IONIZATION PHENOMENA IN A GAS - PARTICLE PLASMA

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ABSTRACT

Several interesting and practically important types of ionization phenomena which occur in a plasma that is composed of thermionically-emitting particles and an ionized gas, have been investigated. There are many interactions that can take place between the particles and the gas which are able to alter the electron density of the plasma appreciably from what it would be in the absence of the particles. Several of these interactions have been explored. Throughout all of the analysis, the emphasis has been placed on gaining a physical understanding of the basic phenomena which are involved.

In order to determine the nature of the potential and charge distributions which exist in a gas - particle plasma, the problem in which there is no gaseous ionization and equilibrium prevails has first been thoroughly investigated. Using a family of numerical solutions to Poisson's equation, it has been shown that these distributions can be divided into two characteristic regimes and that a simple algebraic expression, which has been derived, is a good approximation to the potential distribution in one of them. A readily applied method of calculation of the electron density in the plasma and a study of the dependence of this density on the initial parameters which enter the problem have been presented.

The relations which are required in order to analyze non-equilibrium ionization phenomena in a gas - particle plasma have been formulated and then applied to various special cases. The case which has received the major emphasis is that in which the particles are

hotter than the gas and an enhancement in the gaseous ionization results. It has been shown that this enhancement could be quite large. Electron absorption by the particles, particle quenching of the gaseous ionization, and the supression of either particle or gas ionization, due to the presence of the other, have also been investigated.

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I. INTRODUCTION

A. Background

At present, there is a considerable amount of interest in the academic, military, and commercial fields in the physics of plasmas. This interest centers on the capability of a plasma, a gas with high concentrations of electrically-charged species, to interact with an electromagnetic field. The attention is usually directed towards either the action of the electromagnetic field on the plasma; as in a plasma accelerator, a plasma power generator, or the modification of re-entry flow fields; or the attenuation and scattering of the field by the plasma, as in signal transmission through a plasma structure, radar tracking and discrimination of re-entry bodies, or microwave diagnostics. In all cases, the strength of the interaction is largely determined by the magnitude of the electron density in the plasma, and any phenomenon which could appreciably alter this density is worth considerable attention.

Several such phenomena arise directly from the fact that the degree of ionization of a slightly-ionized gas in equilibrium varies approximately as $e^{-V/kT}$ where V is the ionization potential, k is Boltzmann's constant, and T is the temperature. Variations in the temperature of the entire system, variations in just the electron temperature (produced by the application of an electrogmanetic field) or changes in the effective ionization potential (produced by seeding with an alkali metal vapor) are all able to produce relatively large changes in the electron density. Recently, attention has been directed towards phenomena arising in plasmas that are composed of two states of matter, gas-solid or gas-liquid, in which the non-gaseous state is in the form of small particles, 10^{-6} to 10^{-3} cm in size[1]. The primary consideration here is the magnitude of the electron density existing in this type of heterogeneous plasma relative to that which would exist in a pure gaseous plasma. In many situations, some in which the thermionic emission from the particles is of importance and others in which it is not, the electron density may be considerably altered due to the presence of the particles. The analysis of ionization phenomena in a gas-particle plasma is the subject of this investigation.

Gas-particle plasmas arise in many areas in which the interaction with an electromagnetic field is, or could be, of large importance. Several of these areas are mentioned below.

1. The combustion products of a solid rocket motor contain alumina particles approximately 10^{-4} cm in diameter. The detection and communication difficulties associated with the high electron densities in the exhaust plume are well known.

2. In MHD power generators which employ an alkali metal vapor as the working fluid, condensation of a fraction of the gas into liquid particles can occur in the nozzle used to accelerate the gas into the electrode section. The problem here is with the alteration of the electrical conductivity of the plasma due to the presence of these

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[1] The word "particle" will always be used here to denote particles of this magnitude rather than those of atomic dimensions.

particles.

3. In fuel-rich hydrocarbon flames, the unusually large electron concentrations observed have been attributed in some cases to the thermionically-emitting carbon particles that were formed during combustion.

4. In astrophysical investigations of interstellar matter, the interaction between small dust particles ($\sim 10^{-5}$ cm) and free electrons has been considered.

5. The electron density in wakes of re-entry bodies may be influenced by the presence of particles.

One of the first studies which examined an interaction between charged particles and an electron gas was done by Spitzer [references 1 and 2, 1941 and 1947, respectively][2], who was concerned with the physical properties of the interstellar medium, in particular, those which are influenced by the presence of small dust particles. In the course of this work, he calculated a kinetic-theory capture cross section, taking a "sticking factor" into account, for the electron-charged particle collision.

In 1949, Parker and Wolfhard [3] investigated the formation of carbon particles in hydrocarbon diffusion flames and found them to be of graphitic structure with diameters of $10^{-6} \rightarrow 5 \times 10^{-6}$ cm. Sugden and Thrush [4] experimentally investigated the unusually high electron density in fuel-rich hydrocarbon flames and attributed it to

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[2] The references indicated are listed at the end of the text.

thermionic emission from the carbon particles formed. They noted that an additional term should be added to the thermionic work function to account for the positive potential built up on the particle due to its loss of electrons. Shuler and Weber [5] also measured the electron density in flames containing carbon particles and attributed their findings to thermionic emission. However, in their analysis they erroneously assumed that the energy for an electron to escape a particle was a constant; that is, that the charge of the particle could be neglected.

In 1956, Einbinder [6] pointed out that in a system of thermionically-emitting particles, even if all particles are identical, the charge on any two particles need not be the same, and he calculated the equilibrium statistical distribution of particles over the "ionized states". Unfortunately though, in his analysis he neglected the possibility of electron absorption by a particle, which was in essence a neglect of all of the "negatively ionized states". Smith [7, 8] corrected this oversight, made a more rational definition of the first ionization potential of a particle, and greatly simplified the statistical calculation of the electron density. He showed that the distribution of particles over the ionized states is a Gaussian and also derived a relation for the electron density in a system in which both thermionically-emitting particles and an ionized gas are present. Millikan [9] measured the gas and particle temperatures in an atmospheric pressure,premixed ethylene-air flame and found that, within experimental accuracy, they were almost the same. He thus concluded that the radiant-energy loss from the particles and the energy transferred to the particles by surface reactions were not large enough to displace the gas and particles far from thermal equilibrium. Also, from electron microscope pictures of the particles, one of which is displayed in his paper [9], he obtained particle sizes of 10^{-6} to 4×10^{-6} cm. From the picture presented, it is seen that the particles were approximately spherical in shape.

Sodha [10, 11] used a kinetic theory approach to analyze the thermal and photoelectric ionization of a system of solid particles and studied the propagation of electromagnetic waves through such a medium. Rosen [12] has proposed that in a plasma composed of an ionized gas and particles which do not thermionically emit, it is possible for the particles to absorb or capture an appreciable fraction of the total number of electrons produced by gaseous ionization and thereby greatly reduce the electron density below the value it would assume if the particles were not present. However, Allport and Rigby [13] have pointed out that, in practice, it is extremely difficult to find particles which could act as electron absorbers in a high-temperature gas plasma ($\sim 3000^{\circ}$K) because, at high temperatures, most materials become good thermionic emitters or introduce metal vapors into the system which ionize and actually augment the electron density. Soo [14, 15] has recognized that, in many situations, the potential around a particle cannot be assumed to vary simply as $1/r$ and that, when the electron distribution outside the particles can be assumed to be continuous, the potential distribution should really be found by treating the situation as a shielding problem.

In addition, he considered the interesting and practically-important problems of electron-density variation due to initially-charged particles, the electrical conductivity of a gas-particle medium, and the time for the equilibrium-charge distribution to be reached. However, there are several fundamental errors in his work; a major one is found in equation 1.1 in reference 15. This error is discussed in Appendix B.

Carlson [16] has experimentally demonstrated that in a gas-particle nozzle flow, an appreciable lag may exist between the particle temperature and the gas temperature, the latter of which is changing rapidly. Spokes and co-workers [17] have been studying in detail the degree of particle ionization to be expected in a gas-alumina particle plasma which is formed by the combustion products of a solid rocket engine.

The capability of the particles to alter the electron density of the gaseous medium is not the only interaction of importance in a gas-particle plasma. When a system's temperature and momentum are changing in time, as in the case of nozzle flow, the changes in the particle properties will lag those of the gas, and the transfer of energy between random thermal motion and directed motion will be modified. Extensive studies on this type of gas-particle interaction have been carried out by Kliegel [18], Hogland [19], Rannie [20], and Marble [21, 22]. However, in all of the analytical studies so far mentioned on the level of ionization in a gas-particle plasma, it has been assumed that fluid-mechanical interactions of this type and interactions directly affecting the ionization are uncoupled so that the temperatures and densities of the particles and the gas are considered to be known parameters. This assumption will be discussed in Part II.

B. Objective of this Investigation

The main objective of this investigation is to develop both qualitative and quantitative understanding of ionization phenomena in a gas-particle plasma. Of special interest are those phenomena in which the electron density of the plasma may be appreciably altered due to the presence of the particles.

In Part II, the physical phenomena of primary importance are discussed. Part III considers the special case in which the particles are ionized by thermionic emission but where there is no gaseous ionization. In addition to developing a means by which the level of ionization can readily be determined, a great deal of emphasis is placed on understanding the nature of the potential and charge distributions which surround a particle. In Part IV, the sensitivity of the level of ionization, in the special case just considered, to variations in the parameters of the system is investigated. The parameters which are considered are the thermionic work function, the temperature, the particle size, and the particle number density.

The special case in which there is no gaseous ionization is one of two extreme cases, the other being that in which there is no thermionic emission from the particles and only the gas is ionized. In Part V, the additional relations which are required to analyze this second limiting case, as well as several other interesting cases lying between the two extremes, are introduced. This formulation is directed primarily towards the analysis of a gas-particle interaction by which the electron density in the plasma may be made much larger than that which would be expected from either pure gas or pure particle ionization alone. That is, when the thermionically-emitting particles are hotter than the gas, the electron temperature is also higher than that of the gas, and the gaseous ionization is enhanced. The assumptions and model used to investigate this phenomenon will first be discussed. Then, the calculation of the electron number and energy fluxes to and from a particle, a problem interesting in its own right, will be investigated. In Part VI, the relationships which have just been derived and discussed will be employed to analyze several special cases of interest, including that of gaseous-ionization enhancement as well as two others in which the electron density may be severely reduced by the presence of the particles.

In order that this investigation be limited to a tractable length and scope, the gas-particle plasma is treated as a closed system. That is, ionization phenomena which arise due to the action of outside influences are neglected. Interesting interactions thus excluded are photoionization of the gas, photoelectric ionization of the particles, and enhanced gaseous ionization through electron heating with an electromagnetic field. An attempt has been made to carry out a realistic investigation and, at the same time, to keep the formulation of each problem simple in order that greater physical understanding may be obtained from the results.

II. PHYSICAL PHENOMENA OF PRIMARY IMPORTANCE

In order to further introduce the problems which are to be investigated, several of the more important physical phenomena encountered in analyzing the ionization in a gas-particle plasma will be considered. The discussion will be directed mainly toward the problem in which only particle ionization is important. The additional effects of gaseous ionization will be considered more fully in Part V.

A. Thermionic Emission

Thermionic emission may be pictured crudely as a "boiling out" of electrons from within a solid or liquid. Due to their thermal motion, a certain fraction of the electrons in the conduction band will have sufficient energy to penetrate the potential barrier existing at the surface of the material and will escape.

The methods of statistical mechanics may be applied to the calculation of the electron density immediately outside the surface of a conducting material; however, four main assumptions are involved. First, complete thermodynamic equilibrium is assumed. Second, the interelectronic forces are ignored, and the electrons are regarded as a free electron gas in which each electron has three translational degrees of freedom. Third, the electrons are treated as Fermi particles so that the Pauli exclusion principle is applicable. Only two electrons may be in the same quantum state, and these must have anti-parallel spin vectors. Last, each quantum state occupies an extension or volume in phase space equal to $1/h^3$. The calculation of the electron density immediately outside of a thermionically-emitting surface, n_E , is now straightforward [see reference 23, pp. 18-22], and yields

$$n_E^{(th)} = 2\left[\frac{2\pi mkT}{h^2}\right]^{3/2} e^{-e\Phi/kT} \quad (2.1)$$

where

$n_E^{(th)}$ = theoretical value of electron density resulting from thermionic emission,

m = electron mass,

h = Planck's constant,

$e\Phi$ = thermionic work function.

It is seen that the density of electrons outside the surface is proportional to the Boltzmann factor, $e^{-e\Phi/kT}$, even though the electrons inside the material have a Fermi-Dirac energy distribution. This arises from the fact that the high-energy tail of the Fermi-Dirac distribution is nearly equivalent to a Maxwellian distribution in its energy dependence and, since $e^{e\Phi/kT} >> 1$, those electrons which escape are from this high-energy tail.

Several objections may be raised to the direct application of this formula.

1) Some electrons which classically have enough energy to escape will be reflected by the potential barrier.

2) There will be some anisotropy of electron motion introduced due to crystal lattice effects.

3) In many cases, surface contamination and surface irregularities exist.

4) The work function is found to be temperature dependent.

Despite these objections, an equation having the general form of equation (2. 1) is found to be valid, for most metals and many non-metals, within temperature increments large enough to be useful; however, the work function and multiplicative constant must be found from experiment. Thus, the experimental value of n_E is given by

$$n_E^{(ex)} = B^{(ex)} T^{3/2} e^{-e\Phi/kT} \qquad (2.2)$$

where $B^{(ex)}$ and $e\Phi$ are experimental values.

The experiments which are carried out measure the saturation current due to thermionic emission from a flat, clean surface. Theoretically,

$$J = \frac{e}{4} n_E \bar{C}_e , \qquad (2.3)$$

where J is the saturation value of the current density, e is the electron charge, $\bar{C}_e$ is the mean electron speed, and n_E is the electron density for the full distribution which is assumed to be isotropic. Hence, n_E is directly proportional to J, and from

$$J^{(ex)} = A^{(ex)} T^2 e^{-e\Phi/kT} \qquad (2.4)$$

$B^{(ex)}$ may be determined:

$$B^{(ex)} = \frac{A^{(ex)}}{A^{(th)}} B^{(th)} . \qquad (2.5)$$

Relatively small errors will be introduced in this indirect solution for $B^{(ex)}$ because, in practice, the velocity distribution is not isotropic and the mean electron speed is not exactly equal to $\left(\frac{8kT}{\pi m}\right)^{\frac{1}{2}}$, usually due to a deficiency in low-energy electrons [reference 23, p. 17].

In applying equation (2. 2) to emission from particles in a gaseous medium, several additional factors must be considered.

First, when there is a net loss of electrons from a particle due to thermionic emission, the number of electrons in the conduction band will be decreased. It will be assumed here that this loss is negligible. That is,

$$z << a^3 n_e^{(cond)} \tag{2.6}$$

where

z = net number of electrons lost from the initially neutral particle,

a = particle radius,

$n_e^{(cond)}$ = electron density in the conduction band of the particle material.

Assuming that $n_e^{(cond)} = 10^{19}$ cm^{-3} for an insulator and $a = 10^{-4}$ cm, the loss from the conduction band will be negligible if z is very much less than 10^7, a condition which is always well satisfied. However, in the case of very small particles at high degrees of ionization, this assumption should be re-examined.

Second, the surface charge on a particle will affect its emission properties. But, if the number of charges is much less than the number of atoms at the particle surface, this effect will be negligible. That is, if

$$z << (a/d)^2 , \tag{2.7}$$

where d is the interatomic spacing, the work function will be unaltered due to this effect. For $d = 10^{-8}$ cm and $a = 10^{-4}$ cm, this requires that z be much less than 10^8, a requirement that, as in the previous consideration, is easily satisfied. The means by which the emission properties are changed is through the alteration of the individual short-range atomic forces acting on the electron as it passes through the surface. However, the surface charge, when acting as a uniform continuous layer, would not alter the work function because the potential across a single layer of charge is continuous.

Third, if a gas is absorbed on the particle surface, the work function may be altered by the formation of a dipole layer, across which the potential is not continuous. At high temperatures, it would be expected that gas absorption would be small, but further investigation of this point for each gas-particle system is required.

Last, n_E has been defined as the electron density "immediately outside" the surface. To what degree may the long-range potential contributing to the work function, the mirror-image potential, be separated from the potential which arises due to the net charge on the particle so that this definition is physically meaningful? To gain an estimate of the magnitudes involved, the characteristic length of the mirror-image potential will be compared with the particle radius. The distance from the surface at which the mirror-image potential falls off to .1 ev is calculated to be 3.6×10^{-7} cm. In most cases, the particle potential will be much larger than .1 volt, so that when the characteristic length of the potential sheath around the particle is much larger than 10^{-7} cm, a distinct separation of the two fields may be made. In Part III, it will be shown that the characteristic length of the potential sheath around very small particles is usually the particle radius. This point is further investigated in Appendix F.

From this discussion, it is seen that the calculation of the density of electrons being thermionically emitted from a surface may be complicated by many factors, and in some cases, because of the lack of complete knowledge of the phenomena involved, the results are only speculative. Thus, it may appear that the results of an analysis, which predicts the value of the electron density in the plasma using n_E as an input parameter, would be open to serious question. However, it will be shown that, due to the shielding of the particles, the value of the mean electron density is usually very insensitive to the value of n_E, so that errors made in the calculation of n_E will introduce only relatively small errors in the estimation of the degree of particle ionization in the system.

B. Equilibrium

When complete thermodynamic equilibrium exists in the gas-particle plasma, the value of the local electron density outside of a particle will be proportional to the Boltzmann factor, $e^{e\Phi/kT}$, where Φ is the local value of the electrostatic potential. In theory, the establishment of a Maxwellian energy distribution for the electrons is not dependent upon electron-electron collisions outside of a particle because the energy distribution of the electrons thermionically emitted from the particles is already Maxwellian and will not change as long as the gas is in thermodynamic equilibrium at the particle temperature. Also, the velocity distribution at the particle surface will be isotropic because the half-Maxwellian distribution of the emitted electrons will be complemented by the half-Maxwellian of the electrons which reach the surface from the gas.

C. Potential and Electron Charge Distribution

The potential and electron charge distribution within a gas-particle plasma are closely coupled to one another. The existence of a charge distribution automatically implies the existence of a potential distribution. Also, because the electrons are free to continually interchange their translational and potential energies, the charge distribution will be directly related to the potential distribution; in the case of equilibrium, this relation is given by the Boltzmann factor. Thus, in order to describe the properties of a gas-particle plasma, the potential and charge distributions must be determined simultaneously.

In the actual physical situation, the potential and charge distributions are likely to be quite complex in nature because the properties of the particles themselves are, to some degree, random (size, shape, spatial distribution, and composition). In general, there are two distinct approaches to the problem of describing the properties of plasmas, each of which idealizes the actual situation to some extent.

The statistical approach makes use of the fact that, when there is a very large number of particles in any one system, a statistical distribution of particles over the possible "ionized states" can be determined. The electron density is then found by adding up the number of electrons contributed by each particle in a unit volume. The possible ionized states must include negative as well as positive degrees of ionization since electrons may attach themselves to particles as well as be thermionically emitted from them. In the statistical approach, no assumptions about the distribution of the particles in space need be made. Also, the size, shape, and composition of the particles do not have to be uniform; however, it is reasonable as a first approach to assume that all the particles are identical in these respects. Thus, the ionization of a system of thermionically-emitting particles may be treated in the same way as the ionization of a system of atoms in a gaseous state which can be multiply ionized, both positively and negatively.

The major difficulty encountered in the statistical approach is that the ionization potential for each degree of ionization must be known. If it is assumed that each electron which is external to a particle is effectively an infinite distance away from it, these potentials may be easily determined. However, in the actual situation, the free electrons may be considered to be at most an interparticle distance away and, in some cases, there could be an appreciable fraction of the net total number of electrons emitted by a particle located very close to the particle's surface. This last situation would be analogous to a multiply-ionized atom in which there were many more electrons in the "excited states" than in the ionized state. It follows that if the statistical approach is to be used in a rigorous way to calculate the electron density in a system of thermionically-emitting particles, the continuum of energy levels and the corresponding statistical weights must first be known. The difficulty encountered here is traced directly back to the coupling which exists between the potential and the charge distributions.

The second approach to the problem is one which begins with the examination of a one-particle system. That is, each particle in the plasma is assumed to be identical to all the others in every respect (size, shape, composition, and degree of ionization) , and the particles are assumed to be uniformly distributed throughout the plasma so that a volume equal to one over the number density may be allotted to each particle. In this approach, once the properties of a particle are assumed to be known and the volume allotted to it is given, the degree of ionization is considered to be determined. There is no statistical distribution over the available ionized states taken into account; that is, the ionization is assumed to be uniform. However, non-uniformities in the particle properties may be introduced, but each particle is still assumed to be ionized to the degree specified by its own individual properties. The one characteristic which all particles will have in common, however, is the electron density at the "adjoining surfaces" of their allotted volumes.

The advantage of the uniform ionization approach is that the coupling which exists between the potential and charge distribution can be treated in an accurate and straightforward manner by finding the solution to Poisson's equation. It is reasonable to first assume that the ionization per particle is large with respect to unity so that the charge distribution can be treated as a continuum. In the analysis to be carried out here, the results of this uniform ionization - continuous charge approach will be compared with the statistical approach in a regime in which they should agree. This will indicate what restriction, if any, should be placed on the magnitude of the degree of particle ionization with respect to unity so that the continuum approximation is valid.

The examination of a one-particle system under the assumptions of a continuous charge distribution and spherical symmetry should clarify several points. First, it should point out under what conditions the potential around a spherical particle is given only by a $1/r$ field arising from the net positive charge on the particle, and when the divergence introduced into the electric field by the electron charge is also of importance. Second, it should indicate when it is safe to equate the mean electron density to that which exists far away from a particle, and, when a significant fraction of the total number of emitted electrons is packed close to the particle's surface at relatively high densities. A detailed examination of this one-particle system will be carried out in Part III. However, before the problem is approached in a formal way, it is informative to see what general statements may be made about the type of solution to be expected.

Initially, it may be anticipated that the nature of the potential distribution existing in this one-particle system could be divided into two distinctly different regimes, depending upon the relative magnitudes of the particle radius, a, and the Debye shielding distance, defined by the properties at the surface of the particle, λ_{Da}.

$$\lambda_D \sim (T/n_e)^{\frac{1}{2}} \tag{2.8}$$

When

$$a >> \lambda_{Da} , \tag{2.9}$$

the curvature of the particle should have little effect on the potential distribution close to the particle's surface, and the potential sheath will be one-dimensional in nature. That is, the characteristic length of the potential sheath will be λ_{Da} .

When

$$a << \lambda_{Da} , \tag{2.10}$$

the effects of the spherical geometry will be dominant, and the potential close to the particle will vary approximately as $1/r$. If the particle radius is also very much smaller than the interparticle distance, most of the potential drop between the particle and the plasma will occur within the first few radii away from the particle, and the potential of the particle will be given approximately by

$$ez/(4\pi\epsilon_o a) , \tag{2.11}$$

where z is the number of positive charges on the particle. z , in turn, is given by

$$z = \overline{n_e}/N , \tag{2.12}$$

where $\overline{n_e}$ is the mean electron density of the plasma and N is the particle density.

When the potential varies approximately as $1/r$ close to the particle, the electric field will vary approximately as $1/r^2$, and it would be inconsistent to expect that the number of electrons located within a few radii away from the particle could be an appreciable fraction of z . Therefore, the mean electron density of the plasma will approximately equal the electron density far from the particle.

Hence, using the Boltzmann factor,

$$\overline{n_e} = n_E \, e^{-(e\overline{n_e})/(4\pi\varepsilon_o aNkT)} , \tag{2.13}$$

where n_E is the electron density at the surface of the particle due to thermionic emission. Thus, in the special case in which the particle radius is very much less than both the Debye shielding distance defined at the surface of the particle and the interparticle distance, the potential distribution becomes uncoupled from the charge distribution, because of the dominance of geometrical effects, and the approximate value of the electron density in the plasma may be readily determined.

D. Characteristic Times

If the particles and gas are not in thermodynamic equilibrium and the characteristic time for charge redistribution is very much shorter than the characteristic times for particle and gas temperature changes, the electron charge distribution will follow the changes in the gas and particle temperatures with negligible lag. The assumption that this lag is zero is, in essence, the application of the adiabatic or quasi-steady state approximation to this system; the system is analyzed assuming the particle and gas temperatures to be external parameters that are unchanging in time. In order to determine the validity of this application of the adiabatic approximation, six characteristic times must be examined.

1) <u>Time for Charge Redistribution</u> - When gaseous ionization is negligible, an upper bound on the charge redistribution time is readily estimated. If it is assumed that most of the free electrons

are well outside the high-potential region around each particle and that these high-potential regions are collisionless, kinetic theory can be used to show that the rate at which electrons escape the particles per unit volume is

$$\dot{n}_e^{(escape)} = (4\pi a^2 N)(\tfrac{1}{4}\overline{n_e})(\overline{C}_e)\left(1 + \frac{e\Phi_a}{kT}\right) \qquad (2.14)$$

where $\overline{C}_e$ is the mean electron speed, and Φ_a is the potential difference between the particle and the plasma. The first three terms on the right correspond to those which would be calculated for the analogous one-dimensional case; (total surface area) $\times \frac{1}{4}$(number density outside sheath) $\times$ (mean electron speed). The last factor arises because of the spherical geometry and will be discussed in Part V-C. The characteristic time for charge redistribution will be defined by

$$\tau_1 \equiv \frac{\overline{n_e}}{\dot{n}_e^{(escape)}} \quad . \qquad (2.15)$$

Hence,

$$\tau_1 = \left[(\pi a^2 N\overline{C}_e)\left(1 + \frac{e\Phi_a}{kT}\right)\right]^{-1} \qquad (2.16)$$

Numerical values will be calculated for this and the other characteristic times using values which are representative of the conditions found in the exhaust nozzle of a solid rocket motor. The time rates of change of gas properties in this system are of the same magnitude or much larger than those encountered in most problems of engineering interest. Thus, the relative magnitude of the values calculated for the characteristic times should give a pessimistic estimate of the applicability of the adiabatic approximation to most physical situations of engineering interest.

When $a = 10^{-4}$ cm, $N = 10^{8}$ cm^{-3} , $\overline{C}_e = 3 \times 10^{7}$ cm/sec , and $(e\Phi_a)/kT = 10$, the time for charge redistribution is $\tau_1 = 1 \times 10^{-9}$ sec. It is shown in the following paragraphs that this time is very much smaller than the characteristic times calculated for the gas or particle temperature changes.

2) Time for Gas Temperature Change - In a high-velocity flow system, the characteristic time for gas temperature change may be defined by

$$\tau_2 \equiv L/v \, , \tag{2.17}$$

where L is a characteristic length of the system, and v is a characteristic gas speed. For the rocket nozzle flow, assume $L = 1$ ft , and $v = 10^4$ fps . Thus, $\tau_2 = 10^{-4}$ sec , a time which is larger by a factor of 10^5 than the representative value of τ_1 just calculated.

3) Time for Particle Thermal Equilibration - Before an estimate of the characteristic time for the particle temperature change is made, it will be assumed that the temperature distribution within a particle is uniform. This is a valid assumption if the time for the establishment of a uniform temperature is very much shorter than the time in which appreciable changes in the particle temperature can occur. The characteristic time for the establishment of particle temperature uniformity is

$$\tau_3 = a^2/\alpha_T \, , \tag{2.18}$$

where a is the particle radius and α_T is the thermal diffusivity of the particle material. The following table gives estimates of τ_3 for various particle sizes and materials.

TABLE I

Time for Establishment of Particle Temperature Uniformity

Material	α_T cm^2/sec^{-1}	a cm	τ_3 sec
graphite	.01	10^{-6}	1×10^{-10}
alumina	.003	10^{-4}	3×10^{-6}
aluminum	.3	10^{-4}	3×10^{-8}

It will be seen that these times are small relative to the times for particle temperature change.

4) Time for Particle Temperature Change - The temperature change of a particle will usually be due to the direct energy exchange between it and the gas. The radiative energy loss of a particle, except in a high temperature - low gas density system, is relatively small. Also, the net energy transported by the electron gas to or from a particle may be neglected except under some conditions which will be discussed. Therefore, the characteristic time for gas temperature change should serve as a lower limit on the time for particle temperature change. Since it has been shown that $\tau_2 >> \tau_1$, for the flow system investigated, it may be concluded that the time for particle temperature change will also be much larger than the time for charge redistribution in most engineering applications.

It is also of interest to explore the characteristic time for particle temperature change in a system through which a shock wave has been passed; that is, in a system in which the gas experiences a step-function change in temperature. Of importance here, in analyzing the conditions behind the shock, is the time for particle-gas thermal equilibration. If this time is very much larger than the time for charge redistribution, the adiabatic approximation is still applicable.

Marble [21] has shown that when Stokes' drag law correctly describes the gas-particle interaction force, the gas-particle thermal equilibration time, for most gases and metal particles, is given by

$$\tau_4 = \frac{2}{9}\left(\frac{\rho_s}{\rho_G}\right)\left(\frac{a^2}{\nu_G}\right), \tag{2.19}$$

where ρ_s and ρ_G are the mass densities of the particle material and the gas, respectively, and ν_G is the kinematic viscosity of the gas. From a simplified kinetic theory description of the transport processes [24], it is seen that

$$\nu_G = \frac{1}{3}\,\overline{C}_G\,\lambda_{GG} \tag{2.20}$$

where $\overline{C}_G$ is the mean speed of a gas molecule and λ_{GG} is its mean free path. Thus,

$$\tau_4 = \frac{2}{3}\left(\frac{m_s}{m_G}\right)\left(\frac{n_s}{n_G}\right)\left(\frac{a^2}{\overline{C}_G\,\lambda_{GG}}\right) \tag{2.21}$$

where m_s and n_s are the mass and number density of the atoms in the particle, and m_G and n_G are the corresponding quantities for the gas. If it is assumed that $m_s/m_G = 1$, $n_s = 10^{22}$ cm^{-3}, $n_G = 10^{18}$ cm^{-3}, $a = 10^{-4}$ cm, $\overline{C}_G = 10^4$ cm/sec^{-1}, and $\lambda_{GG} =$

10^{-4} cm , then equation (2.21) yields $\tau_4 = \frac{2}{3} \times 10^{-4}$ sec. This time is very much shorter than the previous estimates made for τ_1 and τ_3 , but approximately equal to the estimate made for τ_2 .

If the gas mean free path is very much larger than the particle radius, the characteristic time for the particle-gas thermal equilibration may be obtained from the energy equation for a single particle derived from kinetic theory considerations. When it is assumed that the specific heat per unit volume of the particle material is $3n_s k$ and the accommodation coefficient for the thermal energy equilibration is unity, the rate of thermal energy change of a particle is

$$\left(\frac{4}{3}\pi a^3\right)(3n_s k\dot{T}_p) = (\pi a^2)(n_G\bar{C}_G)(2k)(T_G - T_p) \quad . \tag{2.22}$$

The characteristic time for particle-gas thermal equilibration is defined by

$$\tau_5 \equiv (T_G - T_p)/\dot{T}_p \quad . \tag{2.23}$$

Thus,

$$\tau_5 = 2\left(\frac{n_s}{n_G}\right)\left(\frac{a}{\bar{C}_G}\right) \quad . \tag{2.24}$$

If it is assumed that $a = 10^{-4}$ cm , $\bar{C}_G = 10^4$ cm/sec^{-1} , $n_s = 10^{22}$ cm^{-3} , and $n_G = 10^{18}$ cm^{-3} , equation (2.24) yields $\tau_5 = 2 \times 10^{-4}$ sec, approximately the same value obtained for τ_4 using the same densities and particle radius.

The characteristic time for particle-electron gas thermal equilibration, τ_6 , may be calculated in a similar manner. When it is assumed that the electron mean free path is much larger than the particle radius, the particle potential is much larger than kT , most of the free electrons are outside the potential sheath, and the accommodation coefficient is unity, it may be shown that (see Section V-C):

$$(\frac{4}{3}\pi a^3)(3n_s k\dot{T}_p) = (4\pi a^2)(\frac{1}{4}\overline{n_e})(\overline{C}_e)\left(\frac{e\phi_a}{kT_p}\right)(kT_e - kT_p) \ . \qquad (2.25)$$

Defining τ_6 by

$$\tau_6 = (T_e - T_p)/\dot{T}_p \ ,$$

it is seen that

$$\tau_6 = 4\left(\frac{n_s}{\overline{n_e}}\right)\left(\frac{a}{\overline{C}_e}\right)\left(\frac{kT_p}{e\phi_a}\right) \ . \qquad (2.26)$$

When $a = 10^{-4}$ cm, $\overline{C}_e = 3 \times 10^7$ cm/sec^{-1}, $(kT_p)/(e\phi_a) = .1$, $n_s = 10^{22}$ cm^{-3}, and $\overline{n_e} = 10^{10}$ cm^{-3}, equation (2.26) yields $\tau_6 =$ 1.3 sec. Thus, for the numerical example considered here, the electrons will transport a negligible amount of thermal energy relative to that which is transported by the gas.

A lower limit on the value of $\overline{n_e}$ at which the electron energy transport becomes of relative importance is obtained by setting $\tau_5 = \tau_6$. Thus, the "break even" point will occur when

$$\frac{\overline{n_e}}{n_G} = 2\left(\frac{\overline{C}_G}{\overline{C}_e}\right)\left(\frac{kT_p}{e\phi_a}\right)$$

$$(\sim 10^{-3} \rightarrow 10^{-4}) \qquad (2.27)$$

In most engineering problems, unless the gas pressure is low, $\overline{n_e}/n_G$ will rarely be this large. Also, if the gas pressure is low, the radiative energy loss of a particle may be of major importance.

From this discussion of the characteristic times to be found in a gas-particle plasma, several general statements, applicable to most of the current problems of engineering interest, can be made.

1. The electron density will always follow the changes in the gas and particle temperatures with negligible lag. In the rocket exhaust system numerically investigated, the ratio of the characteristic time for charge redistribution to the characteristic times for gas or particle temperature changes was 10^{-5}. It is felt that the use of a rocket exhaust system for the numerical examples should give an estimate of the upper limit on this ratio for most situations of interest. Even if this estimate is a factor of 10^2 or 10^3 too low, the adiabatic approximation is still applicable.

2. The temperature distribution within a particle will be uniform.

3. Relative to the gas, the electrons will rarely be effective in changing the thermal energy of the particles.

III. EQUILIBRIUM PARTICLE IONIZATION

A. Introduction

The first problem to be investigated is that of equilibrium particle ionization. The gas and particles are assumed to be in thermal equilibrium, and the gaseous ionization is assumed to be negligible; all of the existing free-electron charge is the result of thermionic emission from the particles.

In the Fundamental Shielding Problem, the system to be investigated will be reduced to a one-particle, spherically-symmetrical system by the assumption that all of the particles are spherical and identical in all respects. Also, it will be assumed initially that the net number of electrons emitted from a particle is large with respect to unity. The coupling which exists between the potential and charge distributions is taken directly into account by finding the solution to Poisson's equation.

This approach to the problem will clarify several points. First, it will indicate under what conditions the potential around a spherical particle is given only by a $1/r$ field arising from the net positive charge on the particle and when the divergence of the electric field produced by the electron charge is of relative importance. Second, it will determine what fraction of the total number of emitted electrons is packed close to the particle at relatively high densities over a wide range of initial parameters. In addition to yielding a qualitative understanding of the shielding phenomena, the solution to the one-particle problem will provide a means by which the degree of equilibrium particle ionization can be readily calculated.

In the Comparison with the Statistical Method, the results of the two approaches to the problem of particle ionization will be compared. The requirement, if any, which must be placed on the degree of particle ionization, in order for the continuous charge approximation to be valid, will be determined.

B. Fundamental Shielding Problem

1) Assumptions and Model - The assumptions which define the model to be used in this problem will now be stated. The motivations and justifications for these assumptions have been previously discussed in Part II, Physical Phenomena of Primary Importance.

(a) Uniform particle properties. The assumption is made that all of the parameters which determine the degree of particle ionization in the gas-particle plasma are the same for all particles. These parameters are the temperature of the system, T ; the particle radius, a ; and the electron density at the surface of the particle, n_{ea} .

(b) Uniform particle ionization. It is assumed that the particle ionization is uniform. That is, the net number of electrons emitted from an initially uncharged particle, z , is the same for all particles.

(c) Negligible gas ionization. The contribution of gaseous ionization to the electron density of the system is assumed to be negligible.

(d) Spherical symmetry. It is assumed that the particles are spherical and that the surrounding potential and charge distributions are spherically symmetric.

The problem has now been reduced to the investigation of a one-particle system of inner radius a, the particle radius, and outer radius b, the radius of the spherical volume in which the total net charge is zero. Here, b is defined by

$$\frac{4}{3}\pi b^3 = \frac{1}{N} , \tag{3.1}$$

where N is the particle number density. From Gauss' law, it follows that the electric field at $r = b$ is zero.

(e) Equilibrium. The system is assumed to be in thermodynamic equilibrium. Therefore, the electron density at radius r is given by

$$n_e(r) = n_{eb}\, e^{\frac{e\phi(r)}{kT}} , \tag{3.2}$$

where n_{eb} is the electron density at $r = b$, ϕ is the electrostatic potential, k is Boltzmann's constant, and $\phi(b)$ has been set equal to zero.

(f) Electron density at the particle surface. It is assumed that the electron density at the particle's surface is known; later, it will be equated to the electron density due to thermionic emission.

2) Mathematical Statement of the Problem - The relation which determines the potential is Poisson's equation:

$$\nabla^2\phi = \frac{e}{\epsilon_o} n_e \tag{3.3}$$

For a spherically-symmetric problem it becomes, with the aid of equation (3.2),

$$\frac{1}{r^2}\frac{d}{dr}\left[r^2\frac{d\phi}{dr}\right] = \frac{e}{\varepsilon_o} n_{eb}\, e^{\frac{e\phi}{kT}} \qquad (3.4)$$

with the boundary conditions

$$\phi = 0 \quad \text{and} \quad \frac{d\phi}{dr} = 0 \quad \text{at} \quad r = b\,.$$

Once T, n_{ea}, a, and b are specified, the problem is determined. In order to reduce the number of parameters required to define the solution, the variables are made non-dimensional.

$$y \equiv \frac{e\phi}{kT} \qquad \text{and} \qquad x \equiv \frac{r}{b} \qquad (3.5)$$

Hence

$$\frac{1}{x^2}\frac{d}{dx}\left[x^2\frac{dy}{dx}\right] = \left(\frac{b}{\lambda_{Db}}\right)^2 e^y\,; \qquad (3.6)$$

$$y = 0 \quad \text{and} \quad \frac{dy}{dx} = 0 \quad \text{at} \quad x = 1$$

where

$$\lambda_{Db} = \left(\frac{\varepsilon_o kT}{e^2 n_{eb}}\right)^{\frac{1}{2}}\,, \qquad (3.7)$$

the Debye shielding distance at $r = b$.

The solution is now defined by the two ratios b/a and b/λ_{Db}. Since

$$\lambda_{Da} = \lambda_{Db}\, e^{-y_a/2}\,, \qquad (3.8)$$

b/a and b/λ_{Da} might equally well be used. It is the latter two ratios which are known at the outset of the problem and are therefore preferable to use in plotting the interesting quantities arising from the solution.

3) Method of Solution - Equation (3.6) is a transcendental

equation of the Emden type[3] and no solutions are known to the author. Therefore, an IBM 7090 was used to integrate this equation and to obtain a family of curves for twenty different values of b/λ_{Db} between 0.1 and 2.0. In order to use equally-spaced increments in b/r, it was found convenient to change the independent variable to $\xi = 1/x$ and to solve the following differential equation:

$$\frac{d^2y}{d\xi^2} = \left(\frac{b}{\lambda_{Db}}\right)^2 \frac{1}{\xi^4} e^y \tag{3.9}$$

$$y = 0 \quad \text{and} \quad dy/d\xi = 0 \quad \text{at} \quad \xi = 1 .$$

The integration was run between the limits $1 \leq b/r \leq 10^3$ or $0 \leq y \leq 18.5$, whichever limit was reached first. Therefore, each curve gave the solution to all problems having the corresponding value of b/λ_{Db} and b/a between 1 and the limiting value. Further details of the method of solution are given in Appendix D.

4) Approximate Solution - By examining the nature of the solution, it is possible to divide this shielding problem into two characteristic regimes and to obtain an approximate solution which is very good in one of them. This approximate solution will facilitate further analytical investigation of the problem.

The manner in which an approximate solution arises for this problem may be understood by rewriting and examining equation (3.6):

$$\frac{d^2y}{dx^2} = \left(\frac{b}{\lambda_{Db}}\right)^2 e^y - \frac{2}{x}\frac{dy}{dx} . \tag{3.10}$$

---

[3] Equation (3.6) may be written in the form of a first-order equation (see Appendix C). For a further discussion of this type of equation, see reference 25.

The relative magnitude of the two terms on the right hand side of this equation will determine the nature of the solution, and it is therefore informative to examine the ratio

$$R \equiv \frac{-\frac{2}{x}\left(\frac{dy}{dx}\right)}{\left(\frac{b}{\lambda_{Db}}\right)^2 e^y} \quad . \tag{3.11}$$

When x is close to 1, $R \simeq 0$ and $e^y \simeq 1$. As x decreases, R becomes very much larger than 1 before e^y changes appreciably from 1 for the range of b/λ_{Db} investigated. This suggests that an approximate solution is found by solving the equation

$$\frac{d^2y}{dx^2} = \left(\frac{b}{\lambda_{Db}}\right)^2 - \frac{2}{x}\frac{dy}{dx} \tag{3.12}$$

$$y = 0 \quad \text{and} \quad dy/dx = 0 \quad \text{at} \quad x = 1 ,$$

the solution of which is

$$y = \frac{1}{3}\left(\frac{b}{\lambda_{Db}}\right)^2 \left\{\frac{b}{r} - \frac{3}{2} + \frac{1}{2}\left(\frac{r}{b}\right)^2\right\} \quad . \tag{3.13}$$

As x decreases further, R will continue to increase until the exponential behavior of the solution becomes significant, and it will then decrease and pass again through unity. The above solution will be a good approximation up to the latter point. For low values of b/λ_{Db}, R becomes very large and the approximate solution is extremely accurate. (For example, for $b/\lambda_{Db} = .1$, R reaches a maximum value of 2.5×10^8 at $b/r = 900$.) When the approximate solution is valid, the potential at the surface of the particle is easily determined. From equations (3.8) and (3.13),

$$y_a\, e^{y_a} = \frac{1}{3}\left(\frac{b}{\lambda_{Da}}\right)^2 \left\{\frac{b}{a} - \frac{3}{2} + \frac{1}{2}\left(\frac{a}{b}\right)^2\right\} \quad . \tag{3.14}$$

Also, the ratio of n_{eb} to n_{ea} can be found from

$$\left(\frac{n_{ea}}{n_{eb}}\right)\left(\ell n \frac{n_{ea}}{n_{eb}}\right) = \left(\frac{n_{ea}}{N}\right)\left(\frac{e^2}{4\pi\epsilon_o akT}\right)\left\{1 - \frac{3}{2}\left(\frac{a}{b}\right) + \frac{1}{2}\left(\frac{a}{b}\right)^3\right\} . \qquad (3.15)$$

5) <u>Prominent Features of the Solution</u> - The numerical solutions to equation (3.6), in which no approximation is made, are presented in Figure 1 as a plot of the quantity y, normalized by $\frac{1}{3}\left(\frac{b}{\lambda_{Db}}\right)^2$, versus (b/r). It is noted that the curves coincide with the function $\{\frac{b}{r} - \frac{3}{2} + \frac{1}{2}(\frac{r}{b})^2\}$ up to limiting values of b/r; the curves having lower values of b/λ_{Db} correspond to higher limiting values of b/r. In Figure 2, $-(dy)/d(r/b)$, normalized by $\frac{1}{3}(b/\lambda_{Db})^2$, is plotted as a function of b/r. It is seen that there is also agreement, but to a lesser degree, between the derivatives of the exact and approximate solutions. In Figure 3, the range of b/λ_{Db} and b/a in which the approximate solution is valid is shown; this region has been termed the "weak shielding" regime, for reasons which will soon become clear.

Physically, the approximate solution may be derived from the concept of a "sea" of electrons of almost constant density around the particles. In moving in toward the particle from $r = b$, most of the electron charge is passed before the electron density has changed appreciably, because of the geometrical or r^3 effect. When the electron density is no longer constant, the field is determined almost entirely by the large positive charge on the particle. For $b/r >> 1$, this is the dominant effect and $\phi \sim 1/r$ which may be seen by rewriting the approximation solution [equation (3.13)]

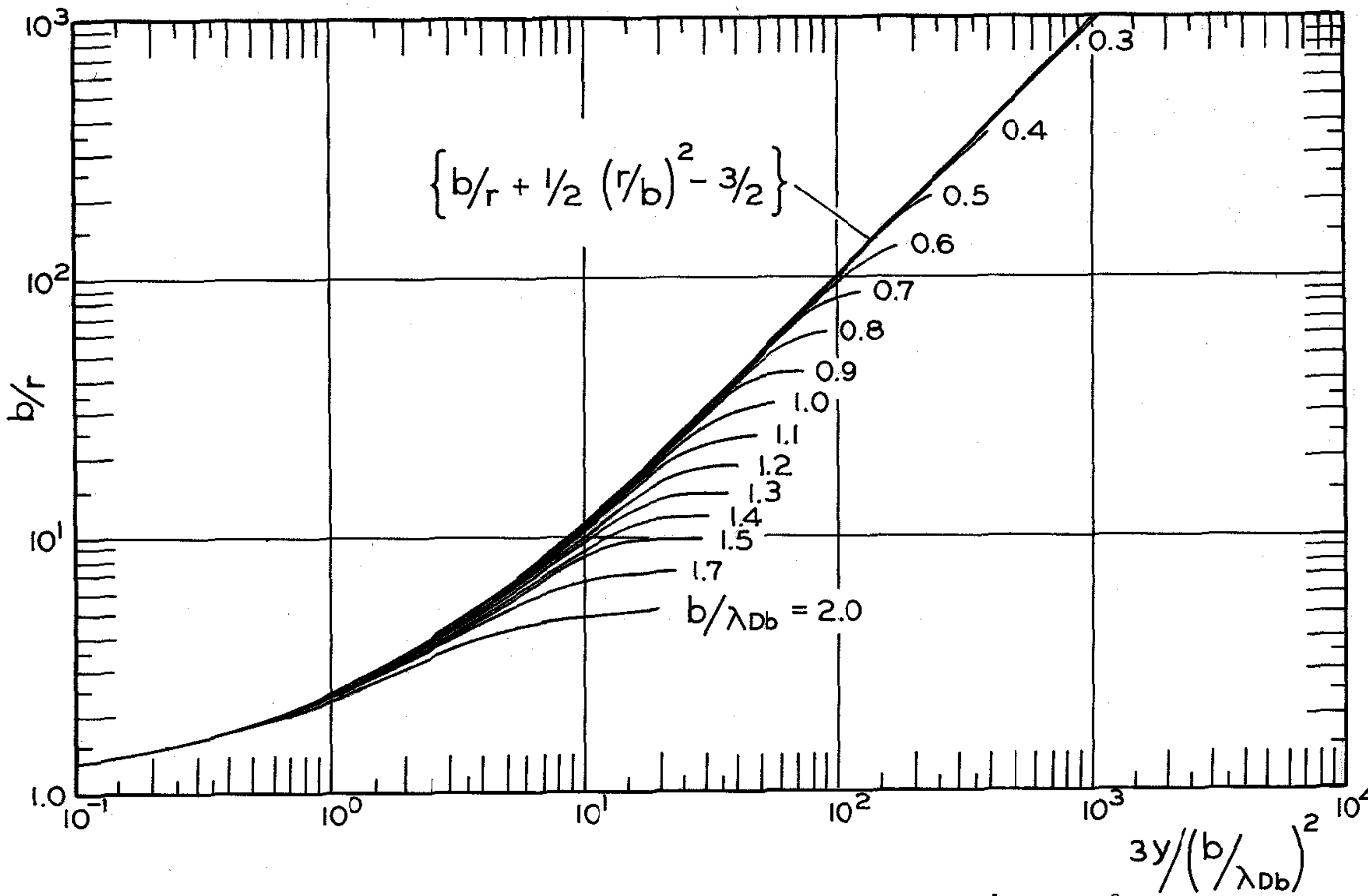

Figure 1. Numerical Solution to Poisson's Equation Normalized by $\frac{1}{3}(b/\lambda_{Db})^2$.

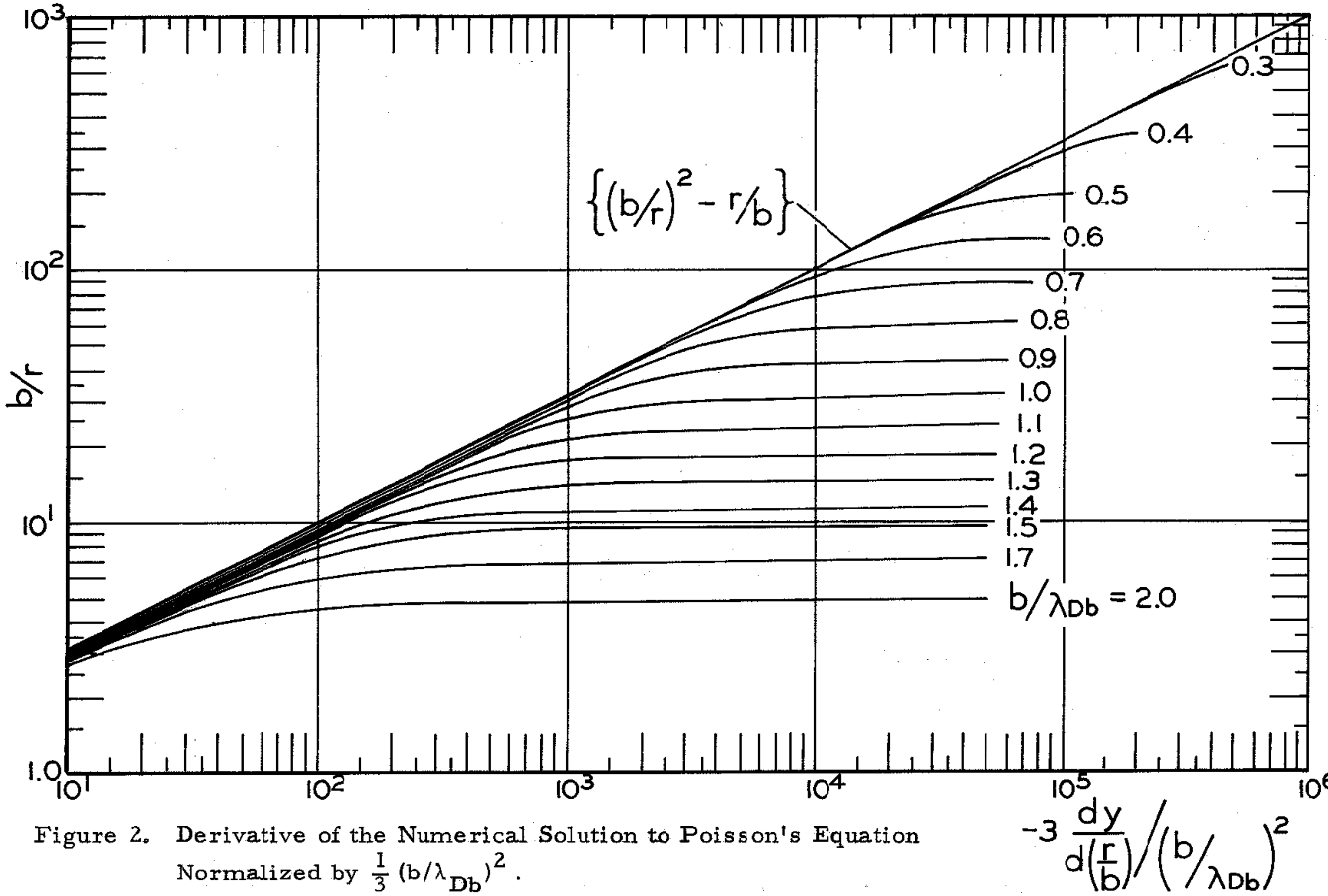

Figure 2. Derivative of the Numerical Solution to Poisson's Equation Normalized by $\frac{1}{3}\,(b/\lambda_{Db})^2$.

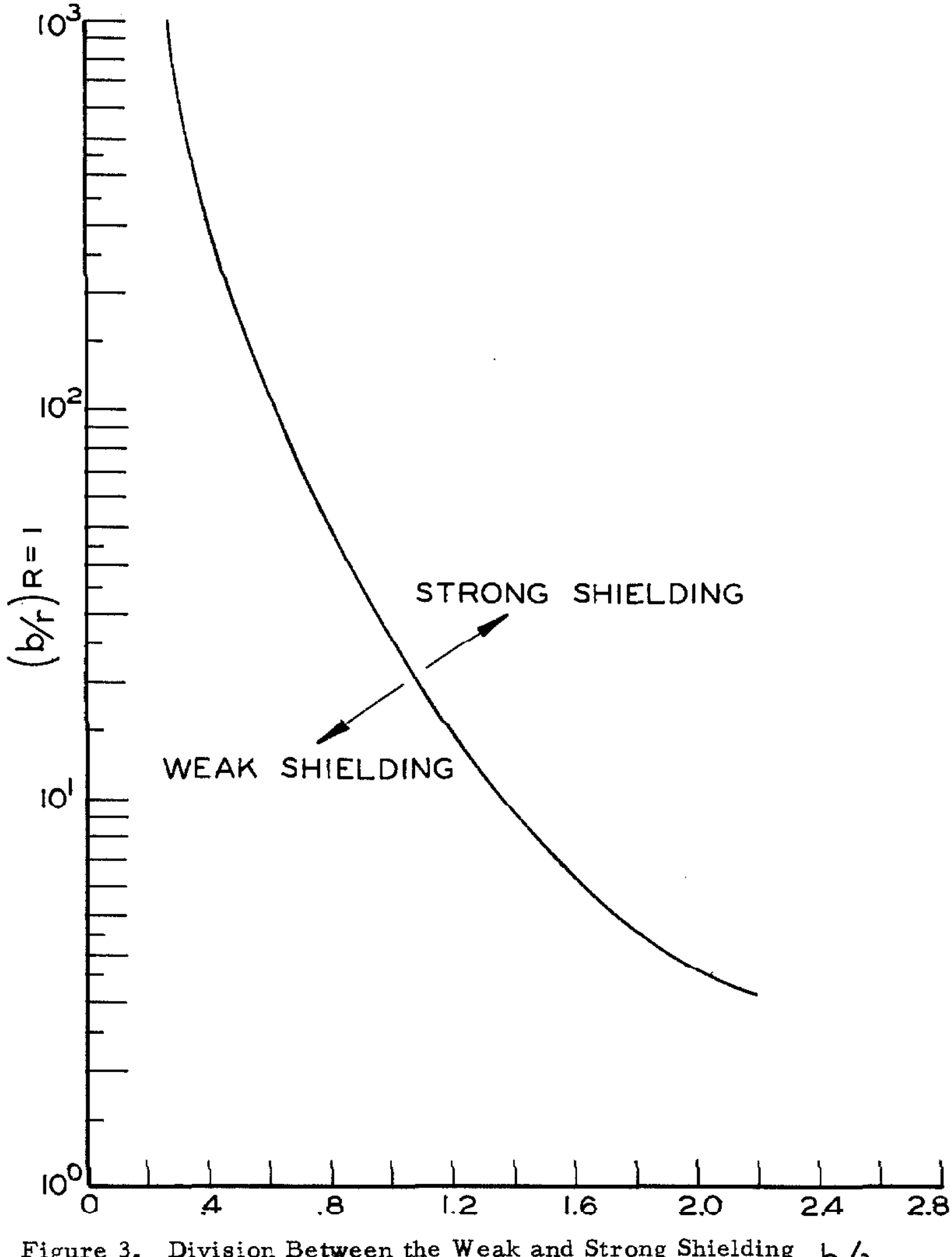

Figure 3. Division Between the Weak and Strong Shielding Regimes in the $(\frac{b}{r}, \frac{b}{\lambda_{Db}})$ Plane.

$$\phi = \frac{e\, n_{eb}}{(4\pi\varepsilon_o r)N} \left\{1 - \frac{3}{2}\left(\frac{r}{b}\right) + \frac{1}{2}\left(\frac{r}{b}\right)^3\right\} \tag{3.16}$$

One point about the weak shielding regime, which may appear inconsistent at first, is that the inter-electron distance may be on the order of the particle radius, and the abstraction to a continuous charge distribution close to the particle is therefore not possible. However, in this region the potential is determined by the geometrical effect or the strong field arising from the positive charge on the particle and not by the shielding of the electrons.

The weak shielding approximation is no longer valid when the effect of the shielding is larger than that of the geometry and a significant fraction of the emitted electrons are close to the particle. To clarify this, consider the quantity

$$Z(r) = \int_r^b 4\pi r^2 n_e(r) dr \quad . \tag{3.17}$$

Problems falling into what is appropriately termed the "strong shielding regime" would have the following qualitative features, shown in the following diagram.

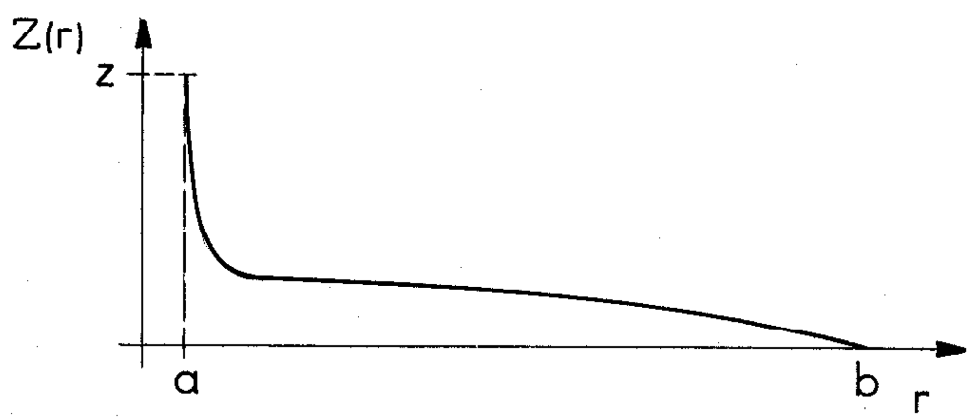

Figure 4. Total Electron Charge Outside r for a Problem in the Strong Shielding Regime.

This physical interpretation suggests that a quantity that may be used in differentiating the strong and weak shielding regimes is the ratio

$$\sigma = \frac{\overline{n_e}}{n_{eb}} \tag{3.18}$$

where $\overline{n_e}$ is the mean electron density defined by

$$\overline{n_e} = zN . \tag{3.19}$$

When σ is close to unity, most of the electrons are in the regions of nearly zero potential and the problem is characterized by weak shielding. When σ is appreciably larger than unity, the reverse is true. From Figure 5, it is seen that the divisions between the two types of shielding regimes defined by R and σ are essentially the same.

The requirement, for the weak shielding approximation to be valid, may be stated in fairly simple terms. Using Gauss' law,

$$eZ(r) = -4\pi r^2 \varepsilon_o \frac{d\phi}{dr} , \tag{3.20}$$

$R(r)$ may be written as

$$R(r) = \frac{Z(r)}{2\pi r^3 n_e(r)} . \tag{3.21}$$

Therefore, for the approximation to be valid, it is necessary that

$$z > 2\pi a^3 n_{ea} . \tag{3.22}$$

Also, from Figure 5, the line dividing the strong and weak shielding regimes by the requirement $R = 1$ may be approximated by

$$\frac{b}{\lambda_{Da}} [R=1] = 5.0(b/a) . \tag{3.23}$$

Therefore, the approximation is valid if

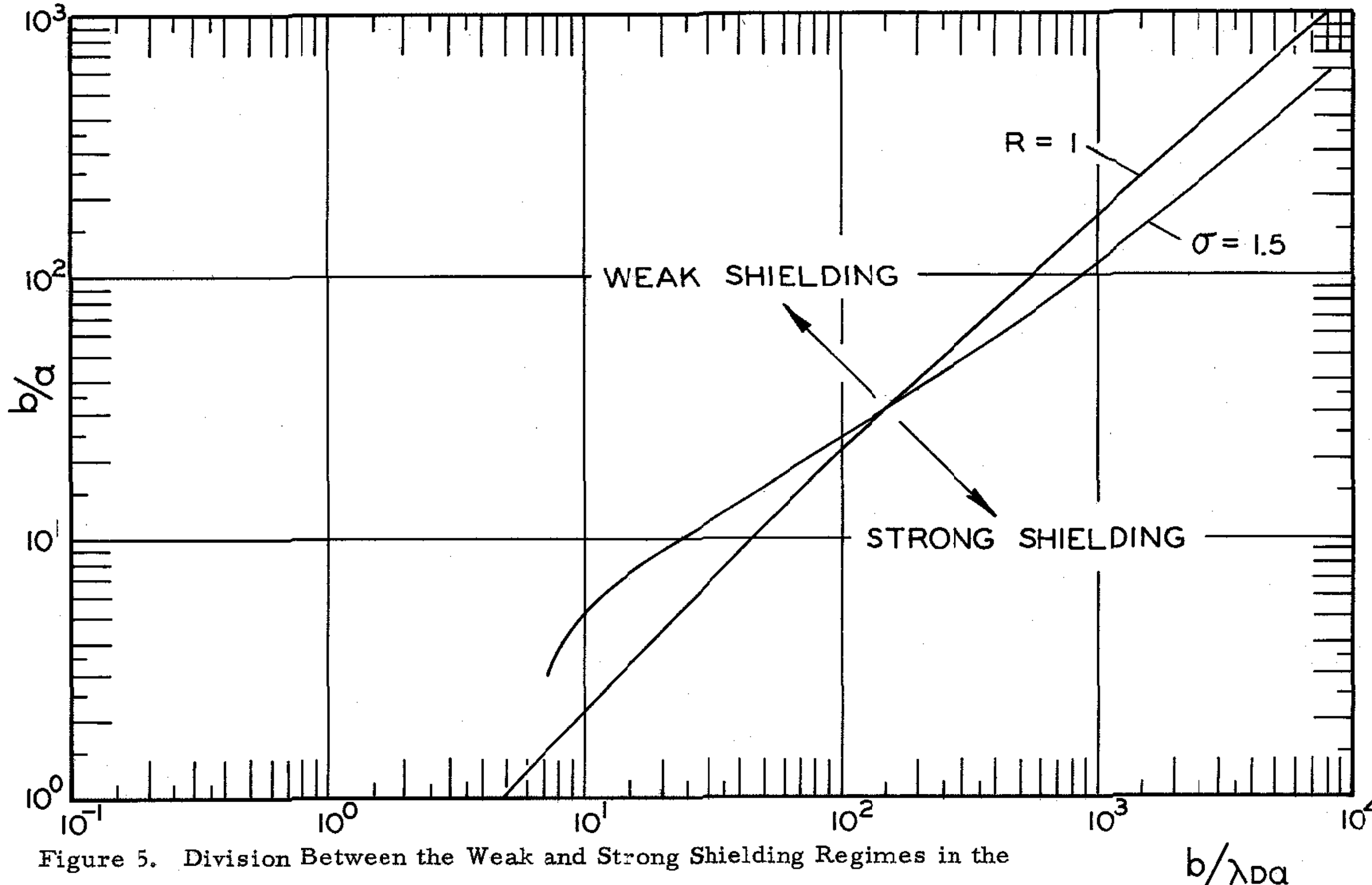

Figure 5. Division Between the Weak and Strong Shielding Regimes in the (b/a , b/λ_{Da}) Plane.

$$\frac{a}{\lambda_{Da}} < 5 \quad . \tag{3.24}$$

This confirms the idea that, in the weak shielding regime, the solution is determined primarily by geometrical rather than shielding effects. The shielding that is present is primarily due to a widespread sea of electrons of almost constant density rather than a thin, high-density sheath of electrons of thickness λ_{Da} at the particle's surface.

It should also be made clear that the weak shielding approximation does not place any restriction on y_a ; that is, y_a may be very much larger than unity.

Since the property of this problem that makes the approximation possible is the spherical geometry, it is to be expected that the same approximation may be made in the case of cylindrical geometry. This case corresponds to the problem of finding the potential around a thermionically-emitting wire of radius a for which the potential and electric field are set equal to zero at the radius b . The analogous approximation leads to

$$\phi = \frac{e\, n_{eb}(\pi b)^2}{2\pi\epsilon_o} \{ \ln\frac{b}{r} - \frac{1}{2}[1 - (\frac{r}{b})^2] \} \tag{3.25}$$

Again, a conservative criterion for the validity of this approximation is that $\underline{a}$ should be much less than λ_{Da} .

When the geometrical effects are small, that is, in the region of strong shielding, the spherical and cylindrical cases become one-dimensional in nature, and

$$\frac{d^2y}{dx^2} = (\frac{b}{\lambda_{Db}})^2 e^y \quad . \tag{3.26}$$

A general feature of many shielding problems, including the one studied here, may be illustrated by inspection of the solution to the above one-dimensional problem.[4] Here, the distance from the wall is r and the potential and electric fields are zero at $r = b$. The solution is[5]

$$y = \ln\left[\sec^2\left(\frac{b - r}{\sqrt{2}\,\lambda_{Db}}\right)\right] \tag{3.27}$$

As the potential at the wall becomes large, $b/(\sqrt{2}\,\lambda_{Db})$ approaches $\pi/2$ and n_{eb} has the limiting value

$$\frac{1}{2}\left(\frac{\pi}{b}\right)^2 \frac{\epsilon_o}{e^2}\, kT \ . \tag{3.28}$$

Therefore, once n_{eb} has reached a value close to the limiting one, it is very insensitive to changes in the electron density at the wall. Changes in the potential at the wall accompany the changes in the electron density and the net variation of n_{eb} is small.

This same type of behavior is to be expected in the strong shielding regime of particle ionization. In the strong shielding regime, it has been found that, over a wide range of values for the initial parameters (b/a and b/λ_{Da}), the value of b/λ_{Db} is always on the order of unity. That is, the electron charge distribution will adjust itself so that λ_{Db} is approximately equal to b. Thus, an approximate upper limit on n_{eb}, for the range of initial parameters investigated here, may be obtained. If an upper limit on b/λ_{Db} of two is assumed, the maximum value of n_{eb} is

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[4] For a more complete discussion of the problem, see reference 23, pp. 18-22.

[5] See Appendix E.

$$n_{eb}^{(max)} = \frac{4\varepsilon_o kT}{e^2 b^2} \quad . \tag{3.29}$$

When $T = 2900^oK$ and $N = 10^8$ ($b = 1.34 \times 10^{-3}$ cm), equation (3.29) yields $n_{eb}^{(max)} = 3.1 \times 10^{11}$ cm^{-3}. Hence, once the strong shielding regime is encountered, large increases in n_{eb} can only be accomplished by decreasing b; that is, by increasing the particle number density, N.

6) Utilization of the Numerical Solution - As stated previously, once the ratios b/λ_{Da} and b/a have been specified, the solution is defined. Hence, all of the quantities which describe the ionization of the particulate system may be plotted as functions of these two ratios only.

y_a, the non-dimensional potential at the surface of the particle, is plotted in Figures 6a, 6b, and 6c. The mean electron density, $\overline{n_e}$, divided by n_{ea} is plotted in Figures 7a and 7b. From this and equation (3.19), the degree of ionization of a particle is easily found. Figure 8 shows the dependence of σ, the shielding ratio, on b/λ_{Da} and b/a. In Figure 9, where σ is plotted as a function of b/λ_{Db} and b/a, the sharpness of the transition between the weak and strong shielding regimes is illustrated. The electric field may be found from Figure 2, and

$$E = -\frac{kT}{eb}\left[\frac{dy}{d(r/b)}\right] . \tag{3.30}$$

If the electric field at the surface of the particle is desired, it may be calculated more accurately from

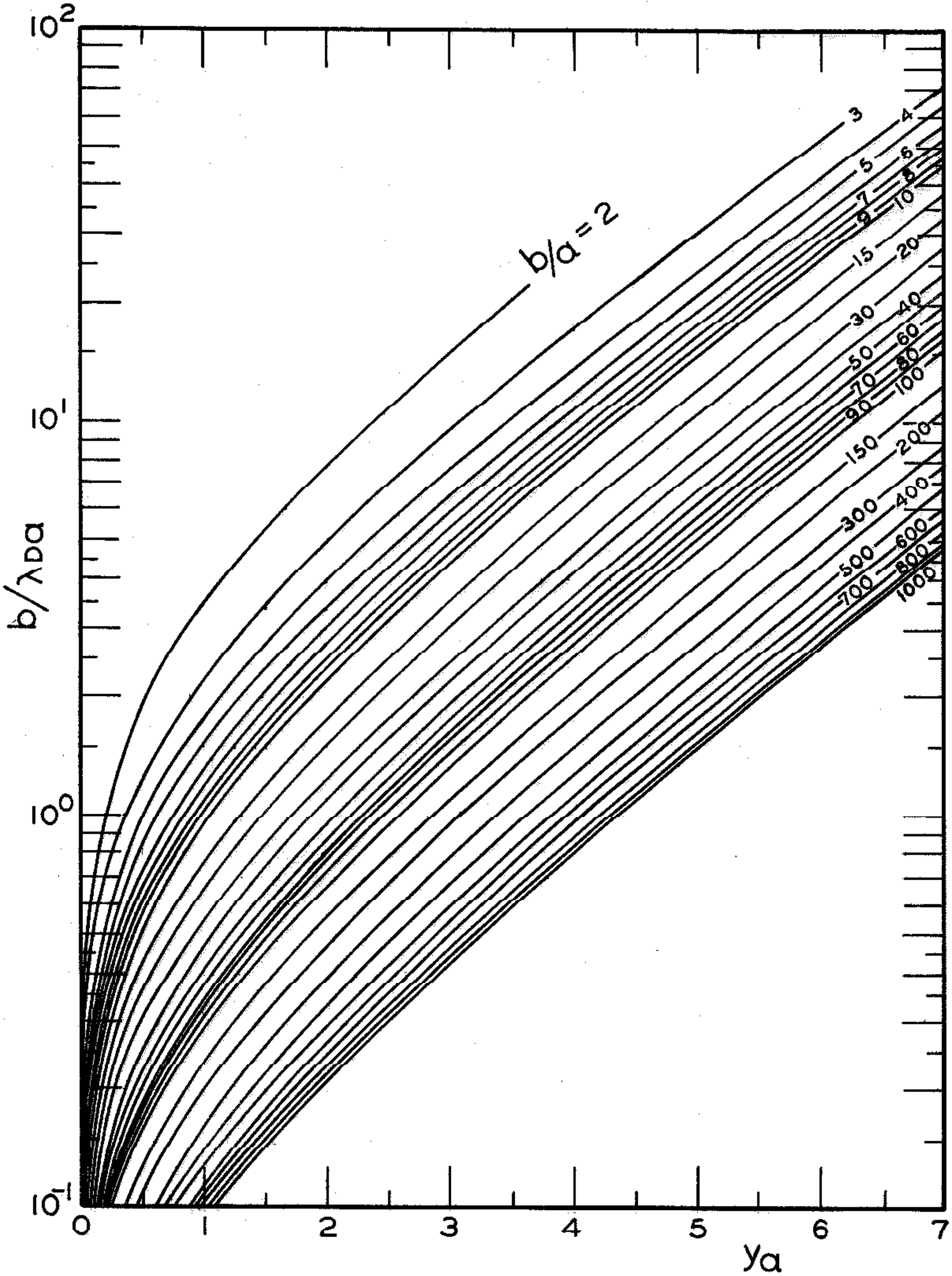


Figure 6a. Numerical Solution for the Particle Potential.

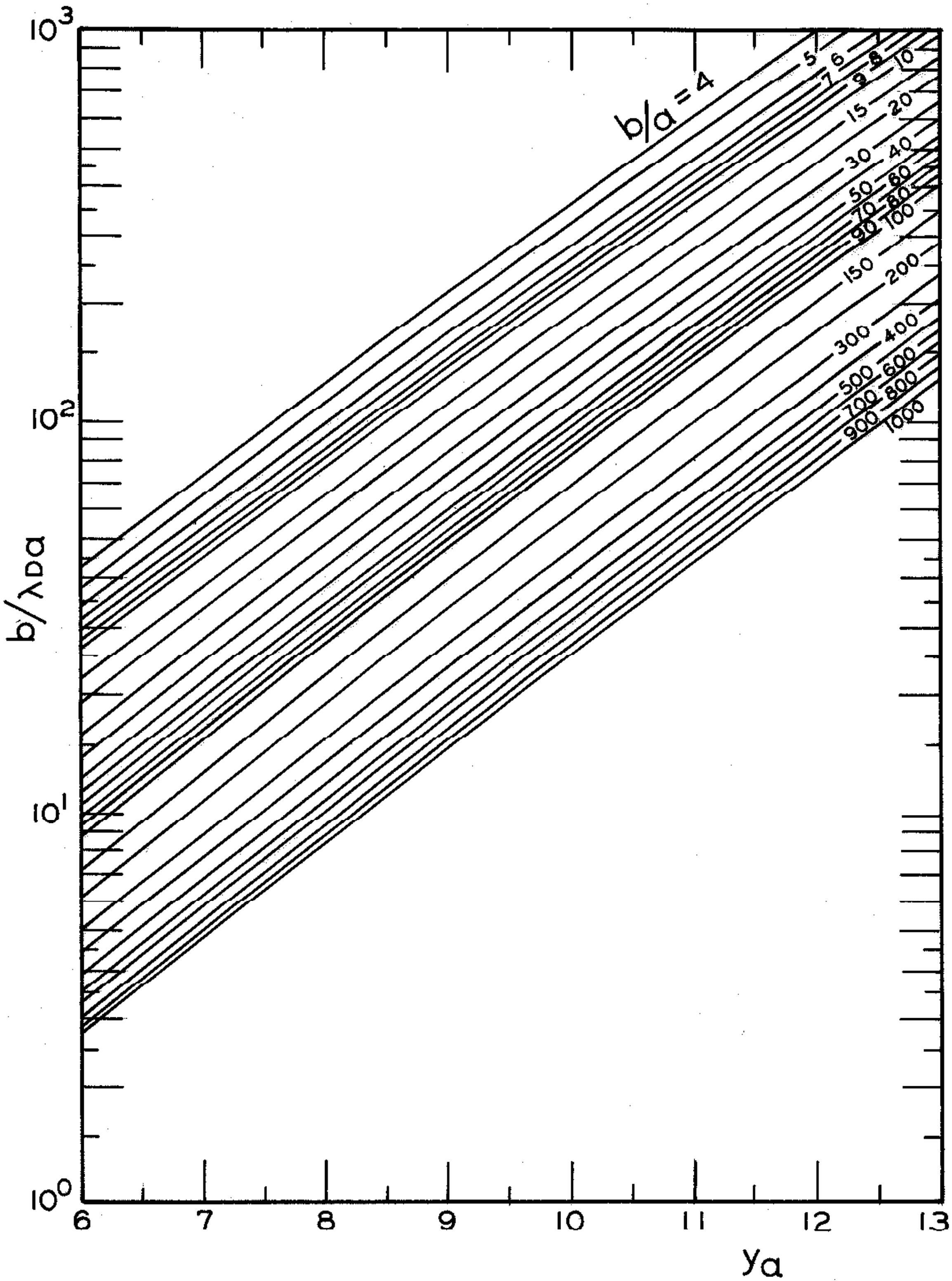


Figure 6b. Numerical Solution for the Particle Potential.

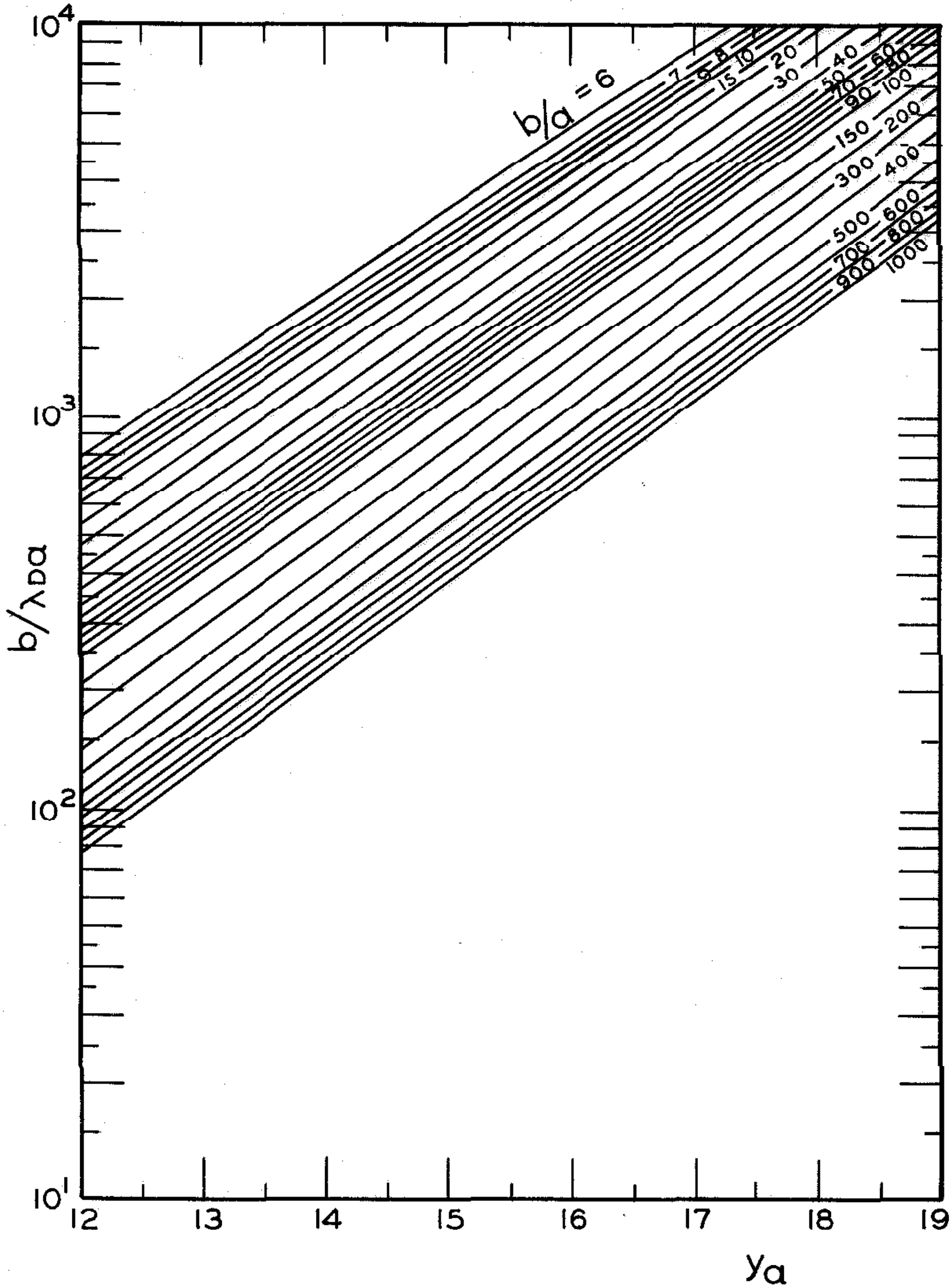


Figure 6c. Numerical Solution for the Particle Potential.

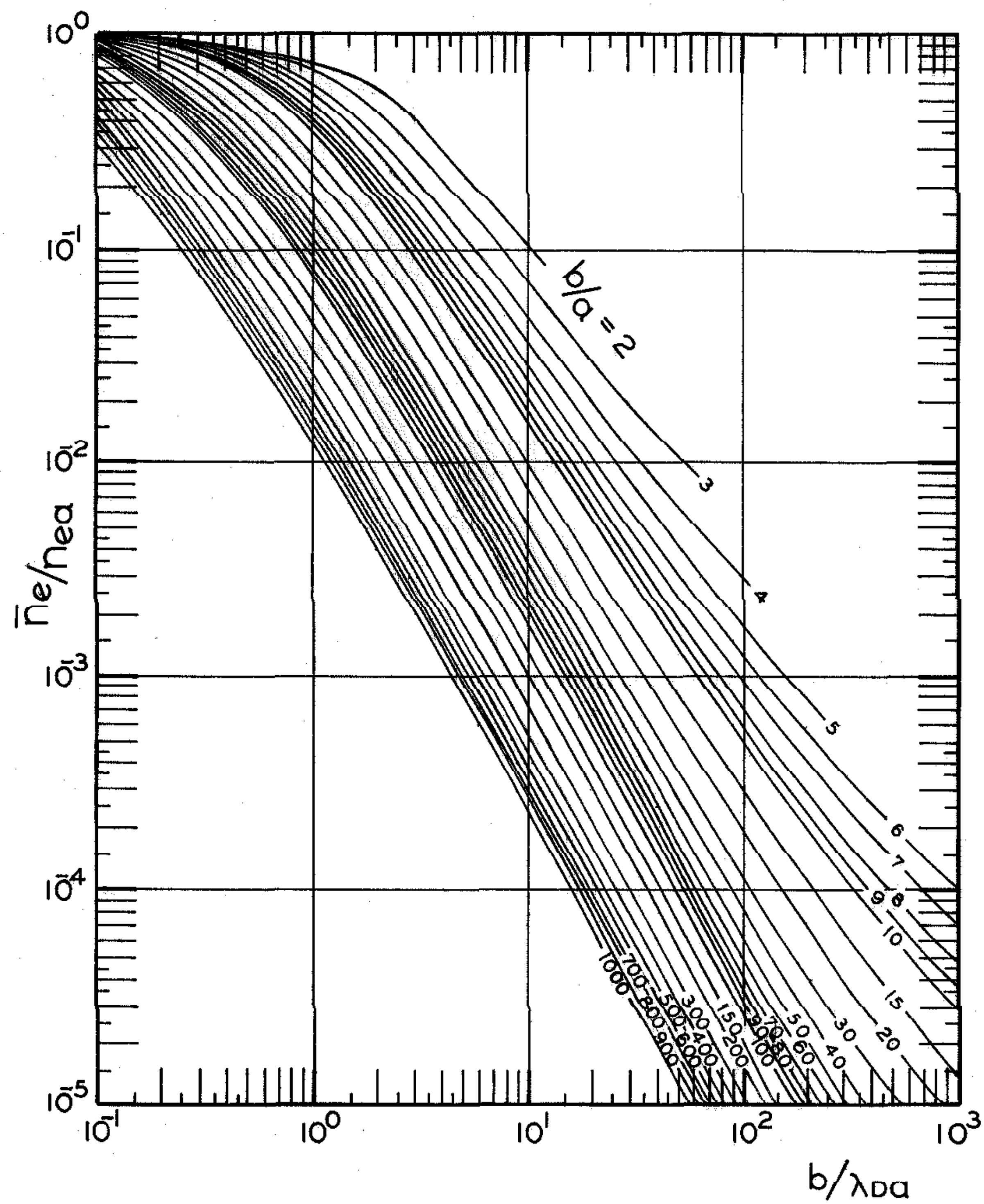


Figure 7a. Numerical Solution for the Mean Electron Density Normalized by the Electron Density at the Particle Surface.

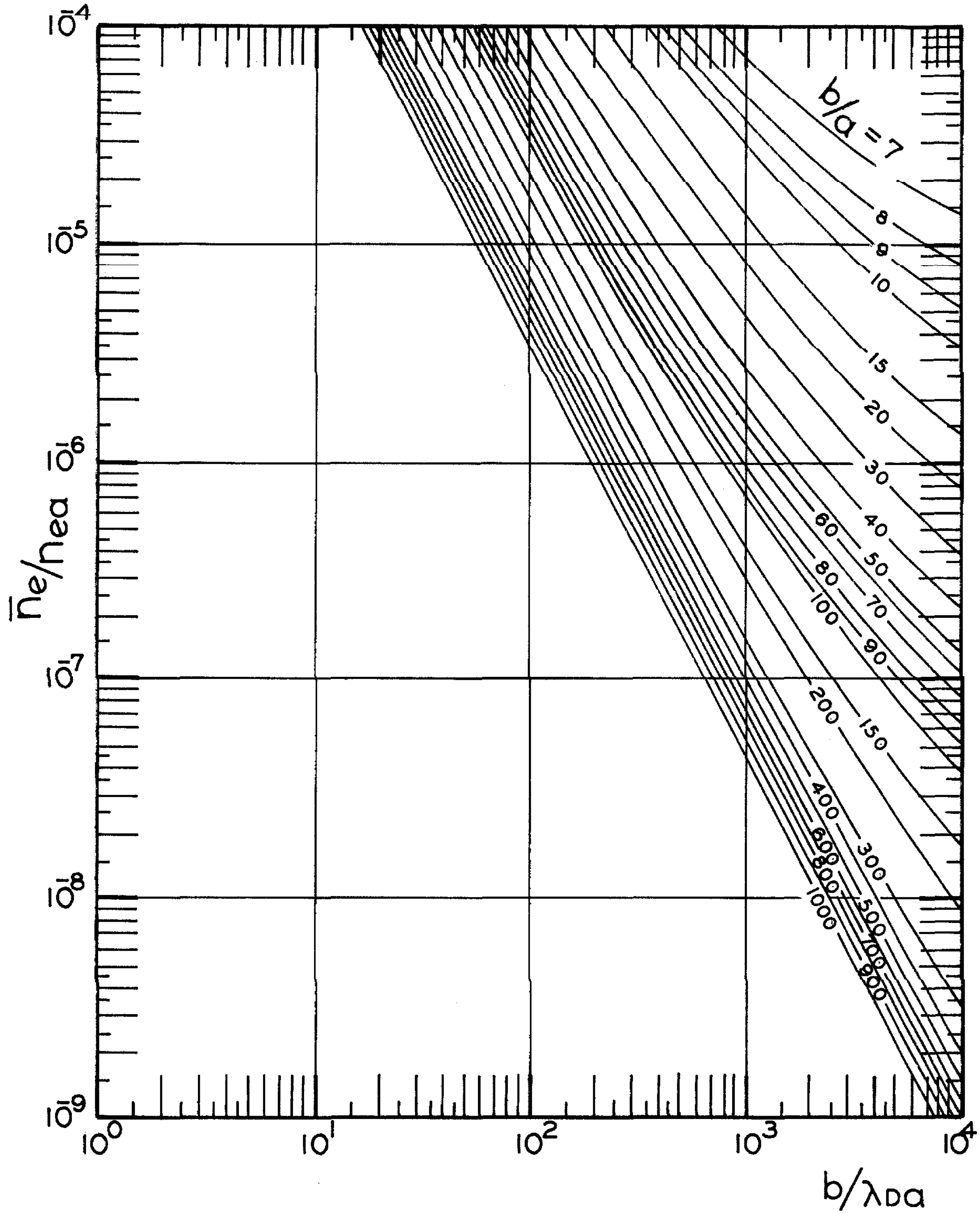


Figure 7b. Numerical Solution for the Mean Electron Density Normalized by the Electron Density at the Particle Surface.

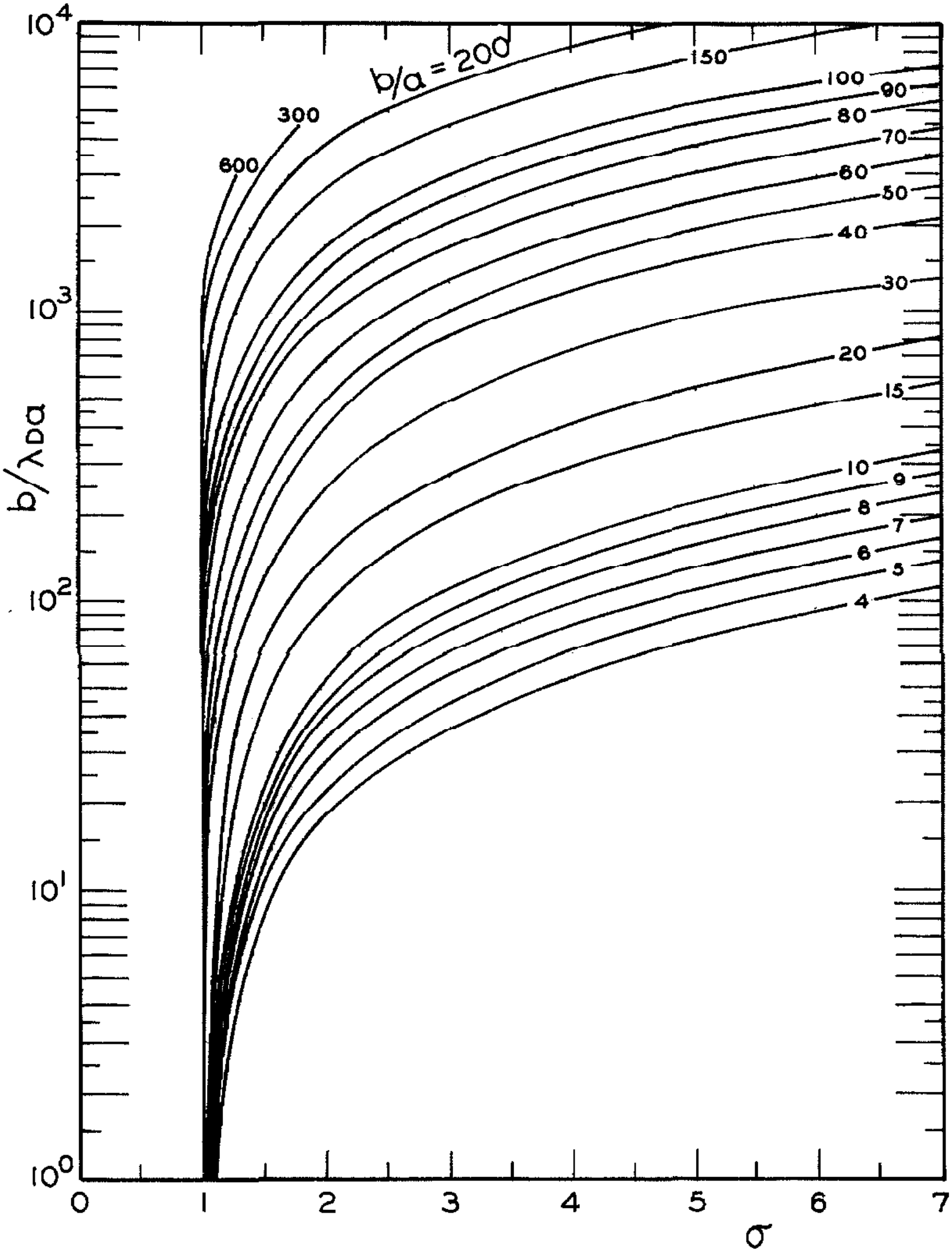


Figure 8. Numerical Solution for the Shielding Ratio.

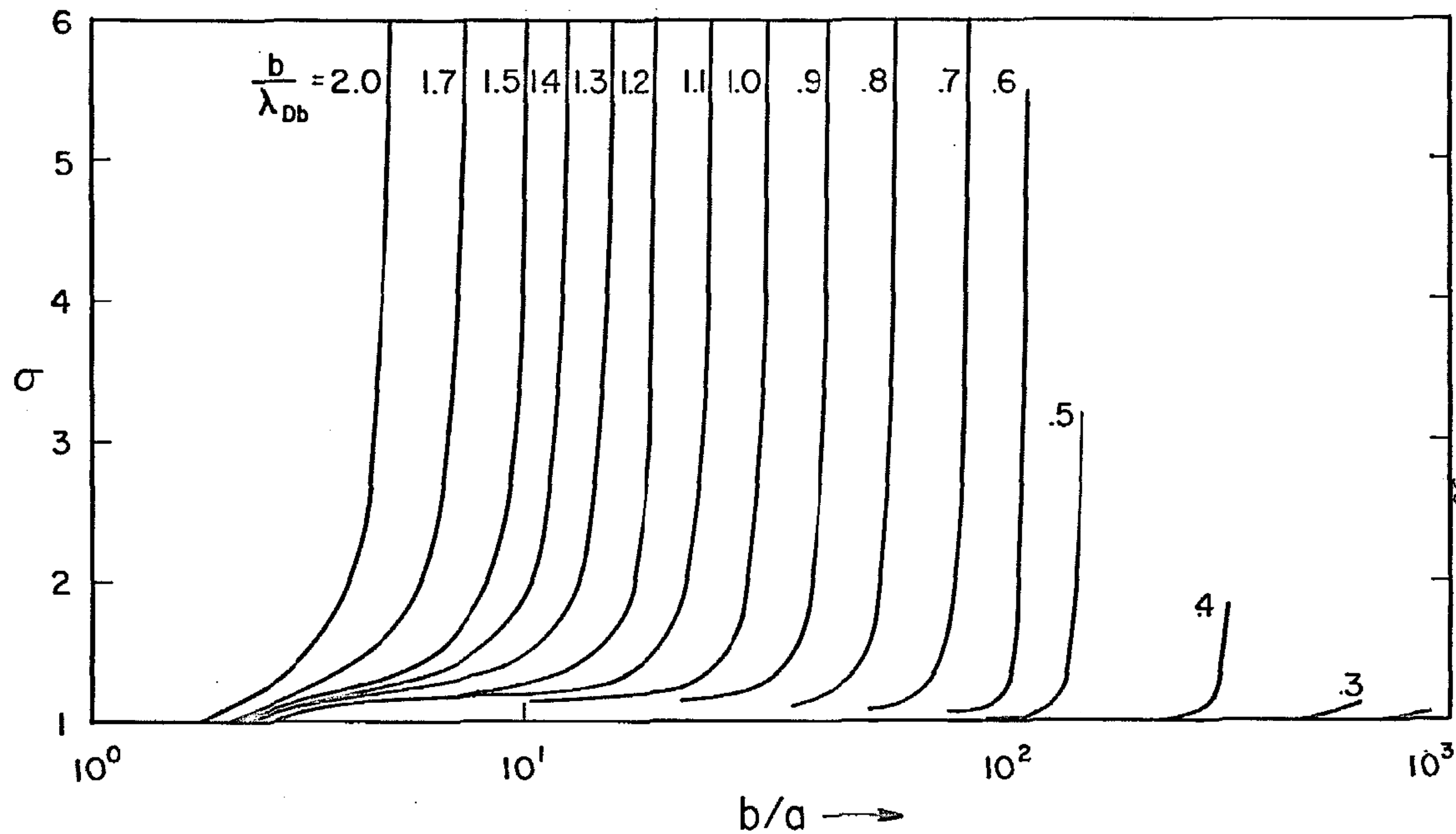


Figure 9. Illustration of the Sharp Transition Between the Weak and Strong Shielding Regimes.

$$E_a = \frac{ez}{4\pi\epsilon_o a^2} \tag{3.31}$$

In order to specify the ratios b/λ_{Da} and b/a for a system of particles, the four quantities n_{ea}, the electron density at the surface of a particle; T, the temperature of the system; a, the particle radius; and N, the particle number density, must be known. T, a, and N are usually calculated or measured in accordance with the details of a particular system. On the other hand, for all systems, n_{ea} is given by n_E, the value of the electron density due to thermionic emission at the surface of the particles. Hence, an extension of the discussion of the method of estimating n_E found in Section II-A is appropriate.

The experimental value of n_E may be found conveniently from

$$n_E^{(ex)} = \varkappa n_E^{(th)} \tag{3.32}$$

where

$$\varkappa = \frac{A^{(ex)}}{A^{(th)}} . \tag{3.33}$$

The value of $A^{(th)}$, the theoretical value of the constant coefficient in the Richardson-Dushman equation, is 120 amps/cm^2-^{o}K. Two good sources of values for $A^{(ex)}$ are reference 23, page 175, and reference 26, page 109. Reference 23 also lists many other sources. $n_E^{(th)}$ may be calculated from

$$n_E^{(th)} = (4.83 \times 10^{15})\, T^{3/2}\, e^{-e\Phi/kT}\ \text{cm}^{-3} \tag{3.34}$$

where T is in ^{o}K.

In Figures 10a and 10b, $n_E^{(th)}$ is plotted as a function of $e\Phi$ for the range of temperatures between 1500°K and 3500°K. Given the value of n_{ea}, λ_{Da} is found from

$$\lambda_{Da} = 6.90\,(T/n_{ea})^{1/2} \text{ cm} \tag{3.35}$$

where T is in °K and n_{ea} is in cm^{-3}. Since $n_{ea} = n_E^{(ex)}$ in this case, the corresponding experimental value of λ_{Da} is

$$\lambda_{Da}^{(ex)} = (1/\varkappa)^{\frac{1}{2}}\,\lambda_{Da}^{(th)} \; . \tag{3.36}$$

In Figure 11, $\lambda_{Da}^{(th)}$ is also plotted as a function of $e\Phi$ for the temperature range of 1500°K to 3500°K.

7) <u>Illustration</u> - In order to demonstrate the method of calculation and to illustrate the main features of the weak shielding regime, a system composed of alumina particles will be considered. The parameters chosen for computation are: T = 2900°K, a = 10^{-4} cm, N = 10^8 cm^{-3}. From reference 26, page 109,

$$e\Phi = 3.77 \text{ ev} \quad , \quad \varkappa = 1.17 \times 10^{-2} \; .$$

Using the figures and relations just discussed in Section II-B-6, the following are obtained:

$n_{ea} = 2.35 \times 10^{12}$ cm^{-3}	$\lambda_{Da} = 2.42 \times 10^{-4}$ cm
$b/a = 13.4$	$b/\lambda_{Da} = 5.52$
$y_a = 3.58$	$\overline{n_e}/n_{ea} = 3.1 \times 10^{-2}$
$\overline{n_e} = 7.29 \times 10^{10}$ cm^{-3}	$n_{eb} = 6.53 \times 10^{10}$ cm^{-3}
$\sigma = 1.11$	$z = 729$
$b/\lambda_{Db} = .921$	$a/\lambda_{Da} = .413$

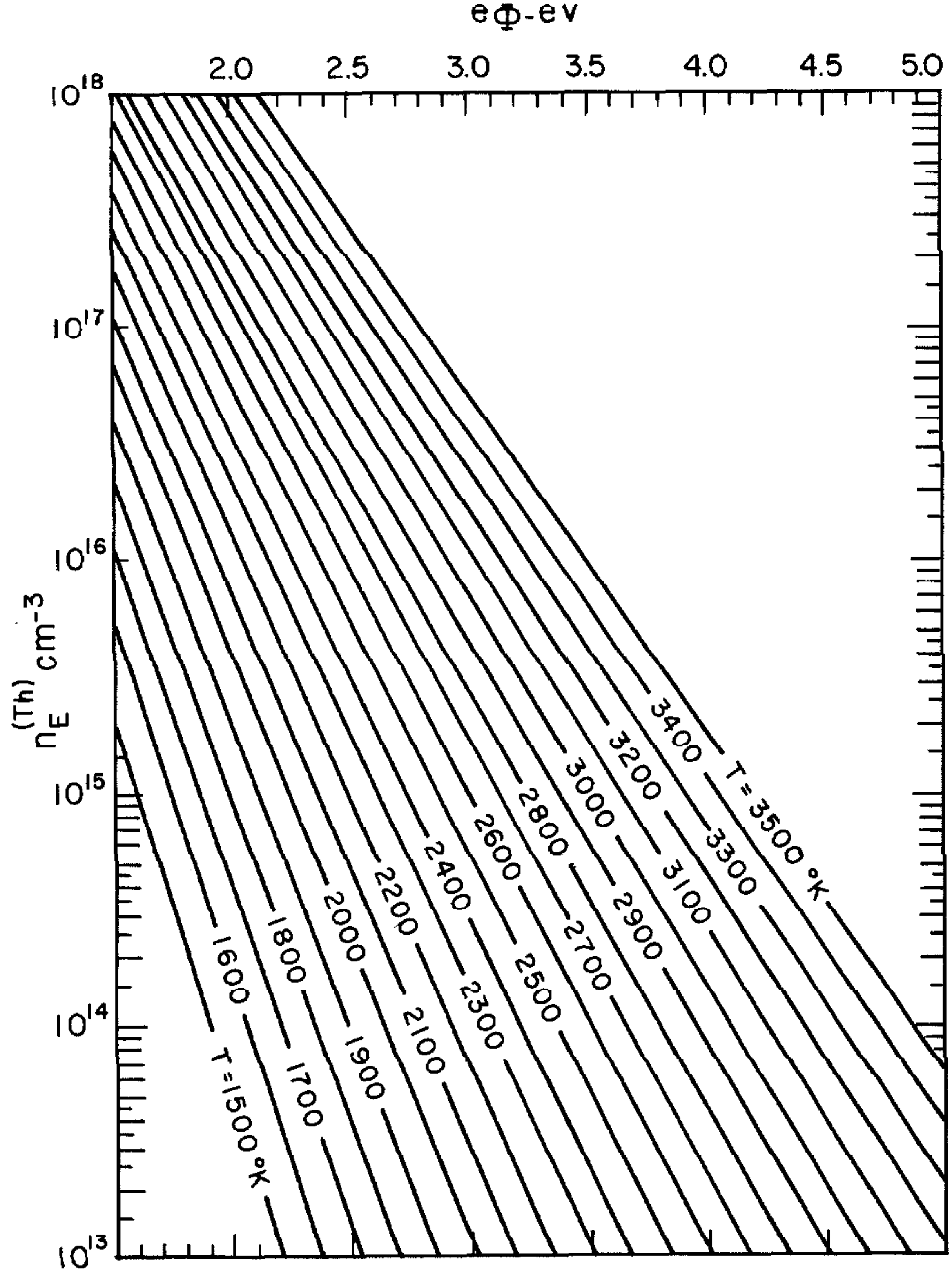


Figure 10a. Theoretical Value of the Electron Density for Thermionic Emission.

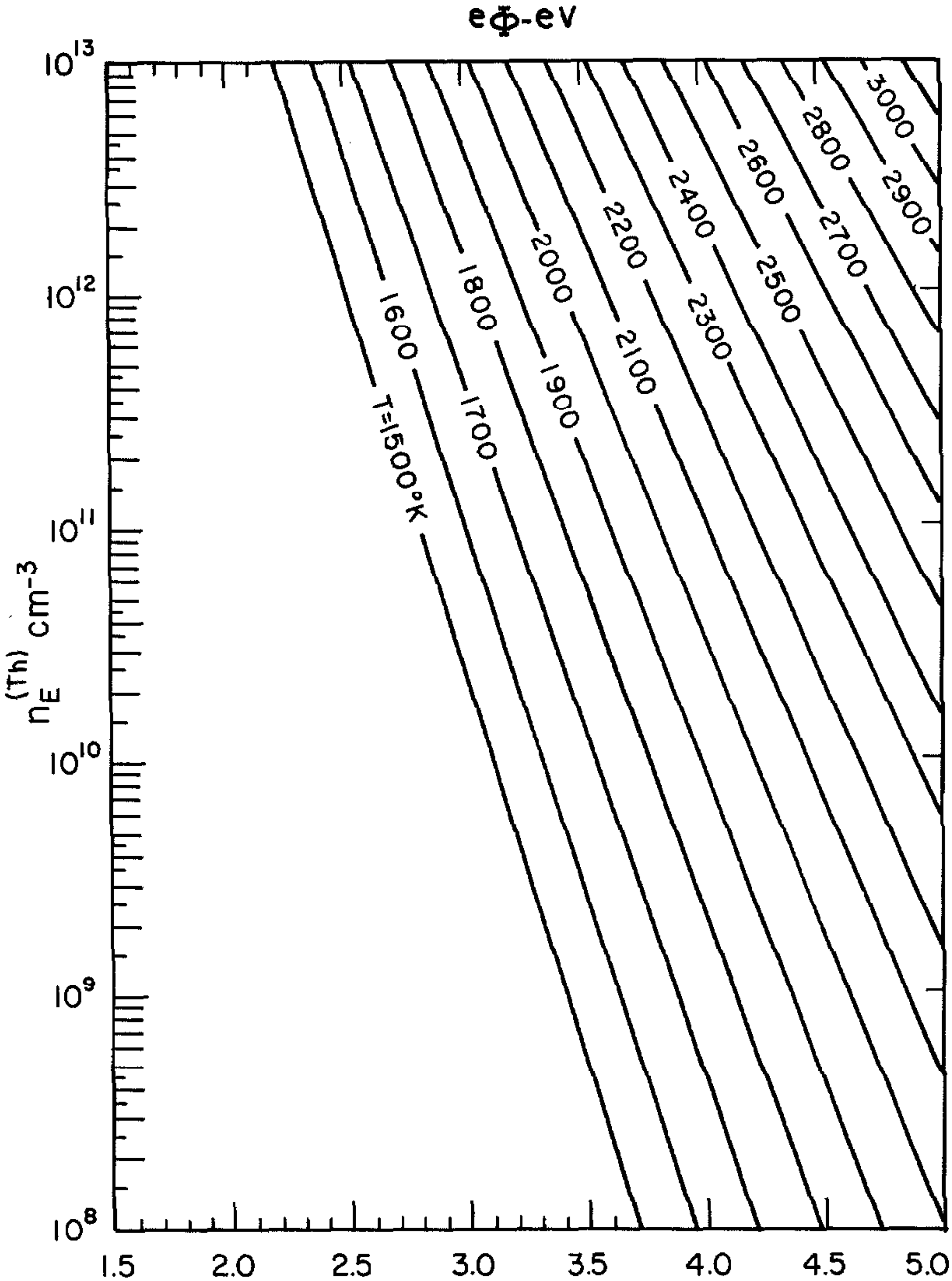


Figure 10b. Theoretical Value of the Electron Density for Thermionic Emission.

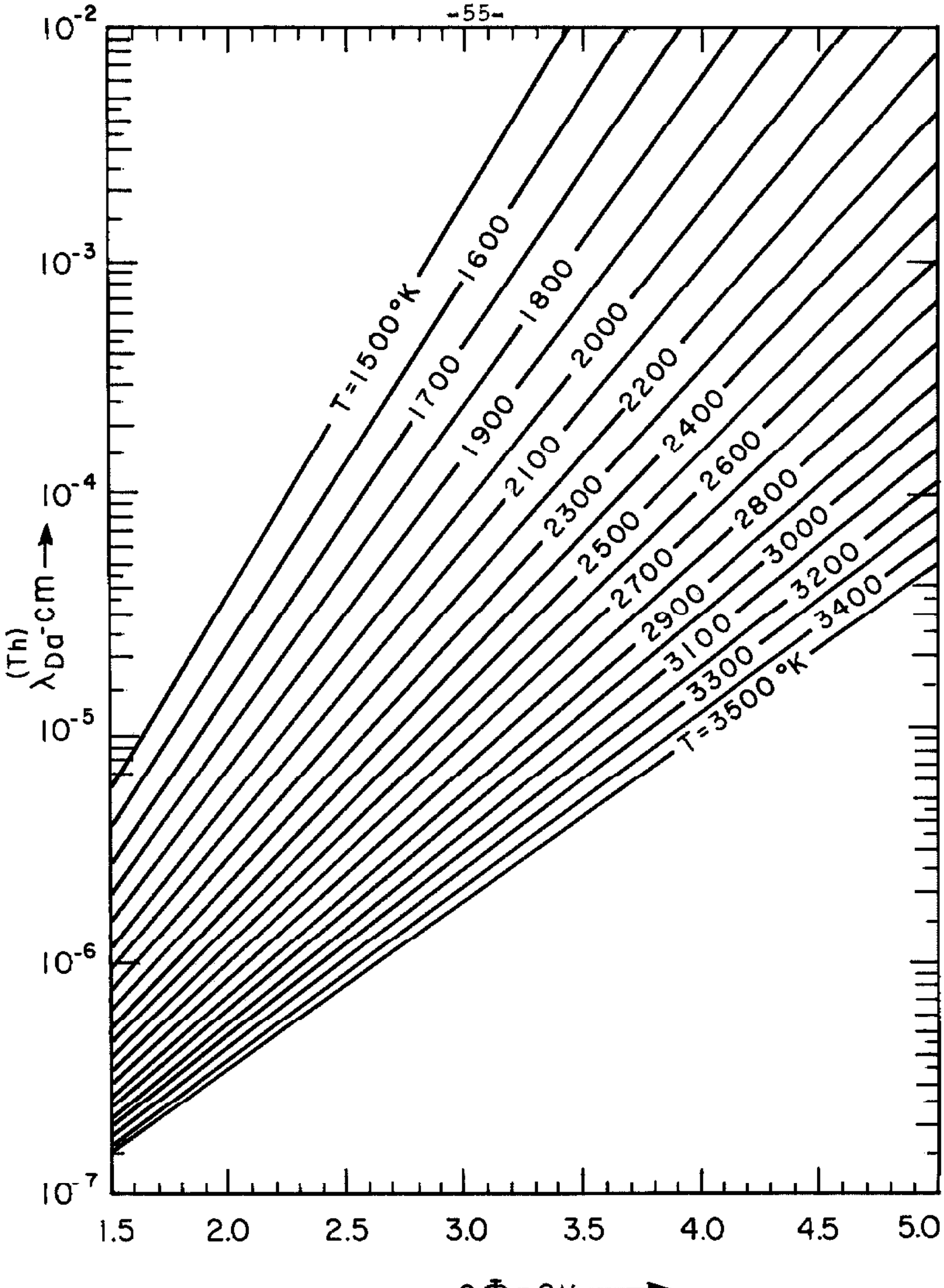


Figure 11. Theoretical Value of the Debye Shielding Distance at the Particle Surface.

The approximate solution for y_a, equation (3.14), is y_a = 3.53. The requirement for the validity of the approximate solution, $z > 2\pi a^3 n_{ea}$, is well satisfied since

$$2\pi a^3 n_{ea} = 14.8 << 729 .$$

Also,

$$a/\lambda_{Da} = .413 << 5.$$

In Figure 12, the variables of interest, normalized by their values at $r = a$, are plotted as functions of r/a. The important feature to note is that almost all of the electrons are in the region of zero potential and constant density. That is, Z/z, the fraction of total electron charge outside a given radius, is at a value of .90, while ϕ/ϕ_a is as low as .12, and n_e is only 1.5 times its value at $r = b$. Thus, the approximation of a uniform density sea of electrons, for this case, is a good one.

C. Comparison with the Statistical Method

At the beginning of the Fundamental Shielding Problem, it was explicitly assumed that the particle ionization was uniform. However, in an actual system, there will be a distribution of ionized states over the particles even though the particle parameters (T, $e\xi$, and a) are uniform. It is instructive, therefore, to see what effect this distribution has on the determination of $\overline{n_e}$. Also, when the majority of particles are not ionized many times, the assumption of charge continuity breaks down and the statistical approach is the one which must be used. Smith [7, 8] has shown that for small degrees of ionization, the Saha equation in which the work function assumes the role of the

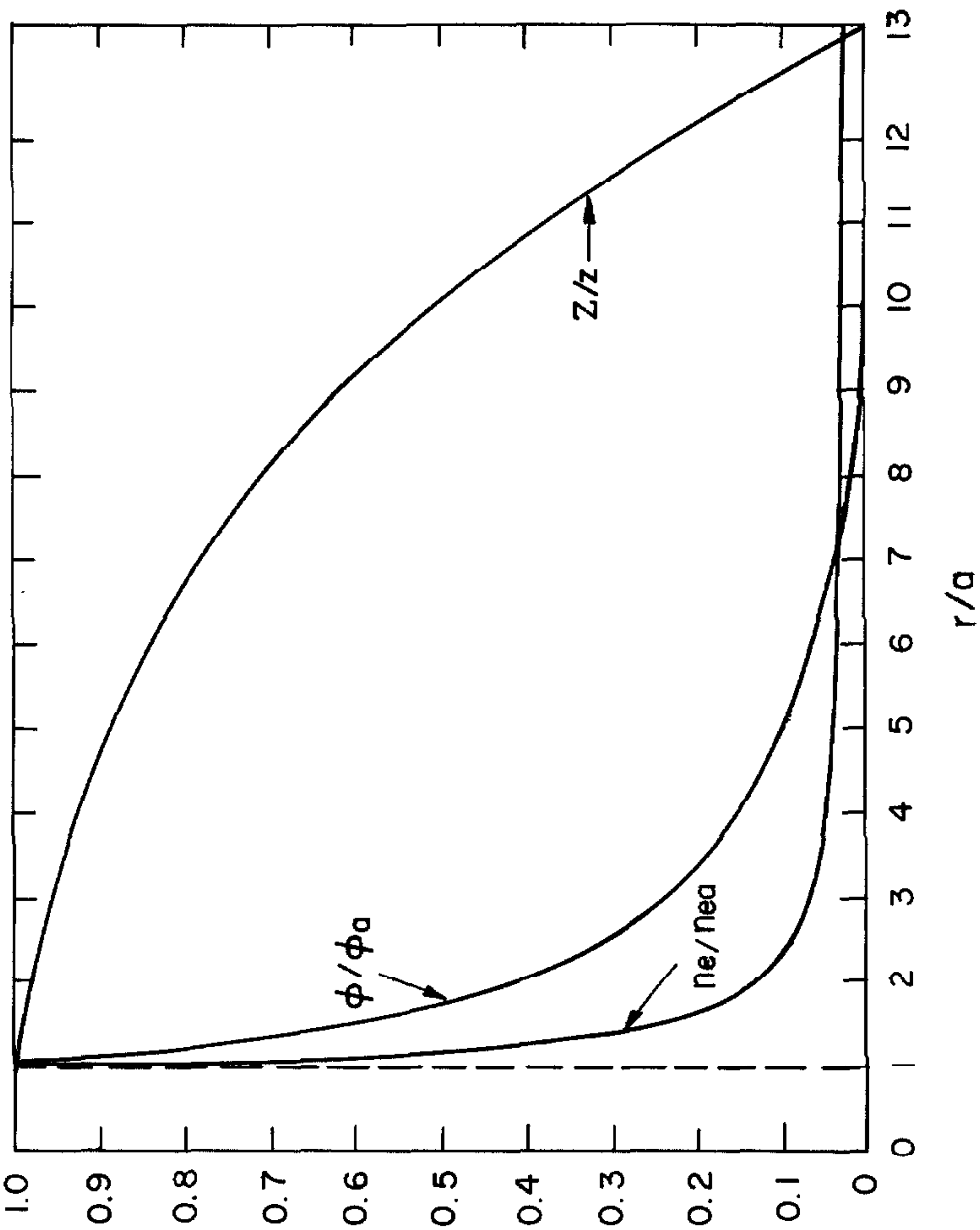


Figure 12. Spatial Distribution of the Potential, Electron Density, and Total Electron Charge Outside r for an Illustrative Problem in the Weak Shielding Regime.

ionization potential of the particle is not valid, as might have been anticipated. Rather, due to the attachment of electrons to particles, the maximum value of n_{eb} that may be obtained is $n_E^{(th)}$, the saturation electron density.

In the treatment of this problem by Einbinder [6] and Smith [7, 8], several assumptions have been made. They are:

(1) All of the electrons which are external to the particles are at zero potential. Hence, $\sigma = 1.0$, and the calculations are confined to the weak shielding regime.

(2) The particle parameters (T, $e\Phi$, and a) are uniform.

(3) The statistical weight of a particle is independent of its degree of ionization. This is a valid assumption because the particles are macroscopic in size.

(4) In order to obtain numerical results, the capacitance of the isolated particle, ze/ϕ_a, must be assumed. Einbinder and Smith, by treating each free electron as though it were an infinite distance away from a particle, have implicitly assumed $b/a >> 1$. If this is assumed, the capacitance of a spherical particle is given by

$$C = 4\pi\varepsilon_o a . \tag{3.37}$$

The starting point of the statistical analysis is the expression of the equilibrium constant for the reaction:

$$P_{\ell-1} \rightleftharpoons P_\ell + e - [e\phi_\ell] . \tag{3.38}$$

P_ℓ denotes a particle which is ionized ℓ times, and $e\phi_\ell$ is the increase in the potential energy of the one-particle system. The equilibrium constant defined by

$$K_\ell \equiv N_\ell \, \overline{n_e}/N_{\ell-1} \quad , \tag{3.39}$$

where N_ℓ is the number density of particles ionized ℓ times, is given by the ratio of the appropriate partition functions,

$$K_\ell = (F_\ell F_e)/F_{\ell-1} \quad . \tag{3.40}$$

The partition function for the electron is the translational partition function times a factor of 2 resulting from the degeneracy of the electron spin.

$$F_e = 2[2\pi mkT/h^2]^{3/2} \tag{3.41}$$

The ratio of the particle partition functions is given by

$$\frac{F_\ell}{F_{\ell-1}} = e^{-e\phi_\ell/kT} \quad . \tag{3.42}$$

Here, $e\phi_\ell$ is the sum of the energy required for the electrons to penetrate the potential barrier at the surface, the work function, and the energy required to move the electron from a position just outside the particle to one of zero potential. The latter is the change in the energy in the electrostatic field surrounding the particle. Hence[6]

$$e\phi_\ell = e\Phi + (\ell - \tfrac{1}{2})\frac{e^2}{C} \quad . \tag{3.43}$$

Equations (3.39) to (3.43) combine to give

$$\frac{N_\ell}{N_o} = \left\{\frac{n_E^{(th)}}{n_{eb}}\right\}^\ell \prod_{j=1}^{\ell} e^{-(j-\frac{1}{2})\frac{e^2}{C}} \quad . \tag{3.44}$$

---

6 An alternate derivation of this total "escape energy" is given in Appendix F, where the effect of the curvature of the surface on the long-range mirror-image potential of the work function is investigated.

The values of y_a and z which correspond to the assumption of uniform ionization are

$$y_a = \ln \frac{n_e^{(th)}}{n_{eb}} \tag{3.45}$$

$$z = y_a C kT/e^2 \tag{3.46}$$

Using these relations, equation (3.44) may be written in the form

$$\frac{N_\ell}{N_o} = e^{+(zy_a)/2} e^{-(y_a/2z)(\ell - z)^2} . \tag{3.47}$$

The distribution of ionized states is seen to be a Gaussian function which is peaked at the value of ℓ corresponding to uniform ionization. In addition, the width of the distribution is given by

$$\delta\ell = (z/y_a)^{\frac{1}{2}} \tag{3.48}$$

and

$$\frac{\delta\ell}{\ell_{peak}} = (1/zy_a)^{\frac{1}{2}} . \tag{3.49}$$

Hence, as zy_a becomes large, the relative width of the distribution approaches zero, and the ionization becomes essentially uniform. Also, according to Smith, the difference between z and the actual mean value of the distribution is negligible when

$$e^2/(2CkT) < 3 . \tag{3.50}$$

For spherical particles with $a/b \ll 1$ and at a temperature of 2000°K, the above requirement means that <u>a</u> must be larger than 14 Å, a requirement that, practically speaking, is not very restrictive. It is to be noted that this requirement does not depend upon the magnitude of z. That is, even though z is on the order of unity, the uniform ionization and continuous charge approximations lead to the correct value of n_{eb}.

This discussion has pointed out the following facts.

(1) In the weak shielding regime, the values of n_{eb} predicted by the uniform ionization - continuous charge approach and the statistical approach are in good agreement, except for extremely small particles.

(2) When $y_a z >> 1$, the relative width of the Gaussian distribution of the particles over the ionized states is very much less than unity; that is, the ionization is essentially uniform. Using the values of y_a and z calculated for the alumina-particle system studied in Section III-B-7, the relative half-width is found to be $\delta\ell/\ell_{peak}$ = .020 and hence the distribution curve is very sharply peaked.

IV. DEPENDENCE OF THE ELECTRON DENSITY ON THE INITIAL PARAMETERS

A. Introduction

Two primary objectives of an investigation of the ionization in a gas-particle plasma are first, to arrive at a prediction of the electron density, and second, to obtain an estimate of the extent to which the level of this electron density may be controlled. Hence, it is profitable to determine the dependence of the electron density on the parameters which define the solution for the particle ionization problem which has been studied in Part III.

Since an approximate solution for the potential distribution is known, it is possible to carry out a detailed investigation for problems which fall into the weak shielding regime. However, rough estimates of what will occur for problems that fall into the strong shielding regime can also be made.

B. Strong Shielding Regime

In the strong shielding regime, there are two electron densities of interest. One is the "free" electron density, or the density outside the charge sheath around each particle, n_{eb}. These electrons are not bound close to a particle's surface by strong fields and are relatively free to move throughout the plasma. The second density, which may be of interest, is $\overline{n_e}$. This density is representative of the degree of particle ionization $(z = \overline{n_e}/N)$. These two densities will now be briefly considered.

It has previously been pointed out (see Section III-B-5) that, once the strong shielding regime is encountered, the Debye shielding distance far from a particle, λ_{Db} , will always adjust itself to a value which is approximately equal to b . Hence, from equation (3.7),

$$n_{eb} \sim T/b^2 \\ \sim TN^{3/2} \tag{4.1}$$

That is, as long as the shielding remains in the strong-shielding regime, n_{eb} will be insensitive to changes in a or n_{ea} but will be directly proportional to $TN^{2/3}$. If the total content of liquid or solid phase remains the same in the plasma, but the size of the particles is varied,

$$Na^3 = \text{const.} \tag{4.2}$$

and

$$n_{eb} \sim T/a^2 . \tag{4.3}$$

Therefore, the electron density outside the sheath may most efficiently be increased by dividing the non-gaseous phase into finer particles.

If it is assumed that the situation of interest is well into the strong shielding regime ($\sigma >> 1$), then the value of the mean electron density, $\overline{n_e}$, will be approximately equal to the number of electrons packed close to a particle in its charge sheath times the number density of particles. Since the characteristic distance which the sheath extends from the surface of a particle is λ_{Da} , the mean electron density is given by

$$\overline{n_e} \sim a^2 \lambda_{Da} n_{ea} N \\ \sim a^2 N (T n_{ea})^{\frac{1}{2}} \tag{4.4}$$

Again, if Na^3 is held constant,

$$\overline{n_e} \sim \frac{1}{a}(T n_{ea})^{\frac{1}{2}} \quad . \tag{4.5}$$

Hence, $\overline{n_e}$ will be an exponential function of T and $e\Phi$ because it is directly proportional to $n_{ea}^{\frac{1}{2}}$. In addition, $\overline{n_e}$ may also be increased by dividing the non-gaseous material into finer particles.

C. Weak Shielding Regime

The dependence of the electron density on the initial parameters for the problems which fall into the weak shielding regime is of interest for two reasons. First, for the majority of the current engineering problems, the condition, $a/\lambda_{Da} < 5$, is satisfied and the weak shielding regime is therefore primarily the one of practical interest. Second, the existence of the shielding tends to nullify the strong dependence of the electron density on emission phenomena and, since an approximate solution is known for the potential distribution, this effect can be readily analyzed.

The main objective here is to determine the sensitivity of the electron density of the plasma, n_{eb}, to variation in the parameters of the system. These parameters are taken to be $e\Phi$, the thermionic work function of the particle material; T, the temperature; a, the particle radius; and N, the particle number density. The variations of n_E alone are done in essence by the variations of $e\Phi$.

Variations in the parameters will take two forms. Each parameter will be assumed to be uniform throughout the plasma but the value of one or more of them changed from some initial value. Or, a non-uniformity in one or more of the parameters will be introduced.

The main interest is in the variation of $\overline{n_e}$. However, by the assumption of weak shielding, $n_{eb} \simeq \overline{n_e}$, so that it will serve equally well to study n_{eb}. It will be seen that the variations of ϕ_a which accompany those of n_{eb} may also be readily calculated.

1) Uniform Variations

(a) Formulation. The relationships which define the value of n_{eb} are the following: the potential at the surface of the particle,

$$\phi_a = \left(\frac{n_{eb}}{N}\right)\left(\frac{e}{4\pi\epsilon_o a}\right)\left\{1 - \frac{3}{2}\left(\frac{a}{b}\right) + \frac{1}{2}\left(\frac{a}{b}\right)^3\right\} \;; \tag{4.6}$$

the electron density at the surface of the particle,

$$n_{ea} = BT^{3/2} e^{-e\Phi/kT} \;; \tag{4.7}$$

and the relation resulting from the assumption of equilibrium,

$$n_{ea} = n_{eb}\, e^{e\phi_a/kT} \;. \tag{4.8}$$

In order to calculate the changes which occur in n_{eb} and ϕ_a resulting from changes made in one or more of the parameters of the system, it is convenient to refer all changes to a reference state. That is, all the variables will be non-dimensionalized by reference state values. The selection of the parameters for this reference state is arbitrary. Letting primes denote reference state conditions, the dependent variables in the variation process are

$$\nu = n_{eb}/n_{eb}' \;, \tag{4.9}$$

$$\eta = \phi_a/\phi_a' \;; \tag{4.10}$$

the independent variables are

$$\omega = e\Phi/e\Phi' \;, \tag{4.11}$$

$$\tau = T/T' \;, \tag{4.12}$$

$$\alpha = a/a' , \tag{4.13}$$

$$\mu = N/N' , \tag{4.14}$$

and the normalized particle potential and work function of the reference state are

$$y_a = e\phi_a' / kT' , \tag{4.15}$$

$$w = e\Phi' / kT' . \tag{4.16}$$

(b) Variation of the work function. By setting all of the independent variables equal to unity except ω and dividing equations (4.6) to (4.8) by those corresponding to the reference state, the following are obtained.

$$\eta = \nu \tag{4.17}$$

$$\nu e^{y_a(\eta-1)} = e^{w(1-\omega)} \tag{4.18}$$

Hence

$$\nu = 1 + \frac{w}{y_a}(1-\omega) - \frac{1}{y_a} \ln \nu , \tag{4.19}$$

$$\left.\frac{d\nu}{d\omega}\right)_{\omega=1} = - \frac{w}{y_a+1} . \tag{4.20}$$

Equations (4.19) and (4.20) clearly show the manner in which the effects of the shielding tend to dominate those of emission as the potential of the reference state is increased. When y_a is less than or on the order of unity, the value of n_{eb} changes in an exponential manner with the work function. That is, for n_{eb} on the order of n_{ea}, the level of ionization of the system is determined by emission phenomena. However, when y_a is of the same magnitude as w, $(e\phi_a' \approx e\Phi')$, the level of ionization is controlled by shielding effects and is relatively insensitive to the value of n_{ea}. For this case, $-d\nu/d\omega\big|_{\omega=1} \approx 1$, and the relation between the level of ionization and the work function is not an exponential one but nearly a linear one. It is clear that the reason for this is the changes in ϕ_a which accompany those of n_{ea}. This second situation is the one which prevails in most systems in which the particles are composed of a good thermionically-emitting material and are at a high temperature. For these systems, errors made in the estimation of $e\Phi$ or changes made in the actual value of $e\Phi$ will not greatly affect the level of ionization. These effects are clearly shown in Figure 13, where ν is plotted as a function of ω for $w = 15$ and $y_a = 1, 5, 10, 15$, and ∞.

In the illustration considered in Section III-B-7, $y_a = 3.58$ and $w = 15.75$. If the work function is increased from 3.77 ev to 4.50 ev, $\omega = 1.195$, $n_{ea}/n_{ea}' = 0.0514$, and $\nu = n_{eb}/n_{eb}' = 0.40$. That is, the electron density of plasma is decreased only by a factor of 1/2.5, even though the electron density due to thermionic emission at the surfaces of the particles has been decreased by a factor of 1/19.5.

It is seen that as the particle potential becomes large, the presence of the shielding has the same effect as that found in the strong shielding regime but to lesser degree; the changes which occur in the value of λ_{Da} will produce relatively small changes in the value of λ_{Db}.

(c) <u>Variation of the temperature.</u> The same procedure that was carried out for the variation of the work function is followed here, but τ, the ratio of temperature of the actual state to that of the reference state, is the independent variable. The equations thus

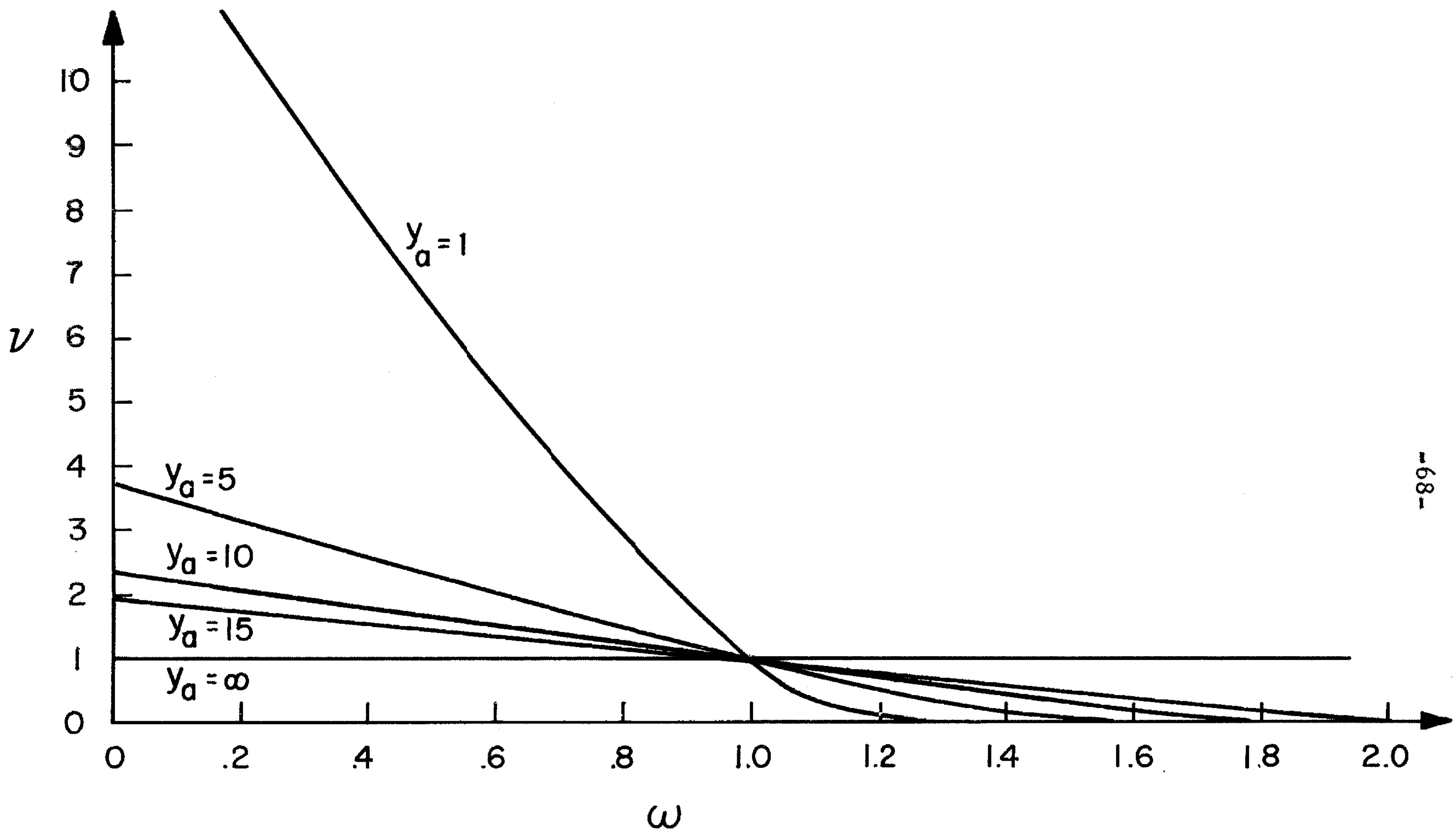


Figure 13. Fractional Changes in the Electron Density Versus Fractional Changes in the Work Function, w = 15.

obtained are:

$$\eta = \nu \ , \tag{4.21}$$

$$\nu e^{y_a(\frac{\nu}{\tau} - 1)} = \tau^{3/2} e^{w(1 - \frac{1}{\tau})} \ . \tag{4.22}$$

Hence,

$$\nu = \tau + \frac{w}{y_a} (\tau - 1) + \frac{\tau}{y_a} \ln \left(\frac{\tau^{3/2}}{\nu}\right) , \tag{4.23}$$

$$\left.\frac{d\nu}{d\tau}\right)_{\tau=1} = \frac{w + y_a + 3/2}{y_a + 1} \ . \tag{4.24}$$

Again, the same general behavior is exhibited. When y_a is small with respect to w, emission phenomena dominate. When y_a is of the same magnitude as w, shielding effects are more important. However, in this second case the effect is twofold, as is evidenced by equation (4.24). As the temperature is increased, the thermal energy of the electrons increases and more electrons are able to penetrate the potential barriers of both the work function and the particle potential. Opposing this is the associated increase in the particle potential. Chiefly because of this twofold effect, the ionization of the system is more sensitive to fractional changes of the temperature than of the work function.

In Figure 14, ν is plotted as a function of τ for the same values of w and y_a used in the case of the work function variation.

(d) <u>Variation of the particle radius and density.</u> For the majority of practical cases, b/a is much larger than unity. Therefore, it is not very restrictive to rewrite equation (4.6) as

$$\phi_a = \left(\frac{n_{cb}}{N}\right)\left(\frac{e}{4\pi\varepsilon_o a}\right) \tag{4.25}$$

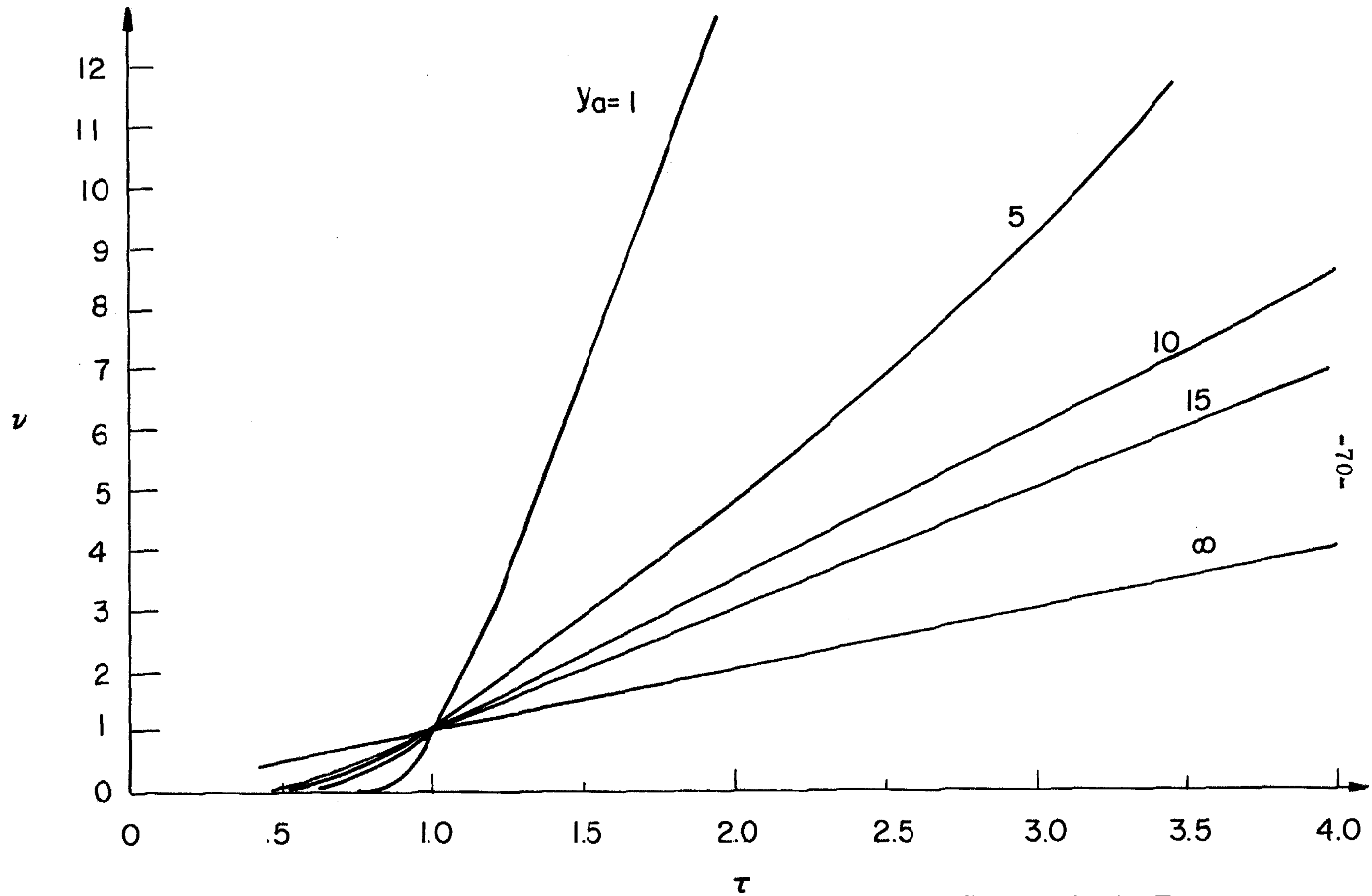


Figure 14. Fractional Changes in the Electron Density Versus Fractional Changes in the Temperature.

Since this is the only governing relation in which a and N appear explicitly and they appear as a product, any fractional variation of a will have the same effect as the corresponding fractional variation of N. Therefore, only the variation of a will be treated, it being understood that the conclusions drawn hold equally well for N.

The same procedure is followed as in the previous two cases but with the independent variable being α, the ratio of the actual radius to the reference radius. The resulting equations are

$$\eta = \nu / a \tag{4.26}$$

$$\nu = e^{y_a(1-\nu/a)} \quad . \tag{4.27}$$

Thus,

$$\nu = \alpha \left[1 - \frac{1}{y_a} \ln \nu \right] \tag{4.28}$$

$$\left. \frac{d\nu}{d\alpha} \right)_{\alpha=1} = \frac{y_a}{y_a + 1} \quad . \tag{4.29}$$

When y_a is small, it is seen that the size of the particle has little effect on the ionization since the values of a and n_{ea} are unrelated. For the case in which shielding is dominant and y_a is much larger than unity, $n_{eb} \sim a$ because of the small decrease in ϕ_a which accompanies an increase in a. Hence, the level of ionization is at most directly proportional to the particle radius or the particle density.

These trends are illustrated in the plot of ν versus α given in Figure 15 for $y_a = 1$, 5, 10, 15, and ∞.

(e) Variation of the particle radius with the mass fraction held constant. For many physical situations, it is possible to obtain a

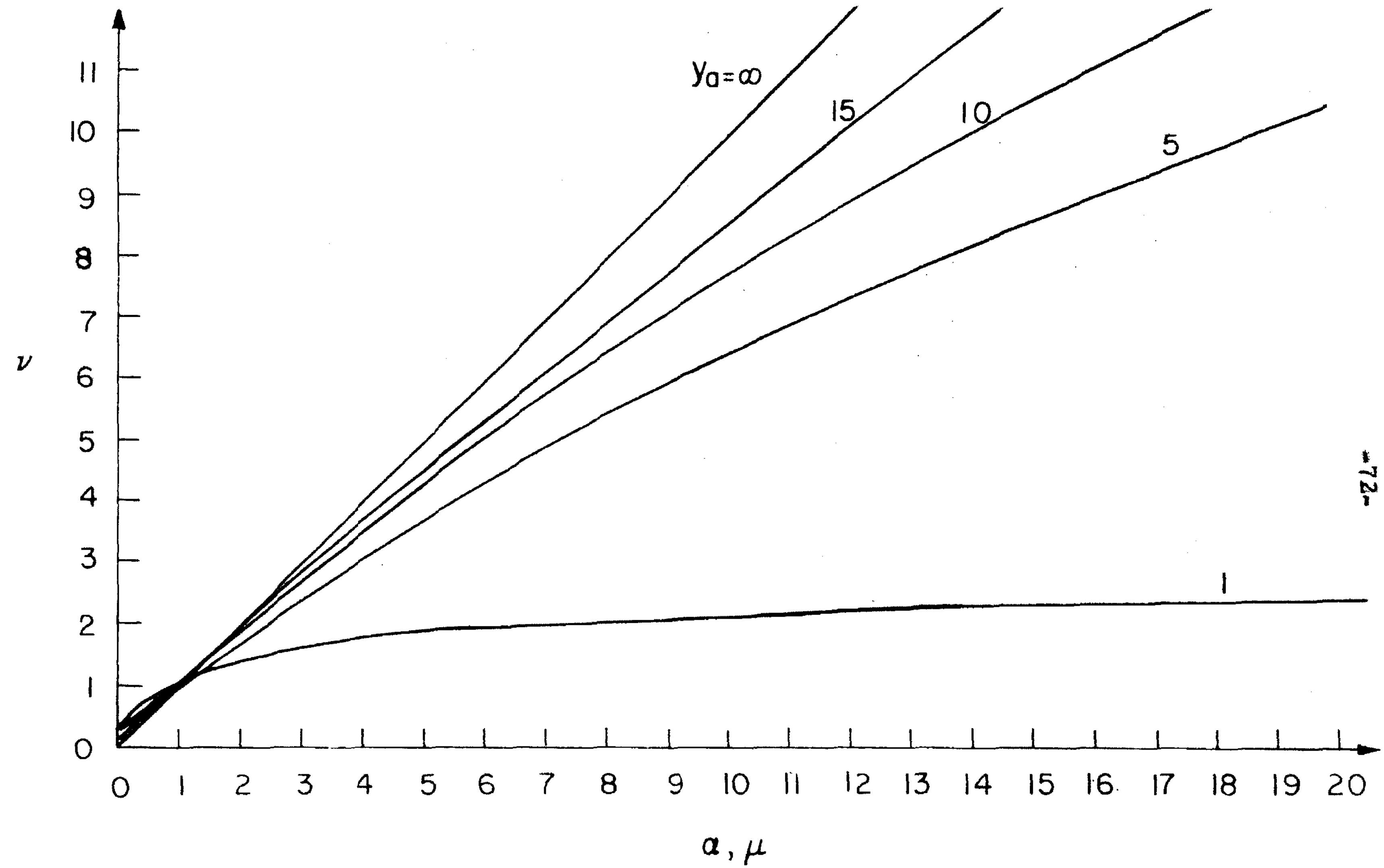


Figure 15. Fractional Changes in the Electron Density Versus Fractional Changes in the Particle Radius or the Particle Number Density.

fairly good estimate for the total solid or liquid mass in the system but not for the particle radius. It is therefore instructive to consider the variation of the particle radius when the total mass of the non-gaseous phase in the system is held constant.

The total mass of the non-gaseous material is proportional to Na^3 ; therefore, an additional relation is

$$\mu\alpha^3 = 1 . \tag{4.30}$$

When the same procedure used in previous cases is followed, and it is again assumed that b/a is much larger than unity, the following are obtained:

$$\eta = \nu , \tag{4.31}$$

$$\nu = e^{-y_a(\nu\alpha^2 - 1)} . \tag{4.32}$$

Hence,

$$\nu = \frac{1}{\alpha^2}\left[1 - \frac{1}{y_a} \ln \nu \right] , \tag{4.33}$$

$$\left.\frac{d\nu}{d\alpha}\right|_{\alpha=1} = - \frac{2 y_a}{y_a + 1} \tag{4.34}$$

As in the previous case, where <u>a</u> alone was varied, the ionization is not sensitive to the particle radius when y_a is small. In contrast, however, when y_a is much larger than unity, $n_{eb} \sim 1/a^2$, which is the same result as that obtained in the case of strong shielding. Thus, the smaller the size of the particles into which a given amount of non-gaseous material is divided, the larger will be the level of ionization. The limiting value of electron density is, of course, n_E . These effects are illustrated in Figure 16, where ν is plotted as a function of α for the condition of $\mu\alpha^3 = 1$.

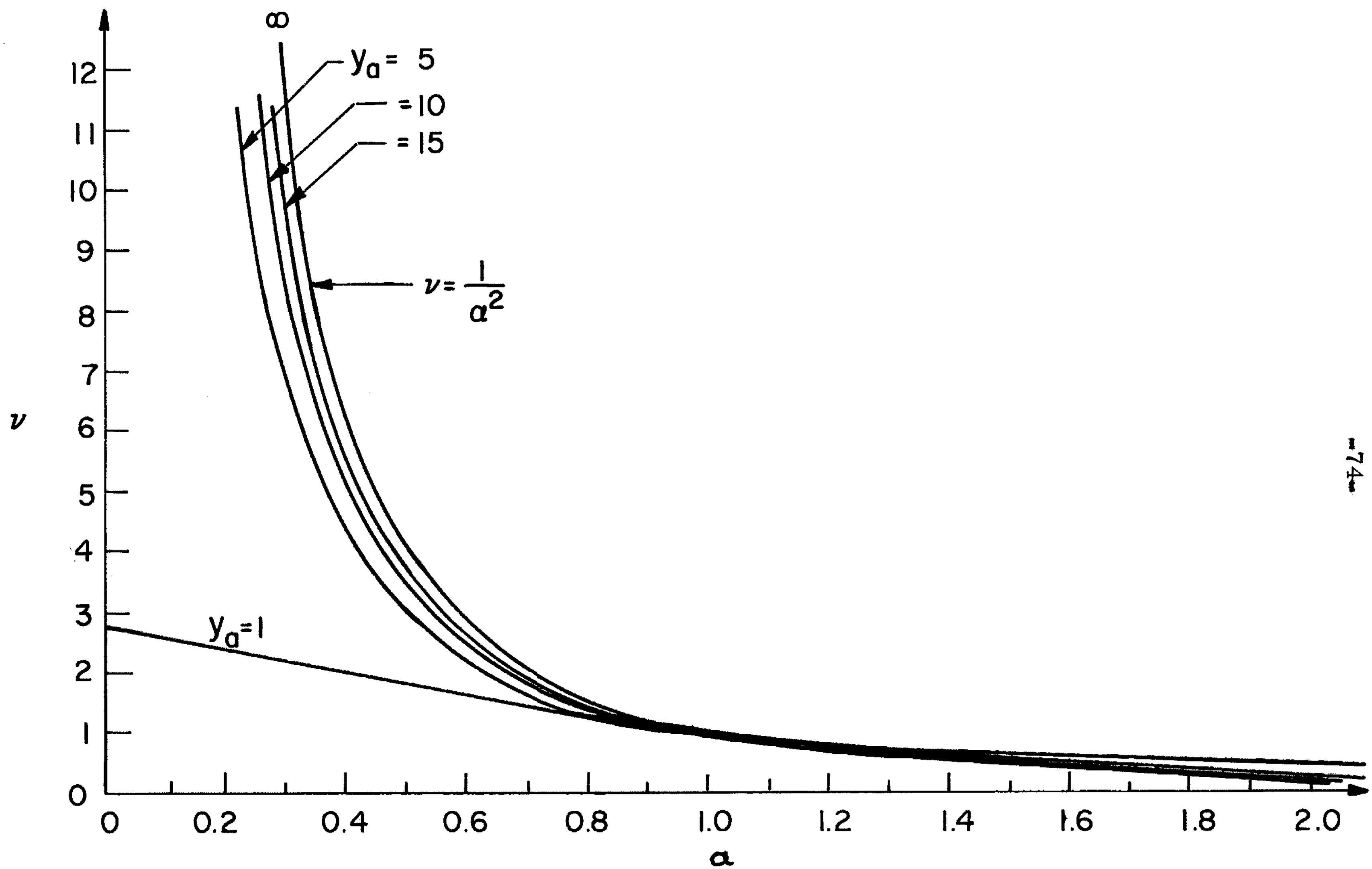


Figure 16. Fractional Changes in the Electron Density Versus Fractional Changes in the Particle Radius for a Constant Mass Fraction.

These variations may be demonstrated in a second way. $(b/a)^3$ is proportional to the ratio of gaseous to non-gaseous material and n_{ea} is independent of the value of $\underline{a}$. Thus, Figures 7a and 7b, in which $\overline{n_e}/n_{ea}$ is plotted as a function of b/λ_{Da} for fixed values of b/a, may also be used to find the dependence of $\overline{n_e}$ on $\underline{a}$. That is, $\overline{n_e} \sim \overline{n_e}/n_{ea}$ and, for fixed b/a, $a \sim b/\lambda_{Da}$. This plot also covers the range in which b/a is on the order of unity as well as the strong shielding regime.

It is also of interest to see how the level of ionization varies with the total exposed surface area of the particles. Denoting this area by S,

$$S = N(4\pi a^2), \tag{4.35}$$

$$S \sim 1/a, \tag{4.36}$$

$$n_{eb} \sim S^2 . \tag{4.37}$$

(f) Summary. Several general statements may be made as a result of this study of the uniform variations in the parameters of the system for weak shielding.

(1) When the particle potential, $e\phi_a$, is much less than the work function, $e\Phi$, the electron density is exponentially dependent on the work function and the temperature. Also, it is most sensitive to fractional changes in the temperature.

(2) If the particle potential, $e\phi_a$, is much less than or approximately equal to kT, the electron density is very insensitive to changes in the $\underline{a}$ or N.

(3) When $e\phi_a$ is of the same magnitude as $e\Phi$, the electron density is linearly related to the work function and temperature, and

it is still more sensitive to fractional changes in the temperature.

(4) If $e\phi_a$ is much larger than kT, the electron density is directly proportional to a or N, but varies as $1/a^2$ when Na^3 is held constant.

In situations in which these last two statements are applicable, it is probable that the largest error made in predicting the value of n_{eb} will be due to errors made in determining a, since the percentage error made in the estimation of a is likely to be much larger than those made in the estimation of $e\Phi$ or T. Also, since it is a which may be most easily changed by large factors for a given mass fraction, it is the particle size which can be most effectively used in controlling the electron density of the plasma.

2) Unequal Gas and Particle Temperatures - It has been seen in Section II-D that it is possible for the characteristic time for gas-particle thermal equilibration to be of the same magnitude as the characteristic time for gas temperature change. Thus, in a plasma in which the gas temperature is changing rapidly, the particle temperature can be appreciably different than that of the gas. It will now be shown that it is the particle temperature, rather than the gas temperature, which is of primary importance in establishing the magnitude of the electron density.

In order to determine the maximum possible effect which unequal gas and particle temperatures may have on the particle ionization, it will be assumed that the electrons in the plasma are completely thermalized with the gas. That is, the electrons are assumed to have a Maxwellian energy distribution at the gas temperature. If the electrons were thermalized at the particle temperature, there would be no effect due to the presence of a gas at a different temperature.

The equations which are applicable are the following:

$$\phi_a = \left(\frac{n_{eb}}{N}\right)\left(\frac{e}{4\pi\varepsilon_o a}\right)\left\{1 - \frac{3}{2}\left(\frac{a}{b}\right) + \frac{1}{2}\left(\frac{a}{b}\right)^3\right\} \tag{4.38}$$

$$n_E = B\, T_p^{3/2}\, e^{-e\Phi/(kT_p)} \tag{4.39}$$

$$n_{ea} = n_{eb}\, e^{(e\phi_a)/(kT_G)} \tag{4.40}$$

$$n_E\, T_p^{1/2} = n_{ea}\, T_G^{1/2} \tag{4.41}$$

where T_G and T_p are the gas and particle temperatures, respectively. The last relation is the equation of steady state in which the electron fluxes to and from a particle are equated to one another. In this relation, it has been assumed that the electron-gas mean free path, λ_{eG}, is very much less than the particle radius. It will be shown in Section V-C-1 that, if $\lambda_{eG} \gg a$, this relation would be altered at most by an interchange of T_p and T_G. Also, the Boltzmann factor would contain the particle temperature rather than the gas temperature. Thus, since the major effect is in the Boltzmann factor, this analysis should lead to results whose magnitude represents an upper bound for the results to be expected under a wide range of conditions.

The same procedure that was used in the case of uniform variations will be followed here except that the reference temperature will be taken to be that of the particles.

$$\tau = T_G/T_p \tag{4.42}$$

The resulting equations are

$$\eta = \nu \ , \tag{4.43}$$

$$\tau^{\frac{1}{2}} \nu \, e^{y_a\left(\frac{\nu}{\tau} - 1\right)} = 1 \ , \tag{4.44}$$

and

$$\nu = \tau - \frac{\tau}{y_a} \ln \left(\tau^{\frac{1}{2}} \nu\right) , \tag{4.45}$$

$$\left.\frac{d\nu}{d\tau}\right|_{\tau=1} = \frac{y_a - \frac{1}{2}}{y_a + 1} \ . \tag{4.46}$$

As may have been anticipated, the gas temperature does not alter the ionization in an exponential manner. For y_a small, the level of ionization decreases with increasing T_G because of the increased electron flux to the particle. When y_a is large, the Boltzmann factor determines the value of n_{eb}, and the level of ionization is therefore proportional to T_G. By comparison of equation (4.46) with the corresponding one for the case of uniform temperature variation, equation (4.24), it is concluded that the "effective temperature" of the system is that of the particles, and the effect of unequal gas and particle temperatures is relatively small.

Again, these changes are shown in Figure 17, where ν is plotted as a function of τ for $y_a = \frac{1}{2}$, 1, 5, 10, 15, and ∞.

3) <u>Non-uniform Distribution of Particle Radius and Temperature</u> - At the beginning of the fundamental shielding problem, it was explicitly assumed that all of the particle parameters which influence the ionization of the system were uniform. However, in an actual physical situation, this is not the case. In most systems, there will be a distribution of particle sizes and, if the temperature of the system is not static, there will be an associated non-uniformity in the

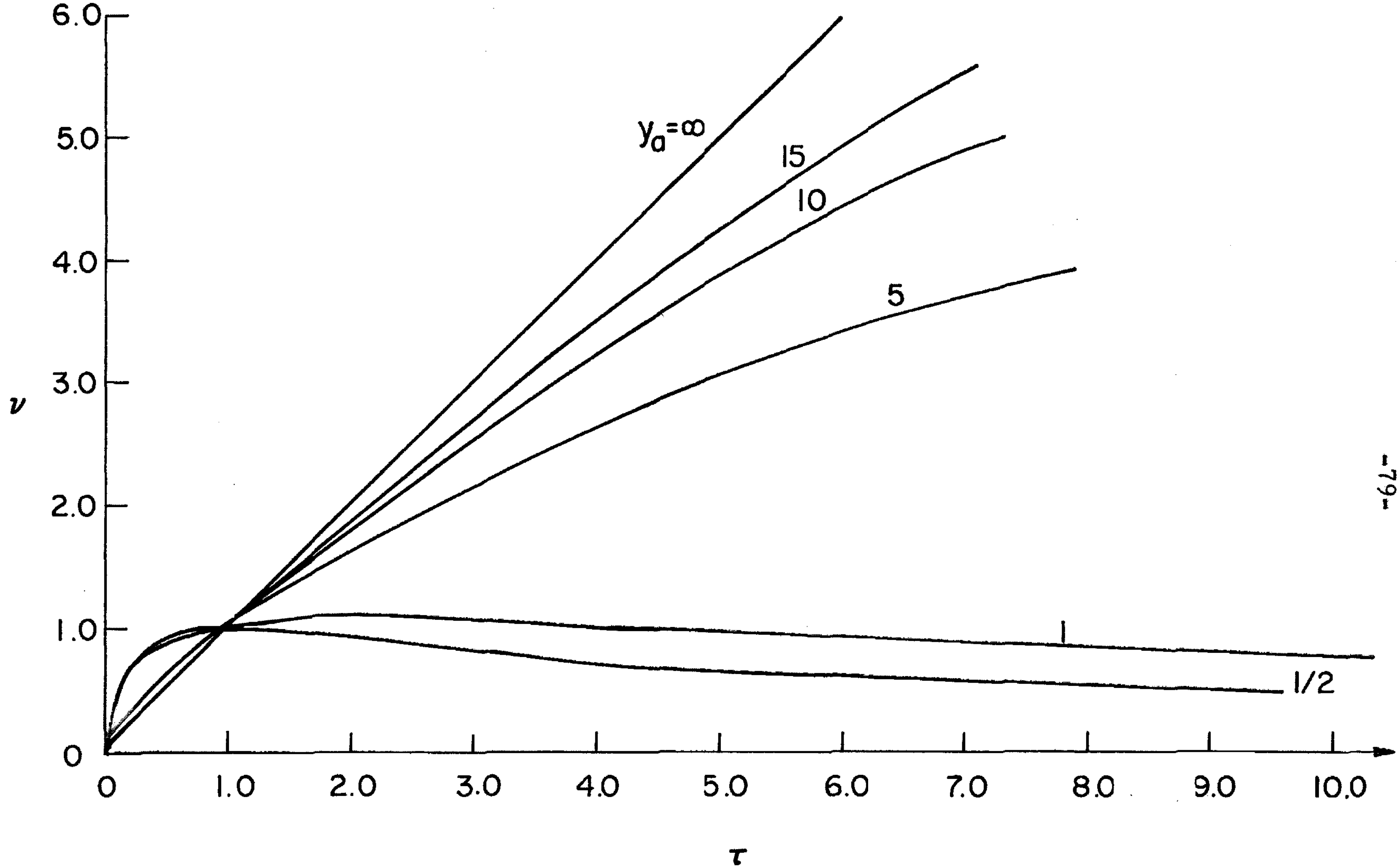


Figure 17. Fractional Changes in the Electron Density Versus Fractional Changes in the Gas Temperature.

particle temperature due to the dependence of particle - gas thermal equilibration time on particle size. Therefore, it is instructive to consider an example in which a spectrum of particle sizes and the related spectrum of particle temperatures are taken into account. The major point which is made in this section is that, if the values of $\underline{a}$, N, and T_p are properly chosen for problems in the weak shielding regime, the relations which correspond to the uniform particle property assumption may be used to calculate n_{eb} with negligible error.

It will again be assumed that the temperature of the electrons external to the particles is equal to the gas temperature. The approximate solution for the potential is still applicable, since it results from a property of the spherical geometry and not uniform conditions. However, an equal volume may not be allotted to each particle since the radius, surface density of electrons, and degree of ionization of each particle are not the same. One quantity which each particle does have in common, though, is the electron density at the outer radius, n_{eb}. Hence, the volume allotted to each particle will be proportional to its degree of ionization. Also, for very large particles where the strong shielding regime is encountered, or for very small particles where the emission phenomena may be altered, the use of the approximate solution and a uniform work function is invalid. However, because the number of particles at these two extremes approaches zero, this procedure is justified.

Letting i denote the i^{th} type of particle and assuming $b_i/a_i >> 1$, the potential of the i^{th} particle is

$$\phi_{ai} = \frac{e\, n_{eb}\, b_i^3}{3\, \varepsilon_o a_i} \quad . \tag{4.47}$$

From the conservation of volume,

$$\frac{4}{3}\pi \sum N_i\, b_i^3 = 1 \, . \tag{4.48}$$

The remaining three relations as used previously are

$$n_E = B\, T_i^{3/2}\, e^{-e\Phi/kT_i} \tag{4.49}$$

$$n_{Ei}\, T_i^{1/2} = n_{eai}\, T_G^{1/2} \tag{4.50}$$

$$n_{eai} = n_{ebi}\, e^{+(e\phi_{ai})/(kT_G)} \tag{4.51}$$

In order that the changes in the ionization which occur due to a distribution of $\underline{a}$ and T may be measured with respect to some base, a reference state will again be defined. The mean particle radius and the total content of the non-gaseous phase of this state will be taken to be the same as in the actual system. That is,

$$N'a' = \sum N_i a_i \quad , \tag{4.52}$$

$$N'a'^3 = \sum N_i a_i^3 \; . \tag{4.53}$$

The reference state temperature is the gas temperature

$$T' = T_G \, , \tag{4.54}$$

and the work function is that which is common to all the particles.

The same variables and reference state parameters will be used as were defined in equations (4.9) to (4.16) with the addition

$$\beta_i = b_i / b' \; . \tag{4.55}$$

Upon eliminating n_{Ei} and n_{eai} and dividing the remaining equations by the corresponding ones for the reference state, the following are obtained:

$$\eta_i = \nu\beta_i^{\,3}/\alpha_i \tag{4.56}$$

$$\tau_i^{\,2}\, e^{-w(1/\tau_i - 1)} = \nu\, e^{+y_a(\eta_i - 1)} \tag{4.57}$$

$$\sum \beta_i^{\,3}\mu_i = 1 \tag{4.58}$$

$$\sum \alpha_i \mu_i = 1 \tag{4.59}$$

$$\sum \alpha_i^{\,3}\mu_i = 1 \tag{4.60}$$

Using equation (4.59), equations (4.56) to (4.58) reduce to

$$y_a(\nu - 1) + \ln \nu = w - \sum \left\{ \frac{w}{\tau_i} - 2 \ln \tau_i \right\} \alpha_i \mu_i \tag{4.61}$$

which may be solved for ν once $\mu_i(\alpha_i, \tau_i)$ is known. For the problem at hand, μ_i will be assumed to be a continuous function of α_i and the subscript i will be dropped.

The case to be considered here is one in which the particles are hotter than the gas. Also, the particle temperature will be related to the particle radius in a way which is motivated by the assumption that the particle temperature change is governed by convection heat transfer to the gas. Since the thermal equilibration time between a small particle and its gaseous surroundings varies as the square of the particle radius when Stokes' drag law is applicable, a reasonable relation to assume between τ and α is

$$\tau - 1 = \epsilon_\tau \alpha^2 \tag{4.62}$$

where ϵ_τ defines the magnitude of the temperature spread. Equation (4.61) becomes

$$y_a(\nu-1)+\ell n\nu = w - \int_0^\infty \{\frac{w\alpha}{(1+\epsilon_T\alpha^2)} - 2\alpha\ell n(1+\epsilon_T\alpha^2)\}\mu(\alpha)d\alpha \quad . \tag{4.63}$$

Without further information, it is also reasonable to assume that $\mu(\alpha)$ is peaked at a value close to $\alpha = 1$ and falls off to zero as α approaches zero or infinity. A function of this form is

$$\mu(\alpha) = \frac{C_1}{\alpha} \exp\{-\frac{1}{\epsilon_\alpha}(\frac{C_2}{\alpha^2} + \alpha^2)\} , \tag{4.64}$$

where ϵ_α defines the width of the distribution. The constants of this particle size distribution are determined by the normalization conditions [equations (4.54) and (4.55)]

$$\int_0^\infty \alpha\mu(\alpha)\, d\alpha = 1 , \tag{4.65}$$

$$\int_0^\infty \alpha^3\mu(\alpha)\, d\alpha = 1 . \tag{4.66}$$

Hence (see Appendix G),

$$\mu(\alpha) = \frac{2}{(\pi\epsilon_\alpha)^{\frac{1}{2}}} \frac{1}{\alpha} \exp\left\{-\frac{1}{\epsilon_\alpha}\left[\frac{\left(1-\frac{\epsilon_\alpha}{2}\right)^2}{\alpha^2} + \alpha^2 - (2-\epsilon_\alpha)\right]\right\} \tag{4.67}$$

Plots of this distribution for various values of ϵ_α are shown in Figure 18. In the limit of $\epsilon_\alpha \to 0$, $\mu(\alpha)$ becomes a delta function located at $\alpha = 1$.

When the above expression for $\mu(\alpha)$ is introduced into equation (4.63), two integrations must be performed. The first may be done by setting up and solving a differential equation with the integral as the dependent variable. The second integral is performed, with negligible error introduced, by expanding $\ell n(1+\epsilon_T\alpha^2)$ about the maximum value

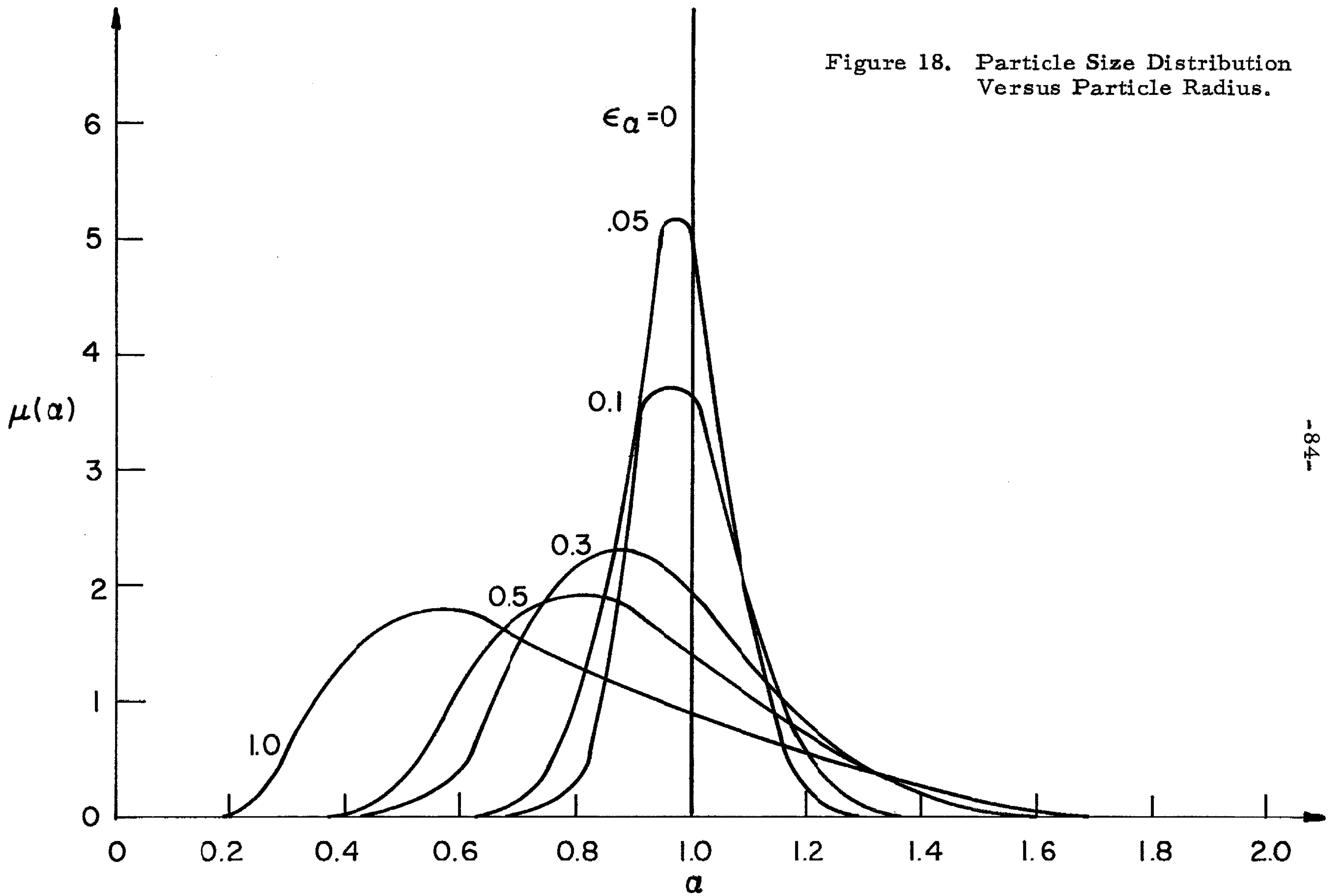


Figure 18. Particle Size Distribution Versus Particle Radius.

of $\alpha\mu(\alpha)$. These integrations are performed in Appendix H. The resulting expression which determines $\nu(y_a, w, \varepsilon_\alpha, \varepsilon_T)$ is

$$y_a(\nu-1)+\ln\nu = w\left[1-\left(\frac{\pi}{\varepsilon_\alpha\varepsilon_T}\right)^{\frac{1}{2}}\left(1-\mathrm{erf}\left\{\left[\frac{1}{\varepsilon_\alpha\varepsilon_T}\right]^{\frac{1}{2}}+\left[1-\frac{\varepsilon_\alpha}{2}\right]\left[\frac{\varepsilon_T}{\varepsilon_\alpha}\right]^{\frac{1}{2}}\right\}\right)\right.$$

$$\left.\left(\exp\left\{\frac{1}{\varepsilon_\alpha}\left[\frac{1}{\varepsilon_T}+\left(1-\frac{\varepsilon_\alpha}{2}\right)^2\varepsilon_T+2\right]-1\right\}\right)\right]+2\ln\left\{1+\varepsilon_T\left[1-\frac{\varepsilon_\alpha}{2}\right]\right\}+\frac{\varepsilon_T\varepsilon_\alpha}{1+\varepsilon_T\left(1-\frac{\varepsilon_\alpha}{2}\right)}. \tag{4.68}$$

When the error function is close to unity, equation (4.68) reduces to:

$$y_a(\nu-1)+\ln\nu \simeq w\left(\frac{\zeta}{\zeta+1}\right)+2\ln(1+\zeta)+\frac{\varepsilon_T\varepsilon_\alpha}{\zeta+1}, \tag{4.69}$$

where

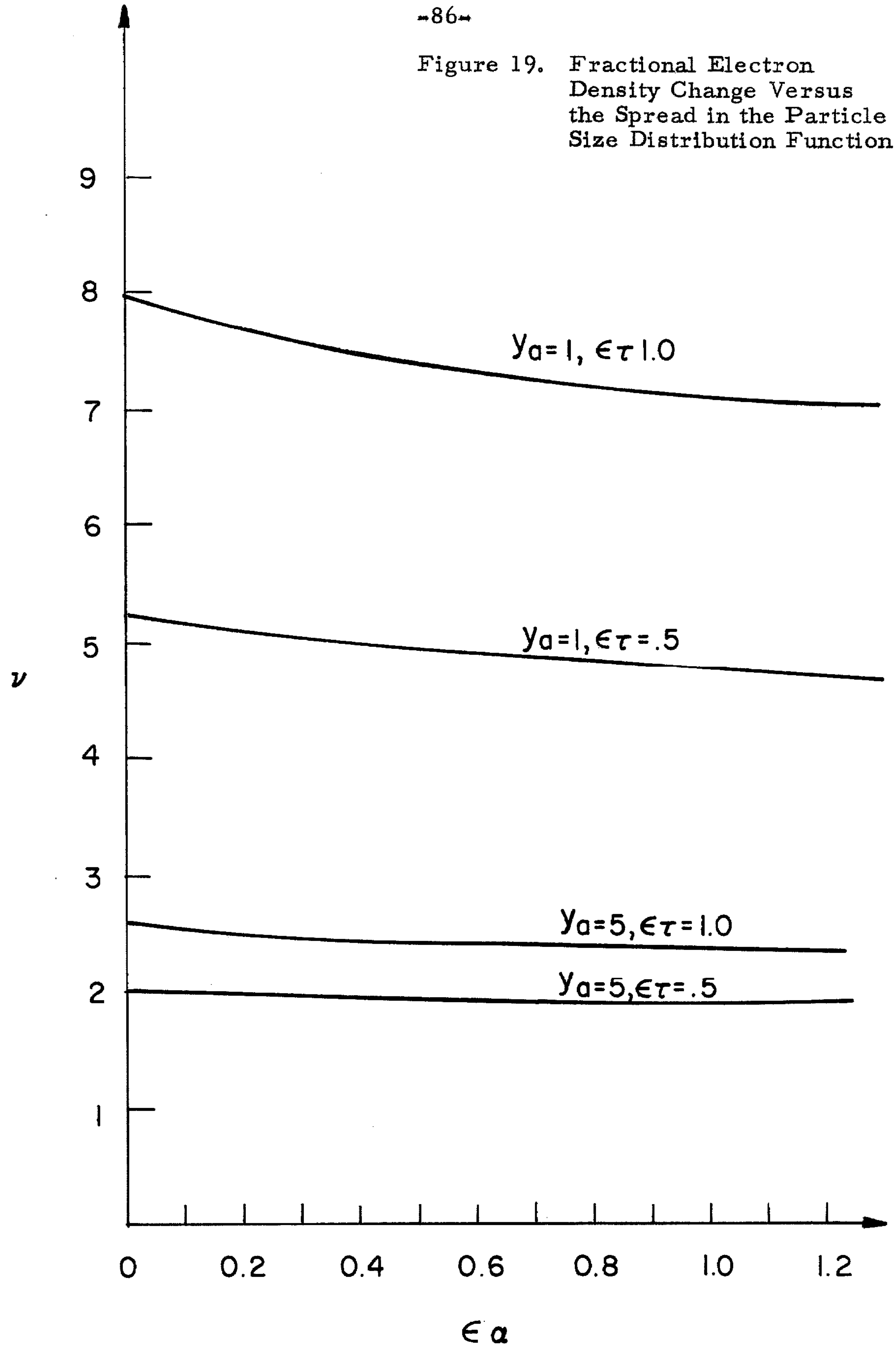
$$\zeta = \left(1-\frac{\varepsilon_\alpha}{2}\right)\varepsilon_T . \tag{4.70}$$

When the particle temperature is equal to the gas temperature everywhere, $\varepsilon_T = 0$, equation (4.69) reduces to $\nu = 1$. The reason for this may be traced back to the normalization conditions, equations (4.65) and (4.66), and is not related to the form of $\mu(\alpha)$ chosen. Therefore, when the reference state values of a' and N' defined by equations (4.52) and (4.53) are used in calculating n_{eb} in the fundamental shielding problem, no error within the limits of accuracy imposed by the assumptions of this analysis will be introduced by the assumption of uniform particle size.

For the case in which both the particle radius and temperature are uniform but the particle temperature is $(1+\varepsilon_T)$ greater than the gas temperature, that is, for $\varepsilon_\alpha = 0$, equation (4.69) becomes

$$y_a(\nu-1)+\ln\nu = \frac{w\varepsilon_T}{1+\varepsilon_T}+2\ln(1+\varepsilon_T) \tag{4.71}$$

Figure 19. Fractional Electron Density Change Versus the Spread in the Particle Size Distribution Function



and

$$\left.\frac{d\nu}{d\epsilon_\tau}\right)_{\epsilon_\tau=0} = \frac{w+2}{y_a+1} \, . \qquad (4.72)$$

In Figure 19, ν is plotted as a function of the width of the particle size distribution, ϵ_α, for various values of ϵ_τ and y_a and $w = 15$. The most striking feature is the almost complete lack of dependence of ν upon ϵ_α. This is true even when the problem is dominated by emission phenomena, $y_a = 1$, and the corresponding temperature spread is large, $\epsilon_\tau = 1$. Of course, there is the strong dependence on y_a and ϵ_τ as predicted by equations (4.71) and (4.72). It is to be expected that if a different form for the particle size distribution was used, and the particle radius and temperature were related in a different manner, the details of the effect might be somewhat altered; however, its magnitude should remain the same.

Thus, the difference between the electron density found in the reference state system and the actual system will be small when the parameters of the reference state are properly chosen. The values of $\underline{a}$ and N should be defined such that the mean particle radius and total non-gaseous content are the same for both systems. The value of T_p, for the reference state, should be that which corresponds to the value of the reference state radius.

V. NON-EQUILIBRIUM IONIZATION IN A GAS - PARTICLE PLASMA

A. Introduction

When thermionically-emitting particles are present in an ionized gas, there are many interesting interactions which may take place between the two types of ionization. In fact, there is a whole spectrum of problems to be associated with the many situations which fall between the two extreme cases of negligible gas ionization, the case just investigated, and negligible particle ionization. Some of the special cases located between these two extremes will now be considered. However, the major phenomenon under study will be a mechanism by which an electron density may exist in a gas-particle system which can be several orders of magnitude larger than that which would be expected from either particle or gas ionization alone.

When hot particles are in a relatively cool but ionizable background gas, the possibility exists that the net energy flux in the electrons emitted from and returning to the particles is sufficient to maintain the temperature of the electron gas outside the particles at a value significantly greater than the temperature of the heavy species and thereby enhance the gaseous ionization. The physical property which makes this phenomenon possible is the relatively large differences in electron and atomic masses. When elastic collisions with heavy species constitute the major power loss from the electron gas, this large mass discrepancy will effectively insulate the electrons from the heavy species, and the electron temperature can be almost equivalent to that of the particles. The role of the hot particles in elevating the electron temperature in this situation is not unlike that of the electric field which gives rise to non-equilibrium conductivity in a plasma [29, 30, 31]. Both the hot particles and electric field provide a power input to the electron gas which is balanced by the power lost to the heavy species through collisions.

In this section, Part V, the equations which are required in order to analyze non-equilibrium ionization phenomena in a gas - particle plasma will be formulated. Although the major interest is directed toward the special case of gaseous ionization enhancement, the formulation will possess enough generality so that other non-equilibrium and equilibrium situations may be readily considered.

The assumptions and model which are to be employed will first be discussed. Then, since the calculation of the electron or ion number and energy fluxes to and from the particles is important in this analysis, but yet complex under some conditions, the electron and ion fluxes will next be investigated in some detail. In addition to its importance to this analysis, the calculation of these fluxes is an interesting problem in its own right. Then, in Part VI, the relations which have been derived in this section will be used to consider several special cases of both equilibrium and non-equilibrium ionization.

Throughout both the formulation and the application of non-equilibrium theory in a gas - particle plasma, the main effort will be to investigate, in a relatively simple way, the physically important and interesting phenomena.

B. Assumptions and Model

1) Quasi-Steady Approximation - The concept of a static system with three distinct temperatures, particle, gas, and electron, is a realistic one for the situations in which the characteristic times for the particle or the gas temperature changes are long compared to the characteristic time for the establishment of a static charge distribution. When this is the case, the system may be analyzed using the adiabatic or quasi-steady approximation in which the particle and gas temperatures are considered as external parameters that are unchanging in time. This approximation has been considered in greater detail in Section II-E.

2) Spatially Uniform Conditions - The model to be used here for the particles is identical to that used in the fundamental shielding problem. The particles are assumed to be spherical with uniform values of $\underline{a}$, the particle radius, $e\Phi$, the particle work function, and T_p, the particle temperature. Also, the degree of ionization of all the particles is assumed to be the same.

The assumption is made that all the heavy species in the gas have a common and uniform value of translational temperature, T_G. Because the density of the heavy species is usually much larger than that of the electrons, and the heavy species have masses which are all roughly of the same magnitude, this is a very good assumption.

The electron gas is also assumed to have a uniform temperature. Implicit in this is the additional assumption that a temperature for the electrons can be defined. A discussion of these two assumptions involves the consideration of four characteristic lengths: λ_{ee}, the electron-electron mean free path; λ_{eG}, the electron - heavy species mean free path; a, the particle radius; and b, a length approximately equal to the interparticle distance and which is given by equation (3. 1).

A Maxwellian temperature, T_e, for the electrons may be defined if the length of travel required for an electron to thermalize with other electrons is much smaller than the thermalization length for an electron with the heavy species. That is, the following requirement should be met:

$$\left(\frac{m}{m_G}\right)\left(\frac{\lambda_{ee}^{(b)}}{\lambda_{eG}}\right) << 1 \quad , \qquad (5.1)$$

where m and m_G are the electron and heavy-species masses respectively, and $\lambda_{cc}^{(b)}$ is defined in the main body of the plasma, that is, in the region of relatively small fields many radii away from a particle. In the event it is not possible to define a Maxwellian temperature, some of the relations which follow will be in error, but the magnitude of the effects explored should be the same.

For many of the problems of current interest, including that of ionization in a solid-rocket exhaust plume, the following are true: $\lambda_{ee}^{(a)} >> a$, where $\lambda_{ee}^{(a)}$ is defined at the particle surface; and $\lambda_{ee}^{(b)} \gtrsim b$, where $\lambda_{ee}^{(b)}$ is defined in the main body of the plasma. When it is also assumed that $\lambda_{eG} \gtrsim a$, there will be a collisionless region of appreciable magnitude around each particle. Under these conditions, an electron, which is emitted from a particle and which escapes this collisionless region, will thermalize with the other electrons in the main body of the plasma but not necessarily in the vicinity of the particle from which it was emitted. For these conditions, the assumption of a uniform electron temperature in the main body of the plasma is reasonable. Also, since the particle radii for most applications are on the order of $10^{-6} \rightarrow 10^{-4}$ cm, the assumption, $\lambda_{eG} \gtrsim a$, is not greatly restrictive.

3) Weak Shielding - The approximate solution, that was found to be valid in the weak shielding regime in the fundamental shielding problem, is assumed to be valid here. That is, the physical concept of a nearly constant-density sea of charge with abrupt changes occurring close to the particles is still applicable. However, rather than the electron density at $r = b$ being the important quantity in determining the charge on a particle, it is the net negative charge density at $r = b$, $e(n_{eb}-n_{ib})$, where n_{ib} is the ion density at $r = b$, which must be used. Since the plasma is considered to be macroscopically neutral, the charge per particle in the weak shielding approximation is then $e(n_{eb}-n_{ib})/N$, which may be either positive or negative. The normalized potential at the surface of a particle is thus assumed to be

$$\frac{e\phi_a}{kT_p} = \left(\frac{n_{eb}-n_{ib}}{N}\right) P(a, b, T_p) , \tag{5.2}$$

where

$$P(a, b, T_p) = \left\{ \frac{e^2}{4\pi\varepsilon_o akT_p} \right\} \left\{ 1 - \frac{3}{2}\left(\frac{a}{b}\right) + \frac{1}{2}\left(\frac{a}{b}\right)^3 \right\} \tag{5.3}$$

which can be expressed as:

$$P(a, b, T_p) = \left\{ \frac{.144\times 10^{-2}}{akT_p} \right\} \left\{ 1 - \frac{3}{2}\left(\frac{a}{b}\right) + \frac{1}{2}\left(\frac{a}{b}\right)^3 \right\} , \tag{5.4}$$

where $\underline{a}$, in the first factor, is in microns, and kT_p is in electron volts.

The criterion for the validity of the approximate solution may be established, as in the fundamental shielding problem, by a consideration of the relative magnitude of the characteristic lengths at a particle's surface of the two types of potential distribution involved, a and λ_{Da}. Here, the charge density of the species which actually shields the plasma from the potential of the particle must be used; this species will have a charge of opposite sign than the charge on the particle. In the fundamental shielding problem, it was found that the approximate solution was applicable when $a/\lambda_{Da} < 5$. It is to be expected that this same criterion is applicable here, and in some cases it would be a conservative one since the presence of the second charged species tends to increase the characteristic length of the one-dimensional Debye sheath.

As in the previous instances where the approximate solution has been used, the analysis is greatly simplified because $e\phi_a$ is not coupled to $n_e(r)$ and $n_i(r)$ but is determined by the conditions at $r = b$ and the spherical geometry.

4) Ionization Equilibrium - The assumption is made that the ionization of the heavy species is in a state of equilibrium corresponding to the electron temperature at $r = b$, where all the spatial gradients are zero. However, since the charge densities and fields do not change appreciably from their value at $r = b$, except in regions which are on the order of a close to a particle, the ionization equilibrium assumption should be fairly good over the main body of the plasma. The equilibrium relation is

$$\frac{n_{eb}n_{ib}}{n_o - n_{ib}} = K_G(T_e) \tag{5.5}$$

where

$$K_G(T_e) = 2\left\{\frac{2\pi mkT_e}{h^2}\right\}^{3/2} \frac{g_i}{g_o} e^{-V/kT_e} ; \tag{5.6}$$

g_i and g_o are the statistical weights of the ion and atom ground states, respectively. For simplicity, it has been assumed that only one of the heavy gas species is capable of ionization. The density before ionization of this species is n_o and its ionization potential is V.

The assumption of equilibrium ionization of the gas at the electron temperature requires that the electron density be high enough so that the forward and backward excitation and ionization rates are dominated by electron collisional processes. That is, radiative processes or processes which involve collisions of only heavy species must be unimportant relative to those collisional processes which involve electrons and heavy species. Since both the electron - heavy species inelastic cross-sections and the electron thermal speed are relatively large, this is not an unreasonable assumption.

5) Electron Energy Loss - The assumption is made that the dominant power loss from the electron gas is by elastic collisions with with the heavy species.[7] Hence, the power loss of the electron gas per unit volume, which is to be balanced by the net energy input from the particles, is

[7] For further discussion, see references 29 and 30.

$$(\frac{3}{2}k)(T_e - T_G)(n_{eb})(\frac{8kT_e}{\pi m})^{\frac{1}{2}} \{ \overline{(\frac{m}{m_G})(\frac{\delta}{\lambda_{eG}})} \} , \tag{5.7}$$

where

$$\overline{(\frac{m}{m_G})(\frac{\delta}{\lambda_{eG}})} = \frac{m}{2} \sum_j \{ (\delta n_G/m_G)_j \int_0^\infty Q_{ej}(\varepsilon)\varepsilon^2 e^{-\varepsilon} d\varepsilon \} \tag{5.8}$$

in which δ_j is an energy transfer coefficient of species j, shortly to be discussed; n_{Gj} is the density of species j; $Q_{ej}(\varepsilon)$ is the momentum transfer cross-section for electron - species j collisions; and ε is the normalized electron energy, $(mv_e^2)/(2kT_e)$.

By using only the above term to represent all of the power loss of the electron gas, it has been assumed that the net power loss due to inelastic collisions with heavy species is negligible. Actually, the electrons undergo inelastic collisions which give rise to a power input to the inelastic degrees of freedom, excitation, and ionization, of the heavy species. If these inelastic losses are to be negligible in the steady state, there must be an equivalent power input to the electrons from the inelastic degrees of freedom resulting from three-body recombination and super-elastic collisions. When radiation from free - bound and bound - bound transitions escapes the system, the power return to the electron gas will be decreased, and an additional power loss from the electron gas will result. In this instance, an additional term equal to the radiation power loss per unit volume should be added to expression (5.7). If some of the heavy species are molecules, vibrational and rotational degrees of freedom may also be excited. Here, too, there will generally not be an equivalent power return to the electron gas. Losses may result again from radiative de-excitation or through the transfer of energy from the rotational to the translational degrees of freedom. An approximate way to include this power loss in expression(5.7) is by increasing the value of δ_j above 8/3, it value for purely elastic collisions; a value of δ whose magnitude is 10^2 or 10^3 is not uncommon. However, a more rigorous way would be to add an additional term to expression (5.7) to account for such losses. Also neglected in expression (5.7) is the direct radiative power loss due to free - free transitions (bremsstrahlung).

The elevation of electron temperature above that of the gas and the accompanying enhancement of gaseous ionization has recently been experimentally demonstrated [30, 31]. In this experiment, a D.C. electric field supplied the power input to the electron gas. Good agreement was obtained between the experimental results and the theory, which utilized many of the assumptions and relations employed here.

C. Number and Energy Flux Problem

1) Introduction - In order to complete the set of relations required to analyze the non-equilibrium ionization in a gas - particle plasma, the net current to and from a particle must be computed and set equal to one another to satisfy the requirement of steady state. Also, the net power emitted from the particles into the electron gas must be calculated and set equal to the power loss by elastic collisions in order to define the electron temperature. Hence, the number and energy fluxes to and from a particle will be calculated.

Only the electron fluxes will be directly considered since the ion fluxes are easily obtained by analogy. Also, from this point on, the ion flux to the particles will be neglected with respect to emitted electron flux from the particles except in the last three special cases to be considered in Part VI. The justification for this lies in the large differences in the values of electron and ion mean thermal speeds. The inclusion of both the ion flux to a particle and the emitted electron flux in the current conservation equation is straightforward but considerably increases the algebraic complexity of the problem.

The magnitude of the electron number and energy fluxes to and from a particle will be influenced by the size of the collisionless region which exists around it. When this region is very small, $\lambda_{eG} \ll a$, the motion of the electrons will be characteristic of a one-dimensional diffusion process. When this region becomes of significant extent, $\lambda_{eG} \gtrsim a$, the three-dimensional nature of the free-fall problem must be taken into account. A problem, interesting in its own right, will now be considered that is representative of this situation and which should give an insight into the effects of this three-dimensional collisionless region.

2) Model - The actual situation will be idealized, to some extent, in order to obtain a workable model.

The outer limit of the collisionless region which exists around a particle is defined by the radius $r = c$ (see Figure 20), where $c \approx a + \lambda_{eG}$. It will be assumed that most of the potential drop between the particle and the plasma occurs in this region, so that it is valid to assume that $\phi_c = 0$. Thus, if the potential distribution which exists around the particle falls into the weak shielding regime, the

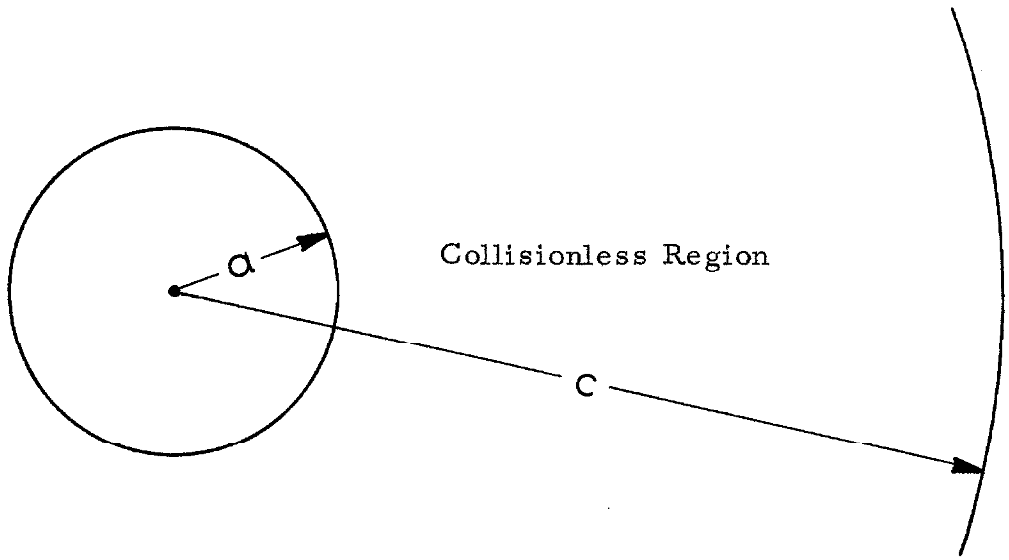


Figure 20. Extent of the Collisionless Region.

lower limit on the radius c must be at least a few particle radii. However, if $\lambda_{Da} \ll a$, so that the potential distribution falls well into the strong shielding regime, and $\lambda_{eG} \ll a$, the ratio c/a will nearly be equal to unity. In this case, the one-dimensional problem will be recovered.

The velocity distribution of the electrons emitted from the particle will be assumed to be a half-Maxwellian with the parameters (n_E, T_p). Also, it will be assumed that the electrons which strike the surface of a particle do not rebound but remain with the particle.[8]

It will be further assumed that all the electrons which enter the collisionless region from $r > c$ have had their velocities completely

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[8] A discussion of the "sticking probability" for electrons which strike the surfaces of various materials is given in reference 2, pp. 25-26.

randomized by electron - heavy species collisions and have also thermalized with the electrons in the main body of the plasma. Thus, the velocity distribution of the electrons which enter the collisionless region from $r < c$ is a half-Maxwellian with the parameters (n_{eb}, T_e).

Thus, the problem has been reduced to that of two concentric spheres, each of which emits electrons into the collisionless region between them. The fluxes which exist in this region will be obtained by finding the distribution function for the electrons in phase space and then taking the appropriate moments.

3) Distribution in Phase Space - In the collisionless region between the two emitting spheres, under the requirement of steady state, Boltzmann's equation for the density $f(\vec{q}, \vec{p})$ in the six-dimensional phase space $(\vec{q}, \vec{p})$ is

$$(\dot{\vec{p}} \cdot \nabla_p + \dot{\vec{q}} \cdot \nabla_q) f(\vec{q}, \vec{p}) = 0 . \tag{5.9}$$

That is, the density along streamlines in phase space is constant. Therefore, if $f(\vec{q}, \vec{p})$ is known on the surfaces in phase space defined by $r = a$ or $r = c$, from which all streamlines originate, then $f(\vec{q}, \vec{p})$ is known in all occupied regions of phase space. The specification of the half-Maxwellians serves this purpose.

The density in phase space may be divided into two parts; one which accounts for the electrons emitted from $r = a$, and one which accounts for the electrons emitted from $r = c$.

$$f(\vec{q}, \vec{p}) = f_a(\vec{q}, \vec{p}) + f_c(\vec{q}, \vec{p}) . \tag{5.10}$$

Either f_a or f_c, but not both, may be non-zero at any point $(\vec{q}, \vec{p})$. Hence, equation (5.9) is satisfied by either function. In particular,

$$(\vec{p} \cdot \nabla_p + \vec{q} \cdot \nabla_q) f_a(\vec{q}, \vec{p}) = 0 \tag{5.11}$$

In the following, only f_a will be considered in detail, since the same considerations apply to f_c.

In order that equation (5.11) can be satisfied, $f_a(\vec{q}, \vec{p})$ must be a function of the constants of the motion. From the condition specified at $r = a$, it is seen that it can only be a function of $\mathcal{H}(\vec{q}, \vec{p})$, the Hamiltonian, and

$$f_a(\vec{q}, \vec{p}) = C_a e^{-\mathcal{H}(\vec{q}, \vec{p})/kT_p} . \tag{5.12}$$

For this system, the Hamiltonian is equal to the total energy of an electron and is constant along a streamline.

Using the spherical coordinates, r, θ, and ψ, the origin of which is located at the center of the particle, the Lagrangian for a single electron is

$$\mathcal{L} = \frac{m}{2} \{\dot{r}^2 + (r\dot{\theta})^2 + (r \sin\theta \dot{\psi})^2\} + e\phi . \tag{5.13}$$

The canonical momenta, defined by

$$p_i = \partial \mathcal{L} / \partial q_i , \tag{5.14}$$

are

$$p_r = m\dot{r} , \quad p_\theta = mr^2\dot{\theta} , \quad p_\psi = mr^2 \sin\theta \dot{\psi} , \tag{5.15}$$

and the Hamiltonian is

$$\mathcal{H} = \frac{1}{2m} \left\{ p_r^2 + \frac{p_\theta^2}{r^2} + \frac{p_\psi^2}{r^2 \sin\theta} \right\} - e\phi . \tag{5.16}$$

Since ψ is a cyclic coordinate, p_ψ is a constant of the motion. It is also easily shown, with the aid of Lagrange's equations, that the total angular momentum vector, $\vec{p}_\Omega$, defined by

$$\vec{p}_\Omega = \vec{r} \times \frac{d}{dt}(m\dot{\vec{r}}) \, , \tag{5.17}$$

is also a constant of the motion. Since

$$p_\Omega^{\,2} = p_\theta^{\,2} + \frac{p_\psi^{\,2}}{\sin^2\theta} \quad , \tag{5.18}$$

the Hamiltonian becomes

$$\mathcal{H} = \frac{1}{2m}\{p_r^{\,2} + \frac{p_\Omega^{\,2}}{r^2}\} - e\phi \quad . \tag{5.19}$$

$f(\vec{q}, \vec{p})$ is the density in phase space; therefore, the density in physical space of electrons emitted from $r = a$, $(d^3N_e^{\,a})/(d^3r)$, is given by

$$n_e^{\,a} = \frac{1}{r^2\sin\theta} \frac{d^3N_e^{\,a}}{d^3q} \quad , \tag{5.20}$$

where

$$\frac{d^3N_e^{\,a}}{d^3q} = \int_{\vec{p}} f_a(\vec{q}, \vec{p}) d^3p \quad , \tag{5.21}$$

or

$$n_e^{\,a} = \frac{2\pi}{r^2} \int_{p_r}\int_{p_\Omega} f_a(r, p_r, p_\Omega) p_\Omega dp_\Omega dp_r \quad , \tag{5.22}$$

The constant, C_a, is evaluated from the conditions at $r = a$. Thus,

$$f_a(r, p_r, p_\Omega) = \frac{n_E \, e^{-\frac{e(\phi_a - \phi)}{kT_p}}}{(2\pi mkT_p)^{3/2}} \exp\{-\frac{1}{2mkT_p}(p_r^{\,2} + \frac{p_\Omega^{\,2}}{r^2})\} \, , \tag{5.23}$$

and, in the same manner

$$f_c(r, p_r, p_\Omega) = \frac{n_{eb} \, e^{+e\phi/kT_e}}{(2\pi mkT_e)^{3/2}} \exp\{-\frac{1}{2mkT_e}(p_r^{\,2} + \frac{p_\Omega^{\,2}}{r^2})\} \, . \tag{5.24}$$

The calculation of the fluxes is now reduced to taking the desired moments over the allowed regions in momentum space. That is, f_a is non-zero only at points in (p_r, p_Ω) space which are consistent with the constants of motion and the conditions at $r = a$. Since momentum space has been reduced to one which is two-dimensional, these allowed regions may be specified by marginal contours in (p_r, p_Ω) space. The nature of these contours will depend upon whether the potential of the particle is positive or negative. Hence, these two separate cases will now be considered.

4) Positive Particle Potential - When the thermionic emission from the particles is dominant over the gaseous ionization, a particle will have emitted a net number of electrons, and its potential with respect to the plasma will be positive. Hence, the electrons will be attracted to the particle and a certain fraction of those emitted from $r = a$ will not escape and cross $r = c$. Also, because of a relatively high initial angular momentum, many of the electrons emitted from $r = c$ will leave the collisionless region rather than striking the particle surface at $r = a$.

The marginal contours are defined from the conditions at $r = a$ or $r = c$ and the constants of the motion, the energy, and angular momentum.

$$p_r^2 + p_\Omega^2/r^2 - 2me\phi = \text{const.} \tag{5.25}$$

$$p_\Omega^2 = \text{const.} \tag{5.26}$$

For the electrons emitted from $r = a$, one contour is defined by the condition that $p_r(r=a) \geq 0$. Letting the superscript (1) denote the initial condition $p_r(r=a) = 0$, this marginal contour for $a \leq r \leq c$ is given by

$$p_r^{(1)^2} = \left(\frac{1}{a^2} - \frac{1}{r^2}\right) p_\Omega^{(1)^2} - 2me(\phi_a - \phi), \qquad (5.27)$$

which describes a hyperbola in (p_r, p_Ω) space. For all r, it must be true that $p_r^2 > p_r^{(1)^2}$. In Figure 21, the path in physical space of a marginal electron is illustrated and is denoted by (1). Also shown are the paths which correspond to the marginal contours yet to be defined. In Figure 22, the marginal contour $p_r^{(1)}$ and three other contours in (p_r, p_Ω) space are shown.

A second marginal contour is defined by those electrons which reverse their radial momentum at $r = c$ and return to the particle. It is clear that the conservation of angular momentum as well as energy must be considered in the general case, because electrons may have a kinetic energy greater than zero at $r = c$ and still return to the particle. (See Figure 21.) If the superscript (2) corresponds to the condition $p_r(r=c) = 0$, then

$$p_r^{(2)^2} = 2me\phi - \left(\frac{1}{r^2} - \frac{1}{c^2}\right) p_\Omega^{(2)^2}, \qquad (5.28)$$

which describes an ellipse in (p_r, p_Ω) space. Electrons with $p_r < p_r^{(2)}$ will return to the particle, and those with $p_r > p_r^{(2)}$ will contribute to the net number and energy flux leaving the particle (see Figure 22). An implicit assumption made here is that $p_r^2 > 0$ at all r for the electrons with $p_r > p_r^{(2)}$. This really reduces to an assumption about the shape of the potential field about the particle. This is further discussed in Appendix I.

The marginal contours for the electrons emitted from $r = c$

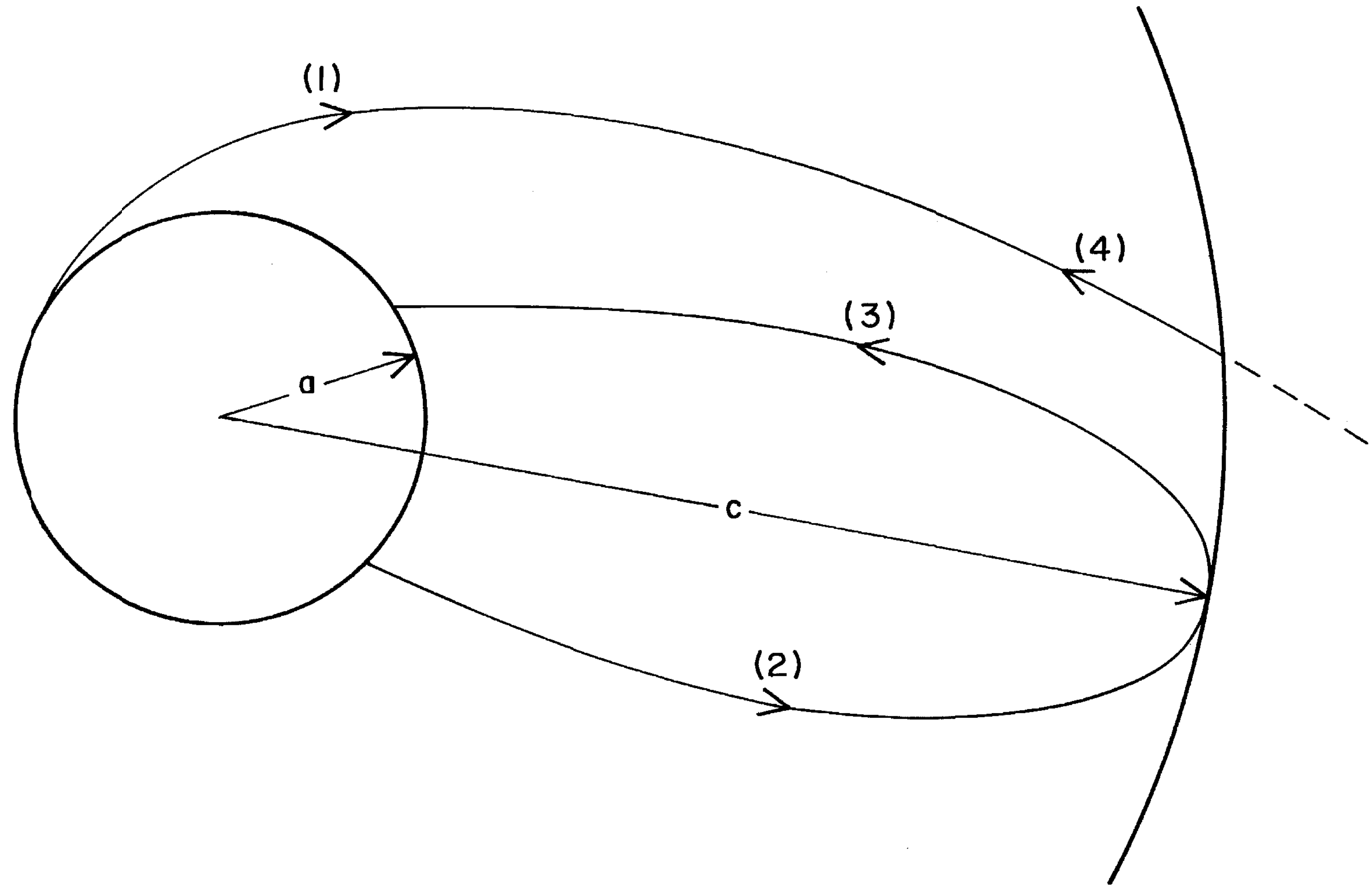

Figure 21. Marginal Electron Paths in Physical Space.

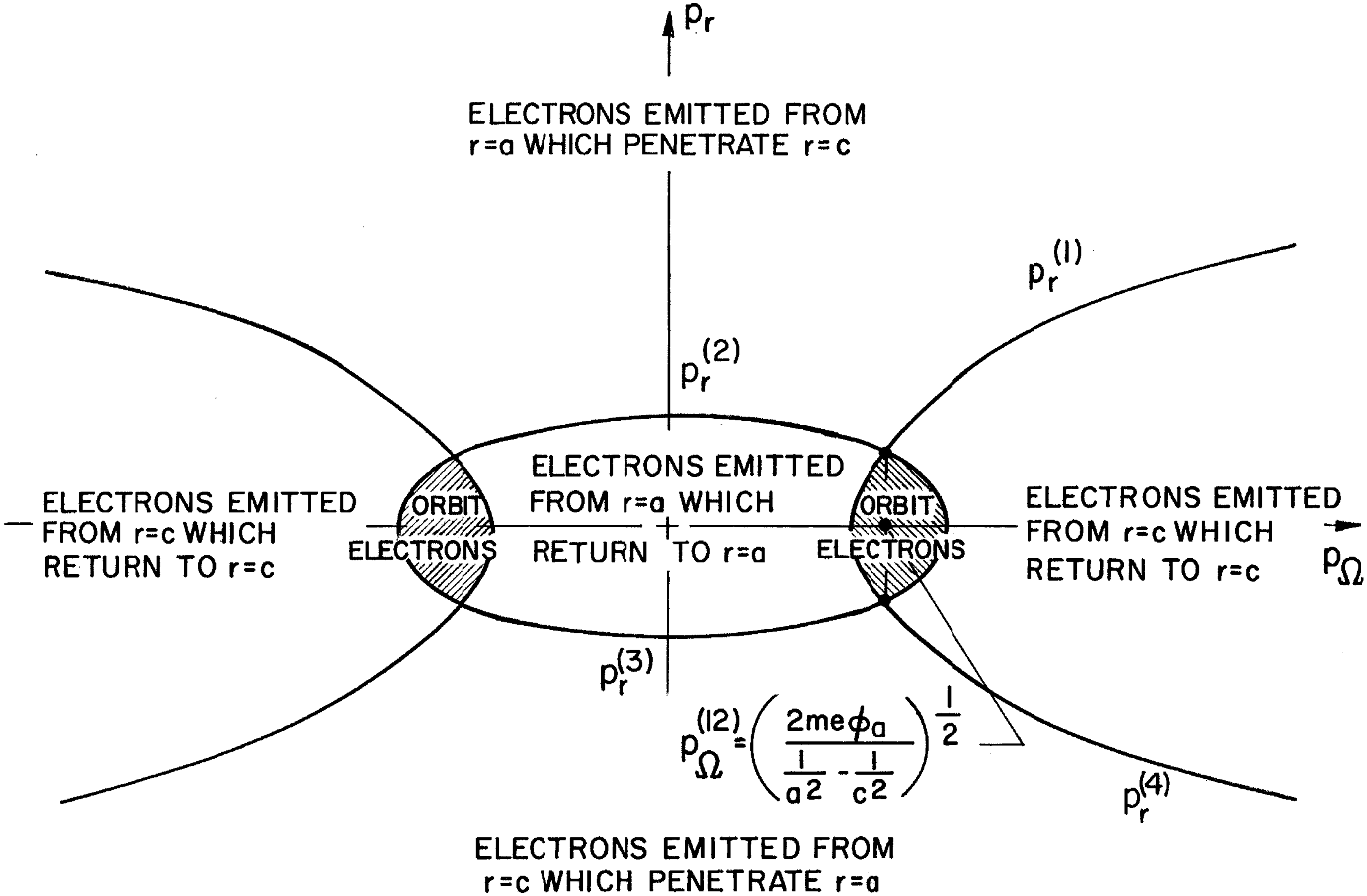

Figure 22. Marginal Contours and Regions of (p_r, p_Ω) Space for a Positive Particle Potential.

are likewise defined. The requirement that $p_r(r=c) \leq 0$ for the emitted electrons defines the contour

$$p_r^{(3)^2} = 2me\phi - \left(\frac{1}{r^2} - \frac{1}{c^2}\right) p_\Omega^{(3)^2} . \tag{5.29}$$

The marginal contour which separates those electrons which strike the particle from those which do not is given by the condition $p_r(r=a) = 0$. Thus,

$$p_r^{(4)^2} = \left(\frac{1}{a^2} - \frac{1}{r^2}\right) p_\Omega^{(4)^2} - 2me(\phi_a - \phi) . \tag{5.30}$$

Those electrons which do not reach the particle are deflected toward it but return to $r = c$. It is seen that $p_r^{(1)} = -p_r^{(4)}$ and $p_r^{(2)} = -p_r^{(3)}$, so that the marginal contours in (p_r, p_Ω) space are symmetric about $p_r = 0$. This symmetry is exhibited in Figure 22, and the regions into which the marginal contours divide (p_r, p_Ω) space are indicated. The regions marked "orbit" denote those electrons which cross neither $r = a$ nor $r = c$, but rather orbit the particle. These orbit electrons, however, do not contribute to the number and energy fluxes.

The desired fluxes may now be calculated, since f_a and f_c are known and the regions in phase space in which each is non-zero are defined. The net electron flux in the $+r$ direction for the electrons emitted from the particle, $\Gamma_{n_e}^{+}$, is given by

$$\Gamma_{n_e}^{+} = \frac{1}{r^2} \iint\limits_{\vec{p}_a} (p_r/m) f_a 2\pi p_\Omega dp_\Omega dp_r , \tag{5.31}$$

where $\vec{p}_a$ is the region of (p_r, p_Ω) space in which f_a is non-zero. Thus,

$$\Gamma_{n_e}^{\ +} = \frac{2\pi n_E\, e^{-\frac{e(\phi_a-\phi)}{kT_p}}}{(2\pi m k T_p)^{3/2}(mr^2)} \left\{ \int_0^{p_r^{(12)}} p_\Omega dp_\Omega \int_{p_r^{(2)}}^{\infty} e^{-\frac{\left(p_r^{\ 2}+\frac{p_\Omega^{\ 2}}{r^2}\right)}{2mkT_p}} p_r dp_r \right.$$

$$\left. + \int_{p_r^{(12)}}^{\infty} p_\Omega dp_\Omega \int_{p_r^{(1)}}^{\infty} e^{-\frac{\left(p_r^{\ 2}+\frac{p_\Omega^{\ 2}}{r^2}\right)}{2mkT_p}} p_r dp_r \right\} \tag{5.32}$$

where $p_\Omega^{(12)}$ is the value of p_Ω at the intersection of contours (1) and (2) [see Figure 22].

$$p_\Omega^{(12)} = \left(\frac{2me\phi_a}{\frac{1}{a^2} - \frac{1}{c^2}} \right)^{\frac{1}{2}} . \tag{5.33}$$

The integrations are straightforward and yield[9]

$$\Gamma_{n_e}^{\ +} = \frac{n_E}{4} \left(\frac{8kT_p}{\pi m} \right)^{\frac{1}{2}} e^{-\frac{e\phi_a}{kT_p}} \left(\frac{a}{r}\right)^2 \left[\gamma - \left\{ (\gamma-1)\, e^{-\frac{e\phi_a}{(\gamma-1)kT_p}} \right\} \right] \tag{5.34}$$

where

$$\gamma = (c/a)^2 . \tag{5.35}$$

The first three factors give the flux which one would calculate for the corresponding one-dimensional collisionless case, and the $1/r^2$ dependence is consistent with total flux conservation, $4\pi r^2 \Gamma_{n_e}^{\ +} = \text{const.}$ The quantity in square brackets introduces the effect of the three-

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[9] The author has recently found that this result is essentially the same as that obtained by Langmuir [32], who, in a conceptually somewhat different way, computed the ion current to a spherical plasma probe which was at a negative potential and was surrounded by a collisionless sheath.

dimensional collisionless region. As $c/a \to 1$, $[\ ] \to 1$. Since ϕ_c has been set equal to zero, this would correspond to the case in which the size of the collisionless potential sheath is very much less than the particle radius; that is, $\lambda_{Da} << \lambda_{eG} << a$, so that the emission through the potential sheath is one dimensional in nature. For the limit in which the geometrical effects are large compared to the shielding effects,

$$\frac{e\phi_a/kT_p}{(c/a)^2-1} << 1, \quad [\ ] \to \left[1 + \frac{e\phi_a}{kT_p}\right].$$

Hence, when $(\lambda_{eG}/a)^2 >> e\phi_a/kT_p$, the rate at which electrons are emitted from a particle is given by

$$(4\pi a^2)\left(\frac{n_E}{4}\right)\left(\frac{8kT_p}{\pi m}\right)^{\frac{1}{2}} e^{-e\phi_a/kT_p}\left(1 + \frac{e\phi_a}{kT_p}\right). \tag{5.36}$$

Thus, the effect of a large collisionless region around the particle is to increase the net electron emission by the factor $[1 + (e\phi_a/kT_p)]$ over what would be estimated from one-dimensional considerations. Curves of $[\ ]$ versus $(c/a)^2$, which illustrate this transition from the one- to the three-dimensional case, are shown in Figure 23.

The energy flux in the electrons leaving the particle is readily computed. The kinetic energy of an electron which has escaped the field of the particle and enters the electron gas at $r = c$ is $\mathcal{H}$, the Hamiltonian. Therefore, the energy flux in the electrons escaping the particle, $\Gamma_{\mathcal{H}_e}^{+}$, is

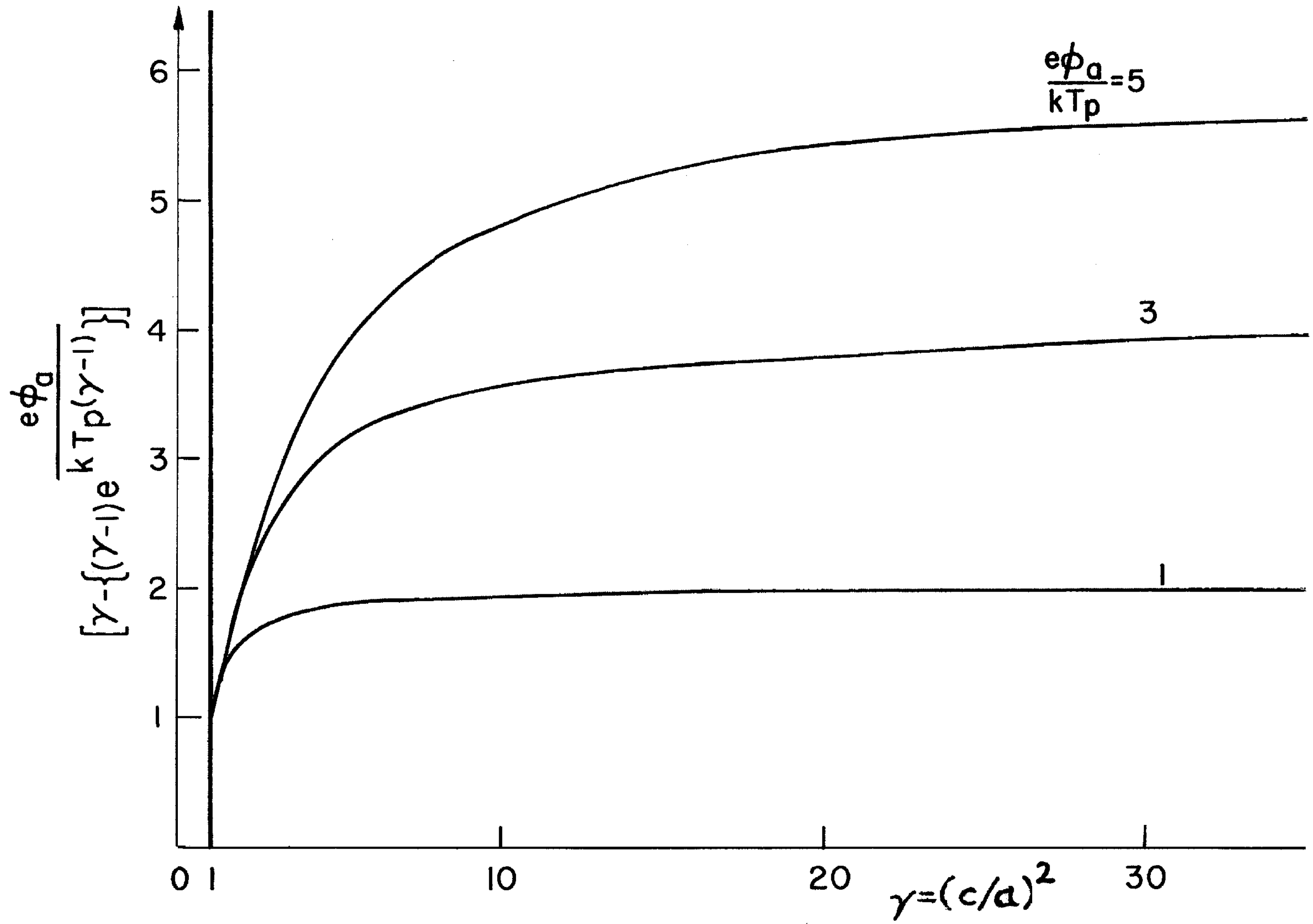

Figure 23. Fractional Increase of the Net Number Flux Versus Increases in the Size of the Collisionless Region.

$$\Gamma_{\mathcal{H}_e}^{\ +} = \iint\limits_{p_a} \left(\frac{p_r}{m}\mathcal{H}\right) f_a 2\pi p_\Omega dp_\Omega dp_r \ , \tag{5.37}$$

which is related to the previously evaluated integral by

$$\Gamma_{\mathcal{H}_e}^{\ +} = C_a kT_p{}^2 \frac{\partial}{\partial T_p}\{\Gamma_{n_e}^{\ +}/C_a\} \tag{5.38}$$

where C_a is the constant in equation (5.12). Thus,

$$\Gamma_{\mathcal{H}_e}^{\ +} = 2kT_p\Gamma_{n_e}^{\ +}\left[1 - \frac{e\phi_a/2kT_p}{\gamma e^{+\frac{e\phi_a}{(\gamma-1)kT_p}} - (\gamma-1)}\right] \ . \tag{5.39}$$

As would be expected, the one-dimensional result is recovered in the limit $c/a \to 1$; [ ] $\to 1$; and the mean energy carried per electron is $2kT_p$. When the collisionless region is large, $(c/a)^2 >> 1$, and

$$\frac{e\phi_a/kT_p}{(c/a)^2-1} << 1 \ , \quad [\ ] \to \left[\frac{1+(e\phi_a/2kT_p)}{1+(e\phi_a/kT_p)}\right] \ .$$

Therefore, the mean energy carried in the three-dimensional case is less than $2kT_p$, and when $(e\phi_a)/(kT_p) >> 1$, it is only kT_p . However, the energy flux is still larger than would be estimated using a one-dimensional approach because of the augmentation of $\Gamma_{n_e}^{\ +}$; in the three-dimensional limit

$$\Gamma_{\mathcal{H}_e}^{\ +} = \left(\frac{n_E}{4}\right)\left(\frac{8kT_p}{\pi m}\right)^{\frac{1}{2}} e^{-e\phi_a/kT_p}\left(\frac{a}{r}\right)^2 (2kT_p)\left[1 + \frac{e\phi_a}{2kT_p}\right] \ . \tag{5.40}$$

The calculation of the fluxes to the particle from the electron gas proceeds exactly in the same manner and, since the marginal contour in (p_r, p_Ω) is symmetric about $p_r = 0$, the integrals to be carried out are of the same form as those previously evaluated.

Thus,

$$\Gamma_{n_e}^{-} = \left(\frac{n_{eb}}{4}\right)\left(\frac{8kT_e}{\pi m}\right)^{\frac{1}{2}}\left(\frac{a}{r}\right)^2\left[\gamma - (\gamma-1)\,e^{-\frac{e\phi_a}{(\gamma-1)kT_e}}\right] \tag{5.41}$$

and

$$\Gamma_{\mathcal{H}_e}^{-} = 2kT_e\Gamma_{n_e}^{-}\left[1 - \frac{e\phi_a/2kT_e}{\gamma e^{+\frac{e\phi_a}{(\gamma-1)kT_e}} - (\gamma-1)}\right]. \tag{5.42}$$

Of course, the discussion of $\Gamma_{n_e}^{+}$ and $\Gamma_{\mathcal{H}_e}^{+}$ applies equally well to the above quantities. Also, in the three-dimensional limit, a relation analogous to equation (5.36) is obtained, and is the same as that which Spitzer [1] derived from kinetic theory considerations.

If the value of the ion flux to a negative particle is desired, a relation analogous to equation (5.41) can be used; both e and ϕ_a change in sign.

The steady-state requirement is that the net current to a particle should be zero. That is,

$$\Gamma_{n_e}^{+} = \Gamma_{n_e}^{-} \tag{5.43}$$

or

$$n_E T_p^{\frac{1}{2}} e^{-\frac{e\phi_a}{kT_p}}\left[\gamma - (\gamma-1)\,e^{-\frac{c\phi_a}{(\gamma-1)kT_p}}\right] = n_{eb}T_e^{\frac{1}{2}}\left[\gamma - (\gamma-1)\,e^{-\frac{e\phi_a}{(\gamma-1)kT_e}}\right] \quad . \tag{5.44}$$

When the shielding effect is small compared to the geometrical effect,

$$\frac{e\phi_a/kT_p}{(c/a)^2-1} \ll 1 \; ,$$

but $(e\phi_a)/(kT_p) \gg 1$, the requirement of steady state becomes

$$n_E T_p^{-\frac{1}{2}} e^{-e\phi_a/kT_p} = n_{eb} T_e^{-\frac{1}{2}} \; . \qquad (5.45)$$

The net power per unit volume into the electron gas, which is to be equated to the power lost per unit volume by electron-heavy species elastic collisions [expression (5.7)] , is given by

$$[4\pi a^2 N][\Gamma_{\mathcal{H}_e}^{+}(a) - \Gamma_{\mathcal{H}_e}^{-}(a)] \; . \qquad (5.46)$$

In general, this expression is an algebraically complicated one. However, for the case in which

$$\frac{e\phi_a/(kT_{p,e})}{(c/a)^2 - 1} \ll 1 \; ,$$

but $e\phi_a/(kT_{p,e}) \gg 1$, it reduces to

$$[4\pi a^2 N]\left[\left(\frac{n_E}{4}\right)\left(\frac{8kT_p}{\pi m}\right)^{\frac{1}{2}} e^{-\frac{e\phi_a}{kT_p}}\right]\left[\left(\frac{e\phi_a}{kT_p}\right)(kT_p - kT_e)\right] , \qquad (5.47)$$

which is a factor of $e\phi_a/(2kT_p)$ larger than the corresponding one-dimensional estimate.

5) Negative Particle Potential - When gaseous ionization is dominant over thermionic emission from the particles, a particle will absorb electrons, and its potential with respect to the plasma will be negative. The calculation of the fluxes in this case is relatively simple. It is easily seen that all of the electrons emitted from the particle will leave the collisionless region. However, not all the electrons emitted from the electron gas will reach the particle. The situation may again be examined through marginal contours.

For the electrons which are emitted from $r = a$, the requirement that $p_r(r=a) \doteq 0$ leads to the contour

$$p_r^{(5)^2} = \left(\frac{1}{a^2} - \frac{1}{r^2}\right) p_r^{(5)^2} + 2me(\phi - \phi_a) \quad , \tag{5.48}$$

where the superscript (5) denotes the condition $p_r(r=a) = 0$. The contour which separates the electrons which reach the particle from those which do not, after emission from $r = c$, is also given by the condition $p_r(r=a) = 0$. Therefore, equation (5.45) again is valid for this contour, but $p_r^{(6)} = -p_r^{(5)}$. These contours are illustrated in Figure 24.

Using equations (5.31) and (5.38), the fluxes are

$$\Gamma_{n_e}^{+} = \frac{n_E}{4}\left(\frac{8kT_p}{\pi m}\right)^{\frac{1}{2}}\left(\frac{a}{r}\right)^2 \tag{5.49}$$

$$\Gamma_{n_e}^{-} = \frac{n_{eb}}{4}\left(\frac{8kT_e}{\pi m}\right)^{\frac{1}{2}}\left(\frac{a}{r}\right)^2 e^{+\frac{e\phi_a}{kT_e}} \tag{5.50}$$

and

$$\Gamma_{\mathcal{H}_e}^{+} = \Gamma_{n_e}^{+}(2kT_p - e\phi_a) \tag{5.51}$$

$$\Gamma_{\mathcal{H}_e}^{-} = \Gamma_{n_e}^{-}(2kT_e - e\phi_a) \quad . \tag{5.52}$$

It is seen that the fluxes for the case of negative particle potential are just what one would calculate for the one-dimensional problem times the geometrical factor, $(a/r)^2$. The requirement of steady state yields

$$n_E T_p^{\frac{1}{2}} = n_{eb} T_e^{\frac{1}{2}} e^{+e\phi_a/kT_e} \quad , \tag{5.53}$$

and the expression for the power input to the electron gas per unit volume is

$$[4\pi a^2 N]\left[\frac{n_E}{4}\left(\frac{8kT_p}{\pi m}\right)^{\frac{1}{2}}\right][2k(T_p - T_e)] \, , \tag{5.54}$$

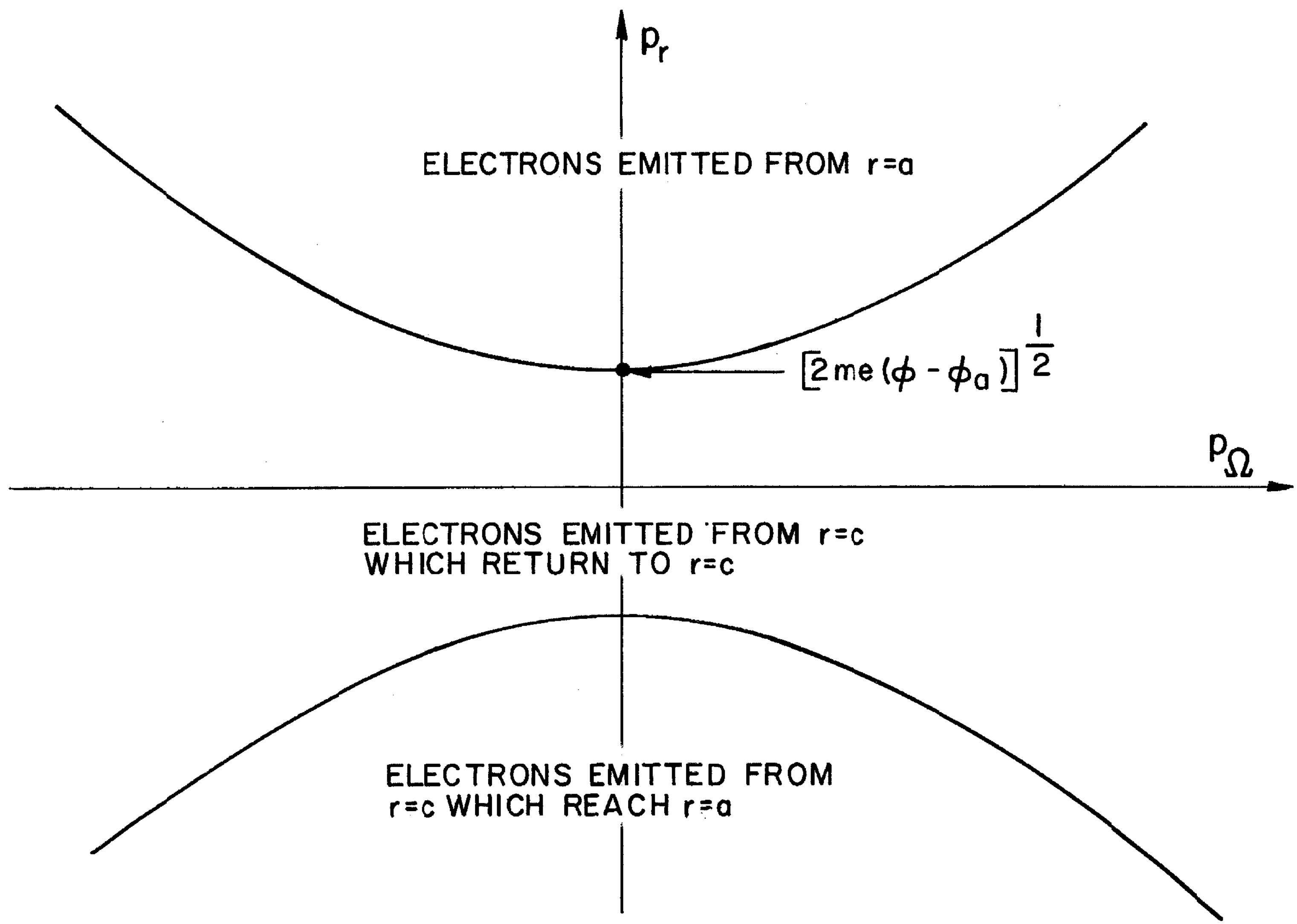

Figure 24. Marginal Contours and Regions of (p_Ω, p_r) Space for a Negative Particle Potential.

both of which are relatively simple compared to the results of the positive potential case.

VI. SPECIAL CASES

A. Introduction

All of the relationships that are necessary to define the electron density in a gas - particle system have been derived and discussed in Part V. These relationships will now be employed to study various special cases. First, however, the order that exists between these special cases will briefly be considered.

In the problem of ionization in a gas - particle plasma, there are two extreme cases; one of negligible gas ionization, where the particle potential is always positive; and one of negligible particle ionization, where the thermionic emission from a particle is relatively small so that the ion and electron fluxes to a particle are almost equivalent and the particle potential is negative. The first extreme case, in which only the particles are ionized, has been considered in Parts III and IV. If the amount of ionization of the gas in this extreme case is now increased from zero, a point will be reached where the contributions to the electron density by the gas and the particles will be of comparable magnitudes, and there will be a noticeable supression of one form of ionization due to the presence of the other. The two special cases of a slightly ionized gas and a fully ionized gas will be used to illustrate this direct interaction. In this range, the particle potential will be positive, but may be close to zero.

The situation which is of major interest here, that in which the enhancement of the gaseous ionization is large and the particle ionization is negligible, will next be considered. In this case, the interesting interaction between the gas and particles is the indirect one of electron heating leading to enhanced ionization rather than the more direct interaction between the two types of ionization just considered. The particle potential in this case will generally be negative. A numerical example will be presented to illustrate the possible magnitude of the ionization enhancement.

The last three special cases to be investigated are at the second extreme where the gaseous ionization is completely dominant and the thermionic emission from the particles can be assumed to be equal to zero. One phenomenon of interest here is the electron absorption by the particles which, under some conditions, can greatly reduce the number of free electrons in the system. The possible magnitude of this effect will be explored by again considering the two special cases of a slightly ionized gas and a fully ionized gas. A second phenomenon which is of interest, when the thermionic emission from the particles is negligible, is that of quenching of the gaseous ionization. This results when the particles are cooler than the gas so that the rate of recombination exceeds the rate of ionization at the surface of the particles and the electron gas is therefore cooled. This, in turn, will decrease the level of ionization.

In all of these special cases, it is convenient to reduce the number of relationships required to define the electron density to two. This will now be done for the special case of primary interest, gaseous ionization enhancement. The treatment of this special case will not only illustrate this reduction, but the resulting relations will also be applicable to the first two special cases to be considered.

In the special case of gaseous ionization enhancement, it will be assumed that the particle potential is negative and that the electron flux emitted from a particle is balanced by the flux emitted back from the electron gas. Hence, equation (5.53) and expression (5.54) are applicable. The approximate solution for the particle potential, equation (5.2); Saha's equation, equation (5.5); and the steady state requirement, equation (5.53); may be combined to yield

$$\frac{\rho^2 + \rho\left(\frac{N}{Pn_E}\right)\left(\frac{T_e}{T_p}\right)\left[\ell n\, \rho - \frac{1}{2}\ell n \frac{T_p}{T_e}\right]}{\frac{n_o}{n_E} - \rho - \left(\frac{N}{Pn_E}\right)\left(\frac{T_e}{T_p}\right)\left[\ell n\, \rho - \frac{1}{2}\ell n \frac{T_p}{T_e}\right]} = \frac{K_G(T_e)}{n_E} \tag{6.1}$$

where

$$\rho = n_{eb}/n_E \,. \tag{6.2}$$

By equating the power lost by elastic collisions with the heavy species, expression (5.7), to the power gained by thermal replenishment from the particles, expression (5.54), the power balance on the electron gas yields

$$\left(\frac{T_p - T_e}{T_e - T_G}\right)\left(\frac{T_p}{T_e}\right)^{\frac{1}{2}} = \left(\frac{3}{4\pi a^2 N}\right)\{\overline{(m/m_G)(\delta/\lambda_{eG})}\}\, \rho \,. \tag{6.3}$$

This relation determines the value of T_e relative to T_p and T_G.

If the right hand side of equation (6.3) is very much less than unity, T_e will be close to T_p. This is a very possible situation in view of the relative magnitudes of the electron and atomic masses. For the case in which the electron energy loss is accomplished through purely elastic collisions, an average electron emitted from a particle would have to make at least $10^4 \rightarrow 10^5$ collisions with the heavy species before returning to another particle if the electron temperature is to differ appreciably from that of the particles. The transitinn of the temperature of the electrons from the particle temperature to the gas temperature is shown in Figure 25 where T_e/T_g is plotted as a function of the right hand side of equation (6. 3) for many values of T_p/T_G between 1. 0 and 2. 0.

In general, both equations (6. 1) and (6. 3) must be solved simultaneously for T_e and ρ . A straightforward iteration procedure would be to first assume a value for T_e , calculate ρ , and then obtain T_e from equation (6. 3) or Figure 25. From this value of T_e , a new estimate may be made and the cycle repeated until an acceptable agreement between the assumed and calculated values of T_e is reached.

The same general procedure for solution may be followed in the case of a positive particle potential, although the algebraic complexity is somewhat greater.

The special cases mentioned previously will now be investigated.

B. Particle and Gas Ionizations of Comparable Magnitudes, $T_e = T_p$.

When the parameters of the particles and heavy species are such that the electron temperature may be assumed equal to the particle temperature, the electron density may be calculated directly from equation (6. 1); this is equivalent to assuming $T_e = T_p$ in the above iteration scheme, and it should be checked once ρ is determined.

The calculation of the electron density in this case is interest-

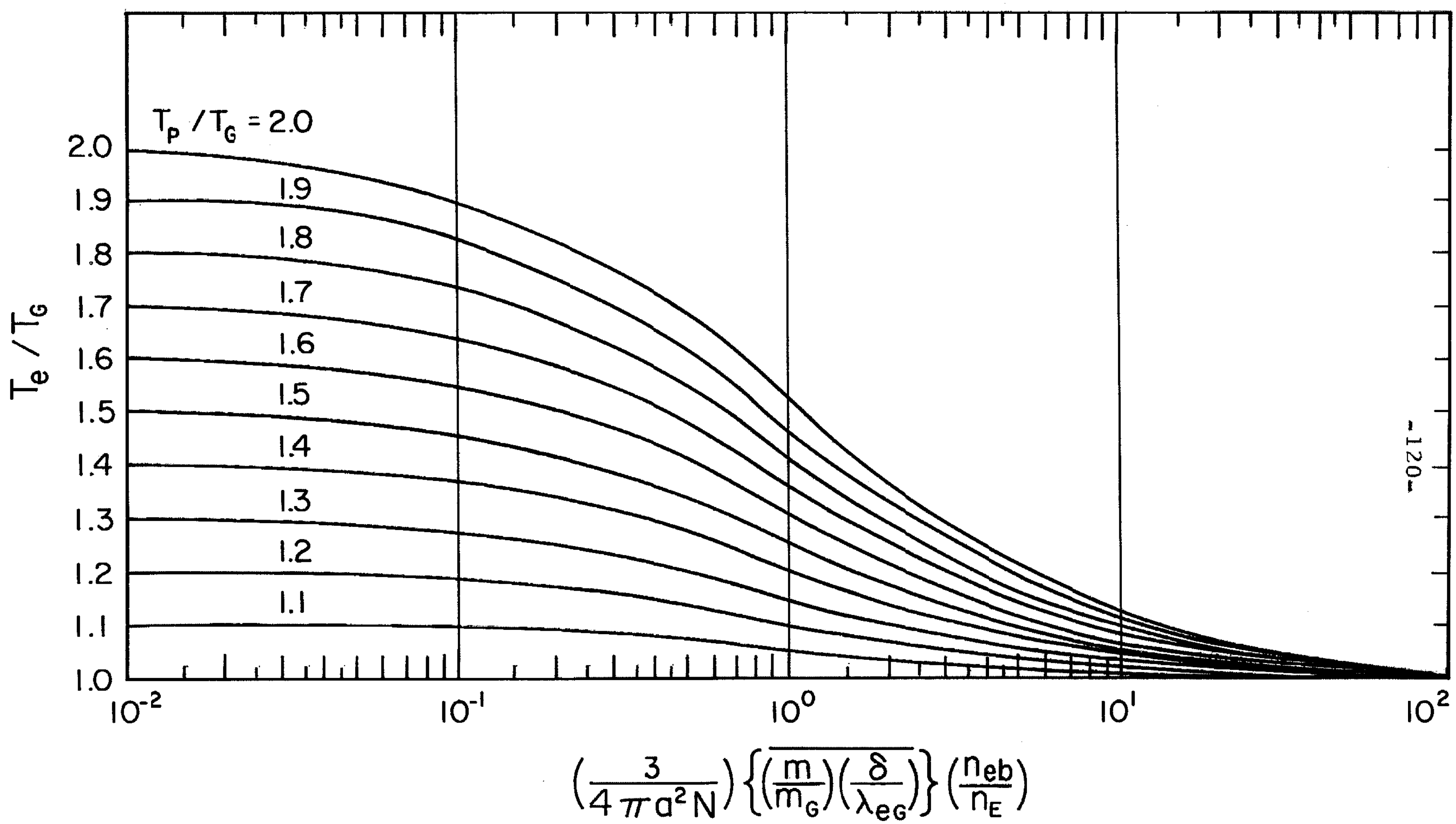

Figure 25. Electron Temperature Divided by the Gas Temperature Versus the Relative Strength of the Electron - Heavy Gas Species Collision Loss.

ing for several reasons. First, when the particle and gaseous ionizations are of the same magnitude, the direct interaction which occurs between them produces effects which are also of this magnitude and are easily demonstrated. Second, it describes the case of complete thermodynamic equilibrium; T_e and T_p are equivalent, and T_G does not enter. Third, it really applies to both positive and negative particle potentials because the requirement of steady state, equation (5.44) or (5.53), is the same for both cases when $T_p = T_e$. It should be pointed out, however, that the changes in the electron density due to this direct interaction are small relative to the other phenomena to be investigated (< 50 per cent).

1) <u>Slightly Ionized Gas</u> - Assuming that the gas is slightly ionized, and $T_e = T_p$, equation (6.1) becomes

$$\rho^2 + \left(\frac{N}{Pn_E}\right)\rho \ln \rho = \frac{n_o K_G(T_p)}{n_E^2} \quad . \tag{6.4}$$

In order to better visualize the direct interaction which occurs between the two types of ionization, two characteristic values of the relative electron density, ρ_p and ρ_G, will be defined. When the particle density is zero, so that all the existing electrons are due only to gaseous ionization,

$$\rho_G^2 = \frac{n_o K_G(T_p)}{n_E^2} \quad . \tag{6.5}$$

When the density of ionizable gas species is zero, so that all the electrons are from the particles,

$$\frac{1}{\rho_p} \ln \frac{1}{\rho_p} = \frac{Pn_E}{N} \quad . \tag{6.6}$$

In terms of these two characteristic quantities, ρ is given by

$$\rho^2 + \rho\,\rho_p \frac{\ln \rho}{\ln \frac{1}{\rho_p}} = \rho_G^{\,2} . \tag{6.7}$$

The anticipated effect here is that the presence of one type of ionization tends to supress the level of the other so that ρ is always less than $(\rho_p + \rho_G)$ as predicted by the principle of Le Chatelier. This effect is clearly illustrated in Figure 26.

2) Fully Ionized Gas - In the case of a slightly ionized gas, there were two degrees of freedom; both the particle and gas ionizations were variable. If the gas is assumed to be fully ionized, there exists only one degree of freedom, and it would be expected that the total supression of the ionization would not be as large.

When the gas is fully ionized and $T_e = T_p$, equation (6.1) becomes

$$\rho + \frac{N}{Pn_E} \ln \rho = \frac{n_o}{n_E} . \tag{6.8}$$

When characteristic quantities are defined as done previously,

$$\rho_G = n_o/n_E , \tag{6.9}$$

and

$$\frac{1}{\rho_p} \ln \frac{1}{\rho_p} = \frac{Pn_E}{N} , \tag{6.10}$$

equation (6.8) becomes

$$\rho + \rho_p \frac{\ln \rho}{\ln \frac{1}{\rho_p}} = \rho_G . \tag{6.11}$$

This equation is also plotted in Figure 26. Again, ρ is less than $\rho_p + \rho_G$, but the supression is not as large, and the net ionization is

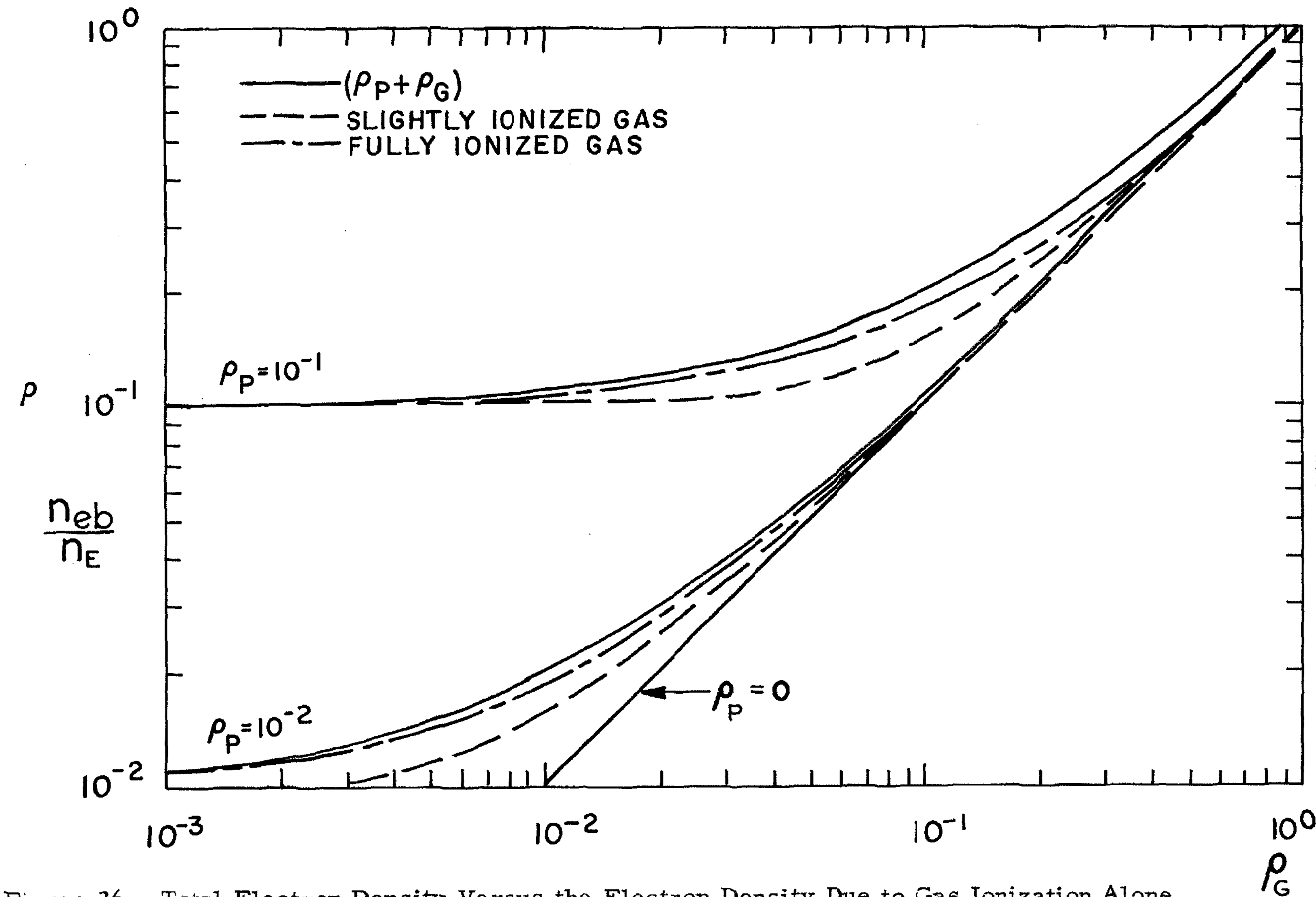

Figure 26. Total Electron Density Versus the Electron Density Due to Gas Ionization Alone.

greater than in the case of a slightly ionized gas. When the exact form of Saha's equation is used, the resulting value of ρ will be somewhere between the two limiting values which correspond to a slightly or a fully ionized gas.

C. Negligible Particle Ionization

When the electron density that would exist in the case of only particle ionization is very much less than the electron density that would exist in the case of only gaseous ionization at temperature T_e, the electron flux to the particles in the composite system will be of such a magnitude that, for a steady state to exist, the particles must absorb electrons and be at a negative potential. This defines the range in which the particle ionization is said to be negligible.

When the particle potential is negative, the ion flux to the particles should be included in the equation of steady state that requires that the net current to the particles be zero. Equation (5.34) will be used for the ion flux, and it will be assumed that the collisionless region around a particle is large; that is,

$$\left(-\frac{e\phi_a}{kT_p}\right)\left(\frac{a}{\lambda_{iG}}\right)^2 << 1 ,$$

where λ_{iG} is the ion - gas mean free path. Equations (5.49) and (5.50) will be used for the electron fluxes. Thus, the requirement of steady state is

$$n_E T_p^{\frac{1}{2}} + n_{ib} T_G^{\frac{1}{2}}\left(\frac{m}{m_i}\right)^{\frac{1}{2}}\left(1 - \frac{e\phi_a}{kT_G}\right) = n_{eb} T_e^{\frac{1}{2}} e^{+\frac{e\phi_a}{kT_e}} , \qquad (6.12)$$

where m_i and m are the masses of the ion and electron, respectively.

Rather than pursuing this general case further, which would involve a great deal of algebraic complexity, the range of negative particle potential will be broken up into two parts. The first part is that in which the electron fluxes to and from a particle are almost equivalent. The second part, that in which the thermionic emission from the particles is negligible compared to the gaseous ionization ($n_E \simeq 0$), is that in which the ion and electron fluxes to a particle are nearly the same.

The phenomenon of enhancement of gaseous ionization would most likely fall into the first part of this range.

The second part includes some situations in which the gas - particle interaction is capable of greatly reducing the number of free electrons in the system. A direct interaction, by which this is done, is the one in which the particles absorb an appreciable fraction of the free electrons generated by the gas and thereby reduce the electron density well below that which would exist if the particles were not present. An indirect interaction, similar in nature to that of gaseous ionization enhancement, but also able to vastly reduce the electron density, is that in which the electron gas suffers a power loss because the rate of recombination exceeds the rate of ionization at a particle's surface. This power loss reduces the electron temperature, and, therefore, the gaseous ionization.

1) <u>Enhancement of Gaseous Ionization</u> - As stated previously, it is assumed in analyzing the effect of gaseous ionization enhancement, that the particle potential is negative and that the ion flux to the

particles may be neglected. Also, in order to have this effect considered under conditions where it may be large, it is further assumed that the gas is slightly ionized and that there is only a negligible fraction of electrons absorbed on the particles.

Using these assumptions, equation (6.1) reduces to Saha's equation:

$$\rho = \left\{ \frac{n_o K_G(T_e)}{n_E^2} \right\}^{\frac{1}{2}} . \tag{6.13}$$

Combining this with equation (6.3), it is seen that

$$\left(\frac{T_p - T_e}{T_e - T_G} \right) \left(\frac{T_p}{T_e} \right)^{\frac{1}{2}} = \psi \, \frac{\{n_o K_G(T_e)\}^{\frac{1}{2}}}{n_E(T_p)} \tag{6.14}$$

where

$$\psi = \left(\frac{3}{4\pi a^2 N} \right) \{ \overline{(m/m_G)(\delta/\lambda_{eG})} \} . \tag{6.15}$$

Before considering a numerical example, it is instructive to analyze equation (6.14) in order to determine the dependence of the electron temperature on the particle temperature.

It might initially be anticipated that by increasing the strength of the thermionic emission from the particles by increasing the particle temperature, the ratio T_p/T_e could be driven toward unity. However, the power lost by the electron gas through elastic collisions has a corresponding increase proportional to the increase in n_{eb}. Equation (6.14) shows that

$$\left(\frac{T_p - T_e}{T_e - T_G} \right) \left(\frac{T_p}{T_e} \right)^2 T_e^{3/4} \sim e^{-\frac{1}{kT_p} \left[\frac{V}{2} \frac{T_p}{T_e} - e\Phi \right]} . \tag{6.16}$$

Thus, when $T_p/T_e \simeq 1$, the work function has twice the "weight" of the gas ionization potential. If $(V/2)(T_p/T_e) < e\Phi$ and T_p is increased, T_p/T_e will be driven closer to unity. However, if $(V/2)(T_p/T_e) > e\Phi$, and T_p is increased, T_p/T_e will decrease and, as long as the gas remains slightly ionized, T_p/T_e cannot be driven closer to unity by increasing T_p. Physically, it is a question of which will increase faster as T_p is raised; the electron density due to the thermionic emission, n_E, which increases the power input to the electron gas, or the electron density due to gaseous ionization, n_{eb}, which increases the power lost by the electron gas through elastic collisions.

The numerical example of gaseous ionization enhancement which is to be considered illustrates the magnitude of the enhancement that can occur in the exhaust plume of a solid rocket motor. The combustion products in a plume may contain both thermionically-emitting alumina particles, whose temperature lags that of the gas, and alkali metal impurities. It is the interaction between these particles and the other products of combustion which determines the enhancement of the ionization of the alkali metal vapors.

The observed electron densities are usually on the order of $10^{10} \rightarrow 10^{11}$ cm^{-3}, but equilibrium calculations of the particle ionization at the particle temperature or the alkali metal ionization at the gas temperature yield considerably lower values. One possibility is that the chemical reactions, including recombination of electrons and ions, become frozen upstream of the exit plane and thereby give electron densities which are higher than the equilibrium values. A second possibility, and one which may be attributed directly to the presence of the particles, is the type of gaseous ionization enhancement studied here. This phenomenon is one of equilibrium in the sense that the ionization is assumed to be in equilibrium at the electron temperatures but one of non-equilibrium in that the particles and gas temperatures are unequal. It is felt that, even though the model used to study the ionization enhancement is much simpler than the actual system, the results are representative of the correct magnitude of the enhancement to be expected.

Some of the parameters which will be used to describe the conditions in the plume have been chosen in an arbitrary, but is is hoped, realistic manner. Others have been taken directly from reference 33. The average properties of the gas are taken to be

$$T_G = 1500^{\circ}K$$
$$n_G = 10^{17}\ cm^{-3} \qquad \Rightarrow \text{pressure} \simeq .02 \text{ atm.}$$
$$0 \leq \delta \leq 100 \quad (\delta = \tfrac{8}{3} \text{ for purely elastic collisions})$$
$$\text{molecular weight} = 27$$
$$\text{electron-gas total collision cross-section} = 10^{-15}\ cm^2$$

The alkali metal impurity is assumed to be 300 ppm of potassium vapor which has the following characteristics:

$$n_o = 3 \times 10^{13}\ cm^{-3}, \qquad V = 4.34 \text{ ev},$$
$$g_i = 1, \qquad g_o = 2.$$

The parameters of the alumina particles are taken to be

$$N = 10^7\ cm^{-3}, \qquad a = 10^{-4}\ cm,$$
$$e\Phi = 3.77 \text{ ev}, \qquad \varkappa = 1.17 \times 10^{-2}.$$

Direct calculations yield

$$\lambda_{eG} = 10^{-2} \text{ cm} , \qquad m/m_G = 2 \times 10^{-5} ,$$

$$\left(\frac{3}{4\pi}\right)\left(\frac{1}{Na^2\lambda_{eG}}\right)\left(\frac{m}{m_G}\right)\delta = (4.77 \times 10^{-3})\,\delta .$$

In the calculation of the ionization enhancement, equations (3.32), (5.6), and (6.13), and Figures 10a, 10b, and 25 are used. The results are shown in Figures 27 and 28.

In Figure 27, the electron temperature is plotted as a function of the particle temperature for $\delta = 10$, 100, and the case of no electron energy loss, $\delta = 0$. When T_p is close to T_G, n_E is very much less than n_{eb}, so that the energy input to the electrons is not sufficient to displace them far from thermal equilibrium with the gas. However, since $V/2 < e\Phi$, increases made in T_p tend to drive T_p/T_e closer to unity at higher values of T_p. In Figure 28, the enhanced ionization is plotted as a function of T_p for $\delta = 0$, 10, and 100. The approximate magnitudes to be expected for intermediate values of δ are readily obtained by extrapolation. The electron density due to gas ionization alone at the gas temperature, and the electron density due to particle ionization alone at the particle temperature, are also shown. It is seen that for $\delta = 10$ and a particle thermal lag of 500°C, the ionization is enhanced by a factor of 10. A lag of 800°C gives a factor of enhancement 10^2, that is, an electron density of 3.5×10^{11} cm^{-3}. Thus, the ionization enhancement in this system can be quite large.

In order to further define the magnitude of the phenomenon in

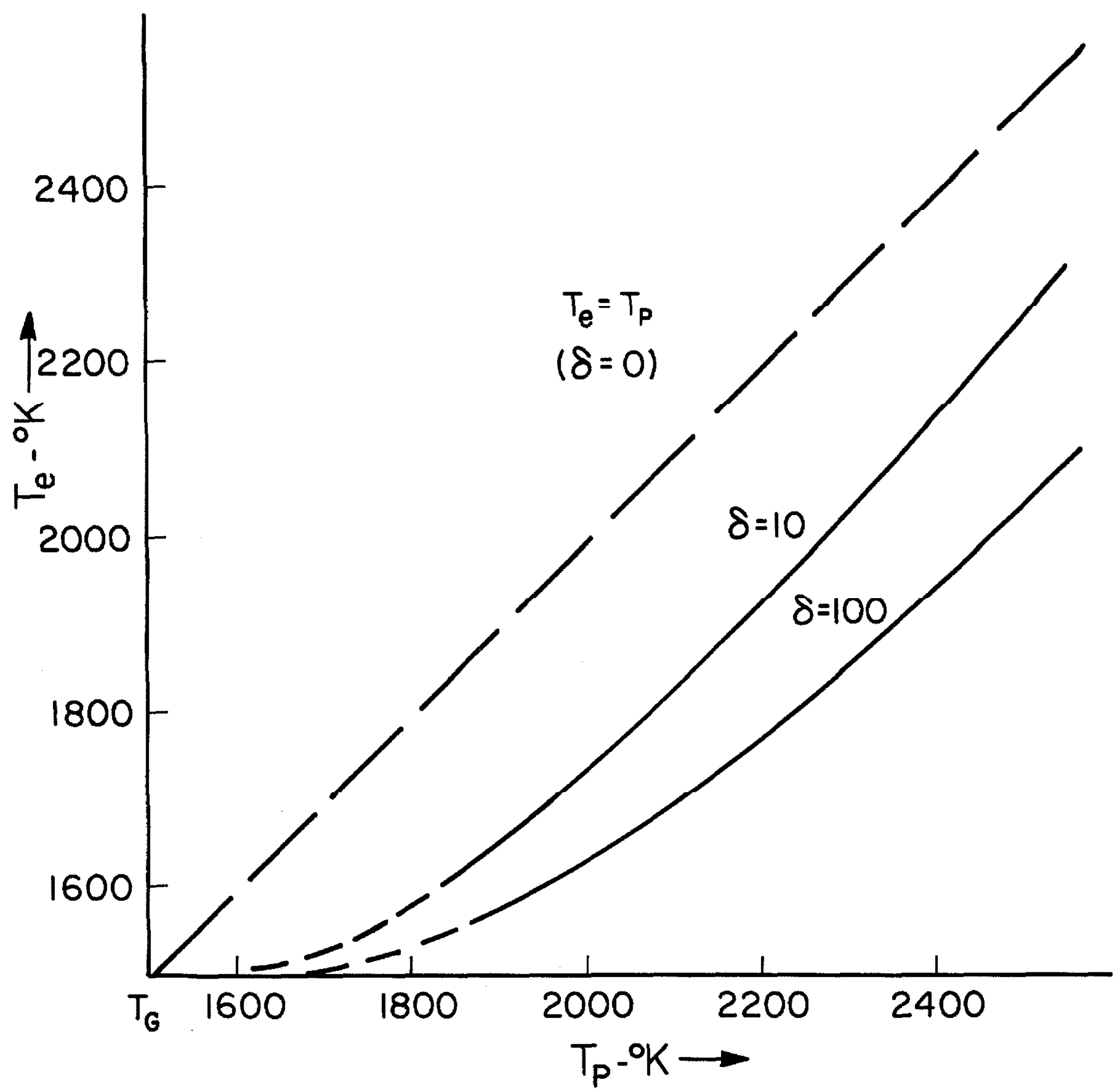

Figure 27. Electron Temperature Versus the Particle Temperature.

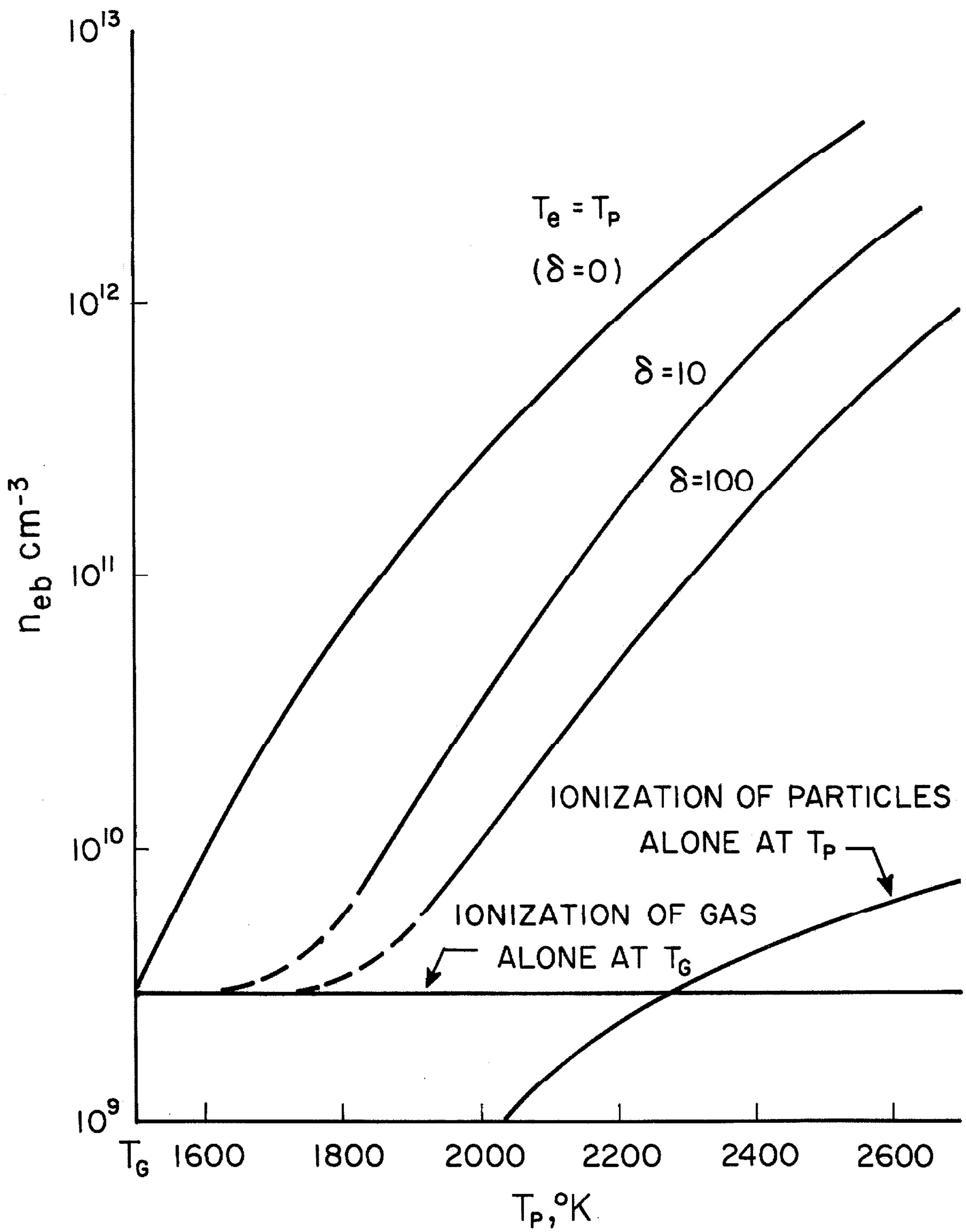

Figure 28. Electron Density Versus the Particle Temperature.

an actual system, the particle thermal lag, the rate of ion - electron recombination, the electron energy loss mechanisms, and the physical properties of the particles and the gas should be more fully examined.

2) Electron Absorption - Rosen [12] has proposed that in some gas - particle systems, it is possible for the particles to absorb an appreciable fraction of the electrons produced by gaseous ionization. The purpose of this discussion is to further explore this possibility under the assumption that the thermionic emission from the particles is negligible.

The requirement of steady state, equation (6.12), now reduces to

$$\left(\frac{n_{ib}}{n_{eb}}\right)\left(1 - \frac{e\phi_a}{kT_G}\right) e^{-\frac{e\phi_a}{kT_e}} = \left(\frac{m_i T_e}{m T_G}\right)^{\frac{1}{2}} . \qquad (6.17)$$

That is, the ion and electron fluxes to a particle are equal to one another. From equation (6.17), it is seen that there are two limiting and essentially different ways by which the requirement of steady state may be satisfied. First, when the system is dominated by the absorption phenomena, the particle potential will be close to zero, and the electron and ion fluxes are equivalent by virtue of the large discrepancy in the electron and ion densities. That is, equation (6.17) becomes

$$\left(\frac{n_{ib}}{n_{eb}}\right) \simeq \left(\frac{m_i T_e}{m T_G}\right)^{\frac{1}{2}} . \qquad (6.18)$$

At the other limit, the gaseous ionization will dominate, the particle potential will be large and negative, and there will be only a negligible loss of free electrons from the plasma. In order to gain an estimate of the magnitude of the particle potential at this limit, assume $T_e = T_G$, $m/m_i = 10^{-5}$, and $n_{eb}/n_{ib} = 1$. The result, using equation (6.17), is $(e\phi_a)/(kT_e) = -4.12$.

The ability of the gas to produce more free electrons to partially compensate for those absorbed on the particles is taken into account by the Saha equation.

$$\frac{n_{eb}n_{ib}}{n_o - n_{ib}} = K_G(T_e) \tag{5.5}$$

This equation, together with the equation of steady state, equation (6.17), and the relation for the particle potential

$$\frac{e\phi_a}{kT_p} = \left(\frac{n_{eb}n_{ib}}{N}\right) P(a, b, T_p), \tag{5.2}$$

must be solved simultaneously to determine the value of the electron density. In order to identify the important parameters of the problem and to predict the magnitude of this phenomena, two special cases will be considered.

(a) Slightly ionized gas and thermodynamic equilibrium. When the system is in a state of complete thermodynamic equilibrium ($T_p = T_G = T_e$) and the gas is slightly ionized, the analysis is relatively simple and the essential features of the absorption phenomena are readily apparent. It is convenient to define a quantity, χ, termed the absorption ratio, as the ratio of the actual free-electron density in the gas - particle system, n_{eb}, to the electron density which would exist if the particles were not present, n'_{eb}. Thus,

$$\chi = n_{eb}/n'_{eb} \tag{6.19}$$

and

$$n'_{eb} = (n_o K_G)^{\frac{1}{2}} . \tag{6.20}$$

Using equations (6.17), (5.5), and (5.2), the relation which determines χ is

$$\chi^2 = \left(\frac{m}{m_i}\right)^{\frac{1}{2}}\left[1 + Q\left(\frac{1}{\chi} - \chi\right)\right] e^{+Q\left(\frac{1}{\chi} - \chi\right)} , \tag{6.21}$$

where

$$Q = \left(\frac{n'_{eb}}{N}\right)\left(\frac{e^2}{4\pi\epsilon_o akT}\right)\left\{1 - \frac{3}{2}\left(\frac{a}{b}\right) + \frac{1}{2}\left(\frac{a}{b}\right)^3\right\} . \tag{6.22}$$

$\chi(Q)$ is plotted in Figure 29 for $m/m_i = 10^{-4}$ and 10^{-5}. When $Q \gg 1$, the case of large electron density and small particle density and radius, it is seen that the absorption ratio is close to unity, implying that the system is only slightly perturbed. However, when $Q \ll 1$, absorption phenomena will dominate, and $\chi \to (m/m_i)^{1/4}$. That is, the electron density is reduced by $(m/m_i)^{1/4}$, the degree of ionization is increased by $(m_i/m)^{1/4}$, and the ratio of the free electron density to the ion density is $(m/m_i)^{1/2}$. In this instance, the gas - particle system is in a state almost exactly opposite to that found when the gaseous ionization was negligible; a particle has a negative charge and the charged species surrounding it is almost entirely positive.

An estimate of the magnitude of the absorption parameter, Q, for representative values of the temperature and particle radius is informative. When $(a/b) \ll 1$, Q is given by

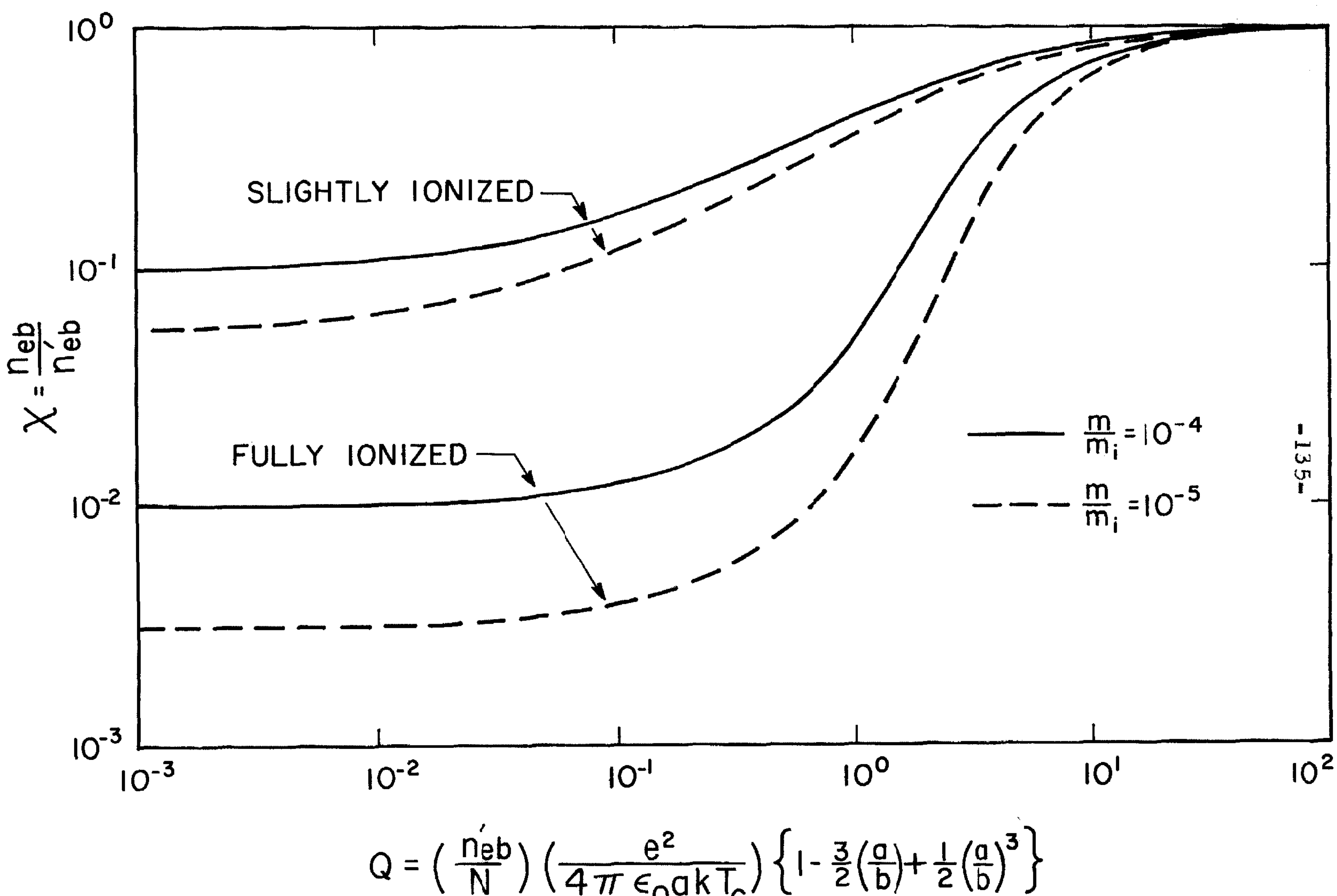

Figure 29. Absorption Ratio Versus the Absorption Parameter.

$$Q = \left(\frac{n'_{eb}}{N}\right)\left(\frac{.144 \times 10^{-2}}{akT}\right) \tag{6.23}$$

where a is in microns and kT is in electron volts. For $a = 1$ micron and $kT = .172$ ev ($2000^{o}K$), $Q = .00837\ (n'_{eb}/N)$. Thus, it is evident that for appreciable absorption to occur, the particle density must be at least on the order of $n'_{eb}/100$. For one micron size particles and high values of n'_{eb}, this is a very difficult requirement. It is seen, however, that for fixed mass fraction (Na^3 = const.), $Q \sim a^2$, so that the absorption effect may be increased by dividing the non-gaseous material into finer particles. Thus, if the particle radius was reduced to .1 micron while the mass fraction was held constant in this example, the absorption phenomena would become important if the particle density were only as large as $n'_{eb} \times 10^{-4}$.

(b) Fully ionized gas and thermodynamic equilibrium. It is to be expected that the absorption phenomena in a fully ionized gas would be of a larger magnitude than in a slightly ionized gas, because the gas does not have the freedom to generate more electrons to partially compensate for those absorbed.

The absorption ratio, χ, may be used as defined previously by equation (6.19), but

$$n'_{eb} = n_o . \tag{6.24}$$

Thus, for a plasma in which thermodynamic equilibrium prevails and the ionizable species is fully ionized, the absorption ratio is determined by

$$\chi = (m/m_G)^{\frac{1}{2}}[1 + Q(1-\chi)]e^{Q(1-\chi)}, \tag{6.25}$$

where Q, the absorption parameter, is given by equation (6.22).

This relation is also plotted in Figure 29. The behavior exhibited here is of the same nature as that found in the previous case, but the magnitude is much larger. When $Q >> 1$, the absorption phenomena are negligible. But when $Q << 1$, $\chi \rightarrow (m/m_i)^{1/2}$, the square of the value found for the reduction in the case of a slightly ionized gas. However, the total fraction of free electrons, n_{eb}/n_{ib}, will be the same, $(m/m_i)^{1/2}$, because the ion density is unaltered by the electron absorption.

The investigation of these two special cases has shown that the absorption parameter, Q, defined by equation (6.25), is the appropriate one for estimating the magnitude of the absorption phenomena and that for this phenomena to be appreciable, it is necessary for the absorption parameter to be of the magnitude of or very much less than unity. Also, the maximum factor by which the electron density may be reduced is $(m/m_i)^{1/4}$, for a gas which is slightly ionized, and $(m/m_i)^{1/2}$ for one which is fully ionized.

Attention should be called to the fact that the importance of the phenomena of absorption as well as that of quenching of gaseous ionization, yet to be discussed, rests on the assumption that the electron current emitted thermionically from a particle is negligible with respect to the ion current returning to it. Allport and Rigby [13] have pointed out that in high-temperature gas - particle systems ($\sim 3000^{\circ}$K), it is exceedingly difficult in practice to find materials that are not good thermionic emitters or which do not introduce metal vapor into the system which will ionize and add substantially to the electron density. Hence, the application of these analyses to plasmas in which the particles are at high temperatures ($\sim 3000^{o}K$) is speculative, and the difficulties mentioned by Allport and Rigby should be considered. The application to systems of lower temperature is more realistic.

3) Quenching of Gaseous Ionization - The purpose of this section is to briefly point out the existence of several additional phenomena which may appreciably alter the degree of gaseous ionization by changing the electron temperature. The discussion will center primarily on the quenching of the gaseous ionization which results when the particles, rather than the electrons, act as energy-absorbing third bodies in recombination collisions, and thereby induce a cooling of the electron gas.

When the particles and the gas are at the same temperature and none of their properties are changing with time, a state of thermodynamic equilibrium prevails in which the electron temperature is equal to the particle and gas temperatures, and the rates of ionization and recombination at the surface of each particle are equal to one another. If now the particle temperature is reduced, the rate of surface ionization will be exceeded by the rate of surface recombination and there will be a net rate of removal of electrons and ions at the surfaces of the particles. If a steady state is to exist, there must be a net rate of ion and electron generation in the plasma and, if it is assumed that the ionization is controlled by electron - atom collisions, there will also exist an associated net power drain on the electron gas. The compensating power input to the electron gas, which balances this power loss, will result from elastic and super-elastic electron collisions with the heavy species. Thus, when the particles are at a lower temperature than the gas and are not thermionically emitting an electron current, the temperature of the electrons will be reduced below the gas temperature by this mechanism, and a reduction in the gaseous ionization will result. Also, since the rate of heating of the particles by surface recombination is limited by the rate at which energy may be transferred from the gas by collisional processes, it is to be expected that the rate of particle temperature change will be very small, except when the particles themselves are very small (see Section II-D-4).

When the thermionic emission from the particles is negligible, the condition of steady state is

$$\Gamma_{n_e}^{-} = \Gamma_{n_i}^{-} \; . \tag{6.26}$$

The power balance on the electron gas will be written in a simplified form by assuming that for every ion that strikes the surface of a particle, a net amount of energy is absorbed by the particle which is equal to $\alpha_R V$, where α_R is an accommodation coefficient equal to or less than unity, and V is the ionization potential of the gas. This accommodation coefficient is representative of the difference between the rates of surface recombination and surface ionization as well as the efficiency of energy transfer to a particle's surface. When $T_p = T_G$, $\alpha_R = 0$.

A second effect which should be included here is that which arises due to the negative potential of the particles. When the rate of surface ionization is lower than the rate of surface recombination, there will not be as many high-energy electrons expelled by the field around a particle as penetrate this field and reach the particle's surface. Thus, for each electron that reaches the surface of a particle, an amount of energy equal to $\alpha_R e\phi_a$ is lost from the electron gas. For simplicity, the coefficient α_R has been assumed to be the same for each of these two energy-loss processes. Thus, the power balance on the electron gas is

$$(4\pi a^2 N)(\alpha_R)[V\Gamma_{n_i}^-(a) - e\phi_a \Gamma_{n_e}^-(a)]$$

$$= n_{eb}\overline{C_e}\{\overline{(m/m_G)(\delta/\lambda_{eG})}\}(\tfrac{3}{2}K)(T_G - T_e) , \qquad (6.27)$$

where δ is the energy transfer coefficient now for elastic and superelastic collisions.

If equations (6.26) and (6.27) are combined, equation (5.50) is used for $\Gamma_{n_e}^-(a)$, and it is assumed that the energy restoration to the electrons is controlled by one heavy species only, the following is obtained:

$$\frac{k(T_G - T_e)}{V - e\phi_a} = \left(\frac{2\pi}{3}\right)\left(Na^2\lambda_{eG}\right)\left(\frac{\alpha_R}{\delta}\right)\left(\frac{m_G}{m}\right)e^{+\frac{e\phi_a}{kT}} . \qquad (6.28)$$

Once the right hand side, which varies as the total surface of the particles, is increased to a value on the order of $kT_G/(V - e\phi_a)$, further increases will decrease the electron temperature abruptly. However, the electron temperature can be decreased only so far before the atom-atom collisions become dominant in the electron - ion generation reactions. In order to gain an estimate of the magnitude of the right hand side, assume the following:

$N = 10^7\ cm^{-3}$ $\qquad a = 10^{-4}\ cm$

$\lambda_{eG} = 10^{-2}\ cm$ $\qquad \alpha_R = 10^{-1}$

$\delta = 10$ $\qquad m_G/m = 10^5$

$(e\phi_a)/kT_e = -4.6$ (negligible electron absorption by the particles).

Upon substitution of these values into equation (6. 28), the following is obtained:

$$\frac{k(T_G - T_e)}{V - e\phi_a} = 2.09 \times 10^{-2} .$$

If $V = 4$ ev and $T_G = 3000^\circ K$, a reduction of T_e to $1850^\circ K$ is predicted.

Two additional phenomena which could also produce significant changes in the degree of gaseous ionization by altering the electron temperature become apparent when the inverse of some of the situations already investigated are considered. The situation in which the particles are again cooler than the gas but the thermionically-emitted current from a particle is now much larger than the ion current to the particle, is really just the inverse of the situation considered when gaseous ionization enhancement was analyzed. The particles absorb hot electrons from the plasma and emit relatively cooler ones, and the temperature of the electron gas and the degree of gaseous ionization are thereby reduced. The formalism developed in Section VI-A is applicable here, but $T_p \leq T_e \leq T_G$. The case of quenching of gaseous ionization also has associated with in an inverse situation. If the particles are hotter than the gas but not thermionically emitting, the rate of surface ionization will exceed the rate of surface recombination, and a net energy input to the electron gas will result. This also could appreciably increase the electron density in the plasma.

VII. SUMMARY

It has been shown that, when thermionically-emitting particles are present in an ionized gas, there are many interesting and practically important interactions which may take place between the gaseous and the particle ionizations. Some of the interactions which have been explored, the case of gaseous ionization enhancement in particular, are able to produce extremely large changes in the electron density of the plasma from what would normally be expected from either particle or gas ionization alone.

An understanding of the basic ionization phenomena which occur in a gas - particle plasma has been developed through an analysis of various special cases. The order of presentation of these cases in the analysis has followed the transition from the case of pure particle ionization alone, with no gaseous ionization, to the second extreme case of pure gaseous ionization, with no thermionic emission from the particle. However, before the analysis was undertaken, the physical phenomena which are of primary importance were discussed (Part II). Thermionic emission from the particles, the coupling which exists between the charge and potential distributions, the equilibrium charge distribution, and the application of the quasi-steady state approximation to the plasma were considered.

The first special case, which has been investigated (Part III), is that in which there is no gaseous ionization and the plasma is in equilibrium. The coupling between the charge and potential distributions was taken into account in a direct manner by finding a family of numerical solutions to Poisson's equation for a one-particle system. These solutions cover the range in which almost all problems of current engineering interest are located. From the results, it has been shown that the potential distribution which exists around a particle will fall into one of two distinctly different characteristic regimes; the weak shielding regime, or the strong shielding regime.

In the weak shielding regime, the potential distribution is determined primarily by geometrical effects, and it has therefore been possible to obtain a relation which is a very good approximation to the actual potential distribution. It was shown that in this regime almost all of the electrons which are external to the particles are located in a region of nearly uniform density and potential, and that the picture of a "uniform sea" of electrons is appropriate. Also, in the weak shielding regime, there is a limiting case that can be easily analyzed by statistical methods which take the distribution of particles over the available ionized states into account. It has been pointed out that the results of such an analysis are in good agreement with the approximate solution. In addition, it was demonstrated that under certain conditions, which are well satisfied in many problems of current interest, the degree of ionization of each particle in the plasma is nearly the same.

The essential features of the strong shielding regime were found to be in sharp contrast to those of the weak shielding regime. First, a large fraction of the electrons in the plasma are located close to the surfaces of the particles so that there are relatively few "free electrons". Second, the potential distribution close to a particle's surface is one-dimensional in nature. Also, it was found that the transition which occurs in going from the weak to the strong shielding regime is a very sharp one and that it takes place as the particle radius becomes larger than five times the Debye shielding distance defined by the properties at the particle's surface.

Values of the particle potential and other quantities of interest have been plotted as functions of two parameters which are initially known and which completely define the solution. Both regimes are covered by these plots.

In order to gain an estimate of the degree of accuracy with which the value of the electron density in the plasma may be calculated or the extent to which it can be controlled, an analysis was carried out (Part IV) to determine the functional dependences of the electron density on the various parameters which define the solution. Again, the special case of negligible gas ionization was explored.

It has been found that, in the strong shielding regime, the value of the Debye shielding distance far from a particle adjusts itself to a value which is approximately equal to the interparticle distance so that the "free" electron density is limited by the temperature and the particle number density. On the other hand, the mean electron density, which is almost directly proportional to the number of electrons packed close to the surfaces of the particles, is strongly dependent on the properties of the thermionic emission from the particle.

In the weak shielding regime, two types of limiting behavior were found. First, when the particle potential is small with respect to the work function of the particle material and kT , the electron density of the plasma is exponentially dependent on the temperature and work function in the same manner in which the thermionic emission law would predict. Also, it is very insensitive to the particle radius or number density. However, when the particle potential becomes of the same magnitude as the work function and much larger than kT, the electron density is linearly related to the work function, temperature, particle radius, and particle number density. When the total non-gaseous content of the plasma remains the same but the size of the particles is varied, the electron density varies as the inverse square of the particle radius. Hence, the effects of shielding tend to nullify the strong dependence of the electron density on the thermionic emission properties, and changes in the particle size and number density become the means by which the electron density can most efficiently be altered by large factors.

Two additional features of the weak shielding regime have been established in Part IV. First, it has been demonstrated that it is the particle temperature, rather than the gas temperature, which is of primary importance in fixing the degree of particle ionization. Second, it has been shown that, when there is a distribution of particle sizes and a related distribution of particle temperatures, the electron density will be nearly the same as that which would be found in another system which had the same amount of particle material, the same mean particle radius, and the temperature which corresponded to this mean radius.

The additional effects which non-equilibrium gaseous ionization introduces into the problem were next analyzed (Parts V and VI). The special case of primary interest, in these last two sections, has been that of gaseous ionization enhancement. This occurs when the particles are hotter than the gas, so that the electrons which are emitted from a particle are hotter than those which return to it. This causes a heating of the electron gas which, in turn, enhances the level of gaseous ionization. Before any special cases were considered, however, the additional relations which are required in order to analyze these non-equilibrium effects have been formulated.

The assumptions and model which were used in the analysis were first considered. This included a discussion of the following assumptions:

(1) quasi-steady state,

(2) spatially uniform conditions,

(3) the validity of the weak shielding approximation for the case in which both electrons and ions are present,

(4) ionization equilibrium of the gas at the electron temperature.

The energy transfer processes which exist between the electrons and the gas were also discussed.

The electron number and energy fluxes to and from a particle were next considered in some detail. A problem in which a three-dimensional collisionless region exists around the particle, has been solved in order to obtain expressions for these fluxes for both positive and negative particle potentials.

In the last section (Part VI), some special cases in which gaseous ionization is of importance have been investigated.

The first situation which was considered is that in which the particle and gas ionizations are of the same magnitude. The supression of one type of ionization due to the presence of the other has been clearly illustrated by two special cases. In both of these, the gas - particle plasma was in equilibrium; but in the first, the gas was slightly ionized, while in the second, it was fully ionized. However, the magnitude of the effects which were explored here were found to be small relative to the magnitudes which have been encountered in the other special cases studied.

The special case of primary interest, that of gaseous ionization enhancement, was next considered. A straightforward method of calculation of the ionization enhancement, which is induced by the relatively hot particles, has been presented. This method was then applied to the calculation of the ionization enhancement that could be expected in the exhaust plume of a solid rocket motor. It was found that this enhancement could be quite large (a factor of 10^2) if the temperature change of the particles appreciably lagged that of the gas. It has also been pointed out that this process has an inverse; when the temperature of the particles is less than that of the gas, the degree of gaseous ionization will be reduced.

The last two types of interaction between the particles and the gas, which were investigated, are located at the second extreme where the thermionic emission from the particles is negligible.

The first of these two interactions, which has been considered, was the direct one of electron absorption by the particles. The magnitude of this effect was explored by again investigating the two special cases of an equilibrium plasma in which the gas was either slightly or fully ionized. It was shown that the magnitude of the reduction in the free electron density can be expressed in terms of one parameter only, and this parameter has been termed the absorption parameter. Plots of the magnitude of the reduction in free electron density as functions of this absorption parameter have been presented.

It was shown that when the absorption parameter is very much less than unity, the maximum absorption occurs. In the case of a slightly ionized gas, the electron density can be reduced by a factor of $(m/m_i)^{1/4}$, where m and m_i are the masses of the electron and ion respectively. In the case of a fully ionized gas, the factor of reduction could be $(m/m_i)^{1/2}$. It has been concluded that, for one micron size particles and high electron densities, it would be difficult to make the absorption parameter of magnitude of less than unity. However, it was shown that the value of this parameter can be greatly decreased by dividing the non-gaseous material into finer sized particles.

The other interaction between the gas and particles, that is located at the second extreme where the thermionic emission from the particles is negligible and which was investigated, is the indirect one of quenching of gaseous ionization. It was pointed out that if the particles are cooler than the gas, the rate of recombination will exceed the rate of ionization at the surface of a particle and a net power loss from the electron gas will result. This, in turn, will produce a decrease in the degree of gaseous ionization. This process, like that of ionization enhancement previously examined, also has an inverse;

when the temperature of the particles is greater than that of the gas, the degree of gaseous ionization will be increased.

In general, the results of this analysis have been twofold. First, the problem of equilibrium particle ionization, in which there is no gaseous ionization, has been thoroughly analyzed and a good understanding of the nature of the potential and charge distributions has been obtained. Second, many gas - particle interactions, by which the electron density of the plasma can be greatly altered due to the presence of the particles, have been investigated.

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APPENDIX A

Nomenclature

Each symbol which is used repeatedly in the text is defined below. Following each definition is the number of the equation where the symbol is first used or quantitatively defined.

Any quantity which is primed is to be evaluated at reference state conditions [see equations (4.9) - (4.16) or equation (6.20)] .

The notation $f(T_{p,e})$ means $f(T_{p,e}) = f(T_p)$ or $f(T_e)$.

$A^{(th)}, A^{(ex)}$	theoretical (2.5) and experimental (2.4) values of the constant coefficient in the Richardson-Dushman equation
a	particle radius (2.6)
$B^{(th)}, B^{(ex)}$	theoretical (2.5) and experimental (2.2) values of the constant in the equation for the electron density due to thermionic emission
b	outer radius which defines the volume allotted to each particle (3.1)
C	particle capacitance (3.37)
$\overline{C_e}$	mean electron speed (2.3)
$\overline{C_G}$	mean gas molecule speed (2.20)
C_i	i^{th} constant (used repeatedly)
c	outer radius of collisionless region (see Figure 20)
E	electric field (3.30)
e	absolute value of electron charge (2.1)
$e\Phi$	thermionic work function (2.1)

$e\phi_\ell$	particle ionization potential of ℓ^{th} level (3.38)
$f(\vec{q}, \vec{p})$	number density in phase space (5.9)
f_a, f_c	number density in phase space of the electrons emitted from $r = a$ and $r = c$ (5.10)
g_i, g_o	statistical weight of ion and ground state atom (5.6)
$\mathcal{H}$	Hamiltonian (5.12)
h	Planck's constant (2.1)
J	saturation electron current (2.3)
$J^{(ex)}$	experimental value of saturation electron current (2.4)
$K_G(T)$	ionization equilibrium constant (5.6)
k	Boltzmann's constant (2.1)
$\mathcal{L}$	Lagrangian (5.13)
m	electron mass (2.1)
m_G	gaseous molecule mass (2.21)
m_i	ion mass (6.12)
m_s	atom mass for atoms in particle (2.21)
N	number density of particles (2.12)
N_e	number of electrons due to emission from the particle
N_ℓ	number density of particles ionized ℓ times (3.39)
n_E	electron density due to thermionic emission (2.3)
$n_E^{(th)}, n_E^{(ex)}$	theoretical (2.1) and experimental (2.2) values of the electron density due to thermionic emission
$n_e(r)$	electron density as a function of r (3.2)
n_{ea}, n_{eb}, n_{ec}	electron density at $r = a$, b, and c (3.2)
n_G	density of gaseous heavy species (2.21)
n_{ib}	ion density at $r = b$ (5.2)

n_o	density of ionizable species at zero degree of ionization (5.5)
n_s	number density of atoms in particle material (2.21)
$\overline{n_e}$	mean electron density (2.12, 3.19)
P	non-dimensional potential of a particle with one electron charge (5.3)
P_ℓ	particle ionized ℓ times (3.38)
$\vec{p}$	conjugate momentum vector (5.9)
p_r, p_θ, p_ψ	momentum conjugate to r, θ, and ψ coordinates (5.15)
$p_r^{(i)}$	radial momentum for i^{th} marginal contour (5.27)
p_Ω	total angular momentum (5.17)
Q	absorption parameter (6.22)
$\vec{q}$	generalized coordinate vector (5.9)
R	ratio of geometrical to charge density terms in Poisson's equation (3.11)
r	radius whose origin is at the particle center (3.4)
T	temperature of plasma in equilibrium (2.1)
T_e, T_G, T_p	temperature of electrons (2.25), gas (2.22), and particles (2.22)
V	ionization potential of ionizable species (5.6)
w	non-dimensional work function (4.16) of reference state
x	non-dimensional radius (3.5)
y	non-dimensional potential (3.5)
y_a	non-dimensional potential at $r = a$ (3.8) or of reference state (4.15)

$Z(r)$	total number of electrons outside radius r (3.17)
z	positive charge on particle in electron charges (2.6)
α	ratio of actual to reference state particle radius (4.13)
α_R	accommodation coefficient (6.27)
β	ratio of actual to reference state outer radius (4.55)
$\Gamma^+(r)$	quantity per unit time and area at radius r which is emitted from the particle and leaves the collisionless region (5.31, 5.37)
$\Gamma^-(r)$	quantity per unit time and area at radius r which is emitted into the collisionless region and reaches the particle (5.41, 5.42, 6.26)
γ	square of the ratio of the radius of the collisionless region to the radius of the particle (5.7)
ϵ_α	spread in the particle size distribution (4.64)
ϵ_o	permittivity of a vacuum (2.11)
ϵ_τ	spread in the particle temperature distribution (4.62)
η	ratio of actual to reference state particle potential (4.10)
θ	angle between $\vec{r}$ and z axis in spherical coordinates (5.13)
$\varkappa$	ratio of experimental to theoretical values of the electron density due to thermionic emission (3.33)
$\lambda_{Da}, \lambda_{Db}$	Debye shielding distance at radius a and radius b (2.8, 3.7)
$\lambda_{Da}^{(th)}, \lambda_{Da}^{(ex)}$	theoretical and experimental values of the Debye shielding distance corresponding to $n_E^{(th)}$ and $n_E^{(ex)}$ (3.36)

λ_{ee}	electron - electron mean free path (5. 1)
λ_{eG}	electron - gas mean free path (5. 1)
μ	ratio of actual to reference state particle number density (4. 14)
ν	ratio of actual to reference state electron density at radius b (4. 9)
ξ	independent variable of the numerical integration (3. 9)
ρ	ratio of the electron density at radius b to the electron density due to thermionic emission (6. 2)
ρ_G, ρ_p	values of ρ with pure gas ionization alone (6. 5, 6. 9) and pure particle ionization alone (5. 6, 6. 10)
σ	shielding ratio (3. 18)
τ	ratio of actual to reference state temperature (4. 12)
$\tau_1 \rightarrow \tau_6$	characteristic times (2. 15)
$e\Phi$	thermionic work function (2. 1)
ϕ	electrostatic potential (3. 2)
ϕ_a	particle potential (2. 14)
$e\phi_\ell$	particle ionization potential of ℓ^{th} level (3. 38)
χ	absorption ratio (6. 19)
ψ	azimuthal angle in spherical coordinates (5. 13) and electron energy loss parameter (6. 15)
ω	ratio of actual to reference state work function (4. 11)

APPENDIX B

Integral-Relation for the Potential Distribution About a Spherical Particle

Soo has attempted to solve for the potential distribution about a positively-charged spherical particle which is surrounded by a continuous electron charge. This charge extends out to an outer radius defined by the requirement that the net charge contained within this radius is zero. This is the same problem which is solved in Part III by the use of Poisson's equation. Soo has elected to use the integral relation for the potential which, using an opposite sign convention and to within an additive constant, he has written as (reference 15, equation 1.1)

$$\phi(r) = \frac{e}{4\pi\varepsilon_o r}\left[z - \int_a^r n_e(r')\, 4\pi r'^2\, dr'\right] \qquad \text{(b-1)}$$

where r is the position radius, a is the particle radius, z is the number of positive electron charges on the particle, and $n_e(r)$ is the electron density. $\phi(b)$ is set equal to zero, b being the outer radius. This equation is in error in that it assumes that the potential may be determined, to within an additive constant, by knowing only the value of the net charge within the sphere of radius r. This is true for the electric field, but not for the potential. In particular, the potential at the surface of the particle is not given just by $ez/(4\pi\varepsilon_o a)$, but it also depends upon the manner in which the electrons are distributed about the particle.

The correct relation for $\phi(r)$ may be found from Gauss' law

$$\int_S \vec{D} \cdot d\vec{S} = \int_V \rho \, dv \tag{b-2}$$

where $\vec{D}$ is the electric displacement vector and ρ is the charge density. For this problem, it may be written as

$$- 4\pi\varepsilon_o r^2 \frac{d\phi}{dr} = e\left[z - \int_a^r n_e(r')4\pi r'^2 dr'\right] \tag{b-3}$$

$$= e \int_r^b n_e(r')4\pi r'^2 dr' \quad . \tag{b-4}$$

Thus,

$$\phi(r) = \frac{e}{\varepsilon_o} \int_{r''=r}^{b} \int_{r'=r''}^{b} \{r'/r''\}^2 n_e(r')dr'dr'' \tag{b-5}$$

Interchanging the order of integration, by examination of the area integrated over in the (r', r'') plane,

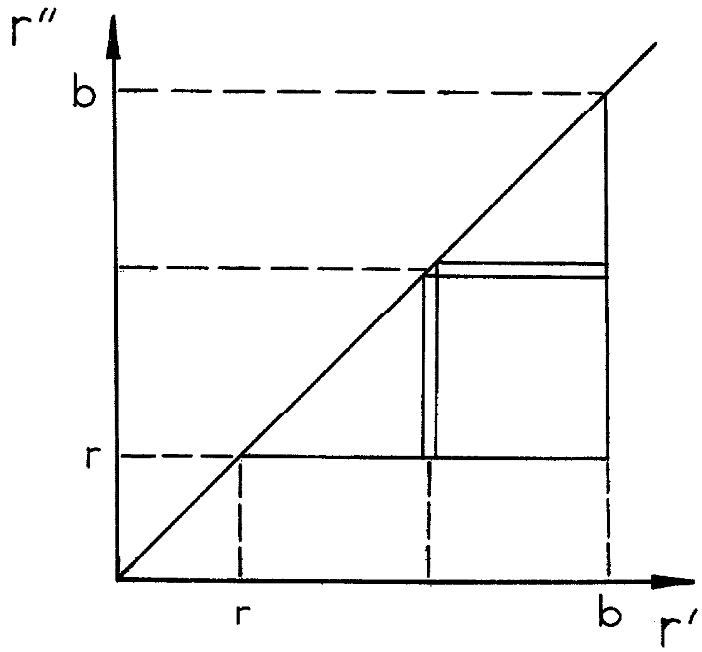

Figure 30. Area of Integration in the (r', r'') Plane.

yields

$$\phi(r) = \frac{e}{\varepsilon_o} \int_{r'=r}^{b} \int_{r''=r}^{r'} \{r'/r''\}^2 n_e(r') dr'' dr' \tag{b-6}$$

$$= \frac{e}{\varepsilon_o} \int_{r}^{b} \left(\frac{1}{r} - \frac{1}{r'}\right) n_e(r') r'^2 dr' \quad , \tag{b-7}$$

or

$$\phi(r) = \frac{e}{4\pi\varepsilon_o r}\left[z - \int_a^r n_e(r') 4\pi r'^2 dr\right] - \int_r^b \left[\frac{e n_e(r')}{4\pi\varepsilon_o r'}\right] 4\pi r'^2 dr' \quad . \tag{b-8}$$

Comparing this with equation (b-1), it is seen that the actual potential will be lower by the amount

$$\int_r^b \left[\frac{e\, n_e(r')}{4\pi\varepsilon_o r'}\right] 4\pi r'^2 dr' \quad . \tag{b-9}$$

The origins of this and the other terms in equation (b-8) are easily seen by dividing the charge distribution up into thin, spherical shells and summing over the potential due to each.

APPENDIX C

Transformation of Poisson's Equation

Poisson's equation, equation (3.6), may be written in the form of a first-order equation by using the following transformation:

$$u = x\frac{dy}{dx} + 2 \tag{c-1}$$

$$z = \left(\frac{b}{\lambda_{Db}}\right)^2 x^2 e^y \ . \tag{c-2}$$

This yields

$$u\frac{du}{dz} + \frac{1}{z}(u - 2) = 1 \ , \tag{c-3}$$

which, although relatively simple in appearance, has not been of assistance to the author in obtaining an analytical solution.

APPENDIX D

Computer Solution to the Fundamental Problem

The DEQ subroutine of an IBM 7090 computer was used to integrate the following differential equation:

$$\frac{d^2y}{d\xi^2} = (b/\lambda_{Db})^2 \frac{1}{\xi^4} e^y \tag{d-1}$$

$$y = 0 \text{ and } dy/d\xi = 0 \text{ at } \xi = 1 .$$

In order to make use of this subroutine, the problem was reduced to two simultaneous first-order differential equations.

$$y_1 \equiv y \tag{d-2}$$

$$y_2 \equiv dy/d\xi \tag{d-3}$$

$$dy_1/d\xi = y_2 \tag{d-4}$$

$$dy_2/d\xi = (b/\lambda_{Db})^2(1/\xi^4)\, e^{y_1} \tag{d-5}$$

$$y_1 = 0 \text{ and } y_2 = 0 \text{ at } \xi = 1 .$$

y_1 and y_2 were calculated as functions of ξ and increments of $\Delta\xi = +1$ were used. The error per interval was less than 10^{-6} and the maximum error for a given value of ξ was less than $\xi \times 10^{-6}$. Twenty values of (b/λ_{Db}) between .1 and 2.0 were used.

Several quantities of interest were calculated at each value of ξ and are given below.

1. $$-\frac{dy}{dx} \Big/ [\tfrac{1}{3}(b/\lambda_{Db})^2] = \tfrac{1}{3}(b/\lambda_{Db})^2\, \xi^2\, y_2 \tag{d-6}$$

2. $$\frac{b}{\lambda_{Dr}} = \frac{b}{\lambda_{Db}}\, e^{+y_1/2} \tag{c-7}$$

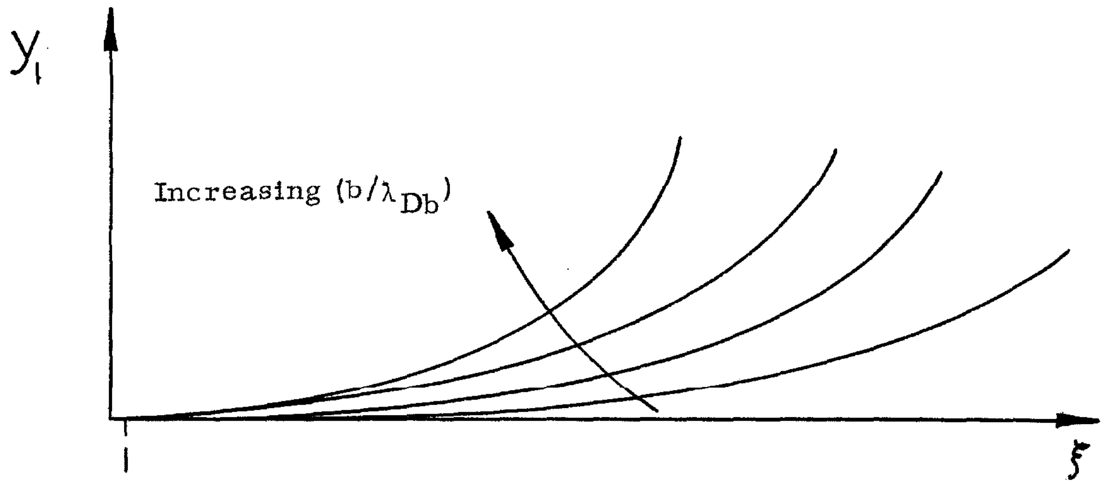

Figure 31. Family of Curves Found in the Numerical Solution.

3. $$R = \frac{2\,\xi^3 y_2}{\left(\frac{b}{\lambda_{Db}}\right)^2 e^{y_1}} \tag{d-8}$$

4. $$\sigma = \overline{n_e}/n_{eb} \tag{d-9}$$

Using Gauss' law,

$$eZ(r) = -4\pi\varepsilon_o r^2 \frac{d\phi}{dr} \tag{d-10}$$

$$\sigma = 3\,y_2 \Big/ (b/\lambda_{Db})^2 \tag{d-11}$$

5. $$\overline{n_e}/n_{ea} = \sigma\, e^{-y_1} \tag{d-12}$$

APPENDIX E

Solution to the One-Dimensional Problem

The one-dimensional problem is defined by

$$\frac{d^2y}{dx^2} = (b/\lambda_{Db})^2 \, e^y \tag{e-1}$$

$$y = 0 \text{ and } dy/dx = 0 \text{ at } x = 1 .$$

Multiplying both sides by dy/dx and integrating from $x = 1$ to $x = x$ gives

$$\frac{dy}{dx} = -\sqrt{2}\,(b/\lambda_{Db})(e^y - 1)^{\frac{1}{2}} . \tag{e-2}$$

Letting

$$p = e^y , \tag{e-3}$$

$$\frac{dp}{p\,(p-1)^{\frac{1}{2}}} = -\sqrt{2}\,(b/\lambda_{Db})\,dx . \tag{e-4}$$

Integrating again from $x = 1$ to $x = x$ gives

$$y = \ell n \left[\sec^2\left(\frac{b - r}{\sqrt{2}\,\lambda_{Db}}\right)\right] . \tag{e-5}$$

APPENDIX F

Effective Work Function for a Spherical Particle

It is of interest to explore the effect which the curvature of the surface of a spherical particle may have on the value of the work function for thermionic emission. A good portion of the work function arises from the image force felt by the electron as it escapes the surface, and it is possible that the image force felt by an electron which is emitted from a sphere will not lead to the same contribution to the work function as that which exists in the case of a flat surface.

An approximation which is many times made is that the work function for a flat surface is the sum of two terms.[10] One term, $e\Phi_S$, is the integral of all the short-range forces acting on the electron as it leaves the surface. The range of these forces, X_o, is a length whose magnitude is equal to one or two interatomic distances of the solid ($\sim 10^{-8}$ cm). The second term, $e\Phi_L$, is the integral of the long-range forces felt by the electron in moving from X_o to infinity. The most important of the long-range forces has been identified as the "mirror-image" force through field emission experiments. This mirror-image force, which is induced by an electron as it leaves a semi-infinite conducting solid, is a consequence of the fact that a perfectly conducting medium must be at a uniform potential; hence, all electric field lines must be perpendicular to its surface. The potential distribution set up by the charge - image charge system in Figure 32 will satisfy this boundary condition of an equi-

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[10] Reference 27, pp. 199-211; reference 23, pp. 28-33.

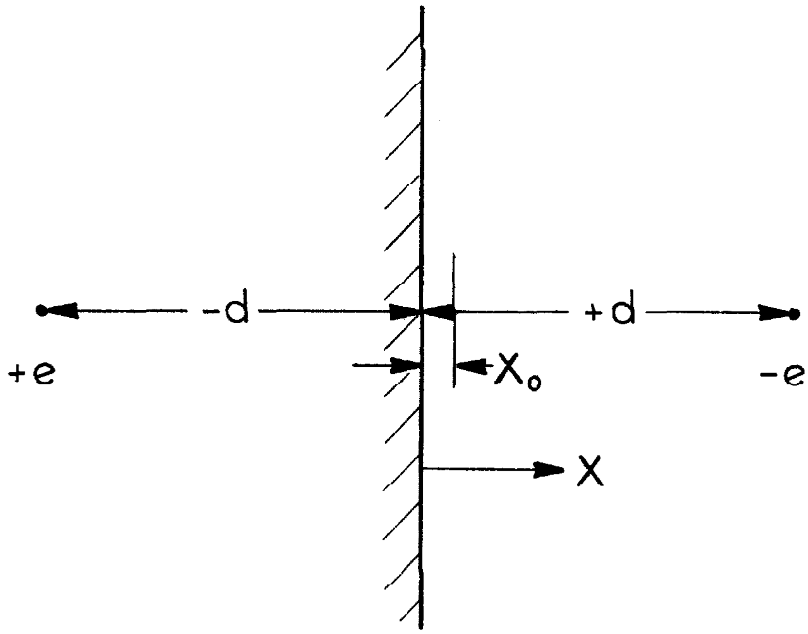

Figure 32. Charge - Image Charge System for a Flat Wall.

potential surface at X = 0 as well as the one of zero potential at infinity. Hence, this potential distribution will be the correct one for $X \geq 0$ and will lead to the actual force felt by the electron as it moves away from the surface. This force is easily calculated, since it is the force between the electron and its image charge. Thus, the value of the long-range contribution to the work function corresponding to this potential distribution is

$$e\Phi_L = e^2/(16\pi\varepsilon_o X_o) \quad . \tag{f-1}$$

These same concepts may be applied to the case of a spherical emitter for which a , the radius, is much larger than X_o (see Figure 33). When an image charge of magnitude q' is placed at a position r' , defined by equation (f-2), the spherical surface r = a will be

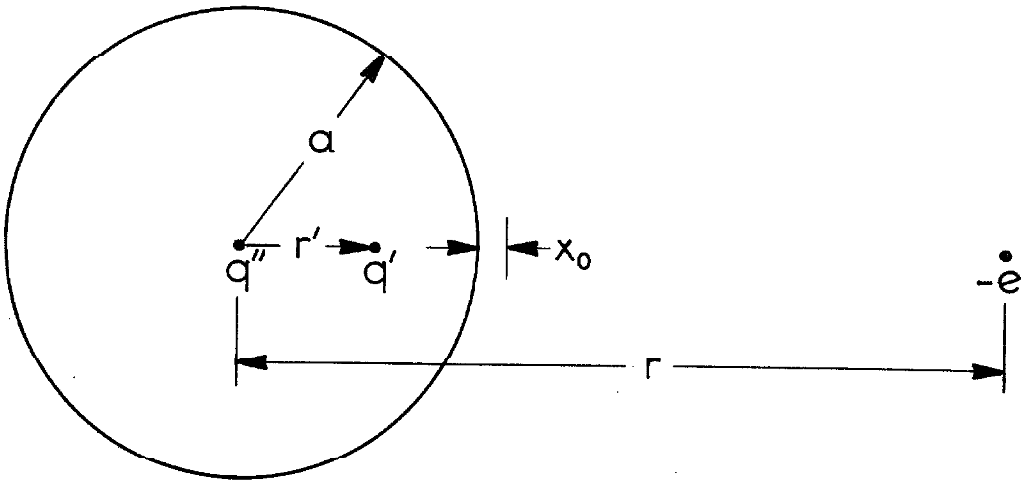

Figure 33. Charge - Image Charge System for a Sphere.

$$q' = \frac{a}{r} e \quad , \quad q'' = (\ell - \frac{a}{r}) e \quad , \quad r' = \frac{a^2}{r} \tag{f-2}$$

an equipotential one.[11] The resulting electric field must also obey Gauss' law for any surface enclosing the sphere. Thus, for the case in which the degree of ionization goes from $\ell-1$ to ℓ, a charge of magnitude $(\ell - \frac{a}{r})e$ must also be placed inside the sphere. It is placed at $r = 0$ so that the surface $r = a$ remains at an equipotential. The potential energy gain of an electron as it moves from inside the sphere to $r = \infty$, W_ℓ, may be calculated in the same manner as was done in the plane case.

$$W_\ell = \int_a^{a+X_o} \vec{F} \cdot d\vec{r} + \int_{a+X_o}^{\infty} \vec{F} \cdot d\vec{r} \tag{f-3}$$

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11 See reference 28, pp. 27-30.

$$W_\ell = e\Phi_S + \frac{e^2}{4\pi\varepsilon_o}\int_0^{a+X_o} \left(\ell - \frac{a}{r}\right)\left(\frac{1}{r^2}\right)dr + \frac{e^2}{4\pi\varepsilon_o}\int_{a+X_o}^{\infty} \left\{\left(\frac{a}{r}\right)\left(\frac{1}{r - \frac{a^2}{r}}\right)^2 + \left(\ell - \frac{a}{r}\right)\left(\frac{1}{r^2}\right)\right\} dr \; . \qquad \text{(f-4)}$$

It is noted that the integral of the forces experienced by the electron in moving from $r < a$ to $r = a + X_o$ includes not only the short-range contribution to the work function, as in the case of a flat surface, but also the work done against the initial positive charge on the sphere which appears to the electron to be of magnitude q'' located at $r = 0$. As was done previously, the image force is assumed not to exist up to the distance X_o outside the surface. Integration of equation (f-4) yields

$$W_\ell = e\Phi_S + \frac{e^2}{16\pi\varepsilon_o X_o}\left(\frac{1}{1 + \frac{X_o}{2a}}\right) + \left(\ell - \tfrac{1}{2}\right)\left(\frac{e^2}{4\pi\varepsilon_o a}\right) . \qquad \text{(f-5)}$$

Using equation (f-1) and neglecting terms of magnitude $e\phi_L(X_o/a)^2$ or smaller,

$$W_\ell = \left\{e\Phi_S + e\Phi_L - \frac{e^2}{32\pi\varepsilon_o a}\right\} + \left(\ell - \tfrac{1}{2}\right)\left(\frac{e^2}{4\pi\varepsilon_o a}\right) . \qquad \text{(f-6)}$$

The last term in equation (f-6) is that which one would compute for the change in the energy of the electromagnetic field surrounding the particle and arises directly from the work done against the force between q'' and the electron as the electron moves from $r = a$ to $r = \infty$. The first term is identified as the work function for a flat surface minus the small quantity

$$\frac{e^2}{32\pi\varepsilon_o a} = \frac{1.80}{a} \text{ ev} , \qquad \text{(f-7)}$$

where a is in angstroms. Hence, there is a slight reduction in the long-range contribution to the work function due to the curvature of the particle. It exists because the force between the electron and its image is slightly less than that for the plane case when the electron is the same distance from the surface. However, this reduction would be negligible for particle sizes of practical interest ($a > 100\text{Å}$). In addition, other effects due to the discontinuous nature of the particle material may become of comparable magnitude for very small particles.

APPENDIX G

Evaluation of the Constants in $\mu(\alpha)$

The assumed form of $\mu(\alpha)$ is

$$\mu(\alpha) = \frac{C_1}{\alpha} \exp\{-\frac{1}{\epsilon_\alpha}\left(\frac{C_2^2}{\alpha^2} + \alpha^2\right)\} \tag{g-1}$$

and the normalization conditions which determine C_1 and C_2 are

$$\int_0^\infty \alpha\mu(\alpha)\, d\alpha = 1 \quad , \tag{g-2}$$

$$\int_0^\infty \alpha^3\mu(\alpha)\, d\alpha = 1 \quad . \tag{g-3}$$

The integral resulting from the first normalization condition is a tabulated one:

$$I_o(s,t) = \int_0^\infty \exp\{-(\frac{s}{\alpha^2} + t\alpha^2)\}\, d\alpha \quad , \tag{g-4}$$

$$I_o(s,t) = \tfrac{1}{2}\left(\frac{\pi}{t}\right)^{\frac{1}{2}} e^{-2\sqrt{st}} \quad . \tag{g-5}$$

The second normalization condition gives rise to the integral

$$I_1(s,t) = \int_0^\infty \alpha^2 \exp\{-(\frac{s}{\alpha^2} + t\alpha^2)\}\, d\alpha \,, \tag{g-6}$$

which may be evaluated from the recursion relation

$$I_1(s,t) = -\,\partial I_o/\partial t \,. \tag{g-7}$$

Thus,

$$I_1(s,t) = \tfrac{1}{2}\left(\frac{\pi}{t}\right)^{\frac{1}{2}} e^{-2\sqrt{st}} \left[\frac{1}{2t} + (s/t)^{\frac{1}{2}}\right] \quad . \tag{g-8}$$

The relations which determine C_1 and C_2 are

$$\tfrac{1}{2}C_1(\pi\epsilon_\alpha)^{\frac{1}{2}}\, e^{-2C_2/\epsilon_\alpha} = 1\,, \tag{g-9}$$

$$\tfrac{1}{2}C_1(\pi\epsilon_\alpha)^{\frac{1}{2}}\, e^{-2C_2/\epsilon_\alpha}\,[\,(\epsilon_\alpha/2) + C_2\,] = 1\,. \tag{g-10}$$

Hence,

$$C_1 = \frac{2}{(\pi\epsilon_\alpha)^{\frac{1}{2}}}\, e^{\left(\frac{2}{\epsilon_\alpha} - 1\right)}\,, \tag{g-11}$$

$$C_2 = 1 - \frac{\epsilon_\alpha}{2}\,. \tag{g-12}$$

APPENDIX H

Evaluation of the Integrals in Equation (4.63)

The first integral to be evaluated is of the form

$$I_2(s,t) = \int_0^{\infty} \left(\frac{1}{1+\epsilon_T \alpha^2}\right) \exp\{-\left(\frac{s}{\alpha^2} + t\alpha^2\right)\} \, d\alpha \, , \qquad \text{(h-1)}$$

which obeys the following differential equation:

$$I_2 - \epsilon_T \frac{\partial I_2}{\partial t} = I_o \, , \qquad \text{(h-2)}$$

or

$$\frac{\partial I_2}{\partial t} - \frac{1}{\epsilon_T} I_2 = -\frac{1}{2\epsilon_T} \left(\frac{\pi}{t}\right)^{\frac{1}{2}} e^{-2\sqrt{st}} \, . \qquad \text{(h-3)}$$

This is a first-order linear differential equation with the solution

$$I_2(s,t) = C_3 \, e^{t/\epsilon_T} - e^{t/\epsilon_T} \int e^{-t/\epsilon_T} \left(\frac{1}{2\epsilon_T}\right) \left(\frac{\pi}{t}\right)^{\frac{1}{2}} e^{-2\sqrt{st}} \, . \qquad \text{(h-4)}$$

The integration indicated in this solution is accomplished by completing the square. When the following boundary condition is then applied,

$$I_2(s,t) \to 0 \qquad \text{as} \qquad t \to \infty \, ,$$

the result is

$$I_2(s,t) = \frac{\pi}{2 \, \epsilon_T} \, e^{\left(\frac{t}{\epsilon_T} + s\epsilon_T\right)} \left[1 - \operatorname{erf}\{\left(\frac{t}{\epsilon_T}\right)^{\frac{1}{2}} + (s\epsilon_T)^{\frac{1}{2}}\}\right] \, . \qquad \text{(h-5)}$$

As a check, it is seen that in the limit

$$I_2(s,t) \to I_o(s,t) \qquad \text{as} \qquad \epsilon_T \to 0 \, .$$

Using the above, the first integral in equation (4.63) becomes

$$w \int_0^\infty \frac{\alpha\mu(\alpha)}{(1+\epsilon_T\alpha^2)}\, d\alpha = w\left(\frac{\pi}{\epsilon_\alpha\epsilon_T}\right)^{\frac{1}{2}}\left[1 - \operatorname{erf}\{\left(\frac{1}{\epsilon_\alpha\epsilon_T}\right)^{\frac{1}{2}} + (1-\frac{\epsilon_\alpha}{2})(\frac{\epsilon_T}{\epsilon_\alpha})^{\frac{1}{2}}\}\right]$$

$$\times \left[\exp\{\frac{1}{\epsilon_\alpha}\left(\frac{1}{\epsilon_T} + [1 - \frac{\epsilon_\alpha}{2}]^2 \epsilon_T + 2\right) - 1\}\right] \quad . \qquad \text{(h-6)}$$

The second integral in equation (4. 63) is of the form

$$I_3 = \int_0^\infty \ln(1 + \epsilon_T\alpha^2)\,\alpha\mu(\alpha)\,d\alpha \quad . \qquad \text{(h-7)}$$

Compared to the first integral, the contribution of this second one is small because of the logarithmic variation of the term in the integrand. This is also evidenced by tracing the second integral back to the T^2 dependence in the thermionic emission law and the first integral back to the exponential dependence. Because of this and the peaked nature of $\alpha\mu(\alpha)$, little error in the calculation of ν will be introduced by expanding $\ln(1+\epsilon_T\alpha^2)$ about the value of α corresponding to the maximum value of $\alpha\mu(\alpha)$.

$$(\alpha\mu)_{max} \Rightarrow \alpha_M = \left(1 - \frac{\epsilon_\alpha}{2}\right)^{\frac{1}{2}} \qquad \text{(h-8)}$$

Rewriting $\ln(1+\epsilon_T\alpha^2)$ and expanding about α_M,

$$\ln(1+\epsilon_T\alpha^2) = \ln(1+\epsilon_T\alpha_M^2) + \ln\left(1 + \frac{\epsilon_T[\alpha^2-\alpha_M^2]}{1+\epsilon_T\alpha_M^2}\right) \quad , \qquad \text{(h-9)}$$

$$\ln(1+\epsilon_T\alpha^2) = \ln(1+\epsilon_T\alpha_M^2) + \frac{\epsilon_T(\alpha^2-\alpha_M^2)}{1+\epsilon_T\alpha_M^2} + \ldots \quad . \qquad \text{(h-10)}$$

Retaining only the first two terms in the expansion and using the normalization conditions to do the resulting integrals, the second integral in equation (4. 63) becomes

$$2\int \ell n\,(1+\varepsilon_\tau \alpha^2)\,\alpha u(\alpha)\,d\alpha \simeq 2\,\ell n\,(1+\varepsilon_\tau[1-\frac{\varepsilon_\alpha}{2}]\,) + \frac{\varepsilon_\tau \varepsilon_\alpha}{1+\varepsilon_\tau[1-\frac{\varepsilon_\alpha}{2}]} \quad . \tag{h-11}$$

APPENDIX I

The Requirement on the Shape of the Potential Distribution

The marginal contour which divides those electrons which escape from those which do not has been found to be

$$P_r^{(2)^2} = 2me\phi - \left(\frac{1}{r^2} - \frac{1}{c^2}\right) P_\Omega^{(2)^2} \qquad \text{(i-1)}$$

by solving for the condition which corresponds to $P_r(r=c) = 0$. However, if the force due to the electric field which is felt by an electron as it moves away from the particle decreases faster than the centrifugal force, a point may be reached where the electron no longer experiences a net force toward the particle, but rather, is accelerated toward $r = c$. The motion of a marginal electron in this case would be one in which the radial velocity comes to zero at some value of $r < c$, and then either returns to the particle or is accelerated to $r = c$. This effect would tend to decrease the total amount of emission.

The restriction which must be placed on the shape of the field in order for equation (5.28) to be valid, may be found from the requirement that $P_r^2 > 0$ for all possible electron trajectories. Thus, from the conservation relations, equations (5.25) and (5.26), it is necessary that

$$P_r^2 = P_{ra}^2 + \left(\frac{1}{a^2} - \frac{1}{r^2}\right) P_\Omega^2 - 2me(\phi_a - \phi) > 0 \qquad \text{(i-2)}$$

or

$$2me\phi > 2me\phi_a - P_{ra}^2 + \left(\frac{1}{a^2} - \frac{1}{r^2}\right) P_\Omega^2, \qquad \text{(i-3)}$$

where $P_{ra} = P_r(r = a)$. The trajectories that are in question are those of $P_r^{(2)}$. That is, it may be necessary for P_r^2 to become less than zero at $r < c$ in order that it can be equal to zero at $r = c$. Thus, the lower limit on P_{ra}^2 is $P_{ra}^{(2)^2}$:

$$P_{ra}^2 = 2me\phi_a - \left(\frac{1}{a^2} - \frac{1}{c^2}\right) P_{\Omega a}^2 . \tag{i-5}$$

The maximum value of $P_{\Omega a}^2$ corresponds to $P_{ra}^2 = 0$. Hence, the requirement which must be placed on the shape of the potential distribution, if equation (i-1) is to be used to define the marginal contour for escape, is [12]

$$\frac{\phi}{\phi_a} > \frac{(c/r)^2 - 1}{(c/a)^2 - 1} . \tag{i-6}$$

This requirement, relation (i-3), may be shown to be equivalent to the requirement that the electric field should decrease less rapidly than the centrifigal force in moving away from the particle. That is, upon differentiation of relation (i-6) with respect to r, it is seen that

$$-\frac{d\phi}{dr} > \frac{\text{const.}}{r^3} . \tag{i-7}$$

The left hand side of relation (i-7) is the absolute value of the electric field and, since the centrifugal force is equivalent to P_Ω^2/mr^3, the right hand side varies as the centrifugal force.

The inequality, relation (i-6), is satisfied in the weak shielding regime since the potential close to a particle varies approximately as $1/r$. It is clear, though, that in the strong shielding regime

[12] This is analogous to the requirement found by Langmuir [32], p. 120.

this requirement is, in general, not satisfied, since the potential falls off more rapidly than $1/r^2$. However, $P_r^{(2)^2}$, as defined by equation (i-1), is valid in the limit, $c/a \to 1$. That is, as $c/a \to 1$, $(1/r^2 - 1/c^2) \to 0$, and $P_r^{(2)^2} \to 2me\phi$, which is the correct marginal contour for the one-dimensional problem. Therefore, the correct one-dimensional relations should be obtained for the fluxes derived in Section V-C in the limit of $c/a \to 1$.