TWO PROBLEMS IN PLANE FINITE ELASTOSTATICS

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SUMMARY

In this paper the fully nonlinear equilibrium theory of homogeneous and isotropic incompressible elastic solids is used to study the elastostatic fields in plane strain near the point of application of a concentrated force on a deformed half plane and near the vertex of a circular sector whose plane deformed faces are subjected to prescribed tractions.

In the concentrated force problem, restricting only the form of the elastic potential at large extensional deformations, it is shown that, for materials which "harden" in simple shear, the displacement is bounded at the point of application of the load. This is not the case for materials which "soften" in shear. Estimates of the true stress tensor near the singular point are given.

In the sector problem, for a class of the materials mentioned, the deformation and stress field near the vertex of the deformed cross-section are derived and discussed.

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INTRODUCTION

Under certain circumstances, the description of the elastostatic field furnished by the classical linearized theory of elasticity may be inadequate, even when the applied loads are small. Such breakdowns in the linearized theory are ordinarily local in nature and are brought about, for example, by stress concentrations such as those induced by holes or cracks in the interior of the loaded solid. The most extreme examples of problems of this kind involve a singular point in the elastostatic field--the tip of a crack, for example--near which the displacement gradient is unbounded. Since the basic approximative assumption underlying the linear theory requires that this gradient be negligibly small in comparison with unity, it is hardly surprising that results based on this theory may be in error near such a singular point.

Problems involving large displacement gradients properly fall within the scope of the finite theory of elasticity. In recent years there have been several investigations within the framework of the finite theory of the local structure of the elastostatic field near a geometrically-induced singular point. Much of this work is summarized in the review articles [1,2], where references are given. In general, the analyses of singular problems reviewed in [1,2] are necessarily *local* in character; they reveal that the results from linear theory near the singular point are invariably incorrect quantitatively, and in some instances may be qualitatively misleading as well. Since it is often the field *near* the singular point which is of primary physical interest, analyses based on finite elasticity are of considerable significance.

In the present paper, a further singular problem in elastostatics is considered within the scope of the theory of finite elasticity. This is the plane strain problem of a concentrated uniform normal line force applied to an elastic body which, in the undeformed state, occupies a half-space. Here the singularity arises because of the character of the applied load, rather than from the geometry of the undeformed body, as is the case in all singular problems previously treated within the finite theory [1,2]. The present analysis aims at the asymptotic determination of the displacements and stresses near the point of application of the load. We deal with the fully nonlinear equilibrium theory for homogeneous, isotropic incompressible materials that possess an elastic potential. The only restriction on this potential is one which pertains to its asymptotic behavior at large deformations; it is this regime of deformation which dominates the local field near the singular point. Again, it is found that the structure of the stress and displacement fields near the singular point differs from that predicted by the linear theory.

The only previous works devoted to the effect of nonlinearity on the elastostatic field near the point of application of a concentrated force are those of Arutiunian [3] and Atkinson [4]. Both of these authors retain the assumption of infinitesimal displacement gradients appropriate to the linearized theory, but replace the constitutive law of the latter theory by a nonlinear one.

The second problem treated here is the local analysis of the plane strain equilibrium field near the vertex of a body whose undeformed shape is that of a circular sector. The deformation is brought about by the application of tractions to the deformation images of the plane faces of the sector.

The only directly related work in finite elastostatics for the sector problem is that of Klingbeil and Shield [5]. We discuss their results in Section 3.

Section 1 contains a review of some prerequisites from the theory of finite plane elastostatics for homogeneous, isotropic, incompressible elastic solids. We also introduce in Section 1 the class of elastic solids underlying the subsequent analysis.

Section 2 is devoted to the formulation, analysis, and discussion of the concentrated force problem. Finally, the problem of the sector is treated in Section 3.

1. PRELIMINARIES FROM PLANE FINITE ELASTOSTATICS

In this work we shall be concerned with the analysis, within the finite theory, of plane elastostatic fields in incompressible, homogeneous, and isotropic elastic materials in the absence of body forces.[1]

Consider an elastic body which--in the undeformed state--is an infinite cylinder, and let Π denote a plane open cross-section of this cylinder perpendicular to its generators. Let (x_1,x_2) be the coordinates of a generic point in Π relative to a fixed two-dimensional rectangular cartesian coordinate system in the plane of Π .

A plane deformation of the body is given by the transformation

$$y_\alpha = \hat{y}_\alpha(x_1,x_2) = x_\alpha + u_\alpha(x_1,x_2) \quad \text{on } \Pi \ , \quad \alpha = 1,2 \ , \tag{1.1}$$

where y_α are the components of the position vector $\underset{\sim}{y}$ of the particle in the deformed body whose position vector in the undeformed configuration is $\underset{\sim}{x}$; u_α are the components of the displacement vector $\underset{\sim}{u}$, all with respect to the rectangular coordinate system. The function $\hat{\underset{\sim}{y}}$ is required to be twice continuously differentiable on Π, and it is further required that the mapping $\underset{\sim}{x} \leftrightarrow \underset{\sim}{y}$ be one-to-one and that its inverse $\hat{\underset{\sim}{x}}$ have the same smoothness.

The deformation gradient tensor $\underset{\sim}{F}$ associated with $\underset{\sim}{y}$ has components

$$F_{\alpha\beta} = \frac{\partial y_\alpha}{\partial x_\beta} \ . \tag{1.2}$$

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[1] For a discussion of the foundations of finite elasticity see Gurtin [6]. For further reading on plane finite elastostatics of incompressible materials, see [7].

Since the material is presumed to be incompressible, the deformation (1.1) must be locally volume-preserving, whence the Jacobian determinant of the mapping must satisfy

$$J = \det[F_{\alpha\beta}] = 1 \quad \text{on } \Pi \ . \tag{1.3}$$

Define the right and left two-dimensional Cauchy-Green tensors $\underset{\sim}{C}$ and $\underset{\sim}{G}$, respectively, by

$$\underset{\sim}{C} = \underset{\sim}{F}^T \underset{\sim}{F} \quad , \qquad \underset{\sim}{G} = \underset{\sim}{F}\,\underset{\sim}{F}^T \ . \tag{1.4}$$

These deformation tensors have common fundamental scalar invariants given by

$$\begin{aligned} I_1 &= \text{tr}\ \underset{\sim}{C} = F_{\alpha\beta}F_{\alpha\beta} = I \ , \text{ say,} \\ I_2 &= \det\ \underset{\sim}{C} = J^2 = 1 \ . \end{aligned} \tag{1.5}$$ [1]

The invariant I is found to obey

$$I \geq 2 \quad \text{on } \Pi \ . \tag{1.6}$$

Moreover, $I = 2$ if and only if $\underset{\sim}{F} = \underset{\sim}{1}$, where $\underset{\sim}{1}$ is the two-dimensional unit tensor.

Let $\underset{\sim}{\tau}$ be the two-dimensional true (Cauchy) stress tensor regarded as a function of position on the deformation image Π^* of Π. Its components $\tau_{\alpha\beta}$ represent forces per unit deformed area. If $\underset{\sim}{\sigma}$ is the associated nominal (Piola) stress tensor field on Π, whose components $\sigma_{\alpha\beta}$

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[1] Repeated subscripts are summed over the range (1,2).

represent forces per unit undeformed area, one has

$$\underset{\sim}{\sigma} = \underset{\sim}{\tau}(\underset{\sim}{F}^T)^{-1} \quad . \tag{1.7}$$

For an equilibrium deformation in the absence of body forces, it is necessary that $\underset{\sim}{\tau}$ satisfy

$$\operatorname{div} \underset{\sim}{\tau} = 0 \quad , \quad \underset{\sim}{\tau} = \underset{\sim}{\tau}^T \quad \text{on } \Pi^* . \tag{1.8}$$

It follows from (1.7),(1.8) that

$$\operatorname{div} \underset{\sim}{\sigma} = 0 \quad , \quad \underset{\sim}{\sigma}\underset{\sim}{F}^T = \underset{\sim}{F}\underset{\sim}{\sigma}^T \quad \text{on } \Pi \ . \tag{1.9}$$

Suppose that Γ is a regular arc in Π which is mapped onto Γ^* in Π^* by the deformation (1.1), and denote by $\underset{\sim}{n}$ and $\underset{\sim}{n}^*$ unit normal vectors of Γ and Γ^*, respectively. The true traction vector $\underset{\sim}{t}$ and the associated nominal traction vector $\underset{\sim}{s}$ are given by

$$\begin{aligned} \underset{\sim}{s} &= \underset{\sim}{\sigma}\underset{\sim}{n} \quad \text{on } \Gamma \ , \\ \underset{\sim}{t} &= \underset{\sim}{\tau}\underset{\sim}{n}^* \quad \text{on } \Gamma^* . \end{aligned} \tag{1.10}$$

It can be shown that

$$\underset{\sim}{s} = \underset{\sim}{0} \ \text{on } \Gamma \text{ if and only if } \underset{\sim}{t} = \underset{\sim}{0} \ \text{on } \Gamma^* \ . \tag{1.11}$$

Moreover, (1.11) continues to hold true for an arc Γ on the boundary of Π if the deformation and nominal stress field are suitably regular on the closure $\overline{\Pi}$ of Π. This important fact allows the boundary condition for a traction-free surface Γ^* in the deformed body to be specified on the known pre-image Γ of Γ^* in the undeformed body.

The mechanical response of the homogeneous, isotropic, incompressible material under consideration is governed by the strain energy density W per unit undeformed volume. For a plane deformation of the type described above, W depends only on the deformation invariant I:

$$W = W(I) \quad . \tag{1.12}$$

The stress-deformation relation is

$$\tau_{\alpha\beta} = 2W'(I)\, F_{\alpha\rho}F_{\beta\rho} - p\delta_{\alpha\beta} \quad \text{on } \Pi^*, \tag{1.13}$$

where $\delta_{\alpha\beta}$ is the Kronecker delta and the scalar field p is an arbitrary hydrostatic pressure whose presence is necessary because of the constraint of incompressibility. Because of the presence of p, the true stress tensor is not completely determined by the deformation for an incompressible material. From (1.13), (1.7) it follows that

$$\sigma_{\alpha\beta} = 2W'(I)F_{\alpha\beta} - p\varepsilon_{\beta\gamma}\varepsilon_{\alpha\rho}F_{\rho\gamma} \quad \text{on } \Pi, \tag{1.14}$$

provided $\varepsilon_{\alpha\beta}$ are the components of the two-dimensional alternator. In the foregoing, W' denotes the derivative of W with respect to I; we assume that W is twice continuously differentiable for $I \geq 2$. It is further assumed that W vanishes in the undeformed state, so that

$$W(2) = 0 \quad , \tag{1.15}$$

and that

$$W'(I) > 0 \quad , \quad I \geq 2 \; , \tag{1.16}$$

so that the Baker-Ericksen inequality is not violated.[1]

The linear theory of elastostatic plane strain is recovered from the finite deformation theory briefly described above by a systematic linearization with respect to the displacement gradients $u_{\alpha,\beta}$. Under this linearization, the distinction between true and nominal stresses disappears, and the constitutive law passes over into

$$\tau_{\alpha\beta} = \sigma_{\alpha\beta} = 2\mu\, \gamma_{\alpha\beta} - p\delta_{\alpha\beta} \quad , \tag{1.17}$$

where

$$\gamma_{\alpha\beta} = \frac{1}{2}\,(u_{\alpha,\beta} + u_{\beta,\alpha}) \tag{1.18}$$

are the components of the infinitesimal strain tensor, and

$$\mu = 2W'(2) \tag{1.19}$$

is the infinitesimal shear modulus. The incompressibility condition $J = 1$ linearizes to

$$\gamma_{\alpha\alpha} = u_{\alpha,\alpha} = \operatorname{div} \underset{\sim}{u} = 0 \ . \tag{1.20}$$

The approximate form of W for infinitesimal deformations is found by linearization to be

$$W = \frac{\mu}{2}\, \gamma_{\alpha\beta}\, \gamma_{\alpha\beta} \ . \tag{1.21}$$

A deformation (1.1) of the form

[1] See [8].

$$y_\alpha = B_{\alpha\beta}\, x_\beta \quad , \tag{1.22}$$

where the $B_{\alpha\beta}$ are constants satisfying

$$\det[B_{\alpha\beta}] = 1 \quad , \tag{1.23}$$

is a homogeneous deformation of the incompressible body. Two particular homogeneous deformations are of special interest: uniaxial stress and simple shear. For the former one takes

$$y_1 = \lambda x_1 \ , \qquad y_2 = \frac{1}{\lambda}\, x_2 \ , \qquad \lambda > 0 \ , \tag{1.24}$$

with λ constant. From (1.13), one then finds that $\tau_{12} = \tau_{21} = 0$, and, if p is chosen to be

$$p = 2W'(I)\, \lambda^{-2} \ , \tag{1.25}$$

where

$$I = \lambda^2 + \lambda^{-2} \quad , \tag{1.26}$$

one has

$$\tau_{22} = 0 \quad , \tag{1.27}$$

as well. The only nonvanishing stress component is then found from (1.13) to be

$$\tau_{11} = 2W'(I)(\lambda^2 - \lambda^{-2}) \ . \tag{1.28}$$

For simple shear, one has the homogeneous deformation

$$y_1 = x_1 + kx_2 \quad , \qquad y_2 = x_2 \ , \tag{1.29}$$

where the constant k is the amount of shear. From (1.13) one finds the relation between the true shear stress τ_{12} and the amount of shear k to be

$$\tau_{12} = 2W'(I)\,k \ , \tag{1.30}$$

where now

$$I = 2 + k^2 \ . \tag{1.31}$$

We shall assume throughout that W has the following property:

$$W(I) = AI^n + o(I^n) \quad \text{as} \quad I \to \infty \ , \tag{1.32}$$

where A and n are material constants satisfying

$$A > 0 \ , \qquad n > 1/2 \ . \tag{1.33}$$

For an incompressible material satisfying (1.32), one sees from (1.26),(1.28) that in extreme uniaxial stress ($\lambda \to \infty$), one has

$$\tau_{11} \sim 2nA\lambda^{2n} \ , \quad \lambda \to \infty \ . \tag{1.34}$$

For severe simple shear ($k \to \infty$), one obtains from (1.31),(1.30),(1.32)

$$\tau_{12} \sim 2nA\,k^{2n-1} \ , \quad k \to \infty \ . \tag{1.35}$$

Since $n > 1/2$, the stress response in uniaxial stress is, by (1.34), always asymptotically <u>hardening</u> as $\lambda \to \infty$, in the sense that $d\tau_{11}/d\lambda$ is increasing with increasing λ. In shear, the stress response of (1.35) is <u>hardening</u> as $k \to \infty$ for those materials with $n > 1$, softening for $\frac{1}{2} < n < 1$, and asymptotically linear if $n = 1$. The asymptotic forms of the stress response curves in uniaxial stress and simple shear for materials satisfying (1.32),(1.33) are shown in Figure 1.

If one were to permit $n < 1/2$ in (1.32) one would find that the field equations of the equilibrium theory would cease to be an elliptic system at sufficiently severe deformations; see [7].

Before proceeding to the specific problems to be discussed, it is useful to take note of an implication of the field equations (1.3), (1.9), (1.14). One can show that $\det \underset{\sim}{F} \equiv 1$ implies that

$$\varepsilon_{\beta\gamma}\, \varepsilon_{\alpha\rho}\, F_{\rho\gamma,\beta} \equiv 0 \quad \text{on } \Pi \,. \tag{1.36}$$

Substitution from (1.14) into the equilibrium equations (1.9) then gives, with the help of (1.36), the equation

$$[2W'(I)\, F_{\alpha\beta}]_{,\beta} = p_{,\beta}\, \varepsilon_{\beta\gamma}\, \varepsilon_{\alpha\rho}\, F_{\rho\gamma} \qquad \text{on } \Pi \,. \tag{1.37}$$

If one multiplies (1.37) by $F_{\alpha\lambda}$, makes use of the fact that $\det \underset{\sim}{F} \equiv 1$ as well as of the definitions $(1.4)_1$,$(1.5)_1$, one finds that

$$\underset{\sim}{\nabla} p = 2W'(I)\, \underset{\sim}{F}^T \nabla^2 \underset{\sim}{y} + 2W''(I)\, \underset{\sim}{F}^T \underset{\sim}{F}\, \underset{\sim}{\nabla} I \qquad \text{on } \Pi \,. \tag{1.38}$$

We will find this form of the equilibrium equations helpful in the sequel.

2. THE HALF-PLANE DEFORMED BY A CONCENTRATED FORCE

A. Formulation of the Problem

We consider the case in which the open cross-section Π of the undeformed body is the half-plane $x_1 > 0$, $-\infty < x_2 < \infty$, and we denote by $\overset{\circ}{\bar{\Pi}}$ the closure of Π with the origin deleted. Given the plane strain elastic potential $W(I)$ of the homogeneous, isotropic, incompressible material to be considered, we seek a deformation $y_\alpha = \hat{y}_\alpha(x_1,x_2)$ on $\overset{\circ}{\Pi}$ such that the nominal stresses $\sigma_{\alpha\beta}$ generated by the deformation through (1.2)-(1.5) and (1.14) conform to the equation of equilibrium (1.9). We further assume that the free-surface conditions

$$\sigma_{11}(0,x_2) = \sigma_{21}(0,x_2) = 0 \quad , \quad |x_2| > 0 \tag{2.1}$$

hold and we require that, as $|\underset{\sim}{x}| \to \infty$, the true stress field should tend to zero:

$$\tau_{\alpha\beta}(x_1,x_2) \to 0 \quad \text{as} \quad |\underset{\sim}{x}| \to \infty \quad , \quad x_1 \geq 0 . \tag{2.2}$$

Further, we impose the requirement that

$$\sigma_{\alpha\beta} = O(r^{-1}) \;\text{ as } r \to 0, \;\text{ uniformly in } \theta, \quad -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2} , \tag{2.3}$$

where r,θ are polar coordinates at the origin: $x_1 = r\cos\theta$, $x_2 = r\sin\theta$. We next prescribe that

$$\int_{-\pi/2}^{\pi/2} \sigma_{\alpha\beta}\, n_\beta\, r\, d\theta = F\,\delta_{1\alpha} \quad , \qquad r > 0 \tag{2.4}$$

corresponding to a concentrated force of magnitude $|F|$ acting on the boundary of the deformed body in a direction parallel to the x_1-axis. We shall limit our attention to the case $F > 0$, so that the force is in the negative x_1-direction and is therefore tensile. In (2.4), $\underset{\sim}{n}$ is the unit vector in the radial direction.

We finally assume the elastostatic field to be symmetric about the x_1-axis. This in particular rules out a concentrated moment at the origin; symmetry also implies that (2.4) holds automatically for $\alpha = 2$.

B. The Elastostatic Field near the Origin--Lowest Order Asymptotic Analysis

We now assume that the elastic potential $W(I)$ satisfies (1.32), (1.33), and we investigate the local structure of the field near the point of application of the force. We begin by making the Ansatz

$$y_\alpha = r^{m_\alpha} v_\alpha(\theta) + o(r^{m_\alpha - 1}) \quad \text{as } r \to 0, \text{ (no sum on } \alpha) , \tag{2.5}$$

uniformly for $-\pi/2 \leq \theta \leq \pi/2$, where m_1 and m_2 are constants restricted by

$$m_1 < 1 \quad , \quad m_2 > 1 \tag{2.6}$$[1]

and neither of the unknown functions $v_\alpha(\theta) \in C^2([-\frac{\pi}{2}, \frac{\pi}{2}])$ vanish identically.[2] Moreover, in view of the prevailing symmetry one has

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[1]One can show systematically that (2.6) are necessary when the applied force is tensile. The hypothesis (2.6) must be altered when the applied force is compressive. Note that we do not assume $m_1 \geq 0$.

[2]We actually need the slightly stronger assumption that $v_1(\theta) \not\equiv 0$ and v_2 has a finite number of zeros in $[-\frac{\pi}{2}, \frac{\pi}{2}]$.

$$v_1(-\theta) = v_1(\theta) \quad , \qquad v_2(-\theta) = -v_2(\theta) \ . \tag{2.7}$$

It is assumed that (2.5) may be formally differentiated twice.

From (2.5) one obtains for the deformation gradient tensor $\underset{\sim}{F}$ the local asymptotic representation

$$F_{\alpha\beta} \sim f_{\alpha\beta}\, r^{m_\alpha - 1} \quad \text{as } r \to 0 \text{ (no sum on } \alpha) \ , \tag{2.8}$$

provided

$$f_{\alpha\beta} = m_\alpha v_\alpha(\theta)\, c_\beta(\theta) + \varepsilon_{\gamma\beta} c_\gamma(\theta)\, \dot{v}_\alpha(\theta) \quad \text{(no sum on } \alpha) \ . \tag{2.9}$$

Here the dot denotes differentiation with respect to θ and we have introduced the abbreviations.

$$c_1(\theta) = \cos\theta \quad , \qquad c_2(\theta) = \sin\theta \ . \tag{2.10}$$

From (2.8),(2.9) there follows

$$\begin{aligned} J \equiv \det \underset{\sim}{F} = (m_1 v_1 \dot{v}_2 - m_2 v_2 \dot{v}_1)\, r^{m_1 + m_2 - 2} \\ + o(r^{m_1 + m_2 - 2}) \ , \quad r \to 0 \end{aligned} \tag{2.11}$$

Since incompressibility requires $J \equiv 1$, we must have

$$m_1 + m_2 - 2 \leq 0 \quad , \tag{2.12}$$

and either

$$m_1 v_1 \dot{v}_2 - m_2 v_2 \dot{v}_1 = 0 \qquad \text{if } \ m_1 + m_2 < 2 \ , \tag{2.13}$$

or

$$m_1 v_1 \dot{v}_2 - m_2 v_2 \dot{v}_1 = 1 \quad \text{if} \quad m_1 + m_2 = 2 . \tag{2.14}$$

From (1.5),(2.8),(2.9) we obtain

$$I \sim r^{2(m_1-1)} G(\theta) \quad \text{as} \quad r \to 0 \tag{2.15}$$

where

$$G(\theta) = \dot{v}_1^2(\theta) + m_1^2 \, v_1^2(\theta) . \tag{2.16}$$

In view of the assumption (2.6) concerning m_1, one has from (2.15) that $I \to \infty$ as $r \to 0$. The material assumption (1.32) then yields

$$\left.\begin{aligned} W(I) &\sim AG^n(\theta) \, r^{2n(m_1-1)} , \\ W'(I) &\sim nAG^{n-1}(\theta) \, r^{2(n-1)(m_1-1)} , \\ W''(I) &\sim n(n-1) \, AG^{n-2}(\theta) \, r^{2(n-2)(m_1-1)} , \end{aligned}\right\} \text{as } r \to 0 . \tag{2.17}$$

We now recall the field equations in the form (1.38); with the help of (2.17),(2.8),(2.9),(2.6),(2.15) we find from (1.38) that

$$\frac{\partial p}{\partial r} \sim 2n A \, m_1 v_1(\theta) \, Z(\theta) \, r^{2(m_1-1)n-1} , \tag{2.18}$$

and

$$\frac{1}{r}\frac{\partial p}{\partial \theta} \sim 2n A \, \dot{v}_1(\theta) \, Z(\theta) \, r^{2(m_1-1)n-1} , \tag{2.19}$$

as $r \to 0$, where

$$Z(\theta) = G^{n-2}(\theta)\ \{G(\theta)[\ddot{v}_1(\theta) + m_1^2\ v_1(\theta)]$$

$$+ (n-1)[\dot{G}(\theta)\ \dot{v}_1(\theta) + 2m_1(n-1)\ G(\theta)\ v_1(\theta)]\} . \qquad (2.20)$$

Compatibility of (2.18),(2.19) requires that v_1 and Z satisfy

$$m_1 v_1 \dot{Z} + [1 - (2n-1)(m_1-1)]\ \dot{v}_1 Z = 0 . \qquad (2.21)$$

Once (2.21) is fulfilled, one finds from either (2.18) or (2.19) that

$$p \sim \frac{m_1}{m_1-1} A\ v_1(\theta)\ Z(\theta)\ r^{2n(m_1-1)} \qquad \text{as} \quad r \to 0 . \qquad (2.22)$$

We next consider the boundary conditions (2.1). Because of (1.14) these are

$$\left.\begin{aligned} 2W'(I)F_{11} - pF_{22} &= 0 \\ 2W'(I)F_{21} + pF_{12} &= 0 \end{aligned}\right\} \quad \text{at } \theta = \pm\frac{\pi}{2}, \quad r > 0 . \qquad (2.23)$$

Multiplying the first of (2.23) by F_{11}, the second by F_{21}, adding the results, and using (1.3), we obtain

$$p = 2W'(I)(F_{11}^2 + F_{21}^2) \qquad \text{at} \quad \theta = \pm\frac{\pi}{2}, \quad r > 0 . \qquad (2.24)$$

On the other hand, eliminating p between the two equations (2.23) yields, in view of (1.16),

$$F_{11}F_{12} + F_{21}F_{22} = 0 \qquad \text{at} \quad \theta = \pm\frac{\pi}{2}, \quad r > 0 . \qquad (2.25)$$

Making use of (2.17),(2.8), and (2.9), we obtain from (2.24) the result

$$p \sim 2n\,A\,G^{n-1}(\theta)\,\dot{v}_1(\theta)\,r^{2n(m_1-1)} \quad \text{as} \quad r \to 0,\ \theta = \pm\frac{\pi}{2}\ . \tag{2.26}$$

Comparing (2.22) at $\theta = \pm\frac{\pi}{2}$ and (2.26) leads to

$$2nG^{n-1}(\pm\tfrac{\pi}{2})\,\dot{v}_1(\pm\tfrac{\pi}{2}) = \frac{m_1}{m_1-1}\,v_1(\pm\tfrac{\pi}{2})\,Z(\pm\tfrac{\pi}{2})\ . \tag{2.27}$$

The second boundary condition (2.25), with the help of (2.8),(2.9),(2.15) gives

$$\dot{v}_1(\pm\tfrac{\pi}{2})\ v_1(\pm\tfrac{\pi}{2}) = 0\ . \tag{2.28}$$

We now show that the two boundary conditions (2.27) and (2.28), together with the differential equation (2.21), imply that

$$\dot{v}_1(\pm\tfrac{\pi}{2}) = 0 \tag{2.29}$$

and

$$Z(\pm\tfrac{\pi}{2}) = 0\ . \tag{2.30}$$

We first establish (2.30) by showing that the hypothesis $Z(\pi/2) \neq 0$ leads to a contradiction. If $Z(\pi/2) \neq 0$, there is an interval $[\theta_o,\pi/2]$, $\theta_o < \pi/2$, on which $Z(\theta)$ vanishes nowhere, by continuity. Equation (2.21) can then be integrated on $[\theta_o,\pi/2]$ to give

$$v_1(\theta) = C\left|Z(\theta)\right|^{\frac{m_1}{(2n-1)(m_1-1)-1}}\ , \qquad \theta_o \leq \theta \leq \frac{\pi}{2}\ , \tag{2.31}$$

where C is a constant. If $C = 0$, then $v_1(\theta) \equiv 0$ on $[\theta_o,\pi/2]$, whence by (2.16),(2.20), $Z(\theta) \equiv 0$ on $[\theta_o,\pi/2]$, contradicting the hypothesis $Z(\pi/2) \neq 0$. Thus $C \neq 0$, and so by (2.31), $v_1(\theta) \neq 0$ for all

$\theta \in [\theta_0, \pi/2]$. In particular, $v_1(\pi/2) \neq 0$, so that by (2.28), $\dot{v}_1(\pi/2) = 0$. Thus (2.27) has been violated unless $m_1 = 0$. But if $m_1 = 0$ we conclude from (2.21) that $v_1 \equiv$ constant on $[\theta_0, \pi/2]$, which leads via (2.20),(2.16) to $Z \equiv 0$ on $[\theta_0, \pi/2]$, again contradicting the hypothesis. Thus indeed, $Z(\pi/2) = 0$, and, since $Z(\theta)$ is even, (2.30) holds. From (2.27) and (2.16) it then follows that (2.29) holds as well.

An argument similar to that just used to establish (2.30) can now be constructed to show that (2.21),(2.30) imply that

$$Z(\theta) \equiv 0 \quad \text{on } [-\pi/2, \pi/2] \tag{2.32}$$

From (2.32),(2.20),(2.29) we then obtain a nonlinear eigenvalue problem for $m_1, v_1(\theta)$:

$$[G^{n-1}(\theta)\, \dot{v}_1(\theta)]^{\cdot} + [m_1^2 + 2m_1(m_1-1)(n-1)]G^{n-1}(\theta)\, v_1(\theta) = 0\ , \qquad -\pi/2 \leq \theta \leq \pi/2 \tag{2.33}$$

$$\dot{v}_1(\pm \tfrac{\pi}{2}) = 0\ . \tag{2.34}$$

The differential equation (2.33) is identical with one which has arisen in the local analysis of the elastostostatic field near the tip of a crack; see [9],[10],[11], [12]

In view of (2.32), we have from (2.22) that

$$p = o(r^{2n(m_1-1)}) \quad \text{as} \quad r \to 0\ . \tag{2.35}$$

Suppose that $m_1 = 0$. Then (2.33),(2.34) imply that $v_1(\theta) \equiv$ constant and the leading term $r^{m_1} v_1(\theta)$ in the expansion of y_1 near $r = 0$ may be viewed as a rigid body translation parallel to the x_1-axis. Since the boundary value problem determines the elastostatic field at best to within an arbitrary translation of this kind, we shall discard the case $m_1 = 0$ and assume henceforth that, in addition to (2.6),

$$m_1 \neq 0 \;, \tag{2.36}$$

also holds.

It is now possible to prove that (2.36),(2.6) and the assumptions made concerning v_1 and v_2 imply that (2.13) leads to the contradiction $v_1 \equiv 0$. Thus (2.14) must hold, and thus

$$m_1 + m_2 = 2 \;. \tag{2.37}$$

From (1.14) we have

$$\sigma_{1\beta} = 2W'(I)\, F_{1\beta} - p\, \varepsilon_{\beta\gamma}\, F_{2\gamma} \;. \tag{2.38}$$

Making use of (2.17),(2.8),(2.9),(2.35)-(2.38) we can show that the first term on the right in (2.38) dominates the second, and hence that

$$\sigma_{1\beta} \sim 2n\, A\, m_1 G^{n-1}(\theta)\; f_{1\beta}(\theta)\; r^{(2n-1)(m_1-1)} \quad \text{as} \quad r \to 0 \;. \tag{2.39}$$

From (2.3) we conclude that $(2n-1)(m_1-1) \geq -1$. In order to use (2.4) with $\alpha = 1$, we first observe that $n_\beta = c_\beta$ (see (2.10)), and from (2.39), (2.9) that

$$\int_{-\pi/2}^{\pi/2} \sigma_{1\beta} n_\beta \; r \, d\theta \sim 2nm_1 A \int_{-\pi/2}^{\pi/2} G^{n-1}(\theta) \; v_1(\theta) \; d\theta \; r^{(2n-1)(m_1-1)+1} \qquad \text{as } r \to 0 . \qquad (2.40)$$

It follows that

$$m_1 = \frac{2(n-1)}{2n-1} < 1 \quad , \qquad n \neq 1 . \qquad (2.41)^1$$

From the boundary value problem (2.33),(2.34) with m_1 given by (2.41), one finds

$$v_1(\theta) = k_1 = \text{constant} \quad , \qquad -\pi/2 \leq \theta \leq \pi/2 \; , \quad n \neq 1 . \qquad (2.42)$$

Since (2.37) holds, we have

$$m_2 = 2 - m_1 = \frac{2n}{2n-1} > 1 \quad , \qquad n \neq 1 . \qquad (2.43)$$

With the help of (2.42) and the fact that $v_2(\theta)$ is odd, we can now determine v_2 from (2.14) as

$$v_2(\theta) = \frac{1}{k_1 m_1} \theta = k_2 \theta \quad , \qquad -\pi/2 \leq \theta \leq \pi/2 \; , \quad n \neq 1 . \qquad (2.44)$$

Finally, we return to (2.4) with $\alpha = 1$ to determine k_1 in terms of F. Using (2.40)-(2.42) and (2.16), we obtain

$$[(m_1 k_1)^2]^{n-1} \; (m_1 k_1) = \frac{F}{2n A\pi} \quad , \qquad n \neq 1 \quad , \qquad (2.45)$$

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[1]Recall that $m_1 = 0$ has been excluded.

and hence

$$k_1 = \frac{2n-1}{2(n-1)} \left(\frac{F}{2n A\pi}\right)^{\frac{1}{2n-1}} \quad , \qquad n \neq 1 \quad . \tag{2.46}$$

Thus, for $n \neq 1$, we have determined the first terms (2.5) in the approximation to the deformation near $r = 0$ as follows:

$$\left.\begin{aligned} y_1 &\sim k_1\, r^{\frac{2(n-1)}{2n-1}} \quad , \\ y_2 &\sim \frac{2n-1}{2(n-1)} \frac{1}{k_1}\, r^{\frac{2n}{2n-1}}\, \theta \quad , \end{aligned}\right\} \text{as } r \to 0,\ -\pi/2 \leq \theta \leq \pi/2\ ,\quad n \neq 1\ , \tag{2.47}$$

with k_1 related to F through (2.46). The deformation image of the boundary $\theta = \pm\pi/2$ of the half-plane is then given in first approximation by

$$y_1 \sim \frac{2n-1}{n-1}\ \frac{1}{\pi}\left(\frac{F}{4nA}\right)^{1/n} |y_2|^{1-\frac{1}{n}} \quad , \qquad |y_2| \to 0\ , \qquad n \neq 1\ . \tag{2.48}$$

We note that if $n > 1$, so that the material is asymptotically hardening in simple shear [see (1.35) and Figure 1], the displacement under the load is finite, while this is not the case for softening materials ($n < 1$). A sketch of the deformed surface based on (2.48) is given in Figure 2.

The case $n = 1$ (a material which is asymptotically linear in shear (Figure 1)) has been excluded in the results (2.47),(2.48). To treat this case, it is necessary to replace the Ansatz (2.5), (2.6) by

$$\left.\begin{aligned} y_1 &\sim (\log r)\, v_1(\theta) \quad , \\ y_2 &\sim r^{m_2}\, v_2(\theta) \quad , \qquad m_2 > 1 \end{aligned}\right\} \text{as } r \to 0\ . \tag{2.49}$$

The special nature of the case $n=1$ arises because $m_1 = 0$ and $m_1 = 2(n-1)/(2n-1)$ are both eigenvalues (adjacent ones in fact) of the problem (2.33),(2.34). They are distinct as long as $n \neq 1$, but coalesce[1] as $n \to 1$. This coalescence may be used to motivate the form of the new Ansatz (2.49) for $n = 1$; we omit the details. One finds from (2.49) that $m_2 = 2$, $v_1(\theta) = k_1 = \text{constant}$, $v_2(\theta) = (1/k_1)\theta$, where $k_1 = F/2A\pi$. The counterparts of (2.47) are

$$\left.\begin{aligned} y_1 &\sim \frac{F}{2A\pi}\log r \quad , \\ y_2 &\sim \frac{2A\pi}{F} r^2\,\theta \quad , \end{aligned}\right\} \text{ as } r \to 0,\ -\pi/2 \leq \theta \leq \pi/2\ ,\quad n=1, \tag{2.50}$$

while the deformed boundary is now given approximately by

$$y_1 \sim \frac{F}{4A\pi}\ \log\left(\frac{F|y_2|}{A\pi^2}\right) \qquad \text{as } |y_2| \to 0\ ,\quad n = 1\ . \tag{2.51}$$

The displacement is unbounded near the point of application of the load, as it is for the softening material ($n < 1$).

Although the nominal stresses σ_{11}, σ_{12} are fully determined to leading order as $r \to 0$ at this stage, the fact that, as yet, only the weak estimate (2.35) is available for the hydrostatic pressure p makes the asymptotic determination of σ_{22}, σ_{21} impossible without higher order considerations. In view of the relationship (1.7) between the nominal stresses $\sigma_{\alpha\beta}$ and the true stresses $\tau_{\alpha\beta}$, the full asymptotic determinatior

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[1]A similar but more complicated coalescence of eigenvalues arises in crack problems, see [9],[10].

of the latter must also await such higher order results.

C. Higher-Order Asymptotic Considerations

For the present, suppose that $n \neq 1$ and replace (2.5) by the two-term asymptotic representations

$$\left.\begin{array}{l} y_1 \sim k_1 r^{m_1} + w_1(\theta) r^{s_1} \quad , \\ \\ y_2 \sim k_2 \theta r^{m_2} + w_2(\theta) r^{s_2} \quad , \end{array}\right\} \text{ as } r \to 0 \ , \quad -\pi/2 \leq \theta \leq \pi/2 \ , \tag{2.52}$$

with the stipulation that

$$s_1 > m_1 \ , \quad s_2 > m_2 \ , \quad w_\alpha \varepsilon\, C^2([-\tfrac{\pi}{2}, \tfrac{\pi}{2}]) \ , \quad w_\alpha \not\equiv 0 \ \text{ on } [-\tfrac{\pi}{2}, \tfrac{\pi}{2}] \ , \tag{2.53}$$

and that w_1, w_2 have the respective parity of v_1 and v_2. Equations (1.2), (2.52) lead to the following representation for the components of the deformation gradient tensor:

$$F_{\alpha\beta} = f_{\alpha\beta}\, r^{m_\alpha - 1} + g_{\alpha\beta}\, r^{s_\alpha - 1} + o(r^{s_\alpha - 1}) \ , \ (\text{no sum on } \alpha), \tag{2.54}$$

provided $f_{\alpha\beta}$ is given by (2.9) and

$$g_{\alpha\beta}(\theta) = s_\alpha c_\beta(\theta)\, w_\alpha(\theta) + \varepsilon_{\gamma\beta} c_\gamma(\theta)\, \dot{w}_\alpha(\theta) \quad (\text{no sum on } \alpha). \tag{2.55}$$

The asymptotic representation of the deformation invariant I depends on the value of s_1. One can show after some calculation that necessarily

$$m_1 < s_1 < 4 - 3m_1 \quad . \tag{2.56}$$

Then the asymptotic representation for I is

$$I \sim m_1^2 \; v_1^2 \; r^{2(m_1-1)} + 2m_1 s_1 v_1 w_1 \; r^{m_1+s_1-2} \quad . \tag{2.57}$$

Equations (2.17),(2.54),(2.55),(2.9),(2.57) and (1.38) yield

$$\left.\begin{aligned} \frac{\partial p}{\partial r} &\sim 2n\,A\,G_1^{2n-1} \; Y(\theta) \; r^{s_1-3} \quad , \\ \frac{1}{r}\frac{\partial p}{\partial \theta} &= o(r^{s_1-3}) \quad , \end{aligned}\right\} \; r \to 0 \; , \tag{2.58}$$

where

$$\left.\begin{aligned} Y(\theta) &= \ddot{w}_1 + \kappa w_1 \quad , \\ \kappa &= s_1[(2n-1)s_1 - 2(n-1)] \quad , \end{aligned}\right\} \tag{2.59}$$

and

$$G_1(\theta) = m_1 v_1(\theta) = m_1 k_1 \quad . \tag{2.60}$$

On the other hand, the boundary conditions (2.23) lead to

$$w_1(\pm \tfrac{\pi}{2}) = 0 \quad , \qquad p(r, \pm \tfrac{\pi}{2}) \sim o(r^{2(s_1-1)-m_1}) \; . \tag{2.61}$$

Integrate the first of (2.58) with respect to r to get

$$p \sim \frac{2n\,A\,G_1^{2n-1}}{s_1 - 2} \; Y(\theta) \; r^{s_1-2} \quad , \qquad (s_1 \neq 2) \quad . \tag{2.62}$$

Comparing (2.62) with the second of (2.61), one deduces that

$$Y(\pm \tfrac{\pi}{2}) = 0 \quad . \tag{2.63}$$

The compatability of (2.58),(2.62) together with (2.63) then gives

$$Y(\theta) \equiv 0 \qquad \text{on } [-\pi/2, \pi/2] \quad . \tag{2.64}$$

Equations (2.59) and the first of (2.61) imply

$$w_1(\theta) = B \cos \sqrt{\kappa}\,\theta \quad , \text{ B constant,} \quad -\pi/2 \le \theta \le \pi/2 \quad , \tag{2.65}$$

$$\sqrt{\kappa}\,\frac{\pi}{2} = j\pi \qquad , \quad j = 0, \pm 1, \pm 2, \cdots \quad . \tag{2.66}$$

The second of (2.59) and (2.66) imply that

$$s_1 = \frac{2(n-1) \pm \sqrt{[2(n-1)]^2 + 16(2n-1)\,j^2}}{2(2n-1)} \; ; \quad j = \pm 1, \pm 2, \cdots \; . \tag{2.67}$$

One seeks the smallest value of s_1 satisfying (2.56). This occurs for $j = 1$, so that

$$s_1 = \frac{n-1}{2n-1} + \sqrt{\left(\frac{n-1}{2n-1}\right)^2 + \frac{4}{2n-1}} \quad . \tag{2.68}$$

The asymptotic results for the spatial coordinates deduced this far may be summarized as follows:

$$y_1 \sim k_1\, r^{\frac{2(n-1)}{2n-1}} + B \cos 2\theta\, r^{s_1} \quad , \quad n \neq 1 \quad , \tag{2.69}$$

$$y_2 \sim k_2\, \theta\, r^{\frac{2n}{2n-1}} \quad , \quad n \neq 1 \quad , \tag{2.70}$$

$$p \sim o(r^{s_1 - 2}) \quad , \quad n > 1/2 \quad . \tag{2.71}$$

The case $n = 1$ is treated at the end. Substituting from (2.54) into (1.3), using (2.11),(2.14) gives that

$$J \sim 1 + (k_2 s_1 w_1 - m_2 k_2 \dot{w}_1 \theta)\, r^{s_1 + m_2 - 2} + m_1 k_1 \dot{w}_2\, r^{s_2 + m_1 - 2} + O(r^{s_1 + s_2 - 2}) \; . \tag{2.72}$$

A simple analysis gives

$$s_2 = s_1 + m_2 - m_1 = s_1 + \frac{2}{2n-1} \quad , \tag{2.73}$$

and

$$\dot{w}_2 = \frac{m_2}{m_1}\frac{k_2}{k_1}\theta\,\dot{w}_1 - \frac{k_2 s_1}{k_1 m_1} w_1 \quad , \tag{2.74}$$

which on integration, using (2.65), yields

$$w_2 = \frac{B}{(m_1 k_1)^2}\left\{m_2\theta\cos 2\theta - \frac{1}{2}(m_2 + s_1)\sin 2\theta\right\} \quad . \tag{2.75}$$

For n = 1, we can similarly show that

$$y_1 \sim k_1 \log r + B\cos 2\theta\, r^2 \;, \qquad n = 1 \quad . \tag{2.76}$$

In an effort to find a strong estimate for the pressure field, we now assume the following three-term asymptotic representation for the deformation:

$$y_\alpha \sim v_\alpha r^{m_\alpha} + w_\alpha r^{s_\alpha} + z_\alpha r^{t_\alpha} \quad , \quad (\text{no sum on } \alpha), \tag{2.77}$$

with the stipulation that

$$t_1 > s_1 > m_1 \quad , \qquad t_2 > s_2 > m_2 \quad , \tag{2.78}$$

and z_1, z_2 are functions possessing derivatives of second order on $[-\pi/2, \pi/2]$, which fail to vanish identically and have the same parity as v_1 and v_2.

For n > 1, it can be shown that

$$z_1(\theta) = D\cos 2\sqrt{t_1}\,\theta + k_2^3(\mu_1\theta^2 + \mu_2) \quad , \tag{2.79}$$

and

$$t_1 = 2m_2 - m_1 = \frac{2(n+1)}{2n-1} \quad , \tag{2.80}$$

where

$$\left.\begin{aligned}
D &= \frac{(m_2 + 2\mu_1)\ k_2^3\ \frac{\pi}{2}}{2\sqrt{t_1}\ \sin\sqrt{t_1}\,\pi} \quad , \qquad 1 < \sqrt{t_1} < 2 \ , \\
\mu_1 &= \frac{1}{8t_1}\left[(2 - 3m_1)(m_2 - m_1) - 4m_2\right] m_2 \quad , \\
\mu_2 &= \frac{1}{4t_1}\left[\nu - 2(m_1 + \mu_1)\right] \quad , \\
\nu &= (2 - 3m_1)\left[1 - (m_2 - m_1)\ \frac{m_2}{2}\ (\tfrac{\pi}{2})^2\right] \quad ,
\end{aligned}\right\} \tag{2.81}$$

and we arrive at a strong estimate for the pressure field,

$$p \sim 2n\,A\,G_1^{2(n-1)}\ k_2^2\{(m_2 - m_1)\ \frac{m_2}{2}\ [\theta^2 - (\tfrac{\pi}{2})^2] + 1\}\ r^{\frac{2(2-n)}{2n-1}} \quad , \quad n > 1 \ . \tag{2.82}$$

For <u>$n < 1$</u>, we can show that

$$z_1(\theta) = B^2(\hat{\mu}_1 \cos 4\theta - \hat{\mu}_2) \quad , \tag{2.83}$$

and

$$t_1 = 2s_1 - m_1 \quad , \tag{2.84}$$

where

$$\hat{\mu}_1 = -\frac{1}{4}\,(2n-1)t_1 s_1 k_2 \ , \qquad \hat{\mu}_2 = \frac{1}{2}\,(n-1)s_1 k_2 \quad , \tag{2.85}$$

but for the pressure field we have only the weak estimate

$$p \sim o(r^{2s_1 - m_1 - 2}) \tag{2.86}$$

At this stage we can determine z_2, t_2 from condition (1.3); however, it is not necessary to record the results here.

To find a strong estimate for the pressure field for $\underline{n < 1}$ we assume the deformation admits the representation

$$y_\alpha \sim v_\alpha r^{m_\alpha} + w_\alpha r^{s_\alpha} + z_\alpha r^{t_\alpha} + q_\alpha r^{\ell_\alpha} \quad , \text{ (no sum on } \alpha), \tag{2.87}$$

with the stipulation that

$$\ell_1 > t_1 > s_1 > m_1 \quad , \qquad \ell_2 > t_2 > s_2 > m_2 \quad ,$$

$$q_\alpha \in C^2([-\pi/2, \pi/2]) \ , \qquad q_\alpha \equiv 0 \quad \text{on } [-\pi/2, \pi/2] \quad .$$

We can now show that

$$\ell_1 = 2m_2 - m_1 \quad , \qquad 7/12 < n < 1 \quad , \tag{2.88}$$

and that $q_1(\theta)$ is given by the value of $z_1(\theta)$ in (2.79). Thus, we can see a trade in dominance between the third and fourth term of (2.87), for $\alpha = 1$, as n passes through $n = 1$. Condition (1.3) is again used to determine ℓ_2 and q_2. For $7/12 < n < 1/2$, equation (2.82) is found to give a strong estimate for the pressure field. The value $n = 7/12$ is a transition point for the pressure field. A strong estimate for n in the range (1/2, 7/12) requires much further analysis.

At this point, we will record the results for n = 1.

$$\left.\begin{aligned} y_1 \sim {} & k_1 \log r + B\cos 2\theta \; r^2 + \frac{3}{8} k_2^3 \cos 4\theta \; r^4 \log r \\ & + [E\cos 4\theta - \frac{k_3^2}{16}(3\theta \sin 4\theta + 4\theta^2 + \pi^2 - \frac{5}{2})] r^4 \quad , \end{aligned}\right\}$$

$$\left.\begin{aligned} y_2 &\sim k_2\theta\, r \qquad , \\ p &\sim Ak_2^2(4\theta^2 - \pi^2 + 2)\, r^2 \qquad , \end{aligned}\right\} \quad (2.89)$$

where B and E are constants.

D. Summary of Results for Deformation and Stresses

The asymptotic results for the spatial coordinates and pressure field may be summarized as follows:

$$\left.\begin{aligned}
y_1 &\sim k_1\, r^{\frac{2(n-1)}{2n-1}} + B\cos 2\theta\, r^{s_1} + B^2(\hat{\mu}_1\cos 4\theta - \hat{\mu}_2)r^{2s_1-m_1} \quad , \ \tfrac{1}{2} < n \le \tfrac{7}{12} \ , \\
y_1 &\sim k_1\, r^{\frac{2(n-1)}{2n-1}} + B\cos 2\theta\, r^{s_1} + B^2(\hat{\mu}_1\cos 4\theta - \hat{\mu}_2)r^{2s_1-m_1} \\
&\qquad + [D\cos 2\sqrt{t_1}\,\theta + k_2^3(\mu_1\theta^2 + \mu_2)]r^{4-3m_1} \quad , \quad \tfrac{7}{12} < n < 1 \quad , \\
y_1 &\sim k_1^{\frac{2(n-1)}{2n-1}} + B\cos 2\theta\, r^{s_1} + [D\cos 2\sqrt{t_1}\,\theta + k_2^3(\mu_1\theta^2 + \mu_2)]r^{4-3m_1} \quad , \ n > 1 \ , \\
y_2 &\sim k_2\theta\, r^{\frac{2n}{2n-1}} + w_2(\theta)\, r^{s_1 + \frac{2}{2n-1}} \quad , \quad n \neq 1 \\
p &\sim \frac{F}{\pi}k_2^3\left\{\frac{2n}{(2n-1)^2}\left[\theta^2 - \left(\frac{\pi}{2}\right)^2\right] + 1\right\} r^{\frac{2(2-n)}{2n-1}} \quad , \quad \tfrac{7}{12} < n < \infty \ , \ n \neq 1 \ ,
\end{aligned}\right] \quad (2.90)$$

where k_1, k_2 are given in (2.44),(2.46); $w_2(\theta)$ in (2.75); s_1 and t_1 in (2.68),(2.80); D, μ_1, μ_2 in (2.81); and $\hat{\mu}_1, \hat{\mu}_2$ in (2.85). B, a constant, is left undetermined by the local analysis.

We now turn to the asymptotic determination of the actual stresses $\tau_{\alpha\beta}$. From (1.13),(2.17),(2.8),(2.9),(2.41)-(2.44), and (2.35), one finds

$$\tau_{11} \sim \frac{F}{\pi} (m_1 k_1)\, r^{\frac{-2n}{2n-1}} \quad , \qquad n \neq 1 \quad , \tag{2.91}$$

$$\tau_{12} = \tau_{21} \sim \frac{F}{\pi} (m_2 k_2)\, r^{\frac{-2(n-1)}{2n-1}} \theta, \ n \neq 1 \quad , \tag{2.92}$$

and using (2.82),

$$\tau_{22} \sim \frac{F}{\pi} m_2 k_2^3 \{\theta^2 + \frac{1}{2n-1} (\frac{\pi}{2})^2\} \, r^{\frac{2(2-n)}{2n-1}} \quad , \ \frac{7}{12} < n < \infty \ , \quad n \neq 1 \ . \tag{2.93}$$

The true stresses here are referred to the material polar coordinates (r,θ). The stress component τ_{11}, which is of primary physical interest, becomes unbounded at the origin for all admissible values of the hardening parameter, the singularity becoming more severe with decreasing values of n, and for the range of n under consideration is always stronger than that predicted by the linear theory. The other normal stress component τ_{22} remains bounded for $n \leq 2$, but for $n > 2$ it becomes unbounded, while the actual shearing stress τ_{12} is bounded for $n < 1$, and becomes unbounded at the origin for hardening materials ($n > 1$). In both stresses τ_{22}, τ_{12}, the severity of the singularity increases with increasing values of n; however, the order of the singularity is less than that predicted by the linear theory for all allowable n.

For $n = 1$ we have, in summary, that

$$\left.\begin{aligned} y_1 &\sim k_1 \log r + B \cos 2\theta\, r^2 + \frac{3}{8} k_2^3 \cos 4\theta \; r^4 \log r \\ &\qquad + [E \cos 4\theta + \frac{k_2^3}{16} (6\theta \sin 4\theta + 4\theta^2 + \pi^2 - \frac{5}{2})] \quad , \\ y_2 &\sim k_2 \theta\, r^2 + 2k_2^2 B\, (\theta \cos \theta - \sin \theta)\, r^4 \quad , \end{aligned}\right] \tag{2.94}$$

$$p \sim Ak_2^2(4\theta^2 - \pi^2 + 2)\, r^2 \quad .$$

Here, $k_1 = 1/k_2 = \frac{F}{2A\pi}$, while B and E are constant, undetermined by the local analysis. The components of the actual stress tensor are, using (1.13), (2.17), (2.8), (2.9), (2.50), and (2.35),

$$\tau_{11} \sim \frac{F}{\pi} k_1 r^{-2} \quad , \qquad \tau_{22} \sim \frac{F}{\pi} 2k_2^3\{\theta^2 + (\tfrac{\pi}{2})^2\} r^2 \quad ,$$
$$\tau_{12} = \tau_{21} \sim \frac{F}{\pi}(2k_2)\,\theta \quad . \tag{2.95}$$

At this stage, a comparison of the results given in (2.94),(2.95) with those predicted by the classical linear theory reveals the presence of a $\log r$ singularity in the dominant term of the deformation in the x_1 direction in both cases. This is the only similarity! According to linear theory all the components of the stress tensor possess a $1/r$ singularity at the origin, but from (2.95) we see the stress component τ_{11} is more singular, while the other components are, in fact, bounded there.

3. A CLASS OF PROBLEMS FOR AN ELASTIC SECTOR

A. Formulation of the Problem

We now turn to a different class of problems in finite plane elastostatics. Let Π^* denote the deformation image of the cross-section Π of the undeformed body. We assume that Π^* is an infinite sector whose closure $\bar{\Pi}^*$ occupies the region $R \geq 0$, $\beta^- \leq \phi \leq \beta^+$, where R and ϕ are the polar coordinates of a point in the deformed state (see Figure 3):

$$y_1 = R\cos\phi \ , \quad y_2 = R\sin\phi \ , \quad \beta^- \leq \phi \leq \beta^+ \ , \quad R \geq 0 \ . \tag{3.1}$$

We suppose that the boundaries $\Gamma^{*(\pm)}$: $\phi = \beta^{(\pm)}$ $R \geq 0$ of Π^* are acted upon by prescribed distributions of true traction $\underset{\sim}{t}^{(\pm)}$:

$$\text{on } \Gamma^{*(\pm)}: \ \tau_{\alpha\gamma}\, n_\gamma = t_\alpha^{(\pm)}(R) \quad , \qquad 0 \leq R < \infty \quad , \tag{3.2}$$

where n is the unit outward normal on $\Gamma^{*(\pm)}$. It is assumed that the given functions $t_\alpha^{(\pm)}$ vanish for all sufficiently large R, and that they are continuously differentiable for $R \geq 0$. Moreover, it is required that

$$\int_0^\infty t_\alpha^{(\pm)}(R)\, dR = 0 \qquad , \tag{3.3}$$

and that

$$\int_0^\infty \varepsilon_{\alpha\gamma}\, y_\gamma\, t_\alpha^{(\pm)}(R)\, dR = 0 \ . \tag{3.4}$$

Conditions (3.3) and (3.4) express the fact that $t_\alpha^{(\pm)}$ contribute no resultant force or moment, respectively, to the overall force balance on $\bar{\Pi}^*$.

We suppose that the deformation which carries the undeformed cross-section $\bar{\Pi}$ to $\bar{\Pi}^*$ is such that the vertex of $\bar{\Pi}^*$ has remained fixed and hence is a point of the boundary of $\bar{\Pi}$ as well. Let $\mathring{\bar{\Pi}}$ stand for $\bar{\Pi}$ with the origin deleted. We seek a deformation $\hat{\underset{\sim}{y}}(x)$ which is continuous on $\bar{\Pi}$, twice continuously differentiable on $\mathring{\bar{\Pi}}$, and for which the deformation gradient tensor $\underset{\sim}{F}$ is bounded near the origin. We suppose that the true stress tensor $\underset{\sim}{\tau}$ associated with $\hat{\underset{\sim}{y}}$ through (1.13) satisfies the equilibrium equations (1.8), the boundary conditions (3.2), and the condition

$$\tau_{\alpha\beta} = o(R^{-1}) \text{ as } R \to \infty \text{ , uniformly in } \phi, \text{ for } \beta^- \leq \phi \leq \beta^+ \text{ .} \qquad (3.5)$$

Condition (3.5) assures that no resultant force or moment is applied to Π^* "at infinity."

We are interested in the local structure of such an elastostatic field near $R = 0$. To study this question, it is natural to investigate possible elastostatic fields in the sector Π^* for which the associated true tractions applied to the rays $\phi = \beta^{(\pm)}$ have the <u>constant</u> values $\underset{\sim}{t}^{(\pm)}(0)$, and for which the associated deformation satisfies the smoothness requirements and the field equations spelled out above, but for which no conditions are specified at infinity. Such elastostatic fields would presumably include among them one which describes the local behavior near $R = 0$ of the solution to the global problem described above. We thus inquire into the existence of such local fields. For this purpose, we consider now the case in which the undeformed cross-section $\bar{\Pi}$ is a sector $r \geq 0$, $-\alpha \leq \theta \leq \alpha$, where r,θ are polar coordinates in the undeformed state:

$$x_1 = r\cos\theta \quad , \quad x_2 = r\sin\theta \quad , \quad r \geq 0 \text{ , } -\alpha \leq \theta \leq \alpha \quad . \qquad (3.6)$$

We shall study deformations of this form:

$$y_\alpha = r^{m_\alpha} w_\alpha(\theta) \ , \qquad m_\alpha \geq 1 \ , \qquad \text{(no sum on } \alpha) \ , \tag{3.7}$$

and we shall show that, under certain circumstances, such deformations carry the sector Π to a sector Π^* whose boundary rays carry uniform tractions. The opening angle $2\beta(=\beta^- + \beta^+)$ of Π^* is then related to the opening angle 2α of Π via the deformation.

For our purposes it is more convenient to work with the normal and tangential components of the applied tractions. Thus we set

$$\overline{N}_1 = t_\alpha^{(+)}(0)\, n_\alpha \qquad , \qquad \overline{N}_2 = t_\alpha^{(-)}(0)\, n_\alpha \quad , \tag{3.8}$$

$$\overline{S}_1 = t_\alpha^{(+)}(0)\, e_\alpha(\beta^+) \quad , \qquad \overline{S}_2 = t_\alpha^{(-)}(0)\, e_\alpha(\beta^-), \tag{3.9}$$

where $\underset{\sim}{n}$ is the outward normal on $\Gamma^{*(\pm)}$, and $\underset{\sim}{e}(\phi)$ is the radial unit vector associated with the polar coordinates R,ϕ.

We turn now to the analysis of such local fields.

B. Analysis

Since the material is assumed to be incompressible, substituting from (3.7) into (1.3), using (1.2) gives

$$J = (m_1 w_1 \dot{w}_2 - m_2 w_2 \dot{w}_1)\, r^{m_1 + m_2 - 2} = 1 \quad , \tag{3.10}$$

from which we find, recalling $m_\alpha \geq 1$, that

$$m_1 = m_2 = 1 \quad , \tag{3.11}$$

for otherwise $J \to 0$ as $r \to 0$. Let u_r and u_θ be the radial and circumferential components of the displacement field, then

$$u_1 = c_1 u_r + c_2 u_\theta \quad , \qquad u_2 = c_2 u_r + c_1 u_\theta \quad , \tag{3.12}$$

where u_α are given in (1.1) and

$$c_1 = \cos\theta \quad , \qquad c_2 = \sin\theta \quad . \tag{3.13}$$

Letting

$$v_1 = c_1 w_1 + c_2 w_2 \quad , \qquad v_2 = c_1 w_2 - c_2 w_1 \quad , \tag{3.14}$$

one finds using (3.7), (3.11)-(3.14), and (1.1) that

$$u_r = (v_1 - 1)\, r \quad , \qquad u_\theta = v_2\, r \quad , \tag{3.15}$$

and the incompressibility constraint (3.10) is now

$$v_1^2 + v_2^2 + v_1 \dot{v}_2 - \dot{v}_1 v_2 = 1 \quad . \tag{3.16}$$

From (3.7), (3.11)-(3.16), we get

$$\begin{aligned} R^2 &= y_1^2 + y_2^2 = (v_1^2 + v_2^2)\, r^2 \equiv R(r,\theta) \quad , \\ \phi &= \tan^{-1} \frac{y_2}{y_1} = \tan^{-1} \left\{ \frac{v_1 c_2 + v_2 c_1}{v_1 c_1 - v_2 c_2} \right\} \equiv \phi(\theta) \quad . \end{aligned} \tag{3.17}$$

From the second of (3.17) we see that an undeformed sector Π of opening angle 2α is deformed to Π^* which is also a sector of opening angle 2β where

$$2\beta = \phi(\alpha) + \phi(-\alpha) \ , \qquad \phi(\pm\alpha) = \beta^{(\pm)} \ . \tag{3.18}$$

It is necessary to assume that the hydrostatic pressure field which occurs in the constitutive relationship (1.13) is of the form

$$p(r,\theta) = r^{\ell}\, q(\theta) \quad , \tag{3.19}$$

where $q(\theta) \in C^2([-\alpha,\alpha])$, $q(\theta) \not\equiv 0$ on $[-\alpha,\alpha]$, and ℓ is a constant to be determined. In light of the boundary conditions (3.8),(3.9) and the second of (1.10), one finds on transforming the stress-deformation relationship (1.13) into polar coordinates that in fact

$$\ell = 0 \tag{3.20}$$

and then

$$\begin{bmatrix} \tau_{RR} & \tau_{R\phi} \\ \tau_{R\phi} & \tau_{\phi\phi} \end{bmatrix} = \frac{2W'(I)}{v_1^2 + v_2^2} \begin{bmatrix} (v_1^2 + v_2^2)^2 + (v_1\dot{v}_1 + v_2\dot{v}_2)^2 & v_1\dot{v}_1 + v_2\dot{v}_2 \\ v_1\dot{v}_1 + v_2\dot{v}_2 & 1 \end{bmatrix} - q\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \ , \tag{3.21}$$

using (3.7), (3.11)-(3.16), (3.19), and (3.20). The components of the true stress are given as functions of the material polar coordinates r,θ. Computing the components of the deformation gradient tensor defined in (1.2), using (3.7), (3.11)-(3.16), enables us to find the scalar invariant I defined in (1.5) as

$$I = 2 + \dot{v}_1^2 + \dot{v}_2^2 \tag{3.22}$$

which is independent of r. A particular class of strain energy functions which has the property laid down in (1.32) is

$$W(I) = A(I-2)^n \quad ; \quad A > 0 \, , \quad 1/2 < n < \infty \quad . \tag{3.23}$$

For n = 1, the strain energy function in (3.23) reduces to that of a neo-Hookean material, and except for n = 1, the stress-strain curve has no linear range.

The components of the nominal stress tensor are found, using (3.7), (3.11)-(3.15) in (1.14) and the appropriate transformation law from cartesian to polar coordinates, to be

$$\begin{bmatrix} \sigma_{rr} & \sigma_{r\theta} \\ \sigma_{\theta r} & \sigma_{\theta\theta} \end{bmatrix} = 2nAG^{n-1} \begin{bmatrix} v_1 & \dot{v}_1 - v_2 \\ v_2 & \dot{v}_2 + v_1 \end{bmatrix} - q \begin{bmatrix} \dot{v}_2 + v_1 & -v_2 \\ -\dot{v}_1 + v_2 & v_1 \end{bmatrix} \, , \tag{3.24}$$

where

$$G(\theta) = I - 2 = \dot{v}_1^2 + \dot{v}_2^2 > 0 \quad . \tag{3.25}$$

The nominal stress equations of equilibrium (1.9) in polar coordinates, in the absence of body forces, are

$$\frac{\partial \sigma_{r\theta}}{\partial \theta} + \sigma_{rr} - \sigma_{\theta\theta} = 0 \, , \qquad \frac{\partial \sigma_{\theta\theta}}{\partial \theta} + \sigma_{r\theta} + \sigma_{\theta r} = 0 \, , \tag{3.26}$$

since $\underset{\sim}{\sigma}$ is independent of r. Substituting from (3.24) into (3.26) yields the pair of equations,

$$2nA[G^{n-1}(\ddot{v}_1 - 2\dot{v}_2) + (n-1)\,G^{n-2}\,\dot{G}(\dot{v}_1 - v_2)] + \dot{q}v_2 = 0 \quad ,$$
$$2nA[G^{n-1}(\ddot{v}_2 + 2\dot{v}_1) + (n-1)\,G^{n-2}\,\dot{G}(\dot{v}_2 + v_1)] - \dot{q}v_1 = 0 \quad , \tag{3.27}$$

which with (3.16) give three equations for the unknown functions v_1, v_2, and q. Eliminate q from (3.27), using (3.16), to find

$$G\{v_1(\ddot{v}_1 - 2\dot{v}_2) + v_2(\ddot{v}_2 + 2\dot{v}_1)\} + (n-1)\,\{v_1\dot{v}_1 + v_2\dot{v}_2\} = 0 \quad , \tag{3.28}$$

and

$$\dot{q} = 2n(n-1)\,A\,\frac{G^{n-2}\,\dot{G}}{v_1^2 + v_2^2} \quad . \tag{3.29}$$

To solve for v_1 and v_2 from (3.16) and (3.28), introduce polar coordinates in the $v_1 - v_2$ plane:

$$v_1(\theta) = \xi(\theta)\sin\psi(\theta) \quad , \qquad v_2(\theta) = \xi(\theta)\cos\psi(\theta) \quad , \tag{3.30}$$

then (3.16) and (3.28) give

$$\xi^2(1 - \dot{\psi}) = 1 \quad , \tag{3.31}$$

$$(\dot{\xi}^2 + \xi^2\dot{\psi})\;\{\ddot{\xi} - \xi\dot{\psi}^2 + 2\xi\dot{\psi}\} + 2(n-1)\{\dot{\xi}(\ddot{\xi} + \xi\dot{\psi}^2) + \xi^2\dot{\psi}\ddot{\psi}\}\dot{\xi} = 0 \; . \tag{3.32}$$

Substitute from (3.31) into (3.32) to get

$$\{(2n-1)\dot{\xi}^2 + \xi^2\dot{\psi}^2\}\;\{\ddot{\xi} - \xi\dot{\psi}^2 + 2\xi\dot{\psi}\} = 0 \quad , \tag{3.33}$$

the first term of which is strictly positive, since n > 1/2, therefore one has

$$\ddot{\xi} - \xi\dot{\psi}^2 + 2\xi\dot{\psi} = 0 \quad , \tag{3.34}$$

for all $n > 1/2$. The pair (3.16) and (3.28) are easily solved for $n = 1$ and one readily verifies that this solution indeed satisfies these equations for all $n > 1/2$. We find that

$$v_1(\theta) = B \sin 2\theta - C \cos 2\theta + \frac{k_1}{2} \quad , \tag{3.35}$$

$$v_2(\theta) = B \cos 2\theta + C \sin 2\theta - \frac{k_2}{2} \quad , \tag{3.36}$$

$$q(\theta) = q_o \quad , \tag{3.37}$$

where B, C, k_1, k_2, and q_o are constants, which, because of (3.16), are required to satisfy

$$\frac{k_1^2 + k_2^2}{4} = B^2 + C^2 + 1 \quad . \tag{3.38}$$

One easily verifies now that, in fact, the true stress equations of equilibrium (1.8) are satisfied on Π^*, in the absence of body forces.

We now turn to the boundary conditions (3.8) and (3.9) to determine the unknown constants. A simple computation based on (3.8),(3.9) and the second of (1.10) gives that

$$\begin{aligned} &\tau_{R\phi}(R,\phi(+\alpha)) = \overline{S}_1 \quad ; \qquad \tau_{R\phi}(R,\phi(-\alpha) = \overline{S}_2 \; ; \\ &\tau_{\phi\phi}(R,\phi(+\alpha)) = \overline{N}_1 \quad ; \qquad \tau_{\phi\phi}(R,\phi(-\alpha) = \overline{N}_2 \; . \end{aligned} \tag{3.39}$$

so from (3.21), (3.23), (3.25), (3.35)-(3.37), (3.39) we find,

$$\Lambda(\nu_1,\nu_2)\,\frac{\nu_1\sin 2\alpha + \nu_2\cos 2\alpha}{\mu - \nu_1\cos 2\alpha + \nu_2\sin 2\alpha} = S_1 \quad , \tag{3.40}$$

$$\Lambda(\nu_1,\nu_2)\,\frac{-\nu_1\sin 2\alpha + \nu_2\cos 2\alpha}{\mu - \nu_1\cos 2\alpha - \nu_2\sin 2\alpha} = S_2 \quad , \tag{3.41}$$

$$\frac{\Lambda(\nu_1,\nu_2)}{\mu - \nu_1\cos 2\alpha + \nu_2\sin 2\alpha} - \frac{q_o}{2A} = N_1 \quad , \tag{3.42}$$

$$\frac{\Lambda(\nu_1,\nu_2)}{\mu - \nu_1\cos 2\alpha - \nu_2\sin 2\alpha} - \frac{q_o}{2A} = N_2 \quad , \tag{3.43}$$

where

$$2A\,N_\gamma = \overline{N}_\gamma \quad , \quad 2A\,S_\gamma = \overline{S}_\gamma \quad , \quad (\gamma = 1,2) \quad , \tag{3.44}$$

$$\nu_1 = k_1 C + k_2 B \ , \qquad \nu_2 = k_1 B - k_2 C \ , \tag{3.45}$$

$$\Lambda(\nu_1,\nu_2) = n[2(\sqrt{\nu_1^2 + \nu_2^2 + 1} - 1)]^{n-1} > 0 \quad , \tag{3.46}$$

while (3.38) and (3.45) imply

$$\mu^2 = \nu_1^2 + \nu_2^2 + 1 \quad . \tag{3.47}$$

For all n, the cases $\alpha = \pi/4, \pi/2, 3\pi/4$, are given separate treatment at the end. For $\underline{n = 1}$ from (3.46) one has $\Lambda = 1$, and the equations (3.40)-(3.43) can be solved readily. After some algebraic manipulation, one finds

$$\nu_1 = \frac{(S_2 - S_1)\cot 2\alpha - 2S_1S_2}{\pm 2\sqrt{-\left\{(S_2-S_1)\cos 2\alpha - S_1S_2\sin 2\alpha + \frac{\cos^2 2\alpha}{\sin 2\alpha}\right\}\left\{(S_2-S_1)\cos 2\alpha - (S_1S_2+1)\sin 2\alpha\right\}}}, \tag{3.48}$$

$$\nu_2 = \frac{-(S_1 + S_2)}{\{(S_2-S_1)\cot 2\alpha - 2S_1S_2\}}\,\nu_1 \quad . \tag{3.49}$$

Since (3.40)-(3.43),(3.47) are five equations in four unknowns μ, ν_1, ν_2, and q_o, one has the following constraint on the prescribed tractions,

$$N_2 - N_1 = \pm(S_1 + S_2)\sqrt{\frac{\{(S_2 - S_1)\cos 2\alpha - (S_1S_2 + 1)\sin 2\alpha\}}{-\left\{(S_2-S_1)\cos 2\alpha - S_1S_2\sin 2\alpha + \frac{\cos^2 2\alpha}{\sin 2\alpha}\right\}}} \quad . \tag{3.50}$$

From equations (3.40)-(3.43), we can see that when

$$N_2 = N_1 , \qquad \text{then} \quad S_1 = -S_2 \quad , \tag{3.51}$$

This case is given separate attention, so assume here that $N_2 - N_1 \neq 0$, $S_1 + S_2 \neq 0$. Solving for q_o, after some manipulation, we find

$$\frac{q_o}{2A} = \frac{(N_2 - N_1)\cot 2\alpha - (N_2S_1 + N_1S_2)}{(S_1 + S_2)} \tag{3.52}$$

Thus the components of the <u>true</u> stress field are

$$\tau_{RR}(r,\theta) = 2A\left[2\mu - \frac{1}{\mu + \nu_2\sin 2\theta - \nu_1\cos 2\theta}\right] - q_o \quad ,$$

$$\tau_{R\phi}(r,\theta) = 2A\left[\frac{\nu_1\sin 2\theta + \nu_2\cos 2\theta}{\mu + \nu_2\sin 2\theta - \nu_1\cos 2\theta}\right] , \tag{3.53}$$

$$\tau_{\phi\phi}(r,\theta) = 2A\left[\frac{1}{\mu + \nu_2\sin 2\theta - \nu_1\cos 2\theta}\right] - q_o .$$

Thus the prescribed N_1, N_2, S_1, S_2 and the opening angle 2α, must be such that traction constraint (3.50) is satisfied, then with (3.48),(3.49) in (3.53) the true stress field is fully determined. The displacement field, which is determined to within a rigid displacement, follows from (3.35),(3.36) on solving (3.38),(3.45) for B, C, k_1, and k_2.

For $n \neq 1$, equations (3.40)-(3.43) are algebraically nonlinear in the unknowns ν_1 and ν_2. Eliminating ν_1 and ν_2, we find a nonlinear algebraic equation for $q_o/2A$,

$$\delta_o + \delta_1\left(\frac{q_o}{2A}\right) = n\left[\frac{\beta_2\left(\frac{q_o}{2A}\right)^2 + \beta_1\left(\frac{q_o}{2A}\right) + \beta_o}{\left(\frac{q_o}{2A} + N_1\right)\left(\frac{q_o}{2A} + N_2\right)}\right]^{n-1} > 0 \quad , \tag{3.54}$$

where

$$\left.\begin{aligned}
\delta_o &= \frac{N_1S_2 + N_2S_1}{N_2 - N_1}\tan 2\alpha , & \delta_1 &= \frac{S_1 + S_2}{N_2 - N_1}\tan 2\alpha , \\
\gamma_o &= N_1N_2 , & \gamma_1 &= N_1 + N_2 \\
\beta_o &= (N_2S_1 - N_1S_2)\cot 2\alpha + \gamma_1\delta_o - \gamma_o , & & \\
\beta_1 &= (S_1 - S_2)\cot 2\alpha + \gamma_1\delta_1 + 2\delta_o - \gamma_1 ; & \beta_2 &= 2\delta_1 - 1 .
\end{aligned}\right\} \tag{3.55}$$

In general there is more than one value of q_o which satisfies (3.54). Suppose q_* is an admissible value of q_o, then Λ, as defined in (3.46) equals the right-hand side of (3.54), so

$$\Lambda = \delta_o + \delta_1 q_* \quad . \tag{3.56}$$

Defining

$$\tilde{S}_\gamma = \frac{S_\gamma}{\Lambda} \quad , \qquad \tilde{N}_\gamma = \frac{N_\gamma}{\Lambda} \quad , \tag{3.57}$$

we can write equations (3.40)-(3.43), after eliminating q_o, as

$$\left.\begin{aligned} \frac{\nu_1 \sin 2\alpha + \beta_2 \cos 2\alpha}{\mu - \nu_1 \cos 2\alpha + \nu_2 \sin 2\alpha} &= \tilde{S}_1 \quad , \\ \frac{-\nu_1 \sin 2\alpha + \nu_2 \cos 2\alpha}{\mu - \nu_1 \cos 2\alpha - \nu_2 \sin 2\alpha} &= \tilde{S}_2 \quad , \\ \frac{2\nu_2 \sin 2\alpha}{(\mu - \nu_1 \cos 2\alpha)^2 - \nu_2 \sin^2 2\alpha} &= \tilde{N}_2 - \tilde{N}_1 \quad , \end{aligned}\right\} \tag{3.58}$$

from which we find ν_1 and ν_2 as in (3.48),(3.49) and a constraint as in (3.50), all with N_γ, S_γ replaced by $\tilde{N}_\gamma, \tilde{S}_\gamma$. From (3.57) this constraint can be written,

$$N_2 - N_1 = \pm(S_1 + S_2)\sqrt{\frac{-\{(S_2 - S_1)\Lambda \cos 2\alpha - (S_1 S_2 + \Lambda^2) \sin 2\alpha\}}{\{(S_2 - S_1)\Lambda \cos 2\alpha - S_1 S_2 \sin 2\alpha + \Lambda^2 \frac{\cos^2 2\alpha}{\sin 2\alpha}\}}} \quad . \tag{3.59}$$

Thus the true stress can be completely determined if we can find q_*. Let us briefly address the important question of when an admissible value of q_* exists and if it does, is it unique for a given set of prescribed tractions and opening angles? Because of the Baker-Ericksen inequality

(1.16), from (3.54) admissible values of q_o are such that

$$q_o > -\frac{\delta_o}{\delta_1} \quad . \tag{3.60}$$

It can be readily shown that

$$q_o \notin [-N_1, -N_2] \quad , \tag{3.61}$$

in order that the right-hand side of (3.61) is real; further, the interval

$$[I^-, I^+] \quad , \tag{3.62}$$

where

$$I^{\pm} = -\frac{\delta_o}{\delta_1} + \frac{(N_2 - N_1)(S_2 - S_1)\cot^2 2\alpha}{2(S_1 + S_2)}\left[1 \pm \sqrt{1 - \frac{4S_1 S_2 \tan^2 2\alpha}{(S_2 - S_1)^2}}\right], \tag{3.63}$$

must be excluded from the range of admissible q_o for the same reason.

From (3.54) define

$$y_T(q_o/2A) = \beta_2\left(\frac{q_o}{2A}\right)^2 + \beta_1\left(\frac{q_o}{2A}\right) + \beta_o \quad , \tag{3.64}$$

then, one finds that

$$\begin{aligned} y_T(-N_1) &= (N_2 - N_1)\, S_1 \sec^2 2\alpha \cot 2\alpha \quad , \\ y_T(-N_2) &= (N_2 - N_1)\, S_2 \sec^2 2\alpha \cot 2\alpha \quad . \end{aligned} \tag{3.65}$$

From (3.65) and (3.54) we find, on close examination, the following subcases should be considered:

I: $\quad \mathrm{sgn}(S_1) = -\mathrm{sgn}(S_2)$

II: $\quad \mathrm{sgn}(S_1) = \mathrm{sgn}(S_2)$

and that for $n > 1$ and $1/2 < n < 1$ different behavior is expected from the right-hand side of (3.54). With a certain amount of analysis one can now say that, for example, when $1 < n < \infty$ in case II, if

$$(N_2 - N_1)(S_1 + S_2) \tan 2\alpha > 0 \tag{3.66}$$

then indeed, a unique value of q_* exists. However, one cannot find a simple condition or a collection of conditions to guarantee the existence of a unique admissible q_*. The problem, posed in a numerical setting is not as bad; there with a quite simple program one could easily determine q_*, if it exists, for a given set N_1, N_2, S_1, S_2, and 2α. In the examples looked at it was found that a value of q_* did not always exist, but when it existed it was unique.

We now turn to the special cases mentioned earlier. First consider $\alpha = \pi/4$, then boundary conditions (3.40)-(3.43) reduce to

$$\begin{aligned} &\Lambda(\nu_1,\nu_2)\,\frac{\nu_1}{\mu+\nu_2} = S_1 \quad , \qquad \Lambda(\nu_1,\nu_2) = \frac{S_2}{\mu-\nu_2} \quad ; \\ &\frac{\Lambda(\nu_1,\nu_2)}{\mu+\nu_2} - \frac{q_0}{2A} = N_1 \quad , \qquad \frac{\Lambda(\nu_1,\nu_2)}{\mu-\nu_2} - \frac{q_0}{2A} = N_2 ; \end{aligned} \tag{3.67}$$

from which we easily find that

$$\frac{q_0}{2A} = -\,\frac{N_2 S_1 + N_1 S_2}{S_1 + S_2} \quad , \tag{3.68}$$

and that

$$\nu_1 = -\frac{(S_1 + S_2)}{N_2 - N_1} \quad , \qquad \nu_2 = \sqrt{\frac{1 + \nu_1^2}{\left(\frac{S_2 - S_1}{S_1 + S_2}\right)^2 - 1}} \quad , \tag{3.69}$$

while the traction constraint becomes

$$N_2 - N_1 = \pm(S_1 + S_2)\sqrt{\frac{1 + S_1 S_2}{1 - S_1 S_2}} \quad . \tag{3.70}$$

The true stress field is given by (3.53) with ν_1, ν_2, and q_o given by (3.68),(3.69) together with (3.47). For $\alpha = 3\pi/4$, the analysis is similar.

For $\alpha = \pi/2$, the boundary conditions (3.40)-(3.43) become

$$\begin{aligned} -\Lambda(\nu_1,\nu_2)\frac{\nu_2}{\mu + \nu_1} &= S_1 \quad , \qquad -\Lambda(\nu_1,\nu_2)\frac{\nu_2}{\mu + \nu_1} = S_2 \quad , \\ \frac{\Lambda(\nu_1,\nu_2)}{\mu + \nu_1} - \frac{q_o}{2A} &= N_1 \quad , \qquad \frac{\Lambda(\nu_1,\nu_2)}{\mu + \nu_1} - \frac{q_o}{2A} = N_2 \quad , \end{aligned} \tag{3.71}$$

from which we see that

$$S_1 = S_2 = S \text{ , say ,} \qquad N_1 = N_2 = N \text{ , say.}$$

Thus we have three equations in four unknowns ν_1, ν_2, μ and q_o, two in (3.71) and (3.47). This leaves one unknown arbitrary and thus the resulting solution is not unique.

For $N_1 = N_2 = N$, say, and $S_1 = -S_2 = S$, say, the boundary conditions (3.40)-(3.43) give

$$\Lambda(\nu_1,\nu_2)\,\frac{\nu_1\sin 2\alpha + \nu_2\cos 2\alpha}{\mu - \nu_1\cos 2\alpha + \nu_2\sin 2\alpha} = S \quad , \tag{3.72}$$

$$\Lambda(\nu_1,\nu_2)\,\frac{-\nu_1\sin 2\alpha + \nu_2\cos 2\alpha}{\mu - \nu_1\cos 2\alpha - \nu_2\sin 2\alpha} = -S \;, \tag{3.73}$$

$$\frac{\Lambda(\nu_1,\nu_2)}{\mu - \nu_1\cos 2\alpha + \nu_2\sin 2\alpha} - \frac{q_o}{2A} = N \quad , \tag{3.74}$$

$$\frac{\Lambda(\nu_1,\nu_2)}{\mu - \nu_1\cos 2\alpha - \nu_2\sin 2\alpha} - \frac{q_o}{2A} = N \quad , \tag{3.75}$$

From (3.74),(3.75) we find

$$\nu_2 = 0 \quad , \tag{3.76}$$

and (3.72)-(3.75) and (3.47) now become

$$\frac{\Lambda(\nu_1)\ \nu_1\sin 2\alpha}{\mu - \nu_1\cos 2\alpha} = S \quad , \tag{3.77}$$

$$\frac{\Lambda(\nu_1)}{\mu - \nu_1\cos 2\alpha} - \frac{q_o}{2A} = N \quad , \tag{3.78}$$

$$\mu^2 = \nu_1^2 + 1 \quad . \tag{3.79}$$

From (3.77),(3.79) we get that

$$f(\nu_1) = S \quad , \tag{3.80}$$

where

$$f(\nu_1) = \frac{n[2\sqrt{\nu_1^2+1} - 1]^{n-1}\ \nu_1 \sin 2\alpha}{\sqrt{\nu_1^2 + 1} - \nu_1 \cos 2\alpha} \quad , \tag{3.81}$$

Equation (3.80) has a unique solution $\nu_1(S)$ for $n \geq 1$, but not so for $1/2 < n < 1$, in which case a numerical investigation is necessary in order that the stress distribution can be calculated.

C. Summary of Results

From (3.35),(3.36),(3.15), we can now conclude that the problem formulated in section A of this chapter has a displacement field wnose radial and circumferential components are given by

$$\begin{aligned} u_r(r,\theta) &= \{B \sin 2\theta - C \cos 2\theta + (\frac{k_1}{2} - 1)\}\, r \quad , \\ u_\theta(r,\theta) &= \{B \cos 2\theta + C \sin 2\theta - \frac{k_2}{2}\}\, r \quad , \end{aligned} \tag{3.82}$$

provided the prescribed tractions N_1, N_2, S_1, S_2 and the opening angle 2α satisfy the traction constraint (3.59), in which Λ is calculated from (3.56) once q_* has been determined, numerically in general, as an admissible solution of (3.54). If $\tilde{\nu}_1$ and $\tilde{\nu}_2$ are then determined from (3.48), (3.49) with N_γ and S_γ replaced by $\tilde{N}_\gamma$ and $\tilde{S}_\gamma$ according to (3.57), we can solve for B, C, k_1, and k_2 from (3.38) and the equations

$$\tilde{\nu}_1 = k_1 C + k_2 B \quad , \qquad \tilde{\nu}_2 = k_1 B - k_2 C \quad . \tag{3.83}$$

From (3.53) the components of the actual stress tensor are

$$\left.\begin{aligned}
\tau_{RR}(r,\theta) &= 2A\left\{2\tilde{\mu} - \frac{1}{\tilde{\mu}+\tilde{\nu}_2\sin 2\theta - \tilde{\nu}_1\cos 2\theta} - q_*\right\}, \\
\tau_{R\phi}(r,\theta) &= 2A\left\{\frac{\tilde{\nu}_1\sin 2\theta + \tilde{\nu}_2\cos 2\theta}{\tilde{\mu}+\tilde{\nu}_2\sin 2\theta - \tilde{\nu}_1\cos 2\theta}\right\}, \\
\tau_{\phi\phi}(r,\theta) &= 2A\left\{\frac{1}{\tilde{\mu}+\tilde{\nu}_2\sin 2\theta - \tilde{\nu}_1\cos 2\theta} - q_*\right\},
\end{aligned}\right\} \tag{3.84}$$

where

$$\tilde{\mu}^2 = \tilde{\nu}_1^2 + \tilde{\nu}_2^2 + 1 \quad . \tag{3.85}$$

Notice that from (3.85), the denominator in (3.84)

$$\tilde{\mu} + \tilde{\nu}_2 \sin 2\theta - \tilde{\nu}_1 \cos 2\theta > 0 \tag{3.86}$$

for all $\theta \in [-\alpha,\alpha]$; moreover, in view of the incompressibility constraint, the denominators in (3.48),(3.49) are never zero. Thus the components of the true stress tensor are finite for all values of α in the interval $(0,\pi)$.

Klingbeil and Shield [5] investigate a class of equilibrium problems in finite plane strain for which the deformed cross-section Π^* is a sector of infinite radius. They study true stress fields which are functions of the polar angle ϕ only, and arrive at deformation fields for which

$$\rho = rf(\theta) \quad , \qquad \phi = A\log r + \hat{\phi}(\theta) \quad , \tag{3.87}$$

where A is a constant and $f,\hat{\phi}$ are suitable functions of θ. For the case

$A = 0$, the structure of the stress and deformation fields are the same as those determined here, and they observe for this case the necessity of a constraint of the type (3.59) on the applied (uniform) tractions. They do not, however, analyze this condition in detail, in general, and do not discuss conditions on the applied tractions which assure the fulfillment of this constraint when the opening angle 2α of the undeformed sector is given.

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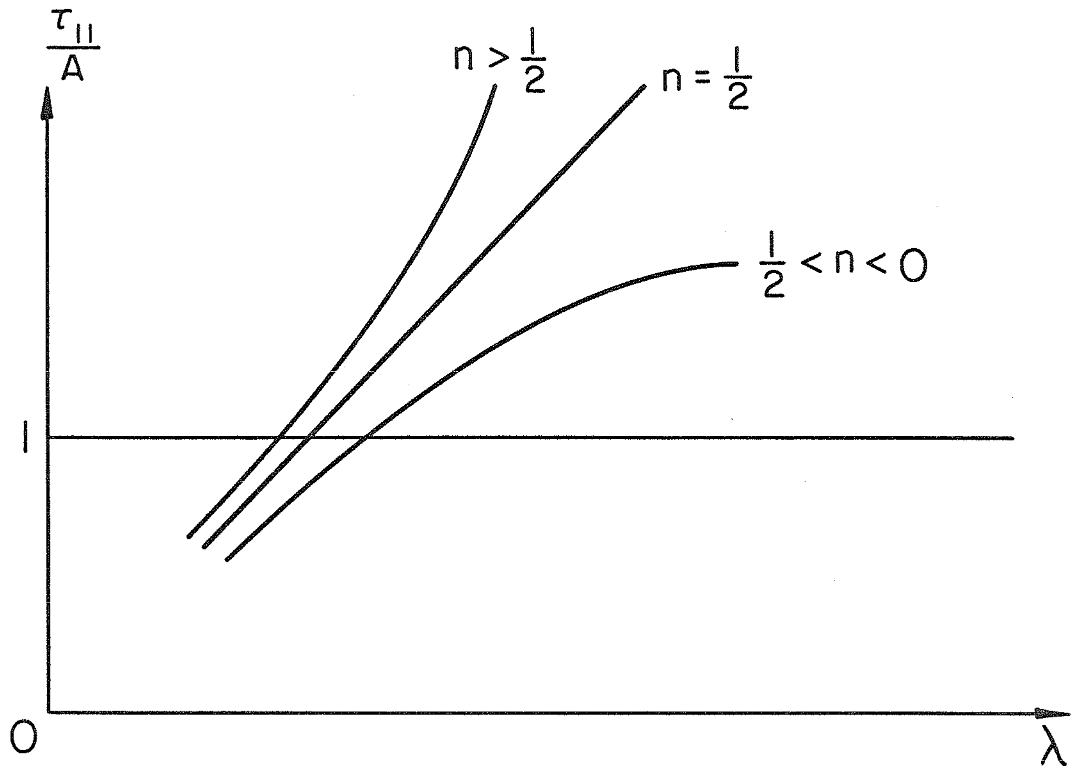
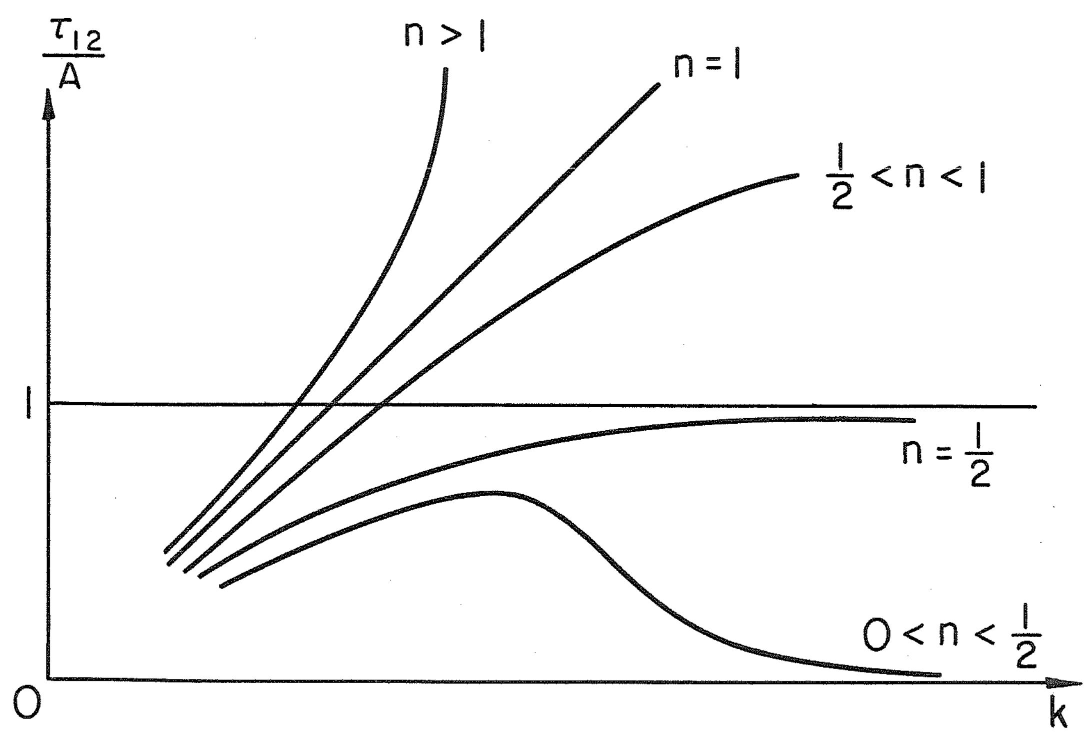
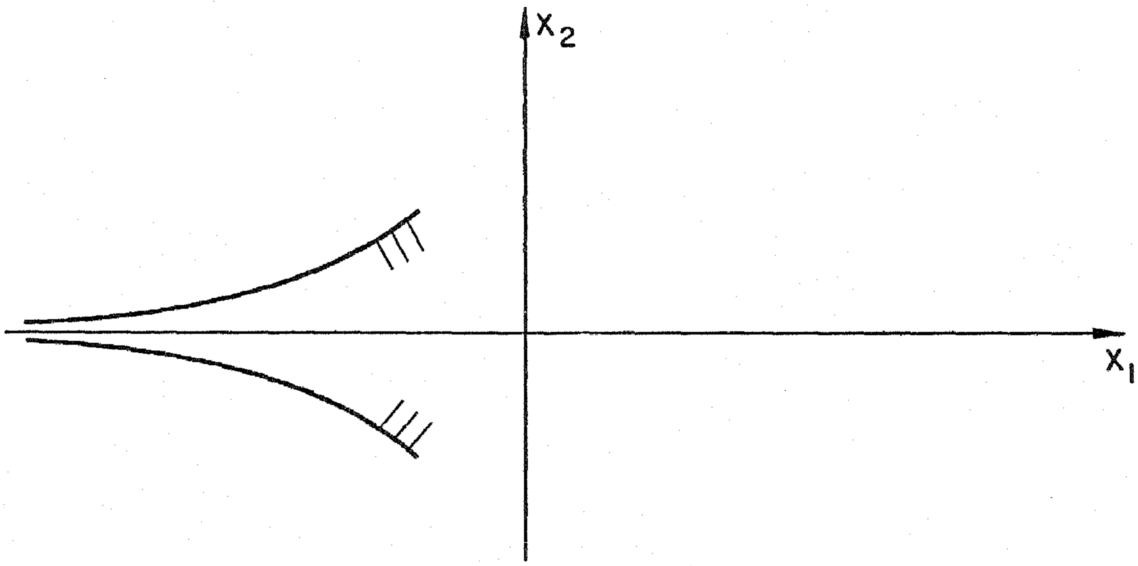


Power-law material response curves for extreme uniaxial stress.

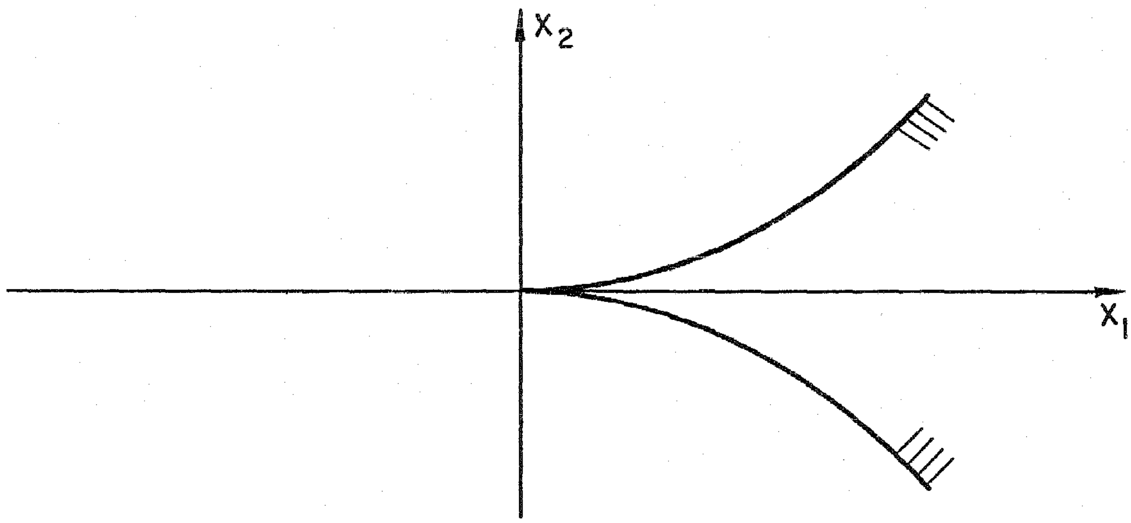


Power-law material response curves for severe simple shear.

FIGURE 1



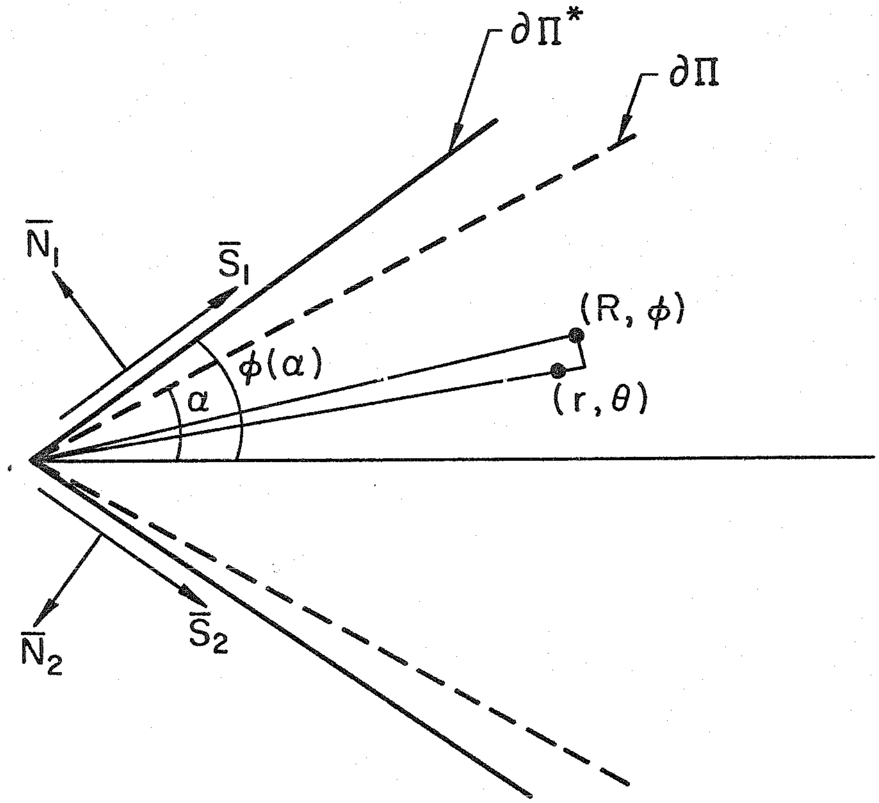
$\frac{1}{2} < n \leq 1$ (SOFTENING MATERIAL)



$n > 1$ (HARDENING MATERIAL)

Local image of half-plane deformed by a tensile concentrated force.

FIGURE 2



Undeformed and deformed sector.

FIGURE 3