RADIATIVE CORRECTIONS TO NEUTRINO-ELECTRON PROCESSES

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ABSTRACT

The lowest order electromagnetic radiative corrections to electron-positron annihilation into neutrino and antineutrino and to neutrino scattering by electrons are calculated. The pair annihilation corrections are used to calculate the radiative corrections to stellar neutrino luminosities in the temperature range in which the zero order process gives the dominant contribution. The corrections to the neutrino-electron scattering cross section are those relevant to a proposed experiment detecting scattered electrons with more than a minimum recoil energy whether or not a bremsstrahlung photon is emitted.

The results depend logarithmically on an ultraviolet cut-off in the same way as does the lowest order vacuum polarization diagram of electrodynamics. When the cut-off is taken to be on the order of the nucleon mass the luminosity is enhanced by as much as 10 percent below $T \sim 10^9\,^{o}K$ and depressed by ~ 1 percent for $T > 10^9\,^{o}K$. The scattering cross section is depressed by ~ 4 percent for the incident neutrino energies ~ 8 - 14 MeV of the proposed experiment.

A characteristic distance, the neutrino charge radius, is associated with a charge distribution of the neutrino and depends on the cut-off. With the cut-off at the nucleon mass the charge radius of the electron neutrino is estimated to be two orders of magnitude smaller than the experimental limit. The possible effects of a charge radius larger than the estimate are also considered. It is found that interference between the electromagnetic and weak

couplings could depress both the non-relativistic stellar luminosity and the scattering cross section, but that if the scattering cross section is found to be as large as expected, the luminosity must be also.

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I. INTRODUCTION

The Feynman - Gell-Mann theory[1] of the weak interactions predicts that electrons and electron neutrinos interact through the square term

$$H_W = \frac{G}{\sqrt{2}} (\bar{\nu}_e \gamma_\mu (1 + \gamma_5) e)(\bar{e} \gamma_\mu (1 + \gamma_5) \nu_e), \quad {}^{2)}$$

in the interaction Hamiltonian $-J_\mu^+ J_\mu^-$, where

$$J_\mu^- = i\sqrt{\frac{G}{\sqrt{2}}} \left\{ (\bar{e}\ \nu_e) + (\bar{\mu}\ \nu_\mu) + (\bar{n}\ p) + \dots \right\}.$$

Here, the gamma matrices are implicit and the remainder contains currents of the other strongly interacting particles. The cross terms in the product of currents seem to describe well β-decay, μ-decay, and μ-capture, for example. Evidence for a parity violating term in the Hamiltonian of nucleii, which the square term $\frac{G}{\sqrt{2}} (\bar{p}\ n)(\bar{n}\ p)$ would imply, has been detected in nuclear transitions[3]. It appears probable that the theory is essentially correct and that the neutrino-electron interaction occurs as predicted.

The predicted interaction has been shown to have important consequences for the cooling of massive stars[4,5,6]. If the temperature of stellar material rises to $\sim 10^9\ ^oK$, the photons in thermal equilibrium have enough energy to produce electron-positron pairs through reactions like $2\gamma \rightarrow e^- + e^+$. Equilibrium between the radiation and the pairs should be reached in less than a second, which is much less than the time the star is expected to take to evolve from one stage to another. In stars of mass $\gtrsim 10\ M_\odot$ the density of the hot core, where the pairs would be produced, can be sufficiently low that the production is not suppressed by degeneracy of the electrons[7]. The square term in the weak interaction Hamiltonian predicts that the pairs can then annihilate into a neutrino and an antineutrino. Although this mode of annihilation would occur only $\sim 10^{-20}$ times as frequently as annihilation into photons, the neutrinos have a mean free path several hundred times the stellar radii[8,9] compared to the photon mean free path of less than a centimeter, so the energy of the neutrinos, but not that of the photons, is lost from the star.

The rate of energy loss from stellar material due to pair annihilation to neutrinos and antineutrinos is on the order of $(N^+ N^- c\sigma)(2mc^2)$, where $N^\pm$ is the number density of electrons or positrons. Neglecting the residual electrons, for $kT \sim mc^2$ the number density of pairs given by the Fermi distribution is approximately $.2 \left(\frac{kT}{\hbar c}\right)^3 \sim 10^{30}/\mathrm{cm}^3$. With an annihilation cross section of $\sim 10^{-44}\ \mathrm{cm}^2$, $\frac{du}{dt} \sim 5 \times 10^{20}\ \mathrm{erg/(cm^3 sec)}$, which could deplete the star's thermal energy in less than a year[5]. According to Fowler and Hoyle[7] oxygen burning could provide for the luminosity for about a day and the burning of higher mass nucleii up to iron for a similar length of time. Without the neutrino losses, this sequence of events would apparently take about 10^7 years longer.

In the model of Fowler and Hoyle, after the core exhausts its nuclear fuel the star can become a supernova and the heavy elements are disseminated in the explosion. Fowler and Hoyle have argued that the neutrino losses predicted by the current-current theory are of the right order of magnitude to give detailed agreement between the predicted and observed abundances of elements[7]. However, the course of evolution of massive stars is controversial[10,11].

There are other processes besides $e^- + e^+ \to \nu_e + \bar{\nu}_e$ which can produce neutrino luminosities in stars[12]. A number of them involve β-decay and $e^{\pm}$-capture and do not depend on the neutrino-electron interaction. But they would be dominated by many orders of magnitude by processes which do depend on it, if those occur as predicted. The reaction $\gamma + e \to e + \nu_e + \bar{\nu}_e$ does not require the production of pairs and is more important than $e^- + e^+ \to \nu_e + \bar{\nu}_e$ for $T < 5 \times 10^8$ °K. For very high densities, when the electrons become degenerate, the decay of a plasmon is supposed to dominate. In this paper we calculate the effect of the process $e^- + e^+ \to \nu_e + \bar{\nu}_e + \gamma$ together with the radiative corrections to $e^- + e^+ \to \nu_e + \bar{\nu}_e$.

The radiative corrections to $e^- + e^+ \to \nu_e + \bar{\nu}_e$ should be of order $\alpha = \frac{1}{137}$ smaller than the process with no photon involved, though corrections of order 10 percent to some weak interaction cross sections are known. When calculated in detail the amplitude to order α for pair annihilation into neutrino and antineutrino depends on the charge radius of the neutrino, which can only be estimated, since the field theory calculations diverge. But with what are thought to be reasonable estimates we find that the corrections indeed produce an effect of only a few percent on the stellar neutrino luminosities in the temperature range of interest, from $\sim 5 \times 10^8$ °K to $\sim 7 \times 10^9$ °K.

In Chapter II we calculate the necessary amplitudes and in Chapter III the corrections to the annihilation cross section. Radiative corrections to neutrino-electron scattering, $\nu_e + e \to \nu_e + e$, including the emission of soft photons only, have been discussed by Lee and Sirlin[13,14]. We can use the crossing relations between the scattering and annihilation processes to check the amplitudes involving virtual photons and soft real photons. The high energy limit to the radiative corrections can be checked against the general results of Yennie, Frautschi, and Suura[15]. The contributions of the diagrams are unambiguous except for the one in which the photon interacts with the charge distribution of the neutrino. If the weak interaction is local or the mass of the intermediate boson is much greater than the energies involved (as is the case for a boson of mass $m_W \gtrsim 3$ BeV[16]), to lowest order the charge distribution is effectively due to a virtual electron-positron pair. The amplitude for this diagram can be calculated with a cut-off mass Λ, on which the dependence is only logarithmic. The ambiguous term is proportional to the square of the charge radius of the neutrino. We will take Λ of the order of the mass of the nucleon as a reasonable estimate. In Chapter IV we give the corrections to the neutrino luminosity to which the cross section corrections give rise.

The neutrino-electron interaction could be detected in neutrino scattering by electrons in the laboratory. Reines and Kropp[17] have proposed an experiment to look for this reaction. The experiment will attempt to use as a neutrino source the decay of

B^8 in the sun. Though the expected flux of $10^7/cm^2$sec from this decay, $B^8 \to Be^{8*} + e^+ + \nu_e$, is too small to be important in the dynamics of the sun[18], the energy of up to $\sim$ 14 MeV makes these neutrinos convenient for several detection experiments. The flux can be measured independently in experiments which have been proposed using $\nu_e + Cl^{37} \to Ar^{37} + e^-$ [19,20] or the reactions $\nu_e + Li^7 \to Be^7 + e^-$ and $\nu + B^{11} \to C^{11} + e^-$ [21]. In the scattering experiment the B^8 neutrinos could give the recoil electron more than the $\sim$ 8 MeV of energy necessary for it to be distinguished from background.

Experiment has not ruled out the possibility that the neutrino charge radius is larger than expected from the estimates with the cut-off. The choice of Λ on the order of the mass of the nucleon gives a charge radius of $\sim 5 \times 10^{-17}$ cm as compared to the present experimental limit of $\sim 4 \times 10^{-15}$ cm [22]. A large charge radius could give the neutrino an electromagnetic interaction with electrons which could simulate the local weak interaction or partially cancel it. The effects that such an electromagnetic interaction would have on the stellar neutrino luminosity and on the cross-section for neutrino scattering by electrons are considered in Chapter VI. We also discuss there the information that could be obtained about the charge radius from several processes which would be expected to vanish if the neutrino had no electromagnetic coupling.

II. THE AMPLITUDES FOR THE RADIATIVE CORRECTIONS

A. The Basis of the Calculation

The Hamiltonian given by $-J_\mu^+ J_\mu^-$ for the weak interaction between neutrinos and electrons can be put in the form $H_W = -\frac{G}{\sqrt{2}} (\bar{\nu}_e \gamma_\mu (1 + \gamma_5) \nu_e)(\bar{e} \gamma_\mu (1 + \gamma_5) e)$ with a Fierz transformation. If there were a fundamental local interaction of neutral leptonic currents or if the neutral currents $(\bar{e}\, e)$ and $(\bar{\nu}_e \nu_e)$ were coupled to a heavy neutral intermediate boson, the contribution to the effective S matrix element would be of the same form and could possibly cancel the term from the product of the charged currents. But if the coupling G_0 of neutral leptonic currents to each other is not greater than their coupling to neutral currents of strongly interacting particles, G_0 appears to be much smaller than the coupling G of charged currents. The reaction $\nu_\mu + p \to \nu_\mu + p$ gives the limit $|G_0| < \frac{1}{10} |G|$ [9]. A similar limit is obtained from the ratio of the $K_2^0 \to \mu^+ + \mu^-$ and $K^+ \to \mu^+ + \nu_\mu$ decays[23], and a limit $|G_0| < \frac{1}{100} |G|$ from the branching ratio of $K^+ \to \pi^+ + e^+ + e^-$ to $K^+ \to \pi^0 + e^+ + \nu_e$ [24], when the ratios are assumed to be proportional to $(G_0/G)^2$; this would be the case if a neutral intermediate boson were coupled to the mesonic and leptonic currents with about equal strengths.

The effect of an intermediate boson on electron-positron annihilation into neutrino and antineutrino has been given by Levine[4]. The cross section for total center of mass energy E_T and electron or positron velocity β is

$$\sigma = \frac{G^2}{6\pi\beta}\left\{(E_T{}^2 - m_e{}^2) + \frac{1}{m_W{}^2}\left[-\frac{E_T{}^4}{2} + E_T{}^2\, m_e{}^2 + m_e{}^4\right] + 0\left(\frac{E_T{}^6}{m_W{}^4}\right)\right\} \tag{II.1}$$

With $m_W \geq 3$ BeV according to the present experimental limit[16], the correction due to finite boson mass would not be as large as the expected correction due to electromagnetic effects for center-of-mass energies less than $\sim$ 300 MeV. Since $kT \sim \frac{T_9}{12}$ MeV, where T_9 is the temperature in 10^9 °K, and the stars in which the annihilation process should be significant would collapse before reaching $T_9 \sim 10$, the electrons and positrons will have average energies $\lesssim 1$ MeV. In the proposed scattering experiments the momentum transfer can only be on the order of 10 MeV. In both cases the electromagnetic effects of order α are larger than would be the effects of an intermediate boson.

We calculate the electromagnetic correction using the interaction Hamiltonian $H_W + H_{em}$, where $H_{em} = ie(\bar{e}\,\gamma_\mu e)A_\mu$. Although H_W is not the basis of a renormalizable field theory, its use to first order, including also first order radiative corrections, has appeared to have some validity. In μ-decay no ambiguities arise in the computation of the radiative corrections[25] and the corrections to the electron spectrum are in agreement with experiment[26]. In β-decay, when the nucleons are considered as point particles, the divergent wave function renormalization term appears in the result, though for an ultraviolet cut-off of the order of the nucleon mass, the divergent term does not give the main effect[25,27]. The difference of a few percent between the β-decay and μ-decay vector coupling constants, which remains when the radiative corrections are taken into account (with any reasonable choice of cut-off), seems to be consistent with the Cabibbo theory[28]. The cut-off procedure at least does not give rise to an obviously wrong effect. It is argued that because of the low momentum transfer in β-decay the effects on the radiative corrections of the structure of the strongly interacting particles are small[29]. Corrections to decays like $\pi \rightarrow e + \nu$, on the other hand, are complicated by the structure of the pion[30]. The diagrams for the corrections to the neutrino-electron processes, except for the contribution of the charge distribution of the neutrino, correspond to those for μ-decay (with appropriate reversal of particle lines). It seems plausible that at least the effects corresponding to those of μ-decay should be correct.

We assume that the neutrino charge is exactly zero and that the two-component theory of the neutrino is correct[22]. With the square term in the current-current product the diagram in Fig. 1a is of first order in G, whereas processes like those indicated in the diagrams of Figs. 1b, 1c, and 1d would be of second order in G. To lowest order muon pairs would not contribute to the charge distribution of the electron neutrino and the absence of neutral currents would rule out pion or nucleon pairs. If an intermediate boson exists the diagrams in Fig. 1f and 1g give the contributions of first order in G.

The electron loop is like the vacuum polarization of electrodynamics and the result contains the ultraviolet cut-off in a term proportional to log Λ^2/m_e^2. The neutrino form factor in a theory with intermediate bosons has been calculated by several authors[31]. The diagram in Fig. 1f in which the photon interacts with the electron, is finite and gives a term log m_W^2/m_e^2 which corresponds to the log Λ^2/m_e^2 from the electron loop. The diagram in Fig. 1g, in which the photon interacts with the boson, diverges and depends on the method of cut-off. When terms of the order of m_e^2/m_W^2 and q^2/m_W^2 are neglected, the form factor reduces to $F(q^2) = q^2 f(q^2)$, where $f(q^2) - f(0)$ is finite and the same as given by the electron loop and $f(0)$ depends on the method of evaluation. The electron loop gives

$$f_F(0) = \frac{G}{16\pi^2\sqrt{2}} \left[\frac{4}{3} \log \frac{\Lambda^2}{m_e^2}\right] .$$

Lee and Bernstein argue that intermediate boson theory gives (with anomolous moment K = 0)

$$f_W(0) = \frac{G}{16\pi^2\sqrt{2}} \left[\frac{4}{3} \log \frac{m_W^2}{m_e^2} - \frac{5}{3} \log \alpha^{-1} + 2\right] .$$

Numerically $f_F(0) = 1.8 \times 10^{-34} \mathrm{cm}^2$ [20.4] for $\Lambda = m_N$, and $f_W(0) = 1.8 \times 10^{-34} \mathrm{cm}^2$ [20.4 - 3.3] for $m_W = 3m_N$. As has been emphasized[31], only an order of magnitude estimate of f(0) is warranted. Electrodynamics has been checked only to distances corresponding to a mass of $\sim \frac{1}{2}$ BeV[38]. The weak interaction would violate unitarity unless damped before $\sim$ 300 BeV. It therefore seems reasonable to take Λ between $\frac{1}{2}$ and 300 BeV. Because the dependence on Λ or m_W is logarithmic, f(0) is not sensitive to the particular values chosen. f(0) would be halved for $\Lambda \sim 20$ MeV and doubled for $\Lambda \sim 2 \times 10^3$ BeV.

The self-energy and vertex diagrams in the intermediate boson theory as in the Fermi theory would correspond to those of μ-decay. Therefore, as in μ-decay[32] they should give a contribution to the cross section for pair annihilation or neutrino-electron scattering which can be written, after renormalization, as the sum of the result of the Fermi theory, a part proportional to m_e^2/m_W^2 and terms of higher order. Except at high energies an intermediate boson would make little difference to the radiative corrections just as it would make little difference to the zero order process.

Before continuing to the details of the calculations for electron neutrino interactions with electrons we comment that electron-positron pairs could annihilate to a muon neutrino and antineutrino as in the diagram in Fig. 1e through the charge distribution of the muon neutrino. The cross section for that will contribute to the total cross section for annihilation into electron or muon neutrinos, which is the relevant quantity for the luminosity calculations. However, if the estimates of the charge radii are reasonable (for the muon neutrino the estimated f(0) is about a factor of 10 smaller than for the electron neutrino and somewhat more dependent on the method of calculation[31]), the cross section for $e^- + e^+ \rightarrow \nu_\mu + \bar{\nu}_\mu$ is of order $G^2\alpha^2$, an order α smaller than the lowest order radiative corrections to $e^- + e^+ \rightarrow \nu_e + \bar{\nu}_e$.

B. <u>The Contributions of Virtual Photons</u>

We will keep only the electron loop contribution to the neutrino charge distribution. The diagrams which give the radiative corrections of order α to the amplitude for $e^- + e^+ \to \nu_e + \bar{\nu}_e$ are shown in Fig. 2, with the labelling of the momenta in the annihilation channel. From the results for $e^-(p^-) + e^+(p^+) \to \nu_e(k) + \bar{\nu}_e(\bar{k})$, the results for neutrino scattering by electrons $\nu_e(k_1) + e^-(k_2) \to \nu_e(k_2) + e^-(p_2)$ are obtained by the substitutions $k \to k_2$, $\bar{k} \to - k_1$, $p^- \to p_1$, and $p^+ \to - p_2$. The results for anti-neutrino scattering, $\bar{\nu}_e(\bar{k}_1) + e_1^-(p_1) \to \bar{\nu}_e(\bar{k}_2) + e^-(p_2)$ are obtained by the substitutions $\bar{k} \to \bar{k}_2$, $k \to - \bar{k}_1$, $p^- \to p_1$, and $p^+ \to - p_2$.

The electron self energy and mass renormalization of the diagrams in Figs. $2a_1$ and $2a_2$ give the amplitude

$$T_{a_1 + a_2} = - \frac{G}{\sqrt{2}} \left[\bar{U}_k \Gamma_\mu V_{-\bar{k}}\right] \left[\bar{V}_{-p^+} \Gamma_\mu \frac{-i}{i\not{p} + m} (-i)(\Sigma(p^-) - \delta m) U_{p^-}\right] \tag{II.2}$$

where $\Gamma_\mu = \gamma_\mu(1 + \gamma_5)$, and

$$\Sigma(p^-) = - ie^2 \int \frac{d^4 r}{(2\pi)^4} \frac{1}{r^2 + \lambda^2} \gamma_\nu \frac{1}{i(\not{p} - \not{r}) + m} \gamma_\nu$$

$$= \delta m + \left[Z_2^{-1} - 1 + C(p^-)\right] \left[i\not{p} + m\right] . \quad ^{34)} \tag{II.3}$$

The photon has been given a small mass λ to avoid the divergence associated with the intermediate electron being on the mass shell in the limit of low virtual photon energy. m is the physical mass of the electron and $\delta m = m - m_0$, where m_0 is the bare mass. These corrections are the same as in electrodynamics[33]. $C(p^-) = 0$ for $i\not{p}^- = m$, so that

$$T_{a_1} + {}_{a_2} = -\frac{G}{\sqrt{2}} \left[\bar{U}_k \Gamma_\mu V_{-\bar{k}}\right]\left[\bar{V}_{-p^+} \Gamma_\mu (-1)(Z_2^{-1} - 1) U_{p^-}\right] . \quad \text{(II.4)}$$

Expressing the bare field theory wave functions in terms of the wave functions of the physical particles and including the self-energy and mass renormalization diagrams for the positron line, we have

$$T_0 + T_a = -\frac{G}{\sqrt{2}} \left[\bar{U}_k \Gamma_\mu V_{-\bar{k}}\right]\left[\bar{V}_{-p^+} \Gamma_\mu \left[1 - (Z_2^{-1} - 1)\right] U_{p^-}\right] . \quad \text{(II.5)}$$

The amplitude for the vertex correction of Fig. 2b is

$$T_b = -\frac{G}{\sqrt{2}} \left[\bar{U}_k \Gamma_\mu V_{-\bar{k}}\right]\left[\bar{V}_{-p^+} \Lambda_\mu(-p^+, p^-) U_{p^-}\right] , \quad \text{(II.6)}$$

with

$$\Lambda_\mu(-p^+, p^-) = ie \int \frac{d^4 r}{(2\pi)^4} \frac{1}{r^2 + \lambda^2} \gamma_\nu \frac{1}{-i(\not{p}^+ + \not{r}) + m} \Gamma_\mu \frac{1}{i(\not{p}^- - \not{r}) + m} \gamma_\nu . \quad \text{(II.7)}$$

This function contains a vector part $\Lambda_\mu^V(-p^+, p^-)$ and an axial vector part $\Lambda_\mu^A(-p^+, p^-)$ arising from the γ_μ or $\gamma_\mu \gamma_5$ in Γ_μ, respectively. Λ_μ^V is the same as the vertex correction in electrodynamics and can be written as $(Z_1^{-1} - 1)\gamma_\mu + \Lambda_\mu^{cV}(-p^+, p^-)$, where Λ_μ^{cV} is independent of the cut-off necessary in calculation of Z_1.

Rationalizing the electron propagators and combining denominators, we have for general Γ_μ,

$$\Lambda_\mu(p',p) = 2ie^2\int_0^\infty dz_1\int_0^\infty dz_2\int_0^\infty dz_3\ \delta(1 - \sum_1^3 z_i)\int\frac{d^4r}{(2\pi)^4}$$

$$\frac{\{\gamma_\nu[m - i(\not p' - \not r)]\Gamma_\mu[m - i(\not p - \not r)]\gamma_\nu\}}{[(r - p'z_2 - pz_3)^2 + c]^3} \qquad \text{(II.8)}$$

where between electron spinors $p^2 = p'^2 = - m^2$,

$$c = c(Q^2) = m^2(1 - z_1)^2 + \lambda^2 z_1 + Q^2 z_2 z_3 - i\epsilon,\ Q = p' - p,$$

and ϵ is a small positive number. The integration over r can be changed to integration over $r' = r - p'z_2 - pz_3$ [35]. Dropping the terms in the numerator linear in r'_μ, which by symmetry will not contribute, we have

$$\Lambda_\mu(p',p) = \frac{i\ \alpha}{2\pi^3}\int_0^\infty dz_1'\int_0^\infty dz_2\int_0^\infty dz_3\ \delta(1 - \sum_1^3 z_i)\int d^4r\ \frac{[0_\mu(p',p) - \gamma_\nu\not r\ \Gamma_\mu\not r\ \gamma_\nu]}{(r^2 + c)^3}\ . \qquad \text{(II.9)}$$

$$0_\mu(p',p) = \gamma_\nu[m - i\not p'(1 - z_2) + i\not p z_3][m - i\not p(1 - z_3) + i\not p' z_2]\gamma_\nu, \qquad \text{(II.10)}$$

is independent of r_μ. Again by symmetry the term $\gamma_\nu\not r\ \Gamma_\mu\ \not r\ \gamma_\nu$ is effectively $1/4\ \gamma_\nu\gamma_\kappa\gamma_\mu\gamma_\lambda\gamma_\kappa\ \delta_{\kappa\lambda}r^2$ in the integral and the numerator is effectively $0_\mu(p',p) - 1/4\ \gamma_\nu\gamma_\kappa\ \Gamma_\mu\ \gamma_\kappa\gamma_\nu r^2$. The term with r^2 gives a logarithmic divergence.

For $\Lambda_\mu^V(p',p)$ the divergent part is proportional to $-1/4\ \gamma_\nu\gamma_\kappa\gamma_\mu\gamma_\kappa\gamma_\nu = -\gamma_\mu$ and gives the divergent part of $(Z_1^{-1} - 1)\gamma_\mu$. By Ward's identity $Z_1 = Z_2$ [36], so that the divergent contributions of the electron self-energy and wave function renormalization cancel the divergent integral in $(Z_1^{-1} - 1)$. Ward's identity is due to the relation

$$\frac{\partial}{\partial p_\mu} \frac{-i}{i\not{p} + m} = \frac{-i}{i\not{p} + m} \Gamma_\mu \frac{-i}{i\not{p} + m} \qquad \text{(II.11)}$$

for $\Gamma_\mu = \gamma_\mu$, which gives

$$\frac{\partial}{\partial p_\mu} \Sigma(p) = i\, \Lambda_\mu^V(p,p). \qquad \text{(II.12)}$$

For $\Gamma_\mu = \gamma_\mu\gamma_5$, $\frac{\partial}{\partial p_\mu} \Sigma(p)$ and $\Lambda_\mu^A(p,p)$ are not related so simply, since the relation (II.11) does not hold. However, for large momenta r_μ in the integral (II.7) the mass m in the propagators can be neglected and the γ_5 in Λ_μ^A can be commuted to the right, so that the largest part of $\Lambda_\mu^A(p',p)$ should equal the largest part of $\Lambda_\mu^V(p',p)$ times γ_5. In terms of the integral in the form (II.9), for $\Gamma_\mu = \gamma_\mu\gamma_5$, $-1/4\ \gamma_\nu\gamma_\kappa\gamma_\mu\gamma_5\gamma_\kappa\gamma_\nu = -\gamma_\mu\gamma_5$, so that we have the same result, that the divergent part of $\Lambda_\mu^A(p',p)$ is the same as that of $(Z_1^{-1} - 1)\ \gamma_\mu\gamma_5$ and the cancellation of divergences will occur in this case too.

For any other Γ_μ the cancellation would not occur. The amplitude contributed by the electron self-energy and the wave function renormalization of (II.4) holds for any Γ_μ. But $-1/4\ \gamma_\nu\gamma_\kappa(1,\gamma_5,\sigma_{\lambda\mu})\gamma_\kappa\gamma_\nu = (-4, -4\gamma_5, 0)$, so that the coefficient of $(Z_1^{-1} - 1)\ \Gamma_\mu$ contributed by the vertex correction would not be 1 [37]. If the coupling in $H_W = \frac{G}{2} (\bar{\nu}_e e)(\bar{e}\ \nu_e)$ is vector and axial, it must be $\gamma_\mu(1 + \gamma_5)$ for only γ_μ and $\gamma_\mu\gamma_5$ to appear between the electron spinors in the Hamiltonian when it is put in the form $(\bar{e}\ e)(\bar{\nu}_e \nu_e)$. The cancellation would not occur for right-handed neutrinos.

That the cancellation of divergent parts occurs for $\gamma_\mu\gamma_5$ as well as for γ_μ can be seen as a consequence of a γ_5 invariance of the free Lagrangian for a Dirac particle[29]. If the masses of the two charged fermions involved in the vertex are different, as in μ-decay, the vertex function is $\Lambda_\mu(p_2,m_2;\ p_1,m_1)$. Under the transformation $\psi_1 \to \gamma_5\psi_1$ and $m_1 \to -\ m_1$ the free Lagrangian $\mathcal{L}_{free} = -\ \bar{\psi}_1(\not\partial + m_1)\psi_1$ is invariant and the vector and axial vector currents interchange, that is, $\bar{\psi}_1\gamma_\mu\psi_2 \rightleftarrows \bar{\psi}_1\gamma_\mu\gamma_5\psi_2$. Thus, we should have $\Lambda_\mu{}^A(p_2,m_2;p_1,m_1) = \Lambda_\mu{}^V(p_2,m_2;p_1,-m_1)$. The divergent parts of Z_1 and Z_2 are independent of m_1 so that the cancellation should occur for both vector and axial couplings.

Returning to the calculation of $\Lambda_\mu(p',p)$, for $\not p = \not p' = im$, we have

$$0_\mu(p,p) = -\ 2m^2(1 - 4z_1 + z_1{}^2)\gamma_\mu + 2m^2(1 + z_1{}^2)\gamma_\mu\gamma_5\,, \qquad \text{(II.13)}$$

and we obtain

$$\Lambda_\mu(p,p) = (Z_1^{-1} - 1)\Gamma_\mu - \frac{\alpha}{2\pi}\,\gamma_\mu\gamma_5. \qquad \text{(II.14)}$$

There remains $\Lambda_\mu(p',p) - \Lambda_\mu(p,p)$ to determine, which we denote by $\Lambda_\mu{}^c(p',p)$. The integrals over r in $\Lambda_\mu{}^c$ are convergent and give

$$\Lambda_\mu{}^c(p',p) = -\frac{\alpha}{4\pi}\int_0^1 dz_1\int_0^{1-z_1} dz_2 \left\{ \frac{\Delta 0_\mu{}^V(p',p) + \Delta 0_\mu{}^A(p',p)}{c(Q^2)} - \frac{Q^2 z_2 z_3 0_\mu(p,p)}{c(Q^2)\ c(0)} - 2\,\Gamma_\mu \log\frac{c(0)}{c(Q^2)} \right\}. \qquad \text{(II.15)}$$

Here $z_3 = 1 - z_1 - z_2$ and

$$\triangle 0_\mu{}^V(p',p) + \triangle 0_\mu{}^A(p',p) = 0_\mu(p',p) - 0_\mu(p,p),$$

where

$$\triangle 0_\mu{}^V(p',p) = [2Q^2(1 - z_2)(z_1 + z_2) - 4m^2\, z_1(1 - z_1)]\gamma_\mu$$
$$- 2im\, z_1(1 - z_1)\Sigma_\mu + 2im\,[z_2 + z_2{}^2 - z_3 - z_3{}^2]Q_\mu \qquad \text{(II.16a)}$$

and

$$\triangle 0_\mu{}^A(p',p) = 2Q^2(1 - z_2)(z_1 + z_2)\gamma_\mu\gamma_5$$
$$+ 2im\,[1 - z_1 - 2z_2z_3 + z_3{}^2 + z_2{}^2]Q_\mu\gamma_5 \qquad \text{(II.16b)}$$
$$+ 2im\,[z_3 - z_3{}^2 - z_2 + z_2{}^2]\Sigma_\mu\gamma_5\ ,$$

with $Q_\mu = (p' - p)_\mu$ and $\Sigma_\mu = (p' + p)_\mu$. Between spinors $\Lambda_\mu{}^c$ can be expressed in general as

$$\Lambda_\mu{}^c(p',p) = \Lambda_\mu{}^c(S) + \Lambda_\mu{}^c(PS)\gamma_5 + \Lambda^c(V)\gamma_\mu + \Lambda^c(A)\gamma_\mu\gamma_5, \qquad \text{(II.17)}$$

where

$$\Lambda_\mu{}^c(S) = \Lambda^c(S,\Sigma)\Sigma_\mu + \Lambda^c(S,Q)Q_\mu \qquad \text{(II.18a)}$$

and

$$\Lambda_\mu{}^c(PS) = \Lambda^c(PS,\Sigma)\Sigma_\mu + \Lambda^c(PS,Q)Q_\mu. \qquad \text{(II.18b)}$$

$\Lambda^c(V)$, $\Lambda^c(A)$, $\Lambda^c(S,\Sigma)$, ... are scalar functions of Q^2 given by the coefficients of the appropriate vector or axial vector in (II.15). We will show later that $\Lambda^c(S,Q)$ and $\Lambda^c(PS,\Sigma)$ must be zero.

Finally, the amplitude for the electron loop diagram in

Fig. 2c is given by

$$T_c = -\frac{G}{\sqrt{2}}\,[\bar{U}_k\Gamma_\mu V_{-\bar{k}}][\bar{V}_{-p^+}\gamma_\nu \frac{-i}{Q^2+\lambda^2}\,\Pi_{\nu\mu}(Q)\,U_{p^-}], \qquad \text{(II.19)}$$

where in the annihilation channel $Q = -p^+ - p^-$.

$$\Pi_{\nu\mu}(Q) = 4\pi\alpha\int\frac{d^4r}{(2\pi)^4}\,\frac{\mathrm{Tr}\{\gamma_\nu(m - i\not{r})\Gamma_\mu[m - i(\not{r} - \not{q})]\}}{(r^2+m^2)[(r-q)^2+m^2]}\,. \qquad \text{(II.20)}$$

Since there is only the one vector, Q_μ, available, a pseudotensor cannot be constructed, so the $\gamma_\mu\gamma_5$ part of Γ_μ cannot contribute. Then mathematically $\Pi_{\nu\mu}(Q)$ is the same as the contribution of the electron-positron pair to the vacuum polarization tensor of electrodynamics. The difference between the integral for fermion mass m and that for mass Λ is finite and gauge invariant. The result[33] of combining denominators and integrating over r is

$$\bar{\Pi}_{\nu\mu}(Q) = -\frac{2i\alpha}{\pi}\,(\delta_{\nu\mu}Q^2 - Q_\nu Q_\mu)\int_0^1 dz\; z(1-z)\,\log\frac{\Lambda^2 + z(1-z)Q^2 - i\epsilon}{m^2 + z(1-z)Q^2 - i\epsilon}\,. \qquad \text{(II.21)}$$

Conservation of the electron current is sufficient to imply that the term proportional to $Q_\nu Q_\mu$ does not contribute. If the cut-off mass Λ is taken to be much greater than $|Q^2|$

$$\gamma_\nu\,\frac{-i}{Q^2}\,\bar{\Pi}_{\nu\mu} \sim \gamma_\mu\,[Z_3 - 1 + \Pi^c(Q^2)], \qquad \text{(II.22)}$$

where

$$Z_3 - 1 \sim -\frac{\alpha}{3\pi}\log\frac{\Lambda^2}{m^2}\,, \qquad \text{(II.23)}$$

and

$$\Pi^c(Q^2) = \frac{2\alpha}{\pi}\int_0^1 dz\; z(1 - z)\log\left[1 - i\epsilon + \frac{z(1 - z)Q^2}{m^2}\right] \quad . \qquad (II.24)$$

are the same as in electrodynamics.

Including all the contributions the total amplitude to order α is

$$T = -\frac{G}{\sqrt{2}}[\bar{U}_k\Gamma_\mu V_{-\bar{k}}][\bar{V}_{-p^+}(\Gamma_\mu + \Delta\Gamma_\mu)U_{p^-}], \qquad (II.25)$$

where

$$\Delta\Gamma_\mu = -\frac{\alpha}{2\pi}\gamma_\mu\gamma_5 + \Lambda_\mu^c(-p^+,p^-) + \gamma_\mu[Z_3 - 1 + \Pi^c(Q^2)]. \qquad (II.26)$$

Before calculating the sum over electron spins of $|T|^2$, it is convenient to note that $\Lambda^c(S,Q)$ and $\Lambda^c(PS,\Sigma)$ of (II.18) are identically zero. For instance, from (II.15) and (II.16b),

$$\Lambda^c(PS,\Sigma) = -\frac{im\alpha}{2\pi}\int_0^1 dz_1\int_0^{1-z_1} dz_2\,\frac{z_3(1 - z_3) - z_2(1 - z_2)}{c(Q^2)} .$$

Changing variables to $x = 1 - z_1$ and u given by $z_2 = xu$ and integrating over x, we are left with

$$\Lambda^c(PS,\Sigma) = -\frac{im\alpha}{2\pi}\int_0^1 du\,\frac{(\frac{1}{2} - u)}{m^2 + Q^2u(1 - u) - i\epsilon} = 0,$$

since the integrand is antisymmetric about $u = 1/2$. The vanishing of these functions follows from the invariance of $H_W + H_{em}$ under time reversal. To first order in G the S matrix element for the process $\nu + e \to \nu' + e'$ has the form

$$S_{f,i} = - i(2\pi)^4\delta^4(p_f - p_i)\langle\nu'|J_\mu^{(\nu)}|\nu\rangle\langle e'|J_\mu^{(e)}|e\rangle,$$

where

$$\langle\nu'|J_\nu^{(\nu)}|\nu\rangle = i\sqrt{\frac{G}{\sqrt{2}}}\,[\bar{\nu}'\Gamma_\mu\nu],$$

and

$$\langle e'|J_\mu^{(e)}|e\rangle = i\sqrt{\frac{G}{\sqrt{2}}}\;[\bar{e}'(\Gamma_\mu + \Delta\Gamma_\mu)e] \quad .$$

Since $H_W + H_{em}$ is invariant under time reversal, the relation $S_{f,i} = S_{Ti,Tf}$ must hold. Then the transformation property of $J_\mu^{(\nu)}$

$$\langle\nu'|T^+J_\mu^{(\nu)}T|\nu\rangle = - \langle\nu'|J_\mu^{(\nu)}|\nu\rangle,$$

implies the same transformation property of $J_\mu^{(e)}$ and $\Delta\Gamma_\mu$ can have no terms of the form Q_μ or $\Sigma_\mu\gamma_5$. We also notice that the term $[\bar{V}_{-p^+}\,\Lambda^c(PS,Q)Q_\mu\gamma_5\,U_{p^-}]$ does not contribute when contracted with $[\bar{U}_k\,\Gamma_\mu V_{-\bar{k}}]$.

With these results the effective Γ_μ can be written as

$$\Delta\Gamma_\mu = S_\Sigma\Sigma_\mu + V\,\gamma_\mu + A\,\gamma_\mu\gamma_5, \tag{II.27}$$

with

$$S_\Sigma = \Lambda^c(S,\Sigma), \tag{II.28a}$$

$$V = \Lambda^c(V) + [Z_3 - 1 + \Gamma^c(Q^2)], \tag{II.28b}$$

and

$$A = -\frac{\alpha}{2\pi} + \Lambda^c(A). \tag{II.28c}$$

The sum over initial and final spins of $|T|^2$ gives to first order in α

$$\sum_{\text{spins}} |T|^2 = \frac{G^2}{2} Tr^{(\nu)} [Tr_0^{(e)} + \Delta Tr^{(e)}], \qquad (II.29)$$

where

$$Tr^{(\nu)} = Tr\{\bar{\Gamma}_\nu(-i\not{k}) \Gamma_\mu(-i\bar{\not{k}})\}, \qquad (II.30)$$

$$Tr^{(\nu)} Tr_0^{(e)} = 256\ k\cdot p^+\ \bar{k}\cdot p^-, \qquad (II.31)$$

and

$$\Delta Tr^{(e)} = Tr\ \{\Gamma_\mu(m - i\not{p}^-)\overline{\Delta\Gamma}_\nu(-m - i\not{p}^+) + \Delta\Gamma_\mu(m - i\not{p}^-)\bar{\Gamma}_\nu(-m - i\not{p}^+)\}\ . \qquad (II.32)$$

In these expressions $\bar{\Gamma}_\nu = -(-)^{\delta_{\nu 4}} \Gamma_\nu$ and

$$\overline{\Delta\Gamma}_\nu = -(-)^{\delta_{\nu 4}}\{-S_\Sigma^* \Sigma_\nu + V^*\gamma_\nu + A^*\gamma_\nu\gamma_5\}.$$

We find

$$\begin{aligned} \frac{1}{64} Tr^{(\nu)}\Delta\ Tr^{(e)} &= m\ Im\ S_\Sigma\ (2\Sigma\cdot\bar{k}\ \Sigma\cdot k - \Sigma^2\ k\cdot\bar{k}) \\ &+ 2\ Re\ V(2\ p\cdot\bar{k}\ p^+\cdot k - m^2\ k\cdot\bar{k}) \qquad (II.33) \\ &+ 2\ Re\ A\ (2\ p\cdot\bar{k}\ p^+\cdot k + m^2 k\cdot\bar{k}). \end{aligned}$$

This will be used for the virtual photon corrections to the cross sections for annihilation in Chapter III and for scattering in Chapter V.

C. The Contributions of Real Photons

The amplitude for the bremsstrahlung diagrams in Fig. 3 is

$$T^b = -\frac{G}{\sqrt{2}} [\bar{U}_k \Gamma_\mu V_{-\bar{k}}][\bar{V}_{-p^+}\left(\Gamma_\mu \frac{-i}{i(\not{p}^- - \not{q}) + m} \quad e\not{\epsilon}^*\right.$$

$$\left. + e\not{\epsilon}^* \quad \frac{-i}{i(-\not{p}^+ + \not{q}) + m} \Gamma_\mu\right) U_{p^-}], \qquad \text{(II.34)}$$

where $\epsilon^*_\beta = (-)^{\delta_{\alpha 4}} \epsilon^\dagger_\beta$. T^b is invariant under the transformation $\epsilon_\mu \to \epsilon_\mu + q_\mu$ as required by gauge invariance.

Summation over fermion spins gives

$$\sum_{e\ \text{spins}} |T^b|^2 = \frac{G^2}{2} e^2 \, Tr^{(\nu)} Tr^{(e\gamma)}, \qquad \text{(II.35)}$$

where $Tr^{(\nu)}$ is given in (II.30) and

$$Tr^{(e\gamma)} = (-)^{\delta_{\nu 4}} \, Tr \left\{ \left[\frac{(2\epsilon^* \cdot p^+ - \not{\epsilon}^* \not{q})}{-2q \cdot p^+} \Gamma_\mu + \Gamma_\mu \frac{(-2\bar{p} \cdot \epsilon^* + \not{q} \not{\epsilon}^*)}{-2q \cdot p^-} \right] [m - i\not{p}^-] \right.$$

$$\left. \times \left[\Gamma_\nu \frac{(2\epsilon \cdot p^+ - \not{q}\not{\epsilon})}{2q \cdot p^+} + \frac{(-2p^- \cdot \epsilon + \not{\epsilon}\not{q})}{2q \cdot p^-} \Gamma_\nu \right] [- m - i\not{p}^*] \right\}. \qquad \text{(II.36)}$$

The usual bremsstrahlung infrared divergence will occur for low energy zero mass photons. The order α corrections due to virtual photons have meaning only together with the cross section for emitting one photon, since any observable cross section will include soft photons. Having cut off the infrared divergence in the elastic cross section by giving the photon a finite mass, we must cut off the divergence in the bremsstrahlung cross section in the same manner. Hence we give the photon the small mass λ. The term $q^2 = -\lambda^2$ in the denominator of (II.36), however, can be neglected compared to $-q\cdot p_i$ which becomes $\lambda\, E_i$ when $|\vec{q}| \to 0$.

Finally, we sum over the photon polarizations with the relation $\sum_{pol} \epsilon_\alpha \epsilon_\beta^* = \delta_{\alpha\beta} + \frac{q_\alpha q_\beta}{\lambda^2}$. The $q_\alpha q_\beta$ term does not contribute because of current conservation. The result is

$$(128\ G^2 e^2)^{-1} \sum_{spins} |T^b|^2 = p^-\cdot k\ p^+\cdot k \left(\frac{p^-}{q\cdot p^-} - \frac{p^+}{q\cdot p^+}\right)^2$$

$$+ p^-\cdot\bar{k}\ p^+\cdot k \left(\frac{1}{q\cdot p^-} + \frac{1}{q\cdot p^+}\right) + \frac{p^-\cdot p^+}{q\cdot p^- q\cdot p^+}\,(q\cdot\bar{k}\ p^+\cdot k + q\cdot k\ p^-\cdot\bar{k})$$

$$- \frac{\bar{p}^-\cdot\bar{k}\ p^-\cdot k}{q\cdot p^-} - \frac{p^+\cdot\bar{k}\ p^+\cdot k}{q\cdot p^+} + \frac{p^-\cdot\bar{k}\ q\cdot k}{(q\cdot p^+)^2}\,(m^2 - q\cdot p^+) + \frac{p^+\cdot k\ q\cdot k}{(q\cdot p^-)^2}\,(m^2 - q\cdot p^-). \tag{II.37}$$

This result checks with the $\sum_{e\ spins} |T|^2$ which Chin and Stabler[6] give for $\gamma + e \to e + \nu_e + \bar{\nu}_e$, when we carry out the sum over photon polarizations and make the substitutions $q \to -q$ and $p' \to -p^+$ in their expression.

III. THE PAIR ANNIHILATION CROSS SECTIONS

To obtain the virtual photon corrections to the cross section for $e^- + e^+ \rightarrow \nu_e + \bar{\nu}_e$ we first write the correction to $\sum_{spins} |T|^2$ given by (II.29) and (II.33) as

$$\Delta \sum_{spins} |T|^2 = 32\ G^2\ k_\alpha \bar{k}_\beta M_{\alpha\beta}\ , \qquad \text{(III.1)}$$

where

$$\begin{aligned} M_{\alpha\beta} = m\ \text{Im}\ S_\Sigma [2\ \Sigma_\alpha \Sigma_\beta - \Sigma^2\ \delta_{\alpha\beta}] + 2\ \text{Re}\ V[2p^+_\alpha\ p^-_\beta - m^2\ \delta_{\alpha\beta}] \\ + 2\ \text{Re}\ A[2p^+_\alpha p^-_\beta + m^2\ \delta_{\alpha\beta}] \end{aligned} \qquad \text{(III.2)}$$

does not depend on k or $\bar{k}$. The correction to the cross section is

$$\begin{aligned} \Delta\sigma &= \frac{1}{16(2\pi)^2 \sqrt{(p^- \cdot p^+)^2 - m^4}} \int \frac{d^3k}{k} \frac{d^3\bar{k}}{\bar{k}}\ \delta^4(p^- + p^+ - k - \bar{k})\ \frac{1}{4} \Delta \sum_{spins} |T|^2 \\ &= \frac{G^2}{8\pi^2 \sqrt{(p^- \cdot p^+)^2 - m^4}} \int \frac{d^3k}{k} \frac{d^3\bar{k}}{\bar{k}}\ \delta^4(p^- + p^+ - k - \bar{k}) k_\alpha \bar{k}_\beta\ M_{\alpha\beta}\ . \end{aligned} \qquad \text{(III.3)}$$

Using

$$\int \frac{d^3k}{k} \frac{d^3\bar{k}}{\bar{k}}\ \delta^4(-Q - k - \bar{k}) k_\alpha \bar{k}_\beta = \frac{\pi}{6}\ [2Q_\alpha Q_\beta + \delta_{\alpha\beta}\ Q^2]\ , \qquad \text{(III.4)}$$

we obtain

$$\Delta\sigma = \frac{G^2}{6\pi\beta}\ \{m\ \text{Im}\ S_\Sigma(t - 4m^2) + \text{Re}\ V(t + 2m^2) + \text{Re}\ A(t - 4m^2)\}. \qquad \text{(III.5)}$$

Here $t = -Q^2 = -(p^+ + p^-)^2$ and $\beta = \sqrt{\frac{t - 4m^2}{t}}$.

In Appendix A the functions S_Σ, V, and A are evaluated for a photon of small mass λ. For $t > 4m^2$ the results give

$$\mathrm{Im}\ S_\Sigma = -\frac{\alpha}{4m\pi}\frac{(1-\beta^2)}{\beta}\tanh^{-1}\beta, \qquad \text{(III.6a)}$$

$$\mathrm{Re}\ V = \mathrm{Re}\ \Lambda^c(V) + (Z_3 - 1) + \mathrm{Re}\ \Pi^c(Q^2), \qquad \text{(III.6b)}$$

$$\mathrm{Re}\ A = \mathrm{Re}\ \Lambda^c(V) + \frac{\alpha}{\pi}\frac{(1-\beta^2)}{\beta}\tanh^{-1}\beta, \qquad \text{(III.6c)}$$

with

$$\mathrm{Re}\ \Lambda^c(V) = \frac{\alpha}{2\pi}\left\{-\left[\frac{1+\beta^2}{\beta}\tanh^{-1}\beta - 1\right]\log\frac{m^2}{\lambda^2} + \frac{1+\beta^2}{\beta}\frac{\pi^2}{2} - 2 + 3\beta\tanh^{-1}\beta + \frac{1+\beta^2}{2\beta}\left[L\left(\frac{2\beta}{1+\beta}\right) - L\left(\frac{-2\beta}{1-\beta}\right)\right]\right\}, \qquad \text{(III.6d)}$$

and

$$\mathrm{Re}\ \Pi^c(Q^2) = \frac{\alpha}{3\pi}\left\{-\frac{5}{3} - (1-\beta^2) + 2\beta\left[1 + \frac{1-\beta^2}{2}\right]\tanh^{-1}\beta\right\}. \qquad \text{(III.6e)}$$

$L(x) = \int_0^x dt\,\frac{\log(1-t)}{t}$ is the Spence function[42]. The limit $\lambda \to 0$ has been taken except in the term in $\Lambda^c(V)$ in which the infrared divergence appears.

Combining terms we have for the order α corrections to the cross section for $e^- + e^+ \to \nu_e + \bar{\nu}_e$

$$\left(\frac{G^2}{6\pi\beta}\right)^{-1}\Delta\sigma = (t - m^2)\frac{\alpha}{\pi}\left\{-\left(\frac{1+\beta^2}{\beta}\tanh^{-1}\beta - 1\right)\log\frac{m^2}{\lambda^2} + \frac{1+\beta^2}{2\beta}\left[\pi^2 + L\left(\frac{2\beta}{1+\beta}\right) - L\left(\frac{-2\beta}{1+\beta}\right)\right] - 2 + 3\beta\tanh^{-1}\beta\right\}$$

$$+ \frac{3\alpha}{\pi} m^2 \beta \tanh^{-1} \beta$$

$$+ (t + 2m^2) \left\{ Z_3 - 1 + \frac{\alpha}{3\pi} \left[- \frac{8}{3} + \beta^2 + \beta(3 - \beta^2)\tanh^{-1}\beta \right] \right\} . \qquad \text{(III.7)}$$

The cross section for the annihilation with emission of a photon is

$$\sigma^b = \frac{1}{32(2\pi)^5 \sqrt{(p^- \cdot p^+)^2 - m^4}} \int \frac{d^3q}{q_0} \frac{d^3k}{k} \frac{d^3\bar{k}}{\bar{k}} \delta^4(p^- + p^+ - k - \bar{k} - q)$$

$$\frac{1}{4} \sum_{\text{spins}} |T^b|^2. \qquad \text{(III.8)}$$

Writing $\sum_{\text{spins}} |T^b|^2 = 64\ G^2 e^2\ k_\alpha \bar{k}_\beta\ M^b_{\alpha\beta}$, we have

$$\sigma^b = \frac{G^2 \alpha}{(2\pi)^4 \sqrt{(p^- \cdot p^+)^2 - m^4}} \int \frac{d^3q}{q_0} \frac{d^3k}{k} \frac{d^3\bar{k}}{\bar{k}} \delta^4(K - k - \bar{k}) k_\alpha \bar{k}_\beta\ M^b_{\alpha\beta}. \qquad \text{(III.9)}$$

$K = - Q - q$, and

$$M^b_{\alpha\beta} = p^+_\alpha p^-_\beta \left(\frac{p^-}{q \cdot p^-} - \frac{p^+}{q \cdot p^+} \right)^2$$

$$+ p^+_\alpha p^-_\beta \left(\frac{1}{q \cdot p^-} + \frac{1}{q \cdot p^+} \right) + \frac{p^- \cdot p^+}{q \cdot p^- q \cdot p^+} (p^+_\alpha q_\beta + q_\alpha p^-_\beta) \qquad \text{(III.10)}$$

$$- \frac{p^-_\alpha p^-_\beta}{q \cdot p^-} - \frac{p^+_\alpha p^+_\beta}{q \cdot p^+} + \frac{q_\alpha p^-_\beta}{(q \cdot p^+)^2} (m^2 - q \cdot p^+) + \frac{p^+_\alpha q_\beta}{(q \cdot p^-)^2} (m^2 - q \cdot p^-).$$

The integrals over k and $\bar{k}$ give

$$\sigma^b = \frac{G^2}{6\pi \sqrt{(p^- \cdot p^+)^2 - m^4}} \frac{\alpha}{(2\pi)^2} \frac{1}{2} \int \frac{d^3q}{q_0} \left\{ K^2 (K^2 + m^2) \left(\frac{p^-}{q \cdot p^-} - \frac{p^+}{q \cdot p^+} \right)^2 \right.$$

$$+ 4K^2 - \frac{2(q\cdot K)^2(K^2 - m^2)}{q\cdot p^- \; q\cdot p^+}\Bigg\} \quad . \tag{III.11}$$

For soft photons the dominant contribution comes from the term proportional to $(p^-/q\cdot p^- - p^+/q\cdot p^+)^2$. The cross section for emitting a soft photon of energy q in the range dq and solid angle $d\Omega$ is approximately

$$d\sigma^{\text{soft }\gamma} = \frac{\alpha}{(2\pi)^2}\,\sigma_0\,\frac{q^2 dq d\Omega}{q_0}\left(\frac{p^-}{q\cdot p^-} - \frac{p^+}{q\cdot p^+}\right)^2 , \tag{III.12}$$

where

$$\sigma_0 = \frac{G^2}{6\pi\beta}\,(t - m^2) \tag{III.13}$$

is the total cross section of zero order in α for pair annihilation into neutrino and antineutrino. $d\sigma^{\text{soft }\gamma}$ describes the expected classical bremsstrahlung due to the decceleration of the electron and positron.

The total energy of frequency ω per interval $d\omega$ emitted into solid angle $d\Omega$ for each annihilation is given in terms of the Fourier transform of the electric field radiated by accelerated particles. With the particle velocities $c\,\vec{\beta}_i$ and the unit vector $\vec{n}$ from the region of acceleration to the observer

$$dI(\omega) = \frac{d\Omega}{(2\pi)^2 4\pi c}\left|\sum_i e_i\int_{-\infty}^{\infty} dt\; e^{i\omega t}\left\{\frac{\vec{n}\times[(\vec{n}-\vec{\beta}_i)\times\dot{\vec{\beta}}_i]}{(1-\vec{n}\cdot\vec{\beta}_i)^3}\right\}_{\text{ret.}}\right|^2 , \tag{III.14}$$

where the velocities are evaluated at the retarded times and the summation is over the particles of charge e_i [39]. For the electron and positron with velocities $c\,\vec{\beta}^{\pm}$ before decceleration and zero after the result can be written as

$$dI(\omega) = \frac{d\Omega}{(2\pi)^2}\frac{e^2}{4\pi c}\sum_{j=1}^{2}\left|\frac{\vec{\epsilon}_j\cdot\vec{\beta}^-}{1-\vec{n}\cdot\vec{\beta}^-} - \frac{\vec{\epsilon}_j\cdot\vec{\beta}^+}{1-\vec{n}\cdot\vec{\beta}^+}\right|^2 . \qquad \text{(III.15)}$$

The summation here is over the transverse polarizations of the radiation in the direction $\vec{n} = \vec{q}/q_0$. The quantity $(p^-/q\cdot p^- - p^+/q\cdot p^+)^2$ in $d\sigma^{\text{soft }\gamma}$ came from $\sum_{\gamma\ \text{pol}}\left|\frac{\epsilon\cdot p^-}{q\cdot p^-} - \frac{\epsilon\cdot p^+}{q\cdot p^+}\right|^2$.

We may choose the radiation gauge. Then

$$\frac{q_0}{\sigma_0}\frac{d\sigma^{\text{soft }\gamma}}{dq\,d\Omega} = \frac{\alpha}{(2\pi)^2}\, q^2 \sum_{j=1}^{2}\left|\frac{\vec{\epsilon}_j\cdot\vec{p}^-}{q\cdot p^-} - \frac{\vec{\epsilon}_j\cdot\vec{p}^+}{q\cdot p^+}\right|^2 . \qquad \text{(III.16)}$$

We see that, with e fixed at the physical value e_{ph} while $\hbar \to 0$,

$$\frac{dI(\omega)}{d\Omega} = \frac{\hbar q_0}{\sigma_0}\frac{d\sigma^{\text{soft }\gamma}}{dq\,d\Omega} = \lim_{\substack{\hbar\to 0\\ e = e_{ph}}} \frac{\hbar q_0}{\sigma_0}\frac{d\sigma}{dq\,d\Omega} . \qquad \text{(III.17)}$$

For the stellar neutrino luminosities it is the cross section including the emission of photons of all possible energies which is of interest. The integration over the photon momentum for σ^b is done in Appendix C. We find the result

$$\left(\frac{G^2}{6\pi\beta}\frac{\alpha}{\pi}\right)^{-1}\sigma^b = (t - m^2)\left\{\left(\frac{1+\beta^2}{\beta}\tanh^{-1}\beta - 1\right)\log\frac{m^2}{\lambda^2} + \left(\frac{1+\beta^2}{2\beta}\tanh^{-1}\beta - 1\right)\log\frac{t}{m^2}\right.$$

$$\left. + \frac{2}{\beta}\tanh^{-1}\beta - \frac{1+\beta^2}{2\beta}\left[L\left(\frac{1+\beta}{2}\right) - L\left(\frac{1-\beta}{2}\right) - 2L(\beta) + 2L(-\beta)\right]\right\}$$

$$- (3t - 2m^2)\left(\frac{1+\beta^2}{\beta}\tanh^{-1}\beta - 1\right) + \left(\frac{t}{3} + m^2\right)\frac{1}{\beta}\tanh^{-1}\beta - \frac{t}{6}. \tag{III.18}$$

From $\Delta\sigma$ in (III.7) and σ^b in (III.18), the total order α cross section for pair annihilation to neutrinos plus photons is

$$\begin{aligned}\Delta\sigma + \sigma^b = \frac{G^2}{6\pi\beta}\frac{\alpha}{\pi}\Bigg\{ & (t - m^2)\left[\pi^2 + 4L\left(\frac{2\beta}{1+\beta}\right) + 4\tanh^{-1}\beta \ \log\frac{2}{1+\beta}\right. \\ & \left. - \log\frac{t}{m^2} - 2 + (2 + 3\beta^2)\frac{1}{\beta}\tanh^{-1}\beta\right] \\ & + 3m^2\beta\tanh^{-1}\beta - (3t - 2m^2)\left(\frac{1+\beta^2}{\beta}\tanh^{-1}\beta - 1\right) \\ & + \left(\frac{t}{3} + m^2\right)\frac{1}{\beta}\tanh^{-1}\beta - \frac{t}{6} \\ & + (t + 2m^2)\left[\left(\frac{\alpha}{\pi}\right)^{-1}(Z_3 - 1) - \frac{8}{9} + \frac{\beta^2}{3} + \frac{\beta}{3}(3 - \beta^2)\tanh^{-1}\beta\right]\Bigg\} .\end{aligned} \tag{III.19}$$

The result has been somewhat simplified using the relations between Spence functions in Appendix B. The infrared divergent terms proportional to $\log m^2/\lambda^2$ have cancelled.

In the non-relativistic limit, in which $\beta \to 0$, the total cross section becomes

$$\sigma \sim \sigma_0 \left[1 + \frac{\alpha\pi}{2\beta}\right] . \tag{III.20}$$

The dominant correction comes from the Coulomb attraction between the electron and positron, which at low energies the vertex diagram in Fig. 2b represents. A non-relativistic approach gives the same result. The non-relativistic electron and positron can be described by a

Schrödinger equation for the reduced mass and the relative coordinates with the Coulomb potential $-\alpha/r$. The cross section for the annihilation into neutrinos through the local weak interaction should be enhanced by the ratio of the density of the Coulomb wave at the origin to that of a plane wave; we should have

$$\frac{\sigma}{\sigma_0} \sim \left| \frac{U_c(0)}{U(0)} \right|^2 = \frac{-2\pi\alpha/V}{e^{-2\pi\alpha/V} - 1} \quad . \qquad \text{(III.21)}$$

For relative velocities V not too small this can be expanded in powers of α as $1 + \pi\alpha/V$, which is the same as the non-relativistic limit of our result, since in the limit, $\beta \to V/2$.

For high energies, where $-p^-\cdot p^+ \gg m^2$, $\beta \to 1$, and $\tanh^{-1}\beta \to \frac{1}{2} \log \frac{-2p^-\cdot p^+}{m^2}$. We find

$$\frac{\Delta\sigma}{\sigma_0} \sim \frac{\alpha}{\pi} \left\{ - \left[\log \frac{-2p^-\cdot p^+}{m^2} - 1\right] \log \frac{m^2}{\lambda^2} - \frac{1}{2} \left(\log \frac{-2p^-\cdot p^+}{m^2}\right)^2 + \frac{11}{6} \log \frac{-2p^-\cdot p^+}{m^2} \right\} \qquad \text{(III.22)}$$

Yennie, Frautschi, and Suura in their treatment of radiative corrections at high energies write the factor $\frac{-i}{i(\not p - \not k) + m} \gamma_\nu U_p$, which is contributed to an amplitude when an electron of momentum p emits a photon of momentum k and polarization ϵ_ν, as the sum of contributions from a convection current and a magnetic interaction,

$$\frac{-\left\{ 2p_\nu - k_\nu + \frac{1}{2}\left[\not k, \gamma_\nu\right] \right\}}{(p - k)^2 + m^2} \, U_p .$$

The contribution of the convection current is independent of spin and can be treated in general. According to their results the convection currents of the electron and positron in our case should give a high energy order α contribution to $\Delta\sigma/\sigma_0$ of

$$2\alpha \text{ Re } B = -\frac{\alpha}{\pi}\left\{\log\frac{-2^{-}\cdot p^{+}}{m^2}\left[\log\frac{m^2}{\lambda^2}+\frac{1}{2}\log\frac{-2p^{-}\cdot p^{+}}{m^2}-\frac{1}{2}\right]-\log\frac{m^2}{\lambda^2}\right\} \tag{III.23}$$

We calculate that the magnetic terms should contribute $\frac{\alpha}{\pi}\log\frac{-2p^{-}\cdot p^{+}}{m^2}$ to the ratio. A contribution of $\frac{\alpha}{3\pi}\log\frac{-2p^{-}\cdot p^{+}}{m^2}$ comes from the vacuum-polarization-like diagram in Fig. 2c. The sum of 2α Re B and the contributions of the magnetic interaction and the electron loop do give our result for the limit of $\Delta\sigma/\sigma_0$.

The contribution to σ^b/σ_0 of

$$\frac{1}{\sigma_0}\int d\sigma^{\text{soft }\gamma} = \frac{\alpha}{(2\pi)^2}\int\frac{d^3q}{q_0}\left(\frac{p^-}{q\cdot p^-}-\frac{p^+}{q\cdot p^+}\right)^2 \tag{III.24}$$

is what Yennie, Frautschi, and Suura call $2\alpha\,\widetilde{B}$. In the electron-positron center-of-momentum (c.m.) system our maximum photon energy is isotropic. It is equal to the c.m. energy w of the electron or positron. Yennie, Frautschi, and Suura give the high energy limit of $2\alpha\,\widetilde{B}$ for the scattering of an electron in a potential when the maximum photon energy is isotropic and small. Our integrand is the same with p^+ in the place of p_2. The only difference between $2\alpha\,\widetilde{B}$ in our case and theirs is that our maximum photon energy is large at high energies, whereas theirs is stipulated to be small. However, they do not discard terms because of this requirement and we find that with the appropriate substitutions the high-energy limit of our result for $1/\sigma_0 \int d\sigma^{\text{soft }\gamma} = 2\alpha \tilde{B}$ checks with their limit. The contribution at high energies is

$$\frac{\alpha}{\pi}\left\{\left[\log \frac{-2p^-\cdot p^+}{m^2} - 1\right] \log \frac{m^2}{\lambda^2} + \frac{1}{2}\left(\log \frac{-2p^-\cdot p^+}{m^2}\right)^2\right\}.$$

Because *our maximum photon energy is large at high energies* $\int d\sigma^{\text{soft }\gamma}$ does not give the only logarithmic contributions. The other parts of σ^b give to σ^b/σ_0

$$\frac{\alpha}{\pi}\left(-\frac{17}{6}\log \frac{-2p^-\cdot p^+}{m^2}\right).$$

In this discussion we have dropped contributions to $\Delta\sigma/\sigma_0$ and σ_b/σ_0 of order 1 and smaller. However, $\log -2p^-\cdot p^+/m^2$ is not of order 10 until the c.m. electron energy w is $\gtrsim 10^2$ m and we will be interested in the cross section at stellar temperatures corresponding to lower energies than that. Keeping terms of order 1 also, we obtain for the total cross section the limit

$$\sigma \sim \frac{G^2}{6\pi}\, t \left\{1 + (Z_3 - 1) + \frac{\alpha}{\pi}\left[\frac{\pi^2}{3} + \frac{10}{36} - \log \frac{t}{m^2}\right]\right\}. \qquad \text{(III.25)}$$

We note that when all photons are included in the cross section there are no double logarithm terms in the high-energy limit.

IV. THE CORRECTIONS TO STELLAR NEUTRINO LUMINOSITIES

The neutrino luminosity of stellar material due to the process $e^- + e^+ \rightarrow \nu_e + \bar{\nu}_e$ is given by $du/dt = \sum \mathcal{P}(E^+ + E^-)$, where the summation is over initial electron and positron states, $\mathcal{P}$ is the reaction rate per cubic centimeter from a given initial pair state, and du/dt is calculated in the (lab) frame in which the star's center of mass is at rest. $\mathcal{P}$ is an invariant given in the center-of-momentum (c.m.) system for the initial pair by $n^+(p^+)n^-(p^-)|\vec{v}^+ - \vec{v}^-|\sigma$. The positron and electron number densities $n^{\pm}(p^{\pm})$ are the fourth components of the four-vectors $(n^{\pm}\vec{v}^{\pm}, in^{\pm})$ and σ is an invariant, so that $E^+E^-|\vec{v}^+ - \vec{v}^-|$ should be the invariant $\sqrt{(p^-\cdot p^+)^2 - m^4}$ which it equals in the c.m. system. Then

$$\frac{du}{dt} = \int \frac{d^3p^+}{(2\pi)^3} \frac{d^3p^-}{(2\pi)^3} \frac{n^+(p^+)}{E^+} \frac{n^-(p^-)}{E^-} \sqrt{(p^+\cdot p^-)^2 - m^4}\, \sigma(E^+ + E^-), \tag{IV.1}$$

where now $p^{\pm} = (\vec{p}^{\pm}, i\, E^{\pm})$ are the four-momenta in the lab system.

In the annihilation with bremsstrahlung, $e^- + e^+ \rightarrow \nu_e + \bar{\nu}_e + \gamma$, the energy of the photon is trapped in the star. For the total rate of energy loss to neutrinos we will need

$$\epsilon^b = \int d\sigma^b(k_0 + \bar{k}_0), \tag{IV.2}$$

where $d\sigma^b$ is the cross section for producing a neutrino of energy k_0 and an antineutrino of energy $\bar{k}_0$ in the lab system. ϵ^b is the fourth component of the four vector $\int d\sigma^b(k + \bar{k})_\mu$. This is $\int d\sigma^b(p^- + p^+ - q)_\mu$ by energy-momentum conservation. By Lorentz invariance

$$\int d\sigma^b q_\mu = a\ (p^- + p^+)_\mu + b\ (p^- - p^+)_\mu, \qquad \text{(IV.3)}$$

where a and b are scalar functions of $p^- \cdot p^+$. a and b are easily evaluated in the c.m. system[4]. There $\int d\sigma^b q_0 = a\ 2w$ and $\int d\sigma^b q_z = b\ 2p$. These integrals contain no infrared divergence because of the additional power of q. b is identically zero as shown in Appendix C, since $d\sigma^b$ is even in $x = (\hat{q} \cdot \hat{p}^-)$. That is, the photon has an equal probability of being emitted at a given angle with respect to the positron or electron in the c.m. system. The two situations are related by a reversal of the sign of the current and a rotation. The integration for a is straightforward. We have

$$a = \frac{G^2}{6\pi\beta} \frac{\alpha}{\pi} \frac{t}{8} \left\{ \left[\frac{1+\beta^2}{\beta} \tanh^{-1}\beta - 1 \right] \left[\frac{5}{3} + \beta^2 \right] + \frac{1}{3} \left[-1 + \frac{2(2-\beta^2)}{\beta} \tanh^{-1}\beta \right] \right\}, \qquad \text{(IV.4a)}$$

and

$$b = 0 \qquad \text{(IV.4b)}$$

Using (IV.4b),

$$\int d\sigma^b (k + \bar{k})_\mu = (\sigma^b - a)(p^+ + p^-)_\mu . \qquad \text{(IV.5)}$$

The total neutrino luminosity due to pair annihilation to order α is

$$\frac{du}{dt} = \int \frac{d^3p^+}{(2\pi)^3} \int \frac{d^3p^-}{(2\pi)^3} \frac{n^+(p^+)}{E^+} \frac{n^-(p^-)}{E^-} \sqrt{(p^+ \cdot p^-)^2 - m^4} \left\{ (\sigma_0 + \Delta\sigma)(E^+ + E^-) + \int d\sigma^b (k_0 + \bar{k}_0) \right\}$$

$$= \int \frac{d^3p^+}{(2\pi)^3} \int \frac{d^3p^-}{(2\pi)^3} \frac{n^+(p^+)}{E^+} \frac{n^-(p^-)}{E^-} \sqrt{(p^+ \cdot p^-)^2 - m^4} \left[\sigma_0 + \Delta\sigma + \sigma^b - a\right] (E^+ + E^-). \quad \text{(IV.6)}$$

The average energy lost to neutrinos through both $e^- + e^+ \to \nu_e + \bar{\nu}_e$ and $e^- + e^+ \to \nu_e + \bar{\nu}_e + \gamma$ is

$$\bar{\epsilon} = \frac{[\sigma_0 + \Delta\sigma + \sigma^b - a][E^+ + E^-]}{[\sigma_0 + \Delta\sigma + \sigma^b]} . \quad \text{(IV.7)}$$

Physically, a must be positive. This is so for the expression in (IV.5) $\left(\frac{1}{\beta} \tanh^{-1}\beta = \frac{1}{2\beta} \log \frac{1+\beta}{1-\beta} > 1\right)$.

The densities $n^\pm(p^\pm)$ for electrons and positrons in thermal equilibrium are

$$n^\pm(p^\pm) = \frac{2}{e^{\frac{E_\pm - \mu_\pm}{kT}} + 1} \quad \text{(IV.8)}$$

Since the electron and positrons are in equilibrium with photons, their chemical potentials $\mu^\pm$ must satisfy the relation

$$\mu^+ + \mu^- = \mu_\gamma = 0 \;^{43)}. \quad \text{(IV.9)}$$

The conservation of lepton number gives the additional relation

$$N^- - N^+ = \int \frac{d^3p^-}{(2\pi)^3} n^-(p^-) - \int \frac{d^3p^+}{(2\pi)^3} n^+(p^+) = N_0 \quad \text{(IV.10)}$$

where N_0 is the number density of residual electrons associated with the nucleii. (IV.8), (IV.9) and (IV.10) determine $\mu^\pm$.

According to Fowler and Hoyle[7)], the ρ,T paths of type II presupernovae with mass $M \sim 10 - 35\ M_\odot$ lie in a ρ,T region in which the electrons and positrons are non-degenerate. The approximation that the fermions are non-degenerate corresponds to assuming, that for energies $E^{\pm}$ important in integrals over the distributions,

$$e^{\frac{E^{\pm} - \mu^{\pm}}{kT}} \gg 1. \tag{IV.11}$$

With this approximation

$$n^{+}(p^{+})n^{-}(p^{-}) \sim 4\, e^{-\frac{E_{+} + E_{-}}{kT}}, \tag{IV.12}$$

independent of $\mu^{\pm}$ because of the condition (IV.9).

For low temperatures emphasizing non-relativistic electron and positron energies the integrand in the integral for du/dt (IV.7) can be expanded in powers of $p_{\pm}/m$. In the non-relativistic limit, in which $\beta \ll 1$,

$$\Delta\sigma + \sigma^{b} - a = (\sigma_0)_{n.r.} \frac{\alpha}{\pi}\left[\frac{\pi^2}{2\beta} + C_0 + O(\beta)\right], \tag{IV.13}$$

where

$$(\sigma_0)_{n.r.} = \frac{G^2 m^2}{2\pi\beta}\left[1 + O(\beta^2)\right], \tag{IV.14}$$

and

$$C_0 = 2(Z_3 - 1)\left(\frac{\alpha}{\pi}\right)^{-1} - \frac{97}{18}. \tag{IV.15}$$

From (II.23), in terms of the cut-off Λ, $Z_3 - 1 = -\frac{\alpha}{3\pi} \log \frac{\Lambda^2}{m^2}$.

Recalling that $\beta = \sqrt{\frac{t - 4m^2}{t}} = \sqrt{\frac{-p^{-}\cdot p^{+} - m^2}{-p^{-}\cdot p^{+} + m^2}}$, we see that

$$\frac{\sqrt{(p^{-}\cdot p^{+})^2 - m^4}}{\beta} = -p^{-}\cdot p^{+} + m^2. \tag{IV.16}$$

For non-relativistic energies this is $2m^2 + \frac{1}{2}(\vec{p}^- - \vec{p}^+)^2 + \ldots$. Using (IV.12), (IV.13), and (IV.14) in (IV.6) for du/dt, we have

$$\left(\frac{du}{dt}\right)_{n.r.} = \frac{4}{(2\pi)^6}\int d^3p_+\, d^3p_-\, e^{-\frac{2m}{kT}}\, e^{-\frac{p^{+2}+p^{-2}}{2mkT}}\, 2\left(\frac{G^2m^2}{2\pi}\right)\left[1 + \frac{\alpha\pi}{\pi\beta} + \frac{\alpha}{\pi}C_0\right] 2m. \qquad \text{(IV.17)}$$

In the order α correction both the β^{-1} and β^0 terms can be kept since the higher orders of the rest of the integrand would contribute terms of order β^2 times the lowest order. In the non-relativistic limit $\frac{1}{2\beta} = \frac{1}{\sqrt{(\vec{v}_+ - \vec{v}_-)^2}}\,(1 + 0(v^2))$.

We obtain

$$\left(\frac{du}{dt}\right)_{n.r.} = \frac{G^2m^9}{\pi^4}\left(\frac{kT}{m}\right)^3 e^{-\frac{2m}{kT}}\left\{1 + \frac{\alpha}{\pi}C_0 + \alpha\left(\frac{kT}{m}\right)^{-3}\int_0^\infty v_+^2dv_+\int_0^\infty v_-^2dv_-\int_{-1}^{1}dx \text{ times } \frac{e^{-\frac{m}{2kT}(v_+^2+v_-^2)}}{\sqrt{(\vec{v}^+ - \vec{v}^-)^2}}\right\}, \qquad \text{(IV.18)}$$

where $x = \hat{v}^+\cdot\hat{v}^-$. We rewrite (IV.18) as

$$\left(\frac{du}{dt}\right)_{n.r.} = \left(\frac{du}{dt}\right)_{0,n.r.}\left\{1 + \frac{\alpha}{\pi}C_0 + \alpha\pi\left\langle\frac{1}{v}\right\rangle\right\}, \qquad \text{(IV.19)}$$

where

$\left(\frac{du}{dt}\right)_{0,n.r.} = \frac{G^2m^9}{\pi^4}\left(\frac{kT}{m}\right)^3 e^{-\frac{2m}{kT}}$ is the luminosity with no photon corrections found by Levine[4)] and by Chiu and Stabler[6)]. In Appendix D we find

$$\langle \frac{1}{v} \rangle = \sqrt{\frac{m}{\pi kT}} \,. \qquad \text{(IV.20)}$$

Using (IV.15), (IV.19), and (IV.20), we have finally

$$\left(\frac{du}{dt}\right)_{n.r.} = \left(\frac{du}{dt}\right)_{0,n.r.} \left\{1 + \alpha\sqrt{\frac{\pi m}{kT}} - \frac{\alpha}{\pi}\frac{97}{18} + 2(Z_3 - 1)\right\} \qquad \text{(IV.21a)}$$

$$= \left(\frac{du}{dt}\right)_{0,n.r.} \left\{1 + \frac{0.035}{\sqrt{T_9}} - 0.034\right\} . \qquad \text{(IV.21b)}$$

In the numerical result $\Lambda = m_N$ has been used.

$\left(\frac{du}{dt}\right)_{n.r.}$ in (IV.21) gives the corrected neutrino luminosity for $\frac{2kT}{m} \ll 1$ or $T_9 \ll 3$. The non-relativistic energy loss is enhanced by about 30 percent for temperatures as low as $T_9 \sim 10^{-2}$; the enhancement is due to the Coulomb attraction between the electron and positron. As mentioned in Chapter III, in the non-relativistic limit the ratio of the cross sections with and without the Coulomb interaction should be $(-2\pi\alpha/v)/(e^{(-2\pi\alpha/v)} - 1)$. For temperatures lower than $T_9 \sim 10^{-2}$ the expansion in powers of α, which gives to the energy loss the correction $\pi\alpha \langle \frac{1}{v} \rangle$, is not valid and the exact expression for the cross section ratio should be used.

In the high energy limit appropriate for $T_9 \gg 3$,

$$\Delta\sigma + \sigma^b - a = (\sigma_0)_{e.r.} \frac{\alpha}{\pi}\left[-\frac{11}{8}\log\frac{t}{m^2} + C_1 + \text{lower order terms}\right] , \qquad \text{(IV.22)}$$

where

$$(\sigma_0)_{e.r.} = \frac{G^2}{6\pi}\, t, \qquad \text{(IV.23)}$$

and

$$C_1 = (Z_3 - 1)\left(\frac{\alpha}{\pi}\right)^{-1} + \frac{\pi^2}{3} + \frac{47}{72}. \tag{IV.24}$$

With the approximation $p^{\pm} \gg m$ in (IV.7) and $x = \hat{p}_+ \cdot \hat{p}_-$,

$$\left(\frac{du}{dt}\right)_{e.r.} = \frac{4}{(2\pi)^6}\frac{G^2}{3\pi}\int d^3p_+ \, d^3p_- \, e^{-\frac{p^+ + p^-}{kT}} p^+p^-(p^+ + p^-)(1 - x)^2$$
$$\left[1 + \frac{\alpha}{\pi} C_1 - \frac{11}{8}\frac{\alpha}{\pi} \log \frac{2p_+p_-(1 - x)}{m^2}\right]. \tag{IV.25}$$

The integrals are done in Appendix D. They give

$$\left(\frac{du}{dt}\right)_{e.r.} = \left(\frac{du}{dt}\right)_{0,e.r.}\left\{1 - \frac{11}{4}\frac{\alpha}{\pi} \log \frac{2kT}{m} + \frac{\alpha}{\pi}\left[C_1 + \frac{11}{4} C - \frac{473}{96}\right]\right\}, \tag{IV.26}$$

where $C = .5772$ is Euler's constant, and $\left(\frac{du}{dt}\right)_{0,e.r.} = \frac{128\, G^2(kT)^9}{\pi^5}$ as found by Levine[4)] and by Chiu and Stabler[6)]. With C_1 from (IV.24),

$$\left(\frac{du}{dt}\right)_{e.r.} = \left(\frac{du}{dt}\right)_{0,e.r.}\left\{1 + (Z_3 - 1) + \frac{\alpha}{\pi}\left[\frac{\pi^2}{3} + \frac{11}{4} C - \frac{1231}{288}\right] - \frac{11}{4}\frac{\alpha}{\pi} \log \frac{2kT}{m}\right\}$$
$$= \left(\frac{du}{dt}\right)_{0,e.r.}\left\{1 - 0.010 - 0.0064 \log \frac{T_9}{2.97}\right\}. \tag{IV.27}$$

Again $\Lambda = m_N$ has been used in the numerical result.

In the region of $T_9 \sim 1$, the radiative correction changes sign and depresses the luminosity by a few percent. The Coulomb term $\alpha\pi \langle \frac{1}{v} \rangle$ diminishes in importance at high energies, leaving the $(Z_3 - 1)$ term dominant (if it is not much smaller than the estimates) in the region $1 < T_9 < 10$ and the log 2kT/m term dominant for $T_9 \gtrsim 10^3$.

V. NEUTRINO SCATTERING BY ELECTRONS

The radiative corrections for the cross section measured in an experiment on the scattering reaction, $\nu_e + e \rightarrow \nu_e + e$, depend on how many of the events in which a bremsstrahlung photon is emitted would contribute to the measured cross section. The experiment proposed by Reines and Kropp[17)] would measure for the elastic scattering the total cross section for an electron to recoil with energy $> E_0 \sim 8 - 10$ MeV. In an experiment done with a liquid scintillator it appears that the criterion for including in the count an event in which a bremsstrahlung photon is emitted remains that the recoil electron have energy $> E_0$. The spectrum of B^8 neutrinos ends at $\sim$ 14 MeV. Thus less than 14 MeV would be available to a photon produced by a neutrino from B^8 decay. As the scintillator efficiency for converting gamma rays of such energies is low[44)], most of the bremsstrahlung photons would not deposit energy in the detector.

The total cross section for the electron to recoil with energy $> E_0$, including the cross section for emitting in the process one photon of any kinematically allowed energy is

$$\sigma(E_2{}^L > E_0) = \int_{E_0}^{E_m} dE_2{}^L \frac{d\sigma}{dE_2{}^L} , \qquad (V.1)$$

where $E_2{}^L$ is the recoil electron energy in the lab system, in which the initial electron is at rest, and E_m is the maximum recoil energy kinematically available. ($E_m = k + \frac{m(m + k)}{m + 2k}$, where k is the incident neutrino energy in the lab.) With $t = -(p_2 - p_1)^2 = -2m(E_2{}^L - m)$,

$$\sigma(E_2^L > E_0) = \int_{t_m}^{t_0} dt \frac{d\sigma}{dt} . \qquad (V.2)$$

$t_0 = -2m(E_0 - m)$ and $t_m = -2m(E_m - m)$. The total invariant differential cross section is

$$\frac{d\sigma}{dt} = \frac{d\sigma_0}{dt} + \frac{\Delta d\sigma}{dt} + \frac{d\sigma^b}{dt} . \qquad (V.3)$$

$\frac{d\sigma_0}{dt} = \frac{G^2}{\pi}$. $\Delta \frac{d\sigma}{dt}$ is the order α contribution of virtual photons.

$\frac{d\sigma^b}{dt}$ is the contribution of $\nu_e + e \to \nu_e + e + \gamma$.

For $\frac{d\sigma_0}{dt} + \frac{\Delta d\sigma}{dt}$, $\sum_{\text{spins}} |T^b|^2$ is given by (II.31) and (II.33) with the substitutions $k \to k_2$, $\bar{k} \to -k_1$, $p^- \to p_1$, and $p^+ \to -p_2$. Using $s = -(p_1 + k_1)^2$, $Q^2 = (p_2 - p_1)^2$ and energy-momentum conservation, we obtain

$$\sum_{\text{spins}} |T^b|^2 = \frac{G^2}{2}\left\{64(s - m^2)^2 + Tr^{(\nu)} \Delta Tr^{(e)}\right\} , \qquad (V.4)$$

where

$$\begin{aligned} \frac{1}{64} Tr^{(\nu)} \Delta Tr^{(e)} = & -2m \, Im \, S_\Sigma[(s - m^2)^2 - Q^2 s] \\ & + Re \, V \, [(s - m^2)^2 - Q^2 m^2] \qquad (V.5) \\ & + Re \, A \, [(s - m^2)^2 + Q^2 m^2]. \end{aligned}$$

When V and A are expressed in terms of the contributions of the vertex and electron loop diagrams in (II.28), this becomes

$$\frac{1}{64}\,\mathrm{Tr}^{(V)}\triangle\mathrm{Tr}^{(e)} = 2\,\mathrm{Re}\,\Lambda^c(V)(s-m^2)^2 - 2m\,\mathrm{Im}\,S_\Sigma\,[(s-m^2)^2 - Q^2 s]$$
$$+ \mathrm{Re}\,[\,-\frac{\alpha}{2\pi} + (\Lambda^c(A) - \Lambda^c(V))][(s-m^2)^2 + Q^2 m^2]$$
$$+ \mathrm{Re}\,[Z_3 - 1 + \Pi^c(Q^2)][(s-m^2)^2 - Q^2 m^2]. \qquad (V.6)$$

Since only $\Lambda^c(V)$, and not $(\Lambda^c(A) - \Lambda^c(V))$, contains an infrared divergence, and gives a contribution proportional to the uncorrected cross section, as does the soft photon bremsstrahlung, the cancellation of the infrared divergence that occurred for the annihilation will occur for the crossed reactions.

The virtual photon correction to the invariant differential cross section is

$$\triangle\frac{d\sigma}{dt} = \frac{1}{64\pi^2(-p_1\cdot k_1)}\int\frac{d^3p_2}{E_2}\,\frac{d^3k_2}{k_2}\,\delta^4(p_1+k_1-p_2-k_2)\,\delta(t+(p_2-p_1)^2)$$
$$\frac{1}{2}\triangle\sum_{\text{spins}}|T|^2$$
$$= \frac{G^2}{\pi}\,\frac{\alpha}{\pi}\,\mathcal{M}, \qquad (V.7)$$

where

$$\mathcal{M} = \frac{1}{64(s-m^2)^2}\left(\frac{\alpha}{\pi}\right)^{-1}\mathrm{Tr}^{(\nu)}\triangle\,\mathrm{Tr}^{(e)}. \qquad (V.8)$$

Using the results of Appendix A and (V.6),

$$\mathcal{M} = -\left[\theta\ \mathrm{ctnh}\,\theta - 1\right]\log\frac{m^2}{\lambda^2} - \frac{\theta}{2\sinh\theta}\left[1 - \frac{4m^2 s}{(s-m^2)^2}\sinh^2\varphi\right]$$
$$- \frac{\theta}{\sinh\theta}\left[1 + \frac{4m^4}{(s-m^2)^2}\sinh^2\varphi\right] - 2 + \frac{3\,\theta\,\cosh^2\varphi}{\sinh\theta}$$

$$+ \text{ctnh}\ \theta \left[\bar{L}\left(\frac{2\cosh\varphi}{e^{\varphi}}\right) - \bar{L}\left(\frac{2\cosh\varphi}{e^{-\varphi}}\right)\right] \tag{V.9}$$

$$+ \left(\frac{\alpha}{\pi}\right)^{-1} (Z_3 - 1) - \frac{5}{9} + \frac{1}{3\sinh^2\varphi} + \frac{\theta}{3}\,\text{ctnh}\,\varphi \left(1 - \frac{1}{2\sinh^2\varphi}\right)$$

Here $t = - Q^2 = - 4m^2\sinh^2\varphi$ and $\theta = 2\varphi$. This is the same as the result of the Lee and Sirlin[13)].

The dominant terms of $\mathcal{M}$ at high energies, which will be used for the corrections relevant to the proposed experiment, can be checked with the results of Yennie, Frautschi, and Suura. For incident neutrino energies $k \gg m$, we can have $- p_1 \cdot p_2 \gg m^2$. Then $\theta \sim \log - 2p_1 \cdot p_2/m^2$. For $\theta \gg 1$,

$$\mathcal{M} \sim (\theta - 1) \log \frac{m^2}{\lambda^2} - \frac{1}{2}\theta^2 + \frac{11}{6}\theta. \tag{V.10}$$

For an electron scattering in a potential, Yennie, Frautschi, and Suura found

$$2\alpha B \sim - \frac{\alpha}{\pi} \left\{\log \frac{-2p_1 \cdot p_2}{m^2} \left[\log \frac{m^2}{\lambda^2} + \frac{1}{2} \log \frac{-2p_1 \cdot p_2}{m^2} - \frac{1}{2}\right] - \log \frac{m^2}{\lambda^2}\right\} . \tag{V.11}$$

$\frac{\alpha}{\pi}\mathcal{M}$ is the sum of $2\alpha B$, the contribution $\frac{\alpha}{\pi} \log \frac{-2p_1 \cdot p_2}{m^2}$ from the magnetic interaction, and the contribution $\frac{\alpha}{3\pi} \log \frac{-2p_1 . p_2}{m^2}$ from the electron loop. The contribution of the magnetic interaction to $\Delta \frac{d\sigma}{dt} / \frac{d\sigma_0}{dt}$ is the same in our case as for electron scattering in a potential. The high energy part proportional to $\gamma_\mu\gamma_5$ comes out to be the same as that for γ_μ. It would be different for other Fermi couplings (which as mentioned in Chapter II are also different in that the wave function and vertex renormalization constants do not cancel). The contribution of the electron loop in our case is half that of the vacuum polarization in electron scattering in a potential. Including only the electron loop correction the amplitude is proportional to $[\bar{U}_{k_2}\gamma_\mu(1 + \gamma_5)U_{k_1}][\bar{U}_{p_2}(\gamma_\mu(1 + \gamma_5) + \eta\gamma_\mu)U_{p_1}]$, where η contains the vacuum polarization-like effect. Then

$$\sum_{\text{spins}} |T|^2 \propto \mathrm{Tr}^{(\nu)} \Big\{ (1 + \eta)^2 \mathrm{Tr}\ \gamma_\mu(m - i\not{p}_1)\gamma_\nu(m - i\not{p}_2) + \mathrm{Tr}\ \gamma_\mu\gamma_5 (m - i\not{p}_1)\gamma_\nu\gamma_5(m - i\not{p}_2)$$

$$+ (1 + \eta)\mathrm{Tr}[\gamma_\mu(m - i\not{p}_1)\gamma_\nu\gamma_5(m - i\not{p}_2) + \gamma_\mu\gamma_5(m - i\not{p}_1)\gamma_\nu(m - i\not{p}_2)] \Big\} .$$

At high energies, when m can be neglected relative to $\not{p}_i$, the contribution of a pure vector or a pure axial vector operator between the electron spinors become the same and

$$\sum_{\text{spins}} |T|^2 \propto \mathrm{Re}^{(\nu)} \Big\{ - [1 + (1 + \eta)^2]\mathrm{Tr}\ \gamma_\mu\not{p}_1\ \gamma_\nu\not{p}_2 + 2(1 + \eta)\ \mathrm{Tr}\ \gamma_\mu\not{p}_1\gamma_\nu\not{p}_2\gamma_5 \Big\} ,$$

so that to first order in η, $\sum_{\text{spins}} |T|^2 = (1 + \eta) \sum_{\text{spins}} |T_0|^2$.

This is to be compared to $T = A_\mu \bar{U}_{p_2}\gamma_\mu(1 + \eta)U_{p_1}$ which implies

$$\sum_{\text{spins}} |T|^2 = (1 + 2\eta) \sum_{\text{spins}} |T_0|^2 \text{ to first order in } \eta.$$

For the cross section for scattering with bremsstrahlung, $\nu_e + e \rightarrow \nu_e + e + \gamma$, the appropriate momentum substitutions in $\sum_{\text{spins}} |T^b|^2$ of (II.37) give

$$(128G^2e^2)^{-1} \sum_{\text{spins}} |T^b|^2 = p_1\cdot k_1\; p_2\cdot k_2 \left(\frac{p_1}{q\cdot p_1} - \frac{p_2}{q\cdot p_2}\right)^2$$

$$+ p_1\cdot k_1\; p_2\cdot k_2\left(\frac{1}{q\cdot p_1} - \frac{1}{q\cdot p_2}\right) + \frac{p_1\cdot p_2}{q\cdot p_1\; q\cdot p_2}\,(q\cdot k_1\; p_2\cdot k_2 - q\cdot k_2 p_1\cdot k_1)$$

$$+ \frac{p_1\cdot k_1 p_1\cdot k_2}{q\cdot p_1} - \frac{p_2\cdot k_1 p_2\cdot k_2}{q\cdot p_2} - \frac{p_1\cdot k_1 q\cdot k_2}{(q\cdot p_2)^2}\,(m^2 + q\cdot p_2)$$

$$+ \frac{p_2\cdot k_2 q\cdot k_1}{(q\cdot p_1)^2}\,(m^2 - q\cdot p_1). \qquad \text{(V.12)}$$

The cross section is

$$\frac{d\sigma^b}{dt} = \frac{1}{(4\pi)^5(-p_1\cdot k_1)}\int \frac{d^3p_2}{E_2}\,\frac{d^3q}{q_0}\,\frac{d^3k_2}{k_2}\,\delta(t + (p_2 - p_1)^2)$$

$$\delta^4(p_1 + k_1 - p_2 - k_2 - q)\,\frac{1}{2}\sum_{\text{spins}} |T^b|^2 . \qquad \text{(V.13)}$$

The integration can be carried out exactly. We sketch here our procedure and in Appendix E give further details. First we do the integrals over the final neutrino and photon momenta in the frame in which $\vec{q} + \vec{k}_2 = \vec{p}_1 + \vec{k}_1 - \vec{p}_2 = 0$. Expressing the results of the invariant

$$\int \frac{d^3q}{q_0} \frac{d^3k_2}{k_2} \; \delta^4(p_1 + k_1 - p_2 - k_2 - q) \sum_{\text{spins}} |T^b|^2$$

in terms of the invariants which can be constructed from p_1, k_1, and p_2, we then do the integrals over the final electron momenta in the c.m. system for the initial neutrino and electron. The integral over $\hat{p}_1 \cdot \hat{p}_2$ can be done with the delta function of t. The condition that $|\hat{p}_1 \cdot \hat{p}_2| \leq 1$, together with the condition $-(p_1 + k_1 - p_2)^2 \geq \lambda^2$ from the delta function of energy restrict the final integration over E_2^c. We find

$$E_2^- \leq E_2^c \leq E_1^c - \frac{\lambda^2}{2\sqrt{s}} \quad , \tag{V.14}$$

where

$$E_2^- = E_1^c \cosh\theta - p_1^c \sinh\theta. \tag{V.15}$$

θ is defined as for the elastic reaction by $t = -4m^2 \sinh^2 \theta/2$. When the integrals are done in this order the infrared divergence appears in the integration over E_2^c at the upper limit; the photon cannot have zero energy unless the electron satisfies the kinematics of the elastic reaction.

$d\sigma/dt$ can be expressed in terms of the two parameters s/m^2 and e^θ. For an incident neutrino of 14 MeV energy in the lab, $s/m^2 = 57$, and for $E_2^L \gtrsim 8$ MeV, $e^\theta \gtrsim 32$. Thus both parameters are large compared to 1 for the energies relevant to the proposed experiment. Since log 57 $\sim$ 4, the logarithms are not dominant over the terms of order 1, though double logarithms can be of order 10. We will neglect terms of lower order than 1, including terms like $(m^2/s)(\log s/m^2)^2$, although the latter could give a 10 percent correction to the terms of order 1.

For the contribution of $\mathcal{M}$ to the high energy approximation for $d\sigma/dt$ we need to add the terms of order 1 to the limit given in (V.10). With these included

$$\mathcal{M} \sim (1 - \theta)\log \frac{m^2}{\lambda^2} + \left(\frac{\alpha}{\pi}\right)^{-1}(Z_3 - 1) - \frac{17}{6} + \frac{\pi^2}{6} + \frac{11}{6}\theta - \frac{\theta^2}{2} . \quad (V.16)$$

In Appendix E $\left(\frac{G^2}{\pi}\frac{\alpha}{\pi}\right)^{-1}\frac{d\sigma^b}{dt}$ is written as $\mathcal{I} + \mathcal{J}$, where $\mathcal{I} = \mathcal{I}^{IR} + \mathcal{I}^F$ and $\mathcal{J} = \sum_{i=1}^{6} \mathcal{J}_i$. Here we give the high energy results for these parts. We define $a = s/m^2$ and $\gamma = a - e^\theta$. Then

$$\mathcal{I}^{IR} \sim (\theta - 1)\log \frac{m^2}{\lambda^2} + \left(\frac{3}{2} - 2\theta\right)\frac{\gamma}{a} - \frac{\pi^2}{6} + (\theta - 1)\log\frac{\gamma}{a} + \log\frac{1+\gamma}{\gamma}$$

$$+ \frac{(a + e^\theta)\gamma}{8a}\log\left(\frac{1+\gamma}{1-\gamma}\right)^2 - L\left(\frac{e^\theta}{a}\right) + L\left(-\frac{1}{\gamma}\right) - L\left(\frac{-e^\theta}{\gamma}\right) + \frac{\theta^2}{2} . \quad (V.17)$$

When $\mathcal{M}$ and $\mathcal{I}^{IR}$ are added the infrared divergent terms cancel as they should. For neutrino energies k such that $(k - E_0) \gg m$, the approximation $\gamma \gg 1$ is good over most of the range of t, or $E_2{}^L$, for which we need $d\sigma/dt$ to compute $\sigma(E_2{}^L > E_0)$. Making this additional approximation

$$\mathcal{M}+\mathcal{J}^{IR} \sim \left(\frac{\alpha}{\pi}\right)^{-1}(Z_3 - 1) - \frac{17}{6} + \frac{11}{6}\,\theta + \left(\frac{3}{2} - 2\theta\right)\frac{\gamma}{a} + (\theta - 1)\log\frac{\gamma}{a}$$

$$+ \frac{a + e^{\theta}}{2a} - L\left(\frac{e^{\theta}}{a}\right) - L\left(-\frac{e^{\theta}}{\gamma}\right). \tag{V.18}$$

The remaining contributions are

$$\mathcal{J}^{F} \sim -\frac{e^{\theta}}{a}\,(1 + \log\gamma), \tag{V.19}$$

$$\sum_{i=1}^{4} \mathcal{J}_i \sim \frac{\theta\gamma^2}{2a^2} + \frac{1}{2}\left[\frac{\pi^2}{6} + L\left(\frac{e^{\theta}}{a}\right) + \theta\log a - \theta^2\right] + \frac{1}{2}\left[\log a - \frac{\theta e^{2\theta}}{a^2}\right]$$

$$- \left[1 - \frac{e^{\theta}}{2a}\right]\log\gamma - \frac{3}{4} - \frac{\gamma}{a}, \tag{V.20}$$

and

$$\sum_{i=5}^{6} \mathcal{J}_i \sim \frac{1}{12}\log\gamma + \frac{a^2 - e^{2\theta}}{8a^2} + \frac{1}{72a^2}\left[23a^2 + 6\,ae^{\theta} + 12\,e^{2\theta}\right]. \tag{V.21}$$

We have calculated the limits for $\mathcal{J}$ by making the approximations both before the integration over $E_2^{\,c}$ and after doing it exactly and obtained the same results.

For γ not large a correction to $\mathcal{J}$ and $\mathcal{J}$ is necessary. The approximation $\gamma \gg 1$ allowed the expansion $\log(1 \pm 1/\gamma) \sim \pm 1/\gamma + \ldots$ If in the difference between the exact result and the above approximations we use $e^{\theta} \sim a$ except where a divergence results, we obtain for the correction to $\mathcal{J}$ when $\gamma \ll 1$

$$\Delta\mathcal{J} \sim -\frac{5}{12}\log\left(1+\frac{1}{\gamma}\right)+\frac{13}{72}\,. \tag{V.22}$$

$\mathcal{J}^{F} \to 0$ as $\gamma \to 0$. $\mathcal{J}^{IR}$ is given in (V.17) for all γ.

If only low energy photons were included in the cross section, the term $i = p_1 \cdot k_1 p_2 \cdot k_2 \left(\frac{p_1}{q \cdot p_1} - \frac{p_2}{q \cdot p_2} \right)^2$ in $\sum_{\text{spins}} |T^b|^2$ would dominate over the rest, as the corresponding term did for $e^- + e^+ \to \nu_e + \bar{\nu}_e + \gamma$.

For a maximum photon energy much less than the kinematic limit we would have

$$\frac{d\sigma^b}{dt} \sim \frac{G^2}{\pi}\frac{\alpha}{\pi}\mathcal{J}^{\text{soft }\gamma}\,, \tag{V.23}$$

where $\mathcal{J}^{\text{soft }\gamma}$ is the approximation to $\mathcal{J}$ obtained by neglecting the photon energy and momentum in the delta functions in $\mathcal{J}$.

$$\mathcal{J}^{\text{soft }\gamma} = \frac{1}{4\pi}\int\frac{d^3q}{q_0}\left(-\frac{m^2}{(q\cdot p_1)^2} - \frac{2p_1\cdot p_2}{q\cdot p_1 q\cdot p_2} - \frac{m^2}{(q\cdot p_2)^2}\right) \tag{V.24}$$

where p_2 satisfies the kinematics of the elastic reaction, $\nu_e + e \to \nu_e + e$. In the lab system for example, $E_2^L = m \cosh\theta$ and $\hat{p}_2 \cdot \hat{k}_1 = \frac{m+k}{k}\tanh\frac{\theta}{2}$. The azimuthal angle of $\vec{p}_2$ can be taken as fixed in (V.24). For a maximum photon energy q_m isotropic in the lab system $\mathcal{J}^{\text{soft }\gamma}$ gives the bremsstrahlung contribution which Lee and Sirlin use[13]. $\mathcal{J}^{\text{soft }\gamma}$ corresponds to $2\pi\,\tilde{B}$ for the scattering of an electron with small energy loss in the treatment by Yennie, Frautschi, and Suura[15].

The integral for $\mathcal{J}^{\text{soft }\gamma}$ or $2\pi\,\tilde{B}$ in (V.24) is defined even if the integration over the photon momentum is extended to include all photons kinematically allowed. The integral with the extended limits on the photon integration is $2\pi\,\tilde{B}$ for our problem. With the correct kinematics for the reaction $\nu_e + e \rightarrow \nu_e + e + \gamma$, the factor $p_2 \cdot k_2$ in i is $p_1 . k_1 - q\cdot(p_1 + k_1)$. The first term, $p_1\cdot k_1$, gives the contribution $\mathcal{J}'$ of photons with spectrum proportional to dq/q for small q. In two cases of high energy and large momentum transfer we can find the dominant (log and $\log^2$) terms of $2\pi\,\tilde{B}$ and compare them to the dominant terms of $\mathcal{J}'$ from our results for $d\sigma^b/dt$. The case which we will discuss first is that of nearly maximum momentum transfer, when $e^\theta \sim a$. The second is that of momentum transfer large, yet far from the maximum, when $a \gg e^\theta \gg 1$. We find that in both cases the dominant terms of $\mathcal{J}'$ are the same as those of $2\pi\,\tilde{B}$. In Appendix F we argue that the correspondence between the high energy limits of $\mathcal{J}'$ and $2\pi\,\tilde{B}$ should obtain, so that the agreement of the results is a check on our calculation of $d\sigma^b/dt$. Here we discuss the limits and the extent to which the contributions of $\mathcal{J}'$(or $2\pi\,\tilde{B}$) dominate the other contributions to $d\sigma^b/dt$.

In the limit of maximum momentum transfer the photon energy in the lab cannot get larger than m/2 in any direction and no logarithmic terms arise besides those that correspond to the limit of $\mathcal{J}^{\text{soft }\gamma}$. We find

$$\mathcal{I} + \mathcal{J} \sim \mathcal{J}^{IR} \sim (\theta - 1) \left[\log \frac{m^2}{\lambda^2} + \log \frac{\gamma^2}{a}\right] . \qquad (V.25)$$

In Appendix F we show that the photon energy is restricted to be less than $m\gamma/2$ in the forward direction, which is the direction the matrix element emphasizes, and that the limit obtained from $2\pi\,\tilde{B}$ is the same as the limit in (V.25) of the exact results. $\mathcal{J}$ and $\mathcal{J}^F$ actually vanish in the limit of maximum momentum transfer; they contain no divergent terms and the phase space for the situation vanishes. We note that when the infrared divergent term $(\theta - 1)\log m^2/\lambda^2$ is cancelled with that in $\mathcal{M}$, a logarithmic divergence is left in $d\sigma/dt$ at maximum momentum transfer. The electron can only have its maximum momentum transfer by scattering without loss of energy and the divergence reflects the need to take into account the emission of more than one low energy photon. Since the divergence is only logarithmic, it does not invalidate neglect of the correction terms for $e^\theta \sim a$ in the integration over t, or alternatively e^θ, to obtain $\sigma(E_2^L > E_0)$.

In the second case, when $a \gg e^\theta \gg 1$, we find for the logarithmic contributions

$$\mathcal{J} \sim \mathcal{J}^{IR} \sim (\theta - 1)\log\frac{m^2}{\lambda^2} + \frac{\theta^2}{2} - 2\theta. \qquad \text{(V.26)}$$

In this case the photon energy can be large, on the order of the incident neutrino energy in the forward direction. The term -2θ in (V.26) for $\mathcal{J}$ comes from the $-q\cdot p_1$ term in $p_2\cdot k_2$. In Appendix F we show that the limit of $2\pi\,\tilde{B}$ corroborates the term $\theta^2/2$. Because the photon can have a large energy, the contributions of $\mathcal{J}$ are of the same order as those of $\mathcal{J}$. For comparison, in this limit we have

$$\mathcal{J} \sim \frac{1}{2}(\theta - 1)(\log a - \theta) + \frac{1}{12}\log a. \qquad \text{(V.27)}$$

Finally, we turn to the integral over $d\sigma/dt$ for the energy range relevant to the experiment proposed by Reines and Kropp. The integration over E_2^L or t can be transformed to an integration over θ. With θ_m given by $\cosh\theta_m = E_m/m$ and θ_0 by $\cosh\theta_0 = E_0/m$,

$$\sigma(E_2^L > E_0) = 2m^2 \frac{G^2}{\pi}\int_{\theta_0}^{\theta_m} d\theta \sinh\theta \left\{1 + \frac{\alpha}{\pi}\left[\mathcal{M} + \mathcal{J} + \mathcal{J}\right]\right\}. \qquad (V.28)$$

In terms of the electron's minimum recoil energy E_0 and the incident neutrino lab energy k, we obtain for the total cross section when $k - E_0 \gg m$,

$$\sigma(E_2^L > E_0) = \frac{2G^2 m}{\pi}(k - E_0)\left\{1 + (Z_3 - 1) + \frac{\alpha}{\pi} f(k,E_0)\right\}, \qquad (V.29)$$

where

$$f(k,E_0) = a(k,E_0)\log\frac{2k}{m} + b(k,E_0)\log\frac{2(k-E_0)}{m} + c(k,E_0)\log\frac{2E_0}{m}$$
$$+ d(k,E_0) - L\left(\frac{E_0}{k}\right) - L\left(-\frac{E_0}{k-E_0}\right) + \frac{2k + E_0}{2(k-E_0)} L\left(1 - \frac{E_0}{k}\right), \qquad (V.30)$$

with

$$a(k,E_0) = \frac{7}{3} + \frac{E_0}{2k} + \log\left(1 - \frac{E_0}{k}\right) \qquad (V.31a)$$

$$b(k,E_0) = -\frac{8}{3} - \frac{E_0}{4k}, \qquad (V.31b)$$

$$c(k,E_0) = -\frac{1}{2}\frac{E_0}{k - E_0}\log\frac{k}{E_0}, \qquad (V.31c)$$

and

$$d(k,E_0) = \left[\frac{1}{3} + \frac{E_0}{2k} + \frac{3}{2}\log\left(1 - \frac{E_0}{k}\right)\right]\frac{E_0}{k - E_0}\log\frac{k}{E_0}$$

$$- \frac{3}{8} - \frac{41}{72} \frac{E_0}{k} + \frac{1}{72} \left(\frac{E_0}{k}\right)^2 . \qquad \text{(V.31d)}$$

For k = 14 MeV, f(k, 8 MeV) = - 6.4 and f(k, 12 MeV) = - 10.6. $\left(\frac{\alpha}{\pi}\right)^{-1} (Z_3 - 1) = - \frac{2}{3} \log \frac{\Lambda}{m} \sim - 5.1$ for $\Lambda = m_N$. Thus $\sigma(E_2^L > E_0)$ should be depressed by on the order of 3 - 4 percent. Curves of $f(k,E_0)$ are given in Fig. 4.

The term $(Z_3 - 1)$ in $\sigma(E_2^L > E_0)$ and a term - 1/2 contained in $f(k,E_0)$ would contribute a correction to neutrino scattering by electrons for any energy and momentum transfer. For zero momentum transfer the bremsstrahlung cross section vanishes. As pointed out by Lee and Sirlin[13), there would be a residual correction due mainly to the neutrino charge radius. When $\theta \to 0$ in the limit $Q^2 \to 0$, $d\sigma^b/dt \to 0$ exactly. The term corresponding to the classical bremsstrahlung of an accelerated particle would be expected to vanish. In this case the entire bremsstrahlung amplitude vanishes. Furthermore the phase space for the configuration also goes to zero. In $\mathcal{M}$, $\Lambda^c(p_1,p_1) = 0$ and $\Pi^c(0) = 0$ by definition and the only remaining contributions come from $(Z_3 - 1)$ and the term $-\alpha/2\pi$ due to the finite difference between the vector and axial vector renormalization constants.

$$\frac{d\sigma}{dt} \xrightarrow[t \to 0]{} \frac{G^2}{\pi} \left[1 + (Z_3 - 1) - \frac{\alpha}{2\pi}\right] . \qquad \text{(V.32)}$$

Numerically, for $\Lambda = m_p$ the correction is about - 1.3 percent, about a third of the radiative corrections to $\sigma(E_2^L > E_0)$ for the proposed experiment.

VI. THE DEPENDENCE ON THE NEUTRINO CHARGE RADIUS

We have given in Chapter IV the effect of radiative corrections of order α on stellar neutrino luminosities due to pair annihilation. The pair annihilation is supposed to be the dominant mechanism for cooling massive stars for T_9 between 0.5 and 7. For Λ on the order of the nucleon mass the corrections range from an enhancement of $\sim$ 8 percent at $T_9 = 0.5$ (IV.21b) to a depression of $\sim$ 1 percent at $T_9 \sim 7$ (IV.27). At $T_9 \sim 1$ the negative contribution of $(Z_3 - 1)$ cancels the effect of the Coulomb enhancement; so the radiative corrections in the temperature range of interest should be very small.

In Chapter V we have given the corrections to the cross section for neutrino scattering by electrons when a bremsstrahlung photon of any energy is included. The corrections to antineutrino scattering would be similar and could be obtained from the amplitudes for neutrino scattering. For small momentum transfer there is a residual correction which should decrease the cross section by about 1.3 percent. As Lee and Sirlin have pointed out[13) this correction to the differential cross section is the same for antineutrinos. Thus it would be the correction affecting experiments using reactor antineutrinos which have low energies. The corrections for incident neutrino energy much greater than the electron mass are given in (V.29). For B^8 neutrinos of $\lesssim$ 14 MeV the corrections to $\sigma(E_2^L > E_0)$ should be 3 - 4 percent. The dependence of the radiative correction to this cross section on k and E_0 is indicated in Fig. 4. The form we have used for the electron loop contribution assumes $|Q^2| \ll \Lambda^2$. For $\Lambda \sim M_N$ the results would not need modification for $k \lesssim 10^3$ BeV.

In conclusion we consider the effects there could be on the annihilation and scattering if the estimates for f(0), or equivalently the quantity we have called $Z_3 - 1$, were incorrect. We do not know, after all, how to calculate f(0).

In a γ_5 invariant neutrino theory there is only one form factor. The matrix element of the electromagnetic current must be of the form

$$\langle \nu' | J_\mu(0) | \nu \rangle = -ie\, F(q^2)\, [\bar{U}_{\nu'}\, \gamma_\mu (1 + \gamma_5) U_\nu] \;, \qquad \text{(VI.1)}$$

for F a scalar function of q^2. Here $q^2 = (k' - k)^2$ for initial and final neutrino momenta k and k'. A zero total charge for the neutrino implies F(0) = 0. In a frame in which q^2 is space-like it is customary to define

$$\rho(r) = \int \frac{d^3q}{(2\pi)^3}\, e^{i\vec{q}\cdot\vec{r}}\, e\, F(q^2), \qquad \text{(VI.2)}$$

where it is assumed that $F(q^2)$ falls off sufficiently fast for (VI.2) to exist. Assuming also that $F(q^2)$ can be written as $F(q^2) = q^2 f(q^2)$, where $f(0) = \left.\frac{\partial F(q^2)}{\partial q^2}\right|_{q^2 = 0}$ is finite,

$$f(0) = -\frac{1}{6} \langle r^2 \rangle, \qquad \text{(VI.3)}$$

with

$$\langle r^2 \rangle = \frac{1}{e}\int d^3r\ r^2\ \rho(r). \qquad \text{(VI.4)}$$

$\rho(r)$ is called the charge distribution of the neutrino even though there is no frame in which the neutrino is at rest and $\sqrt{|\langle r^2\rangle|}$ is called the charge radius even though $\langle r^2\rangle$ can be negative. With the electron loop in Fig. 1a giving the charge distribution,

$$\langle r^2 \rangle = \frac{G}{\sqrt{2}\pi^2}\,\frac{3}{2}\,\frac{\pi}{\alpha}\,(Z_3 - 1) = -\frac{G}{\sqrt{2}\pi^2}\log\frac{\Lambda}{m}\ . \qquad \text{(VI.5)}$$

(The $F(q^2) \propto q^2[Z_3 - 1 + \pi^c(q^2)]$ which the electron loop contributes does not fall off for large q^2. But the expression is only valid for $|q^2| \ll \Lambda^2$. The rationale for the cut-off is the expectation that $F(q^2)$ decreases for $|q^2| \gtrsim \Lambda^2$). For $\Lambda = m_N$, (VI.5) gives $\sqrt{|\langle r^2\rangle|} \sim 5 \times 10^{-17}$ cm. In contrast with this estimated charge radius for the electron neutrino, the experimental upper limit is $\sim 4 \times 10^{-15}$ cm, which Bernstein, Ruderman, and Feinberg extracted from the experiment of Reines and Cowan[45] using reactor antineutrinos. The limit implied by the preliminary experiment on $\nu_e + e^- \rightarrow \nu_e + e^-$ by Reines and Kropp[17] is about the same .

With $F(q^2) \sim q^2\ f(0) = -\frac{1}{6}\,q^2\langle r^2\rangle$, the amplitude for a neutrino and an electron to interact through exchange of a photon differs from the amplitude to interact through the local Fermi coupling only in having γ_μ rather than $\gamma_\mu\gamma_5$ between the electron spinors when the weak interaction Hamiltonian is put in the order $(\bar{e}\ e)(\bar{\nu}_e\ \nu_e)$. The cross section for annihilation due to the charge radius alone would be

$$\sigma_{c.r.} = \frac{e^4\langle r^2\rangle^2}{36}\,\frac{1}{6\pi\beta}\,(t + 2m^2), \qquad \text{(VI.6)}$$

as compared to $\sigma_0 = G^2 \frac{1}{6\pi\beta}(t - m^2)$ for annihilation due to the weak interaction. In the low and high energy limits the energy dependence would be the same for the two cases. In the non-relativistic limit the neutrino luminosity due to the charge radius annihilation would be the same as that which the weak interaction would give with the substitution $G^2/\pi \to \frac{8}{9}\alpha^2 \langle r^2\rangle^2$; in the extreme relativistic limit the luminosity would be the same with the substitution $G^2/\pi \to \frac{4}{9}\alpha^2 \langle r^2\rangle^2$. If the neutrino charge radius were actually as large as the present limits on it, the neutrino luminosities would be $\sim 10^3$ times the prediction of the weak interaction. This appears implausible on astrophysical grounds[7)]. A charge radius of $\sim 4 \times 10^{-16}$ cm, about a factor of 10 smaller than the present limit, would correspond to cross sections for $e^- + e^+ \to \nu_e + \bar{\nu}_e$ and $\nu_e + e^- \to \nu_e + e^-$ of the magnitude which the weak interaction is expected to give.

A charge radius and weak interaction of the same order of magnitude would give a cross section for pair annihilation which can be written as

$$\sigma = \frac{1}{6\pi\beta}\left\{\left[\frac{G^2}{2} + \left(\frac{G}{\sqrt{2}} + \frac{e^2\langle r^2\rangle}{6}\right)^2\right](t - m^2) + 3m^2\left[\left(\frac{G}{\sqrt{2}} + \frac{e^2\langle r^2\rangle}{6}\right)^2 - \frac{G^2}{2}\right]\right\}. \qquad \text{(VI.6)}$$

G and $\langle r^2\rangle$ are real if the weak Hamiltonian and the electromagnetic current are Hermitian operators. At high energies the interference can only decrease the cross section by a factor of 2 from σ_0. This occurs when the charge radius amplitude just cancels the vector part of the weak interaction. At low energies the axial vector interaction by itself gives a cross section vanishing as $(t - 4m^2)$. (We recall that in this reaction in the c.m. system $\sqrt{t}$ is the total energy.) This would lead to a luminosity for $t \ll 3$ of

$$\left(\frac{du}{dt}\right)_{n.r.} = \left(\frac{du}{dt}\right)_{0,n.r.} \frac{1}{6} \langle (\vec{v}_+ - \vec{v}_-)^2 \rangle$$

$$= \left(\frac{du}{dt}\right)_{0,n.r.} \left(\frac{kT}{m}\right) . \qquad \text{(VI.7)}$$

$(du/dt)_{0,n.r.}$ here is the uncorrected luminosity as in Chapter IV. For $T_9 = 1$ the cancellation would decrease the luminosity by a factor of ~ 6.

The cross section for scattering due to a charge radius and a weak interaction of the same order of magnitude would be, for $k \gg E_0 \gg m$,

$$\sigma(E_2^{\,L} > E_0) = \frac{1}{\pi} \left\{ \left(2 \frac{G}{\sqrt{2}} + \frac{e^2 \langle r^2 \rangle}{6} \right)^2 + \frac{e^4 \langle r^2 \rangle^2}{108} \left(1 - \frac{E_0}{k}\right)^2 \right\} m(k - E_0) . \qquad \text{(VI.8)}$$

For $E_0 > 1/2\, k$ the functional form would be close to that for a pure weak interaction unless $\frac{e\langle r^2 \rangle}{6} \sim - \frac{2G}{\sqrt{2}}$, in which case, the cross section could be very small. For this special choice of the charge radius, the luminosity would be exactly the same as for the weak interaction alone. The behavior of $\sigma(E_2^{\,L} > E_0)$ and $(du/dt)_{n.r.}$ as functions of $\frac{e^2 \langle r^2 \rangle}{6} \Big/ \frac{G}{\sqrt{2}}$ is sketched in Fig. 5. For $\sqrt{|\langle r^2 \rangle|} \sim 5 \times 10^{-16}$ cm both the scattering and the luminosity could be small. There is a range of $\langle r^2 \rangle$ for which the cross section could be smaller than expected, but the luminosity larger . In this model the opposite situation of a small luminosity but a large cross section could not occur.

Experiments on neutrino processes which, in the lack of weak neutral current couplings, should occur only through an electromagnetic coupling of the neutrino are most feasible for the muon neutrinos produced by accelerators. The present limit on the cross section for muon neutrino scattering by protons can be used to obtain an upper limit on the muon neutrino charge radius, as well as on a weak coupling of neutral currents. Electron neutrinos from nuclear β-decay could give the proton only a very small recoil energy. Limits could also be obtained for the muon neutrino charge radius from processes involving the interaction of muon neutrinos with electrons.

The exchange of a photon between a muon neutrino and a proton would give

$$\frac{d\sigma}{dt} = \frac{1}{\pi}\left(\frac{e^2F(t)}{t}\right)^2 \left\{\left[F_1{}^2(t) - \frac{t}{4m_p^2}F_2(t)^2\right]\left[1 + \frac{ts}{(s-m_p^2)^2}\right] + \left[F_1(t) + F_2(t)\right]^2 \frac{t^2}{2(s-m_p^2)^2}\right\}, \qquad \text{(VI.9)}$$

with $s = -(k_1 + p_1)^2$ and $t = -(p_2 - p_1)^2$. $F(t)$ is the muon neutrino form factor. $F_1(t)$ and $F_2(t)$ are the proton electromagnetic form factors. Except for the factor $F(t)^2$, a factor of 2 because of the two-component theory, and a factor of 2 because of the spinology,this is the same as the Rosenbluth formula. For $F(t) = t/6\ \langle r^2\rangle$ and incident neutrinos of ~ 2 BeV we find[46], with $r = \tilde{r} \times 10^{-15}$ cm,

$$\sigma = 10^{-37} \langle \tilde{r}^2 \rangle^2 \text{ cm}^2. \qquad \text{(VI.10)}$$

The limit on this cross section for neutrinos of a few BeV energy is $\sim 10^{-40}$ cm^2 [47]. Then $\sqrt{|\langle r^2 \rangle|} < 2 \times 10^{-16}$ cm. This limit is better than the 10^{-15} cm which Bernstein, Ruderman, and Feinberg[22] obtained for the muon neutrino from the comparison $\nu_\mu + p \rightarrow \nu_\mu + p + \pi^0$ and $e^- + p \rightarrow e^- + p + \pi^0$. However, if a muon loop model, as in Fig. 1e, and the estimated charge radius of $\sim 10^{-17}$ cm are correct, the cross section would not be as large as (VI.10) for σ would imply. The variation in f(t) given by the muon loop would reduce the cross section by a factor of $\sim 10^{-4}$ to $\sim 10^{-49}$ cm^2. [48].

The cross section for the scattering of muon neutrinos by electrons should be for $k \gg E_0$,

$$\sigma \sim \frac{4}{27} \alpha^2 \langle r^2 \rangle^2 m(k - E_0). \qquad \text{(VI.11)}$$

For $k \sim 1$ BeV $\gg E_0$

$$\sigma \sim 5 \times 10^{-42} \langle \tilde{r}^2 \rangle^2 \text{ cm}^2. \qquad \text{(VI.12)}$$

An electromagnetic interaction between muon neutrinos and electrons would lead to electron-positron production in the field of a nucleus by muon neutrinos. Czyz and Walecka[49] calculated the cross sections for $\nu_e + Z \rightarrow \nu_e + Z + e^- + e^+$. Their result for an incident neutrino of energy ~ 1 BeV in a cross section $\sigma \sim 3 \times 10^{-45} Z^2 \text{cm}^2$, for production off a nucleus of atomic number Z. Assuming that the difference in the matrix element for the charge radius interaction and the weak interaction at high energies leads only to a reduction of the cross section by a factor of ~ 2, the corresponding charge radius cross section would be

$$\sigma \sim 10^{-43} \langle \tilde{r}^2 \rangle^2 Z^2 \mathrm{cm}^2. \qquad \text{(VI.13)}$$

If the form factor of the muon neutrino can be investigated and the estimate given by the muon loop with a cut-off on the order of the nucleon mass is correct, it would be plausible that the electron loop with cut-off gives a correct estimate for the form factor of the electron neutrino. If the estimate is correct, the radiative corrections to stellar neutrino luminosities are those we have given in Chapter IV and the corrections to the scattering cross section are those given in Chapter V. A neutrino charge radius ~ 10 times larger than the estimate, would have effects of the same order as the weak interaction, rather than give a correction of a few percent. If the cross section for electron neutrino scattering by electrons is found to be small, it is possible that the leptonic square term in the product $J_\mu^+ J_\mu^-$ does not occur or does not give a valid first order matrix element for the process. It would also be possible that the neutrino charge radius partially cancels the square term. In the latter case, since the cancellation cannot be complete, the process would be found to occur in other experiments, for instance, in the scattering of low energy antineutrinos by electrons. Data on the charge radius of the muon neutrino would be helpful in deciding the likelihood of large charge radius effects for the electron neutrino.

APPENDIX A. THE FUNCTIONS S_Σ, V, AND A

From (II.15), (II.16a), and (II.28a)

$$S_\Sigma = -\frac{\alpha}{4\pi}\int_0^1 dz_1 \int_0^{1-z_1} dz_2 \, \frac{[-2imz_1(1 - z_1)]}{c(Q^2)}, \tag{A.1}$$

where

$$c(Q^2) = m^2(1 - z_1)^2 + \lambda^2 z_1 + Q^2 z_2(1 - z_1 - z_2) - i\epsilon. \tag{A.2}$$

It is convenient to transform the variables z_1 and z_2 to x and u given by

$$x = 1 - z_1 \tag{A.3a}$$

$$xu = z_2. \tag{A.3b}$$

Then $\int_0^1 dz_1 \int_0^{1-z_1} dz_2 = \int_0^1 du \int_0^1 x\, dx$, and we can write

$$c(Q^2) = [m^2 + Q^2\, u(1 - u) - i\epsilon]\, x^2 + \lambda^2(1 - x). \tag{A.4}$$

In S_Σ no divergence occurs in the limit $\lambda \to 0$. The integration over z gives

$$S_\Sigma \underset{\lambda \to 0}{=} \frac{i\alpha m}{4\pi}\int \frac{du}{m^2 + Q^2 u(1 - u) - i\epsilon}. \tag{A.5}$$

The integral over u will be required for V and A also. It is

$$K = \int_0^1 \frac{du}{m^2 + Q^2 u(1-u) - i\epsilon} = \frac{1}{\sqrt{Q^2(Q^2 + 4m^2)}}$$

$$\log \left\{ \frac{-2Q^2 u + Q^2 - \sqrt{Q^2(Q^2 + 4m^2)} + i\epsilon Q^2}{-2Q^2 u + Q^2 + \sqrt{Q^2(Q^2 + 4m^2)} - i\epsilon Q^2} \right\}_{u=0}^{u=1} , \qquad (A.6)$$

where we have used the consideration that in the regions of physical interest $Q^2(Q^2 + 4m^2) > 0$. For scattering of either neutrinos or antineutrinos by electrons this follows from the fact that $Q^2 = (p_2 - p_1)^2 \underset{c.m.}{=} 2p_1^{\,2}(1 - \hat{p}_1 \cdot \hat{p}_2) > 0$. In pair annihilation, $Q^2 = (p^+ + p^-)^2 \underset{c.m.}{=} - 4w^2$, so that $Q^2(Q^2 + 4m^2)_{c.m.} = 16w^2 p^2 > 0$ again.

For the scattering reactions it is convenient to define φ and θ by

$$Q^2 = 4m^2 \sinh^2 \varphi \qquad (A.7a)$$

$$\theta = 2\varphi. \qquad (A.7b)$$

For the annihilation we will use the center-of-momentum (c.m.) electron energy w and velocity β. Then from (A.6)

$$K = \begin{cases} \dfrac{\theta}{m^2 \sinh \theta} & Q^2 > 0 \qquad (A.8a) \\[2ex] -\dfrac{1}{w^2 \beta} \left[\tanh^{-1}\beta - \dfrac{i\pi}{2}\right] & Q^2 < -4m^2. \qquad (A.8b) \end{cases}$$

Using (A.8)

$$S_\Sigma = \begin{cases} \dfrac{i\alpha}{4\pi m} \, \dfrac{\theta}{\sinh \theta} & Q^2 > 0 \qquad (A.9a) \\[2ex] -\dfrac{i\alpha m}{4\pi w^2 \beta} \left[\tanh^{-1}\beta - \dfrac{i\pi}{2}\right] & Q^2 < -4m^2. \qquad (A.9b) \end{cases}$$

For V and A we need $\Lambda^c(V)$ and $\Lambda^c(A)$. Again from (II.15) and (II.16)

$$\Lambda^c(V) = -\frac{\alpha}{4\pi}\int_0^1 dz_1 \int_0^{1-z_1} dz_2 \left\{ \frac{2Q^2(1 - z_2)(z_1 + z_2) - 4m^2 z_1(1 - z_1)}{c(Q^2)} \right.$$

$$+ \frac{2m^2Q^2 z_2(1 - z_1 - z_2)(1 - 4z_1 + z_1^2)}{c(0)\ c(Q^2)}$$

$$\left. - 2 \log \frac{c(0)}{c(Q^2)} \right\}, \qquad \text{(A.10)}$$

and

$$\Lambda^c(A) = \Lambda^c(V) - \frac{\alpha}{\pi}\int_0^1 dz_1 \int_0^{1-z_1} dz_2 \left\{ \frac{m^2 z_1(1 - z_1)}{c(Q^2)} - \frac{m^2Q^2 z_2(1 - z_1 - z_2)(1 - z_1)^2}{c(0)\ c(Q^2)} \right\}$$

(A.11)

$\Lambda^c(A) - \Lambda^c(V)$ contains no divergence in the limit $\lambda \to 0$.

We obtain

$$\Lambda^c(A) - \Lambda^c(V) = \frac{\alpha}{2\pi} \left\{ 1 - 2m^2\ K \right\}$$

$$= \begin{cases} \dfrac{\alpha}{2\pi} \left[1 - \dfrac{2\theta}{\sinh\ \theta} \right] & Q^2 > 0 \qquad \text{(A.12a)} \\ \dfrac{\alpha}{2\pi} \left[1 + \dfrac{2m^2}{w^2\beta} \left(\tanh^{-1}\beta - \dfrac{i\pi}{2} \right) \right] & Q^2 < -4m^2. \qquad \text{(A.12b)} \end{cases}$$

For $\Lambda^c(V)$ in (A.10) we need

$$\int_0^1 dz_1 \int_0^{1-z_1} dz_2 \log \frac{c(0)}{c(Q^2)} = -\frac{1}{2}\int_0^1 du \log \left[1 + \frac{Q^2}{m^2} u(1 - u) - i\epsilon \right]$$

$$= \begin{cases} 1 - \dfrac{\theta \cosh^2\varphi}{\sinh \theta} & Q^2 > 0 \quad \text{(A.13a)} \\ 1 - \beta \left[\tanh^{-1}\beta - \dfrac{i\pi}{2}\right] & Q^2 < -4m^2. \quad \text{(A.13b)} \end{cases}$$

In terms of x and u the remaining integral for $\Lambda^c(V)$ is

$$\int_0^1 du \int_0^1 x\, dx \left\{ \frac{2Q^2 + x[2Q^2(-1 + xu(1-u)) - 4m^2(1-x)]}{c(Q^2)} + \frac{2m^2Q^2u(1-u)x^2[-2 + x(2+x)]}{c(0)\; c(Q^2)} \right\}. \quad \text{(A.14)}$$

In the first term of (A.14) 2 powers of x prevent divergence, in the second term 4 powers. The part containing no divergence when $\lambda \to 0$ is

$$\int_0^1 du \left\{ \frac{-2(Q^2 + m^2) + 6Q^2 u(1-u)}{m^2 + Q^2u(1-u) - i\epsilon} \right\}$$

$$= \begin{cases} 6 - 8\,\dfrac{\theta \cosh^2\varphi}{\sinh \theta} & Q^2 > 0 \quad \text{(A.15a)} \\ 6 - 8\beta \left[\tanh^{-1}\beta - \dfrac{i\pi}{2}\right] & Q^2 < -4m^2. \quad \text{(A.15b)} \end{cases}$$

There remain two integrals which we will call K_1 and K_2, in which the limit $\lambda \to 0$ produces a divergence

$$K_1 = 2Q^2 \int_0^1 du \int_0^1 dx \frac{x}{(m^2 + Q^2u(1-u) - i\epsilon)x^2 + \lambda^2(1-x)}$$

$$= Q^2 \int_0^1 du \frac{1}{m^2 + Q^2u(1-u) - i\epsilon} \left\{ \log \frac{m^2 + Q^2u(1-u) - i\epsilon}{\lambda^2} + \lambda^2 \int_0^1 dx \frac{1}{(m^2 + Q^2u(1-u) - i\epsilon)x^2 + \lambda^2(1-x)} \right\}. \quad \text{(A.16)}$$

The contribution of the second term in K_1 vanishes in the limit $\lambda \to 0$. Before doing the u integration we reduce K_2 similarly.

$$K_2 = -4m^2Q^2\int_0^1 du\int_0^1 dx\,\frac{x^3u(1-u)}{[m^2x^2+\lambda^2(1-x)][(m^2+Q^2u(1-u)-i\epsilon)x^2+\lambda^2(1-x)]}$$

$$= -4m^2\int_0^1 du\int_0^1 dx\; x\left\{\frac{1}{m^2x^2+\lambda^2(1-x)} - \frac{1}{(m^2+Q^2u(1-u)-i\epsilon)x^2+\lambda^2(1-x)}\right\}$$

$$= -2m^2\int_0^1 du\left\{\frac{1}{m^2}\log\frac{m^2}{\lambda^2} - \frac{1}{m^2+Q^2u(1-u)-i\epsilon}\log\frac{m^2+Q^2u(1-u)-i\epsilon}{\lambda^2}\right\}, \qquad (A.17)$$

where only the contributions to the integral over x which contribute in the limit $\lambda \to 0$ have been kept as for K_1 in (A.16). Together

$$K_1 + K_2 = -2\log\frac{m^2}{\lambda^2} + (Q^2+2m^2)\int_0^1 du\,\frac{1}{m^2+Q^2u(1-u)-i\epsilon}\log\frac{m^2+Q^2u(1-u)-i\epsilon}{\lambda^2}. \qquad (A.18)$$

For $Q^2 = 4m^2\sinh^2\varphi$, $1+\frac{Q^2}{m^2}u(1-u) = 4\sinh^2\varphi(u_+-u)(u-u_-)$, where $u_\pm = \pm\, c^{\pm\varphi}/(2\sinh\varphi)$. Then

$$K_L = \int_0^1 du\,\frac{1}{m^2+Q^2u(1-u)-i\epsilon}\log\frac{m^2+Q^2u(1-u)-i\epsilon}{m^2}$$

$$= \frac{1}{4m^2\sinh^2\varphi}\int_0^1 du\,\frac{1}{(u_+-u)(u-u_-)}\log 4\sinh^2\varphi(u_+-u)(u-u_-). \qquad (A.19)$$

The integral can be done in terms of the Spence function $L(x) = \int_0^x dt \, \frac{\log(1 - t)}{t}$. Using (B.7) the result can be written as

$$K_L = \frac{1}{m^2 \sinh \theta} \left[\bar{L} \left(\frac{2 \cosh \varphi}{e^{-\varphi}} \right) - \bar{L} \left(\frac{2 \cosh \varphi}{e^{\varphi}} \right) \right], \qquad \text{(A.20a)}$$

where $\bar{L}(x) = \int_0^x dt \, \frac{\log|1 - t|}{t}$. Then for $Q^2 > 0$, using (A.8) and (A.20a) in (A.18) we have

$$K_1 + K_2 = [-2 + 2\theta \operatorname{ctnh} \theta] \log \frac{m^2}{\lambda^2} + 2 \operatorname{ctnh} \theta \left[\bar{L} \left(\frac{2 \cosh \varphi}{e^{-\varphi}} \right) - \bar{L} \left(\frac{2 \cosh \varphi}{e^{\varphi}} \right) \right]. \qquad \text{(A.21a)}$$

In the case of $Q^2 = -4w^2 < -4m^2$,

$1 - \frac{4w^2}{m^2} u(1 - u) - i\varepsilon = \frac{4}{1 - \beta^2} (u - u_+)(u - u_-)$, with $u_\pm = \frac{1}{2}(1 \pm \beta) \pm i\varepsilon$. Now

$$K_L = \frac{1 - \beta^2}{4m^2} \int_0^1 du \, \frac{1}{(u - u_+)(u - u_-)} \log \frac{4}{1 - \beta^2} (u - u_+)(u - u_-)$$

$$= \frac{1 - \beta^2}{4m^2 \beta} \left\{ \left[\left(\log \frac{4}{1 - \beta^2} \right) \log \frac{u - u_+}{u - u_-} + \frac{1}{2} (\log(u - u_+))^2 - \frac{1}{2} (\log(u - u_-))^2 \right]_{u=0}^{u=1} + \int_0^1 du \, \frac{\log(u - u_-)}{(u - u_+)} - \int_0^1 du \, \frac{\log(u - u_+)}{(u - u_-)} \right\}$$

$$= \frac{1 - \beta^2}{4m^2\beta}\left\{ - \log \frac{4}{1 - \beta^2} \log\left(\frac{1 + \beta}{1 - \beta} e^{-i\pi}\right) + \int_0^1 du \frac{\log(u - u_-)}{(u - u_+)} - \int_0^1 du \frac{\log(u - u_+)}{(u - u_-)} \right\} . \qquad (A.22)$$

$$\int_0^1 du \frac{\log(u - u_-)}{(u - u_+)} = \int_0^1 du \frac{\log(u - |u_-| + i\epsilon)}{u - |u_+| - i\epsilon} = \int_{-|u|_+}^{|u_-|} dt \frac{\log(t + \beta + i\epsilon)}{t - i\epsilon}$$

$$= \log \beta \left(\log \left|\frac{u_-}{u_+}\right| e^{i\pi}\right) + L\left(- \frac{|u_-|}{\beta}\right) - L\left(\frac{|u_+|}{\beta} - i\epsilon\right). \qquad (A.23a)$$

Similarly

$$\int_0^1 du \frac{\log(u - u_+)}{u - u_-} = \int_0^1 du \frac{\log(u - |u_+| - i\epsilon)}{u - |u_-| + i\epsilon} = \int_{-|u_-|}^{|u_+|} dt \frac{\log(t - \beta - i\epsilon)}{t + i\epsilon}$$

$$= (\log \beta e^{-i\pi}) \left(\log \left|\frac{u_+}{u_-}\right| e^{-i\pi}\right) + L\left(\frac{|u_+|}{\beta} - i\epsilon\right) - L\left(- \frac{|u_-|}{\beta}\right). \qquad (A.23b)$$

Using (A.23) in (A.22) and simplifying the result with the help of (B.8), (B.6a), and (B.7) we obtain

$$K_L = \frac{1 - \beta^2}{2m^2\beta}\left\{ L\left(\frac{2\beta}{1 + \beta}\right) - L\left(\frac{-2\beta}{1 - \beta}\right) + 2i\pi\left[\tanh^{-1}\beta - \frac{i\pi}{2}\right]\right\}. \qquad (A.20b)$$

With (A.8) and (A.20b) in (A.18) we have for $Q^2 < - 4m^2$

$$K_1 + K_2 = \left[- 2 + \frac{2(1+\beta^2)}{\beta}\left(\tanh^{-1}\beta - \frac{i\pi}{2}\right)\right]$$

$$- \frac{1+\beta^2}{\beta}\left[L\left(\frac{2\beta}{1+\beta}\right) - L\left(\frac{-2\beta}{1-\beta}\right) + 2i\pi\left(\tanh^{-1}\beta - \frac{i\pi}{2}\right)\right]. \quad \text{(A.21b)}$$

Gathering the contributions to $\Lambda^c(V)$ from (A.13), (A.15) and (A.21), we have

$$\Lambda^c(V) = - \frac{\alpha}{2\pi}\left[\theta \operatorname{ctnh}\theta - 1\right] \log \frac{m^2}{\lambda^2} + \frac{\alpha}{2\pi}\left\{ \operatorname{ctnh}\theta \left[\bar{L}\left(\frac{2\cosh\varphi}{e^{\varphi}}\right)\right.\right.$$

$$\left.\left. - \bar{L}\left(\frac{2\cosh\varphi}{e^{-\varphi}}\right)\right] - 2 + \frac{3\theta\cosh^2\varphi}{\sinh\theta}\right\}, \quad Q^2 > 0, \quad \text{(A.24a)}$$

$$\Lambda^c(V) = - \frac{\alpha}{2\pi}\left[\frac{1+\beta^2}{\beta}\left(\tanh^{-1}\beta - \frac{i\pi}{2}\right) - 1\right] \log \frac{m^2}{\lambda^2}$$

$$+ \frac{\alpha}{2\pi}\left\{\frac{1+\beta^2}{2\beta}\left[L\left(\frac{2\beta}{1+\beta}\right) - L\left(\frac{-2\beta}{1-\beta}\right) + 2i\pi\left(\tanh^{-1}\beta - \frac{i\pi}{2}\right)\right]\right.$$

$$\left. - 2 + 3\beta\left(\tanh^{-1}\beta - \frac{i\pi}{2}\right)\right\} \quad Q^2 < - 4m^2. \quad \text{(A.24b)}$$

From (A.12) and (II.28c)

$$A = \Lambda^c(V) - \begin{cases} - \dfrac{\alpha}{\pi}\dfrac{\theta}{\sinh\theta} & Q^2 > 0 \quad \text{(A.25a)} \\[2ex] \dfrac{\alpha}{\pi}\dfrac{m^2}{w^2\beta}\left(\tanh^{-1}\beta - \dfrac{i\pi}{2}\right) & Q^2 < - 4m^2. \quad \text{(A.25b)} \end{cases}$$

The $\Lambda^c(V)$ computed here is the same for $Q^2 > 0$ as occurs in the vertex correction to electron scattering in a potential. S_Σ, $\Lambda^c(V)$, and A for $Q^2 > 0$ agree with the results of Behrends, Finklestein, and Sirlin[40), for the corrections to μ-decay, when we put $m_1 = m_2$ in their results.

We include here the results for $\Pi^c(Q^2)$[41) which we use in V.

$$\Pi^c(Q^2) = \begin{cases} \dfrac{\alpha}{3\pi}\left[-\dfrac{5}{3} + \dfrac{1}{\sinh^2\varphi} + \dfrac{\cosh\varphi}{\sinh\varphi}\left(1 - \dfrac{1}{2\sinh^2\varphi}\right)\theta\right] & Q^2 > 0 \quad \text{(A.26a)} \\ \dfrac{\alpha}{3\pi}\left[-\dfrac{5}{3} - (1-\beta^2) + 2\beta\left(1 + \dfrac{1-\beta^2}{2}\right)\left(\tanh^{-1}\beta - \dfrac{i\pi}{2}\right)\right] & Q^2 < -4m^2 \quad \text{(A.26b)} \end{cases}$$

APPENDIX B. SPENCE FUNCTIONS[42)]

We define

$$L(x) = \int_0^x \frac{dt}{t} \log(1 - t). \tag{B.1}$$

For $|x| \leq 1$, L(x) has the power series expansion

$$L(x) = -\sum_1^\infty \frac{x^m}{n^2}. \tag{B.2}$$

The special values

$$L(1) = -\frac{\pi^2}{6}, \tag{B.3}$$

and

$$L(-1) = \frac{\pi^2}{12} \tag{B.4}$$

are useful. The function L(x) has a cut for real $x > 1$. We define

$$\bar{L}(x) = \int_0^x dt \frac{\log|1 - t|}{t}. \tag{B.5}$$

Then for real $x > 1$,

$$L(x - i\epsilon) = \int_0^x dt \frac{\log(1 - t + i\epsilon)}{t} = \bar{L}(x) + i\pi \log x \tag{B.6a}$$

and

$$L(x + i\epsilon) = \int_0^x \frac{dt}{t} \log(1 - t - i\epsilon) = \bar{L}(x) - i\pi \log x. \tag{B.6b}$$

$\bar{L}(x)$ is the average of the values above and below the cut. For real $x < 1$, $\bar{L}(x) = L(x)$. The following relations are useful

$$\bar{L}(x) + \bar{L}\left(\frac{1}{x}\right) = -\frac{\pi^2}{3} + \frac{1}{2}(\log x)^2 \quad x > 0 \tag{B.7}$$

$$L(x) + L\left(\frac{1}{x}\right) = \frac{\pi^2}{6} + \frac{1}{2}(\log |x|)^2 \quad x < 0. \tag{B.8}$$

$$L(1 - x) + L(x) = -\frac{\pi^2}{6} + \log x \log(1 - x) \quad 0 \leq x \leq 1. \tag{B.9}$$

$$L(x) + L(-x) = \frac{1}{2} L(x^2) \quad - \leq x \leq 1. \tag{B.10}$$

$$L(1 - x) - L(-x) = -\frac{\pi^2}{12} - \log x \log(1 + x) + \frac{1}{2} L(1 - x^2)$$

$$0 \leq x \leq 2. \tag{B.11}$$

$$L\left(\frac{1}{1 - x}\right) - \bar{L}(x) = \frac{\pi^2}{3} - \frac{1}{2}\log|1 - x| \log\frac{x^2}{|1 - x|} \quad x > 1. \tag{B.12}$$

$$2[L(x) - L(-x)] - \left[L\left(\frac{1 + x}{2}\right) - L\left(\frac{1 - x}{2}\right)\right] = \bar{L}\left(\frac{1 + x}{1 - x}\right)$$

$$- L\left(\frac{1 - x}{1 + x}\right) + \log\frac{1 + x}{1 - x}\log\frac{\sqrt{1 - x^2}}{2x^2} \quad 0 \leq x \leq 1. \tag{B.13}$$

$$-4\int_0^{\varphi} \alpha \tanh\alpha \, d\alpha = \bar{L}\left(\frac{2\cosh\varphi}{e^{\varphi}}\right) - \bar{L}\left(\frac{2\cosh\varphi}{e^{-\varphi}}\right). \tag{B.14}$$

APPENDIX C. INTEGRALS FOR $e^- + e^+ \rightarrow \nu_e + \bar{\nu}_e + \gamma$

In the integrand for σ^b in (III.11), $K = p^+ + p^- - q$. We do the remaining integrations over the photon variables in the center-of-momentum system, keeping a mass λ for the photon wherever a divergence would arise in the limit $\lambda \rightarrow 0$. p and w are the momentum and energy of the electron and $\vec{p}^- = p\,\hat{e}_z$. Then $K^2 = -t + 2\sqrt{t}\,q$ and $q\cdot K = -\sqrt{t}\,q$, with $t = -(p^+ + p^-)^2$.

For σ^b we need

$$I = \int \frac{d^3q}{q_0}\left\{K^2(K^2 + m^2)\left(\frac{p^-}{q\cdot p^-} - \frac{p^+}{q\cdot p^+}\right)^2 + 4K^2 - \frac{2(q\cdot K)^2(K^2 - m^2)}{q\cdot p^-\; q\cdot p^+}\right\}. \tag{C.1}$$

The second and third terms are finite for $\lambda = 0$ and give

$$I' = 4\pi\int_{-1}^{1} dx \int_0^w q\, dq \left\{2(-t + 2\sqrt{t}\,q) + \frac{t(t + m^2 - 2q\sqrt{t})}{(w^2 - p^2x^2)}\right\}, \tag{C.2}$$

where $x = \hat{p}^- \cdot \hat{q}$. We obtain

$$I' = \frac{32\pi}{3} w^4\left\{\frac{1}{\beta}\left(\log\frac{1+\beta}{1-\beta}\right)\left[1 + \frac{3m^2}{4w^2}\right] - 1\right\}. \tag{C.3}$$

For the first term

$$\left(\frac{p^-}{q\cdot p^-} - \frac{p^+}{q\cdot p^+}\right)^2 = \frac{4\beta^2}{q_0^2}\;\frac{(1 - v^2x^2)}{(1 - v^2\beta^2x^2)^2}, \tag{C.4}$$

with $\vec{v} = \vec{q}/q_0$, and

$$K^2(K^2 + m^2) = t(t - m^2) + 2q\sqrt{t}\,(-2t + m^2) + 4t\,q^2. \tag{C.5}$$

The contributions due to the terms in $K^2(K^2 + m^2)$ proportional to q are finite for $\lambda = 0$. They give

$$I'' = 32\pi \left[- 6w^2 + m^2\right] \left[- 1 + \frac{1 + \beta^2}{2\beta} \log \frac{1 + \beta}{1 - \beta}\right]. \qquad (C.6)$$

There remains

$$2\pi \int_{-1}^{1} dx \int_{0}^{w} q^2 \frac{dq}{q_0^3}\, t\ (t - m^2)\ \frac{4\beta^2(1 - v^2x^2)}{(1 - v^2\beta^2x^2)^2} = 8\pi\beta^2\ t(t - m^2)\ J, \qquad (C.7)$$

for

$$J = \int_{-1}^{1} dx \int_{0}^{w} \frac{q^2 dq}{q_0^3}\ \frac{(1 - v^2x^2)}{(1 - v^2\beta^2x^2)^2} = \int_{-1}^{1} dx \int_{0}^{v_m = 1 - \frac{\lambda^2}{2w^2}} \frac{v^2 dv}{1 - v^2}\ \frac{1 - v^2x^2}{(1 - v^2\beta^2x^2)^2}\ . \qquad (C.8)$$

$$K(\beta^2x^2) = \int_0^{v_m} \frac{v^2 dv}{1 - v^2}\ \frac{1 - v^2x^2}{(1 - v^2\beta^2x^2)^2} = \frac{\partial}{\partial(\beta^2x^2)} \int_0^{v_m} \frac{dv}{1 - v^2}\ \frac{1 - v^2x^2}{(1 - v^2\beta^2x^2)}$$

$$= \frac{1 - x^2}{(1 - \beta^2x^2)^2} \log \frac{2w}{\lambda} + \frac{1}{2}\left[\frac{1 - \beta^2}{\beta}\ \frac{x}{(1 - \beta^2x^2)^2} - \frac{1 + \beta^2}{2\beta^3 x(1 - \beta^2x^2)}\right] \log \frac{1 + \beta x}{1 - \beta x}$$

$$+ \frac{1 - \beta^2}{2\beta^2}\ \frac{1}{(1 - \beta^2x^2)^2}\ . \qquad (C.9)$$

$$J = \frac{1}{\beta^2}\left[\frac{1 + \beta^2}{2\beta} \log \frac{1 + \beta}{1 - \beta} - 1\right] \log \frac{2w}{\lambda} + \frac{1}{2\beta^3} \log \frac{1 + \beta}{1 - \beta} - \frac{1 + \beta^2}{8\beta^3} \left(\log \frac{4}{1 - \beta^2}\right) \left(\log \frac{1 + \beta}{1 - \beta}\right)$$

$$- \frac{1 + \beta^2}{4\beta^3} \left\{L\left(\frac{1 + \beta}{2}\right) - L\left(\frac{1 - \beta}{2}\right) - 2\left[L(\beta) - L(-\beta)\right]\right\}. \qquad (C.10)$$

From (C.2), (C.6), (C.7), (C.10) and (III.11), we obtain the result for σ^b in (III.18).

If we multiply the integrand in (C.1) by $q_z = qx$, we see from (C.2), (C.4), and (C.5) that the new integrand is odd in x, so that the integral vanishes. For a in Chapter IV the integrand in (C.1) is multiplied by q_0. We have done first the q and second the x integration to obtain (IV.4a).

APPENDIX D. INTEGRALS FOR THE LUMINOSITY CORRECTIONS

For non-relativistic electrons and positrons in thermal equilibrium with radiation

$$\left\langle \frac{1}{v} \right\rangle = \left\langle \frac{1}{\sqrt{(\vec{v}^+ - \vec{v}_-)^2}} \right\rangle = \frac{1}{\pi}\left(\frac{kT}{m}\right)^{-3} \int_0^\infty v^{+2} dv^+ \int_0^\infty v^{-2} dv^- \int_{-1}^{1} dx \, \frac{e^{-\frac{m(v^{+2} + v^{-2})}{2kT}}}{\sqrt{(\vec{v}^+ - \vec{v}^-)^2}} , \tag{D.1}$$

where $x = \hat{v}^+ \cdot \hat{v}^-$. The integration over x gives

$$\left\langle \frac{1}{v} \right\rangle = \frac{1}{\pi}\left(\frac{kT}{m}\right)^{-3} \int_0^\infty v^+ dv^+ \int_0^\infty v^- dv^- \, e^{-\frac{m(v^{+2} + v^{-2})}{2kT}} \left[\sqrt{(v^+ + v^-)^2} - \sqrt{(v^+ - v^-)^2}\right]. \tag{D.2}$$

Let $y = v^+ + v^-$ and $z = v^+ - v^-$. Then $v^+ = \frac{y + z}{2}$, $v^- = \frac{y - z}{2}$, and $dv^+ \, dv^- = \frac{1}{2} \, dydz$. Making the transformation of variables,

$$\left\langle \frac{1}{v} \right\rangle = \frac{1}{\pi}\left(\frac{kT}{m}\right)^{-3} I, \tag{D.3}$$

where

$$I = \frac{1}{8}\int_0^\infty dy \int_{-y}^{y} dz \, e^{-\frac{m(y^2 + z^2)}{4kT}} (y^2 - z^2)(y - |z|),$$
$$= \frac{1}{4}\int_0^\infty dy \int_0^y dz \, e^{-a(y^2 + z^2)} (y^2 - z^2)(y - z), \tag{D.4}$$

with $a = m/4kT$. We need

$$J_1 = \int_0^y dz \, e^{-az^2} (y^2 - z^2), \tag{D.5a}$$

and

$$J_2 = \int_0^y dz\ e^{-az^2}(y^2 - z^2)z. \qquad \text{(D.6a)}$$

$$J_1 = \left(y^2 + \frac{\partial}{\partial a}\right)\int_0^y e^{-az^2}\,dz = \left(y^2 + \frac{\partial}{\partial a}\right)\frac{1}{\sqrt{a}}\int_0^{\sqrt{a}\,y} dt\ e^{-t^2}$$

$$= \left(y^2 + \frac{\partial}{\partial a}\right)\frac{\sqrt{\pi}}{2\sqrt{a}}\,\mathrm{erf}(\sqrt{a}\,y) = \frac{\sqrt{\pi}}{2}\left(\frac{y^2}{\sqrt{a}} - \frac{1}{2a^{3/2}}\right)\mathrm{erf}(\sqrt{a}\,y) + \frac{y}{2a}\,e^{-ay^2}, \qquad \text{(D.5b)}$$

$$J_2 = \frac{1}{2}\int_0^{y^2} dt\ (y^2 - t)e^{-at} = \frac{1}{2a^2}\left[e^{-ay^2} + (ay^2 - 1)\right]. \qquad \text{(D.6b)}$$

From (D.4), (D.5), and (D.6)

$$I = \frac{1}{4}\int_0^\infty dy\ e^{-ay^2}\left\{\frac{\sqrt{\pi}\,y}{2a^{3/2}}(y^2a - 1/2)\,\mathrm{erf}(\sqrt{a}\,y) + \frac{(ay^2 - 1)}{2a^2}(e^{-ay^2} - 1)\right\}$$

$$= 4\sqrt{\pi}\left(\frac{kT}{m}\right)^{5/2}\int_0^\infty du\ e^{-u^2}\left\{u(u^2 - 1/2)\,\mathrm{erf}(u) + \frac{1}{\sqrt{\pi}}(u^2 - 1)(e^{-u^2} - 1)\right\}. \qquad \text{(D.7)}$$

The first term of the integral in (D.7) can be done by parts and the second is straightforward. We find

$$\int_0^\infty du\ e^{-u^2}\,u(u^2 - 1/2)\mathrm{erf}(u) = \frac{3}{8\sqrt{2}}, \qquad \text{(D.8)}$$

and

$$\frac{1}{\sqrt{\pi}}\int_0^\infty du\ e^{-u^2}(u^2 - 1)(e^{-u^2} - 1) = \frac{1}{4} - \frac{3}{8\sqrt{2}}. \qquad \text{(D.9)}$$

With (D.8) and (D.9), we have

$$I = \sqrt{\pi}\left(\frac{kT}{m}\right)^{5/2}, \qquad .10)$$

and finally substituting (D.10) in (D.3)

$$\langle\frac{1}{v}\rangle = \sqrt{\frac{kT}{\pi m}} . \qquad (D.11)$$

In the extreme relativistic limit, from (IV.25)

$$\left(\frac{du}{dt}\right)_{e.r.} = \frac{16G^2}{(2\pi)^4 3\pi}\int_0^\infty p^{+4}\, dp^+ \int_0^\infty p^{-3}\, dp^-\, e^{-\frac{p^+ + p^-}{kT}} \int_{-1}^{1} dx(1 - x)^2$$

$$\left[1 + \frac{\alpha}{\pi}C_1 - \frac{11}{8}\frac{\alpha}{\pi}\log\frac{2p^+p^-(1 - x)}{m^2}\right] . \qquad (D.12)$$

$$\int_{-1}^{1} dx\ (1 - x)^2 \left[A - B\log\frac{2p^+p^-(1 - x)}{m^2}\right]$$

$$= \frac{8}{3}\left(A + B/3 - \frac{8}{3}B\log\frac{4p_1p_2}{m^2}\right) . \qquad (D.13)$$

Using (D.13) in (D.12),

$$\left(\frac{du}{dt}\right)_{e.r.} = \frac{8G^2}{9\pi^5}\left\{\left(1 + \frac{\alpha}{\pi}C_1 + \frac{11}{24}\frac{\alpha}{\pi}\right)3!4!(kT)^9\right.$$

$$\left. - \frac{11}{8}\frac{\alpha}{\pi}\left[4!(kT)^5\, J_3 + 3!(kT)^4\, J_4\right]\right\} , \qquad (D.14)$$

where

$$J_3 = \int_0^\infty p^3 dp \ \log \frac{2p}{m} \ e^{-\frac{p}{kT}} = 6 \left[\log \frac{2kT}{m} + \frac{11}{6} - C \right] , \qquad \text{(D.15)}$$

and

$$J_4 = \int_0^\infty p^4 dp \ \log \frac{2p}{m} \ e^{-\frac{p}{kT}} = 24 \left[\log \frac{2kT}{m} + \frac{25}{12} - C \right] , \qquad \text{(D.16)}$$

with Euler's constant $C = .5772$. From (D.14), (D.15), and (D.16), the result (IV.26) follows.

APPENDIX E. INTEGRALS FOR $\nu_e + e^- \to \nu_e + e^- + \gamma$

We write the bremsstrahlung cross section as the sum of two parts,

$$\frac{d\sigma}{dt} = \frac{G^2}{\pi}\frac{\alpha}{\pi}\left[\mathcal{I} + \mathcal{J}\right], \tag{E.1}$$

where in $\mathcal{J}$ no divergence occurs when $\lambda \to 0$

$$\begin{pmatrix}\mathcal{I}\\ \mathcal{J}\end{pmatrix} = \frac{1}{(2\pi)^2}\int\frac{d^3p_2}{E_2}\,\delta(t + (p_2 - p_1)^2)\int\frac{d^3q}{q_0}\,\frac{d^3k_2}{k_2}\,\delta^4(p_1 + k_1 - p_2 - k_2 - q)\,\frac{1}{(-p_1\cdot k_1)}\begin{pmatrix} i\\ j\end{pmatrix}, \tag{E.2}$$

where

$$i = p_1\cdot k_1\; p_2\cdot k_2\left(\frac{p_1}{q\cdot p_1} - \frac{p_2}{q\cdot p_2}\right)^2 \tag{E.3}$$

and

$$j = p_1\cdot k_1\; p_2\cdot k_2\left(\frac{1}{q\cdot p_1} - \frac{1}{q\cdot p_2}\right) + \frac{p_1\cdot p_2}{q\cdot p_1\; q\cdot p_2}\,(p_2\cdot k_2\; q\cdot k_1 - p_1\cdot k_1\; q\cdot k_2)$$
$$+ \frac{p_1\cdot k_1 p_1\cdot k_2}{q\cdot p_1} - \frac{p_2\cdot k_1 p_2\cdot k_2}{q\cdot p_2} - \frac{p_1\cdot k_1 q\cdot k_2}{(q\cdot p_2)^2}\,(m^2 + q\cdot p_2) + \frac{p_2\cdot k_2 q\cdot k_1}{(q\cdot p_1)^2}$$
$$(m^2 - q\cdot p_1). \tag{E.4}$$

As discussed in Chapter V

$$\begin{pmatrix}\mathcal{I}\\ \mathcal{J}\end{pmatrix} = \int_{E_2^-}^{E_1^c - \lambda^2/2\sqrt{s}} \frac{dE_2^c}{2p_1^c}\begin{pmatrix} I\\ J\end{pmatrix}, \tag{E.5}$$

where

$$\begin{pmatrix} I \\ J \end{pmatrix} = \frac{1}{2\pi} \int \frac{d^3q}{q_0} \frac{d^3k_2}{k_2} \delta^4(p_1 + k_1 - p_2 - k_2 - q) \frac{1}{(-p_1 \cdot k_1)} \begin{pmatrix} i \\ j \end{pmatrix}$$

$$= \frac{1}{2\pi} \frac{q}{\epsilon} \int d\Omega_q \frac{1}{(-p_1 \cdot k_1)} \begin{pmatrix} i \\ j \end{pmatrix} \qquad \text{(E.6)}$$

The last form in (E.6) holds in the $\vec{q} + \vec{k}_2 = 0$ frame with $\epsilon = E_1 + k_1 - E_2$ and $q = (\epsilon^2 - \lambda^2)/2\epsilon$. $E^c_{1,2}$ in (E.5) are the initial and final electron energies in the c.m. system of the initial neutrino and electron.

We do first the integration for I. We have

$$I = \frac{1}{2\pi} \frac{q^2}{\epsilon} \int d\Omega_q (\vec{p}_2 \cdot \hat{q} + E_2) \left\{ \frac{-m^2}{(E_2 q_0 - \vec{p}_2 \cdot \vec{q})^2} - \frac{2p_1 \cdot p_2}{(E_2 q_0 - \vec{p}_2 \cdot \vec{q})(E_1 q_0 - \vec{p}_1 \cdot \vec{q})} - \frac{m^2}{(E_1 q_0 - \vec{p}_1 \cdot \vec{q})^2} \right\}$$

$$= \frac{q^2}{\epsilon} \left\{ - m^2 \int_{-1}^{1} dx \frac{(E_2 + p_2 x)}{(E_2 q_0 - p_2 q x)^2} - m^2 \int_{-1}^{1} dx \frac{(E_2 + \vec{p}_2 \cdot \hat{p}_1 x)}{(E_1 q_0 - p_1 q x)^2} + 2m^2 \cosh\theta \int_0^1 d\eta \int_{-1}^{1} \frac{E_2 + \vec{p}_2 \cdot \hat{p}_\eta x}{(E_\eta q_0 - p_\eta q x)^2} \right\}, \qquad \text{(E.7)}$$

where $p_\eta = p_2 + \eta(p_2 - p_1)$. Completing the angular integration

$$I = \frac{q^2}{\epsilon} \left\{ - m^2 \left[\frac{2E_2 \epsilon}{q} \frac{1}{E_2^2 q_0^2 - p_2^2 q^2} - \frac{1}{p_2 q^2} \log \frac{E_2 q_0 + p_2 q}{E_2 q_0 - p_2 q} \right] \right.$$

$$- m^2 \left[2\left(E_2 + \frac{\vec{p}_1\cdot\vec{p}_2 q_0 E_1}{p_1^2 q}\right) \frac{1}{E_1^2 q_0^2 - p_1^2 q^2} - \frac{\vec{p}_1\cdot\vec{p}_2}{p_1^3 q^2} \log \frac{E_1 q_0 + p_1 q}{E_1 q_0 - p_1 q} \right]$$

$$+ 2m^2 \cosh\theta \int_0^1 d\eta \left[2\left(E_2 + \frac{\vec{p}_2\cdot\vec{p}_\eta q_0 E_\eta}{q\, p_\eta^2}\right) \frac{1}{E_\eta^2 q_0^2 - p_\eta^2 q^2} - \frac{\vec{p}_2\cdot\vec{p}_\eta}{q^2 p_\eta^3} \log \frac{E_\eta q_0 + p_\eta q}{E_\eta q_0 - p_\eta q} \right] \Bigg\} . \qquad (E.8)$$

The integral over η can be simplified. Using the relation

$$\frac{\vec{p}_2\cdot\vec{p}_\eta}{p_\eta^3} = \frac{d}{d\eta}\left(\frac{\eta}{p_\eta}\right) ,$$

$$\int_0^1 d\eta \frac{\vec{p}_2\cdot\vec{p}_\eta}{p_\eta^3} \log \frac{E_\eta q_0 + p_\eta q}{E_\eta q_o - p_\eta q} = \frac{1}{p_1} \log \frac{E_1 q_0 + p_1 q}{E_1 q_0 - p_1 q} + \int_0^1 d\eta \frac{\eta}{p_\eta} \frac{2qq_0 \left[p_\eta \frac{dE_\eta}{d\eta} - E_\eta \frac{dp_\eta}{d\eta}\right]}{E_\eta^2 q_0^2 - p_\eta^2 q^2} , \qquad (E.9)$$

with which we obtain for the η integral in (E.8)

$$\frac{1}{q^2 p_1} \log \frac{E_1 q_0 + p_1 q}{E_1 q_0 - p_1 q} - 2\int_0^1 d\eta \frac{\epsilon E_2}{q} \frac{1}{E_\eta^2 q_0^2 - p_\eta^2 q^2} . \qquad (E.10)$$

To do the remaining integration it is necessary to express the variables of the $\vec{q} + \vec{k}_2 = 0$ frame in terms of invariants and those in terms of the c.m. variables. The following relations will be used, in which several new variables are defined.

$$\epsilon^2 = -(p_1 + k_1 - p_2)^2 = 2\sqrt{s}\,(E_1^c - E_2^c) = 2\sqrt{s}\,x,$$

where $x = E_1^c - E_2^c$,

$$\epsilon E_2 = -(p_1 + k_1 - p_2)\cdot p_2 = \sqrt{s}\,E_2^c - m^2 = y_2 = \sqrt{s}(p_1^c - x)$$

$$\epsilon p_2 = \epsilon\sqrt{E_2^2 - m^2} = \sqrt{s}\,p_2^c,$$

$$\epsilon E_1 = -(p_1 + k_1 - p_2)\cdot p_1 = \sqrt{s}\,p_1^c - 2m^2\sinh^2\varphi = y_1,$$

$$\epsilon p_1 = \epsilon\sqrt{E_1^2 - m^2} = z. \qquad \text{(E.11)}$$

The terms in I proportional to $\log\dfrac{E_i q_0 + p_i q}{E_i q_0 - p_i q}$ have no divergence when $\lambda \to 0$. They contribute

$$\mathcal{J}^F = \frac{m^2}{2\sqrt{s}\,p_1^c}\mathcal{L}(s,\varphi) + \frac{1}{2\sqrt{s}\,p_1^c m^2}\Bigg\{\frac{1}{2}(\sqrt{s}\,p_1^c + m^2\sinh\theta)\left[z_0\log\frac{4z_0^2}{y_1^2 - z_0^2} - y_1\log\frac{y_1 + z_0}{y_1 - z_0}\right]$$

$$+\frac{1}{2}\left[\sqrt{s}\,p_1^c + 2m^2\cosh^2\varphi - 4m^2\cosh\theta\right]\left[y_1\log\frac{4y_1^2}{y_1^2 - z_0^2} - z_0\log\frac{y_1 + z_0}{y_1 - z_0}\right]\Bigg\}. \qquad \text{(E.12)}$$

Here $z_0 = \sqrt{s}\,p_1^c - m^2\sinh\theta$. (This can be negative, in contrast to ϵp_1; the integration gives a function invariant under the transformation $(\epsilon p_1)_0 \to -(\epsilon p_1)_0$, so that we can take z_0 instead of the correct $(\epsilon p_1)_0 = |z_0|$.)

$$\mathcal{L}(s,\varphi) = \bar{L}\left(\frac{s}{m^2}\right) + L(1) - \bar{L}(e^{\theta}) - \bar{L}\left(\frac{s}{m^2}\,e^{-\theta}\right). \tag{E.13}$$

The remainder of I, which contains the infrared divergence, is

$$I^{IR} = \frac{-2m^2\, q\, E_2}{E_2^2 q_0^2 - p_2^2 q^2} - \frac{2m^2 q^2}{\epsilon}\left[E_2 + \frac{\vec{p}_1\cdot\vec{p}_2 q_0 E_1}{p_1^2 q}\right]\frac{1}{E_1^2 q_0^2 - p_1^2 q^2}$$
$$+\int_0^1 d\eta\, \frac{4m^2 \cosh\theta\, \epsilon\, E_2}{q(E_\eta^2 q_0^2 - p_\eta^2 q^2)}\ . \tag{E.14}$$

It is convenient to use the variable x, which ranges from $\lambda^2/2\sqrt{s}$ to $x_m = E_1^c - E_2^-$. For $\lambda^2/m^2 \ll 1$, the first term in (E.14) is

$$I^{IR_2} \sim -\frac{2\,sx(p_1^c - x)}{sx^2 + \frac{\lambda^2}{m^2}\left[s(p_1^c - x)^2 - m^2\sqrt{s}x\right]}\ , \tag{E.15}$$

and contributes

$$\mathcal{J}^{IR_2} = \frac{x_m}{p_1^c} - \log\frac{m}{\lambda}\frac{x_m}{p_1^c}\ . \tag{E.16}$$

Similarly, we obtain the contributions of the second and third terms of I^{IR} in (E.14),

$$\mathcal{J}^{IR_1} = \frac{x_m}{2p_1^c} - \log\frac{m\, x_m\sqrt{s}}{\lambda\, y_1} + \frac{(sp_1^{c2} - m^4\sinh^2\theta)}{4m^2\sqrt{s}p_1^c}\ \log\frac{y_1^2}{z_0^2}\ , \tag{E.17}$$

and

$$\mathcal{J}^{IR_{12}} = \frac{2\cosh\theta}{p_1^{\,c}}\int_0^1 d\eta\, \frac{1}{\left[1+4\eta(1-\eta)\sinh^2\varphi\right]}\left\{-x_m + \frac{p_1^{\,c}}{2}\right.$$

$$\left.\log \frac{m^2 x_m^{\,2} s\left[1+4\eta(1-\eta)\sinh^2\varphi\right]}{\lambda^2\left[\sqrt{s}\,p_1^{\,c} - 2\eta m^2\sinh^2\varphi\right]^2}\right\} \tag{E.18}$$

Finally, the integral over η gives

$$\mathcal{J}^{IR_{12}} = 2\theta\cosh\theta\left\{-\frac{x_m}{p_1^{\,c}} + \log\frac{m\,x_m}{\lambda\,p_1^{\,c}} + \frac{1}{2}\log(2\cosh\varphi)\right.$$

$$\left. - \frac{1}{2}\log\left[1 - \frac{2m^2}{\sqrt{s}\,p_1^{\,c}}\sinh^2\varphi\left(1+\frac{m^2}{2\sqrt{s}\,p_1^{\,c}}\right)\right]\right\}$$

$$+ \operatorname{ctnh}\theta\left\{\left[L\left(\frac{e^{\varphi}}{2\cosh\varphi}\right) - L\left(\frac{e^{-\varphi}}{2\cosh\varphi}\right)\right] - \left[L(\alpha_+) - L(\alpha_-)\right]\right.$$

$$\left. + \left[L(\zeta_-) - L(\zeta_+)\right]\right\}, \tag{E.19}$$

where

$$\alpha_\pm = \frac{-m^2}{\sqrt{s}\,p_1^{\,c}}\sinh\varphi\, e^{\pm\varphi} \Big/ \left[1 + \frac{m^2}{\sqrt{s}\,p_1^{\,c}}\sinh\varphi\, e^{-\varphi}\right],$$

$$\zeta_\pm = \frac{m^2}{\sqrt{s}\,p_1^{\,c}}\sinh\varphi\, e^{\pm\varphi} \Big/ \left[1 - \frac{m^2}{\sqrt{s}\,p_1^{\,c}}\sinh\varphi\, e^{\varphi}\right].$$

For the computation of $\mathcal{J}$, we write $j = \sum_{i=1}^{6} j_i$ to which $J = \sum_{i=1}^{6} J_i$ corresponds. The j_i are as follows:

$$j_1 = p_1\cdot k_1\, p_2\cdot k_2 \left(\frac{1}{q\cdot p_1} - \frac{1}{q\cdot p_2}\right),$$

$$j_2 = \frac{p_1\cdot p_2}{q\cdot p_1\, q\cdot p_2} \left[p_2\cdot k_2\, q\cdot k_1 - p_1\cdot k_1\, q\cdot k_2\right],$$

$$j_3 = \frac{p_1\cdot k_1\, p_1\cdot k_2}{q\cdot p_1} - \frac{p_2\cdot k_1\, p_2\cdot k_2}{q\cdot p_2},$$

$$j_4 = -\frac{p_1\cdot k_1\, q\cdot k_2 (m^2 + q\cdot p_2)}{(q\cdot p_2)^2},$$

$$j_5 = \frac{m^2\, p_2\cdot k_2\, q\cdot k_1}{(q\cdot p_1)^2},$$

$$j_6 = -\frac{p_2\cdot k_2\, q\cdot k_1}{q\cdot p_1}. \qquad \text{(E.20)}$$

From these the following J_i are obtained:

$$J_1 = -1 + \frac{\vec{p}_1\cdot\vec{p}_2}{p_1^{\,2}} + \frac{E_2}{p_2}\log\frac{E_2+p_2}{E_2-p_2} - \left(E_2 + \frac{\vec{p}_1\cdot\vec{p}_2 E_1}{p_1^{\,2}}\right)\frac{1}{2p_1}\log\frac{E_1+p_1}{E_1-p_1},$$

$$J_2 = \frac{-m^2\cosh\theta}{2\sqrt{s}\, p_1^{\,c}}\left\{\frac{4\theta}{m^2\sinh\theta}(y_2 - \sqrt{s}\, p_1^{\,c}) - \frac{2E_2}{p_2}\log\frac{E_2+p_2}{E_2-p_2} - 2\left(\frac{\vec{p}_1\cdot\vec{p}_2}{p_1^{\,2}} - 1\right)\right.$$
$$\left. + \left[\frac{2E_2}{p_1} + \frac{m^2(\epsilon + E_2 - E_1) - \sqrt{s}\, p_1^{\,c} E_1}{p_1^{\,3}}\right]\log\frac{E_1+p_1}{E_1-p_1}\right\},$$

$$J_3 = 1 - \frac{E_1}{p_1}\log\frac{E_1+p_1}{E_1-p_1} + \frac{1}{\sqrt{s}\, p_1^{\,c}}\left[\sqrt{s}\, E_2^{\,c} - m^2\cosh\theta\right]\left[\frac{E_2}{p_2}\log\frac{E_2+p_2}{E_2-p_2} - 1\right],$$

$$J_4 = -2 + \frac{\epsilon}{2p_2} \log \frac{E_2 + p_2}{E_2 - p_2}$$

$$J_5 = \frac{1}{\sqrt{s}\, p_1^{\ c}} \Bigg\{ E_2(\epsilon + E_2) - \frac{m^2 E_2}{2p_1} \log \frac{E_1 + p_1}{E_1 - p_1} + \frac{\vec{p}_1 \cdot \vec{p}_2}{p_1^{\ 2}} \Bigg[(y_1 + m^2) - \frac{(\epsilon + E_1)m^2}{2p_1} \log \frac{E_1 + p_1}{E_1 - p_1} \Bigg]$$

$$- \frac{(\vec{p}_1 \cdot \vec{p}_2)^2}{p_1^{\ 4}} \Bigg[E_1^{\ 2} + m^2 - \frac{E_1 m^2}{p_1} \log \frac{E_1 + p_1}{E_1 - p_1} \Bigg]$$

$$+ \frac{m^2}{p_1^{\ 4}} \Big[p_1^{\ 2} p_2^{\ 2} - (\vec{p}_1 \cdot \vec{p}_2)^2 \Big] \Bigg[1 - \frac{E_1}{2p_1} \log \frac{E_1 + p_1}{E_1 - p_1} \Bigg] \Bigg\},$$

$$J_6 = \frac{\epsilon}{4\sqrt{s}\, p_1^{\ c}} \Bigg\{ \frac{E_2(\epsilon + E_2)}{p_1} \log \frac{E_1 + p_1}{E_1 - p_1} - 2E_2 + \frac{\vec{p}_1 \cdot \vec{p}_2}{p_1^{\ 2}} \left(\frac{\epsilon E_1}{p_1} \log \frac{E_1 + p_1}{E_1 - p_1} - 2\epsilon \right)$$

$$- \frac{(\vec{p}_1 \cdot \vec{p}_2)^2}{p_1^{\ 4}} E_1 \left(\frac{E_1}{p_1} \log \frac{E_1 + p_1}{E_1 - p_1} - 2 \right)$$

$$- \frac{1}{p_1^{\ 4}} \left(p_1^{\ 2} p_2^{\ 2} - (\vec{p}_1 \cdot \vec{p}_2)^2 \right) \Bigg[E_1 - \frac{m^2}{2p_1} \log \frac{E_1 + p_1}{E_1 - p_1} \Bigg] \Bigg\}. \qquad \text{(E.21)}$$

The integration over $E_2^{\ c}$ can be done exactly, but as we do not use the exact results and they are very long, tabulating them here does not seem warranted.

APPENDIX F. THE HIGH ENERGY CONTRIBUTION OF $\tilde{B}$ FOR $\nu_e + e \to \nu_e + e + \gamma$

In the lab system, when the integrations over p_2 and k_2 have been done first,

$$\mathcal{J}' = \frac{1}{4\pi}\int_{-1}^{1} dx_{qk} \int_0^{q_m(x_{qk})} \frac{q^2 dq}{q_0} \int_{\varphi_2 \in R} d\varphi_2 \left| 1 - \frac{q}{k}\frac{\partial x_{2q}}{\partial x_{2k}} \right|^{-1} \left(- \frac{m^2}{(q\cdot p_1)^2} - \frac{2p_1\cdot p_2}{q\cdot p_1\; q\cdot p_2} - \frac{m^2}{(q\cdot p_2)^2} \right). \qquad \text{(F.1)}$$

$x_{qk} = \hat{q}\cdot\hat{k}_1$, $x_{2q} = \hat{p}_2\cdot\hat{q}$, and $x_{2k} = \hat{p}_2\cdot\hat{k}_1$. In $\mathcal{J}'$ x_{2k} is given by the conservation of energy and momentum for specified x_{qk}, q, and φ_2. (We have taken $\varphi_q = 0$.) The criterion that x_{2k} be real restricts the range of the subsequent integrations, φ_2 to some interval R contained in the interval $[0, 2\pi]$ and q to be $\leq q_m(x_{qk})$, when the order of the next integration is as specified in (F.1).

We wish to compare $\mathcal{J}'$ with

$$2\pi\,\tilde{B} = \frac{1}{4\pi}\int_{-1}^{1} dx_{qk} \int_0^{q_m(x_{qk})} \frac{q^2 dq}{q_0} \int_0^{2\pi} d\varphi_q \left(- \frac{m^2}{(q\cdot p_1)^2} - \frac{2p_1\cdot p_2}{q\cdot p_1\; q\cdot p_2} - \frac{m^2}{(q\cdot p_2)^2} \right), \qquad \text{(F.2)}$$

where p_2 is fixed by the elastic kinematics, $E_2^L = m\cosh\theta$, and $x_{2k} = (m + k)/k \tanh\theta/2$. (The arbitrarily chosen $\varphi_2 = 0$ here is interchangeable with φ_q.) Correspondence of the dominant terms at high energy and large momentum transfer of $\mathcal{J}'$ and $2\pi\,\tilde{B}$ means that the restriction of the φ_2 integration in $\mathcal{J}'$, the phase space factor

$\left| 1 - \frac{q}{k} \frac{\partial x_{2q}}{\partial x_{2k}} \right|^{-1}$, and the dependence on q in x_{2k} in $\mathcal{J}'$ do not affect the logarithmic terms.

First let us demonstrate the correspondence for the two cases discussed in Chapter V. The limits of $\mathcal{J}'$ are given there. We need to compute the appropriate $q_m(x_{qk})$ and the dominant terms of $2\pi\,\widetilde{B}$. Yennie, Frautschi, and Suura give the limit of $2\pi\,\widetilde{B}$ for q_m isotropic when the initial and final electron have the same energy.

In the lab system, in which the initial electron is at rest, most of the integrations in $2\pi\,\widetilde{B}$ can be done for general $q_m(\Omega_q)$. We let $\vec{\beta} = \frac{\vec{p}_2}{E_2}$, $\vec{v} = \frac{\vec{q}}{q_0}$, and $x = \hat{v}\cdot\hat{\beta}$. Then we have

$$2\pi\,\widetilde{B} = \frac{1}{4\pi}\int d\Omega_q \int_0^{q_m} \frac{q^2 dq}{q_0} \frac{\beta^2(1 - v^2x^2)}{(1 - v\beta x)^2} \; . \qquad \text{(F.3)}$$

If the q integration is transformed to integration over v, this becomes

$$2\pi\,\widetilde{B} = \frac{1}{4\pi}\int d\Omega_q \beta^2 \int_0^{1-\epsilon} \frac{v^2 dv}{1 - v^2} \frac{1 - v^2x^2}{(1 - v\beta x)^2} \; , \qquad \text{(F.4)}$$

where $\epsilon = \lambda^2/2q_m^2$, λ being the small photon mass. For the integral over v we have

$$\frac{\partial}{\partial(\beta x)} \int_0^{1-\epsilon} \frac{vdv}{1 - v^2} \frac{1 - v^2x^2}{(1 - v\beta x)}$$

$$= \frac{1}{2}\left(\log \frac{2q_m^2}{\lambda^2}\right) \frac{1 - x^2}{(1 - \beta x)^2} + \left[\frac{1}{2}(\log 2) \frac{1 - x^2}{(1 + \beta x)^2} - \frac{(1 - x^2)}{(1 - \beta x)(1 - \beta^2x^2)} + \frac{2\beta x(1 - x^2)}{(1 - \beta^2x^2)} \log (1 - \beta x)\right.$$

$$+ \frac{1}{\beta^2}\left(1 + \frac{1}{1-\beta x}\right) + \frac{2}{\beta^3 x}\log(1-\beta x)\Big] . \qquad \text{(F.5)}$$

Then

$$2\alpha\,\widetilde{B} = \frac{\beta^2}{4\pi}\int d\Omega_q \left(\log\frac{q_m}{\lambda}\right)\frac{1-x^2}{(1-\beta x)^2} + C, \qquad \text{(F.6)}$$

where in terms of θ defined by $t = -4m^2\sinh^2\theta/2$ and using (B.13) to simplify the expression,

$$C = C(\theta) = 1 - 2\log 2 + (2\tanh\theta)^{-1}\Big\{\bar{L}(e^{2\theta}) - L(e^{-2\theta})$$

$$+ 2\theta\left[1 - 2\log(\sinh\theta)\right]\Big\} \qquad \text{(F.7)}$$

This is the result Lee and Sirlin[13)] give for C in $\mathcal{J}^{\text{soft }\gamma}$ $(= 2\alpha\,\widetilde{B})$.

For $\theta \gg 1$

$$C(\theta) \sim \theta + 2\theta\log 2 - \theta^2. \qquad \text{(F.8)}$$

The remaining integral is

$$\left[\theta\,\text{ctnh}\,\theta - 1\right]\log\frac{m^2}{\lambda^2} + \mathcal{J}_0$$

where

$$\mathcal{J}_0 = \frac{\beta^2}{4\pi}\int d\Omega_q\left(\log\frac{q_m}{m}\right)\frac{1-x^2}{(1-\beta x)^2} . \qquad \text{(F.9)}$$

Altogether

$$2\alpha\,\widetilde{B} = \left[\theta\,\text{ctnh}\,\theta - 1\right]\log\frac{m^2}{\lambda^2} + \mathcal{J}_0 + C(\theta). \qquad \text{(F.10)}$$

We found it easiest to compute $q_m(x_{qk})$ from the conditions set by the delta functions when the p_2 and k_2 integrations are done in the frame in which $\vec{p}_2 + \vec{k}_2 = 0$. The delta functions of energy,

$\delta(E_1 + k_1 - q_0 - E_2 - k_2)$, and of t, $\delta(t - (p_2 - p_1)^2)$ restrict the integrations over the q momenta to leave, in the $\vec{p}_2 + \vec{k}_2 = 0$ frame,

$$\mathcal{E} = E_1 + k_1 - q_0 \geq m \qquad \text{(F.11a)}$$

and

$$\left|\hat{p}_1 \cdot \hat{p}_2\right| \leq 1. \qquad \text{(F.11b)}$$

In terms of invariants these conditions become

$$-(p_1 + k_1 - q)^2 \geq m^2 \qquad \text{(F.12a)}$$

and

$$\mathcal{Y}_1^2 + \mathcal{Y}_2^2 - 2\mathcal{Y}_1\mathcal{Y}_2 \cosh\theta + m^2\mathcal{E}\sinh^2\theta \leq 0, \qquad \text{(F.12b)}$$

where

$$\mathcal{Y}_1 = -p_1 \cdot (p_1 + k_1 - q) \text{ and } \mathcal{Y}_2 = \frac{\mathcal{E}^2 + m^2}{2} .$$

In terms of the lab system variable (photon energy q, incident neutrino energy k, $x_{qk} = \hat{q} \cdot \hat{k}$), the conditions are

$$q \leq \frac{mk}{m + k(1 - x_{qk})} \qquad \text{(F.13a)}$$

and

$$-\left[m \sinh\varphi\, e^{\varphi} - \frac{k(1 - x_{qk})}{2}\right]\left[q - q^{+}\right]\left[q - q^{-}\right] \leq 0, \qquad \text{(F.13b)}$$

where

$$q^{\pm} = \frac{m\, e^{\mp\varphi} \sinh\varphi \left[k \pm m\, e^{\mp\varphi} \sinh\varphi\right]}{m\, e^{\mp\varphi} \sinh\varphi \pm \frac{k(1 - x_{qk})}{2}} . \qquad \text{(F.14)}$$

The conditions are satisfied for $0 \leq q \leq q_m(x_{qk})$, where

$$q_m(x_{qk}) = \begin{cases} q^+(x_{qk}) & x_{qk} \leq x_0 \\ q^-(x_{qk}) & x_0 \leq x_{qk}, \end{cases} \tag{F.15}$$

with

$$x_0 = 1 - \frac{2m^2}{k} \sinh^2\varphi / \left[k - 2m^2 \sinh^2\varphi\right]. \tag{F.16}$$

$q^+(x_0) = q^-(x_0)$ is the maximum photon energy at any angle and corresponds to the final neutrino having zero energy.

In the limit $a \gg 1$ and $\gamma \ll 1$ for high energy and maximum momentum transfer, the maximum lab photon energy is

$$q_m(x_{qk}) \sim \begin{cases} \frac{m}{2} & x_{qk} \leq x_0 = -1 + 2\gamma \\ \frac{m\gamma}{1 + x_{qk}} & x_0 = -1 + 2\gamma \leq x_{qk}. \end{cases} \tag{F.17}$$

In the c.m. system even more than in the lab system, $q_m(x_{qk})$ is peaked at $x_{qk} = -1$.

$$\mathcal{J}_0 = \frac{\beta^2}{2} \int_0^{2\pi} \frac{d\varphi_q}{2\pi} \left\{ \int_{-1}^{x_0} dx_{qk} \left(\log \frac{q^+(x_{qk})}{m}\right) \frac{1 - x_{2q}^2}{(1 - \beta x_{2q})^2} + \int_{x_0}^{1} dx_{qk} \left(\log \frac{q^-(x_{qk})}{m}\right) \frac{1 - x_{2q}^2}{(1 - \beta x_{2q})^2} \right\}. \tag{F.18}$$

Here

$$x_{2q} = x_{qk}\, x_{2k} + \sqrt{1 - x_{qk}^2}\,\sqrt{1 - x_{2k}^2}\, \cos\varphi_q,$$

and

$$x_{2k} = \frac{m + k}{k} \tanh\frac{\theta}{2} \sim 1 - \frac{2\gamma}{a}.$$

The contribution of the first integral in $\mathcal{J}_0$ vanishes as $\gamma \to 0$. Taking the limit $\gamma \to 0$ for fixed large a, $x_{2k} \to 1$ and $x_{2q} \to x_{qk}$, so that

$$\mathcal{J}_0 \to \frac{\beta^2}{2} \int_{-1}^{1} dx \left(\log \frac{\gamma}{1+x}\right) \frac{1 - x^2}{(1 - \beta x)^2} .$$

$$= (\theta \operatorname{ctnh} \theta - 1) \log \frac{\gamma^2}{4} + \frac{1}{2} \int_{-1}^{1} dx \left(\log \frac{1+x}{2}\right) \frac{1 - x^2}{(1 - \beta X)^2}. \tag{F.19}$$

The last term can contribute no logarithmic terms when $\beta \to 1$ for $\theta \sim \log a \gg 1$. With $C(\theta)$ from (F.8) and (F.10) for $2\alpha \tilde{B}$ we have in this limit

$$2\pi \tilde{B} \to (\theta - 1) \log \frac{m^2}{\lambda^2} + (\theta - 1) \log \frac{\gamma^2}{4} + \theta + 2\theta \log 2 - \theta^2$$

$$= (\theta - 1) \left[\log \frac{m^2}{\lambda^2} + \log \frac{\gamma^2}{a}\right], \tag{F.20}$$

which is the same as (V.25) for $\mathcal{J}'$.

Turning to the case where not only $a \gg 1$ and $e^{\theta} \gg 1$, but also $a - e^{\theta} \gg 1$, we have, with $x_0 \to 1$,

$$q_m(x_{qk}) \sim \begin{cases} \dfrac{m}{1 - x_{qk}} & x_{qk} \leq x_0 \\ \dfrac{ma}{2} & x_0 \leq x_{qk} . \end{cases} \tag{F.21}$$

This maximum photon energy distribution, peaked in the forward direction, corresponds to an isotropic distribution in the center-of-momentum (c.m.) system of the initial neutrino and electron. In the c.m. system the maximum photon energy would be $\sqrt{s}/2$, which is the same as $E_1^{\,c}$ in the limit $a \gg 1$. With this value the results of Yennie, Frautschi, and Suura give

$$2\pi\,\widetilde{B} \sim (\theta - 1)\,\log \frac{m^2}{\lambda^2} + \frac{\theta^2}{2}\,. \qquad \text{(F.22)}$$

This is the same as the contribution of $\mathcal{J}'$ to $\mathcal{J}$ in (V.26).
The dominant terms of $\mathcal{J}_0$ can also be found in the lab system for this case and with the other contributions to $2\pi\,\widetilde{B}$ give (F.22).

In these limits $\mathcal{J}'$ corresponds to $2\pi\,\widetilde{B}$ because over most of the important range of integration the high energy and very forward electron is little influenced by the kinematic restrictions due to the presence of the photon. The φ_2 integration in $\mathcal{J}'$ begins to be restricted when $q(x_{qk})$ is large enough for $x_{2k} = 1$ to be possible. (This can be seen graphically.) The restrictions begin for

$$q'(x_{qk}) = \frac{m \sinh\varphi\, e^{-\varphi}\left[k - m \sinh\varphi\, e^{\varphi}\right]}{m \sinh\varphi\, e^{-\varphi} + \frac{k - m \sinh\theta}{2}\,(1 - x_{qk})}\,.$$

Here, as before, $\varphi = \theta/2$. For $a \gg e^{\theta} \gg 1$, $q'(x_{qk}) \to q_m(x_{qk})$.
For $a \sim e^{\theta} \gg 1$, $q'(x_{qk}) \to q_m(1)$. Because of the matrix element it is the forward direction (about parallel to $\vec{p}_2$) which is important. Thus, in these limits, over most of the q range, φ_2 is unrestricted. The restriction of φ_2 corresponds to the fact that for $q(x_{qk}) > q'(x_{qk})$ the phase space factor $\left|1 - \frac{q}{k}\frac{\partial x_{2q}}{\partial x_{2k}}\right|^{-1}$ has a singularity at the boundary of the allowed φ_2 variables. But for $q \lesssim q'$ the phase space is about that of a photon indpendent of the final electron and neutrino. In the matrix element the dependence of x_{2q} on q is small

for $\vec{q}$ approximately parallel to $\vec{p}_2$ and the integration over φ_2 tends to average out the difference.

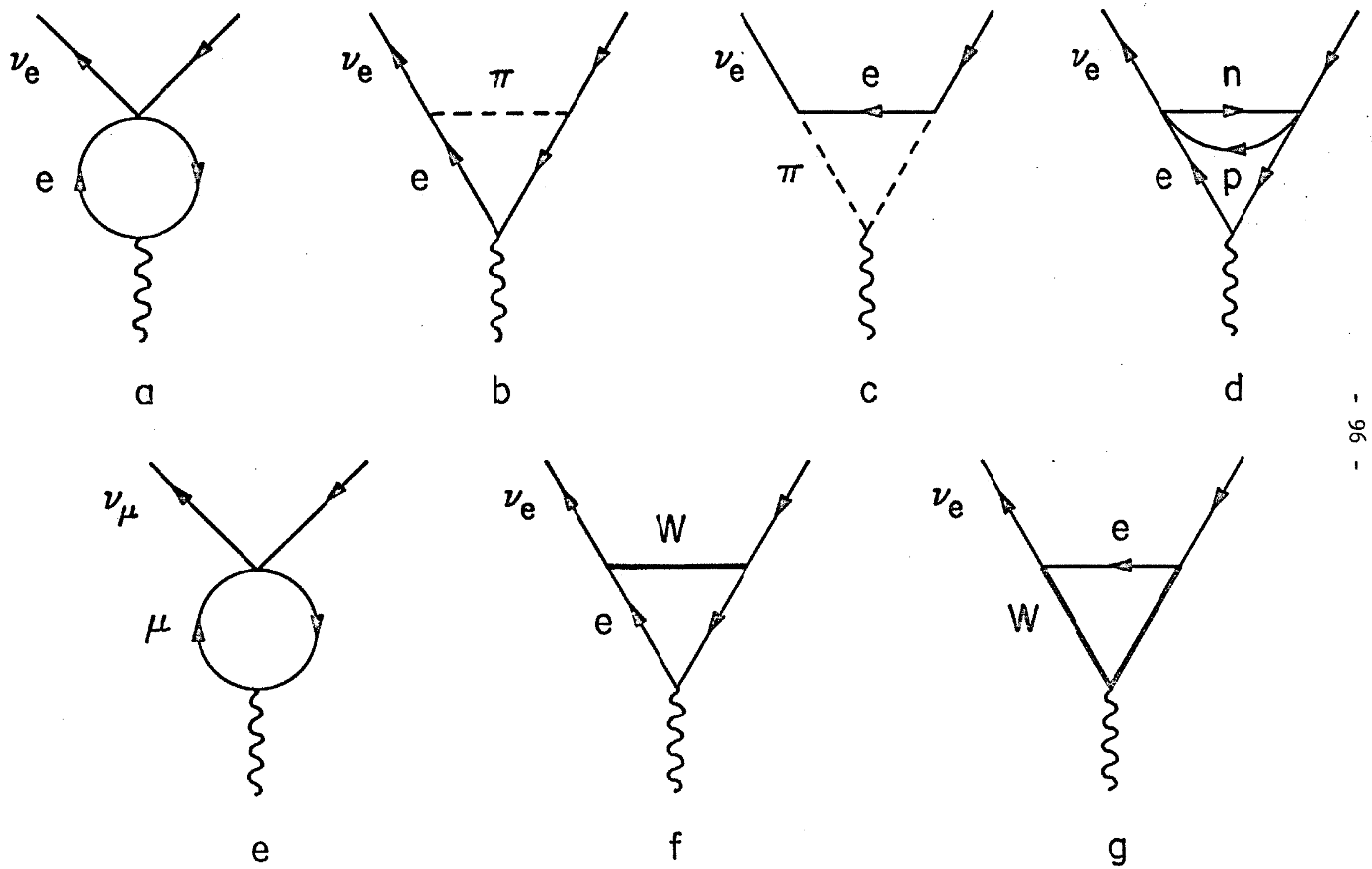

Figure 1

Processes which could contribute to the charge distribution of neutrinos.

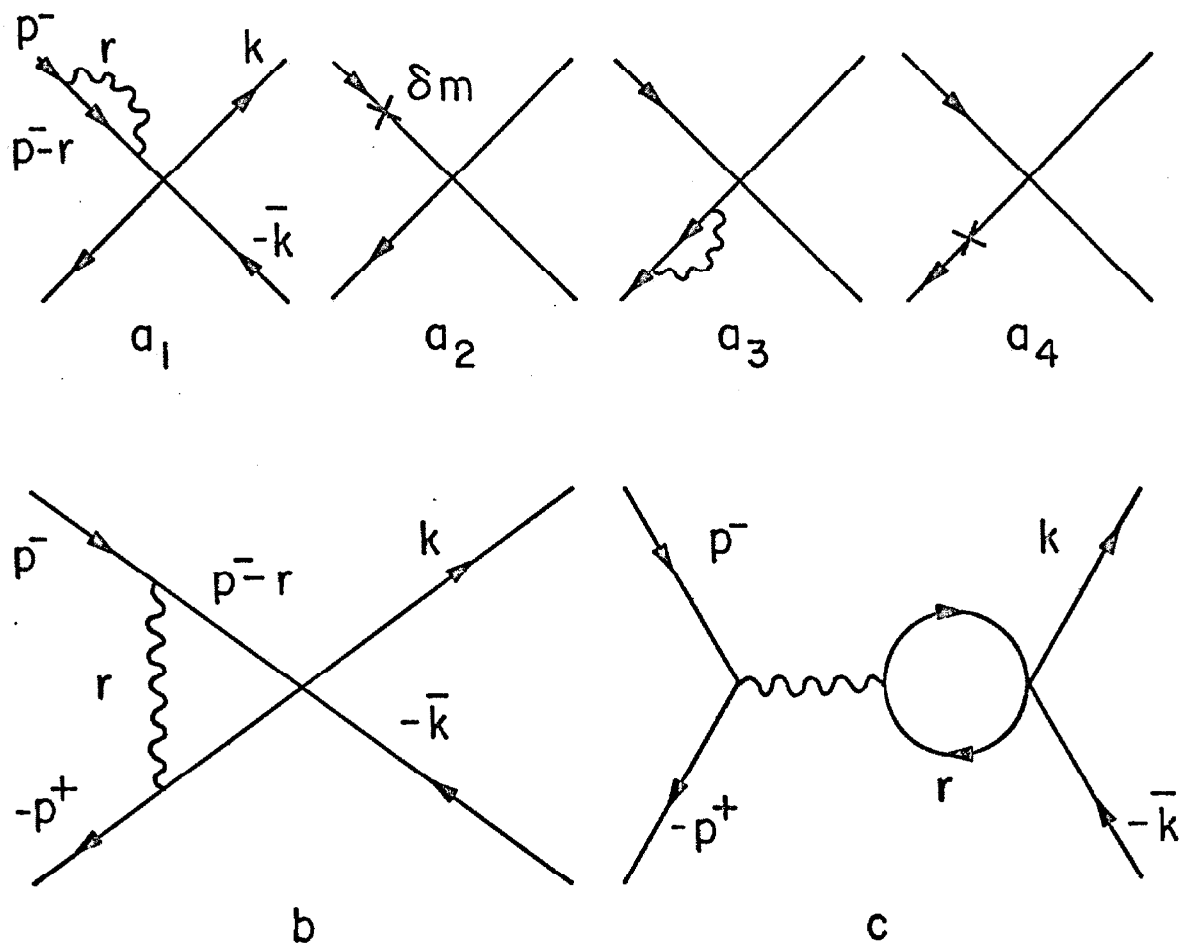

Figure 2

Diagrams for the virtual photon corrections to $e^- + e^+ \rightarrow \nu_e + \bar{\nu}_e$.

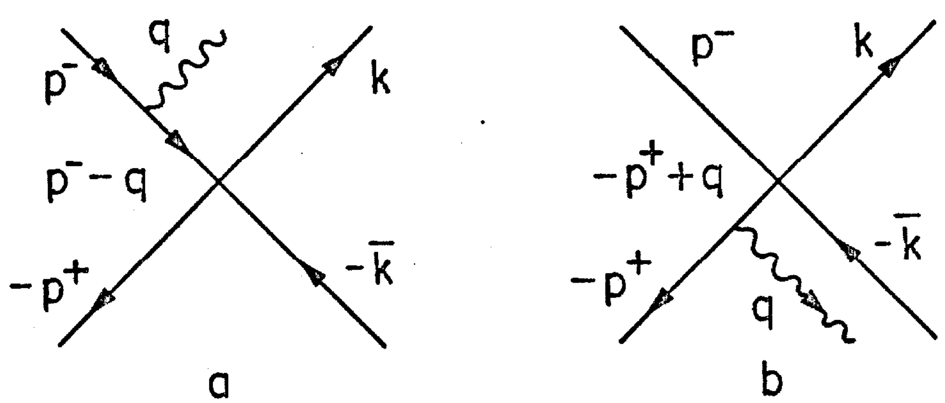

Figure 3

Diagrams for the bremsstrahlung corrections to $e^- + e^+ \rightarrow \nu_e + \bar{\nu}_e$.

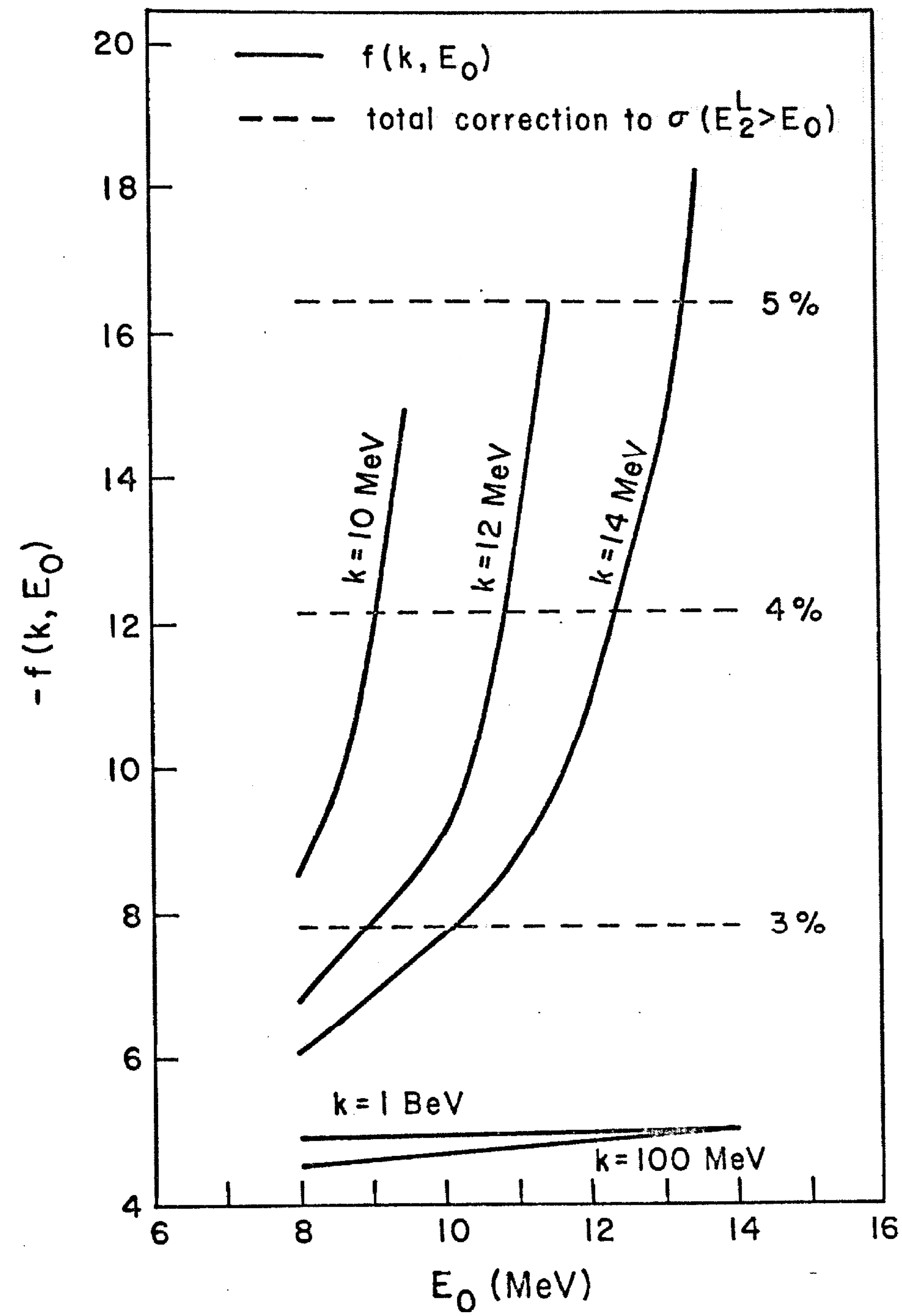

Figure 4

Radiative corrections to $\sigma(E_2^L > E_0)$ for $\nu_e + e \rightarrow \nu_e + e$. $f(k,E_0)$ needs modification when $k - E_0 \sim m$. The percent total correction is for $\Lambda \sim m_N$.

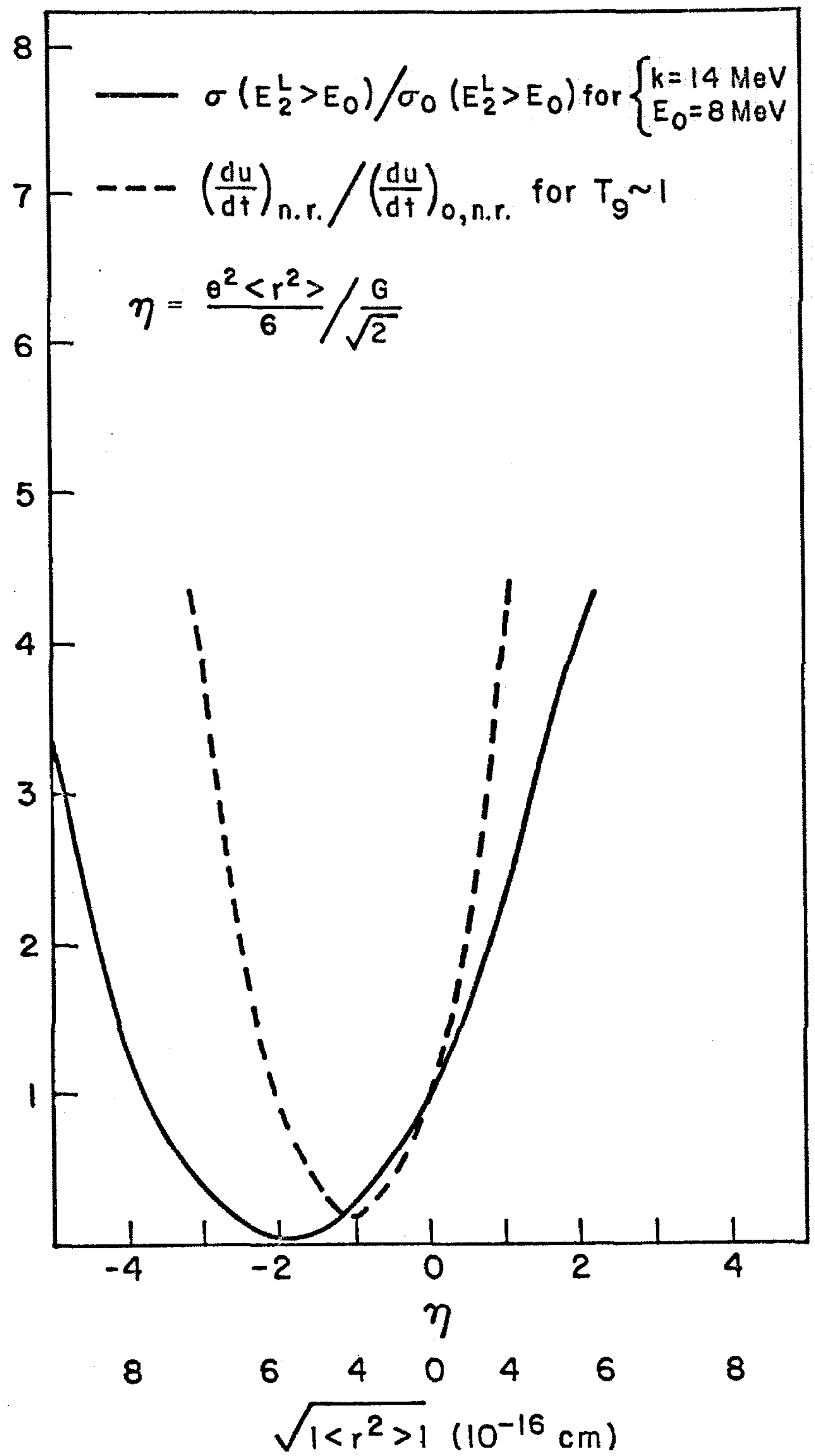

Figure 5

The dependence of $\sigma(E_2^L > E_0)$ and $(du/dt)_{n.r.}$ on the neutrino charge radius.

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