COLLISIONLESS ABSORPTION AND

EMISSION OF ELECTROMAGNETIC WAVES

BY A BOUNDED PLASMA

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ABSTRACT

The absorption and emission of electromagnetic waves by a hot bounded plasma can be greatly influenced by the collective behavior of the plasma. A homogeneous slab model of a hot plasma is used in a theoretical study of these collective effects. A boundary-value problem is solved to find the fields in the plasma when a plane wave is incident on the plasma boundary. Power is found to be absorbed from the incident wave because of the Landau damping of longitudinal plasma waves excited in the plasma. If the incident wave frequency is close to one of the plasma wave resonance frequencies, the absorption may be substantial, but only if the Landau damping rate is approximately equal to the radiation damping rate, i.e. if the plasma is "matched" to the radiation field. The width of an absorption line is the sum of the Landau damping and radiation damping widths. Electromagnetic noise radiation is emitted from a collisionless plasma because of the coupling of the field fluctuations in the plasma to the external radiation field at the plasma boundary. Plasma wave fluctuations are excited by fast electrons in the plasma and are reinforced by the collective response of the plasma at the resonance frequencies. Emission and absorption are related by Kirchhoff's law for a thermal plasma. For a nonthermal plasma the emission may be enhanced if there are many electrons in a high-energy tail of the distribution function.

CONTENTS

I.   INTRODUCTION

A plasma is a quasi-neutral collection of electrically charged particles. As such, it interacts strongly with electromagnetic fields. By observing the interaction of a plasma with electromagnetic waves, it is possible to learn something about the behavior of this collection of particles. Of particular interest in this investigation is the collective behavior, in which large numbers of particles participate, which results in the propagation of longitudinal waves with frequencies close to the plasma frequency ω_p . These waves may be coupled to the electromagnetic waves if the plasma has boundaries. The existence of such longitudinal plasma waves may be manifested in peaks in the amount of energy absorbed from an incident electromagnetic wave, as a function of frequency. At these resonance frequencies, the amount of energy scattered, or reflected, by the plasma may also have maxima.

These properties have been observed experimentally in gaseous discharge plasmas (1). The positions of the resonance frequencies have been calculated, in good agreement with experiment, by Parker, Nickel, and Gould (2), but the mechanism of power absorption has not been fully explained. Also observed, in microwave experiments by Kerzar and Weissglas (3), and by Gould (4), are similar peaks in the noise emission, which are not completely understood.

The purpose of this investigation is to show how the power absorption at the resonance frequencies of a bounded plasma is related to Landau damping (5), and to discuss the corresponding noise emission mechanism. Expressions for the absorption coefficient and the emissivity are derived theoretically for a bounded plasma which is not

necessarily in thermal equilibrium. In the special case of thermal equilibrium the expression for the emissivity agrees with the result obtained from Kirchhoff's law using the expression for the absorption coefficient.

The type of plasma which is of most interest here is the nearly collisionless hot plasma. The mean kinetic energy of the particles is assumed to be much larger than their mean potential energy due to the Coulomb forces between them. In a thermal equilibrium plasma this may be expressed as

$$\frac{\kappa T}{n^{1/3} e^2} \equiv \Lambda^{2/3} \gg 1$$

which is satisfied in most cases of interest. The basis of the theoretical description is then the quasi-free motions of the charged particles in the macroscopic electric field of the plasma, with particle encounters treated as small perturbations. This contrasts with the fluid description which is often used, and which is usually justified (in the absence of a strong magnetic field) only if collisions are very frequent (6).

We are concerned here with observable quantities, which might be measured in a laboratory experiment. Since the plasma must be produced in the laboratory, it is necessarily bounded in spatial extent. This feature is quite important for the electromagnetic properties of the plasma, but only a few problems involving hot bounded plasmas have been solved. Most of the theoretical work on hot plasmas is concerned with the case where there are no boundaries (infinite plasma) or where boundary effects are not of primary importance. One problem which has

been solved, however, and which is closely related to the problems considered in this investigation, is the "plasma capacitor" problem (7). A plasma of uniform density is contained between the plates of an infinite plane capacitor, and the particles of the plasma are supposed to undergo specular reflection at the plates. The problem is to find the specific impedance of the capacitor, i.e., the voltage drop across the plates divided by the displacement current per unit area as a function of frequency. The solution shows that there are resonances in the impedance, at the natural frequencies of standing waves in the plasma.

A similar model of the plasma is used in this investigation, mainly to obtain a tractable mathematical problem. The plasma is again supposed to be homogeneous and contained between two parallel planes, the boundaries of the "plasma slab". It is the interaction with electromagnetic waves, propagating at an angle to the slab normal, which is of interest here, so this is not a one-dimensional problem in contrast to the "plasma capacitor". The electrons are again supposed to be confined to the slab by specular reflection at the slab boundaries. This can be thought of as due to a large static electric field very near the boundaries, similar to the boundary sheath field in a laboratory discharge plasma. The ions can be supposed motionless; they simply provide a uniform neutralizing background of charge on the time scale of interest (the plasma oscillation period, $2\pi/\omega_p$). Because of the quasi-neutrality of the plasma, there is no static electric field in the plasma except very near the boundaries. The electrons, therefore, travel on trajectories which are very nearly straight lines until

they are specularly reflected at the boundaries, being slightly per-
turbed by the electric fields of plasma waves and by particle
encounters.

An essential point in the present calculation of the absorption
and emission from the plasma slab is that a boundary-value problem has
been solved for the electromagnetic fields. The usual method of calcu-
lating emission from a plasma makes use of the radiative transfer
equation (8), which is based upon the geometrical optics approximation.
In the case of most interest in this investigation, the thickness of
the plasma slab is less than the wavelength of the electromagnetic
waves. Hence, the approximation of geometrical optics is invalid and
the radiative transfer equation cannot be used. Also, the reflection
of waves at the boundaries of a plasma cannot be correctly accounted
for in the usual approach unless the waves lose phase information at
each reflection. Furthermore, there is implicit in the radiative
transfer approach the assumption of small damping. The solution of the
boundary-value problem used here has none of these shortcomings.

In Part II the boundary-value problem for the plasma slab is
solved. The self-consistent electric field in the plasma is obtained
in terms of the boundary values of the tangential component of
magnetic field. Maxwell's equations are Fourier analyzed in Section A
and the particle equations of motion solved approximately in Section B.
Then the Fourier coefficients of the current density are obtained in
Section C, in terms of the electric field in the plasma, from the par-
ticle motions. In Section D the self-consistent field is solved for.

This is the field which causes the particles to produce a current density which generates the same field, through the solution of Maxwell's equations. The steps in Sections B and C are usually carried out by means of the Vlasov equation in most theoretical treatments of the self-consistent field in a hot plasma (9). The treatment given here brings out more clearly the physical content of the theory and expresses the assumptions and approximations involved in terms of the particle trajectories, rather than the distribution function. This method is equivalent to the usual method, however, because the differential equations for the particle trajectories are the same as the Lagrangian subsidiary equations, or characteristics, of the Vlasov equation (10).

The solution of the boundary-value problem is used in Part III to find the absorption coefficient of the plasma slab, i.e., the fraction of the power in an incident electromagnetic wave which is absorbed. The wave is assumed to be polarized with $\vec{E}$ in the plane of incidence so that it excites longitudinal plasma waves in the slab. In Section A the concept of surface impedance is introduced which is a ratio of fields at the surface of the plasma and is obtained from the solution of the boundary-value problem. The absorption coefficient is related to it in Section B and, therefore, to the properties of the plasma. The general expression for the absorption coefficient, equation 3.24, is interpreted in terms of Landau damping of plasma waves. If the frequency of the incident wave is close to the natural frequency for a longitudinal plasma wave with an integral number of half-wavelengths between the slab boundaries, then such a standing wave can

be excited in the plasma by the incident wave. Because the plasma wave
is Landau-damped, the plasma absorbs energy from the incident wave.
Under some conditions, discussed in Section C, the absorption is
appreciable but only for frequencies very near these "normal mode"
frequencies. The absorption spectrum then contains narrow lines at
these frequencies, with Lorentzian shapes. The reflectivity of the
plasma slab, i.e. the fraction of power in the incident wave which is
reflected, also has resonances at these frequencies. In Section D the
notion of Landau damping in the plasma slab is discussed critically,
as well as the conditions under which it can be the primary mechanism
for power absorption in a steady state.

In Part IV the statistical properties of the noise radiation
from a plasma slab are calculated. In Section A the surface emissivity
is defined, and related to the autocorrelation function of the radia-
tion field. It is then shown to be related to the Laplace transform
of the radiation field which is more easily obtained than the time-
dependent field and which is related to the Laplace transform of the
field fluctuations at the plasma boundary. These boundary values are
determined in Section B by the solution of a boundary-value problem for
the electric field fluctuations in the plasma. The results of Part II
are simply carried over, with a reinterpretation of some of the vari-
ables. The field fluctuations are produced by "dressed test particles"
i.e., individual charged particles and their screening clouds, which
are assumed to be statistically independent. The general expression
for the emissivity, equation 4.18, is then derived in a straightforward
way. The radiation from the slab is just a result of continuing the

field fluctuations in the plasma into the region external to the plasma
by requiring continuity of the fields at the boundary. The emission
mechanism is therefore the excitation of electric field fluctuations
by the motions of the individual charged particles. Emission lines
may occur, under some conditions, at the plasma wave resonance fre-
quencies, the emissivity being proportional to the number of particles
with velocities close to the phase velocities of the waves. The
enhancement of emission lines, caused by an increase in the number of
fast electrons over a thermal equilibrium distribution, is discussed
in Section C. The effect of particle encounters, or collisions, is
examined briefly in Section D.

Finally, the results are summarized in Part V and compared with
the experimental results on microwave scattering, absorption, and
emission by gaseous discharge plasmas.

II.   SELF-CONSISTENT SOLUTION OF MAXWELL'S EQUATIONS AND
THE PARTICLE EQUATIONS OF MOTION

A.   Fourier Analysis of Maxwell's Equations

As a first step in the self-consistent solution of the differen-
tial equations for the electromagnetic fields and the charged particles,
we suppose the current density $\vec{j}(\vec{x},t)$ to be known; the electromagne-
tic fields are then determined by the solution of Maxwell's equations:

$$\text{curl } \vec{B} = \frac{4\pi}{c}\vec{j} + \frac{1}{c}\frac{\partial\vec{E}}{\partial t} \tag{2.1}$$

$$\text{curl } \vec{E} = -\frac{1}{c}\frac{\partial\vec{B}}{\partial t} \quad . \tag{2.2}$$

The solution of these equations in the plasma volume V for all times
t > 0 can be made unique (11) by specifying (in addition to the cur-
rent density in V for $t \geqq 0$) the tangential components of $\vec{B}$ on the
surface S of the plasma volume, for $t \geqq 0$, and the initial values
(at t = 0) of $\vec{E}$ and $\vec{B}$ at all points of V whose distances from
the observation point are $\leqq$ ct .

Although we have to solve an initial boundary-value problem in
principle, we are most interested in the dependence of the solution
upon the boundary values of tangential $\vec{B}$. This is because we are
interested in steady-state, or time-stationary phenomena, for which
the initial values have no effect. For simplicity we will assume that
curl $\vec{E}$ and $\vec{B}$ are zero everywhere in V at t = 0 . The initial
value of div $\vec{E}$ is determined by the charge density ρ at t = 0
by the relation div $\vec{E}$ = $4\pi\rho$., which we assume is satisfied at t = 0.

Laplace transforms of the field variables with respect to time may be introduced as follows:

$$\vec{E}(\vec{x};\omega) = \int\limits_{0}^{\infty} dt\ e^{i\omega t}\ \vec{E}(\vec{x},t)$$

and similarly for $\vec{B}$, where the usual Laplace transform variable s is replaced by $-i\omega$, and we take $\text{Im}\ \omega > 0$ to insure convergence of the integral. Laplace transforming Maxwell's equations yields the following equations for the transforms of the field variables:

$$\text{curl}\ \vec{B} = \frac{1}{c}(4\pi\vec{j} - \vec{E}(0)) - ik_o\vec{E} \tag{2.3}$$

$$\text{curl}\ \vec{E} = ik_o\vec{B} \tag{2.4}$$

where $k_o = \omega/c$, and $\vec{E}(0) \equiv \vec{E}(\vec{x},t=0)$ is the initial condition on the electric field.

Now the plasma volume V will be chosen, for simplicity, to be a slab of thickness L, described by rectangular coordinates x,y,z in the range $0 \leqq x \leqq L$, $-\infty < y < \infty$, $-\infty < z < \infty$. It is convenient to use Fourier integral representations of the field variables with respect to y and z, and Fourier series representations with respect to x in the above interval. For the variables E_x, B_y, B_z, and j_x we use a sine-series representation, e.g.,

$$E_x(x,y,z;\omega) = \int\limits_{-\infty}^{\infty} \frac{dk_y}{2\pi} \int\limits_{-\infty}^{\infty} \frac{dk_z}{2\pi}\ e^{i(k_y y + k_z z)}\ \frac{2}{L} \sum_{n=1}^{\infty} E_{xs}(\vec{k}_n,\omega)\sin k_n x$$

where $\vec{k}_n = (k_n,k_y,k_z)$ and $k_n = n\pi/L$. For the variables E_y, E_z,

B_x, j_y, j_z and ρ , we use a cosine series representation, e.g.,

$$E_y(x,y,z;\omega) = \int_{-\infty}^{\infty} \frac{dk_y}{2\pi} \int_{-\infty}^{\infty} \frac{dk_z}{2\pi} e^{i(k_y y + k_z z)} \frac{2}{L} \sum_{n=0}^{\infty}{}' E_{yc}(\vec{k}_n,\omega)\cos k_n x$$

where the prime on the summation means the $n = 0$ term is multiplied by $\frac{1}{2}$. The Fourier coefficients of the variables are obtained by integrating over the slab volume, e.g.

$$\left\{ \begin{array}{c} E_{xs}(\vec{k}_n,\omega) \\ E_{yc}(\vec{k}_n,\omega) \end{array} \right\} = \int_{-\infty}^{\infty} dy \int_{-\infty}^{\infty} dz\, e^{-i(k_y y + k_z z)} \int_{0}^{L} dx \left\{ \begin{array}{c} E_x(\vec{x};\omega)\sin k_n x \\ E_y(\vec{x};\omega)\cos k_n x \end{array} \right\}$$

Introduction of these Fourier representations into the x,y and z components of equation 2.4 yields the following equations:

$$k_y E_{zc} - k_z E_{yc} = k_o B_{xc}$$

$$ik_z E_{xs} + k_n E_{zc} = ik_o B_{ys}$$

$$-k_n E_{yc} - ik_y E_{xs} = ik_o B_{zs}$$

Similarly, from the x component of equation 2.3 we obtain

$$ik_y B_{zs} - ik_z B_{ys} = \frac{1}{c}(4\pi j_{xs} - E_{xs}(0)) - ik_o E_{xs} \quad .$$

If the   y   and   z   components of equation 2.3 are multiplied by $e^{-i(k_y y + k_z z)}$ $\cos k_n x$ and integrated over the slab volume, the derivatives can be transformed by integration by parts. We obtain, in this way, the following equations:

$$ik_z B_{xc} - k_n B_{zs} + B_{zo} - (-1)^n B_{zL} = \frac{1}{c}(4\pi j_{yc} - E_{yc}(0)) - ik_o E_{yc}$$

$$k_n B_{ys} - ik_y B_{xc} - B_{yo} + (-1)^n B_{yL} = \frac{1}{c}(4\pi j_{zc} - E_{zc}(0)) - ik_o E_{zc}$$

where $B_{zo} \equiv B_z(0;k_y,k_z,\omega)$ and $B_{zL} \equiv B_z(L;k_y,k_z,\omega)$ are the Fourier transforms with respect to y and z of the boundary values of B_z at $x = 0$ and $x = L$. The occurrence of these boundary terms is a result of integration by parts in the term containing the derivative $\frac{\partial B_z}{\partial x}$.

The Fourier coefficients of $\vec{B}$ may be eliminated from the above six equations. The resulting three equations for the Fourier coefficients of $\vec{E}$ can be written as follows (where we consider only the $k_z = 0$ components):

$$(k_o^2 - k_y^2) E_{xs} + ik_n k_y E_{yc} = - \frac{ik_o}{c}(4\pi j_{xs} - E_{xs}(0)) \quad (2.5)$$

$$-ik_n k_y E_{xs} + (k_o^2 - k_n^2)E_{yc} = - \frac{ik_o}{c}(4\pi j_{yc} - E_{yc}(0))$$

$$+ ik_o[B_{zo} - (-1)^n B_{zL}] \quad (2.6)$$

$$[k_o^2 - (k_n^2 + k_y^2)] E_{zc} = - \frac{ik_o}{c} 4\pi j_{zc} - ik_o[B_{yo} - (-1)^n B_{yL}]$$

$$(2.7)$$

The last equation is uncoupled from the first two by the choice $k_z = 0$, which means simply that the fields do not vary in the z direction. Also, $E_{zc}(0) = 0$ in that case.

If the Fourier coefficients of $\vec{j}$ were known, then the solution of these equations for E_{xs}, E_{yc} and E_{zc} would give the Fourier

representation of the electric field inside the slab in terms of the
boundary values of tangential $\vec{B}$ at the surface. But the current
density is determined by the motions of the charged particles of the
plasma, which are influenced by the electric field in the plasma.
Therefore, the next step in the self-consistent determination of the
field is to find the current density $\vec{j}$ produced by a known field $\vec{E}$.

B. Perturbation Analysis of the Particle Equations of Motion

Before the current density can be found, the motions of the
charged particles in a known field $\vec{E}$ must be determined. The posi-
tion $\vec{r}(t)$ of a particle with (charge/mass) $= \eta$ in an electric field
$\vec{E}(\vec{x},t)$ is determined by the solution of the differential equation of
motion:

$$\ddot{\vec{r}} = \eta \, \vec{E}(\vec{r}(t),t) \tag{2.8}$$

The solution is made unique by specifying initial conditions:

$$\vec{r}(0) = \vec{r}_o \quad , \quad \dot{\vec{r}}(0) = \vec{v}_o \quad . \tag{2.9}$$

We assume the electric field to be given by

$$\vec{E}(\vec{x},t) = \vec{E}_o(\vec{x},t) + \vec{E}_1(\vec{x},t)$$

so that

$$\ddot{\vec{r}}(t) = \eta \, \vec{E}_o(\vec{r}(t),t) + \eta \, \vec{E}_1(\vec{r}(t),t) \tag{2.10}$$

where $\eta \vec{E}_1$ is a small perturbation in the equation of motion.
Because this differential equation is nonlinear, we need to use
approximation methods. The solution may be obtained as a perturbation
series:

$$\vec{r}(t) = \vec{r}^{(0)}(t) + \vec{r}^{(1)}(t) + \cdots \qquad (2.11)$$

where $\vec{r}^{(0)}(t)$ is the solution of the equation of motion with $\eta\vec{E}_1$ neglected:

$$\ddot{\vec{r}}^{(0)}(t) = \eta\vec{E}_0(\vec{r}^{(0)}(t),t) \quad . \qquad (2.12)$$

For simplicity we require that $\vec{r}^{(0)}(t)$ have the same initial conditions (equation 2.9) as $\vec{r}(t)$, so that the other terms in the perturbation series satisfy zero initial conditions, e.g.

$$\vec{r}^{(1)}(0) = 0 \quad , \quad \dot{\vec{r}}^{(1)}(0) = 0 \qquad (2.13)$$

The solution of the nonlinear equation 2.12 with initial conditions 2.9 is assumed to be a known function. For definiteness we denote this function as follows:

$$\vec{r}^{(0)}(t) = \vec{R}(t|\vec{r}_0,\vec{v}_0,0) \qquad (2.14)$$

where the initial conditions are indicated explicitly.

At this point we digress to discuss in some detail the trajectory function $\vec{R}$ for a particular case to which the analysis will be applied. We will deal with a model of a nearly-collisionless hot plasma confined to a slab with boundary planes $x = 0$ and $x = L$. The ions can be supposed nearly motionless (on the time scales of interest) and the electrons confined by an equilibrium static field $\vec{E}_0(\vec{x})$ given by

$$\eta\vec{E}_0(\vec{x}) = -\hat{x}\frac{d\phi_0(x)}{dx} \quad , \quad \text{where} \quad \phi_0(x) = \phi_0\cot^2(\frac{\pi x}{L})$$

and $\eta = -e/m_e$ for electrons. The potential energy/unit mass $\phi_0(x)$

has the form of a well, with sides which become infinitely high as the boundaries are approached from inside the slab.

Since the electrons are not confined in the y and z directions, the trajectory function may be written

$$\vec{R}(t|\vec{r}_o,\vec{v}_o,0) = [X(t|x_o,v_{xo},0), y_o + v_{yo}t, z_o + v_{zo}t] \quad . \qquad (2.15)$$

The function X may be obtained from the energy integral,

$$\mathcal{U} = \frac{v_x^2}{2} + \phi_o \cot^2(\frac{\pi x}{L}) = \frac{v_{xo}^2}{2} + \phi_o \cot^2(\frac{\pi x_o}{L}) \quad . \qquad (2.16)$$

It is given explicitly by

$$X(t|x_o,v_{xo},0) = \frac{L}{\pi} \cos^{-1} \left\{ \cos(\frac{\pi x_o}{L}) \cos \Omega t \right.$$
$$\left. \mp \sqrt{\frac{1}{1 + \phi_o/\mathcal{U}} - \cos^2(\frac{\pi x_o}{L})} \sin \Omega t \right\} \qquad (2.17)$$

where $\Omega(\mathcal{U}) = \sqrt{2} \frac{\pi}{L} (\mathcal{U} + \phi_o)^{1/2}$ is the fundamental oscillation frequency in the potential well, as a function of energy, and $\mathcal{U}$ is given in terms of x_o and v_{xo} by equation 2.16. The value of the inverse cosine is chosen to lie always between 0 and π , and the upper (lower) sign in front of the square root is taken for $v_{xo} > 0$ ($v_{xo} < 0$).

We will treat only the limiting case $\phi_o \to 0$ for which the potential approaches a square well, and the oscillation frequency becomes $\Omega = \frac{\pi}{L}|v_{xo}|$. Then equation 2.17 becomes, in this limit,

$$X(t|x_o,v_{xo},0) \ = \ \frac{L}{\pi} \cos^{-1} \left\{ \cos \frac{\pi}{L} (x_o + v_{xo}t) \right\} \quad . \quad (2.18)$$

Since the confining field acting on the particle is negligible except very near the boundaries, the particle moves on a straight-line trajectory with a constant speed until it reaches one of the boundary surfaces, where it is specularly reflected. The x component of velocity appears to change sign discontinuously in this limit, since differentiation of equation 2.18 gives

$$\dot{X}(t|x_o,v_{xo},0) \ = \ v_{xo} \ \frac{\sin \frac{\pi}{L}(x_o + v_{xo}t)}{\left| \sin \frac{\pi}{L}(x_o + v_{xo}t) \right|} \qquad (2.19)$$

Whenever a continuous function is needed to justify the analysis below, we can always return, in principle, to equation 2.17 with $\varphi_o/\mathcal{U} \ll 1$.

Returning now to the discussion of the particle equation of motion including the small perturbation $\eta \, \vec{E}_1$, we substitute the perturbation series 2.11 into equation 2.10. After expanding $\vec{E}_o(\vec{r}(t),t)$ in a Taylor series about $\vec{r}(t) = \vec{r}^{(0)}(t)$, we find that $\vec{r}^{(1)}(t)$ satisfies the following equation (in Cartesian tensor notation):

$$\ddot{r}_\mu^{(1)}(t) - \eta \, \nabla_\alpha \, E_{o\mu}(\vec{r}^{(0)}(t),t) \, r_\alpha^{(1)}(t) = \eta \, E_{1\mu}(\vec{r}^{(0)}(t),t) \ .$$
$$(2.20)$$

[The higher order terms in the perturbation series satisfy linear differential equations like 2.20, but with right-hand sides of correspondingly higher order in $\eta \, E_1$; these will not be considered here.]

The solution of equation 2.20 with zero initial conditions can be written in the form

$$r_\mu^{(1)}(t) = \eta \int_0^t dt' \, T_{\mu\beta}(t,t') \, E_{1\beta}(\vec{r}^{(0)}(t'),t') \ . \qquad (2.21)$$

The tensor $T_{\mu\beta}$ is assumed to satisfy the differential equation

$$\ddot{T}_{\mu\beta}(t,t') - \eta \, \nabla_\alpha \, E_{o\mu}(\vec{r}^{(0)}(t),t) \, T_{\alpha\beta}(t,t') = 0 \qquad (2.22)$$

(where dots indicate differentiation with respect to t) and the initial conditions

$$T_{\mu\beta} = 0 \ , \quad \dot{T}_{\mu\beta} = \delta_{\mu\beta} \qquad \text{at} \quad t = t' \ . \qquad (2.23)$$

Now we recall that the trajectory function $\vec{R}(t|\vec{r}',\vec{v}',t')$ satisfies the differential equation

$$\ddot{R}_\mu - \eta \, E_{o\mu}(\vec{R},t) = 0 \qquad (2.24)$$

and the initial conditions $R_\mu = r_\mu'$, $\dot{R}_\mu = v_\mu'$ at $t = t'$. Differentiating with respect to v_β' gives the differential equation 2.22 and the initial conditions 2.23. Hence $T_{\mu\beta}$ is given explicitly in terms of the known trajectory function $\vec{R}$ by

$$T_{\mu\beta}(t,t') = \frac{\partial}{\partial v_\beta'} R_\mu(t|\vec{r}',\vec{v}',t') \ . \qquad (2.25)$$

We have assumed, of course, that $\vec{r}' = \vec{r}^{(0)}(t')$, $\vec{v}' = \dot{\vec{r}}^{(0)}(t')$, so that $\vec{R}(t|\vec{r}',\vec{v}',t') = \vec{r}^{(0)}(t)$.

Thus the solution of the linearized equation of motion 2.20 for the deviation of the particle trajectory from the unperturbed trajectory $\vec{r}^{(0)}(t)$ is given by equations 2.21 and 2.25 as an integral of the perturbation $\eta \vec{E}_1$ experienced by a particle which moves along the unperturbed trajectory from time 0 to t . How much the trajectory is deviated by a given value of $\eta \vec{E}_1$ is determined by the influence tensor $T_{\mu\beta}$, which is determined by the unperturbed trajectory. For the example considered above, we have

$$T_{x\beta}(t,t') = \delta_{x\beta} \frac{\partial}{\partial v'_x} X(t|x',v'_x,t')$$

$$= \delta_{x\beta}(t-t') \frac{\sin \frac{\pi}{L}(x' + v'_x(t-t'))}{|\sin \frac{\pi}{L}(x' + v'_x(t-t'))|} . \qquad (2.26)$$

If the particle undergoes an even number of reflections at the boundaries between times t' and t , then a positive perturbation $\eta E_{1x}(\vec{r}',t')$ causes a positive contribution to $x^{(1)}(t)$ but if the number of reflections is odd, the contribution is negative.

Finally, a few remarks are necessary concerning the linearization of the particle equations of motion. The electric field perturbation $\vec{E}_1$ will be interpreted as a wave with frequency close to the plasma frequency ω_p . In order that collective behavior with this characteristic frequency be correctly described by the linearized equations, the wave amplitude must be small enough that the particles are not deviated appreciably from their unperturbed trajectories during a time interval of many plasma oscillation periods. Furthermore, close-range encounters between particles must be assumed to occur infrequently enough,

and the cumulative effect of long-range encounters assumed slow enough that the unperturbed trajectories provide a good description of the motion of a particle for many plasma periods. A rough criterion is that $\omega_p t_c \gg 1$, where t_c is the collisional relaxation time (see Section IV,D.) On the other hand, the collision rate must be sufficiently high to prevent the invalidation of the linearized equations due to particle trapping. This will be discussed further in Section III,D.

C. Fourier Coefficients of the Current Density

The current density due to the electrons is

$$\vec{j}(\vec{x},t) = (-e) \left\langle \sum_j \vec{v}_j(t) \; \delta(\vec{x} - \vec{r}_j(t)) \right\rangle \tag{2.27}$$

where $\vec{r}_j(t)$ and $\vec{v}_j(t)$ are the position and velocity of the jth particle. The ensemble average, indicated by the angular brackets, is taken because we are interested here in the macroscopic current produced by a known field $\vec{E}$, and not in the fluctuations due to particle discreteness. Hence, the positions and velocities of the particles at time t are treated as random variables with a known joint probability distribution, and the brackets indicate that we use the statistical mean over all possible positions and velocities of all particles.

The Fourier coefficients of the current density are given by

$$\vec{j}(\vec{k}_n,t) = \int\limits_{-\infty}^{\infty} dy \int\limits_{-\infty}^{\infty} dz \; e^{-i(k_y y + k_z z)} \int\limits_{0}^{L} dx \left\{ \begin{array}{l} j_x(\vec{x},t)\sin k_n x \\ j_y(\vec{x},t)\cos k_n x \\ j_z(\vec{x},t)\cos k_n x \end{array} \right\}$$

Using equation 2.27 this may be written

$$j_\alpha(\vec{k}_n,t) = (-e) \left\langle \sum_j v_{j\mu}(t) \; g_{\mu\alpha}(\vec{r}_j(t),\vec{k}_n) \right\rangle$$

where $g_{\mu\alpha} = 0$ for $\alpha \neq \mu$ and

$$g_{xx}(\vec{x},\vec{k}_n) = e^{-ik_y y} \sin k_n x$$

$$g_{yy}(\vec{x},\vec{k}_n) = g_{zz}(\vec{x},\vec{k}_n) = e^{-ik_y y} \cos k_n x \qquad \text{for } k_z = 0$$

Now the perturbation series of the previous section can be used for $\vec{r}_j(t)$ and $\vec{v}_j(t) \equiv \dot{\vec{r}}_j(t)$, and if we retain terms up to first order in ηE_1 , we have approximately

$$j_\alpha(\vec{k}_n,t) = (-e) \left\langle \sum_j \left\{ v_{j\mu}^{(0)}(t) \; g_{\mu\alpha}(\vec{r}_j^{(0)}(t),\vec{k}_n) + v_{j\mu}^{(1)}(t) \; g_{\mu\alpha}(\vec{r}_j^{(0)}(t),\vec{k}_n) \right. \right.$$

$$\left. \left. + \; v_{j\mu}^{(0)}(t) \; r_{j\gamma}^{(1)}(t) \; \nabla_\gamma \; g_{\mu\alpha}(\vec{r}_j^{(0)}(t),\vec{k}_n) \right\} \right\rangle \qquad (2.28)$$

The first term on the right-hand side of this equation is a current density Fourier component for particles unperturbed by the field ηE_1 and may be denoted by $\vec{j}^{(0)}(\vec{k}_n,t)$:

$$j_\alpha^{(0)}(\vec{k}_n,t) = (-e) \left\langle \sum_j v_{j\mu}^{(0)}(t) \; g_{\mu\alpha}(\vec{r}_j^{(0)}(t),\vec{k}_n) \right\rangle$$

The second and third terms contain the perturbation in the position and velocity due to the field $\vec{E}_1$. From equations 2.21 and 2.25 these perturbation quantities are

$$r_\mu^{(1)}(t) = \eta \int_0^t dt' \ E_{1\beta}(\vec{r}',t') \ \frac{\partial}{\partial v_\beta'} \ R_\mu(t|\vec{r}',\vec{v}',t')$$

and

$$v_\mu^{(1)}(t) = \eta \int_0^t dt' \ E_{1\beta}(\vec{r}',t') \ \frac{\partial}{\partial v_\beta'} \ \dot{R}_\mu(t|\vec{r}',\vec{v}',t')$$

where $\quad \vec{r}' = \vec{R}(t'|\vec{r}_o,\vec{v}_o,0) \quad , \quad \vec{v}' = \dot{\vec{R}}(t'|\vec{r}_o,\vec{v}_o,0)$ \hfill (2.29)

are the unperturbed position and velocity at time t' . (The particle index j has been dropped for simplicity). The second term in equation 2.28 may be written as follows:

$$v_\mu^{(1)}(t) \ g_{\mu\alpha}(\vec{r}^{(0)}(t),\vec{k}_n) = \eta \int_0^t dt' \ E_{1\beta}(\vec{r}',t')$$

$$\frac{\partial}{\partial v_\beta'} \ \dot{R}_\mu(t|\vec{r}',\vec{v}',t') \ g_{\mu\alpha}(\vec{R}(t|\vec{r}',\vec{v}',t'),\vec{k}_n)$$

$$\hfill (2.30)$$

The argument of the functions $g_{\mu\alpha}$ is $\vec{r}^{(0)}(t) = \vec{R}(t|\vec{r}',\vec{v}',t')$ where t' is the integration variable. Note that the value of $\vec{R}(t|\vec{r}',\vec{v}',t')$ is constant (for t fixed) and does not change with t' as long as $\vec{r}'$ and $\vec{v}'$ are given by equation 2.29. The value of the integral is therefore not altered by formally including the functions $g_{\mu\alpha}$, with the above argument under the integration. The third term in equation 2.28 may be treated similarly, and combined with the second term by identifying the derivative of a product. The result is

$$v_\mu^{(1)}(t)\, g_{\mu\alpha}(\vec{r}^{(0)}(t),\vec{k}_n) + v_\mu^{(0)}(t)\, r_\gamma^{(1)}(t)\, \nabla_\gamma\, g_{\mu\alpha}(\vec{r}^{(0)}(t),\vec{k}_n)$$

$$= \eta \int_0^t dt'\, E_{1\beta}(\vec{r}',t')\, \frac{\partial}{\partial v'_\beta}\Big[\dot{R}_\mu(t|\vec{r}',\vec{v}',t')\, g_{\mu\alpha}(\vec{R}(t|\vec{r}',\vec{v}',t'),\vec{k}_n)\Big] \quad (2.31)$$

When equation 2.31 is used in equation 2.28, we sum over all particles and average over positions and velocities at time t' . We can use the probability density of unperturbed positions and velocities, $f_0(\vec{r}',\vec{v}',t')$, to first order in $\eta\, E_1$. To make the probability density properly normalized we must also divide by V , the volume of plasma under consideration; the sum runs over the N particles in this volume. Since the particles are all identical, the ensemble average of the sum is just N times the ensemble average for any one particle. Passing to the limit $N \to \infty$, $V \to \infty$, $N/V = n_0$ (finite), we obtain the following from equation 2.28:

$$j_\alpha(\vec{k}_n,t) = j_\alpha^{(0)}(\vec{k}_n,t) + \frac{n_0 e^2}{m}\int_0^t dt'\int d\vec{r}'\int d\vec{v}'\, f_0(\vec{r}',\vec{v}',t')$$

$$E_{1\beta}(\vec{r}',t')\, \frac{\partial}{\partial v'_\beta}\Big[\qquad\Big],$$

where $\Big[\qquad\Big] = [\dot{R}_\mu(t|\vec{r}',\vec{v}',t')\, g_{\mu\alpha}(\vec{R}(t|\vec{r}',\vec{v}',t'),\vec{k}_n)]$ \quad (2.32)

The same result can be obtained from the linearized Vlasov equation (9) by integrating along unperturbed trajectories, integrating over velocity $\vec{v}$ to get $\vec{j}$, integrating over position $\vec{r}$ to get the Fourier coefficients, making a change of integration variables from $\vec{r},\vec{v}$ to $\vec{r}',\vec{v}'$ and integrating by parts with respect to $\vec{v}'$. The

method used above is somewhat more direct, however, and brings out the
physical meaning of the linearization process, in terms of particle
trajectories, more clearly.

To proceed further it is necessary to make some further assump-
tions about the equilibrium state of the plasma. The model which will
be used here is that of a homogeneous slab, with a time-independent
equilibrium state. Then the particle position $\vec{r}'$ is distributed
(statistically) uniformly within the slab volume, and its probability
density is independent of time:

$$f_0(\vec{r}',\vec{v}',t') = \begin{cases} f_0(\vec{v}') \ , & 0 \leq x' \leq L \\ 0 \ , & \text{otherwise} \end{cases}$$

The trajectory function depends only on the difference between the
initial and final times: $\vec{R}(t|\vec{r}',\vec{v}',t') = \vec{R}(t - t'|\vec{r}',\vec{v}',0)$ and is
given explicitly by equations 2.15 and 2.18 for the square well con-
fining potential. Using these equations and equation 2.19 the compon-
ents of the square bracket in equation 2.32 (except for the factor
$e^{-ik_y(y'+ v_y'\tau)}$) can be written as follows:

$\alpha = x$ component $\quad \dot{X}(\tau|x',v_x',0)\sin k_n X(\tau|x',v_x',0) = v_x' \sin k_n(x'+ v_x'\tau)$

$$= v_x' \left\{ \sin k_n x' \cos k_n v_x'\tau + \cos k_n x' \sin k_n v_x'\tau \right\} \quad ,$$

where $\quad \tau = t - t'$, and

$\alpha = y$ component $\quad v_y' \cos k_n X(\tau|x',v_x',0) = v_y' \cos k_n(x' + v_x'\tau)$

$$= v_y' \left\{ \cos k_n x' \cos k_n v_x'\tau - \sin k_n x' \sin k_n v_x'\tau \right\}$$

(and similarly for the $\alpha = z$ component, with v_y' replaced by v_z' .)

Now the square bracket in equation 2.32 is differentiated with respect to the β component of $\vec{v}'$, multiplied by $f_o(\vec{v}')$, and integrated over all $\vec{v}'$. Assuming f_o to be a function of the magnitude of $\vec{v}'$ only (i.e., _isotropic_ in velocity space), the terms which are odd functions of v_x, v_y or v_z will drop out upon integration. The surviving terms are proportional to either $\sin k_n x'$ or $\cos k_n x'$, and so the integration over $\vec{x}'$ in equation 2.32 transforms the components of $\vec{E}_1$ into its Fourier coefficients. We obtain, finally

$$j_\alpha(\vec{k}_n,t) = j_\alpha^{(0)}(\vec{k}_{n,t}) + \int_0^t dt' \, \sigma_{\alpha\beta}(\vec{k}_n;t-t') \, E_{1\beta}(\vec{k}_n,t') \qquad (2.33)$$

where

$$\left[\sigma_{\alpha\beta}(\vec{k}_n;\tau)\right] = \frac{\omega_n^2}{4\pi} \int d\vec{v} f_o(\vec{v}) e^{-i(k_n v_x + k_y v_y)\tau} \begin{bmatrix} (1-ik_n v_x \tau) & k_y v_x \tau & 0 \\ -k_n v_y \tau & (1-ik_y v_y \tau) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The fact that the Fourier coefficients of the current density are linearly related to those of the electric field, with the same wave vector $\vec{k}_n$, is a result of the choice of a homogeneous plasma model, with simple specularly reflecting trajectories.

The Laplace transform of equation 2.33 can be obtained by using the convolution theorem; using $E_\beta(\vec{k}_n,\omega)$ for the Laplace transform of $E_{1\beta}(\vec{k}_n,t)$, we have

$$j_\alpha(\vec{k}_n,\omega) = j_\alpha^{(0)}(\vec{k}_n,\omega) + \sigma_{\alpha\beta}(\vec{k}_n,\omega) \, E_\beta(\vec{k}_n,\omega) \qquad (2.34)$$

where

$$\left[\sigma_{\alpha\beta}(\vec{k}_n,\omega)\right] = \frac{i\omega_p^2}{4\pi} \int d\vec{v} \; \frac{f_0(\vec{v})}{(\omega - k_n v_x - k_y v_y)^2} \begin{bmatrix} \omega - k_y v_y & ik_y v_x & 0 \\ -ik_n v_y & \omega - k_n v_x & 0 \\ 0 & 0 & \omega - k_n v_x - k_y v_y \end{bmatrix}$$

(2.35)

for $\mathrm{Im}\,\omega > 0$.

This is the frequency and wave vector dependent conductivity tensor of the homogeneous plasma slab.

Using the assumed isotropy of $f(\vec{v})$, the following relations are easily derived from equation 2.35:

$$\sigma_{xx} + \sigma_{yy} = \sigma_\ell + \sigma_t \; , \; \sigma_{xy} = -\sigma_{yx} \; , \; k_n \sigma_{xx} - ik_y \, \sigma_{xy} = k_n \, \sigma_\ell \; ,$$

$$ik_y \, \sigma_{yy} + k_n \, \sigma_{xy} = ik_y \, \sigma_\ell$$

where

$$\sigma_\ell(\vec{k}_n,\omega) = \frac{i\omega\omega_p^2}{4\pi} \int d\vec{u} \; \frac{f_0(\vec{u})}{(\omega - \vec{k}_n \cdot \vec{u})^2}$$

(2.36)

and

$$\sigma_t(\vec{k}_n,\omega) = \sigma_{zz}(\vec{k}_n,\omega) = \frac{i\omega_p^2}{4\pi} \int d\vec{u} \; \frac{f_0(\vec{u})}{(\omega - \vec{k}_n \cdot \vec{u})}$$

(2.37)

are the "longitudinal" and "transverse" conductivities. In the limit $(\mathrm{Im}\,\omega) \to 0$, both σ_t and σ_ℓ have real parts, which are obtained by using the Dirac relation $1/x+i\varepsilon = P\frac{1}{x} - i\pi\delta(x)$ for $\varepsilon \to 0$.

The above relations can be transformed into the following:

$$\sigma_{xx} = \alpha^2 \sigma_\ell + \beta^2 \sigma_t \quad , \quad \sigma_{yy} = \beta^2 \sigma_\ell + \alpha^2 \sigma_t$$

$$\sigma_{xy} = -\sigma_{yx} = i\alpha\beta(\sigma_\ell - \sigma_t) \quad ,$$

$$(2.38)$$

where

$$\alpha = \frac{k_n}{\sqrt{k_n^2 + k_y^2}} \quad , \quad \beta = \frac{k_y}{\sqrt{k_n^2 + k_y^2}}$$

These are equivalent to the statement that the matrix $\begin{pmatrix} \sigma_{xx} & \sigma_{xy} \\ \sigma_{yx} & \sigma_{yy} \end{pmatrix}$

can be diagonalized by a unitary transformation defined by

the equations

$$\left\{ \begin{array}{l} E_\ell = \alpha E_{xs} + i \ \beta E_{yc} \\ \\ E_t = i\beta \ E_{xs} + \alpha E_{yc} \end{array} \right\}$$

$$(2.39)$$

its eigenvalues being σ_ℓ and σ_t . Note that these eigenvalues are

not purely imaginary; the matrix is not anti-hermitian, as it would be

for a dissipationless medium.

It is convenient to make the following <u>approximation</u>:

$$\sigma_t(\vec{k}_n,\omega) \simeq \frac{i\omega_p^2}{4\pi\omega}$$

$$(2.40)$$

This approximation can be justified in the following way. The denomi-

nator in the integrand of equation 2.37 can be expanded, for small

$\vec{k}_n \cdot \vec{u}/\omega$. Since $f_o(\vec{u})$ is isotropic, $\langle \vec{u} \rangle = 0$, and the terms neg-

lected are of order $|\vec{k}_n|^2 \langle v^2 \rangle / \omega^2$. For large n such that

$|\vec{k}_n|^2 \sim \omega^2 / \langle v^2 \rangle$, the approximation 2.40 is not valid, but then

$|\vec{k}_n|^2 / k_o^2 \sim c^2 / \langle v^2 \rangle \gg 1$, and $|1 + \frac{4\pi i}{\omega} \sigma_t| \lesssim 1$ for $\omega^2 \gtrsim \omega_p^2$. Hence

the expression $[|\vec{k}_n|^2/k_o^2 - (1 + \frac{4\pi i}{\omega}\sigma_t)]^{-1}$ in which σ_t will

always occur, can be well approximated by using 2.40 for all n .

Although we will be interested in frequencies fairly close to the

plasma frequency, it is interesting to note that for very low frequen-

cies, $\omega^2 \sim \frac{\langle v^2 \rangle}{c^2}\omega_p^2$, we have $|1 + \frac{4\pi i}{\omega}\sigma_t| \sim \frac{\omega_p^2}{\omega^2} \sim |\vec{k}_n|^2/k_o^2$ for

$|\vec{k}_n|^2 \sim \omega^2/\langle v^2 \rangle$, and so the approximation 2.40 cannot be used for all

n . For frequencies this low, particles can pass through the layer

of width c/ω_p near the boundaries, where the field is not zero, in

one period of the field variation or less. The result is that

absorption of energy can occur in collisionless plasmas which corres-

ponds to the anomalous skin effect in metals (12).

D. Self-Consistent Solution of the Boundary Value Problem

The requirement that the electric field in the plasma be deter-

mined self-consistently means that the field--determined by equations

2.5, 2.6 and 2.7 in terms of the current density, the initial condi-

tions, and the boundary conditions--must be the same field which

produces the current density by equation 2.34. Eliminating the

Fourier components of the current density from these equations, we

obtain the following:

$$(k_o^2 - k_y^2 + \frac{4\pi i k_o}{c}\sigma_{xx}) E_{xs} + (ik_n k_y + \frac{4\pi i k_o}{c}\sigma_{xy}) E_{yc} =$$

$$- ik_o(\frac{4\pi}{c} j_{xs}^{(o)} - \frac{1}{c} E_{xs}(0)) \qquad (2.41)$$

$$(-ik_n k_y + \frac{4\pi i k_o}{c} \sigma_{yx}) E_{xs} + (k_o^2 - k_n^2 + \frac{4\pi i k_o}{c} \sigma_{yy}) E_{yc}$$

$$= -ik_o(\frac{4\pi}{c} j_{yc}^{(o)} - \frac{1}{c} E_{yc}(0) - B_{zo} + (-1)^n B_{zL}) \quad (2.42)$$

$$(k_o^2 - k_n^2 - k_y^2 + \frac{4\pi i k_o}{c} \sigma_{zz}) E_{zc} = -ik_o(\frac{4\pi}{c} j_{zc}^{(o)} + B_{yo} - (-1)^n B_{yL}) \quad (2.43)$$

The last equation can be solved immediately for E_{zc}. Using the approximation 2.40 for σ_{zz} $(= \sigma_t)$ we have

$$E_{zc} = \frac{ik_o}{k_n^2 - k_p^2} [\frac{4\pi}{c} j_{zc}^{(o)} + B_{yo} - (-1)^n B_{yL}] \quad (2.44)$$

where $\quad k_p^2 = k_o^2(1 - \omega_p^2/\omega^2) - k_y^2 \quad (2.45)$

We recall that $j_{zc}^{(o)}$ is the Fourier component of the z component of the macroscopic current density which would exist in the absence of the field disturbance in the plasma, due to particles moving on their unperturbed trajectories. We will assume that this is zero in problems involving excitation of macroscopic waves by boundary fields.

We now multiply by $(2/L) \cos k_n x$ and sum over all n, leaving k_y fixed, to get

$$E_z(x; k_y, \omega) = \frac{2}{L} \sum_{n=0}^{\infty}{}' E_{zc}(\vec{k}_n, \omega) \cos k_n x =$$

$$\frac{2ik_o}{L} \sum_{n=0}^{\infty}{}' \frac{(B_{yo} - (-1)^n B_{yL}) \cos k_n x}{k_n^2 - k_p^2} \quad .$$

This is the Fourier transform with respect to y and z (with wave numbers k_y and $k_z = 0$) of the Laplace transform of the field E_z at x. This Fourier series can be summed by complex variable techniques (13) with the result

$$E_z(x;k_y,\omega) = \frac{-ik_o}{k_p \sin k_p L} [B_{yo} \cos k_p (x - L) - B_{yL} \cos k_p x] \quad (2.46)$$

This result could have been obtained without the use of Fourier series methods by assuming that an electric field which is polarized perpendicular to the plane of incidence (the xy plane) propagates as though it were in a dielectric medium with index of refraction $\sqrt{1 - \omega_p^2/\omega^2}$. This result is based on the approximation 2.40 and is therefore not valid for very low frequencies.

Equation 2.46 cannot be used when $k_p L$ is close to π times an integer, because of the $\sin k_p L$ in the denominator. These combinations of ω and k_y correspond to electromagnetic standing wave modes in the slab, such that a nonzero field can exist in the slab with zero boundary values. The response to given nonzero boundary values then becomes very large, in general, as $k_p L \to m\pi$ (unless $B_{yo} = (-1)^m B_{yL}$). It is limited only by dissipative effects which have been neglected in the approximations leading to equation 2.46.

The z-component of the electric field will not be considered further, because it does not couple to the longitudinal plasma waves which are of primary interest.

We now return to the task of solving equations 2.41 and 2.42. Using the relations 2.38 and the approximation 2.40, the determinant

of the system of two equations can be shown to be

$$d = k_o^2 \left(1 + \frac{4\pi i}{\omega}\sigma_\ell\right)(k_p^2 - k_n^2)$$

where k_p^2 is given by equation 2.45. It is again necessary to require

that $k_p^2 \neq k_n^2$ so that the determinant is not zero; the solution is

then straightforward. Defining the longitudinal dielectric function

$\varepsilon = 1 + \frac{4\pi i}{\omega}\sigma_\ell$, we obtain the following:

$$E_{xs}/ik_o = \frac{(4\pi/c)\, i\beta j_t^{(o)}}{k_n^2 - k_p^2} + \frac{4\pi k_n\, \rho_c^{(o)}}{ik_o(k_n^2 + k_y^2)\,\varepsilon(\vec{k}_n,\omega)}$$

$$+ i\alpha\beta(B_{zo} - (-1)^n B_{zL})\left[\frac{1}{k_o^2\,\varepsilon(\vec{k}_n,\omega)} + \frac{1}{k_n^2 - k_p^2}\right] \qquad (2.47)$$

$$E_{yc}/ik_o = \frac{-(4\pi/c)\,\alpha j_t^{(o)}}{k_n^2 - k_p^2} - \frac{4\pi k_y\,\rho_c^{(o)}}{k_o(k_n^2 + k_y^2)\,\varepsilon(\vec{k}_n,\omega)}$$

$$+ (B_{zo} - (-1)^n B_{zL})\left[\frac{\beta^2}{k_o^2\,\varepsilon(\vec{k}_n,\omega)} - \frac{\alpha^2}{k_n^2 - k_p^2}\right] \qquad (2.48)$$

where $j_t^{(o)} = i\beta\, j_{xs}^{(o)} + \alpha j_{yc}^{(o)}$, α and β are defined in equation

2.38 and $\rho_c^{(o)}$ is the Fourier coefficient of a charge density for

particles unperturbed by the electric field disturbance. We have used

the continuity equation for this charge density

$$\text{div }\vec{j}^{(o)} + \frac{\partial\rho^{(o)}}{\partial t} = 0 \text{ , or } k_n j_{xs}^{(o)} + ik_y j_{yc}^{(o)} - i\omega\rho_c^{(o)} - \rho_c(0) = 0 \text{ ,}$$

the initial condition $\text{div }\vec{E} - 4\pi\rho = 0$ at $t = 0$, or

$k_n E_{xs}(0) + ik_y E_{yc}(0) - 4\pi\rho_c(0) = 0$ and the initial condition

curl $\vec{E} = 0$ at $t = 0$, or $k_n E_{yc}(0) + ik_y E_{xs}(0) = 0$.

We now assume that $j_t^{(o)}$ and $\rho_c^{(o)}$ are zero. The Fourier

series for a fixed value of k_y are then

$$E_x(x;k_y,\omega) = -\frac{2k_y}{k_o L} \sum_{n=1}^{\infty} \frac{k_n(B_{zo} - (-1)^n B_{zL}) \sin k_n x}{(k_n^2 + k_y^2)\ \epsilon(\vec{k}_n,\omega)}$$

$$-\frac{2k_y k_o}{L} \sum_{n=1}^{\infty} \frac{k_n(B_{zo} - (-1)^n B_{zL}) \sin k_n x}{(k_n^2 + k_y^2)(k_n^2 - k_p^2)} \qquad (2.49)$$

$$E_y(x;k_y,\omega) = \frac{2ik_y^2}{k_o L} \sum_{n=0}^{\infty}{}' \frac{(B_{zo} - (-1)^n B_{zL}) \cos k_n x}{(k_n^2 + k_y^2)\ \epsilon(\vec{k}_n,\omega)}$$

$$-\frac{2ik_o}{L} \sum_{n=1}^{\infty} \frac{k_n^2(B_{zo} - (-1)^n B_{zL}) \cos k_n x}{(k_n^2 + k_y^2)\ (k_n^2 - k_p^2)} \qquad (2.50)$$

The second infinite series in the expression for E_y can be summed so

that

$$E_y(x;k_y,\omega) = \frac{2ik_y^2}{k_o L} \sum_{n=0}^{\infty}{}' \frac{(B_{zo} - (-1)^n B_{zL}) \cos k_n x}{(k_n^2 + k_y^2)\ \epsilon(\vec{k}_n,\omega)}$$

$$-\frac{ik_y}{k_o} \frac{B_{zo}\cosh k_y(L-x) - B_{zL}\cosh k_y x}{(1 - \omega_p^2/\omega^2)\sinh k_y L}$$

$$+\frac{ik_p}{k_o} \frac{B_{zo}\cos k_p(L-x) - B_{zL}\cos k_p x}{(1 - \omega_p^2/\omega^2)\sin k_p L} \qquad (2.51)$$

The infinite series of terms involving the longitudinal dielec-

tric function $\epsilon(\vec{k}_n,\omega)$ represents the *field due to the longitudinal*

plasma waves which are coupled to the electromagnetic field polarized

with $\vec{E}$ in the xy plane. For frequencies close to the plasma frequency these waves are important, and are the main feature of interest here. In the result of the linearized calculation of the self-consistent electric field, the dielectric function $\varepsilon(\vec{k}_n,\omega)$ appears in the denominator. The field is therefore enhanced due to the <u>collective behavior</u> of the plasma under those conditions for which $\varepsilon(\vec{k}_n,\omega)$ is small.

The last two terms in equation 2.51 were obtained by summing the infinite series of modes of the "transverse" part of the electric field (the divergence-free part). The characteristic lengths for the variation of these terms are comparable with or greater than the wavelength of electromagnetic waves in vacuum. Their effective phase velocities are comparable with or greater than that of light, and hence much greater than that of the particles in the plasma, so they are not greatly affected by the thermal motions of the particles.

In summary, we have obtained the self-consistent electric field in a homogeneous plasma slab in terms of the boundary values of the tangential components of the magnetic field at the surfaces of the slab. The Laplace transforms with respect to time and Fourier transforms with respect to the y and z coordinates (with $k_z = 0$) are given by equations 2.46, 2.49 and 2.51. The dependence of the solution on the initial conditions has been discarded; only the part of the current density proportional to the field disturbance E_1 has been kept, and this has been assumed to be a small perturbation to the particle motions.

III. COLLISIONLESS ABSORPTION IN A PLASMA SLAB

A. Surface Impedance

In the previous section the solution to a boundary value problem was obtained, giving the electric field in a plasma slab in terms of the boundary values of the tangential magnetic field. In practical applications these boundary values are unknown, however, and must be solved for. The procedure which must be used is to assume the form of the field solution outside the plasma, and match it to the solution inside the plasma by requiring that the tangential electric and magnetic fields be continuous at the boundary. To avoid carrying an unimportant multiplicative constant in this procedure, it is convenient to match the ratio of field components, e.g., E_y/B_z, at the boundary. The parameters in the field solution outside the plasma can then be related to the properties of the plasma.

Suppose we assume the fields outside the plasma to be those appropriate to a plane, monochromatic wave incident on the plasma at angle θ, polarized with the electric vector in the plane of incidence (the xy plane), with corresponding reflected and transmitted waves. Strictly speaking, the amplitudes of these plane waves should be regarded as Fourier transforms with respect to y and z, and Laplace transforms with respect to time, of arbitrary fields satisfying Maxwell's equations in vacuum. To insure convergence of the integral defining the Laplace transform, it is necessary to require that $\mathrm{Im}\,\omega > 0$. The results for a nearly monochromatic wave (not exactly monochromatic because it was turned on at some time in the past) can

be obtained at the end of the calculation by passing to the limit $(\text{Im } \omega) \to 0$.

The fields of the incident and reflected waves on one side of the slab, and the transmitted wave on the other side, may be written as follows (except for an unimportant multiplicative factor and the factor $e^{i(k_y y - \omega t)}$):

$$E_y = \begin{cases} (e^{ik_x x} - Re^{-ik_x x})\cos\theta \; , & x \leq 0 \\ Te^{ik_x(x-L)} \cos\theta \; , & x \geq L \end{cases} \qquad (3.1)$$

$$B_z = \begin{cases} e^{ik_x x} + Re^{-ik_x x} \; , & x \leq 0 \\ Te^{ik_x(x-L)} \; , & x \geq L \end{cases} \qquad (3.2)$$

where $k_y = k_o \sin\theta$, $k_x = k_o \cos\theta$.

The ratio E_y/B_z at the boundary $x = L$ is equal to $\cos\theta$, with the assumed field solution for $x \geq L$. Using this boundary condition the ratio E_y/B_z at the boundary $x = 0$ can be determined from the field solution in the plasma slab. Denoting this ratio by Z , the surface impedance, the continuity condition at $x = 0$ takes the form

$$\frac{(1 - R)\cos\theta}{1 + R} = Z \; .$$

Hence, the reflection coefficient R is given in terms of the plasma properties (contained in the surface impedance) by

$$R = \frac{1 - Z/\cos\theta}{1 + Z/\cos\theta}$$

Although the above procedure is fairly simple in principle, it can become algebraically complicated. An alternate procedure can be

used, which takes advantage of the symmetry of the slab with respect to reflection through its midplane $x = L/2$. In this procedure we consider plane waves incident on the plasma slab from <u>both</u> sides, with the same amplitude and angle of incidence, so that the ratio $|E_y| / |B_z|$ is the same at both boundaries. If the phase and polarization ($\vec{E}$ in the plane of incidence) are specified for one wave, there are two possibilities for the other wave. We call the resulting field configurations "symmetric", or "antisymmetric", according to whether the ratio E_{yo}/E_{yL} is $+1$ or -1. In the <u>symmetric</u> case the fields may be written as follows:

$$E_y^{(s)} = \begin{cases} \frac{1}{2} (e^{ik_x x} - R_s e^{-ik_x x}) \cos \theta \quad , \quad x \lesssim 0 \\ \frac{1}{2} (e^{ik_x(L-x)} - R_s e^{-ik_x(L-x)}) \cos \theta, \quad x \gtrsim L \end{cases} \tag{3.3}$$

$$B_z^{(s)} = \begin{cases} \frac{1}{2} (e^{ik_x x} + R_s e^{-ik_x x}) \quad , \quad x \lesssim 0 \\ -\frac{1}{2} (e^{ik_x(L-x)} + R_s e^{-ik_x(L-x)}), \quad x \gtrsim L \end{cases} \tag{3.4}$$

The amplitude R_s of the scattered wave relative to the incident wave is the same on both sides of the slab, by symmetry. It is related to the surface impedance Z_s defined by

$$Z_s = [E_y^{(s)}/B_z^{(s)}]_{x=0} = - [E_y^{(s)}/B_z^{(s)}]_{x=L} \tag{3.5}$$

as follows:

$$R_s = \frac{1 - Z_s/\cos\theta}{1 + Z_s/\cos\theta} \quad . \tag{3.6}$$

In the <u>antisymmetric</u> case the fields may be written

$$E_y^{(a)} = \begin{cases} \frac{1}{2}(e^{ik_x x} - R_a e^{-ik_x x})\cos\theta \quad , \quad x \leqslant 0 \\ -\frac{1}{2}(e^{ik_x(L-x)} - R_a e^{-ik_x(L-x)})\cos\theta, \quad x \geqslant L \end{cases} \tag{3.7}$$

$$B_z^{(a)} = \begin{cases} \frac{1}{2}(e^{ik_x x} + R_a e^{-ik_x x}) \quad , \quad x \leqslant 0 \\ \frac{1}{2}(e^{ik_x(L-x)} + R_a e^{-ik_x(L-x)}) , \quad x \geqslant L \end{cases} \tag{3.8}$$

The scattering coefficient R_a is related to the surface impedance Z_a , defined by

$$Z_a = [E_y^{(a)}/B_z^{(a)}]_{x=0} = -[E_y^{(a)}/B_z^{(a)}]_{x=L}$$

as follows:

$$R_a = \frac{1 - Z_a/\cos\theta}{1 + Z_a/\cos\theta} \quad . \tag{3.9}$$

The fields in these symmetric and antisymmetric cases are the same as the <u>symmetric and antisymmetric parts of the unsymmetric fields</u> 3.1 and 3.2:

$$E_y^{(s)}(x) = \frac{1}{2}[E_y(x) + E_y(L-x)] \quad , \quad E_y^{(a)}(x) = \frac{1}{2}[E_y(x) - E_y(L-x)]$$

(and similarly for B_z) provided we make the identifications $R_s = R - T$, $R_a = R + T$. That is, the fields for the <u>unsymmetric</u> case,

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equations 3.1 and 3.2, can be considered as the superposition of the symmetric and antisymmetric fields:

$$E_y = E_y^{(a)} + E_y^{(s)} \quad , \quad B_z = B_z^{(a)} + B_z^{(s)} \quad \text{(see Figure 1)}$$

where

$$R = \frac{1}{2}(R_a + R_s) \quad \text{and} \quad T = \frac{1}{2}(R_a - R_s) \tag{3.10}$$

Hence, the scattering coefficients R and T are related to Z_a and Z_s by equations 3.10, 3.9 and 3.6.

The surface impedances Z_a and Z_s for a homogeneous plasma slab can be obtained from the solution for the field in the plasma, given by equation 2.51. In the "symmetric" case we have $B_{zL} = -B_{z0}$, so we obtain, by setting $x = 0$ in 2.51:

$$Z_s = \frac{4ik_y^2}{k_o L} \sum_{n \text{ even}}' \frac{1}{(k_n^2 + k_y^2)\, \varepsilon(\vec{k}_n, \omega)} - \frac{ik_y}{k_o}\left[\frac{\cosh k_y L + 1}{(1 - \omega_p^2/\omega^2)\sinh k_y L}\right]$$

$$+ \frac{ik_p}{k_o}\left[\frac{\cos k_p L + 1}{(1 - \omega_p^2/\omega^2)\sin k_p L}\right] \tag{3.11}$$

Similarly, in the "antisymmetric" case we have $B_{zL} = B_{z0}$, so

$$Z_a = \frac{4ik_y^2}{k_o L} \sum_{n \text{ odd}} \frac{1}{(k_n^2 + k_y^2)\varepsilon(\vec{k}_n, \omega)} - \frac{ik_y}{k_o}\left[\frac{\cosh k_y L - 1}{(1 - \omega_p^2/\omega^2)\sinh k_y L}\right]$$

$$+ \frac{ik_p}{k_o}\left[\frac{\cos k_p L - 1}{(1 - \omega_p^2/\omega^2)\sin k_p L}\right] \tag{3.12}$$

Ey GIVEN BY (3.1)   Ey GIVEN BY (3.3)   Ey GIVEN BY (3.7)
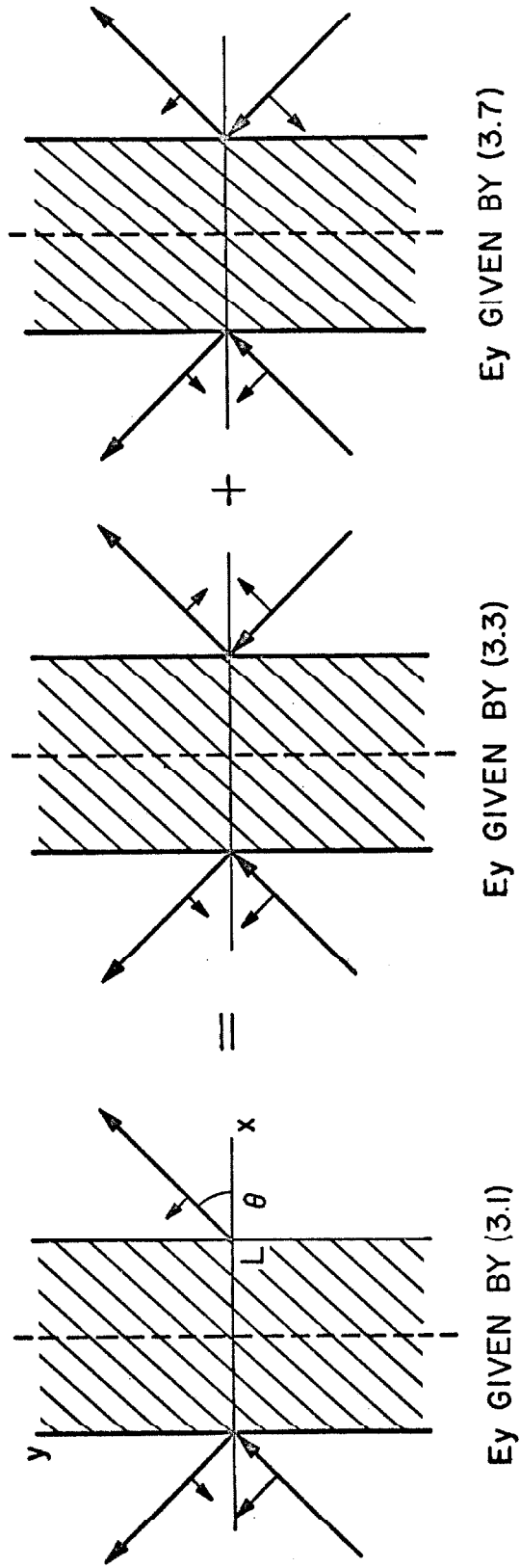
Figure I: DECOMPOSITION OF THE PLANE WAVE ELECTRIC FIELDS
INTO SYMMETRIC AND ANTISYMMETRIC PARTS.

In equations 3.11 and 3.12, $k_p = k_o \sqrt{\cos^2\theta - \omega_p^2/\omega^2}$, and $k_y = k_o \sin\theta$.

The last two terms in each of equations 3.11 and 3.12 appear to become infinite for $\omega = \omega_p$, but the infinite parts of the two terms cancel. That is, the sum of the last two terms has a finite limit for $\omega \to \omega_p$. It will be sufficient in the following to approximate equations 3.11 and 3.12 by replacing the last two terms by their limiting values for $\omega \to \omega_p$:

$$Z_s \simeq \frac{4ik_y^2}{k_o L} \sum_{n\ even}' \frac{1}{(k_n^2 + k_y^2)\ \epsilon(\vec{k}_n,\omega)}$$

$$- \frac{i(k_o L)(\cosh k_y L + 1)(\sinh k_y L/k_y L - 1)}{2 \sinh^2 / k_y L} \qquad (3.13)$$

$$Z_a \simeq \frac{4ik_y^2}{k_o L} \sum_{n\ odd} \frac{1}{(k_n^2 + k_y^2)\ \epsilon(\vec{k}_n,\omega)}$$

$$- \frac{i(k_o L)(\cosh k_y L - 1)(\sinh k_y L/k_y L + 1)}{2 \sinh^2 k_y L} \qquad (3.14)$$

These approximations are valid for frequencies close to the plasma frequency, which is the case of most interest. In the $n = 0$ term in equation 3.13, the dielectric function is real, $\epsilon(\hat{y}k_y,\omega) \simeq 1 - \omega_p^2/\omega^2$, since $f_o(\vec{u}) = 0$ for $u_y = c/\sin\theta$ in 2.36; no particles can travel faster than light.

B. The Absorption Coefficient of a Collisionless Plasma Slab

The absorption coefficient α of a plasma slab is defined as the ratio of power absorbed to power flow into the slab in the incident wave. For the fields given by equations 3.1 and 3.2 the fraction of the incident power which is reflected is $|R|^2$, and the fraction transmitted is $|T|^2$, so the fraction absorbed is $\alpha = 1 - |R|^2 - |T|^2$.

Using equation 3.10 this can be written as

$$\alpha = 1 - \frac{1}{2}|R_s|^2 - \frac{1}{2}|R_a|^2 \qquad (3.15)$$

where R_s and R_a are the reflection coefficients for the symmetric and antisymmetric cases considered in the last section. For these cases there is no power flow across the midplane of the slab because the x-component of the Poynting vector is zero there: $B_z = 0$ in the symmetric case, and $E_y = 0$ in the antisymmetric case, at $x = L/2$. Hence, the absorption coefficients in these cases are

$$\alpha_s = 1 - |R_s|^2 \qquad \text{and} \qquad \alpha_a = 1 - |R_a|^2 \qquad (3.16)$$

Combining equations 3.15 and 3.16 gives $\alpha = \frac{1}{2}(\alpha_s + \alpha_a)$. $\qquad (3.17)$

Hence, the fraction of power absorbed from the wave is half the sum of the fractions which would be absorbed from the symmetric and anti-symmetric parts of the fields separately.

Using equations 3.6 and 3.9, equations 3.16 may be written

$$\alpha_s = \frac{4 \text{ Re } Z_s/\cos \theta}{|1 + Z_s/\cos \theta|^2} \quad , \quad \alpha_a = \frac{4 \text{ Re } Z_a/\cos \theta}{|1 + Z_a/\cos \theta|^2} \quad (3.18)$$

which, with 3.17, gives the absorption coefficient α in terms of the symmetric and antisymmetric surface impedances.

The real parts of the surface impedances which determine the absorption coefficients by equations 3.18 are obtained using the expressions 3.13 and 3.14 for Z_s and Z_a :

$$\text{Re } Z_s = \frac{4k_y^2}{k_o L} \sum_{n=2,4,\cdots} \frac{\text{Im } \epsilon(\vec{k}_n,\omega)}{|\vec{k}_n|^2 |\epsilon(\vec{k}_n,\omega)|^2} \quad (3.19)$$

$$\text{Re } Z_a = \frac{4k_y^2}{k_o L} \sum_{n=1,3,\cdots} \frac{\text{Im } \epsilon(\vec{k}_n,\omega)}{|\vec{k}_n|^2 |\epsilon(\vec{k}_n,\omega)|^2} \quad (3.20)$$

Using equation 2.36, the explicit formula for the longitudinal dielectric function is

$$\epsilon(\vec{k}_n,\omega) = 1 + \frac{4\pi i}{\omega} \sigma_\ell (\vec{k}_n,\omega) = 1 - \omega_p^2 \int \frac{d\vec{u} \, f_o(\vec{u})}{(\omega - \vec{k}_n \cdot \vec{u})^2}$$

for $\text{Im } \omega > 0$. Passing to the limit $(\text{Im } \omega) \to 0$, and integrating over the two components of velocity perpendicular to the direction of $\vec{k}_n$, we obtain

$$\text{Re } \epsilon(\vec{k}_n,\omega) = 1 - \frac{\omega_p^2}{|\vec{k}_n|^2} \, \mathrm{P} \int_{-\infty}^{\infty} \frac{dv \, F(v)}{(v - \omega/|\vec{k}_n|)^2} \quad , \quad (3.21)$$

and

$$\text{Im } \epsilon(\vec{k}_n,\omega) = - \frac{\pi \omega_p^2}{|\vec{k}_n|^2} \, F'(\frac{\omega}{|\vec{k}_n|}) \quad (3.22)$$

where

$$F(v) = \int d\vec{u} \, f_o(\vec{u}) \, \delta(v - \frac{\vec{k}_n \cdot \vec{u}}{|\vec{k}_n|})$$

is the one-dimensional velocity distribution, i.e., the function

$f_o(\vec{u})$ integrated over the two components of velocity perpendicular to

$\vec{k}_n$. But from the assumed *isotropy* of $f_o(\vec{v})$, F(v) does not depend

upon the *direction* of $\vec{k}_n$. In equation 3.21 the principal part is

taken with respect to the pole in the integrand. The integral in

equation 3.22 gives the result of integrating around a small semi-

circle, passing below the pole in the complex v plane. It is pro-

portional to the derivative of the one-dimensional probability density

for the component of velocity in the direction of propagation of the

n^{th} normal mode of the slab, evaluated at its phase velocity.

Combining equations 3.18, 3.20 and 3.22, we have

$$\alpha_a = \frac{\omega \, \omega_p^2 \sin^2\theta}{c \cos\theta \, |1 + Z_a/\cos\theta|^2} \frac{16\pi}{L} \sum_{n=1,3,\cdots} \frac{-F'(\frac{\omega}{|\vec{k}_n|})}{|\vec{k}_n|^4 |\epsilon(\vec{k}_n,\omega)|^2} \quad (3.23)$$

α_s is given by a similar expression, with Z_a replaced by Z_s , and

the summation being over $n=2,4,6,\cdots$. Finally, the *general expres-*

sion for the absorption coefficient is given, according to equation

3.17, as follows:

$$\alpha = \frac{1}{2}(\alpha_s + \alpha_a) = \frac{\omega \, \omega_p^2 \sin^2\theta}{c \cos\theta} \frac{8\pi}{L} \sum_{n=1}^{\infty} \frac{-F'(\frac{\omega}{|\vec{k}_n|})}{|1 + Z_n/\cos\theta|^2 \, |\vec{k}_n|^4 \, |\epsilon(\vec{k}_n,\omega)|^2}$$

$$(3.24)$$

where $\quad Z_n = \left\{ \begin{array}{l} Z_s \ , \ n \text{ even} \\ Z_a \ , \ n \text{ odd} \end{array} \right\}$ $\qquad (3.25)$

The absorption coefficient is given as an infinite series, cor-
responding to the infinite number of normal modes of the plasma slab.
The nth term determines the damping of the plasma waves excited in the
slab by the incident electromagnetic waves in the nth mode. The damp-
ing of this mode is due to electrons whose velocities happen to be
nearly the same as the phase velocity of the mode, and is proportional
to the derivative of the velocity distribution at this phase velocity.
This type of damping was first discussed by Landau (5) in connection
with the initial-value problem for plasma waves in an infinite homo-
geneous medium. It will be discussed in more detail in connection
with the present boundary-value problem in Section D.

For any given frequency ω all plasma wave modes will be
present in the slab. As indicated by equation 3.24, the damping of all
these modes contributes to the power absorption from the external
fields. It is interesting to note that particles contribute to the
collisionless absorption if they belong to any one of a large number
of velocity classes. This is characteristic of bounded or inhomo-
geneous plasmas.

Each plasma wave mode has a definite wavelength. If the fre-
quency is chosen to be the natural frequency for plasma waves with one
of these wavelengths, then that particular mode will be strongly
excited. Thus there is a sequence of frequencies $\{\omega_n\}$ at which the
response of the plasma to the incident wave may be quite large, and
the power absorption may be expected to be correspondingly large.
These natural frequencies are defined by the condition that

$|\varepsilon(\vec{k}_n, \omega_n)| \simeq 0$, which is the dispersion relation for longitudinal

plasma waves. Although $\mathrm{Im}\ \varepsilon$, given by equation 3.22, cannot be zero

in general for $\omega/|\vec{k}_n|$ finite, it can be very small for $\omega/|\vec{k}_n|$ much

larger than the electron mean thermal speed. (This assumes that the

derivative of the electron velocity distribution function goes to zero

reasonably fast for large velocities.)

Assuming $\omega/|\vec{k}_n|$ to be large, the following asymptotic expres-

sion for $\mathrm{Re}\ \varepsilon$ can be obtained from equation 3.21:

$$\mathrm{Re}\ \varepsilon(\vec{k}_n, \omega) \simeq 1 - \frac{\omega_p^2}{\omega^2}\left[1 + 3\frac{|\vec{k}_n|^2 \langle v^2 \rangle}{\omega^2}\right]$$

for $(|\vec{k}_n|^2 \langle v^2 \rangle / \omega^2) \ll 1$, where $\langle v^2 \rangle = \int_{-\infty}^{\infty} dv\ v^2 F(v)$. The fre-

quencies $\{\omega_n\}$ may then be determined approximately by setting

$\mathrm{Re}\ \varepsilon = 0$ which gives

$$\omega_n^2 \simeq \omega_p^2 + 3|\vec{k}_n|^2 \langle v^2 \rangle \ . \tag{3.26}$$

Assuming that $\mathrm{Im}\ \varepsilon(\vec{k}_n, \omega_n) \ll 1$, we have $|\varepsilon(\vec{k}_n, \omega_n)| \ll 1$. It

follows that the amplitude of the nth mode of the electric field E_x

(equation 2.49), which varies like $\sin k_n x$, is quite large at the

frequency $\omega = \omega_n$. This is because of the $\varepsilon(\vec{k}_n, \omega)$ in the

denominator of the nth term in the first series in equation 2.49. The

plasma wave with wavelength corresponding to the nth mode of the slab

has been assumed to be only slightly damped, so that its amplitude may

be quite large when it is driven by external fields at its natural

frequency. It follows also that the real part of the surface impedance

Z_s or Z_a may be quite large at the frequency $\omega = \omega_n$, because of

the $|\varepsilon(\vec{k}_n,\omega)|^2$ in the denominator of the nth term of equation 3.19 for n even, or 3.20 for n odd. The absorption coefficient given by equation 3.24 might also be expected to be large at the frequencies $\{\omega_n\}$, but there are further conditions that this be so, which are discussed in Section C.

The above discussion is based on the assumption that the plasma wave modes have small damping. This can only be true if the mode number n is not too large. This can be made more precise by considering the important special case of a Maxwellian electron velocity distribution

$$F(u) = \frac{1}{\pi^{1/2} v_e} e^{-u^2/v_e^2} \tag{3.27}$$

The mean-square velocity in one direction is

$$\langle u^2 \rangle = \frac{1}{2} v_e^2 \equiv \kappa T_e/m$$

and the natural frequency of the nth mode of the slab is given by

$$\omega_n^2 = \omega_p^2 \left[1 + \frac{3}{2\zeta_n^2} \right] \text{ for } \zeta_n^2 \gg 1 , \tag{3.28}$$

where

$$\zeta_n^2 = \frac{\omega_p^2}{|\vec{k}_n|^2 v_e^2} = \left[2(\frac{n\pi\lambda_D}{L})^2 + \frac{v_e^2}{c^2} \sin^2\theta \right]^{-1} \tag{3.29}$$

and $\lambda_D = (\kappa T_e/4\pi n e^2)^{1/2}$ is the Debye length. With the wave-vector and frequency of the nth mode, the imaginary part of the dielectric function has a value given approximately by

$$\text{Im } \varepsilon(\vec{k}_n, \omega_n) \simeq 2\pi^{1/2} \zeta_n^3 e^{-3/2} e^{-\zeta_n^2} \tag{3.30}$$

Therefore, the nth mode will have small damping, i.e. $\text{Im } \varepsilon(\vec{k}_n, \omega_n) \ll 1$ if $n \ll (L/\sqrt{2}\,\pi\,\lambda_D)$. We will assume that the slab thickness is many Debye lengths: $L/\lambda_D \gg 1$ so that there will be several modes whose damping is small. As a numerical example, if $n \stackrel{\sim}{<} (0.3)(L/\sqrt{2}\,\pi\,\lambda_D)$, say, then $\text{Im } \varepsilon(\vec{k}_n, \omega_n) \stackrel{\sim}{<} 0.04$. For $(L/\sqrt{2}\,\pi\,\lambda_D) = 20$, this condition is satisfied for $n \stackrel{\sim}{<} 6$. Larger values of n correspond to wavelengths comparable to the Debye length, which have more Landau damping.

C. Absorption Lines and Scattering Resonances

The behavior of the absorption coefficient, as a function of frequency, is not immediately apparent from equation 3.24. If we go back to equation 3.18 we have

$$\alpha_a = \frac{4 \text{ Re } Z_a/\cos\theta}{(1 + \text{Re } Z_a/\cos\theta)^2 + (\text{Im } Z_a/\cos\theta)^2} \tag{3.31}$$

and a similar expression for α_s . It is true that $\text{Re } Z_a$ may be quite large at the natural frequencies $\{\omega_n\}$ of the plasma wave normal modes, and its frequency behavior may take the form of narrow resonance lines at these frequencies if the wave damping is small (see Figure 2). But the denominator of equation 3.31 contains $\text{Re } Z_a$ to a higher power than the numerator, so the behavior of α_a near the resonance frequencies is not clear.

To determine the behavior of the absorption coefficient for ω in the vicinity of ω_n , it is convenient to use the following approximation to the dielectric function:

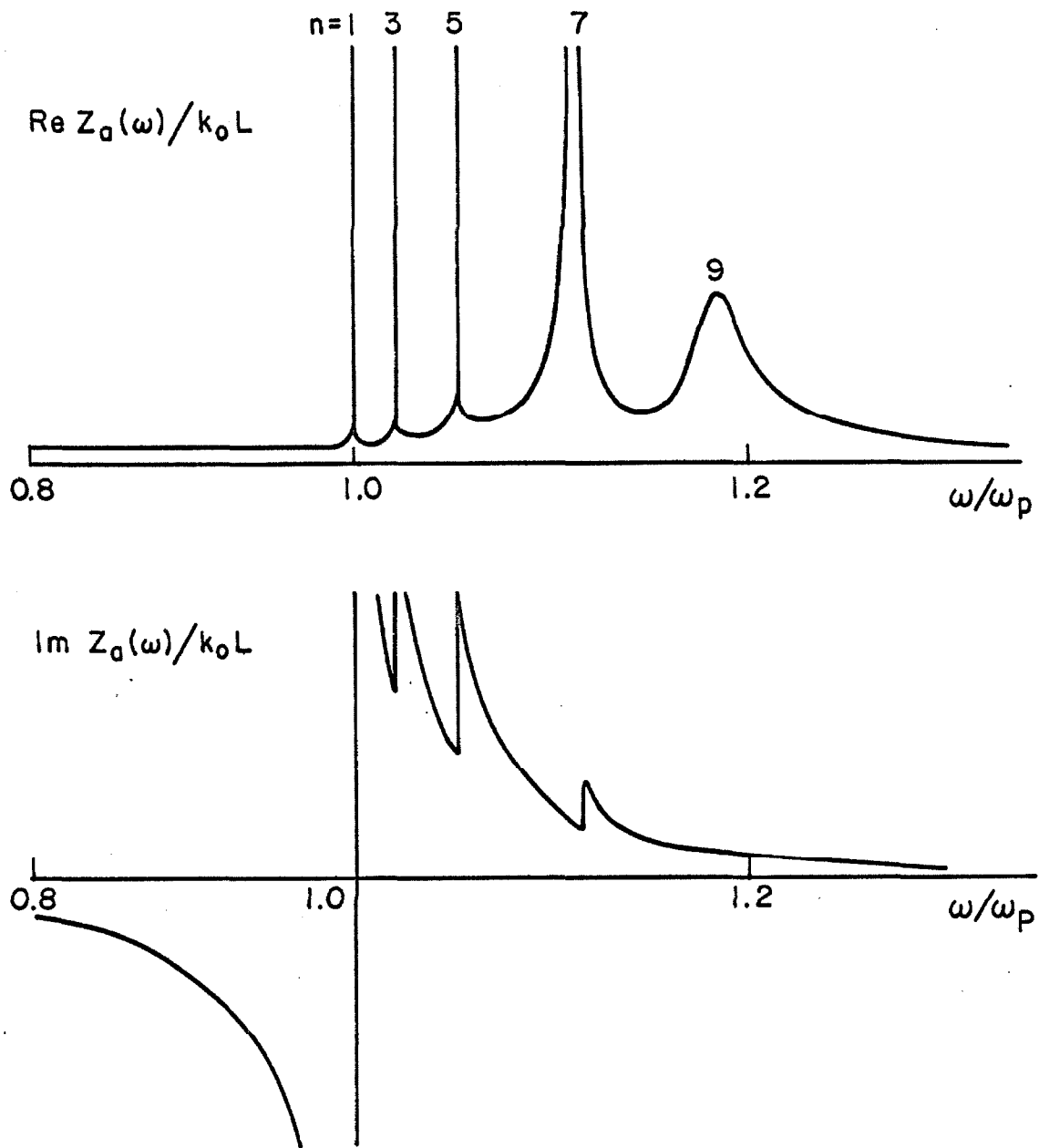Figure 2 : REAL AND IMAGINARY PARTS OF NORMALIZED ANTISYMMETRIC SURFACE IMPEDANCE $Z_a(\omega)/k_0L$ FOR A "THIN"PLASMA SLAB $(k_0L \ll 1)$.THE VERTICAL SCALE IN THE UPPER CURVE IS MAGNIFIED BY A FACTOR OF ABOUT 45 COMPARED WITH THE LOWER CURVE. SLAB THICKNESS: $L/\sqrt{2\pi}\lambda_D = 20$. ANGLE OF INCIDENCE: $\theta = 45°$ [BASED ON NUMERICAL CALCULATIONS BY R.W. GOULD, FOR THE "PLASMA CAPACITOR" PROBLEM.]

$$\varepsilon(\vec{k}_n, \omega) \simeq 2[(\omega/\omega_n - 1) + i\,\delta_n] \qquad . \qquad (3.32)$$

This is valid for $\omega \simeq \omega_n$, provided $\delta_n \ll 1$, where

$$\delta_n = \frac{1}{2}\, \text{Im}\; \varepsilon(\vec{k}_n, \omega_n) \equiv \gamma_n/\omega_n \qquad . \qquad (3.33)$$

(γ_n is the Landau damping rate of the nth mode).

Now using only the nth term in equation 3.14 (where n is odd) we have

$$\text{Re}\; Z_a(\omega) \simeq \frac{4k_y^2}{k_o L(k_n^2 + k_y^2)}\; \text{Im}\left[\frac{-1}{\varepsilon(\vec{k}_n, \omega)}\right], \quad \text{for}\quad \omega \simeq \omega_n \; .$$

The other terms in the series can be neglected for $\omega \simeq \omega_n$. Using equation 3.32 and setting $k_y = k_o \sin\theta$, we have

$$\text{Re}\; Z_a(\omega)/\cos\theta \simeq \frac{r_n\,\delta_n}{(\omega/\omega_n - 1)^2 + \delta_n^2} \;, \quad \text{for}\quad \omega \simeq \omega_n \qquad (3.34)$$

where

$$r_n = \frac{2(k_o L)\sin^2\theta}{\cos\theta(n^2\pi^2 + k_o^2 L^2 \sin^2\theta)} \qquad (3.35)$$

is a measure of the strength of coupling between the nth mode and the electromagnetic field. Equation 3.34 expresses the fact that, near the resonance frequency ω_n , $\text{Re}\; Z_a(\omega)$ can be approximated by a narrow Lorentzian, provided the relative width δ_n is small compared with unity.

Now equations 3.14 and 3.32 may be used to obtain an approximate expression for $\text{Im}\; Z_a(\omega)$, for $\omega \simeq \omega_n$. Let us assume at first,

that n is odd but not equal to one, i.e. n = 3,5,$\cdots$. From Figure

2 one sees that both the first and the nth terms in the series in equa-

tion 3.14 are equally important numerically. The first term varies

relatively slowly with frequency near ω_n , so it can be replaced by

its value at $\omega = \omega_n$. We find that

$$\text{Im } Z_a(\omega)/\cos\theta \simeq \frac{r_n(\omega/\omega_n - 1)}{(\omega/\omega_n - 1)^2 + \delta_n^2} + c_n \text{ , for } \omega \simeq \omega_n \qquad (3.36)$$

where

$$c_n = \frac{r_1(\omega_n/\omega_1 - 1)}{(\omega_n/\omega_1 - 1)^2 + \delta_1^2} - \frac{(k_oL)(\cosh k_yL - 1)(\sinh k_yL/k_yL + 1)}{2\cos\theta \sinh^2 k_yL} \qquad (3.37)$$

The restriction that $n \neq 1$ can now be removed. These expressions

can also be used for $n = 1$ because the first term in the expression

for c_n is zero then, as it should be.

Now equations 3.34 and 3.36 can be substituted into equation

3.31 to give an approximate expression for α_a . After some algebra

we find that (for $\omega \simeq \omega_n$)

$$\alpha_a(\omega) \simeq \frac{4\delta_n r_n/(1 + c_n^2)}{(\omega/\omega_n - 1 + \frac{r_n c_n}{1 + c_n^2})^2 + \Delta_n^2} \qquad (3.38)$$

where $\quad \Delta_n = \delta_n + \frac{r_n}{(1 + c_n^2)} \qquad . \qquad (3.39)$

Thus the absorption coefficient α_a , as a function of frequency

has a Lorentzian line shape near the frequency ω_n . But two new fea-

tures are present in equations 3.38 and 3.39. First, the relative

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width of the line Δ_n is the sum of the Landau damping width δ_n and a "radiation damping width" $r_n/(1+c_n^2)$ which is the energy radiated by the nth mode in one period of the wave divided by 4π times the energy stored. Second, there is a relative shift of $-r_n c_n/(1+c_n^2)$ in the resonance frequency because of the radiation damping and because of the effective coupling to the transverse modes of the electromagnetic field and (for $n \neq 1$) to the $n=1$ mode.

The absorption coefficient $\alpha_a(\omega)$ will be very small for all frequencies, because $\delta_n \ll 1$, unless $r_n \ll 1$ also. Thus, a sharp line structure in $Re\ Z_a(\omega)$ does not lead to a similar structure for $\alpha_a(\omega)$ unless the radiation damping is as small as the Landau damping. This will be true, for a given value of θ, only* if the slab thickness is small compared with the wavelength of the electromagnetic waves: $k_o L \ll 1$. In this case c_n^2 and $r_n c_n$ can be neglected, compared with unity, and we have (for $\omega \simeq \omega_n$)

$$\alpha_a(\omega) \simeq \frac{4 r_n \delta_n}{(\omega/\omega_n - 1)^2 + (\delta_n + r_n)^2} \ , \quad \text{if } k_o L \ll 1 \qquad (3.40)$$

---

* From equation 3.35 it is clear that r_n is also small for a thick slab ($k_o L \gg 1$). But in that case the resonance frequencies of the first N modes are all approximately equal to ω_p, where $N \sim k_o L$, so these should be considered together, as a single resonance. The effective coupling parameter for this combined resonance is about N times the value of r_n for a single one, and is of order unity. Hence, the radiation damping cannot be made as small as the Landau damping, and the absorption is negligible.

The absorption coefficient α_a will then consist of the sum of several such narrow lines, centered at the frequencies ω_1, ω_3, ω_5, $\cdots$. For large values of n, the lines will blend into each other when their widths exceed their spacing.

The absorption line given by equation 3.40 has a peak value of $4r_n \delta_n / (\delta_n + r_n)^2$ whose maximum with respect to r_n is equal to one, when $r_n = \delta_n$. For a given value of n, i.e. a given line, this maximum can be obtained by changing either Θ or $k_o L$, so that the radiation damping is <u>matched</u> to the Landau damping for that line. This is the <u>condition for maximum power transfer from the incident wave to the plasma.</u>

Now recall that δ_n increases with n, while r_n decreases. If $r_1 = \delta_1$, then the peak value of the $n=1$ line will be greater than that of the others, and the peak values will decrease monotonically with n. But if the matching condition $r_n = \delta_n$ is more nearly satisfied for an odd value of n not equal to one, the peak values of the lines will not decrease monotonically with n.

One of the sharp resonances in the real part of the <u>symmetric</u> surface impedance Z_s can be analyzed in a similar way, to determine the behavior of α_s in the vicinity of one of the symmetric resonance frequencies. The constant term corresponding to c_n in equation 3.36 must contain the $n=0$ term in $\text{Im } Z_s$ evaluated at ω_n $(n \neq 0)$, which is proportional to $(k_o L)^{-1}$. The reactive part of Z_s thus increases as $k_o L$ decreases, so the symmetric absorption coefficient α_s cannot be made large enough to be observable by making the slab

thin. In fact, for a thin slab ($k_oL \ll 1$) we have $\text{Re } Z_s \lesssim 1$,

$\text{Im } Z_s \gg 1$, so from 3.18 it follows that $\alpha_s \ll 1$.

Since α_s is negligible for a <u>thin slab</u>, we have, from equation

3.17, $\alpha \simeq \frac{1}{2} \alpha_a$. This result can be understood by regarding half the

power in the incident wave to be carried by the symmetric parts of the

fields, and half by the antisymmetric parts. The symmetric parts are

not absorbed, so the absorption coefficient α is just half of the

fraction of the power absorbed from the antisymmetric parts. In parti-

cular, since the maximum value of α_a is 1, <u>the maximum value of α </u>

<u>is 1/2 </u>.

For a thin slab it follows from equation 3.6 that $R_s \simeq -1$,

because $\text{Re } Z_s \lesssim 1$, $\text{Im } Z_s \gg 1$. For symmetric excitation of a thin

slab, the magnetic fields of the incident waves tend to cancel, while

the tangential electric fields tend to add. This causes the reactive

part of Z_s to be very large, and most of the power is reflected.

Using equation 3.10, the fraction of the power reflected in the

unsymmetric case, i.e. the <u>reflectivity</u>, is given by

$$|R|^2 \simeq \left| \frac{R_a - 1}{2} \right|^2 \quad , \quad \text{or} \quad |R|^2 \simeq \frac{|Z_a/\cos \theta|^2}{|1 + Z_a/\cos \theta|^2}$$

using 3.9.

Using equations 3.34 and 3.36, the reflectivity for $\omega = \omega_n$

corresponding to the absorption line 3.40 is found to be

$$|R(\omega)|^2 \simeq \frac{r_n^2}{(\omega/\omega_n - 1)^2 + (\delta_n + r_n)^2} \quad , \text{ if } k_oL \ll 1 \qquad (3.41)$$

This represents a <u>scattering resonance</u> with a Lorentzian profile, at

the same resonance frequency as the corresponding absorption line, and
with the same line width. The peak value is $r_n^2/(\delta_n + r_n)^2$, which
decreases monotonically with increasing n . Its value is 1/4 when
the radiation damping is matched to the Landau damping, i.e. $r_n = \delta_n$.
The absorption coefficient α then has its maximum value of 1/2 at
the center of the line, so the half of the power in the antisymmetric
part of the incident fields is completely absorbed. The other half, in
the symmetric part, is scattered by the plasma, symmetrically to either
side. Hence one would observe 1/4 of the incident power to be reflected
and 1/4 to be transmitted.

These scattering resonances are due to the excitation of plasma
waves in the slab, at their resonance frequencies. They are accom-
panied by absorption lines at the same frequencies for the following
reason. The ratio of power dissipated to energy stored by a given
plasma wave mode is fixed. When one of the modes is excited to a
larger amplitude at its resonance frequency, it stores more energy
and must therefore absorb correspondingly more power from the incident
wave.

D. Landau Damping in a Bounded Plasma

In this section we attempt to explain the physical mechanism by
which energy is absorbed in a plasma slab. Consider a single normal
mode of the slab, with electric field E_x proportional to $e^{ik_y y}$
sin $k_n x$. Inserting the time factor $e^{-i\omega t}$ and taking the real part,
we may represent the field as

$$E_x = E_o \sin k_n x \cos(\omega t - k_y y) \quad .$$

But the wavelength in the y direction is much longer than the distance the particles can travel during one period of the wave, so we may neglect $k_y y$ compared to ωt . The field can be written as the sum of two traveling waves, with phase velocities equal in magnitude but opposite in direction:

$$E_x = \frac{E_o}{2} [\sin(k_n x - \omega t) + \sin(k_n x + \omega t)] \tag{3.42}$$

The particles which are responsible for the damping are, according to equation 3.22, those with velocities about equal to ω/k_n . A particle traveling in a certain direction is essentially affected only by the wave traveling in the same direction, i.e. one or the other of the terms in equation 3.42. If its velocity is nearly equal to ω/k_n , its unperturbed trajectory keeps in step with one of the traveling waves until it is reflected, with the other until it is reflected again, and with the first until the next reflection, etc. The phase relations are preserved, on alternate reflections from the boundaries, if the oscillation frequency of the particle Ω is approximately equal to ω/n , where ω is the frequency of the incident wave and n is the mode number. The mechanism of collisionless damping in a plasma slab can therefore be discussed in much the same way as for an infinite plasma. The theory of this damping in an infinite plasma will be reviewed briefly.

If there is a sinusoidal disturbance in the electron density at some time, then there will be a coherent wave propagating through the plasma at later times. The linearized theory (5) predicts that the wave will be damped, and its energy is transferred to those

particles with nearly the same velocity as the wave. Some of the particles are accelerated by the electric field of the wave, and some are decelerated, according to their positions relative to the field maxima and minima in the wave. This causes a slow rearrangement, or flow, of the particles in phase space (14).

Because there are more particles traveling slightly slower than the wave than are traveling slightly faster, this flow in phase space results in a net transfer of energy to the resonant particles.

Of course, this can only be true for a time which is short compared to the "trapping time". The periods of oscillation of the resonant particles which are trapped in the potential energy minima in a coordinate system moving with the wave, are greater than or equal to the "trapping time",

$$\tau_{tr} = (\frac{m}{e\epsilon_n k_n})^{1/2} ,$$

where ϵ_n is the amplitude of the wave. For times comparable to τ_{tr}, the net effect of the wave on the resonant particles cannot be considered a small perturbation, and the linearized theory becomes invalid (14).

If close-range encounters with other particles are frequent enough, however, the electrons will be scattered out of the potential wells of the wave before the linearized theory becomes invalid. Then the initial-value problem for the motion of a particle may be considered to be started over again after each scattering encounter. Also, if the cumulative effect of long-range encounters is rapid enough, the energy taken from the wave by the resonant electrons during the

first half of their oscillation periods in the potential wells, will be shared with the other electrons before it can be returned coherently to the wave in the second half of their oscillation periods. Thus the presence of some collisions, or dynamical friction, is necessary to make the linearized theory uniformly valid for all times.

In order that the linearized theory be valid in a steady state, where power is supplied by sources external to the plasma, we must assume that the mean collision rate is much larger than the "trapping rate", or $\bar{\nu}\tau_{tr} \gg 1$. This places an upper limit on the amplitudes of the normal modes in the plasma, in order that the linearized theory of power absorption be valid.

It should be clear from this discussion that the so-called "collisionless" absorption mechanism is really an indirect mechanism by which the plasma absorbs energy from an incident electromagnetic wave and converts it into heat. It is indirect because the energy is first transferred to groups of resonant particles and then to the plasma as a whole. It can be much more efficient than the direct collisional mechanism when longitudinal waves are excited in the plasma, especially if there are appreciable numbers of particles with nearly the same velocities as the longitudinal waves.

If the nearly collisionless theory is to provide a good description of the damping, the Landau damping rate must be much larger than the mean collision rate. We must therefore assume the following double inequality to hold: $\gamma_n \gg \bar{\nu} \gg \tau_{tr}^{-1}$. The effects of collisions will be discussed further in Section IVD.

IV. COLLISIONLESS NOISE RADIATION FROM A PLASMA SLAB

A. Surface Emissivity

A bounded plasma contains charged particles in motion, which are accelerated by their interactions with each other and with the confining fields and which therefore emit electromagnetic radiation. The radiation fields of different electrons have no definite phase relations between them, so the total radiation field of the plasma has a noise-like character. Although the radiation intensity can be obtained from Kirchhoff's law if the plasma is in thermal equilibrium (this will be done in Section C), we are concerned here with the more general case.

The radiation fields are determined by the current density in the plasma, which is due to the motions of the charged particles, and these in turn depend upon the initial conditions of all of the particles at some instant of time. But the initial conditions for this extremely large number of particles are unknown. The best that can be done is to assume the joint probability distribution of these initial conditions to be known. Then the statistical properties of the radiation field can be determined by averaging over the initial conditions of the particles. This procedure will be carried out in Section B.

In this section the relation between the surface emissivity of a plasma slab and the Laplace transforms of the radiation field and the field at the plasma boundary will be derived.

A quantity which might be measured in an idealized experiment is the surface emissivity $J(\Theta,\omega)$. Let $J(\Theta,\omega) \, d\Omega \, d\omega$ be defined

as the power radiated in a direction making an angle θ with the slab

normal, within solid angle $d\Omega$, in frequency interval $d\omega$, per unit

surface area of the slab. Only the single polarization with electric

vector in the plane of the radiation direction and the slab normal is

counted, because it is the only one which is coupled to the longitudi-

nal fields in the plasma. The emissivity is closely related to the

spectral flux density of the radiation field, which will now be defined.

The Poynting vector of the electromagnetic field outside the

plasma is

$$\vec{S}(\vec{r},t) = \frac{c}{4\pi} \, \delta\vec{E}(\vec{r},t) \times \delta\vec{B}(\vec{r},t) \quad ,$$

where the notation for the electric and magnetic fields is meant to

emphasize that these are fluctuation quantities having zero expecta-

tion values. The expected value of the component of the Poynting

vector in the direction $\hat{r}$ (per cm^2 of surface area) is

$$\frac{1}{A} \left\langle \hat{r} \cdot \vec{S}(\vec{r},t) \right\rangle = \frac{c}{4\pi A} \left\langle \hat{r} \cdot \delta\vec{E}(\vec{r},t) \times \delta\vec{B}(\vec{r},t) \right\rangle$$

$$= \int_0^\infty d\omega \, S(\omega) \tag{4.1}$$

which defines the spectral flux density $S(\omega)$. The total surface

area of the plasma slab is $A = V/L$, where V is the volume and L

the width. (The limit $V \to \infty$ will be taken later.)

Suppose the direction $\hat{r}$ makes an angle θ with the slab

normal, and the fields $\delta\vec{E}, \delta\vec{B}$ are radiation fields propagating in

the direction $\hat{r}$, with the polarization of interest. Then $S(\omega)d\Sigma \, d\omega$

is the power radiated in frequency interval $d\omega$ in a direction making

an angle θ with the slab normal which crosses an area $d\Sigma$ at the

observation point $\vec{r}$ with its normal in the direction of propagation $\hat{r}$ per unit surface area of the slab. Because $d\Sigma = r^2 d\Omega$, where $d\Omega$ is the solid angle subtended by the area $d\Sigma$ at a distance r , the relation between the surface emissivity and the spectral flux density is

$$J = r^2 S(\omega) \qquad (4.2)$$

The spectral flux density is (except for a constant factor) the Fourier transform of the radiation field autocorrelation function. Hence,

$$J = r^2 S(\omega) = (\frac{cLr^2}{4\pi^2 V}) \int_{-\infty}^{\infty} d\tau \; e^{i\omega\tau} \left\langle \delta E(\vec{r},t) \; \delta E(\vec{r},t + \tau) \right\rangle \qquad (4.3)$$

Now J can also be obtained directly from the Laplace transform of the radiation field as a function of time:

$$J = (\frac{cLr^2}{4\pi^2 V}) \lim_{\gamma \to 0} 2\gamma \left\langle \delta E(\vec{r},-\omega + i\gamma) \; \delta E(\vec{r},\omega + i\gamma) \right\rangle \qquad (4.4)$$

where $\qquad \delta E(\vec{r},\omega) = \int_{0}^{\infty} dt \; e^{i\omega t} \; \delta E(\vec{r},t) \quad . \qquad (4.5)$

To prove 4.4, substitute the definition 4.5 of the Laplace transform into the right-hand side of equation 4.4. After interchanging the order of integrations over time and the statistical average, we have

$$\left\langle \delta E(\vec{r},-\omega + i\gamma) \; \delta E(\vec{r},\omega + i\gamma) \right\rangle$$

$$= \int_{0}^{\infty} dt \; e^{i(-\omega+i\gamma)t} \int_{0}^{\infty} dt' e^{i(\omega+i\gamma)t'} \left\langle \delta E(\vec{r},t) \; \delta E(\vec{r},t') \right\rangle$$

Now we replace the integration variable t' by $\tau = t' - t$, and change the order of integration in the double integral over t and τ . Then we obtain

$$\left\langle \delta E(\vec{r},-\omega+i\gamma) \ \delta E(\vec{r},\omega+i\gamma) \right\rangle =$$

$$\int_0^\infty d\tau \ e^{i(\omega+i\gamma)\tau} \int_0^\infty dt \ e^{-2\gamma t} \left\langle \delta E(\vec{r},t) \ \delta E(\vec{r},t+\tau) \right\rangle$$

$$+ \int_{-\infty}^0 d\tau \ e^{i(\omega+i\gamma)\tau} \int_{-\tau}^\infty dt \ e^{-2\gamma t} \left\langle \delta E(\vec{r},t) \ \delta E(\vec{r},t+\tau) \right\rangle \ .$$

Assuming that the radiation field is a stationary random function, the autocorrelation function is a function of the time difference only, i.e. $\left\langle \delta E(\vec{r},t) \ \delta E(\vec{r},t+\tau) \right\rangle$ is a function of τ only (15). The integrals over t can therefore be carried out, with the result that

$$\left\langle \delta E(\vec{r},-\omega+i\gamma) \ \delta E(\vec{r},\omega+i\gamma) \right\rangle =$$

$$\frac{1}{2\gamma} \int_{-\infty}^\infty d\tau \ e^{i\omega\tau} \ e^{-\gamma|\tau|} \left\langle \delta E(\vec{r},t) \ \delta E(\vec{r},t+\tau) \right\rangle$$

Hence

$$\lim_{\gamma \to 0} 2\gamma \left\langle \delta E(\vec{r},-\omega+i\gamma) \ \delta E(\vec{r},\omega+i\gamma) \right\rangle = \int_{-\infty}^\infty d\tau \ e^{i\omega\tau} \left\langle \delta E(\vec{r},t) \ \delta E(\vec{r},t+\tau) \right\rangle \ ,$$

and using 4.3 we obtain the required relation, 4.4.

The surface emissivity can now be obtained from equation 4.4 if the Laplace transform of the radiation field is given. This can be related to the Laplace transform of the field fluctuation δE_y at the slab boundary, which in turn is related to the plasma properties

through the solution of the boundary value problem for the field fluctuations. If we are interested in the radiation field in the region $x < 0$, i.e., to the left of the slab, we first Fourier analyze the field $\delta E_y(\vec{r},\omega)$ at the boundary $x = 0$:

$$\delta E_y(0,y,z,\omega) = \int\limits_{-\infty}^{\infty} \frac{dk_y}{2\pi} \int\limits_{-\infty}^{\infty} \frac{dk_z}{2\pi} \, e^{i(k_y k + k_z z)} \, \delta E_{yo}(k_y,k_z,\omega) \ . \quad (4.6)$$

These field fluctuations must now be "analytically continued" away from the surface of the plasma.

The radiation field in vacuum which is a solution of

$$(\nabla^2 + k_o^2) \, \delta E_y(\vec{r},\omega) = 0 \quad , \quad \text{for} \quad x < 0 \quad ,$$

which has the boundary values given by equation 4.6 at $x = 0$, and which represents waves traveling away from the plasma for $x \to -\infty$, is given by

$$\delta E_y(\vec{r},\omega) = \int\limits_{-\infty}^{\infty} \frac{dk_y}{2\pi} \int\limits_{-\infty}^{\infty} \frac{dk_z}{2\pi} \, e^{ik_o r f(k_y,k_z)} \, \delta E_{yo}(k_y,k_z,\omega) \quad (4.7)$$

where

$$f(k_y,k_z) = \frac{k_x x + k_y y}{k_o r} = \frac{k_y}{k_o} \sin\theta + \sqrt{1 - (k_y/k_o)^2 - (k_z/k_o)^2} \, \cos\theta \ .$$

The observation point $\vec{r}$ has been chosen to be in the x-y plane at a distance r from the origin: $\vec{r} = (-r\cos\theta, \, r\sin\theta, 0)$. If we assume this distance is many wavelengths away from the origin, so that $k_o r \gg 1$, then the radiation field δE_y given by equation 4.7 can be approximated by a spherical wave propagating out from the origin.

To show this we evaluate the double integral in 4.7 approximately, using the stationary phase approximation. Because of the exponential

factor in the integrand, which oscillates rapidly as a function of k_y and k_z, there is no significant contribution from most of the region of integration. Only the region near the stationary phase point (k_{ys}, k_{zs}) where $\partial f/\partial k_y = \partial f/\partial k_z = 0$ needs to be considered, because the exponential factor oscillates most slowly in that region. Assuming that $\delta E_{yo}(k_y, k_z, \omega)$ is a slowly varying function of k_y and k_z, it can be replaced by its value at the stationary phase point. The exponential factor can be integrated approximately by expanding $f(k_y, k_z)$ in a Taylor series about the stationary phase point, which is found to be $k_{ys} = k_o \sin \theta$, $k_{zs} = 0$. The stationary phase approximation to the double integral in 4.7 is found to be

$$\delta E_y(\vec{r}, \omega) \simeq \frac{-ik_o}{2\pi} \cos \theta \, \frac{e^{ik_o r}}{r} \, \delta E_{yo}(k_o \sin \theta, \omega) \quad , \text{ for } k_o r \gg 1 \quad (4.8)$$

where the argument $k_z = 0$ is to be understood. This represents a spherical wave going out from the origin whose amplitude is proportional to the Fourier coefficient of the plane wave component in 4.7 which propagates in the direction from the origin to the observation point $\vec{r}$.

Thus result can be expressed in terms of $\delta E(\vec{r}, \omega)$, the component perpendicular to the direction of propagation, since $\delta E_y = \delta E \cos \theta$. By substituting the result into equation 4.4, the surface emissivity can be obtained from the field fluctuations at the plasma boundary, as follows:

$$J(\theta,\omega) = (\frac{ck_o^2 L}{16\pi^4}) \frac{1}{V} \lim_{\gamma \to 0} 2\gamma \left\langle \delta E_{yo}(-k_o \sin\theta, -\omega+i\gamma) \; \delta E_{yo}(k_o \sin\theta, \omega+i\gamma) \right\rangle$$

$$(4.9)$$

B. Electric Field Fluctuations in a Plasma Slab

Since the emissivity of a plasma slab is determined by the boundary values of the field fluctuation δE_y, it is necessary to determine these boundary values in terms of the plasma properties. This can be done by using the solution of the boundary value problem which was derived in Part II. Although we interpreted all of the fields as ensemble average, or macroscopic fields, an alternate interpretation is possible and is necessary to find the field fluctuations. In the perturbation analysis of the particle equations of motion (Section II.B), $\vec{E}_1$ is now to be interpreted as the fluctuation of the field from its ensemble average value $\vec{E}_o \equiv \langle \vec{E} \rangle$:

$$\delta\vec{E} \equiv \vec{E} - \langle \vec{E} \rangle \; . \tag{4.10}$$

No macroscopic disturbances, such as the plasma waves considered in Sections II and III, are to be present.

The current density calculated in Section II.C is then due to particles moving nearly on their unperturbed trajectories in the confining field of the slab, being slightly perturbed by the field fluctuation $\delta\vec{E}$. The "unperturbed part" of the current density should not be averaged as in equation 2.31, since it represents the source of the field fluctuations:

$$\delta \vec{j}^{(o)}(\vec{x},t) = q \sum_{j} \vec{v}_{j}^{(o)} \delta(\vec{x} - \vec{r}_{j}^{(o)}(t)) \qquad . \qquad (4.11)$$

The ensemble average can be used, however, for that part of the current density fluctuation $\delta \vec{j}$ which is caused by the field $\delta \vec{E}$, with the condition that $\delta \vec{E}(\vec{x},t)$ is a given function. This conditional expectation still has a fluctuation nature because $\delta \vec{E}$ is to be determined by solving Maxwell's equations with the current density fluctuation $\delta \vec{j}$ as the source. The neglect of higher-order fluctuation quantities (16) in this procedure is justified if there are many particles per Debye sphere: $n\lambda_D^3 \gg 1$. The fluctuation $\delta \vec{j}$ is then given by equation 2.32 with $j_{\mu}^{(o)}$ replaced by $\delta j_{\mu}^{(o)}$ and $E_{1\beta}$ by δE_{β} .

The part of $\delta \vec{j}$ due to the perturbation of the particle trajectories, caused by the field $\delta \vec{E}$, represents the linearized response of the plasma to the superposition of the fields of all the individual particles. The response is calculated as though these individual particles moved independently on trajectories determined by the confining fields of the slab and their initial conditions. It is the response of a dielectric medium having the average properties of the collection of charged particles which is the plasma, and it tends to shield the fields of the individual charges.

The field fluctuation $\delta \vec{E}$ can therefore be thought of as being produced by non-interacting, independent quasi-particles, i.e. the individual charged particles with their associated shielding clouds. This is the <u>principle of superposition of the fields of "dressed test particles"</u>, first discussed by Hubbard (17). Rostoker showed that the

use of these statistically independent quasi-particles is equivalent to a correct treatment of particle correlations (18).

After making the other assumptions and approximations discussed in Section II.C, the electric field fluctuations can be solved for, self-consistently as in Section II.D. The result, corresponding to equation 2.51, is as follows:

$$\delta E_y(x;k_y\omega) = \frac{2ik_y^2}{k_o L}\sum_{n=0}^{\infty}{}' \frac{\cos k_n x}{|\vec{k}_n|^2\, \epsilon(\vec{k}_n,\omega)} \left\{ \delta B_{zo} - (-1)^n\, \delta B_{zL} \right.$$
$$\left. - \frac{4\pi k_o}{k_y}\delta\rho_c^{(o)}(\vec{k}_n,\omega)\right\} - \frac{(i/k_o)}{(1-\omega_p^2/\omega^2)}\left\{k_y\left[\frac{\delta B_{zo}\cosh k_y(L-x)-\delta B_{zL}\cosh k_y x}{\sinh k_y L}\right]\right.$$
$$\left. + k_p\left[\frac{\delta B_{zo}\cos k_p L - \delta B_{zL}}{\sin k_p L}\right]\right\} . \tag{4.12}$$

The "unperturbed charge density fluctuations" $\delta\rho_c^{(o)}$ cannot be neglected because they are now the source of the field fluctuations. We have neglected $\delta j_t^{(o)}/c$ in comparison with $\delta\rho_c^{(o)}$, since the unperturbed particle trajectories have velocities which are much smaller than c. This is consistent with the neglect of Re σ_t in the approximation 2.40.

In addition to the fields induced in the plasma by the fields at the boundary, there are the fields induced by the particle discreteness. The screening effect of the plasma is indicated in equation 4.12, where the charge density of the "bare" particles $\delta\rho_c^{(o)}$ is divided by the dielectric function $\epsilon(\vec{k}_n,\omega)$ to obtain the charge density of the "dressed" particles.

The boundary values of the tangential magnetic field are determined by the condition that the fields δE_y, δB_z be continuous at the boundary. These fields must therefore have the same ratio as the fields outside the plasma which represent outgoing plane waves. This is the radiation condition, which takes the form

$$\delta E_{yo}/\delta B_{zo} = - \cos \theta \quad , \quad \delta E_{yL}/\delta B_{zL} = \cos \theta \qquad (4.13)$$

These relations can be substituted into equation 4.12, but δE_{yo} and δE_{yL} must still be solved for. By setting $x = 0$ and $x = L$ in equation 4.12, we obtain two simultaneous algebraic equations for δE_{yo} and δE_{yL} . The solution of these gives

$$\delta E_{yo}(k_y,\omega) = \frac{-8\pi i}{L} \sum_{n=0}^{\infty} {}' \frac{k_y \; \delta\rho_c^{(o)}(\vec{k}_n,\omega)}{[1 + Z_n/\cos \theta]| \vec{k}_n |^2 \; \varepsilon(\vec{k}_n,\omega)} \qquad (4.14)$$

where Z_n is given by equation 3.25. A similar expression is obtained for δE_{yL} , but with the nth term in the series multiplied by $(-1)^n$. The electric field fluctuation δE_y in the plasma can now be obtained by substituting these expressions, and the relations 4.13, into equation 4.12.

Equation 4.14 expresses the basic test-particle property of the plasma slab. The electric field fluctuations are due to the discreteness of the individual charges, and are modified by the collective behavior of the plasma, as indicated by the dielectric function $\varepsilon(\vec{k}_n,\omega)$ in the denominator. They are also modified by the coupling with the radiation field outside the plasma, as indicated by the presence of the surface impedances. The symmetric surface impedance

Z_s appears with the symmetric modes of the charge density fluctuation (even n), and the antisymmetric surface impedance Z_a with the antisymmetric modes (odd n).

Using equation 4.14, the emissivity given by equation 4.9 can be written as follows:

$$J(\theta,\omega) = \frac{4ck_o^4 \sin^2\theta}{\pi^2 L} \sum_{m=0}^{'} \sum_{n=0}^{'}$$

$$\times \; \frac{\frac{1}{V} \lim_{\gamma \to 0} 2\gamma \langle \delta\rho_c^{(o)}(-\vec{k}_n,-\omega+i\gamma) \; \delta\rho_c^{(o)}(\vec{k}_m,\omega+i\gamma) \rangle}{(1+Z_n^*/\cos\theta)(1+Z_m/\cos\theta)|\vec{k}_n|^2 |\vec{k}_m|^2 \; \varepsilon^*(\vec{k}_n,\omega) \; \varepsilon(\vec{k}_m,\omega)}$$

$$(4.15)$$

where $\vec{k}_n = (n\pi/L, k_o \sin\theta, 0)$. The emissivity is thus finally related to the statistical properties of the unperturbed part of the charge density:

$$\delta\rho_c^{(o)}(\vec{k}_n,\omega) = \int_0^\infty dt \; e^{i\omega t} \int_{-\infty}^\infty dy \int_{-\infty}^\infty dz \; e^{-i(k_y y + k_z z)}$$

$$\times \int_0^L dx \; \cos k_n x \; \delta\rho^{(o)}(\vec{x},t)$$

where $\delta\rho^{(o)}(\vec{x},t) = q \sum_{j=1}^N \delta(\vec{x} - \vec{R}(t|\vec{r}_j,\vec{v}_j,0))$, and $\vec{R}(t|\vec{r}_j,\vec{v}_j,0)$ is the unperturbed trajectory function discussed in Section II.B. Carrying out the indicated integration we have (for $k_z = 0$)

$$\delta\rho_c^{(o)}(\vec{k}_n,\omega) = q \sum_{j=1}^{N} e^{-ik_y y_j} \left[\frac{i(\omega-k_y v_{yj})\cos k_n x_j + k_n v_{xj}\sin k_n x_j}{(\omega - k_y v_{yj})^2 - k_n^2 v_{xj}^2} \right] \quad (4.16)$$

Since the effect of particle correlations is included in equation 4.15 through the use of the dielectric function $\epsilon(\vec{k}_n,\omega)$, the initial conditions $\{\vec{r}_j,\vec{v}_j\}$ for all the particles must now be assumed to be <u>independent</u> random variables. Each of the initial positions $\vec{r}_j$ is assumed to be uniformly distributed within the plasma volume, so that the probability density for $\vec{r}_j$ is constant within the slab and zero outside. The initial velocities are assumed to be identically distributed with probability density $f_o(v_x,v_y)$. We then find, in the limit $N,V \to \infty$, $N/V = n_e$, that

$$\frac{1}{V}\left\langle \delta\rho_c^{(o)}(-\vec{k}_n,-\omega')\, \delta\rho_c^{(o)}(\vec{k}_m,\omega) \right\rangle$$

$$\to \frac{n_e e^2}{2}\delta_{nm}\int_{-\infty}^{\infty}dv_x \int_{-\infty}^{\infty}dv_y f_o(v_x,v_y)\left\{\frac{(\omega'-k_y v_y)(\omega-k_y v_y) + k_n^2 v_x^2}{[(\omega'-k_y v_y)^2-k_n^2 v_x^2][(\omega-k_y v_y)^2-k_n^2 v_x^2]}\right\}$$

for $n \neq 0$ with an additional factor of 2 for $n = 0$.

We now replace ω' by $\omega-i\gamma$ and ω by $\omega+i\gamma$. After some algebraic simplification we multiply by 2γ and take the limit $\gamma \to 0$, making use of the formal relation

$$\frac{2\gamma}{x^2+\gamma^2} \to 2\pi\delta(x) \quad .$$

The result can be expressed as follows:

$$\frac{1}{V} \lim_{\gamma \to 0} 2\gamma \left\langle \delta\rho_c^{(o)}(-\vec{k}_n,-\omega+i\gamma) \; \delta\rho_c^{(o)}(\vec{k}_m,\omega+i\gamma) \right\rangle = \frac{\pi n_e e^2}{|\vec{k}_n|} \delta_{nm} F(\frac{\omega}{|\vec{k}_n|})$$

$$(4.17)$$

for $n \neq 0$, with an additional factor of 2 for $n = 0$, where $F(u)$ is the one-dimensional velocity distribution.

Combining equations 4.15 and 4.17, we have the following general expression for the emissivity:

$$J(\theta,\omega) = \frac{4n_e e^2 \omega^4 \sin^2\theta}{\pi L c^3} \sum_{n=1}^{\infty} \frac{F(\frac{\omega}{|\vec{k}_n|})}{|1+ Z_n/\cos \theta|^2 \; |\vec{k}_n|^5 \; |\epsilon(\vec{k}_n,\omega)|^2} \qquad (4.18)$$

(The $n=0$ term has been dropped, since $F(\frac{c}{\sin \theta})$ must be zero; particles cannot travel at the speed of light or faster.) This is one of the primary results of this investigation. Under some conditions, which are discussed in the next section, the emissivity is large at those frequencies for which the dielectric function $\epsilon(\vec{k}_n,\omega)$ is small. These are the natural frequencies of standing waves in the slab, which are excited by the motions of the charged particles. The fast electrons with velocities close to the phase velocity of a plasma wave are responsible for its excitation. Hence the nth term in the emissivity expression is proportional to $F(\frac{\omega}{|k_n|})$, the number of electrons with the phase velocity of the nth plasma wave.

Now recall that the absorption coefficient is given in general by a very similar expression (equation 3.24), but the nth term is proportional to $F'(\frac{\omega}{|k_n|})$, the derivative of the one-dimensional velocity distribution function. This is because the absorption is due

to Landau damping. In a thermal equilibrium plasma there is a relation between F and F' . Hence, there is also a relation between the emissivity and the absorption coefficient, which agrees with Kirchhoff's law. This is discussed in the next section.

Since the emissivity corresponding to a given plasma wave is proportional to the number of electrons with the same velocity, the emissivity can be enhanced if the number of such electrons is increased. The enhancement of emission lines caused by a non-thermal high-energy tail in the one-dimensional distribution function F is discussed in the next section.

C. Enhancement of Emission Lines in Non-Thermal Plasmas

For purposes of comparison we first discuss the case of thermal equilibrium. If a plasma slab is in thermal equilibrium at temperature T_e , the electron velocity distribution is a Maxwellian distribution, equation 3.27, and so

$$F(u) \; = \; \frac{-\kappa T_e}{mu} \, F'(u) \qquad\qquad (4.19)$$

Equation 4.18 for the emissivity can therefore be written as follows for the case of thermal equilibrium:

$$J(\theta,\omega) \; = \; \frac{\omega_p^2 \, \omega^3 \sin^2\theta (kT_e)}{\pi^2 L \, c^3} \sum_{n=1}^{\infty} \frac{-F'(\frac{\omega}{|\vec{k}_n|})}{|1 + Z_n/\cos\theta|^2 \; |\vec{k}_n|^4 \, |\varepsilon(\vec{k}_n,\omega)|^2} \cdot \qquad (4.20)$$

It is of interest to compare this result with the emissivity calculated by using Kirchhoff's law (19). Let us imagine the plasma slab to be enclosed in a large cavity, and to be in thermal equilibrium with the walls of the cavity which are at temperature T_e . The

radiation intensity at any point in the cavity external to the slab

must have the black-body value in every direction. In particular, the

power (ergs/sec) in the radiation field which flows away from the

slab in a direction making an angle θ with the slab normal, per unit

solid angle per unit frequency interval per unit surface area, must be

$I_{BB} \cos \theta$, where $I_{BB} = \dfrac{\omega^2}{8\pi^3 c^2} \kappa T_e$. This is the black-body intensity

in the Rayleigh-Jeans limit $(\hbar\omega \ll \kappa T_e)$ for the single polarization

of interest.

If we neglect incoherent scattering[*] this intensity is due to

the radiation from the plasma, and the reflection and transmission of

the black-body radiation incident on the slab:

$$ J + (|R|^2 + |T|^2) \, I_{BB} \cos \theta = I_{BB} \cos \theta \qquad . $$

Since the absorption coefficient of the slab is $\alpha = 1 - |R|^2 - |T|^2$, the

emissivity must be

$$ J = \alpha \, I_{BB} \cos \theta = \alpha \, \frac{\omega^2 \cos \theta}{8\pi^3 c^2} \kappa T_e \qquad . \qquad (4.21) $$

With the expression for α given by equation 3.24, equation 4.21

becomes identical with 4.20.

Thus the emissivity calculated by using the thermal equilibrium

argument is the same as that obtained from the more general expression

4.18 for the emissivity in the special case where the electron velocity

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[*] The intensity due to incoherent (Thomson) scattering is smaller than
that due to the processes considered by many orders of magnitude. If
it were not negligible, the detailed balance argument given here
would be invalid, because the incoherent scattering of electromagnetic
waves can change the angle θ and the frequency ω .

distribution is Maxwellian. It should be clear that these two methods for obtaining the emissivity in thermal equilibrium are essentially independent, so that the comparison provides a consistency check on the more general method, which leads to equation 4.18.

Also, it is worth pointing out that the assumption of "thermal equilibrium at temperature T_e" really involves the electron distribution function only. The radiation, at frequencies as high as the plasma frequency, is not affected by the motion of the ions, so no assumption is necessary about the ion distribution function.

Because the emissivity J is proportional to the absorption coefficient α in thermal equilibrium, we expect that J will have a line structure as a function of frequency, under the same conditions that α has such a line structure. For a thin slab ($k_o L \ll 1$), the emissivity is given in the vicinity of the nth line ($\omega \simeq \omega_n$), by equation 4.21, where $\alpha = 1/2 \, \alpha_a$, and α_a is given by equation 3.40. The emission lines have Lorentzian shapes, provided they are narrow, and their widths are determined by the Landau damping and the radiation damping of the respective plasma wave modes. It should be emphasized that these are not modes of the macroscopic charge density of the plasma, but of the microscopic density fluctuations which are responsible for the noise radiation. One can speak of the same dispersion relation and damping mechanisms as for macroscopic waves, however.

If the radiation damping is matched to the Landau damping for the nth emission line, then the emissivity from a thermal equilibrium plasma is 1/2 of the black-body value at the center of the line.

[Recall from Section III.C that the maximum value of the absorption

coefficient α for a thin slab is 1/2.] Otherwise, the emissivity is

less than this value. There is no such restriction on the emissivity

of a nonthermal plasma. It is therefore of interest to investigate

the properties of the emission lines from nonthermal plasmas.

The emission lines can be discussed within the framework of the

discussion of absorption lines in Section III.C. The emissivity

formula 4.18 can be written in a form which is formally similar to the

thermal equilibrium expression 4.20 by using the following relation:

$$F(u) = \frac{-\Theta(u)}{mu} F'(u) \tag{4.22}$$

which defines the function $\Theta(u)$. If the distribution function is

Maxwellian, then $\Theta(u) = \kappa T_e$, the temperature of the electrons (in

energy units), and is independent of u. If the distribution function

is not Maxwellian, then $\Theta(u)$ is a function of u, and is not a tem-

perature. It can be thought of as a "pseudo-temperature" which

characterizes the distribution function $F(u)$. It gives the tempera-

ture for that Maxwellian distribution which would have the same

logarithmic derivative at that value of u .

An emission line is obtained from the corresponding absorption

line [equation 3.40 for α_a] for a nonthermal plasma as follows:

$$J(\omega) \simeq \frac{1}{2} \alpha_a(\omega) \frac{\omega^2 \cos \theta}{8\pi^3 c^2} \Theta(\frac{\omega}{|\vec{k}_n|}) \quad . \tag{4.23}$$

The emissivity depends on the velocity distribution function F and

is not restricted to be less than half a black-body value (which has a

meaning only in thermal equilibrium.)

As an example of a nonthermal distribution function, suppose $F(u)$ has the following properties:

$$F(u) = \begin{cases} \dfrac{1}{\pi^{1/2}v_e} e^{-u^2/v_e^2} & , \quad u \lesssim v_e \\[4mm] \dfrac{1}{\pi^{1/2}v_e} e^{-u^2/v_e^2} + Cu^{-s}, & u \gg v_e \end{cases} \qquad (4.24)$$

This is a Maxwellian with temperature $\kappa T_e = \frac{1}{2} m v_e^2$ for low velocities. For high velocities there is, in addition, a power-law distribution which gives the distribution function a "high-energy tail". The corresponding "pseudo-temperature" has the following properties:

$$\theta(u) = \begin{cases} \kappa T_e & , \quad u \lesssim v_e \\[4mm] \dfrac{1}{s} m u^2 & , \quad u \gg v_e \end{cases} \qquad (4.25)$$

(We assumed both the magnitude and slope of the Maxwellian part to be much smaller than for the power law part of the distribution function for high velocities.)

The emission lines which correspond to modes with very large phase velocities will be enhanced because of the presence of the high-energy tail. The emissivity of the nth such line will be multiplied by the <u>enhancement factor</u>

$$f_n = \theta\left(\frac{\omega}{|\vec{k}_n|}\right) \Big/ \kappa T_e \qquad (4.26)$$

This factor is much larger than unity in the example considered above

[with $\theta(u)$ given by equation 4.25] if the phase velocity is much larger than the thermal velocity v_c, and s is of order unity. In this example the Landau damping is also increased by the high-energy tail. Hence, the emission line is widened and the value of radiation damping for matching to the radiation field is increased. To gain an understanding of this enhancement, it is necessary to discuss the mechanism of the emission.

The collective response of the plasma tends to screen out the field of a charged particle for distances greater than a Debye length, if the speed of the particle is less than the mean thermal speed. For a faster particle the effective screening distance is somewhat larger than a Debye length and, in addition, collective oscillations are excited in its wake (20). This excitation of plasma waves is similar to the Cerenkov radiation caused by a fast electron in a dielectric material. The more fast electrons there are, with the phase velocity of a given plasma wave mode, the greater will be the amplitude to which this mode is excited. Since the electromagnetic radiation from this mode is simply a result of continuing the electric field fluctuations into the region outside the plasma, it is clear why the corresponding emissivity should be proportional to the expected number of electrons with the phase velocity of the plasma waves, i.e. the distribution function evaluated at that velocity.

Although there is a relationship between the emissivity and the absorption coefficient for a thermal equilibrium plasma corresponding to the relation between the distribution function and its derivative, there is no such relationship for a nonthermal plasma. Since the

emissivity is determined only by the electron velocity distribution, it can be enhanced over the thermal equilibrium value by high-velocity electrons in the tail of the distribution.

D. The Effect of Collisions on "Collisionless" Absorption and Emission

We have been concerned in this investigation with plasmas which can be considered nearly collisionless. We pointed out in Section III.D that some collisions are necessary to justify the use of the linearized theory of plasma waves. Therefore, the effects of collisions should be investigated.

Encounters between many charged particles take place simultaneously in a plasma, because of the long-range nature of the Coulomb force. The resulting electric field fluctuations cause the trajectory of each particle to undergo an almost continuous series of small deflections. The fraction of close encounters which cause large deflections is of order $(e^2/\kappa T)^3 n \equiv \Lambda^{-2}$, in a thermal equilibrium plasma, and is very small for temperatures and densities of most gaseous plasmas. The parameter Λ is about ten times the number of particles in a sphere whose radius is the Debye length $\lambda_D = (\kappa T/4\pi n e^2)^{1/2}$. Assuming Λ to be very large, the vast majority of particle encounters are distant encounters, which produce very small deflections.

The result is that a particle exhibits a Brownian motion, or diffusion, in velocity space and tends to slow down because of the effective frictional forces acting on it. The time scale for a substantial deviation of the particle trajectory because of particle encounters is the collision time t_c , where

$$t_c \simeq \frac{\Lambda}{\omega_p \ln \Lambda} \quad .$$

Therefore, in a plasma where the parameter Λ is large, the collision time is many plasma wave periods, so the effect of collisions upon the plasma waves is small.

Particle encounters, or collisions, tend to randomize any coherent wave motions in the plasma and to transform the energy of the waves into heat. The information about the location of the energy is thus lost, and we say that the energy has been absorbed by the plasma. The collisional absorption in a fully-ionized plasma can be attributed basically to dynamical friction. Because of the electric field fluctuations and the polarization force exerted on a particle by its screening cloud, the particle tends to slow down. More precisely, its expected velocity decreases in time from a given initial value. Because the diffusion is slow, the position of a particle remains fairly close to its expected position for many plasma periods. The theory of Brownian motion (21) indicates that in equations 2.18 and 2.19 we should replace $x_o + v_{xo} t$ by $x_o + v_{xo}(1 - e^{-\bar{\nu} t})/\bar{\nu}$, and multiply the right-hand side of 2.19 by $e^{-\bar{\nu} t}$ where $\bar{\nu}$ is an average collision frequency. For long wavelengths the approximate effect of collisions can be obtained by multiplying $\dot{R}_\gamma$ in equation 2.32 by $e^{-\bar{\nu}(t-t')}$, which replaces ω in equation 2.35 by $\omega + i\bar{\nu}$.

For partially-ionized gases, electron-neutral collisions must also be considered. These can be treated correctly by using the Boltzmann equation. The result is again to replace ω by $\omega + i\nu_m$, where

ν_m is the momentum-transfer collision frequency (22). The approximate results for a fully-ionized plasma can therefore be applied to a partially ionized gas by replacing $\overline{\nu}$ by $\overline{\nu} + \nu_m$.

When collisions are taken into account, the approximation 2.40 must be modified; assuming $\overline{\nu} \ll \omega$, we have

$$\sigma_t(\overline{k}_n,\omega) \simeq \frac{\omega_p^2(i\omega + \overline{\nu})}{4\pi \, \omega^2}$$

[A more rigorous derivation of this result, and the expression for $\overline{\nu}$, was given by Chang (23)]. The "transverse conductivity" therefore has a real part and so the divergence-free part of the electric field can be dissipated. The corresponding modification in the surface impedance and the absorption coefficient α of the slab shows that energy can be absorbed from an incident wave by this process.

The modification of the "longitudinal conductivity" σ_ℓ due to collisions introduces the collisional damping of longitudinal plasma waves. This process is important for high phase-velocity waves, when the Landau damping rate is small. The approximate effect is then to add $2\overline{\nu}/\omega$ to the right-hand side of equation 3.22 and to replace the Landau damping rate γ_n in equation 3.33 by $\gamma_n + \overline{\nu}$. The relative width δ_n is therefore determined by the collisional damping rate $\overline{\nu}$ when the Landau damping rate is very small. The widths of the absorption and emission lines of the plasma slab, which are associated with the plasma wave modes, are increased by the effects of collisions. Also, the value of the "radiation damping width" r_n for which the plasma is "matched" to the radiation field, is increased, since it is

now the sum of the Landau damping width and the collisional damping width. These are the modifications which are introduced by collisions; the main results of the previous work remain the same for absorption or for emission in thermal equilibrium.

The emission from a nonthermal plasma cannot be discussed quite as simply. The test-particle model of Section B can be used to calculate the emission directly, without using Kirchhoff's law. To include the effects of collisions, however, the sideways deflections must be included, as well as the slowing down of the mean velocity. This corresponds to acceleration of a particle because of encounters with other particles, and results in radiation. If these accelerations are included in $\delta j_t^{(o)}$, which can then not be neglected in equation 4.12, the emissivity will have a contribution due to the bremsstrahlung emission into the transverse field. If the accelerations are included in $\delta \rho_c^{(o)}$ in equation 4.12 as well, there is emission due to collisional excitation of plasma waves.

This procedure has been carried out by Birmingham, Dawson and Oberman for an infinite homogeneous plasma (24). They found that, although the <u>longitudinal</u> emission is restricted to the frequency range $\omega_p \lesssim \omega \lesssim 1.4\ \omega_p$, the emissivity in this range is <u>much larger</u> <u>than that for transverse waves</u> by a factor of order c^3/v_e^3 . For higher frequencies, where longitudinal waves do not propagate readily because of Landau damping, the usual transverse bremsstrahlung should be the dominant radiation process.

The excitation of plasma waves by electron-ion collisions may be more important than the "collisionless" excitation by fast electrons

for very large phase velocities. Since collisional effects are mainly
due to particles of average energy, their effect on the emission is
substantially the same for a nonthermal velocity distribution as for a
Maxwellian, with the same mean thermal energy (25). The collisional
contribution to the emissivity can therefore be calculated by using
Kirchhoff's law. The result is that, in addition to the distribution
function F in the numerator of each term of equation 4.18, there
should be a term proportional to the mean collision frequency. If the
collisional excitation and damping of plasma waves are more important
than the excitation by fast electrons and Landau damping, then the
emissivity has the thermal equilibrium value. It can be enhanced by
nonthermal electrons, however, if the latter processes are more impor-
tant than the former.

V.   SUMMARY AND CONCLUSIONS

The effect of collective behavior on absorption and emission of electromagnetic radiation by a bounded plasma has been investigated. A boundary-value problem has been solved in order to determine the properties of the absorption coefficient and the reflectivity near the collective resonance frequencies. The statistical properties of the noise emission near these same frequencies have also been determined by reinterpreting the solution of the boundary-value problem in terms of fluctuation quantities.

The model which has been used to represent a bounded plasma is that of a plane slab with uniform electron and ion densities. The electron velocity distribution is supposed known and independent of time. The electrons are specularly reflected when they strike the boundaries of the slab, which provides the means of containing the plasma. On the short time scales which are characteristic of plasma waves, the ions are supposed motionless.

The main assumptions which have been made are as follows:

(1) The plasma is nearly collisionless: $\omega_p t_c \gg 1$

(2) The collective oscillations are only small perturbations on the particle trajectories, so linearization is valid

(3) The distribution function is isotropic: $f_o = f_o(|\vec{v}|)$

(4) The plasma is non-relativistic: $\langle v^2 \rangle \ll c^2$

(5) The slab thickness is many Debye lengths: $L/\lambda_D \gg 1$

(6) The mean collision rate is much smaller than the Landau damping rate, but much larger than the trapping rate:
$\gamma_n \gg \bar{\nu} \gg \tau_{tr}^{-1}$

(7) Fluctuation quantities of higher order than those kept
in the dressed test particle approach are negligible.

The specific _results_ which have been obtained are the _following._
If a plane electromagnetic wave is incident on the plasma slab at an
angle $\theta \neq 0$, and is polarized with the electric field in the plane of
incidence, then it can be partially absorbed by Landau damping of the
plasma waves which it excites. If the incident wave frequency is close
to one of the plasma resonance frequencies, the absorption may be sub-
stantial, but only if the slab is thin compared with the electromagnetic
wavelength: $k_o L \ll 1$. The width of the absorption line is the sum of
the widths due to Landau damping and radiation damping. If the radia-
tion damping is equal to the Landau damping for one of the lines, then
the plasma is matched to the radiation field in that line, and the
absorption coefficient has its maximum value of 1/2 at the center of
that line. The half of the incident power in the antisymmetric part of
the field is totally absorbed at that frequency, and the other half in
the symmetric part of the field is scattered symmetrically to either
side (if the slab is thin: $k_o L \ll 1$). The peak values of the absorption
lines do not necessarily decrease monotonically with n . The reflec-
tivity also has resonance lines at the same frequencies as the
absorption lines, and with the same linewidths. The peak values of
these scattering resonance lines decrease monotonically with n . When
collisions are taken into account, the Landau damping rate γ_n which
appears in the theoretical results for the nth resonance must be
replaced by $\gamma_n + \bar{\nu}$, where $\bar{\nu}$ is the mean collision frequency.

Furthermore, a nearly-collisionless plasma slab emits noise radiation which is polarized with the electric vector in the plane of the slab normal and the direction of propagation. The emissivity depends on the angle θ between these two directions, and goes to zero like $\sin^2\theta$ for $\theta \to 0$. Emission lines occur under the same conditions as absorption lines and at the same resonance frequencies. In thermal equilibrium the emissivity is related to the absorption coefficient by Kirchhoff's law. If the plasma is also matched to the radiation field in one of the lines, then (for a thin slab) the emissivity at the center of that line is 1/2 of the black-body value. The noise radiation results from continuation of the electric field fluctuations away from the plasma, as a solution of the wave equation in free space. The fluctuations arise in the plasma because of the motions of the electrons, which can be treated using the "dressed test particle" approach. Emission lines occur at those frequencies for which the fluctuations are reinforced by the collective behavior of the plasma. The emissivity in a given line is proportional to the number of electrons with velocities about equal to the phase velocity of the plasma wave which is resonant at that frequency. The emissivity can therefore be enhanced for a nonthermal plasma with many such electrons in a high-energy tail of the distribution function. The collisional excitation of plasma waves may be the dominant mechanism for very large phase velocities.

Although high-frequency fluctuations can be observed in a plasma by incoherent scattering techniques, the scattering cross section is

very small and it may be easier to observe them using noise emission measurements from a bounded plasma. In general, one would expect that the properties of the radiation from any bounded plasma would be strongly influenced by the spectrum of collective wave motions in the plasma.

The comparison of the above theoretical results with the results of the microwave experiments which were mentioned in the Introduction can be of a qualitative nature only. This is because of the differences between the discharge plasmas used in the experiments and the idealized plasma model used in the theoretical calculation. The most important difference is that the discharge plasma is inhomogeneous, so that the electron density is not constant in the plasma. The standing wave normal modes are reflected against the density gradient and exist only in a thin region near the plasma boundary. Since this reduces the effective width of the plasma, the separation of the resonances in frequency is considerably larger than it would be for a homogeneous plasma of the same actual width, as used in the theoretical calculation. Another difference is that the discharge plasmas are usually circular cylinders, while the theory has been worked out for a plane slab. However, the experimental results and the above theoretical results have the following features in common:

(1) The frequencies of the emission lines are approximately the same as the frequencies of the corresponding scattering resonances.

(2) The widths of the emission lines and of the corresponding scattering resonances are also approximately the same.

(3) The scattering maxima decrease monotonically with increasing mode number n (or with increasing frequency ω_n) .

(4) The emission maxima do not necessarily decrease monotonically, depending on the size of the plasma.

For the larger of the two discharge tubes used by Kerzar and Weissglas (3), the emission maximum of the second resonance line is the largest, while for the smaller tube it is the first (or main) resonance line which has the largest peak value. This is understandable in terms of matching of the radiation damping to the internal damping of the plasma, since the radiation damping decreases with increasing mode number n , but increases with the size of the plasma.

The observed line widths decrease with increasing mode number similar to the behavior of the radiation damping. But the internal damping rate must be of the same order of magnitude as the radiation damping rate, in order that the plasma be approximately matched in the first or second line. Hence the internal damping rate must be approximately independent of mode number. If we knew that the Landau damping rate for an inhomogeneous plasma increases sharply with mode number as it does for a homogeneous plasma, then we could conclude that the internal damping is not Landau damping, and must be collisional. A better reason for this conclusion is that the absorption and emission in the experiments of Kerzar and Weissglad increase in proportion to the neutral gas density. Therefore, the primary dissipative mechanism in these discharge plasmas must be electron-neutral collisions. The corresponding radiation mechanism is probably the excitation of plasma waves by these collisions, which is consistent with the fact that the experimental absorption and emission spectra are very similar.

The theory of "collisionless" absorption and emission of electro-magnetic waves is therefore not entirely applicable to the experiments mentioned above. The theory cannot be completely confirmed until similar experiments are carried out with nearly-collisionless plasmas.

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