STABILITY AND RELATED PROBLEMS IN RANDOMLY EXCITED SYSTEMS

Thesis by

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In Partial Fulfillment of the Requirements

For the Degree of

Doctor of Philosophy

California Institute of Technology

Pasadena, California

1964

ACKNOWLEDGEMENTS

The author wishes to express his great appreciation to Dr. T. K. Caughey for his suggestions, help, and encouragement during the duration of his work.

Most of this work was completed while he held Fellowships from the Garrett Corporation, the National Science Foundation, and it was finally completed while he held an Institute Scholarship.

Thanks are also due to Mrs. Elizabeth Fox who did an excellent job of typing and correcting the author's spelling errors.

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1. THE CONTINUOUS MARKOV PROCESS

1.1.0 INTRODUCTION

Since the first treatments of Brownian Motion as an example of a continuous Markov Process, the applications of Markov Processes in physical situations have extended over a wide range which includes such extremes as barometric pressure distributions and structural responses to earthquakes.

In this part, the notion of a continuous Markov Process is presented and described in terms of a transition probability and a Fokker-Planck Equation. Two uniqueness theorems are presented here, as well as a heuristic discussion of the large time behavior of such a process.

1.2.0 THE N-DIMENSIONAL, FIRST ORDER, CONTINUOUS MARKOV PROCESS

The first order Markov process is a random process for which only the first conditional probability is needed for a complete description. In the case of a Markov process the first conditional probability is given the name "transition probability." Let ${}_a y$ represent an n dimensional vector whose components are ${}_a y_i$ $(i = 1, 2, \ldots n)$ and let y represent an n dimensional vector with components y_i. The conditional probability density for y at time t, on the condition that y was equal to ${}_1 y$ at time t_1, ${}_2 y$ at time t_2, ..., and ${}_k y$ at time t_k will be denoted by $P_k(y, t/{}_1 y, t_1; {}_2 y, t_2, \ldots; {}_k y, t_k)$. A first order Markov process is one for which

$$P_k(y, t/{}_1 y, t_1; {}_2 y, t_2; \ldots; {}_k y, t_k) = P_1(y, t/{}_1 y, t_1) , \tag{1.1}$$

when $t > t_1 > t_2 \ldots > t_k$. This first conditional probability density is often called the transition probability, however for convenience, the transition probability will here refer to P_T, where

$$P_T(y/x,\ t;t_o) = P_1(y, t + t_o/x, t_o)\ . \qquad (1.2)$$

When the process is a stationary one, the conditional probabilities can depend only upon the time differences, and one may write in this case

$$P_T(y/x, t) = P_T(y/x,\ t;t_o)\ . \qquad (1.3)$$

1.2.1 THE FIRST PASSAGE PROBABILITY

The probability of y equaling z at least once in the time interval from t_1 to t, given that y was equal to x at time t_1 (where $t_1 < t$) will be denoted by $F(z/x,\ t-t_1;t_1)$. When the time derivative of this exists, it will be denoted by $T(z/x,\ t-t_1;t_1)$ so that

$$T(z/x,\ t-t_1;t_1) = \frac{\partial}{\partial t} F(z/x,\ t-t_1;t_1)\ .$$

T will be called the first passage probability density. This name arises from the fact that the probability of y equaling z at some time in the time interval from t to t + dt, on the condition that y was x at a prior time t_1 and $y \neq z$ for all time from t_1 to t, is given by $T(z/x,\ t-t_1;t_1)\ dt$.

As it is obvious that for y to get from the point (in phase space) x to the point z, it must at some time equal z for the first time, one may formally write

$$P_1(z,t/x,t_o) = \int_{t_o}^{t} dt_1 \; T(z/x,t_1-t_o;t_o) \; P_2(z,t/z,t_1;x,t_o) \; ,$$

for $t > t_o$. For a stationary Markov process the dependence on the initial time, t_o, disappears, and the conditional probabilities become transition probabilities. Thus, when the process is a stationary Markov process, and $T(z/x,t_1-t_o;t_o) = T(z/x,t_1;0)$ is replaced by $T(z/x,t_1)$

$$P_T(z/x,t) = \int_0^t dt_1 \; T(z/x,t_1) \; P_T(z/z, \; t-t_1) \; . \qquad (1.4)$$

The applications of the convolution equation given by Eq. 1.4 will be discussed later, in Part 4 of this thesis.

1.2.2 THE SMOLUCHOWSKI EQUATION

For simplicity of notation, the integral expressed as a single integral in z, that is $\int dz$, will be used to denote an n-fold volume integration over all the z_i,

$$\int dz \;\; f(z) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} f(z) \; dz_1 dz_2 \; \ldots \; dz_n .$$

Using this notation, and utilizing the joint probability densities, one can show that for any random processes having such densities,

$$P_1(y,t_1/x,t_o) = \int dz \; P_1(z,t_2/x,t_o) \; P_2(y,t_1/z,t_2;x,t_o) \; . \qquad (1.5)$$

This equation is a general form of what is often called the Smoluchowski Equation. If the process is a Markov process, the conditional probabilities become transition probabilities, and Eq. 5 becomes

$$P_T(y/x,t;t_o) = \int dz \; P_T(z/x, \tau;t_o) \; P_T(y/z, \; t-\tau; \; t_o + \tau), \qquad (1.6)$$

provided that $t > \tau > 0$. Eq. 1.6 is the Smoluchowski Equation. The Smoluchowski Equation is used in the derivation of the so-called Fokker-Planck or Kolmogorov Equations, which will be discussed in the following section.

1.2.3 THE FOKKER-PLANCK EQUATIONS

The Smoluchowski Equation is often used as a tool for obtaining the Kolmogorov or Fokker-Planck Equations, which are partial differential equations governing the behavior of the transition probability in a Markov Process.* For brevity, only the name Fokker-Planck Equation shall be used in this thesis.

As derivations of the Fokker-Planck Equations are common to much of the literature,** a derivation will not be given here. Given certain assumptions, an n-dimensional, first order Markov Process has a transition probability $P_T(y/x,t;t_o)$ which satisfies a forward Fokker-Planck Equation given by

$$\frac{\partial}{\partial t}[P_T(y/x,t;t_o)] = -\sum_{k=1}^{n}\frac{\partial}{\partial y_k}[A_k(y,t+t_o)P_T(y/x,t;t_o)] + \frac{1}{2}\sum_{k=1}^{n}\sum_{i=1}^{n}\frac{\partial^2}{\partial y_k \partial y_i}[D_{ki}(y,t+t_o)P_T(y/x,t;t_o)] \,, \tag{1.7}$$

and a reverse Fokker-Planck equation given by

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* A derivation not utilizing the Smoluchowski Equation was presented by J. K. Dienes (1).

** See for example S. Chandrasekhar (2), pp. 31-33, or M. C. Wang and G. E. Uhlenbeck (3), pp. 331-332.

$$\frac{\partial}{\partial t} P_T(y/x,t;t_o) - \frac{\partial}{\partial t_o} P_T(y/x,t;t_o) = \sum_{k=1}^{n} A_k(x,t_o) \frac{\partial}{\partial x_k} P_T(y/x,t;t_o)$$

$$+ \frac{1}{2} \sum_{k=1}^{n} \sum_{i=1}^{n} D_{ki}(x,t_o) \frac{\partial^2}{\partial x_i \partial x_k} P_T(y/x,t;t_o) \ , \tag{1.8}$$

where the A_k and D_{ki} are to be determined from the so-called "incremental moments."

Let y have the initial value x at time t_o, and let the vector Δy, whose components are Δy_k, be defined by

$$\Delta y = y(t_o + \Delta t) - x \ . \tag{1.9}$$

The incremental moments will be given by the mean or expected value of products of the form $\Delta y_i \Delta y_k \ . \ . \ . \ \Delta y_m$. If the taking of the mean or expected value is denoted by brackets, $\langle \ \rangle$, then one can define the incremental moments by $M_{i,j \ldots m}(x,t_o;\Delta t)$, where

$$M_{i,j,\ldots m}(x,t_o;\Delta t) = \langle \Delta y_i \Delta y_j \ . \ . \ . \ \Delta y_m \rangle \ , \tag{1.10}$$

or expressed as an integral,

$$M_{i,j,\ldots k}(x,t_o;\Delta t) = \int dy (y_i - x_i)(y_j - x_j) \ . \ . \ . \ (y_m - x_m) P_T(y/x, \Delta t; t_o) \ . \tag{1.11}$$

The basic assumption used in deriving the Fokker-Planck Equations is that the limits given by

$$\lim_{\Delta t \to 0} \frac{M_{i,j,\ldots,k}(x,t_o;\Delta t)}{\Delta t}$$

each exist. Further, it is assumed that when the incremental moment is higher than a second moment, that is $M_{i,j,k}$, $M_{i,j,k,m}$, etc., this limit will be zero. In other words, it is assumed that

$$\lim_{\Delta t \to 0} \frac{\langle \Delta y_i \Delta y_j \Delta y_k \rangle}{\Delta t} = 0 ,$$

$$\lim_{\Delta t \to 0} \frac{\langle \Delta y_i \Delta y_j \Delta y_k \Delta y_m \rangle}{\Delta t} = 0 ,$$

etc. In this case the limits taken using the first and second incremental moments will yield the coefficients A_k and D_{ki},

$$A_k(x,t_o) = \lim_{\Delta t \to 0} \frac{M_k(x,t_o;\Delta t)}{\Delta t} = \lim_{\Delta t \to 0} \frac{\langle \Delta y_k \rangle}{\Delta t} , \qquad (1.12)$$

$$D_{ki}(x,t_o) = \lim_{\Delta t \to 0} \frac{M_{ki}(x,t_o;\Delta t)}{\Delta t} = \lim_{\Delta t \to 0} \frac{\langle \Delta y_k \Delta y_i \rangle}{\Delta t} . \qquad (1.13)$$

The reverse Fokker-Planck Equation, given by Eq. 1.8, has certain applications in first passage time problems. It is basically the forward Fokker-Planck Equation that will be of interest here. In discussing properties of its solutions, the term Fokker-Planck Equation used here will in general refer to the equation given by

$$\frac{\partial P}{\partial t} = -\sum_{k=1}^{n} \frac{\partial [A_k(y,t)P]}{\partial y_k} + \frac{1}{2} \sum_{k=1}^{n} \sum_{i=1}^{n} \frac{\partial^2 [D_{ki}(y,t)P]}{\partial y_k \partial y_i} . \qquad (1.14)$$

Thus, any Markov process having a transition probability that satisfies Eq. 1.7, and an initial probability density f(y), will have a probability density P(y,t) that will satisfy the Fokker-Planck Equation, Eq. 1.14, and have the obvious initial condition

$$P(y,t_1) = f(y) \ ,$$

provided that the initial time is considered to be t_1.*

Further, if this initial condition is an n-dimensional delta function,

$$f(y) = \delta(y-x) = \delta(y_1-x_1)\,\delta(y_2-x_2)\ .\ .\ .\ \delta(y_n-x_n) \ ,$$

then P(y,t) will be equal to the transition probability,

$$P(y,t) = P_T(y/x, t-t_1; t_1) \ .$$

When the Markov Process is a stationary one, the coefficients A_k and D_{ki} no longer depend upon time, and the Fokker-Planck Equation, Eq. 1.14, is greatly simplified.

Before proceeding to applications of the Fokker-Planck Equation, certain properties of solutions to the equation will be discussed.

1.3.0 SOLUTIONS TO THE FOKKER-PLANCK EQUATION

A number of properties of solutions to the Fokker-Planck

* This fact may be seen by noting that the transition probability is simply the first conditional probability, so that

$$P(y,t) = \int dx\, f(x)\, P_T(y/x, t-t_1; t_1) \ .$$

Equation are to be considered here, particularly uniqueness, existence, and large time behavior. For simplicity of notation, the FP operator will be used, where the FP operator is defined by

$$FP(g) = \frac{\partial g}{\partial t} + \sum_{k=1}^{n} \frac{\partial[A_k(y,t)g]}{\partial y_k} - \frac{1}{2}\sum_{k=1}^{n}\sum_{i=1}^{n} \frac{\partial^2[D_{ki}(y,t)g]}{\partial y_k \partial y_i} \quad . \tag{1.15}$$

The Fokker-Planck Equation will often be considered in specific cases where certain restrictions are placed on the coefficients A_k and D_{ki}. The two definitions to follow will cover these cases.

Definition 1: The Fokker-Planck Equation is stationary if the coefficients A_k and D_{ki} do not depend on time, so that $A_k(y,t) = A_k(y)$ and $D_{ki}(y,t) = D_{ki}(y)$.

Definition 2: The Fokker-Planck Equation is steady if the coefficients A_k and D_{ki} are such that

(i) The $D_{ki}(y,t)$ are zero for i or k less than m + 1 (where m is an integer less than n, $0 \leqslant m < n$),

(ii) The equation

$$\sum_{k=m+1}^{n}\sum_{i=m+1}^{n} D_{ki}(y,t)x_i x_k = 0$$

has as its only solution $x_k = 0$ for all $k \geqslant m+1$,

(iii) The set of equations

$$\frac{\partial g}{\partial t} = - \sum_{k=1}^{m} A_k(y,t) \frac{\partial g}{\partial y_k} ,$$

$$\frac{\partial g}{\partial y_k} = 0 \text{ for } k = m + 1, m - 2, \ldots n,$$

has as its only solution g = constant.

As in all uniqueness proofs, there are certain restrictions on the class of functions involved. To avoid continual repetition of these requirements in the sections on uniqueness to follow, the restrictions will be combined into a single definition. For lack of a better name, a function which satisfies these will be called "well behaved."

Definition 3: A probability density, P, is well-behaved if and only if each of the following requirements is satisfied:

(i) $FP(P) = 0$ for all $t > t_1$ and all y;

(ii) $P \geqslant 0$ for all $t > t_1$ and all y;

(iii) The multiple improper Riemann Integrals given by $\int dy\ P$, $\int dy\ A_k\ \partial P/\partial y_k$, $\int dy\ D_{ki}\ \partial^2 P/\partial y_k \partial y_i$, $\int dy\ P\ \partial A_k/\partial y_k$, $\int dy\ (\partial D_{ki}/\partial y_k)\ (\partial P/\partial y_i)$, and $\int dy\ P\ \partial^2 D_{ki}/\partial y_k \partial y_i$ are each absolutely and uniformly convergent for t in every closed interval lying strictly between t_1 and ∞;

(iv) $\int dy\ P(y,t_1) = 1$;

(v) $\partial P/\partial t$ is uniformly continuous in y and t for all y and for t in every closed interval lying strictly between t_1 and ∞;

(vi) The limit as $y_k \to \pm\infty$ of each of the following functions exists and is zero:

$a_k P$, $D_{ki}\partial P/\partial y_i$, $P\partial D_{ki}/\partial y_i$.

The requirements defining a "well-behaved" probability density appear quite restrictive, however they are only slightly more so than the requirements that would be needed for a rigorous derivation of the Fokker-Planck Equation.

1.3.1 UNIQUENESS OF SOLUTIONS TO THE FOKKER-PLANCK EQUATION

It will be proved here that any two well-behaved* solutions to the Fokker-Planck Equation having the same initial condition are identical. This proof will be two intermediate theorems, to be stated here, and proved in Sections 1.3.1.1 and 1.3.1.2

Theorem 1: If P is a well-behaved* probability density, then for all $t \geq t_1$,

$$\int dy\, P(y,t) = 1 \quad (1.16)$$

Theorem 2: Given that P_1 and P_2 are each well-behaved* probability densities. Define P_3, P_4 and x by

* See Definition 3, page 9.

$$P_3 = aP_1 + (1-a)P_2 \ , \tag{1.17a}$$

$$P_4 = bP_1 + (1-b)P_2 \ , \tag{1.17b}$$

$$x \ = P_3/P_4 \ , \tag{1.17c}$$

where a and b are any numbers such that

$$0 < a < b < 1 \ .$$

Let $g(x)$ be any function of x, such that for x lying in the range $a/b \leqslant x \leqslant (1-a)/(1-b)$, $g'(x)$ and $g''(x)$ exist and

$$|g(x)| \leqslant M,$$

$$|g'(x)| \leqslant M,$$

$$0<c\leqslant g''(x) \leqslant M,$$

are satisfied for some value of M and c.

In this case, the integral

$$A(t) = \int dy \, g(x) \, P_4, \tag{1.18}$$

exists, and satisfies the inequality

$$A(t) \geqslant g(1), \tag{1.19}$$

with strict equality holding if and only if $P_1 = P_2$. Further, the derivative $dA(t)/dt$ exists, and

$$dA(t)/dt \leqslant 0. \tag{1.20}$$

Uniqueness follows readily from Theorem 2. If P_1 and P_2 are each "well-behaved" and are identical at t_1 (so that at t_1, x is 1), then $A(t_1) = g(1)$. As $A(t)$ can only decrease (Eq. 1.20) and is bounded below by g(1), it must be g(1) for all $t \geqslant t_1$. From the Theorem, it can be seen that this implies $P_1 = P_2$. This may be expressed in a uniqueness theorem as follows.

<u>Theorem 3:</u> UNIQUENESS: Given that P_1 and P_2 are each well-behaved* probability densities, having the same initial condition, $P_1(y,t_1) = P_2(y,t_2)$. Then for all $t \geqslant t_1$,

$$P_1(y,t) = P_2(y,t) \; .$$

1.3.1.1 Proof of Theorem 1:

By using the uniform continuity of $\partial P/\partial t$, and uniform convergence of the integral $\int dy \; \partial P/\partial t$, it is possible to write

$$\frac{d}{dt}\int dy \; P = \int dy \; \partial P/\partial t \; .$$

From the Fokker-Planck Equation, this may be converted to the form

$$\frac{d}{dt}\int dy \; P = -\sum_{k=1}^{n}\int dy \frac{\partial[A_k(y,t)P]}{\partial y_k} + \frac{1}{2}\sum_{k=1}^{n}\sum_{i=1}^{n}\int dy \frac{\partial^2[D_{ki}(y,t)P]}{\partial y_k \partial y_i}$$

From Definition 3, it may be noted that each of the integrals on the right hand side of the above equation is absolutely and uniformly convergent, and hence the order of integration may be interchanged. By doing so, and using requirement vi of Definition 3, one sees that

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* See Definition 3, page 9.

they are each 0, so that

$$\frac{d}{dt} \int dy\ P = 0 \ .$$

Hence, as $\int dyP$ is one at time t_1, it must be one for all $t \geq t_1$.

1.3.1.2 Proof of Theorem 2:

To prove this theorem, four intermediate lemmas will be utilized.

Lemma 1: A(t) exists, and

$$A(t) \geq g(1) + \tfrac{1}{2}\, c \int dy\ (x-1)^2 P_4. \tag{1.21}$$

Proof: From the continuity of the P's, the bound on g(x), and the definition of g(x) it is obvious that g(x) is a bounded, continuous function. This, coupled with the integrability of P_1 and P_2, and hence P_4 implies the existence of the integral defining A(t).

If a finite Taylor Expansion is used for g(x), and the lower bound on g''(x) (see Eq. 1.17) is used, then

$$g(x) \geq g(1) + (x-1)g'(1) + \tfrac{1}{2}(x-1)^2\ c.$$

Multiplying by P_4 and integrating leads to

$$\int dy\ g(x)\ P_4 \geq g(1) \int dy\ P_4 + g'(1) \int dy\ (P_3 - P_4) + \tfrac{1}{2} c \int dy\ (x-1)^2\ P_4 \ . \tag{1.22}$$

If now Theorem 1 is applied to this, the inequality of Eq. 1.21 is obtained.

Lemma 2: dA(t)/dt exists and

$$\frac{dA(t)}{dt} = \frac{d}{dt} \int dy\ P_4\ g(x) = \int dy\ \partial[P_4 g(x)]/\partial t \ .$$

Proof: From Lemma 1, the integral defining A(t), $\int dy\ P_4 g(x)$, exists. By noting that

$$\frac{\partial}{\partial t}[P_4\, g(x)] = g(x)\, \partial P_4/\partial t + g'(x)[\partial P_3/\partial t - x\, \partial P_4/\partial t], \tag{1.22}$$

one can combine the continuity and boundedness of $g(x)$ and $g'(x)$ with the uniform integrability of $\partial P_3/\partial t$ and $\partial P_4/\partial t$ to show that the integral

$$\int dy\, \partial[P_4\, g(x)]/\partial t$$

is uniformly convergent. Similarly, one can use Eq. 1.22 to show that the integrand, $\partial[P_4\, g(x)]/\partial t$, is uniformly continuous. Thus the differentiation under the integral sign, as indicated by the statement of the lemma, is justified.

Lemma 3: With the FP operator as defined by Eq. 1.15,

$$dA(t)/dt = \int dy\, FP\,[P_4\, g(x)] \;. \tag{1.23}$$

Proof: By combining the definition of the FP operator with the result of Lemma 2, one finds

$$\int dy\, FP\,[P_4 g(x)] - \frac{dA(t)}{dt} = \sum_{k=1}^{n} \int dy\, \frac{\partial[A_k(y,t)P_4 g(x)]}{\partial y_k}$$
$$-\frac{1}{2}\sum_{k=1}^{n}\sum_{i=1}^{n} \int dy\, \frac{\partial^2[D_{ki}(y,t)P_4 g(x)]}{\partial y_k \partial y_i} \;. \tag{1.24}$$

As in the proof of Lemma 2, the continuity and boundedness of $g(x)$, $g'(x)$, and $g''(x)$ can be used along with the integrability of terms involving the "well-behaved" probability (requirement iii of Definition 3) to show that each of the integrals on the right hand side of Eq. 1.24 is absolutely and uniformly convergent. The absolute convergence justifies an interchange in the order of integration, and the limits stated in requirement vi of Definition 3 shows that each of these integrations is zero. Hence,

$$\int dy\, FP\,[P_4 g(x)] - dA(t)/dt = 0.$$

Lemma 4: With the FP operator as defined by Eq. 1.15, one has

$$FP[P_4 g(x)] = -P_4\, g''(x) \sum_{k=1}^{n}\sum_{i=1}^{n} \tfrac{1}{2} D_{ki}(y,t)\partial x/\partial y_i \partial x/\partial y_k. \tag{1.25}$$

Proof: The proof of this is a straightforward substitution into the definition for the FP operator. One must also utilize the definitions of x and g(x), as well as the fact that both P_3 and P_4 will each satisfy the Fokker-Planck Equation.

Lemmas 1-4 can now be utilized to complete the proof of Theorem 2. From Lemma 1, it is seen that the integral defining A(t) exists, and from Lemma 2, dA(t)/dt exists. Eq. 1.19 of the theorem follows directly from Lemma 1. Equality can hold in Eq. 1.19, if and only if

$$c\int dy\ (x-1)^2\ P_4 = 0,$$

as seen from Lemma 1. As c is positive and the integrand is non-negative, the integrand must be zero. For the integrand to be zero, either $x = 1$ in which case $P_1 = P_2$, or $P_4 = 0$ in which case $P_1 = P_2 = 0$. Thus there is equality in Eq. 1.19 if and only if $P_1 = P_2$.

By using Lemmas 3 and 4, one may express dA(t)/dt as

$$\frac{dA(t)}{dt} = -\int dy\ P_4\ g''(x) \sum_{k=1}^{n}\sum_{i=1}^{n} \tfrac{1}{2}D_{ki}(y,t)\,\partial x/\partial y_k\,\partial x/\partial y_i\ . \qquad (1.26)$$

From the definition of the D_{ki} as incremental second moments, Eq. 1.13, it is seen that the D_{ki} must be elements of a positive definite matrix. Thus, as P_4 and $g''(x)$ are non-negative, and the double sum must be non-negative, it follows that

$$\frac{dA(t)}{dt} \leqslant 0\ ,$$

which is Eq. 1.20 of the Theorem.

1.3.2 LARGE TIME BEHAVIOR OF SOLUTIONS TO THE STEADY FOKKER-PLANCK EQUATION

In most physical problems that lead to a Fokker-Planck Equation, one would intuitively think that the effect of initial conditions would disappear as time went on. Such a tendency can be formally shown in the case of a steady Fokker-Planck Equation, but cannot be rigorously proved. To demonstrate this, an additional theorem is needed.

<u>Theorem 4:</u> Given that P_1 and P_2 are each well-behaved* solutions to a steady** Fokker-Planck Equation, and that $P_1 > 0$. Then equality can hold in Eq. 1.20 of Theorem 2,

$$\frac{dA(t)}{dt} \leqslant 0,$$

if and only if $P_1 = P_2$.

Proof: If $x = 1$, then equality will hold in Eq. 1.20, as $A(t)$ in this case will be identically $g(1)$. If equality holds in Eq. 1.20, then by using Eq. 1.26, one has

$$\int dy\ P_4 g''(x) \sum_{i=1}^{n} \sum_{k=1}^{n} D_{ki}(y,t) \partial x/\partial y_k \partial x/\partial y_i = 0. \qquad (1.27)$$

But as the integrand is non-negative, it must be zero itself. Further as $P_1 > 0$, then $P_4 > 0$. One also has $g''(x) \quad 0$. Thus, Eq. 35 yields the result

$$\sum_{k=1}^{n} \sum_{i=1}^{n} D_{ki}(y,t)\ \partial x/\partial y_k\ \partial x/\partial y_i = 0\ .$$

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* See Definition 3, page 9.

** See Definition 1, page 8.

According to the definition of a "steady" Fokker-Planck Equation, Definition 3, this implies that

$$\partial x/\partial y_k = 0 \quad \text{for } k = m+1,\ m+2, \ldots n. \qquad (1.28)$$

As x is defined by P_3/P_4, the Fokker-Planck Equation may be combined with Eq. 1.28 to yield

$$\partial x/\partial t = -\sum_{k=1}^{m} A_k(y,t)\ \partial x/\partial y_k \ .$$

But, according to the definition of a "steady" Fokker-Planck Equation, this implies that x is a constant. Further, Theorem 1 may be utilized to show that because

$$\int dy\ P_3 = \int dy\ x\ P_4 = x\int dy\ P_4 \ ,$$

the value of x must be one, so that $P_1 = P_2$. Hence equality can hold in Eq. 1.20 if and only if $P_1 = P_2$.

The result of Theorem 4 can now be used to show (formally) the tendency of solutions of a steady Fokker-Planck Equation to have the same asymptotic behavior. Assume that there is at least one solution, $P_1(y,t)$, such that $P_1(y,t) > 0$. Let $P_2(y,t)$ represent any other well-behaved solution. As in Theorem 2, define $g(x)$ and $A(t)$. Then, from Theorems 2 and 4, one has

$$A(t) \geqslant g(1) \ , \qquad (1.29)$$

$$\frac{dA(t)}{dt} \leqslant 0, \qquad (1.30)$$

with equality holding in each case if and only if $P_1 = P_2$.

Rigorously, one can conclude only two facts from this, and these are that the limits of $A(t)$ and $dA(t)/dt$ exist and satisfy

$$\lim_{t\to\infty} A(t) \geqslant g(1), \qquad (1.31)$$

$$\lim_{t\to\infty} dA(t)/dt = 0. \qquad (1.32)$$

if one uses Definition 3 to show continuity of $dA(t)/dt$.

Thus, $A(t)$, which is in a loose way a measure of the differences in P_1 and P_2, will decrease to a limit.

Formally, one might conclude that because $dA(t)/dt$ can be zero if and only if $P_1 = P_2$ and the limit of $dA(t)/dt$ is itself zero, then

$$\lim_{t \to \infty} (P_1 - P_2) = 0. \tag{1.33}$$

This formal conclusion cannot be considered a rigorous proof, but should be considered a heuristic argument.

Eq. 1.33 implies then, that solutions to the steady Fokker-Planck Equation appear the same asymptotically as time goes to infinity, provided that it is known that there is at least one solution P_1 such that $P_1 > 0$.

1.3.3 UNIQUENESS OF STEADY STATE SOLUTIONS TO THE FOKKER-PLANCK EQUATION

When the Markov Process is a stationary process, the Fokker-Planck Equation becomes stationary,* so that the coefficients are independent of time. In this case, it is often possible to find a solution to the Fokker-Planck Equation that does not depend upon time. Such a solution is called a steady state solution.

Definition 4: A <u>steady state</u> solution to a stationary Fokker-Planck Equation is any well-behaved** solution P, such that $\partial P/\partial t = 0$.

It is possible to use the theorems developed thus far to prove that the steady state solution is unique.

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* See Definition 1, page 8.

** See Definition 3, page 9.

Theorem 5: UNIQUENESS: If there is one steady state solution, P, to the stationary*, steady** Fokker-Planck Equation such that

$$P > 0,$$

then P is the only steady state solution to the equation.

Proof: Assume that there are two solutions P and P_2 that are both steady state solutions. As in Theorem 2, define A(t), using $P_1 = P$. As both P and P_2 are independent of time, $dA(t)/dt = 0$. But from Theorem 4, this implies that $P = P_2$.

1.3.4 LARGE TIME BEHAVIOR OF SOLUTIONS TO THE STATIONARY FOKKER-PLANCK EQUATION

In the case of a stationary Fokker-Planck Equation (one whose coefficients are independent of time), it would seem, on the basis of physical reasoning, that the solutions to the Fokker-Planck Equation would approach limits of some sort as time goes to infinity. This is heuristically implied in the case of the steady,** stationary Fokker-Planck Equation where it is known that any steady state solution must be unique.

For example, if there is a steady state solution $P_s(y)$, then by using the formal arguments of Section 1.3.2, one has (formally) that for any well-behaved solution $P(y,t)$

$$\lim_{t \to \infty} P(y,t) = P_s(y) \tag{1.34}$$

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* See Definition 1, page 8

** See Definition 2, page 8

Even when no steady state solution can be found, the fact that, for any T, P(y,t+T) must satisfy the stationary Fokker-Planck Equation if P(y,t) does, may be used with the formal arguments of Section 1.3.2 (Eq. 1.33) to show (formally) that

$$\lim_{t\to\infty} [P(y,t+T) - P(y,t)] = 0 \tag{1.35}$$

If requirement v of the definition of a "well behaved" probability density is strengthened to make $\partial P/\partial t$ uniformly continuous in the open interval from t_1 to infinity, then Eq. 1.35 can be used to show that

$$\lim_{t\to\infty} \frac{\partial P(y,t)}{\partial t} = 0 \ .$$

This implies in a formal way that P(y,t) approaches a limit as t goes to infinity. If this limit happens to be a solution to the Fokker-Planck Equation, and is non-zero for all y, then it is possible to continue the formal arguments to show formally that this limit is the steady state solution. Similarly, if this limit is zero, for all y, it can be formally shown that there is no steady state solution (in the sense of Definition 4). In such a case, regardless of the initial conditions, the system will diverge (y will go to infinity in probability).

If the stationary Fokker-Planck Equation is non-steady, very few conclusions can be drawn, as neither the formal results of Section 1.3.2 nor the uniqueness theorem for the steady state, Theorem 4, are applicable. Such situations arise physically when there are points or regions in phase space that are either unattainable or "traps" so that

$P_T(y/x,t) = 0$ for certain values of x, certain values of y, and all t.

A singular point in the Fokker-Planck Equation (a y for which all of the $D_{ki}(y)$ vanish) is often an indication of a "trap." For example, if at a point $y = z$, all of the coefficients $A_k(y)$ and $D_{ki}(y)$ vanish, then a possible formal solution to the differential equation is

$$P(y,t) = \delta(y-z).$$

If there is more than one such point, it is obvious that there is nothing unique about time independent (formal) solutions to the stationary Fokker-Planck Equation.

The behavior of solutions to the stationary equation, where there is only one such "trap" is discussed by Khas'minskii (4), where a method analogous to Liapunov's second method is used to obtain necessary and sufficient conditions for "stability" of the Markov Process. However, in the treatment by Khas'minskii, it is assumed that the $D_{ki}(y)$ are the elements of a strictly positive definite matrix for $y \neq 0$. This requirement is too stringent for most of the physical problems that utilize the Fokker-Planck Equation, and hence no discussion of his results will be given here.

1.4.0 SUMMARY

In this section the basic idea of a continuous Markov process, its transition probability, and the Fokker-Planck Equation has been introduced. It has been shown that solutions to the Fokker-Planck Equation with prescribed initial conditions are unique, and, for the special case of a steady stationary Fokker-Planck Equation, the steady state solutions are unique.

It has been demonstrated that well-behaved solutions to the steady Fokker-Planck Equation have a tendency to converge towards each other, in that a difference between any pair of solutions can be "measured," as in Eq. 1.18 of Theorem 2, by a function of time, A(t), which can only decrease. This leads in a formal way, as described in Section 1.3.2, to the heuristic conclusion that any pair of solutions to the steady Fokker-Planck Equation will asymptotically behave the same. A problem yet to be solved is that of finding the restrictions on the coefficients of the Fokker-Planck Equation for which it can be rigorously proved that all solutions will asymptotically behave the same.

Another unsolved problem is that of determining the possible existence of steady state solutions to the Fokker-Planck Equation. Does a stationary steady Fokker-Planck Equation always have a steady state solution?

Finally we come to the relation of the Fokker-Planck Equation and the Markov Process to physical problems, and also the application of the Fokker-Planck Equation for obtaining statistical results. The type of system which yeilds a Markov Process as a solution, as well as the derivation of the coefficients of the Fokker-Planck Equation, is discussed in the following section.

2. THE MARKOV PROCESS AS GENERATED BY DIFFERENTIAL EQUATIONS WITH RANDOM COEFFICIENTS

2.1.0 INTRODUCTION

In Part 1, the Markov Process was treated from the point of view of the Fokker-Planck Equation alone, and no discussion was presented treating the process itself. In this part, it will be demonstrated that a system of differential equations can define a Markov Process, and the Fokker-Planck Equation for such a process will be derived.

Particular emphasis is given to a discussion of the differing results of various authors in the case of "parametric white noise." These differing results have led to a controversy concerning the coefficients A_K of the Fokker-Planck Equation, Eq. 1.14, even in some of the simplest examples.

Among the examples given will be the general linear differential equation with "parametric white noise," and an "equivalent" linear differential equation with no parametric white noise will be derived.

2.2.0 DISCUSSION OF WHITE NOISE

The classical example of an n-dimensional Markov Process is a system of differential equations in which random parameters are present and satisfy certain restrictions. These restrictions lead to a definition of Gaussian "White Noise," or as referred in some literature, the "Formal Derivative of a Wiener Process." Though Gaussian White Noise is often only considered as being a fictitious entity, it is a convenient tool for representing Gaussian Noise whose correlation time is far smaller than the smallest characteristic time of a system being analyzed.

Consider the n first order differential equations of the form

$$y_k' = a_k(y,t) + \sum_{i=1}^{m} h_{ki}(y,t)\, n_i(t) \;, \tag{2.1}$$

where k ranges from 1 to n, $a_k(y,t)$ and $h_{ki}(y,t)$ are known functions of y and t, and the $n_i(t)$ are stationary random variables such that $n_i(t)$ and $n_j(t_1)$ are completely independent when $t \neq t_1$ (for all possible i and j). Assume further that the statistical properties of the $n_i(t)$ are known. In this case, if information about y (the n dimensional vector whose components are y_k) is desired at time t, and its value at time t_o is known, <u>no</u> additional information is obtained by knowledge of y at any time prior to t_o. This would not be true if there was any correlation between the $n_i(t)$ at two different times. Hence, only the first conditional probability density is needed to describe the system, and the process is an n-dimensional, first order Markov Process.

The $n_i(t)$, which are stationary and independent of themselves and each other at differing times, are called white noise. Without loss of generality, it can be assumed that they have a zero mean. Using brackets, $\langle\ \rangle$, to denote the expectation or mean, one can express these properties as

$$\langle n_i(t) \rangle \;=\; 0, \tag{2.2a}$$

$$\langle n_i(t) n_j(t_1) \rangle = \;0 \;\text{ for } t \neq t_1. \tag{2.2b}$$

Thus, the correlation functions $\varphi_{ij}(t)$, where

$$\varphi_{ij}(t) = \langle n_i(s)\, n_j(t+s)\rangle \quad , \tag{2.3}$$

must be zero, except perhaps when the argument is zero. If the correlation function existed for zero argument, then the well-known Wiener-Knintchine relations* could be used to show that the power density spectrum of the noise is zero. In this case, the noise will be considered to be trivial. If, however, one desires non-trivial white noise, so that the power density spectrum is not zero, then impulse functions are necessary in the correlation function.

Thus, the correlation functions for white noise will be of the form

$$\varphi_{ij}(t) = 2\, D_{ij}\, \delta(t) \ , \tag{2.4}$$

so that the power density spectrum will be a constant. From this it is obvious that the mean square of non-trivial white noise is infinite.

It must be noted that satisfying Eq. 2.4 is not sufficient to describe "white noise." This is because a lack of correlation does not necessarily imply independence. If, however, it is required that the $n_i(t)$ be Gaussian, then a lack of correlation does imply independence, and Eq. 2.4 is sufficient to describe Gaussian White Noise.

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* These relate the correlation functions to power density spectrums through the use of the Fourier Cosine Transform.

2.2.1 THE CONTROVERSY IN THE DETERMINATION OF COEFFICIENTS FOR THE FOKKER-PLANCK EQUATION

As shown in Part 1 of this thesis, to calculate the coefficients in the Fokker-Planck Equation, it is necessary to compute the expectation or mean of the moments of the incremental changes in the y_k, Δy_k, where

$$\Delta y_k = y_k\,(t + \Delta t) - y_k(t) \ ,$$

and the $y_k(t)$ are known. Two techniques have been utilized so far, neither of which is rigorous, and only one of which yields consistent results.

The first, as illustrated by Bogdanoff and Kozin (6), involves a free interchange of incremental and differential changes. For example, in attempting to calculate the coefficients $A_k(y,t)$ in the Fokker-Planck Equation, where

$$A_k(y,t) \ = \ \lim_{\Delta t \to 0} \frac{\langle \Delta y_k \rangle}{\Delta t} \ , \tag{2.5}$$

Eq. 2.1 would be used to show that for the system defined by Eq. 2.1

$$\langle dy_k \rangle \ = \ a_k(y,t)\, dt,$$

as the $n_i(t)$ each have zero means. Thus, if one assumes that the interchange of increments and differentials is allowable, it would appear that

$$A_k(y,t) = a_k(y,t) \ .$$

It may be noted that this is equivalent to assuming that

$$\lim_{\Delta t \to 0} \left\langle \frac{\Delta y_k}{\Delta t} \right\rangle = \left\langle \lim_{\Delta t \to 0} \frac{\Delta y_k}{\Delta t} \right\rangle = \left\langle \frac{dy_k}{dt} \right\rangle .$$

The basic fallacy here is the assumption that such interchanges may be done. The inconsistencies arising from such an interchange will be pointed out in the example to be treated in Section 2.2.2.

The second technique, though often expressed in many different ways, always involves an integration of the Gaussian White Noise. This is not surprising, as Δy_k is given by

$$\Delta y_k = y_k(t+\Delta t) - y_k(t) = \int_t^{t+\Delta t} y_k'(t_1)\, dt_1 ,$$

and the equation for y_k', Eq. 2.1, involves the noise terms. Mr. Kozin (7) has pointed out that white noise may not be integrated, as it does not exist in the mean square sense. Rigorously, if one accepts the unintegrability of white noise, the incremental changes Δy_k cannot in general be found and the Fokker-Planck Equation cannot be obtained. Formally, however, the integral of white noise is simply the Wiener Process, and can be treated as such. It might be noted that Kozin, in conjunction with Bogdanoff (6), has utilized the term "Formal Derivative of a Wiener Process" to describe the white noise. If the term "formal derivative" is used to imply the inverse of an integration, then white noise is integrable, and its integral is a Wiener Process.

To avoid this difficulty concerning the integrability of white noise, one may either ignore the fact of its lack of integrability and use the results of a formal integration, or one may use Gaussian

Noise whose correlation time is far smaller than any characteristic time of the system. Physically it would seem quite reasonable to assume that a system could not differentiate between pure "Gaussian White Noise" and Gaussian Noise with an extremely short correlation time, any more than the naked eye could distinguish between a point source of light and a small light source if both are at a great distance from the observer. This is a difficult thing to justify mathematically, and so will be demonstrated in an example in Section 2.2.2, and postulated in the more general derivation to follow in Section 2.3.

Approaching the problem from this latter viewpoint yields results that are consistent with each other and physically meaningful. Approaching the problem from the former viewpoint (interchanging of differentials and increments) yields inconsistent results. Both approaches yield the same results in the case where

$$\sum_{k=1}^{n} h_{kr}(y,t)\,\frac{\partial h_{ij}(y,t)}{\partial y_k} = 0 \;,$$

which takes in a large number of physical problems. It is possible, in these cases where both viewpoints yield the same result, to change variables to obtain an apparently different problem for which the two viewpoints yield differing results. It is only the latter viewpoint that yields results consistent with the variable change in these cases. The following example will demonstrate this (see Section 2.2.2.1).

2.2.2 THE FIRST ORDER LINEAR SYSTEM WITH PARAMETRIC WHITE EXCITATION

Because it illustrates a number of points just discussed, and because it has been the source of some controversy, the first order linear system with parametric white excitation will be discussed here.

Consider the system whose differential equation is given by

$$y'(t) + a_o\, y(t) - n(t)\, y(t) = 0, \tag{2.6}$$

where a_o is a constant and $n(t)$ is Gaussian White Noise with a correlation function given by

$$\langle n(t)\, n(t_o) \rangle = 2\, D\delta(t-t_o) , \tag{2.7}$$

or, as some authors prefer, $n(t)$ is the formal time derivative of a Wiener Process, $z(t)$, for which

$$\langle [z(t) - z(t_o)]^2 \rangle = 2\, D\, |t-t_o| . \tag{2.8}$$

From the earlier discussion, it is seen that $y(t)$ is the result of a one-dimensional continuous Markov Process. From Part 1 of this thesis, it is seen that the Fokker-Planck Equation (assuming there is one) must be of the form

$$\frac{\partial P}{\partial t} = - \frac{\partial a(y)P}{\partial y} + \frac{1}{2} \frac{\partial^2 b(y)P}{\partial y^2} , \tag{2.9}$$

where the coefficients $b(y)$ and $a(y)$ are to be determined by the limits

$$a(y) = \lim_{\Delta t \to 0} \frac{\langle \Delta y \rangle}{\Delta t} , \tag{2.10}$$

$$b(y) = \lim_{\Delta t \to 0} \frac{\langle (\Delta y)^2 \rangle}{\Delta t} . \tag{2.11}$$

If one assumes a free interchange of differentials and increments, then he may obtain the results:*

$$a(y) = -a_o y,$$
$$b(y) = 2Dy^2 \quad . \qquad (2.12)$$

However, as mentioned earlier, this cannot be justified. If one utilizes a formal integration of the white noise, or treats it as non-white and then lets it approach white noise (before taking the limits indicated in Eqs. 2.10 and 2.11), he will obtain the results:**

$$a(y) = (D - a_o)y \; ,$$
$$b(y) = 2Dy^2 \quad . \qquad (2.13)$$

As there are faults to be found in either derivation, the author will attempt here to demonstrate the validity of the latter result (Eq. 2.13) by observing the implications of the derivation, and its relation to known results. It will first be shown that a change of variables in the well-known Brownian Motion problem leads to inconsistencies in the former result, and secondly it shall be shown that by using a non-white n(t) and letting it approach Gaussian White Noise also leads to inconsistencies in the former result.

2.2.2.1 Parametric Excitation Obtained by a Variable Change in the Brownian Motion Problem

One of the first applications of the Fokker-Planck Equation was the subject of Brownian Motion of free particles. In the presence of

* See for example Bogdanoff and Kozin (8).

** See for example Caughey and Dienes (9).

no damping, the velocity of the particles will satisfy the Langevin Equation,

$$\frac{dx}{dt} = n(t) \ , \tag{2.14}$$

where n(t) is the white noise described by Eq. 2.7. The transition probability is known to have a Fokker-Planck Equation identical to the one-dimensional diffusion equation,

$$\frac{\partial P_1}{\partial t} = D \frac{\partial^2 P_1}{\partial x^2} \ , \tag{2.15}$$

and the transition probability is Gaussian and given by

$$P_1(x/x_o, t) = \frac{1}{\sqrt{4\pi D t}} \exp\left[- \frac{(x-x_o)^2}{4Dt} \right] . \tag{2.16}$$

If one now changes to a new variable y, where the change of variables is given as

$$y = y_o \exp(-x + x_o - a_o t) \ , \tag{2.17}$$

then one formally obtains as a differential equation for y the result

$$y' + a_o y + n(t)\, y = 0 \ , \tag{2.18}$$

which is identical to Eq. 2.6. Is such a procedure justified? The only argument against it might lie in the differentiability of y. However, y can be no less differentiable than x, and thus Eq. 2.18 is satisfied to the same degree that Eq. 2.14 is.

The change of variables of Eq. 2.17 can be used with elementary techniques to find the transition probability for y,

$$P_2(y/y_o,t) = \frac{1}{\sqrt{4\pi Dt}\ |y|} \exp\left\{-\frac{[\log(y/y_o) + a_o t]^2}{4Dt}\right\} \qquad (2.19)$$

for $y/y_o > 0$ and zero for $y/y_o < 0$. Similarly the Fokker-Planck Equation in the new variable can be found to be

$$\frac{\partial P_2}{\partial t} = \frac{\partial (a_o - D)yP_2}{\partial y} + \frac{\partial^2 Dy^2 P_2}{\partial y^2} \quad . \qquad (2.20)$$

A direct comparison of Eq. 2.20 with the results obtained by an integration of white noise (see Eq. 2.13) shows they are identical, and a comparison with results obtained by interchanging of differentials and increments (see Eq. 2.12) shows a distinct inconsistency.

2.2.2.2 Almost White Parametric Noise:

Consider the system defined by the differential equation

$$y' + a_o y + n(t)y = 0 \ , \qquad (2.21)$$

where the n(t) is not white, but a Gaussian variable with zero mean, and an autocorrelation function $\varphi(t)$ where

$$\varphi(t) = \langle n(t+s)n(s) \rangle \ . \qquad (2.22)$$

It is possible to solve explicitly Eq. 2.21 for y, and if its initial value is y_o at time zero, then the Gaussian character of log (y/y_o) can be used to find the conditional probability density for y (there is no transition probability in this case). Thus, if one denotes by u(t) the variance of $\log(y/y_o)$, then

$$u(t) = \int_0^t \int_0^t \varphi(t_1 - t_2)\, dt_1\, dt_2 = 2 \int_0^t dt_1 \int_0^{t_1} dt_2\, \varphi(t_2), \tag{2.23}$$

and the conditional probability is

$$P(y/y_o, t) = \frac{1}{\sqrt{2\pi u(t)}\ |y|} \exp\left\{ - \frac{[\log(y/y_o) + a_o t]^2}{2\, u(t)} \right\}, \tag{2.24}$$

for $y/y_o > 0$, and zero for $y/y_o < 0$.

A direct substitution shows that this conditional probability density will satisfy the differential equation

$$\frac{\partial P}{\partial t} = \frac{\partial[a_o - \frac{1}{2}u'(t)]\, y\, P}{\partial y} + \frac{\partial^2[\frac{1}{2}u'(t)\, y^2\, P]}{\partial y^2} \tag{2.25}$$

As discussed in Part 1 of this thesis, it may be noted that for times large enough to make the correlation, $\varphi(t)$, negligible, this conditional probability density becomes a transition probability. Thus, one might say that Eq. 2.25 represents a Fokker-Planck Equation for the process, when times are much larger than the correlation time.

For purposes of comparison, let the correlation function, $\varphi(t)$, be expressed as

$$\varphi(t) = 2\, D\psi(t)\ , \tag{2.26}$$

where $\psi(t)$ is "almost" a unit impulse. To be more specific,

$$\int_{-\infty}^{\infty} \psi(t)\, dt = 1$$

and if $e_1(t)$ and $e_2(t)$ are defined by

$$e_1(t) = 2\int_t^{\infty} \psi(t_1)\, dt_1 \ , \tag{2.27}$$

$$e_2(t) = \frac{1}{t}\int_0^{t} e_1(t_1)\, dt_1 \ , \tag{2.28}$$

then there is a "correlation time," t_c, such that for $t >> t_c$, one has

$$|e_1(t)| << 1 \ ,$$

and

$$|e_2(t)| << 1.$$

By using the definitions of Eqs. 2.26-2.28, one can show that u(t) and u'(t) are given by

$$u(t) = 2Dt\,[1 - e_2(t)] \ ,$$

$$u'(t) = 2D\,[1 - e_1(t)] \ .$$

Thus for t much larger than the correlation time, t_c, the conditional probability density, P(y,t), becomes a transition probability given by

$$P_T(y/y_o, t) = \frac{1}{\sqrt{4\pi Dt}\ |y|} \ \exp\left\{ - \frac{[\log(y/y_o) + a_o t]^2}{4\,D\,t} \right\} \ , \tag{2.29}$$

when $y/y_o > 0$ and zero for $y/y_o < 0$. Further the Fokker-Planck Equation for this transition probability is given by

$$\frac{\partial P_T}{\partial t} = \frac{\partial (a_o - D) P_T}{2y} + \frac{\partial^2 D\, y^2\, P_T}{\partial y^2} \ . \tag{2.30}$$

The results of Eqs. 2.29 and 2.30 are identical to those found for white noise, when the coefficients in the Fokker-Planck Equation are based on an integration.

It should be pointed out that, in the example just discussed, no characteristic time of the system arose. This is not true in general. Similarly, in general, the definition of an "almost white" function would have to be adapted to suit the problem. However, the main notion of an almost white function remains as one for which the correlation time is small.

2.3.0 A GENERAL EXAMPLE OF AN N-DIMENSIONAL MARKOV PROCESS

As discussed in Section 2.2.0, the system of n differential equations, given by

$$y_k' = a_k(y,t) - \sum_{i=1}^{m} h_{ki}(y,t)\, n_i(t) \, , \tag{2.31}$$

where k runs from one to n, $a_k(y,t)$ and $h_{ki}(y,t)$ are known functions of y and t, and the $n_i(t)$ are Gaussian White Noise having correlation functions

$$\langle n_i(t)\, n_j(t_1) \rangle = 2B_{ij}\, \delta(t-t_1) \, , \tag{2.32}$$

defines an n-dimensional Markov Process. The Fokker-Planck Equation for this process will be given by Eq. 1.14, rewritten here as Eq. 2.33,

$$\frac{\partial P}{\partial t} = -\sum_{k=1}^{n} \frac{\partial A_k(y,t)P}{\partial y_k} + \tfrac{1}{2}\sum_{k=1}^{n}\sum_{i=1}^{n} \frac{\partial^2 D_{ki}(y,t)\, P}{\partial y_k \partial y_i} \quad . \tag{2.33}$$

It will be shown here that the coefficients $A_k(y,t)$ and $D_{ki}(y,t)$ are given by the relations

$$A_k(y,t) = a_k(y,t) + \sum_{i=1}^{n}\sum_{j=1}^{m}\sum_{r=1}^{m} B_{rj}h_{ir}(y,t)\frac{\partial h_{kj}(y,t)}{\partial y_i}, \tag{2.34}$$

and

$$D_{ki}(y,t) = 2\sum_{j=1}^{m}\sum_{r=1}^{m} B_{rj}\, h_{ir}\,(y,t)\, h_{kj}(y,t)\,. \tag{2.35}$$

Some authors in deriving the coefficients for the Fokker-Planck Equation, have found $A_k(y,t)$ to be simply $a_k(y,t)$, as discussed in Section 2.2.0. The effects of such a result were discussed in that section, and will not be further treated here. Similar results to those given by Eqs. 2.34 and 2.35 were found for simpler problems by Dienes (1), Caughey and Dienes (9), and Leibowitz (10). Each of these authors used a different approach to obtain their result.

Using summation convention, so that repeated indices imply a summation over all possible values of the index, one may combine Eqs. 2.33-2.35 to give the Fokker-Planck Equation

$$\begin{aligned}\frac{\partial P}{\partial t} = &-\frac{\partial a_k(y,t)P}{\partial y_k} - B_{rj}\frac{\partial}{\partial y_k}\left\{h_{ir}(y,t)\left[\frac{\partial h_{kj}(y,t)}{\partial y_i}\right]P\right\} \\ &+ B_{rj}\frac{\partial^2}{\partial y_k \partial y_i}\left[h_{ri}(y,t)\, h_{jk}(y,t)\, P\right]\,.\end{aligned} \tag{2.36}$$

For some purposes, it is convenient to combine the second and third terms on the right hand side of Eq. 2.36, and write the Fokker-Planck Equation as

$$\frac{\partial P}{\partial t} = -\frac{\partial [a_k(y,t)P]}{\partial y_k} + B_{rj}\frac{\partial}{\partial y_k}\left\{h_{kj}(y,t)\frac{\partial[h_{ir}(y,t)P]}{\partial y_i}\right\} , \qquad (2.37)$$

where summation convention is again applied.

The derivation of the Fokker-Planck Equation for the system of Eq. 2.31 will be effected by first calculating the incremental moments when the noise terms, the $n_i(t)$, are Gaussian but not white. The results obtained from this are unambiguous, and either procedure discussed in the section concerning the controversy of the coefficients, Section 2.2.0, will yield the same results. Secondly, the noise will be allowed to approach Gaussian White Noise, and the results of this limiting procedure will be used to find the limits defining the $A_k(y,t)$ and the $D_{ki}(y,t)$.

An alternate derivation, utilizing "crossing times" may be found in Dienes, Reference (9). It has the advantage of deriving the coefficients without any calculation of incremental moments as is done in most such derivations.

2.3.1 ALMOST WHITE PARAMETRIC NOISE

Consider the n + m dimensional, first order Markov Process defined by the system of equations

$$y_k' = a_k + \sum_{i=1}^{m} h_{ki}\, n_i \qquad \text{for } k = 1,2,\ldots n, \qquad (2.38a)$$

$$\tau n_i' + n_i = d_i(t) \qquad \text{for } i = 1,2,\ldots m, \qquad (2.38b)$$

where the a_k and h_{ki} may be functions of y and t, and the d_i are White Gaussian Noise such that

$$\langle d_i(t) d_j(t_1) \rangle = 2 B_{ij} \delta(t-t_1) . \tag{2.39}$$

As the time constant τ goes to zero, the $n_i(t)$ will approach white noise. Standard procedures will yield the Fokker-Planck Equation for this system with $\tau \neq 0$, and an identical result will be obtained by either of the two procedures discussed in Section 2.2. For this reason, the derivation is omitted, and only the result is given here. Using summation convention, this is

$$\frac{\partial P}{\partial t} = \frac{-\partial[(a_k + h_{ki} n_i) P]}{\partial y_k} + \frac{\partial(n_i P)}{\tau \, \partial n_i} + \frac{B_{ik}}{\tau^2} \frac{\partial^2 P}{\partial n_i \partial n_k} . \tag{2.40}$$

The expectation or mean of a function of y, n, and t, is given by the n + m fold integral

$$\langle f(y,n,t) \rangle = \int\int \cdots \int f(y,n,t) \, P \, dy_1 dy_2 \cdots dy_n dn_1 dn_2 \cdots dn_m .$$

If Eq. 2.40 is multiplied by f(y,n,t), and the resulting equation is integrated over all y and n, then integration by parts can be formally used to derive the equation

$$\frac{d\langle f \rangle}{dt} = \langle \frac{\partial f}{\partial t} \rangle + \langle (a_k + h_{ki} n_i) \frac{\partial f}{\partial y_k} \rangle - \frac{1}{\tau} \langle n_i \frac{\partial f}{\partial n_i} \rangle + \frac{D_{ik}}{\tau^2} \langle \frac{\partial^2 f}{\partial n_i \partial n_k} \rangle . \tag{2.41}$$

It is this equation which will be used to derive incremental moments.

Eq. 2.41 and the Gaussian nature of the n_i will be used to show that when g(y) is a known function of the y_k satisfying certain requirements, then for sufficiently small τ and Δt, one has

$$\frac{\langle g\rangle_{t+\Delta t}-\langle g\rangle_t}{\Delta t} = \langle a_k \frac{\partial g}{\partial y_k} + B_{rj} h_{ir} \frac{\partial}{\partial y_i} h_{kj} \frac{\partial g}{\partial y_k} \rangle_t$$
$$+ 0(1) \exp(-\Delta t/\tau) + 0(1)\, \tau^{\frac{1}{2}} + 0(1)\, \Delta t \ , \qquad (2.42)$$

where the subscripts on the brackets denote the time at which the mean is taken, and the terms written as 0(1) denote bounded terms. This equation, Eq. 2.42, will be derived in Section 2.3.2.

Once Eq. 2.42 is derived, the coefficients in the Fokker-Planck Equation follow naturally. For if the time constant τ is allowed to approach zero, thus making the $n_k(t)$ approach those of the original problem defined by Eqs. 2.31 and 2.32, then one finds that for the original problem,

$$\frac{\langle g\rangle_{t+\Delta t}-\langle g\rangle_t}{\Delta t} = \langle a_k \frac{\partial g}{\partial y_k} + B_{rj} h_{ir} \frac{\partial}{\partial y_i} h_{kj} \frac{\partial g}{\partial y_k} \rangle_t$$
$$+ 0(1)\, \Delta t \ .$$

Hence, if the value of the y_k at time t is known, so that the brackets may be removed when followed by the subscript t, one has, after taking the limit as Δt goes to zero,

$$\lim_{\Delta t\to 0} \frac{\langle g\rangle_{t+\Delta t}-g}{\Delta t} = a_k \frac{\partial g}{\partial y_k} + B_{rj}\, h_{ir} \frac{\partial}{\partial y_i} h_{kj} \frac{\partial g}{\partial y_k} \ . \qquad (2.43)$$

If one denotes Δy_k by the expression

$$\Delta y_k = y_k - y_k(t) = y_k(t + \Delta t) - y_k(t) ,$$

then Eq. 2.43 can be used with a g(y) given by

$$g(y) = \Delta y_i \, \Delta y_j \, \Delta y_k \, . \, . \, . \, \Delta y_p ,$$

to show that

$$A_k(y,t) = \lim_{\Delta t \to 0} \frac{\langle \Delta y_k \rangle}{\Delta t} = a_k + B_{rj} h_{ir} \frac{\partial h_{kj}}{\partial y_i} , \qquad (2.44a)$$

$$D_{ki}(y,t) = \lim_{\Delta t \to 0} \frac{\langle \Delta y_i \Delta y_k \rangle}{\Delta t} = 2B_{rj} h_{ir} h_{kj} , \qquad (2.44b)$$

and all limits of higher moments, such as

$$\lim_{\Delta t \to 0} \frac{\langle \Delta y_i \, \Delta y_k \, \Delta y_j \rangle}{\Delta t}$$

will be zero. Thus one finds the coefficients and the Fokker-Planck Equation as given by Eqs. 2.34-2.37.

2.3.2 DERIVATION OF EQUATION 2.42

The basic assumption utilized for this derivation is that well-behaved functions of y and t have a mean square that is bounded for some small interval of time following the application of initial conditions. To be more explicit, the following definition is given:

Definition: A function of y and t, f(y,t), is said to be MSB (mean square bounded) if there exists a bound A and a time T_f such that for

$$t_o \leq t \leq t_o + T_f ,$$

and

$$\tau \leq T_f ,$$

one has

$$\langle f^2(y,t)\rangle \leq A ,$$

provided that y has a finite initial value at t_o. Both A and T_f may depend upon t_o, the functional form of $f(y,t)$, and the initial values of y.

It is assumed now that g(y) is sufficiently well behaved so that each of the following is MSB (in the sense of the preceding definition):

i. g(y),

ii. a_k,

iii. h_{ki},

iv. The first five partial derivatives of items i, ii, and iii, with respect to the variables t and the y_k.

v. Products of each of the preceding items up to and including five terms.

This assumption is not unreasonable, in that the only g(y)'s for which it need hold are polynomials in y. In general the a_k and h_{ki} will be well-behaved functions of the y and t, thus it would be an extraordinary situation if the assumption were not satisfied.

The proof to follow will be divided into seven parts.

(1) By first noting that the variances of the $n_i(t)$ in the problem defined by Eqs. 2.38 and 2.39 are of the order of $1/\tau$, or

$$\langle n_i^2\rangle = 0(1)/\tau ,$$

and utilizing the Gaussian nature of the n_i terms, one may state that if f(y,t) is MSB, then for sufficiently small $t-t_o$ (where t_o is the initial time at which finite initial conditions are applied) and sufficiently small τ, one has

$$\langle f(y,t)\, n_i\rangle = 0(1)/\tau^{\frac{1}{2}} , \tag{2.45a}$$

$$\langle f(y,t)\, n_i\, n_j\rangle = 0(1)/\tau , \tag{2.45b}$$

$$\langle f(y,t)\, n_i\, n_j\, n_k\rangle = 0(1)/\tau^{3/2}, \text{ etc.} \tag{2.45c}$$

(2) If g(y) satisfies the basic assumption, then each of the following is MSB:

i. $g(y)$

ii. $g_i(y,t) = h_{ki} \partial g/\partial y_k$

iii. $g_{ri}(y,t) = h_{sr} \partial g_i/\partial y_s$.

Further, if $G(y,t)$ is used to denote either $g(y)$, $g_i(y,t)$ or $g_{ri}(y,t)$, then each of the following is MSB:

iv. $a_j \ \partial G/\partial y_j$,

v. $h_{jp} \ \partial G/\partial y_j$,

vi. $\partial G/\partial t$.

Each of the above follows from the basic assumption following the definition of MSB.

(3) Using the mean equation, Eq. 2.41, one may write

$$\frac{d}{dt} \langle n_v G \rangle + \frac{1}{\tau} \langle n_v G \rangle = \langle n_v \ \partial G/\partial t \rangle$$

$$+ \langle n_v \ a_j \ \partial G/\partial y_j \rangle + \langle n_v \ n_i \ h_{ki} \ \partial G/\partial y_k \rangle \ .$$

Thus, using the results of Eqs. 2.45, this becomes

$$\frac{d}{dt} \langle n_v G \rangle + \frac{1}{\tau} \langle n_v G \rangle = 0(1)/\tau \quad ,$$

for sufficiently small $t\text{-}t_o$ and τ. Thus, for sufficiently small $t\text{-}t_o$ and τ, one has

$$\langle n_v G \rangle_t = \langle n_v G \rangle_{to} \ \exp[-(t-t_o)/\tau] \quad ,$$

or, using the finite initial conditions, simply

$$\langle n_v G \rangle_t = 0(1) \ , \tag{2.46}$$

for sufficiently small $t\text{-}t_o$ and τ.

(4) As in the derivation of Eq. 2.47, one may write,

$$\frac{d\langle G \rangle}{dt} = 0(1) \quad ,$$

by utilizing Eqs. 2.41, 2.45, and 2.46. This then leads to the result

$$\langle G\rangle_t = \langle G\rangle_{t_o} + (t-t_o)0(1) , \qquad (2.47)$$

for sufficiently small $t-t_o$ and τ.

(5) Similarly, by using Eqs. 2.41, 2.45, 2.46, and 2.47, one can show that

$$\langle n_b\, n_v\, G\rangle_t = B_{bv} \langle G\rangle_{t_o} + 0(1)/\tau^{\frac{1}{2}} + 0(1)(t-t_o)/\tau + 0(1)\{\exp[-2(t-t_o)/\tau]\}/\tau , \qquad (2.48)$$

for sufficiently small $t-t_o$ and τ.

(6) In a similar manner, but now including the results of Eq. 2.48, one can show that

$$\langle n_p G\rangle_t = B_{iq}\langle h_{ki}\, \partial G/\partial y_k\rangle_{t_o} + 0(1)\tau^{\frac{1}{2}} + (t-t_o)0(1) + 0(1)\exp[-(t-t_o)/\tau] . \qquad (2.49)$$

for sufficiently small $t-t_o$ and τ.

(7) Finally, by combining the results of Eq. 2.41 and Eqs. 2.45-2.49, one arrives at the result

$$\langle g\rangle_t = \langle g\rangle_{t_o} + (t-t_o)\langle a_k \frac{\partial g}{\partial y_k} + B_{rj}\, h_{ir} \frac{\partial}{\partial y_i} h_{kj} \frac{\partial g}{\partial y_k}\rangle_{t_o} + 0(1)(t-t_o)\exp[-(t-t_o)/\tau] + 0(1)(t-t_o)\tau^{\frac{1}{2}} + (t-t_o)^2\, 0(1) ,$$

for sufficiently small $t-t_o$ and τ. If one replaces t by $t + \Delta t$, and t_o by t, this yields Eq. 2.42, as desired.

2.4.0 A LINEAR SYSTEM WITH PARAMETRIC WHITE NOISE

Consider the (n + 1) th order linear differential equation given by

$$\frac{d^{n+1}y}{dt^{n+1}} + \sum_{k=0}^{n} [b_k + a_k(t)] \frac{d^k y}{dt^k} = a_d(t) + f(t), \tag{2.50}$$

where the b_k are constants, f(t) is a deterministic signal possessing a power density spectrum, and the a(t) are Gaussian White Noise such that

$$\langle a_i(t) \rangle = 0, \tag{2.51}$$

$$\langle a_i(t) a_k(t_1) \rangle = 2D_{ik}\,\delta(t-t_1) . \tag{2.52}$$

The Fokker-Planck Equation will be utilized to analyze the mean and mean square of y, as well as the autocorrelation function. In particular, the mean will be determined by an (n + 1) th order differential equation,

$$\frac{d^{n+1}\langle y \rangle}{dt^{n+1}} + \sum_{k=0}^{n} (b_k - D_{kn}) \frac{d^k \langle y \rangle}{dt^k} = f(t) - D_{dn}, \tag{2.53}$$

and the variance will be determined by a system of $\frac{1}{2}(n+1)(n+2)$ first order linear differential equations with constant coefficients. Hence the stability of the mean and variance can be ascertained by the standard technique of finding the sign of the real part of the roots of an (n+1) th and a $\frac{1}{2}(n+1)(n+2)$ th polynomial. It is the mean equation, Eq. 2.53, that is the basis for the controversy discussed in Section 2.2.

Further, it will be shown that when the system is mean square stable, it possesses the same mean and the same "average" power density spectrum as the output of the linear system,

$$\frac{d^{n+1}x}{dt^{n+1}} + \sum_{k=0}^{n} (b_k - D_{kn}) \frac{d^k x}{dt^k} = f(t) - D_{dn} + a(t), \qquad (2.54)$$

where a(t) is Gaussian White Noise, with an autocorrelation function of the form $2A\delta(t)$, and the constant A is determined by the coefficients of the original differential equation and the D_{ki}. This "equivalent" system can be analyzed using standard Fourier techniques.

2.4.1 THE FOKKER-PLANCK AND MOMENT EQUATIONS

The differential equation for y, Eq. 2.50, can be expressed as n first order differential equations by defining y_k as

$$y_k = \frac{d^k y}{dt^k} \quad k = 0, 1, 2, \ldots, n. \qquad (2.55)$$

Thus one may write

$$y_k' = y_{k+1} \text{ for } k = 0, 1, 2, \ldots, n-1, \qquad (2.56a)$$

$$y_n' = -\sum_{k=0}^{n} [b_k + a_k(t)]\, y_k + a_d(t) + f(t) . \qquad (2.56b)$$

As described in Section 2.1.0, this represents a Markov Process in n + 1 dimensions, and utilizing the techniques of Section 2.3, one can write the Fokker-Planck Equation as

$$\frac{\partial P}{\partial t} = -\sum_{k=0}^{n-1} \frac{\partial (y_{k+1} P)}{\partial y_k} - \frac{\partial}{\partial y_n} \left[f(t)P - \sum_{k=0}^{n} (b_k - D_{nk}) y_k P - D_{dn} P \right]$$

$$+ \frac{\partial^2}{\partial {y_n}^2} \left\{ \left[\sum_{k=0}^{n} \sum_{i=0}^{n} D_{ki} y_k y_i - 2 \sum_{k=0}^{n} D_{dk} y_k + D_{dd} \right] P \right\} . \qquad (2.57)$$

Though in general this Fokker-Planck Equation is too complex to solve for the probability distribution, it can be used to determine some of the statistical properties of the y_k. In particular, if the expected value or mean of a function of the y_k, $M(y_o, y_1, \ldots, y_n)$, is desired, where

$$\langle M \rangle = \int\int \cdots \int M\, P\, dy_o\, dy_1 \ldots dy_n \quad ,$$

then by multiplying the Fokker-Planck Equation, Eq. 2.57, by M and formally integrating by parts, one obtains the equation

$$\frac{d\langle M\rangle}{dt} = \sum_{k=0}^{n-1} \langle y_{k+1} \frac{\partial M}{\partial y_k} \rangle + [\, f(t) - D_{dn}] \langle \frac{\partial M}{\partial y_n} \rangle$$
$$- \sum_{k=0}^{n} (b_k - D_{nk}) \langle y_k \frac{\partial M}{\partial y_n} \rangle + D_{dd} \langle \frac{\partial^2 M}{\partial y_n^2} \rangle \qquad (2.58)$$
$$+ \sum_{k=0}^{n} \sum_{i=0}^{n} D_{ik} \langle y_i y_k \frac{\partial^2 M}{\partial y_n^2} \rangle - 2 \sum_{k=0}^{n} D_{dk} \langle y_k \frac{\partial^2 M}{\partial y_n^2} \rangle \; .$$

2.4.1.1 The Mean of y(t)

By setting $M = y_k$ in Eq. 2.58, one obtains the result

$$\frac{d\langle y_k\rangle}{dt} = \langle y_{k+1}\rangle \quad \text{for } k = 0, 1, 2, \ldots n, \qquad (2.59a)$$

$$\frac{d\langle y_n\rangle}{dt} = - \sum_{k=0}^{n} (b_k - D_{nk}) \langle y_k \rangle + f(t) - D_{dn} \; . \qquad (2.59b)$$

If these two equations, Eqs. 2.59, are combined, they yield the equation

$$\frac{d^{n+1}\langle y_o\rangle}{dt^{n+1}} + \sum_{k=0}^{n} (b_k - D_{nk}) \frac{d^k \langle y_o \rangle}{dt^k} = f(t) - D_{dn} . \qquad (2.60)$$

This equation for the mean is quite similar to the original differential equation, Eq. 2.50, with the basic difference lying in the shift of the coefficients due to the D_{nk} terms. When the noise term in the coefficient of the next to highest derivative, $a_n(t)$, is zero then the equation for the mean is identical to the original equation without the noise terms. It is this shift in the coefficients that has played a major role in the controversy discussed in Section 2.

2.4.1.2 The Variance of y(t):

Let u_{ik} be defined by the equation

$$u_{ik} = u_{ki} = \langle y_i y_k \rangle - \langle y_i \rangle \langle y_k \rangle \quad , \tag{2.61}$$

so that u_{kk} represents the variance of y_k. As $u_{ik} = u_{ki}$ and both k and i will take on all integer values from 0 to n, there are a total of $\frac{1}{2}(n+1)(n+2)$ of the u_{ik}. These may be found by utilizing the moment equation, Eq. 2.58, with M given by

$$M = y_i y_k - \langle y_i \rangle \langle y_k \rangle \quad .$$

If this is done, one obtains

$$\frac{d\,u_{ik}}{dt} = u_{i,k+1} + u_{i+1,k} \quad \text{for } i \neq n,\ k \neq n \quad ,$$

(2.62a)

$$\frac{d\,u_{in}}{dt} = u_{i+1,n} - \sum_{k=0}^{n} (b_k - D_{nk})\, u_{ik} \quad \text{for } i \neq n \; , \tag{2.62b}$$

$$\frac{1}{2} \frac{du_{nn}}{dt} = -\sum_{k=0}^{n} (b_k - D_{nk}) u_{kn} + \sum_{k=0}^{n} \sum_{i=0}^{n} D_{ik} u_{ik}$$

$$+ \sum_{k=0}^{n} \sum_{i=0}^{n} D_{ik} \langle y_k \rangle \langle y_i \rangle - 2 \sum_{k=0}^{n} D_{dk} y_k + D_{dd} . \tag{2.62c}$$

Equation 2.62a represents $\frac{1}{2}n(n+1)$ equations, Eq. 2.62b represents n equations, and Eq. 2.62c represents one equation, giving a total of $\frac{1}{2}(n+1)(n+2)$ equations for the same number of unknown u_{ik}. In principle, Eqs. 2.59 could be solved for the means, and the result used in Eq. 2.62c to lead to the solution for the u_{ik}. In fact, this can be very complex, however, standard Laplace techniques can be used to test for stability of the means and the u_{ik}. If both are stable, then the mean squares, $\langle {y_k}^2 \rangle$, given by

$$\langle {y_k}^2 \rangle = u_{kk} + \langle y_k \rangle^2 ,$$

will also be stable.

The variance of y, u_{oo}, can be used with the well-known Chebyshev Inequality to give a conservative upper bound on the probability of y differing from its mean by more than a given amount,

$$\text{Probability} \{ |y - \langle y \rangle | \geq c \} \leq u_{oo}/c^2 .$$

2.4.1.3 Autocorrelation and the Power Density Spectrum:

Often of interest in the analysis of time functions is the time average autocorrelation function (henceforth time averages will be by a bar over the quantity, $\overline{\quad}$) given by

$$T(\tau) = \overline{y(t)y(t+\tau)} = \lim_{T\to\infty} \frac{1}{T} \int_{t_o}^{t_o+T} y(t)y(t+\tau)dt \ , \qquad (2.63)$$

and its Fourier Transform, often called the power density spectrum. In a system of the sort being discussed, however, the time average correlation function and the power density spectrum (if they exist), are random functions. In fact, for the system being discussed, the limit indicated by Eq. 2.63 does not exist in the normal sense of a limit. In many instances, the stochastic correlation function, $R(\tau)$, where

$$R(\tau) = \langle y(t)y(t+\tau)\rangle \ , \qquad (2.64)$$

is used. To eliminate the effect of initial conditions, either it is assumed that they are applied at $-\infty$, or t is allowed to approach infinity. If all inputs are stationary, this leads to a possible meaningful result, and further if the process is ergodic and the time average correlation function exists, then

$$R(\tau) = T(\tau) \ .$$

To avoid difficulties in having an "autocorrelation function" that is either a random variable or a time function, the time average of the stochastic correlation function will be used, and denoted by $R_a(\tau)$ where

$$R_a(\tau) = \overline{R(\tau)} = \overline{\langle y(t)y(t+\tau)\rangle} \ . \qquad (2.65)$$

If the time average autocorrelation function exists, and the averaging processes can be interchanged, this will equal the stochastic average of the time average correlation function,

$$R_a(\tau) = \langle T(\tau)\rangle = \langle \overline{y(t)y(t+\tau)}\rangle \ .$$

Thus, the Fourier Transform of $R_a(\tau)$ will be called the average power density spectrum, as it will represent the stochastic average of the power density spectrum (if it exists). For a stationary, ergodic system (which will occur if f(t) in Eq. 2.53 is constant) the "average power density spectrum" will be identical to the usual notion of a power density spectrum.

Assume that initial conditions are applied at the time t_o, and that $\tau \geqslant 0$. In this case, the stochastic autocorrelation function for $t \geqslant t_o$ is given by

$$\langle y(t)y(t+\tau)\rangle = \int\int x_o z_o \ P_T(x/z,\tau;t)P_T(z/u,t-t_o,t_o)\ dxdz, \qquad (2.66)$$

where the double integral represents a 2(n+1) fold integration over all the x_i and z_i, the P_T represents the transition probability, and the initial conditions are $y_i = u_i$ at time t_o. In a manner similar to the generation of the equations for the mean and variance, it is possible to show that for $t \geqslant t_o$, $\tau \geqslant 0$, $\langle y(t)y(t+\tau)\rangle = \langle y_o(t)y_o(t+\tau)\rangle =$

$$\langle y_o(t)\rangle \int_t^{t+\tau} H_n(t+\tau-t_1)\,[\,f(t_1) - D_{nk}\,]\ dt_1 + \sum_{k=0}^{n} \langle y_k(t)y_o(t)\rangle\, H_k(\tau)\ , \qquad (2.67)$$

where the $H_k(t)$ are solutions to the equation

$$\frac{d^{n+1}H_k(t)}{dt^{n+1}} + \sum_{i=0}^{n} (b_i - D_{ni}) \frac{d^i H_k(t)}{dt^i} = 0 , \tag{2.68a}$$

with the initial conditions

$$\left[\frac{d^i H_k(t)}{dt^i}\right]_{t=0} = \begin{cases} 0 & \text{for } k \neq i \\ 1 & \text{for } k = i \end{cases} \quad . \tag{2.68b}$$

The form of Eq. 2.67 will be particularly useful for the comparison with an "equivalent" system. As it stands, it is not particularly useful for evaluation of the correlation function, for one must first calculate the mean, $\langle y_o(t)\rangle$, and also the cross moments, $\langle y_o(t)y_k(t)\rangle$, to obtain results.

2.4.2 AN EQUIVALENT SYSTEM

Consider the system represented by the differential equation

$$\frac{d^{n+1}x}{dt^{n+1}} + \sum_{k=0}^{n} (b_k - D_{nk}) \frac{d^k x}{dt^k} = f(t) - D_{dn} + a(t), \tag{2.69}$$

where the b_k, D_{nk}, D_{dn}, and f(t) are all as defined in the original system of Eqs. 2.50-2.52. a(t) is Gaussian White Noise, with zero mean, and a correlation function given by

$$\langle a(t)\, a(t_1)\rangle = 2\, D\, \delta(t-t_1) \quad . \tag{2.70}$$

As in the original system, equations for the mean and variance similar to Eqs. 2.59 and 2.62 may be derived for the x_k, where

$$x_k = \frac{d^k x}{dt^k} \qquad \text{for } k = 0,1,2,\ldots n.$$

The mean equations for the x_k will be identical to those for the y_k, Eqs. 2.59, so if they have the same initial conditions, one finds that

$$\langle x_k \rangle = \langle y_k \rangle \quad . \tag{2.71}$$

Similarly, if one defines w_{ik} by

$$w_{ik} = \langle x_i x_k \rangle - \langle x_i \rangle \langle x_k \rangle \quad , \tag{2.72}$$

then it can be shown that the w_{ik} will satisfy Eqs. 2.62a and 2.62b with the u_{ik} replaced by w_{ik}. Only the equation corresponding to Eq. 2.62c, that for $\frac{dw_{nn}}{dt}$, will be different.

Further, one can show that the autocorrelation function, $\langle x(t)x(t+T) \rangle$, will be given by

$$\begin{aligned} \langle x(t)x(t+T) \rangle &= \langle x_o(t)x_o(t+T) \rangle = \\ & \langle x_o(t) \rangle \int_t^{t+\tau} H_n(t+\tau-t_1) \; [f(t_1) - D_{nk}] \, dt_1 \\ & + \sum_{k=0}^{n} \langle x_k(t)x_o(t) \rangle \; H_k(\tau) \quad , \end{aligned} \tag{2.73}$$

where the $H_k(t)$ are defined by Eqs. 2.68. Thus, when the initial values of the x_k and y_k are identical, so that Eq. 2.71 holds, one may subtract Eq. 2.73 from Eq. 2.67 to obtain

$$\begin{aligned} & \langle y(t)y(t+\tau) \rangle - \langle x(t)x(t+\tau) \rangle = \\ & \sum_{k=0}^{n} (u_{ko} - w_{ko}) \, H_k(\tau) \quad . \end{aligned} \tag{2.74}$$

If the time average of Eq. 2.74 is taken, then one obtains

$$\overline{\langle y(t)y(t+\tau)\rangle} - \overline{\langle x(t)\, x(t+\tau)\rangle} = + \sum_{k=0}^{n} (\overline{u_{ko}} - \overline{w_{ko}})\, H_k(\tau) \ . \tag{2.75}$$

It will be shown that when the original system has a bounded mean square, then it is possible to choose a value of D for which $\overline{u_{ko}} = \overline{w_{ko}}$ for all k, so that in this case x and y not only have the same mean, but have identical average power density spectrums.

If the original system has a bounded mean square, then the means, the u_{ik}, and the w_{ik} must all be bounded. As the derivative of a bounded time function must have a zero time average, one can average Eqs. 2.59 to show that

$$\overline{\langle y_k\rangle} = 0 \qquad \text{for} \quad k = 1, 2, \ldots n, \tag{2.76a}$$

$$\overline{\langle y\rangle} = \overline{\langle y_o\rangle} = [f(t) - D_{dn}]/(b_o - D_{no}) \ . \tag{2.76b}$$

These equations will also hold with y_k replaced by x_k.

Similarly, if the time average is taken of Eqs. 2.62a and 2.62b, one obtains the result

$$\overline{u_{i,k+1}} + \overline{u_{i+1,k}} = 0 \quad \text{for } i \neq n,\ k \neq n, \tag{2.77a}$$

$$\overline{u_{i+1,n}} - \sum_{k=0}^{n} (b_k - D_{nk})\, \overline{u_{ik}} = 0 \ \text{ for } i \neq n \ . \tag{2.77b}$$

Eqs. 2.77 will also hold if the u_{ik} are replaced by the w_{ik}. When the mean square is stable, Eqs. 2.77 will represent $\frac{1}{2}(n+1)(n+2)-1$ independent equations for the $\frac{1}{2}(n+1)(n+2)$ unknowns, so that they may be used to solve for the $\overline{u_{ik}}$ in terms of $\overline{u_{oo}}$, giving

$$\overline{u_{ik}} = c_{ik}\,\overline{u_{oo}}\,, \tag{2.78}$$

and as Eqs. 2.77 also hold for the w_{ik}, one has

$$\overline{w_{ik}} = c_{ik}\,\overline{w_{oo}}\,. \tag{2.79}$$

Thus, if D is chosen such that y and x have identical time average variances, $\overline{u_{oo}} = \overline{w_{oo}}$, Eqs. 2.78 and 2.79 may be combined with Eq. 2.75 to yield

$$\overline{\langle y(t)y(t+\tau)\rangle} = \overline{\langle x(t)x(t+\tau)\rangle}\,. \tag{2.80}$$

To summarize, it can be said that the two systems defined by

$$\frac{d^{n+1}y}{dt^{n+1}} + \sum_{k=0}^{n} [b_k + a_k(t)]\,\frac{d^k y}{dt^k} = a_d(t) + f(t),$$

and

$$\frac{d^{n+1}x}{dt^{n+1}} + \sum_{k=0}^{n} (b_k - D_{kn})\,\frac{d^k x}{dt^k} = a(t) + f(t) - D_{dn}\,,$$

where the $a_k(t)$ and $a(t)$ are white noise such that

$$\overline{\langle x^2(t)\rangle} = \overline{\langle y^2(t)\rangle}\,,$$

have identical means, identical average autocorrelation functions, and hence identical average power density spectrums. If f(t) is a constant, then the two systems will also have identical power density spectrums.

2.4.3 EXAMPLE: THE SECOND ORDER LINEAR SYSTEM WITH PARAMETRIC WHITE EXCITATION

Consider the system defined by the differential equation

$$\frac{d^2y}{dt^2} + [b + a_1(t)] \frac{dy}{dt} + [w_o{}^2 + a_o(t)] y = f(t) + a_d(t) , \tag{2.81}$$

where $a_1(t)$, $a_o(t)$, and $a_d(t)$ are Gaussian White Noise such that

$$\langle a_i(t)a_k(t_1)\rangle = 2 D_{ik} \delta(t-t_1) .$$

This is a specific example of the general linear system of Eq. 2.50. Thus its Fokker-Planck Equation can be found by using Eq. 2.57, and similarly the means, variances, and equivalent system can be found by using the earlier results of this section.

2.4.3.1 The Mean:

As in the general case, the mean, $\langle y\rangle$, will satisfy the differential equation

$$\frac{d^2\langle y\rangle}{dt^2} + (b - D_{11}) \frac{d\langle y\rangle}{dt} + (w_o{}^2 - D_{10})\langle y\rangle = f(t) - D_{1d} .$$

Hence the mean will be stable if and only if

$$b > D_{11} \quad \text{and} \quad w_o{}^2 > D_{10} .$$

2.4.3.2 The Variance:

The equations for the u_{ik}, where the u_{ik} are defined as in the general case by

$$u_{ik} = \langle y_i y_k\rangle - \langle y_i\rangle \langle y_k\rangle ,$$

are given by

$$\frac{du_{00}}{dt} = 2\,u_{01}\ ,$$

$$\frac{du_{01}}{dt} = u_{11} - (w_o^{\,2} - D_{10})u_{00} - (b - D_{11})u_{01}\,,$$

$$\frac{1}{2}\frac{du_{11}}{dt} = -(w_o^2 - 3D_{10})\,u_{01} - (b - 2D_{11})u_{11} + D_{00}u_{00}$$

$$+ D_{00}\langle y_o\rangle^2 + 2\,D_{01}\,\langle y_o\rangle\langle y_1\rangle + D_{11}\,\langle y_1\rangle^2$$

$$-2D_{d0}\langle y_o\rangle \qquad - 2D_{d1}\langle y_1\rangle + D_{dd}\ .$$

Standard techniques, utilizing Laplace Transforms, yield three inequalities that are necessary and sufficient for the stability of the u_{ik} (having once established stability of the means, $\langle y_k\rangle$). It is possible to use these inequalities, along with the physical requirement

$$D_{11}D_{00} > D_{01}^{\;2}\ ,$$

to show that stability of the variance occurs if and only if

$$b - 2D_{11} > 0\ ,$$

and

$$(w_o^{\,2} - D_{01})\,(b - 2D_{11}) > D_{00}\ .$$

2.4.3.3 The Equivalent System:

As in the general case, this system will be equivalent, in the sense of the mean and the average power density spectrum, to a system defined by

$$\frac{d^2x}{dt^2} + (b-D_{11})\frac{dx}{dt} + (w_o^2 - D_{01})x = a(t) + f(t) - D_{1d} , \tag{2.82}$$

where a(t) is Gaussian White Noise such that

$$\langle a(t)a(t_1)\rangle = 2A\,\delta(t-t_1) ,$$

with A to be determined as follows.

Let the constants m, F_{oo}, and F_{11} be defined by

$$m = \frac{\overline{f(t)} - D_{1d}}{w_o^2 - D_{10}} ,$$

$$F_{oo} = \overline{[f(t) - F(t)]^2} ,$$

$$F_{11} = \overline{\left[\frac{df}{dt}\right]^2} .$$

In this case, A will be given by

$$A = \frac{(b-D_{11})(w_o^2-D_{10})}{(b-2D_{11})(w_o^2-D_{10})-D_{00}}(D_{00}F_{oo}+D_{11}F_{11}+D_{dd}-2D_{d0}m+D_{00}m^2). \tag{2.83}$$

Thus, if the mean and variances are stable (Secs. 2.4.3.1 and 2.4.3.2), and A is chosen so that it satisfies Eq. 2.83, one finds that an "equivalent system" for the system defined in Eq. 2.81 is given by Eq. 2.82.

2.5.0 A NONLINEAR EXAMPLE: THE MAXWELL DISTRIBUTION

Assume that there is a system of N identical particles whose coordinates are q_k and momenta are p_k, such that the potential energy is a function of the q_k alone, V(q). If all particles are subjected to a damping force proportional to the velocity and each particle is further subjected to a random force that is Gaussian and white (as in Brownian Motion) then the equations of motion will be given by

$$\frac{1}{m}\frac{dp_k}{dt} = -\frac{\partial V(q)}{\partial q_k} - b\,p_k + n_k(t)\ , \tag{2.84}$$

$$\frac{dq_k}{dt} = \frac{1}{m}\,p_k\ , \tag{2.85}$$

where the $n_k(t)$ are Gaussian White Noise such that

$$\langle n_k(t) n_i(t_1)\rangle = \begin{cases} 0 & \text{for } i \neq k \\ 2D\delta(t-t_1) & \text{for } i = k. \end{cases} \tag{2.86}$$

As described in Sections 2.1 and 2.3, this system comprises a continuous Markov Process of order 2N, and will have a Fokker-Planck Equation given by

$$\frac{dP}{dt} = \sum_{k=1}^{N}\left\{-\frac{1}{m}\frac{\partial(p_k P)}{\partial q_k} + \frac{\partial}{\partial p_k}\left[\left(\frac{\partial V(q)}{\partial q_k} + b\,p_k\right)P\right] + D\frac{\partial^2 P}{\partial {p_k}^2}\right\}. \tag{2.87}$$

If the potential energy, V(q), increases rapidly enough as the q_k go to infinity to make the integral

$$f(a) = \int\!\!\int \cdots \int \exp\,[-aV(q)]\; dq_1 dq_2 \ldots dq_N \tag{2.88}$$

converge for positive values of a, then a possible steady state solution to the equation will be given by

$$P = \frac{\exp\left[-\frac{b}{2D}\sum_{k=1}^{N} p_k^{\,2} - \frac{bmV(q)}{D}\right]}{f(bm/D)\ (2\pi D/b)^{\frac{1}{2}N}} \quad . \tag{2.89}$$

When the potential energy V(q) is such that the integral defining f(a), Eq. 2.88, converges for positive a, and further V(q) is infinite for no finite value of q, then Theorem 5 of Part 1 of this thesis can be used to show that this steady state solution is unique. When there are forbidden regions, such that V(q) may go to infinity, modifications in the uniqueness proof are necessary to rigorously show that Eq. 2.89 represents the only steady state solution. However, if the potential is considered to be very large for certain values of the q, so that they are not forbidden but are highly improbable, then the uniqueness proof may be used.

From the form of the steady state probability density of Eq. 2.89 it is obvious that the momenta and displacements are independent. Further, the probability density for the momenta alone is seen to be the well-known Maxwell distribution.

2.6.0 COMMENTS

In this section the Fokker-Planck Equation has been derived for the system defined by the set of differential equations

$$y_k{}' = a_k(y,t) + \sum_{i=1}^{m} h_{ki}(y,t)n_i(t) \qquad \text{for } k = 1, 2, \ldots, n \ ,$$

where the $n_i(t)$ are Gaussian White Noise. For the case where the $n_i(t)$

are not white, but simply Gaussian produced by the passage of white noise through a linear filter, then a Fokker-Planck Equation can also be derived. However, the new Fokker-Planck Equation contains not only t and the y_k as independent variables, but now also the n_k (and perhaps their derivatives).

As an example, consider the equation

$$\frac{d^2y}{dt^2} + 2z\frac{dy}{dt} + [1 + n(t)]y = 0 , \tag{2.90}$$

where n(t) is Gaussian White Noise. As in Section 2.4, if one defines $y_o = y$ and $y_1 = dy/dt$, then the Fokker-Planck Equation for the probability density $P(y_o, y_1; t)$ will be of the form

$$\frac{\partial P}{\partial t} = \frac{\partial(y_1 P)}{\partial y_o} - \frac{\partial}{\partial y_1}[(y_o + 2zy_1)P] + Dy_o^2\frac{\partial^2 P}{\partial y_1^2} . \tag{2.91}$$

Though this equation cannot be readily solved, it can be used to find equations for the moments as shown in Section 2.4.1. In particular, this is a special case of the example treated in Section 2.4.3, and from the results of that section it is seen that the system is mean square stable if and only if $z > 0$ and $D < 1$.

Now, if n(t) is not white, but the result of white noise passed through a filter, such as

$$\tau\frac{dn}{dt} + n = d(t) ,$$

where d(t) is white, such that

$$\langle d(t)d(t_1)\rangle = 2D\delta(t-t_1) ,$$

then the process becomes a three-dimensional Markov Process (if the filter had been second order it would be four dimensional, etc.), and associated with the three-dimensional probability density $P(y_o, y_1, n;t)$ is the Fokker-Planck Equation

$$\frac{\partial P}{\partial t} = \frac{\partial(y_1 P)}{\partial y_o} - \frac{\partial}{\partial y_1}\left[(y_o + 2zy_1 + ny_o)P\right] - \frac{1}{\tau}\frac{\partial}{\partial n}(nP) + \frac{D}{\tau^2}\frac{\partial^2 P}{\partial n^2} . \tag{2.92}$$

This equation not only cannot be readily solved, it yields no useful moment equations similar to Eqs. 2.59 and 2.61 because of the presence of the term $\partial(ny_o P)/\partial y_1$. Thus, the obtaining of the Fokker-Planck Equation in this case is of dubious importance.

One unsolved problem is thus that of eliminating the n dependence from Eq. 2.92. As the integral of $P(y_o, y_1, n;t)$ over all n will yield a two-dimensional probability in y_o and y_1 (if P is the transition probability, then its integral will be the first conditional probability in the variables y_o and y_1), an integration of the equation might seem worthwhile. However, the presence of the term $\partial(ny_o P)/\partial y_1$ again manages to complicate matters.

Another unsolved problem is that of determining the stability in a situation of this sort without utilizing the Fokker-Planck Equation. Thus far only sufficient conditions for stability have been found. Attempts at solving such a problem and the evaluation of some sufficient stability conditions are discussed in the section to follow.

3. SOME SUFFICIENT STABILITY CONDITIONS FOR LINEAR SYSTEMS WITH RANDOM (NON-WHITE) COEFFICIENTS

3.1.0 INTRODUCTION

As pointed out in Part 2, the stability and instability of linear systems whose coefficients are sums of constants and Gaussian White Noise can be determined by Laplace transforms applied to appropriate moment equations. This will lead explicitly to stability boundaries for the various moments, and is most often useful in determining "mean square stability." Unfortunately, such a procedure cannot be extended to cover non-white parameters. At present, there is no general method that may be used to show instability when the coefficients are random but not white, and only conservative sufficient conditions for various forms of stability can be obtained.

Some attempts at determining stability boundaries have been published by Chelpanov (11) and Samuels (12). In the former paper, correlation times of the random parameters were assumed to be much smaller than the natural times of the system, thus, as described in Part 2 of this thesis, making the random signals essentially white. The latter paper is unfortunately erroneous in parts, and its results are questionable.

In a later paper, F. Kozin (7) treated sufficient stability conditions by utilizing an ergodic property of the random terms and by using the Gronwall-Bellman Lemma (14). Less conservative conditions have been obtained by T. K. Caughey* by using an appropriate Lyaponof function and the same ergodic property.

* Communicated verbally to the author.

Caughey's Lyaponof function was quadratic in form, and suggested the possibility of using a general quadratic Lyaponov function. Herein is presented an approach for obtaining sufficient conditions for stability, utilizing a general quadratic Lyaponov function.

3.2.0 STATEMENT OF THE PROBLEM

Consider the first order vector differential equation represented by

$$\frac{dY}{dt} + [A]\ Y + f_i(t)\,[F_i]\ Y = G(t), \tag{3.1}$$

where $[A]$ and $[F_i]$ are n by n matrices, Y and G(t) are n dimensional column vectors, and the $f_i(t)$ are random scalar time functions whose statistical properties will be discussed in Section 3.3. Summation convention is implied, so that repeated subscripts imply a summation over all values of the subscript.

It is assumed that the system described by

$$\frac{dY}{dt} + [A]\ Y = 0\ ,$$

is absolutely stable, so that the Eigen values of $[A]$ will all have positive real parts.

In the notation to follow, the subscript $_T$ will be used to denote the transpose of a matrix or vector. $[I]$ will represent the unit matrix. The matrix $[W]$ will represent a "weighting" matrix used in determining a "norm." $[W]$ will be strictly positive definite, and the norm of a vector Z will be denoted by $||Z||$, where

$$||Z||^2 = Z_T\,[W]\ Z\ . \tag{3.2}$$

This definition of a norm will satisfy the triangle inequality,

$$\|Z_1 + Z_2\| \leq \|Z_1\| + \|Z_2\| \quad , \tag{3.3}$$

and further,

$$\left\|\int_0^t Z \, dt_1 \right\| \leq \int_0^t \|Z\| \, dt_1 \quad \text{for } t \geq 0. \tag{3.4}$$

Bounds on the components of the vector Z can be found in terms of the norm. In particular, if $[W_j]$ represents the n-1 by n-1 matrix formed by removing the jth row and jth column of the matrix $[W]$, then one has

$$z_j^2 \leq \|Z\|^2 \frac{\det[W_j]}{\det[W]} \quad , \tag{3.5}$$

where $\det[W_j]$ and $\det[W]$ represent the determinants of the respective matrices.

3.2.1 THE HOMOGENEOUS EQUATION AND THE LYAPONOV FUNCTION

Let $X(t;R,t_1)$ represent the solution to

$$\frac{dX}{dt} + [A] \, X + f_1(t) \, [F_i] \, X = 0 \, , \tag{3.6}$$

with the initial condition

$$X(t_1;R,t_1) = R. \tag{3.7}$$

Since it was assumed that the eigen values of the matrix $[A]$ all had positive real parts, it is possible to show* that given any real, symmetric, strictly positive definite matrix $[P]$, there is one unique

* This is a trivial extension of Theorem 2, page 245, of reference (14).

matrix [W] for which

$$[W]\ [A] + [A]_T\ [W] = [P] \ , \tag{3.8}$$

and this matrix [W] will be real, symmetric, and strictly positive definite. Let it be assumed that a choice is made to obtain the real, symmetric, strictly positive matrix [P], and this choice is then used to determine the matrix [W] according to Eq. 3.8. The Lyaponov function to be used, will be the norm square of $X(t;R,t_1)$, as given by

$$\|X\|^2 = X_T[W]\ X \ . \tag{3.9}$$

By a straightforward differentiation, one finds

$$\frac{d}{dt}\ \|X\|^2 = -X_T[P]\ X + f_i(t)X_T\,\{[W]\ [F_i] + [F_i]_T[W]\}\,X \ . \tag{3.10}$$

As the matrix [P] is strictly positive definite, the matrix [P] - a[W] will be positive definite for some positive values of a. Let a be chosen as the largest of these. In terms of eigen values, a may be expressed as

$$a = \text{smallest eigen value of } [W]^{-\frac{1}{2}}[P]\ [W]^{-\frac{1}{2}} \ , \tag{3.11}$$

or

$$a = (\text{largest eigen value of } [P]^{-\frac{1}{2}}\ [W]\ [P]^{-\frac{1}{2}})^{-1} \ . \tag{3.12}$$

As both [P] and [W] are strictly positive definite, there is no inherent obstacle to the obtaining of the square roots. The Lyaponov function utilized by Caughey was found using a [P] equal to the unit matrix, so that in that case,

$$a = (\text{largest eigen value of } [W])^{-1} .$$

Thus, with an a chosen as above so that P - a W is positive definite, one has

$$-X_T[P]\ X \leq -a\ X_T[W]\ X = -a\ \|X\|^2 . \tag{3.13}$$

Similarly, one can obtain the smallest value of b_i for which both the matrix $b_i\,[W] + [W]\ [F_i] + [F_i]_T[W]$ and the matrix $b_i[W] - [W]\ [F_i] - [F_i]_T[W]$ are positive definite. If λ_i is the eigen value of the matrix $[W]^{\frac{1}{2}}[F_i]\,[W]^{-\frac{1}{2}} + [W]^{-\frac{1}{2}}[F_i]_T[W]^{\frac{1}{2}}$ having the largest magnitude, then

$$b_i = |\lambda_i| . \tag{3.14}$$

Thus, with the b_i as chosen above, one finds that

$$|X_T\ \ [W][F_i]+[F_i]_T[W]\ \ X| \leq b_i\ X_T[W]\ X = b_i\ \|X\|^2. \tag{3.15}$$

Combining the inequalities of Eqs. 3.13 and 3.15 with Eq. 3.10 leads to the result

$$\frac{d}{dt}\ \|X\|^2 \leq -a\,\|X\|^2 + b_i\ |f_i(t)|\ \|X\|^2 . \tag{3.16}$$

Thus, integrating and using the initial value from Eq. 3.7, one obtains

$$\|X(t;R,t_1)\|^2 \leq \|R\|^2 \exp\left\{-a(t-t_1) + b_i \int_{t_1}^{t} |f_i(t_2)|\ dt_2\right\} .$$

And by taking the square root, this becomes

$$\|X(t;R,t_1)\| \leq \|R\| \exp\left\{-\tfrac{1}{2}a(t-t_1) + \tfrac{1}{2}b_i \int_{t_1}^{t} |f_i(t_2)|dt_2\right\} . \qquad (3.17)$$

3.2.2 THE INHOMOGENEOUS EQUATION

With the vector $X(t;R,t_1)$ as defined in the previous section one may express the general solution to Eq. 3.1 as

$$Y(t) = X\{t;Y(t_o),t_o] + \int_{t_o}^{t} X[t;G(t_1),t_1] \, dt_1 .$$

If one now uses the inequalities that the norm must satisfy, Eqs. 3.3 and 3.4, one obtains the fact that

$$\|Y(t)\| \leq \|X[t,Y(t_o),t_o]\| + \int_{t_o}^{t} \|X[t;G(t_1),t_1]\| \, dt_1 ,$$

and by applying the results of Eq. 3.17, this yields

$$\|Y(t)\| \leq \|Y(t_o)\| \exp\left\{-\tfrac{1}{2}a(t-t_o) + \tfrac{1}{2}b_i \int_{t_o}^{t} |f_i(t_2)| \, dt_2\right\}$$
$$+ \int_{t_o}^{t} \|G(t_1)\| \exp\left\{-\tfrac{1}{2}a(t-t_1) + \tfrac{1}{2}b_i \int_{t_1}^{t} |f_i(t_2)| \, dt_2\right\} dt_1. \qquad (3.18)$$

This bound on the norm of $Y(t)$ can be used in a discussion of stability conditions for various forms of stability.

3.2.3 TYPES OF STABILITY

Before describing the various notions of stability, the various forms of convergence should be outlined.

i. $h(t)$ will be said to converge to h_o if and only if

$$\lim_{t\to\infty} h(t) = h_o . \qquad (3.19)$$

ii. $h(t)$ will be said to converge in mean square to h_o if and only if

$$\lim_{t\to\infty} \langle (h(t) - h_o)^2 \rangle = 0. \tag{3.20}$$

iii. $h(t)$ will be said to converge in probability to h_o if and only if for every positive ϵ, one has

$$\lim_{t\to\infty} \{\text{Prob.}\,[\,|h(t)-h_o| > \epsilon\,]\} = 0. \tag{3.21}$$

iv. $h(t)$ will be said to converge to h_o with probability one if and only if

$$\text{Prob.}\,[\lim_{t\to\infty} h(t) = h_o\,] = 1\,. \tag{3.22}$$

As pointed out by Parzen (15), this is equivalent to

$$\lim_{T\to\infty} \{\text{Prob.}\,[\operatorname*{supremum}_{t \geq T} |h(t)-h_o| > \epsilon\,]\} = 0 \tag{3.23}$$

for every positive ϵ.

To help keep these various forms of convergence in their proper perspective, the following quotation from page 416 of Parzen (15) is given:

> "One thus sees that convergence in probability is implied by both convergence with probability one and convergence in mean square. However, without additional conditions, convergence in probability implies neither convergence in mean square nor convergence with probability one neither implies nor is implied by convergence in mean square."

The types of stability most often used are the following:

i. Absolute Stability: If Y is the solution vector to a homogeneous differential equation, the system is said to be absolutely stable if $\|Y\|$ converges to zero. If Y is the solution vector to an inhomogeneous differential equation, the system is said to be stable if $\|Y\|$ is bounded.

ii. Mean Square Stability: If Y is the solution vector to a homogeneous differential equation, the system is said to be absolutely stable if $\|Y\|$ converges in mean square to zero. If Y is the solution vector to an inhomogeneous equation, the system is said to be mean square stable if the mean square of $\|Y\|$ is bounded.

iii. Probable Stability: If Y is the solution vector to a homogeneous differential equation, the system is said to be stable in probability if $\|Y\|$ converges in probability to zero.

iv. Almost Sure Stability: If Y is the solution vector to a homogeneous differential equation, the system is said to be almost surely stable if $\|Y\|$ converges to zero with probability one. If Y is the solution vector to an inhomogeneous differential equation, the system is said to be almost surely stable if $\|Y\|$ is bounded with probability one.*

Thus it may be noted that probable stability is implied by almost sure stability, absolute stability, and mean square stability. Almost sure stability neither implies nor is implied by mean square stability.

The "stability" used in the analysis of the linear systems of Part 2 is mean square stability, for in the case of parametric white excitation, it was shown that Laplace techniques can be used to determine stability of the mean and mean square.

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* This implies that given any solution in the ensemble of possible solutions, the probability is one that this solution is bounded (the unbounded solutions form a set of measure zero). There may not be any common bound for the bounded solutions.

3.2.4 THE ERGODIC REQUIREMENT ON THE $f_i(t)$

It is to be assumed that the $f_i(t)$ are random, stationary, and ergodic time functions for which the expectation of their magnitudes,

$$E\,[\,|f_i(t)|\,] = \langle\, |f_i(t)| \,\rangle \;,$$

are known. As they are ergodic and stationary, in some sense the random functions $h_i(t)$, given by

$$h_i(t) = \frac{1}{t}\int_{t_o}^{t_o+t} |f_i(t_1)|\, dt_1 \;, \tag{3.24}$$

will converge to $E\,[|f_i(t)|]$ as t goes to infinity. To keep a clear division of the ways in which this convergence can occur, the following definitions are given.

<u>Definition 1:</u> The $f_i(t)$ will be called <u>strictly ergodic</u> if the $h_i(t)$ converge to $E[\,|f_i(t)|]$.

<u>Definition 2:</u> The $f_i(t)$ will be called <u>ergodic in probability</u> if the $h_i(t)$ converge in probability to $E[\,|f_i(t)|]$.

<u>Definition 3:</u> The $f_i(t)$ will be called <u>almost surely ergodic</u> if the $h_i(t)$ converge with probability one to $E\,[\,|f_i(t)|]$.

The application of each of these different types of ergodicity can be shown as follows. First, if the $f_i(t)$ are strictly ergodic, the limit of the $h_i(t)$ as defined by Eq. 3.24 must exist, and be uniform in t_o (as the $f_i(t)$ are also stationary). Hence, given any positive ϵ_i, there exists a T_i such that

$$\left|\left\{\frac{1}{t}\int_{t_o}^{t_o+t} |f_i(t_3)|dt_3 - E[\,|f_i(t)|]\right\}\right| \leq \epsilon_i \;, \tag{3.25}$$

provided that $t \geq T_i$. One may use this then to bound any integral of the $f_i(t)$ by writing

$$\int_{t_1}^{t_2} |f_i(t)|dt = \int_{t_1}^{t_2+T_i} |f_i(t)| \, dt - \int_{t_2}^{t_2+T_i} |f_i(t)| \, dt, \tag{3.26}$$

where it is assumed that $t_2 > t_1$. By using the inequality of Eq. 26 on the integrals on the right hand side of Eq. 3.26, one obtains

$$\int_{t_1}^{t_2} |f_i(t)|dt \leq (t_2-t_1)\left\{E[\,|f_i(t)|\,] + \epsilon_i\right\} + 2\epsilon_i T_i \ ,$$

provided that $t_2 > t_1$. From this it follows that if the $f_i(t)$ are strictly ergodic, given any positive ϵ one can find a T such that

$$b_i \int_{t_1}^{t_2} |f_i(t)| \, dt \leq (t_2-t_1) \left\{b_i E[\,|f_i(t)|\,] + \epsilon\right\} + 2\epsilon T. \tag{3.27}$$

If the $f_i(t)$ are ergodic in probability, one can show that for any positive ϵ ,

$$\lim_{t \to \infty} \left\{ \text{Prob.} \left[\int_{t_o}^{t_o+t} b_i |f_i(t_1)| \, dt_1 - b_i E[\,|f_i(t)|\,] \, t - \epsilon t \geq 0 \right] \right\} = 0. \tag{3.28}$$

Finally, if the $f_i(t)$ are almost surely ergodic, it implies that the $h_i(t)$ of Eq. 3.24 converge with probability one. Thus, Eq. 3.25 and Eq. 3.27 will be valid with probability one. Hence, given any $f_i(t)$ in the ensemble of possible $f_i(t)$, the probability is one that for every positive ϵ there will be a T such that

$$b_i \int_{t_1}^{t_2} |f_i(t) \, dt \leq (t_2-t_1) \left\{b_i E[\,|f_i(t)|\,] + \epsilon\right\} + 2\epsilon T. \tag{3.29}$$

There may be no common T, as the T will not only depend upon ϵ, but will also depend upon the $f_i(t)$ chosen.

Eqs. 3.27, 3.28, and 3.29 can be used with Eq. 3.18 to help specify sufficient requirements for certain types of stability.

3.2.5 SUFFICIENT STABILITY CRITERIA

It was found in Eq. 3.18 that the norm of the solution vector to the system in question could be bounded as shown

$$\|Y(t)\| \leq \|Y(t_o)\| \exp[-\tfrac{1}{2}a(t-t_o) + \tfrac{1}{2}b_i \int_{t_o}^{t} |f_i(t_2)|dt_2] \;. + \int_{t_o}^{t} \|G(t_1)\| \exp\left\{-\tfrac{1}{2}a(t-t_1) + \tfrac{1}{2}b_i \int_{t_1}^{t} |f_i(t_2)|\, dt_2\right\} dt_1. \tag{3.30}$$

From the ergodic requirement on the $f_i(t)$, one has that in some sense the integral given by

$$\int_{t_1}^{t} |f_i(t_2)|\, dt_2$$

will approach $(t-t_1)E[\,|f_i(t)|\,]$, so that an inspection of Eq. 3.30 would lead one heuristically to conclude that stability of some sort might be implied if

$$a > b_i\, E\,[\,|f_i(t)|\,] \;.$$

Eqs. 3.27, 3.28, and 3.29 can be used with Eq. 3.30 to verify this in certain instances. The results of this are given below:

<u>The Homogeneous Equation</u>

Assume that the $f_i(t)$ are stationary random such that

$$a > b_i\, E\,[\,|f_i(t)|\,] \quad , \tag{3.31}$$

and that G(t) is the zero vector.

(i) If the $f_i(t)$ are strictly ergodic, the system is absolutely stable.

(ii) If the $f_i(t)$ are ergodic in probability, the system is stable in probability.

(iii) If the $f_i(t)$ are almost surely ergodic, the system is almost surely stable.

The Inhomogeneous Equation

Assume that the $f_i(t)$ are stationary random variables such that

$$a > b_i E\,[|f_i(t)|] \quad . \tag{3.32}$$

(i) If the $f_i(t)$ are strictly ergodic and $\|G(t)\|$ is bounded, then the system is absolutely stable.

(ii) If the $f_i(t)$ are almost surely ergodic and $\|G(t)\|$ is bounded with probability one (almost surely bounded), then the system is almost surely stable.

3.3.0 EXAMPLE: THE SECOND ORDER DIFFERENTIAL EQUATION

Consider the system defined by the differential equation

$$\frac{d^2y}{dt^2} + 2z\,\frac{dy}{dt} + [1 + f(t)]\;y = g(t), \tag{3.33}$$

where f(t) is a stationary, almost surely ergodic random variable and g(t) is bounded with probability one. It will be shown that a sufficient condition for almost sure stability is given by

$$E[\,|f(t)|\,] < \frac{2}{1 + (1 + 1/z^2)^{\frac{1}{2}}} \quad . \tag{3.34}$$

This is demonstrated by defining the vector Y as the two-dimensional vector with components y and dy/dt. Thus, Eq. 3.33 can be put in the form of Eq. 3.1 by defining $f_1(t)$ as f(t), G(t) as the two-dimensional column vector with components 0 and g(t), and the matrices [A] and $[F_1]$ as

$$[A] = \begin{bmatrix} 0 & -1 \\ 1 & 2z \end{bmatrix} ,$$

$$[F_1] = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} .$$

Using the identity matrix for [P], one finds from Eq. 3.8 that [W] is given by

$$[W] = \begin{bmatrix} z+1/2z & \frac{1}{2} \\ \frac{1}{2} & 1/2z \end{bmatrix} ,$$

and from Eqs. 3.11 or 3.12 and Eq. 3.13 one finds that a and b_1 are given by

$$a = \frac{2}{z + 1/z + (1 + z^2)^{\frac{1}{2}}} , \tag{3.35}$$

$$b_1 = (1 + z^2)^{-\frac{1}{2}} \quad . \tag{3.36}$$

Thus Eqs. 3.35 and 3.36 can be used with Eq. 3.32 to yield the stability requirement given by Eq. 3.34.

The result obtained as described can be compared with that obtained by Kozin (7) using the Gronwall-Bellman Lemma. Figure One represents a sketch of the upper bounds on $E[|f(t)|]$ for which a sufficient condition for stability is satisfied. The solid line gives the upper bound of Eq. 3.54, and the dashed line gives the result of Kozin.

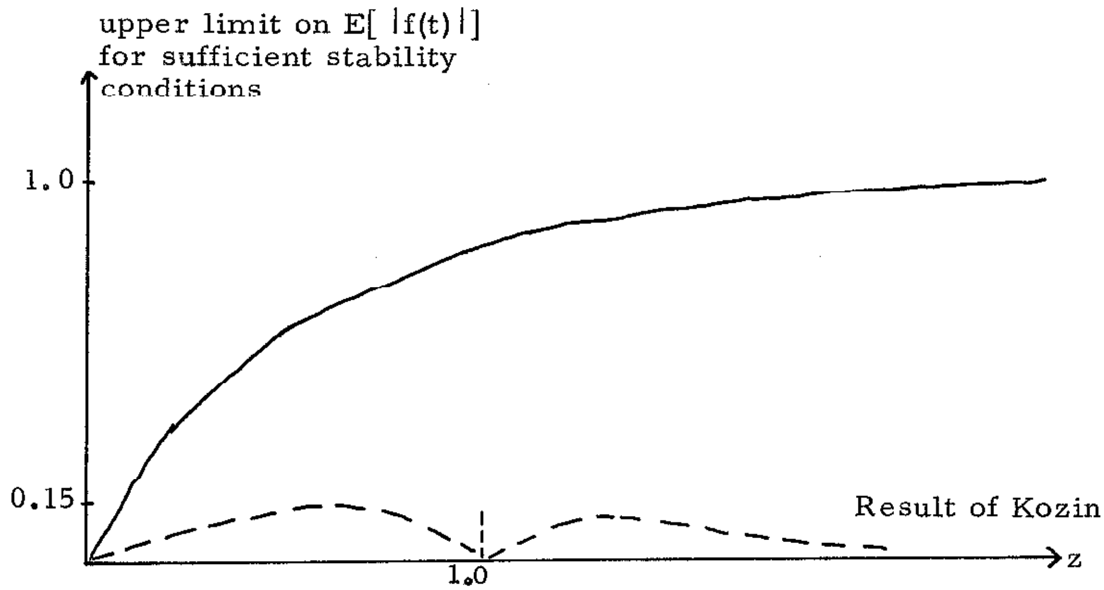

Figure One: Comparison of Sufficient Stability Criteria

The result obtained by using the methods of this section contain no singular behavior at the point of critical damping (z = 1) as does the result of Kozin.

3.4.0 BOUNDED PARAMETRIC NOISE

Consider the vector equation

$$\frac{dY}{dt} + [A]\ Y + f_i(t)\,[F_i]\ Y = G(t)\ , \tag{3.37}$$

where, as in Eq. 3.1, $[A]$ and $[F_i]$ are n by n matrices, Y and G(t) are n dimensional column vectors, and summation convention is implied, so that repeat subscripts imply a summation over all values of the subscript. However, the scalar time functions $f_i(t)$ now are considered as bounded such that

$$|f_i(t)| \leq p_i\ . \tag{3.38}$$

It is further assumed, as in Eq. 3.1, that the eigen values of the real matrix $[A]$ all have positive real parts. One can then choose a real, symmetric, strictly positive definite matrix $[P]$, and proceed as in section 3.2.0 to find the strictly positive definite matrix $[W]$ given by Eq. 3.8, and the coefficient a given by Eqs. 3.11 or 3.12. One could also find the b_i from Eq. 3.13, and use Eq. 3.17 to show that

$$\|X(t;R,t_1)\| \leq \|R\|\ \exp\left[-\tfrac{1}{2}(a-b_ip_i)(t-t_1)\right]\ .$$

However, a better bound can be obtained in many cases.

Proceeding in a manner similar to the determination of the b_i, one can find the smallest c_i for which the matrices $c_i[P] + [W][F_i] + [F_i]_T[W]$ and $c_i[P] - [W][F_i] - [F_i]_T[W]$ are both positive definite. If β_i is the eigen value of the matrix

$$[P]^{-\frac{1}{2}}\{[W][F_i] + [F_i]_T[W]\}\,[P]^{-\frac{1}{2}}$$

having the largest magnitude, then

$$c_i = |\beta_i| \tag{3.39}$$

It is a simple matter to show that

$$c_i \leqslant b_i/a \tag{3.40}$$

Then, similarly to the derivation of Eq. 3.7, one can show that if

$$1 \geqslant c_i p_i , \tag{3.41}$$

one has

$$\|X(t;R,t_1)\| \leqslant \|R\| \exp\left[-\tfrac{1}{2}a(1-c_i p_i)(t-t_1)\right] . \tag{3.42}$$

Thus, as in the derivation of Eq. 3.18, one can use Eq. 3.42 and find

$$\|Y(t)\| \leqslant \|Y(t_o)\| \exp\left[-\tfrac{1}{2}a(1-c_i p_i)(t-t_o)\right]$$
$$\int_{t_o}^{t} \|G(t_1)\| \exp\left[-\tfrac{1}{2}a(1-c_i p_i)(t-t_1)\right] dt_1 , \tag{3.43}$$

provided that $c_i p_i \leqslant 1$.

From Eq. 3.43 it can be seen that not only is the sufficient stability criteria obtained from this equivalent to that for the first order system

$$\frac{dy}{dt} + \tfrac{1}{2}a(1-c_i p_i)y = \|G(t)\| , \tag{3.44}$$

but when $c_i p_i \leqslant 1$, the solution to Eq. 3.44 can be used as an upper bound on $\|Y\|$.

3.4.1 SUFFICIENT STABILITY CRITERIA

From Eq. 3.43, the following obvious conclusions can be drawn:

i. If $c_i p_i < 1$, and $\|G(t)\|$ is bounded, then the system of Eq. 3.37 is absolutely stable.

ii. If $c_i p_i < 1$, and $\|G(t)\|$ has a bounded mean square, then the system of Eq. 3.37 is mean square stable.

iii. If $c_i p_i < 1$, and $\|G(t)\|$ is bounded with probability one, then the system of Eq. 3.37 is almost surely stable.

3.4.2 EXAMPLE: THE SECOND ORDER SYSTEM

Consider the system defined by

$$\frac{d^2y}{dt^2} + 2z\frac{dy}{dt} + [1 + f(t)]\ y = g(t), \tag{3.45}$$

where g(t) has a bounded mean square and f(t) is bounded, such that

$$|f(t)| \leqslant p. \tag{3.46}$$

Assume further that $z > 0$, so that without the parametric term the system would be stable. This equation can be put in the form of Eq. 3.37 if one defines Y as the two-dimensional column vector with components y and dy/dt, G as the two-dimensional column vector with components 0 and g(t), $f_1(t)$ as f(t), p_1 as p, and the matrices [A] and $[F_1]$ as

$$[A] = \begin{bmatrix} 0 & -1 \\ 1 & 2z \end{bmatrix}, \tag{3.44}$$

$$[F_1] = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}. \tag{3.45}$$

Let the [P] matrix be defined by

$$[P] = \begin{cases} \begin{bmatrix} 2 & 2z \\ 2z & 2 \end{bmatrix} & \text{for } z \leqslant 2^{-\frac{1}{2}} \\ \\ \begin{bmatrix} 2 & 1/z \\ 1/z & 2 \end{bmatrix} & \text{for } z \geqslant 2^{-\frac{1}{2}} \end{cases} \tag{3.46}$$

so that the [W] matrix (from Eq. 3.8) will be given by

$$[W] = \begin{cases} \begin{bmatrix} 1/z & 1 \\ 1 & 1/z \end{bmatrix} & \text{for } z \leqslant 2^{-\frac{1}{2}} \\ \\ \begin{bmatrix} 2z & 1 \\ 1 & 1/z \end{bmatrix} & \text{for } z \geqslant 2^{-\frac{1}{2}} \end{cases} \tag{3.47}$$

This will yield for the constants a and c_1 the values of

$$a = \begin{cases} 2z & \text{for} \quad z \leqslant 2^{-\frac{1}{2}} \\ \\ 1/z & \text{for} \quad z \geqslant 2^{-\frac{1}{2}} \end{cases} \quad , \tag{3.48}$$

$$c_1 = \begin{cases} \dfrac{1}{2z(1-z^2)^{\frac{1}{2}}} & \text{for} \quad z \leqslant 2^{-\frac{1}{2}} \\ 1 & \text{for} \quad z \geqslant 2^{-\frac{1}{2}} \end{cases} \quad . \tag{3.49}$$

Thus, as g(t) has a bounded mean square, the system will be mean square stable if

$$p_1 c_1 = p c_1 < 1,$$

or if simply

$$|f(t)| < 1/c_1 \,, \qquad (3.50)$$

where c_1 is given by Eq. 3.49. This region of stability is sketched in Figure Two, below.

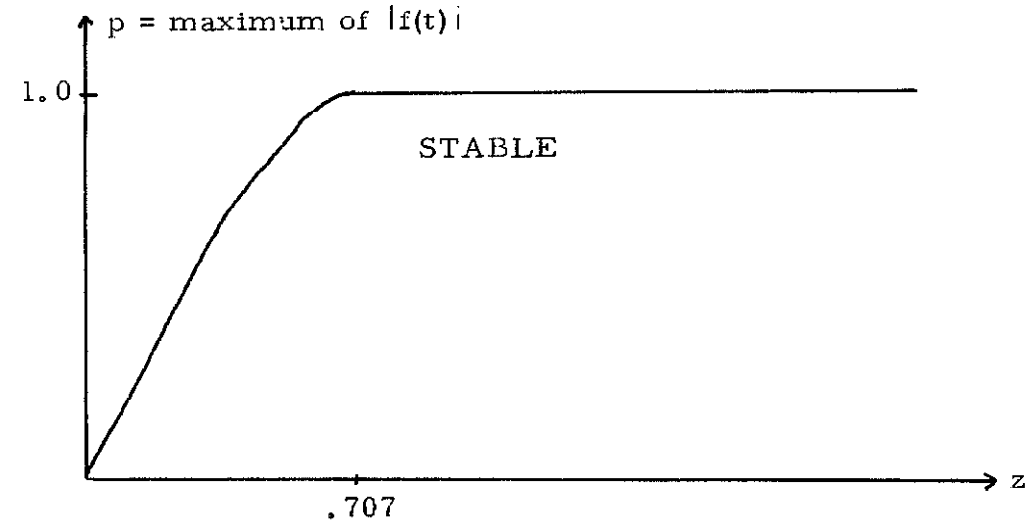

Figure Two: Sufficient Stability Region for Eq. 3.37

One can use the results of Section 3.4.0 to place a bound on the solution. For simplicity, let it be assumed that the initial conditions are zero, so that from Eq. 3.43 one has

$$\|Y(t)\| \int_{t_0}^{t} \|G(t_1)\| \exp\left[-\tfrac{1}{2}a(1-c_1 p)(t-t_1)\right] dt_1 \,, \qquad (3.51)$$

provided that $c_1 p \leq 1$. From the definition of the norm, Eq. 3.2, and the [W] of Eq. 3.47, one has

$$\|G(t)\| = |g(t)|/z^{\frac{1}{2}} \,. \qquad (3.52)$$

Further, one can use Eq. 3.5 to show that

$$y^2 \leq \begin{cases} \|Y\|^2 \, z/(1-z^2) & \text{for } z \leq 2^{-\frac{1}{2}} \\ \|Y\|^2 / z & \text{for } z \geq 2^{-\frac{1}{2}} \end{cases} \,. \qquad (3.53)$$

Eqs. 3.51, 3.52, and 3.53 can be combined with Eqs. 3.48 and 3.49 to show that when $p < 1/c_i$ (one lies in the stable region of Fig. 2), one has

$$|y| \leqslant (1-z^2)^{-\frac{1}{2}} \int_0^t |g(t_1)| \exp\{-z[1- \frac{p}{2z(1-z^2)^{\frac{1}{2}}}] (t-t_1)\} dt_1 \qquad (3.54)$$

for $z \leqslant 2^{-\frac{1}{2}}$, and similarly when $z \geqslant 2^{-\frac{1}{2}}$ one has

$$|y| \leqslant \frac{1}{z} \int_0^t |g(t_1)| \exp[-\frac{1}{2z} (1-p)(t-t_1)] dt_1 \quad . \qquad (3.55)$$

3.5.0 THE SELECTION OF THE LYAPONOV FUNCTION OR NORM

The question naturally arises, how does one choose the best [P] matrix used in the determination of the [W] matrix, which defines the norm or Lyaponov function? In the example of Section 3.3.0, the [P] matrix used is the identity matrix. Is it the best choice?

There appears to be no answer to these questions. If one could use an arbitrary positive definite, symmetrical matrix [P] and use this to minimize the products b_i/a, or in the case of bounded parameters minimize c_i, then this [P] would be the best. No general procedure for this has been found. Hence, one can only guess at good choices for the [P] matrix.

3.6.0 COMMENTS

In this section an approach for obtaining sufficient conditions for stability of a linear system has been given. It still remains for someone to develop a method of determining necessary conditions for stability, and thus to obtain stability boundaries.

One other related problem is that of extending the techniques developed here to cover possibly some nonlinear situations. The following example, due to Caughey,* demonstrates how this can be done in one specific situation. Let y satisfy the differential equation given by

$$\frac{d^2y}{dt^2} + 2\,z\,\frac{dy}{dt} + [1 + f(t)]\;y + g(y) = 0, \tag{3.56}$$

where z is positive, f(t) is almost surely ergodic, and g(y) is a function of y having the same sign as y, and satisfying the inequality

$$2\int_0^y g(x)\,dx \leq yg(y)\;. \tag{3.57}$$

This covers such cases as $g(y)=y^3$, y^5, etc. As will be seen, the sufficient conditions for almost sure stability are identical to those for the problem when g(y) = 0, and are thus given by Eq. 3.53 found in the example of Section 3.3.0, that is

$$E\;[|f(t)|] < \frac{2}{1 + (1+1/z^2)^{\frac{1}{2}}}\;. \tag{3.58}$$

As before, let the vector Y be defined as the two-dimensional column vector with the components y and dy/dt. Let the Lyaponov function V(t) be defined as

$$V(t) = \|Y\|^2 + \frac{1}{z}\int_0^y g(x)\,dx, \tag{3.59}$$

where the norm squared, $\|Y\|^2$, is as defined by Eq. 3.2. Further let the matrices [A], $[F_1]$, and [W], as well as the scalar a and b_1 be as defined in the example of Section 3.3.0. In this case the analog to Eq. 3.16 will have the form

* Communicated verbally to the author.

$$\frac{dV(t)}{dt} \leq -a \; \|Y\|^2 + b_1 \; |f(t)| \, \|Y\|^2 - y \, g(y) \; . \tag{3.60}$$

If one solves Eq. 3.59 for $\|Y\|^2$ and uses this in the inequality of Eq. 3.60 along with the fact that $\int_0^y g(x)\, dx$ must be positive, then

$$\frac{dV(t)}{dt} \leq -aV(t) + b_1 \; |f(t)| \; V(t) + \frac{a}{z} \int_0^y g(x)dx - yg(y) \; . \tag{3.61}$$

But for the a given by Eq. 3.35, it is a simple matter to show that $a \leq 2z$, so that using this along with Eq. 3.57 yields

$$\frac{a}{z} \int_0^y g(x) \, dx - y \, g(y) \; \leq \; 0 \; . \tag{3.62}$$

Thus, from Eqs. 3.61 and 3.62, one has

$$\frac{dV(t)}{dt} \leq -aV(t) + b_1 \; |f(t)| \; V(t) \; . \tag{3.63}$$

Thus, by applying the techniques of this part of the thesis, this shows that the system is almost surely stable for

$$E[|f(t)|] < a/b_1 \; ,$$

which leads to Eq. 3.58.

Once it is established that a system is stable in the sense of being mean square stable, stable in probability, or almost surely stable, other problems arise. For example, in some systems it is of paramount importance to know the probability of the output exceeding a given value. In such a case it does no good to demonstrate, for example, that the mean square is bounded. A bound on the mean square can certainly be used to give an upper bound on the probability of exceeding a fixed level (by use of the Chebyshev inequality) at a given time, but one is usually interested not in one given time, but all times in some interval. Thus we come to the problems of first passage. What is the probability that y will not exceed some given value in a fixed time interval, or what is the mean and mean square time taken for y to get from some initial value to another given value? These are questions that are of great importance, and unfortunately cannot usually be answered. This first passage problem is discussed in the following section.

4. FIRST PASSAGE TIMES IN A SECOND ORDER SYSTEM

4.1.0 INTRODUCTION

Of major interest in vibrational systems with random excitation is the mean and mean square time for the system to get from one state to another. A more general problem is that of determining the probability distribution of the elapsed time in getting from one state to another.

In principle, this problem can be solved for a stationary Markov Process. Using the notation of Part 1 (Sections 1.2.0 and 1.2.1), if one defines $T(z/x,t)$ as the probability density for the time of passage, t, to get from the point x to the point z in n dimensional phase space, and if $P_T(z/x,t)$ represents the transition probability for the variable z, then Eq. 1.4 of Part 1 states that

$$P_T(z/x,t) = \int_0^t dt_1 \; T(z/x,t_1) \; P_T(z/z,t-t_1). \qquad (4.1)$$

By using Laplace Transform techniques, the convolution equation indicated by Eq. 4.1 can be readily solved, giving $T(z/x,t)$ as the inverse Laplace Transform of a ratio of Laplace Transforms of $P_T(z/x,t)$ and $P_T(z/z,t)$. In general, the calculations necessary are far too difficult to perform, except in the simple case of a one-dimensional Markov Process, generated by the linear differential equation

$$\frac{dy}{dt} + ay = n(t),$$

where $n(t)$ is white noise.

A similar problem, that of the evaluation of the frequency with which a variable $y(t)$ crosses a given value z, has been worked out by

Rice (16), for the case where y(t) is a stationary random variable. In particular, when y(t) is a Gaussian variable with zero mean, then the mean number of times per unit time that y is equal to z is given by $\overline{N}_z$, where

$$\overline{N}_z = \frac{\langle y^2 \rangle}{\langle (dy/dt)^2 \rangle} \frac{1}{\pi} \exp\left[-z^2/2\langle y^2 \rangle\right] . \tag{4.2}$$

Consider the resonant system defined by

$$y'' + 2by' + w^2 y = n(t), \tag{4.3}$$

where n(t) is white noise, with an autocorrelation function given by

$$\langle n(t)\, n(t_1) \rangle = 2\, D\delta(t-t_1) . \tag{4.4}$$

Such a system is used as an approximation of a structural response to an earthquake.* A desired result is that of the "probability of failure" which is the probability that in a given interval of length T, $|y|$ will have exceeded a fixed value of X at least once. Another desired result is the "mean time to failure," or the mean time to get from one value of $|y|$ to the fixed level X. Neither of these desired results fall into the type of problems just discussed.

Though this problem does not appear complex on the surface, there is no known technique for solving it. Messrs. Rosenblueth and Bustamante (17) have utilized approximations and boundary value techniques to obtain approximate solutions for the probability of failure. Their approach will be justified here by the obtaining of the same results in another manner, and further the mean and mean square times to failure will be calculated. It will be assumed that the system is highly

* See Rosenblueth and Bustamante (17).

resonant, so that

$$b << w,$$

and that the fixed level X will be such that

$$X >> \frac{(D/w)^{\frac{1}{2}}}{w} .$$

This latter condition will be satisfied if X is much larger than the steady state standard deviation of y, for then

$$X >> \frac{(D/b)^{\frac{1}{2}}}{w} >> \frac{(D/w)^{\frac{1}{2}}}{w} .$$

4.1.1 THE BOUNDARY VALUE PROBLEM APPROACH

As described in Part 2 of this thesis, the process defined by Eq. 4.3 represents a two-dimensional Markov Process. If p represents the time derivative of y, and P is the transition probability for the process, then the Fokker-Planck Equation will be given by

$$\frac{\partial P}{\partial t} = -\frac{\partial (pP)}{\partial y} + \frac{\partial (2bp + w^2 y)P}{\partial p} + D\frac{\partial^2 P}{\partial p^2} . \tag{4.5}$$

It is suggested by some that Eq. 4.5 must be satisfied not only when P is the transition probability, but when P is the conditional probability density on the condition that a given region in phase space has not been left for all time from the application of the initial conditions to the present. The basic difference being that P must satisfy some sort of "absorbing" boundary condition on the boundary of the given region.

If one attempts to use this assumption to calculate the probability of y not exceeding a fixed value Y, one approach might be to use the absorbing boundary condition

$$P = 0 \quad \text{for} \quad y = Y.$$

One might then use a Fourier Transform on the p coordinate and a Laplace transform on the variable Y-y. Formally this leads to a result for which no inverse transform exists. Other attempts at absorbing boundary conditions have been used for this problem and none of them have led to a solution.

However, it is possible to approximate the problem by working with a new variable, and obtain a Fokker-Planck Equation for the transition probability in this new variable as a one-dimensional problem, and this equation can be solved as a boundary value problem with an absorbing boundary condition. For example, if $w > b$, and one defines the cylindrical coordinates r and θ by the relations

$$b\,y + p = r \cos\theta\ ,$$

$$(w^2-b^2)^{\frac{1}{2}}\, y = r \sin\theta\ ,$$

then the Fokker-Planck Equation, Eq. 4.5, takes on the form

$$\frac{\partial P}{\partial t} = 2\,b\,P + br\,\frac{\partial P}{\partial r} + (w^2-b^2)^{\frac{1}{2}}\,\frac{1}{r}\,\frac{\partial P}{\partial r}$$

$$+ \tfrac{1}{2}D\left(\frac{\partial^2 P}{\partial r^2} + \frac{1}{r}\,\frac{\partial P}{\partial r} + \frac{1}{r^2}\,\frac{\partial^2 P}{\partial \theta^2}\right)$$

$$+ \tfrac{1}{2}D\left(\frac{\partial^2 P}{\partial r^2} - \frac{1}{r}\,\frac{\partial P}{\partial r} - \frac{1}{r^2}\,\frac{\partial^2 P}{\partial \theta^2}\right)\cos 2\theta$$

$$+ D\left(\frac{1}{r^2}\,\frac{\partial P}{\partial \theta} - \frac{1}{r}\,\frac{\partial^2 P}{\partial r\,\partial \theta}\right)\sin 2\theta\ .$$

If now one defines $\overline{P}$ as the probability density in r alone so that

$$\overline{P} = \int_0^{2\pi} P \, d\theta \ ,$$

and uses some sort of symmetry assumptions* to neglect those terms in the equation involving sin 2θ and cos 2θ, then an integration of the Fokker-Planck Equation in cylindrical coordinates over the variable θ from θ = 0 to θ = 2π yields

$$\frac{\partial \overline{P}}{\partial t} = 2b\overline{P} + br\frac{\partial \overline{P}}{\partial r} + \frac{D}{2r}\frac{\partial}{\partial r}\left(r\frac{\partial \overline{P}}{\partial r}\right) \ .$$

This equation for $\overline{P}$ is the one used by Rosenblueth and Bustamante for a boundary value problem approach to the first passage problem. The problem itself and its solution are given in Section 4.3.2.2.

4.1.2 THE INTEGRAL EQUATION APPROACH

In principle, it is possible to convert this problem to the solution of an integral equation. For example consider the problem of determining the probability that $|y|$ will not have exceeded the value X in the time interval from 0 to t, on the condition that at t = 0, y and dy/dt are zero.

One may utilize the transition probability, $P_T(y, p/y_o, p_o; t)$, found by solving Eq. 4.5 with appropriate impulsive initial conditions, and utilize this to obtain the first and second conditional probabilities in terms of the variable $r = |y|$. As in the notation of Part 1, Section

*No attempt is made here to justify this assumption as the partial differential equation for $\overline{P}$ is derived in another manner in Section 4.3.0 based on assumptions outlined in Sections 4.2.0 through 4.2.2.

1.2 of this thesis, let these be indicated by $P_1(r,t/r_o,t_o)$ and $P_2(r,t/r_1,t_1;r_o,t_o)$. It may be noted that due to the stationary property of the white noise, n(t), both P_1 and P_2 can depend only upon the time differences, $t-t_o$ and $t-t_1$. Let $T(r/x,t-t_o)$ be the first passage probability density (as in Part 1, Section 1.2) so that the probability that r has not exceeded the value X in the time interval from zero to t will be given by

$$1 - \int_0^t T(X,0,t_1)\,dt_1 ,$$

when r = 0 at t = 0.

In a manner similar to the derivation of Eq. 1.4 of Part 1, one may note that for r to get from an initial value R_o to a new value X, it must pass through any given intermediate value* R_1 for the first time. This may be expressed by the integral equation,

$$P_1(X,t/R_o,0) = \int_{t_o}^t T(R_1/R_o,t_1)P_2(X,t/R_1,t_1;R_o,0)\,dt_1 , \qquad (4.6)$$

provided that R_1 either lies strictly between R_o and X or is equal to X. Thus, if one could solve Eq. 4.6 for $T(R/R_o,t)$, one could find the desired first passage statistics.

4.2.0 THE GENERAL PROBLEM AND A SOLVABLE APPROXIMATION

Consider the highly resonant second order differential equation

$$y'' + 2b\,y' + w^2 y = n(t) , \qquad (4.7)$$

---

* R_1 here lies either strictly between R_o and X or is equal to X. It may be noted that if X and R_o were vectors instead of scalars, there would be no such thing as an intermediate value, so that in that case R_1 would have to equal X, giving an equation analogous to Eq. 4.1.

where b and w are constants such that

$$b << w, \tag{4.8}$$

and n(t) is a stationary Gaussian variable, with zero mean and an auto-correlation function $\varphi(t)$ given by

$$\langle n(t_1+t)n(t)\rangle = \varphi(t). \tag{4.9}$$

It is assumed that n(t) is either white, or nearly white, such that there exists a correlation time t_c, much smaller than 1/w, for which

$$\int_{t_c}^{\infty} |\varphi(t)|\ dt << \int_0^{t_c} \varphi(t)\ dt = D\ . \tag{4.10}$$

The variable r will be defined as the positive square root of

$$r^2 = (py)^2 + (by + y')^2, \tag{4.11}$$

where p is given by

$$p^2 = w^2 - b^2\ . \tag{4.12}$$

The quantity r can be used as a bound on the displacement, velocity, or energy, for

$$|y| \leqslant r/p\ ,$$

$$|y'| \leqslant wr/p\ ,$$

$$\frac{r^2}{1 + b/w} \leqslant \tfrac{1}{2}(wy)^2 + \tfrac{1}{2}(y')^2 \leqslant \frac{r^2}{1 - b/w}\ .$$

It will be shown that the first passage problem when applied to the variable r can be approximated by one where the variable is the result of a one-dimensional Markov Process, and can be solved either by a boundary value approach of Section 4.1.1, or an integral equation approach of Section 4.1.2.

As in the notation used previously, let $T(R/R_o;t)$ represent the first passage probability density, so that $T(R/R_o;t)dt$ is the probability that $r = R$ for the first time in the time interval from t to t+dt, on the condition that $r = R_o$ at $t = 0$. From this first passage probability density, one can find the probability that $r = R$ at least once in the time interval from 0 to t (probability of failure), as well as the mean and mean square times to get from R_o to R. It will be shown in the following sections, that when

$$(R - R_o)^2 >> D/w,$$

the problem may be approximated by another problem where r is the result of a one-dimensional Markov Process, whose transition probability* is given by

$$P_T(r/r_o, t) =$$

$$\frac{2}{\frac{D}{b}(1-e^{-2bt})} \exp\left[-\frac{r^2+r_o^2 e^{-2bt}}{\frac{D}{b}(1-e^{-2bt})}\right] I_o\left[\frac{2rr_o e^{-bt}}{\frac{D}{b}(1-e^{-2bt})}\right]. \qquad (4.13)^{**}$$

The solution of this approximate problem is worked out in Section 4.3.

---

* The area increment associated with this probability density is rdr, so that in the strict sense it might not be considered a probability density. As r must be non-negative, P_T is zero for negative r.

** I_o is the modified Bessel function, $I_o(x) = J_o(ix)$.

4.2.1 DERIVATION OF THE APPROXIMATE TRANSITION PROBABILITY

As the noise, n(t), in Eq. 4.7, is Gaussian, y and y', as well as linear combinations of the two, must be Gaussian. Define x and z by the equations

$$x = by + y' \; , \tag{4.14a}$$

$$z = py \; , \tag{4.14b}$$

so that x and z have the first conditional probability density of the form

$$P_1(x, z/x_o, z_o;t) = \frac{\exp\left[-\dfrac{u_{xx}(z-m_z)^2 - 2u_{xz}(z-m_z)(x-m_x) + u_{zz}(x-m_x)^2}{2(u_{xx}u_{zz}-u_{xz}^2)}\right]}{2\pi[u_{xx}u_{zz}-u_{xz}^2]^{\frac{1}{2}}} \; , \tag{4.15}$$

where one has

$$m_x = \langle x \rangle \; , \tag{4.16a}$$

$$m_z = \langle z \rangle \; , \tag{4.16b}$$

$$u_{xx} = \langle x^2 \rangle - m_x^2 \tag{4.16c}$$

$$u_{xz} = \langle xz \rangle - m_x m_z \; , \tag{4.16d}$$

$$u_{zz} = \langle z^2 \rangle - m_z^2 \; . \tag{4.16e}$$

Each of the means and moments of Eqs. 4.16 are determined on the assumption that at t = 0, x and z have the values x_o and z_o respectively.

One can explicitly solve for x and z in terms of the initial conditions x_o and z_o and the noise term n(t). This solution may be represented in terms of a single complex equation,

$$x+iz = (x_o+iz_o)e^{-(b-ip)t} + \int_0^t n(t-t_1)\, e^{-(b-ip)t_1}\, dt_1 \, . \tag{4.17}$$

Eq. 4.17 can then be used to calculate the means and moments of Eqs. 4.16.

If now, for convenience, one defines the complex variable q by

$$q = x + iz, \tag{4.18}$$

and the quantities q_o, m_q, u_1, and u_2 by

$$q_o = x_o + iz_o, \tag{4.19a}$$

$$m_q = m_x + im_z, \tag{4.19b}$$

$$u_1 = \langle \, |q-m_q|^2 \rangle = u_{xx} + u_{zz} \, , \tag{4.19c}$$

$$u_2 = \langle (q-m_q)^2 \rangle = u_{xx} - u_{zz} + 2iu_{xz}, \tag{4.19d}$$

then the conditional probability, P_1 of Eq. 4.15, may be expressed as

$$P_1(x, z/x_o, z_o;t) = \frac{\exp\left\{-\dfrac{u_1|q-m_q|^2 - \text{Real}\ \overline{u_2}\ (q-m_q)^2}{u_1^2 - |u_2|^2}\right\}}{\pi[u_1^2 - |u_2|^2]^{\frac{1}{2}}} \, . \tag{4.20}$$

From Eq. 4.17, one has

$$q = q_o\, e^{-(b-ip)t} + \int_0^t n(t-t_1)e^{-(b-ip)t_1}\, dt_1 \, .$$

Thus, if it is assumed that the averaging process and integrations may be interchanged, it is possible to find m_q, u_1, and u_2. These are given by

$$m_q = q_o e^{-(b-ip)t} , \tag{4.21}$$

$$u_1 = \int_0^t dt_1 \int_0^t dt_2 \, \varphi(t_1-t_2) e^{-b(t_1+t_2)+ip(t_1-t_2)} , \tag{4.22a}$$

$$u_2 = \int_0^t dt_1 \int_0^t \varphi(t_1-t_2) e^{-(b-ip)(t_1+t_2)} . \tag{4.22b}$$

If only times larger than the correlation time, t_c of Eq. 4.10, are considered, then for $t > t_c$, Eqs. 4.22 yield

$$u_1 = \frac{D}{b}(1 - e^{-2bt}) , \tag{4.23a}$$

$$u_2 = \frac{D}{b-ip} (1 - e^{-2bt} e^{2ipt}) . \tag{4.23b}$$

One may use Eqs. 4.23 to show that

$$|u_2|/u_1 \leqslant \frac{b}{w} \coth bt,$$

and as the hyperbolic cotangent of x is bounded above by $1 + 1/x$,

$$|u_2| \leqslant u_1 \left(\frac{b}{w} + \frac{1}{wt}\right) . \tag{4.24}$$

Thus, for $b << w$ and $t >> 1/w$, one has

$$|u_2| << u_1 . \tag{4.25}$$

Similarly, as the real part of a complex function is less than or equal to its magnitude, one has for $b << w$ and $t >> 1/w$,

$$\left|\text{Real}\left[\overline{u_2}\,(q-m_q)^2\right]\right| << u_1\,|q-m_q|^2 \,. \tag{4.26}$$

Thus, for $t >> 1/w$, the inequalities of Eqs. 4.25 and 4.26 may be used with Eq. 4.20 to give the conditional probability density as

$$P_1(x, z/x_o, z_o;t) = \frac{1}{\pi u_1} \exp\left[-\frac{|q-m_q|^2}{u_1}\right]. \tag{4.27}$$

If one now changes to cylindrical coordinates, where the radius r is identical to the r defined by Eq. 4.11,

$$x = r \cos\theta \,,$$

$$z = r \sin\theta \,,$$

one may find the first conditional probability density for r and θ (where now the area differential is $rdrd\theta$) from Eq. 4.27, by noting that

$$q = r\,e^{i\theta} \,.$$

Denoting this first conditional probability by P_c, one has

$$P_c(r, \theta/r_o, \theta_o;t) = \frac{\exp\left[-\dfrac{r^2 - 2rr_o e^{-bt}\cos(\theta-\theta_o+pt)+r_o^2 e^{-2bt}}{\frac{D}{b}(1-e^{-2bt})}\right]}{\pi\,\frac{D}{b}\,(1-e^{-2bt})}\,, \tag{4.28}$$

provided that $t >> 1/w$.

If one integrates the θ dependence out of Eq. 4.28, what is obtained is the probability density for the variable r on the condition that at $t = 0$, r is r_o and θ is θ_o. However, the result of such an integration is independent of θ_o, so that the result yields the first conditional probability in the variable r alone. Performing this indicated integration yields

$$P_c(r/r_o,t) = \frac{\exp\left[-\frac{r^2+r_o^2 e^{-2bt}}{\frac{D}{b}(1-e^{-2bt})}\right] I_o\left[\frac{2rr_o e^{-bt}}{\frac{D}{b}(1-e^{-2bt})}\right]}{\frac{D}{2b}(1-e^{-2bt})} \quad , \tag{4.29}$$

where I_o is the modified Bessel function, $I_o(x) = J_o(ix)$.

Eq. 4.29 represents the first conditional probability for $t \gg 1/w$. If only time intervals much larger than $1/w$ are considered, then as the correlation time of the noise, t_c, is much less than $1/w$, no additional information is gained if values of r for t less than zero is known. This implies that for time intervals much larger than $1/w$, Eq. 4.29 not only represents the first conditional probability, but a transition probability as well. Thus for time intervals much larger than $1/w$,

$$P_T(r/r_o,t) = P_c(r/r_o,t) \ . \tag{4.30}$$

4.2.2 APPLICATION OF THE APPROXIMATE TRANSITION PROBABILITY

Thus far it has been shown that for large enough times, $t \gg 1/w$, the variable r appears as though generated by a first order, one-dimensional Markov Process, and hence its statistical properties are determined by a transition probability. However, for the solution of the approximate problem, the behavior for all time will be utilized. It is therefore necessary to make some comment about the small time behavior.

One may utilize the results of Section 4.2.1 to find a bound on the variance of r. In the notation of Section 4.2.1 the variance is given by u_1 (Eq. 4.22a). It is possible to use Eq. 4.10 with Eq. 4.22a to show that

$$u_1 \leqslant 2Dt \; . \tag{4.31}$$

If one is interested in only changes of r greater than some value Q, such that there is a t_1 for which

$$Q^2/2D >> t_1 >> 1/w, \text{ and } 1/b >> t_1 >> 1/w,$$

then for times less than t_1 the probability that any significant change in r (in comparison to Q) can take place is very small. This is because the mean of r cannot change (as $bt << 1$ for $t < t_1$) and the variance, 2Dt, is much less than Q^2. This suggests that time differences smaller than t_1 are microscopic enough to be ignored on a larger scale, in analogy with the distance between molecules as compared with the size of objects in mechanics.

For times larger than t_1, the behavior of r is governed by the transition probability of Eq. 4.29. Hence, in solving the first passage problem for r to get from R_o to R_1 where

$$(R_1 - R_o)^2/2D >> 1/w \; , \tag{4.32}$$

the problem may be approximated by a first passage problem where r is governed by the transition probability*

* The area differential for this probability density is rdr, and P_T is zero for negative values of r.

$$P_T(r/r_o,t) = \frac{2}{\frac{D}{b}(1-e^{-2bt})} \exp\left[-\frac{r^2+r_o^2 e^{-2bt}}{\frac{D}{b}(1-e^{-2bt})}\right] I_o \left[\frac{2rr_o e^{-bt}}{\frac{D}{b}(1-e^{-2bt})}\right] . \tag{4.33}$$

4.3.0 SOLUTION OF THE APPROXIMATE PROBLEM

Let $T(r/r_o,t)$ represent the first passage probability density for r. Hence, $T(R/r_o,t)dt$ represents the probability that $r = R$ for the first time in the time interval $[t,t+dt)$ on the condition that $r = r_o$ at time $t = 0$. As in the derivation of Eq. 4.6, when R_1 lies strictly between R_o and R or when R_1 is equal to R, one may write

$$P_T(R/R_o,t) = \int_0^t T(R_1/R_o,t_1)\, P_T(R/R_1,t-t_1)\, dt_1 \; ; \tag{4.34}$$

where P_T is the transition probability of Eq. 4.33.

The convolution equation, Eq. 4.34, is most easily solved through the use of Laplace transforms. Let $H(r/r_o,s)$ and $L(r/r_o,s)$ represent the Laplace transforms of $T(r/r_o,t)$ and $P_T(r/r_o,t)$ so that

$$H(r/r_o,s) = \int_0^\infty T(r/r_o,t)\, e^{-st}\, dt,$$

$$L(r/r_o,s) = \int_0^\infty P_T(r/r_o,t)\, e^{-st}\, dt.$$

By transforming the convolution equation, Eq. 4.34, one obtains the result

$$H(R_1/R_o,s) = \frac{L(R/R_o,s)}{L(R/R_1,s)} , \tag{4.35}$$

provided that R_1 lies strictly between R_o and R or R_1 equals R. Thus, to find the first passage probability density, one need only transform the transition probability, and use Eq. 4.35 to find the transform of the first passage probability density. An inverse transform then will yield the first passage probability density.

From the $H(R_1/R_o, s)$ one can calculate the mean and mean square times to get from R_o to R_1. If these are denoted by $\langle t_p \rangle$ and $\langle t_p^2 \rangle$, then when the limits and integrals exist,

$$\langle t_p \rangle = \int_0^\infty t\, T(R_1/R_o, t)dt = -\lim_{s\to 0} \frac{\partial H(R_1/R_o, s)}{\partial s} , \tag{4.36}$$

$$\langle t_p^2 \rangle = \int_0^\infty t^2 T(R_1/R_o, t)dt = \lim_{s\to 0} \frac{\partial^2 H(R_1/R_o, s)}{\partial s^2} . \tag{4.37}$$

Further, one can calculate the "probability of failure." If $Q(R_1/R_o, t)$ is the probability that $r = R_1$ at least once in the time interval from 0 to t, on the condition that $r = R_o$ at $t = 0$, then

$$Q(R_1/R_o, t) = \int_0^t T(R_1/R_o, t_1)\, dt_1 ,$$

or in terms of the transform of T, one has

$$Q(R_1/R_o, t) = \int_{\sigma - i\infty}^{\sigma + i\infty} \frac{H(R_1/R_o, s)\, e^{st}}{2\pi i s}\, ds . \tag{4.38}$$

It will be shown that the probability of failure can also be determined by solving an appropriate boundary value problem. The Fokker-Planck Equation for the transition probability of Eq. 4.33 is given by

$$\frac{\partial P}{\partial t} = 2bP + br \frac{\partial P}{\partial r} + \frac{D}{2r} \frac{\partial}{\partial r} [r \frac{\partial P}{\partial r}] . \tag{4.39}$$

Assume now that $R_1 > R_o$. Though Eq. 4.39 is an equation for the transition probability (and must be satisfied for all r), if one finds a solution $P(r/R_o,t)$ which satisfies Eq. 4.39 (for $r < R_1$), along with the initial condition

$$P(r/R_o,0) = \frac{1}{R_o}\,\delta(r-R_o) \; ,$$

and the "absorbing" boundary condition

$$P(R_1/R_o,t) = 0 \; ,$$

then a purely physical argument might lead to the following: "$P(r/R_o,t)$ will represent the probability density for r, on the condition that $r = R_o$ when $t = 0$, where the absorbing boundary at $r = R_1$ acts in a way to keep r from ever decreasing below R_1 once it has reached it. Thus the probability that $r < R_1$ for all time from $t = 0$ to $t = T$ will be given by

$$\int_0^{R_1} P(r/R_o,T)\, r\, dr \; ."$$

If one follows this line of thought further, then one would conclude that "the probability of failure, $Q(R_1/R_o,t)$, will be given by

$$Q(R_1/R_o,t) = 1 - \int_0^{R_1} P(r/R_o,t)\, r\, dr." \qquad (4.40)$$

The validity of Eq. 4.40 will be demonstrated in Section 4.3.2.2, by a direct comparison with $Q(R_1/R_o,t)$ as found by Eq. 4.38. Results obtained by using this boundary value problem approach may be found in reference (17).

4.3.1 EVALUATION OF THE TRANSFORM OF THE TRANSITION PROBABILITY

To find $L(r/r_o,s)$ one must evaluate the integral

$$L(r/r_o, s) = \int_0^\infty P_T(r/r_o, t)\, e^{-st}\, dt, \tag{4.41}$$

where P_T is given by Eq. 4.33. If one makes the change of variables,

$$e^{-2bt} = a/(1+a),$$

$$a = 1/(e^{2bt} - 1),$$

the integral becomes

$$L(r/r_o, s) = \int_0^\infty \frac{2b}{D} \frac{a^{s/2b}}{(1+a)^{s/2b}} \frac{1}{a} \exp\left[-\frac{r^2(1+a)+ar_o^2}{D/b}\right]$$

$$\times\ I_o\left[\frac{2brr_o\ \sqrt{a(1+a)}}{D}\right] da\ .$$

The result of this integration has been found by MacRobert,* and is given by

$$L(r/r_o, s) = \frac{{}_1F_1\ (s/2b, 1; br^2/D)\ E(s/2b; s/2b: :br_o^2/D)}{D\ \ e^{br^2/D}\ (br^2/D)^{s/2b}\ \ \Gamma(s/2b)} \tag{4.41a}$$

when $r < r_o$. When $r > r_o$ the integration yields

$$L(r/r_o, s) = \frac{{}_1F_1(s/2b, 1; br_o^2/D)\ E(s/2b; s/2b: :br^2/D)}{D\ e^{br^2/D}\ (br^2/D)^{s/2b}\ \ \Gamma(s/2b)} \tag{4.41b}$$

The ${}_1F_1$ represents the confluent hypergeometric function, and the E is the MacRobert's E function. For simplicity of notation, ${}_1F_1$ will be

*See Eq. 15, p. 470 of Reference (18).

denoted by Humbert's symbol Φ where*

$$ {}_1F_1(a,c;x) = \Phi(a,c;x) \ . \qquad (4.42) $$

Further the ψ function will be used instead of the MacRobert's E function. They are related by*

$$ E(a,b::x) = \Gamma(a)\Gamma(b)\, x^a \psi(a, a-b+1; x) \ . \qquad (4.43) $$

Thus, the transform of Eq. 4.41 may be expressed as

$$ L(r/r_o, s) = \frac{1}{D} e^{-br^2/D}\, \Phi(s/2b, 1; br^2/D)\Gamma(s/2b)\, \psi\, (s/2b, 1; br_o^2/D) \qquad (4.44a) $$

when $r < r_o$, and

$$ L(r/r_o, s) = \frac{1}{D} e^{-br^2/D}\, \Phi(s/2b, 1; br_o^2/D\ \Gamma(s/2b)\, \psi(s/2b, 1; br^2/D) \qquad (4.44b) $$

when $r > r_o$.

4.3.1.1 Properties of $\Phi(a,1;x)$**

$\Phi(a,1;x)$ or ${}_1F_1(a,1;x)$ is given by the series

$$ \Phi(a,1;x) = \sum_{k=0}^{\infty} \frac{\Gamma(a+k)}{\Gamma(a)} \frac{x^k}{(k!)^2} \ , $$

and has the property that it is a solution to the differential equation

* See Reference (19).
**Many of these, and other properties, may be found in Reference (19).

$$x\,y''(x) + (1-x)\,y'(x) - a\,y(x) = 0,$$

that is well behaved as x goes to zero.

For small a, one may write

$$\Phi(a,1;x) = 1 + a\sum_{k=1}^{\infty} \frac{x^k}{(k!)k} + a^2\sum_{k=2}^{\infty}\left[\sum_{r=1}^{k-1} \frac{1}{r}\right]\frac{x^k}{(k!)k} + 0(a^3), \tag{4.45}$$

where $0(a^3)$ represents terms of the order of a^3 or higher. Further, this function has the limiting property

$$\lim_{x\to 0} \Phi(b/x,1;x) = \sum_{k=0}^{\infty} \frac{b^k}{(k!)^2} = I_o(2b^{\frac{1}{2}}), \tag{4.46}$$

where I_o is the modified Bessel function.

The Sturm-Liouville problem defined by

$$x\,y''(x) + (1-x)\,y'(x) + \lambda^2 y(x) = 0 \tag{4.47a}$$

$$y(L) = 0, \tag{4.47b}$$

yields a complete orthogonal set of solutions given by $\Phi(-\lambda_k{}^2,1;x)$, where the Eigen values given by the λ_k are real and positive. The Eigen functions are orthogonal with respect to the weighting function e^{-x}, so that

$$\int_0^L \Phi(-\lambda_k{}^2,1;x)\,\Phi(-\lambda_n{}^2,1;x)\,e^{-x}\,dx = 0 \quad \text{for } k \neq n.$$

It is possible to use the differential equation and end condition that $\Phi(-\lambda_k{}^2,1;x)$ must satisfy, Eqs. 4.47, to show that

$$\int_0^L e^{-x}\Phi^2(-\lambda_k{}^2,1;x)dx = -Le^{-L}\left[\frac{\partial\Phi(r,1;x)}{\partial r}\;\frac{\partial\Phi(r,1;x)}{\partial x}\right]_{\substack{x=L\\ r=-\lambda_k^2}} \tag{4.48}$$

and further that

$$\int_0^L e^{-x}\Phi(-\lambda_k^2,1;x)dx = -\frac{Le^{-L}}{\lambda_k^2}\left[\frac{\partial\Phi(r,1;x)}{\partial x}\right]_{\substack{x=L\\ r=-\lambda_k^2}} \tag{4.49}$$

4.3.1.2 Properties of $\psi(a,1;x)$*

$\psi(a,1;x)$ is related to the MacRobert's E function, $E(a,a::x)$, and to the hypergeometric function, ${}_2F_0(a,a;-1/x)$, by the equations

$$E(a,a::x) = \Gamma^2(a)\, x^a\psi(a,1;x),$$

$${}_2F_0(a,a;-1/x) = x^a\psi(a,1;x) .$$

$\psi(a,1;x)$ is defined by the series,

$$\psi(a,1;x) = \frac{-1}{\Gamma(a)}\ \Phi(a,1;x)\log x$$

$$+\sum_{k=0}^{\infty}\frac{\Gamma(a+k)}{\Gamma(a)}\ [\psi(a+k) - 2\psi(1+k)]\ \frac{x^k}{(k!)^2} ,$$

where $\psi(r)$ represents the logarithmic derivative of the gamma function $\Gamma(r)$,

$$\psi(r) = \frac{d}{dr}\log\ \Gamma(r) .$$

For small a, one may write

$$a\Gamma(a)\psi(a,1;x) = 1 - a(\gamma + \log x)$$

$$+ a^2\sum_{k=1}^{\infty}\left[-\frac{1}{k^2} + \frac{2x^k}{k^2(k!)} + \frac{\psi(k)x^k}{k(k!)} - \frac{(\log x)x^k}{k(k!)}\right] + 0(a^3), \tag{4.50}$$

* Many of these, and other properties, may be found in Reference (19).

where γ is Euler's or Masheroni's constant, 0.5772...

The function $\psi(a,1;x)$ has the limiting property

$$\lim_{x\to 0} \Gamma(b/x)\,\psi(b/x,1;x) = 2\,K_o(2\,b^{\frac{1}{2}}) \,, \tag{4.51}$$

where K_o is the modified Bessel function defined by

$$K_o(x) = -(\log \tfrac{1}{2}x)\,I_o(x) + \sum_{k=0}^{\infty} \psi(k+1)\,\frac{(\frac{1}{2}x)^{2k}}{(k!)^2} \,.$$

4.3.1.3 Special Case of Zero Damping

When the damping term, b, is zero, the transition probability of Eq. 4.33 takes on the simpler form

$$P_T(r/r_o,t) = \frac{1}{Dt}\,\exp\left[-\frac{r^2+r_o^{\,2}}{2Dt}\right]\,I_o\left[\frac{r\,r_o}{Dt}\right] \,. \tag{4.52}$$

The Laplace transform of this may be found in tables (20), and is identical to the limit of the transform of Eqs. 4.41 as b goes to zero. This transform is

$$L(r/r_o,s) = \begin{cases} \frac{2}{D}\,I_o\,[(2s/D)^{\frac{1}{2}}\,r]\;K_o\,[(2s/d)^{\frac{1}{2}}\,r_o] & \text{for} \quad r < r_o, \\ \\ \frac{2}{D}\,I_o\,[(2s/D)^{\frac{1}{2}}\,r_o]\;K_o\,[(2s/D)^{\frac{1}{2}}\,r] & \text{for} \quad r > r_o\,. \end{cases}$$

As the results for zero damping are identical to those where the damping tends to zero, such results will not be treated separately.

4.3.2 PROBABILITY OF FAILURE

Assume that the system starts with an initial condition, $r = R_o$ at $t = 0$. The probability of failure, $Q(R/R_o,t)$, will be defined as the probability that $r = R$ at least once during the time interval from zero to t, on the condition that at $t = 0$, $r = R_o$. When $R = R_o$, $1-Q(R/R_o,t)$ will represent the probability that $r < R$ for all time from 0 to t. This is the only case that will be treated here.

4.3.2.1 Inverse Transform Solution

From Eq. 4.35, the Laplace transform of the first passage probability density, $T(r/r_o,t)$ will be given by

$$H(r/r_o,s) = \frac{L(r_1/r_o,s)}{L(r_1/r,s)} \quad \text{for } r_1 \geqslant r > r_o \; .$$

Thus, from Eq. 4.44b, one finds

$$H(r/r_o,s) = \frac{\Phi(s/2b,1;br_o^2/D)}{\Phi(s/2b,1;br^2/D)} \, , \tag{4.53}$$

for $r > r_o$. Using Eq. 4.38 with Eq. 4.53 yields

$$Q(R/R_o,t) = \int_{\sigma-i\infty}^{\sigma+i\infty} \frac{e^{st}}{s} \, \frac{\Phi(s/2b,1;bR_o^2/D)}{\Phi(s/2b,1;bR^2/D)} \, \frac{ds}{2\pi i} \, . \tag{4.54}$$

The only singularities of the integrand are at $s = 0$, and at $s = -2b\lambda_k^2$ for $k = 1, 2$, etc., where the λ_k are solutions to the equation

$$\Phi(-\lambda_k^2,1;bR^2/D) = 0 \, . \tag{4.55}$$

From Section 4.3.1.1 it is seen that the λ_k are real (they are identical to the eigen values called λ_k in that section, when $L = bR^2/D$). Thus the singularities of the integrand all have a non-positive real part. One can use residue calculus to show that the integration of Eq. 4.56 yields

$$Q(R/R_o,t) = 1 - \sum_{k=1}^{\infty} \frac{\Phi(-\lambda_k^2,1;bR_o^2/D)\, e^{-2b\lambda_k^2 t}}{\lambda_k^2 \; A_k} \quad , \tag{4.56}$$

where the A_k are defined by

$$A_k = \left[\frac{\partial\Phi(x,1;bR^2/D)}{\partial x} \right]_{x=-\lambda_k^2} \quad , \tag{4.57}$$

and the λ_k are given by Eq. 4.55. As will be demonstrated in Section 4.3.2.2, this result is identical to the result obtained by a boundary value approach. As the behavior of such a result has been discussed by Rosenblueth and Bustamante (17), it will not be discussed here.

4.3.2.2 The Boundary Value Solution

As pointed out in Section 4.3.0, an artificial approach to the problem is to define the probability of failure as in Eq. 4.40 by

$$Q(R/R_o,t) = 1 - \int_0^R P(r/R_o,t)\; r\, dr, \tag{4.58}$$

where $P(r/R_o,t)$ is a solution to the boundary value problem

$$\frac{\partial P}{\partial t} = 2\,b\,P + b\,r\,\frac{\partial P}{\partial r} + \tfrac{1}{2} D \frac{1}{r} \frac{\partial}{\partial r}\left(r \frac{\partial P}{\partial r}\right) , \tag{4.59a}$$

$$P(R/R_o,t) \;=\; 0, \tag{4.59b}$$

$$P(r/R_o,0) \;=\; \frac{1}{R_o}\,\delta(r-R_o)\;. \tag{4.59c}$$

Using standard techniques, and the λ_k as defined by Eq. 4.55, the orthogonality of the $\Phi(-\lambda_k^2,1;bR^2/D)$ leads to*

* See Section 4.3.1.1 for the properties of the Φ function.

$$P(r/R_o,t) = \frac{2b}{D}\, e^{-br^2/D} \sum_{k=1}^{\infty} C_k \Phi(-\lambda_k{}^2, 1; br^2/D)\, e^{-2b\lambda_k t}, \tag{4.60}$$

where the C_k are given by

$$C_k = \frac{\Phi(-\lambda_k{}^2, 1; bR_o{}^2/D)}{\int_0^{bR^2/D} e^{-x}\, \Phi^2(-\lambda_k{}^2, 1; x)\, dx} \tag{4.61}$$

Using Eqs. 4.58, 4.60, and 4.61, one finds that

$$Q(R/R_o,t) = 1 - \sum_{k=1}^{\infty} \Phi(-\lambda_k{}^2, 1; bR_o{}^2/D)\, B_k\, e^{-2b\lambda_k{}^2 t}, \tag{4.62}$$

where the B_k are given by

$$B_k = \frac{\int_0^{bR^2/D} e^{-x}\, \Phi(-\lambda_k{}^2, 1; x)\, dx}{\int_0^{bR^2/D} e^{-x}\, \Phi^2(-\lambda_k{}^2, 1; x)\, dx}\ . \tag{4.63}$$

If one now uses Eqs. 4.48 and 4.49, then with B_k given by Eq. 4.63 and with the A_k given by Eq. 4.57, one finds

$$B_k = 1/A_k \lambda_k{}^2\ .$$

Thus the $Q(R/R_o,t)$ obtained by solving the boundary value problem, given by Eq. 4.62, is identical to the $Q(R/R_o,t)$ found by solving the integral equation, Eq. 4.56.

4.3.3 MEAN AND MEAN SQUARE TIME TO FAILURE

Assume again that $r > r_o$, so that the transform of the first passage probability is given by Eq. 4.53, rewritten here as Eq. 4.64:

$$H(r/r_o, s) = \frac{\Phi(s/2b, 1; br_o^2/D)}{\Phi(s/2b, 1; br^2/D)} \quad . \tag{4.64}$$

Using the results of Eqs. 4.36 and 4.37, it is seen that when the mean and mean square times to failure are given by $\langle t_p \rangle$ and $\langle t_p^2 \rangle$ then

$$H(r/r_o, s) = 1 - \langle t_p \rangle \, s + \tfrac{1}{2} \langle t_p^2 \rangle \, s^2 + o(s^2) \ , \tag{4.65}$$

where $o(s^2)$ or "little o of s^2" vanishes faster than s^2 as s goes to zero.

If one expands the hypergeometric functions in Eq. 4.64 with the aid of Eq. 4.45, one finds that

$$\begin{aligned} H(r/r_o, s) = 1 &- \frac{s}{2b} [g(br^2/D) - g(br_o^2/D)] \\ &+ \tfrac{1}{2}(s/2b)^2 \{h(br^2/D) - h(br_o^2/D) + [g(br^2/D) - g(br_o^2/D)]^2\} \\ &+ 0(s^3) \ , \end{aligned} \tag{4.66}$$

where g(x) and h(x) are series defined by

$$g(x) = \sum_{k=1}^{\infty} \frac{x^k}{k(k!)} \quad , \tag{4.67a}$$

$$h(x) = 2 \sum_{n=2}^{\infty} \sum_{k=1}^{n-1} \left[\binom{n}{k} - 1 \right] \frac{1}{k} \frac{x^n}{n(n!)} \quad . \tag{4.67b}$$

From this, one may use Eq. 4.65 to find expressions for the mean and mean square times to failure,*

---

* Time to increase from r_o to r.

$$\langle t_p \rangle = \frac{1}{2b} \left[g(br^2/D) - g(br_o^2/D) \right] \tag{4.68a}$$

$$\langle t_p^2 \rangle = \langle t_p \rangle^2 + \frac{1}{4b^2} \left[h(br^2/D) - h(br_o^2/D) \right] . \tag{4.68b}$$

When the damping, b, is small, so that $br^2 << D$, one obtains the result

$$\langle t_p \rangle = (r^2 - R_o^2)/2D ,$$

$$\langle t_p^2 \rangle = \langle t_p \rangle^2 + (r^4 - r_o^4)/8D^2.$$

For non-zero damping, the results for the mean time to failure are easily tabulated, as the function g(x) defined by Eq. 4.67a can be expressed in terms of the exponential integral, $\overline{Ei}(x)$, which can be found in various tables such as Jahnke and Emde (21). In particular,

$$g(x) = \sum_{k=1}^{\infty} \frac{x^k}{k(k!)} = \int_0^x (e^t - 1) \frac{dt}{t} = \overline{Ei}(x) - \gamma - \log x , \tag{4.69}$$

where γ is Euler's or Masheroni's constant, 0.5772... .

From Eq. 4.68a, the mean time to failure may be expressed as

$$\langle t_p \rangle = \frac{r^2}{2D} \frac{g(br^2/D)}{br^2/D} - \frac{r_o^2}{2D} \frac{g(br_o^2/D)}{br_o^2/D} ,$$

and thus the function g(x)/x plays a major role in determining $\langle t_p \rangle$. The table below gives some representative values of g(x)/x as a function of x

x	g(x)/x	x	g(x)/x
0.01	1.003	1	1.318
0.02	1.005	2	2.035
0.03	1.008	3	3.485
0.04	1.010	4	5.110
0.05	1.013	5	8.244
0.06	1.015	6	14.50
0.07	1.018	7	29.55
0.08	1.020	8	55.23
0.09	1.023	9	115.23
0.1	1.026	10	249.40
0.2	1.052	11	552.1
0.3	1.080	12	1246.8
0.4	1.110	13	2861.3
0.5	1.140	14	6656.8
0.6	1.172	15	15664
0.7	1.206		
0.8	1.242		
0.9	1.279		

For large values of x, one can use an asymptotic expansion for g(x)/x given by

$$\frac{g(x)}{x} = (e^x - 1 - x)/x^2 + (e^x - 1 - x - \tfrac{1}{2}x^2)/x^3 + \ldots$$

$$+ \frac{(n-2)!}{x^n}\left(e^x - \sum_{k=0}^{n-1} \frac{x^k}{k!}\right) + \ldots ,$$

and thus approximate g(x)/x by $x^{-2}e^x$ for very large x.

4.3.4 MEAN TIME TO DECREASE FROM r_o TO r

When $r_o > r$, as in the previous case, one can derive the Laplace transform for the first passage probability. This yields

$$H(r/r_o, s) = \frac{\psi(s/2b, 1; br_o^2/D)}{\psi(s/2b, 1; br^2/D)} .$$

As in the previous case, one may calculate the mean and mean square time for passage from r_o to r, this time by utilizing the expansion given by Eq. 4.50. Proceeding in this manner, if the mean time of passage is denoted by $\langle t_p \rangle$, then one finds

$$\langle t_p \rangle = \frac{1}{b} \log(r_o/r) \quad .$$

If it is noted that this expression is independent of D, then it can be seen that the mean time to decrease from r_o to r will be independent of n(t), the noise for which D is a measure (see Eqs. 4.9 and 4.10). In fact, if one considers the case for which there is no random term n(t), then it is possible to show, without assumptions, that

$$r = r_o \, e^{-bt} \quad ,$$

and thus the time taken to decrease from r_o to r, in the presence of the noise term, is exactly $\frac{1}{b} \log(r_o/r)$. Hence, the addition of the random term n(t) will not, in the framework of our approximations, change the mean time for a decrease.

4.4.0 SUMMARY

In this section the first passage problem for the variable r, where

$$r^2 = (by + dy/dt)^2 + (w^2 - b^2)\, y^2$$

and

$$\frac{d^2y}{dt^2} + 2b\,\frac{dy}{dt} + w^2 y = n(t)$$

has been solved on the basis of certain assumptions. In particular, n(t) is white with the autocorrelation function

$$\langle n(t)n(t_1)\rangle = 2D\delta\ (t-t_1)\ ,$$

or almost white as described in Section 4.2.0. Further, the system is highly resonant so that $b << w$ and only variations r much larger than $\sqrt{D/w}$ and time intervals much larger than $1/w$ are to be considered. For this case, the probability of failure is found in Sections 4.3.2.1 and 4.3.2.2 and its behavior is discussed in more detail in Reference (17). The mean and mean square times to failure are given in Section 4.3.3.

The example considered here is only one example of the problems of this sort, and unfortunately the approximations used here cannot be further extended to cover other cases. There remains to be solved the whole field of first passage problems, in that present techniques are applicable only to one-dimensional Markov processes or one-dimensional approximations of higher order processes.

In principle one can always solve for the first passage probability $T(z/x,t)$ by using Laplace transforms and the convolution equation, Eq. 4.1. However two basic problems arise. The first is strictly a computational problem, that of finding the Laplace transforms needed. Secondly, even if one could find $T(x/z,t)$, it would not be particularly useful when x and z are vectors. It yields, to be sure, the statistics for the passage time between two points in phase space, but in practical problems one is usually interested only in a single coordinate in phase space, such as the displacement, and other coordinates, such as velocity, are only in the way.

5.0 CONCLUSIONS AND SUMMARY

It has been shown that the system of differential equations given by

$$\frac{dy_k}{dt} = a_k(y,t) + h_{ki}(y,t)n_i(t) \quad \text{for} \quad k = 1, 2, \ldots, n, \tag{5.1}$$

where summation convention is implied, the a_k and h_{ki} are known functions of y and t, and the $n_i(t)$ are Gaussian White Noise such that

$$\langle n_i(t)n_j(t_1)\rangle = 2\, B_{ij}\, \delta(t-t_1)\ , \tag{5.2}$$

represents a first order, n dimensional, continuous Markov Process, whose Fokker-Planck Equation, according to Eq. 2.37, is given by

$$\frac{\partial P}{\partial t} = -\frac{\partial(a_k P)}{\partial y_k} + B_{rj}\frac{\partial}{\partial y_k}\left[h_{kj}\frac{\partial(h_{ir}P)}{\partial y_i}\right]. \tag{5.3}$$

As pointed out in the discussion of Section 2.2.1, this is contradictory to the result arrived at by some authors.

As proven in Section 1.3.1, well-behaved solutions to Eq. 5.3 with given initial conditions are unique. Further, if the h_{kj} are zero for $k \leqslant p$, where p is any integer $0, 1, 2, \ldots, n-1$, less than n, then one can show that Eq. 5.3 represents a "steady" Fokker-Planck Equation in the sense of Definition 2, Section 1.3.0, if and only if the coefficients $c_{ik}(y,t)$ defined by

$$c_{ik} = B_{rj}\, h_{kj}\, h_{ir}$$

represent the elements of a strictly positive definite matrix such that for any vector with components x_k,

$$c_{ik} x_i x_k \geqslant 0 ,$$

with equality holding if and only if all the x_k for $p < k \leqslant n$ are zero. When the equation is steady, solutions will tend asymptotically towards each other in the sense of Section 1.3.2, and if there is a steady state solution to the Fokker-Planck Equation, it will be unique.

It has been demonstrated that the linear differential equation

$$\frac{d^{n+1}y}{dt^{n+1}} + \sum_{k=0}^{n} [b_k + a_k(t)] \frac{d^k y}{dt^k} = a_d(t) + f(t) , \qquad (5.4)$$

where the $a_k(t)$ are Gaussian White Noise, represents a simple form of a (n+1)-dimensional Markov Process, for which the stability of the moments, $\langle y \rangle$, $\langle y^2 \rangle$, etc., can be readily determined using appropriate moment equations and Laplace transform techniques. Further when the system is mean square stable so that $\langle y^2 \rangle$ is bounded, then it is "equivalent," in the sense of having the same mean and same average power density spectrum, to the system defined by the equation

$$\frac{d^{n+1}y}{dt^{n+1}} + \sum_{k=0}^{n} (b_k - D_{nk}) \frac{d^k y}{dt^k} = a(t) + f(t) - D_{dn} , \qquad (5.5)$$

where a(t) is a Gaussian White noise term, chosen as described in Section 2.4.2, and the D_{ik} are coefficients arising from the correlation of the $a_i(t)$ terms, according to Eq. 2.52.

In Section 3, some sufficient conditions for stability of linear systems with non-white parametric excitation were derived by assuming that the excitation was ergodic. The techniques used, unfortunately, give no insight into the problem of finding necessary conditions for stability, and only in the case of white parametric excitation can stability boundaries actually be obtained, and then only in simple cases, such as the linear problem of Eq. 5.4.

Related to the problem of determination of stability requirements is the first passage problem, discussed in Part 4. Therein it is pointed out that the problems of this sort can only be solved exactly in simple cases involving one-dimensional Markov Processes. In Part 4, the resonant system defined by

$$\frac{d^2y}{dt^2} + 2b\frac{dy}{dt} + w^2 y = n(t),$$

where n(t) is Gaussian White (or almost white) and $b \ll w$ is converted approximately to a one-dimensional Markov Process in terms of a new random variable, r, where

$$r^2 = (w^2 - b^2)y^2 + \left(by + \frac{dy}{dt}\right)^2 ,$$

and the first passage problem for this one-dimensional Markov Process is solved.

A number of unsolved problems are discussed in the concluding sections of each part (Sections 1.4.0, 2.6.0, 3.6.0, and 4.4.0), and these include such problems as the proving of the existence of steady state solutions of the Fokker-Planck Equation, reducing the order of the Fokker-Planck Equation, evaluating stability boundaries for linear systems with non-white excitation, finding solutions to more general first passage problems, etc. There are other unsolved problems in addition to these. One of the most basic is determining the statistics of the output of a linear filter, when the input is random but not Gaussian.

One can, in principle, find the moments of the output in terms of the correlation moments of the input, but for any moments higher than the second this becomes a burdensome task.

The literature in the field of stochastic processes is extensive, and no attempt at a bibliography will be made here. A recent bibliography listing other bibliographies as well as the articles themselves may be found in Reference (22).

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