Heavy Hadrons In The Large N_c Limit

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Abstract

In the heavy quark and large N_c limit, heavy baryons can be identified as bound states of heavy mesons and light baryons. The binding potential can be calculated under chiral perturbation theory, and is simple harmonic when $N_c \to \infty$. The spectra and properties of these bound states agree reasonably well with the observed heavy baryons. In this framework, some non-perturbative quantities, like the orbital excitation energy and the slope of the Isgur–Wise form factor, can be evaluated. Moreover, the same universal Isgur–Wise form factor describes the semileptonic decays $\Lambda_b \to \Lambda_c$ and $\Sigma_b^{(*)} \to \Sigma_c^{(*)}$. The formalism can also be used to study the spectra, stabilities and decay modes of exotic multiquark states.

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I. Introduction

It has been more than twenty years since the firm establishment of the standard $SU(3)_c \times SU(2)_L \times U(1)_Y$ model as the appropriate description of particle interactions for energies below 100 GeV. The predictions of the standard model agree well with the results of accelerator experiments. On the other hand, due to the non-perturbative nature of QCD at low energies, a full understanding of hadron properties is yet to be achieved. The strong coupling constant α_s becomes larger than unity for energies below $\Lambda_{QCD} \simeq 250$ MeV, and the perturbation series breaks down. The absence of a true expansion parameter makes the first principle calculations of hadron spectra and dynamics from first principles intractable.

This, however, does not prevent physicists from investigating the theoretical aspect of hadron properties. One direction is to build tractable models which capture the "essential" physics of the systems. From such models concrete predictions of hadron properties can be made, which can be compared with experiments. Notable examples of such schemes include the non-relativistic quark model, bag models, and potential models, and they often provide valuable insights to our understanding of the systems. Yet these models are not logically connected to any fundamental theories (like QCD). Different models give different results, and there are no theoretical resolutions of such disagreements. In this light some more general approaches to the study of hadron physics are necessary.

Better connected to the underlying theory of QCD are the symmetry-induced schemes. For example, the well-known chiral perturbation theory is based on chiral symmetry $SU(N_f)_L \times SU(N_f)_R$, which is a symmetry of QCD Lagrangian with N_f light flavors. The symmetry is explicitly broken only by the small (compared to Λ_{QCD}) but non-zero light quark mass terms, and spontaneously broken to $SU(3)_V$ by the vacuum expectation values of quark bilinears.

$$SU(3)_L \times SU(3)_R \to SU(3)_V. \tag{1.1}$$

As a result, hadrons fall into different multiplets of $SU(3)_V$, where the lowest lying ones are the (π, K, η) pseudoscalar octet and the (ρ, K*, ω) vector octet in the meson sector, and the (N, Λ, Σ, Ξ) spin-$\frac{1}{2}$ octet, and the (Δ, Σ, Ξ, Ω) spin-$\frac{3}{2}$ decuplet in the baryon sector. In chiral perturbation theory, the interactions between different hadrons are dictated by the tranformation properties under chiral symmetry. This provides severe constraints on the dynamics of the hadrons. The explicit symmetry breaking effects can be incorporated as small perturbations.

Another symmetry-induced scheme, heavy quark effective theory, is very similar in spirit. When the masses of the heavy quarks (heavy when compared to $\Lambda_{\rm QCD}$) go to infinity, a new SU($2N_Q$) spin-flavor symmetry arises, where N_Q is the number of heavy flavors. This new symmetry, commonly known as heavy quark symmetry, relates the decay rates and form factors of different channels of heavy hadrons decays. The effect of finite heavy quark masses are again incorporated as small perturbations.

Another possible direction of investigation calls for alternate expansion parameters. The famous large N_c expansion is an expansion about $N_c = \infty$, where N_c is the number of colors. It has been proven that a certain class of Feynman diagrams, known as the planar diagrams, dominates when $N_c \to \infty$. Though the set of planar diagrams is still too complicated to be summed explicitly, it can reproduce many qualitative features of low energy hadron interactions like the Zweig's rule. Since, however, the color gauge group SU(N_c) itself is being changed when $N_c \to \infty$, the gauge degrees of freedom A_μ of this many-colored QCD keep changing. As a result, no one has yet succeeded in writing something like

$$\mathcal{L}_{\rm QCD} = \mathcal{L}_0 + \frac{1}{N_c}\mathcal{L}_1 + \frac{1}{N_c^2}\mathcal{L}_2 + \ldots. \tag{1.2}$$

The situation is even worse when we consider baryons in the large N_c limit. Baryons are N_c-quark hadrons under color SU(N_c), and when $N_c \to \infty$, there is an infinite number of degenerate baryon representations of isospin and spin.

Still, we can study large N_c baryons under different approaches. For the past few years, I have been studying the application of one of these approaches, the chiral soliton approach, in the investigation of the properties of heavy hadrons. Chiral solitons are topologically non-trivial pion configurations carrying a topological charge which can be identified with baryon number. By identifying heavy baryons as bound states of chiral solitons to heavy mesons, we can examine their properties through the usual pion-heavy meson Lagrangian, which is given by chiral perturbation theory and heavy quark effective theory. Some of the properties are independent of the couplings in the chiral Lagrangian and these are thought to be predictions of the large N_c limit.

In this thesis, I shall summarize my studies [1-6] on heavy hadrons properties through this rich interplay of chiral perturbation theory, heavy quark effective theory and large N_c limit. We will begin with a brief review on heavy quark symmetry in section II. In section III, we will discuss the chiral soliton model of baryons in the large N_c limit. The bound state picture will be introduced to describe heavy baryons, and various properties of heavy baryons will be studied within this framework. Multiquark exotic states will be examined in section IV, and lastly, we will conclude in section V with a discussion of possible directions of further investigations.

II. Heavy Quark Effective Theory

Due to the existence of several excellent review papers [7-9] on heavy quark symmetry and heavy quark effective theory, I will not attempt to make a comprehensive summary here. Instead I will focus on the issues relevant to this thesis, namely the universality of the Isgur–Wise form factors and their intepretations under the "atomic" or "source and brown muck" picture. In this picture, we can interpret the well known Bjorken and Voloshin sum rules as statements of conservation of probability and energy respectively. We will also obtain a new sum rule from conservation of parity, which gives non-trivial constraints on the behavior of Isgur–Wise form factors.

1. Heavy Quark Symmetry

The physics of a heavy hadron is very similar to that of an atom, which can be viewed as an electron cloud residing in the electromagnetic field of a massive nucleus. The mass of the nucleus M can enter the Hamiltonian in two ways. In the electro-kinetic part of the Hamiltonian, it may appear in the combination of the reduced mass μ,

$$\frac{1}{\mu} = \frac{1}{M} + \frac{1}{m_e}. \tag{2.1}$$

On the other hand, in the magnetic part of the Hamiltonian, M enters through the nuclear magneton μ_n,

$$\mu_n = \frac{Ze}{2M}. \tag{2.2}$$

When $M \to \infty$, $\mu = m_e$, $\mu_n = 0$ and M drops out. Hence the Hamiltonian will be simply that describing an electron cloud in the static electric field of the nucleus, and the hyperfine Hamiltonian vanishes. Moreover, the spin of the nucleus, which enters only through the hyperfine Hamiltonian, also disappears from the Hamiltonian when $M \to \infty$.

Since the physics is independent of the mass and the spin of the nucleus, we can replace the nucleus with another of the same charge, but different mass and spin, without affecting the state of the atom. In fact the old and new masses can be widely different as long as both of them are large enough (compared to the typical mass scale of the problem, which is $\alpha^2 m_e$.) In this way we have got a "heavy nucleus symmetry" which relates nuclei with different masses and spins.

Can we carry this analogy to the physics of heavy hadrons and prove that the non-perturbative QCD Lagrangian should be independent of the masses and spins of any heavy quarks present? N. Isgur and M.B. Wise [7, 10-12] were among the first to explore this possibility and the result is the now well known heavy quark effective theory.

The part of the QCD Lagrangian density that contains a heavy quark Q is

$$\mathcal{L} = \bar{Q}(i\not{D} - m_Q)Q. \tag{2.3}$$

In the limit where $m_Q \to \infty$, we can write

$$Q(x) = \exp(im_Q v \cdot x) h_v(x), \tag{2.4}$$

where $h_v(x)$ satisfies the constraint

$$\not{v} h_v(x) = h_v(x). \tag{2.5}$$

Then the Lagrangian is simplified to

$$\mathcal{L} = \bar{h}_v i v \cdot D h_v. \tag{2.6}$$

Note that the mass and the spin of the heavy quark has dropped out as expected.

The effective Lagrangian above exhibits the "heavy quark symmetry" which applies to heavy quarks with masses much larger than the typical energy scale of hadron physics, which is $\Lambda_{\rm QCD}$. In the real world two of the quarks, the c quark and b quark, satisfy the criterion. Each heavy quark has two spin states, up and down. Hence the heavy quark symmetry is an SU(4) spin-flavor symmetry.

Just like the electron cloud has different eigenstates in the electric field of the nucleus, the "brown muck" (light degrees of freedom) around a heavy quark will also have different eigenstates in the color field of the heavy quark. Such eigenstates are, of course, the hadronic resonances. In the heavy quark limit, different hadrons related by heavy quark spin symmetry should be degenerate. For example, the B and B^* mesons have the same "brown muck" with spin-$\frac{1}{2}$ aligned anti-parallel and parallel respectively to the heavy quark spin. They are, therefore, degenerate in the heavy quark limit. In the baryon sector, the ground state Λ_b is a spinless "brown muck" in the color field of the b quark, while the degenerate Σ_b and Σ_b^* have the same spin-1 "brown muck."

2. Isgur–Wise Form Factors

One of the most important consequences of heavy quark symmetry is the reduction of the numbers of form factors describing exclusive $b \to c$ decays like $B \to D^{(*)}$ and $\Lambda_b \to \Lambda_c$. As it will be shown below, each of these decays can be described by just one form factor. Since N. Isgur and M.B. Wise were the first to discover this reduction [10-12], these universal form factors are commonly known as Isgur–Wise form factors.

As our first example, we will consider Λ_b, which is a ground state spinless "brown muck" in the color field of a b quark. When a b-quark in a Λ_b with velocity v decays into a c-quark with velocity v', all the "brown muck" notice is the change in velocity. The ground state $|0\rangle$ in the color field of a heavy quark with velocity v is in general not an eigenstate of the color field of a quark with a different velocity v'. In the special case of $v = v'$, however, the color field is unchanged and the "brown muck" stay in the same state. This gives the normalization of the Isgur–Wise form factors at the point of zero recoil.

Denote the eigenstates of the "brown muck" in the color field of the charm quark with velocity v' by $|n'\rangle$, with $|0'\rangle$ the ground state Λ_c. The decay amplitude $\mathcal{M}(\Lambda_b(v) \to X_c^{n'}(v')e\bar{\nu})$ is proportional to the overlap of the initial and final states

$\varphi_{n'}(w) = \langle n'|0\rangle$, where $w = v \cdot v'$ and the "brown muck" of $X_c^{n'}$ is in the state $|n'\rangle$. This clearly hints that $\varphi_{n'}(w)$ are closely related to the weak form factors, and indeed it is the case. For example, consider the baryonic Isgur-Wise form factor $\eta(w)$, defined by [12-15]

$$\langle \Lambda_c(v', s')|\bar{c}\Gamma b|\Lambda_b(v, s)\rangle = \eta(w)\bar{u}(v', s')\Gamma u(v, s). \tag{2.7}$$

With

$$|\Lambda_b(v, s)\rangle = |b(v, s)\rangle \otimes |0\rangle \tag{2.8}$$

and

$$|\Lambda_c(v', s')\rangle = |c(v', s')\rangle \otimes |0'\rangle, \tag{2.9}$$

we have

$$\begin{aligned} \langle \Lambda_c(v', s')|\bar{c}\Gamma b|\Lambda_b(v, s)\rangle &= \langle 0'|0\rangle\bar{u}(v', s')\Gamma u(v, s) \\ &= \varphi_{0'}(w)\bar{u}(v', s')\Gamma u(v, s). \end{aligned} \tag{2.10}$$

Immediately we get

$$\varphi_{0'}(w) = \eta(w), \tag{2.11}$$

i.e., the Isgur–Wise form factor is the overlap of initial and final "brown mucks" as expected. As mentioned before, the initial and final "brown mucks" are identical when $v = v'$, i.e., $w = 1$. The complete overlap of wave functions gives the normalization of $\eta(w)$ at the point of zero recoil.

$$\eta(1) = 1. \tag{2.12}$$

One would expect that, due to final quark masses, this normalization will receive corrections of order $1/m_Q$. Luke's theorem [16], however, states that the $1/m_Q$ corrections vanish and the leading corrections start at the order of $1/m_Q^2$.

Weak form factor of excited baryons can also be obtained in a similar way. In particular, for P-wave baryons,

$$\varphi_{n'}(w) = (w+1)^{1/2}\sigma^{n'}(w). \tag{2.13}$$

Where $\sigma^{n'}(w)$ is defined in Ref. [18]. In general, for a state with orbital angular momentum $l > 0$, $\varphi_{n'}(w) \sim |v'-v|^l \sim (w-1)^{l/2}$. For $l = 0$, $\varphi_{n'}(w) \sim (w-1)^0$ for the ground state and $\sim (w-1)^1$ for excited states. This determines the behavior of φ near the point of zero recoil.

Similar analysis can be made for the meson sector. For a b-quark, the ground states with $s_\ell = \frac{1}{2}$ are the B and B^* mesons. The formulas are more complicated, however, as the mesonic Isgur–Wise form factor $\xi(w)$ traditionally defined by [10,11]

$$\frac{< D(v')|\bar{c}\gamma_\mu b|B(v) >}{\sqrt{m_B m_D}} = \xi(w)(v+v')_\mu \ , \tag{2.14a}$$

$$\frac{< D^*(v',\epsilon)|\bar{c}\gamma_\mu\gamma_5 b|B(v) >}{\sqrt{m_B m_{D^*}}} = \xi(w)((1+w)\epsilon^*_\mu - (\epsilon^*\cdot v)v'_\mu) \ , \tag{2.14b}$$

$$\frac{< D^*(v',\epsilon)|\bar{c}\gamma_\mu b|B(v) >}{\sqrt{m_B m_{D^*}}} = \xi(w) i\epsilon_{\mu\nu\lambda\sigma}\epsilon^{*\nu}v'^\lambda v^\sigma \ . \tag{2.14c}$$

is not exactly $\varphi'_0(w)$. In fact, it turns out that

$$|\varphi_{0'}(w)|^2 = \left(\frac{w+1}{2}\right)|\xi(w)|^2. \tag{2.15}$$

Again, due to the complete overlap of initial and final "brown mucks" at the point of zero recoil, we have

$$\xi(1) = 1, \tag{2.16}$$

and the leading corrections start at order $1/m_Q^2$ by Luke's theorem [16].

For the P-wave excited mesons,

$$|\varphi^{(q)}(\tfrac{1}{2}^+;w)|^2 = 2(w-1)|\tau_{1/2}^{(q)}(w)|^2, \tag{2.17a}$$

$$|\varphi^{(r)}(\tfrac{3}{2}^+;w)|^2 = (w-1)(w+1)^2|\tau_{3/2}^{(r)}(w)|^2, \tag{2.17b}$$

where the τ's are defined in the Ref. [19].

(Note that the $\varphi_{0'}$ in Eq. (2.11) and Eq. (2.15) are different objects. It will be wrong, for example, to conclude that $|\eta(w)|^2 = (\frac{w+1}{2})|\xi(w)|^2$.)

We have established the interpretation of the Isgur–Wise form factors as overlaps of initial and final light degrees of freedom. Due to the intractability of non-perturbative QCD, we cannot calculate these form factors from first principles. On the other hand, the completeness of $|v';n'\rangle$

$$\sum_{n'} |n'\rangle\langle n'| = 1 \tag{2.18}$$

can lead to sum rules like

$$\sum_{n'} |\varphi_{n'}(w)|^2 = \sum_{n'} \langle 0|n'\rangle\langle n'|0\rangle = \langle 0|0\rangle = 1, \tag{2.19}$$

which relate different $\varphi(w)$ and which in turn can give us non-trivial information about the form factors. These sum rules will be the topic of the rest of this section.

3. The Bjorken Sum Rule

Traditionally [17-19], the Bjorken sum rule is based on consideration of the four-point function of b, $\bar{b}\Gamma c$, $\bar{c}\Gamma b$ and $\bar{b}$, where $\Gamma = \gamma^\mu(1-\gamma_5)$. The four-point function may be evaluated in quark language by perturbative QCD, or in hadron language in the framework of heavy quark symmetry. By duality the two results should be equal. Hence,

$$h^{b\to c}(w) = \sum_{n'} h^{n'}(w) \tag{2.20}$$

where

$$h^{n'}(w) = \sum_{s'} \langle \Lambda_b(v)|\bar{b}\Gamma c|X_c^{n'}(v',s')\rangle\langle X_c^{n'}(v',s')|\bar{c}\Gamma b|\Lambda_b(v)\rangle, \tag{2.21}$$

where $X^{n'}$ are multiplets in the heavy quark symmetry, and

$$h^{b\to c}(w) = \sum_{s,s'} \langle b(v,s)|\bar{b}\Gamma c|c(v',s')\rangle\langle c(v',s')|\bar{c}\Gamma b|b(v,s)\rangle. \tag{2.22}$$

Since the $\bar{b}\Gamma c$ and $\bar{c}\Gamma b$ in the definition of $h^{n'}$ act on heavy quarks but not on the light degrees of freedom, we expect $h^{n'}$ can be factorized into contributions from the heavy quark sector and those from the light degrees of freedom. The heavy quark sector will just reproduce $h^{b\to c}$, while the contribution from the light degrees of freedom can be expressed in terms of $\varphi_{n'}$. Hence we end up with

$$h^{n'}(w) = h^{b\to c}(w)\langle 0|n'\rangle\langle n'|0\rangle = h^{Q_i\to Q_j}(w)|\varphi_{n'}(w)|^2. \tag{2.23}$$

After summation over all n' and canceling the common factor of $h^{Q_i\to Q_j}$ off both sides of Eq. (2.23), we end up with

$$1 = \sum_{n'} |\varphi_{n'}(w)|^2 \tag{2.24}$$

which is just Eq. (2.17) reproduced. Replacing φ's with the weak form factor, the equation becomes

$$1 = |\eta(w)|^2 + (w^2-1)\sum_q |\sigma^{(q)}(w)|^2 + \mathcal{O}^2(w-1) \tag{2.25}$$

which is just the usual Bjorken sum rule.

Expanding about the point of zero recoil, the Bjorken sum rule can be simplified to

$$\bar{\rho}^2 = \sum_q |\sigma^{(q)}(1)|^2 \tag{2.26}$$

where the charge radius $\bar{\rho}$ of $\eta(w)$ is defined by

$$\eta(w) = 1 - \bar{\rho}^2(w-1) + ... \tag{2.27}$$

Similar analysis can be made in the meson sector. Eq. (2.23) still follows from the completeness of states. The relations between $\varphi_{n'}(w)$ and the mesonic Isgur–Wise form factors as shown in Eq. (2.15) and (2.17) are, however, more complicated than the baryonic counterpart Eq. (2.11) and (2.13). In this case, Eq.(2.23) becomes

$$1 = \left(\frac{w+1}{2}\right) |\xi(w)|^2 + (w-1)\left[2\sum_q |\tau_{1/2}^{(q)}(w)|^2 + (w+1)^2 \sum_r |\tau_{3/2}^{(r)}(w)|^2\right] + \mathcal{O}^2(w-1). \tag{2.28}$$

Defining the charge radius ρ of $\xi(w)$ by

$$\xi(w) = 1 - \rho^2(w-1) + ... \tag{2.29}$$

and the Bjorken sum rule is simplified to

$$\rho^2 = \tfrac{1}{4} + \sum_q |\tau_{1/2}^{(q)}(1)|^2 + 2\sum_r |\tau_{3/2}^{(r)}(1)|^2. \tag{2.30}$$

The extra $\frac{1}{4}$ in Eq. (2.30) when compared to Eq. (2.25) is intriguing. In this formalism it is clear that the $\frac{1}{4}$ results from the "uncanonical" definition of $\xi(w)$ [8]. If the factor of $\frac{w+1}{2}$ in Eq. (2.15) is absorbed into the definition of $\xi(w)$, the equation will have the same form as Eq. (2.11), and the $\frac{1}{4}$ will not appear in the expansion.

4. The Voloshin Sum Rule

Returning to Eq. (2.18), it is noted that a more general sum rule holds for an arbitrary operator $\mathbf{X}$:

$$\sum_{n'} \langle 0|\mathbf{X}|n'\rangle\langle n'|0\rangle = \langle 0|\mathbf{X}|0\rangle. \tag{2.31}$$

In particular, if we put $\mathbf{X} = 1$ in Eq. (2.31), Eq. (2.19) is recovered.

Another case of interest is when $\mathbf{X} = H'$ the Hamiltonian in the color field of a heavy quark with velocity v'. Without loss of generality we choose the *final* velocity $v' = (1, \mathbf{0})$. Then $E_{n'} = \Delta m_{n'} = m_{X_c^{n'}} - m_c$ are just the excitation energies of the resonances $X_c^{n'}$. For the ground state, $\Delta m_0 = \Lambda = m_D - m_c$ in the meson sector and $\bar{\Lambda} = m_{\Lambda_c} - m_c$ in the baryon sector. The right-handed side $\langle 0|H'|0\rangle$ is the energy expectation for a moving ground state "brown muck" under the color field of a stationary heavy quark of velocity v'. By dimensional analysis we know that

$$\langle 0|H'|0\rangle = \Delta m_0 k(w), \tag{2.32}$$

where $k(w)$ is a kinematic factor which depends on w only. Hence the whole sum rule reads as

$$\sum_{n'} \Delta m_{n'} |\varphi_{n'}(w)|^2 = \Delta m_0 k(w). \tag{2.33}$$

The functional form of $k(w)$ can be obtained in some definite scenarios. For example, if we can regard H' as completely kinetic, then $k(w)$ is just the Lorentzian boost factor γ, which is just w. A more realistic scenario sees H' with two parts: the mass of the "brown muck" in v' frame and the potential energy of the "brown muck" in the color field of a heavy quark with velocity v'. If we ignore the quantum fluctuation of the color field, i.e., the heavy quark is a classical, static source, we can choose the color potential of the heavy quark such that

$$A^a_\mu(\mathbf{x}) = A^a(r)v'_\mu. \tag{2.34}$$

The potential energy is of the form

$$U = \int d^3\mathbf{x}\, j^{a\mu}(\mathbf{x})A^a_\mu(\mathbf{x}), \tag{2.35}$$

where $j^{a\mu}(\mathbf{x})$ is the color current density of the "brown muck" and a being the $SU(3)$ index. By symmetry we have

$$j^{a\mu}(\mathbf{x}) = j^a(r)v^\mu + t^\mu(r) \tag{2.36}$$

where $t^\mu(r)$, the component of $j^{a\mu}(\mathbf{x})$ transverse to v^μ, is radially symmetric. On integration, this transverse term vanishes by radial symmetry, and

$$U = w\int d^3\mathbf{x}\, j^a(r)A^a(r). \tag{2.37}$$

Hence when the "brown muck" is boosted from v' to v, the potential energy is increased by just the Lorentzian factor w. Since the mass of the "brown muck" also increases by the same factor under boost, we have $\langle 0|H'|0\rangle = \langle 0'|H'|0'\rangle w$, i.e., $k(w) = w$ if we assume the color field of the heavy quark is purely classical. It is probable that the statement is still valid if we take into account the effects due to quantum fluctuation, though the author has not yet succeeded in proving it.

If we assume $k(w) = w$, then Eq. (2.33) becomes

$$\sum_{n'} \Delta m_{n'}|\varphi_{n'}(w)|^2 = \Delta m_0 w \tag{2.38}$$

which can be recast into

$$\Delta m_0(w-1) = \sum_{n'}(\Delta m_{n'} - \Delta m_0)|\varphi_{n'}(w)|^2. \tag{2.39}$$

The quantity $E_{n'} = \Delta m_{n'} - \Delta m_0$ is just the mass difference over the ground state. In particular it is zero for $n' = 0$, i.e., the term proportional to the Isgur–Wise form factors vanishes.

For the meson sector, $\Delta m_{0'} = \Lambda = m_B - m_b$. Substituting in the weak form factors, we obtain

$$\Lambda(w-1) = \sum_q E^{(q)}_{1/2} 2(w-1)|\tau^{(q)}_{1/2}(1)|^2 + \sum_r E^{(r)}_{3/2}(w-1)(w+1)^2|\tau^{(r)}_{3/2}(1)|^2 + \mathcal{O}^2(w-1). \tag{2.40}$$

Canceling $(w-1)$ off both sides, and putting $w=1$, the sum rule becomes

$$\Lambda = \sum_q 2E^{(q)}_{1/2}|\tau^{(q)}_{1/2}(1)|^2 + \sum_r 4E^{(r)}_{3/2}|\tau^{(r)}_{3/2}(1)|^2 \tag{2.41}$$

which is just the Voloshin sum rule derived in Ref. [21]. On the other hand, in the baryonic sector, $\Delta m_{0'} = \bar{\Lambda} = m_{\Lambda_b} - m_b$, and the sum rule reads as

$$\bar{\Lambda} = \sum_q 2E^{(q)}_1 |\sigma^{(q)}(1)|^2. \tag{2.42}$$

I am not aware of any appearance of this sum rule in the literature before Ref. [3], where this sum rule is obtained in the large N_c limit. In fact, the results of Ref. [3] can be reproduced in our formalism by choosing a definite potential energy function, namely the isotropic harmonic potential $V(r) = \frac{1}{2}\kappa r^2$. We are going to see more about this in following sections.

5. The Parity Sum Rule

In this section, we will consider the case when we put $\mathbf{X} = P'$, the parity operator in the v' frame, into Eq. (2.31). The left-handed side becomes $\sum(-1)^{\pi_{n'}}|\varphi_{n'}(w)|^2$; where $\pi_{n'}$ are the intrinsic parities of $|n'\rangle$. $P'|0\rangle$ up to a phase is a ground state "brown muck" in the color field of a heavy quark with velocity $\bar{v} = (w, -w\mathbf{v})$; hence the right-handed side of Eq. (2.31) becomes $\langle 0|P'|0\rangle = (-1)^{\pi_{0'}}\varphi_{0'}(W)$, $W = v \cdot \bar{v}$. As result, Eq. (2.31) is simplified to

$$\sum_{n'}(-1)^{\pi_{n'}-\pi_{0'}}|\varphi_{n'}(w)|^2 = \varphi_{0'}(W). \tag{2.43}$$

This sum rule is remarkable in the sense that it relates form factors at two different kinematic points, w and W.

Together with the Bjorken sum rule Eq. (2.24), we have

$$2\sum_{n'}^{+}|\varphi_{n'}(w)|^2 - 1 = \varphi_{0'}(W). \tag{2.44}$$

The + above the summation means that the sum runs over the states with the same parity as the ground states only. Denoting the contribution to the sum from excited states as $R(w)$, the sum rule reads

$$2\varphi_{0'}^2(w) - 1 + R(w) = \varphi_{0'}(W). \tag{2.45}$$

Since $\varphi_{0'}(w)$ is the Isgur–Wise form factors up to possibly a known kinematic factor, a bound on $R(w)$ may give a model-independent bound on the Isgur–Wise form factors.

Since $R(w)$ is a sum of absolute squares, $R(w) \geq 0$, and

$$2\varphi_{0'}^2(w) - 1 \leq \varphi_{0'}(W). \tag{2.46}$$

We will change the independent variable from w to the "boost angle" α, which is related to w by $w = \cosh(\alpha)$. We will also change the dependent variable from $\varphi_{0'}(\alpha)$ to $f(\alpha)$, which is related to $\varphi_{0'}(\alpha)$ by $\varphi_{0'}(\alpha) = \cos(f(\alpha)\alpha)$. This greatly simplifies the equation as

$$W = 2w^2 - 1 = \cosh(2\alpha) \tag{2.47}$$

and

$$2\varphi_{0'}^2(\alpha) - 1 = 2\cos^2(f(\alpha)\alpha) - 1 = \cos(2f(\alpha)\alpha). \tag{2.48}$$

Hence Eq. (2.46) becomes

$$\cos(2f(\alpha)\alpha) \leq \cos(2f(2\alpha)\alpha). \tag{2.49}$$

We expect $\varphi_{0'}$ to be a decreasing function. Hence Eq. (2.49) implies

$$f(\alpha) \geq f(2\alpha). \tag{2.50a}$$

Since Isgur–Wise form factors are continuous, this simply states that $f(\alpha)$ is also a decreasing function. Moreover, since $\varphi_{0'}$ is expected to be nodeless and approaches zero as $w \to \infty$ $(\alpha \to \infty)$, we have

$$f(\alpha) \leq \frac{\pi}{2\alpha}; \tag{2.50b}$$

$$f(\alpha) \to \frac{\pi}{2\alpha}, \quad \alpha \to \infty. \tag{2.50c}$$

Finally, the boundary condition at $w = 1$ $(\alpha = 0)$ can be given in terms of the derivative of the Isgur–Wise form factor at the point of zero recoil.

$$f(\alpha = 0) = \bar{\rho}. \tag{2.50d}$$

The unique maximal $f(\alpha)$ satisfying the conditions above is

$$f^{\max}(\alpha) = \begin{cases} \bar{\rho}, & \alpha < \frac{\pi}{2\bar{\rho}}; \\ \frac{\pi}{2\alpha}, & \alpha > \frac{\pi}{2\bar{\rho}}. \end{cases} \tag{2.51}$$

Putting into the original form of $\varphi_{0'}(w)$, a model-independent lower bound for $\varphi_{0'}(w)$ can be obtained.

$$\varphi_{0'}^{\min}(w) = \begin{cases} \cos(\bar{\rho}\cosh^{-1}(w)), & w < \cosh(\pi/2\bar{\rho}); \\ 0, & w > \cosh(\pi/2\bar{\rho}). \end{cases} \tag{2.52}$$

This is a lower bound for all possible forms of $\varphi_{0'}(w)$ *with the same* $\bar{\rho}$.

Since $\eta(w) = \varphi_{0'}(w)$ in the baryon sector, the lower bound above can be applied to the baryon case directly. Plots of $\eta^{\min}(w)$ for different $\bar{\rho}$ are shown in Fig. 1. This lower bound rules out some particular forms of $\eta(w)$ like the piecewise linear model

$$\eta(w) = \begin{cases} 1 - \bar{\rho}^2(w-1), & w < 1 + \bar{\rho}^{-2}; \\ 0, & w > 1 + \bar{\rho}^{-2}. \end{cases} \tag{2.53}$$

We shall see in later sections that, in the large N_c limit, the baryonic Isgur–Wise form factor has an exponential form [3,55],

$$\begin{aligned}\eta(w) &= \exp(-\bar{\rho}^2(w-1)) \\ &= 1-\bar{\rho}^2(w-1)+\frac{\bar{\rho}^4}{2}(w-1)^2+\ldots\ldots\end{aligned} \tag{2.54}$$

while our lower bound, in a Taylor series, is

$$\eta^{\min}(w) = 1-\bar{\rho}^2(w-1)+(\frac{\bar{\rho}^2}{6}+\frac{\bar{\rho}^4}{6})(w-1)^2+\ldots\ldots \tag{2.55}$$

In the large N_c limit, $\bar{\rho}^2 \sim N_c^{3/2}$ is large, and the bound is satisfied.

On the other hand, the mesonic Isgur–Wise form factor $\xi(w)$ is given by the slightly more complicated Eq. (2.15). Hence, the lower bound for $\xi(w)$ is

$$\xi^{\min}(w) = \begin{cases} \left(\frac{2}{1+w}\right)^{1/2}\cos((\rho^2-\frac{1}{4})^{1/2}\cosh^{-1}(w)), & w<\cosh(\pi/2(\rho^2-\frac{1}{4})^{1/2}); \\ 0, & w>\cosh(\pi/2(\rho^2-\frac{1}{4})^{1/2}). \end{cases} \tag{2.56}$$

Plots of $\xi^{\min}(w)$ for different $\bar{\rho}$ are shown in Fig. 2.

Expanding in a Taylor series, we have

$$\xi^{\min}(w) = 1-\rho^2(w-1)+(\frac{\rho^2}{3}+\frac{\rho^4}{6})(w-1)^2+\ldots\ldots \tag{2.57}$$

When compared with the Isgur–Scora–Grinstein–Wise (ISGW) [23], Bauer–Stech–Wirbel (BSW) [24,25] and pole [26] parametrizations of the mesonic Isgur–Wise form factor,

$$\begin{aligned}\xi_{\rm ISGW}(w) &= \exp(-\rho^2_{\rm ISGW}(w-1)) \\ &= 1-\rho^2_{\rm ISGW}(w-1)+\frac{\rho^4_{\rm ISGW}}{2}(w-1)^2+\ldots\ldots,\end{aligned} \tag{2.58}$$

$$\begin{aligned}\xi_{\rm BSW}(w) &= \frac{2}{w+1}\exp\left((1-2\rho^2_{\rm BSW})\frac{w-1}{w+1}\right) \\ &= 1-\rho^2_{\rm BSW}(w-1)+(-\frac{1}{4}+\frac{\rho^2_{\rm BSW}}{2}+\frac{\rho^4_{\rm BSW}}{8})(w-1)^2+\ldots\ldots\end{aligned} \tag{2.59}$$

$$\begin{aligned}\xi_{\text{pole}}(w) &= \left(\frac{2}{w+1}\right)^{2\rho^2_{\text{pole}}} \\ &= 1 - \rho^2_{\text{pole}}(w-1) + (\frac{\rho^2_{\text{pole}}}{4} + \frac{\rho^4_{\text{pole}}}{2})(w-1)^2 + \ldots\ldots\end{aligned} \tag{2.60}$$

We found that, in order to satisfy the lower bound, we must have $\rho^2_{\text{ISGW}} \geq 1$, $\rho^2_{\text{BSW}} \geq \frac{\sqrt{7}-1}{4}$, and $\rho^2_{\text{pole}} \geq \frac{1}{4}$.

Last of all, it's also worth mentioning that the Bjorken–Suzuki upper bound of $\xi(w)$ [17,22]

$$\xi(w) \leq \left(\frac{2}{1+w}\right)^{1/2} \tag{2.61}$$

can be recovered by putting the obvious inequality $|\varphi_{0'}(w)| \leq 1$ into Eq. (2.15). The counterpart of this upper bound in the baryon sector is the trivial statement $\eta(w) \leq 1$.

We have discussed the Bjorken, Voloshin, and parity sum rules within the same framework. This is not meant to be a rigorous derivation (as in Ref. [18,19,21]) but just an intuitive picture making the relationship between the sum rules and the conservation laws behind them more transparent. We have treated the case for heavy mesons and the Λ_Q-type baryons, but similar analysis can be made in the Σ_Q-type baryon sector [3,20] as well.

III. Heavy Baryons In The Large N_c Limit

1. A Historical Review of the Large N_c Limit

The large N_c limit was discovered in 1973, when G. 't Hooft published his classic essay "A Planar Diagram Theory For Strong Interactions" [27]. In this paper, it was observed that the numbers of color degrees of freedom grow at different rates for quarks and gluons when $N_c \to \infty$. A quark, which transforms in the fundamental representation of SU(N_c), can fall in any of the N_c color states, while a gluon lives in the $(N_c^2 - 1) \sim N_c^2$ dimension adjoint representation. By counting the multiplicity of Feynamn diagrams in this way, 't Hooft had shown that a certain class of diagram, known as the "planar diagrams" dominates in the large N_c limit. Although the set of planar diagrams are still too large to be summed explicitly, the theory is quite successful in reproducing meson phenomenologies like the Zweig's rule, Regge phenomenology and the absence of $qq\bar{q}\bar{q}$ exotics.

Due to reasons related in Section I, the description of baryons in the large N_c limit is more subtle. In Ref. [28], E. Witten initiated several approaches to this problem. The discussion of baryon large N_c counting rules was recently discussed by R. Dashen, E. Jenkins and A.V. Manohar [29] in the "current algebra approach," where non-trivial constraints are derived from the preservation of unitarity in scattering precesses like $\pi + \mathrm{N} \to \pi + \mathrm{N}$. The "Hartree–Fock approach," in which the complicated interactions between the N_c quarks in a baryon is approximated by a mean Hartree–Fock field, was also lately pursued by C. Carone, H. Georgi and S. Osofsky in Ref. [30]. Lastly, the suggestion made in the last section of Ref. [28] that regards "baryons as monopoles of QCD" are furthered by G.S. Adkins, C.R. Nappa and E. Witten [31-34] himself in relation to the Skyrme model [35-38], resulting in the "chiral soliton approach." The present direction of the field is to study the relationship between these different approaches, and to make approach-independent predictions. In particular, an SU($2N_f$) light quark spin-flavor symmetry has arisen in all these approaches and is now expected to be the essence of the large N_c limit in the baryon sector.

Since most of my previous studies of the large N_c limit has followed the chiral soliton approach, this thesis will proceed with a brief description of its formalism.

2. The Chiral Soliton Model

Meson dynamics in the large N_c limit is a weakly coupled theory, with the meson-meson coupling of order $1/N_c$. Weakly coupled theories sometimes have states whose masses diverge, for weak couplings, like the inverse of the couplings. For example, the Polyakov–'t Hooft monopoles have masses of order $1/\alpha$ where α is the fine structure constant. In the closing lines of Ref. [28], E. Witten conjectures that baryons can be understood as monopoles of QCD, with masses of order $1/(1/N_c) = N_c$. In fact, the analogy goes much deeper. Both the monopole and the baryon arise from non-perturbative interaction and do not appear at all orders in the perturbative expansion. Both of them carry quantized conserved charges (magnetic and baryonic charges respectively). And even in the weak coupling limit, the S matrices of processes involving a baryon or a monopole will have non-trivial limits. In the case of monopoles, all these properties can be traced back to the fact that monopoles are topological solitons. Naturally we are led to ask: can baryons be regarded as topological solitons as well?

The chiral soliton model [32] studied the possibility of identifying baryons as topological solitons in a non-linear sigma model. For example, consider two-flavor QCD where chiral $SU(2)_L \times SU(2)_R$ is spontaneously broken into $SU(2)_V$. The Goldstone bosons, which are the pions, live on the SU(2) manifold.

$$U = \exp\left(\frac{2i\tau^a\pi^a}{f}\right) \in SU(2), \tag{3.1}$$

where the τ^a are the SU(2) generators and f, the pion decay constant.

The Lagrangian of the non-linear sigma model reads

$$\mathcal{L} = \mathcal{L}_{kin} + \mathcal{L}_{stab}. \tag{3.2}$$

with

$$\mathcal{L}_{kin} = \frac{f^2}{8}\mathrm{Tr}\,(\partial_\mu U \partial^\mu U), \tag{3.3}$$

where $f = 132\,\mathrm{MeV}$ is the pion decay constant. $\mathcal{L}_{stab}$ is a stablization term which will be discussed later. The finiteness of energy implies that, for a stable configuration,

$$U(r \to \infty) = U_0, \tag{3.4}$$

and without loss of generality we will take $U_0 = 1 \in \mathrm{SU}(2)$.

The pion field U is a mapping from three-dimensional space $\mathbf{R}^3$ to the pion field manifold SU(2), which is homeomorphic (topologically equivalent) to $\mathbf{S}^3$. On the other hand, Eq. (3.4) compactifies the $\mathbf{R}^3$ also to $\mathbf{S}^3$. As a result, we have

$$U : \mathbf{S}^3 \to \mathbf{S}^3. \tag{3.5}$$

Such mappings can be classified in the homotopy theory. In this particular case, $\pi_3(\mathbf{S}^3) = \mathbf{Z}$ implies that pion configurations can be characterized by an integer, which can be interpreted as the winding number of the field manifold around the spatial $\mathbf{S}^3$. Such a topological charge is conserved under continuous deformation of the field, and hence a constant of motion. Consequently, a pion configuration of non-zero topological charge cannot decay into the vacuum, which has zero topological charge. Moreover, the topological charge has been shown, through the anomaly-induced coupling of the baryon current to Goldstone bosons, to be nothing but the ordinary baryon current [39,40]. Such a baryon can be quantized as a fermion when $N_f = 2$ [41], and must be fermionic when $N_f > 2$ as long as N_c is odd [32] and a Wess–Zumino term is included [42]. Such results invite the identification of the ground state baryons like N and Δ as the ground state configuration of these chiral solitons.

Let $U(\mathbf{x}) \in \mathrm{SU}(2)$ be a ground state configuration of the chiral soliton. It can be easily seen that a field invariant under rotation

$$i(\mathbf{x} \times \nabla)_a U(\mathbf{x}) = 0, \qquad a = 1,2,3 \tag{3.6}$$

or invariant under isorotation

$$\left[\frac{\tau_a}{2}, U(\mathbf{x})\right] = 0, \qquad a = 1,2,3 \tag{3.7}$$

cannot carry any topological charge, as they can be continuously deformed into the vacuum configuration $U(\mathbf{x}) \equiv 1$. Hence the topological soliton is invariant under neither the rotational $\mathrm{SU}(2)_J$ nor the isospin $\mathrm{SU}(2)_I$. The maximum symmetry it can enjoy is the diagonal subgroup $G = \mathrm{diag}\,[\mathrm{SU}(2)_J \otimes \mathrm{SU}(2)_I]$, i.e., a rotation can be compensated by an isorotation.

$$\left[\frac{\tau_a}{2}, U(\mathbf{x})\right] - i(\mathbf{x} \times \nabla)_a U(\mathbf{x}) = 0. \tag{3.8}$$

Solutions satisfying Eq. (3.8) are of the form

$$U(\mathbf{x}) = \exp\left(\frac{iF(r)\tau_a x_a}{r}\right) \tag{3.9}$$

with the profile function (also called the chiral angle in some literature) $F(r)$ satisfying the boundary conditions:

$$F(r \to \infty) = 0, \tag{3.10}$$

as demanded by Eq. (3.4). Such solutions are said to obey the hedgehog ansatz. Continuity of $U(\mathbf{r})$ at the origin leads to

$$F(r = 0) = -B\,\pi, \tag{3.11}$$

where B is the topological charge, i.e., the baryon number. In particular, configurations with unit topological charge have $F(r = 0) = -\pi$.

(For flavor symmetry SU(N_f) with $N_f > 2$, there are solutions of Eq. (3.8) which are inequivalent to a hedgehog configuration. Such new solutions, however, are of higher topological charge. The H di-baryon, which will be discussed in later parts of this thesis, can be interpreted as such a soliton state.)

Hedgehog configurations are extraordinary as the spatial index of x_a is contracted to the isospatial index of τ_a. Rotations and isorotations are related, and hence when quantized, the chiral solitons have $K = 0$, where $K = I + J$. The state with $I = J = \frac{1}{2}$ can be identified with the nucleon N, and the $I = J = \frac{3}{2}$ state a Delta Δ.

Note that the results obtained above (the existence of topological solitons, the identification of the topological charge as the baryon number, the spin-statistics, the hedgehog ansatz, *etc.*) are independent of the underlying dynamics. Nowhere have we used the Lagrangian in Eq. (3.2). The dynamics, however, is crucial when the stability of the topological soliton is considered. Just by considering the stability of a hedgehog configuration under scale deformations, R.H. Hobart [43] and G.H. Derrick [44] proved that the Lagrangian $\mathcal{L} = \mathcal{L}_{kin} + \mathcal{L}_V$, where $\mathcal{L}_V$ is an arbitary potential of the pion fields, cannot support stable soliton solutions. To stablize the soliton, terms of higher power in field derivatives must be added [43,45]. The most notable example of such stabilization terms is the Skyrme term:

$$\mathcal{L}_{stab} = \mathcal{L}_{Sk} = \frac{1}{32e^2}\mathrm{Tr}\,[(\partial_\mu U)U^\dagger, (\partial_\nu U)U^\dagger]^2, \tag{3.12}$$

where e is the Skyrme parameter [36].

The Skyrme term is well-discussed in the literature. It is the unique term with four derivatives which leads to a positive Hamiltonian, and also the unique term with four derivatives which leads to a Hamiltonian second order in time derivatives. Moreover, starting from a meson Lagrangian with pions and ρ mesons, the kinetic term for the ρ meson becomes exactly the Skyrme term when the limit $m_\rho \to \infty$ is taken and the ρ degrees of freedoms are integrated out [46,47], with $e = g_{\rho\pi\pi}$. In the Skyrme model, where $\mathcal{L} = \mathcal{L}_{kin} + \mathcal{L}_{Sk}$ is assumed, one can calculate the static properties, like the mass relations, magnetic moments and axial couplings of the chiral soliton (called a skyrmion in this context) as a function of f and e [33]. The predictions agree with experimental results in the order of magnitude. The typical deviation is about 40%.

In general, however, the low energy effective Lagrangian in the large N_c limit may be much more complicated than that of the Skyrme model. Integrating out other mesons like ω and σ may produce other terms in $\mathcal{L}_{stab}$, which may have significant effects on the dynamics. Our ignorance of the exact form of $\mathcal{L}_{stab}$ does not prevent us from using the chiral soliton model in making predictions of heavy baryon properties, since it will be shown below that the dynamics will just depend on a few parameters, which can be viewed as generic functional of $\mathcal{L}_{stab}$. We will now proceed to see how one can study heavy baryons in the large N_c limit.

3. The Bound State Picture

Different approaches to baryons in the large N_c limit have different generalizations to accommodate heavy baryons in their framework. For the "current algebra approach" [29], a suitable representation of the SU($2N_f$) spin-flavor group can be chosen to describe heavy baryons. For the "Hartree–Fock approach" [30], wave equations for light quarks living under the combined influence of the Hartree–Fock mean field and an extra color field due to the presence of a heavy quark are considered. And for the "chiral soliton model," heavy baryons are described as bound states of heavy mesons and chiral solitons.

The bound state picture was first suggested in Ref. [48,49] to describe hyperons as bound states of nucleons and K mesons. Naive attempts of using the same formalism to describe heavy baryons like Λ_b as bound states of the B mesons and nucleons [50-52] are not valid, as the inclusion of B and exclusion of B^* violates heavy quark symmetry. The remedy is, of course, to include both B and B^* in the framework [53-55]. We will see below that such an attempt yields a reasonable heavy baryon spectrum.

Recall that a chiral soliton is a pion configuration satisfying the hedgehog ansatz. The interactions between a chiral soliton and a heavy meson should be derivable from the effective theory describing the interactions of heavy mesons with pions. Such an effective theory must respect chiral symmetry, which is a symmetry of the underlying theory of QCD when light quark masses are ignored. It turns out that, under this constraint, the leading term of the effective Lagrangian in a derivative expansion is unique up to an unknown coupling constant.

Eq. (3.1) can be recast in the following way.

$$U = \exp\left(\frac{2iM}{f}\right), \tag{3.13}$$

where

$$M = \begin{pmatrix} \pi^0/\sqrt{2} & \pi^+ \\ \pi^- & -\pi^0/\sqrt{2} \end{pmatrix}. \tag{3.14}$$

Under chiral $SU(2)_L \times SU(2)_R$

$$U \rightarrow LUR^\dagger, \tag{3.15}$$

with $L \in SU(2)_L$ and $R \in SU(2)_R$. Under parity

$$U(\mathbf{x}) \rightarrow U^\dagger(-\mathbf{x}), \tag{3.16}$$

as pions are pseudoscalars.

$$M(\mathbf{x}) \rightarrow -M(-\mathbf{x}). \tag{3.17}$$

Similarly, we can study the transformations of the heavy meson fields. Under heavy quark symmetry, the ground state meson fields D and D^* can be combined into a bispinor field

$$H^{(c)} = \frac{(1+\not{v})}{2}\left[D^*_\mu \gamma^\mu - D\gamma_5\right], \tag{3.18}$$

where v^μ is the heavy quark four-velocity, and $v^2 = 1$. Similarly $H^{(b)}$ can be defined by the $\overline{B}$ and $\overline{B}^*$ fields. The transformation property of H under $SU(2)_L \times SU(2)_R$

chiral symmetry has an arbitrariness associated with field redefinitions. We will follow the choice made in Ref. [54].

$$H \to H R^{\dagger}. \tag{3.19}$$

This unusual choice (only $SU(2)_R$ generators are involved) dictates an also unusual transformation law of H under parity.

$$H(\mathbf{x}) \to \gamma^0 H(-\mathbf{x}) \gamma^0 U^{\dagger}(-\mathbf{x}). \tag{3.20}$$

It is also convenient to introduce the field

$$\overline{H}^{(c)} = \gamma^0 H^{(c)\dagger} \gamma^0 = [D_\mu^{*\,\dagger} \gamma^\mu + D^{\dagger} \gamma_5] \frac{(1 + \not{v})}{2}. \tag{3.21}$$

The interaction Lagrangian respecting chiral symmetry and parity is [56-58]

$$\mathcal{L} = -i \operatorname{Tr} \overline{H} v^\mu \partial_\mu H + \tfrac{i}{2} \operatorname{Tr} \overline{H} H v^\mu U^{\dagger} \partial_\mu U + \tfrac{i}{2}\, g \operatorname{Tr} \overline{H} H \gamma^\mu \gamma_5 U^{\dagger} \partial_\mu U + \ldots, \tag{3.22}$$

where the ellipsis denotes the contribution of terms containing more derivatives. Neither the first nor the second term is invariant under parity, and requiring their sum to be parity invariant fixes the ratio of their coefficients. This Lagrangian can describe many processes in heavy meson phenomenology. For example, the decay $D^{*+} \to D^0 \pi^+$ is determined by the last term,

$$\Gamma(D^{*+} \to D^0 \pi^+) = \frac{1}{6\pi} \frac{g^2}{f^2} |\mathbf{p}_\pi|^3, \tag{3.23}$$

where g is a coupling constant of unspecified value. Present experimental measurements on D^* width and $\mathrm{Br}(D^{*+} \to D^0 \pi^+)$ give $\Gamma(D^{*+} \to D^0 \pi^+) \leq 72\,\mathrm{keV}$ and hence $g^2 < 0.4$ [59]. The constituent quark model predicts that g is positive.

In general, processes involving only heavy mesons and low energy pions can be described by the chiral Lagrangian in Eq. (3.22). Another of its many applications is predicting the corrections to the mesonic Isgur–Wise form factor $\xi(w)$ due to soft pion loops. This has been done at the point of zero recoil by L. Randall and M.B. Wise [101] and away from the point of zero recoil by M.B. Wise and the author [1]. (Corresponding calculations for baryons are done by M. Savage in Ref. [102].) This work, however, is not closely related to the theme of this thesis and hence will be covered in the Appendix.

Returning to the bound state picture, one can, using the chiral Lagrangian, calculate the binding energy of a heavy meson due to a hedgehog configuration of background pion field. This has been done in Ref. [53-55], and the results can be summarized as follows:

1) The binding potential V is a function of $K = I + s_\ell$, where I is the isospin and s_ℓ is the spin of the light degrees of freedom of the bound state. The dependences of V on I and s_ℓ enter solely through its dependence on K.

2) In general the binding potential can be expanded as a Taylor series in x, the relative distance between the heavy meson and (the center of) the chiral soliton.

$$V(x;K) = V_0(K) + \tfrac{1}{2}\kappa(K)x^2 + \dots. \tag{3.24}$$

It is found that the terms of quartic or higher powers in x of the potential (the ellipsis in Eq. (3.24)) are subleading in $1/N_c$. Hence, the potential is *exactly simple harmonic* in the large N_c limit.

3) When the higher-derivative terms are neglected, the truncated chiral Lagrangian (3.22) gives

$$V_0(K=0) = -\tfrac{3}{2}gF'(0), \tag{3.25a}$$

and

$$\kappa(K=0) = \kappa = g\left[\tfrac{1}{3}[F'(0)]^3 - \tfrac{5}{6}F'''(0)\right], \tag{3.25b}$$

where $F'(0)$ and $F'''(0)$ are respectively the first and third derivative of the profile function $F(r)$ at $r = 0$. We expect $F'(0) > 0$ and $F'''(0) < 0$, giving $V_0(K = 0) < 0$, $\kappa > 0$ and hence stable bound states. The exact shape of the profile function depends on the stabilization Lagrangian $\mathcal{L}_{stab}$, and the expression for κ in general changes when more terms are added to the Lagrangian (3.22). We may, however, regard $V_0(K = 0)$ and κ as parameters of the theory, which are functionals of $F(r)$. In particular, under truncated chiral Lagrangian, $V_0(K = 0)$ and κ are *odd* functional of $F(r)$. In general, however, when more terms are included in the chiral Lagrangian, $V_0(K = 0)$ and κ have no definite symmetry under $F \to -F$. The exact form of these functionals depend on how exactly the large N_c limit is realized but are independent of the heavy quark species (by heavy quark symmetry) and the baryon spin (by the large N_c limit). The value of the spring constant κ can be determined to be $(530\text{MeV})^3$ in the Skyrme model and $(440\text{MeV})^3$ from $\Lambda_c^* - \Lambda_c$ splitting. The ground states have orbital momentum $L = 0$ and $I = s_\ell$. They can be identified as Λ_Q, with $I = s_\ell = 0$ and $\Sigma_Q^{(*)}$, with $I = s_\ell = 1$.

4) For states with $K = 1$ the potential is opposite in sign:

$$V(x; K = 1) = -\tfrac{1}{3} V(x; K = 0), \tag{3.26}$$

and the resulting states are unbound.

The successful prediction of the spins and isospins of ground state heavy baryons (point (3) above) is a huge triumph for the bound state picture with the truncated chiral Lagrangian. Moreover, the simple harmonicity of the potential makes the calculation of many non-perturbative quantities possible. For example, the Isgur–Wise form factors for $\Lambda_b \to \Lambda_c$ and $\Sigma_b \to \Sigma_c^{(*)}$ can be evaluated in this framework, as shown in the section below.

4. Evaluation of the Isgur–Wise Form Factors

Recall that, in the heavy quark limit, the semileptonic $\Lambda_b \to \Lambda_c$ decay depends on a universal form factor $\eta(w)$.

$$\langle\Lambda_c(v',s')|\,\bar{c}\Gamma b\,|\Lambda_b(v,s)\rangle = \eta(w)\,\bar{u}_{\Lambda_c}(v',s')\,\Gamma\,u_{\Lambda_b}(v,s), \tag{3.27}$$

Similar results hold for the semileptonic $\Sigma_b^{(*)} \to \Sigma_c^{(*)}$ decay. In this case we have two Isgur–Wise form factors, $\zeta_1(w)$ and $\zeta_2(w)$ [12-15].

$$\langle\Sigma_c^{(*)}(v',s')|\,\bar{c}\Gamma b\,|\Sigma_b^{(*)}(v,s)\rangle = (\zeta_1(w)g_{\mu\nu} + \zeta_2(w)v_\nu v'_\mu)\,\bar{u}^\nu_{\Sigma_c^{(*)}}(v',s')\,\Gamma\,u^\mu_{\Sigma_b^{(*)}}(v,s), \tag{3.28}$$

where $u^\nu_{\Sigma_b^*}(v',s')$ is the Rarita–Schwinger spinor vector for a spin-$\frac{3}{2}$ particle, and $u^\mu_{\Sigma_b}(v,s)$ is defined by

$$u^\mu_{\Sigma_b}(v,s) = \frac{(\gamma^\mu + v^\mu)\gamma_5}{\sqrt{3}} u_{\Sigma_b}(v,s) \tag{3.29}$$

and similar for $u^\nu_{\Sigma_c^{(*)}}(v',s')$.

As mentioned above, Λ_Q and $\Sigma_Q^{(*)}$ can be viewed as the bound states of chiral solitons (N, Δ) to heavy mesons. We will describe the light degrees of freedom of a heavy baryon by $|I,a;s_\ell,m\rangle$, where I and s_ℓ are the isospin and spin of the light degrees of freedom, while a and m are their 3-components respectively. Hence the light degrees of freedom of Λ_Q is denoted by $|0,0;0,0\rangle$, while that of Σ_Q is $|1,a;1,m\rangle$.

Following the example of Ref. [53], the light degrees of freedom of a chiral soliton is denoted by $|R,b;R,n)$. A nucleon N has $R=\frac{1}{2}$ while Δ has $R=\frac{3}{2}$. On the other hand, the light degrees of freedom of a heavy meson is denoted by $|\frac{1}{2},c;\frac{1}{2},p\}$.

Since we are working in the $K = I + s_\ell = 0$ sector, the binding potential between the chiral soliton and the heavy meson is independent of both the isospins and the spins of the particles. We will denote the ground state wavefunction in momentum space as $\phi(\mathbf{q})$. Then we have these decompositions:

$$|0,0;0,0\,(v)\rangle = \int d^3\mathbf{q}\,\phi(\mathbf{q})\,(\tfrac{1}{2},b;\tfrac{1}{2},c\,|0,0)\;(\tfrac{1}{2},n;\tfrac{1}{2},p\,|0,0)\\ |\tfrac{1}{2},b;\tfrac{1}{2},n\,(\mathbf{v}-\mathbf{q}/M_B))\;|\tfrac{1}{2},c;\tfrac{1}{2},p\,(\mathbf{v}+\mathbf{q}/M_H)\}, \tag{3.30}$$

$$|1,a;1,m\,(v)\rangle = \int d^3\mathbf{q}\,\phi(\mathbf{q})\Bigg[\sqrt{\tfrac{1}{3}}(\tfrac{1}{2},b;\tfrac{1}{2},c\,|1,a)\;(\tfrac{1}{2},n;\tfrac{1}{2},p\,|1,m)\\ |\tfrac{1}{2},b;\tfrac{1}{2},n\,(\mathbf{v}-\mathbf{q}/M_B))\;|\tfrac{1}{2},c;\tfrac{1}{2},p\,(\mathbf{v}+\mathbf{q}/M_H)\}\\ +\sqrt{\tfrac{2}{3}}(\tfrac{3}{2},b;\tfrac{1}{2},c\,|1,a)\;(\tfrac{3}{2},n;\tfrac{1}{2},p\,|1,m)\\ |\tfrac{3}{2},b;\tfrac{3}{2},n\,(\mathbf{v}-\mathbf{q}/M_B))\;|\tfrac{1}{2},c;\tfrac{1}{2},p\,(\mathbf{v}+\mathbf{q}/M_H)\}\Bigg]. \tag{3.31}$$

The $(j_1,m_1;j_2,m_2\,|J,M)$'s are the Clebsch–Gordon coefficients. M_B and M_H are the masses of the chiral soliton and the heavy meson respectively. (Note that in the $K=1$ sector an $I=s_\ell=1$ state also appears. This state is orthogonal to that described by Eq. (3.31).)

The spin-$\frac{1}{2}$ Λ_Q is composed of a heavy quark with spin-$\frac{1}{2}$ and light degrees of freedom with spin-0. Hence

$$\langle\Lambda_c(v',s')|\,\bar{c}\Gamma b\,|\Lambda_b(v,s)\rangle = \langle 0,0;0,0\,(v')|0,0;0,0\,(v)\rangle\,\bar{u}_c\,\Gamma\,u_b. \tag{3.32}$$

Comparing with Eq. (3.27), we get [54]

$$\eta(w) = \langle 0,0;0,0\,(v')|0,0;0,0\,(v)\rangle\\ = \int d^3\mathbf{q}'\int d^3\mathbf{q}\,\phi^*(\mathbf{q}')\,\phi(\mathbf{q})\\ (\tfrac{1}{2},b';\tfrac{1}{2},c'\,|0,0)^*\;(\tfrac{1}{2},n';\tfrac{1}{2},p'\,|0,0)^*\;(\tfrac{1}{2},b;\tfrac{1}{2},c\,|0,0)\;(\tfrac{1}{2},n;\tfrac{1}{2},p\,|0,0)\\ (\tfrac{1}{2},b';\tfrac{1}{2},n'\,(\mathbf{v}'-\mathbf{q}'/M_B)|\tfrac{1}{2},b;\tfrac{1}{2},n\,(\mathbf{v}-\mathbf{q}/M_B))\\ \{\tfrac{1}{2},c';\tfrac{1}{2},p'\,(\mathbf{v}'+\mathbf{q}'/M_H)|\tfrac{1}{2},c;\tfrac{1}{2},p\,(\mathbf{v}+\mathbf{q}/M_H)\}. \tag{3.33}$$

Both the chiral soliton and the heavy meson matrix elements in Eq.(3.33) can be evaluated.

$$(\tfrac{1}{2}, b'; \tfrac{1}{2}, n' \, (\mathbf{v}' - \mathbf{q}'/M_B) | \tfrac{1}{2}, b; \tfrac{1}{2}, n \, (\mathbf{v} - \mathbf{q}/M_B)) = \delta_{bb'} \delta_{nn'} \delta^3(\mathbf{v} - \mathbf{v}' - (\mathbf{q} - \mathbf{q}')/M_B), \tag{3.34}$$

$$\{\tfrac{1}{2}, c'; \tfrac{1}{2}, p' \, (\mathbf{v}' + \mathbf{q}'/M_H) | \tfrac{1}{2}, c; \tfrac{1}{2}, p \, (\mathbf{v} + \mathbf{q}/M_H)\} = \delta_{cc'} \delta_{pp'}. \tag{3.35}$$

In Eq. (3.35) there should be an extra factor dependent on $\xi(w)$, the meson Isgur–Wise form factor. $\xi(w)$, however, is a slowly varying function and can be set to unity.

The Kronecker delta's make all Clebsch–Gordon coefficients vanish, and hence

$$\eta(w) = \int d^3\mathbf{q} \, \phi^*(\mathbf{q}) \, \phi(\mathbf{q} + M_B(\mathbf{v} - \mathbf{v}')) \tag{3.36}$$

and the result in Ref. [54] is reproduced.

Similarly, we can evaluate $\zeta_1(w)$ and $\zeta_2(w)$. For concreteness we will focus on the decay $\Sigma_b^* \to \Sigma_c^*$. The spin-$\frac{3}{2}$ Σ_Q is composed of a heavy quark with spin-$\frac{1}{2}$ and light degrees of freedom with spin-1. Hence,

$$\langle \Sigma_c^{(*)}(v', s') | \, \bar{c} \Gamma b \, | \Sigma_b^{(*)}(v, s) \rangle = \langle 1, a'; 1, m' \, (v') | 1, a; 1, m \, (v) \rangle \bar{u}_c \, \Gamma \, u_b. \tag{3.37}$$

The light matrix element can be evaluated in a way similar to Eq. (3.33).

$$\begin{aligned} \langle 1, a'; 1, m' \, (v') | 1, a; 1, m \, (v) \rangle = & \int d^3\mathbf{q}' \int d^3\mathbf{q} \, \phi^*(\mathbf{q}') \, \phi(\mathbf{q}) \\ & \Big[\tfrac{1}{3} (\tfrac{1}{2}, b'; \tfrac{1}{2}, c' \, |1, a')^* \, (\tfrac{1}{2}, n'; \tfrac{1}{2}, p' \, |1, m')^* \, (\tfrac{1}{2}, b; \tfrac{1}{2}, c \, |1, a) \, (\tfrac{1}{2}, n; \tfrac{1}{2}, p \, |1, m) \\ & \quad (\tfrac{1}{2}, b'; \tfrac{1}{2}, n' \, (\mathbf{v}' - \mathbf{q}'/M_B) | \tfrac{1}{2}, b; \tfrac{1}{2}, n \, (\mathbf{v} - \mathbf{q}/M_B)) \\ & \quad \{\tfrac{1}{2}, c'; \tfrac{1}{2}, p' \, (\mathbf{v}' + \mathbf{q}'/M_H) | \tfrac{1}{2}, c; \tfrac{1}{2}, p \, (\mathbf{v} + \mathbf{q}/M_H)\} \quad . \\ & + \tfrac{2}{3} (\tfrac{3}{2}, b'; \tfrac{1}{2}, c' \, |1, a')^* \, (\tfrac{3}{2}, n'; \tfrac{1}{2}, p' \, |1, m')^* \, (\tfrac{3}{2}, b; \tfrac{1}{2}, c \, |1, a) \, (\tfrac{3}{2}, n; \tfrac{1}{2}, p \, |1, m) \\ & \quad (\tfrac{3}{2}, b'; \tfrac{3}{2}, n' \, (\mathbf{v}' - \mathbf{q}'/M_B) | \tfrac{3}{2}, b; \tfrac{3}{2}, n \, (\mathbf{v} - \mathbf{q}/M_B)) \\ & \quad \{\tfrac{1}{2}, c'; \tfrac{1}{2}, p' \, (\mathbf{v}' + \mathbf{q}'/M_H) | \tfrac{1}{2}, c; \tfrac{1}{2}, p \, (\mathbf{v} + \mathbf{q}/M_H)\} \Big] \end{aligned} \tag{3.38}$$

The cross terms vanish as

$$(\tfrac{3}{2}, b'; \tfrac{3}{2}, n' \, (\mathbf{v}' - \mathbf{q}'/M_B) | \tfrac{1}{2}, b; \tfrac{1}{2}, n \, (\mathbf{v} - \mathbf{q}/M_B)) = 0. \tag{3.39}$$

As before, the chiral soliton and heavy meson matrix elements can be calculated using Eq. (3.34), Eq. (3.35) and

$$(\tfrac{3}{2}, b'; \tfrac{3}{2}, n' \, (\mathbf{v}' - \mathbf{q}'/M_B) | \tfrac{3}{2}, b; \tfrac{3}{2}, n \, (\mathbf{v} - \mathbf{q}/M_B)) = \delta_{bb'} \delta_{nn'} \delta^3(\mathbf{v} - \mathbf{v}' - (\mathbf{q} - \mathbf{q}')/M_B). \tag{3.40}$$

By the identities

$$(j_1, m_1; j_2, m_2 \,| J, M')^* \, (j_1, m_1; j_2, m_2 \,| J, M) = \delta_{MM'}, \tag{3.41}$$

Eq. (3.38) can be simplified to

$$\begin{aligned} \langle 1, a'; 1, m' \, (v') | 1, a; 1, m \, (v) \rangle &= \delta_{aa'} \delta_{mm'} \int d^3\mathbf{q} \, \phi^*(\mathbf{q}) \, \phi(\mathbf{q} + M_B(\mathbf{v} - \mathbf{v}')) \\ &= \delta_{aa'} \delta_{mm'} \eta(w). \end{aligned} \tag{3.42}$$

Notice that $\eta(w)$ has reappeared. The expected $\delta_{aa'}$ is just a consequence of isospin conservation in the weak decay. On the other hand, the $\delta_{mm'}$ demands that the initial and final light degrees of freedom are in the same spin state. In terms of the polarization vectors ϵ and ϵ', we have

$$\langle \Sigma_c^*(v', \epsilon', s') | \, \bar{c} \Gamma b \, | \Sigma_b^*(v, \epsilon, s) \rangle = \frac{\eta(w)}{1 + w} \left[(1 + w) g_{\mu\nu} - v_\nu v'_\mu \right] \epsilon'^{*\nu} \epsilon^\mu \bar{u}_c \Gamma \, u_b. \tag{3.43}$$

The combination in the square brackets should be familiar as it also appears in the $B \to D^*$ decay.

$$\langle D^*(v', \epsilon') | \, \bar{c} \gamma_\mu \gamma_5 b \, | B(v) \rangle = \xi(w) \left[(1 + w) g_{\mu\nu} - v_\nu v'_\mu \right] \epsilon'^{*\nu}. \tag{3.44}$$

Comparing Eq. (3.43) to Eq. (3.28), we conclude that, in the large N_c limit,

$$\zeta_1(w) = -(1 + w)\zeta_2 = \eta(w). \tag{3.45}$$

This is the main result of Ref. [3]. The same universal form factor $\eta(w)$ describes the weak decays of Λ_Q and Σ_Q. An equivalent result has been obtained in

Ref. [60] under different assumptions. Our formalism is applicable in the kinematic region $w-1 \leq N_c^{3/2}$, through which $\eta(w)$ drops from unity to a small quantity. Outside this kinematic region anharmonic terms in the binding potential may be significant, and the universality may be broken by such corrections.

In the real world, $\Sigma_b^{(*)}$ decays strongly and the decays $\Sigma_b^{(*)} \to \Sigma_c^{(*)}$ are hardly ever observed. On the other hand, Ω_b, the chiral SU(3) partner of Σ_b, does decay weakly to $\Omega_c^{(*)}$. Then chiral SU(3) predicts that the $\Omega_b \to \Omega_c^{(*)}$ decay is also described by the same universal form factor $\eta(w)$. This statement may be put to experimental test in the future. The deviation from Eq. (3.45) is an indication of how good the large N_c limit is.

The exact form of $\eta(w)$ depends solely on $\phi(\mathbf{q})$. As shown above, the binding potential between a heavy meson and a chiral soliton is simple harmonic in the large N_c limit. Hence $\phi(\mathbf{q})$ is just the ground state momentum wave function.

$$\phi(\mathbf{q}) = \frac{1}{(\pi^2 M_B \kappa)^{3/8}} \exp\left(-\mathbf{q}^2/2\sqrt{M_B\kappa}\right). \tag{3.46}$$

With this particular form of $\phi(\mathbf{q})$, $\eta(w)$ can be calculated [3,55].

$$\eta(w) = \exp\left(-\frac{(w-1)}{2}\sqrt{\frac{M_B^3}{\kappa}}\right). \tag{3.47}$$

This form factor satisfies the lower bound given in Eq. (2.52) when N_c is large. Expanding η about zero recoil

$$\eta(w) = 1 - \bar{\rho}^2(w-1) + \dots , \tag{3.48}$$

we see that

$$\bar{\rho}^2 = \frac{1}{2}[M_B^3/\kappa]^{1/2}. \tag{3.49}$$

It should be noted that Eq. (3.45) does not depend on any particular form of $\phi(\mathbf{q})$ or $\eta(w)$. It does depend, however, on the crucial assumption that the binding potential is independent of the isospins and spins of the particles.

We have proved that, in the large N_c limit, a new universality appeared for the baryon weak form factor $\eta(w)$. This same form factor describes weak decays of both Λ_Q and Ω_Q in the exact chiral $SU(3)$ limit. In the real world, where chiral $SU(3)$ is broken, the leading correction is expected to come from kaon loops. In particular, it is expected that, near the point of zero recoil

$$|\zeta_1(w) - \eta(w)| \sim \frac{g_3^2 \bar{\Delta}^2}{(4\pi f)^2} \ln\left(\frac{m_K^2}{\mu^2}\right), \tag{3.50}$$

where $\bar{\Delta}$ is the $\Omega_Q^* - \Omega_Q$ splitting, μ the subtraction point and g_3, the $\Sigma_Q \Sigma_Q \pi$ coupling constant [61]. The correction is expected to be about 1%, but the relevant loop integrals must be calculated and the counterterms known to get the exact magnitude of the correction.

5. Excited Baryons In The Bound State Picture

Experimental evidence for a doublet of excited charm baryons has recently been obtained at ARGUS [62], CLEO [63] and E687 [64]. (For a summary on excited charm states, see Ref. [65].) They have masses 340 MeV and 308 MeV above the Λ_c. It is natural to interpret these states as the spin 3/2 and 1/2 isospin zero members of a doublet that has spin parity of the light degrees of freedom, $s_\ell^{\pi_\ell} = 1^-$.

Properties of these excited Λ_c baryons can be estimated using the nonrelativistic constituent quark model [66]. In this phenomenological model the observed excited Λ_c baryons have quark content cud with the ud pair in an isospin zero and spin zero state like the ground state Λ_c. However, unlike the ground state, in these excited Λ_c baryons the ud pair has a unit of orbital angular momentum about the charm quark.

In the bound state picture, a Λ_Q-type excited baryon is a nucleon harmonically bound to a heavy meson in an excited bound state. When the orbital angular momentum of the bound state is non-zero, they occur (in the $m_Q \to \infty$ limit) in degenerate doublets that arise from combining the orbital angular momentum of the bound state with the heavy quark spin. The harmonic oscillator potential in Eq. (3.24) gives rise to an infinite tower of Λ_Q-type baryons with excitation energies

$$\Delta E^{(Q)}_{(n_1,n_2,n_3)} = (n_1 + n_2 + n_3)\sqrt{\kappa/\mu_Q}, \tag{3.51}$$

where μ_Q is the reduced mass

$$\frac{1}{\mu_Q} = \frac{1}{m_Q} + \frac{1}{M_B}. \tag{3.52}$$

In Eq. (3.51) (n_1, n_2, n_3) are the quantum numbers that specify the bound states when the Schrödinger equation is solved by separating variables in cartesian coordinates.

For states with the same quantum numbers (n_1, n_2, n_3), but different heavy quarks, Eq. (3.51) gives

$$\Delta E^{(c)}/\Delta E^{(b)} = \left(\frac{1 + M_B/m_c}{1 + M_B/m_b}\right)^{\frac{1}{2}} \simeq 1 + \frac{1}{2}\left(\frac{M_B}{m_c} - \frac{M_B}{m_b}\right) + \ldots. \tag{3.53}$$

Eq. (3.51) was obtained by solving the Schrödinger equation including the kinetic energy of the heavy meson. This corresponds to taking simultaneously the limits $m_Q \to \infty$ and $N_c \to \infty$ with the ratio M_B/m_Q held fixed (recall M_B is of order N_c). If m_Q was taken to infinity first, then effects of order M_B/m_Q are neglected, and heavy quark flavor symmetry determines the ratio of excitation energies in Eq. (3.53) to be unity. In the large N_c limit, the leading corrections to heavy quark symmetry [10,11] arise from including the kinetic energy of the heavy meson in the Schrödinger equation for the soliton-heavy meson bound state [55]. This violates heavy quark flavor symmetry but leaves heavy quark spin symmetry intact. Despite the fact that M_B/m_c is not particularly small, the ratio of excitation energies in Eq. (3.53) differs from unity by less than 20%.

The excitation energies given in Eq. (3.51) are of order $N_c^{-1/2}$. The first excited states have quantum numbers $(1,0,0)$, $(0,1,0)$, $(0,0,1)$. There is another basis of quantum numbers $[N,\ell,m]$, $N = n_1 + n_2 + n_3$, that is also useful. Here ℓ is the orbital angular momentum of the bound state and m is the component of the orbital angular momentum along the spin-quantization axis. In this basis $N \geq \ell$ and even values of ℓ occur for N even while odd values of ℓ occur for N odd. The first excited states have $N = 1$, $\ell = 1$, and $m = 0, +1, -1$ giving $s_\ell^{\pi_\ell} = 1^-$ for the spin parity of the light degrees of freedom. Combining this with the spin of the heavy quark gives a doublet of negative parity states with total spins 3/2 and 1/2. For $Q = c$ these states correspond to the observed doublet of excited Λ_c states. Comparing Eq. (3.51) with the experimental value of the excitation energy ($\simeq$ 340MeV) gives $\kappa \simeq (440\text{MeV})^3$.

In general, Λ_Q-type states have total spins $s = \ell \pm 1/2$ formed by combining the spin of the heavy quark with the orbital angular momentum ℓ. We label them by the quantum numbers $\{N,\ell;s,m\}$, where now m is the component of the total spin along the spin-quantization axis. In this notation the ground state Λ_Q baryon has quantum numbers $\{0,0;1/2,m\}$ and the first excited Λ_Q doublet contains the states $\{1,1;1/2,m\}$ and $\{1,1;3/2,m\}$.

Semileptonic decay amplitudes of the Λ_b to excited Λ_c baryons are determined by matrix elements of the vector and axial vector heavy quark currents. These matrix elements can be calculated using the bound state picture we have outlined above. It is found that

$$\begin{aligned}&\langle \Lambda_c^{\{N,\ell;s',m'\}}(v')|\ \bar{h}_{v'}^{(c)}\gamma^\mu h_v^{(b)}\ |\Lambda_b^{\{0,0;1/2,m\}}(v)\rangle \\ &\quad = \delta^{\mu,0}(\ell, m'-m; 1/2, m|s', m')\mathcal{F}^{[N,\ell,m'-m]}(w)\end{aligned} \tag{3.54a}$$

and

$$\begin{aligned}&\langle \Lambda_c^{\{N,\ell;s',m'\}}(v')|\ \bar{h}_{v'}^{(c)}\gamma^\mu\gamma_5 h_v^{(b)}\ |\Lambda_b^{\{0,0;1/2,m\}}(v)\rangle \\ &\quad = \delta^{\mu,j}\sum_{m''}(\ell, m'-m''; 1/2, m''|s', m')\left[\chi^\dagger(m'')\sigma^j\chi(m)\right]\mathcal{F}^{[N,\ell,m'-m'']}(w),\end{aligned} \tag{3.54b}$$

where $\mathcal{F}^{[N,\ell,m]}(w)$ is an overlap of momentum space harmonic oscillator wave functions

$$\mathcal{F}^{[N,\ell,m]}(w) = \int d^3q\, \phi_c^{*[N,\ell,m]}(\mathbf{q})\, \phi_b^{[0,0,0]}(\mathbf{q} - M_B(\mathbf{v} - \mathbf{v}')). \tag{3.55}$$

In Eq. (3.54b) χ is a two-component Pauli spinor. The sum over m'' in Eq. (3.54b) collapses to a single term since $\chi^\dagger(m'')\sigma^3\chi(m)$ vanishes for $m'' = -m$ and $\chi^\dagger(m'')\sigma^{1,2}\chi(m)$ vanishes for $m'' = m$. In Eq. (3.55) $\phi_Q^{[N,\ell,m]}(\mathbf{q})$ denotes the normalized momentum space harmonic oscillator wave function. Its dependence on the type of heavy quark arises from the dependence of the reduced mass μ_Q on the heavy quark mass.

Eqs. (3.54) and (3.55) are valid in the kinematic region $(w-1) \leq \mathcal{O}(N_c^{-3/2})$. For recoil velocities greater than this, the overlap $\mathcal{F}^{[N,\ell,m]}(w)$ is very small and terms subdominant in N_c that we have neglected may be important [55]. When $w \neq 1$ the operator $\bar{h}_{v'}^{(c)}\Gamma h_v^{(b)}$ requires renormalization [67]. However, in the kinematic regime very near zero recoil where Eqs. (3.53) and (3.55) apply, the subtraction point dependence of $\bar{h}_{v'}^{(c)}\Gamma h_v^{(b)}$ is negligible.

It is easiest to evaluate $\mathcal{F}^{[N,\ell,m]}(w)$ in the case where $\mathbf{v} - \mathbf{v}'$ is directed along the spin-quantization axis. The expression is particularly simple when the limit $m_Q \to \infty$ is taken first so that $\mu_Q = M_B$ independent of heavy quark type. Then

$$\mathcal{F}^{[N,\ell,m]}(w) = \delta^{m,0}\frac{C^{N\ell}}{\sqrt{N!}}[M_B^3/\kappa]^{N/4}(w-1)^{N/2}\exp\left(-\frac{1}{2}[M_B^3/\kappa]^{1/2}(w-1)\right) \tag{3.56}$$

where

$$C^{N\ell} = \int d^3q\phi^{*[N,\ell,0]}(\mathbf{q})\phi^{(0,0,N)}(\mathbf{q}). \tag{3.57}$$

The ground state $\Lambda_b \to \Lambda_c$ transition corresponds to the case $N = 0,\ \ell = 0$ and Eq. (3.57) is reduced to Eq. (3.47). Transition matrix elements to excited Λ_c

states with $\ell = 1$ are constrained by heavy quark symmetry to have the form [18]

$$\langle \Lambda^{\{N,1;1/2,m'\}}(v')|\bar{h}_{v'}^{(c)}\Gamma h_v^{(b)}|\Lambda_b^{\{0,0;1/2,m\}}(v)\rangle = \frac{\sigma^{(N)}(w)}{\sqrt{3}}\bar{u}(v',s')\gamma_5(\not{v}+w)\Gamma u(v,s), \tag{3.58a}$$

$$\langle \Lambda_c^{\{N,1;3/2,m'\}}(v')|\bar{h}_{v'}^{(c)}\Gamma h_v^{(b)}|\Lambda_b^{\{0,0;1/2,m\}}(v)\rangle = \sigma^{(N)}(w)\bar{u}_\mu(v',s')v^\mu\Gamma u(v,s). \tag{3.58b}$$

Comparing these expressions with Eqs. (3.52), (3.53), and (3.54) gives

$$\sigma^{(N)}(w) = \frac{C^{N1}}{\sqrt{2(N!)}}[M_B^3/\kappa]^{N/4}(w-1)^{(N-1)/2}\exp\left[-\frac{1}{2}[M_B^3/\kappa]^{1/2}(w-1)\right]. \tag{3.59}$$

Note that fractional powers of $(w-1)$ do not occur in Eq. (3.59) because N must be odd. In particular for $N=1$,

$$\begin{aligned}\mathcal{F}^{[1,1,0]}(w) &= \int d^3\mathbf{q}\,\phi_1^*(\mathbf{q})\,\phi(\mathbf{q}+M_B(\mathbf{v}-\mathbf{v}'))\\ &= [M_B^3/4\kappa]^{1/4}(w-1)\exp\left[-\frac{1}{2}[M_B^3/\kappa]^{1/2}(w-1)\right],\end{aligned} \tag{3.60}$$

where $\phi_1(\mathbf{q})$ is the momentum wavefunction of the first orbital excited state of the simple harmonic potential.

At zero recoil $\sigma^N(1)$ is zero for $N>1$, while for the first excited state

$$\sigma^{(1)}(1) = [M_B^3/4\kappa]^{1/4}, \tag{3.61}$$

using $C^{11}=1$.

For simplicity we have derived our expressions for the Isgur–Wise functions $\sigma^{(N)}(w)$ by taking the limit $m_Q \to \infty$ followed by the limit $N_c \to \infty$. However, Eqs. (3.52) and (3.53) can be used to include corrections to the heavy quark limit to all orders in M_B/m_Q. As we have noted, these corrections do not violate heavy quark spin symmetry. Therefore, the form of the matrix elements given in Eqs. (3.58) still holds, but the functions $\sigma^{(N)}(w)$ become dependent on the heavy quark masses.

The fact that $\sigma^{(N)}(1)$ is zero for $N > 1$ means that in the large N_c limit the Bjorken sum rule (2.26) for the slope $\bar{\rho}^2$ of the Isgur–Wise function $\eta(w)$ and Voloshin sum rule (2.42) for the mass of the light degrees of freedom $\bar{\Lambda}$ are saturated by the first doublet of excited Λ_c states. In fact, the Bjorken sum rule

$$\bar{\rho}^2 = |\sigma^{(1)}(1)|^2 \tag{3.62}$$

is reproduced when Eq. (3.49) and (3.61) are combined, where the Voloshin sum rule is reduced to

$$\bar{\Lambda} = M_B \tag{3.63}$$

which is true in leading order of $1/N_c$.

It is straightforward to repeat the exercise for Ω_b decays. Heavy quark symmetry allows four independent form factors for Ω_b decays into the first excited Ω_c pentalet [20]. Yet in the large N_c limit, all of them are expressible in $\mathcal{F}^{[1,1,0]}(w)$. Likewise, both the Bjorken [20] and the Voloshin sum rules are again saturated by this pentalet.

In this section, we have discussed how heavy baryons arise as bound states of heavy mesons and chiral solitons, and how Isgur–Wise form factors can be evaluated for ground state and excited baryons. There are other properties of excited heavy baryons that can be examined in the large N_c limit. For example, at the leading order in chiral perturbation theory [61], the strong couplings of the ground state Λ_Q to $\Sigma_Q\pi$ or $\Sigma_Q^*\pi$ are of order $N_c^{1/2}$ and can be related to the pion-nucleon coupling [53]. However, because of the orthogonality of the harmonic oscillator wave functions, the analogous couplings for excited Λ_Q states [68] are only of order $N_c^{-1/2}$.

IV. Heavy Multiquark Exotics In The Large N_c Limit

One of the early triumphs of the quark model is its success in describing the hadron spectrum. By regarding hadrons as $q\bar{q}$ or qqq configurations, their quantum numbers (charge, spin, isospin, strangeness, etc.) are well accounted for. On the other hand, the question of the existence of exotics has attracted more attention in recent years. In particular, since R.L. Jaffe's classic papers [69-71] on di-meson and di-baryon states in the quark-bag model, the existence of multiquark states has been investigated through many other approaches.

In this section, we are going to investigate the existence of multiquark states containing one or two heavy quarks (for $N_c = 3$). In the large N_c limit, they are states with one or $N_c - 1$ heavy quarks. Two types of interactions are going to play important roles in our discussion. They are the heavy meson-chiral soliton binding, which has been introduced in the previous section, and the Coulombic attraction between heavy quarks.

In the heavy quark limit, the color potential between two heavy quarks are Coulombic. As the result, $N_c - 1$ heavy antiquarks can form a small colored complex of size $(N_c\,\alpha_s(m_Q)m_Q)^{-1}$ which transforms just like a heavy quark under color SU(N_c). (Note that we keep $N_c\,\alpha_s$ constant when taking the large N_c limit; the size of this "fake heavy quark" has a finite limit when $N_c \to \infty$.) Hence, for any hadron with one heavy quark, we can replace the heavy quark with this "fake heavy quark" to obtain another hadron [72,73]. We will see below that some of these states are exotics.

We will begin this section by reviewing the Coulombic attraction by considering the properties of normal baryons with $N_c - 1$ heavy quarks. After that, hadrons with $2N_c - 2$, $N_c + 2$ and $2N_c$ quarks, which for brevity will be called tetraquarks, pentaquarks and hexaquarks respectively, will be investigated. Last of all, the validity of the results for finite quark masses and possible generalizations of this framework will be discussed.

1. Heavy Baryons With $\mathbf{N}_c-1$ Heavy Anti-Quarks

As discussed before, N_c-1 heavy antiquarks can form a small colored object $\bar{Q}^{N_c-1}$ which transforms just like a heavy quark under color SU(N_c). As long as $N_c\alpha_s(m_Q)m_Q \gg \Lambda_{\rm QCD}$, the light degrees of freedom cannot resolve the individual heavy antiquarks within this "fake heavy quark." Hence, for any heavy hadron containing a single heavy quark, we can replace this heavy quark with the "fake heavy quark" and obtain another hadron which contains N_c-1 heavy antiquarks. In particular, by replacing the heavy quark Q in the heavy meson $Q\bar{q}$ with $\bar{Q}^{N_c-1}$, we get the non-exotic baryon $\bar{Q}^{N_c-1}\bar{q}$.

$\bar{Q}^{N_c-1}$ has binding energy of the order of $N_c\,\alpha_s^2(m_Q)m_Q$. When $m_Q\to\infty$, the binding energy grow to infinity. The heavy antiquarks are very tightly bound in the heavy quark limit, and $\bar{Q}^{N_c-1}\bar{q}$ is safe from dissociations like $\bar{Q}^{N_c-1}\bar{q}\to \bar{Q}q+\bar{Q}^{N_c-2}\bar{q}\bar{q}$. As a result, $\bar{Q}^{N_c-1}$ must decay weakly.

To describe the weak decays, again we need an Isgur–Wise form factor. For the decay $B_{a_1\ldots a_{N_c-2}b}\to B_{a_1\ldots a_{N_c-2}c}$, where $B_{a_1\ldots a_{N_c-2}b}\equiv \bar{Q}_{a_1}\ldots\bar{Q}_{a_{N_c-2}}\bar{Q}_b\bar{q}$ and $B_{a_1\ldots a_{N_c-2}c}\equiv \bar{Q}_{a_1}\ldots\bar{Q}_{a_{N_c-2}}\bar{Q}_c\bar{q}$, the Isgur–Wise form factor $\eta_{a_1\ldots a_{N_c-2}(b\to c)}(w)$ is dominated by the overlap of the Hartree–Fock wave functions, which describes the interaction of $\bar{Q}_b$ or $\bar{Q}_c$ with the other N_c-2 heavy antiquarks [28]. In the real world, $N_c=3$ and the Hartree–Fock wave functions become Coulombic wave functions [74]. Hence we can calculate the Isgur–Wise form factor $\eta_{a(b\to c)}(w)$ for the $\bar{Q}_a\bar{Q}_b\bar{q}\to\bar{Q}_a\bar{Q}_c\bar{q}$ transition by evaluating the overlap of Coulombic wave functions. (Note that $\eta_{a(b\to c)}(w)$ is called $\eta_{abc}(w)$ in Ref. [74].) For $B=\mu_{ab}\,\alpha_s(\mu_{ab})$ and $C=\mu_{ac}\,\alpha_s(\mu_{ac})$, we have

$$\eta_{a(b\to c)}(w)=\int d^3\mathbf{p}\,\psi^*(C;\mathbf{p})\,\psi(B;\mathbf{p}+\mathbf{q}), \tag{4.1}$$

where

$$\mathbf{q}=m_a(\mathbf{v}-\mathbf{v}'), \tag{4.2}$$

and $\psi(B;\mathbf{p})$ is the ground state Coulombic wave function with Bohr radius B^{-1}

and similar for $\psi(C;\mathbf{p}+\mathbf{q})$. The exact form of $\eta_{a(b\to c)}(w)$ is given in Ref. [74]. Note that in general the Isgur–Wise form factor is not normalized at the point of zero recoil. In fact,

$$\eta_{a(b\to c)}(1) = \left(\frac{2\sqrt{BC}}{B+C}\right)^3 , \tag{4.3}$$

which is not equal to unity unless $B = C$, i.e., $m_b = m_c$. This is very different from the normalization of $\eta(w)$, which holds regardless of the size of $m_b - m_c$ as long as both m_b and $m_c \gg \Lambda_{\text{QCD}}$.

In our calculations of $\eta_{a(b\to c)}$, we have neglected the overlap of the "brown muck," which includes the single light valence antiquark and the quark-antiquark sea. Including that would lead to an extra factor of $\xi(w)$. The meson Isgur–Wise form factor $\xi(w)$ is a slowly varying function as its slope at the point of zero recoil is of the order of unity. On the other hand, $\psi(B,\mathbf{p})$ has a short range of the order of $B \sim m_Q\alpha_s(m_Q)$. Hence, near the point of zero recoil, the slope $\eta_{a(b\to c)}$ is of the order of $\alpha_s^{-2}(m_Q)$. In the heavy quark limit, this slope is large ($\alpha_s^{-2}(m_b) \sim 22.5$) and the w-dependence of $\eta_{a(b\to c)}$ does overwhelm that of $\xi(w)$. This justifies the neglect of the $\xi(w)$ factor in our calculations.

Unlike $\eta(w)$ the normal Isgur–Wise form factor, which is non-perturbative in nature, the form factor $\eta_{a_1\ldots a_{N_c-2}(b\to c)}(w)$ results from the perturbative attraction between the heavy antiquarks. As we will see below, to describe the semileptonic decays of the multiquark exotics, we need just the two form factors we have discussed, namely $\eta(w)$ and $\eta_{a_1\ldots a_{N_c-2}(b\to c)}(w)$.

2. Tetraquarks

The large N_c analog of tetraquark states $\bar{Q}\bar{Q}qq$ is ambiguous. Both $\bar{Q}\bar{Q}qq$ and $\bar{Q}^{N_c-1}q^{N_c-1}$ reduce to $\bar{Q}\bar{Q}qq$ when $N_c = 3$. In his classic paper on baryons in the $1/N_c$ expansion [28], E. Witten showed that $\bar{Q}\bar{Q}qq$ states are absent in the large N_c limit. On the other hand, stable $\bar{Q}^{N_c-1}q^{N_c-1}$ states, which he called "baryonium" states, exist. In this section, we will discuss their properties and decay modes.

We can obtain a "baryonium" $\bar{Q}^{N_c-1}q^{N_c-1}$ by replacing the heavy quark Q in a heavy baryon Qq^{N_c-1} with the "fake heavy quark" $\bar{Q}^{N_c-1}$. In the real world, where $N_c = 3$, it is just the di-meson, or tetraquark hadron $\bar{Q}\bar{Q}qq$ which Jaffe discussed in Ref. [69,70] in the context where all the quarks are light, and subsequently discussed in Ref. [5,28,75-80].

Since non-perturbative QCD cannot resolve the individual heavy antiquarks in the "fake heavy quark," the tetraquark has similar spectroscopic properties with the normal baryon. In particular, the lowest-lying configurations will have $(I, S_\ell) = (0,0)$. These states are safe from dissociations like $\bar{Q}^{N_c-1}q^{N_c-1} \to \bar{Q}^{N_c-1}\bar{q} + q^{N_c}$ and hence must decay weakly.

The Isgur–Wise form factor $\eta^{(4)}_{a_1\ldots a_{N_c-2}(b\to c)}(w)$, which describes the tetraquark weak decays $T_{a_1\ldots a_{N_c-2}b} \to T_{a_1\ldots a_{N_c-2}c}$, where $T_{a_1\ldots a_{N_c-2}b} \equiv \bar{Q}_{a_1}\ldots\bar{Q}_{a_{N_c-2}}\bar{Q}_b qq$ and $T_{a_1\ldots a_{N_c-2}c} \equiv \bar{Q}_{a_1}\ldots Q_{a_{N_c-2}}\bar{Q}_c qq$, can be expressed as the product of two terms with different physical origins [5]. A perturbative term comes from the overlap of the initial and final "fake heavy quark" wave function. Since this "fake heavy quark" transition is identical with what happens in the $B_{a_1\ldots a_{N_c-2}b} \to B_{a_1\ldots a_{N_c-2}c}$ decay, the perturbative term is exactly $\eta_{a_1\ldots a_{N_c-2}(b\to c)}(w)$. On the other hand, a non-perturbative term describes the overlap of the initial and final light degrees of freedom under the color field of the "fake heavy quark." Since the light degrees of freedom of a tetraquark is identical to that of a normal baryon, the non-perturbative term is just $\eta(w)$. As a result, we get

$$\eta^{(4)}_{a_1\ldots a_{N_c-2}(b\to c)}(w) = \eta(w)\ \eta_{a_1\ldots a_{N_c-2}(b\to c)}(w). \tag{4.4}$$

When $N_c = 3$ we have

$$\eta^{(4)}_{a(b\to c)}(w) = \eta(w)\ \eta_{a(b\to c)}(w), \tag{4.5}$$

which has been proved in Ref. [5]. Note that the normalization of $\eta^{(4)}_{a(b\to c)}(w)$ is given by the normalization of each of its factors.

$$\eta^{(4)}_{a(b\to c)}(1) = \left(\frac{2\sqrt{BC}}{B+C}\right)^3 . \tag{4.6}$$

We will see this factorization of perturbative and non-perturbative contributions also in the case of hexaquark decays. Also note that this factorization is purely a consequence of the decoupling of the dynamics of the heavy and light degrees of freedom in the heavy quark limit and does not depend on the large N_c limit [5]. In particular, Eq. (4.5) is still valid away from of large N_c limit. In such cases, however, $\eta(w)$ is also expected to be slowly varying and the behavior of $\eta^{(4)}_{a(b\to c)}(w)$ is dominated by that of $\eta_{a(b\to c)}(w)$.

3. Pentaquarks

In Ref. [54], the authors have remarked upon the possibility of the existence of exotic bound states of heavy mesons and chiral *anti*-solitons. In the large N_c limit, a chiral anti-soliton, which has baryon number -1, is also a configuration of the pion field satisfying the hedgehog ansatz. The profile function satisfying the boundary condition $F(0) = \pi$ and $F(\infty) = 0$. In other words, a chiral anti-soliton is obtained when we flip the sign of $F(x)$ of a chiral soliton, and *vice versa.* Since the binding potential $V(x)$ is odd in $F(x)$, the binding of a chiral anti-soliton to a heavy meson will be the same in magnitude but opposite in sign with that of a chiral soliton. Denoting the binding of a chiral anti-soliton to a heavy meson by $\tilde{V}(x;K)$, we have

$$\tilde{V}(x;K) = -V(x;K). \tag{4.7}$$

Hence $K = 0$ states are unbound and the $K = 1$ states are bounded. The stable bound states are those with $(I, S_\ell) = (0,1)$, $(1,1)$, $(1,0)$ and *etc.* In the quark model, such states are exotic $Q\bar{q}^{N_c+1}$ multiquarks. When $N_c = 3$, these $Q\bar{q}\bar{q}\bar{q}\bar{q}$ states are just the pentaquarks discussed in Ref. [78,81-85].

Combining Eq. (3.26) and Eq. (4.7), we get the relation

$$\tilde{V}(x;1) = \tfrac{1}{3}V(x;0), \tag{4.8}$$

which means that the binding energy of the pentaquark is just a third of that of a normal heavy baryon. Still, the pentaquark is below the $Q\bar{q}^{N_c+1} \to Q\bar{q} + \bar{q}^{N_c}$ threshold and hence must decay weakly. Here again the Isgur–Wise function $\eta^{(5)}(w)$ is given by the overlap of simple harmonic wave functions $\tilde{\phi}(\mathbf{p})$. The only difference is that for the pentaquark system the spring constant is just $\frac{1}{3}\kappa$, i.e., a third of that of a normal heavy baryon. Since the natural unit for momentum of a simple harmonic oscillator is $(m_S\kappa)^{1/4}$, we have

$$\tilde{\phi}(\mathbf{p}) = \phi(3^{1/4}\mathbf{p}). \tag{4.9}$$

With

$$\eta^{(5)}(w) = \int d^3\mathbf{p}\, \tilde{\phi}^*(\mathbf{p})\, \tilde{\phi}(\mathbf{p}+\mathbf{k}), \tag{4.10}$$

we finally obtain

$$\eta^{(5)}(w) = \eta(\sqrt{3}(w-1)+1). \tag{4.11}$$

We have succeeded in relating the Isgur–Wise form factors of pentaquarks and that of normal heavy baryons. Like $\eta(w)$, $\eta^{(5)}(w)$ also non-perturbative in nature. It obeys Luke's theorem and is normalized at the point of zero recoil,

$$\eta^{(5)}(1) = 1. \tag{4.12}$$

We can also consider the hadron obtained by replacing the heavy quark in a pentaquark system by a "fake heavy quark." The resultant hadron is the famous H-dibaryon, which is also known as the hexaquark. This will be the topic of the next section.

4. Hexaquarks

As suggested above, when we replace the heavy quark inside the pentaquark system with a "fake heavy quark," the resultant system $\bar{Q}^{N_c-1}\bar{q}^{N_c+1}$ has baryon number 2. When $N_c = 3$, the hadron $\bar{Q}\bar{Q}\bar{q}\bar{q}\bar{q}\bar{q}$ is just the H particle first suggested by Jaffe [71] in the context of a *uuddss* complex, and subsequently discussed by Ref. [28,78,83,86-97]. It is one of the most well-discussed exotics as it arises in many different scenarios like the bag models, large N_c Hartree-Fock model, $\Lambda\Lambda$ molecule, Skyrme models, potential models and lattice gauge calculations. Noteworthy are Ref. [86], [89] and [93], in which H arises as topological chiral solitons under $SU(3)_L \times SU(3)_R$ chiral symmetry. Under light flavor SU(3) generated by $\{\lambda_a : a = 1, \ldots, 8\}$, the G-invariance equation (3.8) which expresses the equivalence of a rotation in real space and that in the flavor space becomes

$$[\Lambda_a, U(\mathbf{x})] - i(\mathbf{x} \times \nabla)_a U(\mathbf{x}) = 0, \tag{4.13}$$

where $\{\Lambda_a : a = 1, 2, 3\}$ is an SU(2) subalgebra of light flavor SU(3), i.e.,

$$[\Lambda_a, \Lambda_b] = i\epsilon_{abc}\Lambda_c. \tag{4.14}$$

There are two inequivalent sets of $\{\Lambda_a\}$'s satisfying Eq. (4.14). The trivial choice is $\{\Lambda_a\} = \{\frac{\lambda_1}{2}, \frac{\lambda_2}{2}, \frac{\lambda_3}{2}\}$, where the solutions to Eq. (4.13) are hedgehog-like and can be identified with the normal chiral solitons (N and Δ). The non-trivial choice $\{\Lambda_a\} = \{\lambda_2, \lambda_5, \lambda_7\}$ leads to non-hedgehog soliton solutions [86,89]. Due to this relation between the structure constants of SU(3),

$$f_{257} = \tfrac{1}{2} f_{123}, \tag{4.15}$$

the baryon number of H is twice that of a normal chiral soliton. This SU(3) group theoretical approach is applicable when there are three light flavors and H can be viewed as a *uuddss* complex in the quark model. It is interesting to see that, in our formalism, heavy hexaquarks also arise as bound states of normal SU(2) chiral solitons to heavy mesons.

Since the light degrees of freedom of a hexaquark is identical to that of a pentaquark, we have $K = 1$ stable configurations. The discussion on the stability of the hexaquark runs parallel to that of the tetraquark. The large chromoelectric binding energy prevents the dissociation $\bar{Q}^{N_c-1}\bar{q}^{N_c+1} \longrightarrow \bar{Q}\bar{q}^{N_c-1}+\bar{Q}^{N_c-2}\bar{q}\bar{q}$ or other decay modes which involves the splitting up of the "fake heavy quark" system. On the other hand, the stability of the pentaquark system guarantees the stabililty of the hexaquark against $\bar{Q}^{N_c-1}\bar{q}^{N_c+1} \longrightarrow \bar{Q}^{N_c-1}\bar{q} + \bar{q}^{N_c}$. Hence the hexaquark is stable with respect to strong interactions and decays weakly.

Just like its tetraquark counterpart, the hexaquark Isgur–Wise weak form factor $\eta^{(6)}_{a_1\ldots a_{N_c-2}(b\to c)}(w)$ of the hexaquark decay $H_{a_1\ldots a_{N_c-2}b} \longrightarrow H_{a_1\ldots a_{N_c-2}c}$, where $H_{a_1\ldots a_{N_c-2}b} \equiv \bar{Q}_{a_1}\ldots\bar{Q}_{a_{N_c-2}}\bar{Q}_b\bar{q}^{N_c+1}$ and $H_{a_1\ldots a_{N_c-2}c} \equiv \bar{Q}_{a_1}\ldots Q_{a_{N_c-2}}\bar{Q}_c\bar{q}^{N_c+1}$ can also be expressed as the product of a perturbative factor and a non-perturbative factor. The perturbative part, which results from the chromoelectric attraction between the heavy antiquarks, is again given by $\eta_{a_1\ldots a_{N_c-2}(b\to c)}(w)$. The non-perturbative part, which describes the overlap between the initial and final light degrees of freedom, is given by $\eta^{(5)}(w)$ as the light degrees of freedom of a hexaquark is identical with that of a pentaquark. As a result,

$$\eta^{(6)}_{a_1\ldots a_{N_c-2}(b\to c)}(w) = \eta^{(5)}(w)\ \eta_{a_1\ldots a_{N_c-2}(b\to c)}(w). \tag{4.16}$$

When $N_c = 3$ we have

$$\eta^{(6)}_{a(b\to c)}(w) = \eta^{(5)}(w)\ \eta_{a(b\to c)}(w). \tag{4.17}$$

Last of all, we again have the normalization condition

$$\eta^{(6)}_{a(b\to c)}(1) = \left(\frac{2\sqrt{BC}}{B+C}\right)^3. \tag{4.18}$$

5. Discussion

In summary, the following results have been obtained.

1) In the large N_c limits, there exists heavy tetraquark, pentaquark, and hexaquark exotic states which are stable with respect to strong interactions. It is noted that the stabilities of tetraquarks depends simply on the heavy quark limit and the stabilities of normal Λ_Q baryons, while the stabilities of pentaquarks and hexaquarks have been shown in the context of the chiral soliton model, which is a crucial assumption in our discussion.

2) The (I, s_ℓ) of these multiquark states are known.

3) The "heavy to heavy" weak decay of each of these multiquark states are described by a single Isgur–Wise form factor. Moreover, these Isgur–Wise form factors can be expressed in terms of the Isgur–Wise form factors of normal baryons.

Naturally the question of whether in this framework one can generate higher multiquark states like heptaquarks and octaquarks arises. To answer this question, note that all the hadrons considered in this paper can be reduced to two-body bound states, of either a heavy meson or "fake heavy meson" in one hand, and a chiral soliton or anti-soliton in the other. It is exactly the possibility of such reduction to a two-body problem which simplifies the systems and enable us to get the results listed above. When we move on higher multiquark states, such reductions are impossible. For example, the heptaquark is a "fake heavy meson" $\bar{Q}\bar{Q}\bar{q}$ bounded to *two* chiral solitons qqq, and the octaquark is a heavy meson $Q\bar{q}$ bounded to *two* chiral anti-solitons $\bar{q}\bar{q}\bar{q}$. In general, such three-body systems are intractable. Hence we do not expect a simple generalization of our framework to describe these higher multiquark states.

Our discussion on hadrons with $N_c - 1$ heavy antiquarks depends crucially on the assumption that the light degrees of freedom cannot resolve the "fake heavy quark" or equivalently $N_c \alpha_s(m_Q) m_Q \gg \Lambda_{\mathrm{QCD}}$. In the real world, since the top quark does not live long enough to form hadrons, we just have two "hadronizable" heavy quarks, the b quark and the c quark. The assumption above, however, holds for neither of them, and our results cannot be applied directly. Still, it is possible that the picture above is at least qualitatively correct and can serve as the starting point of quantitative investigations of the heavy multiquark systems by including the effects of $1/m_Q$ corrections.

V. Conclusion

In previous sections, we have seen how the bound state picture provides a useful framework for the studies of heavy hadrons. The basic ingredients of our formalism are: chiral symmetry, heavy quark symmetry, and the large N_c limit. It is instructive to review how these different ingredients simplify the physics, and examine the scope of applicability of our framework.

The most crucial simplifications come from the simplest observations. In the heavy quark limit, heavy hadrons are infinitely massive. Similarly, in the large N_c limit, baryons are also infinitely massive. Pair productions are infinitely suppressed, and the corresponding Fock spaces become disjoint Hilbert spaces with conserved particle (heavy quark or baryon) numbers. In other words, *the quantum field theory can been reduced to quantum mechanics!* In the heavy quark and large N_c limit, heavy mesons and light baryons are quantum mechanical entities. And in this thesis, we have been studying the Hilbert space with both heavy quark number and baryon number equal to unity. It must be emphasized that this suppression of pair productions is the consequence of the infinite massiveness of the heavy mesons and baryons, and has nothing to do with the spin symmteries.

The next step is the bound state model, which proposed that a heavy baryon can be treated as the bound state of a heavy meson to a light baryon. Hence the Hilbert space $\mathcal{H}$ can be decomposed into the product of several subspaces.

$$\mathcal{H} = \mathcal{H}_h \otimes \mathcal{H}_b \otimes \mathcal{H}_V. \tag{5.1}$$

$\mathcal{H}_h$ and $\mathcal{H}_b$ are respectively the Hilbert spaces of the heavy mesons and light baryons at rest, and $\mathcal{H}_V$ is the quantum mechanical Hilbert space for their relative motion in a given binding potential $V(\mathbf{x})$. This decomposition is a consequence of the conservation of heavy meson and baryon number mentioned above, and is independent of the origin or form of the binding potential.

The physics of heavy hadrons in the large N_c limit is simple because the relevant Lagrangians (or Hamiltonians) are block-diagonal under decomposition (5.1).

$$\mathcal{L} = \mathcal{L}_h + \mathcal{L}_b + \mathcal{L}_V. \tag{5.2}$$

For example, the mass M of a heavy baryon can be written as

$$M = M_h + M_b + M_V \tag{5.3}$$

where M_h and M_b are the masses of the heavy meson and the light baryon respectively, and M_V is the eigenenergy of the particle in the potential $V(\mathbf{x})$.

Similar decomposition happens for interaction Lagrangians. For example, heavy baryons couples to pions through the axial current,

$$\mathcal{L}_A = \partial_\mu \pi A^\mu, \tag{5.4}$$

where the heavy baryon axial current is defined in Ref. [61].

$$A^\mu = i g_2 \epsilon^{\mu\nu\sigma\lambda} \overline{S}_\nu v_\sigma S_\lambda + g_3[\overline{T} S^\mu + \overline{S}^\mu T], \tag{5.5}$$

with T and S denoting the Λ_Q and Σ_Q fields respectively. (For the exact definitions, please see Ref. [61].) In the bound state picture [53], the heavy baryon axial current can be decomposed as

$$g_i = (g_i)_h + (g_i)_b + (g_i)_V, \quad i = 2, 3, \tag{5.6}$$

with

$$(g_2)_h = -\sqrt{\tfrac{3}{2}}(g_3)_h = \tfrac{1}{2} g, \tag{5.7a}$$

$$(g_2)_b = -\sqrt{\tfrac{3}{2}}(g_3)_b = -\tfrac{3}{2} g_A, \tag{5.7b}$$

and $(g_i)_V \equiv 0$. In this way, heavy baryon axial couplings are expressed in terms of g, the heavy meson axial coupling constant defined in Eq. (3.22), and $g_A \sim 1.25$, the axial coupling for nucleons. Similar decompositions are applicable to electric charge and magnetic moments of heavy baryons.

In this thesis, we are mostly interested in the weak decay of heavy baryons. In other words, we are interested in the matrix elements of $J_\mu = \bar{c}\Gamma b$ between different states of heavy baryons. Since J_μ acts only on the heavy meson Hilbert subspace, we have this decomposition:

$$\langle \Lambda_c^{(*)}, \Sigma_c^{(*)} | J_\mu | \Lambda_b^{(*)}, \Sigma_b^{(*)} \rangle = \langle D^{(*)} | J_\mu | B^{(*)} \rangle \langle \mathrm{N}, \Delta | \mathrm{N}, \Delta \rangle \langle \phi_f | \phi_i \rangle \tag{5.8}$$

The second term, the matrix element between baryons, is identically equal to unity. For a normal $b \to c$ decay, the variation of the overlap of the initial and final spatial wave functions over different values of w dominates over the variation of the meson decay matrix element, which is essentially a slow-varying mesonic Isgur–Wise form factor. Hence

$$\langle \Lambda_c^{(*)}, \Sigma_c^{(*)} | J_\mu | \Lambda_b^{(*)}, \Sigma_b^{(*)} \rangle = \langle \phi_f | \phi_i \rangle. \tag{5.9}$$

On the other hand, for the decay of a "fake heavy quark," both the first term and the last term of Eq. (5.8) are important.

The spin symmetries simplify the dynamics within the Hilbert subspaces. For example, there exists an SU(4) spin symmetry in the baryon sector when $N_c \to \infty$, and this leads to the degeneracy of N and Δ as well as relations between the axial couplings $g_{\pi\mathrm{NN}}$, $g_{\pi\mathrm{N}\Delta}$ and $g_{\pi\Delta\Delta}$. (All of them can be expressed in terms of g_A [33].) Similarly, the SU(4) heavy quark spin symmetry leads to degenerate meson doublets like B and B^* [10,11], and their axial couplings are described by the same coupling constant g [56-58]. We can still decompose the heavy baryon Hilbert space $\mathcal{H}$ into subspaces without the spin symmetries, but the results will be much more complicated as more coupling constants are involved. Moreover, without the spin symmetries, the binding potentials of different spin states of heavy mesons and light baryons are in general unrelated. Only the universality of coupling constants gives a universal binding potential, and hence the universality of the baryonic Isgur–Wise form factor [3].

Only at the calculation of the binding potential does chiral symmetry come into play. In the calculation in Ref. [3,55], only pion-heavy meson couplings are taken into account. I will discuss further the validity and possible improvement of this assumption. Finally, the large N_c limit is used to truncate the binding potential into a simple harmonic one, enabling us to calculate some non-perturbative quantities, like the orbital excitation energy $\sqrt{\kappa/M_b}$ and the slope of the Isgur–Wise form factor $\bar{\rho}$.

In this way, through the interplay of chiral symmetry, heavy quark symmetry and the large N_c limit, we have succeeded in obtaining a reasonable description of heavy baryons. In the real world, none of these symmetries are exactly obeyed. Both the large N_c corrections $O(1/N_c) = 0.33$ and the heavy quark corrections $O(\Lambda_{\rm QCD}/m_c) \sim 0.2$ may be significant. Ultimately we have to rely on experiments to tell if the bound state picture is a useful framework for the investigation of the properties of heavy baryons.

We will end this thesis by a discussion of possible directions for further explorations.

One possible improvement of our framework can be done by inclusion of other light mesons, like ρ, ω, σ and a_1, in the dynamics. As mentioned above, in our treatment the chiral soliton is regarded as a purely pionic configuration, and the effects of other light mesons are neglected. In view of the non-perturbative nature of our problem, this approximation is reasonable as the pions are the only hadron with mass below $\Lambda_{\rm QCD}$. The agreement of predictions of the chiral soliton model with experiment at the 40% level supports the validity of this treatment. On the other hand, it is expected that the agreement will improve when more light mesons are included [31]. In particular, there are reasons for us to expect ρ meson to play a significant role in the chiral soliton model. Being one of the lightest hadrons, the ρ mass is not very much above $\Lambda_{\rm QCD}$ ($m_\rho = 770$ MeV). In the non-relativistic quark model, π and ρ fall into the same SU(4) multiplet. The strongest hint comes from the identification of the Skyrme term as the $m_\rho \to \infty$ limit of the ρ meson kinetic term [46.47]. All these provide motivations to incorporate these light mesons into the chiral soliton model.

The most straightforward way [98] to incorporate the light mesons is to describe the configuration of each meson by a different profile function. Then the energy is minimized with respect to most general chiral invariant Lagrangian to obtain the ground state profile functions. The binding with heavy mesons can be calculated by summing the interaction of the heavy meson with each of these background meson configurations. This method is straightforward but can only be done at the expense of including more coupling constants, which may lead to a decline of predictive power.

A completely different approach is suggested by H. Georgi in Ref. [99]. A new symmetry, which is usually referred to as vector symmetry, is introduced to relate pions to (the longitudinal component of) the ρ meson. The symmetry is badly broken in the real world, but may be restored at the large N_c limit. Under this symmetry, pions and ρ mesons are degenerate and are both massless in the chiral limit. No new coupling constant is needed to describe ρ dynamics. On the other hand, there is no evidence that the vector symmetry will yield a reasonable description of the real world. After all, $m_\pi = 140$MeV and $m_\rho = 770$MeV, differing by a factor of 5. Hence, the applicability of the vector symmetry to the studies of hadron dynamics remains questionable.

Secondly, we must try to estimate the effects of large but finite heavy quark masses. A possible line of attack is to see how to connect our formalism, where the b quark is infinitely massive, to the flavor SU(3) limit, where the "b quark" becomes a massless light quark. This is especially important in the understanding of baryons containing an s quark, as neither the chiral nor the heavy quark limit is directly applicable. Pioneering work in this direction has been done in Ref. [100]. Moreover, as mentioned before, the realistic heavy quark masses are not large enough to form "fake heavy quarks" unresolvable by non-perturbative QCD.

For realistic quark masses, multiquark states may arise as molecules, i.e., bound states through pion exchange in an attractive channel. Tetraquark molecules have been studied in Ref. [80] and it is of interest to repeat the exercise for pentaquark and hexaquark molecules as well.

The chiral soliton model and the bound state picture is the most intuitive among the different approaches to the large N_c limit. However, it cannot be readily extended to include $1/N_c$ and $\Lambda_{\rm QCD}/m_Q$ corrections. It is important to see how our results may arise in the other approaches. In fact, in all the approaches arises the SU(4) spin-flavor symmetry, which is now recognized as the crux of our understanding of large N_c baryons. Some of the results of the chiral soliton model, like the mass degeneracy of Λ_Q and $\Sigma_Q^{(*)}$ and the universality of the pion couplings, are clearly just consequences of the SU(4) symmetry, while the status is not clear for some other predictions, like the exponential form of the Isgur–Wise form factor. Ultimately we would like to formulate the large N_c limit as a controlled expansion, where symmetry breaking effect can be studied order of order in $1/N_c$. It is my belief that studies in this direction can lead to a better understanding in both heavy flavor physics and the non-perturbative aspects of non-Abelian gauge theories.

Appendix:

The Application of the Chiral Lagrangian in Calculating The Corrections From Low Momentum Physics to the Mesonic Isgur–Wise Form Factor

This appendix is essentially a reproduction of Ref. [1], in which the correction to the mesonic Isgur–Wise form factor due to pion loops are calculated under an $SU(2)_L \times SU(2)_R$ chiral Lagrangian. It will end with a discussion on related calculations [102,103]. It is logically unrelated to the rest of this thesis and can be skipped without any loss of completeness. Also, no effort have been made to change the notations in conformation to the rest of the thesis.

Semileptonic $B \to D^{(*)}$ decays provide an interesting arena to test the validity of heavy quark symmetry. They may also give a very accurate determination of the element, V_{cb}, in the Cabibbo-Kabayashi-Maskawa matrix. Heavy quark spin-flavor symmetry implies that as m_b and $m_c \to \infty$ the hadronic matrix elements needed for semileptonic $B \to D^{(*)}$ decay have the form [10,11]

$$\frac{< D(v')|\bar{c}\gamma_\mu b|B(v) >}{\sqrt{m_B m_D}} = \beta(v+v')_\mu \ , \qquad (A.1a)$$

$$\frac{< D^*(v',\epsilon)|\bar{c}\gamma_\mu\gamma_5 b|B(v) >}{\sqrt{m_B m_{D^*}}} = \beta((1+w)\epsilon^*_\mu - (\epsilon^*\cdot v)v'_\mu) \ , \qquad (A.1b)$$

$$\frac{< D^*(v',\epsilon)|\bar{c}\gamma_\mu b|B(v) >}{\sqrt{m_B m_{D^*}}} = \beta i \epsilon_{\mu\nu\lambda\sigma}\epsilon^{*\nu} v'^\lambda v^\sigma \ . \qquad (A.1c)$$

In Eqs. (A.1) β depends on $w = v \cdot v'$ and it has a calculable logarithmic dependence on the heavy c and b quark masses that arises from high momentum perturbative QCD effects [67,104-106]. Furthermore, the value of β is known at zero recoil, i.e., $w = 1$ [10,11,107,108]. In the leading logarithmic approximation

[104-106] (valid for $m_b \gg m_c \gg \Lambda_{\rm QCD}$)

$$\beta(1) = \left[\frac{\alpha_s(m_b)}{\alpha_s(m_c)}\right]^{-6/25} . \qquad (A.2)$$

Perturbative QCD corrections to Eqs. (A.1) and (A.2), suppressed by powers of $\alpha_s(m_c)$ and $\alpha_s(m_b)$, are calculable and don't give rise to any loss of predictive power [67]. Since at zero recoil there are no $1/m_Q$ corrections to eqs. (A.1) and (A.2) [16], the exclusive semileptonic $B \to D^{(*)}$ decays may lead to a very precise determination of V_{cb}.

Recently chiral perturbation theory has been used to examine, at zero recoil, the power $(1/m_c)^{2+n}$ $n = 0, 1, 2, \ldots$ corrections to the matrix elements of Eqs. (A.1) [101]. It was found that these corrections have a nonanalytic dependence on the light up and down quark masses of the form $\ell n m_\pi^2$ when $n = 0$ and $(1/m_\pi)^n$ when $n = 1, 2, 3, \ldots$. These corrections arise from one-loop Feynman diagrams with virtual momenta small compared with the chiral symmetry breaking scale.

In this paper we extend the work of Ref. [101] away from zero recoil. The corrections of order $(1/m_c)^{n+2}$ $n = 0, 1, 2, \ldots$ that have a nonanalytic dependence on the light up and down quark masses are calculated for all w.

Unfortunately, away from zero recoil there are power corrections of order $1/m_c$ and $1/m_b$ that are not calculable using chiral perturbation theory. These must be estimated using phenomenological models (e.g., the nonrelativistic constituent quark model or QCD sum rules) or lattice QCD methods. Although the corrections of order $1/m_c$ and $1/m_b$ are larger than those we can calculate, it is still interesting that some power corrections are calculable, and it is important to verify that they are not anomalously large.

The ground state heavy mesons with $Q\bar{q}_a$ flavor quantum numbers (here $a = 1, 2$ and $q_1 = u, q_2 = d$) have $s_\ell^{\pi_\ell} = \frac{1}{2}^-$, for the spin parity of the light degrees of freedom. Combining the spin of the light degrees of freedom with the spin of the heavy quark gives (in the $m_Q \to \infty$ limit) two degenerate doublets consisting of spin zero and spin-one mesons that are denoted by P_a and P_a^* respectively. In the case $Q = c, P_a = (D^0, D^+)$ and $P_a^* = (D^{*0}, D^{*+})$ while for $Q = b, P_a = (B^-, B^0)$ and $P_a^* = (B^{*-}, B^{*0})$. It is convenient to combine the fields P_a and $P_{a\mu}^*$ that destroy these mesons ($v^\mu P_{a\mu}^* = 0$) into a 4×4 matrix H_a given by

$$H_a = \left(\frac{1 + \not{v}}{2}\right)(P_{a\mu}^* \gamma^\mu - P_a \gamma_5) \ . \tag{A.3}$$

(This is a compressed notation. In situations where the type of heavy quark Q and its four-velocity v are important, the 4×4 matrix is denoted by $H_a^{(Q)}(v)$.) It transforms under the heavy quark spin symmetry group $\mathrm{SU}(2)_v$ as

$$H_a \to S H_a \ , \tag{A.4}$$

where $S \in \mathrm{SU}(2)_v$ and under Lorentz transformations as

$$H_a \to D(\Lambda) H_a D(\Lambda)^{-1} \ , \tag{A.5}$$

where $D(\Lambda)$ is an element of the 4×4 matrix representation of the Lorentz group. It is also useful to introduce

$$\begin{aligned} \bar{H}_a &= \gamma^0 H_a^\dagger \gamma^0 \\ &= (P_{a\mu}^{*\dagger} \gamma^\mu + P_a^\dagger \gamma_5) \frac{(1 + \not{v})}{2} \ . \end{aligned} \tag{A.6}$$

For $\bar{H}_a$ the transformation laws corresponding to those in Eqs. (A.4) and (A.5) become $\bar{H}_a \to \bar{H}_a S^{-1}$ and $\bar{H}_a \to D(\Lambda) \bar{H}_a D(\Lambda)^{-1}$.

The strong interaction also has an approximate $\mathrm{SU}(2)_L \times \mathrm{SU}(2)_R$ chiral symmetry that is spontaneously broken to the vector $\mathrm{SU}(2)_V$ isospin subgroup. This symmetry arises because the light up and down quarks have masses that are small compared with the typical scale of the strong interaction. (If the strange quark is also treated as light, the chiral symmetry group becomes $SU(3)_L \times SU(3)_R$.) Associated with the spontaneous breaking of $SU(2)_L \times SU(2)_R$ chiral symmetry are the pions. The low momentum strong interactions of these pseudo Goldstone bosons are described by a chiral Lagrangian that contains the most general interactions consistent with chiral symmetry. The effects of the up and down quark masses are included by adding terms that transform in the same way under chiral symmetry as the quark mass terms in the QCD Lagrangian.

The pions are incorporated in a 2×2 unitary matrix

$$\Sigma = \exp\left(\frac{2iM}{f}\right) \qquad (A.7)$$

where

$$M = \begin{bmatrix} \pi^0/\sqrt{2} & \pi^+ \\ \pi^- & -\pi^0/\sqrt{2} \end{bmatrix} , \qquad (A.8)$$

and $f \simeq 132$ MeV is the pion decay constant. Under a chiral $SU(2)_L \times SU(2)_R$ transformation

$$\Sigma \to L\Sigma R^\dagger , \qquad (A.9)$$

where $L \in SU(2)_L$ and $R \in SU(2)_R$. It is convenient when discussing the interactions of the π mesons with the P_a and P_a^* mesons to introduce

$$\xi = \exp\left(\frac{iM}{f}\right) . \qquad (A.10)$$

Under a chiral $SU(2)_L \times SU(2)_R$ transformation

$$\xi \to L\xi U^\dagger = U\xi R^\dagger , \qquad (A.11)$$

where typically the special unitary matrix U is a complicated nonlinear function of L, R and the pion fields. However, for transformations $V = L = R$ in the unbroken subgroup $U = V$. We assign the heavy meson fields the transformation law

$$H_a \to H_b U^\dagger_{ba} \ , \tag{A.12}$$

under chiral $SU(2)_L \times SU(2)_R$. (In Eq. (A.12) and for the remainder of this paper, repeated subscripts a and b are summed over $1, 2$.)

The low momentum strong interactions of pions with heavy P_a and P^*_a mesons are described by the effective Lagrange density [56-58]

$$\mathcal{L} = -iTr\bar{H}_a v_\mu \partial^\mu H_a + \frac{1}{2} iTr\bar{H}_a H_b v^\mu (\xi^\dagger \partial_\mu \xi + \xi \partial_\mu \xi^\dagger)_{ba}$$

$$+\frac{1}{2} igTr\bar{H}_a \gamma_\nu \gamma_5 H_b (\xi^\dagger \partial^\nu \xi - \xi \partial^\nu \xi^\dagger)_{ba} + \ldots \ , \tag{A.13}$$

where the ellipsis denote terms with more derivatives. This Lagrange density is the most general one invariant under $SU(2)_L \times SU(2)_R$ chiral symmetry, heavy quark spin symmetry, parity and Lorentz transformations. Heavy quark flavor symmetry implies that g is independent of the heavy quark mass. Note that in eq. (2.11) factors of $\sqrt{m_P}$ and $\sqrt{m_{P^*}}$ have been absorbed into the P_a and $P^{*\mu}_a$ fields so they have dimension $\frac{3}{2}$. Present experimental limit gives $g^2 \leq 0.5$.

It is possible to include the symmetry breaking effects of order m_q and $1/m_Q$ into the effective Lagrangian for pion heavy meson strong interactions. In our calculations explicit chiral symmetry breaking effects enter only through the nonzero pion mass. Other chiral symmetry breaking effects are suppressed relative to the leading corrections which we calculate.

The $1/m_Q$ terms that break the spin-flavor heavy quark symmetry give rise to the additional terms

$$\delta\mathcal{L}^{(2)} = \frac{\lambda_2}{m_Q} Tr\bar{H}_a \sigma^{\mu\nu} H_a \sigma_{\mu\nu} + \frac{\lambda'_2}{m_Q} Tr\bar{H}_a H_a + \ldots \tag{A.14}$$

in the Lagrange density. The second term in Eq. (A.14) violates the heavy quark flavor symmetry but not the spin symmetry, and the first term violates both the heavy quark spin and flavor symmetries. The ellipsis denotes terms with derivatives. Included in these are, for example, the $1/m_Q$ correction to g. The second term in Eq. (A.14) can be removed by a spacetime dependent phase transformation on the heavy meson fields. Therefore, at the leading order in chiral perturbation theory, it is only the first term in Eq. (A.14) that produces violations of heavy quark spin-flavor symmetry in the low energy heavy meson Lagrangian. The effect of λ_2 is to shift the mass of the pseudoscalar relative to the vector meson by

$$\Delta^{(Q)} = m_{P^{*(Q)}} - m_{P^{(Q)}} \ , \qquad (A.15)$$

which distinguishes the heavy meson propagators. Explicitly, $\Delta^{(Q)} = -8\lambda_2/m_Q$, which determines $\lambda_2 \approx (170\mathrm{MeV})^2$. Heavy quark mass suppressed operators in which the pion couples can be neglected at leading order in chiral perturbation theory.

Lorentz invariance and parity invariance of the strong interactions implies that the hadronic matrix elements needed for semileptonic $B \to D^{(*)}$ decay have the form

$$\frac{< D(v')|\bar{c}\gamma_\mu b|B(v) >}{\sqrt{m_B m_D}} = \tilde{f}_+(v+v')_\mu + \tilde{f}_-(v-v')_\mu \qquad (A.16a)$$

$$\frac{< D^*(v',\epsilon)|\bar{c}\gamma_\mu\gamma_5 b|B(v) >}{\sqrt{m_B m_{D^*}}} = \tilde{f}\epsilon^*_\mu + \tilde{a}_+(\epsilon^*\cdot v)(v+v')_\mu + \tilde{a}_-(\epsilon^*\cdot v)(v-v')_\mu \qquad (A.16b)$$

$$\frac{< D^*(v',\epsilon)|\bar{c}\gamma_\mu b|B(v) >}{\sqrt{m_B m_{D^*}}} = i\tilde{g}\epsilon_{\mu\nu\lambda\sigma}\epsilon^{*\nu}v'^\lambda v^\sigma \qquad (A.16c)$$

The operator $\bar{c}\Gamma b$ is a singlet under $\mathrm{SU}(2)_L \times \mathrm{SU}(2)_R$ and in chiral perturbation theory its $B(v) \to D^{(*)}(v')$ matrix are given by those of

$$\bar{c}\Gamma b = -\beta(w) Tr \bar{H}_a^{(c)}(v')\Gamma H_a^{(b)}(v) + \dots \ , \qquad (A.17)$$

where the ellipsis denote terms with derivatives, factors of m_q and factors of $1/m_Q$. Evaluating at tree level $B \to D^{(*)}$ matrix elements of the right-hand side of Eq.

(A.17) for $\Gamma = \gamma_\mu$ and $\Gamma = \gamma_\mu\gamma_5$ gives eqs. (A.1). This implies the relations

$$\tilde{f}_+ = \beta \ , \qquad \tilde{f}_- = 0 \ ,$$

$$\tilde{a}_+ + \tilde{a}_- = 0 \ , \qquad \tilde{a}_+ - \tilde{a}_- = -\beta \ ,$$

$$\tilde{f} = (1+w)\beta \ , \qquad \tilde{g} = \beta \ , \tag{A.18}$$

for the form factors in Eqs. (A.16).

The leading corrections to Eqs. (A.18) that have a nonanalytic dependence on the light quark masses arise from the one loop Feynman diagrams in Fig. 3 and wave function renormalization. In Fig. 3 the shaded square denotes a vertex from Eq. (A.17) and a dot is a $P^*P\pi$ or $P^*P^*\pi$ vertex proportional to g from the chiral Lagrangian density in Eq. (A.13). Explicit calculation of these Feynman diagrams gives that their contribution to the form factors is

$$\delta\tilde{f}_+ = -\frac{3ig^2\beta}{f^2}\Big\{[w+2]I_1(\Delta,w) + [w^2-1]I_2(\Delta,w) - \frac{3}{2}I_3(\Delta,w) - \frac{3}{2}I_3(0,w)\Big\} \tag{A.19a}$$

$$\delta\tilde{f}_- = 0 \tag{A.19b}$$

$$\delta\tilde{f} = -\frac{3ig^2\beta}{f^2}[w+1]\Big\{I_1(-\Delta,w) - \frac{1}{2}I_3(-\Delta,w) + [w+1]I_1(0,w) + [w^2-1]I_2(0,w) - \frac{5}{2}I_3(0,w)\Big\} \tag{A.19c}$$

$$\delta(\tilde{a}_+ - \tilde{a}_-) = \frac{3ig^2\beta}{f^2}\Big\{-[w+1]I_2(-\Delta,w) - \frac{1}{2}I_3(-\Delta,w) + [w+2]I_1(0,w) + [w^2+w]I_2(0,w) - \frac{5}{2}I_3(0,w)\Big\} \tag{A.19d}$$

$$\delta(\tilde{a}_+ + \tilde{a}_-) = -\frac{3ig^2\beta}{f^2}\Big\{-I_1(-\Delta,w) - [w+1]I_2(-\Delta,w)$$

$$+ I_1(0,w) + [w+1]I_2(0,w)\Bigg\} \qquad (A.19e)$$

$$\delta\tilde{g} = -\frac{3ig^2\beta}{f^2}\Bigg\{ I_1(-\Delta,w) - \frac{1}{2}I_3(-\Delta,w) + [w+1]I_1(0,w) + [w^2-1]I_2(0,w) - \frac{5}{2}I_3(0,w)\Bigg\} . \qquad (A.19f)$$

In Eqs. (A.19) the integrals I_1, I_2 and I_3 are

$$I_1 = \mu^{4-n}\int_0^\infty d\alpha \int_0^\infty d\beta \int \frac{d^n k}{(2\pi)^n} \frac{k^2}{[k^2 - (\alpha^2+\beta^2+2\alpha\beta w + 2\Delta\alpha + m_\pi^2) + i\epsilon]^3} \quad (A.20)$$

$$I_2 = 4\mu^{4-n}\int_0^\infty d\alpha \int_0^\infty d\beta \int \frac{d^n k}{(2\pi)^n} \frac{\alpha\beta}{[k^2 - (\alpha^2+\beta^2+2\alpha\beta w + 2\Delta\alpha + m_\pi^2) + i\epsilon]^3} (A.21)$$

$$I_3 = \mu^{(4-n)}\int_0^\infty d\alpha \int \frac{d^n k}{(2\pi)^n} \frac{\alpha k^2}{[k^2 - (\alpha^2 + 2\Delta\alpha + m_\pi^2) + i\epsilon]^3} , \qquad (A.22)$$

where μ is the subtraction point and $\Delta = m_{D^*} - m_D$. For simplicity we have taken $m_b \to \infty$, so that corrections suppressed by factors of $(1/m_b)$ are neglected. Note that $\tilde{f}_-$ receives no corrections and $\delta\tilde{f}/\tilde{f} = \delta\tilde{g}/\tilde{g}, \delta\tilde{a}_+ = -\delta\tilde{g}/2$.

In calculating the Feynman diagrams, the amplitudes were reduced to scalar integrals using four-dimensional algebra (e.g., Tr $\gamma_\mu\gamma_\nu = 4g_{\mu\nu}, k_\mu k_\nu \to \frac{1}{4}k^2 g_{\mu\nu}$, etc). The resulting scalar integrals were then continued to n dimensions. This regularization procedure is similar to that used in supersymmetry. It is easy to see that it preserves the heavy quark spin-flavor symmetry when $\Delta = 0$. Then the corrections in Eqs. (A.19) can be absorbed into the redefinition, $\beta \to \beta + \delta\beta$, where

$$\delta\beta = -\frac{3ig^2}{f^2}\beta\left\{(w+2)I_1(0,w) + (w^2-1)I_2(0,w) - 3I_3(0,w)\right\} . \qquad (A.23)$$

Furthermore, since $I_3(0,w) = I_1(0,w), \delta\beta$ vanishes at zero recoil.

In Eqs. (A.19) the terms containing I_3 result from wave function renormalization. The integrations over α and β arise from combining denominators using the trick introduced in Ref. [67].

The integrals have the form

$$I_j(\Delta, w) = \frac{i}{16\pi^2}[m_\pi \Delta E_j(w) + \Delta^2 \ln(m_\pi^2/\mu^2) G_j(w)$$

$$+\Delta^2 F_j(\Delta/m_\pi, w)] + \dots , \qquad (A.24)$$

where $F_j(0, w) = 0$ and the ellipsis denotes terms that are independent of Δ or have an analytic dependence on the light up and down quark masses. Evaluating the integrals gives

$$E_1(w) = \pi/(w+1) , \qquad (A.25a)$$

$$E_2(w) = -\pi/(w+1)^2 , \qquad (A.25b)$$

$$E_3(w) = \pi , \qquad (A.25c)$$

$$G_1(w) = -\frac{1}{2(w^2-1)}[w - r(w)] \qquad (A.26a)$$

$$G_2(w) = \frac{1}{2(w^2-1)^2}[w^2 + 2 - 3wr(w)] , \qquad (A.26b)$$

$$G_3(w) = -1 , \qquad (A.26c)$$

where

$$r(w) = \frac{1}{\sqrt{w^2-1}} \ln(w + \sqrt{w^2-1}) . \qquad (A.27)$$

For $F_{1,2}$ the expressions are more complicated and we leave them as one-dimensional integrals,

$$F_1(x, w) = \frac{1}{x^2} \int_0^{\pi/2} d\theta \frac{a}{(1 + w \sin 2\theta)} \left\{ \pi \left(\sqrt{1-a^2} - 1 \right) \right.$$

$$-2\left(\sqrt{1-a^2}\ \arctan\left(\frac{a}{\sqrt{1-a^2}}\right)-a\right)\Bigg\}\ , \qquad (A.28a)$$

$$F_2(x,w) = \frac{1}{x^2}\int_0^{\pi/2} d\theta \frac{a\sin 2\theta}{(1+w\sin 2\theta)^2}\Bigg\{-\frac{3\pi}{2}\left(\sqrt{1-a^2}-1\right)$$

$$+\frac{\pi a^2}{2\sqrt{1-a^2}}+\left(\left[\frac{3-4a^2}{\sqrt{1-a^2}}\right]\ \arctan\left(\frac{a}{\sqrt{1-a^2}}\right)-3a\right)\Bigg\}\ , \qquad (A.28b)$$

where

$$a = \frac{x\cos\theta}{\sqrt{1+w\sin 2\theta}}\ . \qquad (A.29)$$

Figs. 4, 5, 6 and 7 present plots of $F_1(1,\omega), F_1(-1,\omega), F_2(1,\omega)$ and $F_2(-1,w)$. Finally

$$F_3(x,w) = \frac{1}{x}\Bigg\{\pi\left(\sqrt{1-x^2}-1\right)-2\left(\sqrt{1-x^2}\ \arctan\left(\frac{x}{\sqrt{1-x^2}}\right)-x\right)\Bigg\}\ . \qquad (A.30)$$

At $x=\pm 1$ the above becomes $F_3(1,w)=(2-\pi)$ and $F_3(-1,w)=(2+\pi)$. Combining these results gives the correction to the form factors that have a non-analytic dependence on the up and down quark masses. For the corrections of order $(1/m_c)^{n+2}, n=0,1,2,...$ we have

$$\delta f_+ = \frac{3g^2\beta}{(4\pi f)^2}\Delta^2\Bigg\{\frac{1}{2}\left(\frac{3w+1-2r}{w+1}\right)\ln(m_\pi^2/\mu^2)$$

$$+(w+2)F_1(\Delta/m_\pi,w)+(w^2-1)F_2(\Delta/m_\pi,w)-\frac{3}{2}F_3(\Delta/m_\pi,w)+...\Bigg\} \qquad (A.31a)$$

$$\delta\tilde{f} = \frac{3g^2\beta}{(4\pi f)^2}\Delta^2(w+1)\Bigg\{\frac{1}{2}\left(1-\left(\frac{w-r}{w^2-1}\right)\right)\ln(m_\pi^2/\mu^2)$$

$$F_1(-\Delta/m_\pi, w) - \frac{1}{2}F_3(-\Delta/m_\pi, w) + ...\Big\} \qquad (A.31b)$$

$$\delta(\tilde{a}_+ - \tilde{a}_-) = -\frac{3g^2\beta^2}{(4\pi f)^2}\Delta^2\Big\{\frac{1}{2}\left(1 - \frac{(w+1)(w^2-3wr+2)}{(w^2-1)^2}\right)\ln(m_\pi^2/\mu^2)$$

$$-(w+1)F_2(-\Delta/m_\pi, w) - \frac{1}{2}F_3(-\Delta/m_\pi, w) + ...\Big\} \qquad (A.31c)$$

$$\delta(\tilde{a}_+ + \tilde{a}_-) = \frac{3g^2\beta^2\Delta^2}{(4\pi f)^2}\Bigg\{\frac{1}{2}\left(\frac{r(2w^2+3w+1) - (w^2+3w+2)}{(w^2-1)^2}\right)\ln(m_\pi^2/\mu^2)$$

$$-F_1(-\Delta/m_\pi, w) - (w+1)F_2(-\Delta/m_\pi, w) + ...\Bigg\} \qquad (A.31d)$$

and

$$\delta\tilde{f}_- = 0, \quad \delta\tilde{g} = \delta\tilde{f}/(w+1). \qquad (A.31e)$$

Eqs. (A.31) are the main results of this paper. In these equations the ellipses denote terms that are less singular as the light quark masses go to zero. The subtraction point dependence of these terms cancels that in the logarithm. Terms independent of Δ have been absorbed into a redefinition of β.

We have also computed the corrections of order $(1/m_c)$ that depend on the light up and down quark masses as $m_q^{1/2}$. They are

$$\delta\tilde{f}_+ = -\frac{3g^2\beta m_\pi\Delta}{32\pi f^2}\left(\frac{3(w-1)}{w+1}\right) \qquad (A.32a)$$

$$\delta\tilde{f} = \frac{3g^2\beta m_\pi\Delta}{32\pi f^2}(w-1) \qquad (A.32b)$$

$$\delta(\tilde{a}_+ - \tilde{a}_-) = -\frac{3g^2\beta m_\pi\Delta}{32\pi f^2}\left(\frac{w-1}{w+1}\right) \qquad (A.32c)$$

$$\delta(\tilde{a}_+ + \tilde{a}_-) = 0 \qquad (A.32d)$$

and

$$\delta\tilde{f}_- = 0, \quad \delta\tilde{g} = \delta\tilde{f}/(w+1). \qquad (A.32e)$$

These corrections are less important than the $1/m_c$ corrections that are independent of m_q. Note that eqs. (A.32a) and (A.32b) are consistent with Luke's theorem since they vanish at $w = 1$. In fact at order $1/m_c$, the $m_q^{1/2}$ corrections to all the form factors vanish at $w = 1$.

Experimentally Δ is larger than m_π. The $B \to D^*$ form factors have imaginary parts but because Δ is close to m_π, they are negligible. It is a very good approximation to evaluate the functions F_j at $\Delta/m_\pi = 1$.

In the limit $m_c, m_b \to \infty$ heavy quark symmetry implies that the form factors for semileptonic $B(v) \to D^{(*)}(v')$ decay can be expressed in terms of a single universal function of $w = v \cdot v'$ and that the value of this function at $w = 1$ is known. Here we examined, using chiral perturbation theory, corrections to the heavy quark symmetry relations that arise from the finite value of the charm quark mass. We calculated corrections of order $(1/m_c)^{n+2}$ $n = 0, 1, 2, ...$ that go as $\ln m_\pi^2$ when $n = 0$ and as $(1/m_\pi)^n$ when $n = 1, 2, 3...$. These arise from physics well below the chiral symmetry breaking scale. The factors of $1/m_c$ occur from insertions of the $D^* - D$ mass difference Δ. In our calculations other sources of heavy quark symmetry breaking are less important for very small up and down quark masses.

Since the value of Δ is close to m_π, all the corrections of order $(1/m_c)^{n+2}$ $n = 1, 2, 3, ...$ are equally important. In Eqs. (A.31) these corrections occur in the functions F_j. Our calculation of the effects of this order is enhanced by a factor of $(1/m_\pi)$ over terms we neglected.

At order $1/m_c^2$ the terms we calculated are only enhanced by $\ln m_\pi^2$ over those we neglected. The pion mass is not small enough to have complete confidence in our calculation of the $1/m_c^2$ corrections to the semileptonic decay form factors. For example, other effects that don't arise from the operator $\bar{h}_v^{(c)} g_s \sigma^{\mu\nu} T^A h_v^{(c)} G_{\mu\nu}^A$ may be more important.

At zero recoil there are no $1/m_c$ or $1/m_b$ corrections and Ref. [101] used chiral perturbation theory to compute the corrections to $\tilde{f}_+$ and $\tilde{f}$ of order $(1/m_c)^{n+2}$ $n = 0, 1, 2, \ldots$. At $w = 1$ our expressions for these form factors agree with those in Ref. [101]. This paper contains the extension of the results of Ref. [101] away from zero recoil.

We also computed the order $1/m_c$ corrections that have a non-analytic dependence on the light quark masses of the form $m_q^{1/2}$. However, the dominant corrections (away from zero recoil) are those of order $1/m_c$ (and $1/m_b$) that are independent of the light quark masses. These are not calculable using chiral perturbation theory and must be estimated using phenomenological models [109] or lattice Monte Carlo methods.

For $g^2 = 0.5$ (the present experimental limit) the corrections we calculated are typically a few percent. The constituent quark model suggests that g is around unity but it is certainly possible that it is much smaller [110].

We have used chiral $SU(2)_L \times SU(2)_R$ so the effects discussed in this paper are associated with small virtual momenta of order the pion mass. It is possible to extend the calculations to chiral $SU(3)_L \times SU(3)_R$; however, the kaon mass is too large to have confidence in such calculations [111]. For example, at momentum scales around the kaon mass, it is difficult to justify the neglect of contributions from excited heavy mesons [112] in the loop of Fig. 3.

Above we have calculated the low momentum corrections to the mesonic Isgur–Wise form factor. The correction depends on two symmetry breaking parameters, the mass of pion m_π which measures the deviation from the chiral limit, and the D^*-D mass splitting Δ which quantifies the heavy quark spin symmetry violation.

The same method can be applied to calculate the low momentum corrections to the baryonic Isgur–Wise form factors [102]. The situation is more complicated, however, as axial couplings mix Λ_Q and Σ_Q states. As a result, the corrections to the Λ_Q Isgur–Wise form factor $\eta(w)$ are not proportional to $\eta(w)$ itself but have pieces proportional to $\zeta_1(w)$ and $\zeta_2(w)$, the Σ_Q Isgur–Wise form factors. Here the symmetry breaking parameters are m_π and $\bar{\Delta}$, the Σ_c^*-Σ_c mass splitting.

Despite the large kaon mass discourage a naive extension of our calculations to chiral $SU(3)_L \times SU(3)_R$, we can use the same framework to estimate the size of the light quark dependence of the Isgur–Wise form factors. Under light flavor SU(3), $B \to D$ and $B_s \to D_s$ should be described by the same Isgur–Wise form factor, i.e., $\xi(w) = \xi_s(w)$. Kaon loops, however, break the symmetry and near the point of zero recoil [103]

$$|\xi_s(w) - \xi(w)| \sim \frac{g^2 \Delta^2}{(4\pi f)^2} \ln\left(\frac{m_K^2}{\mu^2}\right), \tag{A.33}$$

which is about 1%. Due to its larger complexity, similar calculations have not been done in the baryon sector. The form of the correction, however, is expected to be similar and this was used in Eq. (3.50) to estimate the difference between $\eta(w)$ (describing $\Lambda_b \to \Lambda_c$ and $\Sigma_b \to \Sigma_c^{(*)}$) and $\zeta_1(w)$ (describing $\Omega_b \to \Omega_c^{(*)}$) in the large N_c limit.

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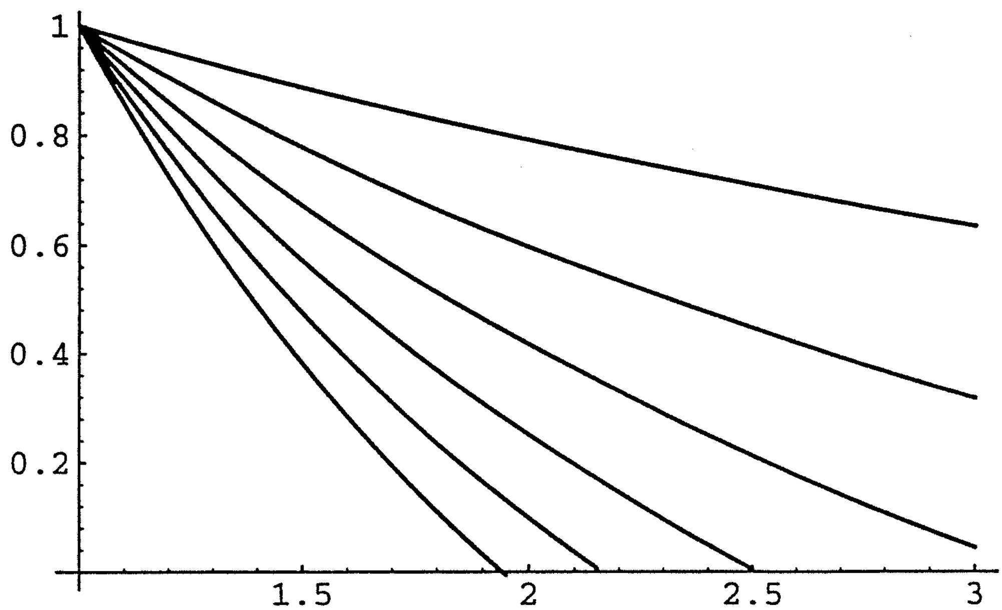

Figure 1:

$\eta^{\min}(w)$ for different values of $\bar{\rho}$.

From top to bottom $\bar{\rho}^2 = 0.25$, 0.50, 0.75, 1.00, 1.25 and 1.50.

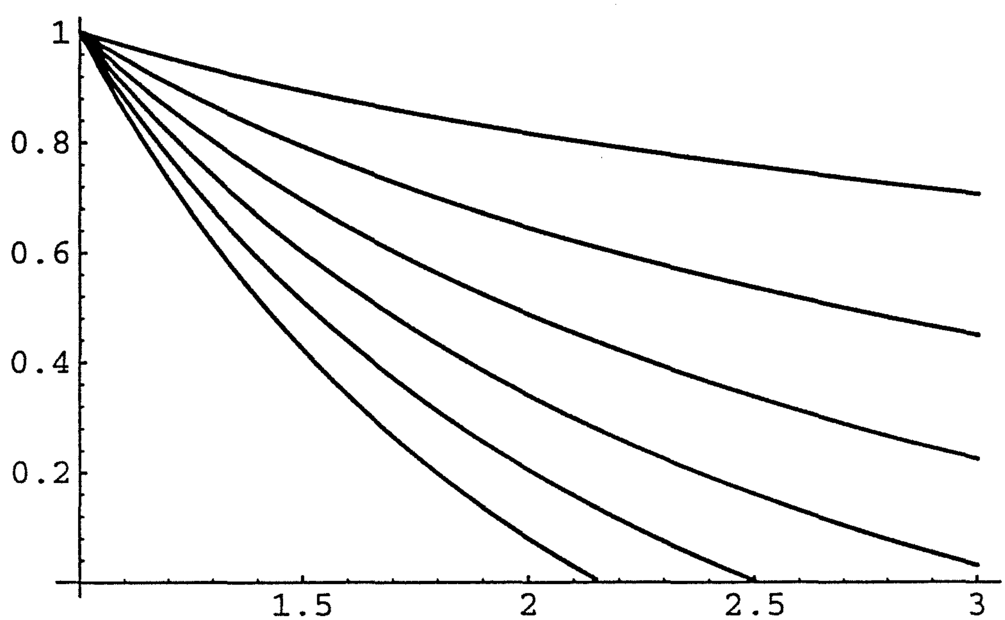

Figure 2:

$\xi^{\min}(w)$ for different values of ρ.

From top to bottom $\rho^2 = 0.25, 0.50, 0.75, 1.00, 1.25$ and 1.50.

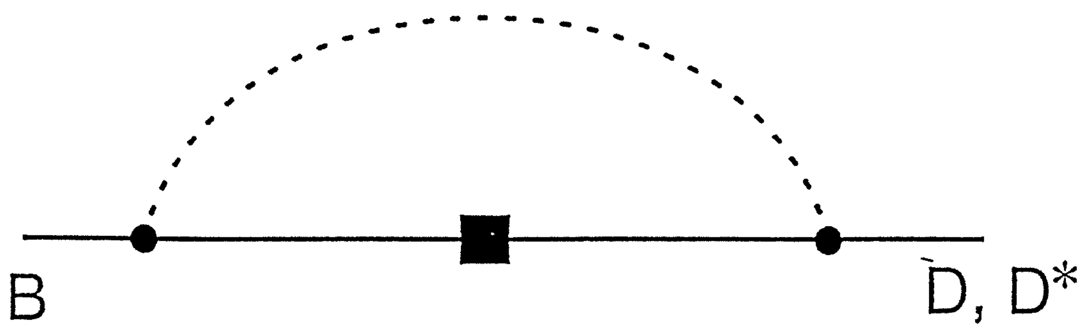

Figure 3:

Feynman diagrams that give corrections to form factors $\tilde{f}_{\pm}, \tilde{f}, \tilde{a}_{\pm}, \tilde{g}$.

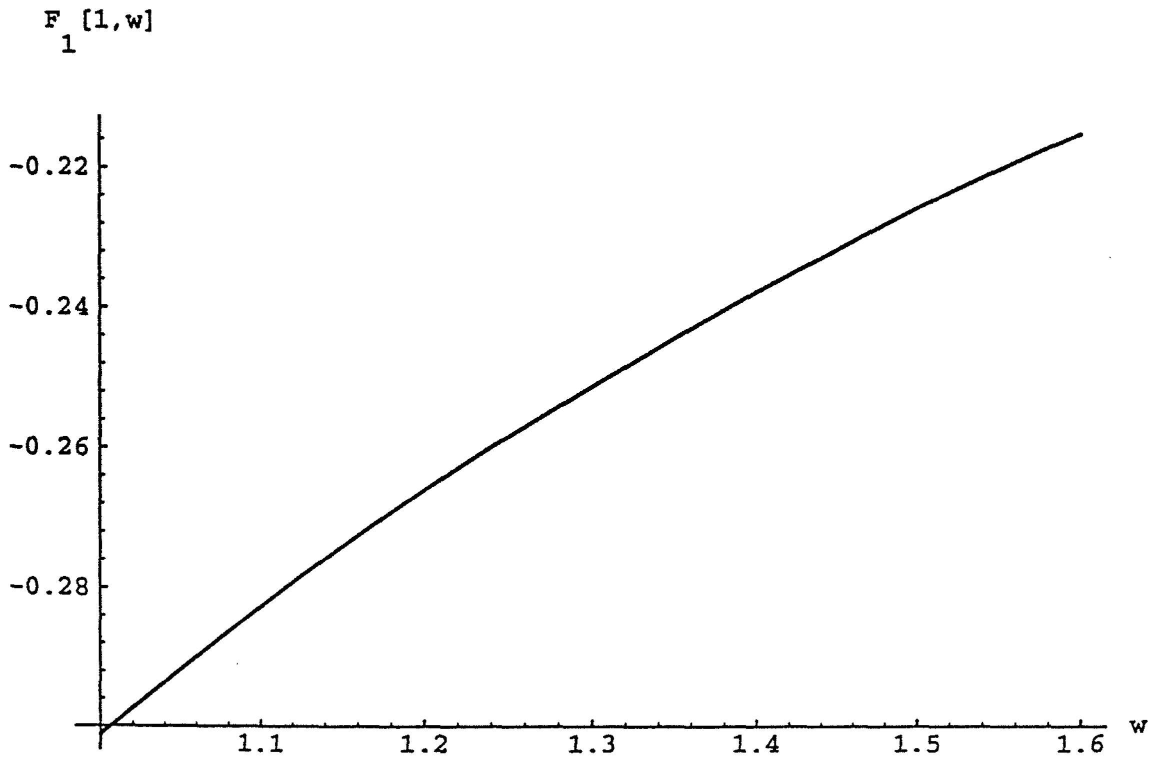

Figure 4:

Plot of $F_1(1, w)$ versus w.

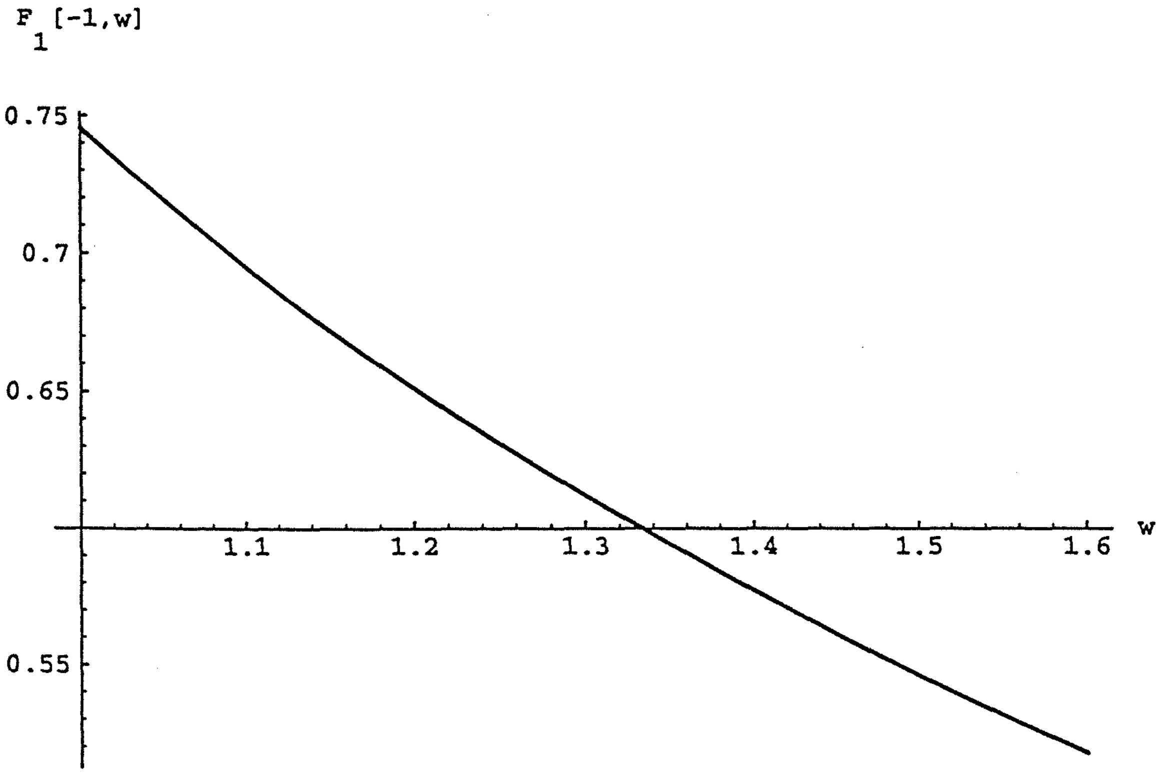

Figure 5:

Plot of $F_1(-1, w)$ versus w.

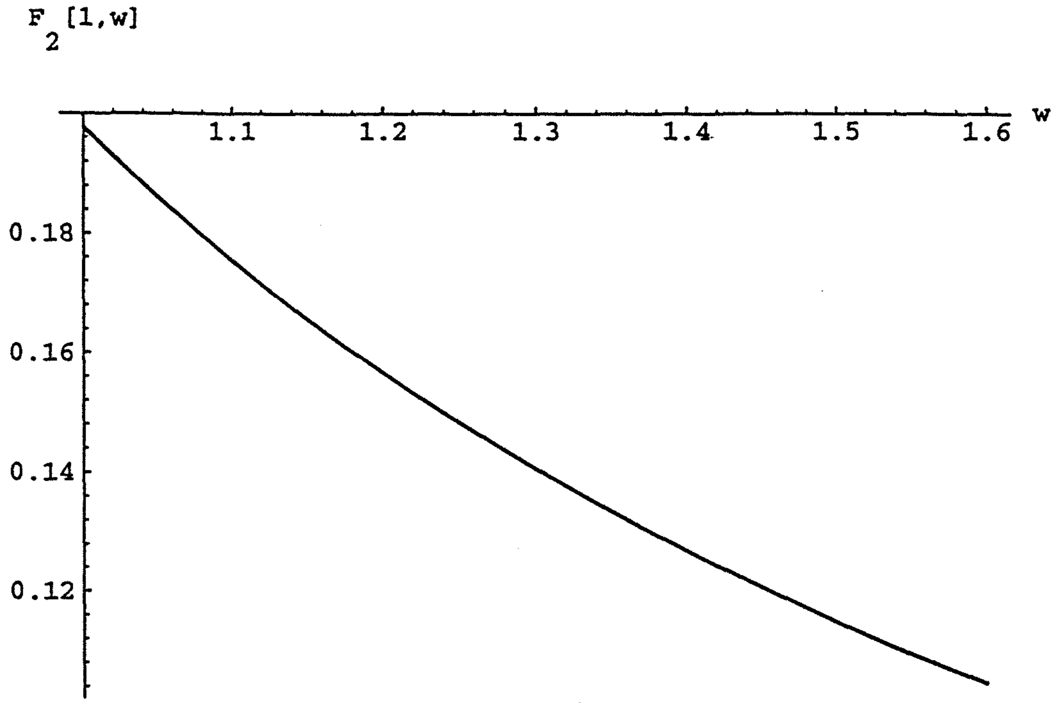

Figure 6:

Plot of $F_2(1, w)$ versus w.

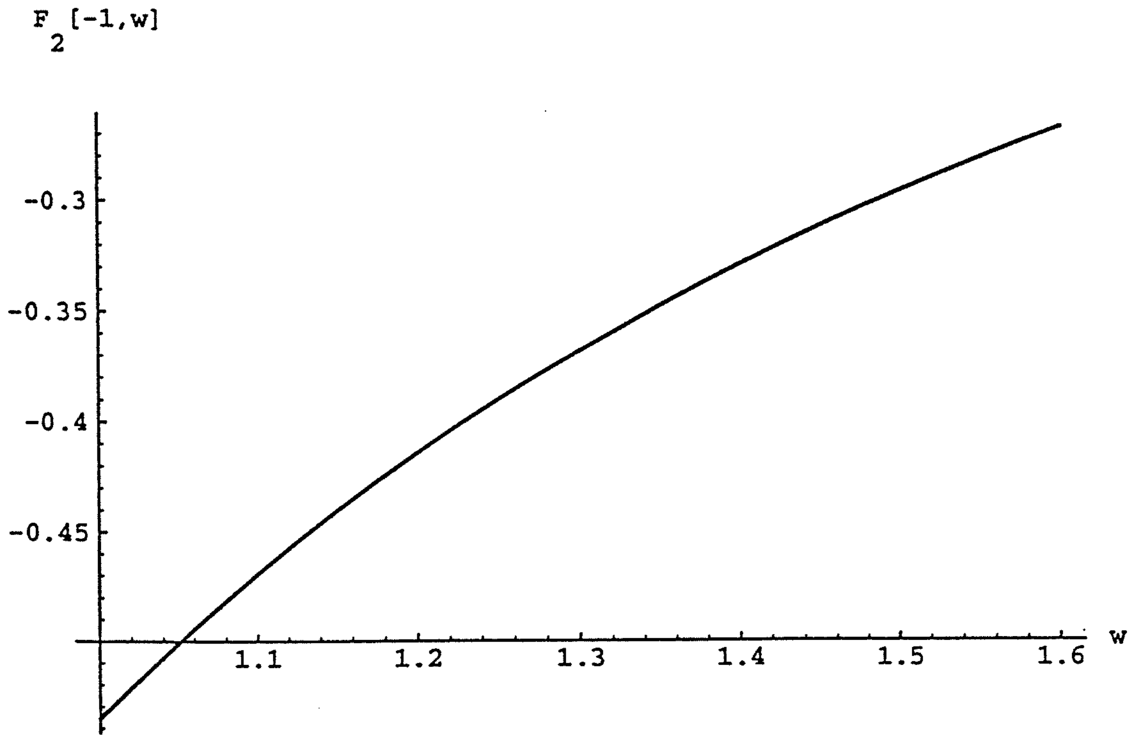

Figure 7:

Plot of $F_2(-1, w)$ versus w.