OPTIMIZATION ISSUES IN WAVELETS AND FILTER BANKS

Thesis by

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ABSTRACT

In the last decade or so, we have witnessed a rapid development of the wavelet and filter bank theory. Wavelets find applications in signal compression, computer vision, geophysics, pattern recognition, numerical analysis, and function theory, just to name a few. Filter banks, on the other hand, offer very efficient implementation of different algorithms in connection with wavelets. The thesis deals with three problems in filter banks and wavelets.

In the first part, we show that perfect reconstruction is equivalent to biorthogonality of the filters. Using this, we examine existence issues in nonuniform filter banks. We show that whenever there exists a rational biorthogonal filter bank, then there is a rational orthonormal filter bank as well. We also derive a number of necessary conditions for the existence of perfect reconstruction nonuniform filter banks. We show how the tools developed in the first part can be used for decorrelation of subband signals.

The second problem deals with optimality issues in wavelet and filter bank theory. We tune scaling function for the analysis of WSS random processes, so that the energy is concentrated in as few transform coefficients as possible. The corresponding problem in the filter bank theory is that of adapting filter responses to a given (discrete time) WSS random process so as to achieve a better energy compaction.

Finally, the last part is devoted to developing sampling theory for multiresolution subspaces. More precisely, we extend existing uniform sampling theory to periodically nonuniform sampling. This extension offers one very important advantage over the existing sampling theory. By allowing for periodically nonuniform sampling grid, it is possible to have compactly supported synthesis functions, which was not the case before. Several variations on the basic theme are considered. Also, an application of the developed techniques to efficient computation of inner products in multiresolution subspaces is presented.

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Introduction

In this section, a short historic overview of filter banks and wavelets will be given. It will be followed by an introduction to multiresolution analysis (MRA) and wavelets. After that, remaining chapters will be described and the main results of the thesis will be stated. At the end, we introduce conventions and notation that will be used in the thesis.

1.1 A brief history of filter banks and wavelets

Multirate filter banks are well known and widely used tools in signal processing community. They find applications in signal compression [Cro76 and Woo86], computer vision [Mal89a], adaptive filtering [Gil87, Gil92, and Sat93], spectrum estimation [Coo80 and Qui83], beamforming [Pri79 and Sch84], etc.

In the absence of quantizers, there are three types of distortion that may occur in filter banks. As in any LTI system, there may be amplitude and phase distortion. Additionally, due to the presence of sub-samplers, there is aliasing as well. At first it was not clear whether aliasing could be eliminated with rational filters. In 1976, Croisier et al. [Cro76] showed that this is indeed possible for a two-channel case. After the aliasing has been eliminated, phase and amplitude distortions were minimized in the design process [Joh80], [Jai84], [Fet85]. A few years later, in 1984, Smith and Barnwell [Smi84] and Mintzer [Min85] found a way to eliminate all three distortions, again for a two-channel filter bank. At the same time very efficient implementation techniques were developed [Gal85].

All the results mentioned so far were for two-channel filter banks only. In a general, M-channel case analysis and synthesis are much more difficult. At first, so called pseudo QMF filter banks were developed. They are called pseudo QMF because only dominant aliasing terms were cancelled [Nus81], [Rot83]. Finally, solutions of the perfect reconstruction problem in the general M-channel case were found by many researchers [Ram84], [Smi85], [Vet85], [Pri86], [Vai87], etc. One solution to the problem stands out, the orthonormal one (also called paraunitary or PU). Not only are the design and analysis simplified in the case of orthonormal filter banks, but many desirable features (useful in subband coding, for example) are achieved for free. The design process is especially simplified due to the complete parameterization of orthonormal filter banks [Vai89]. This technique has roots in the classical network theory [Bel68], where the name paraunitary comes from.

Even though a complete parameterization of M-channel PU filter banks was known, the design was still computationally very involved. There were simply too many parameters to optimize. The design process was drastically simplified by introducing modulated filter banks [Mal90], [Koi91] and [Ram91]. In this case, all the filters in the filter bank are modulated versions of a single filter.

While it is true that the transfer function of a perfect reconstruction (PR) filter bank is unity, the filters themselves do not necessarily have linear phase. In some applications, like image coding, it is important that the filters have linear phase. At first some ad hoc design methods were developed. The first systematic design method covering a large class of linear phase PU filter banks was reported in [Som93]. Unfortunately, this design method was computationally very involved. Finally a design technique for linear phase cosine-modulated PU filter banks was reported in [Lin93].

All the above mentioned advances were made in the area of one-dimensional signal processing. Corresponding problems in the multidimensional (MD) case are much more difficult and, therefore, the results are scarcer. MD filter banks were first considered in [Vet84], [Woo86] and [Wac86]. Even though there have been many results reported so far, there are still no systematic design techniques for designing MD filter banks for arbitrary sampling lattices (see [Vis91]). The problem of perfect reconstruction was considered in [Ans90], [Kar90] and [Kov92]. A very nice and simple design technique for a two-channel case was given in [Pho93].

In the last couple of years, a number of both theoretical and more practical results have been reported. There are many design techniques incorporating additional constraints like minimum overall delay, various time domain constraints, imposition of zeros at aliasing frequencies [Ste93], etc. Also, there are further generalizations of M-channel uniform filter banks to nonuniform, filter banks with non-integer decimation ratios, time-varying filter banks, etc. But even just mentioning of all those results would take us too far afield.

One of the main applications of filter banks is the signal compression using subband coding. Analysis of brick-wall subband coders when the input is a wide sense stationary (WSS) random process is straightforward [Jay84]. It is especially easy in the case of PU filter banks and several interesting results can be derived; see [Vai93]. Results on (un)constrained bit allocation for the optimal quantization appeared in [Hua63], [Seg76], [Ram82], [Wes88], etc. However, the analysis of biorthogonal filter banks has not been done.

During the last decade, researchers from rather diverse fields got interested in wavelets. It is not so common to see quantum physicists, pure mathematicians, geophysicists, engineers, and numerical analysts work on similar problems at the same time. So, what is it that draws so much attention to wavelets? One of the main reasons is their very good joint time-frequency resolution. Wavelets have more desirable properties that other transforms lack (see the next subsection).

Since the invention of the Fourier transform, there have been many attempts to come up with some transform that will have both good time and frequency resolution. Uncertainty principle shows that simultaneous, arbitrarily good resolution in both time and frequency cannot be obtained. But most time-frequency transforms either cannot even get close to the principle of uncertainty (time or frequency resolution is poor) or lack some desirable properties as a transform (orthonormality or stability properties). Finally, there is a transform with all those desirable features – the wavelet transform.

Even though researchers were looking for a transformation with good time-frequency resolution for more than thirty years, the wavelet theory as we know it today has been mostly developed within the last decade or so. The notion of wavelets was introduced by Morlet [Mor82], in the area of geophysics. First smooth orthonormal wavelet bases were constructed by Meyer [Mey85] and Lemarie and Meyer [Lem86]. Wavelet transform can be nicely embedded in the theory of multiresolution analysis, developed by Mallat [Mal89] and Meyer [Mey87]. Multiresolution analysis kind of preprocessing is at the heart of human audio and visual system. It also plays an important role in approximation theory and numerical analysis, computer vision, signal compression, etc. However, it was not until Daubechies' seminal work [Dau88] that wavelets gained popularity they enjoy today.

Daubechies exploited a deep relationship between PU filter banks and wavelet bases to construct an infinite family of compactly supported orthonormal wavelets of arbitrarily high degree of smoothness. Her work initiated a huge activity in both signal processing and mathematical community. It was then realized that some kind of multiresolution analysis had been practiced long before Mallat's and Meyer's formalization (Laplacian pyramid of Burt and Adelson [Bur83], for example). Filter banks are not only used to construct compactly supported orthonormal wavelet bases, but also for computationally extremely efficient implementation of MRA, the so called Fast Wavelet Transform (FWT) [Mal89a].

Wavelets were used for numerical solutions of differential equations in [Bey91]. For the purpose of audio compression, Coifman et al. [Coi90] developed wavelet packets. We mentioned only a few advantages of wavelet bases over other types of bases, but there are many more. They play an important role in harmonic analysis and function theory because they provide unconditional bases for many functional spaces (L^p for $1 < p < \infty$, Hardy spaces, Sobolev spaces, etc.). It is also possible to characterize many of those spaces in terms of their wavelet transform. Most of these nice and desirable properties were lacked by the Fourier transform. Much more on this subject can be found in Meyer's book [Mey92].

As it was mentioned, Daubechies' work initiated mutually beneficial interaction between mathematicians and signal processors. On the one hand, existence of M-channel, biorthogonal, MD, linear phase, etc., filter banks gave rise to construction of M-band, biorthogonal, MD, symmetric, etc., wavelet bases. On the other hand, desire to construct smoother wavelet bases imposed new design constraints on filter banks. It is not enough anymore to satisfy just PR conditions; it is also desirable to put as many zeros at aliasing frequencies as possible. Some preliminary experiments have shown that the imposition of the new constraints improves performances of some image coding systems. Also the role of filter banks when it comes to implementations of MRA and other wavelet algorithms cannot be overemphasized. There is a huge number of references in this area and it would be impossible to mention all of them. [Chu92], [Dau92], [Vai93], [Inf92], and [Sgp93] are good starting points for further exploration and an excellent source of references.

1.2 Wavelets and multiresolution analysis

1.2.1. Wavelet examples

Instead of giving formal definitions, we will present two examples of wavelet transform to develop the intuition. The examples are simple and well-known, yet they show the main features of the wavelet transform.

Example 1.2.1. Littlewood-Paley wavelet. Consider a bandpass signal $x(t)$ with Fourier transform supported in $\omega_1 < |\omega| < \omega_2$. It can be shown that for sampling at the rate 2β there is no overlap of images if and only if one of the edges, ω_1 or ω_2, is a multiple of 2β. This is called the *bandpass sampling theorem.*

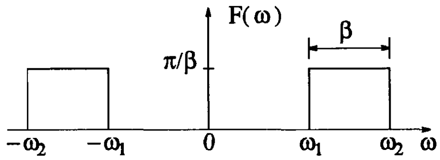


Fig. 1.2.1. Bandpass filter to be used in the reconstruction of the bandpass signal from its samples.

The reconstruction of $x(t)$ from the samples proceeds exactly as in the lowpass case, except that the reconstruction filter $F(\omega)$ is now a bandpass filter (Fig. 1.2.1) occupying precisely the signal bandwidth. The reconstruction formula is $x(t) = \sum_n x(nT)f(t - nT)$ where $T = \pi/\beta$, and $f(t)$ is the bandpass impulse response.

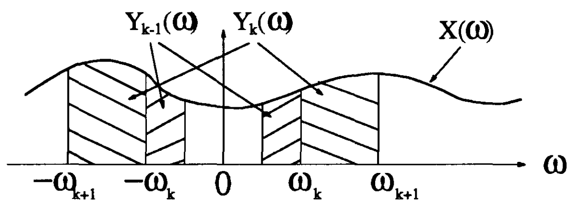


Fig. 1.2.2. Splitting a signal into frequency subbands.

Given a signal $x(t)$, imagine now that we have split its frequency axis into subbands in some manner (Fig. 1.2.2). Letting $y_k(t)$ denote the kth subband signal, we can write $x(t) = \sum_k y_k(t)$. If the subband region $\omega_k \leq |\omega| < \omega_{k+1}$ satisfies the bandpass sampling condition, then the bandpass signal $y_k(t)$ can be expressed as a linear combination of its samples. So, let $\omega_k = 2^k\pi$ ($k = \ldots -1, 0, 1, 2, \ldots$). The bandedges are such that $y_k(t)$ is a signal satisfying the bandpass sampling theorem. It can be sampled at period $T_k = \pi/\beta_k = 2^{-k}$ without aliasing, and we can reconstruct it from samples as

$$y_k(t) = \sum_{n=-\infty}^{\infty} y_k(2^{-k}n)f_k(t - 2^{-k}n). \tag{1.2.1}$$

Since $x(t) = \sum_k y_k(t)$ we see that $x(t)$ can be expressed as

$$x(t) = \sum_{k=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} y_k(2^{-k}n)f_k(t - 2^{-k}n). \tag{1.2.2}$$

Our definition of the filters shows that the frequency responses are scaled versions of each other; that is $F_k(\omega) = 2^{-k}\Psi(2^{-k}\omega)$. The impulse responses are therefore related as $f_k(t) = \psi(2^k t)$, and we can rewrite (1.2.2) as

$$x(t) = \sum_{k=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} y_k(2^{-k}n)\psi(2^k t - n). \tag{1.2.3}$$

In order to make this look like a wavelet expansion, we write it as $x(t) = \sum_k \sum_n c_{kn}\psi_{kn}(t)$, where $c_{kn} = 2^{-k/2}y_k(2^{-k}n)$ and

$$\psi_{kn}(t) = 2^{k/2}\psi(2^k t - n) = 2^{k/2}\psi\Big(2^k(t - 2^{-k}n)\Big). \tag{1.2.4}$$

The function $\psi(2^k t)$ is a dilated version of $\psi(t)$ (squeezed version if $k > 0$ and stretched version if $k < 0$). The dilation factor 2^k is a power of two, so this is said to be a dyadic dilation. The function $\psi(2^k(t-2^{-k}n))$ is a shift of the dilated version. Thus we have expressed $x(t)$ as a linear combination of shifts of (dyadic) dilated versions of a single function $\psi(t)$. This is a typical characteristic of wavelet bases and could be taken as a possible definition (strictly speaking there is no definition of wavelets). (1.2.3) is called the wavelet representation for $x(t)$. The function $\psi(t)$ is called the *ideal bandpass wavelet.* It has also been known as the Littlewood-Paley wavelet.

Example 1.2.2. The Haar wavelet basis. As early as 1910 an orthonormal basis for L^2 functions has been found [Haa10], which satisfies the properties of a wavelet basis given above! That is, the basis functions $\psi_{kn}(t)$ are derived from a single function $\psi(t)$ using dilations and shifts as in (1.2.4). To explain this system first consider a signal $x(t) \in L^2[0,1]$. The Haar basis is built from two functions called $\phi(t)$ and $\psi(t)$, as described in Fig. 1.2.3. The basis function $\phi(t)$ is a constant in $[0,1]$. The basis function $\psi(t)$ is constant on each half interval, and its integral is zero. After this, the remaining basis functions are obtained from $\psi(t)$ by dilations and shifts as indicated. It is clear from the figure that any two of these functions are mutually orthogonal. We have an orthonormal set, and it can be shown that this set of functions is an orthonormal basis for $L^2[0,1]$. However, this is not exactly a wavelet basis yet, because of the presence of $\phi(t)$.†

† We will see later that the function $\phi(t)$ arises naturally in the context of the fundamental idea of multiresolution.

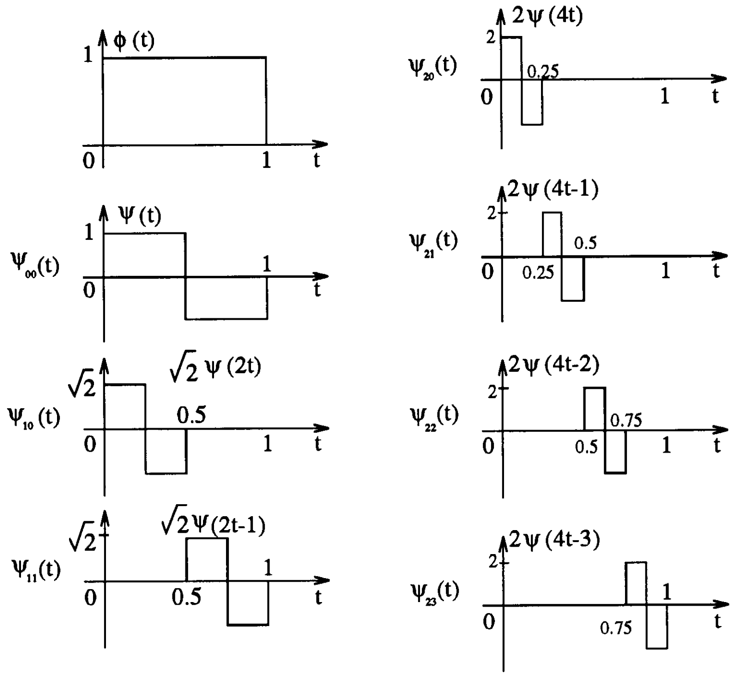


Fig. 1.2.3. Examples of basis functions in the Haar basis for $L^2[0,1]$.

If we eliminate the requirement that $x(t)$ be supported or defined only on $[0,1]$ and consider $L^2(R)$ functions, then we can still obtain an orthonormal basis of the above form by including the shifted versions $\{\psi(2^k t - n)\}$ for *all* integer values of n, and also including the shifted versions $\{\phi(t-n)\}$. An alternative to the use of $\{\phi(t-n)\}$ would be to use stretched (i.e., $\psi(2^k t), k < 0$) as well as squeezed (i.e., $\psi(2^k t), k > 0$) versions of $\psi(t)$. The set of functions can thus be written as in (1.2.4), which has the form of a wavelet basis. It can be shown that this forms an orthonormal basis for $L^2(R)$.

The above two examples are two extreme cases of infinitely many examples of wavelet bases. The first example has a good frequency localization, basis functions are infinitely smooth, but of infinite duration, while the second example has a good time localization, basis functions are compactly supported, but discontinuous. Obtaining basis functions having a good localization in both time and frequency is the aim of the game. In the subsections that follow, we will explain properties and advantages of the wavelet transform over other time-frequency transforms. Now that we know what wavelets are all about, let us review another time-frequency transform, the STFT.

1.2.2. The short time Fourier transform (STFT)

In many applications, we have to accommodate the notion of frequency that evolves or changes with time. For example, audio signals are often regarded as signals with a time varying spectrum, e.g., a sequence of short lived pitch frequencies. This idea cannot be expressed with the traditional FT since $X(\omega)$ for each ω depends on $x(t)$ for all t.

The short time Fourier transform (STFT) was introduced to provide such a time-frequency picture of the signal [Gab46], [Fla66], [Sch73], and [Por80]. Here the signal $x(t)$ is multiplied with a window $v(t-\tau)$ centered or localized around time τ and the FT of $x(t)v(t-\tau)$ computed:

$$X(\omega,\tau)=\int_{-\infty}^{\infty}x(t)v(t-\tau)e^{-j\omega t}dt. \tag{1.2.5}$$

This is then repeated for shifted locations of the window, i.e., for various values of τ. The result is a function of both time τ and frequency ω. In the traditional STFT both ω and τ are discretized on uniform grids $\omega = k\omega_s, \quad \tau = nT_s$. The STFT is thus defined as

$$X_{stft}(k\omega_s, nT_s)=\int_{-\infty}^{\infty}x(t)v(t-nT_s)e^{-jk\omega_s t}dt, \tag{1.2.6}$$

which we abbreviate as $X_{stft}(k,n)$. Thus the time domain is mapped into the time-frequency domain. The quantity $X_{stft}(k\omega_s, nT_s)$ represents the FT of $x(t)$ "around time nT_s" and "around frequency $k\omega_s$."

Optimal time-frequency resolution: the Gabor window. What is the best frequency resolution one can obtain for a given time resolution? That is, for a given duration of the window $v(t)$ how small can the duration of $V(\omega)$ be? If we define duration according to common sense, we are already in trouble because if $v(t)$ has finite duration, then $V(\omega)$ has infinite duration. There is a more useful definition of duration called the *root mean square* (rms) duration. The rms time duration D_t and the rms frequency duration D_f for the window $v(t)$ are defined such that

$$D_t^2=\frac{\int t^2|v(t)|^2dt}{\int |v(t)|^2dt}, \qquad D_f^2=\frac{\int \omega^2|V(\omega)|^2d\omega}{\int |V(\omega)|^2d\omega}. \tag{1.2.7}$$

Intuitively we can see that D_t cannot be arbitrarily small for a specified D_f. The uncertainty principle says that $D_tD_f \geq 0.5$. Equality holds if and only if $v(t)$ has the shape of a Gaussian, i.e., $v(t) = Ae^{-\alpha t^2}, \alpha > 0$. Thus the best joint time-frequency resolution is obtained by using the Gaussian window. Gabor used the Gaussian window as early as 1946! The STFT based on the Gaussian is called the Gabor transform. A limitation of the Gabor transform is that it does not give rise to an orthonormal signal representation.

1.2.3. Wavelet transform versus STFT

We will compare wavelets and STFT on several grounds: time-frequency resolution and localization, stability of the reconstruction from the transform coefficients, existence of orthonormal bases and so forth. The advantage of wavelet transforms over the STFT will be clear after these discussions.

The STFT works with a fixed window $v(t)$. If a high frequency signal is being analyzed, many cycles are captured by the window, and a good estimate of the FT is obtained. But if a signal varies very slowly with respect to the window, then the window is not long enough to capture it fully. The STFT therefore does not provide uniform percentage accuracy for all frequencies — the computational resources are somehow poorly distributed.

In the wavelet case, the frequency resolution gets poorer as the frequency increases, but the fractional resolution (i.e., the filter bandwidth $\Delta\omega_k$ divided by the center frequency ω_k) is constant for all k. That is, the percentage accuracy is uniformly distributed in frequency. In electrical engineering language the filter bank representing wavelet transforms is a *constant Q filter bank,* or an *octave band filter bank.* The nonuniform (constant Q) filter stacking provided by wavelet filters is also naturally suited for analyzing audio signals and sometimes even as components in the modeling of the human hearing system.

Example 1.2.3. Resolution of the wavelet transform and the STFT. This example clearly displays advantages of the wavelet transform over the STFT. Consider the signal $x(t) = \cos(10\pi t) + 0.5\cos(5\pi t) + 1.2\delta_a(t - 0.07) + 1.2\delta_a(t + 0.07)$. It has impulses at $t = \pm 0.07$, in the time domain. There are two impulses (or "lines") in the frequency domain, at $\omega_1 = 5\pi$ and $\omega_2 = 10\pi$. The function is shown in Fig. 1.2.4 (with impulses replaced by narrow pulses). The aim is to try to compute the STFT or WT such that the impulses in time as well as those in frequency are resolved. Figure 1.2.5 shows the STFT plot for three widths of the window $v(t)$ and the wavelet plot. The STFT plots are time-frequency plots, whereas the wavelet plots are (a^{-1}, b) plots where a^{-1} is analogous to "frequency" in the STFT, and b is analogous to "time" in the STFT. The brightness of the plots in Fig. 1.2.5 is proportional to the magnitude of the STFT or WT, so the transform is close to zero in the dark regions. We see that for a narrow window with width $= 0.1$, the STFT resolves the two impulses in time reasonably well, but the impulses in frequency are not resolved. For a wide window with width $= 1.0$, the STFT resolves the "lines" in frequency very well, but not the time domain impulses. For an intermediate window width $= 0.3$, the resolution is poor in both time and frequency. The wavelet transform plot, on the other hand, simultaneously resolves both time and frequency very well. We can clearly see the locations of the two impulses in time, as well as the two lines in frequency.

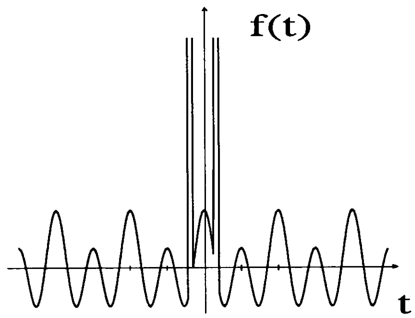


Fig. 1.2.4. Example 1.2.3. The signal to be analyzed by STFT and Wavelet transform.

Orthonormal STFT bases have poor time-frequency localization. It can be shown that if we wish to have an orthonormal STFT basis, the time-frequency density is constrained to be such that $\omega_s T_s = 2\pi$. Under this condition suppose we choose $v(t)$ appropriately to design such a basis. The time frequency localization properties of this system can be judged by computing the mean square durations D_t^2 and D_f^2 defined in (1.2.7). It has been shown by Balian and Low (see [Dau92]) that one of these is necessarily infinite no matter how we design $v(t)$. *Thus an orthonormal STFT basis always satisfies* $D_t D_f = \infty$. That is, either the time or the frequency localization is very poor.
Instability of the Gabor transform. Gabor constructed the STFT using the Gaussian window $v(t) = ce^{-t^2/2}$. In this case the sequence of functions $\{g_{kn}(t)\}$ can be shown to be complete in L^2 as long as $\omega_s T_s \leq 2\pi$. However, the reconstruction of $x(t)$ from $X_{stft}(k\omega_s, nT_s)$ is unstable if $\omega_s T_s = 2\pi$. So even though the Gabor transform has the ideal time frequency localization (minimum $D_t D_f$), it cannot provide a ***stable basis,*** hence certainly not an orthonormal basis, whenever $\omega_s T_s = 2\pi$.

A major advantage of the wavelet transform over the STFT is that it is free from the above difficulties. For example we can obtain an orthonormal basis for L^2 with excellent time-frequency localization (finite, controllable $D_t D_f$).

1.2.4. Wavelets and multiresolution

Daubechies' construction is such that excellent time-frequency localization is possible. Moreover, the smoothness or regularity of the wavelets can be controlled. The construction is based on the two channel paraunitary filter banks. One such filter bank is shown in Fig. 1.2.6.

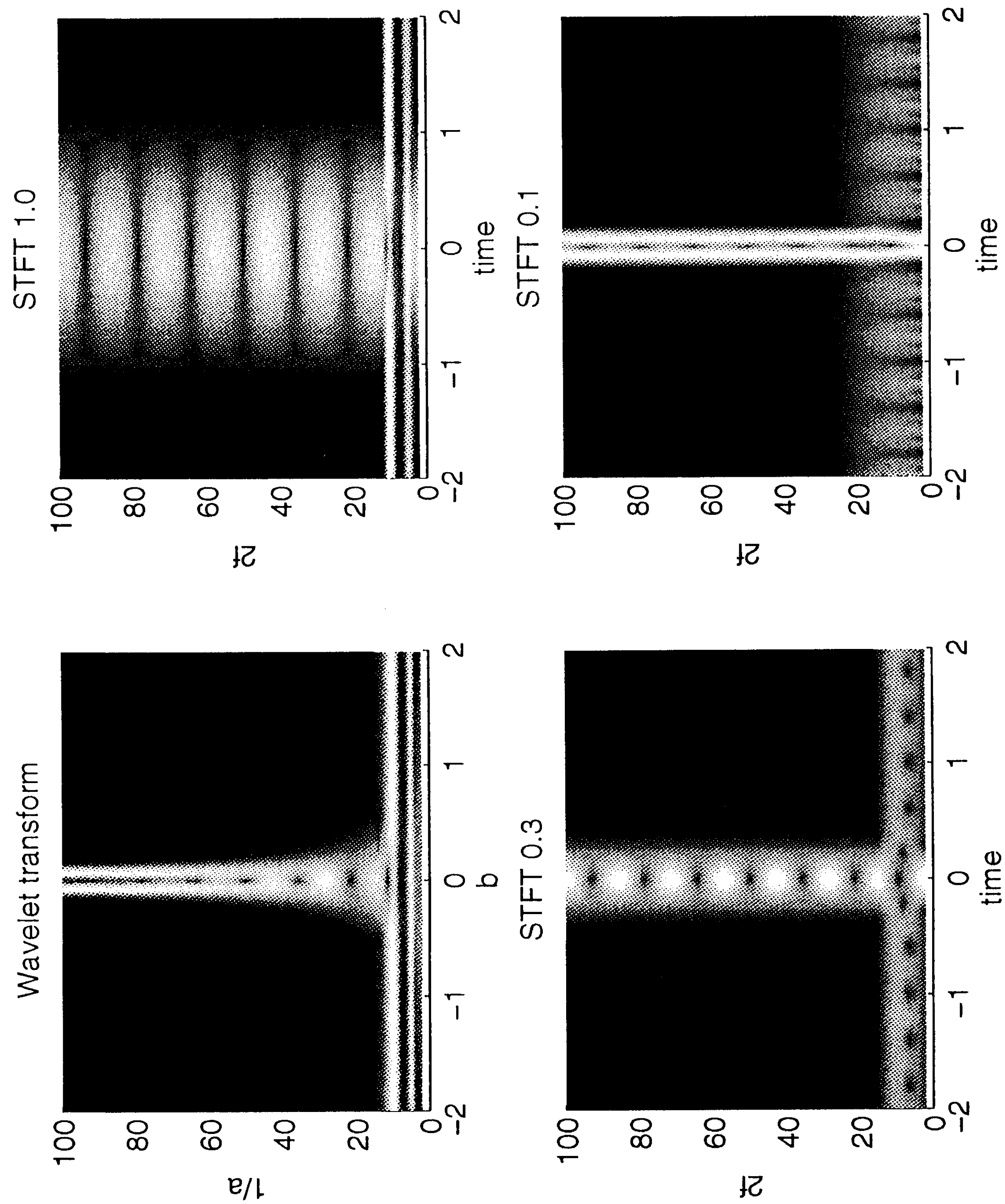


Fig. 1.2.5. Example 1.2.3. STFT plots with window widths of 0.1, 0.3, and 1.0, and Wavelet transform plot.

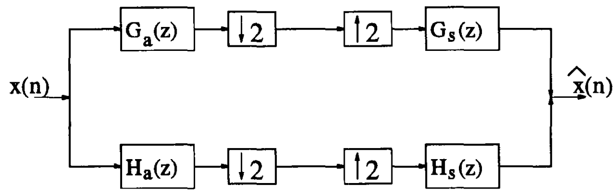


Fig. 1.2.6. A Two-channel PU filter bank.

Let $G_s(z)$ and $H_s(z)$ have impulse responses $g_s(n)$ and $h_s(n)$ respectively. All constructions are based on obtaining the wavelet $\psi(t)$ and an auxiliary function $\phi(t)$ called the scaling function, from the impulse response sequences $g_s(n)$ and $h_s(n)$, using time domain recursions of the form

$$\phi(t) = 2\sum_{n=-\infty}^{\infty} g_s(n)\phi(2t-n), \quad \psi(t) = 2\sum_{n=-\infty}^{\infty} h_s(n)\phi(2t-n), \tag{1.2.8}$$

called *dilation equations.* In the frequency domain we have

$$\Phi(\omega) = G_s(e^{j\omega/2})\Phi(\omega/2), \quad \Psi(\omega) = H_s(e^{j\omega/2})\Phi(\omega/2). \tag{1.2.9}$$

It turns out that if $\{G_s(z), H_s(z)\}$ is a paraunitary pair with further mild conditions (e.g., that the lowpass filter $G_s(e^{j\omega})$ has a zero at π and no zeros in $[0,\pi/3]$), the recursions can be solved to obtain $\psi(t)$ which gives rise to an orthonormal wavelet basis $\{2^{k/2}\psi(2^k t - n)\}$ for L^2. By constraining $G_s(e^{j\omega})$ to have a sufficient number of zeros at π, we can further control the Hölder index (or regularity) of $\psi(t)$. The recursions (1.2.8) are also called the *two-scale equations.* These have origin in the beautiful theory of multiresolution for L^2 spaces [Mey85], [Mal89].

The Idea of Multiresolution

Before giving a more formal definition, let us first review Ex. 1.2.1. Assume for simplicity the wavelets are ideal bandpass functions as in Ex. 1.2.1. The bandpass filters $F_k(\omega) = 2^{-k/2}\Psi(\omega/2^k)$ get narrower and narrower as k decreases (i.e., as k becomes more and more negative). Instead of letting k be negative, suppose we keep only $k \geq 0$ and include a lowpass filter $\Phi(\omega)$ to cover the low frequency region. Imagine for a moment that $\Phi(\omega)$ is an ideal lowpass filter with cutoff $\pm\pi$ (see Fig. 1.2.7). Then we can represent any L^2 function $F(\omega)$ with support restricted to $\pm\pi$ in the form $F(\omega) = \sum_{n=-\infty}^{\infty} a_n \Phi(\omega)e^{-j\omega n}$. This is simply the Fourier series expansion of $F(\omega)$ in $[-\pi,\pi]$. In the time domain this means

$$f(t) = \sum_{n=-\infty}^{\infty} a_n \phi(t-n). \tag{1.2.10}$$

Let us denote by V_0 the closure of the span of $\{\phi(t-n)\}$. Thus, V_0 is the class of L^2 signals that are bandlimited to $[-\pi, \pi]$. We know that $\phi(t)$ is the sinc function, and the shifted functions $\{\phi(t-n)\}$ form an orthonormal basis for V_0.

Consider now the subspace $W_0 \subset L^2$ of bandpass functions bandlimited to $\pi < |\omega| \leq 2\pi$. The bandpass sampling theorem allows us to reconstruct such a bandpass signal $g(t)$ from its samples $g(n)$ by using the ideal filter $\Psi(\omega)$. Denoting the impulse response of $\Psi(\omega)$ by $\psi(t)$, we see that $\{\psi(t-n)\}$ spans W_0. It can be verified that $\{\psi(t-n)\}$ is an orthonormal basis for W_0. Moreover, since $\Psi(\omega)$ and $\Phi(\omega)$ do not overlap, it follows from Parseval's theorem that W_0 is orthogonal to V_0.

Next consider the space of all signals of the form $f(t)+g(t)$ where $f(t) \in V_0$ and $g(t) \in W_0$. This space is called the direct sum (or orthogonal sum) of V_0 and W_0, and is denoted as $V_1 = V_0 \oplus W_0$. It is the space of all L^2 signals bandlimited to $[-2\pi, 2\pi]$. We can continue in this manner and define the spaces V_k and W_k for all k. Then V_k is the space of all L^2 signals bandlimited to $[-2^k\pi, 2^k\pi]$. And W_k is the space of L^2 functions bandlimited to $2^k\pi < |\omega| \leq 2^{k+1}\pi$. The general recursive relation is $V_{k+1} = V_k \oplus W_k$. Fig. 1.2.7 demonstrates this for the case where the filters are ideal bandpass. Only the positive half of the frequency axis is shown for simplicity.

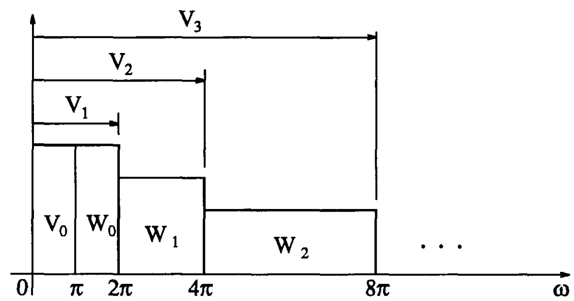


Fig. 1.2.7. Towards multiresolution analysis... The spaces $\{V_k\}$ and $\{W_k\}$ spanned by various filter responses.

It is clear that we could imagine V_0 itself to be composed of subspaces V_{-1} and W_{-1}. Thus $V_0 = V_{-1} \oplus W_{-1}$, $V_{-1} = V_{-2} \oplus W_{-2}$, and so forth. In this way we have defined a sequence of spaces $\{V_k\}$ and $\{W_k\}$ for all integers k such that the following conditions are true:

$$V_{k+1} = V_k \oplus W_k, \qquad \text{and} \qquad W_k \perp W_m, \; k \neq m, \tag{1.2.11}$$

where $\perp$ means "orthogonal." That is, the functions in W_k are orthogonal to those in W_m. It is clear that $V_k \subset V_{k+1}$.

Now, the interesting fact is that *even if the ideal filters* $\Phi(\omega)$ *and* $\Psi(\omega)$ *are replaced with non* *ideal approximations*, we can sometimes define sequences of subspaces V_k and W_k satisfying the above conditions. The importance of this observation is this: *whenever $\Psi(\omega)$ and $\Phi(\omega)$ are such that we can construct such a subspace structure, the impulse response $\psi(t)$ of the filter $\Psi(\omega)$ can be used to generate an orthonormal wavelet basis!* While this might seem too complicated and roundabout, the construction of the function $\phi(t)$ is quite simple and elegant and simplifies the construction of orthonormal wavelet bases.

Definition 1.2.1. Multiresolution analysis. Consider a sequence of closed subspaces $\{V_k\}$ in L^2, satisfying the following six properties.

1. *Ladder property.* $\ldots V_{-2} \subset V_{-1} \subset V_0 \subset V_1 \subset V_2 \ldots$
2. $\bigcap_{k=-\infty}^{\infty} V_k = \{0\}$
3. Closure of $\bigcup_{k=-\infty}^{\infty} V_k$ is equal to L^2.
4. *Scaling property.* $x(t) \in V_k$ if and only if $x(2t) \in V_{k+1}$. Since this implies "$x(t) \in V_0$ if and only if $x(2^k t) \in V_k$", all the spaces V_k are scaled versions of the space V_0. For $k > 0$, V_k is a *finer space* than V_0.
5. *Translation invariance.* If $x(t) \in V_0$ then $x(t-n) \in V_0$; that is, the space V_0 is invariant to translations by integers. By the previous property this means that V_k is invariant to translations by $2^{-k}n$.
6. *Special orthonormal basis.* There exists a function $\phi(t) \in V_0$ such that the integer shifted versions $\{\phi(t-n)\}$ form an orthonormal basis for V_0. By property 4 this means that $\{2^{k/2}\phi(2^k t-n)\}$ is an orthonormal basis for V_k. The function $\phi(t)$ is called the *scaling function* of multiresolution analysis.

Example 1.2.4. The Haar Multiresolution. A simple example of multiresolution where $\Phi(\omega)$ is not ideal lowpass is the Haar multiresolution, generated by the function $\phi(t)$ in Fig. 1.2.8(a) (see Ex. 1.2.2). Here V_0 is the space of all functions that are piecewise constants on intervals of the form $[n, n+1]$. We will see later that the function $\psi(t)$ associated with this example is as in Fig. 1.2.8(b); the space W_0 is spanned by $\{\psi(t-n)\}$. The space V_k contains functions which are constants in $[2^{-k}n, 2^{-k}(n+1)]$. Fig. 1.2.8(c) and (d) show examples of functions belonging to V_0 and V_1. For this example, the six properties in the definition of multiresolution are particularly clear (except perhaps property 3, which can be proved too).

(a)

(b)

(c)

(d)

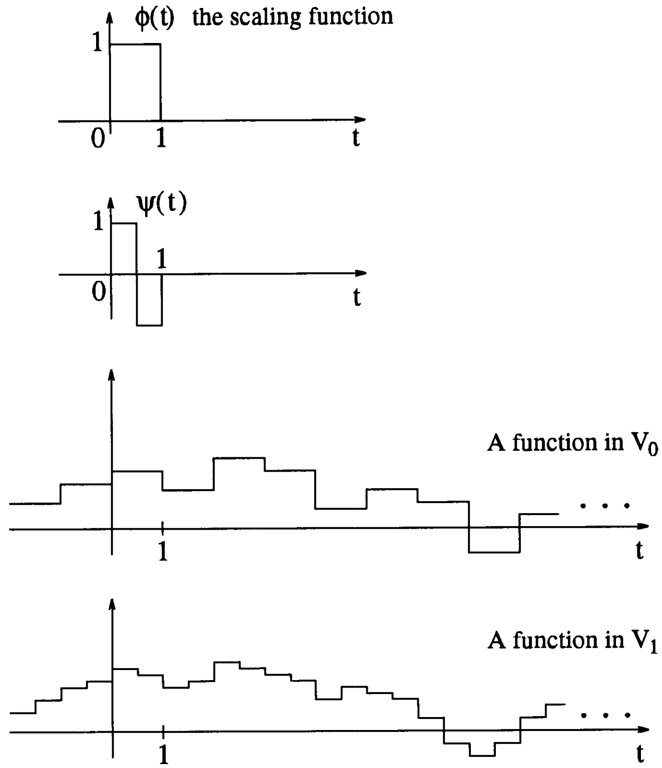


Fig. 1.2.8. The Haar multiresolution example. (a) The scaling function $\phi(t)$ that generates multiresolution, (b) the function $\psi(t)$ which generates W_0, (c) example of a member of V_0 and (d) example of a member of V_1.

Generating Wavelet and Multiresolution Coefficients From Paraunitary Filter Banks

Recall that the subspaces V_0 and W_0 have the orthonormal bases $\{\phi(t-n)\}$ and $\{\psi(t-n)\}$ respectively. By the scaling property, the subspace V_k has the orthonormal basis $\{\phi_{kn}(t)\}$, and similarly the subspace W_k has the orthonormal basis $\{\psi_{kn}(t)\}$, where, as usual, $\phi_{kn}(t) = 2^{k/2}\phi(2^k t - n)$ and $\psi_{kn}(t) = 2^{k/2}\psi(2^k t - n)$. The orthogonal projections of a signal $x(t) \in L^2$ onto V_k and W_k are given, respectively, by

$$P_k[x(t)] = \sum_{n=-\infty}^{\infty} \Big\langle x(t), \phi_{kn}(t) \Big\rangle \phi_{kn}(t), \qquad \text{and} \qquad Q_k[x(t)] = \sum_{n=-\infty}^{\infty} \Big\langle x(t), \psi_{kn}(t) \Big\rangle \psi_{kn}(t). \tag{1.2.12}$$

Denote the scale-k projection coefficients as $d_k(n) = \langle x(t), \phi_{kn}(t) \rangle$ and $c_k(n) = \langle x(t), \psi_{kn}(t) \rangle$ for simplicity. (The notation c_{kn} was used in earlier subsections, but $c_k(n)$ is convenient for the present discussion). We say that $d_k(n)$ are the *multiresolution coefficients* at scale k, and $c_k(n)$ are the *wavelet coefficients* at scale k.

Assume that the projection coefficients $d_k(n)$ are known for some scale, say $k = 0$. We will then show that $d_k(n)$ and $c_k(n)$ for the coarser scales, i.e., $k = -1, -2, \ldots$ can be generated by using a paraunitary analysis filter bank $\{G_a(e^{j\omega}), H_a(e^{j\omega})\}$ corresponding to the synthesis bank $\{G_s(e^{j\omega}), H_s(e^{j\omega})\}$. We know that $\phi(t)$ and $\psi(t)$ satisfy the dilation equations (1.2.8). By substituting the dilation equations into the right-hand sides of $\phi_{kn}(t) = 2^{k/2}\phi(2^k t - n)$ and $\psi_{kn}(t) = 2^{k/2}\psi(2^k t - n)$, we obtain

$$\phi_{kn}(t) = \sqrt{2}\sum_{m=-\infty}^{\infty} g_s(m-2n)\phi_{k+1,m}(t), \qquad \text{and} \qquad \psi_{kn}(t) = \sqrt{2}\sum_{m=-\infty}^{\infty} h_s(m-2n)\phi_{k+1,m}(t). \tag{1.2.13}$$

A computation of the inner products $d_k(n) = \langle x(t), \phi_{kn}(t)\rangle$ and $c_k(n) = \langle x(t), \psi_{kn}(t)\rangle$ then yields

$$\begin{aligned} d_k(n) &= \sum_{m=-\infty}^{\infty} \sqrt{2}g_a(2n-m)d_{k+1}(m), \\ c_k(n) &= \sum_{m=-\infty}^{\infty} \sqrt{2}h_a(2n-m)d_{k+1}(m), \end{aligned} \tag{1.2.14}$$

where $g_a(n) = g_s^*(-n)$ and $h_a(n) = h_s^*(-n)$ are the analysis filters in the paraunitary filter bank.

The beauty of these equations is that they look like *discrete time convolutions!* Thus, if $d_{k+1}(n)$ is convolved with the impulse response $\sqrt{2}g_a(n)$ and the output decimated by two, the result is the sequence $d_k(n)$. A similar statement follows for $c_k(n)$. Because of the perfect reconstruction property of the two channel system (Fig. 1.2.6), it follows that we can reconstruct the projection coefficients $d_{k+1}(n)$ from the projection coefficients $d_k(n)$ and $c_k(n)$.

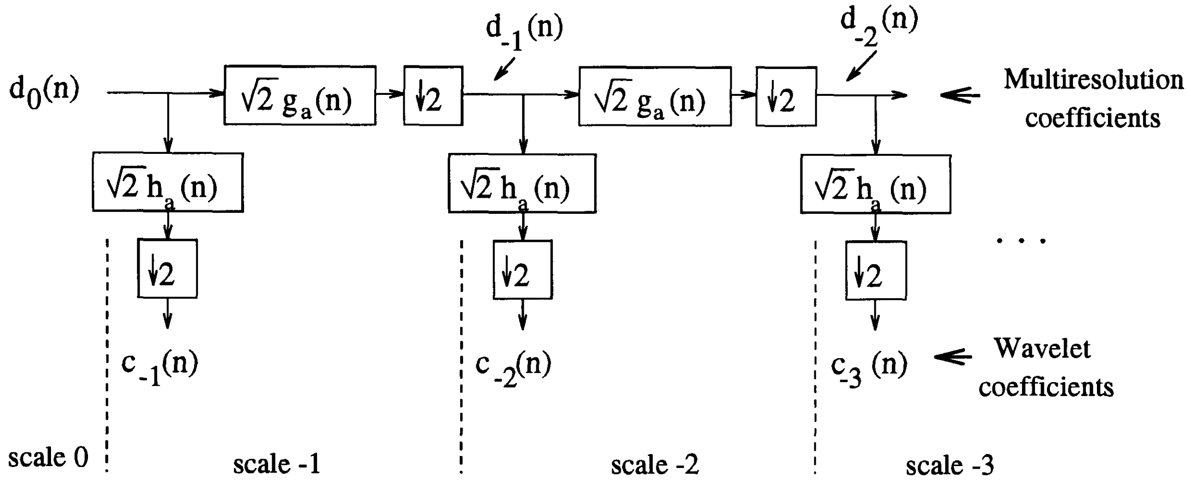


Fig. 1.2.9. Tree structured analysis bank generating wavelet coefficients $c_k(n)$ and multiresolution coefficients $d_k(n)$ recursively.

The Fast Wavelet Transform (FWT). Repeated application of this idea results in Fig. 1.2.9

which is a tree structured paraunitary filter bank with analysis filters $\sqrt{2}g_a(n)$ and $\sqrt{2}h_a(n)$ at each stage. Thus, given the projection coefficients $d_0(n)$ for V_0, we can compute the projection coefficients $d_k(n)$ and $c_k(n)$ for the coarser spaces $V_{-1}, W_{-1}, V_{-2}, W_{-2}, \ldots$ This scheme is sometimes referred to as the *Fast Wavelet Transform (FWT).*

1.3 Thesis overview

The thesis is organized into five chapters. Chapters 2-4 provide the main body of the thesis. They deal with three problems in filter banks and wavelets. The final chapter contains possible directions for future research, open problems and concluding remarks. The rest of this section gives a brief description of Chapters 2-4.

Chapter 2: Results on biorthogonal filter banks

In Chapter 2, we analyze properties of nonuniform maximally decimated filter banks. Even the most trivial problems in uniform filter banks are rather difficult and often impossible to solve in the case of unequal decimation ratios. For example, given a set of decimation ratios, in general, we do not know if there exist rational filters satisfying PR. So we take one step at a time, and first we establish equivalence between PR and biorthogonality. Then, an existence question is answered. If it is known that there are rational biorthogonal filters satisfying PR, we show that there are rational orthonormal filters satisfying PR. The proof is constructive, and we show how to obtain orthonormal filters. However, stability cannot be guaranteed for the causal realizations of filters obtained this way. Next, we give a set of necessary conditions that decimation ratios have to satisfy in order for a rational PR filter bank to exist. Finally, we use techniques developed in the first part to show how to construct filter banks which decorrelate subband signals. These results were reported in [Djo94].

The main results of the chapter are:

1) Equivalence of PR and biorthogonality of filters is established.
2) An orthonormalization technique for nonuniform maximally decimated PR filter banks is derived.
3) A set of necessary conditions on the decimation ratios for existence of rational PR filter banks is derived.
4) A procedure for constructing filters which produce uncorrelated subband signals is derived, for the case of uniform filter banks.

Chapter 3: Statistical wavelet and filter bank optimization

Wavelet transform is a generic term for an infinite number of transformations. The choice of a particular transform depends of the choice of the scaling function (in the case of orthonormal wavelets). It is clear that some wavelet functions are more suitable for a particular task than others. Therefore, if we have some knowledge about the signal, or the class of signals we want to analyze with wavelets, we would like to know how to choose the scaling function. In the deterministic case, the problem had already been analyzed. In this chapter, we find optimal scaling functions for analysis of WSS random processes.

Because of the intimate relation between wavelets and filter banks, there is a corresponding problem in the filter bank theory. Subband coders achieve compression by exploiting unequal energy distribution across the subband. Therefore, it is important that a filter bank has good energy compaction capabilities. The performance of a filter bank to perform energy compaction is expressed in terms of its coding gain. The aim is to optimize filters in the filter bank to give a higher coding gain for a particular input WSS random process [Djo94a]. We show that by putting just one pre- and one post-filter, the coding gain of any PU filter bank can be significantly improved.

For a successful implementation of the MRA, one has to know the approximation of a signal at the finest scale. This is done approximately. In the deterministic case, it had been shown that a simple FIR pre-filter can reduce the approximation error significantly. We extend this to the case of WSS random processes.

The main results of the chapter are:

1) Derivation of the objective function for finding an optimal scaling function.
2) Extension of the pre-filtering technique in MRA approximations to the case of WSS random processes.
3) Optimization of PR filter banks for higher coding gain.

Chapter 4: Generalized sampling in multiresolution subspaces

One of the fascinating features of the MRA is the fact that a number of well known theories can be embedded in its framework. One such example is the sampling theory for band-limited functions. It turns out that band-limited functions form a MRA of $L^2(\mathcal{R})$. The sampling theory for band-limited functions is rather deep and a proper treatment of the problem uses analytic functions, Fourier transform, Reproducing Kernel Hilbert Spaces (RKHS), etc. The notion of RKHS is very useful for the development of sampling theorems. Many spaces allowing sampling theorems have been constructed using the RKHS theory. An important feature of MRA spaces is that under very mild conditions they are RKHS, and sampling theory can be developed. In the fourth chapter, we develop sampling theory for MRA spaces when the sampling grid is periodically nonuniform. This is an extension of the existing uniform sampling theorems. This generalization allows us to have compactly supported synthesis functions, which was not possible previously. Several variations of the basic result are derived. For example, derivative and multi-band samplings are considered, as well as some other extensions. We show how these sampling theorems can be used for efficient computations of inner products in multiresolution analysis [Djo94b].

The main results of the chapter are:

1) Extension of the existing sampling theorems to periodically nonuniform grids.
2) Construction of sampling schemes in which synthesis functions are compactly supported.
3) Various extensions of the basic idea.
4) Application of sampling theorems in efficient computation of inner products in MRA.

1.4. General conventions and notations

1. $\mathcal{R}$ and $\mathcal{Z}$ denote the set of real and integer numbers respectively.
2. $\Re(z)$ and $\Im(z)$ are real and imaginary parts of z.
3. $L^p(I)$ are spaces of functions whose p^{th} power is absolutely integrable over the interval $I \subset \mathcal{R}$. We say that $f \in L^p_{loc}$ if for any finite $I \subset \mathcal{R}$, we have that $f \in L^p(I)$ and the norm is $||f||_{p,I} = \left(\int_I |f(x)|^p dx\right)^{1/p}$.
4. In all the integrals, the limits of integration are $(-\infty, \infty)$ unless it is explicitly indicated.
5. The Fourier transform operation and its inverse are denoted by $\mathcal{F}$ and $\mathcal{F}^{-1}$ respectively.
6. tr $\mathbf{A}$ denotes the trace of $\mathbf{A}$.
7. The quantities $\mathbf{A}^T$ and $\mathbf{A}^\dagger$ stand for transposition and transpose conjugation of the matrix $\mathbf{A}$. The notation $\widetilde{\mathbf{H}}(z) = \mathbf{H}^\dagger(1/z^*)$. Thus $\widetilde{\mathbf{H}}(z) = \mathbf{H}^\dagger(z)$ on the unit circle.
8. $W_N = e^{-2\pi j/N}$. The subscript N is omitted whenever it is clear from the context. $\mathbf{W}$ is the $N \times N$ DFT matrix. It has elements $[\mathbf{W}]_{mn} = W_N^{mn}$. Note that $\mathbf{W}^\dagger\mathbf{W} = N\mathbf{I}$.
9. The AC (alias-component) matrix for analysis filters (defined for uniform M-channel filter banks) is the one with components $[\mathbf{H}(z)]_{mn} = H_n(zW^m)$. For the synthesis filters we define a similar matrix: $[\mathbf{F}(z)]_{mn} = F_n(zW^m)$.
10. A delay chain is a single input multi output system, with transfer function matrix given by $\mathbf{e}(z) = [1 \quad z^{-1} \quad \ldots \quad z^{-M+1}]^T$.

11. The M-fold decimator has input-output relation $y(n) = x(n)\big\downarrow_M = x(Mn)$, or in the z-domain

$$Y(z) = X(z)\downarrow_M = \frac{1}{M}\sum_{i=0}^{M-1} X(z^{1/M}W_M^i). \tag{1.4.1}$$

The M-fold expander's input-output relation is $y(n) = x(n)\big\uparrow_M = \begin{cases} x(n/M), & n = \text{mul. of } M \\ 0, & \text{otherwise} \end{cases}$ or in the z-domain $Y(z) = X(z)\uparrow_M = X(z^M)$.

12. The so-called noble identity for multirate systems [Vai93] can be stated, for our purpose, as follows

$$\Big(A(z^{m_1})B(z)\Big)\Big\downarrow_{m_1m_2} = \Big(A(z)\Big(B(z)\Big)\Big\downarrow_{m_1}\Big)\Big\downarrow_{m_2}. \tag{1.4.2}$$

13. We say that $H(z)$ has the Nyquist(M) property if $(H(z))\big\downarrow_M = 1$.

14. If $f(t)$ is a random process, its autocorrelation function is defined as

$$r_{ff}(t,\tau) = E[f(t)f^*(t-\tau)], \tag{1.4.3}$$

where $E[\cdot]$ denotes the statistical expectation and $*$ denotes complex conjugation. When this autocorrelation function and the mean $E[f(t)]$ do not depend on t, we say that it is a wide sense stationary (WSS) random process. In that case, we define its power spectrum as the Fourier transform of $R_{ff}(\tau) = r_{ff}(0,\tau)$

$$S_{ff}(\omega) = \mathcal{F}(R_{ff}(\tau)) = \int R_{ff}(\tau)e^{-j\omega\tau}d\tau. \tag{1.4.4}$$

When $r_{ff}(t,\tau)$ is a periodic function of t with period T (and if the same is true for the mean), we say that it is a cyclo-wide sense stationary random process $(\text{CWSS})_T$ [Pap65]. Then one usually defines the autocorrelation function of this $(\text{CWSS})_T$ process as the time average

$$R_{ff}(\tau) = \frac{1}{T}\int_{-T/2}^{T/2} r_{ff}(t,\tau)dt. \tag{1.4.5}$$

Now the power spectrum of a $(\text{CWSS})_T$ signal is the Fourier transform of $R_{ff}(\tau)$. Similar definitions are used for discrete time signals.

2

Results on Biorthogonal Filter Banks

2.1. Introduction

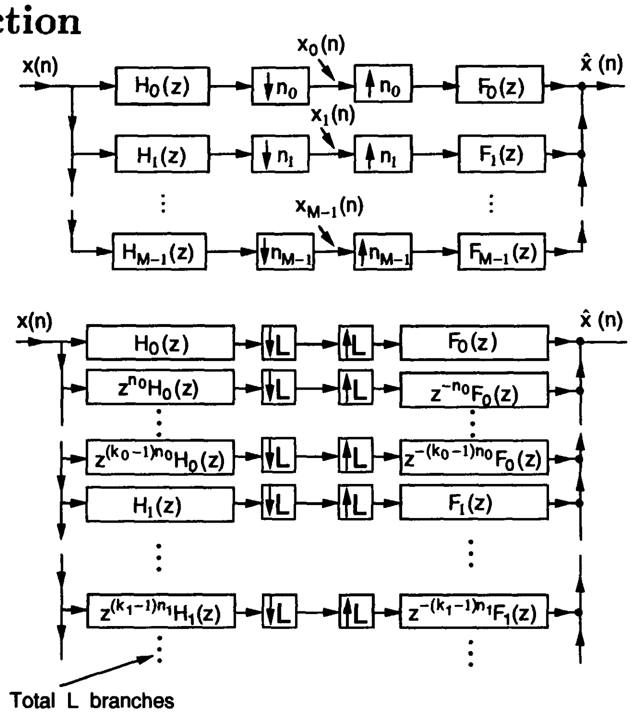


Fig. 2.1.1. (a) A nonuniform filter bank and (b) an equivalent uniform bank.

Fig. 2.1.1(a) shows an M-channel filter bank with integer decimation ratios n_k. The input signal $x(n)$

is split into M signals which are passed through the analysis filters $H_0(z), H_1(z), \ldots, H_{M-1}(z)$ and decimated by n_k (integers) for $k = 0, 1, \ldots, M-1$. At the synthesis end, these signals are expanded, passed through synthesis filters $F_0(z), F_1(z), \ldots, F_{M-1}(z)$ and added. When $\hat{x}(n) = cx(n - n_0)$, this system achieves perfect reconstruction (PR). In this thesis the PR property corresponds only to $\hat{x}(n) = x(n)$, as this eliminates some inconvenient notations without much loss of generality.

When $\sum_k 1/n_k = 1$, we have a maximally decimated filter bank. A special case is when $n_k = M$ for all k. We call it the uniform filter bank. Every nonuniform maximally decimated filter bank can be equivalently represented by a "larger" uniform filter bank as in Fig. 2.1.1(b) (see [Hoa89], [Kov91] and [Nay93]). The theory of uniform filter banks is well developed and such a system is shown in Fig. 2.1.2(a). The analysis and synthesis filters can be expressed in polyphase form as

$$H_i(z) = \sum_{k=0}^{M-1} z^{-k} E_{ik}(z^M) \quad \text{and} \quad F_i(z) = \sum_{k=0}^{M-1} z^{k} R_{ki}(z^M). \tag{2.1.1}$$

With each filter represented like this, the system can be drawn as in Fig. 2.1.2(b) where $\mathbf{E}(z)$ and $\mathbf{R}(z)$ are, respectively, the polyphase matrices of the analysis and synthesis banks. This system has the PR property $\widehat{x}(n) = x(n)$ if and only if $\mathbf{R}(z) = \mathbf{E}^{-1}(z)$. There are different ways to design a uniform filter bank that achieves PR, so the existence of rational filters (i.e., transfer functions which are ratios of two polynomials) satisfying the PR property is trivially guaranteed. But in the nonuniform case, it is not always possible to achieve PR with rational filters [Hoa89] (block decimation [Nay93] is not considered in this chapter). Notice, however, that ideal filters (nonrational, with possibly complex impulse response) can always be found such that the PR property holds for any set $\{n_k\}$ satisfying $\sum_k 1/n_k = 1$. So, whenever we discuss existence of PR systems, the discussion pertains *only to rational filters.*

A set of *necessary and sufficient* conditions on the set $\{n_k\}$ for PR to be possible is not known. On the other hand, we know some sufficient conditions. If the numbers $\{n_k\}$ are coming from a tree structure, for example, then we can have PR with rational filters [Som93a], [Smi86]. Not all decimation ratios allowing PR allow it with a tree structure. †For example consider $M = 23$ and the set

$$\{6, 10, 15, \underbrace{30, \ldots, 30}_{20 \text{ times}}\}.$$

This set satisfies $\sum_{k=0}^{22} 1/n_k = 1$. The filters that achieve PR are $H_i(z) = \widetilde{F}_i(z) = z^{-l_i}$ where the set of l_i's is $\{0, 1, 2, 3, 4, 5, 7, 8, 9, 10, 13, 14, 15, 16, 19, 20, 22, 23, 25, 26, 27, 28, 29\}$. For this, note that

† The authors would like to thank Tsuhan Chen for pointing out this example.

the output of the i^{th} decimator is $x(mn_i - l_i)$. We want every input sample to go through one and only one branch, which is equivalent to saying that $mn_i - l_i \neq kn_j - l_j$ for $i \neq j$ and any choice of m and k. On the other hand, since $\gcd(n_0, n_1, \ldots, n_{22}) = 1$, these numbers cannot come from a tree structure (if there were a tree, the decimation ratio at the first level of the tree would be a factor of this gcd). Because of such possibilities, we will not assume that $\{n_k\}$'s come from a tree. Before we discuss these issues in greater detail, let us explain some conventions and definitions in this chapter.

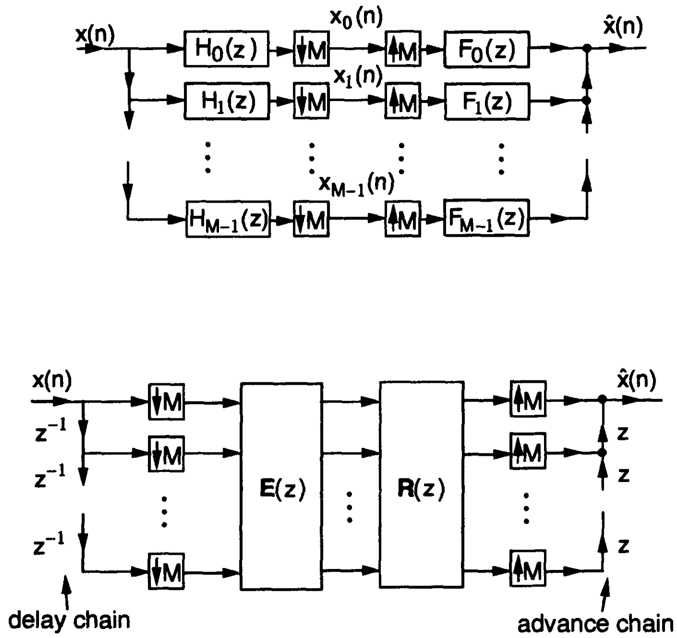


Fig. 2.1.2. (a) A uniform filter bank and (b) its polyphase decomposition.

All our signals are in l_2 space (i.e., finite energy signals). The inner product is defined as

$$< x(n), y(n) >= \sum_{n=-\infty}^{\infty} x(n)y^*(n),$$

and the norm $||x(n)||_2$ will be defined according to $||x(n)||_2^2 =< x(n), x(n) >$.

Biorthogonality and Orthonormality

Definition 2.1.1. A system of sequences $\{h_i(n - mn_i), f_l(n - kn_l)\}$, $0 \leq i, l \leq M - 1$ for all $m, k \in \mathcal{Z}$ is called a biorthogonal system if

$$< h_i(n - mn_i), f_l^*(-n + kn_l) >= \delta(i - l)\delta(k - m) \qquad \text{(biorthogonality)}. \qquad (2.1.2)$$

$\diamondsuit$

In the special case of orthonormal filter banks, the perfect reconstruction property is achieved by setting $f_k(n) = h_k^*(-n)$. In this case, the biorthogonality reduces to

$$< h_i(n - mn_i), h_l(n - kn_l) >= \delta(i-l)\delta(k-m) \qquad \text{(orthonormality)}. \tag{2.1.3}$$

If the above equation holds for some i and l, we often say that "the two filters $H_i(z)$ and $H_\ell(z)$ are orthonormal." It should be borne in mind that the actual meaning depends on n_i and n_l. The set $\{f_i(n - n_i k)\}_{i=0}^{M-1} \quad \forall k$ will be referred to as a filter bank-like system.

A question of interest in nonuniform filter bank theory is the following: suppose the integers $\{n_k\}_{k=0}^{M-1}$ are such that a biorthogonal PR system (with biorthogonal, rational filters) exists. Does it mean that an orthonormal PR system also exists? (Again, for the uniform case, the existence is trivially guaranteed simply by constraining $\mathbf{E}(z)$ to be paraunitary.) We will show by construction that for a given set of integer decimation ratios $\{n_k\}_{k=0}^{M-1}$, the existence of biorthogonal systems implies the existence of orthonormal PR systems as well (Sec. 2.3).

The procedure to convert the biorthogonal system to an orthonormal one is reminiscent of the Gram-Schmidt (GS) procedure, but is not the same for a variety of reasons. First, the orthonormalization of the basis is required to preserve the filter bank-like form of the basis; a conventional GS procedure would not give us this. Furthermore, using z-domain analysis and the special form of our system, we will be able to do the orthonormalization process in a *finite number of steps* (even though l_2 is an infinite dimensional space). This is another point of departure from the traditional GS technique.

At this point, the reader should be warned that this orthonormalization procedure is mostly of theoretical importance. The filters resulting from the orthonormalization not only are IIR in general, but also have huge orders; the proposed orthonormalization is not an alternative design technique for filter banks (after all we do not have biorthogonal filters to start the orthonormalization process). For the purpose of subband coding, there exists a simple scheme to generate inexpensive orthonormal filter banks, based on the so-called power symmetric filters (pp. 204 [Vai93]). These can also be used in a tree structure to obtain a subclass of nonuniform IIR orthonormal systems.

2.1.1. Chapter outline

In Sec. 2.2 we discuss the detailed reasons why biorthogonality and perfect reconstruction (PR) are identical concepts for *maximally* decimated filter banks†. Several corollaries of this result are

† A brief sketch of some of the results has been presented in [Djo93].

derived in Sec. 2.2.2. For example, we show that for PR to be possible, no two decimation ratios can be relatively prime. We also show that if a perfect reconstruction system is such that all the analysis and synthesis filters have unit energy, then the system becomes *orthonormal* (paraunitary in the uniform case). We also show that the shifted filter responses form a Riesz basis for l_2 space.

In Sec. 2.3 we show that whenever the decimation ratios $\{n_i\}$ of a maximally decimated system are such that perfect reconstruction is possible (i.e., such that there exist biorthogonal filters), then in particular, there exists an orthonormal filter bank. The proof is constructive, that is, given a set of biorthogonal filters we show how to find a set of orthonormal filters starting from these. Numerical examples are included. In general, the resulting orthonormal filters turn out to be IIR even if we start with an FIR biorthogonal system. However, the IIR filters are guaranteed to be free from poles on the unit circle. This means that, should they turn out to be unstable, a non-causal implementation can be found which is stable [Opp89], [Ram88].

In Sec. 2.4 we use the techniques of Sec. 2.3 to decorrelate the subband signals. Unlike the KLT, this decorrelation is for all time instants.

In Sec. 2.5 we derive some further necessary conditions on the decimation ratios $\{n_k\}$ for perfect reconstructability. These can be regarded as generalizations of the compatibility condition given in [Hoa89] and [pp. 285 of Vai93]. Some of the technical details which arise in the proofs have been moved to the Appendices (A–C) to provide a smoother reading.

2.1.2. Chapter–specific notations and conventions

Throughout the chapter, we will use the following notations. Integer M denotes the number of channels of the nonuniform system (Fig. 2.1.1(a)). The integer $L = \text{lcm}\ (n_0, n_1, \ldots n_{M-1})$. Also, $g_{ij} = \text{gcd}\ (n_i, n_j)$ throughout the chapter. The integers $\{k_i\}$ are numbers that satisfy

$$L = k_0 n_0 = k_1 n_1 = \cdots = k_{M-1} n_{M-1}. \tag{2.1.4}$$

2.2. Equivalence of biorthogonality and PR property

For the study and design of uniform filter banks, there exist powerful tools such as the polyphase formulation and the AC matrix formulation. In order to use them in a nonuniform filter bank, we have to transform it into the equivalent uniform one [Hoa89], [Kov91]. This is shown in Fig. 2.1.1(b). There are L branches (where L is the lcm of $\{n_i\}$), and each of them has the same decimation ratio.

The analysis filters are numbered as

$$S_0(z), S_1(z), \ldots, S_{k_0-1}(z), S_{k_0}(z), \ldots, \tag{2.2.1}$$

and similarly for the synthesis filters $Q_i(z)$. Thus the analysis and synthesis filters are

$$S_i(z) = z^{pn_l} H_l(z), \qquad Q_k(z) = z^{-rn_m} F_m(z), \tag{2.2.2}$$

where

$$i = p + \sum_{j=0}^{l-1} k_j, \quad 0 \le p \le k_l - 1, \qquad k = r + \sum_{j=0}^{m-1} k_j, \quad 0 \le r \le k_m - 1. \tag{2.2.3}$$

Here $k_j = L/n_j$ and its meaning is clear; we just made k_i delayed filters from each original filter $H_i(z)$, i.e., each new filter comes from one of M original filters. The biorthogonality (2.1.2) can be rewritten as [Som93a]

$$\sum_n h_i(n) f_l(mg_{il} - n) = \delta(i-l)\delta(m), \quad 0 \le i, l \le M-1, \quad m \in \mathcal{Z} \qquad \text{(biorthogonality)}, \tag{2.2.4}$$

where g_{il} =gcd(n_i, n_l). This infinite set of conditions can be compactly written as a finite set of conditions in the z-domain

$$\Big(H_i(z)F_l(z)\Big)\Big\downarrow_{g_{il}} = \delta(i-l), \quad 0 \le i, l \le M-1 \qquad \text{(biorthogonality)}. \tag{2.2.5}$$

Again, if $F_i(z) = \widetilde{H}_i(z)$, then the above property is called the orthonormal property and can be written as

$$\Big(H_i(z)\widetilde{H}_l(z)\Big)\Big\downarrow_{g_{il}} = \delta(i-l) \qquad \text{(orthonormality)}. \tag{2.2.6}$$

In this section, we will show that the most general form of PR for a maximally decimated filter bank is a Riesz basis for l_2 space, formed by the analysis and synthesis filters. This issue has come up in earlier work, but has not been shown or proved this way. The relation between filter banks and wavelets and the role of orthonormality has been discussed in [Vet92], [Rio91], [Gop92], [Dau92] and [Som93a]. This will be followed by the derivation of a number of corollaries.

2.2.1. PR implies biorthogonality of analysis and synthesis filters

Theorem 2.2.1. Let the system in Fig. 2.1.1(a) be a maximally decimated filter bank with decimation ratios $\{n_k\}_{k=0}^{M-1}$. If the filter bank has the perfect reconstruction property, the filters form a biorthogonal system; that is they satisfy (2.2.5).

$\diamond$

Remarks.

1. This is a fairly subtle fact, holding *only because of maximal decimation.* For example, consider a two-channel undecimated system with filters $H_0(z) = 1 + z^{-1}$, $H_1(z) = 1 - z^{-1}$, $F_0(z) = F_1(z) = 1/2$. Then we have PR, but not biorthogonality.
2. The converse of the above theorem also holds; see Appendix 2.B of [Vai93a].

Proof. Let $S_i(z)$ and $Q_k(z)$ be analysis and synthesis filters of the equivalent uniform filter bank. They are related to $H_l(z)$ and $F_m(z)$ as in (2.2.2) and (2.2.3) (see Fig. 2.1.1(b)). Let $\mathbf{S}(z)$ denote the AC matrix of the above analysis bank $S_i(z)$; that is $[\mathbf{S}(z)]_{mn} = S_n(zW^m)$ (Sec. 2.1.2), and let $\mathbf{E}(z)$ be the corresponding polyphase matrix. These are related as [Vai93, pp. 234]

$$\mathbf{S}(z) = \mathbf{W}^\dagger \mathbf{\Lambda}(z)\mathbf{E}^T(z^L), \tag{2.2.7}$$

where $\mathbf{\Lambda}(z) = \text{diag}(1, z^{-1}, \ldots, z^{-(L-1)})$. Similarly for the synthesis bank define $\mathbf{Q}(z)$ such that $[\mathbf{Q}(z)]_{mn} = Q_n(zW^m)$, and let $\mathbf{R}(z)$ be the polyphase matrix as defined in Sec. 2.1 We have

$$\mathbf{Q}(z) = \mathbf{W}\mathbf{\Lambda}^{-1}(z)\mathbf{R}(z^L). \tag{2.2.8}$$

Then

$$\mathbf{S}(z)\mathbf{Q}^T(z) = \mathbf{W}^\dagger\mathbf{\Lambda}(z)\overbrace{\left(\mathbf{R}(z^L)\mathbf{E}(z^L)\right)^T}^{=\mathbf{I}}\mathbf{\Lambda}^{-1}(z)\mathbf{W} = \mathbf{W}^\dagger\mathbf{W} = L\mathbf{I}. \tag{2.2.9}$$

This is because $\mathbf{R}(z)\mathbf{E}(z) = \mathbf{I}$, due to PR property. So we get [since $\mathbf{S}(z)$ and $\mathbf{Q}(z)$ are square matrices]

$$\mathbf{Q}(z)\mathbf{S}^T(z) = \mathbf{S}(z)\mathbf{Q}^T(z) = L\mathbf{I}. \tag{2.2.10}$$

Notice that without the assumption of maximal decimation, the matrix $\mathbf{\Lambda}(z)$ would not be square, $\mathbf{\Lambda}^{-1}(z)$ would not exist and we would not have $\mathbf{S}^T(z)\mathbf{Q}(z) = \mathbf{Q}(z)\mathbf{S}^T(z) = L\mathbf{I}$ (which is an important step in the proof!).

The condition $\mathbf{S}^T(z)\mathbf{Q}(z) = L\mathbf{I}$ implies, in view of the definitions of $\mathbf{S}(z)$, $\mathbf{Q}(z)$ and (1.4.1), that

$$(S_i(z)Q_k(z))\downarrow_L = \delta(i-k). \tag{2.2.11}$$

Now suppose that i, k are such that $l \neq m$ in (2.2.3), i.e., that $S_i(z)$ and $Q_k(z)$ do not come from the same original branch. Then (2.2.11) can be written as

$$\left(z^{pn_l - rn_m}H_l(z)F_m(z)\right)\Big\downarrow_L = 0, \tag{2.2.12}$$

for $0 \le p \le k_l - 1$ and $0 \le r \le k_m - 1$. With

$$n_l = b_{lm} g_{lm} \qquad \text{and} \qquad n_m = b_{ml} g_{lm}, \tag{2.2.13}$$

where g_{lm} =gcd(n_l, n_m), this becomes

$$\left(z^{(pb_{lm} - rb_{ml})g_{lm}} H_l(z) F_m(z)\right) \downarrow_L = 0. \tag{2.2.14}$$

By multiplying (2.2.14) by z^d and using $L = k_m n_m = k_m b_{ml} g_{ml}$, we get

$$z^d \left(\left(z^{(pb_{lm} - rb_{ml})g_{lm}} H_l(z) F_m(z)\right) \downarrow_L \right) = \left(z^{(pb_{lm} - rb_{ml} + dk_m b_{ml})g_{lm}} H_l(z) F_m(z)\right) \downarrow_L = 0, \forall d \in \mathcal{Z}. \tag{2.2.15}$$

It is shown in Appendix 2.A that $(pb_{lm} - rb_{ml} + dk_m b_{ml})$ can take any integer value a, under the conditions $0 \le p \le k_l - 1$, $0 \le r \le k_m - 1$, and $d \in \mathcal{Z}$. Then

$$\left(z^{a g_{lm}} H_l(z) F_m(z)\right) \Big\downarrow_{k_l b_{lm} g_{lm}} = \left(z^a \left(\left(H_l(z) F_m(z)\right) \downarrow_{g_{lm}} \right)\right) \Big\downarrow_{k_l b_{lm}} = 0 \qquad \forall a \in \mathcal{Z}. \tag{2.2.16}$$

Since this holds for all integers a, we can rewrite it as

$$\left(H_l(z) F_m(z)\right) \Big\downarrow_{g_{lm}} = 0. \tag{2.2.17}$$

Let now $S_i(z)$ and $Q_k(z)$ come from the same branch, i.e., $l = m$. Then (2.2.11) means

$$\left(z^{(p-r)n_m} H_m(z) F_m(z)\right) \downarrow_L = \left(z^{(p-r)} \left(\left(H_m(z) F_m(z)\right) \Big\downarrow_{n_m} \right)\right) \Big\downarrow_{k_m} = \delta(p - r). \tag{2.2.18}$$

Now $p - r$ can reach any integer in $[-k_m + 1, k_m - 1]$, so that the last equation is equivalent to

$$\left(H_m(z) F_m(z)\right) \downarrow_{n_m} = 1. \tag{2.2.19}$$

Together with (2.2.17), this implies biorthogonality (2.2.5).

◇

2.2.2. Corollaries

Corollary 2.2.1. No two decimators can be coprime.

If any two n_i's are relatively prime, then their gcd is 1 and we cannot satisfy the conditions for PR with rational filters. This is because (2.2.5) now implies $H_l(z) F_m(z) = 0$ for $l \ne m$ and this cannot be satisfied with rational filters.

Corollary 2.2.2. Completeness.

Definition 2.2.1. A set of vectors $\{\mathbf{x}_i\}_{i=0}^{\infty}$ in an infinite-dimensional Hilbert space is said to be complete if the zero vector is the only vector orthogonal to all of $\mathbf{x}_i$'s (pp. 6. [You80]).

$\diamondsuit$

Assuming that the filter bank has the perfect reconstruction property, the completeness of the filter bank follows immediately. To see this, let us write the reconstructed signal

$$x(n) = \sum_{i=0}^{M-1} \sum_{k=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} h_i(n_i k - m) x(m) f_i(n - n_i k) = \sum_{i=0}^{M-1} \sum_{k} \langle x(-m), h_i^*(m + n_i k) \rangle f_i(n - n_i k). \tag{2.2.20}$$

Assume there is a nonzero input $x(n)$ such that $x(-n)$ is orthogonal to all the analysis filters and their n_i-shifted versions. Then the above sum would be zero and the system would not be a PR system. Similar conclusion can be made for the synthesis filters if we interchange the analysis and synthesis filters (because PR is not violated by such an interchange).

Corollary 2.2.3. Linear independence.

Definition 2.2.2. A set of vectors $\eta_{ik}(n) = \{f_i(n - n_i k)\}_{i=0}^{M-1}, \;\; k \in \mathcal{Z}$ in an infinite-dimensional Hilbert space is said to be linearly independent (or minimal) if none of $\eta_{JK}(n)$'s lie in the closure of the linear span of $\{\eta_{lm}(n)\}_{l=0}^{M-1}$ $m \neq K$ for $l = J$ (see pp. 28. [You80] for the Banach space case).

$\diamondsuit$

Since the synthesis filters form a biorthogonal system, it can be proved [You80] that the set of sequences $\eta_{ik}(n)$ is linearly independent. We will often say "filters $F_i(z)$ and $F_j(z)$ are linearly independent," meaning that the corresponding time sequences and their shifts are linearly independent in the above sense.

Corollary 2.2.4. Basis property.

In an infinite dimensional Hilbert space, completeness and independence of a set of vectors is not sufficient to conclude that these vectors form a Riesz basis†. However, by using the further assumption that the synthesis and analysis filters are stable (i.e., $\sum_n |h_i(n)| < \infty$ and $\sum_n |f_i(n)| < \infty$ or $F_i(e^{j\omega})$, $H_i(e^{j\omega})$ exist and are upper bounded by a finite constant), we show that $\{f_i(n - mn_i)\}_{i=0}^{M-1}$ and $\{h_i(n - mn_i)\}_{i=0}^{M-1}$ $\forall \, m \in \mathcal{Z}$ are bases for l_2 space. For this, we will invoke Theorem 9, p. 32, [You80]. Since completeness and linear independence have been established earlier, it is sufficient,

† For the definition of a Riesz or unconditional basis, see [You80] p. 31. or [Chu92] p. 71.

according to the above theorem, to show that $\exists\, 0 < C_1, C_2 < \infty$ such that

$$\sum_{i=0}^{M-1}\sum_{m} | < x(n), f_i^*(mn_i - n) > |^2 < C_1||x||_2^2 \text{ and } \sum_{i=0}^{M-1}\sum_{m} | < x(n), h_i^*(mn_i - n) > |^2 < C_2||x||_2^2. \tag{2.2.21}$$

In this discussion, all summations are from $-\infty$ to ∞ unless indicated differently. Thus

$$\sum_{i=0}^{M-1}\sum_{m} | < x(n), f_i^*(mn_i - n) > |^2 = \sum_{i=0}^{M-1}\sum_{m}\Big|\sum_{n} x(n) f_i(mn_i - n)\Big|^2 = \sum_{i=0}^{M-1}\sum_{m} |z_i(m)|^2, \tag{2.2.22}$$

where $z_i(n)$ is the n_i-fold decimation of the convolution $x(n) * f_i(n)$. Using Parseval's relation the above can be rewritten as

$$\sum_{i=0}^{M-1}\sum_{m=-\infty}^{\infty} | < x(n), f_i^*(mn_i - n) > |^2 = \sum_{i=0}^{M-1} \frac{1}{2\pi}\int_0^{2\pi} |Z_i(e^{j\omega})|^2 d\omega$$

$$\leq \sum_{i=0}^{M-1} \frac{1}{2\pi}\int_0^{2\pi} \left|X(e^{j\omega})F_i(e^{j\omega})\right|^2 d\omega \leq ||x(n)||_2^2 \sum_{i=0}^{M-1} \max_{\omega\in[0,2\pi]} \left|F_i(e^{j\omega})\right|^2 < C||x(n)||_2^2. \tag{2.2.23}$$

The first inequality follows because the energy of a decimated sequence is no greater than that of the undecimated version. The proof for the analysis filters is similar. So, we really have a biorthogonal basis formed by the set of n_i-shifted versions of the synthesis and analysis filters.

Corollary 2.2.5. Unit energy implies orthonormality!

Consider the maximally decimated system [Fig. 2.1.1(a)]. Suppose the following two properties are satisfied:

1. Perfect reconstruction property, and
2. All analysis and synthesis filters have unit energy, i.e., $\sum_n |h_i(n)|^2 = \sum_n |f_i(n)|^2 = 1$, for $0 \leq i \leq M-1$.

Then the synthesis filters satisfy orthonormality. In other words, eqn. (2.2.6) holds. To prove this, note that the perfect reconstruction property implies biorthogonality (Theorem 2.2.1.), so that, in particular, $\sum_n h_i(n) f_i(-n) = 1$. Now, Cauchy-Schwarz inequality says

$$\sum_n |h_i(n)|^2 \sum_n |f_i(n)|^2 \geq \Big|\sum_n h_i(n) f_i(-n)\Big|^2. \tag{2.2.24}$$

The right-hand side is unity, by the biorthogonality. The left-hand side is also unity if the analysis and synthesis filters have unit energy. But equality in Cauchy-Schwarz inequality implies $h_i(n) = e^{j\theta_i} f_i^*(-n)$ for some θ_i. Substituting in (2.2.4) we readily conclude that $\theta_i = 0$ and that the set of synthesis filters (equivalently, the set of analysis filters) satisfies orthonormality.

Corollary 2.2.6. Generalization of Nyquist and power complementary properties.

For uniform filter banks ($n_i = M$ for all i) it is well-known that if the system is orthonormal (paraunitary), the filters $H_i(z)$ and $F_i(z)$ are spectral factors of Nyquist(M) filters (or Mth band filters) [Vai93, pp. 297]. In the more general (nonuniform, and biorthogonal case), this property is replaced with the property that $H_i(z)F_i(z)$ is a Nyquist(n_i) filter. We can readily see this from the biorthogonality condition (2.2.5) $\Big(H_i(z)F_i(z)\Big)\big\downarrow_{n_i} = 1$.

Next, for the uniform paraunitary filter bank, it is well known that the analysis filters are power complementary, and so are the synthesis filters [Vai93, pp. 296]. For the general case (nonuniform, biorthogonal) we have

$$\sum_i \frac{1}{n_i} H_i(z)F_i(z) = 1, \tag{2.2.25}$$

(see [Vai93a]) which reduces to the power complementary property $\sum_i H_i(z)\widetilde{H}_i(z) = M$ in the uniform paraunitary case.

2.3. Orthonormalization of biorthogonal filter banks

From the second section, we know that if the integers $\{n_k\}_{k=0}^{M-1}$ are such that PR is possible, the analysis and synthesis filters (and their shifted versions) form a *biorthogonal* basis. Under this condition, does there exist a PR system with *orthonormal* filters? The answer to this question is in the affirmative; we will present an orthonormalization process which preserves the filter bank-like form of the system $\{h_i(n - mn_i), f_l(n - kn_l)\}$ for $0 \le i, l \le M-1$ and $m, k \in \mathcal{Z}$. If one wants just to orthonormalize some set of vectors, the Gram-Schmidt technique is one way of doing this, but we want more, namely to preserve the filter bank-like form of the system. We now show how to achieve this aim. Our procedure is reminiscent of the Gram-Schmidt technique, but it converges in a finite number of steps even though the space has infinite dimension.

2.3.1. Normalization condition or Nyquist condition

Let $F_k(z)$ be a rational transfer function. Define

$$G_k(z) = \alpha_k(z^{n_k})F_k(z). \tag{2.3.1}$$

Then $G_k(z)\widetilde{G}_k(z) = \alpha_k(z^{n_k})\widetilde{\alpha}_k(z^{n_k})F_k(z)\widetilde{F}_k(z)$, so that

$$\Big(\alpha_k(z^{n_k})\widetilde{\alpha}_k(z^{n_k})F_k(z)\widetilde{F}_k(z)\Big)\big\downarrow_{n_k} = \alpha_k(z)\widetilde{\alpha}_k(z)\Big(F_k(z)\widetilde{F}_k(z)\Big)\big\downarrow_{n_k}. \tag{2.3.2}$$

Now if we choose $\alpha_k(z)$ such that

$$\alpha_k(z)\widetilde{\alpha}_k(z) = \frac{1}{\left(F_k(z)\widetilde{F}_k(z)\right)\big\downarrow_{n_k}}, \tag{2.3.3}$$

we get $\left(G_k(z)\widetilde{G}_k(z)\right)\big\downarrow_{n_k} = 1$. A function $G_k(z)$ with this property will be called *normalized.* This is different from the usual meaning of normalization of vectors in l_2. In the time domain, the above normalization condition means that the n_k-shifted versions of $g_k(n)$ (i.e., $\{g_k(n - in_k)\}$) form an orthonormal set. Equivalently, $\widetilde{G}_k(z)G_k(z)$ is a Nyquist(n_k) filter [Vai93, pp. 151]; that is its impulse response coefficients $h(n)$ satisfy $h(n_k i) = 0$ for $i \neq 0$.

The existence of $\alpha_k(z)$ satisfying (2.3.3) is assured because of the following. We have

$$\left(F_k(z)\widetilde{F}_k(z)\right)\big\downarrow_{n_k} = \frac{1}{n_k}\sum_{l=0}^{n_k-1} F_k(z^{1/n_k}W^l)\widetilde{F}_k(z^{1/n_k}W^l). \tag{2.3.4}$$

Since $F_k(z)\widetilde{F}_k(z) \geq 0$ on the unit circle, each term in the above expression remains nonnegative. Thus whenever $F_k(z)$ is rational, the function $\left(F_k(z)\widetilde{F}_k(z)\right)\big\downarrow_{n_k}$ is rational and nonnegative on the unit circle. Such functions can always be written as a product $a(z)\widetilde{a}(z)$. The spectral factor $a(z)$ (a rational function, not unique) can be obtained by standard spectral factorization techniques. Now take $\alpha_k(z) = 1/a(z)$ and (2.3.3) is satisfied. The function $\alpha(z)$ can be chosen to have no poles outside the unit circle (by choosing $a(z)$ to have minimum phase), but what if $a(z)$ has a zero on the unit circle? Then $\alpha_k(z)$ will have a pole on the unit circle! This potential instability will be handled later on.

If two filters $F_i(z)$ and $F_k(z)$ are orthogonal, will that property be preserved by the above operation? Let $G_i(z)$ and $G_k(z)$ be the normalized versions of $F_i(z)$ and $F_k(z)$. Then

$$\left(G_i(z)\widetilde{G}_k(z)\right)\big\downarrow_{g_{ik}} = \alpha_i(z^{n_i/g_{ik}})\widetilde{\alpha}_k(z^{n_k/g_{ik}})\underbrace{\left(F_i(z)\widetilde{F}_k(z)\right)\big\downarrow_{g_{ik}}}_{=0} = 0 \tag{2.3.5}$$

showing that orthogonality is preserved (notice that n_k/g_{ik} and n_i/g_{ik} are integers). Summarizing, if we have a set of orthogonal filters $\{F_i(z)\}_{i=0}^{M-1}$, then the above normalization can be used to obtain a set of ortho*normal* filters $\{G_i(z)\}_{i=0}^{M-1}$.

2.3.2. Orthogonalization

Let $\{H_k(z), F_k(z)\}_{k=0}^{M-1}$ be a biorthogonal set of rational analysis and synthesis transfer functions for a maximally decimated PR filter bank with decimation ratios $\{n_k\}_{k=0}^{M-1}$. We now describe a procedure to get a new set of rational transfer functions $\{G_k(z)\}_{k=0}^{M-1}$ which are mutually orthogonal, i.e., satisfy

$$\Big(G_k(z)\widetilde{G}_l(z)\Big)\Big\downarrow_{g_{kl}} = 0 \qquad \text{for } k,l = 0,1,\ldots,M-1 \text{ and } k \neq l. \tag{2.3.6}$$

We start by making $G_0(z) = F_0(z)$ and $G_1(z)$ orthogonal to $G_0(z)$. For this, let us look for $G_1(z)$ of the form

$$G_1(z) = F_1(z) - \beta_{01}(z^{g_{01}})F_0(z). \tag{2.3.7}$$

$\Big(G_1(z)\widetilde{G}_0(z)\Big)\Big\downarrow_{g_{01}} = 0$ can be achieved if we set

$$\beta_{01}(z) = \frac{(F_1(z)\widetilde{G}_0(z))\downarrow_{g_{01}}}{(F_0(z)\widetilde{G}_0(z))\downarrow_{g_{01}}}. \tag{2.3.8}$$

Clearly $\beta_{01}(z)$ is a rational transfer function. Then, $G_1(z)$ as in (2.3.7) remains a rational transfer function. This is how we start this orthogonalization process. Assume that we have made $G_0(z), G_1(z), \ldots, G_{s-1}(z)$ orthogonal to each other in the sense of (2.3.6). In the s^{th} step we want to make $G_s(z)$ orthogonal to $G_0(z), G_1(z), \ldots, G_{s-1}(z)$. Assume $G_s(z)$ in the form

$$G_s(z) = F_s(z) - \sum_{i=0}^{s-1} \beta_{is}(z^{g_{si}})F_i(z). \tag{2.3.9}$$

Let

$$L = g_{s0}c_{s0} = g_{s1}c_{s1} = \cdots = g_{s,s-1}c_{s,s-1}. \tag{2.3.10}$$

After expanding $\beta_{is}(z)$ into c_{si}-fold polyphase components, we get

$$\beta_{is}(z^{g_{si}}) = \sum_{l=0}^{c_{si}-1} z^{-lg_{si}}\beta_{isl}(z^L), \tag{2.3.11}$$

so $G_s(z)$ is of the form

$$G_s(z) = F_s(z) - \sum_{i=0}^{s-1}\sum_{l=0}^{c_{si}-1} \beta_{isl}(z^L)z^{-lg_{si}}F_i(z). \tag{2.3.12}$$

We want to make $G_s(z)$ orthogonal to $G_k(z)$ for $k = 0,1,\ldots,s-1$. In other words we want

$$\Big(G_s(z)\widetilde{G}_k(z)\Big)\Big\downarrow_{g_{sk}} = \Big(F_s(z)\widetilde{G}_k(z)\Big)\Big\downarrow_{g_{sk}} - \sum_{i=0}^{s-1}\sum_{l=0}^{c_{si}-1}\Big(\beta_{isl}(z^L)z^{-lg_{si}}F_i(z)\widetilde{G}_k(z)\Big)\Big\downarrow_{g_{sk}} = 0. \tag{2.3.13}$$

It is easily verified that $(A(z))\downarrow_g = 0$ if and only if $(z^{mg}A(z))\big\downarrow_{L=cg} = 0$ for $m = 0,1,\ldots,c-1$. Then, (2.3.13) can be written as

$$\sum_{i=0}^{s-1}\sum_{l=0}^{c_{si}-1} \beta_{isl}(z)\Big(z^{mg_{sk}-lg_{si}}F_i(z)\widetilde{G}_k(z)\Big)\Big\downarrow_L = \Big(z^{mg_{sk}}F_s(z)\widetilde{G}_k(z)\Big)\Big\downarrow_L, \tag{2.3.14}$$

for $m = 0, 1, \ldots, c_{sk} - 1$ and $k = 0, 1, \ldots, s-1$. So we have $\sum_{i=0}^{s-1} c_{si}$ unknowns $\beta_{isl}(z)$, and the same number of linear equations. If the determinant of the system (which is a rational function of z) is not identically zero, we can solve the system of linear equations for $\beta_{isl}(z)$'s. If this is not the case, we can keep decreasing the number of unknowns until we have a determinant that is not identically zero (see Appendix 2.B). After solving it, we see that $\beta_{is}(z)$'s are rational functions, so $G_s(z)$ will remain a rational transfer function. Using the biorthogonal filter $H_s(z)$ and the time domain equivalent of (2.3.9), one can show that the trivial solution $G_s(z) = 0$ is excluded $[(\beta_{is}(z^{g_{si}})F_i(z)H_s(z)) \downarrow_{n_s} = 0$ for $i = 0, 1, \ldots, s-1$ and $(F_s(z)H_s(z)) \downarrow_{n_s} = 1$, so that $(G_s(z)H_s(z)) \downarrow_{n_s} = 1$ which implies that $G_s(z) \not\equiv 0]$. At the end of this process, we have a new set of rational transfer functions $\{G_k(z)\}_{k=0}^{M-1}$ satisfying (2.3.6).

2.3.3. Stability

In this subsection we show that if the transfer functions resulting from the orthonormalization have poles outside the unit circle, they can be moved inside, preserving the orthonormal and PR property. We also show that in the process of orthogonalization and normalization described in Sec. 2.3.2 and 2.3.1, the poles will *automatically* be excluded from being on the unit circle.

First assume that after the orthonormalization we got $\{G'_k(z)\}_{k=0}^{M-1}$ with some poles outside the unit circle. For example, let z_0 be a pole outside the unit circle. $(1 - z^{-n_k} z_0^{n_k})$ has zeros at $z_0 W_{n_k}^l$ for $0 \le l \le n_k - 1$. Define the product

$$Q_k(z^{n_k}) = (1 - z^{-n_k} z_0^{n_k})(1 - z^{-n_k} z_1^{n_k}) \ldots \tag{2.3.15}$$

where $z_0, z_1 \ldots$ are the poles of $G'_k(z)$ outside the unit circle, and construct the allpass function

$$\gamma_k(z^{n_k}) = \frac{Q_k(z^{n_k})}{\widetilde{Q}_k(z^{n_k})}. \tag{2.3.16}$$

This has all poles inside the unit circle. Now form a new set of functions as $G_k(z) = \gamma_k(z^{n_k})G'_k(z)$. Then $G_k(z)$ has no poles outside the unit circle. The new set satisfies orthogonality because, for $m \neq k$,

$$\begin{aligned} \Big(G_k(z)\widetilde{G}_m(z)\Big)\Big\downarrow_{g_{km}} &= \Big(\gamma_k(z^{n_k})G'_k(z)\widetilde{\gamma}_m(z^{n_m})\widetilde{G}'_m(z)\Big)\Big\downarrow_{g_{km}} \\ &= \gamma_k(z^{n_k/g_{km}})\widetilde{\gamma}_m(z^{n_m/g_{km}})\underbrace{\Big(G'_k(z)\widetilde{G}'_m(z)\Big)\Big\downarrow_{g_{km}}}_{=0} = 0. \end{aligned} \tag{2.3.17}$$

(Recall that n_k/g_{km} and n_m/g_{km} are integers.) Normality is preserved too since

$$\Big(G_k(z)\widetilde{G}_k(z)\Big)\Big\downarrow_{n_k} = \underbrace{\gamma_k(z)\widetilde{\gamma}_k(z)}_{=1}\underbrace{\Big(G'_k(z)\widetilde{G}'_k(z)\Big)\Big\downarrow_{n_k}}_{=1} = 1. \tag{2.3.18}$$

So, we have shown how to replace the poles outside the unit circle with poles inside, without destroying orthonormality.

Avoiding poles on the unit circle.

Let us repeat (2.3.9) below, but call it $P_s(z)$ for notational convenience.

$$P_s(z) = F_s(z) - \sum_{i=0}^{s-1} \beta_{is}(z^{g_{si}})F_i(z). \tag{2.3.19}$$

This function, in general, can have both poles and zeros on the unit circle. First, assume that it has a pole of order r at $z_p = e^{j\omega_p}$. It will be shown that this will be canceled in the process of normalization. Recall that the normalized function $G_s(z)$ is constructed according to $G_s(z) = \alpha_s(z^{n_s})P_s(z)$, where

$$\alpha_s(z)\widetilde{\alpha}_s(z) = \frac{1}{\left(P_s(z)\widetilde{P}_s(z)\right)\downarrow_{n_s}}. \tag{2.3.20}$$

It is shown in Appendix 2.C that if $P_s(z)$ has a pole of order r on the unit circle, then $\alpha_s(z^{n_s})$ defined as per (2.3.20) will have a zero of order at least r at the same point. This zero will cancel out the pole of $P_s(z)$, so that the normalized $G_s(z)$ will not have any pole at that point. We see that $G_s(z)$ cannot have any pole on the unit circle coming from $P_s(z)$.

The other possibility is that $\alpha_s(z^{n_s})$ itself has a pole on the unit circle, i.e., $\left(P_s(z)\widetilde{P}_s(z)\right)\downarrow_{n_s}$ has a *zero* on the unit circle. Assume that $\alpha_s(z^{n_s})$ has a pole of order r at $z_0 = e^{j\omega_0}$, and hence at $z_0W_{n_s}^k$, $0 \le k \le n_s - 1$. We have

$$\alpha_s(z^{n_s})\widetilde{\alpha}(z^{n_s}) = \frac{1}{\left(\left(P_s(z)\widetilde{P}_s(z)\right)\downarrow_{n_s}\right)\uparrow_{n_s}}. \tag{2.3.21}$$

From this equation we conclude that $\alpha_s(z^{n_s})$ can have a pole of order r at some point, on the unit circle, if and only if $\left(\left(P_s(z)\widetilde{P}_s(z)\right)\downarrow_{n_s}\right)\uparrow_{n_s}$ has a zero of order $2r$ at that point. For this to happen $P_s(z)$ must have zeros of order at least r at $z = z_0W_{n_s}^k$ for $k = 0, 1, \ldots, n_s - 1$ (for the proof see Appendix 2.C). These zeros will cancel with the above mentioned poles of $\alpha_s(z^{n_s})$ when $G_s(z)$ is formed. From this we can conclude that $G_s(z)$ cannot have any poles on the unit circle. Together with the fact that poles outside the unit circle can be moved inside, we conclude that the described procedure leads to stable filters.

2.3.4. Numerical examples

Example 2.3.1. Uniform system. As an example of the above described procedure, we orthonormalized a uniform, four-channel filter bank. The filters that we started with were all FIR, linear-phase, obtained from a two-level tree of two-channel filter banks. Each filter in the two-channel module has length 10 ([Ngu89]). The resulting orthonormal filters are IIR and their numerator degrees are $28, 44, 140, 380$ and denominator degrees $25, 41, 77, 377$ respectively. We see that the orders of the filters increase rapidly as we proceed with the orthonormalization process. The magnitude responses [see Fig. 2.3.1 (a) and (b)] are more or less the same before and after orthonormalization. Most of the polynomial coefficients after orthonormalization are very small and can be discarded without harming the frequency response, but it deteriorates the orthonormality property.

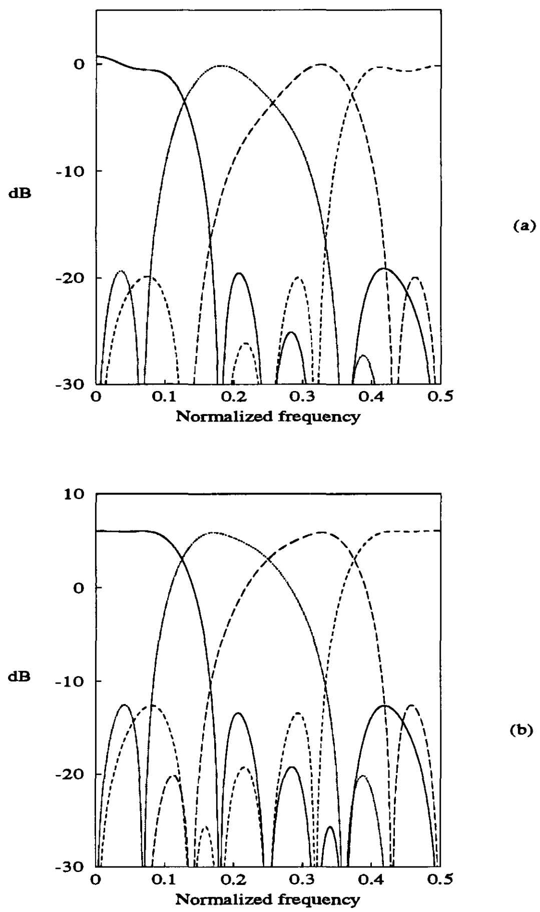


Fig. 2.3.1. Example 2.3.1. Magnitude responses of analysis filters, (a) before orthonormalization, (b) after orthonormalization.

Example 2.3.2. Nonuniform system. We orthonormalized a three-channel filter bank with decimation ratios 4, 4 and 2. The filters that we started with were all FIR with lengths 28, 28 and 10. After the orthonormalization, we got IIR filters with numerator degrees 100, 28, 10 and denominator degrees 33, 25, 9 respectively. Again their magnitude responses [shown in Fig. 2.3.2 (a) and (b)] do not differ much.

The examples show that, while the above procedure is of theoretical interest, the resulting filters are far from being efficient. The main aim of the section is to emphasize the existence of orthonormal systems for nonuniform filter banks where biorthogonal systems exist, and then demonstrate the orthonormalization technique.

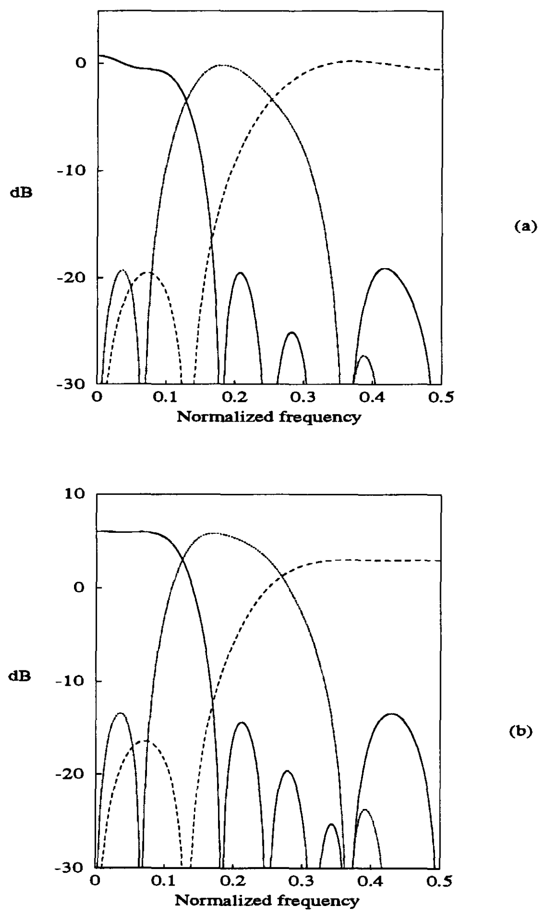


Fig. 2.3.2. Example 2.3.2. Magnitude responses of analysis filters (a) before orthonormalization, (b) after orthonormalization.

2.3.5. Numerical considerations

In actually implementing the orthogonalization algorithm, one faces the problem of decimating IIR transfer functions [(2.3.5), (2.3.8), etc.]. Theoretically, we could expand the rational transfer function into partial fractions, then expand each of them into a power series in z and retain every n_i^{th} term.

This does not yield numerically accurate results. There are several ways to avoid the factorization of polynomials.

The first one is based on a state-space manifestation of the decimation. Namely, if $\mathbf{A}$ is a state transition matrix in some realization of $H(z)$, then $\mathbf{A}^{n_i}$ is a transition matrix of the decimated system $H(z)\downarrow_{n_i}$. Now in order to get the denominator of the decimated system, we need to find the characteristic polynomial of $\mathbf{A}^{n_i}$. Notice that the size of $\mathbf{A}^{n_i}$ grows linearly with the filter order.

Another method, again system theoretical, relies on the fact that a rational transfer function (with no common factors in the numerator and denominator) of order N can be determined from the first $2N+1$ impulse response coefficients (it can be shown that the determinant of the system is nonsingular [Kai80]). The impulse response coefficients of the decimated system can be obtained from the impulse response of the original system, which can be easily obtained from the difference equation described by that transfer function. The problem with this approach is that the matrix of the system of linear equations, even though nonsingular, is typically ill-conditioned.

The third method is based on the frequency manifestation of the decimation. Namely, we know that

$$\left(H(z)\right)\Big\downarrow_{n_i} = (1/n_i)\sum_{k=0}^{n_i-1} H(z^{1/n_i}W^k). \tag{2.3.22}$$

Now if we write $H(z)$ as a ratio $H(z) = N(z)/D(z)$, the denominator of the decimated system can be written as

$$D_d(z) = n_i \prod_{k=0}^{n_i-1} D(z^{1/n_i}W^k). \tag{2.3.23}$$

So in order to get the denominator of the decimated system, we have to find FFT's of the modulated denominators of the original system $D(zW^k)$, multiply them, stretch n_i times (i.e., decimate by n_i) and multiply by n_i. The inverse FFT of the result will give us $D_d(z)$. Notice that here we have a product of polynomials, which is appropriately implemented using the FFT. The critical factor is the number of terms in the product. It depends on n_i only, not on the order of the filter (as opposed to the first method).

After we get the denominator, getting the numerator is easy. We can again use FFT techniques. Calculate the sampled DFT of $H(zW^k)$, $H(e^{j\omega_l}W^k) = \frac{P(e^{j\omega_l}W^k)}{D(e^{j\omega_l}W^k)}$, where ω_l's are the sampling frequencies (frequencies at which DFT is the sampled Fourier transform of the numerator and denominator). We add all of these, divide by n_i, and stretch n_i times, to get the sampled DFT of $H(z)\downarrow_{n_i} = H_d(z)$. Since we already know the DFT of the denominator of this sampled frequency response, we can get the DFT of the numerator. The product of this sampled DFT and the DFT of $D_d(z)$ will give us $P_d(e^{j\omega_l}) = H_d(e^{j\omega_l})D_d(e^{j\omega_l})$. After doing the inverse DFT of this sequence, we get the numerator polynomial $P_d(z)$. One has to pay attention to the number of points of FFT (sampling density of DFT) to avoid aliasing in the frequency.

The third method yielded much better results than the second one, especially for high order filters and large decimation ratios. This method was actually used for producing all the above examples.

2.4. Complete decorrelation of subband signals

In the traditional transform coding, where the polyphase matrices of the corresponding uniform filter bank are just constant unitary matrices (KLT for example [Jay84]), the subband signals $x_i(n)$ are decorrelated for the same time instant. That is, $E\{x_i(n)x_j^*(m)\} = 0$ whenever $i \neq j$ *and* $m = n$. In the other extreme case where the filters are ideal brick wall filters (the polyphase matrix having infinite order) the subband signals are completely uncorrelated, that is $E\{x_i(n)x_j^*(m)\} = 0$ whenever $i \neq j$, for any choice of n, m.

In this section we will consider the problem of complete decorrelation by use of rational (finite order) filters. We will show that the subband signals cannot be decorrelated in this way if we use rational paraunitary filter banks (unless the input signal has severely restricted statistical properties; see below).

2.4.1. Decorrelation with orthonormal filters

Consider a uniform system in which $n_k = M$ for all k (nonuniform systems have subband signals which are not necessarily jointly WSS). Let the filter bank input $x(n)$ be WSS with power spectrum $S_{xx}(z)$, assumed to be a rational function of z. For any scalar input signal $x(n)$ we can form a vector signal $\mathbf{x}(n) = (\,x(nM) \quad x(nM-1) \quad \cdots \quad x(nM-M+1)\,)^T$. This vector signal is the output of the delay chain in Fig. 2.1.2(b) after decimation, and is called the M-fold blocked version of the input signal $x(n)$. It is known [Sat93] that the power spectral matrix of this vector WSS process is pseudo-circulant. Namely

$$\mathbf{S_{xx}}(z) = \begin{pmatrix} S_{xx,0}(z) & S_{xx,1}(z) & \cdots & S_{xx,M-1}(z) \\ z^{-1}S_{xx,M-1}(z) & S_{xx,0}(z) & \cdots & S_{xx,M-2}(z) \\ \vdots & \vdots & \ddots & \vdots \\ z^{-1}S_{xx,1}(z) & z^{-1}S_{xx,2}(z) & \cdots & S_{xx,0}(z) \end{pmatrix}, \tag{2.4.1}$$

where $S_{xx,i}(z)$ is the i^{th} polyphase component of the autocorrelation function $S_{xx}(z)$. After passing

$\mathbf{x}(n)$ through the analysis bank polyphase matrix $\mathbf{E}(z)$, the output signal $\mathbf{y}(n)$ has power spectrum

$$\mathbf{S_{yy}}(z) = \mathbf{E}(z)\mathbf{S_{xx}}(z)\widetilde{\mathbf{E}}(z). \tag{2.4.2}$$

If we want the subband signals to be decorrelated, then $\mathbf{S_{yy}}(z)$ has to be a diagonal matrix. Furthermore, if we use orthonormal systems, the polyphase matrix has to be unitary on the unit circle. Thus, on the unit circle, (2.4.2) can be regarded as a unitary diagonalization of the Hermitian matrix $\mathbf{S_{xx}}(e^{j\omega})$. Now we recall that a pseudo-circulant matrix can be written as [Vai88a]

$$\mathbf{S_{xx}}(z) = \frac{\mathbf{\Gamma}(z)\mathbf{W}}{\sqrt{M}} M\mathbf{D}(z)\frac{\mathbf{W}^{\dagger}\mathbf{\Gamma}^{-1}(z)}{\sqrt{M}}, \tag{2.4.3}$$

where $\mathbf{\Gamma}(z) = \text{diag}(1, z^{-1/M}, z^{-2/M}, \ldots, z^{-\frac{M-1}{M}})$ and $\mathbf{D}$ is a diagonal matrix. Since the matrix $\frac{\mathbf{\Gamma}(z)\mathbf{W}}{\sqrt{M}}$ is unitary on the unit circle, it follows that the diagonal matrix $\mathbf{S_{yy}}(e^{j\omega})$ is identical to the diagonal matrix $M\mathbf{D}(e^{j\omega})$ up to rearrangement of the diagonal elements. Ignoring this rearrangement, we get $\mathbf{S_{yy}}(e^{j\omega}) = M\mathbf{D}(e^{j\omega})$. Now assume that $\mathbf{S_{xx}}(e^{j\omega})$ is rational, and that we wish to diagonalize it with the rational paraunitary matrix $\mathbf{E}(e^{j\omega})$. From (2.4.2) we see that $\mathbf{S_{yy}}(e^{j\omega})$ and, therefore, $\mathbf{D}(e^{j\omega})$ have to be rational. Now Eq. (2.4.3) implies $\mathbf{W}\mathbf{D}(z)\mathbf{W}^{\dagger} = \mathbf{\Gamma}^{-1}(z)\mathbf{S_{xx}}(z)\mathbf{\Gamma}(z)$. Using the pseudo-circulant property of $\mathbf{S_{xx}}(z)$ we conclude that the rationality of $\mathbf{S_{xx}}(z)$ and $\mathbf{D}(z)$ implies that $\mathbf{S_{xx}}(z)$ has the form $C(z)\mathbf{I}$. This means that the power spectrum of the input process $x(n)$ has the form $S_{xx}(z) = C(z^M)$. In other words, the autocorrelation $R(k) = 0$ unless k is a multiple of M.

2.4.2. Biorthogonal decorrelation

Having shown that an orthonormal filter bank cannot in general be used for decorrelation, we will decorrelate the subband signals using a biorthogonal filter bank. Assume again that $x(n)$ is WSS. Then the crosscorrelation between $x_i(n)$ and $x_j(n)$ is

$$r_{ij}(l) = r_{ij}(n,l) = E[x_i(n)x_j^*(n-l)] = \sum_k \sum_m h_i(m)h_j^*(k)r(Ml+k-m). \tag{2.4.4}$$

The subband signals are decorrelated if this is zero for $i \neq j$. Equivalently, in the z-domain,

$$S_{ij}(z) = \left(\widetilde{H}_j(z)H_i(z)S(z)\right)\Big\downarrow_M = 0 \qquad \text{for} \qquad i \neq j, \tag{2.4.5}$$

where $S_{ij}(z)$ is the z-transform of $r_{ij}(l)$.

Given a PR (biorthogonal) system with analysis filters $H_i(z)$, we show how to obtain a new set of analysis filters such that the above holds. For this we apply techniques similar to the ones in

Section 2.3, just simpler. Because the filter bank is uniform, the problem is actually the usual finite-dimensional Gramm-Schmidt orthogonalization. New analysis filters will be $G_i(z)$. Let $G_0(z) = H_0(z)$. Assuming that we have decorrelated $x_i(n)$ and $x_j(n)$ for $j \neq i$ and $0 \leq i, j \leq s-1$, in the s^{th} step we put $G_s(z) = H_s(z) - \sum_{k=0}^{s-1} \beta_{ks}(z^M) G_k(z)$. $x_s(n)$ will not be correlated to any of $x_i(n)$ for $i < s$ if

$$\beta_{ks}(z) = \frac{\left(H_s(z)\widetilde{G}_l(z)S(z)\right)\downarrow_M}{\left(G_k(z)\widetilde{G}_l(z)S(z)\right)\downarrow_M} \quad \text{for} \quad l = 0, 1, \ldots, s-1. \tag{2.4.6}$$

This way we get filters $G_i(z)$'s which decorrelate subband signals. They can be stabilized using the techniques of Sec. 2.3. The corresponding synthesis filters can be obtained by inverting the analysis bank polyphase matrix (stability cannot be guaranteed).

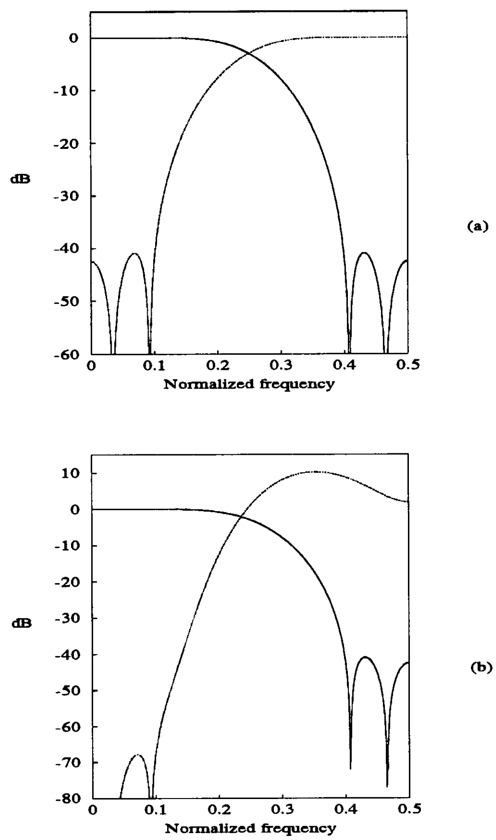


Fig. 2.4.1. Magnitude responses of analysis filters

(a) before decorrelation, (b) after decorrelation.

Example 2.4.1 As an example of the above decorrelation procedure, we take a lowpass AR(6) process [Jay84] and a paraunitary two-channel filter bank [Vai88] with FIR filters of order 7 (filter

8A). Fig. 2.4.1 (a) and (b) show the frequency responses of the original and modified analysis filters that decorrelate subband signals. The resulting analysis filters are FIR of order 7 and 18. The synthesis filters are IIR of orders 18 and 12. If one calculates the coding gain of the new system (see [Jay84] for the definition), it turns out that it is less than that of the original filter bank.

2.5. The compatibility condition, and generalizations

We now present different types of necessary conditions for perfect reconstructability in nonuniform filter banks. These can sometimes be used to quickly reject certain sets of integer decimation ratios $\{n_i\}$ from being considered for perfect reconstruction. In all our discussions we assume maximal decimation, that is $\sum_{i=0}^{M-1} 1/n_i = 1$. The discussions of this section do not apply to the case of the so-called block decimation [Nay93].

2.5.1. Compatibility

In Fig. 2.1.1(a), the reconstructed signal $\widehat{X}(z)$ is given by

$$\widehat{X}(z) = \sum_{k=0}^{M-1} F_k(z) \frac{1}{n_k} \sum_{n=0}^{n_k-1} H_k(zW_{n_k}^n) X(zW_{n_k}^n). \tag{2.5.1}$$

In order for each of the alias terms $X(zW_{n_k}^n), n \neq 0$ to be canceled, it is necessary that for every k and n there exist $\ell \neq k$ and m such that $W_{n_k}^n = W_{n_\ell}^m$. This requirement is called the compatibility condition and is a necessary condition for alias cancelation (see [Hoa89], p. 285 of [Vai93] and the corresponding solution manual for the derivation). If the set of integers $\{n_i\}$ satisfies this, we say that it is a compatible set.

Statement of the Test

Assume that the integers n_i are numbered such that $n_0 \leq n_1 \leq \ldots \leq n_{M-1}$.

Step 1. Check if $n_{M-2} = n_{M-1}$. If *no*, then $\{n_i\}$ is not compatible. If *yes*, continue.

Step 2. Form the smaller set by collecting those n_i that are *not* factors of n_{M-1}. Then imagine that this is the given set, and repeat the test, i.e., go to Step 1.

At some point, if the answer is *no* in Step 1, then the original set is not compatible. If we keep getting *yes* in Step 1, then after a finite number of repetitions, the "smaller set" in Step 2 becomes empty. The original set $\{n_i\}$ is then compatible. Thus, the test always gives a decision in a finite number of steps.

To demonstrate the test, consider the set $(2, 6, 10, 12, 12, 30, 30)$. We have $n_{M-2} = n_{M-1} = 30$ so that Step 1 is successful. The smaller set of numbers that are not factors of 30 is given by $(12, 12)$. This set again passes Step 1 successfully. The next smaller set is empty. Thus the test has been completed, and the given set $(2, 6, 10, 12, 12, 30, 30)$ is indeed compatible. It turns out that this set of integers cannot come from a tree structure (binary or otherwise). For, if it did, then the first level of the tree would have to be a two–channel system with decimators $(2, 2)$. The second level then splits the lower branch of the first level only, with the decimators $(3, 5, 6, 6, 15, 15)$. Since 3 and 5 do not have common factors, this set of numbers could not have come from a tree-structured connection of uniform filter banks.

It turns out that while the above set is compatible, it is still not consistent with another necessary condition for alias cancelation (hence PR). This statement will be elaborated at the end of the next subsection.

2.5.2. Generalizations

We can obtain further necessary conditions by looking deeper into the details of alias cancelation. Thus consider the PR condition, expressed in terms of the L–channel equivalent uniform filter bank (where L is the lcm of n_i's; Sec. 2.1.2); it takes the form

$$\begin{pmatrix} S_0(z) & S_1(z) & \cdots & S_{L-1}(z) \\ S_0(zW_L) & S_1(zW_L) & \cdots & S_{L-1}(zW_L) \\ \vdots & \vdots & \ddots & \vdots \\ S_0(zW_L^{L-1}) & S_1(zW_L^{L-1}) & \cdots & S_{L-1}(zW_L^{L-1}) \end{pmatrix} \begin{pmatrix} Q_0(z) \\ Q_1(z) \\ \vdots \\ Q_{L-1}(z) \end{pmatrix} = \begin{pmatrix} L \\ 0 \\ \vdots \\ 0 \end{pmatrix}. \tag{2.5.2}$$

By substituting for the L pairs of filters $\{S_k(z), Q_k(z)\}$ in terms of the original M pairs filters $\{H_k(z), F_k(z)\}$ using (2.2.8), we can rewrite this in the form

$$\mathbf{H}_L(z)\mathbf{f}(z) = [\,L \quad 0 \quad \cdots \quad 0\,]^T, \tag{2.5.3}$$

where $\mathbf{f}(z) = [\,F_0(z) \quad \ldots \quad F_{M-1}\,]^T$ and $\mathbf{H}_L(z)$ is an $L \times M$ matrix. The i^{th} column of this matrix has the form

$$[\,k_iH_i(z) \quad \mathbf{0} \quad k_iH_i(zW_{n_i}) \quad \mathbf{0} \quad k_iH_i(zW_{n_i}^2) \quad \ldots \quad \mathbf{0} \quad k_iH_i(zW_{n_i}^{n_i-1}) \quad \mathbf{0}\,]^T, \tag{2.5.4}$$

where $\mathbf{0}$ is a string of $k_i - 1$ zeros. (Recall that k_i are integers such that $k_in_i = L$). The compatibility condition says that any nonzero row of $\mathbf{H}_L(z)$ should have at least two nonzero entries.

Now notice that the decimation ratios $n_0, n_1, \dots n_{M-1}$ of the M–channel nonuniform filter bank may not all be distinct. Let us relabel them in terms of distinct integers, for convenience of discussion. Thus let the decimators be

$$\underbrace{n_0, n_0, \dots n_0,}_{N_0 \text{ times}} \underbrace{n_1, n_1, \dots n_1,}_{N_1 \text{ times}} \dots \underbrace{n_{K-1}, n_{K-1}, \dots n_{K-1},}_{N_{K-1} \text{ times}} \tag{2.5.5}$$

In this notation, n_i are distinct integers and $N_0 + N_1 + \dots N_{K-1} = M$.

For example, let us fully understand the 0^{th} column of the matrix $\mathbf{H}_L(z)$. If $N_0 = 2$, then there are two columns (zeroth and first) of the form (2.5.4), with the same decimation ratio n_0 (and the same k_0). More generally, there are N_0 columns of the form (2.5.4) with the same n_0 and the same k_0. These N_0 columns have nonzero entries occurring in the same positions, namely 0^{th}, k_0^{th}, $2k_0^{th}$ and so forth. Consider now another column, say the one corresponding to n_i. This has nonzero elements occurring at the locations 0, k_i, $2k_i$, and so forth. Now compare this with the 0^{th} column, and identify the locations where nonzero elements overlap. With the exception of the 0^{th} location, the first overlap of nonzero elements will occur at the location $\text{lcm}(k_0, k_i)$. Define

$$m_0 = \frac{\min_{i \neq 0} \ \text{lcm}\ (k_0, k_i)}{k_0}. \tag{2.5.6}$$

Then the nonzero elements of the leftmost column in the k_0^{th} $2k_0^{th}$, … $(m_0 - 1)k_0^{th}$ positions do not overlap with any nonzero elements from any other columns, except of course, columns 1, 2, … N_0. We can isolate these nonzero elements in the first N_0 columns of equation (2.5.3), and write

$$\begin{bmatrix} H_0(zW_{n_0}) & H_1(zW_{n_0}) & \dots & H_{N_0-1}(zW_{n_0}) \\ H_0(zW_{n_0}^2) & H_1(zW_{n_0}^2) & \dots & H_{N_0-1}(zW_{n_0}^2) \\ \vdots & \vdots & \ddots & \vdots \\ H_0(zW_{n_0}^{m_0-1}) & H_1(zW_{n_0}^{m_0-1}) & \dots & H_{N_0-1}(zW_{n_0}^{m_0-1}) \end{bmatrix} \underbrace{\begin{bmatrix} F_0(z) \\ F_1(z) \\ \vdots \\ F_{N_0-1}(z) \end{bmatrix}}_{\mathbf{g}(z)} = \mathbf{0}. \tag{2.5.7}$$

A necessary condition. We will now prove that, if the number of rows $m_0 - 1 \geq$ number of columns N_0, then perfect reconstruction is not possible! So the condition

$$m_j - 1 < N_j, \quad 0 \leq j \leq K - 1 \tag{2.5.8}$$

is necessary for perfect reconstruction, where N_j is the number of decimators equal to n_j, and

$$m_j = \frac{\min_{i \neq j} \ \text{lcm}\ (k_i, k_j)}{k_j}. \tag{2.5.9}$$

This is a generalization of the compatibility condition which merely said that any nonzero row of $\mathbf{H}_L(z)$ should have at least two nonzero entries.

Proof of the necessary condition. Eqn. (2.5.7) implies that the columns of the matrix are linearly dependent [unless all the $F_i(z)$'s in that equation are zero, which is not possible in a maximally decimated perfect reconstruction system]. If $m_0 - 1 \geq N_0$, this means that the *rows* are linearly dependent. Denoting the first row of the matrix in (2.5.7) as $\mathbf{h}(zW_{n_0})$, the remaining rows are $\mathbf{h}(zW_{n_0}^2) \ldots \mathbf{h}(zW_{n_0}^{m_0-1})$. The linear dependence implies that

$$\mathbf{h}(zW_{n_0}) = \sum_{i=2}^{m_0-1} \alpha_i(z)\mathbf{h}(zW_{n_0}^i). \tag{2.5.10}$$

Since this holds for all z, we can replace z with $zW_{n_0}^{-1}$ to obtain

$$\mathbf{h}(z) = \sum_{i=1}^{m_0-2} \alpha_{i+1}(zW_{n_0}^{-1})\mathbf{h}(zW_{n_0}^i) = \sum_{i=1}^{m_0-2} \beta_i(z)\mathbf{h}(zW_{n_0}^i). \tag{2.5.11}$$

But eqn. (2.5.7) says that $\mathbf{h}(zW_{n_0}^i)\mathbf{g}(z) = 0$ for $1 \leq i \leq m_0 - 1$. Using this in (2.5.11) we conclude $\mathbf{h}(z)\mathbf{g}(z) = 0$. That is,

$$H_0(z)F_0(z) + H_1(z)F_1(z) + \ldots + H_{N_0-1}(z)F_{N_0-1}(z) = 0. \tag{2.5.12}$$

But this cannot happen in a maximally decimated perfect reconstruction system. To see this note that the biorthogonality condition (2.2.5) implies, in particular,

$$\Big(H_0(z)F_0(z) + H_1(z)F_1(z) + \ldots + H_{N_0-1}(z)F_{N_0-1}(z)\Big)\Big|_{\downarrow n_0} = N_0, \tag{2.5.13}$$

which is not possible if (2.5.12) is true! This completes the proof of (2.5.8) for $j = 0$. The same argument can be used to show that (2.5.8) is true for $j = 1, 2, \ldots, K-1$ as well.

$\diamond$

This test is strictly stronger than the test for compatibility. To demonstrate, consider the same set $(2, 6, 10, 12, 12, 30, 30)$ from the end of the previous subsection. As shown there, it satisfies the compatibility condition. According to the notation in this subsection, n_i's are distinct numbers and we have $K = 5$, $L = 60$ and

i	0	1	2	3	4
N_i	2	2	1	1	1
n_i	30	12	10	6	2
k_i	2	5	6	10	30
m_i	3	2	1	1	1

Since $m_0 - 1 = 2 = N_0$, we conclude that PR is not possible.

2.6. Conclusion

For a maximally decimated nonuniform filter bank, the perfect reconstruction (PR) property is equivalent to biorthogonality. Using this fact we derived a number of properties of PR filter banks. We then showed that whenever the decimation ratios are such that biorthogonality is possible, it is in particular possible to obtain orthonormality. This was done by developing an orthonormalization procedure. While reminiscent of the Gram-Schmidt approach, the procedure converges in a finite number of steps and furthermore preserves the filter bank-like form of the basis functions. We applied this technique to decorrelate the subband signals. Finally we considered the problem of alias cancelation, and obtained a generalization of the so-called compatibility condition which is a necessary condition for perfect reconstruction in maximally decimated systems.

2.7 Appendices

Appendix 2.A: Reaching arbitrary integers

In connection with equation (2.2.15), we will show that the quantity $pb_{lm} - rb_{ml} + dk_m b_{ml}$ can be made to take any integer value by proper choice of the integer d, and the integers p, r in the ranges $0 \leq p \leq k_l - 1$, $0 \leq r \leq k_m - 1$. For this recall the meanings of the integers b_{ml}, b_{lm}, and L, namely, eqns. (2.1.4) and (2.2.13). Since $L = k_l n_l = k_m n_m$ by definition, we have $k_l b_{lm} g_{lm} = k_m b_{ml} g_{lm}$. So $k_l b_{lm} = k_m b_{ml}$. Since b_{lm} and b_{ml} are relatively prime by construction, there exist integers $\widehat{p}$ and $\widehat{r}$ such that

$$\widehat{p}b_{lm} - \widehat{r}b_{ml} = \text{ any desired integer } a. \tag{2.A.1}$$

We can always decompose $\widehat{p}$ and $\widehat{r}$ as $\widehat{p} = p + nk_l$ and $\widehat{r} = r + ik_m$, where $0 \leq p \leq k_l - 1$, and $0 \leq r \leq k_m - 1$. Substituting this into (2.A.1) and rearranging, we get

$$pb_{lm} - rb_{ml} + dk_m b_{ml} = a, \tag{2.A.2}$$

where $d = (n - i)$. Thus, we can write any integer a as above where p and r are in the stated range, provided we can assign any integer value to d.

Appendix 2.B: Eliminating redundant variables

If the system of equations (2.3.14) has a determinant that is identically zero, we can reduce the size of the problem as follows. In this case, there exists $a_{il}(z)$, with at least one of them different from zero, such that

$$\sum_{i=0}^{s-1}\sum_{m=0}^{c_{si}-1} a_{il}(z)\left(z^{mg_{sk}-lg_{si}}F_i(z)\widetilde{G}_k(z)\right)\downarrow_L = 0, \tag{2.B.1}$$

for $i = 0, 1, \ldots, s-1$ and $l = 0, 1, \ldots, c_{sk}-1$. While it is not obvious that there are polynomials $a_{il}(z)$ satisfying (2.B.1), this can be verified to be the case, by use of the Smith-McMillan decomposition for rational functions [Kai80], [Gan59]. The previous equation can be rewritten as

$$\left(z^{mg_{sk}}\widetilde{G}_k(z)\sum_{k=0}^{s-1}\sum_{m=0}^{c_{si}-1} a_{il}(z^L)z^{-lg_{si}}F_i(z)\right)\Bigg\downarrow_L = 0 \qquad \text{for all } \begin{cases} i = 0, 1, \ldots, s-1 \\ l = 0, 1, \ldots, c_{sk}-1 \end{cases}. \tag{2.B.2}$$

This is equivalent to

$$\left(\widetilde{G}_k(z)\sum_{i=0}^{s-1} A_i(z^{g_{si}})F_i(z)\right)\Bigg\downarrow_{g_{sk}} = 0 \qquad \text{for all} \quad k = 0, 1, \ldots, s-1, \tag{2.B.3}$$

where $A_i(z) = \sum_{l=0}^{c_{si}-1} z^{-l}a_{il}(z^{c_{si}})$. Now, (2.B.3) implies that the assumed form (2.3.12) is redundant; namely, we can discard some of $\beta_{isl}(z^L)$'s. To see this, let $a_{lJ}(z) \not\equiv 0$. Then (2.B.3) implies

$$\beta_{lsJ}(z^L)z^{-Jg_{sl}}F_l(z) + \frac{\beta_{lsJ}(z^L)}{a_{lJ}(z^L)}\sum_{i=0}^{s-1}\sum_{\substack{m=0\\ m\neq J,\ \text{for } i=l}}^{c_{si}-1} \widetilde{a}_{lm}(z^L)z^{-mg_{si}}F_i(z) \tag{2.B.4}$$

is orthogonal to $G_k(z)$ for $k = 0, 1, \ldots, s-1$. Then we can drop $\beta_{lsJ}(z^L)z^{-Jg_{sl}}F_l(z)$ from (2.3.12) and form a smaller system of linear equations. We keep doing this till the determinant is not identically zero.

Appendix 2.C: Poles on the unit circle

We will show that when a biorthogonal filter bank is orthonormalized, the resulting filters will naturally be free from poles on the unit circle. We will do this in two parts.

Observation 1.

Let $A(z)$ be a rational function with a pole on the unit circle, at $z_0 = e^{j\omega_0}$. Let r_a be the order of this pole. Then, in the neighborhood of z_0, the function $\widetilde{A}(e^{j\omega})A(e^{j\omega})$ behaves as

$$\widetilde{A}(e^{j\omega})A(e^{j\omega}) \sim \frac{c_a}{(e^{j\omega}-e^{j\omega_0})^{r_a}(e^{-j\omega}-e^{-j\omega_0})^{r_a}}. \tag{2.C.1}$$

This is the behavior of a pole of order $2r_a$. Since $\widetilde{A}(z)A(z) \geq 0$ on the unit circle, we have $c_a > 0$. Now let $B(z)$ be another rational function with a possible pole at the same point z_0, with order r_b. Then $\widetilde{B}(e^{j\omega})B(e^{j\omega})$ can be expressed in a similar way. So

$$\widetilde{A}(e^{j\omega})A(e^{j\omega}) + \widetilde{B}(e^{j\omega})B(e^{j\omega}) \sim \frac{c_a}{(e^{j\omega}-e^{j\omega_0})^{r_a}(e^{-j\omega}-e^{-j\omega_0})^{r_a}} + \frac{c_b}{(e^{j\omega}-e^{j\omega_0})^{r_b}(e^{-j\omega}-e^{-j\omega_0})^{r_b}}. \tag{2.C.2}$$

Since $c_a, c_b > 0$, we see that there can be no cancelations, and as ω approaches ω_0, the result behaves like a pole of order $= \max(2r_a, 2r_b)$. Similarly, if we have a sum of several nonnegative functions having poles of various orders on the unit circle, the sum behaves like a pole of order equal to the largest one.

A consequence of observation 1. Now consider the normalization step (2.3.21). The denominator of $\alpha_s(z^{n_s})\widetilde{\alpha}_s(z^{n_s})$ can be written as

$$\left(\left(P_s(z)\widetilde{P}_s(z)\right)\downarrow_{n_s}\right)\uparrow_{n_s} = \frac{1}{n_s}\sum_{k=0}^{n_s-1} P_s(zW_{n_s}^k)\widetilde{P}_s(zW_{n_s}^k). \tag{2.C.3}$$

Each term on the right-hand side is nonnegative on the unit circle. So if $P_s(z)$ has a pole of order r at $z_0 = e^{j\omega_0}$, then the above summation still has this pole, with order $\geq 2r$. As a result, $\alpha_s(z^{n_s})$ has a *zero* of order $\geq r$. This means that when we form the normalized filter $G_s(z) = \alpha_s(z^{n_s})P_s(z)$, this unit-circle pole will be completely canceled.

Observation 2.

From (2.3.21) we see that $\alpha_s(z^{n_s})$ will have a pole of order r at $z_0 = e^{j\omega_0}$ if and only if

$$\left(\left(P_s(z)\widetilde{P}_s(z)\right)\downarrow_{n_s}\right)\uparrow_{n_s}$$

has a zero of order $2r$ at $z = z_0 = e^{j\omega_0}$. Now consider (2.C.3). Each term in this summation is nonnegative. Suppose the function $P_s(zW_{n_s}^k)$ has a zero of order r_k at z_0. Then

$$P_s(e^{j\omega}W_{n_s}^k)\widetilde{P}_s(e^{j\omega}W_{n_s}^k) = (e^{j\omega} - e^{j\omega_0})^{r_k}(e^{-j\omega} - e^{-j\omega_0})^{r_k} \times \text{(nonnegative function)} \tag{2.C.4}$$

on the unit circle. If the summation in (2.C.3) has the factor $(e^{j\omega} - e^{j\omega_0})^r(e^{-j\omega} - e^{-j\omega_0})^r$, it is therefore necessary that $r_k \geq r$ for each k. That is, each of the quantities $P_s(zW^k)$ has to have a zero of order $\geq r$ at z_0. In particular, therefore, $P_s(z)$ has a zero of order $\geq r$ at z_0.

The conclusion is that if $\alpha_s(z^{n_s})$ has a pole of order r at $z_0 = e^{j\omega_0}$, then $P_s(z)$ has a zero of order at least r at z_0.

3

Statistical Wavelet and Filter Bank Optimization

3.1. Introduction

In [Dau88], Daubechies showed how orthonormal, compactly supported wavelets can be constructed using paraunitary filter banks. They gave another sufficient, but not necessary, condition for the orthonormality of wavelets generated from paraunitary filter banks. The equivalence of paraunitary filter banks and wavelet tight frames was established by Lawton in [Law90]. Finally, two *necessary and sufficient* conditions were given by Lawton and Cohen in [Law91] and [Coh90]. It was clear that the relationship between filter banks and wavelets could be exploited for wavelet optimizations [Zou93], [Pol90], [Vai88], [Tew92], and [Ode92]. In this chapter, we will deal with three different, but somewhat related problems. The first two are concerned with errors in the wavelet decompositions of random signals. The third one is the filter bank optimization for a higher coding gain. The notion of multiresolution analysis was introduced in Ch. 1. Let us now explain the problems we will analyze in this chapter.

Problem 1.

Let $P_m f$ and $Q_m f$ be the projections of $f \in L^2$ onto V_m and W_m respectively. Then the property 3. of MRA (given in Def. 1.2.1) says that $\lim_{m\to\infty} P_m f = f$. The error made by this approximation at the m^{th} level is $f - P_m f = \sum_{k=m}^{\infty} Q_k f$. We can see that $Q_m f$ provides the additional information needed to go from the approximation $P_m f$ to the finer $P_{m+1} f$. If we want to approximate an input signal f with its projection at scale m, then the error is obviously a function of the signal itself and the analyzing scaling function $\phi(x)$. So we can try to choose a scaling function $\phi(x)$ which will make some measure of the error smaller.

The relationship between filter banks and wavelets provides a nice parameterization of the wavelets that can be constructed using filter banks [Zou93], [Pol90], [Vai88]. Using those parameterizations, we can optimize wavelets according to some criteria (smaller error in our case). In [Tew92], the authors tried to tune the wavelets to a given input signal. They derive upper bounds on MSE and try to minimize these bounds, rather than to minimize the MSE itself. Similar problems were analyzed in [Ode92]. Explicit formulae for different norms of errors as functions of the input signal and the analyzing scaling function were derived.

This setting is useful when we know the exact signal we want to represent with wavelets and when we are willing to optimize $\phi(x)$ for every particular input. Sometimes all we know about the input signal are its statistical properties, in which case we want one $\phi(x)$ to be optimal for a *class* of signals. For example, the input can be some wide sense stationary (WSS) random process with a given power spectrum. Then it may be desirable to tune the wavelets to this class of signals with a given power spectrum so as to minimize some appropriately chosen measure of the error. Our first problem is to extend the ideas and results of [Ode92] to WSS random inputs. Statistical aspects of wavelet analysis have also been considered in [Gol93].

Problem 2.

Filter banks offer a very efficient way of getting projections $P_m f$ and $Q_m f$ onto any V_m and W_m for $m < J$, once $P_J f$ is known. So the major computational burden comes from calculating the inner products $\langle f, \phi_{Jk} \rangle$. In order to avoid it, Mallat suggested to approximate $\langle f, \phi_{Jk} \rangle$ with $2^{-J/2} f(2^{-J}k)$, a scaled sample of the input signal (see [Dau92]). A method for the reduction of this type of errors was proposed in [She92]. The idea is to pre-filter samples of the input signal by a digital filter h_n, so that the output $\sum_k h_{n-k} 2^{-J/2} f(2^{-J}k)$ is a better approximation of $\langle f, \phi_{Jn} \rangle$ than $2^{-J/2} f(2^{-J}n)$. A formula for the MSE as a function of the input signal, analyzing scaling function and the pre-filter, was derived in [Xia93]. It was observed in [Xia93] that it may be more appropriate to optimize wavelets to some class of signals. A natural setting for this would be to take into account statistics of the input (assumed WSS random process). So our second problem is to extend the results of [Xia93] to this case.

Problem 3.

One of the main applications of filter banks is data compression using subband coding. Consider a uniform filter bank as shown in Fig. 3.1.1. The input signal $x(n)$ is passed through the analysis filters $H_k(z)$. The outputs of those filters after the decimation are the subband signals $x_k(n)$. Those subband signals are quantized, and errors are caused by this quantization. If we wanted to reduce the quantization error in filter banks with some method similar to the above techniques for wavelet optimizations, we should try to tune frequency responses of the filters to a particular signal or a class of signals. So far, most authors mainly tried to approximate ideal brick-wall filters as much as possible. In [Som93a] and [Aka91] statistical properties of the signals being processed were taken into account.

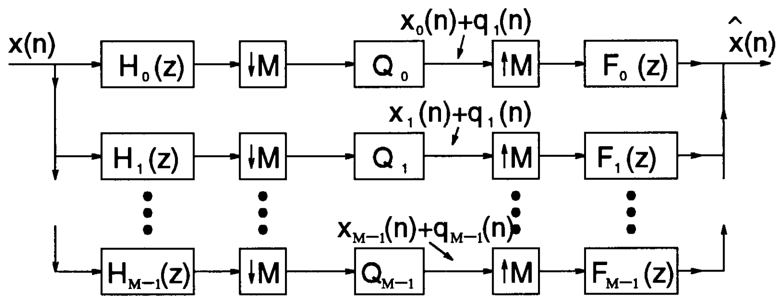


Fig. 3.1.1. A uniform filter bank used for subband coding.

In [Som93a] authors did a direct optimization of the coding gain for an arbitrary M-channel uniform, paraunitary filter bank. The problem with this approach is that the restriction to paraunitary filter banks prevents us from achieving a bigger coding gain, as we will demonstrate later. Similar approach, just for the case of two-channel filter banks, was taken in [Aka91]. In this chapter we use the idea of pre-filtering to improve the coding gain of any paraunitary filter bank. Namely, we put a pre-filter before and a post-filter after some paraunitary filter bank (FB). We then show how to choose the pre- and post-filter so as to improve the coding gain. This way we end up with an IIR stable biorthonormal filter bank which has much better coding gain than the original FIR orthonormal FB. The output noise is not white as in the case of a PU FB. The pre-filter is actually the filter used for half-whitening [Jay84].

We have to mention that the MSE is used in this chapter because of its mathematical tractability. It is not the best measure of the error, but it is some indication of the error and it is commonly used.

3.1.1. Chapter outline

In Sec. 3.2 we deal with the first problem, namely, we extend the ideas and results of [Ode92] to random inputs. The resulting MSE has a form similar to the one in [Ode92], but the derivation has to be modified for proper handling of random signals. As we said, we will consider the case of orthonormal wavelets only. So, the projection of any $f \in L^2$ onto V_m is

$$P_m f(x) = \sum_{k=-\infty}^{\infty} \langle f, \phi_{mk} \rangle \phi_{mk}(x). \tag{3.1.1}$$

There is a subtle technical problem with this setting. Namely, we require that the input random process is WSS, which is contradictory to the requirement that its ensemble members $f(t)$ belong to L^2. However, if we constrain $\phi(x)$ to be compactly supported, $E|\langle f, \phi_{mk} \rangle| < \infty$ (E denotes the statistical expectation of a random variable). For the same reasons, we have no convergence problems in (3.1.1). We will find an expression for the error variance when f is approximated by $P_J f$. The assumptions we need to make in Sec. 3.2 will be explicitly stated at the beginning of the section.

The second problem is dealt with in Sec. 3.3. We derive an expression for the error variance when a WSS random process is being analyzed by wavelets and inner products in (3.1.1) are approximated by $\sum_n h_{k-n} 2^{-J/2} f(2^{-J} n)$, where h_n are filter coefficients of the pre-filter. Then, this pre-filter can be optimized so as to minimize the error variance. At the beginning of the section, we state the assumptions we use in our derivations. They may not be necessary, but they make the derivations simpler and they do not severely restrict the scope of the results anyway. We calculate the autocorrelation function, the power spectrum of the error and then integrate it to get the variance.

In Sec. 3.4, we first derive a coding gain formula for a uniform biorthonormal filter bank. The extension to nonuniform filter banks with integer decimation ratios is easy. We then use the Cauchy-Schwarz inequality to increase the coding gain with a pre-filter. A very efficient implementation scheme is proposed.

At the end of each section we present some examples to illustrate the theory. These examples show that Daubechies' wavelets are nearly optimal in the sense of Sec. 3.2, due to the maximal regularity of the corresponding scaling functions. Examples for the third section show that an FIR pre-filter of fairly low order can reduce the MSE by about 14 dB. The examples for Sec. 3.4 show that we can get a significant improvement in the coding gain over any paraunitary filter bank with just two more IIR filters of *low orders.* The last two examples show that there are finite order systems which have ***higher coding gains than the brick-wall filter banks*** with the same number of channels. This also gives a technique for the design of stable IIR biorthonormal filter banks which optimize the coding gain.

3.2. Statistical wavelet optimization

In this section we consider the wavelet decomposition of a class of signals. Our aim is to find an expression for the error variance when the input $f(t)$ is approximated by its projection $P_J f(t)$. The meaning of projection is now slightly different from the one in Sec. 3.1. Since $f(t)$ is not in L^2 (as an ensemble member of a WSS random process), $P_J f$ is not a projection onto V_J, because $P_J f \notin L^2$ with probability one (i.e., for almost every ensemble member f). However, because $\phi(x)$ is compactly supported, (3.1.1) is locally well defined, namely $f(t)$ and $P_J f(t)$ are in L^2_{loc} with probability one. Then, for any finite $I \subset \mathcal{R}$ $E[\lim_{J \to \infty} ||f - \sum_k < f, \phi_{Jk} > \phi_{Jk}||_{2,I}] = 0$. We consider the case $M = 2$ only, because an extension of the result for $M > 2$ is routine. First, we derive an expression for the autocorrelation function of the error $r_e(t, \tau)$ and show that it is actually a $(\mathrm{CWSS})_T$ random process. After averaging over the period T, we get the variance as the autocorrelation function evaluated at $\tau = 0$.

Assumptions

1. The scaling functions $\phi(t)$ are assumed orthonormal (its integer shifts), compactly supported and in $L^1 \cap L^2$. These are very weak constraints on compactly supported wavelets generated by paraunitary filter banks (see [Dau92]).
2. We will assume that the input random process $f(t)$ is WSS with a given power spectrum $S(\omega)$. We also assume that $f(t)$ has finite mean and variance.
3. We know that $S(\omega) \geq 0$ and that $S(\omega) \in L^1$, but we will also assume that $S(\omega) \in L^2$, so that $R(\tau) \in L^2$ as well.

3.2.1. Derivation of error variance

First, we will elaborate on the meaning of "projection onto V_J." The random variables

$$a_{Jk} = 2^{J/2} \int f(t) \phi^*(2^J t - k) dt \tag{3.2.1}$$

have finite mean and variance. Then, since $\phi(t)$ are compactly supported, for each $t \in I \subset \mathcal{R}$, there are only finitely many nonzero terms in

$$P_J f(t) = 2^{J/2} \sum_k a_{Jk} \phi(2^J t - k). \tag{3.2.2}$$

Then we have $E\left[\lim_{J\to\infty} |f(t) - 2^{J/2} \sum_k a_{Jk}\phi(2^J t - k)|^2\right] = 0$, i.e., we have mean square convergence. We want to get some measure of the error made when J is finite. For this, we form an error function in the following way

$$e(t) = f(t) - P_J f(t) = f(t) - 2^{J/2} \sum_k a_{Jk} \phi(2^J t - k). \tag{3.2.3}$$

This is a CWSS (see below) random process, and because of the mathematical tractability, we will work with MSE $\sigma_e^2 = E|e(t)|^2$. The autocorrelation function of $e(t)$ is $E\left[e(t)e^*(t-\tau)\right]$. To simplify this, we need to substitute from (3.2.3), with a_{Jk} given by (3.2.1). Thus

$$\begin{aligned} r_e(t,\tau) &= E\left[e(t)e^*(t-\tau)\right] \\ &= E[f(t)f^*(t-\tau)] \\ &- 2^J \sum_k E\left[\left(f(t)\int f^*(x)\phi(2^J x - k)dx\right)\right]\phi^*\left(2^J(t-\tau)-k\right) \\ &- 2^J \sum_k E\left[f^*(t-\tau)\left(\int f(x)\phi^*(2^J x - k)dx\right)\right]\phi(2^J t - k) \\ &+ 2^{2J}\sum_k\sum_l E\left[\int\int f(x)f^*(y)\phi^*(2^J x - k)\phi(2^J y - l)dxdy\right] \\ &\times \phi(2^J t - k)\phi^*\left(2^J(t-\tau)-l\right). \end{aligned} \tag{3.2.4}$$

The expectations can be moved under the integrals (because $E[|f(x)|^2]$ is finite $\forall x \in \mathcal{R}$ and the Fubini's theorem [Rud87] can be applied) to get

$$\begin{aligned} r_e(t,\tau) &= E\left[e(t)e^*(t-\tau)\right] \\ &= R(\tau) \\ &- 2^J \sum_k \left(\int R(t-x)\phi(2^J x - k)dx\right)\phi^*\left(2^J(t-\tau)-k\right) \\ &- 2^J \sum_k \left(\int R(x-t+\tau)\phi^*(2^J x - k)dx\right)\phi(2^J t - k) \\ &+ 2^{2J}\sum_k\sum_l \left(\int\int R(x-y)\phi^*(2^J x - k)\phi(2^J y - l)dxdy\right) \\ &\times \phi(2^J t - k)\phi^*\left(2^J(t-\tau)-l\right), \end{aligned} \tag{3.2.5}$$

where $R(\tau)$ is the autocorrelation function of the input signal, i.e., $R(\tau) = E[f(t)f^*(t-\tau)]$.

Lemma 3.2.1: $e(t)$ as defined in (3.2.3) is a $(\text{CWSS})_T$ with $T = 2^{-J}$.

$\diamond$

Proof: See Appendix 3.A.

Now we can average $r_e(t,\tau)$ over the period $T = 2^{-J}$ to get

$$R_e(\tau) = \frac{1}{T}\int_0^T r_e(t,\tau)dt. \tag{3.2.6}$$

Putting all this together, we get

$$\begin{aligned} R_e(\tau) &= R(\tau) \\ &- \underbrace{2^{2J}\int_0^{2^{-J}} \sum_k \left(\int_{-\infty}^{\infty} R(t-x)\phi(2^J x - k)dx\right)\phi^*(2^J(t-\tau)-k)dt}_{R_{e2}(\tau)} \\ &- \underbrace{2^{2J}\int_0^{2^{-J}} \sum_k \left(\int R(x-t+\tau)\phi^*(2^J x - k)dx\right)\phi(2^J t - k)dt}_{R_{e3}(\tau)} \\ &+ \underbrace{2^{3J}\int_0^{2^{-J}} \sum_{k,l} \left(\int\int R(x-y)\phi^*(2^J x - k)\phi(2^J y - l)dxdy\right)\phi(2^J t - k)\phi^*(2^J(t-\tau)-l)dt}_{R_{e4}(\tau)}. \end{aligned} \tag{3.2.7}$$

The variance is $\sigma_e^2 = R(0)$. It is shown in Appendix 3.B that

$$R_{e2}(0) = R_{e3}(0) = \frac{2^{-2J}}{2\pi}\int S(\omega)|\Phi(2^{-J}\omega)|^2 d\omega \tag{3.2.8}$$

and

$$R_{e4}(0) = \frac{2^{-3J}}{2\pi}\int S(\omega)|\Phi(2^{-J}\omega)|^2 d\omega. \tag{3.2.9}$$

Putting (3.2.8) and (3.2.9) together, we get

$$\sigma_e^2 = \frac{1}{2\pi}\int_{-\infty}^{\infty} S(\omega)\left(1 - |\Phi(2^{-J}\omega)|^2\right)d\omega. \tag{3.2.10}$$

Notice that we have not assumed bandlimitedness of $f(t)$ (unlike in [Ode92]).

3.2.2. Discussion and an example

From the simplified expression (3.2.10) it is easy to see the energy distribution of the error and how it depends on the input power spectrum and the Fourier transform of the scaling function. Notice that because $\Phi(\omega) = \prod_{k=1}^{\infty}\frac{1}{\sqrt{2}}H(e^{j\omega 2^{-k}})$ and $H(z)$ is orthonormal, $|\Phi(\omega)| \leq 1\ \forall\omega \in \mathcal{R}$. Ideally, we would like to have $|\Phi(\omega)| = 1\ \forall\omega$, so that $\sigma_e^2 = 0$. This is not possible because $||\Phi||_2 = 1$.

Then the next best thing is to have $\Phi(\omega)$ very close to 1 in the region where $S(\omega)$ has the most energy. This is also difficult to achieve unless $S(\omega)$ is lowpass, because $H(1) = 1$ implies $\Phi(0) = 1$ (and $H(1) = 1$ is a necessary condition, see [Dau92]). These are contradictory requirements on $\phi(x)$.

Example 3.2.1. We first obtain the PU lattice for the Daubechies' ${}_5\phi$ scaling function (see [Dau92] and [Vai93], Chapter 6). Then we re-optimize the lattice coefficients so as to minimize (3.2.10) under the constraint $H(1) = 1$. Fig. 3.2.1 shows the power spectrum of the input random process (in this case a continuous version of the bandpass AR(6) model of speech [Jay84]). The solid line shows the original spectrum, while the dotted one shows the spectrum of the signal approximated with ${}_5\phi$.

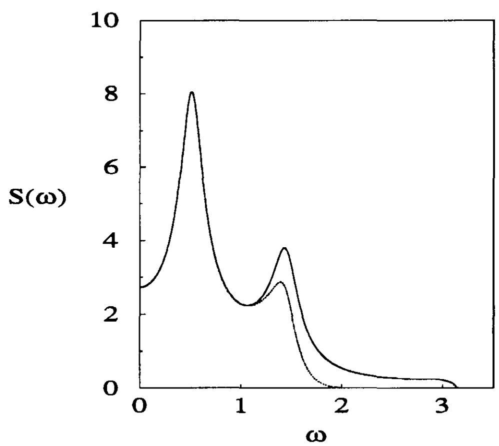


Fig. 3.2.1. Example 3.2.1. A continuous version of the AR(6) model of speech (solid) and its approximation at the finest resolution level (dotted).

Fig. 3.2.2 shows frequency responses of the lowpass filters which define scaling functions (the dotted one is the Daubechies' filter $H_{D5}(z)$ corresponding to ${}_5\phi$, and the solid one is the optimized filter $H(z)$).

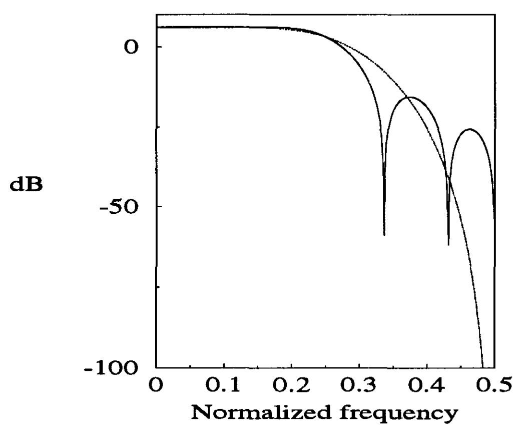


Fig. 3.2.2. Example 3.2.1. Frequency responses of $H_{D5}(z)$ (dotted) and the optimized filter (solid).

From these responses, we can conclude that the Mallat's sufficient condition for the orthonormality of the resulting scaling function is satisfied. Namely, $H(e^{j\omega})$ has no zeros for $|\omega| < \pi/2$, which guarantees that the scaling function given by $\Phi(\omega) = \prod_{k=1}^{\infty} \frac{1}{\sqrt{2}} H(e^{j\omega 2^{-k}})$ generates an orthonormal multiresolution analysis [Mal89].

Finally Fig. 3.2.3 shows the scaling function before (dotted) and after the optimization (solid). We can see that the price paid for an MSE reduction is a worse regularity of the corresponding scaling function. The MSE is now reduced by 10% only. This shows that the Daubechies' wavelets are nearly optimal for representation of lowpass random processes, since they have maximal flatness at $\omega = 0$. All this is intuitively clear from (3.2.10).

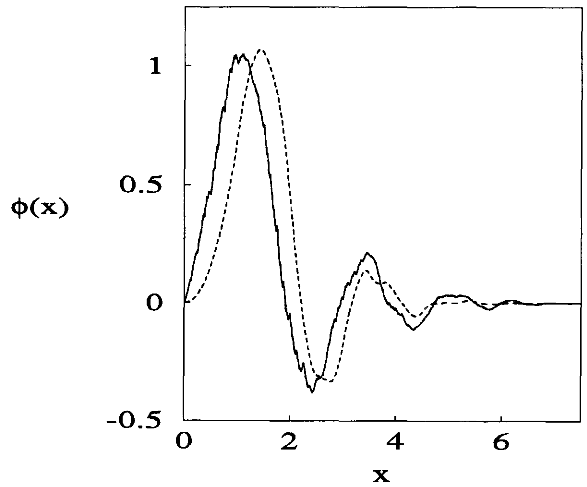


Fig. 3.2.3. Example 3.2.1. Daubechies' ${}_5\phi(x)$ (dotted) and the optimized (solid) scaling functions.

3.3. Statistical pre-filter optimization

As stated in the introduction, we now extend the ideas and results of [She92] and [Xia93] to WSS random inputs. First, we form an error sequence, obtain its autocorrelation, and check that this error is actually a WSS random process. We then obtain its power spectrum and, finally, the error variance.

Assumptions

1. Wavelets are assumed orthonormal, compactly supported and from $L^1 \cap L^2$.
2. We again assume that the input $f(t)$ is a WSS random process with power spectrum $S(\omega)$.

3. This process will be sampled, so we have to assume its bandlimitedness. Since we want to approximate $f(t)$ with $P_J f(t)$, let then $S(\omega)$ be bandlimited to $[-2^J\pi, 2^J\pi]$. In order to avoid the aliasing, we have to sample the input at least with frequency $2^{J+1}\pi$.

4. We will also assume that there exists a real constant C such that $|R(\tau)| < C/|\tau|^{1+\alpha}$ (for some $\alpha > 0$) and that the pre-filter is BIBO stable, i.e., $\sum_n |h_n| < \infty$. These assumptions are sufficient to enable us to use the Poisson's summation formula later in the derivation (see Theorem 2.25. on p. 45 [Chu92]).

3.3.1. Derivation

Let $f_n = f(2^{-J}n)$. The coefficients in the expansion (3.2.2)

$$a_{Jk} = 2^{J/2} \int f(t)\phi^*(2^J t - k)dt, \tag{3.3.1}$$

form a discrete WSS (see below) random process a_{Jk}. We want to approximate a_{Jk} with filtered samples of the input, namely with

$$\widehat{a}_{Jk} = 2^{-J/2} \sum_l h_{k-l} f_l, \tag{3.3.2}$$

where h_n are filter coefficients of the filter that we want to design. The idea is to optimize the filter coefficients h_n so as to minimize some cost function. The reason for inserting the scale factor $2^{-J/2}$ in the above formula is

$$2^J \int f(t)\phi^*(2^J t - k)dt \to f(2^{-J}k) \qquad \text{for} \quad J \to \infty, \tag{3.3.3}$$

so that

$$2^{J/2} \int f(t)\phi^*(2^J t - k)dt \to 2^{-J/2} f(2^{-J}k) \tag{3.3.4}$$

if $f(x)$ is continuous at $2^{-J}k$. We want to find some measure of how good the approximation in (3.3.2) is. The approach is similar to the one in Sec. 3.2, with the difference that we deal with discrete time random processes now. First, we form an error sequence $e_k = a_{Jk} - \widehat{a}_{Jk}$. This is a WSS random process (as will be shown later). Again, because of its tractability, we will examine the variance of this sequence $\sigma_e^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} S_e(e^{j\omega})d\omega$, where $S_e(e^{j\omega})$ is the Fourier transform of the autocorrelation sequence of e_k. Let us start with the autocorrelation sequence

$$\begin{aligned}\epsilon(k,n) &= E[e_k e^*_{k-n}] \\ &= 2^J \int\int E\left[f(x)f^*(y)\right]\phi^*(2^Jx-k)\phi(2^Jy-k+n)dxdy \\ &- \int\sum_l E\left[f(x)f^*(2^{-J}l)\right]h^*_{k-n-l}\phi^*(2^Jx-k)dx \\ &- \int\sum_l E\left[f^*(x)f(2^{-J}l)\right]h_{k-l}\phi(2^Jx-k+n)dx \\ &+ 2^{-J}\sum_{l,m} E\left[f(2^{-J}l)f^*(2^{-J}m)\right]h_{k-l}h^*_{k-n-m}.\end{aligned} \tag{3.3.5}$$

Let $R(\tau) = E[f(t)f^*(t-\tau)]$ be the autocorrelation function of the input signal. The above expression can be written as

$$\begin{aligned}\epsilon(k,n) = 2^J &\int\int R(x-y)\phi^*(2^Jx-k)\phi(2^Jy-k+n)dxdy \\ &- \int\sum_l R(x-2^{-J}l)h^*_{k-n-l}\phi^*(2^Jx-k)dx \\ &- \int\sum_l R(2^{-J}l-x)h_{k-l}\phi(2^Jx-k+n)dx \\ &+ 2^{-J}\sum_{l,m} R(2^{-J}(l-m))h_{k-l}h^*_{k-n-m}.\end{aligned} \tag{3.3.6}$$

If we change indices of summations, we have

$$\begin{aligned}\epsilon_n &= \epsilon(k,n) = \epsilon(0,n) \\ &= 2^J\int\int R(x-y)\phi^*(2^Jx)\phi(2^Jy+n)dxdy \\ &- \int\sum_l R(x-2^{-J}l)h^*_{-n-l}\phi^*(2^Jx)dx \\ &- \int\sum_l R(2^{-J}l-x)h_{-l}\phi(2^Jx+n)dx \\ &+ 2^{-J}\sum_{l,m} R(2^{-J}(l-m))h_{-l}h^*_{-n-m}.\end{aligned} \tag{3.3.7}$$

We see that $\epsilon(k,n)$ does not depend on k at all, so e_n is a WSS process. The power spectrum of the error is

$$S_\epsilon(e^{j\omega}) = \sum_n \epsilon_n e^{-jn\omega}. \tag{3.3.8}$$

Technical details of the derivation are moved to the Appendix 3.C, where it is shown that

$$S_e(e^{j\omega}) = \sum_n S(2^J\omega + 2\pi n 2^J)\left[|\Phi(\omega+2\pi n)|^2 - 2\Re\left(H(e^{j\omega})\Phi(\omega+2\pi n)\right) + |H(e^{j\omega})|^2\right]. \tag{3.3.9}$$

The last expression can be written as

$$S_e(e^{j\omega}) = \sum_n S(2^J\omega + 2\pi n 2^J) \left|\Phi(\omega + 2\pi n) - H(e^{j\omega})\right|^2. \tag{3.3.10}$$

Because of the bandlimitedness of $S(\omega)$ to $[-2^J\pi, 2^J\pi]$, for each $\omega \in \mathcal{R}$ only one term in the above sum is nonzero for any given ω. The error variance is

$$\sigma_e^2 = E|a_{Jk} - \widehat{a}_{Jk}|^2 = \epsilon_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} S_e(e^{j\omega})d\omega \tag{3.3.11}$$

or, using (3.3.10),

$$\sigma_e^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} S(2^J\omega)\left|\Phi(\omega) - H(e^{j\omega})\right|^2 d\omega. \tag{3.3.12}$$

3.3.2. Discussion and an example

Formula (3.3.12) offers us insight into the frequency distribution of the error, what this approximation is all about, and how to improve it. First notice that we can make this error go to zero by letting the order of $H(z)$ increase indefinitely. Moreover, this can be done regardless of the random process. Namely, from (3.3.12), we have

$$\sigma_e^2 \leq \frac{1}{2\pi}\int_{-\pi}^{\pi} S(2^J\omega) \sup_{\omega\in[-\pi,\pi]} |\Phi(\omega) - H(e^{j\omega})|^2 d\omega = \sigma_x^2 \sup_{\omega\in[-\pi,\pi]} |\Phi(\omega) - H(e^{j\omega})|^2, \tag{3.3.13}$$

and we know that an arbitrary continuous function on $[-\pi, \pi]$ can be uniformly approximated by trigonometric polynomials. So the main point of the section is to show how to design finite order systems that minimize the error.

Notice that as J increases, $S(2^J\omega)$ gets concentrated more and more around $\omega = 0$, so that $H(e^{j\omega}) = \Phi(0) = 1$, i.e., running no filter at all, is already a good approximation. Also, if $\phi(x) \approx \delta(x)$, then $H(e^{j\omega}) = 1$ is a good approximation, which is in accordance with (3.3.3) and (3.3.4) (this was Mallat's original motivation). So our problem is reduced to a weighted (by $S(2^J\omega)$) L^2 approximation of $\Phi(\omega)$ by trigonometric polynomials.

Example 3.3.1. In this example, we take the bandpass AR(6) model of speech [Jay84] for the input random process. For the scaling functions, we take Daubechies' ${}_3\phi$ and ${}_4\phi$. We do not want the filter $H(z)$ to be too complex; in particular, we want it to be FIR (then the assumption $\sum_n |h_n| < \infty$ is automatically satisfied). We will design the optimal pre-filter using the eigenfilter approach [Ngu93]. This method is very simple. The idea is that this error can be written as

$$\sigma_e^2 = \mathbf{h}^\dagger \mathbf{P}\mathbf{h}, \tag{3.3.14}$$

where $\mathbf{h} = (\, h_0 \quad h_1 \quad \cdots \quad h_N \,)^T$ and $\mathbf{P}$ is a positive definite symmetric matrix. The error variance will be minimized when $\mathbf{h}$ is the eigenvector of $\mathbf{P}$ corresponding to the minimum eigenvalue. To get $\mathbf{P}$, we will follow [Ngu93]. $H(e^{j\omega})$ can be written as

$$H(e^{j\omega}) = \mathbf{h}^{\dagger}\mathbf{e}(e^{j\omega}) = \mathbf{e}^{\dagger}(e^{j\omega})\mathbf{h}, \tag{3.3.15}$$

where $\mathbf{e}(e^{j\omega}) = (\,1 \quad e^{-j\omega} \quad \cdots \quad e^{-jN\omega}\,)^T$. Then $|\Phi(\omega) - H(e^{j\omega})|^2$ can be written as

$$|\Phi(\omega) - H(e^{j\omega})|^2 = \left|\frac{\Phi(\omega)}{H(1)}(\mathbf{h}^{\dagger}\mathbf{e}(1)) - \mathbf{h}^{\dagger}\mathbf{e}(e^{j\omega})\right|^2$$

$$= \mathbf{h}^{\dagger}\left[\frac{\Phi(\omega)}{H(1)}\mathbf{e}(1) - \mathbf{e}(e^{j\omega})\right]\left[\frac{\Phi^*(\omega)}{H^*(1)}\mathbf{e}^{\dagger}(1) - \mathbf{e}^{\dagger}(e^{j\omega})\right]\mathbf{h} = \mathbf{h}^{\dagger}\mathbf{P}_1(\omega)\mathbf{h}. \tag{3.3.16}$$

Since $\mathbf{P}_1(\omega)$ is a Hermitian matrix and we look for a real solution (real h_n's), we have $\mathbf{h}^{\dagger}\mathbf{P}_1(\omega)\mathbf{h} = \mathbf{h}^{\dagger}\Re\{\mathbf{P}_1(\omega)\}\mathbf{h}$. If we multiply this with $S(2^J\omega)$ and integrate, we get (note that $H(1) = \Phi(0) = 1$)

$$\sigma_e^2 = \mathbf{h}^{\dagger}\mathbf{P}\mathbf{h}, \tag{3.3.17}$$

where

$$[\mathbf{P}]_{km} = \frac{1}{2\pi}\int_{-\pi}^{\pi} S(2^J\omega)\left[|\Phi(\omega)|^2 - \Re\{\Phi(\omega)\}[\cos(k\omega) + \cos(m\omega)]\right.$$

$$\left. + \Im\{\Phi(\omega)\}[\sin(m\omega) + \sin(k\omega)] + \cos((m-k)\omega)\right] d\omega. \tag{3.3.18}$$

It is easy to calculate these elements $[\mathbf{P}]_{km}$ using some method for numerical integration. Then the eigenvector corresponding to the minimum eigenvalue of this positive semidefinite matrix is the solution. It is easy to see that this can be applied to (3.3.12), without restricting $S(\omega)$ to be bandlimited. Fig. 3.3.1 shows the relative error (σ_e^2/σ_x^2) for Daubechies' ${}_3\phi$ (solid line) and ${}_4\phi$ (dotted line) scaling functions [Dau92] as a function of the pre-filter length (σ_x is the variance of the input signal). From the graphs, we can see that the error can be significantly reduced with very low order FIR filters.

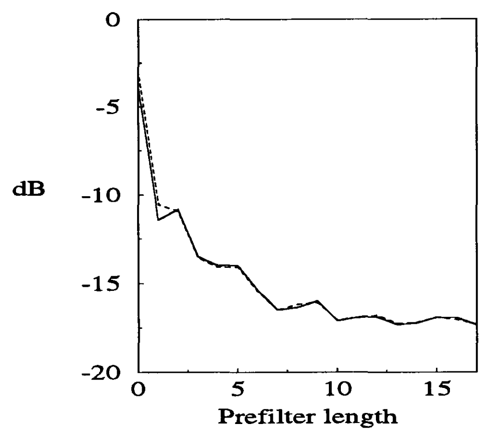


Fig. 3.3.1 Example 3.3.1 Relative error variance σ_e^2/σ_x^2 for Daubechies' ${}_3\phi$ (solid) and ${}_4\phi$ (dotted) scaling functions as a function of the pre-filter length.

3.4. Coding gain optimization

In order to derive an expression for the coding gain of a biorthogonal FB, the noise sources produced by the subband quantizers are assumed white and uncorrelated. This is a reasonable assumption as long as the subband signals are not too coarsely quantized.

3.4.1. Coding gain of a biorthogonal filter bank

Consider a uniform perfect reconstruction (PR) FB as in Fig. 3.4.1. The variances of the subband signals $x_k(n)$ are

$$\sigma_{x_k}^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} S(e^{j\omega})|H_k(e^{j\omega})|^2 d\omega, \qquad k = 0, 1 \ldots, M-1, \tag{3.4.1}$$

where $S(e^{j\omega})$ is the Power Spectral Density (PSD) function of the input random process.

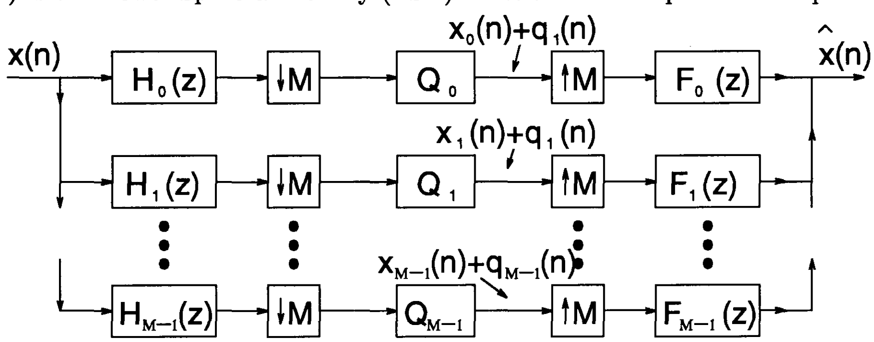


Fig. 3.4.1. A uniform filter bank used for subband coding.

The noise PSD function at the output of the k^{th} quantizer is (see [Jay84])

$$S_{q_k q_k}(e^{j\omega}) = E[q_k(n)q_k^*(n)] = C2^{-2b_k}\sigma_{x_k}^2 = \sigma_{qk}^2, \tag{3.4.2}$$

where b_k is the number of bits allocated to the k^{th} channel, C is some constant which depends on the statistics of $x_k(n)$, and $q_k(n)$ is the noise sequence. After some WSS random process passes through an expander, it becomes a cyclo-WSS process (see [Sat93]). The period of cyclo-stationarity is M. Then we can average the variance over the period to get $\sigma_{out,k}^2 = \frac{\sigma_{qk}^2}{M}||f_k(n)||_2^2$. The subband noises are uncorrelated and they remain such after passing through the synthesis filters. Then the output noise variance is $\sigma_{SBC}^2 = \sum_{k=0}^{M-1}\sigma_{out,k}^2$. Using (3.4.2) and (3.4.1), we get

$$\sigma_{SBC}^2 = \frac{C}{M}\sum_{k=0}^{M-1} 2^{-2b_k}\frac{1}{2\pi}\int_{-\pi}^{\pi}|F_k(e^{j\omega})|^2 d\omega \frac{1}{2\pi}\int_{-\pi}^{\pi} S(e^{j\omega})|H_k(e^{j\omega})|^2 d\omega. \tag{3.4.3}$$

The average bit rate is $b = 1/M\sum_{k=0}^{M-1} b_k$. If we quantized the input signal to this number of bits, without any subband decomposition, i.e., just PCM coding, the noise variance would be [Jay84]

$$\sigma_{PCM}^2 = C2^{-2b}\sigma_x^2 = C2^{-2b}\frac{1}{2\pi}\int_{-\pi}^{\pi} S(e^{j\omega})d\omega. \tag{3.4.4}$$

So the coding gain, defined as the ratio of the above variances is

$$G = \frac{\sigma_{PCM}^2}{\sigma_{SBC}^2} = \frac{2^{-2b}\frac{1}{2\pi}\int_{-\pi}^{\pi} S(e^{j\omega})d\omega}{\frac{1}{M}\sum_{k=0}^{M-1} 2^{-2b_k}\frac{1}{2\pi}\int_{-\pi}^{\pi}|F_k(e^{j\omega})|^2 d\omega\frac{1}{2\pi}\int_{-\pi}^{\pi} S(e^{j\omega})|H_k(e^{j\omega})|^2 d\omega}.$$

One of the optimization steps is an optimal bit allocation. We can make this step now and minimize the denominator. The optimal bit allocation (see [Som93a]) turns the sum in the denominator into a product. So we have the following expression for the coding gain under optimal bit allocation.

$$G = \frac{\sigma_{PCM}^2}{\sigma_{SBC}^2} = \frac{\frac{1}{2\pi}\int_{-\pi}^{\pi} S(e^{j\omega})d\omega}{\left(\prod_{k=0}^{M-1}\frac{1}{2\pi}\int_{-\pi}^{\pi}|F_k(e^{j\omega})|^2 d\omega\frac{1}{2\pi}\int_{-\pi}^{\pi} S(e^{j\omega})|H_k(e^{j\omega})|^2 d\omega\right)^{1/M}}. \tag{3.4.5}$$

This coding gain formula is valid as long as the subband noises are white and uncorrelated.

3.4.2. Pre-filters for PU filter banks

Consider the class of pre-filtered PU (PPU) FBs obtained by putting pre- and post-filters around a PU FB $\{P_k(z), Q_k(z)\}$ (see Fig. 3.4.2). The aim of this subsection is to find a PPU FB which maximizes coding gain.

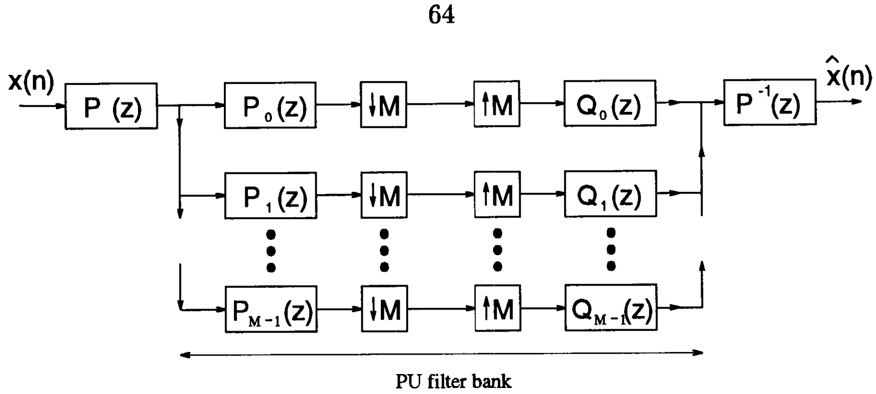


Fig. 3.4.2. Filter bank with pre- and post-filters.

First, notice that maximization of the coding gain is the same as minimization of the denominator of (3.4.5). From the Cauchy-Schwarz inequality it follows that

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}|F_k(e^{j\omega})|^2 d\omega \frac{1}{2\pi}\int_{-\pi}^{\pi} S(e^{j\omega})|H_k(e^{j\omega})|^2 d\omega \geq \left(\frac{1}{2\pi}\int_{-\pi}^{\pi}|H_k(e^{j\omega})F_k(e^{j\omega})|\sqrt{S(e^{j\omega})}d\omega\right)^2, \tag{3.4.6}$$

with equality achieved if and only if

$$|H_k(e^{j\omega})|\sqrt{S(e^{j\omega})} = \lambda_k|F_k(e^{j\omega})|, \qquad \text{for} \quad 0 \leq k \leq M-1, \tag{3.4.7}$$

for arbitrary choice of $\lambda_k \neq 0$ (e.g., $\lambda_k = 1 \; \forall k$). Notice that the PSD function is $S(e^{j\omega}) \geq 0$, so that its square root is well defined.

Since $H_k(z) = P_k(z)P(z)$ and $F_k(z) = Q_k(z)/P(z)$, we have $H_k(z)F_k(z) = P_k(z)Q_k(z)$. So the right-hand side in (3.4.6) depends only on the product $P_k(z)Q_k(z)$ and is independent of the pre-filter $P(z)$. Thus, if the pre-filter $P(z)$ can be chosen to achieve equality in (3.4.6) for all k, it will maximize the coding gain for a fixed filter bank $\{P_k(z), Q_k(z)\}$. This observation is true whether the sandwiched system $\{P_k(z), Q_k(z)\}$ is orthonormal or biorthogonal. However, when $\{P_k(z), Q_k(z)\}$ is orthonormal, equality in (3.4.6) is achievable for all k. To see this note that (3.4.7) can be rewritten as

$$|P(e^{j\omega})P_k(e^{j\omega})|\sqrt{S(e^{j\omega})} = \left|\frac{1}{P(e^{j\omega})}Q_k(e^{j\omega})\right|. \tag{3.4.8}$$

When $\{P_k(z), Q_k(z)\}$ is an orthonormal filter bank, then $Q_k(e^{j\omega}) = P_k^*(e^{j\omega})$ for perfect reconstruction [Vai93]. So the condition (3.4.8) for equality reduces to

$$|P(e^{j\omega})| = [1/S(e^{j\omega})]^{1/4} \tag{3.4.9}$$

and is independent of k. The coding gain (3.4.5) becomes

$$G_{PPU} = \frac{\frac{1}{2\pi}\int_{-\pi}^{\pi} S(e^{j\omega})d\omega}{\left(\prod_{k=0}^{M-1}\frac{1}{2\pi}\int_{-\pi}^{\pi}|P_k(e^{j\omega})|^2\sqrt{S(e^{j\omega})}d\omega\right)^{2/M}}. \tag{3.4.10}$$

Summarizing, we have proved:

Theorem 3.4.1. Consider the class of all PR FBs that can be obtained from the structure in Fig. 3.4.2, where $\{P_k(z), Q_k(z)\}$ is a PU FB. Then, the pre-filter which maximizes coding gain will satisfy (3.4.9).

$\diamond$

The coding gain with the optimal pre-filter is given by (3.4.9). The orthonormal filter bank that maximizes this coding gain is the one that has maximum coding gain for an input with the power spectrum $\sqrt{S(e^{j\omega})}$. There are techniques to identify such a system based on the work by Unser [Uns93]. We shall not go into detail of this here. Summarizing our main point, if we wish to find an optimal filter bank of the form in Fig. 3.4.2, where $\{P_k(z), Q_k(z)\}$ is orthonormal, we construct $P(e^{j\omega})$ according to (3.4.9) and construct $\{P_k(z), Q_k(z)\}$ to be the orthonormal filter bank that maximizes the coding gain for an input with power spectrum $\sqrt{S(e^{j\omega})}$. So the optimization of $P(z)$ has been decoupled from that of $\{P_k(z), Q_k(z)\}$. This establishes the following corollary.

Corollary 3.4.1. Putting pre- and post-filters as given by (3.4.9) around *any* PU FB $\{P_k(z), Q_k(z)\}$ will not decrease its coding gain. It will strictly increase the coding gain if the input spectrum is not piecewise constant.

An insightful way to understand the above corollary is as follows. If we put $F_k(e^{j\omega}) = \widetilde{P}_k(e^{j\omega})$ and $H_k(e^{j\omega}) = P_k(e^{j\omega})$ into (3.4.5), and use the fact that PU filters have unit energy, then after optimal bit allocation we get the coding gain of $\{P_k(z), Q_k(z)\}$ with input $x(n)$

$$G_{PU} = \frac{\frac{1}{2\pi}\int_{-\pi}^{\pi} S(e^{j\omega})d\omega}{\left(\prod_{k=0}^{M-1} \frac{1}{2\pi}\int_{-\pi}^{\pi} |P_k(e^{j\omega})|^2 S(e^{j\omega})d\omega\right)^{1/M}}. \tag{3.4.11}$$

The ratio of the two coding gains

$$\eta = G_{PPU}/G_{PU} = \prod_{k=0}^{M-1}\left(\frac{\frac{1}{2\pi}\int_{-\pi}^{\pi} |P_k(e^{j\omega})|^2 S(e^{j\omega})d\omega}{\left(\frac{1}{2\pi}\int_{-\pi}^{\pi} |P_k(e^{j\omega})|^2 \sqrt{S(e^{j\omega})}d\omega\right)^2}\right)^{1/M} \tag{3.4.12}$$

satisfies $\eta \geq 1$. This is because for each k we have

$$\left(\frac{1}{2\pi}\int_{-\pi}^{\pi} |P_k(e^{j\omega})|^2 \sqrt{S(e^{j\omega})}d\omega\right)^2 \leq \frac{1}{2\pi}\int_{-\pi}^{\pi} |P_k(e^{j\omega})|^2 S(e^{j\omega})d\omega, \tag{3.4.13}$$

with equality if and only if $S(e^{j\omega})$ is a constant over the support of $P_k(e^{j\omega})$. This follows from the Cauchy-Schwarz's inequality. We see that this simple system always outperforms any PU system, as long as $S(e^{j\omega})$ is not constant where $P_k(e^{j\omega}) \neq 0$. This improvement in the performance is more significant as $S(e^{j\omega})$ has more non-constant behavior.

Relation to half-whitening. A well-known data compression technique called half-whitening is described in [Jay84]. Here, a signal $x(n)$ is first pre-filtered with a filter $H(z)$, then quantized and post-filtered with $1/H(z)$. Under mild assumptions on the joint statistics of the signal $x(n)$ and the quantizer noise, the best pre-filter (to maximize the output signal to noise ratio) is such that $|H(e^{j\omega})| = [1/S(e^{j\omega})]^{1/4}$. Our result in Theorem 3.4.1 shows that a similar result is true if the quantizer is replaced with a paraunitary subband coder. If the filter bank $\{P_k(z), Q_k(z)\}$ is not orthonormal, then the preceding results are not true. For example, if $\{P_k(z), Q_k(z)\}$ were biorthogonal rather than orthonormal, then the insertion of $P(z)$ and $1/P(z)$ with $P(z)$ as in (3.4.9) could even decrease the coding gain. Here is a way to visualize such a situation: Suppose $\{P_k(z), Q_k(z)\}$ is itself a biorthogonal filter bank obtained by sandwiching an orthonormal filter bank between an optimal pre-filter (3.4.9) and a post-filter. If we now insert another pair of $P(z)$ and $1/P(z)$ (with $P(z)$ still given as in (3.4.9)), it can only decrease the coding gain! Theorem 3.4.1 and Corollary 3.4.1 should not, therefore, be regarded as a simple extension of the half-whitening result.

Relation to Prediction Gain. In order to give an intuitive feeling why this scheme works, let us look at the following expression

$$\left(\prod_{k=0}^{M-1}\left(\frac{1}{2\pi}\int_{-\pi}^{\pi}|P_k(e^{j\omega})|^2 S^p(e^{j\omega})d\omega\right)^{1/p}\right)^{1/M}. \tag{3.4.14}$$

Notice that when $p = 1$ this is the denominator in (3.4.11) (the case of PU FB), and when $p = 1/2$ it is the denominator in (3.4.10) (the case of PPU). Now consider the theoretical bound on the coding gain, namely the prediction gain.

$$G_\infty = \frac{\sigma_x^2}{\exp\{\frac{1}{2\pi}\int_{-\pi}^{\pi}\log_e(S(e^{j\omega}))d\omega\}}. \tag{3.4.15}$$

The denominator here can be obtained from the expression (3.4.14) as the limit when $p \to 0$. For this, note that $\frac{1}{2\pi}\int |P_k(e^{j\omega})|^2 d\omega = 1$ and $\sum_{k=0}^{M-1}|P_k(e^{j\omega})|^2 = M$, since $\{P_k(z), Q_k(z)\}$ is a PU FB. Then (see [Rud87])

$$\lim_{p\to 0}\left(\prod_{k=0}^{M-1}\left(\frac{1}{2\pi}\int_{-\pi}^{\pi}|P_k(e^{j\omega})|^2 S^p(e^{j\omega})d\omega\right)^{1/p}\right)^{1/M}$$

$$= \left(\prod_{k=0}^{M-1}\exp\left\{\frac{1}{2\pi}\int_{-\pi}^{\pi}|P_k(e^{j\omega})|^2\log_e(S(e^{j\omega}))d\omega\right\}\right)^{1/M}$$

$$= \exp\left\{\frac{1}{2\pi}\int_{-\pi}^{\pi}\underbrace{\frac{\sum_{k=0}^{M-1}|P_k(e^{j\omega})|^2}{M}}_{=1}\log_e(S(e^{j\omega}))d\omega\right\} = \exp\left\{\frac{1}{2\pi}\int_{-\pi}^{\pi}\log_e(S(e^{j\omega}))d\omega\right\}. \quad (3.4.16)$$

So we improved the coding gain of a PU system (which corresponds to $p = 1$) by finding the structure in which $p = 1/2$. If there existed a structure corresponding to $p < 1/2$, it would further improve the coding gain†. The examples below will demonstrate that our technique approximately halves the gap between the performance of a PU system and the prediction gain bound (3.4.15) on a dB scale.

3.4.3. Examples

Example 3.4.1. DCT filter bank with pre-filtering. The above developed technique will be applied to a very simple PU FB. Let $\{P_k(z), Q_k(z)\}$ be a DCT FB, i.e., the one in which the polyphase matrix $\mathbf{E}(z)$ is the DCT IV matrix [Yip87]. The DCT filters have poor attenuations. Fig. 3.4.3 shows $[1/S(e^{j\omega})]^{1/4}$, the test function chosen for this example (dotted curve). The solid curve is its 2^{nd} order rational approximation (i.e., $P_a(z)$ is a 2^{nd} order filter). The input PSD function $S(e^{j\omega})$ was the lowpass AR(5) model of speech [Jay84]. Fig. 3.4.4 shows the coding gain for different FBs. We can see that even pre-filter alone (without any FB) gives some coding gain (see Ch. 7 in [Jay84]). The coding gain changes only slightly if the ideal pre-filter $[1/S(e^{j\omega})]^{1/4}$ is approximated by a 2^{nd} order rational filter.

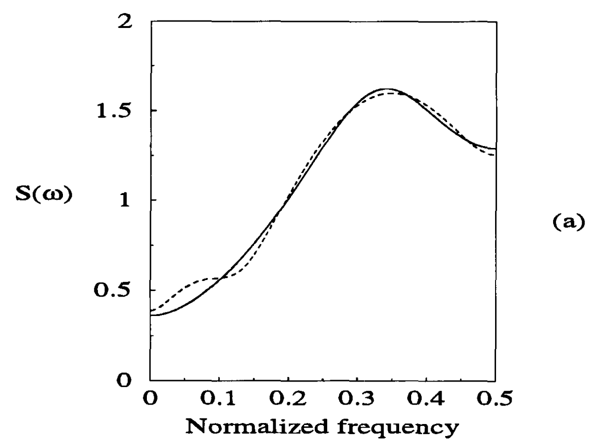
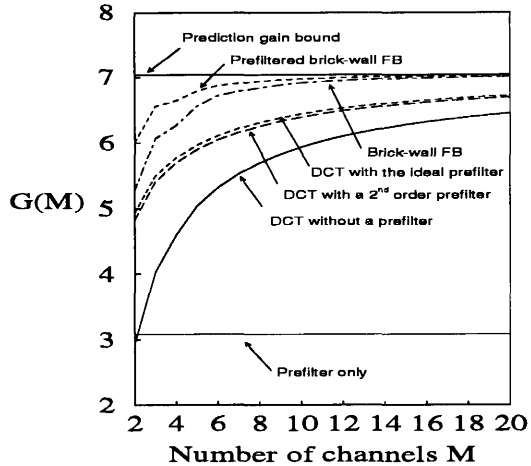


Fig. 3.4.3. Plots of $S^{-1/4}(e^{j\omega})$ for a test example (dotted curve), and a rational approximation $|P_a(e^{j\omega})|$ (solid curve). The approximation filter $P_a(e^{j\omega})$ is a 2^{nd} order IIR filter.

† It can be shown using Jensen's inequality (see [Rud87]), that as p decreases, the coding gain can only increase.

Notice that the coding gain of PPU FB approximately halves the gap (on a dB scale) between the coding gain achieved with the PU FB and the prediction gain bound on the coding gain given by (3.4.15). The next example is striking in the sense that a finite order filter bank performs better than a brick-wall FB.



(b)

Fig. 3.4.4. Example 3.4.1. Coding gain of DCT filter banks as a function of the number of channels, with and without pre-filters.

Example 3.4.2. Tree structured filter bank with pre-filtering. In the previous example, the DCT filters had poor frequency responses. In this example, we design tree structured FBs (number of channels M is a power of 2) using a two-channel PU FB as a basic building block. The filter length of each filter in the two-channel module is 8 (8A from [Hoa89]). We use the same 2^{nd} order approximation of $[1/S(e^{j\omega})]^{1/4}$ as in Fig 3.4.3.

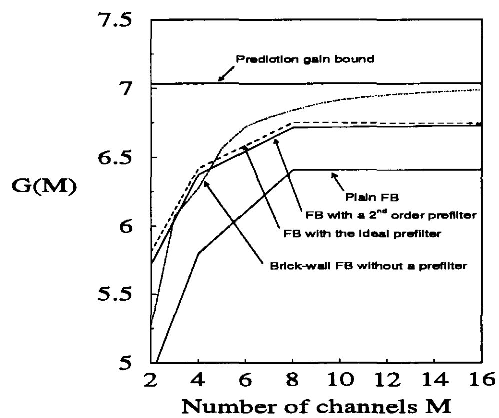


Fig. 3.4.5. Example 3.4.2. Coding gain of tree-structured filter banks as a function of the number of channels, with and without pre-filters.

Now we can see in Fig. 3.4.5 that the PPU finite order FBs (for $M = 2, 4$) perform better than the corresponding ideal brick-wall FBs, which shows that an ideal brick-wall FB does not necessarily maximize the coding gain for a given number of channels.

3.5. Conclusion

We extended the techniques developed in [Tew92], [Ode92] for the design of an optimal scaling function for the representation of a *class* of signals. We derived an expression for the error variance and gave some examples to show how those expressions can be used in the design process. Similarly, an idea of [She92] and a technique of [Xia93] were extended to the statistical case. The idea of pre-filtering is applied to the problem of filter bank optimization to give a higher coding gain when the input is a random process with a given power spectrum. We saw why this pre-filtering improves the coding gain.

3.6. Appendices

Appendix 3.A: Proof of Lemma 3.2.1

In order to check for the cyclo-stationarity, substitute $t+T$ instead of t in (3.2.5) and rearrange it to get

$$\begin{aligned}
&r_e(t+T,\tau) = R(\tau) \\
&- 2^J \sum_k \left(\int R\left(t-(x-T)\right) \phi\left(2^J (x-T) - k + 2^J T\right) dx \right) \phi^*\left(2^J (t-\tau) - k + 2^J T\right) \\
&- 2^J \sum_k \left(\int R\left((x-T) - t + \tau\right) \phi^*\left(2^J (x-T) - k + 2^J T\right) dx \right) \phi(2^J t - k + 2^J T) \\
&+ 2^{2J} \sum_{k,l} \left(\int\!\!\int R\left((x-T)-(y-T)\right) \phi^*\left(2^J (x-T) - k + 2^J T\right) \phi\left(2^J (y-T) - l + 2^J T\right) dx dy \right) \\
&\times \phi(2^J t - k + 2^J T) \phi^*\left(2^J (t-\tau) - l + 2^J T\right).
\end{aligned} \tag{3.A.1}$$

Now, if we change variables in the integrals according to $x - T = u$, and use $y - T = v$ in the third, we get

$$
\begin{aligned}
&r_e(t+T,\tau) = R(\tau) \\
&-2^J \sum_k \left(\int R(t-u)\, \phi\left(2^J u - \left(k - 2^J T\right)\right) du \right) \phi^*\left(2^J (t-\tau) - \left(k - 2^J T\right)\right) \\
&-2^J \sum_k \left(\int R(u-t+\tau)\, \phi^*\left(2^J u - \left(k - 2^J T\right)\right) du \right) \phi\left(2^J t - \left(k - 2^J T\right)\right) \\
&+2^{2J} \sum_{k,l} \left(\int \int R(u-v)\, \phi^*\left(2^J u - \left(k - 2^J T\right)\right) \phi\left(2^J v - \left(l - 2^J T\right)\right) du dv \right) \\
&\times \phi\left(2^J t - \left(k - 2^J T\right)\right) \phi^*\left(2^J (t-\tau) - \left(l - 2^J T\right)\right).
\end{aligned}
\tag{3.A.2}
$$

Using the invariance of the basis $\{\phi(x-n) \mid n \in \mathcal{Z}\}$ under integer translations, we see that whenever $2^J T$ is an integer we can change indices of summations $n = k - 2^J T$ and $m = l - 2^J T$. Then (3.A.2) becomes (3.2.5), i.e., $r_e(t+T,\tau) = r_e(t,\tau)$. The smallest T for which this is true is the period of cyclo-stationarity of $r_e(t,\tau)$ and it is $T_{min} = 2^{-J}$ (see [Wor90]).

Appendix 3.B: Derivations for Section 3.2

We will handle terms one by one. Note that

$$
\mathcal{F}\left(\phi^*(2^J x - k)\right)^* = 2^{-J} \Phi(-2^{-J}\omega) e^{jk\omega 2^{-J}} \tag{3.B.1}
$$

and

$$
\mathcal{F}\left(R(t-x)\right) = e^{-j\omega t} S(-\omega). \tag{3.B.2}
$$

Then by the Parseval's equality, we have

$$
\int R(t-x)\phi(2^J x - k) dx = \frac{2^{-J}}{2\pi} \int e^{-jy(t-k2^J)} S(-y) \Phi(-2^{-J} y) dy. \tag{3.B.3}
$$

When we put this into the second term in (3.2.7), we get

$$
R_{e2}(\tau) = \frac{2^{-J}}{2\pi} \int_0^{2^{-J}} \sum_k \int e^{-jy(t-2^{-J}k)} S(-y) \Phi(-2^{-J} y) \phi^*(2^J(t - 2^{-J}k) - 2^J \tau) dy dt. \tag{3.B.4}
$$

Now, because $|\Phi(\omega)| \leq 1$, we have

$$
\begin{aligned}
&\int_0^{2^{-J}} \int \sum_k |S(-y)\Phi(-2^{-J}y)\phi^*(2^J(t-\tau) - k)| dy dt \\
&\leq \int_0^{2^{-J}} \int S(-y) \sum_k |\phi(2^J(t-\tau) - k)| dy dt \\
&= \int_0^{2^{-J}} \sum_k |\phi(2^J(t-\tau) - k)| dt \int S(-y) dy < \infty,
\end{aligned}
\tag{3.B.5}
$$

so that we can interchange the orders of integrations and summation as we wish, because of Fubini's theorem. Then (3.B.4) becomes

$$R_{e2}(\tau) = \frac{2^{-J}}{2\pi} \int S(-y)\Phi(-2^{-J}y) \sum_k \int_0^{2^{-J}} e^{-jy(t-2^{-J}k)} \phi^*(2^J(t-2^{-J}k) - 2^J\tau) dt dy. \quad (3.B.6)$$

Now notice that

$$\sum_k \int_0^{2^{-J}} e^{-jy(t-2^{-J}k)} \phi^*(2^J(t-2^{-J}k) - 2^J\tau) dt = \int_{-\infty}^{\infty} e^{-jyt} \phi^*(2^J(t-\tau)) dt = 2^{-J}\Phi^*(-2^{-J}y)e^{-jy\tau}, \quad (3.B.7)$$

by putting $k = 2^J\tau$ in (3.B.1). Finally, (3.B.6) becomes

$$R_{e2}(\tau) = \frac{2^{-2J}}{2\pi} \int S(-y)|\Phi(-2^{-J}y)|^2 e^{-jy\tau} dy, \quad (3.B.8)$$

and

$$R_e(0) = \frac{2^{-2J}}{2\pi} \int S(\omega)|\Phi(2^{-J}\omega)|^2. \quad (3.B.9)$$

Now we go to the third term in (3.2.7). In a similar way [see (3.B.5)], if we use the Parseval's identity, we get

$$\begin{aligned} R_{e3}(\tau) &= \frac{2^{-J}}{2\pi} \int_0^{2^{-J}} \sum_k \left(\int S(y) e^{-jy(t-2^{-J}k-\tau)} \Phi^*(2^{-J}y) dy \right) \phi(2^J t - k) dt \\ &= \frac{2^{-J}}{2\pi} \int S(y) e^{jy\tau} \Phi^*(2^{-J}y) \underbrace{\sum_k \int_0^{2^{-J}} e^{-jy(t-2^{-J})} \phi(2^J(t-2^{-J}k)) dt}_{\mathcal{F}(\phi(2^J t)) = 2^{-J}\Phi(2^{-J}y)} dy \\ &= \frac{2^{-2J}}{2\pi} \int S(y) |\Phi(2^{-J}y)|^2 e^{jy\tau} dy, \end{aligned} \quad (3.B.10)$$

i.e., the second and third terms in (3.2.7) are the same.

There is only one term left. Consider the double integral in the fourth term. Using Parseval's equality, it can be written as

$$\begin{aligned} &\int \left(\int R(x-y)\phi(2^J y - l) dy \right) \phi^*(2^J x - k) dx \\ &= \frac{1}{2\pi} \int \mathcal{F}\left(R(x) * \phi(2^J x - l)\right) \left(2^{-J} e^{-jku2^{-J}} \Phi(2^{-J}u)\right)^* du \\ &= \frac{2^{-2J}}{2\pi} \int S(u) e^{-j2^{-J}(l-k)u} |\Phi(2^{-J}u)|^2 du, \end{aligned} \quad (3.B.11)$$

where $f * g$ denotes the convolution of f and g. Substitute this back into the fourth term in (3.2.7) and get

$$R_{e4}(\tau) = \frac{2^{-2J}}{2\pi} \int_0^{2^{-J}} \sum_{k,l} \left(\int S(u)|\Phi(2^{-J}u)|^2 e^{-j2^{-J}u(l-k)} du \right) \phi(2^J t - k) \phi^*(2^J(t-\tau) - l) dt. \quad (3.B.12)$$

Again, because of a compact support of $\phi(t)$, $\exists M \in \mathcal{R}$ such that

$$\sum_{k,l} \left| \phi(2^J t - k)\phi^*(2^J(t-\tau) - l) \right| \leq M \quad \forall t, \tau, u \in \mathcal{R}. \tag{3.B.13}$$

Then, since $|\Phi(u)| \leq 1$ and $S(u) \in L^1$, we can interchange the orders of integration with respect to t and u and summations. So we get

$$R_{e4}(\tau) = \frac{2^{-2J}}{2\pi} \int S(u)|\Phi(2^{-J}u)|^2 \int_0^{2^{-J}} \left(\sum_{k,l} \phi(2^J t - k)\phi^*(2^J(t-\tau) - l)e^{-j2^{-J}u(l-k)} \right) dtdu. \tag{3.B.14}$$

If we now put $l = n + k$ and pull the integral under the summation with respect to k, we have

$$\begin{aligned} R_{e4}(\tau) = &\frac{2^{-2J}}{2\pi} \int S(u)|\Phi(2^{-J}u)|^2 \\ &\times \sum_k \left(\int_0^{2^{-J}} \phi(2^J(t - 2^{-J}k)) \sum_n \phi^*(2^J(t-2^{-J}k) - 2^J\tau - n)e^{-j2^{-J}un}dt \right) du. \end{aligned} \tag{3.B.15}$$

Again

$$\begin{aligned} &\sum_k \int_0^{2^{-J}} \phi(2^J(t - 2^{-J}k)) \sum_n \phi^*(2^J(t-2^{-J}k) - 2^J\tau - n)e^{-j2^{-J}un}dt \\ &= \int \phi(2^J t) \sum_n \phi^*(2^J t - 2^J\tau - n)e^{-j2^{-J}un}dt. \end{aligned} \tag{3.B.16}$$

The order of integration and summation can be interchanged, and if we apply the Parseval's identity to the last expression, it becomes

$$\begin{aligned} &\sum_n e^{-j2^{-J}un}\frac{2^{-2J}}{2\pi} \int |\Phi(2^{-J}y)|^2 e^{-j2^{-J}(2^J\tau+n)y}dy \\ &= \sum_n e^{-j2^{-J}un}\frac{2^{-2J}}{2\pi} \int_{-2^J\pi}^{2^J\pi} \left(\overbrace{\sum_k |\Phi(2^{-J}y + 2\pi k)|^2 e^{-j\tau(y+2\pi k 2^J)}}^{\eta(y)} \right) e^{-j2^{-J}ny}dy \\ &= 2^{-J} \sum_k |\Phi(2^{-J}u + 2\pi k)|^2 e^{-j\tau(u+2\pi k2^J)} \quad \text{a.e.} \end{aligned} \tag{3.B.17}$$

because $\eta(\omega) \in L^p([-\pi, \pi])$ for $\forall p \geq 1$ and its Fourier series converges to it almost everywhere (a.e.). Then

$$R_{e4}(\tau) = \frac{2^{-3J}}{2\pi} \int S(u)|\Phi(2^{-J}u)|^2 \sum_n |\Phi(2^{-J}u + 2\pi n)|^2 e^{j\tau(2^J 2\pi n+u)}du. \tag{3.B.18}$$

So,

$$R_{e4}(0) = \frac{2^{-3J}}{2\pi} \int S(\omega)|\Phi(2^{-J}\omega)|^2 d\omega, \tag{3.B.19}$$

because $\sum_n |\Phi(2^{-J}u + 2\pi n)|^2 = 1$ by the orthonormality (see [Dau92]).

Appendix 3.C: Derivations for Sec. 3.3

We will handle each of the four terms in (3.3.7) separately. Let us start with the first one. By the Parseval's identity, we have

$$2^J \int\int R(x-y)\phi^*(2^J x)\phi(2^J y+n)dxdy = \frac{1}{2\pi}\int\int S(u)e^{-juy}\Phi^*(2^{-J}u)\phi(2^J y+n)dudy. \quad (3.C.1)$$

Now, since $\int\int |S(u)\Phi^*(2^{-J}u)\phi(2^J y+n)|dudy < \infty$, we can integrate with respect to y first and get

$$\begin{aligned} S_1(e^{j\omega}) &= 2^J \sum_n e^{-jn\omega}\int\int R(x-y)\phi^*(2^J x)\phi(2^J y+n)dxdy \\ &= \frac{1}{2\pi}\sum_n e^{-jn\omega}\int S(u)\Phi^*(2^{-J}u)\left(\int e^{-juy}\phi(2^J y+n)dy\right)du \\ &= \frac{2^{-J}}{2\pi}\sum_n e^{-jn\omega}\int S(u)\Phi^*(2^{-J}u)e^{jnu2^{-J}}\Phi(2^{-J}u)du \\ &= \frac{2^{-J}}{2\pi}\sum_n e^{-jn\omega}\mathcal{F}\left(S(u)|\Phi(2^{-J}u)|^2\right)(-2^{-J}n) \\ &= \frac{2^{-J}}{2\pi}\sum_n e^{j(2^{-J}n)(2^J\omega)}\mathcal{F}\left(S(u)|\Phi(2^{-J}u)|^2\right)(2^{-J}n). \end{aligned} \quad (3.C.2)$$

$|R(\tau)| < C/(1+|\tau|^{1+\alpha})$ implies that $S(\omega)$ is continuous. This with the fact that $S(\omega)$ and $\phi(x)$ are compactly supported implies that sufficient conditions for the application of the Poisson's identity are satisfied (see [Chu92] p. 45)

$$2\pi a\sum_n f(\omega+2\pi an) = \sum_n F(n/a)e^{jn\omega/a}, \quad (3.C.3)$$

where $F(\omega) = \mathcal{F}(f)$. If we put $f(\omega) = S(\omega)|\Phi(2^{-J}\omega)|^2$ and $a = 2^J$ into (3.C.3), then (3.C.2) becomes

$$S_1(e^{j\omega}) = \sum_n S(2^J\omega + 2\pi n 2^J)|\Phi(\omega+2\pi n)|^2. \quad (3.C.4)$$

As for the second term in (3.3.7), first note that by the Parseval's identity we have

$$\int R(x-2^J l)\phi^*(2^J x)dx = \frac{2^{-J}}{2\pi}\int S(y)\Phi^*(2^{-J}y)e^{-jy2^{-J}l}dy. \quad (3.C.5)$$

Now the Fourier transform of the second term in (3.3.7) can be written as

$$\begin{aligned} S_2(e^{j\omega}) &= -\frac{2^{-J}}{2\pi}\sum_n e^{-jn\omega}\sum_l h^*_{-n-l}\int S(y)e^{-jyl2^{-J}}\Phi^*(2^{-J}y)dy \\ &= -\frac{2^{-J}}{2\pi}\sum_n e^{-jn\omega}\int e^{jn2^{-J}y}S(y)\Phi^*(2^{-J}y)\sum_l h^*_{-l}e^{-jyl2^{-J}}dy. \end{aligned} \quad (3.C.6)$$

The order of integration and summation could be exchanged because we assumed $\sum_n |h_n| < \infty$ and $S(y)\Phi^*(2^{-J}y) \in L^1$. Let $H(e^{j\omega}) = \sum_n h_n e^{-jn\omega}$, then we have

$$\begin{aligned} S_2(\omega) &= -\frac{2^{-J}}{2\pi}\sum_n e^{-jn\omega}\int e^{jn2^{-J}y}S(y)\Phi^*(2^{-J}y)H^*(e^{j2^{-J}y})dy \\ &= -\frac{2^{-J}}{2\pi}\sum_n e^{jn\omega}\int e^{-jn2^{-J}y}S(y)\Phi^*(2^{-J}y)H^*(e^{j2^{-J}y})dy \\ &= -\frac{2^{-J}}{2\pi}\sum_n e^{j(2^{-J})(2^J n)\omega}\mathcal{F}\left(S(\omega)\Phi^*(2^{-J}\omega)H^*(e^{j2^{-J}\omega})\right)(n/2^J). \end{aligned} \tag{3.C.7}$$

If we put $f(\omega) = S(\omega)\Phi^*(2^{-J}\omega)H^*(e^{j2^{-J}\omega})$ and $a = 2^J$ into (3.C.3) (notice that $f(\omega)$ satisfies the conditions for application of the Poisson's formula), we get

$$\begin{aligned} S_2(e^{j\omega}) &= -\sum_n S(2^J\omega + 2\pi n2^J)\Phi^*(\omega + 2\pi n)H^*(e^{j(\omega+2n\pi)}) \\ &= -H^*(e^{j\omega})\sum_n S(2^J\omega + 2\pi n2^J)\Phi^*(\omega + 2\pi n), \end{aligned} \tag{3.C.8}$$

because $H(e^{j\omega})$ is a periodic function with period 2π. The third term is just a complex conjugate of the second one, i.e.,

$$S_3(e^{j\omega}) = -H(e^{j\omega})\sum_n S(2^J\omega + 2\pi n2^J)\Phi(\omega + 2\pi n). \tag{3.C.9}$$

Finally, for the last term, we can write

$$\begin{aligned} S_4(e^{j\omega}) &= 2^{-J}\sum_n e^{-jn\omega}\sum_{l,m} R(2^{-J}(l-m))h_{-l}h^*_{-n-m} \\ &= 2^{-J}\sum_{l,m} R(2^{-J}(l-m))h_{-l}e^{j\omega m}\left(\sum_n h_{-n-m}e^{-j\omega(-n-m)}\right)^* \\ &= 2^{-J}\sum_{l,m} R(2^{-J}l)h_{-l-m}e^{-j\omega(-l-m)}e^{-j\omega l}H^*(e^{j\omega}) \\ &= 2^{-J}\sum_l R(2^{-J}l)e^{-j\omega l}|H(e^{j\omega})|^2 = 2^{-J}|H(e^{j\omega})|^2\sum_l R(2^{-J}l)e^{-j\omega l}. \end{aligned} \tag{3.C.10}$$

The orders of summations can be switched because of assumed sufficiently fast decay of h_n and $R(n)$. Now we use the Poisson's summation formula [put $F(x) = R(x)$ and $a = 2^J$ in (3.C.3)] to get

$$\sum_l R(2^{-J}l)e^{-j\omega l} = 2^J\sum_l S(2^J(\omega + 2\pi l)). \tag{3.C.11}$$

Substitute this into (3.C.10) and get

$$S_4(e^{j\omega}) = |H(e^{j\omega})|^2\sum_l S(2^J(\omega + 2\pi l)). \tag{3.C.12}$$

The orders of summation could be switched because of the absolute convergence of the corresponding series.

4

Generalized Sampling Theorems In Multiresolution Subspaces

4.1. Introduction

Ever since Mallat and Meyer [Mal89, Mey87] came up with the concept of Multiresolution analysis (MRA), it has been an interesting field for extension of results obtained in other frameworks. One example is the sampling theory. Originally, the theory was developed for uniform sampling of bandlimited signals [Whi15]. A couple of decades later, the research was concentrated on nonuniform sampling [Pal34]. In the second half of this century, those ideas were extended to random processes [Llo59]. All these results hold true for the class of bandlimited signals. What are other classes of signals where we can develop similar theory? A more general setting is the class of so called Reproducing Kernel Hilbert Spaces (RKHS) [Yao67] (see Appendix 4.A). It turns out that the wavelet subspaces (MRA subspaces) are RKHS (under very mild restrictions on the scaling function) [Wal92]. MRA was described in Sec. 1.2.

In sampling theory, there are two problems that one has to deal with. The first one is that of *uniqueness*. Namely, given a sequence of sampling instants $\{t_n\}$, can we have $\{f(t_n)\} = \{g(t_n)\}$ for some $f(t) \neq g(t)$ and $f, g \in \mathcal{H}$ ($\mathcal{H}$ is the underlying RKHS)? If this cannot happen, we say that $\{t_n\}$ is a sequence of uniqueness for $\mathcal{H}$. The other problem is that of finding a *stable* inversion scheme. This means, given some sequence of uniqueness $\{t_n\}$, is it possible to find synthesizing functions

$S_n(t) \in \mathcal{H}$ such that two things are true. First,

$$f(t) = \sum_{n=-\infty}^{\infty} f(t_n) S_n(t) \quad \forall f(t) \in \mathcal{H}, \tag{4.1.1}$$

and, second, if $\{f(t_n)\}$ is close to $\{g(t_n)\}$, then so is $\sum_n f(t_n)S_n(t)$ to $\sum_n g(t_n)S_n(t)$, in norms of the corresponding spaces. It is possible that no stable reconstruction exists even though uniqueness part is satisfied (see [Jer77]).

As we already mentioned, Walter showed in [Wal92] that V_m are RKHS, under very mild conditions on the decay and regularity of $\phi(t)$. He further shows that a stable reconstruction from samples at $t_n = n$ is possible and constructs the synthesizing functions $S_n(t)$'s. Janssen [Jan93] extended Walter's result to the case of uniform non-integer sampling. Neither of these schemes allows for both $\phi(t)$ and synthesizing functions $S_n(t)$'s to be compactly supported[†], unless $\phi(t)$ is of a restricted form (characteristic function of $[0,1]$ or its convolution with itself, for example).

4.1.1. Aims of the chapter

In this chapter, we extend Walter's work in several directions. We extend it to:

1. Periodically nonuniform sampling;
2. Reconstruction from local averages;
3. Oversampling;
4. Reconstruction from under-sampled functions and their derivatives;
5. Multi-band or multiscale sampling;
6. Uniform and nonuniform sampling of WSS random processes.

One of the motivations for these new schemes is the desire to achieve compactly supported synthesizing functions. Periodically nonuniform sampling can guarantee compactly supported $S_n(t)$'s under some restrictions as we explain in subsection 4.2.3. In order to overcome those restrictions of periodically nonuniform sampling, we introduce local averaging. This scheme can guarantee compact supports for $S_n(t)$'s under milder constraints. Local averaging offers some additional advantages. It has good noise sensitivity properties and some compression capabilities, as we show in subsection

† In the future, whenever we talk about compactly supported synthesizing functions, we assume that $\phi(t)$ is compactly supported as well. Cases when $S_n(t)$'s are compactly supported at the expense of not having compactly supported $\phi(t)$ are not considered.

4.2.4. At the expense of a slightly higher sampling rate, oversampling can always guarantee compactly supported $S_n(t)$'s. This is shown in subsection 4.3.1. There are situations where besides $f(t)$, its derivative is available as well. In these cases, we can reconstruct $f(t)$ from samples of $f(t)$ and $f'(t)$, at half the usual rate. In subsection 4.3.2, we show how this can be done. If it is known that $f(t)$ belongs to some subspace of V_m, this additional information can be used to sample $f(t)$ at a lower rate (sampling of bandpass signals, for example). This problem is treated in subsection 4.3.3. An application of some of the above mentioned methods to efficient computation of inner products in MRA subspaces is explained in subsection 4.3.4 (also, see the next subsection). Finally, in subsection 4.3.5, we analyze what happens if we make errors in sampling times, namely, if, instead of sampling at $\{t_n\}$, we sample at $\{t'_n\}$. We will show that this error can be controlled by $\delta = \sup_{n\in\mathcal{Z}} |t_n - t'_n|$.

All the above problems are embedded in the framework of multirate filter banks. We give sufficient conditions for the existence of stable reconstruction schemes and explicitly derive expressions for synthesizing functions. The theory of FIR filter banks is used to obtain compactly supported synthesizing functions.

In section 4.4, Walter's idea is extended to random processes. Things are little different now. First, we have to specify a class of random processes for which we want to develop the theory. So far, mainly wide sense stationary (WSS) random processes were considered [Llo59]. The autocorrelation function of random process $\{f(t), -\infty < t < \infty\}$ is defined as

$$R_{ff}(t,\tau) = E[f(t+\tau)f^*(t)], \tag{4.1.2}$$

where * denotes complex conjugation and $E[\cdot]$ statistical expectation. When $R_{ff}(t,\tau)$ does not depend on t, we call it a WSS random process. We assume that $R_{ff}(\tau)$ is the inverse Fourier transform of the power spectral density (PSD) function $S_{ff}(\omega)$. In Sec. 4.4, we will use assumptions that will insure that the Fourier transform of $R_{ff}(\tau)$ exists (for the relationship in a general case, see [Doo53]). Now we can characterize a random process in terms of its autocorrelation function. For example, a random process is bandlimited if its autocorrelation function is bandlimited. In this chapter, we consider WSS random processes whose autocorrelation functions belong to some space related to wavelet subspaces. The problem of reconstruction has two meanings now. First, we can talk about the reconstruction of a random process itself, i.e., existence of functions $S_n(t)$ such that $f(t) = \sum_n f(t_n)S_n(t)$ in the MS sense, i.e., $\lim_{N\to\infty} E[|f(t) - \sum_{n=-N}^{N} f(t_n)S_n(t)|^2] = 0$. The other interpretation is a reconstruction of the PSD function $S_{ff}(\omega)$. We show that a random process itself cannot be reconstructed if the synthesizing functions are assumed to be integer shifts of one function, unless, of course, the process is bandlimited. However, the PSD function can be reconstructed, and we show how. This is done for uniform sampling in subsection 4.4.1. Deterministic nonuniform sampling of a WSS random process does not give a WSS discrete parameter random process. That is why we introduce randomness into the sampling times (jitter). Nonuniform sampling of WSS random processes is considered in subsection 4.4.2.

4.1.2. The new results in the perspective of earlier work

An actual implementation of the MRA requires computation of the inner products $c_n = c_{0,n} = (f(t), \phi(t-n))$, which is computationally rather involved. Mallat proposed a method that gives an approximation of $c_{0,n}$ by highly oversampling $f(t)$. Daubechies suggested another method (another interpretation of Walter's theorem) that computes $c_{0,n}$ exactly, but involves convolutions with IIR filters. Shensa [She92] proposed a compromise between the above two methods. It has moderate complexity and nonzero error.

All of the above mentioned methods involve sampling of signals in V_0. Since our work is about sampling in MRA subspaces, we apply some of the results obtained in Sec. 4.2 and 4.3 to the problem of computation of $c_{0,n}$'s. In subsection 4.3.4, we give a qualitative comparison of the new and existing methods in terms of complexity, sampling rate and approximation error. While all our methods have zero error and pretty low complexities and sampling rates, periodically nonuniform sampling scheme achieves *zero* error at the *minimal* rate with FIR filters (*lowest complexity*).

4.2. Discrete representations of deterministic signals

In this section, we consider different discrete representations of functions in MRA subspaces V_m. We work in V_0 only, since all the relevant properties are independent of the scale (see [Wal92]). Other V_k and W_k subspaces will be considered in the case of multi-band sampling (next section). Let us first state assumptions and make some preliminary derivations. Certain definitions and theorems from the analysis that we use are given in Appendix 4.A, for the reader's convenience.

4.2.1. Assumptions and preliminary derivations

We assume that $\{\phi(t-n)\}$ forms a Riesz basis for $V_0 \subset L^2(\mathcal{R})$. In order to show that V_0 is a RKHS, Walter assumes that $\phi(t)$ is continuous and that it decays faster than $1/|t|$ for large t, i.e., there exists $C > 0$, such that $|\phi(t)| < C/(1+|t|)^{1+\epsilon}$, for some $\epsilon > 0$ (see [Wal92]). Janssen derives his result under weaker assumptions, namely, that $\phi(t)$ is bounded and that

$$\sum_n |\phi(t-n)| < C_\phi \tag{4.2.1}$$

converges uniformly on $[0,1]$. Note that this assures us that $\phi(t) \in L^1(\mathcal{R})$ and $\{\phi(t-n)\} \in l^1 \subset l^2$ for all t. Also, the Fourier transform of $\phi(t)$ is well defined and is a continuous function (see [Chu92]).

Since $\{\phi(t-n)\}$ is a Riesz basis for V_0, then for any $f(t) \in V_0$ there exists a unique sequence $\{c_n\} \in l^2$ such that

$$f(t) = \sum_n c_n \phi(t-n). \tag{4.2.2}$$

If the sampling times are $t_n = n + u_n$, then

$$f(t_n) = \sum_k c_k \phi(t_n - k) = \sum_k c_k \phi(u_n + n - k). \tag{4.2.3}$$

Let

$$\Phi_{u_n}(e^{j\omega}) = \sum_k \phi(u_n + k) e^{-jk\omega} \tag{4.2.4}$$

be the l^1–Fourier transform of $\{\phi(u_n+k)\}^\dagger$. Note that $\Phi(e^{j\omega})$ is a BIBO stable filter because of (4.2.1). In the rest of the chapter, we will frequently use the following lemma.

Lemma 4.2.1. Samples $f(t_n)$ can be written as follows:

$$f(t_n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} C(e^{j\omega}) \Phi_{u_n}(e^{j\omega}) e^{jn\omega} d\omega. \tag{4.2.5}$$

$\diamondsuit$

Proof: If we substitute $c_k = \frac{1}{2\pi}\int_{-\pi}^{\pi} C(e^{j\omega}) e^{jk\omega} d\omega$ in (4.2.3), we have

$$f(t_n) = \frac{1}{2\pi} \sum_k \int_{-\pi}^{\pi} C(e^{j\omega}) e^{jk\omega} \phi(u_n + n - k) d\omega = \frac{1}{2\pi} \int_{-\pi}^{\pi} C(e^{j\omega}) \Phi_{u_n}(e^{j\omega}) e^{jn\omega} d\omega. \tag{4.2.6}$$

The order of integration and summation can be interchanged because $\sum_k \int_{-\pi}^{\pi} |C(e^{j\omega})\phi(u_n + n - k)| d\omega < \infty$ (remember that $C(e^{j\omega}) \in L^2[-\pi,\pi] \subset L^1[-\pi,\pi]$ and $\{\phi(u_n+k)\} \in l^1)^\ddagger$.

† In our notation, $F(\omega)$ is the Fourier transform of a function $f(t)$ in $L^1(\mathcal{R})$ or $L^2(\mathcal{R})$, whereas, $F(e^{j\omega})$ is the Fourier transform of a sequence $\{f_n\}$ from l^1 or l^2.

‡ This argument for the interchangeability of integrals and/or sums will be often used without explicitly stating it.

◊

Let us say a few words about the type of convergence in (4.2.2). Riesz basis property guarantees L^2–convergence of the sum in (4.2.2), so that $f(t)$ is determined only almost everywhere (a.e.). Even though sampling of functions defined only a.e. is meaningless, it is a well-defined operation in our case because the sum in (4.2.2) converges uniformly on $\mathcal{R}$. To see this, use (4.2.1) and (4.2.2), to get

$$|f(t)| \leq \left(\sum_k |\phi(t-k)|^2 \right)^{1/2} ||c_k||_2 < C||f||_2, \tag{4.2.7}$$

where the second inequality is obtained using the definition of a Riesz basis. This relationship transforms L^2-convergence into uniform convergence. Since we already know that the sum in (4.2.2) converges in L^2 sense, it readily follows that it actually converges uniformly. Therefore, sampling of $f(t)$, as given by (4.2.2), is well-defined. In the rest of the chapter, we will consider the pointwise convergence of (4.2.2) and not worry about this anymore. Our main concern will be to get sequences $\{c_n\}$ from $\{f(t_n)\}$ in a stable way. The next subsection is a review of Walter's and Janssen's work.

4.2.2. Review of uniform sampling in wavelet subspaces

We will use an abstract setting for the sampling theory. It offers us a unified approach to all the problems in this and the following section. So, let us first explain this approach. The idea of sampling in wavelet subspaces is to find an invertible map between $\{c_n\}$ and $\{f(t_n)\}$ in (4.2.3). More generally, we want to find invertible maps between $\{c_n\}$ and some other discrete representations of functions in V_0. Let us define this map.

Definition 4.2.1. By $\mathcal{T}$ we denote an operator from l^2 into l^2. It maps sequence $\{c_n\}$ to $\{d_n\}$, where $\{d_n\}$ is some discrete representation of $f = \sum_n c_n \phi(t-n)$.

◊

When we talk about sampling theorems, we have $\{d_n\} = \{f(t_n)\}$, whereas in subsection 2.4, $\{d_n\}$ will be a sequence of local averages. Because of the Fourier transform isomorphism between l^2 and $L^2[-\pi,\pi]$, we can think of $\mathcal{T}$ as a map of $L^2[-\pi,\pi]$ into itself. In this basis, $\mathcal{T}$ is just a multiplication operator [from (4.2.5), action of $\mathcal{T}$ is multiplication by $\Phi_u(e^{j\omega})$, in the case of uniform sampling]. The following theorem is borrowed from [Jan93] just for the completeness of the presentation (the proof can be found in [Jan93]).

Theorem 4.2.1. (Janssen) Let $t_n = n + u$, for some $u \in [0,1)$. Operator $\mathcal{T} : \{c_n\} \to \{f(t_n)\}$, maps

l^2 into l^2 for any $f \in V_0$. Furthermore, $\mathcal{T}^{-1}$ is bounded, if $\Phi_u(e^{j\omega}) \neq 0$ for all $\omega \in [-\pi, \pi]$.

◊

This theorem can be visualized as in Fig. 4.2.1. So we see that this sampling theorem has a rather simple engineering interpretation in terms of digital filters.†

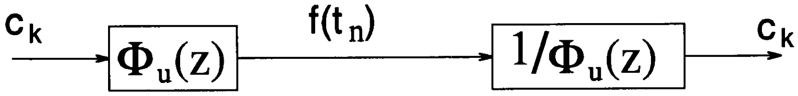


Fig. 4.2.1. Filter representation of uniform sampling.

Remarks.

1. There are some interesting connections to regularity theory. Notice that the condition from the above theorem, in terms of [Jan93], is that the Zak transform of $\phi(t)$, $(\mathcal{ZT}\phi)(u, \omega) \neq 0$ a.e. This condition is the same as Rioul's condition [Rio92] for the optimality of his regularity estimates. So if there is a $u \in [0, 1)$ such that $(\mathcal{ZT}\phi)(u, \omega) \neq 0$, then Rioul's regularity estimates are optimal, and $f(t)$ can be reconstructed from $\{f(n + u)\}$.

2. The synthesizing functions $S_n(t)$ have been derived in [Wal92] and [Jan93]. They have the following shift property

$$S_n(t) = S(t - n), \tag{4.2.8}$$

   where $S(t) = \sum_k \phi_k^{-u} \phi(t - k)$ and $\{\phi_n^{-u}\} = \mathcal{F}^{-1} \left(1/\Phi_u(e^{j\omega})\right)$.

3. When $\Phi_u(e^{j\omega})$ turns out to be a rational filter, it is obvious that if $\Phi_u(e^{j\omega}) \neq 0$, then $1/\Phi_u(e^{j\omega})$ is stable (possibly non-causal).

4. Evidently, we cannot make both $\Phi_u(e^{j\omega})$ and $1/\Phi_u(e^{j\omega})$ trigonometric polynomials, unless $\Phi_u(e^{j\omega}) = e^{-jk\omega}$. Therefore, if $\phi(t)$ is compactly supported [i.e., if $\Phi_u(e^{j\omega})$ is a trigonometric polynomial], $S(t)$ cannot have compact support. This negative conclusion does not hold in the schemes we propose.

4.2.3. Periodically nonuniform sampling

Now we consider the case of periodically nonuniform sampling. For this, let us choose $u_m \in [0, L)$ for

---

† In this chapter, z is just a formal argument, and it stands for $e^{j\omega}$. $\Phi(z)$ should not be interpreted as Z-transform in the conventional sense [Opp89]. Most of our signals are assumed to be in l^2, so that their Fourier transform exists in l^2 sense only.

$m = 0, 1, \ldots, L-1$, and set $t_{kL+m} = kL + u_m$. From the abstract point of view that we developed in the previous subsection, $\mathcal{T}$ can be viewed as the analysis filter bank of an L-channel maximally decimated filter bank (see [Vai93]). To see this, notice that $f(t_{kL+m})$, using Lemma 4.2.1, is

$$f(t_{kL+m}) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{jkL\omega} C(e^{j\omega}) \Phi_m(e^{j\omega}) d\omega, \tag{4.2.9}$$

where $\Phi_m(e^{j\omega}) = \Phi_{u_m}(e^{j\omega})$ for $m = 0, 1, \ldots, L-1$. In terms of multirate filter banks, $\{f(t_{kL+m})\}$ is the m^{th} subband signal in the following filter bank.

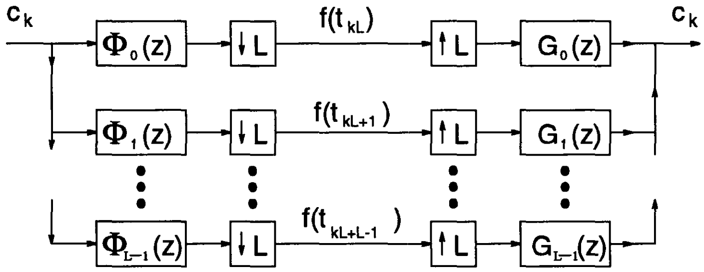


Fig. 4.2.2. Filter bank interpretation of periodically nonuniform sampling.

There is no fundamental difference between this scheme and that of Janssen. Fig. 4.2.1 is a trivial 1-channel filter bank. The aim here is the same. We want to get back $\{c_n\}$ from $\{f(t_n)\}$ in a stable way ($\{f(t_{kL+m})\} \in l^2$ for $m = 0, 1, \ldots, L-1$, therefore, $\{f(t_n)\} \in l^2$ as well). In Fig. 4.2.1, we inverted filter $\Phi_u(z)$. Here, we have to find a stable synthesis filter bank (inverse of the analysis filter bank) †. For this we will use standard techniques from the multirate filter bank theory. Let

$$\Phi_m(z) = \sum_{k=0}^{L-1} z^{-k} E_{mk}(z^L) \qquad \text{and} \qquad G_m(z) = \sum_{k=0}^{L-1} z^{k} R_{km}(z^L)$$

be the polyphase decompositions of analysis and synthesis filters (for more discussion on polyphase decompositions, see [Vai93]). Then we define the polyphase matrices as $[\mathbf{E}(z)]_{kl} = E_{kl}(z)$ and $[\mathbf{R}(z)]_{kl} = R_{kl}(z)$. So, when this filter bank has perfect reconstruction (PR) property, the operator $\mathcal{T}$ and its inverse $\mathcal{T}^{-1}$ can be represented by $\mathbf{E}(z)$ and $\mathbf{R}(z)$ respectively (see Fig. 4.2.3). Now that we have this filter bank interpretation of $\mathcal{T}$, we can return to the problem of uniqueness and stability.

† Stable in our context does not necessarily mean BIBO stable. We want a bounded transformation of l^2 into l^2. However, it turns out that if the synthesis filter bank is BIBO stable, i.e., if the filter coefficients are in l^1, then it also represents a bounded transformation from l^2 into l^2; see Appendix 4.A.

Lemma 4.2.2. In the case of periodically nonuniform sampling, $\{t_n\}$ is a sequence of uniqueness if $\mathbf{E}(e^{j\omega})$, as defined above is nonsingular almost everywhere (a.e.).

$\diamond$

Proof: Let $\mathbf{c}_k = [c_{kL} \quad c_{kL-1} \quad \cdots \quad c_{kL-L+1}]^T$ and $\mathbf{f}_k = [f(t_{kL}) \quad f(t_{kL+1}) \quad \cdots \quad f(t_{kL+L-1})]^T$ be blocked versions of $\{c_k\}$ and $\{f(t_k)\}$. Then, using Noble identities (see [Vai93]), the system from Fig. 4.2.2 can be transformed into that in Fig. 4.2.3.

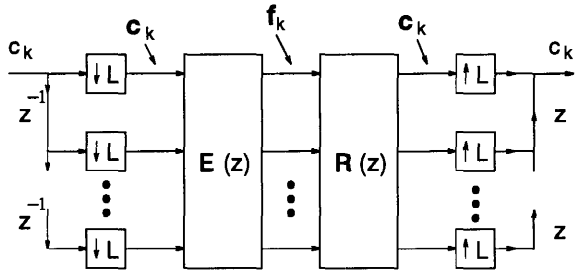


Fig. 4.2.3. Polyphase representation.

Notice that $||c_k||_2^2 = ||\mathbf{c}_k||_2^2 = \sum_k \mathbf{c}_k^\dagger \mathbf{c}_k$ and that $\mathbf{f}(e^{j\omega}) = \mathbf{E}(e^{j\omega})\mathbf{c}(e^{j\omega})$, where $\mathbf{c}(e^{j\omega}) = \sum_k \mathbf{c}_k e^{-jk\omega}$ and $\mathbf{f}(e^{j\omega}) = \sum_k \mathbf{f}_k e^{-jk\omega}$ [$\mathbf{f}(e^{j\omega})$ and $\mathbf{c}(e^{j\omega})$ are elements of $(L^2[-\pi,\pi])^L$]. Again, using Parseval's equality, we have

$$||f_k||_2^2 = ||\mathbf{f}_k||_2^2 = \sum_k \mathbf{f}_k^\dagger \mathbf{f}_k = \frac{1}{2\pi}\int_{-\pi}^{\pi} \mathbf{f}^\dagger(e^{j\omega})\mathbf{f}(e^{j\omega})d\omega$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} \mathbf{c}^\dagger(e^{j\omega})\mathbf{E}^\dagger(e^{j\omega})\mathbf{E}(e^{j\omega})\mathbf{c}(e^{j\omega})d\omega. \tag{4.2.10}$$

If $\mathbf{E}^\dagger(e^{j\omega})\mathbf{E}(e^{j\omega})$ (a positive semidefinite matrix) is nonsingular a.e., $||f_k||_2$ can be zero if and only if $\mathbf{c}^\dagger(e^{j\omega})\mathbf{c}(e^{j\omega}) = 0$ a.e., which implies that $\{c_k\}$ is a zero sequence itself.

$\diamond$

The question of stable reconstruction can be answered using Wiener's theorem (see Appendix 4.A, and [Rud87]). What we want is BIBO stable $\mathbf{E}^{-1}(e^{j\omega}) = \mathbf{R}(e^{j\omega})$.

Lemma 4.2.3. A stable recovery from periodically nonuniform samples $\{f(t_n)\}$ of any $f \in V_0$ is possible if there exist BIBO stable synthesizing filters $G_k(z)$. This will be the case if and only if [det $\mathbf{E}(e^{j\omega})$] $\neq 0$ for all $\omega \in [-\pi,\pi]$.

$\diamond$

Proof: Since entries of $\mathbf{E}(e^{j\omega})$ are Fourier transforms of l^1 sequences, so is the determinant [det $\mathbf{E}(e^{j\omega})$] (operation of convolution is closed in l^1, as explained in Appendix 4.A). Now by the Wiener's theorem, convolutional inverse of [det $\mathbf{E}(e^{j\omega})$] is in l^1, if and only if [det $\mathbf{E}(e^{j\omega})$] $\neq 0$. Then the entries of $\mathbf{E}^{-1}(e^{j\omega})$ are Fourier transforms of l^1 sequences. This means that $\mathbf{R}(z)$ is a MIMO, BIBO stable system and therefore $\mathbf{c}(e^{j\omega})$ can be recovered from $\mathbf{f}(e^{j\omega})$ in a stable way.

◇

Remark. The existence of FIR perfect reconstruction filter banks allows for the possibility of having both $\phi(t)$ and $S_n(t)$'s compactly supported, unlike the schemes in [Wal92] and [Jan93]. In particular, if $\Phi_m(z) = \sum_{k=0}^{L-1} \phi(u_m + n)z^{-n}$ for $m = 0, 1, \ldots, L-1$, then the polyphase matrix will be just a constant matrix. In this case, the inverse filter bank is guaranteed to be FIR. This constraint on the number of channels will not be necessary in the schemes we propose next.

Expressions for synthesis functions

Let us now construct synthesis functions and show that they have some shift property as well. $G_k(e^{j\omega}) = \sum_n g_n^k e^{-jn\omega}$ are the synthesis filters in our perfect reconstruction (PR) filter bank (see Fig. 4.2.2). The PR property implies that (see [Vai93])

$$c_m = \sum_{k=0}^{L-1} \sum_n f(t_{nL+k}) g_{m-nL}^k. \tag{4.2.11}$$

Then the reconstructed function is

$$f(t) = \sum_{k=0}^{L-1} \sum_n f(t_{nL+k}) \sum_m g_{m-nL}^k \phi(t-m) = \sum_{k=0}^{L-1} \sum_n f(t_{nL+k}) S_{kn}(t). \tag{4.2.12}$$

If we define $S_k(t) = \sum_m g_m^k \phi(t-m)$, then

$$S_{kn}(t) = \sum_m g_{m-nL}^k \phi(t - m + nL - nL) = S_k(t - nL), \qquad 0 \leq k \leq L-1. \tag{4.2.13}$$

So all the synthesizing functions are obtained as shifts of L basic functions. The above lemmas are summarized in the following theorem.

Theorem 4.2.2. Let $\Phi_m(z)$ and $\mathbf{E}(z)$ be analysis filters and their polyphase matrix as shown in Fig. 4.2.2 and 4.2.3. A stable reconstruction of $\{c_n\}$ from the samples $\{f(t_n)\}$ exists if the determinant [det $\mathbf{E}(e^{j\omega})$] $\neq 0$ for all $\omega \in [-\pi, \pi]$. Furthermore, all synthesizing functions are shifts of L fixed functions, as given by (4.2.13).

◇

Next, we are going to illustrate the above theory with some practical examples. Before this, let us first give a short summary of the algorithm.

Summary of the algorithm for recovery of $\{c_n\}$ from $\{f(t_n)\}$

1. Choose $u_m \in [0, L-1)$, for $m = 0, 1, \ldots, L-1$.
2. Obtain filters $\Phi_m(e^{j\omega}) = \sum_n \phi(n + u_m)e^{-jn\omega}$ and form $\mathbf{E}(z)$.
3. Find $\mathbf{E}^{-1}(z)$ (provided Theorem 4.2.2 is satisfied) and calculate synthesis filters $G_m(e^{j\omega})$'s.
4. Construct synthesis functions $S_{kn}(t)$'s as given by (4.2.13).

Example 4.2.1. Consider the MRA generated by linear splines (see [Chu92]). The scaling function is

$$\phi(t) = \begin{cases} t, & \text{for } 0 \le t < 1, \\ 2 - t, & \text{for } 1 \le t < 2, \\ 0, & \text{otherwise.} \end{cases}$$

We are interested in the case where there exist compactly supported synthesizing functions. For this, let us choose $L = 2$, $u_0 = 0$ and $u_1 = 0.5$. Fig. 4.2.4 shows $\phi(t)$ and samples $\phi(n + u_m)$ which determine filter coefficients. Namely, from (4.2.4), $\Phi_m(z) = \sum_n \phi(n + u_m)z^{-n} = \sum_n \phi_n^m z^{-n}$, so that $\Phi_0(z) = z^{-1}$ and $\Phi_1(z) = \frac{1}{2}(1 + z^{-1})$.

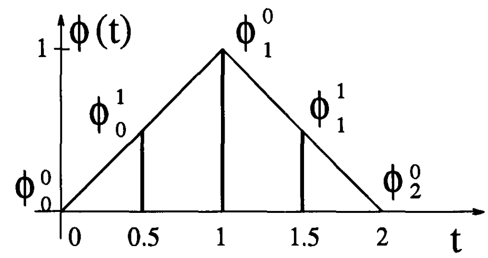


Fig. 4.2.4. Linear spline and its samples at $n + u_m$.

The polyphase matrix is $\mathbf{E}(z) = \begin{pmatrix} 0 & 1 \\ 1/2 & 1/2 \end{pmatrix}$. The inverse of $\mathbf{E}(z)$ is $\mathbf{R}(z) = \mathbf{E}^{-1}(z) = \begin{pmatrix} -1 & 2 \\ 1 & 0 \end{pmatrix}$. Then the synthesis filters are $G_0(z) = z - 1$ and $G_1(z) = 2$. The synthesis functions are $S_0(t - 2n)$ and $S_1(t - 2n)$, where $S_0(t) = \phi(t+1) - \phi(t)$ and $S_1(t) = 2\phi(t)$.

Example 4.2.2. Let us now consider the case of quadratic splines [Chu92]. The scaling function is

$$\phi(t) = \begin{cases} t^2/2, & \text{for } 0 \le t < 1, \\ -(t - 3/2)^2 + 3/4, & \text{for } 1 \le t < 2, \\ \frac{1}{2}(t-3)^2, & \text{for } 2 \le t < 3, \\ 0, & \text{otherwise.} \end{cases}$$

As usual, we want to have compactly supported synthesizing functions. Since $\phi(t)$ is supported on $[0, 3]$, we choose $L = 3$ and $u_0 = 0$, $u_1 = 1/3$, and $u_2 = 2/3$. Then $\Phi_0(z) = \frac{1}{2}(z^{-1} + z^{-2})$, $\Phi_1(z) = \frac{1}{18} + \frac{13}{18}z^{-1} + \frac{2}{9}z^{-2}$, and $\Phi_2(z) = \frac{2}{9} + \frac{13}{18}z^{-1} + \frac{1}{18}z^{-2}$. The polyphase matrix and its inverse are

$$\mathbf{E}(z) = \begin{pmatrix} 0 & 1/2 & 1/2 \\ 1/18 & 13/18 & 2/9 \\ 2/9 & 13/18 & 1/18 \end{pmatrix} \qquad \text{and} \qquad \mathbf{E}^{-1}(z) = \mathbf{R}(z) = \begin{pmatrix} 13/4 & -9 & 27/4 \\ -5/4 & 3 & -3/4 \\ 13/4 & -3 & 3/4 \end{pmatrix}.$$

From $\mathbf{R}(z)$ we get $G_k(z)$'s and synthesizing functions $S_0(t-3n)$, $S_1(t-3n)$ and $S_2(t-3n)$, where $S_0(t) = \frac{13}{4}\phi(t) - \frac{5}{4}\phi(t+1) + \frac{13}{4}\phi(t+2)$, $S_1(t) = -9\phi(t) + 3\phi(t+1) - 3\phi(t+2)$, and $S_2(t) = \frac{27}{4}\phi(t) - \frac{3}{4}\phi(t+1) + \frac{3}{4}\phi(t+2)$.

So, we see that the actual implementation of the algorithm can be simple. The above examples show how we can achieve compactly supported synthesizing functions, but the problem is that if the support of $\phi(t)$ contains interval $[N_1, N_2]$ (N_1, N_2 are integers), then we need $L \geq N_2 - N_1$ channels. In other words, the number of channels L, necessary for $\mathbf{E}(z)$ to be a constant matrix, grows linearly with the length of the support of $\phi(t)$.

4.2.4. Reconstruction from local averages

In this subsection, we consider another discrete representation of functions in V_0. The motivation for this subsection comes from [Fei94], where it was shown that a bandlimited function can be recovered from its local averages. We extend this representation to wavelet subspaces. This scheme offers three advantages over the previous ones.

1. *Compactly supported synthesizing functions.* If $\phi(t)$ is compactly supported, then we can guarantee existence of compactly supported synthesizing functions $S_n(t)$'s. Unlike in the case of nonuniform sampling, we can guarantee this even when we have only two channels, regardless of the length of the support of $\phi(t)$.
2. *Shaping of the frequency response.* We have a complete control over the shape of frequency responses of analysis/synthesis filters. Namely, we can force filters $\Phi_m(z)$ to be anything we want (provided, a length constraint is satisfied).
3. *Robustness.* This scheme has reduced sensitivity towards the input noise, compared to pure sampling.

In the first part of the subsection, we make introductory derivations, similar to those in the case of uniform and periodically nonuniform sampling. Then, at the expense of a small increase in complexity, we modify the scheme in order to gain control over the filter coefficients of $\Phi_m(z)$'s. This discrete representation has the same average rate of sampling as in previous sections.

The main idea is to find ways of reconstructing $f(t)$ not from samples $f(t_k)$, but from local averages $a_k = \int_{t_k-\Delta}^{t_k+\Delta} f(t)dt$ around t_k (as shown in Fig. 4.2.5). In this uniform case we can choose $\Delta = 1/2$ and $t_k = k + 1/2 + u$. Then †

$$a_k = \int_{k+u}^{k+1+u} f(t)dt = \int_{k+u}^{k+1+u} \sum_n c_n \phi(t-n)dt. \tag{4.2.14}$$

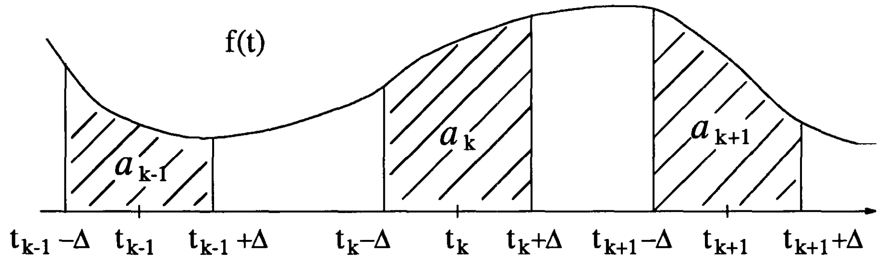


Fig. 4.2.5. Local averaging scheme.

If $\phi_k^u = \int_{k+u}^{k+1+u} \phi(t)dt$, then (4.2.1) implies that $\{\phi_k^u\} \in l^1$. $\{a_k\}$ can be viewed as a convolution of $\{c_k\} \in l^2$ and $\{\phi_k^u\}$, so that $\{a_k\} \in l^2$. Therefore, its Fourier transform $A(e^{j\omega}) = \sum_n a_n e^{-jn\omega}$ is well defined. Then, just as in the case of uniform sampling (subsection 2.2), we have the following relation

$$A(e^{j\omega}) = C(e^{j\omega})\Phi_u(e^{j\omega}),$$

where, now, $\Phi_u(e^{j\omega}) = \sum_n \phi_n^u e^{-jn\omega}$ is the Fourier transform of $\{\phi_n^u\}$. The only difference from the case of uniform sampling is that operator $\mathcal{T}$ now maps sequence $\{c_k\}$ into the sequence of local averages $\{a_k\}$. So, this system is the same as that in Fig. 4.2.1, except for the definition of $\Phi_u(z)$ and the meaning of its output $\{a_k\}$. Therefore, we have the following theorem.

Theorem 4.2.3. If $\Phi_u(e^{j\omega})$ as defined above is nonzero on $[-\pi, \pi]$, then the representation of a function $f \in V_0$ by its local averages a_k is unique. Moreover, there exists a stable reconstruction algorithm.

◇

Remark. This scheme has the same problems as that of uniform sampling. Namely, $S_n(t)$'s and $\phi(t)$ cannot be simultaneously compactly supported.

Let us now consider periodically nonuniform averaging. The idea is to partition the interval $[0, L]$ into L subintervals I_m, $m = 0, 1, \ldots, L-1$. The sequence of local averages is defined as

$$a_{kL+m} = \int_{kL+I_m} f(t)dt. \tag{4.2.15}$$

† This integral exists because $f(t) \in L^2([u, u+1]) \subset L^1([u, u+1])$ for all $u \in \mathcal{R}$.

Let $\phi_n^m = \int_{n+I_m} \phi(t)dt$ and $\Phi_m(e^{j\omega}) = \sum_n \phi_n^m e^{-jn\omega}$. Using this notation, we have

$$a_{nL+m} = \int_{nL+I_m} \sum_k c_k \phi(t-k)dt = \sum_k c_k \phi_{nL-k}^m. \tag{4.2.16}$$

This equation can be viewed as a convolution of $\{c_k\}$ with $\{\phi_k^m\}$ followed by L-fold decimation. Accordingly, in the frequency domain, this means $A_m(e^{j\omega}) = \left(C(e^{j\omega})\Phi_m(e^{j\omega})\right)\downarrow_L$, where $A(z) = \sum_{k=0}^{L-1} z^{-k} A_k(z^L)$ is the polyphase decomposition of $A(z)$, and $\downarrow_L$ denotes L-fold decimation. The situation is completely identical to that in the case of periodically nonuniform sampling (shown in Fig. 4.2.2), and Theorem 4.2.2 holds true for this case (only filters $\Phi_m(e^{j\omega})$'s are obtained in a different way). Again, if $\Phi_m(e^{j\omega})$'s are FIR and such that the synthesis filter bank is FIR as well, the synthesizing functions will be compactly supported.

But if everything is the same as in the case of nonuniform sampling, what is the point of doing all this? The answer is given in the rest of the subsection: *at the expense of a little more complexity, we will be able to force filters $\Phi_m(e^{j\omega})$ to be anything we want.*

Nonuniform sampling gave us very little control over the filters $\Phi_m(e^{j\omega})$. Consequently, it is unlikely that filters $\Phi_m(e^{j\omega})$, in that scheme, will have an FIR inverse filter bank (unless we choose L big enough to make $\mathbf{E}(z)$ a constant matrix). In order to make this happen for any $L \geq 2$, we have to work a little harder. For this, notice that what we have done so far, in this section, is equivalent to finding inner products of $f(t)$ and windows $w_k(t-nL)$ where $w_k(t) = \chi_{I_k}(t)$ is the characteristic function of the interval I_k. If we use some other window, can we use this freedom to make sure that the filter bank has an FIR inverse? We will show that the answer is in the affirmative.

Let $\phi(t)$ be compactly supported on an interval $[0, N]$. We divide $[0,1]$ into N subintervals I_k, $k = 0, 1, \ldots, N-1$. Let the windows be piecewise constant functions

$$w_m(t) = \sum_{l=0}^{N-1} \alpha_{ml} \chi_{I_l}(t). \tag{4.2.17}$$

We are going to show how coefficients α_{ml} can be used to gain more control over filters $\Phi_m(e^{j\omega})$'s.

Let $\{H_m(e^{j\omega})\}$ and $\{G_m(e^{j\omega})\}$ be analysis and synthesis filters of an FIR PR, L-channel filter bank. We assume that filters $H_m(e^{j\omega})$ have lengths N. Note that, except for the length constraint, $\{H_m(z)\}$, $\{G_m(e^{j\omega})\}$ is an arbitrary FIR PR filter bank.

The idea is to choose α_{ml}'s so that $\Phi_m(e^{j\omega}) = H_m(e^{j\omega})$. Let us see how to achieve this. The filter coefficients of $\Phi_m(e^{j\omega})$ are $\phi_n^m = \int w_m(t)\phi(t+n)dt$. Using (4.2.17), this can be written as

$$\phi_n^m = \sum_{l=0}^{N-1} \alpha_{ml} \int_{I_l} \phi(n+t)dt = \sum_{l=0}^{N-1} \alpha_{ml}\gamma_{nl}, \tag{4.2.18}$$

where $\gamma_{nl} = \int_{I_l} \phi(n+t)dt$. Now, there are L systems of linear equations (for $m = 0, 1, \ldots, L-1$), each in N unknowns α_{ml}, $l = 0, 1, \ldots, N-1$. All these systems have the same system matrix $[\mathbf{\Gamma}]_{nl} = \gamma_{nl}$. If $\mathbf{\Gamma}$ is nonsingular, there is a unique solution for α_{ml}'s. If $\mathbf{\Gamma}$ turns out to be singular (which is very unlikely), we can change subintervals I_l and get a nonsingular $\mathbf{\Gamma}$ (almost surely). This way we can shape filters $\Phi_m(e^{j\omega})$ into anything we want. The synthesizing functions can be obtained from $\{G(z)\}$ as in the previous subsection [see (4.2.11-2.13)]. Even though the equations look messy, application of the algorithm is pretty straightforward. Let us summarize the steps.

Summary of the algorithm

1. Partition interval $[0, 1]$ into N arbitrary subintervals I_k.
2. Form the matrix $\mathbf{\Gamma}$ with entries $[\mathbf{\Gamma}]_{n,l} = \int_{I_l} \phi(n+t)dt$. If $\mathbf{\Gamma}$ is singular, go to step 1; otherwise, determine α_{nl}'s from the filter coefficients of desired filters $\{H(z)\}$.
3. Determine synthesizing functions $S_{kn}(t)$'s from $\{G_k(z)\}$ as given by (4.2.13).

We demonstrate the algorithm on simple examples of spline generated MRA's.

Example 4.2.3. Let the scaling function be the linear spline, as in Example 2.1, and let $L = 2$ (two channels), $I_0 = [0, 0.5]$ and $I_1 = [0.5, 1]$. Entries of $\mathbf{\Gamma}$ are easy to calculate. They are the areas shown in Fig. 4.2.6. More precisely, $\gamma_{00} = \gamma_{11} = \int_0^{0.5} \phi(t)dt = 1/8$, $\gamma_{01} = \gamma_{10} = \int_{0.5}^{1} = 3/8$. $\mathbf{\Gamma} = \frac{1}{8}\begin{pmatrix} 1 & 3 \\ 3 & 1 \end{pmatrix}$ is nonsingular and its inverse is $\mathbf{\Gamma}^{-1} = \begin{pmatrix} -1 & 3 \\ 3 & -1 \end{pmatrix}$.

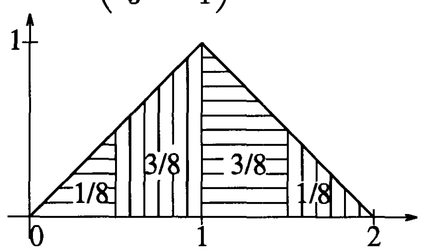


Fig. 4.2.6. Areas under $\phi(t)$ over intervals I_l are entries of $\mathbf{\Gamma}$.

Suppose we wish to choose windows $w_k(t)$ such that $\Phi_0(z) = H_0(z) = \frac{1}{\sqrt{2}}(1+z^{-1})$ and $\Phi_1(z) = H_1(z) = \frac{1}{\sqrt{2}}(1-z^{-1})$ ($\Phi_0(z)$ and $\Phi_1(z)$ are analysis filters of the simplest two-channel paraunitary filter bank). Then the coefficients α_{nl} are obtained as $\begin{pmatrix} \alpha_{00} \\ \alpha_{01} \end{pmatrix} = \mathbf{\Gamma}^{-1}\begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix} = \begin{pmatrix} \sqrt{2} \\ \sqrt{2} \end{pmatrix}$ and $\begin{pmatrix} \alpha_{10} \\ \alpha_{11} \end{pmatrix} = \mathbf{\Gamma}^{-1}\begin{pmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{pmatrix} = \begin{pmatrix} -2\sqrt{2} \\ 2\sqrt{2} \end{pmatrix}$. So the windows are $w_0(t) = \sqrt{2}\ \chi_{[0,1]}(t)$ and $w_1(t) = 2\sqrt{2}(-\chi_{[0,1/2]}(t) + \chi_{[1/2,1]})$. Since $\Phi_k(z)$ form a paraunitary filter bank, the synthesis filters are time reversed versions of $\Phi_k(z)$'s, i.e., $G_0(z) = \frac{1}{\sqrt{2}}(z+1)$ and $G_1(z) = \frac{1}{\sqrt{2}}(-z+1)$. So, the compactly supported synthesizing functions are $S_0(t-2n)$ and $S_1(t-2n)$, where $S_0(t) = \frac{1}{\sqrt{2}}(\phi(t)+\phi(t+1))$ and $S_1(t) = \frac{1}{\sqrt{2}}(\phi(t)-\phi(t+1))$. This example shows how easy it is to shape filters $\Phi(z)$'s. In a similar way, if we consider some scaling function with a bigger support, we can have longer filters with better frequency responses.

Example 4.2.4. In this example we want to show that the number of channels may be smaller than the length of filters $\Phi_m(z)$'s. For demonstration, we choose quadratic splines as given in Example 4.2.2. Choose $L = 3$, and $I_0 = [0, 1/3]$, $I_1 = [1/3, 2/3]$, and $I_2 = [2/3, 1]$; it is straightforward to check that

$$\mathbf{\Gamma} = \frac{1}{162}\begin{pmatrix} 1 & 7 & 19 \\ 34 & 40 & 34 \\ 19 & 7 & 1 \end{pmatrix}$$

is nonsingular. So we can force filters $\Phi_m(z)$ to be any desired filters of length 3. Notice that there are no assumptions about the number of channels. In particular, compactly supported synthesizing functions can be guaranteed with a two-channel filter bank (unlike in the case of periodically nonuniform sampling). All this generalizes for $\phi(t)$'s with bigger supports.

Remarks.

1. *Complexity price.* Notice that the windows $w_k(t)$ are step functions and it is not necessary to do true inner products. A simple "integrate and dump" circuit with weighted output will do.
2. *An additional advantage.* We wanted a complete control over the filters $\Phi_k(z)$ in order to be able to guarantee compact supports for the synthesizing functions $S_k(t)$'s. However, this freedom can be used to achieve even more. Namely, we can design $\Phi_m(z)$'s with good frequency characteristics and then use standard subband coding techniques for signal compression.
3. *Limitations.* This extended local averaging technique works for compactly supported scaling functions only. As the length of the support of $\phi(t)$ increases, we have to subdivide interval $[0, 1]$ into more and more subintervals. This makes the scheme more and more sensitive to errors in limits of integration.

At the beginning of the subsection we announced three main advantages of local averages over the previous schemes. So far, we justified the first two. Intuitively, it is clear that if the input signal $f(t)$ is contaminated with a zero mean noise $n(t)$, local integration will tend to eliminate the effect of the noise. This can be more rigorously justified, and the details are provided in Appendix 4.B.

4.3. Further extensions of sampling in wavelet subspaces

In this section, ideas of sampling in wavelet subspaces are extended to three more cases. Namely, we consider oversampling, derivative sampling and multi-band sampling. As in Sec. 4.2, we mainly work in V_0, except for the multi-band case. All the assumptions from subsection 4.2.1 are kept, and whenever we make some additional assumption, it will be explicitly stated.

4.3.1. Oversampling

We already saw two schemes in which both $\phi(t)$ and $S_n(t)$'s can be compactly supported. Here, we introduce another such scheme, oversampling. So far we considered L-channel maximally decimated filter banks only (Fig. 4.2.1 is a special case with $L = 1$). Let us see what happens in the case of non-maximally decimated filter banks. Instead of doing derivations for the most general case, we will demonstrate the idea on the example of a two–channel non-decimated filter bank. For this, we choose $u_0, u_1 \in [0, 1)$ and $u_0 \neq u_1$. Sampling $f(t) \in V_0$ at all points $n + u_0$, $n + u_1$, we get two sequences, $\{f(n+u_0)\}$ and $\{f(n+u_1)\}$. We know from subsection 4.2.2 that $f(t)$ can be recovered from either of these two sequences (provided conditions of Theorem 4.2.1 are satisfied). The idea is to use the redundancy to achieve reconstruction of $f(t)$ with compactly supported functions. Let $\Phi_i(e^{j\omega}) = \sum_k \phi(k + u_i)e^{-jk\omega}$ for $i = 0, 1$. Using our abstract approach, this situation can be represented as in Fig. 4.3.1

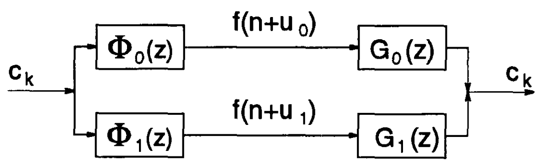


Fig. 4.3.1. Filter bank interpretation of oversampling.

Now, if $\Phi_i(z)$ are polynomials, Euclid's algorithm guarantees existence of polynomial $G_i(z)$'s if and only if $\Phi_0(z)$ and $\Phi_1(z)$ are coprime. In the general case, when $\Phi_i(z)$'s are not necessarily FIR, the following theorem gives us solution to the reconstruction problem.

Theorem 4.3.1. For the system in Fig. 4.3.1, operator $\mathcal{T}$, maps $L^2[-\pi,\pi]$ into $L^2[-\pi,\pi] \times L^2[-\pi,\pi]$. If $|\Phi_0(e^{j\omega})| + |\Phi_1(e^{j\omega})| \neq 0$ for all $\omega \in [-\pi,\pi]$, $\mathcal{T}$ has a bounded inverse. Furthermore, if $\phi(t)$ is compactly supported, there *always* exists an FIR inverse, so that $S_n(t)$'s can be compactly supported.

◇

Proof: In this case, operator $\mathcal{T}$ is a 2×1 matrix whose entries are Fourier series of l^1 sequences. It is easy to see that $\mathcal{T}$ maps $L^2[-\pi,\pi]$ into $L^2[-\pi,\pi] \times L^2[-\pi,\pi]$. What we need is a bounded inverse operator $\mathcal{T}^{-1}$. If we can guarantee that there always exists a 1×2 matrix [$(G_0(z) \quad G_1(z))$ in our notation in Fig. 4.3.1], whose entries are Fourier transforms of sequences in l^1, then we are done (see Appendix 4.A).

For the case of polynomial $\Phi_i(z)$'s, Euclid's algorithm guarantees existence of polynomials $G_0(z)$ and $G_1(z)$ such that $\Phi_0(z)G_0(z) + \Phi_1(z)G_1(z) = 1$, provided $\Phi_0(z)$ and $\Phi_1(z)$ are coprime. The same is true for rational $\Phi_i(z)$'s.

In the general case, an extension of the Wiener's theorem (basic Wiener's theorem is stated in Appendix 4.A) is as follows. If $|\Phi_0(e^{j\omega})| + |\Phi_1(e^{j\omega})| \neq 0$ for all $\omega \in [-\pi,\pi]$ and $\Phi_0(e^{j\omega})$, $\Phi_1(e^{j\omega})$ have absolutely summable Fourier series, then there exist functions $G_i(e^{j\omega})$ $(i = 0,1)$ with absolutely summable Fourier series, such that $\Phi_0(e^{j\omega})G_0(e^{j\omega}) + \Phi_1(e^{j\omega})G_1(e^{j\omega}) = 1$ (see [Vid85] for the proof).

◇

Remarks.

1. *Local averages.* It is clear that we can conceive the idea of oversampled local averages. Namely, let us divide the interval $[0,1)$ by some u $(0 < u < 1)$ and consider following sequences $a_n^0 = \int_n^{n+u} f(t)dt$ and $a_n^1 = \int_{n+u}^{n+1} f(t)dt$. Then, we can have a similar structure as in Fig. 4.3.1, and a theorem analogous to theorem 4.3.1. holds.
2. *Oversampling at a lower rate.* Compact supports of $\phi(t)$ and $S_n(t)$'s can be guaranteed even if we oversample at a lower rate. In the discussions above, we considered a non-decimated two–channel filter bank. However, both Euclid's and Wiener's theorem can be generalized for the matrix case. For example, if we choose L points $u_0, u_1, \ldots, u_{L-1} \in [0, M]$ and sample at t_{kM+m} for $m = 0, 1, \ldots, L-1$, the scheme can be viewed as an L-channel filter bank, with decimation ratio $M < L$. The operator $\mathcal{T}$ is an $L \times M$ matrix $\mathbf{E}(e^{j\omega})$ whose entries are Fourier transforms of l^1 sequences. In the case of polynomial matrices, an extended Euclid's theorem guarantees existence of a polynomial inverse $M \times L$ matrix if rank $[\mathbf{E}(e^{j\omega})] = M$ for all $\omega \in [-\pi,\pi]$. More generally, an extended version of the Wiener's theorem guarantees the existence of a bounded inverse operator $\mathcal{T}^{-1}$ if rank$[\mathbf{E}(e^{j\omega})] = M$ for all $\omega \in [-\pi,\pi]$. Sampling rate in this case is $L/M > 1$. So we see that compact supports for $\phi(t)$ and $S_n(t)$'s can be guaranteed if we sample at rate $1 + \epsilon$ for any $\epsilon > 0$.

4.3.2. Reconstruction from samples of functions and their derivatives

It is well known that a bandlimited function $f(t)$ can be recovered from samples of $f(t)$ and its derivative at *half* the Nyquist rate (see [Jer77] for further references). In this subsection, we want to show how this idea can be extended for the case of wavelet subspaces. This problem can be treated the same way as that of periodically nonuniform sampling. We will derive expressions for the analysis bank filters, since that is the only difference from subsection 4.2.3. Let us demonstrate the idea on the example of reconstruction of $f(t) \in V_0$ from the samples of $f(t)$ and its derivative at rate 1/2.

We assume that the scaling function $\phi(t)$ is compactly supported and that it has a derivative $\phi'(t)$. It is also assumed that $\phi'(t)$ itself satisfies Janssen's conditions stated in subsection 4.2.1. Consider uniform sampling, i.e., $t_n = n + u$. The above assumptions enable us to differentiate (4.2.2) term by term and get

$$f'(t_n) = \sum_k c_k \phi'(t_n - k). \tag{4.3.1}$$

Using Lemma 4.2.1, we have

$$f(t_n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{jn\omega} C(e^{j\omega})\Phi_0(e^{j\omega})d\omega \quad \text{and} \quad f'(t_n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{jn\omega} C(e^{j\omega})\Phi_1(e^{j\omega})d\omega, \tag{4.3.2}$$

where

$$\Phi_0(e^{j\omega}) = \sum_n \phi(u+n)e^{-jn\omega} \quad \text{and} \quad \Phi_1(e^{j\omega}) = \sum_n \phi'(u+n)e^{-jn\omega}. \tag{4.3.3}$$

From [Jan93], we know that $f(t)$ can be reconstructed from $\{f(t_n)\}$. This means that the sequence of derivatives $\{f'(t_n)\}$ is redundant. The idea is to use this redundancy to reconstruct $f(t)$ from subsampled sequences $\{f(t_{2n})\}$ and $\{f'(t_{2n})\}$. Notice that this scheme corresponds to a two–channel maximally decimated filter bank with $\Phi_0(e^{j\omega})$ and $\Phi_1(e^{j\omega})$ as analysis filters. So everything is the same as in Fig. 4.2.2, and Theorem 4.2.2 provides us with conditions for the existence of a stable reconstruction. Namely, stable reconstruction is possible if the polyphase matrix of $\Phi_0(e^{j\omega})$ and $\Phi_1(e^{j\omega})$ is nonsingular for all $\omega \in [-\pi, \pi]$.

This can be easily generalized to the case of higher derivatives. Assume that the scaling function $\phi(t)$ and its $M-1$ derivatives satisfy Janssen's conditions from subsection 4.2.1. Then $f(t)$ can be reconstructed from the samples of $f(t)$ and its $M-1$ derivatives at $1/M^{th}$ Nyquist rate, provided conditions of Theorem 4.2.2 are satisfied. Synthesizing functions can be constructed as in subsection 4.2.3. Let us illustrate the above derivations on the case of quadratic splines.

Example 4.3.1. Consider the MRA generated by the quadratic spline as in Example 4.2.2. Then $\phi(t)$, its derivative and integer samples are shown in Fig. 4.3.2. From the figure it is easy to see that $\Phi_0(z) = \sum_n \phi(n)z^{-n} = 1/2(z^{-1} + z^{-2})$ and $\Phi_1(z) = \sum_n \phi'(n)z^{-n} = z^{-1} - z^{-2}$. The polyphase matrix is $\mathbf{E}(z) = \begin{pmatrix} z^{-1}/2 & 1/2 \\ -z^{-1} & 1 \end{pmatrix}$. Its inverse is $\mathbf{R}(z) = \begin{pmatrix} z & -z/2 \\ 1 & 1/2 \end{pmatrix}$, so that FIR synthesis filters are $G_0(z) = z + z^2$ and $G_1(z) = z/2 - z^2/2$. Finally, synthesis functions are $S_0(t-2n)$ and $S_1(t-2n)$, where $S_0(t) = \phi(t+1) + \phi(t+2)$ and $S_1(t) = \frac{1}{2}\phi(t+1) - \frac{1}{2}\phi(t+2)$.

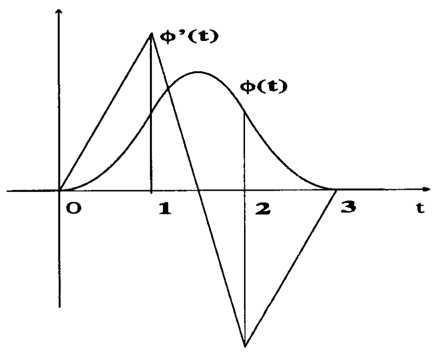


Fig. 4.3.2. $\phi(t)$, $\phi'(t)$ and its samples at integers.

Remark. If one uses longer filters, $\mathbf{E}^{-1}(z)$ may turn out to be IIR. In that case, one can use techniques of nonuniform sampling or reconstruction from local averages together with sampling of derivatives to insure compactly supported synthesizing functions.

4.3.3. Multi-band or multiscale sampling

Consider a signal $F(\omega)$ as in Fig. 4.3.3. If $F(\omega)$ is regarded as a lowpass signal, minimum necessary sampling rate is $2\omega_s$. If it is regarded as a bandpass signal, it can be verified that aliasing copies $F(\omega + k\omega_s)$ caused by sampling at the rate ω_s do not overlap. Therefore, minimum sampling rate in this case is ω_s.

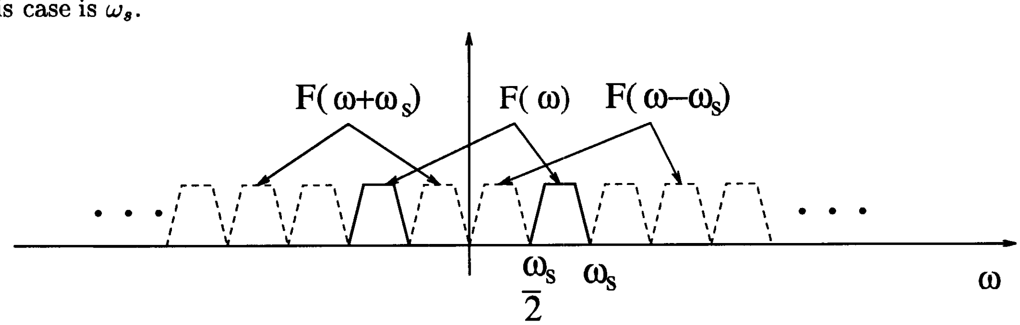


Fig. 4.3.3. An ideal bandpass signal and its aliasing copies.

The aim of this subsection is to find what the equivalent of this situation in wavelet subspaces is.

Spaces V_k, roughly speaking, contain lowpass signals, whereas W_k spaces contain bandpass signals. So far, we have examined lowpass signals only (ones that belong to V_0). Necessary sampling rate for exact reconstruction was unity. We will show that if a signal does not occupy the whole frequency range that V_0 covers, it can be sampled at a lower rate. For example, if $f(t) \in W_{-1}$, we can sample it at the rate 1/2. More generally, assume that $f(t) \in W_{-1} \dot{+} W_{-2} \dot{+} \cdots \dot{+} W_{-J} \subset V_0$. This means that there are sequences $\{c_{-1,n}\}, \{c_{-2,n}\}, \ldots, \{c_{-J,n}\} \in l^2$ such that

$$f(t) = \sum_{k=-J}^{-1} \sum_n c_{k,n} 2^{k/2} \psi(2^k t - n). \tag{4.3.4}$$

From Walter's work, we know that since $f(t) \in V_0$, it can be recovered from its integer samples. Here, the aim is to exploit the fact that $f(t)$ belongs to a subspace of V_0 and sample it at a lower rate. As in previous subsections, the idea is to find an invertible map from the sequences $\{c_{k,n}\}$ to a sequence of samples of $f(t)$. For this, let us sample $f(t)$ at $t_{k,n} = n2^{-k} + u_k$, $k = -J, -J+1 \ldots, -1$, where $u_k \in [0, 2^{-k})$. Intuitively, this rate should be enough, since we can project $f(t)$ onto each of W_k's and then sample those projections at rates 2^k. As in the previous sections, we would like to find a nice interpretation of the operator $\mathcal{T}$ in terms of digital filter banks. The problem is that inputs $\{c_{k,n}\}$ and outputs $\{f(t_{k,n})\}$ operate at different rates for different k's (viz. 2^k). In order to simplify analysis that follows, let us find some equivalent system where all the inputs/outputs operate at the same rate. For this, let

$$C_k(e^{j\omega}) = \sum_n c_{k,n} e^{-j\omega n} \qquad \text{and} \qquad F_m(e^{j\omega}) = \sum_n f(t_{m,n}) e^{-j\omega n} \tag{4.3.5}$$

be the Fourier transforms of $\{c_{k,n}\}$ and $\{f(t_{m,n})\}$. In order to bring all these signals to the same rate, we expand $C_k(e^{j\omega})$'s and $F_k(e^{j\omega})$'s into their 2^{J+k}-fold polyphase components

$$C_k(e^{j\omega}) = \sum_{l=0}^{2^{J+k}-1} e^{-j\omega l} C_{k,l}(e^{j\omega 2^{J+k}}) \quad \text{and} \quad F_k(e^{j\omega}) = \sum_{l=0}^{2^{J+k}-1} e^{-j\omega l} F_{k,l}(e^{j\omega 2^{J+k}}), \quad -J \leq k \leq -1. \tag{4.3.6}$$

Now that all the inputs and outputs are brought to the same rate, our system can be represented as that in Fig. 4.3.4.

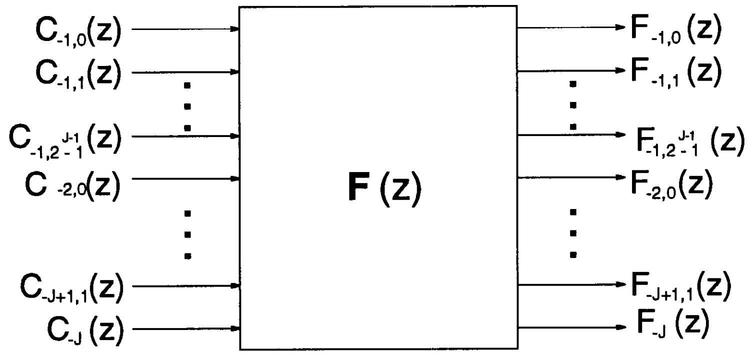


Fig. 4.3.4. Polyphase representation of a MIMO nonuniform filter bank.

Let us find the entries of $\mathbf{F}(z)$. Using (4.3.4), the q^{th} element of $F_m(z)$ is

$$f(2^{-m}q + u_m) = \sum_{k=-J}^{-1} \sum_{l} c_{k,l} 2^{k/2} \psi(2^k(2^{-m}q - 2^{-k}l + u_m)). \tag{4.3.7}$$

Substituting $q = 2^{J+m}n + p$ in the above expression and using $c_{k,l} = \frac{2^{-k}}{2\pi} \int_{-\pi}^{\pi} C_k(e^{j\omega 2^{-k}}) e^{j\omega l 2^{-k}} d\omega$, we get

$$f(2^J n + 2^{-m}p + u_m) =$$

$$= \sum_{k=-J}^{-1} \frac{2^{-k/2}}{2\pi} \int_{-\pi}^{\pi} C_k(e^{j\omega 2^{-k}}) \left(\sum_{l} e^{j\omega(2^{-k}l - 2^J n)} \psi(2^k(2^J n + 2^{-m}p - 2^{-k}l + u_m)) \right) e^{j\omega 2^J n} d\omega. \tag{4.3.8}$$

Finally, after expanding $C_k(e^{k\omega 2^{-k}})$'s into their 2^{J+k}-fold polyphase components, we get

$$f(2^J n + 2^{-m}p + u_m) = \sum_{k=-J}^{-1} \sum_{l=0}^{2^{J+k}-1} \frac{1}{2\pi} \int_{-\pi}^{\pi} C_{k,l}(e^{j\omega 2^J}) H_{k,l}^{m,p}(e^{j\omega 2^{-k}}) e^{j\omega 2^J n} d\omega, \tag{4.3.9}$$

where $H_{k,l}^{m,p}(e^{j\omega 2^{-k}}) = 2^{-k/2} e^{-j\omega 2^{-k}l} \sum_n e^{-j\omega 2^{-k}n} \psi(2^k(2^{-m}p + 2^{-k}n + u_m))$. Therefore, the output polyphase components are

$$F_{m,p}(e^{j\omega}) = \left(\sum_{k=-J}^{-1} \sum_{l=0}^{2^{J+k}-1} C_{k,l}(e^{j\omega 2^J}) H_{k,l}^{m,p}(e^{j\omega 2^{-k}}) \right) \Big\downarrow_{2^J}$$

$$= \sum_{k=-J}^{-1} \sum_{l=0}^{2^{J+k}-1} C_{k,l}(e^{j\omega}) \left(H_{k,l}^{m,p}(e^{j\omega}) \right) \Big\downarrow_{2^{J+k}}. \tag{4.3.10}$$

Hence, the entries of the matrix $\mathbf{F}(z)$ in Fig. 4.3.4 are $[\mathbf{F}(z)]_{i_1,i_2} = \left(H_{k,l}^{m,n}(z) \right) \Big\downarrow_{2^{J+k}}$, where $i_1 = 2^J - 2^{J+m+1} + n$ and $i_2 = 2^J - 2^{J+k+1} + l$. As before, the MIMO version of the Wiener's theorem [Vid85] gives us sufficient conditions for the existence of a stable inversion scheme.

Theorem 4.3.2. If a function $f(t) \in W_{-i_1} \dot{+} W_{-i_2} \dot{+} \cdots \dot{+} W_{-i_J} \subset V_0$ $(1 \leq i_1 < i_2 < \ldots < i_J)$ is sampled at the rate $2^{-i_1} + 2^{-i_2} + \cdots + 2^{-i_J} < 1$, then there exists a stable reconstruction scheme if $\mathbf{F}(e^{j\omega})$, as defined above, is nonsingular for all $\omega \in [-\pi, \pi]$.

◊

Remark. Notice that if the projection of $f(t)$ onto some of W_k's is zero, we can drop the corresponding term $C_k(e^{j\omega})$ and sample at an even lower rate.

Synthesizing functions can be determined from $\mathbf{F}^{-1}(z)$ as follows. Let $\mathbf{G}(z) = \mathbf{F}^{-1}(z)$ and $G^{k,l}_{m,n}(z) = [\mathbf{G}(z)]_{i_1,i_2}$, where $i_1 = 2^J - 2^{J+k+1} + l$ and $i_2 = 2^J - 2^{J+m+1} + n$. Then

$$c_{k,2^{J+k}p+l} = \sum_{m=-J}^{-1} \sum_{n=0}^{2^{J+m}-1} \sum_q g^{k,l}_{m,n}(p-q) f(2^J q + 2^{-m} n + u_m), \tag{4.3.11}$$

where $G^{k,l}_{m,n}(z) = \sum_p g^{k,l}_{m,n}(p) z^{-p}$. Also

$$f(t) = \sum_{k=-J}^{-1} \sum_{l=0}^{2^{J+k}-1} \sum_p c_{k,2^{J+k}p+l} 2^{k/2} \psi(2^k t - 2^{J+k} p - l). \tag{4.3.12}$$

Substituting (4.3.11) into the last equation, we get

$$f(t) = \sum_{m=-J}^{-1} \sum_{n=0}^{2^{J+m}-1} \sum_q f(2^J q + 2^{-m} n + u_m) S^q_{m,n}(t), \tag{4.3.13}$$

where

$$S^q_{m,n}(t) = \sum_{k=-J}^{-1} \sum_{l=0}^{2^{J+k}-1} \sum_p g^{k,l}_{m,n}(p-q) 2^{k/2} \psi(2^k t - 2^{J+k} p - l). \tag{4.3.14}$$

These synthesizing functions have a shift property as well. Notice that $S^{q+1}_{m,n}(t) = S^q_{m,n}(t - 2^J)$, i.e., all the functions are obtained as shifts (for 2^J) of 2^J basic functions $S^0_{m,n}(t) = S_{m,n}(t)$ for $n = 0, 1, \ldots, 2^{J+m}$ and $m = -J, -J+1, \ldots, -1$. Using $S_{m,n}(t)$, the synthesis algorithm (4.3.14) can be written as

$$f(t) = \sum_{m=-J}^{-1} \sum_{n=0}^{2^{J+m}-1} \sum_q f(2^J q + 2^m n + u_m) S_{m,n}(t - 2^J q). \tag{4.3.15}$$

This provides the formula for $f(t)$ in terms of the samples $f(t_{n,k})$.

Example 4.3.2. Let $i_1 = 1$ and $i_2 = 2$ in Theorem 4.3.2. Then our function is $f(t) \in W_{-1} \cup W_{-2} \subset V_0$. The situation is schematically sketched in Fig. 4.3.5.

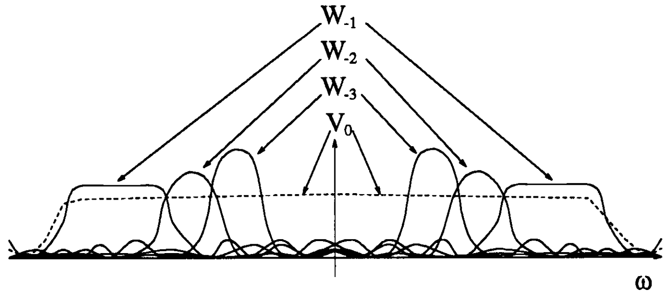


Fig. 4.3.5. $W_{-1} \cup W_{-2} \subset V_0$ in the frequency domain.

In this case, matrix $\mathbf{F}(e^{j\omega})$ is of the form

$$\mathbf{F}(e^{j\omega}) = \begin{pmatrix} \left(H_{-1,0}^{-1,0}(e^{j\omega})\right)\downarrow_2 & \left(H_{-1,1}^{-1,0}(e^{j\omega})\right)\downarrow_2 & H_{-2,0}^{-1,0}(e^{j\omega}) \\ \left(H_{-1,0}^{-1,1}(e^{j\omega})\right)\downarrow_2 & \left(H_{-1,1}^{-1,1}(e^{j\omega})\right)\downarrow_2 & H_{-2,0}^{-1,1}(e^{j\omega}) \\ \left(H_{-1,0}^{-2,0}(e^{j\omega})\right)\downarrow_2 & \left(H_{-1,1}^{-2,0}(e^{j\omega})\right)\downarrow_2 & H_{-2,0}^{-2,0}(e^{j\omega}) \end{pmatrix}.$$

Entries $H_{k,l}^{m,p}(e^{j\omega})$ are functions of the chosen $\phi(t)$ and $\psi(t)$, i.e., the underlying MRA. If [det $\mathbf{F}(e^{j\omega})$] turns out to be nonzero for all $\omega \in [-\pi, \pi]$, then $\mathbf{F}^{-1}(e^{j\omega})$ is stable and we can obtain synthesizing functions as given by (4.3.11-4.3.15).

4.3.4. Efficient computation of inner products in MRA

As one of the inventors of MRA, Mallat was the first to face the problem of computing inner products $c_{k,n} = \langle f(t), \phi_{k,n}(t)\rangle$, where $\phi_{k,n}(t) = 2^{k/2}\phi(2^k t - n)$. There exist computationally a very efficient method for getting $c_{k,n}$ from $c_{0,n}$ for any $k < 0$, the so called "Fast Wavelet Transform" (FWT) (in other words tree structured filter banks). So the problem is to compute $c_{0,n}$. Mallat showed that under mild conditions on regularity of $f(t)$, the samples $f(n/2^J)$ approach $c_{J,n}$ when $J \to \infty$. But obtaining coefficients $c_{0,n}$ was an intermediate step in reconstructing $f(t)$ from its samples (or local averages) in the methods we developed earlier. This means that the schemes we proposed can be used for computation of inner products $\langle f(t), \phi(t-n)\rangle$. In the rest of the subsection we will compare our methods with the existing ones.

Existing schemes. Instead of computing inner products, Mallat samples $f(t)$ at the rate 2^J (notice that this is, usually, a much higher rate than necessary if $f(t) \in V_0$) and then uses FWT to get $c_{0,n}$ and other lower resolution coefficients. So in this case, we have very high sampling rate, moderate complexity and relatively good approximation of true inner products. Daubechies (p. 166 of [Dau92]) outlined a method of getting $c_{0,n}$ from the integer samples $f(n)$. Walter [Wal92] provided detailed derivations for this method in the context of sampling in wavelet subspaces. In terms of subsection 4.2.2, Theorem 4.2.1 says that the coefficients $c_{0,n}$ can be exactly determined from samples of $f(n)$ (unit rate) by filtering them with $1/\Phi(e^{j\omega})$ (provided $\Phi(e^{j\omega})$ has no zeros on the unit circle). However, the problem is that $1/\Phi(e^{j\omega})$ is an IIR filter and one has to make a truncation error. Shensa [She92] proposed a method that is somewhere in between the above two. His idea is to approximate the input $f(t)$ by $\tilde{f}(t) = \sum_n f(n)\chi(t-n)$. It is easy to see that the inner products $(\tilde{f}(t), \phi(t-n))$ are filtered integer samples $f(n)$. The main problem with this method is in finding a good approximation $\tilde{f}(t)$ of $f(t)$, which is the only source of errors. Let us now see how our methods perform in terms of complexity, sampling rate and approximation error.

New schemes. Mallat's algorithm has a rather low complexity. So, from the point of view of complexity, direct computation of inner products and Mallat's algorithm are two extreme cases. The gap between those two extremes is bridged by our local averaging scheme. It has higher complexity than Mallat's algorithm, but it gives zero error and minimal possible rate, and it has some other nice features, as explained in subsection 4.2.4. Mallat's and Daubechies/Walter algorithm are two extreme cases in terms of the sampling rate. While Daubechies/Walter algorithm has a higher complexity, it has minimum possible rate and very small error. The gap between those two methods in terms of the sampling rate is bridged by our oversampling scheme. In its simplest form (for sampling rate 2) it is just Euclid's algorithm, as was described in detail in subsection 4.3.1.

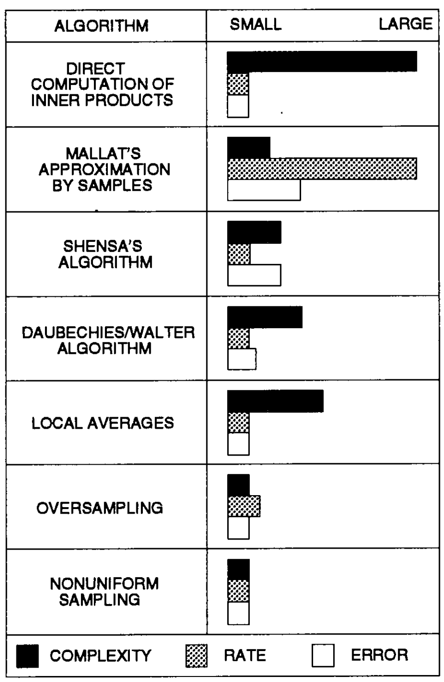


Fig. 4.3.6. Qualitative comparison of different methods for computing $c_{0,n}$'s.

It achieves zero error with negligible complexity (FIR filtering) † at the expense of a slightly higher sampling rate (still small compared to Mallat's $2^J >> 2$). Finally, nonuniform sampling, when $\mathbf{E}(z)$ is forced to be a constant matrix, achieves minimal sampling rate, small complexity (FIR filtering) and zero error.

Depending on the application, one or another scheme that we propose gives results better than any of the previous schemes. In particular, all our schemes achieve zero error and use FIR filters. The above discussion is summarized in Fig. 4.3.6. It shows relative merits of the schemes in terms of computational complexity, sampling rate and approximation error. It should be mentioned that all the above discussion holds true only for the case when we know that $f(t) \in V_0$, otherwise, we make an "aliasing" error [Wal92].

4.3.5. Errors in sampling times

Here, we want to see what happens if we reconstruct $f(t)$ thinking that the sampling times are t_n, but actually, they are $t'_n \neq t_n$. It will be shown that the error is bounded and tends to zero when $t'_n \to t_n$. In this subsection we assume that $\phi(t)$ is compactly supported on an interval of length D and that it satisfies $|\phi(t+h) - \phi(t)| < C|h|^\alpha$ for every t and some $0 < \alpha < 1$ (i.e., $\phi(t)$ is Lipschitz(α) continuous). So let the actual sampling times be

$$t'_n = t_n + \delta_n, \tag{4.3.16}$$

where $|\delta_n| < \delta$. Then we have

$$|f(t_n) - f(t'_n)| = \left|\sum_k c_k(\phi(t_n - k) - \phi(t'_n - k))\right|. \tag{4.3.17}$$

From the last equation and the Cauchy-Schwarz's inequality, we get

$$||f(t_n) - f(t'_n)||_2 = \sum_n |f(t_n) - f(t'_n)|^2 = \sum_n \left|\sum_k c_k(\phi(t_n - k) - \phi(t'_n - k))\right|^2$$

$$\leq D^2 \sup |\phi(t_n) - \phi(t'_n)|^2 \sum_k |c_k|^2 \leq K^2 ||c_n||_2^2 \sup_n |t_n - t'_n|^{2\alpha} \leq K||c_n||_2 \ \delta^\alpha, \tag{4.3.18}$$

where K is some constant. Because of the stability of the reconstruction algorithm, we also have

$$||f(t) - \widetilde{f}(t)||_2 \leq \text{const. } \delta^\alpha, \tag{4.3.19}$$

† When we say FIR filtering, it is assumed that $\phi(t)$ is compactly supported.

where $\widetilde{f}(t) = \sum_n f(t'_n)S_n(t)$. The last inequality means that the L^2 norm of the error can be controlled by δ. As for the supremum norm of the error, we use (4.2.7). Then, it is immediate that

$$\sup_t |f(t) - \widetilde{f}(t)| \leq \text{const'}.\ \delta^\alpha. \tag{4.3.20}$$

We see that both L^2 and supremum norms of the error can be controlled by δ.

4.4. Sampling of WSS random processes

The problem of sampling of random processes was thoroughly investigated by the end of the 1960's. Uniform and nonuniform sampling of WSS and non-stationary bandlimited random processes was considered (for an overview, see [Jer77]). In this section we want to look at the problem of uniform and nonuniform sampling of WSS random processes related to wavelet subspaces.

Let us first specify more precisely the class of random processes that the derivations will apply to. Let $\phi_a(t)$ be the deterministic autocorrelation function of $\phi(t)$, i.e., $\phi_a(\tau) = \int \phi(t+\tau)\phi^*(t)dt$. Since $\phi(t) \in L^1(\mathcal{R}) \cap L^2(\mathcal{R})$, then $\phi_a(\tau) \in L^1(\mathcal{R}) \cap L^2(\mathcal{R})$ as well. Also, the Cauchy-Schwarz's inequality implies $|\phi_a(\tau)| \leq ||\phi(t)||_2^2 < \infty$ for all $\tau \in \mathcal{R}$. We keep assumptions from Sec. 4.2, namely that $\{\phi(t-n)\}$ is a Riesz bases for $V_0 = \overline{\text{span}\{\phi(t-n)\}}$ and that $\phi(t)$ satisfies Janssen's conditions from subsection 4.2.1. We will consider random processes whose autocorrelation functions have the following form

$$R_{ff}(t) = \sum_n c_n \phi_a(t-n), \tag{4.4.1}$$

where $\{c_n\} \in l^1$. The above series converges absolutely and uniformly on $\mathcal{R}$, so that samples of $R_{ff}(\tau)$ are well-defined. Notice, in particular, that $\{c_n\} \in l^1$ implies that $R_{ff}(\tau) \in L^1(\mathcal{R})$ and $R_{ff}(n) \in l^1$. Therefore, the Fourier transform of $R_{ff}(\tau)$ exists and is equal to $S_{ff}(\omega)$ (we also assume that $R_{ff}(\tau)$ is the inverse Fourier transform of $S_{ff}(\omega)$). The PSD function has the following form

$$S_{ff}(\omega) = C(e^{j\omega})\Phi_a(\omega), \tag{4.4.2}$$

where $C(e^{j\omega}) \geq 0$ and $\Phi_a(\omega) = |\Phi(\omega)|^2$.

In the first part we are going to consider uniform sampling of random processes from the specified class. We show that the PSD function can be recovered, but not the random process itself, unless it is bandlimited. In the second part, we consider nonuniform sampling. We have to introduce randomness into the sampling times in order to preserve wide sense stationarity of the sampled random process. As in the case of uniform sampling, we show how to reconstruct the PSD function.

4.4.1. Uniform sampling

Consider random processes $\{f(t), -\infty < t < \infty\}$ with autocorrelation functions $R_{ff}(\tau)$ of the form (4.4.1). The discrete parameter autocorrelation function is

$$r_{ff}(m,n) = E[f(n+m)f^*(m)] = R_{ff}(n). \tag{4.4.3}$$

Since $r_{ff}(m,n)$ is not a function of m, $\{f(n)\}$ is a discrete parameter WSS random process.

Let $s_{ff}(e^{j\omega}) = \sum_n S_{ff}(\omega + 2\pi n)$. Then the Fourier coefficients of $s_{ff}(e^{j\omega})$ are integer samples of the autocorrelation function $R_{ff}(\tau)$, i.e., $r_{ff}(n) = R_{ff}(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} s_{ff}(e^{j\omega})e^{jn\omega}d\omega$. Since $r_{ff}(n) \in l^1$, $s_{ff}(e^{j\omega})$ is the sum of its Fourier series, i.e., $s_{ff}(e^{j\omega}) = \sum_n r_{ff}(n)e^{-jn\omega}$. We will use this relationship later on.

First, we show that the random process $\{f(t)\}$ cannot be reconstructed from the samples $\{f(n)\}$, if the synthesizing functions are restricted to be shifts of one function (unless, of course, the random process is bandlimited). In order to show this, assume the contrary. Let there be a function $g(t) \in L^2(\mathcal{R})$ such that $\{f(t)\}$ is equal to $\sum_n f(n)g(t-n)$ in MS sense. The error random process

$$e(t) = f(t) - \sum_n f(n)g(t-n) \tag{4.4.4}$$

has autocorrelation function

$$R_{ee}(t,\tau) = E\left[\left(f(t+\tau) - \sum_n f(n)g(t+\tau-n)\right)\left(f^*(t) - \sum_n f^*(n)g^*(t-n)\right)\right]$$

$$= R_{ff}(\tau) - \sum_n R_{ff}(t+\tau-n)g^*(t-n) - \sum_n R_{ff}(n-t)g(t+\tau-n) + \sum_{n,m} R_{ff}(n-m)g(t+\tau-n)g^*(t-m). \tag{4.4.5}$$

It is easy to check that $R_{ee}(t+1,\tau) = R_{ee}(t,\tau)$, so the error is a cyclo-WSS random process with period $T = 1$. We can average $R_{ee}(t,\tau)$ over T, to get the autocorrelation function $R_{ee}(\tau)$. Its variance is the value of $R_{ee}(\tau)$ at $\tau = 0$, i.e., $\sigma^2 = R_{ee}(0)$. Now, using the relation $R_{ff}(\tau) = \frac{1}{2\pi}\int e^{j\tau\omega}S_{ff}(\omega)d\omega$, we have (the order of integration and summation in the last term can be switched because $\{R_{ff}(n)\} \in l^1$)

$$\sigma^2 = R_{ff}(0) - \int R_{ff}(t-n)g^*(t-n)dt - \int R_{ff}(n-t)g(t-n)dt + \sum_l R_{ff}(l)\int g(t-l)g^*(t)dt. \tag{4.4.6}$$

Using Parseval's identity, the above expressions can be simplified to

$$\sigma^2 = \frac{1}{2\pi}\int S_{ff}(\omega)\left(1 - G^*(\omega) - G(\omega) + \sum_k |G(\omega+2\pi k)|^2\right)d\omega. \tag{4.4.7}$$

This cannot be zeroed for any choice of $G(\omega)$ unless $\phi(t)$ is bandlimited.

Even though we cannot recover the random process in MS sense, we can still recover the PSD function $S_{ff}(\omega)$. We know that $r_{ff}(n) = R_{ff}(n)$ and that $s_{ff}(e^{j\omega}) = \sum_k S_{ff}(\omega+2\pi k)$. Substituting the special form of $S_{ff}(\omega)$ into the last formula, we get

$$s_{ff}(e^{j\omega}) = \sum_k C(e^{j\omega})|\Phi(\omega+2\pi k)|^2 = C(e^{j\omega})\sum_k |\Phi(\omega+2\pi k)|^2. \tag{4.4.8}$$

Since $s_{ff}(e^{j\omega}) = \sum_n r_{ff}(n)e^{-jn\omega}$, we can recover $C(e^{j\omega})$ as follows

$$C(e^{j\omega}) = \frac{\sum_n r_{ff}(n)e^{-jn\omega}}{\sum_k |\Phi(\omega+2\pi k)|^2}. \tag{4.4.9}$$

Notice that the division is legal, because of the following reason: $\{\phi(t-n)\}$ is assumed to form a Riesz basis for its span. Therefore, there are constants $0 < A \leq B < \infty$ such that $A \leq \sum_k |\Phi(\omega+2\pi k)|^2 \leq B$ a.e., and the result of the division is in $L^2[-\pi,\pi]$. Finally, the reconstructed spectrum is

$$S_{ff}(\omega) = \frac{\sum_n r_{ff}(n)e^{-jn\omega}}{\sum_k |\Phi(\omega+2\pi k)|^2}|\Phi(\omega)|^2. \tag{4.4.10}$$

The above derivation can be summarized in the following theorem.

Theorem 4.4.1. Let autocorrelation function $R_{ff}(\tau)$ of random process $\{f(t)\}$ be of the form (4.4.1). The PSD function $S_{ff}(\omega) = C(e^{j\omega})\Phi(\omega)$ can be recovered from integer samples $\{r_{ff}(n)\}$ of $R_{ff}(\tau)$, as given by equation (4.4.10).

$\diamondsuit$

Remark. If $\{\phi(t-n)\}$ forms an orthonormal basis, then $\sum_k |\Phi(\omega+2\pi k)|^2 = 1$ a.e., and the above equation simplifies to

$$S_{ff}(\omega) = |\Phi(\omega)|^2 \sum_n r_{ff}(n)e^{-jn\omega}. \tag{4.4.11}$$

4.4.2. Nonuniform sampling

It can be easily seen that a deterministic nonuniform sampling of a random process produces a non-stationary discrete parameter random process. In order to preserve stationarity, we introduce randomness (i.e., jitter) into the sampling times. These, so called stationary point random processes, were investigated in [Beu66]. One special case is when the sampling times are $t_n = n + u_n$, where u_n are independent random variables with some distribution function $p(u)$. Let $\gamma(\omega) = E_u[e^{-ju\omega}]$ be its characteristic function. The autocorrelation sequence of the discrete parameter random process $\{f(t_n)\}$ is

$$r_{ff}(m,n) = E_u\left[E[f(t_{n+m})f^*(t_m)]\right] = E_u[R_{ff}(t_{m+n}-t_m)] = \frac{1}{2\pi}\int S_{ff}(\omega)E_u[e^{j(u_{n+m}-u_m)\omega}]e^{jn\omega}d\omega. \tag{4.4.12}$$

Since u_n's are mutually independent, we have

$$E_u[e^{ju_{n+m}\omega}e^{-ju_m\omega}] = \begin{cases} |\gamma(\omega)|^2, & \text{if } n \neq 0 \\ 1, & \text{if } n = 0 \end{cases},$$

so that we finally get

$$r_{ff}(m,n) = \begin{cases} \frac{1}{2\pi}\int e^{jn\omega}|\gamma(\omega)|^2 S_{ff}(\omega)d\omega, & \text{if } n \neq 0; \\ \frac{1}{2\pi}\int S_{ff}(\omega)d\omega & \text{if } n = 0. \end{cases} \tag{4.4.13}$$

Since $r_{ff}(m,n)$ is independent of m, $\{f(t_n)\}$ is a WSS random process and we will just leave out index m in (4.4.13). Our PSD function has a special form $S_{ff}(\omega) = C(e^{j\omega})|\Phi(\omega)|^2$, and putting this in (4.4.13), we get

$$r_{ff}(n) = \frac{1}{2\pi}\int e^{jn\omega}|\gamma(\omega)|^2 C(e^{j\omega})|\Phi(\omega)|^2 d\omega = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{jn\omega}C(e^{j\omega})\sum_k |\gamma(\omega+2\pi k)|^2|\Phi(\omega+2\pi k)|^2 d\omega, \tag{4.4.14}$$

for $n \neq 0$. It is clear now how to recover the PSD function. First, we recover $C(e^{j\omega})$

$$C(e^{j\omega}) = \frac{r(0) + \sum_{n \neq 0}^{n} r_{ff}(n)e^{-jn\omega}}{\sum_k |\gamma(\omega+2\pi k)|^2|\Phi(\omega+2\pi k)|^2}, \tag{4.4.15}$$

where

$$r(0) = \frac{r_{ff}(0) - \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{\sum_k|\Phi(\omega+2\pi k)|^2}{\sum_k|\gamma(\omega+2\pi k)\Phi(\omega+2\pi k)|^2}\sum_{n\neq 0}^{n} r_{ff}(n)e^{-jn\omega}d\omega}{\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{\sum_k|\Phi(\omega+2\pi k)|^2}{\sum_k|\gamma(\omega+2\pi k)\Phi(\omega+2\pi k)|^2}d\omega}.$$

Then the original PSD function is

$$S_{ff}(\omega) = \frac{r(0) + \sum_{n\neq 0}^{n} r_{ff}(n)e^{-jn\omega}}{\sum_k |\gamma(\omega+2\pi k)|^2|\Phi(\omega+2\pi k)|^2}|\Phi(\omega)|^2. \tag{4.4.16}$$

We summarize these derivations in the following theorem.

Theorem 4.4.2. In addition to assumptions of Theorem 4.3.1, assume that there is a random jitter and that its statistics are known. Then the PSD function can be recovered from nonuniform samples $\{f(t_n)\}$ of random process $\{f(t)\}$, as given by equation (4.4.16), where $r_{ff}(n) = E[f(t_{m+n}f^*(t_m)]$.

◇

4.5. Conclusion

In the first part of the chapter, we considered different discrete representations of functions in wavelet subspaces. We examined periodically nonuniform sampling, local averaging, oversampling, sampling of functions and their derivatives, and multi-band sampling. We used some of these new schemes to achieve compact support of both $\phi(t)$ and $S_n(t)$'s which was not possible with previous schemes. We gave sufficient conditions for uniqueness and stability of the inversion schemes. We showed how some of the new schemes can be used for efficient computation of inner products in MRA. In the second part, we considered WSS random processes whose autocorrelation functions belonged to a space related to multiresolution subspaces. First, we considered uniform, integer sampling. We showed that a random process itself cannot be recovered if we want to do it with integer shifts of one synthesizing function (except in the bandlimited case). However, the PSD function can be recovered from the sampled random process. Then we considered jittered nonuniform sampling and showed how to reconstruct the PSD function in this case.

4.6. Appendices

Appendix 4.A: Definitions and Theorems From Analysis

This appendix contains some definitions and theorems from analysis that we use in the paper. It is intended to provide readers with a quick reference. For more detailed discussion, see the corresponding references.

4.A.1. Reproducing Kernel Hilbert Spaces

Some Hilbert spaces have an additional structure built in. Functional Hilbert spaces are one such example. Here is the definition from [You80].

Definition 4.A.1. Let $\mathcal{H}$ be a Hilbert space of functions on X. $\mathcal{H}$ is called a *functional Hilbert* space if "point-evaluation" functionals $\Phi_x(f) = f(x)$ are bounded on $\mathcal{H}, \forall\, x \in X$.

It can be shown that in this case there exists a function $K(x, y)$ on $X \times X$, called reproducing kernel, such that $f(x) = (f(y), K(x, y))$, for all $f \in \mathcal{H}$. Here $(\cdot,\cdot)$ denotes the inner product in $\mathcal{H}$. In this case, $\mathcal{H}$ is called a Reproducing Kernel Hilbert Space (RKHS).

For a more detailed discussion of the role of RKHS in the sampling theory, see [Jer77] and references therein.

4.A.2. Fact From the Analysis

1. *Fourier Transform.* The Fourier transform, as usually defined, exists for functions in $L^1(\mathcal{R})$ only. However, it can also be defined on $L^2(\mathcal{R})$ and in this case it is an isometrical isomorphism from $L^2(\mathcal{R})$ onto itself. In this case, any equality is understood in L^2-sense. Many equalities in this paper are in L^2-sense and it is usually clear from the context. The same is true for the Fourier transform of sequences in l^2 as well [Chu92].

2. *Convolutions.* As we mentioned in Sec. 2, stability in this paper does not mean BIBO stability. All we really need is that $\mathcal{T}^{-1}$ is a bounded linear transformation from l^2 into itself. However, if $F(z)$ is a representation of $\mathcal{T}^{-1}$, and if $F(z)$ is BIBO stable, it implies that $\mathcal{T}^{-1}$ is a bounded transformation of l^2 into itself. This follows from the following theorem.

   Theorem 4.A.1. [Rud87] Let $f(t) \in L^p$ and $g(t) \in L^1$ $(1 \le p \le \infty)$. Then the convolution $f(t)*g(t) \in L^p$. As a special case, the convolution of a sequence from l^1 and a sequence from l^p is a sequence in l^p for all $1 \le p \le \infty$.

3. *Wiener's Theorem* [Rud87]. From the previous theorem, it is clear that the operation of convolution is closed in l^1. Wiener's theorem gives us a necessary and sufficient condition for the existence of a convolutional inverse of some $\{x_n\} \in l^1$.

   Theorem 4.A.2. A sequence $\{x_n\} \in l^1$ has a convolutional inverse $\{y_n\} \in l^1$ if and only if $\sum_n x_n e^{jn\omega} \neq 0$ for all $\omega \in [-\pi, \pi]$. In this case, $\frac{1}{\sum_n x_n e^{-jn\omega}} = \sum_n y_n e^{-jn\omega}$.

Appendix 4.B: Sensitivity to the input noise

In this appendix, we show that "local averages" scheme is indeed less sensitive to the input noise than sampling. In the analysis that follows, we use the additive noise model. The setup is shown in Fig. 4.B.1.

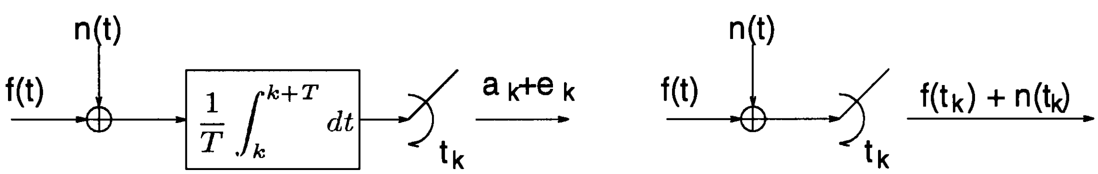


Fig. 4.B.1. Block diagrams of local averaging and sampling schemes.

For fair comparison, we scale outputs of the integrator by $\frac{1}{T}$. We assume that noise $n(t)$ is a zero mean, WSS random process with finite variance and autocorrelation function $R_{nn}(\tau)$. In the case of sampling, the output is not $f(t_k)$ as we expect, but $f(t_k) + n(t_k)$. So the error term is simply a sample of the noise $n(t_k)$. Variance of this error is $\sigma_s^2 = E[|n(t_k)|^2] = R_{nn}(0)$. In the case of local averaging, the noise passes through the integrator. The output is $\frac{1}{T}\int_k^{k+T}(f(t) + n(t))dt$. Then the error term $e_k = \frac{1}{T}\int_k^{k+T} n(t)dt$ has variance

$$\sigma_a^2 = E\left[\frac{1}{T}\int_0^T n(t+k)dt \frac{1}{T}\int_0^T n^*(s+k)ds\right]$$

$$= \frac{1}{T^2}\int_0^T\int_0^T R_{nn}(t-s)dtds = \frac{1}{T^2}\int_{-T}^T (T - |\tau|)R_{nn}(\tau)d\tau. \qquad (4.B.1)$$

We also know that $|R_{nn}(\tau)| \leq R_{nn}(0)$ for all $\tau \in \mathcal{R}$. Therefore, it follows that

$$\frac{\sigma_a^2}{\sigma_s^2} = \frac{1}{T}\int_{-T}^T \left(1 - \frac{|\tau|}{T}\right)\frac{R_{nn}(\tau)}{R_{nn}(0)}d\tau \leq 1, \qquad (4.B.2)$$

i.e., error due to noise in the local averages scheme has variance smaller or equal to that in the case of sampling. It is equal if and only if $R_{nn}(\tau) = R_{nn}(0)$ for all $\tau \in [-T, T]$.

5

Concluding Remarks

In this chapter, first, the main results of the thesis will be summarized. Then, open problems and future research directions will be discussed.

Nonuniform filter banks were the main topic of Chapter 2. Even the most trivial issues in the uniform filter bank theory become very difficult and often intractable in the case of nonuniform filter banks. We restricted our attention to the existence problem and the relationship between biorthogonality and perfect reconstruction. Using an explicit construction technique, it was shown that whenever the decimation ratios are such that a biorthogonal PR is possible, then there exist orthonormal filters which achieve PR as well. The technique developed for the orthonormalization process is used for the decorrelation of subband signals in uniform filter banks. Finally, we presented several necessary conditions that have to be satisfied by the decimation ratios in order that PR can be achieved with rational filters.

In the third chapter, optimality of wavelets and filter banks for signal representation in a stochastic setup was discussed. First, adaptation of a scaling function for the representation of WSS random processes was considered. A cost function was derived. Next, we discussed the design of prefilters for reduction of errors in MRA, again, when the input signal is a WSS random process. In the last part, we analyzed compression capability of PU and biorthogonal filter banks. We showed that a significant improvement in the compression performance of any PU filter bank can be achieved by simply putting a pre- and a post-filter.

Different generalizations of wavelet sampling theorem were considered in the fourth chapter. By allowing for periodically nonuniform sampling grid, we were able to construct compactly supported synthesizing functions. This was not possible with previously existing schemes (in the case of orthonormal MRA). We also showed how functions in MRA subspaces can be recovered from their local averages, derivative samples, etc.

5.1. Open problems

In the area of nonuniform filter banks, there are many unsolved problems. There is no known necessary and sufficient condition on decimation ratios for PR with rational filters. There are no design techniques even when we know that it is possible to have PR with rational filters (except when the decimation ratios come from a tree). It goes without saying that there are many open problems in the theory of MD (non)uniform filter banks. Connection of nonuniform filter banks and the wavelet bases that they give rise to has not been examined. Again, the main reason for this is the lack of nontrivial, nonuniform filter banks. Out of all these open problems, the most important one is that of finding some useful designing techniques.

Elementary analysis of the performance of a filter bank when used as a subband coder is rather straightforward. It is known that PU filter banks are asymptotically optimal as the number of channels goes to infinity. For finite number of channels, not much is known. There is no theoretical bound on the coding gain of an M-channel subband coder. For the case of PU filter banks it is possible to find the best M-channel filter bank. However, the solution over the class of all PR filter banks is not known. Most widely used criterion for the performance of a coding scheme is the MSE, despite the fact that it poorly matches the characteristics of the human perceptual system. Therefore, some more appropriate measure of performances is needed. Then filters in a filter bank should be optimized according to a new criterion.

In the area of mathematical wavelet analysis, every day many questions are being answered and even more new questions are being asked. Therefore, it would be very difficult to mention all of the directions of research in connection with wavelets, so we will mention only a few directly concerned with sampling in MRA subspaces. We extended sampling theory for the case of periodically nonuniform grid. One open problem is to find a sampling theorem for an arbitrary nonuniform sampling grid. All sampling theorems for MRA subspaces assume that the signal belongs to a given subspace. If it does not, then there is an aliasing error. Reduction of this error is an interesting research problem. Another possible research topic would be MD sampling in wavelet subspaces.

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