I. INTERACTIONS OF FAST AND SLOW WAVES IN PROBLEMS WITH TWO TIME SCALES

II. A NUMERICAL EXPERIMENT ON THE STRUCTURE OF TWO-DIMENSIONAL TURBULENT FLOW

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Abstract

I. Interaction of Fast and Slow Waves in Problems with Two Time Scales.

We consider certain symmetric, hyperbolic systems of nonlinear first-order partial differential equations whose solutions vary on two time scales, a 'slow' scale t and a 'fast' scale t/ε. The large ($O(\varepsilon^{-1})$) part of the spatial operator is assumed to have constant coefficients, but a nonlinear term multiplying the time derivatives (a 'symmetriser') is allowed.

In physical applications, it is often the case that the fast scale motion is of little interest, and it is desired to calculate only the slow scale motion accurately. It is known that solutions with arbitrarily small amounts of fast scale motion can be obtained by careful choice of the initial data, and that an error of amplitude $O(\varepsilon^p)$, where $p=2$ for one space dimension or $p=3$ for two or three space dimensions, in this choice is allowable, resulting in fast scale waves of amplitude $O(\varepsilon^p)$ in the solution.

We investigate what happens when the initial data are not prepared correctly for the suppression of the fast scale motion, but contain errors of amplitude $O(\varepsilon)$. We show that then the perturbation in the solution will also be of amplitude $O(\varepsilon)$. Further, we show that if the large part of the spatial operator is nonsingular in the sense that the number of large eigenvalues of the symbol, $P(i\omega)$, of the spatial operator is independent of ω, then the error introduced in the slow scale motion will be of amplitude $O(\varepsilon^2)$, even though fast scale waves of amplitude $O(\varepsilon)$ will be present in the solution. If the symmetriser is a constant, this holds even if the spatial operator is singular, and further if an error $O(\varepsilon^\mu)$ is made in the initial conditions, for any $\mu > 0$, the resulting error in the slow scale motion will be $O(\varepsilon^{2\mu})$.

Our proofs are based on energy estimates which show that spatial derivatives of the solutions are $O(1)$, even if time derivatives are not, and the

development of the solutions in asymptotic expansions.

II. A Numerical Experiment on the Structure of Two-Dimensional Turbulent Flow.

Some previous theories and numerical calculations pertaining to the problem of two-dimensional turburlence are reviewed, and a new numerical experiment is proposed. The purpose of the experiment is to test the hypothesis that narrow regions of concentrated vorticity are produced in two-dimensional flows by advection of vorticity towards dividing streamlines in regions where the local flow is convergent.

The numerical method to be used is described in detail. It integrates the inviscid Euler equations using a Fourier (pseudo-spectral) method for the space derivatives, and a predictor-corrector method due to Hyman (1979) for time stepping. Dissipation is included, following Fornberg (1977), by a chopping of the amplitudes of the higher Fourier modes every few time-steps. This acts as a high-wavenumber energy sink, allowing very high Reynolds number flows to be simulated with relatively little computational effort.

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Part I

INTERACTIONS OF FAST AND SLOW WAVES IN PROBLEMS WITH TWO TIME SCALES

Chapter 1: HYPERBOLIC SYSTEMS WITH TWO TIME SCALES

1.1 Introduction

This thesis is concerned with hyperbolic systems of partial differential equations which have solutions varying on two distinct time scales, a 'slow' scale t, and a 'fast' scale t/ε, where ε is some small parameter. That is, the linearised equations have (at least) two classes of normal mode solutions, one associated with each time scale.

Such systems arise in the description of several physical systems, for example in meteorology, oceanography, acoustics or magnetohydrodynamics. Specific examples are given below. Often it is the case that the fast scale motion is absent from the actual physical system being modelled, or considered relatively unimportant therein, and, in a numerical model of the system, it is desired to compute only the slow scale motion accurately.

Unfortunately, it has been observed that unless care is taken in choice of the initial data for the numerical model, large amplitude fast scale waves are excited early in the calculation, obscuring and possibly destroying the underlying slow scale motion. In particular, data obtained from measurement or observation of a real physical system will excite fast waves not present physically, because of inevitable errors in the collection process.

Various schemes for eliminating these spurious fast scale waves by an appropriate 'initialisation' have been proposed, mostly by meteorologists. Some of these schemes are mentioned in section 1.4 below. In this thesis, we are particularly interested in the 'method of bounded derivatives', proposed by Kreiss [5,12,13], which is described in some detail in section 1.3 below, and which, for reasons outlined there, this author believes to be the most useful scheme so far

suggested.

In brief, the bounded derivative method is based on the observation that time derivatives of the slow scale motion are $O(1)$, whereas those of the fast scale motion $\sim\varepsilon^{-1}$. Thus, solutions in which the fast scale motion is of amplitude $O(\varepsilon^p)$ must have p time derivatives bounded independently of ε at all times, in particular at $t=0$. Kreiss has proven rigorously, for quite general systems, that if the initial data are adjusted to ensure that the solution and a number p of its time derivatives are $O(1)$ at $t=0$, then they will remain so on some finite time interval [0,T], where T is independent of ε, i.e. the fast scale motion present in the solution will have amplitude $O(\varepsilon^p)$ on [0,T].

In nonlinear problems, p is required to be $\geq [\frac{1}{2}n] + 2$, where n is the number of space dimensions, and $[r]$ is the largest integer less than or equal to r. In one space dimension, this means $p \geq 2$, in two or three, $p \geq 3$. However, numerical experiments in a two dimensional problem have found two to be sufficient [4]. This, and considerations concerning the convergence, as $\varepsilon \to 0$, of solutions of the equations with $\varepsilon \neq 0$ to solutions of the limiting system of equations at $\varepsilon=0$ [11], lead us to investigate in this thesis what happens if less than the required number of time derivatives are bounded initially.

First, we consider a model symmetric hyperbolic system of partial differential equations containing two time scales, namely:

$$U_t = \frac{1}{\varepsilon} A U_x + [\Phi(U,V,x,t)]_x$$

$$V_t = [\Psi(U,V,x,t)]_x \qquad (1.1)$$

$$U(x+2\pi,t) \equiv U(x,t) \quad , \quad V(x+2\pi,t) \equiv V(x,t)$$

where A is a constant, non-singular, real, diagonal matrix, and Φ and Ψ are smooth, bounded, periodic functions.

Our first result is that if the initial data for (1.1) are not chosen correctly for the suppression of the fast scale motion, but contain errors of amplitude $O(\varepsilon^{\mu})$, $0<\mu<2$, then fast scale motion of amplitude $O(\varepsilon^{\mu})$ will be present in the solution, but the error in the slow scale motion resulting from interactions of this motion will be of amplitude $O(\varepsilon^{2\mu}) + O(\varepsilon^{\mu+1})$, again on some finite time interval [0,T].

Since the system (1.1) is nonlinear, it is possible that the $O(\varepsilon^{\mu})$ fast scale motions could interact ("resonate") to produce an $O(\varepsilon^{\mu})$ change in the slow scale motion. For example, in the O.D.E. system:

$$\begin{aligned} u_t &= \frac{i}{\varepsilon}\lambda u & u(0) &= \varepsilon u_0 \\ v_t &= \frac{i}{\varepsilon}\mu v & v(0) &= \varepsilon v_0 \\ w_t &= (uv)^{1/2} & w(0) &= 0 \end{aligned} \tag{1.2}$$

the solution has:

$$w(t) = \varepsilon(u_0 v_0)^{1/2} \int_0^t \exp\{\tfrac{1}{2}\varepsilon^{-1}(\lambda+\mu)\tau\}\, d\tau \tag{1.3}$$

which is $O(\varepsilon^2)$, unless $\lambda+\mu=0$, when resonance occurs and w both is $O(\varepsilon)$ and varies on the slow time scale. Our result is that for the system (1.1) this does not happen.

One consequence of our result is that the slow scale motion can be computed with error $O(\varepsilon^2)$ by choosing initial data so that only one time derivative is bounded at $t=0$, carrying the fast scale motion along in the computation, and filtering it out at the end. However, this is probably not a practical method.

A second and possibly more significant consequence is that if fast scale motions of amplitude $O(\varepsilon)$ are in fact present in the physical system being modelled, they can be omitted from the computed solution without introducing

an error greater than $O(\varepsilon^2)$ in the slow scale motion.

In atmospheric modelling, this could be significant, since fast scale motion of amplitude $O(\varepsilon)$ may in fact be present in the atmosphere, but there is little hope of forecasting it accurately. If our result were to extend to the full equations governing atmospheric motion, it would mean that this motion could be omitted entirely from numerical computations without introducing an error greater than $O(\varepsilon^2)$ in the slow scale motion of interest. An error of this size is usually considered acceptable in present day weather prediction.

Interestingly, in [7], it was found that in a simple shallow water equation model, the slow scale motion was relatively unaffected by the presence or absence fast scale motion for periods of up to a day, although no number for the relative amplitude of the fast scale motion was given.

We also consider systems of the form:

$$D^{(1)}(U,V,x,t)\,U_t = \frac{1}{\varepsilon}(AU_x + CU) + [\Phi(U,V,x,t)]_x$$

$$D^{(2)}(U,V,x,t)\,V_t = [\Psi(U,V,x,t)]_x \tag{1.4}$$

$$U(x+2\pi,t) \equiv U(x,t) \quad , \quad V(x+2\pi,t) \equiv V(x,t)$$

where $D^{(1)}$ and $D^{(2)}$ are smooth, bounded, periodic, symmetric, positive-definite, non-singular matrices, C is a constant, real, anti-symmetric matrix, and all other symbols are as in (1.1). The system is again assumed to be symmetric hyperbolic. Again, examples of such systems are given in section 1.2.

The matrix $D = \mathrm{diag}(D^{(1)},D^{(2)})$ often appears in physical applications because of the requirement of symmetry. Hence, in this thesis it will be called a symmetriser, and systems (1.1) and (1.4) will be referred to as 'without symmetriser' and 'with symmetriser' respectively. If D is a constant, independent of (U,V,x,t), then the system behaves as though D were the identity, i.e. like (1.1).

The system (1.4) falls into one of two categories, with regard to its behavior under perturbations of the initial data, according to whether the operator P_0 defined by:

$$P_0 w \equiv A w_x + C w \tag{1.5}$$

is, or is not, singular on the space $\mathbf{S}_p$ of 2π-periodic, once differentiable, square integrable functions of x.

First, suppose P_0 is non-singular. Then we can prove a result similar to that for (1.1), namely that if the initial data for (1.4) are not prepared correctly for the suppression of the fast scale motion, but contain errors of amplitude $O(\varepsilon)$, then fast scale motion of amplitude $O(\varepsilon)$ will be present in the solution, but the resulting change in the slow scale motion will be of amplitude only $O(\varepsilon^2)$.

This result is less general than that for (1.1), in that the error in the initial data is not allowed to be of amplitude $O(\varepsilon^\mu)$ for $\mu < 1$. However, when $\mu < 1$, $D^{(1)}(U,V,x,t)$ and $D^{(2)}(U,V,x,t)$ do not have even one time derivative bounded independently of ε, in general, and so the solution itself may not be bounded independently of ε, and one cannot expect a similar result to be generally true.

Next, if P_0 is singular, for example if $C=0$, since then $P_0 w$ defines w only up to an arbitrary constant, the above result is not in general true, since any eigenfunction of P_0 associated with the eigenvalue zero (in the example $C=0$, this is the mean value over x of w) can vary on the slow time scale and be of amplitude $O(\varepsilon)$. However, we can still show that if the initial data contain errors of amplitude $O(\varepsilon)$, then the error introduced in the solution is also $O(\varepsilon)$ on some finite time interval independent of ε, though it may vary at leading order on both slow and fast time scales.

A third consequence of our results is that solutions of (1.1) or (1.4) for non-zero ε converge as $\varepsilon \to 0$ to a solution of the reduced system obtained by

taking $\varepsilon=0$ in the equations, provided the initial data are chosen to be of the form:

$$(U,V)|_{t=0} = (U_s,V_s)|_{t=0} + O(\varepsilon^{\mu}) \tag{1.6}$$

where $\mu > 0$ for (1.1) and its generalisations, or $\mu \geq 1$ for (1.4) and its generalisations, and (U_s,V_s) is a smooth solution.

In [11], Klainerman and Majda address this question and prove some results for symmetric hyperbolic systems similar to those we consider. Their theorem 3 on the "uniform stability" of solutions under perturbations of the initial data is similar to our result, but is derived under more restrictive assumptions on the coefficients. In particular, their system (4.34), which does not satisfy their structural conditions, and which they conjecture is not uniformly stable, does satisfy our conditions (provided $a_1(u_2) \geq \delta > 0$ for some δ) and, although the large part of the spatial operator is in this case singular, our theorem 5 shows that this system is in fact uniformly stable.

The arguments used in the proofs of our results are quite simple. The crucial step is the demonstration that the x-derivatives of the solutions are bounded independently of ε. This is easily shown for the system (1.1), but is more difficult to show for (1.4). Having shown this, the perturbation in the solution resulting from the error in the initial conditions is found by an asymptotic expansion.

In chapter 2, systems without symmetrisers are considered. Theorem 1, for system (1.1), is stated in section 2.1, bounds on the x-derivatives and some other lemmata are proven in section 2.2, the theorem is proved in section 2.3, and the result generalised to other systems without symmetrisers in section 2.4. Chapter 3 follows the same outline for systems with symmetrisers, except that the singular and nonsingular cases are dealt with separately in sections 3.3 and

3.4 respectively. The crucial proof that the x-derivatives are bounded is given in section 3.2, and generalisations in section 3.5.

The following notation is used throughout the paper:

$$\| w \| = (w,w)^{\frac{1}{2}} \quad , \quad (w,y) = \int_0^{2\pi} w^* y dx$$

$$| B | = sup\{ \| Bw \| : \| w \| = 1 \}$$

$$| w |_\infty = sup\{ | w(x) | : 0 \leq x \leq 2\pi \}.$$

That is, $\| \ \|$ denotes the L_2-norm of a vector, $| \ |$ is the induced matrix norm, and $| \ |_\infty$ is the maximum norm of a vector. Also, the mean value over x of a periodic function is denoted by:

$$< w(t) > = \frac{1}{2\pi} \int_0^{2\pi} w(x,t) \, dt \tag{1.7}$$

while $[r]$ is used to denote the largest integer less than or equal to the real number r.

Finally, a note on distinguishing fast and slow scale motion in a nonlinear system. A function that varies only on the slow time scale is characterised by having all its time derivatives bounded independently of ε. An overbar $\bar{}$ will sometimes be used to denote such a function. On the other hand, a function that 'oscillates on the fast time scale' can be characterised by the existence of some constant δ, independent of ε but with $\varepsilon << \delta << 1$, such that for any (x,t):

$$\int_{t-\delta}^{t+\delta} w(x,\tau) \, d\tau = 0(\varepsilon) \tag{1.8}$$

(where it is assumed the function is scaled so as to ~ 1 as $\varepsilon \to 0$). A tilde $\tilde{}$ will be used to denote such a function.

1.2 Examples of Hyperbolic Systems with Two Time Scales

Many of the systems that stimulated the present work, and most of its references, are drawn from meteorology. There, some systems of equations used to describe atmospheric motions, such as the shallow water equations (scaled as in [4]):

$$\begin{aligned} u_t + uu_x + vu_y + \varepsilon^{-1}(\varphi_x - fv) &= 0 \\ v_t + uv_x + vv_y + \varepsilon^{-1}(\varphi_y + fu) &= 0 \\ \varphi_t + u\varphi_x + v\varphi_y + \varepsilon^{-2}(u_x + v_y) &= 0 \end{aligned} \tag{1.9}$$

are hyperbolic systems having normal modes which fall into two classes. Physically, these are the Rossby modes, which vary on a time scale of a day or so, $0(1)$ in this scaling, and the inertia-gravity modes, which oscillate on a time scale of a few hours, $O(\varepsilon^{-3/2})$ in this scaling. Although both modes are present in the atmosphere, the amplitude of the inertia-gravity modes is smaller than that of the Rossby modes by a factor of ε or ε^2, and it is believed they are of little importance in determining the weather.

Following [4], the system (1.9) can be linearised about a basic (geostrophic, nondivergent) state $(\bar{u},\bar{v},\bar{\varphi})$, and put into symmetric form:

$$\left(\frac{\partial}{\partial t} + A_1\frac{\partial}{\partial x} + A_2\frac{\partial}{\partial y} + A_0\right)Z + F = 0 \tag{1.10}$$

where:

$$A_1 = \begin{bmatrix} \bar{u} & 0 & \varepsilon^{-3/2}d \\ 0 & \bar{u} & 0 \\ \varepsilon^{-3/2}d & 0 & \bar{u} \end{bmatrix}, \qquad A_2 = \begin{bmatrix} \bar{v} & 0 & 0 \\ 0 & \bar{v} & \varepsilon^{-3/2}d \\ 0 & \varepsilon^{-3/2}d & \bar{v} \end{bmatrix}$$

$$A_0 = \begin{bmatrix} 0 & -\varepsilon^{-1}f & 0 \\ \varepsilon^{-1}f & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + O(1), \qquad Z = \begin{bmatrix} u \\ v \\ \varepsilon^{1/2}d^{-1}\varphi \end{bmatrix}$$

and F is an $O(1)$ forcing function depending on the basic state. Here, $d = (1+\varepsilon^2\overline{\varphi})^{1/2}$.

While this does put the system in the symmetric form required by the theory, it does so only for the linearised case. To treat the full nonlinear equations (1.9), a symmetriser multiplying the time derivatives is needed, so this is really a system 'with symmetriser' rather than one 'without symmetriser' as might appear to be the case.

Again following [4], the system (1.10) can be Fourier transformed in the space variables x and y (first 'freezing' the coefficients $A_j(x,y)$), to obtain:

$$\left(\frac{\partial}{\partial t} + A_1 i\omega_1 + A_2 i\omega_2 + A_0\right)\hat{Z} + \hat{F} = 0 \qquad (1.11)$$

where ω_1 and ω_2 are the wave numbers, and $\hat{}$ denotes the Fourier coefficients. The eigenvalues of this system, i.e. of the matrix $|\vec{\omega}|^{-1}(A_1 i\omega_1 + A_2 i\omega_2 + A_0)$, are given by:

$$\lambda_1 = O(1) \quad , \quad \lambda_{2,3} = \pm i\varepsilon^{-3/2}(d^2 + \varepsilon|\vec{\omega}|^{-2}f_0^2)^{1/2} + O(1)$$

Since the general solution of (1.11) is of the form:

$$\hat{Z} = \sum_{j=1}^{3} c_j \exp[-|\vec{\omega}|\lambda_j t]\hat{Z}_j \qquad (1.12)$$

where the c_j are constants determined by the initial conditions, and the $\hat{Z}_j$ are the eigenvectors of the system, it is clear that the eigenvalue λ_1 leads to the Rossby wave solutions, which oscillate on the slow time scale t, and the other two eigenvalues lead to the inertia-gravity modes, which oscillate on a fast scale $\varepsilon^{-3/2}t$.

One final note about this system. If $\vec{\omega} = \vec{0}$, then the eigenvalues reduce to those of A_0, two of which are still large, though only $O(\varepsilon^{-1})$. Thus, the number of

large eigenvalues is independent of x, t and $\vec{\omega}$.

Other examples of hyperbolic systems with two time scales can be found in meteorology, for example the primitive equations, and oceanography.

Also, such systems arise in plasma physics. For example, the 7x7 system governing the motion of a compressible, non-isentropic, magneto-fluid in three space dimensions:

$$p_t + v\cdot\nabla p + \rho\left(\frac{\partial\rho}{\partial p}\right)^{-1}\operatorname{div} v = 0,$$

$$v_t + (v\cdot\nabla)v = \frac{1}{M^2}\frac{\nabla p}{\rho} + \frac{1}{A^2}\frac{H\times\operatorname{curl} H}{\rho}, \tag{1.13}$$

$$H_t + (v\cdot\nabla)H + H\operatorname{div} v - H\cdot\nabla v = 0,$$

has two time scales in circumstances where the Alfven number A is large. These equations can be symmetrised by a simple transformation [11], and then have a diagonal, but nonlinear, symmetriser.

In [9], a specific example is given describing a plasma surrounded by vacuum confined between two infinitely long cylindrical walls. The problem is assumed to be longitudinally uniform, so reduces to a one dimensional problem in the radial direction. The equations are:

$$\rho_t + v\rho_x + \rho v_x = 0$$

$$v_t + \frac{a^2}{p}\rho_x + vv_x + \frac{B}{4\pi\mu\rho}B_x = 0 \tag{1.14}$$

$$B_t + Bv_x + vB_x = 0$$

where ρ is the density, v the velocity, B the magnetic induction, a the sound speed, μ the permeability, and p the pressure, a given function of ρ. Analysis similar to that in [4], outlined above, is performed for this system in [9]. This

system also has two large eigenvalues, and a single $O(1)$ eigenvalue describing the motion of interest. This system can be put in symmetric form, even in the nonlinear case, without the use of a symmetriser.

1.3 Initialisation by the Bounded Derivative Method

As noted above, a simple but very general initialisation scheme, called the bounded derivative method, has recently been proposed by Kreiss, based on the observation that the slow scale motion is characterised by having a number of time derivatives bounded independently of ε. A solution having p time derivatives that are $O(1)$, but whose $(p+1)^{th}$ time derivative $\sim\varepsilon^{-1}$, must necessarily contain fast scale motion of amplitude $\sim\varepsilon^{p}$.

To restrict the amplitude of the fast scale motion to $O(\varepsilon^{p})$, it is therefore necessary to ensure that the solution and p of its time derivatives are $O(1)$ at $t=0$. This can be done by careful choice of the initial data. It remains to be shown that this is sufficient, that the solution will retain bounded time derivatives on some finite time interval [0,T], independent of ε.

In practice, it is not difficult to apply this principle. A constraint on the initial data can be derived for each time derivative that is to be bounded, and these constraints essentially determine the variables associated with the fast scale motion in terms of those associated with the slow scale. It may require a projection in Fourier space to achieve this separation of variables [13]. In [4], the initialisation process for the primitive equations of meteorology, on both mid-latitude and equatorial β-planes, is described in detail, and it is shown that there is more than one way of adjusting initial data obtained from measurements so that the required constraints are satisfied.

This simplicity of application is one advantage of the bounded derivative method. Another is that the method can be applied to both bounded and unbounded regions with a wide variety of boundary conditions. Also, the basic idea of bounding time derivatives can be applied at boundaries to determine appropriate boundary conditions.

We now summarise the theory developed by Kreiss and others [5,12,13,18] which proves rigorously that this initialisation procedure is sufficient to control the amplitude of the fast scale motion. Basically, this theory consists of the derivation of energy estimates, independent of the fast time scale, of the form:

$$\left\| \sum_{\nu=0}^{p} \frac{\partial^\nu u}{\partial t^\nu}(.,t) \right\| \leq K_p e^{\alpha_p t} \left\| \sum_{\nu=0}^{p} \frac{\partial^\nu u}{\partial t^\nu}(.,0) \right\| \tag{1.15}$$

where K_p and α_p are constants independent of ε. [For nonlinear systems such an estimate will in general hold only on finite time intervals, and the constants may depend on the initial data.]

In physical problems, it is natural that such an estimate holds for $p=0$, for if not, the problem is not well-posed, and this probably reflects a deficiency in the model being used rather than a real feature of the actual physical system.

Consider the symmetric hyperbolic system:

$$D(u,\vec{x},t)u_t = \frac{1}{\varepsilon} P_0\left(\vec{x},t,\frac{\partial}{\partial \vec{x}}\right)u + P_1\left(u,\vec{x},t,\frac{\partial}{\partial \vec{x}}\right)u \tag{1.16}$$

where $\vec{x}=(x_1,\ldots,x_n)$, D is a bounded, positive-definite, symmetric, non-singular matrix with bounded inverse, a smooth function of its arguments, and:

$$P_0\left(\vec{x},t,\frac{\partial}{\partial \vec{x}}\right)u \equiv \tfrac{1}{2}\sum_{j=1}^{n}\left[A_j(\vec{x},t)u_{x_j} + (A_j(\vec{x},t)u)_{x_j}\right] + C(\vec{x},t)u \tag{1.17}$$

$$P_1\left(u,\vec{x},t,\frac{\partial}{\partial \vec{x}}\right)u \equiv \tfrac{1}{2}\sum_{j=1}^{n}\left[\bar{A}_j(u,\vec{x},t)u_{x_j} + (\bar{A}_j(u,\vec{x},t)u)_{x_j}\right] + \bar{C}(u,\vec{x},t)u$$

where all the coefficients and their derivatives are bounded independently of ε. Assume that the operators $\varepsilon^{-1}P_0$ and P_1 are semi-bounded, i.e. there exist constants K and $\bar{K}$, independent of ε, such that:

$$\mathrm{Real}(w,P_0 w) \leq \varepsilon K \|w\|^2 \quad , \quad \mathrm{Real}(w,P_1(u)w) \leq \bar{K}\|w\|^2 \tag{1.18}$$

for for any fixed u and all w satisfying the boundary conditions, which may be periodic, 'solid wall', or 'open', but in the last case must be dissipative.

The bounds (1.18) lead to the basic ($p=0$) energy estimate:

$$\|u(.,t)\| \leq K_0 e^{\alpha_0(t-t_0)} \|u(.,t_0)\| \qquad \text{for all } t,t_0 \varepsilon [0,T] \quad (1.19)$$

for constants α_0, K_0 and T, independent of ε, and are therefore natural assumptions.

Finally, assume that the eigenvalues $\kappa(\vec{\omega})$ of the symbol $P_0(\vec{x},t,i\vec{\omega})$ fall into two sets, M_1 and M_2, given by:

$$\kappa(\vec{\omega}) \varepsilon M_1 \Longleftrightarrow \kappa \varepsilon i\mathbf{R} \text{ and } |\kappa(\vec{\omega})| \geq c(|\omega|+1) \qquad \text{for all } \vec{\omega} \qquad (1.20)$$

$$\kappa(\vec{\omega}) \varepsilon M_2 \Longleftrightarrow \kappa(\vec{\omega})=0 \qquad \text{for all } \vec{\omega}$$

where c is some strictly positive constant independent of ε. Further, assume each eigenvalue belongs to the same set M_j for all $\vec{x}$ and t. This ensures that the large part of the operator is nonsingular and elliptic on the subspace spanned by the eigenfunctions associated with M_1, and precludes 'turning-point behaviour'. This is also a natural assumption, since in systems where the fast scale motion is oscillatory in nature, as here, turning-point behaviour is rare.

In some systems of physical interest, the number of large eigenvalues can change at particular values of $\vec{\omega}$. In particular, if $C = 0$ in (1.17), then all eigenvalues of the symbol fall to zero at $\vec{\omega}=\vec{0}$. This is the 'singular case' mentioned in the introduction, and it can lead to genuine differences in the behavior of the system, but for many purposes the condition in (1.20) can be relaxed for $\vec{\omega}$ with $|\vec{\omega}| \leq \beta$, for some constant β (independent of ε, of course).

For the case where the coefficients of P_0 and P_1 are constant, the problem can be reduced to a system of ordinary differential equations by Fourier transformation. Then, the theory developed in [12], in which one finds the

solution by a simple asymptotic expansion, can be applied to show the validity of the method.

When the coefficients are not constant, one must derive energy estimates by the standard method of taking inner products of the various derivatives of the solution with their time derivatives obtained from derivatives of the original equation. When P_0 has constant coefficients, this is not difficult, since the terms involving ε^{-1} always have the form:

$$\varepsilon^{-1}(w, P_0 w)$$

for some w, and, in view of (1.18), these terms are all $O(1)$. Thus the problem is essentially reduced to:

$$D(u,\vec{x},t)u_t = P_1\left(u,\vec{x},t,\frac{\partial}{\partial\vec{x}}\right)u \tag{1.21}$$

which is a standard problem. Estimates of the form (1.15) hold on some finite time interval (which may depend on the initial data), for any non-negative integer p if the system is linear, for any $p \geq [\frac{1}{2}n]+2$ if it is nonlinear. This requirement arises from the need to consider at least this many derivatives, in the nonlinear case, in order to obtain a closed system [5].

If P_0 has variable coefficients, terms apparently of amplitude $\sim\varepsilon^{-1}$ appear in the estimates. For example, taking $D=I$, $P_1 \equiv 0$, and restricting to one space dimension for ease of exposition, if $\dot{u}=u_t$, then

$$\dot{u}_t = \frac{1}{\varepsilon}P_0\dot{u} + \frac{1}{\varepsilon}\dot{P}_0 u \tag{1.22}$$

$$\Rightarrow \qquad \frac{1}{2}\frac{d}{dt}\|\dot{u}\|^2 = \frac{1}{\varepsilon}(\dot{u}, P_0\dot{u}) + \frac{1}{\varepsilon}(\dot{u}, \dot{P}_0 u) \tag{1.23}$$

Here, of course:

$$\dot{P}_0 = \sum_{j=1}^{n} A_t(\vec{x},t)\frac{\partial}{\partial x_j} + [\tfrac{1}{2}A_{xt}(\vec{x},t) + C_t(\vec{x},t)]$$

The first term on the right in (1.23) is $O(1)$, by (1.18), but to show that the second term is also $O(1)$ requires more work. Further suppose (following [12]) that the system has exactly two large eigenvalues, i.e. $P_0(x,t,i\omega)$ has exactly two nonzero eigenvalues for all x, t and ω. This is the case in many applications. Then, Kreiss [12] has shown that P_0 can be transformed into one of two normal forms, namely:

$$P_0 u \equiv \tfrac{1}{2}[A(x,t)u_x + (A(x,t)u)_x] + C(x,t)u \tag{1.24}$$

where either:

$$A = \begin{bmatrix} A_{11} & 0 \\ 0 & 0 \end{bmatrix} \quad , \quad A_{11} = \begin{bmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{bmatrix} \quad , \quad |A_{11}| \neq 0 \tag{1.25a}$$

$$C = \begin{bmatrix} C_{11} & 0 \\ 0 & 0 \end{bmatrix} \quad , \quad C_{11} = \begin{bmatrix} 0 & c_{12} \\ -c_{12} & 0 \end{bmatrix} \quad , \quad c_{12} \neq 0 \tag{1.25b}$$

or:

$$A = \begin{bmatrix} A_{11} & 0 \\ 0 & 0 \end{bmatrix} \quad , \quad A_{11} = \begin{bmatrix} a_{11} & a_{12} \\ a_{12} & 0 \end{bmatrix} \quad , \quad a_{12} \neq 0 \tag{1.26a}$$

$$C = \begin{bmatrix} 0 & c_{12} & c_{13} & 0 & . \\ -c_{12} & 0 & 0 & . & . \\ -c_{13} & 0 & 0 & . & . \\ O & . & . & . & . \\ . & . & . & . & . \end{bmatrix} \quad , \quad c_{13} \neq 0 \tag{1.26b}$$

The special structure of these normal forms can be used to show that the apparently large term in (1.23) is in fact $O(1)$.

In the first case, the fast and slow scale variables separate (to the approximation we have made). No ε^{-1} terms appear in the equation for the slow scale variables, while the first two components u^I of u form the fast scale variables, and satisfy:

$$u_t^I = \frac{1}{\varepsilon}\left[A_{11}u_x^I + (\tfrac{1}{2}A_{11x} + C_{11})u^I\right] \tag{1.27}$$

$$\Rightarrow \qquad u_x^I = -A_{11}^{-1}(\tfrac{1}{2}A_{11x} + C_{11})u^I + \varepsilon A_{11}^{-1}u_t^I \tag{1.28}$$

Thus:

$$\dot{P}_0 u = A_{11t}u_x^I + (\tfrac{1}{2}A_{11xt} + C_{11t})u^I$$

$$= -A_{11t}A_{11}^{-1}(\tfrac{1}{2}A_{11x} + C_{11})u^I + (\tfrac{1}{2}A_{11x} + C_{11})u^I + \varepsilon A_{11t}A_{11}^{-1}u_t^I$$

$$= Bu^I + \varepsilon A_{11t}A_{11}^{-1}\dot{u}^I \tag{1.29}$$

say. Hence:

$$\dot{u}_t^I = \frac{1}{\varepsilon}(P_0^I\dot{u}^I + Bu^I) + A_{11t}A_{11}^{-1}\dot{u}^I \tag{1.30}$$

Now let $\bar{u}$ satisfy:

$$P_0^I\bar{u} + Bu^I = 0 \tag{1.31}$$

By the assumptions (1.20) on the eigenvalues of P_0^I, which ensure that it is a nonsingular elliptic operator, and the boundedness of B:

$$\|\bar{u}\| + \|\bar{u}_x\| \leq \text{const.}\|u^I\| \tag{1.32}$$

So, letting $\dot{u}_1^I = \dot{u}^I - \bar{u}$:

$$\dot{u}_{1t}^I = \frac{1}{\varepsilon}P_0^I\dot{u}_1^I + A_{11t}A_{11}^{-1}\dot{u}^I \tag{1.33}$$

for which an energy estimate is easily obtained. In this way the apparently large term has been eliminated.

For the second normal form, the second and third equations may be written as:

$$\tfrac{1}{2}(a_{12}u_x^{(1)} + (a_{12}u^{(1)})_x) + c_{12}u^{(1)} = \varepsilon f$$

$$c_{13}u^{(1)} = \varepsilon(g_x + h) \tag{1.34}$$

where f, g and h are functions of the other components of the solution and its time derivative. If f, g and h are assumed known, then this is an overdetermined system for $u^{(1)}$, and it can be shown that if a solution exists it must satisfy:

$$\|u_x^{(1)}\| + \|u^{(1)}\| \leq \varepsilon \text{ const.}(\|f\| + \|g\| + \|h\|)$$

$$\leq \varepsilon \text{ const.}(\|\dot{u}\| + \|u\|) \tag{1.35}$$

This shows that $u^{(1)}$ is essentially $O(\varepsilon)$, and since the apparently large term:

$$\frac{1}{\varepsilon}(\dot{u}, \dot{P}_0 u) = \frac{1}{\varepsilon}\dot{u}^* \begin{pmatrix} 0 & \dot{c}_{12} & \dot{c}_{13} & . \\ -\dot{c}_{12} & 0 & 0 & . \\ -\dot{c}_{13} & 0 & 0 & . \\ . & . & . & . \end{pmatrix} u \tag{1.36}$$

involves $u^{(1)}$ in every term, it is not hard to show now that it is in fact $O(1)$.

In more space dimensions, there is only one normal form, the generalisation of the second form. If there are more than two large eigenvalues, as is the case in some magnetohyrodynamic situations for example, then Tadmor [18] has recently shown that there will always be an even number $2q$ of them, and that there are, in one space dimension, $q+1$ normal forms, the two above plus $q-1$ intermediate forms. The same techniques can be used to show that the

solution and its derivatives are bounded in terms of their initial values in these cases also.

Finally, in the case where the coefficients of P_0 are also nonlinear, Kreiss and Browning [5] show the theory still holds if P_0 has the rather special form:

$$P_0(u, \frac{\partial}{\partial x}) = \begin{bmatrix} f_1 & \bar{F}^* \\ \bar{F} & 0 \end{bmatrix} \frac{\partial}{\partial x_1} + \begin{bmatrix} g_1 & \bar{G}^* \\ \bar{G} & 0 \end{bmatrix} \frac{\partial}{\partial x_2} + \begin{bmatrix} h_1 & \bar{H}^* \\ \bar{H} & 0 \end{bmatrix} \frac{\partial}{\partial x_3} \qquad (1.37)$$

where $f_j = \frac{\partial f}{\partial x_j}$, $\bar{F}^* = (f_2, \ldots, f_n)$, and $f(u)$ is a smooth function of u with no explicit x or t dependence, and similarly for g and h.

1.4 Other Initialisation Schemes

In the past, several different schemes for choosing initial data to suppress the fast scale motion have been proposed. A brief summary of most of these schemes, and remarks on their relative merits, may be found in [4]. At present there seem to be three basically different schemes in use, while earlier schemes are in general special cases of one of these three.

The first scheme is called 'dynamic initialisation', see [15] or [16], in which the equations of motion are integrated forwards and backwards for a few time steps from the given initial data, with a small amount of dissipation added. This has the effect of damping out the fast scale waves in the initial data, and so arriving at suitably balanced initial data. This scheme is quite simple to apply, but there are questions concerning its effectiveness in weather prediction, particularly in tropical latitudes.

The second scheme is called 'normal mode initialisation', see [7] or [19], in which the given initial data are expanded as a series in the normal modes of the system. Then the fast scale modes can be removed by setting their coefficients to zero. These modes can return through nonlinear interactions at later times, but it can be shown, using the method of multiple time scales ('two-timing'), that if a suitably chosen small amount of fast scale motion is retained initially, the amplitude of the fast scale motion can be kept as small as is desired, see [1], [2] and [14]. The disadvantage of this scheme is that the normal modes of the system must be found. This is not difficult for, say, a whole-globe weather prediction model, but in limited area forecasting it is in general not possible.

The third scheme is the 'method of bounded derivatives', described in detail in the previous section. This has none of the above-mentioned disadvantages of the previous two schemes, is simple to apply, and quite generally applicable.

Chapter 2: HYPERBOLIC SYSTEMS WITHOUT A SYMMETRISER

2.1 Statement of the Problem

Consider again the system (1.1):

$$U_t = \frac{1}{\varepsilon} A U_x + [\Phi(U,V,x,t)]_x$$

$$V_t = [\Psi(U,V,x,t)]_x \qquad (2.1a)$$

$$U(x+2\pi,t) \equiv U(x,t) \quad , \quad V(x+2\pi,t) \equiv V(x,t)$$

Assume that $A=\mathrm{diag}(\lambda_1,\lambda_2,\ldots,\lambda_m)$, each λ_j, $1\leq j\leq m$, being a non-zero real constant, that Φ and Ψ are C^∞ functions of all their arguments, 2π-periodic in x, having no explicit ε dependence. Also assume that the system is symmetric hyperbolic, i.e. the matrix $\begin{pmatrix} \frac{1}{\varepsilon}A + \Phi_U & \Phi_V \\ \Psi_U & \Psi_V \end{pmatrix}$ is real symmetric.

From [5], solutions of (2.1a) varying on only the slow time scale ('smooth solutions') exist. As discussed above, to obtain such a solution, the initial data must be chosen appropriately: in this case, $V(x,0)$ may be chosen arbitrarily, but then $U(x,0)$ is determined up to $O(\varepsilon^p)$ for any p. In fact, $U(x,t) = O(\varepsilon)$ in any smooth solution.

Thus, suppose $(\varepsilon u_s(x,t),v_s(x,t))$ is a smooth solution of (2.1a). Since we are interested in investigating the behavior of the solution under perturbations of the initial data, take:

$$U(x,0) = \varepsilon u_s(x,0) + \varepsilon^\mu f(x) \qquad (2.1b)$$

$$V(x,0) = v_s(x,0)$$

where $0<\mu\leq 2$ and $f(x) \in C^\infty$ is 2π-periodic and independent of ε, with:

$$\int_0^{2\pi} f(x)dx = 0 \tag{2.1c}$$

This condition is in fact not necessary for our results, as we shall show in section 2.4, but we assume it here to simplify the proof of the theorem in this section.

Since the equation (2.1a) is in conservation form, (2.1c) ensures that the perturbation in its solution resulting from (2.1b) always has mean value zero. That is, the perturbation is restricted to a subspace (S_p^0 say) of S_p on which the large part of the spatial operator $P_0 = A\frac{\partial}{\partial x}$ is nonsingular, and in fact P_0^{-1} is then $O(1)$ in the sense that:

$$P_0 w = f \qquad , \qquad w \in S_p^0$$

$$\Rightarrow \qquad |w|_\infty \leq 2\pi |A^{-1}| |f|_\infty \tag{2.2}$$

Let u,v be the perturbation in the solution:

$$u(x,t) = U(x,t) - \varepsilon u_s(x,t)$$

$$v(x,t) = V(x,t) - v_s(x,t)$$

The equations satisfied by u and v are:

$$u_t = \frac{1}{\varepsilon} A u_x + [B_{11}(x,t)u + B_{12}(x,t)v]_x + [\varphi(u,v,x,t)]_x$$

$$v_t = [B_{21}(x,t)u + B_{22}(x,t)v]_x + [\psi(u,v,x,t)]_x$$

$$u(x,0) = \varepsilon^\mu f(x) \qquad , \quad v(x,0) = 0 \tag{2.3}$$

$$u(x+2\pi,t) \equiv u(x,t) \qquad , \quad v(x+2\pi,t) \equiv v(x,t)$$

where $(B_{11}u + B_{12}v)$ is the linear part and φ the quadratic and higher part of:

$$[\Phi(\varepsilon u_s+u, v_s+v, x, t) - \Phi(\varepsilon u_s, v_s, x, t)],$$

and $(B_{21}u + B_{22}v)$ is the linear part and ψ the quadratic and higher part of:

$$[\Psi(\varepsilon u_s+u, v_s+v, x, t) - \Psi(\varepsilon u_s, v_s, x, t)].$$

Each of B_{ij}, $i,j=1,2$, is C^∞ in x and t, 2π-periodic in x, and, together with all its x and t derivatives, is bounded independently of ε. The same may be assumed of φ and ψ, with bounds uniform in u and v, since such bounds are needed only in a neighborhood of the solution, and φ and ψ may be altered elsewhere without affecting the solution.

The main result of this section is then:

Theorem 1. If u,v is the solution of system (2.3), the perturbation of the solution of (2.1) when (2.2) are used for the initial conditions, then there exist constants ε_0, K_0, K_1, δ and T, independent of ε, and strictly positive, such that:

$$\|u(.,t)\| \leq \varepsilon^\mu K_0 \tag{2.4}$$

$$\|\bar{u}(.,t)\| + \|v(.,t)\| \leq (\varepsilon^{2\mu} + \varepsilon^{\mu+1})K_1$$

where:

$$\bar{u}(x,t) = \int_{t-\delta}^{t+\delta} u(x,\tau)\, d\tau$$

for all $t \in [0,T]$ and all $\varepsilon \leq \varepsilon_0$.

Put another way, the solution of (2.3) has the form:

$$u(x,t) = \varepsilon^\mu \tilde{u}(x,t) + O(\varepsilon^{2\mu}) + O(\varepsilon^{\mu+1})$$

$$v(x,t) = O(\varepsilon^{2\mu}) + O(\varepsilon^{\mu+1})$$

on some $O(1)$ time interval, where $\tilde{u}$ is $O(1)$ and oscillates on the fast time scale.

2.2 Some Useful Lemmata

The first and most important lemma we need states that the x-derivatives of the solution are bounded independently of ε, even if the t-derivatives are not. The lemma is stated here in more general form than is needed for theorem 1, since the more general form will be required in section 2.4.

Lemma 2.1 Suppose $w(x,t)$ satisfies the symmetric hyperbolic system:

$$w_t = \frac{1}{\varepsilon}(Aw_x + Cw) + \varphi(w,x,t)w_x + \gamma(w,x,t)w \tag{2.5}$$

$$w(x,0) = f(x) \quad , \quad w(x+2\pi,t) \equiv w(x,t)$$

where A is constant and symmetric, C is constant and antisymmetric, φ is symmetric, φ, γ and f are periodic in x and C^∞ in all arguments, and there exist constants P_{qrs}, Q_{qrs} and R_s, independent of ε, such that:

$$\left| \frac{\partial^{q+r+s}\varphi}{\partial w_1^{q_1} \cdots \partial w_m^{q_m} \partial x^r \partial t^s}(w,.,t) \right| \le P_{qrs}$$

$$\left| \frac{\partial^{q+r+s}\gamma}{\partial w_1^{q_1} \cdots \partial w_m^{q_m} \partial x^r \partial t^s}(w,.,t) \right| \le Q_{qrs} \tag{2.6}$$

$$\left\| \frac{\partial^s f}{\partial x^s} \right\| \le R_s$$

for all w, all nonnegative r,s and $q_1, \ldots, q_m$ with $q = q_1 + \cdots + q_m$.

Then there exist constants T, ε_0 and K_{rs}, independent of ε, such that:

$$\left\| \frac{\partial^{r+s} w}{\partial x^r \partial t^s}(.,t) \right\| \le \varepsilon^{-s} K_{rs} \tag{2.7}$$

for all $t \in [0,T]$, all $\varepsilon \le \varepsilon_0$, and all nonnegative r,s.

Proof This is a standard result [5], but for completeness a proof is outlined here. It proceeds by estimating the growth of the L_2 norms of the solution and its x-derivatives. First then, from (2.5):

$$\frac{1}{2}\frac{d}{dt}(w,w) = (w,\varphi(w,x,t)w_x) + (w,\gamma(w,x,t)w)$$

since the symmetries of A and C imply that:

$$(w,Aw_x) = 0 \qquad , \qquad (w,Cw) = 0$$

for all w. Now, using the bounds (2.6):

$$(w,\gamma(w,x,t)w) \leq Q_{000}\|w\|^2$$

$$(w,\varphi(w,x,t)w_x) = -\frac{1}{2}\left[w,\frac{\partial\varphi}{\partial x}w\right] = -\frac{1}{2}\sum_{j=1}^{m}(w,\varphi_{w_j}w_{jx}w) - \frac{1}{2}(w,\varphi_x w)$$

$$\leq \tfrac{1}{2}\|w\|^2(P_{010} + mP_{100}|w_x|_\infty)$$

where the subscript notation has been used to denote a partial derivative with respect to x only if all other arguments of the function are to be held constant, while $\frac{\partial}{\partial x}$ has been used when only t is to be held constant. Hence:

$$\frac{d}{dt}(w,w) \leq const.\ \|w\|^2(1 + |w_x|_\infty)$$

Similarly, if $y = w_x$:

$$y_t = \frac{1}{\varepsilon}(Ay_x + Cy) + \varphi(w,x,t)y_x + \gamma(w,x,t)y + \frac{\partial\varphi}{\partial x}(w,x,t)y + \frac{\partial\gamma}{\partial x}(w,x,t)w$$

$$\frac{1}{2}\frac{d}{dt}(y,y) \leq (Q_{000}+\tfrac{1}{2}P_{010}+\tfrac{1}{2}mP_{100}|w_x|_\infty)\|y\|^2 + (Q_{010}+mQ_{100}|w_x|_\infty)\|y\|\,\|w\|$$

$$\leq const.\,(\|y\|^2 + \|y\|\,\|w\|)(1 + |w_x|_\infty)$$

Finally, if $z = w_{xx}$:

$$z_t = \frac{1}{\varepsilon}(Az_x + Cz) + \varphi(w,x,t)z_x + \gamma(w,x,t)z + 2\frac{\partial\varphi}{\partial x}(w,x,t)z$$

$$+ 2\frac{\partial\gamma}{\partial x}(w,x,t)y + \frac{\partial^2\varphi}{\partial x^2}y + \frac{\partial^2\gamma}{\partial x^2}w \qquad (*)$$

Here, the inner product of z with all but the last two terms can be estimated as above, while:

$$\left(z, \frac{\partial^2\varphi}{\partial x^2}y\right) = \sum_{j=1}^{m}(z, \varphi_{w_j}w_{jxx}y) + \sum_{i,j=1}^{m}(z, \varphi_{w_iw_j}w_{ix}w_{jx}y) + \sum_{j=1}^{m}(z, \varphi_{w_jx}w_{jx}y) + (z, \varphi_{xx}y)$$

$$\leq \sum_{j=1}^{m}(z, \varphi_{w_j}w_{jxx}y) + (P_{020} + mP_{110}|w_x|_\infty + \tfrac{1}{2}m^2P_{200}|w_x|_\infty^2)\|y\|\,\|z\|$$

But:

$$\sum_{j=1}^{m}(z, \varphi_{w_j}w_{jxx}y) = \sum_{j=1}^{m}(z, \tilde{\varphi}_j y_j w_{xx}) \leq mP_{100}|y|_\infty \|z\|\,\|w_{xx}\|$$

where the ik^{th} component of $\tilde{\varphi}_j$ is equal to the ij^{th} component of φ_{w_k}. Thus, this term can be bounded in terms of the L_2 norms of w and its first two derivatives and the maximum norm of its first derivative. The last of these can in turn be bounded in terms of the first two. The point here is that by interchanging the positions of w_{xx} and $y = w_x$ in this last term, reference to $|w_{xx}|_\infty$ in the bound is avoided; this is necessary since $|w_{xx}|_\infty$ cannot, in general, be bounded by L_2 norms without introducing the third derivative with respect to x, and if this were to appear, we would not have a closed system.

The final term in (*) can also be estimated by the same means, so, collecting everything together, it follows that:

$$\frac{d}{dt}(z,z) \leq const.\{\|z\|(\|z\| + \|y\| + \|w\|)(1 + |w_x|_\infty) + |w_x|_\infty^2\|z\|(\|y\| + \|w\|)\}$$

In view of Sobolevs inequality:

$$|w_x|_\infty \leq const.(\|w_x\| + \|w_{xx}\|)$$

the three inequalities we have derived form a closed system of nonlinear ordinary differential equations in time for $\|w\|$, $\|y\|$ and $\|z\|$. Further, the system is independent of ε. Therefore, for any $O(1)$ initial data, it has a bounded solution on some finite time interval independent of ε. On this time interval, which may depend on the initial data, the solution is bounded in terms of its initial value. The lemma now follows.

Lemma 2.2 Suppose $w(x,t)$ satisfies:

$$w_t = \frac{1}{\varepsilon} A w_x + [B(x,t)w]_x \tag{2.8}$$

$$w(x,0) = f(x) \quad , \quad w(x+2\pi,t) \equiv w(x,t)$$

where A is a constant, symmetric, nonsingular matrix, B is symmetric, both B and f are 2π-periodic in x and C^∞ in their arguments, f satisfies (2.1c), i.e. it has mean value zero, and there exist constants P_{rs} and R_s, independent of ε, such that:

$$\left| \frac{\partial^{r+s} B}{\partial x^r \partial t^s}(.,t) \right| \leq P_{rs} \quad , \quad \left\| \frac{\partial^s f}{\partial x^s} \right\| \leq R_s$$

for all nonnegative r,s.

Then there exist constants K_1, ε_0 and δ, independent of ε, such that:

$$\|\overline{w}(.,t)\| \leq \varepsilon K_1 \tag{2.9}$$

for all $\varepsilon \leq \varepsilon_0$, where:

$$\overline{w}(x,t) = \int_{t-\delta}^{t+\delta} w(x,\tau)\, d\tau .$$

That is, w is oscillates on the fast time scale.

Proof By the previous lemma, $|w|_\infty$ is $O(1)$. Let:

$$P_0 w \equiv Aw_x + \varepsilon[B(x,t)w]_x$$

As noted above, P_0^{-1} exists and is bounded independently of ε on the subspace S_p^0, and w belongs to this space for all t since it does so initially and the equation has conservation form. Also, since A is constant, P_0^{-1} commutes with $\frac{\partial}{\partial t}$ at leading order. Thus:

$$w_t = \frac{1}{\varepsilon} P_0 w$$

$$\Rightarrow \qquad w = \varepsilon P_0^{-1} w_t = \varepsilon \frac{\partial}{\partial t} P_0^{-1} w + O(\varepsilon^2 w)$$

Thus, for any δ with $\varepsilon << \delta << 1$:

$$\int_{t-\delta}^{t+\delta} w(x,\tau)\, d\tau = \varepsilon \left[P_0^{-1} w \right]_{t-\delta}^{t+\delta} + O(\varepsilon^2 w)$$

$$\leq 2\varepsilon |P_0^{-1}|\, |w|_\infty + 0(\varepsilon^2 w) = O(\varepsilon)$$

whence the result follows.

Lemma 2.3 Suppose $w(x,t)$ satisfies:

$$w_t = \frac{1}{\varepsilon} Aw_x + [B(x,t)w]_x + F(x,t) \tag{2.10}$$

$$w(x,0) = 0 \qquad , \qquad w(x+2\pi,t) \equiv w(x,t)$$

where A and B are as in lemma 2.2, F is C^∞ in x and t and 2π-periodic in x with:

$$\int_0^{2\pi} F(x,t)\, dx = 0 \tag{2.11a}$$

and there exist constants Q_{s0}, independent of ε, such that:

$$\left\| \frac{\partial^s F}{\partial x^s}(.,t) \right\| \leq Q_{s0} \qquad s=0,1 \qquad (2.11b)$$

for all t. Then:

(i) if F varies only on the slow time scale at leading order, i.e. if there exist constants Q_{s1}, independent of ε, such that:

$$\left\| \frac{\partial^{s+1} F}{\partial x^s \partial t}(.,t) \right\| \leq Q_{s1} \qquad s=0,1 \qquad (2.11c)$$

for all t, then for any fixed T, independent of ε, there exist constants K and ε_0, independent of ε, such that:

$$\| w(.,t) \| \leq \varepsilon K \qquad (2.12)$$

for all $t \in [0,T]$ and all $\varepsilon \leq \varepsilon_0$.

(ii) if F oscillates on the fast time scale, in the sense of (1.8), at leading order, i.e. if $\| \bar{F}(.,t) \| \leq \varepsilon \bar{Q}$ for some constant $\bar{Q}$ and all t, then for any fixed T, independent of ε, there exist constants K_0, K_1 and ε_0, independent of ε, such that:

$$\| w(.,t) \| \leq K_0 \quad , \quad \| \overline{w}(.,t) \| \leq \varepsilon K_1 \qquad (2.13)$$

for all $t \in [0,T]$ and all $\varepsilon \leq \varepsilon_0$. Here $\overline{w}(x,t)$ is the mean value of $w(x,\tau)$ over the time interval $[t-\delta, t+\delta]$, $\varepsilon << \delta << 1$. Thus, w also oscillates on the fast time scale at leading order.

Proof (i) As in the proof of lemma 2.1, since A is constant and symmetric:

$$\frac{1}{2} \frac{d}{dt} \| w \|^2 = (w, (Bw)_x) + (w, F)$$

$$\leq const\{\|w\|^2 + \|F\|^2\}$$

$$\frac{1}{2}\frac{d}{dt}\|w_t\|^2 = (w_t,(Bw_t)_x) + (w_t,(B_t w)_x) + (w_t,F_t)$$

$$\leq const\{\|w_t\|^2 + \|w\|^2 + \|F_t\|^2\}$$

Since $\|F\|$, $\|F_t\|$, $w(x,0)$ and $w_t(x,0)$ are bounded independently of ε, it follows that $\|w_t(.,t)\|$ is also so bounded on any $O(1)$ time interval [0,T]. Since F_x is bounded independently of ε, it can be shown in similar manner that $\|w_{xt}(.,t)\|$ is also $O(1)$. Thus by Sobolevs inequality:

$$|w_t(.,t)|_\infty \leq const. = O(1)$$

on [0,T]. Thus, with P_0 as above:

$$w_t = \frac{1}{\varepsilon}P_0 w + F$$

$$\Rightarrow \qquad w = \varepsilon P_0^{-1}(w_t - F)$$

$$\Rightarrow \qquad |w|_\infty \leq \varepsilon const.(|w_t|_\infty + |F|_\infty)$$

$$\leq \varepsilon const.(\|w_t\| + \|w_{xt}\| + \|F\| + \|F_x\|) = 0(\varepsilon)$$

whence the result follows.

(ii) Here:

$$w = \varepsilon P_0^{-1}(w_t - F)$$

$$\Rightarrow \qquad \int_{t-\delta}^{t+\delta} w(x,\tau)\,d\tau = [\varepsilon P_0^{-1} w]_{t-\delta}^{t+\delta} + O(\varepsilon^2 w) - P_0^{-1}\int_{t-\delta}^{t+\delta} F(x,\tau)\,d\tau + O(\varepsilon F)$$

$$\Rightarrow \qquad \|\overline{w}\| = 0(\varepsilon\|w\|) + O(\|\overline{F}\|) + O(\varepsilon\|F\|) = O(\varepsilon)$$

as required.

Lemma 2.4 Suppose $w(x,t)$ satisfies:

$$w_t = [B(x,t)w]_x + [B_1(x,t).F_t(x,t)]_x \tag{2.14}$$

$$w(x,0) = 0 \qquad , \qquad w(x+2\pi,t) \equiv w(x,t)$$

where B and F are as in the previous lemma (either case), except that (2.11b) must hold for $s=0,1,2,3$, i.e. F has three $O(1)$ space derivatives, and B_1 is C^∞ in x and t, 2π-periodic in x, and, together with all its derivatives, is bounded independently of ε. Then there exist constants M, T and ε_0, independent of ε, such that for $\varepsilon \leq \varepsilon_0$ and $t \in [0,T]$:

$$|w(.,t)|_\infty \leq M \tag{2.15}$$

Note this is true whatever the magnitude of the time derivatives of F, which may be $O(\varepsilon^{-1})$.

Proof Let $w^{(1)}$ satisfy:

$$w_t^{(1)} = [B_1(x,t).F_t(x,t)]_x$$

$$w^{(1)}(x,0) = 0 \qquad , \qquad w^{(1)}(x+2\pi,t) \equiv w^{(1)}(x,t)$$

Then:

$$w^{(1)}(x,t) = \left\{\int_0^t B_1(x,t)F_t(x,t)dt\right\}_x$$

$$= \left\{[B_1(x,t)F(x,t)]_0^t - \int_0^t B_{1t}(x,t)F(x,t)dt\right\}_x$$

Thus, $\|w^{(1)}\|$ can be bounded in terms of $\|F\|$ and $\|F_x\|$ (and norms of B_1 and its derivatives of course). Differentiating with respect to x, it follows that $\|w_x^{(1)}\|$ can be bounded in terms of $\|F\|$, $\|F_x\|$ and $\|F_{xx}\|$, while $\|w_{xx}^{(1)}\|$ can be

bounded in terms of these and $\|F_{xxx}\|$. By assumption, all these norms are $O(1)$, so $w^{(1)}$ certainly satisfies a bound of the form (2.15).

Let $w^{(2)} = w - w^{(1)}$:

$$w_t^{(2)} = [B(x,t)w^{(2)}]_x + [B(x,t)w^{(1)}]_x$$

$$w^{(2)}(x,0) = 0 \qquad , \qquad w^{(2)}(x+2\pi,t) \equiv w^{(2)}(x,t)$$

By Duhamels principle, and the bounds on B and its derivatives:

$$\|w^{(2)}(.,t)\| \leq const \sup_{0\leq\tau\leq t} \{\|w^{(1)}(.,\tau)\| + \|w_x^{(1)}(.,\tau)\|\}$$

and:

$$\|w_x^{(2)}(.,t)\| \leq const \sup_{0\leq\tau\leq t} \{\|w^{(1)}(.,\tau)\| + \|w_x^{(1)}(.,\tau)\| + \|w_{xx}^{(1)}(.,\tau)\|\}$$

on any $O(1)$ time interval [0,T]. The result follows.

2.3 Proof of Theorem 1.

In the system (2.3), we are hoping to show that v is an order in ε smaller than u. Therefore, as a first approximation we neglect v. Neglecting also nonlinear terms, since u is expected to be $O(\varepsilon^{\mu})$, let u_0 satisfy:

$$u_{0t} = \frac{1}{\varepsilon} A u_{0x} + [B_{11}(x,t)u_0]_x \tag{2.16}$$

$$u_0(x,0) = \varepsilon^{\mu} f(x) \quad , \quad u_0(x+2\pi,t) \equiv u_0(x,t)$$

By assumption, the bounds required in lemma 2.1 and lemma 2.2 are satisfied by B_{11} and f, so we may write:

$$u_0(x,t) = \varepsilon^{\mu} \tilde{u}_0(x,t) \tag{2.17}$$

where $\tilde{u}_0$ oscillates on the fast time scale, but, together with all its x-derivatives, is $O(1)$. Also, (2.16) may be written as:

$$\varepsilon u_{0t} = P_0 u_0$$

where P_0 differs from the P_0 in (2.2) only by an $O(\varepsilon)$ term. Thus:

$$u_0 = \varepsilon P_0^{-1} u_{0t} = \varepsilon \frac{\partial}{\partial t}(P_0^{-1} u_0) + \varepsilon P_0^{-2} P_{0_t} u_0$$

$$= \varepsilon \frac{\partial}{\partial t}(P_0^{-1} u_0) + O(\varepsilon^2 u_0) \tag{2.18}$$

since (2.2) still holds, and $P_{0t} = O(\varepsilon)$; essentially P_0 and $\frac{\partial}{\partial t}$ commute at leading order.

Next let a first approximation to v be v_1 satisfying:

$$v_{1t} = [B_{22}(x,t)v_1]_x + [B_{21}(x,t)u_0]_x + [\psi(u_0,0,x,t)]_x \tag{2.19}$$

$$v_1(x,0) = 0 \quad , \quad v_1(x+2\pi,t) \equiv v_1(x,t)$$

Using (2.18), it can be seen that the linear forcing term in this equation is:

$$[B_{21}(x,t)u_0]_x = \varepsilon[B_{21}(x,t).(P_0^{-1}u_0)_t]_x + O(\varepsilon^2 u_0) \qquad (2.20)$$

Thus, noting that P_0^{-1} is bounded, and from lemma 2.1 that the x-derivative is of no consequence, lemma 2.4 implies that this term makes a contribution to the solution v_1 of amplitude $O(\varepsilon u_0) = O(\varepsilon^{\mu+1})$. Also, the nonlinear forcing term $[\psi(u_0,0,x,t)]_x$ is of amplitude $O(u_0^2) = O(\varepsilon^{2\mu})$; by Duhamels principle, it therefore makes a contribution to v_1 of amplitude $O(\varepsilon^{2\mu})$. Thus:

$$v_1(x,t) = \varepsilon^{\mu+1} v_1^{\{1\}}(x,t) + \varepsilon^{2\mu} v_1^{\{2\}}(x,t) \qquad (2.21)$$

where $v_1^{\{1\}}$ and $v_1^{\{2\}}$ are both bounded independently of ε but may vary on both the fast and slow time scales at leading order.

Now return to the u equation, and let u_1 satisfy:

$$u_{1t} = \frac{1}{\varepsilon} A u_{1x} + [B_{11}(x,t)u_1]_x + [B_{12}(x,t)v_1]_x + [\varphi(u_0,0,x,t)]_x \qquad (2.22)$$

$$u_1(x,0) = 0 \qquad , \qquad u_1(x+2\pi,t) \equiv u_1(x,t)$$

By lemma 2.1, the x-derivatives are unimportant, so the forcing term in this system has amplitude $O(\varepsilon^{\mu+1}) + O(\varepsilon^{2\mu})$. It may vary on both fast and slow time scales at these orders in epsilon, but, by lemma 2.3, terms that vary only on the slow time scale make a contribution to the solution that is smaller by a factor of ε than the forcing itself. Thus it is sufficient to solve:

$$u_{1t} = \frac{1}{\varepsilon} A u_{1x} + [B_{11}(x,t)u_1]_x + (\mathbf{I}-\mathbf{S})[B_{12}(x,t)v_1 + \varphi(u_0,0,x,t)]_x \qquad (2.23)$$

$$u_1(x,0) = 0 \qquad , \qquad u_1(x+2\pi,t) \equiv u_1(x,t)$$

where $\mathbf{S}$ is the time-averaging operator, given by:

$$Sw(t) = \int_{t-\delta}^{t+\delta} w(\tau)\, d\tau$$

for some δ with $\varepsilon << \delta << 1$. This has solution:

$$u_1(x,t) = \varepsilon^{\mu+1}\tilde{u}_1^{(1)}(x,t) + \varepsilon^{2\mu}\tilde{u}_1^{(2)}(x,t) + \textit{lower order terms} \qquad (2.24)$$

where both $\tilde{u}_1^{(1)}$ and $\tilde{u}_1^{(2)}$ are $O(1)$ and oscillate on the fast time scale.

Also from (2.23), arguing as from (2.17):

$$\varepsilon u_{1t} = P_0 u_1 + \varepsilon G \qquad \text{(say)}$$

$$\Rightarrow \qquad u_1 = \varepsilon P_0^{-1} u_{1t} - \varepsilon P_0^{-1} G$$

$$= \varepsilon \frac{\partial}{\partial t}(P_0^{-1}u_0) - \varepsilon P_0^{-1} G + O(\varepsilon^2 u_1) \qquad (2.25)$$

Here, G is $O(\varepsilon^{2\mu}) + O(\varepsilon^{\mu+1})$. Thus, the next approximation to the v equation:

$$v_{2t} = [B_{22}(x,t)v_2]_x + [B_{21}(x,t)u_1]_x + [\psi(u_0+u_1,v_1,x,t) - \psi(u_0,0,x,t)]_x \qquad (2.26)$$

$$v_2(x,0) = 0 \qquad , \qquad v_2(x+2\pi,t) \equiv v_2(x,t)$$

has solution which, by lemma 2.1, lemma 2.4, and Duhamels principle, is of the form:

$$v_2(x,t) = O(\varepsilon u_1) + O(\varepsilon G) + O(u_0(u_1+v_1))$$

$$= O(\varepsilon^{\mu+2}) + O(\varepsilon^{2\mu+1}) + O(\varepsilon^{3\mu}) \qquad (2.27)$$

This iteration between the two equations can be continued to obtain an asymptotic expansion of the solution to (2.3) to any desired order. All remaining terms will be of the same order in ε as v_2, or smaller. This is so, because the remainder terms $u - u_0 - u_1$ and $v - v_1 - v_2$ satisfy a symmetric system, which is well-posed with a growth constant independent of ε. Thus, by Duhamels

principle, this system will have solution of the same order in ε as the forcing terms, and, by lemma 2.1, these are no larger than v_2.

All terms of amplitude ε^{μ}, $\varepsilon^{2\mu}$ or $\varepsilon^{\mu+1}$ in the solution of (2.3) are thus given by the linear systems (2.16), (2.19) and (2.23).

2.4 Generalisations of Theorem 1

In the proof of theorem 1, the essential features were that all x-derivatives of the solution were bounded independently of ε, so that successive forcing terms in the iteration did indeed become smaller, and that the large part of the spatial operator, P_0, was nonsingular, with inverse bounded independently of ε. The result can be extended to any symmetric hyperbolic system for which these two properties remain true.

(a) Undifferentiated Terms Suppose (2.1) is modified to have the form:

$$U_t = \frac{1}{\varepsilon} A U_x + [\Phi(U,V,x,t)]_x + \Gamma(U,V,x,t)$$

$$V_t = [\Psi(U,V,x,t)]_x + \Omega(U,V,x,t) \qquad (2.28)$$

$$U(x+2\pi,t) \equiv U(x,t) \quad , \quad V(x+2\pi,t) \equiv V(x,t)$$

where Γ and Ω are C^∞ functions of all their arguments, 2π-periodic in x, bounded, together with their derivatives, independently of ε, and all other symbols are as before.

Subtracting out a smooth solution, we obtain, analogously to (2.3):

$$u_t = \frac{1}{\varepsilon} A u_x + [B_{11}u + B_{12}v + \varphi(u,v)]_x + C_{11}u + C_{12}v + \gamma(u,v)$$

$$v_t = [B_{22}v + B_{21}u + \psi(u,v)]_x + C_{22}v + C_{21}u + \omega(u,v) \qquad (2.29)$$

say, with initial and boundary conditions as for (2.3). From lemma 2.1, the x-derivatives of u and v will still be bounded independently of ε, but the mean value $<u>$ of u will no longer be zero for all time, so the spatial operator is not uniquely invertible. However, the mean value of u is a slow scale variable, i.e. it has at least one time derivative bounded independently of ε, so it should really be grouped with the v variables rather than the other u variables.

A separate equation can be formed for $<u>$ by averaging the first equation of (2.29):

$$<u>_t = <C_{11}u> + <C_{12}v> + <\gamma(u,v)>$$

$$=<C_{11}><u> + F(u,<u>,v,<v>) \qquad (2.30)$$

say. Subtracting this from the unaveraged equation, and writing u for $u-<u>$:

$$u_t = \frac{1}{\varepsilon}Au_x + [B_{11}u + B_{12}v + \varphi(u,v)]_x + C_{11}u + C_{12}v + \gamma(u+<u>,v) + C_{11}<u>$$

$$- \{<C_{11}u> + <C_{12}v> + <\gamma(u+<u>,v)> + <C_{11}><u>\} \qquad (2.31)$$

This replaces the first equation in (2.29), while the second equation in (2.29) is augmented by (2.30).

Now the mean value of u is zero, and the proof can proceed as before. The first approximation to u, u_0, is as before, being $O(\varepsilon^{\mu})$, oscillating on the fast time scale, and satisfying (2.18) with slightly modified P_0. (2.18) can now be used in the second equation of (2.29) as before, and also in (2.30), to show that both v and $<u>$ are at leading order $O(\varepsilon^{\mu+1}) + O(\varepsilon^{2\mu})$. The rest of the iteration proceeds as before.

(b) Large Undifferentiated Terms Suppose an undifferentiated term is added to the large part of the spatial operator in (2.1):

$$U_t = \frac{1}{\varepsilon}(AU_x + CU) + [\Phi(U,V,x,t)]_x$$

$$V_t = [\Psi(U,V,x,t)]_x \qquad (2.32)$$

Then, provided C is constant and antisymmetric, so that the x-derivatives remain bounded, the proof goes through as before since the large part of the spatial operator $P_0 = A\frac{\partial}{\partial x} + C$ is nonsingular on S_p, provided any part of u that

is an eigenfunction of P_0 corresponding to eigenvalue zero is subtracted out, just as the mean value was in (a).

(c) More Space Dimensions and Nonseparation of Scales As remarked in [5], lemma 2.1 applies also in more space dimensions, the proof being modified only to the extent that spatial derivatives of higher than second order must be considered before a closed system can be formed, since the Sobolev inequality for the maximum norm in terms of L_2 norms requires higher derivatives. This is not a serious difficulty.

A difficulty that may arise in more space dimensions is that it may not be possible to separate time derivatives of the fast and slow scale variables into two different equations, as we assumed in (2.1). However, such a separation can be carried out in Fourier space, exactly as in [5], by means of a projection. For each fixed $\vec{\omega}$, the eigenvalues, κ, of the symbol, $P_0(i\vec{\omega})$, of the large part of the spatial operator are assumed to fall into two classes:

$$\kappa = 0 \qquad \text{or} \qquad |\kappa| \geq 1$$

(in the latter bound, 1 can be replaced by any $\delta > 0$ by redefining ε). Then, for each $\vec{\omega}$, there exists a unitary matrix $\hat{U}(i\vec{\omega})$ such that:

$$\hat{U}^*(i\vec{\omega})P_0(i\vec{\omega})\hat{U}(i\vec{\omega}) = \begin{bmatrix} \hat{R}(i\vec{\omega}) & 0 \\ 0 & 0 \end{bmatrix} \qquad |\hat{R}^{-1}| \leq 1 \tag{2.33}$$

Then the projection R is defined by:

$$Ru = \sum_{\vec{\omega}} e^{i\vec{\omega}\cdot\vec{x}}\hat{U}(i\vec{\omega})\begin{bmatrix} I_{\vec{\omega}} & 0 \\ 0 & 0 \end{bmatrix}\hat{U}^*(i\vec{\omega})\hat{u}(\vec{\omega}) \tag{2.34}$$

where $\hat{u}(\vec{\omega})$ are the Fourier coefficients of $u(\vec{x})$, and $I_{\vec{\omega}}$ is the unit matrix of the same dimension as $\hat{R}(i\vec{\omega})$. $\mathbf{S}_p$ can be written as the direct sum of:

$$\mathbf{S}_p^I = R\mathbf{S}_p \qquad , \qquad \mathbf{S}_p^{II} = (I-R)\mathbf{S}_p \tag{2.35}$$

and then $P_0 u^{II} = 0$ for all $u^{II} \in \mathbf{S}_p^{II}$ and P_0 is nonsingular on $\mathbf{S}_p^I$ with inverse defined by:

$$P_0^{-1} u^I = \sum_{\vec{\omega}} e^{i\vec{\omega}\cdot\vec{x}} \hat{U}(i\vec{\omega}) \begin{bmatrix} \hat{R}^{-1}(i\vec{\omega}) & 0 \\ 0 & 0 \end{bmatrix} \hat{U}^*(i\vec{\omega}) \hat{u}^I(\vec{\omega}) \tag{2.36}$$

In view of (2.33), $|P_0^{-1}| \leq 1$.

Now, $u^I = Ru$ is the fast scale variable, $u^{II} = (I-R)u$ the slow scale variable, and they satisfy equations of the form:

$$u_t^I = \frac{1}{\varepsilon} P_0\left(\frac{\partial}{\partial \vec{x}}\right) u^I + \cdots$$

$$u_t^{II} = \cdots$$

where the omitted terms are formally $O(1)$.

The remarks in (a), (b) and (c) may be collected together and expressed more formally in the following generalisation of theorem 1:

Theorem 2 Consider the symmetric hyperbolic system:

$$U_t = \frac{1}{\varepsilon} P_0\left(\frac{\partial}{\partial \vec{x}}\right) U + P_1\left(U, \vec{x}, t, \varepsilon, \frac{\partial}{\partial \vec{x}}\right) + F(\vec{x}, t, \varepsilon)$$

$$U(\vec{x}, 0) = U_s(\vec{x}, 0) + \varepsilon^{\mu} f(\vec{x}) \tag{2.37}$$

$$U(\vec{x} + 2\pi\vec{e}_j, t) \equiv U(\vec{x}, t) \qquad 1 \leq j \leq n$$

where $\vec{x} = (x_1, \ldots, x_n)$, $\vec{e}_j$ is the unit vector in the jth direction, $U_s(\vec{x}, t)$ is a smooth solution of the first and third equations, F is C^∞ in $\vec{x}$ and t, 2π-periodic in $\vec{x}$, and, together with all its derivatives, is bounded independently of ε, $\mu > 0$, and:

$$P_0\left(\frac{\partial}{\partial \vec{x}}\right) = \sum_{j=1}^{n} A_j \frac{\partial}{\partial x_j} + C \tag{2.38}$$

$$P_1\left(U,\vec{x},t,\varepsilon,\frac{\partial}{\partial \vec{x}}\right) = \sum_{j=1}^{n} \frac{\partial}{\partial x_j}[\Phi_j(U,\vec{x},t,\varepsilon)] + \Phi_0(U,\vec{x},t,\varepsilon)$$

where $A_j = A_j^*$, $1 \le j \le n$, $C = -C^*$, and Φ_j, $0 \le j \le n$, is C^∞ in all arguments, 2π-periodic in $\vec{x}$, and, together with its $\vec{x}$ and t derivatives, is bounded independently of ε.

Further, we assume that each eigenvalue κ of the symbol $P_0(i\vec{\omega})$ is either zero for all $\vec{\omega}$ or satisfies:

$$|\kappa(\vec{\omega})| \ge \lambda \tag{2.39}$$

for some positive constant λ, independent of ε, for all $\vec{\omega}$, except possibly at a finite number of values where some of these eigenvalues may also be zero.

Then, the result of theorem 1 applies, namely, there exist constants ε_0, K_0, K_1, δ and T, independent of ε, and strictly positive, such that:

$$\|(U - U_s)(.,t)\| \le \varepsilon^\mu K_0 \tag{2.40}$$

$$\|\overline{(U - U_s)}(.,t)\| \le (\varepsilon^{2\mu} + \varepsilon^{\mu+1})K_1$$

for all $t \in [0,T]$ and all $\varepsilon \le \varepsilon_0$, where:

$$\overline{(U - U_s)}(\vec{x},t) = \int_{t-\delta}^{t+\delta} (U - U_s)(\vec{x},\tau)\, d\tau \ .$$

Put another way, the solution of (2.37) has the form:

$$U(\vec{x},t) = U_s(\vec{x},t) + \varepsilon^\mu \tilde{u}(\vec{x},t) + O(\varepsilon^{2\mu}) + O(\varepsilon^{\mu+1}) \tag{2.41}$$

where $\tilde{u}$ is $O(1)$ and oscillates on the fast time scale.

Chapter 3: HYPERBOLIC SYSTEMS WITH A SYMMETRISER

3.1 Statement of the Problem

In this chapter we follow the analysis of chapter 2 for the more general hyperbolic systems with symmetrisers. Thus, consider again the system (1.4):

$$D^{(1)}(U,V,x,t)U_t = \frac{1}{\varepsilon}(AU_x + CU) + [\Phi(U,V,x,t)]_x$$

$$D^{(2)}(U,V,x,t)V_t = [\Psi(U,V,x,t)]_x \tag{3.1a}$$

$$U(x+2\pi,t) \equiv U(x,t) \quad , \quad V(x+2\pi,t) \equiv V(x,t)$$

Assume that $A=\mathrm{diag}(\lambda_1,\lambda_2,\ldots,\lambda_m)$, each λ_j, $1\leq j\leq m$, being a non-zero real constant, that C is a constant, real, anti-symmetric matrix, that Φ, Ψ, $D^{(1)}$ and $D^{(2)}$ are C^∞ functions of all their arguments, 2π-periodic in x, and have no explicit ε dependence, and that each $D^{(j)}$ is symmetric, positive-definite, non-singular and, together with its inverse, bounded uniformly in (U,V). Finally assume that the system is symmetric hyperbolic.

Suppose $(\varepsilon u_s(x,t),v_s(x,t))$ is a smooth solution of (3.1a), see [5]. Take initial conditions:

$$U(x,0) = \varepsilon u_s(x,0) + \varepsilon f(x) \tag{3.1b}$$

$$V(x,0) = v_s(x,0)$$

where $f(x)\ \varepsilon\ C^\infty$ is 2π-periodic and independent of ε.

Set:

$$u(x,t) = \varepsilon^{-1}[U(x,t) - \varepsilon u_s(x,t)] \tag{3.2}$$

$$v(x,t) = \varepsilon^{-1}[V(x,t) - v_s(x,t)]$$

noting that the scaling differs from that in chapter 2. The equations satisfied by the scaled perturbations u and v are:

$$H^{(1)}(u,v,x,t)u_t = \frac{1}{\varepsilon}P_0 u + [B_{11}(x,t)u + B_{12}(x,t)v]_x + \varepsilon[\varphi(u,v,x,t)]_x + \varepsilon F_1(u,v,x,t)$$

$$H^{(2)}(u,v,x,t)v_t = [B_{21}(x,t)u + B_{22}(x,t)v]_x + \varepsilon[\psi(u,v,x,t)]_x + \varepsilon F_2(u,v,x,t)$$

$$u(x,0) = f(x) \quad , \quad v(x,0) = 0 \tag{3.3}$$

$$u(x+2\pi,t) \equiv u(x,t) \quad , \quad v(x+2\pi,t) \equiv v(x,t)$$

Here, $P_0 \equiv A\frac{\partial}{\partial x} + C$, and:

$$H^{(j)}(u,v,x,t) = D_0^{(j)}(x,t) + \varepsilon D_{11}^{(j)}(u,x,t) + \varepsilon D_{12}^{(j)}(v,x,t) + \varepsilon^2 D_2^{(j)}(u,v,x,t)$$

where $D_0^{(j)}(x,t) = D^{(j)}(\varepsilon u_s, v_s, x, t)$ is bounded, symmetric and positive-definite with bounded inverse, while $(D_{11}^{(j)} + D_{12}^{(j)})$ is the linear part and $D_2^{(j)}$ the quadratic and higher part of:

$$D^{(j)}(\varepsilon u_s + \varepsilon u, v_s + \varepsilon v, x, t) - D^{(j)}(\varepsilon u_s, v_s, x, t) \, .$$

Also, $(B_{11}u + B_{12}v)$ is the linear part and φ the quadratic and higher part of:

$$\varepsilon^{-1}\left[\Phi(\varepsilon u_s + \varepsilon u, v_s + \varepsilon v, x, t) - \Phi(\varepsilon u_s, v_s, x, t)\right] ,$$

$(B_{21}u + B_{22}v)$ is the linear part and ψ the quadratic and higher part of:

$$\varepsilon^{-1}\left[\Psi(\varepsilon u_s + \varepsilon u, v_s + \varepsilon v, x, t) - \Psi(\varepsilon u_s, v_s, x, t)\right] ,$$

and:

$$F^{(1)}(u,v,x,t) = -\varepsilon^{-1} u_{st}\left[D^{(1)}(\varepsilon u_s + \varepsilon u, v_s + \varepsilon v, x, t) - D^{(1)}(\varepsilon u_s, v_s, x, t)\right] ,$$

$$F^{(2)}(u,v,x,t) = -\varepsilon^{-1} v_{st}\left[D^{(2)}(\varepsilon u_s + \varepsilon u, v_s + \varepsilon v, x, t) - D^{(2)}(\varepsilon u_s, v_s, x, t)\right] .$$

Each B_{ij}, $i,j=1,2$, is C^∞ in x and t, 2π-periodic in x, and, together with its x and t derivatives, is bounded independently of ε. The same may be assumed of φ, ψ, $D_{ij}^{(k)}$ and F_j, $i,j,k=1,2$, since these bounds are needed only in a neighborhood of the solution, and these functions can be modified for other u,v without affecting the solution.

The main results of this chapter are then:

Theorem 3 If u,v is the solution of system (3.3), the scaled perturbation of the solution of (3.1), then there exist constants ε_0, K and T, independent of ε, and strictly positive, such that:

$$\|u(.,t)\| + \|v(.,t)\| \leq K \tag{3.4}$$

for all $t \in [0,T]$ and all $\varepsilon \leq \varepsilon_0$.

Theorem 4. Further, if P_0 is nonsingular on the space $\mathbf{S}_p$, there exist constants K_1 and δ, independent of ε, and strictly positive, such that:

$$\|\bar{u}(.,t)\| + \|v(.,t)\| \leq \varepsilon K_1 \tag{3.5}$$

where:

$$\bar{u}(x,t) = \int_{t-\delta}^{t+\delta} u(x,\tau)d\tau$$

for all $t \in [0,T]$ and all $\varepsilon \leq \varepsilon_0$.

Put another way, the solution of (3.3) has the form:

$$u(x,t) = \tilde{u}(x,t) + O(\varepsilon)$$

$$v(x,t) = O(\varepsilon)$$

where $\tilde{u}$ is $O(1)$ and oscillates on the fast time scale.

3.2 Boundedness of the x-derivatives.

We again need to show that the x-derivatives of the solution to the system (3.3) are bounded independently of ε, even if the t-derivatives are not. In this case, we need to consider both linearised equations, rather than just the u equation, since both u and v can be $O(1)$ when the large part of the spatial operator is singular.

Note that the coefficient of the x-derivative on the right hand side of (3.6) is assumed to be block-diagonal (i.e. no v_x appears in the u_t equation and vice versa). This is necessary for our proof to go through, but if $O(1)$ off-diagonal blocks are present, they may always be transformed away, since the diagonal blocks differ in magnitude by an order in ε. This is the purpose of the transformation performed at the beginning of section 3.4.

Extra terms $E_{11}(x,t)u$ and $E_{22}(x,t)v$ can be added to the first and second equations in (3.6) respectively with no change in the result. They are omitted here to save writing, and because they do not appear in the transformed system to which we shall eventually apply the lemma.

Lemma 3.1 Suppose $w(x,t) = \begin{bmatrix} u(x,t) \\ v(x,t) \end{bmatrix}$ satisfies the symmetric hyperbolic system:

$$[D_0^{(1)}(x,t) + \varepsilon D_1^{(1)}(w,x,t)]u_t = \frac{1}{\varepsilon}(Au_x + Cu) + [B_{11}(x,t)u]_x + E_{12}(x,t)v$$

$$D_0^{(2)}(x,t)v_t = (B_{22}(x,t)v)_x + E_{21}(x,t)u \qquad (3.6)$$

$$u(x,0) = f_1(x) \quad , \quad u(x+2\pi,t) \equiv u(x,t)$$

$$v(x,0) = f_2(x) \quad , \quad v(x+2\pi,t) \equiv v(x,t)$$

where A is a constant, symmetric, non-singular matrix, C is a constant, anti-symmetric matrix, $D_0^{(j)}$ is a bounded, positive-definite, symmetric matrix, C^∞ in

x and t, 2π-periodic in x, non-singular with bounded inverse, B_{ij}, E_{ij}, $D_1^{(1)}$ and f are C^∞ in their arguments and 2π-periodic in x, B_{ij} and $D_1^{(1)}$ are symmetric, all quantities are real, and there exist constants M_{rs}, N_{qrs}, P_{rs} and Q_{rs}, independent of ε, such that:

$$\left| \frac{\partial^{r+s} D_0^{(j)}}{\partial x^r \partial t^s} \right| \leq M_{rs}$$

$$\left| \frac{\partial^{q+r+s} D_1^{(1)}}{\partial w_1^{q_1} \cdots \partial w_m^{q_m} \partial x^r \partial t^s}(w,.,t) \right| \leq N_{qrs} \tag{3.7}$$

$$\left| \frac{\partial^{r+s} B_{ij}}{\partial x^r \partial t^s} \right| \leq P_{rs} \quad , \quad \left| \frac{\partial^{r+s} E_{ij}}{\partial x^r \partial t^s} \right| \leq Q_{rs}$$

for all w, all non-negative $q_1, \ldots, q_m, r, s$, with $q = q_1 + \cdots + q_m$, and for all $i,j = 1,2$.

Then there exist constants T, K_{rs} and L_{rs}, independent of ε, such that:

$$\left\| \frac{\partial^{r+s} u}{\partial x^r \partial t^s}(.,t) \right\| \leq \varepsilon^{-s} K_{rs} \tag{3.8}$$

$$\left\| \frac{\partial^{r+s} v}{\partial x^r \partial t^s}(.,t) \right\| \leq \varepsilon^{-p(s)} L_{rs}$$

for all $t \in [0,T]$, $r,s = 0,1,2,\ldots$, where $p(s) = 0$ for $s = 0$ and $p(s) = s - 1$ for $s > 0$.

Proof The key to the proof is that, using the fact that A is non-singular, the t-derivatives of u can be estimated independently of its x-derivatives from the first of (3.6). Then this equation can be used to estimate the x-derivatives of u. The second equation of (3.6) is of a more standard form, and estimates of the derivatives of v are easily obtained.

Note that (3.8) is satisfied at $t=0$ (from the equations (3.6) and their derivatives at $t=0$).

Let $H_1^{(1)}(w) = D_0^{(1)} + \varepsilon D_1^{(1)}(w)$. Since $D_0^{(1)}$ is positive-definite and bounded above and below, and $D_1^{(1)}$ is bounded uniformly in w, there exist positive constants δ_1, δ_2, independent of ε and w, such that:

$$\delta_1 \| y \|^2 \leq (y, H_1^{(1)}(w) y) \leq \delta_2 \| y \|^2 \qquad \text{for all y.} \quad (\dagger)$$

Now the first equation of (3.6) is:

$$H_1^{(1)}(w) u_t = \frac{1}{\varepsilon}(Au_x + Cu) + [B_{11}u]_x + E_{12}v \qquad (*)$$

so:

$$(u, H_1^{(1)}(w)u)_t = 2(u, H_1^{(1)}(w)u_t) + (u, H_{1t}^{(1)}(w)u)$$

$$= \frac{2}{\varepsilon}(u, Au_x) + \frac{2}{\varepsilon}(u, Cu) + 2(u, (B_{11}u)_x) + 2(u, E_{12}v) + (u, H_{1t}^{(1)}(u,v)u)$$

where the fact that $H_1^{(1)}$ is real symmetric has been used. Now:

$$(u, Au_x) = 0 \qquad , \qquad (u, Cu) = 0$$

$$2(u, (B_{11}u)_x) = (u, B_{11x}u) \leq P_{10} \| u \|^2$$

$$2(u, E_{12}v) \leq 2Q_{00} \| u \| \| v \|$$

$$(u, H_{1t}^{(1)}(w)u) = \left(u, \frac{\partial D_0^{(1)}}{\partial t} u\right) + \varepsilon \left(u, \frac{\partial D_1^{(1)}(w)}{\partial t} u\right) + \varepsilon \sum_{j=1}^{m} (u, w_{jt} D_{1w_j}^{(1)} u)$$

$$\leq (M_{01} + \varepsilon N_{001} + \varepsilon m N_{100} |w_t|_\infty) \| u \|^2$$

noting that w_{jt} is a scalar, $1 \leq j \leq m$. Hence:

$$(u, H_1^{(1)}(w)u)_t \leq const. \ \| u \|^2 [1 + \varepsilon(|u_t|_\infty + |v_t|_\infty)] \qquad (1)$$

Setting $y = u_t$ and differentiating (*) with respect to t:

$$H_1(w)y_t = \frac{1}{\varepsilon}(Ay_x + Cy) + [B_{11}y + B_{11t}u]_x + E_{12}v_t + E_{12t}v - H_{1t}^{\{1\}}(w)y \quad (**)$$

Hence:

$$(y, H_1(w)y)_t = \frac{2}{\varepsilon}(y, Ay_x) + \frac{2}{\varepsilon}(y, Cy) + 2(y, (B_{11}y)_x) + 2(y, (B_{11t}w)_x)$$

$$+ 2(y, E_{12}v_t) + 2(y, E_{12t}v) - (y, H_{1t}(w)y)$$

Note that:

$$(y, (B_{11t}u)_x) = (y, B_{11xt}u) + (y, B_{11t}u_x) \leq P_{11}\|y\|\|u\| + P_{01}\|y\|\|u_x\|$$

$$(y, E_{12}v_t) \leq Q_{00}\|y\|\|v_t\|$$

$$(y, E_{12t}v) \leq Q_{01}\|y\|\|v\|$$

Thus:

$$(y, H_1^{\{1\}}(w)y)_t \leq const.\{\|y\|^2[1+\varepsilon(|u_t|_\infty + |v_t|_\infty)]$$

$$+ \|y\|(\|u\| + \|u_x\| + \|v_t\| + \|v\|)\} \qquad (2)$$

Finally, setting $z = u_{tt}$ and differentiating (**) with respect to t:

$$H_1^{\{1\}}(w)z_t = \frac{1}{\varepsilon}(Az_x + Cz) + [B_{11}z + 2B_{11t}y + B_{11tt}w]_x + E_{12}v_{tt}$$

$$+ 2E_{12t}v_t + E_{12tt}v - 2H_{1t}^{\{1\}}(w)z - H_{1tt}^{\{1\}}(w)y$$

Now:

$$(z, H_{1tt}^{\{1\}}(w)y) = (z, D_{0tt}^{\{1\}}y) + \varepsilon(z, D_{1tt}^{\{1\}}y) + 2\varepsilon\sum_{j=1}^{m}(z, w_{jt}D_{1tw_j}^{\{1\}}y)$$

$$(z, H^{\{1\}}_{1t}(w)y) = (z, D^{\{1\}}_{0tt}y) + \varepsilon(z, D^{\{1\}}_{1t}y) + 2\varepsilon\sum_{j=1}^{m}(z, w_{jt}D^{\{1\}}_{1w_j}y)$$

$$+ \varepsilon\sum_{j,k=1}^{m}(z, w_{jt}w_{kt}D^{\{1\}}_{1w_jw_k}y) + \varepsilon\sum_{j=1}^{m}(z, w_{jtt}D^{\{1\}}_{1w_j}y)$$

$$\leq (M_{02} + \varepsilon N_{002} + 2\varepsilon m N_{101}|w_t|_\infty + \varepsilon m^2 N_{200}|w_t|_\infty^2)\|z\|\|y\| + \mathbf{R}$$

where:

$$\mathbf{R} = \varepsilon\sum_{j=1}^{m}(z, w_{jtt}D^{\{1\}}_{1w_j}y) = \varepsilon(z, \tilde{D}(y)w_{tt})$$

where:

$$(\tilde{D}(y))_{\mu\nu} = \sum_{j=1}^{m}\frac{\partial D^{\{1\}}_{1\mu j}}{\partial w_\nu}y_j$$

$$\Rightarrow \qquad |\tilde{D}(y)| \leq mN_{100}|y|_\infty$$

$$\Rightarrow \qquad \mathbf{R} \leq \varepsilon m N_{100}|y|_\infty\|z\|^2$$

Since all other terms are easily estimated as before:

$$(z, H_1(w)z)_t \leq (P_{10} + 3M_{01} + 3\varepsilon N_{001} + 5\varepsilon m N_{100}|w_t|_\infty)\|z\|^2$$

$$+ [4P_{01}\|y_x\| + (4P_{11} + 2M_{02} + 2\varepsilon N_{002} + 4\varepsilon m N_{101}|w_t|_\infty + 2\varepsilon m^2 N_{200}|w_t|_\infty^2)\|y\|$$

$$+ 2P_{02}\|u_x\| + 2P_{12}\|u\| + 2Q_{00}\|v_{tt}\| + 4Q_{01}\|v_t\| + 2Q_{02}\|v\|]\|z\|$$

$$\leq const.\ \{\|z\|^2[1+\varepsilon(|u_t|_\infty + |v_t|_\infty)] + \|z\|(\|y\| + \|y_x\| + \|u\| + \|u_x\| + \|v_{tt}\| + \|v_t\| + \|v\|)$$

$$+ \varepsilon(|u_t|_\infty + |v_t|_\infty + |u_t|_\infty + |v_t|_\infty^2)\|y\|\|z\|\} \qquad (3)$$

Similarly, in fact more easily, one finds from the second equation of (3.6) that:

$$(v, D_0^{(2)} v)_t \leq const. \{ \| v \|^2 + \| u \| \| v \| \} \tag{4}$$

$$(v_x, D_0^{(2)} v_x)_t \leq const. \{ \| v_x \|^2 + \| v_x \| (\| v \| + \| u_x \| + \| u \|) \} \tag{5}$$

$$(v_{xx}, D_0^{(2)} v_{xx})_t \leq const. \{ \| v_{xx} \|^2 + \| v_{xx} \| (\| v_x \| + \| v \| + \| u_{xx} \| + \| u_x \| + \| u \|) \} \tag{6}$$

Next note Sobolev's inequality:

$$| w |_\infty \leq const. \{ \| w \| + \| w_x \| \} \tag{7}$$

for any $w \varepsilon S_p$.

From (*):

$$u_x = (A + \varepsilon B_{11})^{-1} \{ -Cu + \varepsilon (H_1^{(1)}(w) u_t - B_{11x} u - E_{12} v) \}$$

$$\Rightarrow \qquad \| u_x \| \leq const. \{ \| u \| + \varepsilon (\| u_t \| + \| v \|) \}$$

$$= const. \{ \| u \| + \varepsilon (\| y \| + \| v \|) \} \tag{8}$$

and from (**):

$$y_x = (A + \varepsilon B_{11})^{-1} \left\{ -Cy + \varepsilon [H_1^{(1)}(w) y_t - B_{11x} y - (B_{11t} u)_x - E_{12} v_t - E_{12t} v + H_{1t}^{(1)}(w) y] \right\}$$

$$\Rightarrow \qquad \| y_x \| \leq const. \left\{ [1 + \varepsilon^2 (| u_t |_\infty + | v_t |_\infty)] \| y \| + \varepsilon (\| y_t \| + \| u_x \| + \| u \| + \| v_t \| + \| v \|) \right\}$$

$$\leq const. \left\{ [1 + \varepsilon^2 (\| y \| + \| y_x \| + \| v_t \| + \| v_{xt} \|)] \| y \|^2 + \varepsilon (\| z \| + \| u \| + \| v_t \| + \| v \|) \right\}$$

using (7) and (8). Hence, provided $\| y \| = o(\varepsilon^{-2})$, as we shall show it is:

$$\| y_x \| \leq const. \left\{ [1 + \varepsilon^2 (\| y \| + \| v_t \| + \| v_{xt} \|)] \| y \|^2 + \varepsilon (\| z \| + \| u \| + \| v_t \| + \| v \|) \right\} \tag{9}$$

Similarly:

$$u_{xx} = (A+\varepsilon B_{11})^{-1}\Big\{-Cu_x + \varepsilon[H_1^{(1)}(w)y_x - 2B_{11x}u_x - B_{11xx}u - E_{12}v_x - E_{12x}v + H_{1x}^{(1)}(w)y]\Big\}$$

$$\Rightarrow \quad \|u_{xx}\| \le const.\{\|u_x\| + \varepsilon[\|u_{xt}\| + \|u_t\| + \|u\| + \|v_x\| + \|v\|$$

$$+\varepsilon(|u_x|_\infty + |v_x|_\infty)\|u_t\|]\}$$

$$\le const.\{\|u_x\| + \varepsilon[\|y_x\| + \|y\| + \|u\| + \|v_x\| + \|v\|$$

$$+\varepsilon(\|u_x\| + \|u_{xx}\| + \|v_x\| + \|v_{xx}\|)\|y\|]\}$$

$$\Rightarrow \quad \|u_{xx}\| \le const.\{\|u_x\| + \varepsilon[\|y_x\| + \|y\| + \|u\| + \|v_x\| + \|v\| \qquad (10)$$

$$+\varepsilon(\|u_x\| + \|v_x\| + \|v_{xx}\|)\|y\|]\}$$

Directly from the v_t equation and its first x- and t-derivatives, one can similarly obtain the bounds:

$$\|v_t\| \le const.\{\|v_x\| + \|v\| + \|u\|\} \qquad (11)$$

$$\|v_{xt}\| \le const.\{\|v_{xx}\| + \|v_x\| + \|v_t\| + \|v\| + \|u_x\| + \|u\|\} \qquad (12)$$

$$\|v_{tt}\| \le const.\{\|v_{xt}\| + \|v_t\| + \|v\| + \|u_t\| + \|u\|\} \qquad (13)$$

The six inequalities (8) - (13) enable us to eliminate the norms of u_x, u_{xx}, $u_{xt}(=y_x)$, v_t, v_{tt} and v_{xt} in favour of the norms of u, $u_t(=y)$, $u_{tt}(=z)$, v, v_x and v_{xx}. Thus, substitution of (8) - (13) in (1) - (6) yields (in view of (†)) a closed system of six nonlinear inequalities for the latter six quantities. Further, if these quantities are scaled by the factors of ε suggested by (3.8), namely if:

$$\zeta_1 = \|u\| \quad , \quad \zeta_2 = \varepsilon\|u_t\| \quad , \quad \zeta_3 = \varepsilon^2\|u_{tt}\|$$

$$\zeta_4 = \|v\| \quad , \quad \zeta_5 = \|v_x\| \quad , \quad \zeta_6 = \|v_{xx}\|$$

then the system (1) - (6) becomes:

$$(\zeta_1^2)_t \le const.\{\zeta_1^2[1+G_1(\zeta)+\varepsilon G_2(\zeta)] + \zeta_1\zeta_4\}$$

$$(\zeta_2^2)_t \le const.\{\zeta_2^2[1+G_1(\zeta)+\varepsilon G_2(\zeta)] + \varepsilon\zeta_2(\zeta_1+\zeta_4+\zeta_5)\}$$

$$(\zeta_3^2)_t \le const.\{\zeta_3^2[1+G_1(\zeta)+\varepsilon G_2(\zeta)] + \varepsilon\zeta_3[\zeta_2+\zeta_3+\varepsilon(\zeta_1+\zeta_4+\zeta_5+\zeta_6+\zeta_2^2)+\varepsilon^2\zeta_2 G_2(\zeta)]$$

$$+ \varepsilon^2\zeta_3 G_1(\zeta)(\zeta_1+\zeta_4+\zeta_5+\zeta_6) + [\varepsilon G_1(\zeta)+\varepsilon^2 G_2(\zeta)+(G_1(\zeta)+\varepsilon G_2(\zeta))^2]\zeta_2\zeta_3\}$$

$$(\zeta_4^2)_t \le const.\{\zeta_4^2+\zeta_4\zeta_1\}$$

$$(\zeta_5^2)_t \le const.\{\zeta_5^2+\zeta_5(\zeta_1+\zeta_2+\zeta_4)\}$$

$$(\zeta_6^2)_t \le const.\{\zeta_6^2+\zeta_6[\zeta_1+\zeta_2+\zeta_3+\zeta_4+\zeta_5+\varepsilon\zeta_2(\zeta_2+\zeta_5+\zeta_6+\varepsilon(\zeta_1+\varepsilon\zeta_4+G_2(\zeta)))]\}$$

where

$$G_1(\vec{\zeta}) = \zeta_2+\zeta_3 + \varepsilon^2(\zeta_1+\zeta_4+\zeta_5) + \varepsilon\zeta_2^2 + \varepsilon^2\zeta_2(\zeta_1+\zeta_4+\zeta_5+\zeta_6)$$

$$G_2(\vec{\zeta}) = (\zeta_1+\zeta_2+\zeta_4+\zeta_5+\zeta_6)$$

Note that this system is regular as $\varepsilon \to 0$. Also, the initial conditions are such that each ζ_j is $O(1)$ at $t=0$. Thus there exists some constant T, independent of ε, such that this system has a finite, bounded, $O(1)$ solution on the time interval [0,T]. Note that T may depend on the initial conditions. This proves that (3.8) holds for the six functions under consideration, i.e. there exist constants $K_{00}, K_{01}, K_{02}, L_{00}, L_{10}, L_{20}$, independent of ε, but possibly dependent on the initial data, such that:

$$\left\| \frac{\partial^j u}{\partial t^j}(.,t) \right\| \le \varepsilon^{-j} K_{0j}$$

$$\left\| \frac{\partial^j v}{\partial x^j}(.,t) \right\| \le L_{j0}$$

for $j=0,1,2$.

Now it is easy to go back to the inequalities (8) - (13) and to show that (3.8) holds for all derivatives of u and v up to second order. For example, from (8):

$$\|u_x\| \leq const.\ \{K_{00} + \varepsilon(\varepsilon^{-1}K_{01}+L_{00})\}$$

$$= K_{10} \qquad \text{(say)} \qquad \text{on } [0,T].$$

while from (11):

$$\|v_t\| \leq const.\{L_{10}+L_{00}+K_{00}\} = L_{01} \qquad \text{(say)} \qquad \text{on } [0,T].$$

and so on. Higher derivatives can be estimated by similar methods.

3.3 Proof of Theorem 3.

The proof of this theorem consists of a transformation of the system (3.3) which reduces the off-diagonal blocks in the coefficient matrix of the x-derivatives to $O(\varepsilon)$, and then the finding of an asymptotic expansion of the solution by an iteration.

First write the system (3.3):

$$u_t = \varepsilon^{-1} H^{(1)-1}(A+\varepsilon B_{11})u_x + H^{(1)-1}B_{12}v_x + \varepsilon^{-1} H^{(1)-1}(C+\varepsilon B_{11x})u + H^{(1)-1}B_{12x}v$$

$$+ \varepsilon H^{(1)-1}\varphi_x + \varepsilon H^{(1)-1}F_1 \tag{3.9}$$

$$v_t = H^{(2)-1}B_{21}u_x + H^{(2)-1}B_{22}v_x + H^{(2)-1}B_{21x}u + H^{(2)-1}B_{22x}v$$

$$+ \varepsilon H^{(2)-1}\psi_x + \varepsilon H^{(2)-1}F_2$$

Note that if:

$$T = \begin{bmatrix} 0 & -A^{-1}B_{12} \\ H^{(2)-1}B_{21}A^{-1}H^{(1)} & 0 \end{bmatrix} \tag{3.10}$$

then:

$$(I+\varepsilon T)^{-1}\begin{bmatrix} \varepsilon^{-1}H^{(1)-1}(A+\varepsilon B_{11}) & H^{(1)-1}B_{12} \\ H^{(2)-1}B_{21} & H^{(2)-1}B_{22} \end{bmatrix}(I+\varepsilon T) = \begin{bmatrix} \varepsilon^{-1}H^{(1)-1}(A+\varepsilon B_{11}) & 0 \\ 0 & H^{(2)-1}B_{22} \end{bmatrix} + O(\varepsilon).$$

Thus the transformation:

$$\begin{bmatrix} u \\ v \end{bmatrix} = (I+\varepsilon T)\begin{bmatrix} \overline{u} \\ \overline{v} \end{bmatrix} \tag{3.11}$$

substituted in (3.9) leads to:

$$\begin{bmatrix}\bar{u}\\ \bar{v}\end{bmatrix}_t = (I+\varepsilon T)^{-1}\begin{bmatrix}\varepsilon^{-1}H^{(1)-1}(A+\varepsilon B_{11}) & H^{(1)-1}B_{12}\\ H^{(2)-1}B_{21} & H^{(2)-1}B_{22}\end{bmatrix}\left\{(I+\varepsilon T)\begin{bmatrix}\bar{u}\\ \bar{v}\end{bmatrix}_x + \varepsilon T_x\begin{bmatrix}\bar{u}\\ \bar{v}\end{bmatrix}\right\}$$

$$+ (I+\varepsilon T)^{-1}\begin{bmatrix}\varepsilon^{-1}H^{(1)-1}(C+\varepsilon B_{11x}) & H^{(1)-1}B_{12x}\\ H^{(2)-1}B_{21x} & H^{(2)-1}B_{22x}\end{bmatrix} \tag{3.12}$$

$$- \varepsilon(I+\varepsilon T)^{-1}T_t(I+\varepsilon T)\begin{bmatrix}\bar{u}\\ \bar{v}\end{bmatrix} + \varepsilon(I+\varepsilon T)^{-1}\begin{bmatrix}H^{(1)-1}(\varphi_x+F_1)\\ H^{(2)-1}(\psi_x+F_2)\end{bmatrix}$$

$$\Longleftrightarrow \qquad \bar{u}_t = \varepsilon^{-1}H^{(1)-1}(A\bar{u}_x+C\bar{u}) + H^{(1)-1}(B_{11}\bar{u})_x - H^{(1)-1}CA^{-1}B_{12}\bar{v} + \varepsilon H^{(1)-1}G_1$$

$$\bar{v}_t = H^{(2)-1}(B_{22}\bar{v})_x + H^{(2)-1}B_{21x}(I+A^{-1}C)\bar{u} + \varepsilon H^{(2)-1}G_2$$

$$\Longleftrightarrow \qquad \bar{H}^{(1)}(\bar{u},\bar{v},x,t)\bar{u}_t = \frac{1}{\varepsilon}(A\bar{u}_x+C\bar{u}) + (B_{11}\bar{u})_x + E_{12}\bar{v}$$

$$+ \varepsilon G_1(\bar{u},\bar{v},\bar{u}_x,\bar{v}_x,\varepsilon\bar{u}_t,\varepsilon\bar{v}_t) \tag{3.13a}$$

$$\bar{H}^{(2)}(\bar{u},\bar{v})\bar{v}_t = (B_{22}\bar{v})_x + E_{21}\bar{u} + \varepsilon G_2(\bar{u},\bar{v},\bar{u}_x,\bar{v}_x,\varepsilon\bar{u}_t,\varepsilon\bar{v}_t)$$

say, where:

$$\bar{H}^{(j)}(\bar{u},\bar{v},x,t) \equiv H^{(j)}(u,v,x,t)$$

$$\equiv D_0^{(j)}(x,t) + \varepsilon\bar{D}_{11}^{(j)}(\bar{u},x,t) + \varepsilon\bar{D}_{12}^{(j)}(\bar{v},x,t) + \varepsilon^2\bar{D}_2^{(j)}(\bar{u},\bar{v},x,t)$$

is of similar structure to $H^{(j)}$, $E_{12}(x,t)$ and $E_{21}(x,t)$ are bounded, C^∞ functions, periodic in x, and G_1 and G_2 are bounded, C^∞ functions of the indicated arguments. The t-derivatives of $\bar{u}$ and $\bar{v}$ appear as arguments because of the presence of the T_t term in (3.12), but it is easy (though a little messy) to verify that they do indeed appear only when multiplied by ε, as indicated.

The initial conditions are now:

$$\begin{bmatrix}\bar{u}(x,0)\\ \bar{v}(x,0)\end{bmatrix} = (I+\varepsilon T)^{-1}\begin{bmatrix}f(x)\\ 0\end{bmatrix} = \begin{bmatrix}f_1(x)\\ \varepsilon f_2(x)\end{bmatrix} \qquad \text{(say)} \tag{3.13b}$$

Now we can take as a first approximation to (3.13) the system:

$$H_1^{(1)}(u_0,v_0,x,t)u_{0t} = \frac{1}{\varepsilon}(Au_{0x}+Cu_0) + (B_{11}(x,t)u_0)_x + E_{12}(x,t)v_0$$

$$D_0^{(2)}(x,t)v_{0t} = (B_{22}(x,t)v_0)_x + E_{21}(x,t)u_0 \qquad (3.14)$$

$$u_0(x,0) = f_1(x), \qquad , \qquad v_0(x,0) = \varepsilon f_2(x)$$

$$u_0(x+2\pi,t) \equiv u_0(x,t) \qquad , \qquad v_0(x+2\pi,t) \equiv v_0(x,t)$$

where:

$$\overline{H}_1^{(1)}(u_0,v_0,x,t) \equiv D_0^{(1)}(x,t) + \varepsilon\overline{D}_{11}^{(1)}(u_0,x,t) + \varepsilon\overline{D}_{12}^{(1)}(v_0,x,t).$$

Since $\overline{D}_{11}^{(1)}$ and $\overline{D}_{12}^{(1)}$ can be modified for large u_0, v_0 to ensure that the bounds (3.7) are met, without the solution being altered, this system has the form (3.6). Thus, by lemma 3.1, there exist constants $K_{rs}^{(0)}$ and T, independent of ε, such that:

$$\left\| \frac{\partial^{r+s}u_0}{\partial x^r \partial t^s}(.,t) \right\| \leq \varepsilon^{-s} K_{rs}^{(0)} \qquad (3.15a)$$

$$\left\| \frac{\partial^{r+s}v_0}{\partial x^r \partial t^s}(.,t) \right\| \leq \varepsilon^{-p(s)} K_{rs}^{(0)} \qquad (3.15b)$$

for all $t \in [0,T]$, $r,s = 0,1,2,\ldots$, where $p(s)=0$ for $s=0$ and $p(s)=s-1$ for $s>0$.

Next let u_1, v_1 satisfy:

$$\bar{H}_1^{(1)}(u_0,v_0,x,t)u_{1t} = \frac{1}{\varepsilon}(Au_{1x}+Cu_1) + (B_{11}(x,t)u_1)_x + E_{12}(x,t)v_1$$

$$- \varepsilon\bar{D}_{11}^{(1)}(u_1,x,t)u_{0t} - \varepsilon\bar{D}_{12}^{(1)}(v_1,x,t)u_{0t} - \varepsilon^2\bar{D}_2^{(1)}(u_0,v_0,x,t)u_{0t}$$

$$+ \varepsilon G_1(u_0,v_0,u_{0x},v_{0x},\varepsilon u_{0t},\varepsilon v_{0t}) \qquad (3.16)$$

$$\bar{H}_1^{(2)}(u_0,v_0,x,t)v_{1t} = (B_{22}(x,t)v_1)_x + E_{21}(x,t)u_1 - \varepsilon\bar{D}_{11}^{(2)}(u_0,x,t)v_{0t}$$

$$- \varepsilon\bar{D}_{12}^{(2)}(v_0,x,t)v_{0t} + \varepsilon G_2(u_0,v_0,u_{0x},v_{0x},\varepsilon u_{0t},\varepsilon v_{0t})$$

$$u_1(x,0) = 0, \qquad , \qquad v_1(x,0) = 0$$

$$u_1(x+2\pi,t) \equiv u_1(x,t) \qquad , \qquad v_1(x+2\pi,t) \equiv v_1(x,t)$$

where $\bar{H}_1^{(2)}(u_0,v_0,x,t) = D_0^{(2)}(x,t)+\varepsilon\bar{D}_{11}^{(2)}(u_0,x,t)+\varepsilon\bar{D}_{12}^{(2)}(v_0,x,t)$.

This system is linear in u_1 and v_1, with coefficients and forcing terms of amplitude $O(\varepsilon)$ each satisfying bounds of the form (3.15a). Thus by Duhamel's principle, there exist constants $K_{rs}^{(1)}$, independent of ε, such that:

$$\left\| \frac{\partial^{r+s}u_1}{\partial x^r \partial t^s}(.,t) \right\| + \left\| \frac{\partial^{r+s}v_1}{\partial x^r \partial t^s}(.,t) \right\| \leq \varepsilon \cdot \varepsilon^{-s} K_{rs}^{(1)} \qquad (3.17)$$

for all $t\,\varepsilon[0,T]$, $r,s=0,1,2,....$

This iteration can be continued indefinitely, each time the correction to the solution being smaller by a factor of ε. Let $d_p=\bar{u}-(u_0+\cdots+u_{p-1})$ and $e_p=\bar{v}-(v_0+\cdots+v_{p-1})$ be the remainders after p steps. The system satisfied by d_p and e_p has forcing terms of amplitude $O(\varepsilon^p)$, and the initial data are such that d_p and e_p and all their derivatives up to second order are $O(\varepsilon^{p-2})$ at $t=0$. Also, the coefficients in the homogeneous part of the system are $O(1)$, though they may vary on the fast time scale so their derivatives up to second order are

$O(\varepsilon^{-2})$.

Now, Browning and Kreiss [5] have derived bounds on the solution of a hyperbolic system of this form in terms of the solution and its first two time derivatives at $t=0$. Their theorem 2.1 states that the norms of the solution and its first two time derivatives will be bounded by a constant times the sum of their initial values on some finite $O(1)$ time interval. The value of this constant depends on the coefficients in the problem. They assumed the coefficients and their derivatives up to second order were $0(1)$, and this implied that the constant would also be $O(1)$. In our case, where the second derivatives of the coefficients could be $O(\varepsilon^{-2})$, the constant will be $O(\varepsilon^{-2})$.

Thus, d_p and e_p and their derivatives up to second order will be $O(\varepsilon^{p-4})$ on some time interval $[0,T]$, independent of ε, so if $p \geq q+5$, the first q terms in the iteration will determine $\bar{u}$ and $\bar{v}$ to within $O(\varepsilon^q)$

This completes the proof of theorem 3.

3.4 Proof of Theorem 4 (The Nonsingular Case).

Note that the only $O(1)$ contribution to the solution of (3.3), or equivalently (3.13), comes from the solution u_0, v_0 of (3.14). This follows from the proof of the previous theorem. Also, from (3.11):

$$v = \bar{v} + \varepsilon H^{(2)-1} B_{21} A^{-1} H^{(1)} \bar{u}$$

$$= \bar{v} + O(\varepsilon) = v_0 + O(\varepsilon)$$

and similarly:

$$u = u_0 + O(\varepsilon)$$

so the theorem is proven if we can show that both v_0 and the mean value of u_0 over a short, but $O(1)$, time interval are $O(\varepsilon)$.

Now, (3.14) is:

$$\bar{H}_1^{(1)}(u_0,v_0)u_{0t} = \frac{1}{\varepsilon}\tilde{P}_0 u_0 + E_{12}v_0$$

$$D_0^{(2)}v_{0t} = (B_{22}v_0)_x + E_{21}u_0 \tag{3.18}$$

$$u_0(x,0) = f_1(x) \quad , \quad v_0(x,0) = \varepsilon f_2(x)$$

$$u_0(x+2\pi,t) \equiv u_0(x,t) \quad , \quad v_0(x+2\pi,t) \equiv v_0(x,t)$$

where $\tilde{P}_0 u_0 = P_0 u_0 + \varepsilon[B_{11}u_0]_x$. Since P_0 is non-singular, so is $\tilde{P}_0$, and so:

$$u_0 = \varepsilon \tilde{P}_0^{-1} \bar{H}_1^{(1)}(u_0,v_0)u_{0t} - \tilde{P}_0^{-1} E_{12} v_0. \tag{3.19}$$

Note that:

$$\tilde{P}_0^{-1}\bar{H}_1^{(1)}(u_0,v_0)u_{0t} = \tilde{P}_0^{-1}\frac{\partial}{\partial t}[\bar{H}_1^{(1)}(u_0,v_0)u_0] - \tilde{P}_0^{-1}\frac{\partial \bar{H}_1^{(1)}(u_0,v_0)}{\partial t}u_0$$

$$= \frac{\partial}{\partial t}\tilde{P}_0^{-1}\bar{H}_1^{(1)}(u_0,v_0)u_0 - \tilde{P}_0^{-1}\frac{\partial \bar{H}_1^{(1)}(u_0,v_0)}{\partial t}u_0 + O(\varepsilon) \tag{3.20}$$

since $\tilde{P}_0$ is independent of t at leading order, and so commutes with $\frac{\partial}{\partial t}$ up to order ε. Hence for any δ with $\varepsilon << \delta << 1$:

$$\int_{t-\delta}^{t+\delta} u_0(x,\tau)\, d\tau = \varepsilon\Big[\tilde{P}_0^{-1}\overline{H}_1^{(1)}(u_0,v_0)u_0\Big]_{t-\delta}^{t+\delta} - \varepsilon\int_{t-\delta}^{t+\delta} \tilde{P}_0^{-1}\,\frac{\partial \overline{H}_1^{(1)}(u_0,v_0)}{\partial t}\,u_0\, d\tau$$

$$- \varepsilon\int_{t-\delta}^{t+\delta} \tilde{P}_0^{-1}E_{12}v_0\, d\tau + O(\varepsilon)$$

$$= O(\varepsilon) \qquad (3.21)$$

since we already know, from the proof of the previous theorem, that u_0, v_0 and $\partial\overline{H}_1^{(1)}/\partial t$ are $O(1)$. This means that u_0 oscillates on the fast time scale, so we write:

$$u_0(x,t) = \tilde{u}_0(x,t)$$

Also from (3.19) and (3.20):

$$D_0^{(2)}v_{0t} = (B_{22}v_0)_x + \varepsilon E_{21}\tilde{P}_0^{-1}\overline{H}_1^{(1)}(u_0,v_0)u_{0t} - \varepsilon E_{21}\tilde{P}_0^{-1}E_{12}v_0 \qquad (3.22)$$

Let $v_0^{(1)}$ satisfy:

$$D_0^{(2)}v_{0t}^{(1)} = \varepsilon E_{21}\tilde{P}_0^{-1}\overline{H}_1^{(1)}(u_0,v_0)u_{0t} \qquad (3.23)$$

$$v_0^{(1)}(x,0) = 0 \quad , \quad v_0^{(1)}(x+2\pi,t) \equiv v_0^{(1)}(x,t)$$

Then:

$$v_0^{(1)}(x,t) = \varepsilon\int_0^t D_0^{(2)\,-1}E_{21}\tilde{P}_0^{-1}\overline{H}_1^{(1)}(u_0,v_0)u_{0t}\, d\tau$$

$$= \varepsilon\int_0^t D_0^{(2)\,-1}E_{21}\,\frac{\partial}{\partial t}\Big[\tilde{P}_0^{-1}\overline{H}_1^{(1)}(u_0,v_0)u_0\Big]\, d\tau$$

$$- \varepsilon\int_0^t D_0^{(2)\,-1}E_{21}\tilde{P}_0^{-1}\,\frac{\partial \overline{H}_1^{(1)}(u_0,v_0)}{\partial t}u_0\, d\tau + O(\varepsilon^2)$$

$$= \varepsilon\Big[D_0^{(2)-1}E_{21}\tilde{P}_0^{-1}\overline{H}_1^{(1)}(u_0,v_0)u_0\Big]_0^t - \varepsilon\int_0^t \frac{\partial}{\partial t}\Big(D_0^{(2)-1}E_{21}\Big)\tilde{P}_0^{-1}\overline{H}_1^{(1)}(u_0,v_0)u_0\,d\tau$$

$$- \varepsilon\int_0^t D_0^{(2)-1}E_{21}\tilde{P}_0^{-1}\frac{\partial\overline{H}_1^{(1)}(u_0,v_0)}{\partial t}u_0\,d\tau + O(\varepsilon^2)$$

$$= O(\varepsilon) \tag{3.24}$$

since $H_1^{(1)}(u_0,v_0)$ has one $O(1)$ time derivative, and $\tilde{P}_0^{-1}$, u_0 and all coefficients are $O(1)$. Similarly, since all x-derivatives of $\overline{H}_1^{(1)}(u_0,v_0)$ and u_0 are $O(1)$:

$$v_{0x}^{(1)}(x,t) = O(\varepsilon). \tag{3.25}$$

Next, if $v_0^{(2)} = v_0 - v_0^{(1)}$:

$$D_0^{(2)}v_{0t}^{(2)} = [B_{22}v_0^{(2)}]_x - \varepsilon E_{21}H_1^{(1)-1}(u_0,v_0)E_{12}v_0^{(2)} + G(x,t) \tag{3.26}$$

$$v_0^{(2)}(x,0) = \varepsilon f_2(x) \qquad , \qquad v_0^{(2)}(x+2\pi,t) \equiv v_0^{(2)}(x,t)$$

where:

$$G(x,t) = [B_{22}v_0^{(1)}]_x - \varepsilon E_{21}H_1^{(1)-1}(u_0,v_0)E_{12}v_0^{(1)}$$

Since $D_0^{(2)}$ is symmetric and bounded above and below, B_{22} is symmetric, and $E_{21}H_1^{(1)-1}(u_0,v_0)E_{12}$ is bounded independently of ε, the homogeneous problem is well-posed. Thus there exist $O(1)$ constants K and α such that, by Duhamel's principle:

$$\|v_0^{(2)}(.,t)\| \le K\left\{\varepsilon\|f_2\|e^{\alpha t} + \left\{\frac{e^{\alpha t}-1}{\alpha}\right\} sup\,\{G(.,\tau):0\le\tau\le t\}\right\} = O(\varepsilon) \tag{3.27}$$

since, in view of (3.24) and (3.25), $\|G\| = O(\varepsilon)$. Thus:

$$\|v_0\| = \|v_0^{(1)}\| + \|v_0^{(2)}\| = O(\varepsilon) \tag{3.28}$$

3.5 Generalisations of the Theorems 3 and 4.

Theorem 5. Suppose the system (3.1) is replaced by the symmetric hyperbolic system:

$$D(U,\vec{x},t)\,U_t = \frac{1}{\varepsilon}P_0\left(\frac{\partial}{\partial\vec{x}}\right)U + P_1\left(U,\vec{x},t,\varepsilon,\frac{\partial}{\partial\vec{x}}\right) + F(\vec{x},t,\varepsilon)$$

$$U(\vec{x},0) = U_s(\vec{x},0) + \varepsilon f(\vec{x}) \tag{3.29}$$

$$U(\vec{x}+2\pi\vec{e}_j,t) \equiv U(\vec{x},t) \qquad 1\le j\le n$$

where $\vec{x}=(x_1,\ldots,x_n)$, $\vec{e}_j$ is the unit vector in the jth direction, $U_s(\vec{x},t)$ is a smooth solution of the first and third equations, F is C^∞ in $\vec{x}$ and t, 2π-periodic in $\vec{x}$, and, together with all its derivatives, is bounded independently of ε, D is a bounded, positive-definite, symmetric, non-singular matrix with bounded inverse, and:

$$P_0\left(\frac{\partial}{\partial\vec{x}}\right) = \sum_{j=1}^{n} A_j\frac{\partial}{\partial x_j} + C$$

$$P_1\left(U,\vec{x},t,\varepsilon,\frac{\partial}{\partial\vec{x}}\right) = \sum_{j=1}^{n}\frac{\partial}{\partial x_j}[\Phi_j(U,\vec{x},t,\varepsilon)] + \Gamma(U,\vec{x},t,\varepsilon)$$

where $A_j=A_j^*$, $1\le j\le n$, $C=-C^*$, and Γ and each Φ_j is C^∞ in all arguments, 2π-periodic in $\vec{x}$, and, together with its $\vec{x}$ and t derivatives, is bounded independently of ε provided U and its derivatives are.

Further, we assume the each eigenvalue κ of the symbol $P_0(i\vec{\omega})$ is either zero for all $\vec{\omega}$ or satisfies:

$$|\kappa(\vec{\omega})| \ge \lambda|\vec{\omega}| \equiv \lambda(\omega_1^2+\ldots+\omega_n^2)^{\frac{1}{2}} \tag{3.30}$$

for all $\vec{\omega}$ and some positive constant λ, except possibly at a finite number of values of $\vec{\omega}$ where some of these eigenvalues may also be zero (i.e. the large part

of the operator is elliptic).

Then, the result of theorem 3 applies, namely, there exist constants k_1 and T, independent of ε, such that:

$$\| U(\vec{x},t) - U_s(\vec{x},t) \| \leq \varepsilon k_1 \tag{3.31}$$

for all $t \in [0,T]$.

Further, if all the eigenvalues in the second class are non-zero for all $\vec{\omega}$ (including $\vec{\omega}=0$), then the result of theorem 4 applies, and there exist constants k_2 and δ, independent of ε, such that:

$$\left\| \int_{t-\delta}^{t+\delta} [U(\vec{x},\tau) - U_s(\vec{x},\tau)] d\tau \right\| \leq \varepsilon^2 k_2 \tag{3.32}$$

for all $t \in [0,T]$. In other words, the perturbation in the smooth solution is of amplitude only $O(\varepsilon^2)$.

Proof The proof given for theorems 3 and 4 generalises to this case. As noted in section 2.4, in more than one space dimension, the fast and slow scale variables cannot, in general, be separated into distinct equations coupled only through undifferentiated or nonlinear terms. However, such a separation can be achieved by a projection in Fourier space, similar to (2.34), though in fact in this case we require the slightly different projection:

$$R'u = \sum_{\vec{\omega}} e^{i\vec{\omega}.\vec{x}} \hat{U}(i\omega) \begin{bmatrix} I_q & 0 \\ 0 & 0 \end{bmatrix} \hat{U}^*(i\vec{\omega}) \hat{u}(\vec{\omega})$$

where $\hat{U}(i\omega)$ is as before. The difference here is that I_q is the identity matrix of dimension q, the number of eigenvalues in the second class, independent of $\vec{\omega}$. Thus, if some of these eigenvalues do go to zero at some values of $\vec{\omega}$ (i e. if the number of large eigenvalues is not a constant), the large part of the spatial

operator will be singular on the space $\mathbf{S}_p^I = \{ R'u : u \in \mathbf{S}_p \}$, while if none do, it will be nonsingular on this space.

Once the separation is done, the system has a form analogous to (3.13), so a first approximation analogous to (3.14) may be made. Lemma 3.1 applies to this system, since all terms in the generalised system have the form of one of the terms in the one-dimensional case, and the condition (3.30) ensures that the x-derivative of the 'fast' part of the solution can be bounded by the solution itself and ε times its t-derivative. This enables its t-derivatives to be bounded independently of its x-derivatives. Thus the iteration of section 3.3 goes through as before, and (3.31) holds.

In the case that the number of large eigenvalues is independent of $\vec{\omega}$, the large part of the spatial operator is nonsingular, and this was all that was needed for the proof of theorem 4. Therefore, (3.32) holds.

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Part II

A NUMERICAL EXPERIMENT ON THE STRUCTURE OF TWO-DIMENSIONAL TURBULENT FLOW

Big whirls have little whirls

That feed on their velocity,

Little whirls have lesser whirls,

And so on 'till viscosity. - Richardson (attrib.).

1. Introduction.

Despite the ubiquity and undoubted importance of turbulence in fluid flows, very little is understood about them from a theoretical viewpoint. Central questions such as how the viscosity acts to dissipate energy, how the rate of dissipation depends on the Reynolds number, or what the qualitative structure of the flow is, remain unanswered.

The equations of motion, the Navier-Stokes equations (in two or three dimensions):

$$\vec{u}_t + (\vec{u}.\vec{\nabla})\vec{u} + \frac{1}{\rho}\vec{\nabla} p = \nu\nabla^2\vec{u}$$

$$\vec{\nabla}.\vec{u} = 0 \qquad (1.1)$$

where $\vec{u}$ is the velocity, ρ the density, p the pressure and ν the viscosity, are too difficult to solve, particularly for flows as complicated as turbulent ones. Also, equations for mean values of velocities and velocity correlations, which are the principle quantities of interest, always contain the mean of some other correlation, so one can never obtain a closed system of equations.

Many different ways of surmounting this difficulty have been proposed, each involving some assumption about the flow which enables one to arrive at a closed system. However, none of these assumptions appears to be based on particularly firm grounds, and all are open to objections of varying degrees of severity. Experimental testing of the consequences of these hypotheses is difficult, because the theory often assumes that the turbulence is homogeneous and isotropic, idealisations which can be only poorly approximated in the laboratory.

Also, numerical solution of the equations of motion in a setting appropriate for testing these theories is still beyond the range of currently available computers, although this may not be the case for much longer. In two space dimensions however, numerical integration of the equations is feasible, and this

provides as good a reason as any to study turbulence in two dimensions.

Unfortunately, 'turbulence' in two dimensions is qualitatively different from that in three in several respects; for example, in the absence of viscosity, both the enstrophy (mean square vorticity) and the energy of the flow are conserved, while in three dimensions only the energy is, and enstrophy is generated by the stretching of vortex lines and tubes. Also, in three dimensions, the rate of dissipation of energy appears to be independent of the viscosity, when it is sufficiently small, and in particular does not tend to zero as the viscosity does. This remarkable property is not shared by two dimensional flow.

Mathematically too, there are differences. It has been known for some time that the two dimensional problem is well-posed, even when the viscosity is zero [26], so that if the initial data are smooth, the solution will be smooth on any finite time interval. In three dimensions, well-posedness has not yet been proven, and it is well within the bounds of possibility that singularities may form in finite time from certain smooth initial conditions (as is the case with Burgers' equation in one dimension).

However, in both cases, the governing equations have the same convective nonlinear terms, and appear to possess similar 'turbulent' solutions in which the nonlinearity is very important and in which the energy of the motion is spread over a very wide range of length scales. Thus it would seem reasonable to expect that the two dimensional case might serve as a useful model for the full problem [1].

This is particularly true for those theories, of which there are many, that make no reference to any properties intrinsic to the three dimensional equations (indeed, some do not make reference to the equations of motion at all), so there is no reason *a priori* to assume that they would not apply equally well to the two dimensional case.

Practical applications of two dimensional turbulent flows are somewhat restricted. It has in the past been suggested that they may have some relevance to large-scale meteorology, where the relative shallowness of the atmosphere and the rotation of the earth restrict motion in the vertical direction, but this seems unlikely to be the case, since at 'small' scales of a few tens of kilometers these effects are unimportant. Another possible application is to magnetofluid situations, where motion in one direction may be inhibited by magnetic forces, with the result that the flow is approximately two dimensional.

In this thesis, it is intended to describe a numerical experiment on the structure of two dimensional turbulence (section 4). Several numerical studies of this problem have been performed before, and these are discussed in section 3. The thrust of these previous studies has usually been to test one or other of the various theories as applied to the two dimensional case, usually by comparison of the energy spectrum of the flow with theoretical predictions. However, as Saffman [2] has pointed out, the use of Fourier space may be a poor way to tackle the problem, and the present study follows Fornberg [3] in looking principally at the structure of the flow in physical space.

We also describe a new idea, due to Kreiss, concerning the structure of two dimensional flows, which the numerical experiment is designed to test. This idea and a few of the other well-known theories are outlined in section 2, but it is not intented to attempt to review all theoretical work on the subject. Reviews of some other theories may be found in the articles by Kraichnan and Montgomery [4] and Saffman [2].

Before proceeding, it is necessary to introduce some terminology. The velocity correlations referred to above are quantities of the form:

$$< u_{i_1}(\vec{x}_1,t_1).....u_{i_m}(\vec{x}_m,t_m)>$$

where $\vec{u}$ is a solution of (1.1), and the angled brackets denote an average, technically an ensemble average over many realisations of flows with initial conditions randomly distributed over some appropriate space, in practice a temporal or spatial average for a single statistically steady or homogeneous flow.

Of particular interest are the second and third order correlations:

$$R_{ij}(\vec{r},t) = < u_i(\vec{x},t) u_j(\vec{x}+\vec{r},t) > \tag{1.2}$$

$$S_{ijk}(\vec{r},t) = < u_i(\vec{x},t) u_j(\vec{x},t) u_k(\vec{x}+\vec{r},t) > \tag{1.3}$$

Correlation tensors can also be formed from the vorticity $\vec{\omega} = \vec{\nabla} \times \vec{u}$, which in two dimensions is a scalar:

$$\omega = \frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2} \tag{1.4}$$

Also frequently used are spectral tensors, the Fourier transforms of the correlation tensors. The most important is the energy spectrum tensor:

$$\varphi_{ij}(\vec{k}) = \frac{1}{(2\pi)^n} \int R_{ij}(\vec{r}) e^{-i\vec{k}.\vec{r}} \, d\vec{r} \tag{1.5}$$

where n is the number of space dimensions, which gives the density of kinetic energy in wavenumber space, and allows an energy spectrum to be defined:

$$E(k) = \tfrac{1}{2} \int_{|\vec{k}|=k} \varphi_{ii}(\vec{k}) \, dA(\vec{k}) \tag{1.6}$$

The total kinetic energy is then:

$$\tfrac{1}{2} < \vec{u}^2 > = \int_0^\infty E(k) \, dk \tag{1.7}$$

Similarly, a spectrum tensor for the vorticity can be found, leading to a vorticity spectrum:

$$\Omega(k) = 2k^2 E(k) \tag{1.8}$$

so that the total enstrophy, by which is meant half the mean square vorticity, is given by:

$$\tfrac{1}{2}\langle \vec{\omega}^2 \rangle = \int_0^\infty k^2 E(k)\, dk \tag{1.9}$$

The flow can alternatively be described in terms of joint probability density functions (j.p.d.f.):

$$P_m(\vec{u}^{(1)}, \ldots, \vec{u}^{(m)}; \vec{x}_1, \ldots, \vec{x}_m) \tag{1.10}$$

giving the probability of finding velocities in the ranges $d\vec{u}^{(1)},\ldots,d\vec{u}^{(m)}$ in the neighborhoods $d\vec{x}_1,\ldots,d\vec{x}_m$. The statistical properties, such as skewness and flatness factors, of these distributions, particularly the second order one, are therefore of interest.

Finally, a couple of remarks on the formulation of the problem. First, in two dimensions, the incompressibility condition means that a stream-function ψ can be introduced with:

$$u = \psi_y \quad , \quad v = -\psi_x \tag{1.11}$$

Then, the vorticity equation becomes:

$$\omega_t + \psi_y \omega_x - \psi_x \omega_y = \nu \nabla^2 \omega \tag{1.12a}$$

where:

$$\nabla^2 \psi = -\omega \ . \tag{1.12b}$$

Secondly, if the equations are nondimensionalised using a length scale L and a velocity scale U, the equations retain the same form with ν replaced by the inverse of the Reynolds number, given by:

$$\mathrm{Re} = \frac{UL}{\nu} \ . \tag{1.13}$$

There is some freedom in the choice of L and U. If the large scale flow is constrained by boundaries or periodicity, these usually determine L, while U may be taken as $< |\vec{u}|^2 >^{1/2}$, or alternatively $< \omega^2 >^{1/2} L$. Otherwise, following [16], L may be defined by:

$$L = \frac{< |\vec{u}|^2 >^{1/2}}{(\nu < |\vec{\nabla}\,\omega|^2 >)^{1/3}} \tag{1.14}$$

The quantity appearing in the denominator here is the cube root of the total rate of dissipation of vorticity. Both these choices of L give a Reynolds number appropriate for the description of the large scale flow. A Reynolds number for the smale scale flow can also be defined. Following [16] again, take:

$$L = l = \frac{< \omega^2 >^{1/2}}{< |\vec{\nabla}\,\omega|^2 >^{1/2}} \quad , \quad U = < \omega^2 >^{1/2} l \ . \tag{1.15}$$

2. Some Theories of Turbulence in Two Dimensions.

Perhaps the simplest and best known theory of turbulence is the 'Universal Equilibrium Theory' of Kolmogorov [5]. He postulated that in turbulence at sufficiently high Reynolds number (i.e. when the fluid viscosity is sufficiently small), the smallest eddies in the flow would become statistically independent of the larger scale eddies. In other words, the large scale motion would merely convect regions small compared with its own length scale, without affecting the relative motion therein.

As a consequence, the small scale motion would be locally homogeneous and isotropic, and all its statistical properties would depend only on the viscosity, ν, and the rate at which it is dissipating energy, ε say. They should thus have universal functional forms, scaling by the length and velocity scales:

$$l = \left(\frac{\nu^3}{\varepsilon}\right)^{1/4} \quad , \quad v = (\nu\varepsilon)^{1/4} \tag{2.1}$$

Also, these forms should apply whether or not the large scale turbulence is homogeneous or isotropic.

To replace that continually being dissipated, energy would have to be transferred from the large scale eddies down to the small. This process is referred to as a 'cascade' of energy. For consistency, the rate of transfer, ε, would have to be independent of ν. If it is further assumed that this transfer is passively carried out by the 'intermediate' scales (provided the Reynolds number is large enough for them to exist), then all statistical properties in this range of scales would be determined solely by ε. On dimensional grounds, the energy spectrum would have the form:

$$E(k) \sim \varepsilon^{2/3} k^{-5/3} \tag{2.2}$$

The range of wavenumbers over which (2.1) would apply, i.e. those corresponding

to length scales larger than the dissipation length but smaller than the scales at which the flow is significantly affected by external forcing or boundaries, is called the 'inertial subrange'. Much experimental and numerical effort has been put into looking for this subrange, and attempting to verify the power law (2.2), with mixed success.

Unfortunately, there are many doubts concerning the validity of this theory [2,6,7 *and others*]. In particular, measurements of flatness factors of the velocity derivatives are in contradiction with the theory [7], which predicts they should be independent of the Reynolds number. Also, observations of turbulent flows suggest that the small scale structure is not simply advected by the larger scale eddies, but rather that the stretching and thinning of the tangled vortex sheets characteristic of high Reynolds number flows by the larger eddies is substantially responsible for the production of the fine structure, and hence that there is a major interaction between eddies of all length scales.

Another objection is that the length scale l is not the natural length scale on which dissipation occurs, that being $O(\nu^{1/2})$. It is not clear than any physical process actually takes place on such a length scale, although interpretations for l have been found under rather speculative conditions [2].

Also, the theory makes no reference to the dimensionality of the problem, or to the nature of the equations of motion. Thus, one might expect it to apply as well to Burgers' equation:

$$u_t + uu_x = \nu u_{xx} \tag{2.3}$$

which was proposed by Burgers (unpublished) as a one-dimensional model for the Navier-Stokes equations, though it is of interest in several other applications also. However, Saffman [2] has pointed out that the Kolmogorov theory is wrong in this case, since the correct form of the energy spectrum for large k (before

the viscous cutoff) is known to be $\sim k^{-2}$, rather than (2.2).

In two dimensions, the situation is somewhat different. Here, both energy and enstrophy are conserved in the inviscid limit. This means that there are two possible 'inertial subranges', one in which energy would be 'cascaded' at a rate, ε, independent of k, and one in which enstrophy would be 'cascaded', say at a rate η. Dimensional arguments can again be used to deduce the form of the energy spectrum, leading to:

$$E(k) \sim \varepsilon^{2/3} k^{-5/3} \tag{2.4}$$

for the energy cascade, and:

$$E(k) \sim \eta^{2/3} k^{-3} \tag{2.5}$$

for the enstrophy cascade.

Kraichnan [8] has shown that there would be no net transfer of enstrophy in wavenumber in such an inertial subrange where energy is transferred, and vice versa. By consideration of interactions of triads of Fourier modes (it is not sufficient to consider pairwise interactions, since one such interaction cannot conserve both energy and enstrophy), he has also shown that in an energy cascade (2.4), the energy would be cascaded towards lower wavenumbers, in contrast to the situation in three dimensions, while in an enstrophy cascade (2.5), the enstrophy would be cascaded to higher wavenumbers.

Thus, in a quasi-steady turbulent flow at sufficiently high Reynolds number, driven at a range of wavenumbers near k_m, where $k_m << k_d$, the dissipation range, and $k_m >> k_0$, the lowest wavenumber allowed by the boundaries, the energy spectrum would consist of an energy inertial range, $E \sim k^{-5/3}$ for $k_0 << k << k_m$, in which energy would be passed towards lower wavenumbers, gradually accumulating in the lowest modes, and an enstrophy inertial range,

with $E \sim k^{-3}$, for $k_m << k << k_d$, in which enstrophy would be passed towards higher wavenumbers until it is dissipated by the action of viscosity.

The above scenario is, of course, dependent on the assumption that the interactions between Fourier modes are in some sense local in wavenumber space. Immediately there is a difficulty, in that the k^{-3} energy spectrum implies a k^{-1} enstrophy spectrum for large k, which diverges logarithmically as the upper limit tends to infinity. Thus, in the limit $\nu \to 0$, the total enstrophy would become infinite. In this situation, it is hard to imagine that nonlocal interactions would in fact be negligible. Logarithmic corrections to the spectrum have been proposed to circumvent this difficulty [8,9], but these do not appear to be based on any firm physical arguments.

Since the purpose of some of the numerical studies described in the next section has been to test some other theories of turbulence, we mention a couple of these here. Kraichnan and his co-workers [8,9,10] have developed a theory, the 'Direct Interaction Theory', based on consideration of direct interactions between triads of Fourier modes. In this theory, the equations for the Fourier coefficients of the solution are replaced by a simpler system in which some triple-interactions are neglected. Further mathematical assumptions are made which enable a linear equation for the second order correlation tensor to be found. The justification for several of the assumptions is not entirely clear, and their validity is hard to assess.

The authors of the theory admit that it has several defects, and their later efforts have been directed towards eliminating some of these, at the expense of considerable complication. The 'Test Field Theory' [10], a more sophisticated version of the model, incorporates 'memory times' for the dynamical interactions, with a free parameter governing the length thereof.

The energy spectra predicted by the Test Field model are very close to those of the Kolmogorov model, although it is appears that this is partly by design, since the fact that the earlier model predicts an inertial range with a different power law seems to have been regarded as one of its defects.

Another theory treats the problem from the point of view of statistical mechanics [4]. Unfortunately, it seems unlikely that a turbulent flow is in any form of statistical equilibrium, since the time taken for the dependence on the initial conditions to be lost would appear to be of the same order as the time in which the turbulence is dissipated [2]. For the record, the equilibrium theory of statistical mechanics predicts an energy spectrum of the form:

$$E(k) \sim \frac{k}{(\alpha + \beta k^2)} \tag{2.6}$$

where α and β are constants determined by the total energy and enstrophy, for a nonviscous formulation of the Navier-Stokes equations truncated to a finite number of Fourier modes [4,15].

An alternative approach to the problem, one that avoids the use of Fourier space, is given by Saffman [2,11]. The motivation for this approach comes from Burgers' equation (2.3), proposed, as noted above, as a one-dimensional model for the Navier-Stokes equation. Exact solutions of Burgers' equation are known, and can be used to predict the behavior of 'turbulent' solutions, i.e. solutions arising from random initial data. It is a property of these solutions that the rate at which energy is dissipated is independent of the viscosity ν, so that this expected feature of turbulence is duplicated.

For small ν, solutions of Burgers' equation evolve quickly to a typical form consisting of 'shocks', regions of thickness:

$$\delta = O\left[\frac{\nu}{V}\right] \quad , \quad V = \langle u^2 \rangle^{1/2} \tag{2.7}$$

across which u jumps by amounts of amplitude $O(V)$, separated by regions of average length L, say, determined by the initial data and independent of ν, over which u varies smoothly. This type of structure is often referred to as 'intermittent'.

In these solutions, the major contribution to the high wavenumber part of the spectrum comes from the shocks, whose local structure depends only on the viscosity, and their amplitudes (related to the rate of dissipation). Since, as noted above, the rate of dissipation is independent of ν, the conditions for Kolmogorov's hypothesis to hold would appear to be satisfied.

However, it can be shown [2] that the predictions of the Kolmogorov theory are incorrect. The small scale statistical properties, such as the velocity correlation (1.2) or the skewness of the two-point j.p.d.f. (1.10), are not functions of ν and ε alone, but take quite different forms. Also, the energy spectrum at large k is proportional to k^{-2}, rather than $k^{-5/3}$.

The reason for the failure of the theory appears to be that the 'cascade' process, which in this case corresponds to the formation and maintainance of the shock regions, is controlled by the large scale motion, rather than being independent of it, as is assumed in the theory. Analytically, the skewness of the two-point velocity j.p.d.f. (1.10), which must be a constant for the theory to hold, varies significantly.

Of course, the failure of the theory in one dimension does not mean that it need fail in two or three dimensions as well. Indeed, there are several significant differences between the two cases. However, this failure does point to the need for a fuller justification of why the basic hypotheses should hold.

There is a further point to note from the one dimensional problem. Because of the intermittent nature of the solutions, it is much easier to perform the analysis in physical space than it would be in Fourier space. There is a good

deal of physical evidence that two or three dimensional turbulence is also intermittent in nature, and this would be expected mathematically from the singular nature of the limit $\nu \to 0$, in which the highest derivatives vanish from the equations. Since most of the theories so far proposed have relied heavily on arguments in Fourier space, this may be one reason why they have met with such little success.

In two dimensions, there are no exact solutions to the equations of motion from which to construct turbulent solutions, but the following analogy may be drawn between the two cases. In one dimension, u is conserved following a fluid particle, and steep gradients arise because convection brings together particles with different values of u. In two dimensions, the vorticity ω is conserved following a fluid particle, and it seems reasonable to suppose that convection will again bring together fluid particles with different values of ω. Of course, there are differences; for one, the characteristics in the two dimensional case do not cross, as they do in one dimension, but their separation can decay exponentially. Thus, it does not seem unreasonable to assume that the vorticity field assumes a piecewise continuous form, with thin 'fronts' separating regions where the vorticity varies smoothly.

Saffman [11] assumes then that the vorticity field contains discontinuities spaced randomly with mean spacing L and width:

$$\delta \sim \left(\frac{\nu L}{< \omega^2 >^{1/2}} \right)^{1/3} \tag{2.8}$$

(obtained by balancing the convection and diffusion terms). It follows that the asymptotic form of the energy spectrum would be:

$$E(k) \sim k^{-4} \quad , \quad L^{-1} << k << \delta^{-1} \tag{2.9}$$

(δ^{-1} is the viscous cutoff scale). For there to be a range of wavenumbers where

this would hold, it must be that:

$$Re^{1/3} = \left(\frac{<\omega^2>^{1/2} L^2}{\nu} \right)^{1/3} >> 1 \qquad (2.10)$$

Hitherto, this has been the only theory of two dimensional turbulence that does not appeal to arguments about the Fourier modes, and also the only one that offers an energy spectrum significantly different from k^{-3} at large k. Unfortunately, as we shall see in the next section, there seems to be no numerical evidence that conclusively favours one form of the spectrum over the other.

Kreiss has recently made a new suggestion concerning the physical structure of the flow, namely that the flow must contain regions of local convergence, i.e. flows locally resembling figure 15, and that these flows will sweep perturbations in the vorticity into the narrow region corresponding to the neighborhood of the dividing streamline S of figure 15. Thus, the vorticity field would, during the initial stages of the flow, contain narrow regions where the vorticity is layered ('layered cakes'), rather than the 'fronts' of Saffman's theory.

Under this hypothesis, the energy in the small scale flow would be swept very rapidly to higher wavenumbers, and so would be rapidly dissipated even if the viscosity were small. Also, the rate of dissipation would be almost independent of the viscosity at small values thereof. The flow would also assume a fairly organised form quite rapidly.

Some of these consequences are consistent with the recent numerical study performed by Fornberg [3], described in the next section, and the numerical experiment described in section 4 has been designed to test this particular idea.

3. Previous Numerical Studies.

In this section, some previous attempts at computing two dimensional turbulent flows are discussed. All these studies use the vorticity-stream-function formulation of the equations (1.12), since it is far more amenable to numerical treatment than the velocity-pressure equations.

Herring *et al.* [16], have done a comparison of the effectiveness of two different numerical methods for integrating the equations (1.12), namely a second order accurate finite-difference scheme based on the Arakawa [17] energy- and enstrophy-conserving approximation to the nonlinear terms in the vorticity equation, and a pseudo-spectral, or Fourier method. Their results suggest that a finite difference scheme with $2n \times 2n$ grid points has about the same resolution as a Fourier method using only $n \times n$ modes. Also, they found that adequate resolution of the flow at a macroscopic Reynolds number (1.14) of 350 required the use of a Fourier method with 128 modes in each direction.

On the basis of this, it appears that the results of early numerical studies of the problem, by Lilly [12,13], and Deem and Zabusky [14], are untrustworthy, since they used schemes with no more resolution than the 128 mode Fourier scheme and Reynolds numbers greater than or equal to 350.

Unfortunately, even with a 128 x 128 mode Fourier scheme, the Reynolds number cannot be made large enough for the inertial ranges (2.4) and (2.5) to have any chance of forming. This is because the viscosity, ν, must be chosen large enough that all modes up to the viscous cut-off are retained by the scheme, and in this case ν is so large as to have a significant effect on almost all modes. Put another way, the enstrophy spectrum $k^2E(k)$ and the dissipation spectrum $k^4E(k)$ will have a substantial region of overlap, while the inertial ranges cannot appear unless they are well separated.

Thus, the main thrust of the computations in [16] is to compare the accuracy of the Direct Interaction Theory with its improved version, the Test Field Model, at the lower Reynolds numbers for which accurate computations can be performed. Their results are indeed in closer agreement with the latter model. It is interesting to note that in the highest Reynolds number experiment, the energy spectrum obtained (fig. 1) is quite close to the k^{-4} predicted by Saffman.

Another feature of the results is that the dependence of the large scale features of the flow on Reynolds number appears to be weak. Figures 2-4, reproduced from [16], show the vorticity contours calculated at a fixed time for runs with the same initial data, but with Reynolds numbers of 138, 349 and 1184 respectively. The flow fields are indeed quite similar. In [16] and [18], it is speculated that it may be possible to simulate quite accurately the larger scale features of flows at high Reynolds number, even if the small scale motion is not accurately resolved.

Seyler *et al.* [15] performed a numerical integration of a truncated version of the inviscid equations in Fourier space, using up to 220 modes in each direction. They found good agreement with the spectrum (2.6) predicted by statistical mechanics for this system, which is surprising. Calculations by Fox and Orszag [19] and Fornberg [3] disagree with their results.

Orszag [20] reports on the construction of a large numerical scheme for computation of flows, which can use up to 1024 Fourier modes in each direction. This should be adequate to test for the existence of inertial ranges in the energy spectrum. Unfortunately, few runs seem to have been performed with this code, interest having shifted to magneto-hydrodynamic situations. Two runs with 512 x 512 modes are reported on in [20], one at Reynolds number 1100, corresponding to the highest Reynolds number run in [16] (fig. 1), and one at Reynolds number 25,000. An energy spectrum closer to k^{-3}, in accordance with the

inertial range prediction, is observed here for wavenumbers in the range $10 \leq k \leq 50$ (see fig. 5). The spectrum for higher wavenumbers (the model must go up to $k = 250$) is not shown. Also, the Reynolds number used would appear to be rather high, even for a model of this resolution, and it is a shame more results are not presented.

An ingenious alternative to the use of high resolution numerical models has been proposed by Fornberg [3]. The viscosity term in the vorticity equation (1.12a) has the effect of multiplying each Fourier mode by a factor:

$$f(\vec{k}) = e^{-4\pi^2\nu(k_1^2+k_2^2)\Delta t} \tag{3.1}$$

at each time step Δt. The problem is that if the coefficient ν is chosen large enough that all modes not resolved by the numerical scheme are sufficiently heavily damped, the modes retained by the model will also suffer significant viscous damping. This prevents any inertial ranges, which by hypothesis are unaffected by dissipation, from forming.

Thus, Fornberg proposed applying dissipation in a different way, by omitting the viscosity term from (1.12a) and instead multiplying the Fourier components at each time step by a different factor, one more dependent on wave number. The factor used was actually:

$$f(\vec{k}) = \begin{cases} 1 & \text{if } k \leq k_{cut} \\ 0 & \text{if } k \geq k_{cut} \end{cases} \tag{3.2}$$

where k_{cut} was adjustable, and the multiplication was performed only only after every n_1 time steps, n_1 also adjustable.

The use of this form of dissipation allows the larger scale motion to evolve as if there were almost no viscosity, while the smaller scale motion may not be resolved but retains its function as an energy sink at the high end of the spectrum. Consequently, conditions for the formation of an inertial range would

appear to be satisfied, even with as few as 64 modes in each direction.

Another advantage of this formulation is that the calculations can be carried for very long times, since the lower, energy-carrying modes are not damped at all, and the rate of dissipation from the system is very small. Thus, with relatively little expense, Fornberg was able to integrate the equations of motion for up to 5000 time steps (the time step scales with the amplitude of the solution, of course), without all the flow structure being dissipated away. Comparison of the flow fields predicted using this form of the dissipation were found to differ very little from those predicted using the 'proper' dissipation, with differences only in the fine scale structure.

A typical series of flow pictures is reproduced from [3] in figures 6-12. The random initial vorticity distribution develops into stringlike patterns which persist for some time. Eventually the flow organises itself into two opposite-signed 'finite area vorticity regions' (FAVRs). In figures 8 and 9, the vorticity field can be seen to have areas of relatively smooth change, separated by fairly narrow regions of large gradients, as suggested by Saffman. However, not all these narrow regions appear to be 'fronts', in the sense that ω varies monotonically through them; in some the vorticity would appear to be 'layered', as suggested by Kreiss. Also, in the final pictures, the two-FAVR vorticity field does not exhibit particularly sharp gradients, though this might, in part, be owing to the relatively low resolution of the numerical scheme.

The energy spectrum seems to develop in each run towards a decay somewhere between k^{-3} and k^{-4} (fig. 13), perhaps starting nearer k^{-3}, and evolving to nearer k^{-4}. However, even runs with radically different initial forms of the spectrum appear to develop this form quite quickly. In one particular run, the spectrum initially conformed to that predicted by the statistical mechanics theory (2.6) (with which Seyler *et al.* [15] found good agreement), but rapidly departed

from this form.

As noted in section 2, cascade arguments imply that energy should propagate towards lower wavenumbers, enstrophy towards higher. This is confirmed by the fact that energy is conserved during the runs, suggesting that none propagates into the higher wavenumbers where it would be dissipated, while the total enstrophy did decay steadily, suggesting a steady transfer into the higher modes. The initial exponential decay of enstrophy observed in some runs is consistent with Kreiss' suggestion.

Another interesting observation made by Fornberg was that the relative phases of the Fourier modes seemed to be important in determining the structure of the flow field (these are of course neglected in most Fourier space models). At the end of one run (fig. 12), the phases of the Fourier modes were randomly redistributed without changing the amplitude of any mode. The flow field changed dramatically (fig. 14), and a 'burst' of energy appears soon afterwards at the high end of the spectrum. Eventually though, the two FAVR pattern reappears, and the energy spectrum returns to normal.

From this, Fornberg speculates that a k^{-3} energy spectrum corresponds to flows in which the 'natural' correlations between the phases of the Fourier modes are not present, and that as these develop, and as the flow takes on a more organised appearance, the spectrum changes to k^{-4}. It is not clear, however, that this latter form is the result of a vorticity field of the form suggested by Saffman. Indeed, Fornberg also suggests that the flow seems so well organised that 'turbulent' is a misnomer; this again fits with Kreiss' suggestion.

4. A Proposed Numerical Investigation.

As mentioned above, Kreiss has suggested that in a turbulent two-dimensional flow, the vorticity field might contain regions of locally convergent flow which would sweep variations in the vorticity into narrow regions, forming 'layered' structures which would be rapidly dissipated by the action of viscosity.

It is proposed to investigate whether this mechanism really could work as suggested. By good fortune, there is a steady solution of the inviscid Euler equations:

$$\omega_t + \psi_y \omega_x - \psi_x \omega_y = 0$$

$$\nabla^2 \psi = -\omega \qquad (4.1)$$

with periodic boundary conditions, representing a converging flow. This solution is:

$$\omega(x,y) = \omega_0(x,y) = \sin(2\pi x)\sin(2\pi y) \qquad (4.2)$$

$$\psi(x,y) = \psi_0(x,y) = (8\pi^2)^{-1}\sin(2\pi x)\sin(2\pi y)$$

and is depicted in figure 15. If Kreiss' hypothesis were correct, integration of the equations (4.1) with periodic boundary conditions, and initial conditions:

$$\omega(x,y,0) = \omega_0(x,y) + \varepsilon f(x,y) \qquad (4.3)$$

where $\varepsilon << 1$, and f is an arbitrary function, would almost certainly lead to layering of the vorticity along the separating streamlines (denoted by S in figure 15). Failure of this to happen would be a strong indication that the hypothesis was not valid.

The numerical scheme that it is proposed to use closely follows that used by Fornberg [3]. It uses a Fourier (pseudo-spectral) method to evaluate spatial derivatives, since this is approximately twice as efficient as a finite-difference or spectral (Galerkin) method of comparable resolution [16,21,22]. Differentiation is done in Fourier space, multiplication of the nonlinear terms in physical space, so that both these operations are 'local'. Fast Fourier Transforms are used to move between the two spaces. Since the variables in physical space are real, only half the Fourier components need be used, and the FFT can be most efficiently implemented using the algorithms given in [23], which exploit this fact.

The vorticity is advanced in time using one of a class of 'Iterative Multi-Step Methods' devised by Hyman [24]. Applied to a simple ordinary differential equation of the form:

$$\omega_t = G(\omega) \, . \tag{4.4}$$

the scheme consists of a predictor step, which is simply leap-frog:

$$\omega^*_{n+1} = \omega_{n-1} + 2\Delta t G(\omega_n) \tag{4.5a}$$

where the subscript n refers to the time level $t_n = n\Delta t$, followed by a corrector step:

$$\omega_{n+1} = \frac{1}{5}\{4\omega_n + \omega_{n-1} + 4\Delta t G(\omega_n) + 2\Delta t G(\omega^*_{n+1})\} \tag{4.5b}$$

The combined predictor-corrector scheme overcomes two disadvantages of the basic leap-frog scheme. These are the restricted stability region, and the tendency of alternate time levels to evolve separately, owing to the weak coupling between them. Additionally, the scheme is third order accurate, one order better than leap-frog.

The stability region of a scheme is defined as follows. Consider the linear problem, (4.4) with $G(\omega) \equiv \lambda\omega$, where lambda is a complex number. The stability region is that part of the complex plane in which $\lambda\Delta t$ must lie for the scheme to be stable. For leap-frog, the stability region consists solely of the segment of the imaginary axis between $-i$ and i. This means that it is stable only for problems whose eigenvalues are purely imaginary, i.e. problems in which there is no dissipation. Any dissipation terms must be treated by a different scheme. As shown in figure 16, the stability region for the Hyman scheme extends further up the imaginary axis, allowing a longer time step to be taken in a problem with purely imaginary eigenvalues, and also covers a considerable region in the left half-plane, which means that problems with dissipation can be treated, or that dissipation may be added to the numerical scheme (this is often necessary, for example in shock calculations).

Now comes the question of dissipation. Some is required for two reasons, firstly to stabilise the numerical method, and secondly because we wish to model flow at a high but finite Reynolds number. On the first reason, it has been observed that in nonlinear problems, the Fourier method as outlined above is unstable, although the calculation may run for some time before an instability develops. The addition of dissipation will prevent this 'nonlinear instability' from occurring.

Since, following Fornberg [3], it is not always desired to include a dissipation term of the form $\nu\nabla^2\omega$, it is proposed to use some form of chopping of the higher Fourier modes. A variety of ways of doing this have been suggested, each with the motivation of stabilising the method. These fall into two classes, first those in which the amplitudes of the Fourier components are multiplied by a factor at each time step (or after every so many time steps). In this class come the sharp cutoff used by Fornberg (3.2) (not applied at every time step), and the

exponential cut-off:

$$f(\vec{k}) = \begin{cases} 1 & \text{if } k \le k_{cut} \\ e^{-\alpha(k-k_{cut})^{2m}} & \text{if } k \ge k_{cut} \end{cases} \tag{4.6}$$

α and m to be chosen, for which Majda, McDonough and Osher [25] proved stability for linear hyperbolic problems. The second class of cut-off functions adjusts the amplitudes of the higher modes so that they are less than some bound depending on the amplitude of the solution. In [22], Kreiss and Oliger prove stability for the Fourier method (for a linear hyperbolic system) when the smoothing operator:

$$H\hat{\omega}(\vec{k}) = \begin{cases} \hat{\omega}(\vec{k}) & \text{if } k \le N_1 \\ \hat{\omega}(\vec{k}) & \text{if } k \ge N_1 \text{ and } |\hat{\omega}(\vec{k})| \le \dfrac{D\,\|\omega_1\|}{(2\pi k)^j} \\ \dfrac{D\,\|\omega_1\|}{(2\pi k)^j}\ \dfrac{\hat{\omega}(\vec{k})}{|\hat{\omega}(\vec{k})|} & \text{otherwise} \end{cases} \tag{4.7}$$

where $N_1 = \left\lfloor 1 - \frac{1}{m} \right\rfloor$ for some integer m, $\omega_1 = \sum_{k \le N_1} \hat{\omega}(\vec{k}) e^{2\pi i \vec{k}\cdot\vec{x}}$, j is an integer greater than 2, and D is a constant.

Cut-offs of the first type are easier to program, and experience suggests that they are effective. Of course, if the results of the computation are to have any relevance, the exact form of the cut-off used should not be important, and it is an important test of the method that the results do not change significantly when the form is changed. Any form would appear to be satisfactory for the purpose of modelling an energy sink at high wavenumber. Comparisons should also be made with runs using the real form of the viscosity.

The result of a preliminary run are presented in figures 17-22. In this run, the initial conditions used were:

$$\omega(x,y,0) = \omega_0(x,y) + .02\ sin(8\pi y)\ . \tag{4.8}$$

Also, 64x64 Fourier modes, i.e. wavenumbers up to 31 in each direction, and a time-step Δt = .0188 were used. The coefficient of viscosity ν was taken to be zero, and the Fourier modes were chopped at each time-step by the linear factor:

$$f(\vec{k}) = \begin{cases} 1 & \text{if } k \le k_{cut} \\ \dfrac{(k_{\max}-k)}{(k_{\max}-k_{cut})} & \text{if } k_{cut} \le k \le k_{\max} \\ 0 & \text{if } k \ge k_{\max} \end{cases} \tag{4.9}$$

with $k_{\max}$ = 31 and k_{cut} = 23. In a separate run, k_{cut} was taken to be 27, with no apparent difference in the results.

In this run, a 'layering' is already present in the initial conditions. As can be seen from the plots (which are contour plots of $\omega-\omega_0$), this layering is indeed increased initially, and a substantial amount of the enstrophy in the perturbation is dissipated in the first 100 time-steps. Clearly, more runs with different initial conditions, different chopping or 'real' viscosity, and probably more wavenumbers are needed before any conclusions may be drawn.

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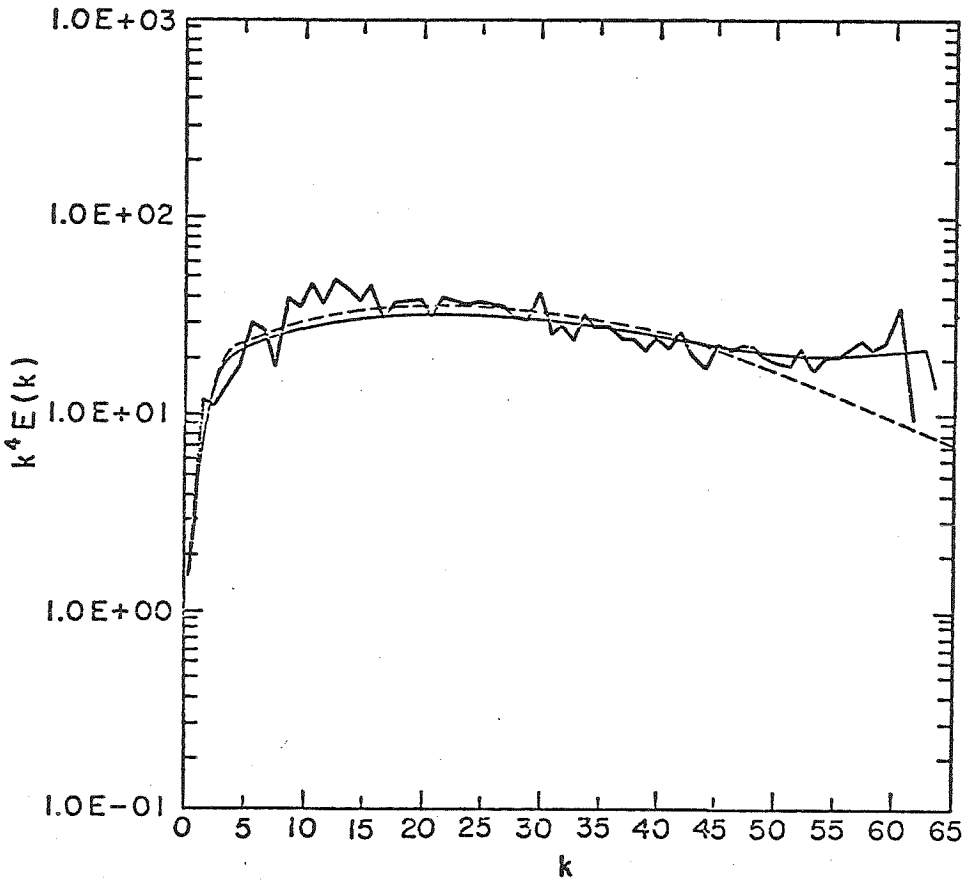

Fig. 1: Spectrum $k^4E(k)$ at $t=2$ obtained by Herring et al. [16] using a 128×128 spectral code (jagged line), with $\nu=.001$ and initial large scale Reynolds number Re=1184. *Smooth lines are the predictions of the Test Field model.*

Fig. 2: Vorticity field obtained at $t=2$ *by Herring et al. [16] with* $\nu=.005$ *and* $Re=138$.

Fig. 3: As figure 2, but with $\nu=.0025$ *and* $Re=349$.

Fig. 4: As figure 2, but with $\nu=.001$ *and* $Re=1184$.

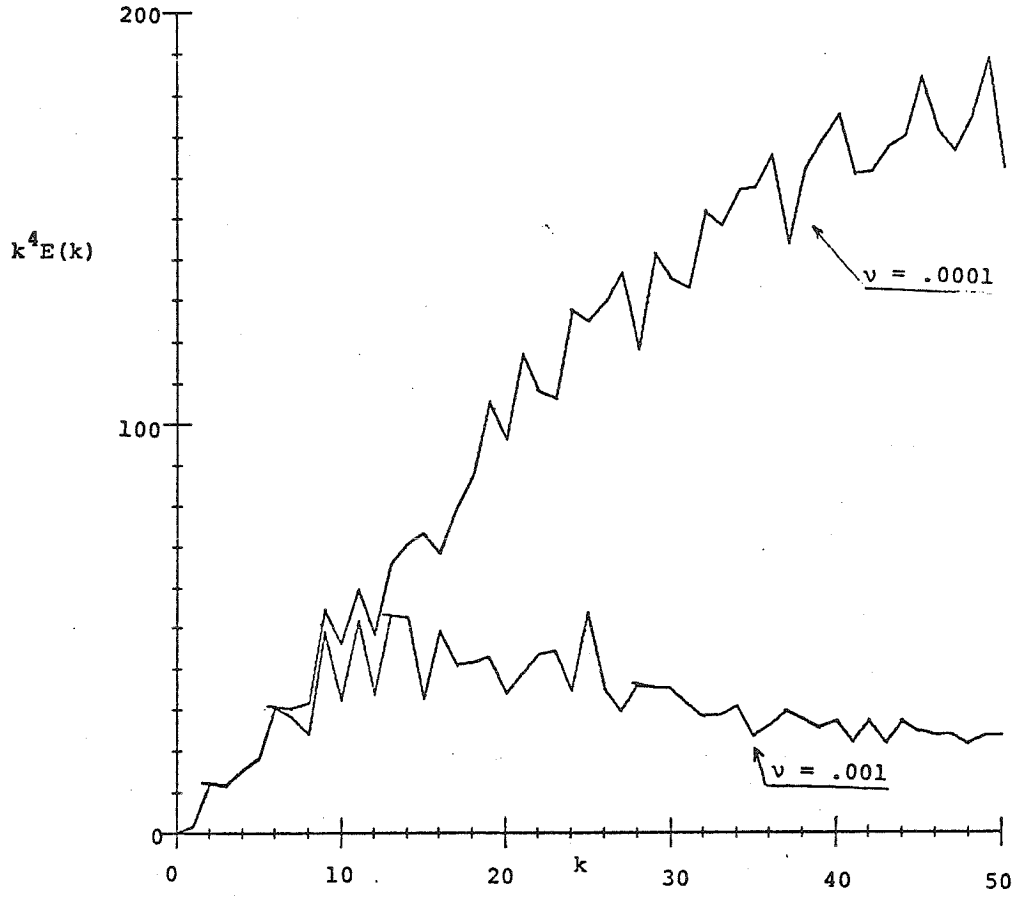

Fig. 5: Spectra of $k^4E(k)$ at $t=2$ obtained by Orszag [20]. Lower line corresponds to the situation in figure 1, upper line to a run with the same initial data, but with ν=.0001 and Re=25,500. *A* 512×512 *spectral code is used for the latter case.*

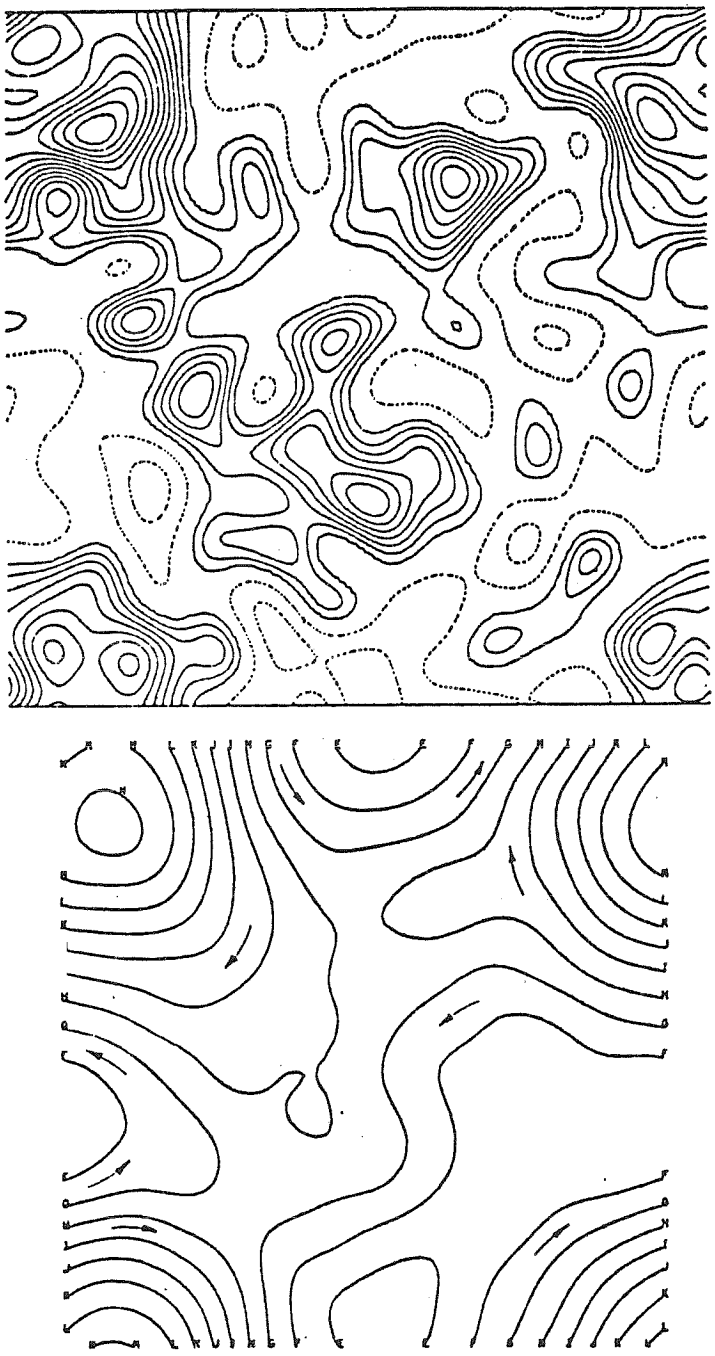

Fig. 6: Vorticity (upper) and stream-function fields from a typical run by Fornberg [3]. Negative vorticity contours are dotted, and spaced three times as far apart as the positive contours. This is the initial field.

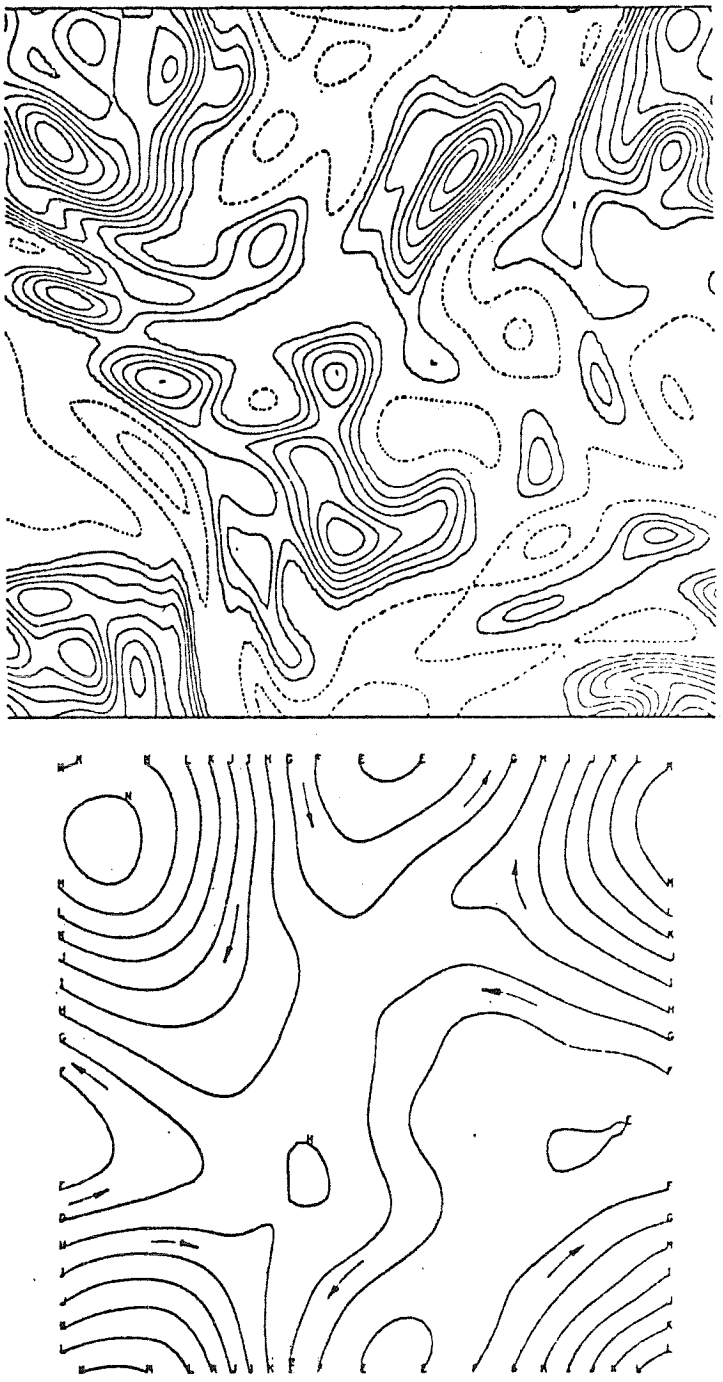

Fig. 7: As figure 6, but after 60 timesteps.

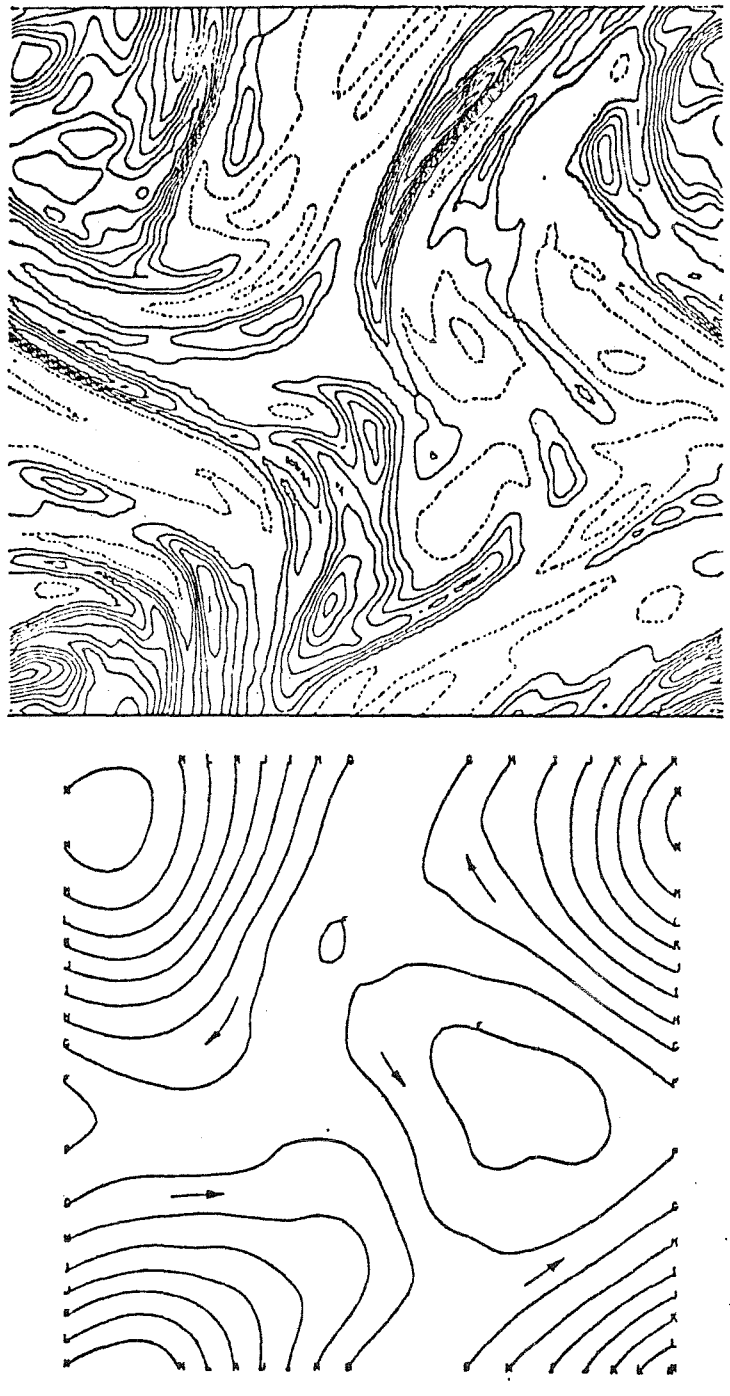

Fig. 8: As figure 6, but after 180 timesteps.

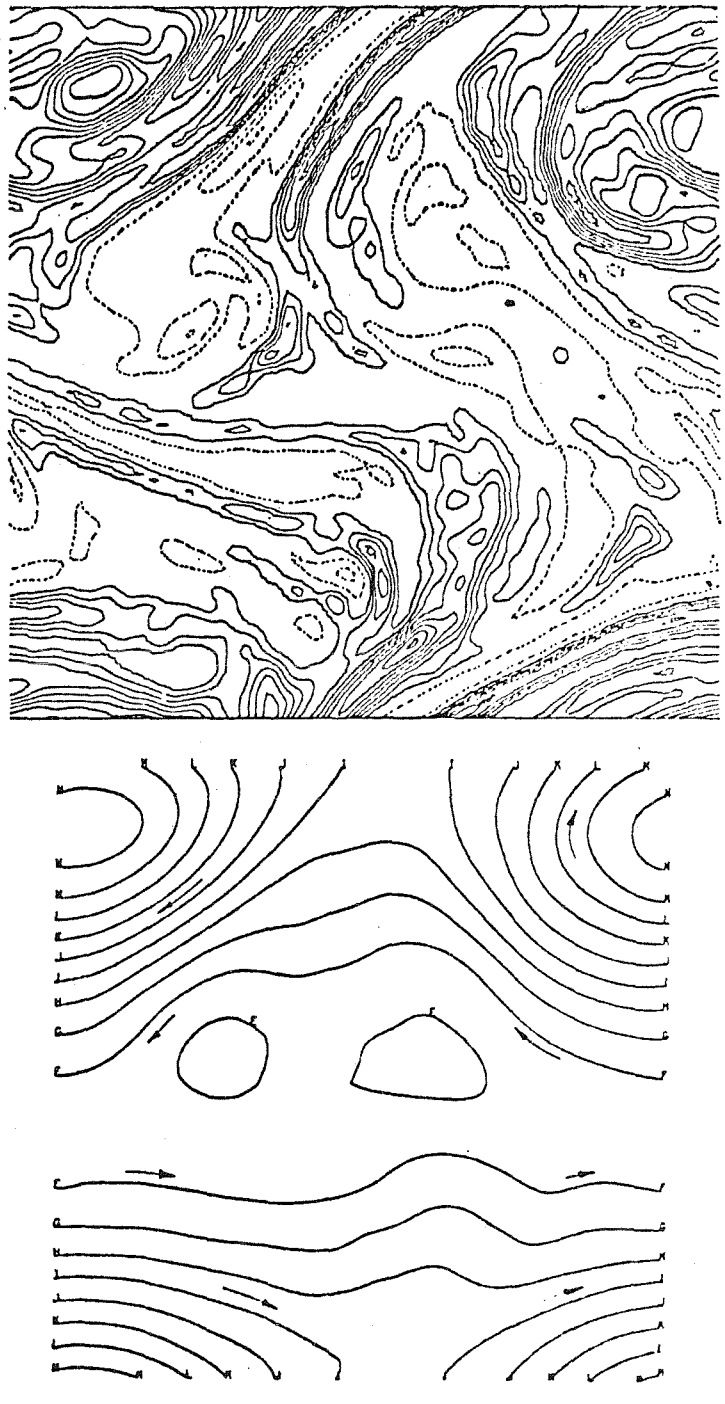

Fig. 9: As figure 6, but after 360 timesteps.

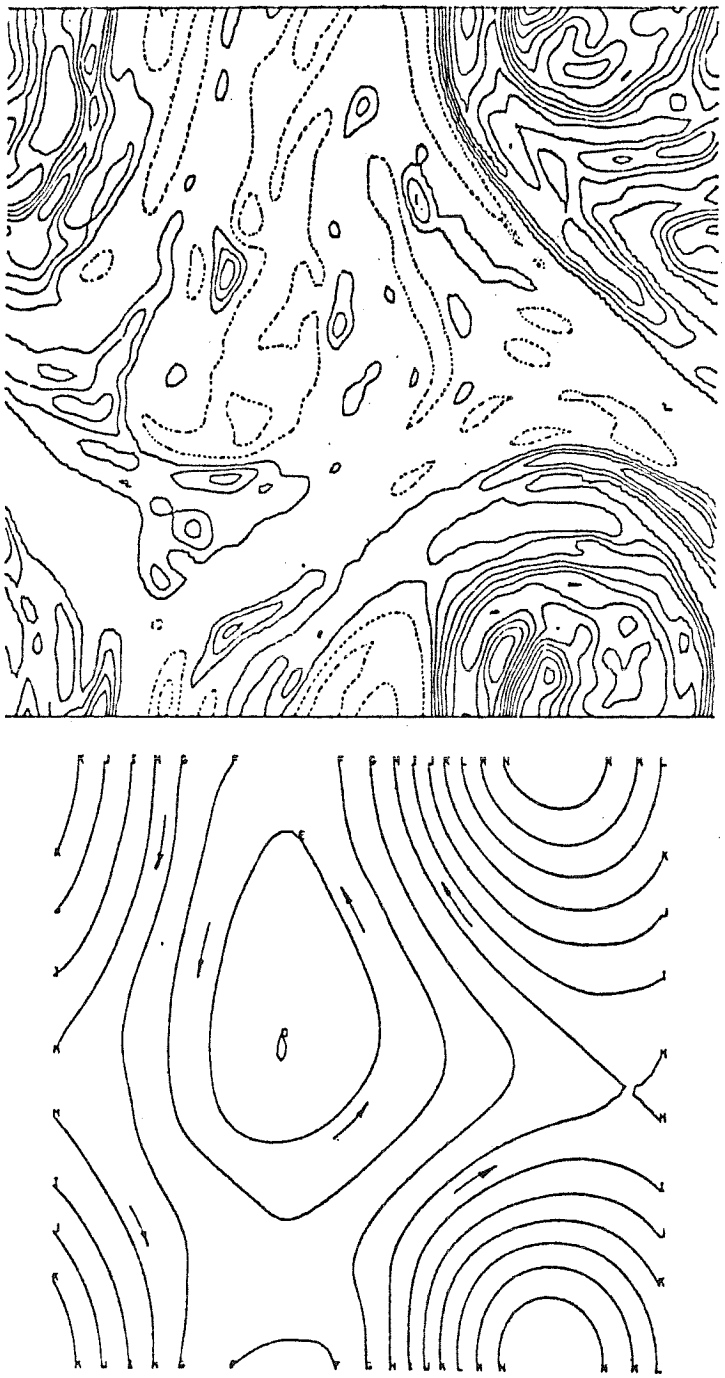

Fig. 10: As figure 6, but after 900 timesteps.

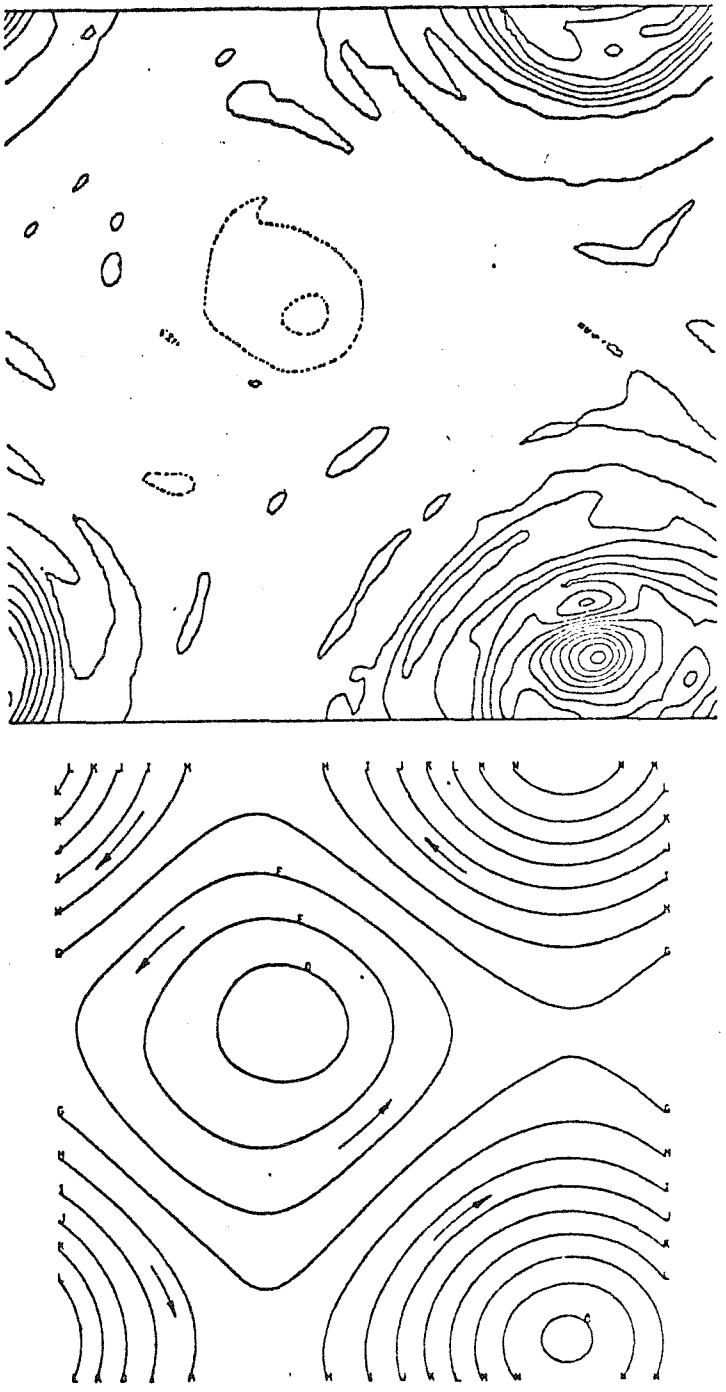

Fig. 11: As figure 6, but after 3060 timesteps.

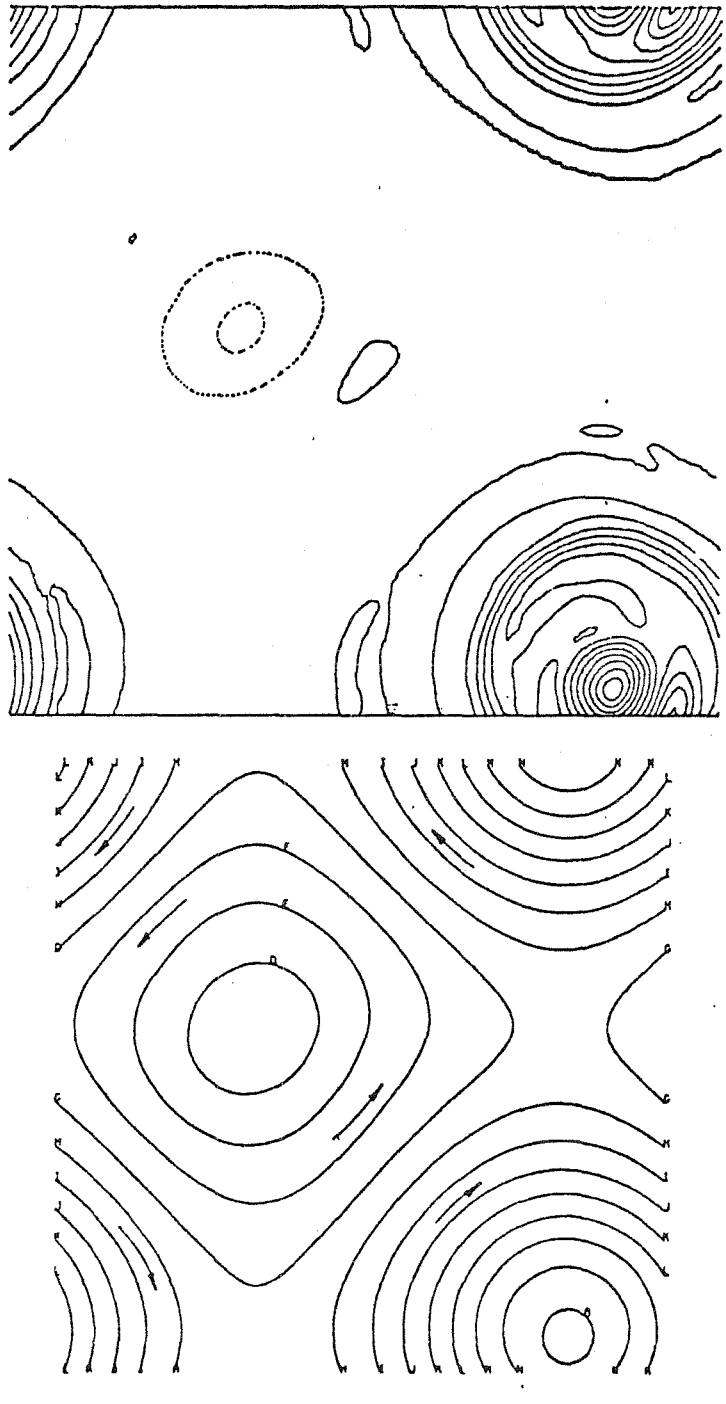

Fig. 12: As figure 6, but after 5220 timesteps.

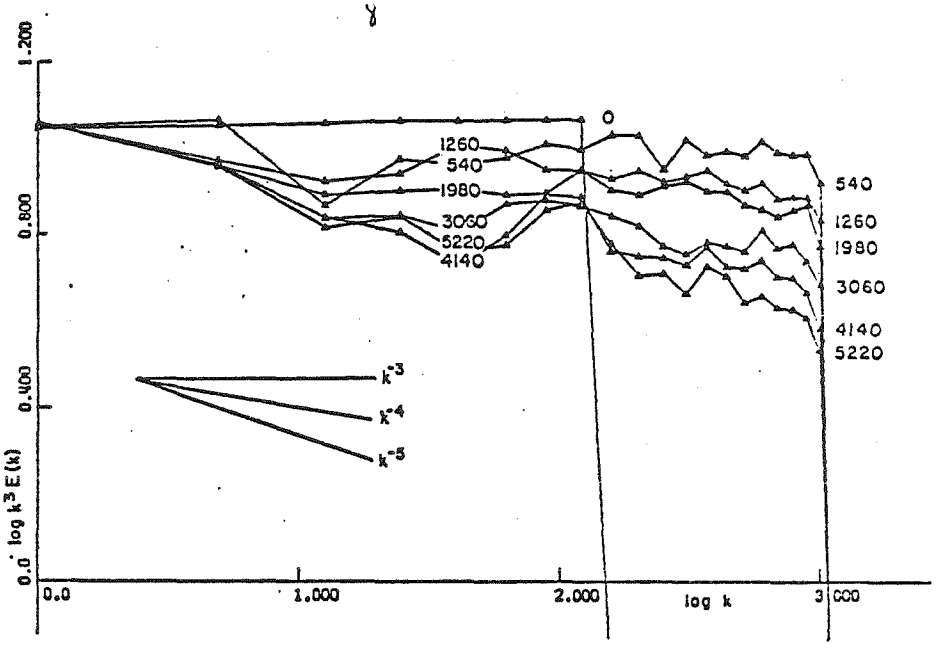

Fig. 13: Development of spectrum $k^3E(k)$ during flow depicted in figures 6-12.

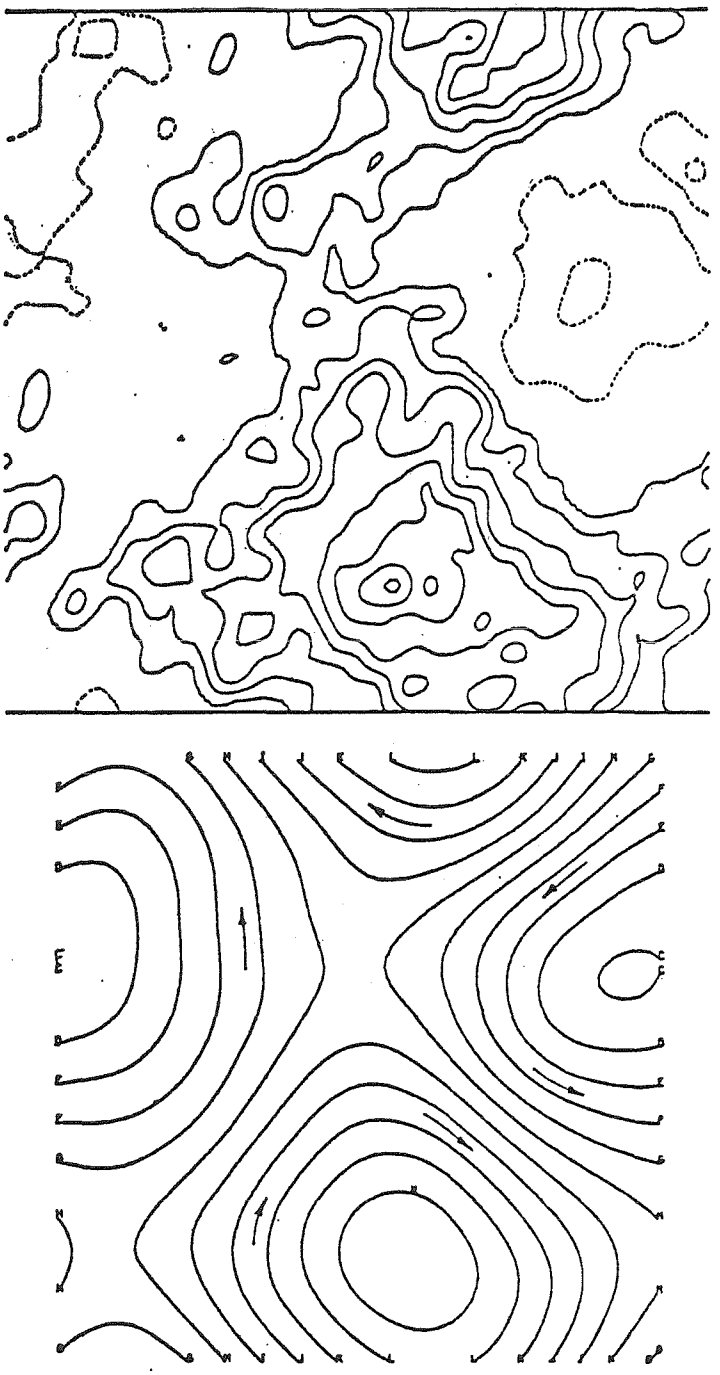

Fig. 14: Vorticity and stream-function contours for the flow obtained from that in figure 12 by randomly rearranging the phases of the Fourier components without altering their amplitudes.

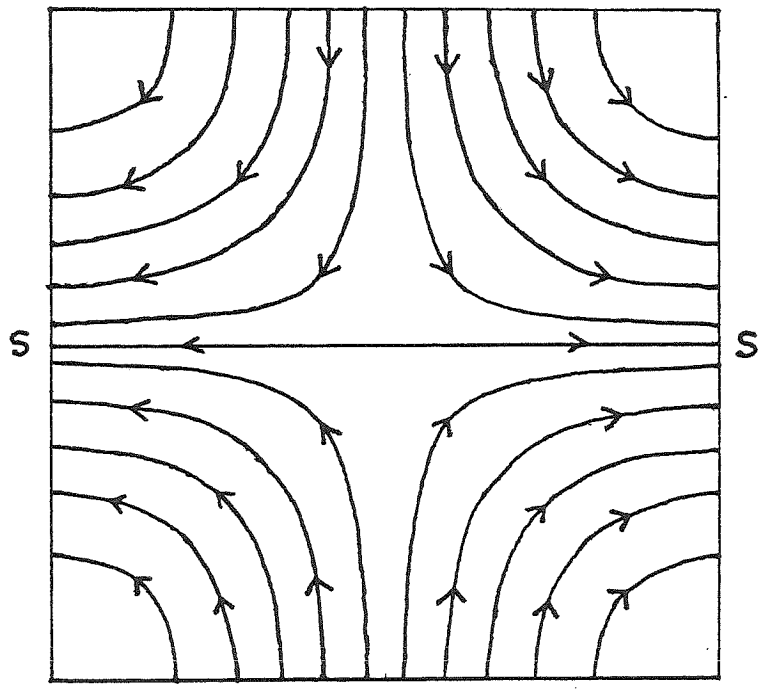

Fig. 15: The steady-state converging flow (4.2).

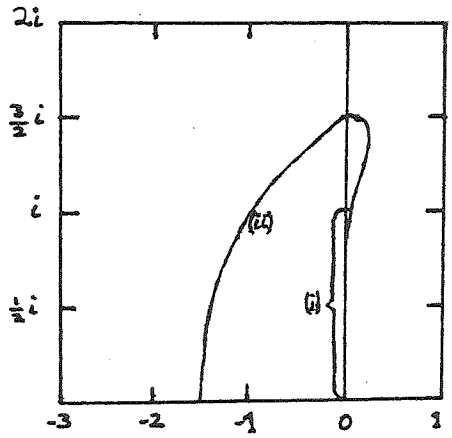

Fig. 16: Stability regions for leap-frog (i) and Hymans predictor-corrector method (ii), from [24].

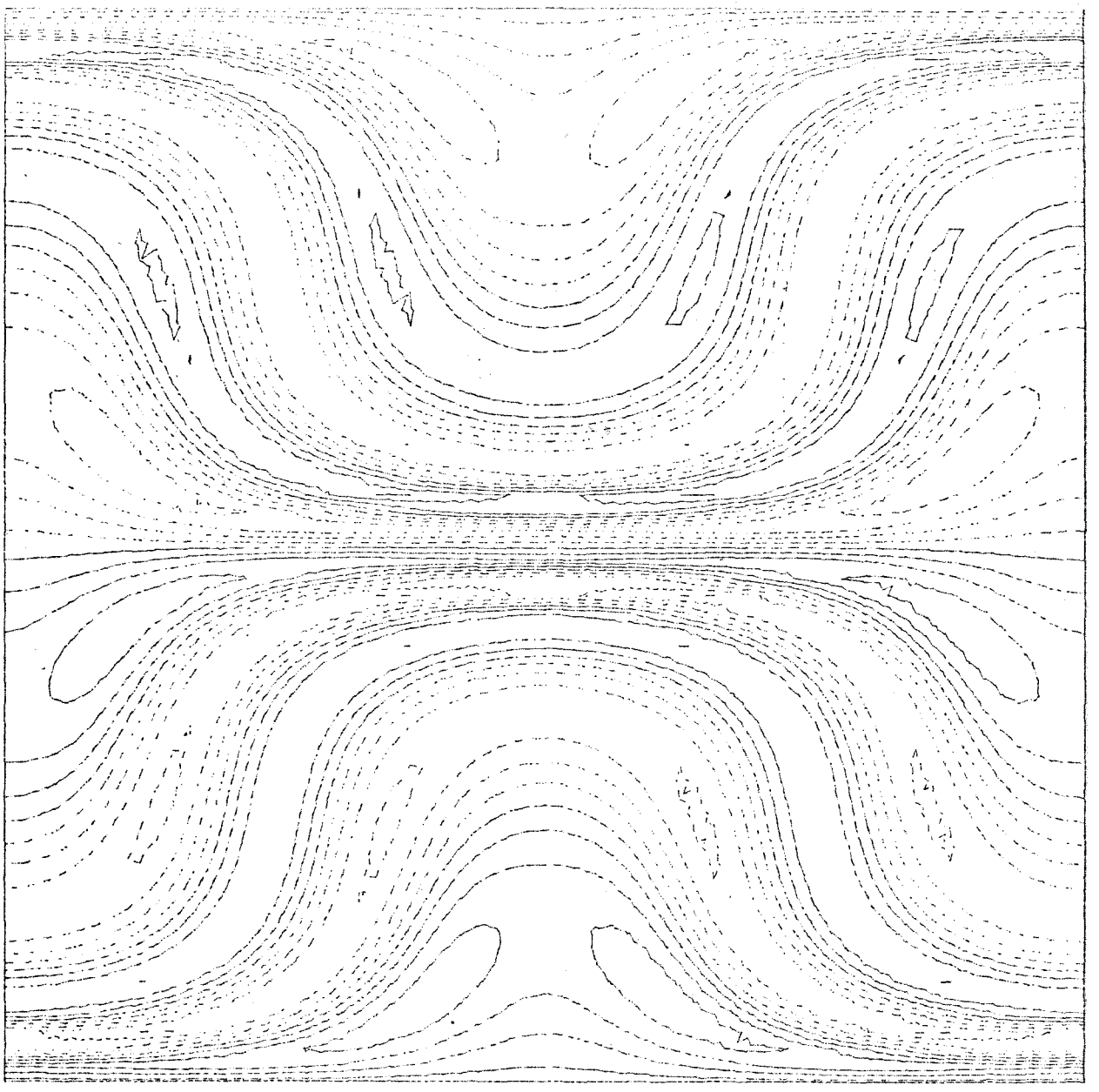

Fig. 17: Contours of the vorticity perturbation after 50 timesteps in the preliminary run described on pages 91-92. Lowest contour value = -.203 x 10^{-1}, *highest contour value = .203 x* 10^{-1}.

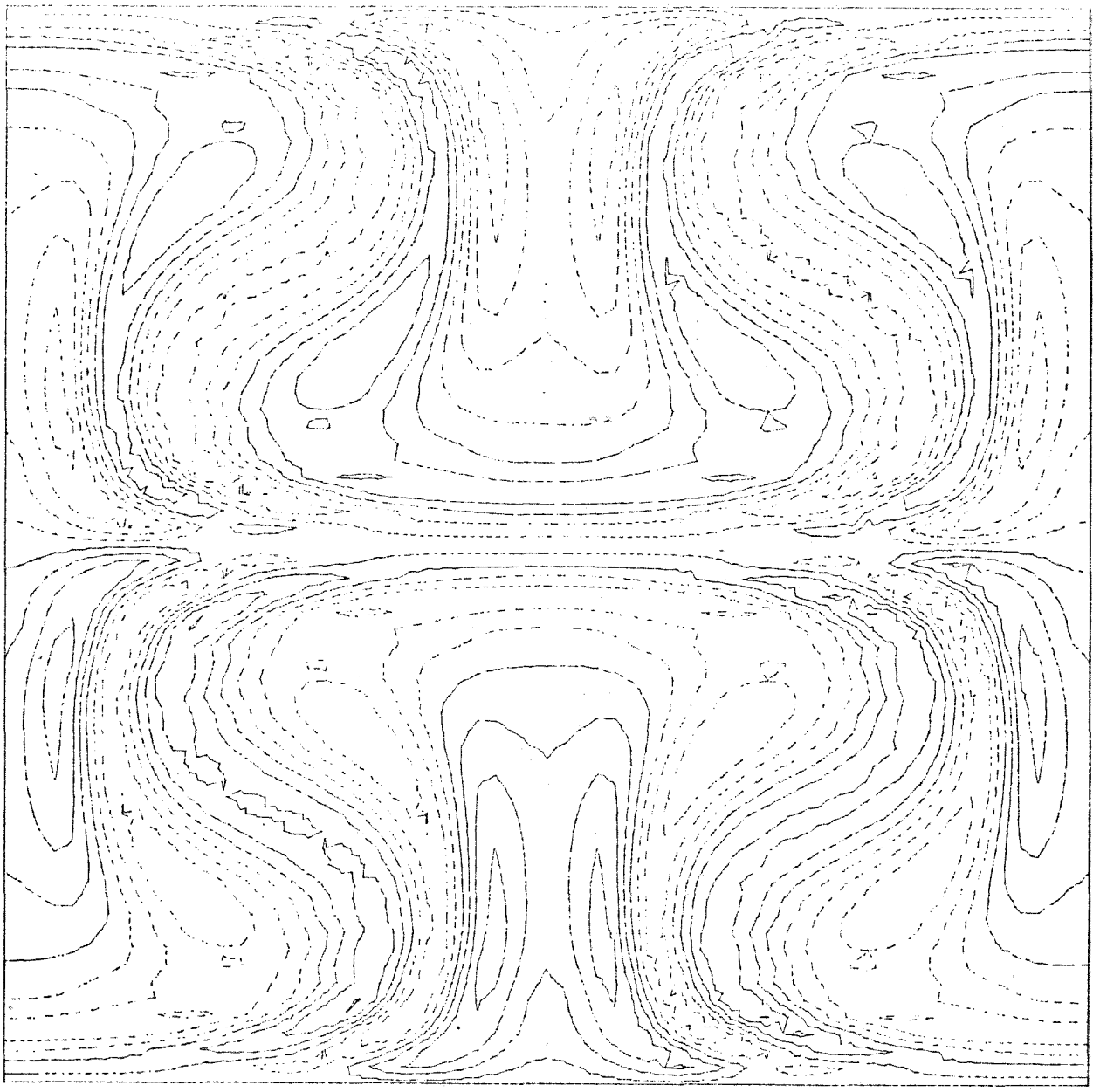

Fig. 18: As figure 17, but after 100 time-steps. Contour values to $\pm .266 \times 10^{-1}$.

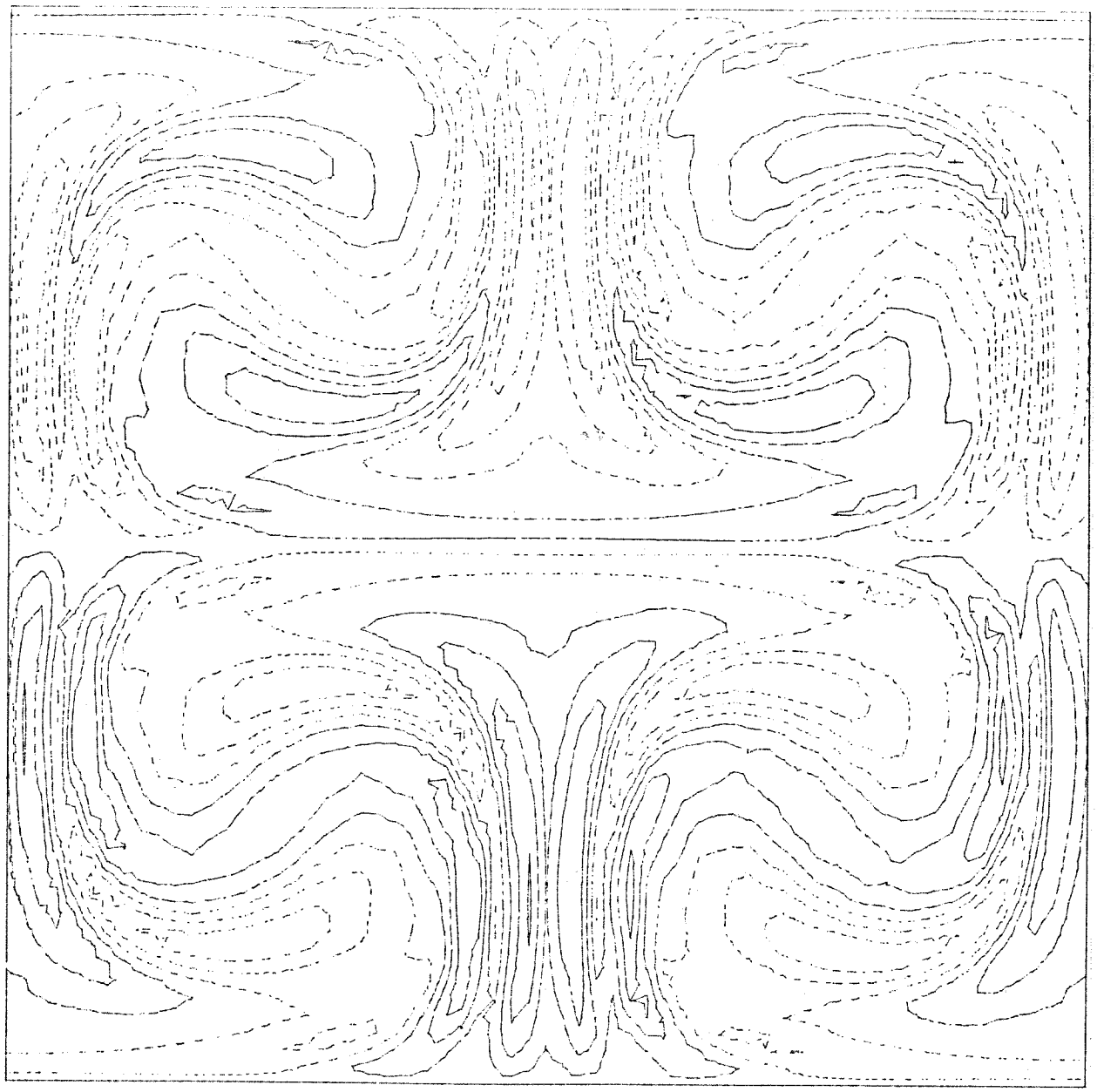

Fig. 19: As figure 17, but after 150 time-steps. Contour values to $\pm .335\times10^{-1}$.

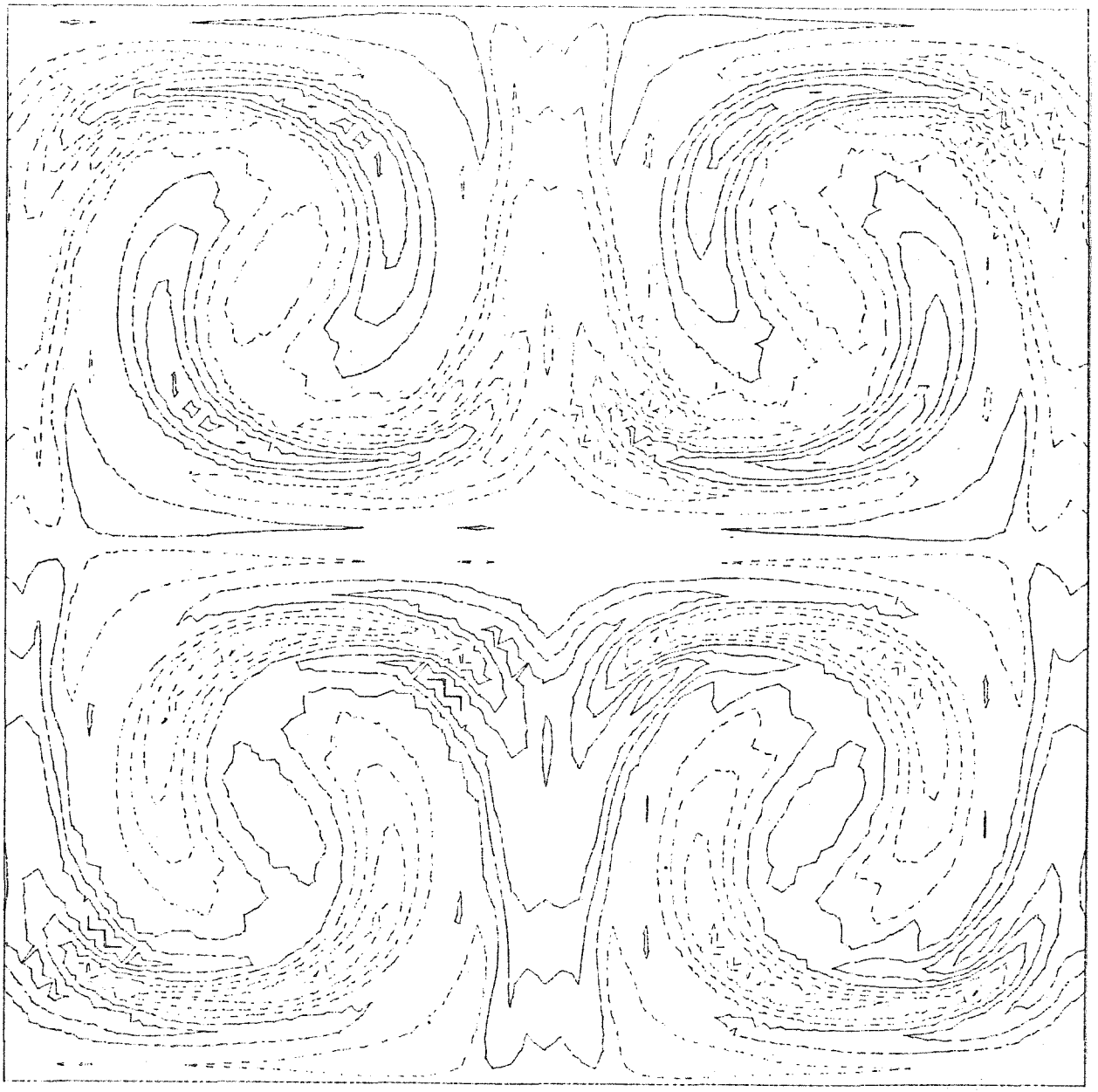

Fig. 20: As figure 17, but after 200 time-steps. Contour values to $\pm .305\times10^{-1}$.

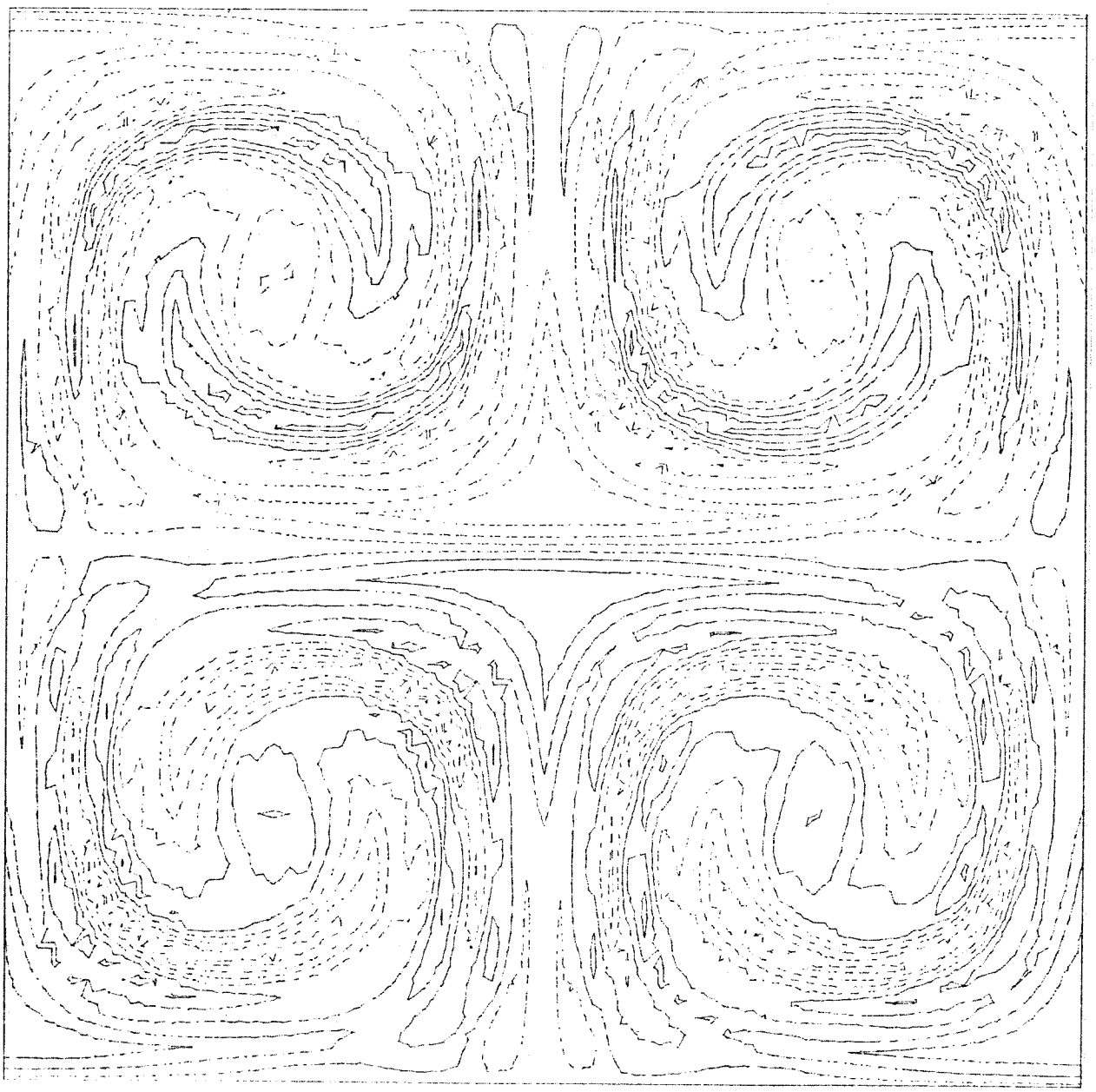

Fig. 21: As figure 17, but after 300 time-steps. Contour values to $\pm .276\times10^{-1}$.

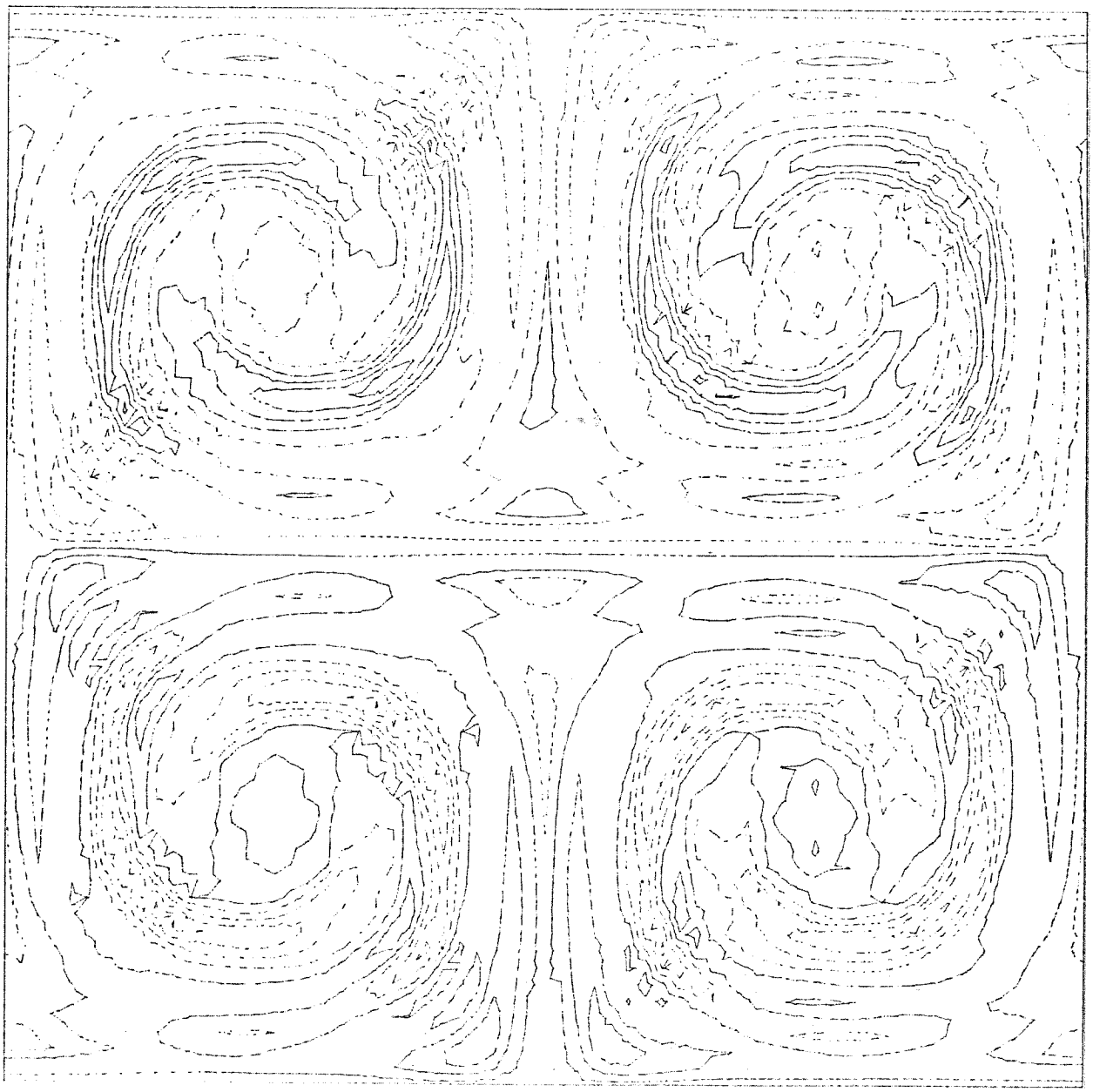

Fig. 22: As figure 17, but after 400 time-steps. Contour values to $\pm .281\times10^{-1}$.