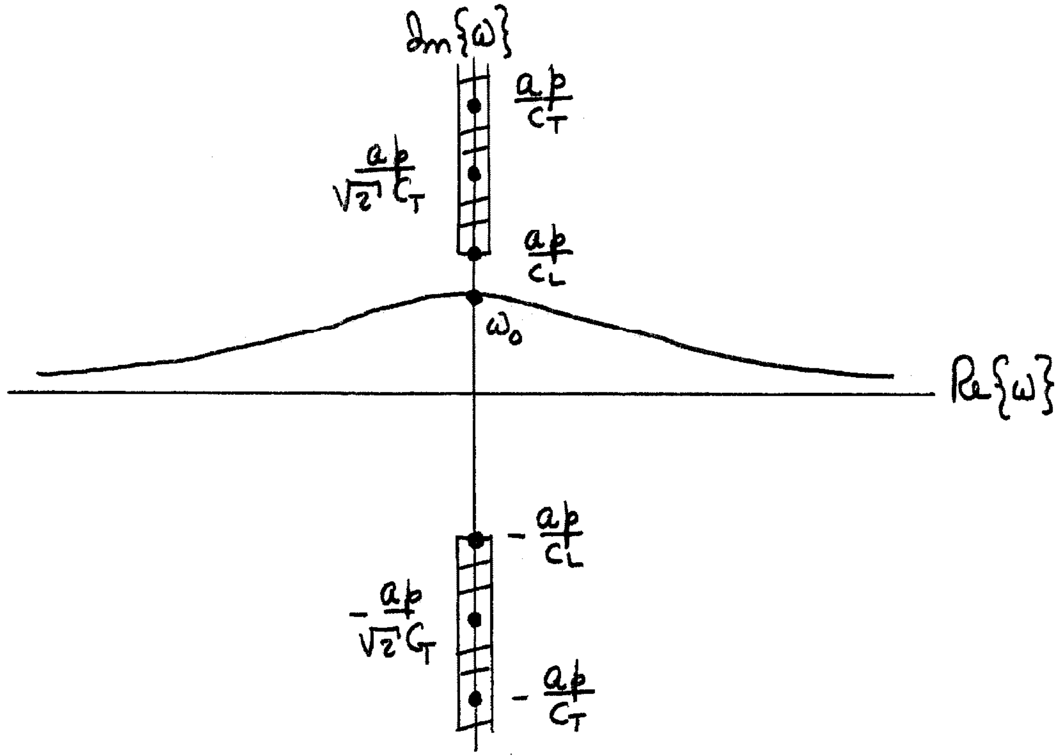


Fig. 9. Contour of Integration in the Complex ω-Plane.

The term $e^{f(\omega)}$ can be written as:

$$e^{f(\omega)} = e^{f(\omega_0) + \cancelto{0}{f'(\omega_0)}(\omega-\omega_0) + f''(\omega_0)\frac{(\omega-\omega_0)^2}{2!} + \cdots}$$

then evaluating equation 7.24 at ω_0, the result is obtained that:

$$\bar{\sigma}_\theta(a,\theta,p) \cong \frac{\sigma}{p} e^{-\frac{ap}{C_L}} \left\{\sqrt{\frac{ap}{C_L}\cos\theta}\right\}\left\{\frac{1}{C_T^2} - \frac{2}{C_L^2}\sin^2\theta\right\}^{-1}$$

$$\text{times}\left\{\left[\frac{\epsilon}{C_T^2} - \frac{2}{C_L^2}\sin^2\theta\right]\left[\frac{1-(1+\epsilon)\sin^2\theta}{\sqrt{\frac{ap}{C_L}\cos\theta}} - (1+\epsilon)\frac{C_L^2}{a^2p^2}\sqrt{\frac{ap}{C_L}\cos\theta}\right]\right.$$

$$-(1+\epsilon)\frac{2\sin^2\theta}{a^2p}\left(-\frac{a^2}{C_L^2}\sin^2\theta + \frac{a^2}{C_T^2}\right)^{\frac{1}{2}}\left[\frac{1}{\sqrt{\frac{ap}{C_L}\cos\theta}}\right.$$

$$\left.\left. - \sqrt{\frac{ap}{C_L}\cos\theta}\right]\right\} e^{\frac{ap}{C_L}\cos\theta} . \quad (7.29)$$

Note that:

$$\int_{-\infty}^{\infty} e^{f''(\omega_0)\frac{(\omega-\omega_0)^2}{2!}} d\omega = \sqrt{2\pi}\sqrt{\frac{ap}{C_L}\cos\theta} \qquad (7.30)$$

which represents the first term in braces in equation 7.29. Equation 7.29 simplifies somewhat to:

$$\bar{\sigma}(a,\theta,p) \cong -\frac{\sigma}{p} e^{-\frac{ap}{C_L}(1-\cos\theta)}$$

$$\text{times}\left\{(\sin^2\theta - \epsilon\cos^2\theta)\left[1 - \frac{(1+\epsilon)\frac{C_L}{ap}\cos\theta}{\cos^2\theta - \epsilon\sin^2\theta}\right]\right.$$

$$\left.+\left[\frac{2C_T(1+\epsilon)\sin^2\theta}{\cos^2\theta - \epsilon\sin^2\theta}\right]\left[1-\left(\frac{1+\epsilon}{2}\right)\sin^2\theta\right]^{\frac{1}{2}}\left[\frac{1}{ap} - \frac{1}{C_L}\cos\theta\right]\right\}. \quad (7.31)$$

The inverse Laplace transform may be obtained from reference (13) as:

$$\sigma_{\theta_{SC}}(a,\theta,t) \cong -\sigma H\left[t - \frac{a}{C_L}(1-\cos\theta)\right]$$

$$\text{times}\left\{\left(\sin^2\theta - \epsilon\cos^2\theta\right)\left[1 - \frac{C_L}{a}\frac{(1+\epsilon)\cos\theta}{(\cos^2\theta - \epsilon\sin^2\theta)}\left(t - \frac{a}{C_L} + \frac{a}{C_L}\cos\theta\right)\right]\right.$$

$$\left.+\left[\frac{2C_T}{a}\frac{(1+\epsilon)\sin^2\theta}{(\cos^2\theta - \epsilon\sin^2\theta)}\right]\left[1-\left(\frac{1+\epsilon}{2}\right)\sin^2\theta\right]^{\frac{1}{2}}\left[t - \frac{a}{C_L}\right]\right\}. \quad (7.32)$$

This expression represents the hoop stress at the cavity boundary resulting from the scattered field. It is only applicable when:

$$t \geq \frac{a}{c_L}(1-\cos\theta)$$

and:

$$|\theta| < \frac{\pi}{2}$$

The total hoop stress at the cavity boundary may be obtained by adding the above result to the stress obtained from the incident field. Equations 1.1 and 2.8 give the incident field stress as:

$$\sigma_{\theta_{in}}(a,\theta,t) = \sigma H\left[t-\frac{a}{c_L}(1-\cos\theta)\right](\sin^2\theta - \epsilon\cos^2\theta) \qquad (7.33)$$

for the left-running wave. Therefore:

$$\sigma_{\theta_{total}} = \sigma_{\theta_{in}} + \sigma_{\theta_{sc}} \cong \sigma H\left[t-\frac{a}{c_L}(1-\cos\theta)\right]$$

$$\text{times}\left\{(\sin^2\theta - \epsilon\cos^2\theta)\left[\frac{c_L}{a}\frac{(1+\epsilon)\cos\theta}{(\cos^2\theta-\epsilon\sin^2\theta)}\right]\left(t-\frac{a}{c_L}+\frac{a}{c_L}\cos\theta\right)\right.$$

$$\left.-\left[\frac{2c_T}{a}\frac{(1+\epsilon)\sin^2\theta}{(\cos^2\theta-\epsilon\sin^2\theta)}\right]\left[1-\left(\frac{1+\epsilon}{2}\right)\sin^2\theta\right]^{\frac{1}{2}}\left(t-\frac{a}{c_L}\right)\right\}. \qquad (7.34)$$

Consider the position $\theta = 0$ on the cavity, then equation 7.34 can be written as:

$$\sigma_{\theta_{total}}(a, 0, t) \cong \sigma H(t)\left\{-\epsilon(1+\epsilon)\frac{c_L t}{a}\right\} \qquad (7.35)$$

which implies that the hoop stress at the leading edge of the cavity is a linear function of time which ranges between 0 and $-\sigma\epsilon(1+\epsilon)$, that is, it approaches $-\sigma\epsilon(1+\epsilon)$ as $t \rightarrow \frac{a}{c_L}$. Suppose Poisson's ratio (ν) is $\frac{1}{3}$ then, from equation 2.5:

$$\epsilon = -\frac{\nu}{1-\nu} = -\frac{1}{2}$$

and the hoop stress at $\theta = 0$ varies linearly with time between the values 0 and $\frac{\sigma}{4}$, approaching $\frac{\sigma}{4}$ as $t \rightarrow \frac{a}{c_L}$ or as the incident wave approaches the position $\theta = \frac{\pi}{2}$.

The long time solution, equation 5.31, yields the following result for $\theta = 0$ at $r = a$ as $t \rightarrow \infty$:

$$\sigma_{\theta_{total}} \cong \sigma\left\{-\epsilon + \frac{1-\epsilon}{2} - \frac{3}{2}(1+\epsilon)\right\} = -\sigma(3\epsilon+1);$$

when $\nu = \frac{1}{3}$:

$$\sigma_{\theta_{total}}(a,0,\infty) = \frac{\sigma}{2} .$$

From the preceding analysis, it may be concluded that the early time behavior is significant. However, the original premise that the early time behavior is dominant has not been verified. Verification of this idea is contingent upon either continuing the solution, based on Friedlander's method, into the diffracted wave zone $\left(|\theta| \geq \frac{\pi}{2}\right)$, or else evaluating equation 6.8 which is based on the Fourier series representation of the potentials and boundary conditions. Neither approach appears to be mathematically tractable at the present time. However, the Fourier series representation does hold some promise for, if the Laplace inversion implied by equation 6.8 were carried out, the resulting summation on "n" could be evaluated for any desired number of terms. In any event, additional study of the early time behavior of the cavity response problem appears to be indicated.

References

1. Lamb, H., "On Waves Due to a Travelling Disturbance, with an Application to Waves in Superposes Fluids", Philosophical Magazine, No. 6, Vol. 31 (1916), pp. 386-399.
2. Lamb, H., "On the Propagation of Tremors over the Surface of an Elastic Solid", Philosophical Transactions of the Royal Society of London, (A), Vol. 203, (1904), pp. 1-42.
3. Lamb, H., "On Wave Patterns due to a Travelling Disturbance", Philosophical Magazine, No. 6, Vol. 31 (1916), pp. 539-548.
4. Sneddon, I. N., "Stress Produced by a Pulse of Pressure moving along a Surface of a Semi-Infinite Solid", Rendiconti Circolo Math. di Palermo, Vol. 2 (1952), pp. 57-62.
5. Cole, J. and Huth, J., "Stresses Produced in a Half-Plane by Moving Loads", Journal of Applied Mechanics, vol. 25, No. 4, (December 1958), pp. 433-436.
6. Ang, D. D., "Transient Motion of a Line Load on the Surface of an Elastic Half-Space", GALCIT Report No. SM 59-14, California Institute of Technology, (June 1959).
7. Love, A. E. H., "The Mathematical Theory of Elasticity, Dover Publications, New York, N. Y., Fourth Edition, (1944), Chapters 5, 8, and 12.
8. Sauter, F., "Der elastische Halbraum bei einer Mechanischen Beinflussung seiner Oberflache (zweidimensionales Problem)", Zeitschrift fur Angewandt Mathematik und Mechanik, Bd. 50, Nr. 7, (July 1950), pp. 203-215.

9. Rayleigh, Lord, "On Waves Propagated along the Plane Surface of an Elastic Solid", Proceedings of the London Mathematical Society, Vol. XVII (1887), pp. 4-11.

10. Ewing, W. M., Jardetzky, U. S. and Press, F., "Elastic Waves in Layered Media, McGraw-Hill, New York, N. Y., (1957), Chapter 3.

11. Biot, M. A., "Propagation of Elastic Waves in a Cylindrical Bore Containing a Fluid", Journal of Applied Physics, Vol. 23, (1952), pp. 997-1005.

12. Viktorov, I. A., "Rayleigh Type Waves on a Cylindrical Surface", Soviet Physics-Acoust., Vol. 4 (1958), pp. 131-136 (American Institute of Physics Translation).

13. Magnus, W. and Oberhettinger, F., "Special Functions of Mathematical Physics, Chelsea Publishing Company, New York (1949), Chapters III and VIII.

14. Gilbert, J. F., "Elastic Wave Interaction with a Cylindrical Cavity", Report to WDIE, AFBMD, Contract No. AF 04(647)-342, (December 1959), E. H. Plesset Associates Inc., Los Angeles, California.

15. Gilbert, F., and Knopoff, L., "Scattering of Impulsive Elastic Waves by a Rigid Cylinder", Journal of the Acoustical Society of America, Vol. 31, (1959), pp. 1169-1175.

16. Friedlander, F. G., "Sound Pulses," Cambridge University Press, (1958), Chapter 6.

17. Soldate, A. and Hook, J., "A Theoretical Study of Structure-Medium Interaction", Report No. AFSWC TN-61-6,(November 1960), Kirtland AFB, Contract No. AF 29(601)-2838, NESCO, Pasadena, California.

18. Miklowitz, J., "Scattering of Elastic Waves by a Circular Cylindrical Cavity; I. Rayleigh Waves and Long Time Solutions", National Engineering Science Co. Tech. Rept. Doc. RTD-TDR 63-3052 for Kirtland AFB (to be published).

19. Miklowitz, J., "Pulse Propagation in a Viscoelastic Solid with Geometric Dispersion", Presented at IUTAM Symposium on Stress Waves in Anelastic Solids at Brown University, (April 1963), (Proceedings forthcoming, Springer-Verlag).

20. Baron, M. E. and Matthews, A. T., "Diffraction of a Pressure Wave by a Cylindrical Cavity in an Elastic Medium", RAND Corporation Report P-2109, (October 1960).

21. Erdelyi, A., Magnus, W., Oberhettinger, F., Tricomi, F. G., "Higher Transcendental Functions", Vol. 2, Bateman Manuscript Project, McGraw-Hill, New York (1953), Chapter 7.

22. Erdelyi, A., Magnus, W., Oberhettinger, F. and Tricomi, F. G., Tables of Integral Transforms, Vol. 1, Bateman Manuscript Project, McGraw-Hill, New York (1954), Chapter V.

23. Huth, J. H. and Cole, J. D., "Elastic Stress Waves Produced by Pressure Loads on a Spherical Shell", Journal of Applied Mechanics, Vol. 22, No. 4 (December 1955), pp. 473-478.

24. Phillips, M. A., "Functions of a Complex Variable", Interscience Publishers, New York, (1947), pp. 133-135.

25. Watson, G. N., "Theory of Bessel Functions", Cambridge University Press, The MacMillan Co., New York (1945).

Appendix A - Integral Evaluation - Subsonic Approximation (Part I)

Examine each integral in equation 6.10 separately. The first is:

$$I_1 = \frac{P}{4\pi^2}\frac{\partial}{\partial z}\int_0^{\infty} e^{-\rho z}\int_0^{2\pi} e^{i\rho r\cos(\theta-\gamma)}\, d\gamma\, d\rho \qquad (A.1)$$

Substitute $\gamma-\theta=\alpha$; $d\gamma = d\alpha$ and:

$$I_1 = \frac{P}{4\pi^2}\frac{\partial}{\partial z}\int_0^{\infty} e^{-\rho z}\int_{-\theta}^{2\pi-\theta} e^{i\rho r\cos\alpha}\, d\alpha\, d\rho \qquad (A.2)$$

The θ-term appearing in the limits of integration is of no significance because of the periodicity in θ and it can be dropped. The resulting integral for α is well known, in fact, it is exactly $2\pi J_0(\rho r)$. Reference Magnus and Oberhettinger (13), pg. 26. Thus, we have:

$$I_1 = \frac{P}{2\pi}\int_0^{\infty} e^{-\rho z} J_0(\rho r)\, d\rho = \frac{P}{2\pi}\frac{\partial}{\partial z}\frac{1}{\sqrt{z^2+r^2}} . \qquad (A.3)$$

See Magnus and Oberhettinger (13), pg. 131.

The second integral can be evaluated from the above results. It is:

$$I_2 = -\frac{Pz}{4\pi^2}\frac{\partial^2}{\partial z^2}\int_0^\infty e^{-\rho z}\int_0^{2\pi} e^{i\rho r\cos(\theta-\gamma)}\, d\gamma\, d\rho \tag{A.4}$$

$$I_2 = -\frac{Pz}{2\pi}\frac{\partial^2}{\partial z^2}\frac{1}{\sqrt{z^2+r^2}}. \tag{A.5}$$

We can write the third integral in equation 6.10 as:

$$I_3 = -\frac{AP}{4\pi^2}\int_0^\infty \rho^2 e^{-\rho z}\int_0^{2\pi} \cos^2\gamma .\, e^{i\rho r\cos(\theta-\gamma)}\, d\gamma\, d\rho \tag{A.6}$$

where:

$$A = \frac{M_T^2 z}{2}\left(\frac{\lambda+2\mu}{\lambda+\mu}\right)$$

After use of a trigonometric identity for $\cos^2\gamma$:

$$I_3 = -\frac{AP}{8\pi^2}\int_0^\infty \rho^2 e^{-\rho z}\int_0^{2\pi} (1+\cos 2\gamma)\, e^{i\rho r\cos(\theta-\gamma)}\, d\gamma\, d\rho \tag{A.7}$$

$$I_3 = -\frac{AP}{4\pi}\frac{\partial^2}{\partial z^2}\frac{1}{\sqrt{z^2+r^2}} - \frac{AP}{8\pi^2}\int_0^\infty \rho^2 e^{-\rho z}\int_0^{2\pi} \cos 2\gamma .\, e^{i\rho r\cos(\theta-\gamma)}\, d\gamma\, d\rho \tag{A.8}$$

where we have used the results from I_2 above. Now we can effect the familiar substitution $\gamma - \theta = \alpha$; $d\gamma = d\alpha$ and expanding $\cos 2(\alpha+\theta)$ obtain:

$$I_3 = -\frac{AP}{4\pi}\frac{\partial^2}{\partial z^2}\frac{1}{\sqrt{z^2+r^2}} - \frac{AP}{8\pi^2}\int_0^\infty \rho^2 e^{-\rho z}\int_{-\theta}^{2\pi-\theta}\left\{\cos 2\alpha \cos 2\theta - \sin 2\alpha \sin 2\theta\right\} e^{i\rho r\cos\alpha}\, d\alpha\, d\rho. \qquad (A.9)$$

The sin 2α terms are odd with respect to $\alpha = \pi$ and give no contribution. Reference to Magnus and Oberhettinger (13), pag. 26 provides the result:

$$I_3 = -\frac{AP}{4\pi}\frac{\partial^2}{\partial z^2}\frac{1}{\sqrt{z^2+r^2}} + \frac{AP}{4\pi}\int_0^\infty \rho^2 e^{-\rho z} J_2(\rho r)\cos 2\theta\, d\rho \qquad (A.10)$$

and the same reference, pg. 131 gives:

$$I_3 = -\frac{AP}{4\pi}\frac{\partial^2}{\partial z^2}\frac{1}{\sqrt{z^2+r^2}} + \frac{AP}{4\pi}\cos 2\theta\left[\frac{3r^2}{(z^2+r^2)^{5/2}}\right]. \qquad (A.11)$$

The fourth integral in equation 6.10 is obtained from

I_3 above as:

$$I_4 = -\frac{BP}{4\pi}\frac{\partial^2}{\partial z^2}\frac{1}{\sqrt{z^2+r^2}} + \frac{BP}{4\pi}\cos 2\theta\left[\frac{3r^2}{(z^2+r^2)^{5/2}}\right] \qquad \text{(A.12)}$$

where:

$$B = \frac{M_T^{\,2} Z^2}{4}\left(\frac{\lambda+3\mu}{\lambda+2\mu}\right).$$

Finally, the sum $I_1 + I_2 + I_3 + I_4$ yields the desired result:

$$\tau_{zz} = \frac{P}{2\pi}\left\{\frac{\partial}{\partial z}\frac{1}{\sqrt{z^2+r^2}} - z\frac{\partial^2}{\partial z^2}\frac{1}{\sqrt{z^2+r^2}}\right.$$

$$- \frac{M_T^{\,2} Z}{4}\left(\frac{\lambda+2\mu}{\lambda+\mu}\right)\left[\frac{\partial^2}{\partial z^2}\frac{1}{\sqrt{z^2+r^2}} - \frac{3r^2\cos 2\theta}{(z^2+r^2)^{5/2}}\right]$$

$$\left.- \frac{M_T^{\,2} Z^2}{8}\left(\frac{\lambda+3\mu}{\lambda+2\mu}\right)\left[\frac{\partial^2}{\partial z^2}\frac{1}{\sqrt{z^2+r^2}} - \frac{3r^2\cos 2\theta}{(z^2+r^2)^{5/2}}\right]\right\} \qquad \text{(A.13)}$$

which reduces to:

$$\tau_{zz} = \frac{P}{2\pi}\left\{-\frac{3z^3}{(z^2+r^2)^{5/2}} + M_T^2\left(\frac{\lambda+2\mu}{\lambda+\mu}\right)\left[\frac{z}{4(z^2+r^2)^{3/2}} - \frac{3z^3}{4(z^2+r^2)^{5/2}}\right.\right.$$

$$\left.+\frac{3zr^2\cos 2\theta}{4(z^2+r^2)^{5/2}}\right] + M_T^2\left(\frac{\lambda+3\mu}{\lambda+2\mu}\right)\left[\frac{z^2}{8(z^2+r^2)^{3/2}}\right.$$

$$\left.\left.-\frac{3z^4}{8(z^2+r^2)^{5/2}} + \frac{3z^2r^2\cos 2\theta}{8(z^2+r^2)^{5/2}}\right]\right\} \qquad (A.14)$$

It is convenient here, as in the evaluation of τ_{zz} (Part I) above, to examine each term in equation 6.20 separately. The first term is:

$$H_1 = \frac{iPz}{8\pi^2\mu}\frac{\partial}{\partial z}\int_0^\infty e^{-\rho z}\int_0^{2\pi}\cos\gamma . e^{i\rho r\cos(\theta-\gamma)}\,d\gamma\, d\rho$$

$$= \frac{iPz}{8\pi^2\mu}\frac{\partial}{\partial z}\int_0^\infty e^{-\rho z}\int_{-\theta}^{2\pi-\theta}(\cos\alpha\cos\theta$$

$$-\sin\alpha\sin\theta)e^{i\rho r\cos\alpha}\,d\alpha\, d\rho \qquad (A.15)$$

where we have used the substitution $\gamma - \theta = \alpha$; $d\gamma = d\alpha$ again. The sin α term is odd with respect to $\alpha = \pi$ and contributes nothing. Reference to Magnus and Oberhettinger (13), pg. 26 gives:

$$H_1 = -\frac{Pz}{4\pi\mu}\frac{\partial}{\partial z}\int_0^\infty e^{-\rho z} J_1(\rho r)\cos\theta \, d\rho \tag{A.16}$$

and we obtain from pg. 131:

$$H_1 = -\frac{Pz}{4\pi\mu}\frac{\partial}{\partial z}\left\{\frac{1}{\sqrt{z^2+r^2}}\cdot\frac{r}{z+\sqrt{z^2+r^2}}\cdot\cos\theta\right\} \tag{A.17}$$

or, since $r\cos\theta = x$:

$$H_1 = -\frac{Pz}{4\pi\mu}\frac{\partial}{\partial z}\left\{\frac{1}{\sqrt{z^2+r^2}}\cdot\frac{x}{z+\sqrt{z^2+r^2}}\right\} \tag{A.18}$$

which reduces to:

$$H_1 = \frac{P}{4\pi\mu}\frac{xz}{(z^2+r^2)^{3/2}} = \frac{P}{4\pi\mu}\frac{xz}{R^3}. \tag{A.19}$$

The second term may be written as:

$$H_z = \frac{iP}{8\pi^2(\lambda+\mu)} \int_0^\infty e^{-\rho z} \int_0^{2\pi} \cos\gamma .\ e^{i\rho r \cos(\theta-\gamma)}\, d\gamma\, d\rho$$

$$= \frac{iP}{8\pi^2(\lambda+\mu)} \int_0^\infty e^{-\rho z} \int_{-\theta}^{2\pi-\theta} (\cos\alpha \cos\theta$$

$$- \sin\alpha \sin\theta)\, e^{i\rho r \cos\alpha}\, d\alpha\, d\rho \qquad (A.20)$$

where the familiar substitution has been made. The sin α term is discarded as before and the previously cited reference gives:

$$H_z = \frac{-P}{4\pi(\lambda+\mu)} \int_0^\infty e^{-\rho z} J_1(\rho r) \cos\theta\, d\rho$$

$$= \frac{-P}{4\pi(\lambda+\mu)} \cdot \frac{1}{\sqrt{z^2+r^2}} \cdot \frac{r\cos\theta}{z+\sqrt{z^2+r^2}} \qquad (A.21)$$

Or equivalently:

$$H_2 = -\frac{P}{4\pi(\lambda+\mu)}\frac{1}{\sqrt{z^2+r^2}}\cdot\frac{x}{z+\sqrt{z^2+r^2}} = -\frac{P}{4\pi(\lambda+\mu)}\cdot\frac{x}{R(z+R)}. \quad (A.22)$$

Evaluation of the remaining terms requires use of the identity:

$$\cos^3\gamma = \frac{1}{4}\left(\cos 3\gamma + 3\cos\gamma\right). \quad (A.23)$$

We can write the third term as:

$$H_3 = i\,\frac{CP}{4\pi^2\mu}\int_0^\infty e^{-\rho z}\int_0^{2\pi}\frac{1}{4}\left(\cos 3\gamma + 3\cos\gamma\right)$$

$$\text{times } e^{i\rho r\cos(\theta-\gamma)}\,d\gamma\,d\rho \quad (A.24)$$

where:

$$C = \frac{M_T^2}{8}\left[\frac{\lambda^2+4\mu\lambda+5\mu^2}{(\lambda+\mu)^2}\right].$$

The familiar substitution yields:

$$H_3 = i\frac{CP}{4\pi^2\mu}\int_0^\infty e^{-\rho z}\int_{-\theta}^{2\pi-\theta}\frac{1}{4}\Big(\cos 3\alpha\cos 3\theta - \sin 3\alpha\sin 3\theta$$

$$+ 3\cos\alpha\cos\theta - 3\sin\alpha\sin\theta\Big)e^{i\rho r\cos\alpha}\,d\alpha\,d\rho. \qquad (A.25)$$

The sin 3α, sin α terms are again odd with respect to $\alpha = \pi$ and yield no contribution. We obtain from the same reference:

$$H_3 = \frac{CP}{2\pi\mu}\int_0^\infty \frac{e^{-\rho z}}{4}\left[\cos 3\theta . J_3(\rho r) - 3\cos\theta . J_1(\rho r)\right]d\rho$$

$$= \frac{CP}{2\pi\mu}\left[\frac{\cos 3\theta}{4}\cdot\frac{1}{\sqrt{z^2+r^2}}\cdot\frac{r^3}{(z+\sqrt{z^2+r^2})^3}\right.$$

$$\left. - \frac{3\cos\theta}{4}\cdot\frac{1}{\sqrt{z^2+r^2}}\cdot\frac{r}{z+\sqrt{z^2+r^2}}\right] \qquad (A.26)$$

or equivalently:

$$H_3 = \frac{P}{64\pi\mu}M_T^2\left[\frac{\lambda^2+4\mu\lambda+5\mu^2}{(\lambda+\mu)^2}\right]\left[\frac{r^3\cos 3\theta}{R(z+R)^3}\right.$$

$$\left. - \frac{3X}{R(z+R)}\right]. \qquad (A.27)$$

The remaining integrals can be evaluated by reference to H_3, e.g.:

$$H_4 = i\frac{Pz}{16\pi^2\mu}\frac{\partial}{\partial z}\int_0^{\infty} M_T^2 e^{-\rho z}\int_0^{2\pi}\cos^3\gamma\, e^{i\rho r\cos(\theta-\gamma)}\,d\gamma\,d\rho \qquad (A.28)$$

$$= \frac{Pz\,M_T^2}{32\pi\mu}\frac{\partial}{\partial z}\left\{\frac{r^3\cos 3\theta}{\sqrt{z^2+r^2}}\cdot\frac{1}{(z+\sqrt{z^2+r^2})^3} - \frac{3x}{\sqrt{z^2+r^2}}\cdot\frac{1}{(z+\sqrt{z^2+r^2})}\right\}. \qquad (A.29)$$

Or equivalently:

$$H_4 = -\frac{Pz\,M_T^2}{32\pi\mu}\left\{r^3\cos 3\theta\left[\frac{3}{(z+\sqrt{z^2+r^2})^3(z^2+r^2)} + \frac{z}{(z+\sqrt{z^2+r^2})^3(z^2+r^2)^{3/2}}\right] - 3x\left[\frac{1}{(z+\sqrt{z^2+r^2})(z^2+r^2)} + \frac{z}{(z+\sqrt{z^2+r^2})(z^2+r^2)^{3/2}}\right]\right\} \qquad (A.30)$$

$$H_4 = -\frac{Pz\,M_T^2}{32\pi\mu}\left\{r^3\cos 3\theta\left[\frac{3}{(z+R)^3R^2} + \frac{z}{(z+R)^3R^3}\right] - 3x\left[\frac{1}{(z+R)R^2} + \frac{z}{(z+R)R^3}\right]\right\}. \qquad (A.31)$$

Similarly, the last integral can be expressed as:

$$H_5 = -i\,\frac{Pz^2M_T^2}{32\pi^2\mu}\left(\frac{\lambda+3\mu}{\lambda+2\mu}\right)\frac{\partial^2}{\partial z^2}\int_0^{\infty} e^{-\rho z}\int_0^{2\pi}\cos^3\gamma\,.\,e^{i\rho r\cos(\theta-\gamma)}\,d\gamma\,d\rho \quad (A.32)$$

$$H_5 = -\frac{Pz^2M_T^2}{64\pi\mu}\left(\frac{\lambda+3\mu}{\lambda+2\mu}\right)\frac{\partial^2}{\partial z^2}\left\{\frac{r^3\cos 3\theta}{\sqrt{z^2+r^2}\,(z+\sqrt{z^2+r^2})^3} - \frac{3x}{\sqrt{z^2+r^2}\,(z+\sqrt{z^2+r^2}}\right\} \quad (A.33)$$

or after differentiation:

$$H_5 = \frac{Pz^2M_T^2}{64\pi\mu}\left(\frac{\lambda+3\mu}{\lambda+2\mu}\right)\left\{-\frac{r^3\cos 3\theta}{(z+\sqrt{z^2+r^2})^3}\left[\frac{9}{(z^2+r^2)^{3/2}} + \frac{6z}{(z^2+r^2)^2} - \frac{1}{(z^2+r^2)^{3/2}} + \frac{3z}{(z^2+r^2)^2} + \frac{3z^2}{(z^2+r^2)^{5/2}}\right] + \frac{3x}{z+\sqrt{z^2+r^2}}\left[\frac{3z}{(z^2+r^2)^2} + \frac{3z^2}{(z^2+r^2)^{5/2}}\right]\right\} \quad (A.34)$$

$$H_5 = -\frac{Pz^2M_T^2}{64\pi\mu}\left(\frac{\lambda+3\mu}{\lambda+2\mu}\right)\left\{\frac{r^3\cos 3\theta}{(z+R)^3}\left[\frac{8}{R^3} + \frac{9z}{R^4} + \frac{3z^2}{R^5}\right] - \frac{3x}{z+R}\left[\frac{3z}{R^4} + \frac{3z^2}{R^5}\right]\right\}. \quad (A.35)$$

Finally the displacement u as expressed by equation 6.20 may be obtained from:

$$u(x, y, z) = H_1 + H_2 + H_3 + H_4 + H_5 . \qquad (A.36)$$

Appendix B - Fourier Series Coefficients (a_n and b_n)

The Fourier coefficients as defined by equations 4.6 and 4.8 of Part III may be written as:

$$a_0 = -\frac{\sigma}{\pi p} e^{-\frac{ap}{c_L}} \int_0^{\pi} \left[1-(1+\epsilon)\sin^2\theta\right] e^{-\frac{ap}{c_L}\cos\theta}\, d\theta$$

$$a_n = -\frac{2\sigma}{\pi p} e^{-\frac{ap}{c_L}} \int_0^{\pi} \left[1-(1+\epsilon)\sin^2\theta\right] \cos n\theta . e^{-\frac{ap}{c_L}\cos\theta}\, d\theta \qquad (B.1)$$

$$b_n = \frac{\sigma}{\pi p} e^{-\frac{ap}{c_L}} (1+\epsilon) \int_0^{\pi} \sin 2\theta \sin n\theta . e^{-\frac{ap}{c_L}\cos\theta}\, d\theta .$$

Coefficient a_0 is readily evaluated using the results of Watson (25), pg. 79 as:

$$a_0 = -\frac{\sigma}{p} e^{-\frac{ap}{c_L}} \left\{ I_0\left(\frac{ap}{c_L}\right) - (1+\epsilon)\frac{c_L}{ap} I_1\left(\frac{ap}{c_L}\right) \right\} . \qquad (B.2)$$

The coefficients a_n and b_n may be rewritten as:

$$a_n = -\frac{2\sigma}{\pi p} e^{-\frac{ap}{c_L}} \int_0^{\pi} \left\{ \left[1 - \left(\frac{1+\epsilon}{2}\right)\cos n\theta\right] + \left(\frac{1+\epsilon}{4}\right)\cos(n+2)\theta \right.$$

$$\left. + \left(\frac{1+\epsilon}{4}\right)\cos(n-2)\theta \right\} e^{-\frac{ap}{c_L}\cos\theta} \, d\theta \qquad (B.3)$$

$$b_n = \frac{\sigma}{2\pi p} e^{-\frac{ap}{c_L}} (1+\epsilon) \int_0^{\pi} \left\{ \cos(n-2)\theta - \cos(n+2)\theta \right\} e^{-\frac{ap}{c_L}\cos\theta} \, d\theta .$$

Noting that coefficients a_n and b_n are of the form:

$$M = \int_0^{\pi} \cos k\theta . \, e^{-\alpha\cos\theta} \, d\theta \qquad (B.4)$$

where $\alpha = \frac{ap}{c_L}$; k = integer, standard contour integration around the unit circle, $z = e^{i\theta}$, in the complex z-plane may be employed to evaluate the coefficients. Let:

$$z = e^{i\theta} ; \; \cos\theta = \frac{1}{2}\left(e^{i\theta} + e^{-i\theta}\right) = \frac{1}{2}\left(z + \frac{1}{z}\right)$$

and $dz = ie^{i\theta} d\theta$. The integral becomes:

$$\eta = \frac{1}{2} \int_0^{2\pi} \cos k\theta . e^{-\alpha\cos\theta} d\theta$$

$$= \frac{1}{4} \int_{|z|=1} (z^k + z^{-k}) e^{-\frac{\alpha}{2}(z+\frac{1}{z})} \frac{dz}{iz}$$

with a pole at z = 0.

Thus:

$$\eta = \frac{1}{4i} \int_{|z|=1} (z^k + z^{-k}) e^{-\frac{\alpha}{2}(z+\frac{1}{z})} \frac{dz}{z}$$

$$= \frac{2\pi i}{4i} \sum_k \text{Residues}\,(a_k) \qquad (B.5)$$

where a_k are poles inside the unit circle.

The form of equation B.4 above suggests that $\eta = fn\,[I_\nu(-\zeta)]$. See Watson (25), page 181, equation 4. Notice that when $\nu = n$, an integer, the second term in the formula of Watson vanishes. The modified Bessel function of the first kind is defined as:

$$I_\nu(\zeta) = \sum_{m=0}^{\infty} \frac{\left(\frac{1}{2}\zeta\right)^{\nu+2m}}{m!\,\Gamma(\nu+m+1)}$$

See reference (25), page 77. Thus, when ν = integer:

$$I_\nu(-\zeta) = (-)^\nu \sum_{m=0}^{\infty} \frac{\left(\frac{1}{2}\zeta\right)^{\nu+2m}}{m!\,\Gamma(\nu+m+1)} = (-)^\nu I_\nu(\zeta) \qquad \text{(B.6)}$$

To obtain the residues at z = 0, the coefficient of $\frac{1}{z}$ in the expansion of the integrand of equation B.5 is required. It is convenient to treat the integrand in two parts. For the first part:

$$z^{k-1} e^{-\frac{\alpha}{2}z} = z^K e^{-\frac{\alpha}{2}z}$$

$$= \sum_{\mu=0}^{\infty} \frac{(-)^\mu \left(\frac{\alpha}{2}\right)^\mu z^{\mu+K}}{\mu!}$$

where:

$$K = k-1 \geq -1$$

$$\mu! = \Gamma(\mu+1).$$

Also:

$$e^{-\frac{\alpha}{2z}} = \sum_{\nu=0}^{\infty} \frac{(-)^{\nu}\left(\frac{\alpha}{2}\right)^{\nu} z^{-\nu}}{\nu!} .$$

Therefore:

$$z^{k-1} \cdot e^{-\frac{\alpha}{2}\left(z+\frac{1}{z}\right)}$$

$$= \sum_{\mu=0}^{\infty} \frac{(-)^{\mu}\left(\frac{\alpha}{2}\right)^{\mu} z^{\mu+K}}{\Gamma(\mu+1)} \cdot \sum_{\nu=0}^{\infty} \frac{(-)^{\nu}\left(\frac{\alpha}{2}\right)^{\nu} z^{-\nu}}{\Gamma(\nu+1)}$$

$$= \sum_{\mu=0}^{\infty} \sum_{\nu=0}^{\infty} \frac{(-)^{\mu+\nu}\left(\frac{\alpha}{2}\right)^{\mu+\nu} z^{\mu+K-\nu}}{\Gamma(\mu+1)\Gamma(\nu+1)} .$$

The coefficient of $\frac{1}{z}$ occurs for $\mu + K - \nu = -1$ or $\nu = \mu + K + 1$; μ and K fixed; $K \geq -1$ always.
Thus, the residue at z = 0 is:

$$\mathrm{Res}(z=0) = \sum_{\mu=0}^{\infty} \frac{(-)^{2\mu+K+1}\left(\frac{\alpha}{2}\right)^{2\mu+K+1}}{\Gamma(\mu+1)\Gamma(\mu+K+2)}$$

$$= (-)^{K+1} \sum_{\mu=0}^{\infty} \frac{\left(\frac{\alpha}{2}\right)^{K+1+2\mu}}{\Gamma(\mu+1)\Gamma(\mu+1+K+1)}$$

$$= (-)^{K+1} I_{K+1}(\alpha) = (-)^{k} I_{k}(\alpha) \qquad (B.7)$$

which accounts for the first term in equation B.5.

The second term may be evaluated in a completely similar fashion, i.e.:

$$z^{-(k+1)} e^{-\frac{\alpha}{2}\left(z+\frac{1}{z}\right)} = z^{-\gamma} e^{-\frac{\alpha}{2}\left(z+\frac{1}{z}\right)} \quad ; \; \gamma = k+1 \geq 1$$

or:

$$z^{-\gamma} e^{-\frac{\alpha}{2}\left(z+\frac{1}{z}\right)} = \sum_{\mu=0}^{\infty} \sum_{\nu=0}^{\infty} \frac{(-)^{\mu+\nu} \left(\frac{\alpha}{2}\right)^{\mu+\nu} z^{\mu-\gamma-\nu}}{\Gamma(\mu+1)\,\Gamma(\nu+1)}$$

and the coefficient of $\frac{1}{z}$ is obtained when:

$\mu - \gamma - \nu = -1$

or $\mu = \gamma + \nu - 1$; ν and γ fixed; $\gamma \geq 1$. Thus:

$$\text{Res}(z=0) = (-)^{\gamma-1} \sum_{\nu=0}^{\infty} \frac{\left(\frac{\alpha}{2}\right)^{\gamma-1+2\nu}}{\Gamma(\gamma-1+\nu+1)\,\Gamma(\nu+1)}$$

$$= (-)^{\gamma-1} I_{\gamma-1}(\alpha) = (-)^{k} I_{k}(\alpha) \qquad \text{(B.8)}$$

which is the same result as obtained for the first term.

The integral $\mathcal{M}$ can be written then as:

$$\mathcal{M} = \frac{1}{4i} \int_{|z|=1} \left(z^{k-1} + z^{-k-1}\right) e^{-\frac{\alpha}{2}\left(z+\frac{1}{z}\right)} dz$$

$$= \frac{\pi}{2} \sum \mathrm{Res}(z=0) = \pi (-)^{k} I_{k}(\alpha). \tag{B.9}$$

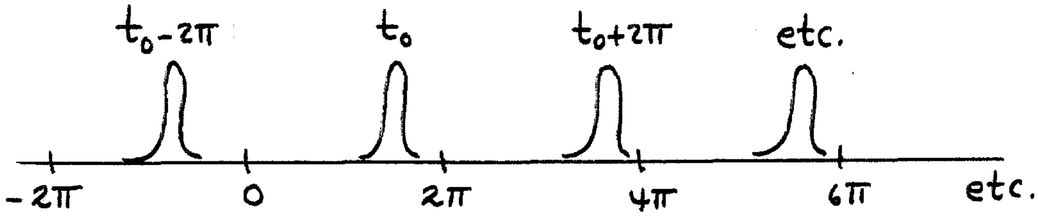
Using the results of equation B.9, the coefficients as expressed by equation B.3 may be written as:

$$a_n = -\frac{2\sigma}{p} e^{-\frac{ap}{c_L}} \left\{ \left[1-\left(\frac{1+\epsilon}{2}\right)\right](-)^{n} I_n\left(\frac{ap}{c_L}\right) + \left(\frac{1+\epsilon}{4}\right)(-)^{n+2} I_{n+2}\left(\frac{ap}{c_L}\right) + \left(\frac{1+\epsilon}{4}\right)(-)^{n-2} I_{n-2}\left(\frac{ap}{c_L}\right) \right\} \tag{B.10}$$

$$b_n = \frac{\sigma}{p} e^{-\frac{ap}{c_L}} \left(\frac{1+\epsilon}{2}\right) \left\{ (-)^{n-2} I_{n-2}\left(\frac{ap}{c_L}\right) - (-)^{n+2} I_{n+2}\left(\frac{ap}{c_L}\right) \right\}.$$

Appendix C - Evaluation of the Fourier Transforms $\tilde{\sigma}_r(a, \omega, p)$ and $\tilde{\tau}_{r\theta}(a, \omega, p)$

Poisson's summation formula may be used conveniently in evaluating the Fourier transforms for $\tilde{\sigma}_r(a, \omega, p)$ and $\tilde{\tau}_{r\theta}(a, \omega, p)$ rather than direct evaluation of equations 7.14 in Part III. This summation formula is easily developed. Suppose there exists an infinite periodic array of delta functions arranged as shown below:



The function $f(t)$ representing this array may be written as:

$$f(t) = \sum_{n=-\infty}^{\infty} \delta(t - t_0 - 2n\pi) \tag{C.1}$$

which is periodic of period 2π and an even function of t.

The function $f(t)$ can be expressed as a Fourier series. In the range $0 \leq t \leq 2\pi$, the function is:

$$f(t) = \delta(t - t_0)$$

therefore:

$$f(t) = \sum_{\nu=-\infty}^{\infty} A_\nu e^{i\nu t}$$

where:

$$A_\nu = \frac{1}{2\pi}\int_0^{2\pi} f(t) e^{-i\nu t} dt = \frac{1}{2\pi}\int_0^{2\pi} \delta(t-t_0) e^{-i\nu t} dt = \frac{1}{2\pi} e^{-i\nu t_0}$$

Thus:

$$\sum_{n=-\infty}^{\infty} \delta(t - t_0 - 2n\pi) = \sum_{\nu=-\infty}^{\infty} \frac{1}{2\pi} e^{i\nu(t-t_0)}$$

or equivalently:

$$\frac{1}{2\pi}\sum_{n=-\infty}^{\infty} \delta\left(\frac{t-t_0}{2\pi} - n\right) = \frac{1}{2\pi}\sum_{\nu=-\infty}^{\infty} e^{i\nu(t-t_0)} \qquad (C.2)$$

Let $t - t_0 = 2\pi k$, say, then

$$\sum_{n=-\infty}^{\infty} \delta(k-n) = \sum_{\nu=-\infty}^{\infty} e^{i\nu(2\pi k)}$$

$$= \sum_{n=-\infty}^{\infty} e^{in(2\pi k)} \qquad (C.3)$$

This result is Poisson's summation formula and is in the appropriate form for use here.

Suppose:

$$\bar{\sigma}_r'(a, \theta, p) = \sum_{n=-\infty}^{\infty} \bar{\sigma}_r(a, \theta + 2n\pi, p)$$

$$= \sum_{n=-\infty}^{\infty} \frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{\bar{\sigma}}_r(a, \omega, p) e^{i\omega(\theta + 2n\pi)} \, d\omega \qquad (C.4)$$

where $(\bar{\ })$ represents the Laplace transform and $(\tilde{\ })$ the Fourier transform. Rewrite equation C.3 as:

$$\sum_{n=-\infty}^{\infty} \delta(\omega - n) = \sum_{n=-\infty}^{\infty} e^{in(2\pi\omega)} \qquad (C.5)$$

Equation C.4 can be rewritten as:

$$\bar{\sigma}_r^{\,1}(a,\theta,p) = \sum_{n=-\infty}^{\infty} \frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{\sigma}_r(a,\omega,p)\, e^{i\omega\theta}\, \delta(\omega-n)\, d\omega$$

$$= \frac{1}{2\pi}\, \tilde{\sigma}_r(a,n,p)\, e^{in\theta} \qquad (C.6)$$

Now $\bar{\sigma}_r^{\,1}(a,\theta,p)$ is given by equation 4.6. It is even in θ and can be expressed as a Fourier series:

$$\bar{\sigma}_r^{\,1}(a,\theta,p) = \sum_{n=-\infty}^{\infty} A_n e^{in\theta} \qquad (C.7)$$

Equating equations C.6 and C.7 yields:

$$\frac{1}{2\pi}\, \tilde{\sigma}_r(a,n,p) = A_n = \frac{a_n}{2} \qquad (C.8)$$

which relates the Fourier transform $\tilde{\sigma}_r$ to the Fourier coefficients of $\bar{\sigma}_r^{\,1}$. Note that:

$$A_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} \bar{\sigma}_r'(a,\theta,p)e^{in\theta}\,d\theta$$

$$= \frac{1}{\pi}\int_{0}^{\pi} \bar{\sigma}_r'(a,\theta,p)\cos n\theta\,d\theta = \frac{a_n}{2} \qquad (C.8a)$$

Similarly, suppose:

$$\bar{\tau}_{r\theta}'(a,\theta,p) = \sum_{n=-\infty}^{\infty} \bar{\tau}_{r\theta}(a,\theta+2n\pi,p) \qquad (C.9)$$

$$= \sum_{n=-\infty}^{\infty} \frac{1}{2\pi}\int_{-\infty}^{\infty} \tilde{\bar{\tau}}_{r\theta}(a,\omega,p)e^{i\omega(\theta+2n\pi)}\,d\omega. \qquad (C.9a)$$

Application of Poisson's summation formula gives:

$$\bar{\tau}_{r\theta}'(a,\theta,p) = \frac{1}{2\pi}\tilde{\bar{\tau}}_{r\theta}(a,n,p)\,e^{in\theta}. \qquad (C.10)$$

The tangential stress $\bar{\bar{\tau}}'_{r\theta}(a,\theta,p)$ is represented by equation 4.6. It is odd in θ and can also be expanded in Fourier series as:

$$\bar{\bar{\tau}}'_{r\theta}(a,\theta,p) = -\sum_{n=-\infty}^{\infty} B_n e^{in\theta} \; ; \; B_n \text{ imaginary} \tag{C.11}$$

Comparison of equations C.10 and C.11 gives:

$$\frac{1}{2\pi}\tilde{\tau}_{r\theta}(a,n,p) = -B_n . \tag{C.12}$$

Equation C.12 must be operated on to give the desired result. Write:

$$B_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} \bar{\bar{\tau}}'_{r\theta}(a,\theta,p)\, e^{in\theta}\, d\theta$$

$$= \frac{i}{\pi}\int_{0}^{\pi} \bar{\bar{\tau}}'_{r\theta}(a,\theta,p)\sin n\theta \, d\theta$$

$$= i\,\frac{b_n}{2} \tag{C.13}$$

by reference to equation 4.8. Substitution of equation C.13 into C.12 yields the desired result:

$$\frac{1}{2\pi}\tilde{\tau}_{r\theta}(a,n,p) = -i\,\frac{b_n}{2}$$

which relates the Fourier transform of $\bar{\tau}_{r\theta}$ to the Fourier coefficient of $\bar{\bar{\tau}}'_{r\theta}$.