THREE STUDIES IN HYDROMAGNETICS

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ABSTRACT

This thesis consists of an introduction to the field of hydromagnetics, followed by three separate studies in this subject.

The first is a study of the small-amplitude hydromagnetic radiation from a localized disturbing source in an unbounded dissipationless fluid permeated by a constant uniform magnetic field. The relevant linearized vector wave equation is treated by Fourier transform methods, utilizing the stationary-phase approximation. Asymptotic solutions are obtained for the wave-zone amplitude in the three modes emitted, and these are discussed in some detail; analytically, geometrically, and physically. Expressions are obtained for the angular distribution of the power radiated into these modes by a distributed source.

The second study concerns itself with some special two-dimensional hydromagnetic steady flows. Various general properties of these flows are discussed. Ten exact solutions of the exact non-linear equations of flow are derived and some of their features noted.

The third study is an investigation of whether, for the case of a deep 'lake' of dissipationless incompressible conducting fluid in a constant uniform magnetic field, there exist characteristic small-amplitude gravity surface-waves different from those known in hydrodynamics. It is concluded that no surface waves exist at all if the magnetic field has a component normal to the undisturbed surface, and that if the field is tangential to the surface, there are no new wave types with a characteristic dispersion law.

TABLE OF CONTENTS

INTRODUCTION

1. Description and History of the Subject.

The subject of hydromagnetics (or magneto-hydrodynamics, as it was originally called) deals with the motion of conducting fluid media under the influence of magnetic forces. In most of the examples encountered in nature, the medium is an ionized gas; in the earth's core and in most of the laboratory experiments to date, it is a conducting liquid. In nearly all the theoretical studies so far published the medium is assumed to be an isotropic "ohmic fluid" describable by the Navier-Stokes equation of ordinary hydrodynamics with the addition of an appropriate magnetic body-force.

The beginnings of this field can be traced at least as far back as 1919, when Sir Joseph Larmor published his speculations on the structure of sunspots. (1) Little or no further development seems to have occurred until 1930, when T. G. Cowling took up, in his doctoral thesis, the problem of the generation and maintenance of magnetic fields by the motion of conducting fluids; the problem of fluid dynamo action in nature still remains the central theme of the present subject and is one of the most difficult problems ever posed in classical theoretical physics.

During the period of the 1930's, further studies basic to hydromagnetics were carried out by a small number of workers, among whom S. Chapman, T. G. Cowling and V. C. A. Ferraro were prominent. During this decade or so basic kinetic-theory studies were made of the properties of ionized gases, and experience was gained in the construction of a number of frequently short-lived theories of stellar and cosmic electromagnetic phenomena. It soon became apparent that, on the mathematical side, the subject is extremely formidable, and on the physical side, as Cowling has remarked, "the probability of being led astray by seductive theories is high".

The recent popularization and active phase of the development of hydromagnetics began shortly after 1942, in which year H. Alfvén (2) pointed out the existence, in a conducting fluid, of wave motions with purely magnetic restoring forces, and subsequently, in a series of boldly imaginative papers, set forth interesting semi-qualitative theories of various solar and geomagnetic phenomena. (3)

Alfvén's pioneer work and that of his colleagues Walén, Lundquist, Lehnert and others in Alfvén's very active Stockholm group brought the subject an increasingly wider measure of attention. In the 1940's, E. C. Bullard in England and W. M. Elsasser and S. Chandrasekhar in the U. S. A. interested themselves in the subject and, with their collaborators have made notable pioneering contributions during the past decade. The whole field is currently under increasingly active development, on both the theoretical and experimental sides.

Hydromagnetic theories have been constructed for such diverse phenomena as the structure of sunspots, solar prominences and flares, the oscillations of magnetic variable stars, terrestrial magnetic storms, acceleration of cosmic rays in interstellar space, the origin and evolution of the earth's magnetic field, the structure of spiral nebulae, oscillations of the earth's ionosphere and, currently, problems of immense potential practical importance centering on the initiation, containment, cooling, and heat insulation of high temperature reactions, chemical and thermonuclear. Electromagnetic pumps and flow-meters based on hydromagnetic effects are already in use, and further technical devices of such kinds are doubtless in the offing.

Until quite recently, the vast majority of the researches in this field were of a purely theoretical nature. Because of the very great difficulties of exact analysis, these studies, especially those directed to problems set by nature, were usually of a partly qualitative character, characterized by a very free use of assumptions and intuitive arguments, and with order-of-magnitude estimates instead of accurate calculations. Experimental studies, on the other hand, have been hampered by the lack of high-conductivity low-density fluid media and, for the best available approximations thereto (mercury and liquid sodium), by the difficulty of arranging sufficiently extended and sufficiently large magnetic fields. Ionized gases furnish much more suitable media, but there are severe experimental difficulties in achieving a uniform sufficiently high ionization over an extended region and under controlled reproducible conditions. Currently, however, such experiments are being gotten under way and the next few years should bring considerably more, and much more helpful, experimental information than has been available in the past.

The physical content of the customary formulation of hydromagnetics, as a simple generalization of hydrodynamics, is very clear. However, the mathematical complexities of a quantitative treatment of even the simplest idealized problems are usually too great to allow a thorough investigation. At the present state of development of the subject, it is thus of interest to study simple problems permitting of detailed mathematical analysis. One may hope, then, that progress will be made in the same way as in fluid dynamics -- that, ultimately, useful reliable approximations may be developed to aid intuitive thinking and to reduce the mathematical complications.

The studies reported on in this thesis are directed to this end. It is hoped these simple examples will illustrate some of the features which must be taken account of in more complex situations.

The rest of this chapter aims to present a brief but self-contained account of the formal basis of hydromagnetics, on the customary assumptions. For further details, discussions of applications, and accounts of experimental work, reference may be made to a number of authoritative review articles and symposia. (4, 5, 6, 7, 8, 9, 10, 11) The entire development given here is a direct generalization of hydrodynamics, based on a macroscopic description of the dynamics and electrical properties of the fluid. It is outside the scope of the present work to give any quantitative discussion of the conditions under which such a description is appropriate for an ionized gas.

2. The Basic Hydrodynamical and Electromagnetic Equations

In hydromagnetics one is concerned with the dynamics of a slowly varying electromagnetic field. The periods of fluid motions and field fluctuations are comparable; loosely speaking, the field is embedded in the fluid and moves with it. The equations describing these effects are those needed in hydrodynamics (the Navier-Stokes equation, the equation of energy balance, and the thermal and caloric equations of state), with the addition of the quasi-static Maxwell equations. No problems have yet been treated with all these relations taken simultaneously into account, but logically all are necessary. We shall show how, from this array of relations we may derive, for small amplitude disturbances of a uniform fluid in a large, constant, uniform magnetic field, a pair of linearized coupled equations for the velocity and temperature fields, from which all the others may be derived. The problems studied in this thesis involve only very special cases of the equations presented here; it seems worthwhile, however, to present a considerably more general formulation to begin with.

Throughout this work, rationalized MKS units are used. We assume a uniform medium, devoid of electric or magnetic polarization, and possessed of the properties assigned to a viscous compressible fluid in hydrodynamics. In addition, the medium is assumed to be isotropically conducting with uniform conductivity σ such that, when it is permeated by fields $\mathbb{E}$ and $\mathbb{B}$ a current $\mathring{\mathbb{J}} = \sigma \mathbb{E}$ flows locally, these quantities being measured in a reference frame moving with the fluid locally. In our system of units, the dielectric constant and permeability are assigned the respective values

$$\epsilon_0 = 8.854 \times 10^{-12} \text{ farad/meter}$$

and

$$\mu_0 = 4\pi \times 10^{-7} \text{ henry/meter}$$

The equations from which we begin, written in the inertial coordinate system with respect to which the fluid is moving with velocity $\mathbb{V}$, may be divided into groups as follows.

(a) Maxwell equations and complements:

$$\nabla \times \mathbb{E} + \frac{\partial \mathbb{B}}{\partial t} = 0 \tag{1}$$

$$\frac{\nabla \times \mathbb{B}}{\mu_0} = \mathbb{J}_{tot} \tag{2}$$

$$\nabla \cdot \mathbb{B} = 0 \tag{3}$$

$$\epsilon_0 \nabla \cdot \mathbb{E} = \eta \tag{4}$$

$$\mathbb{J}_{tot} = \eta \mathbb{V} + \sigma(\mathbb{E} + \mathbb{V} \times \mathbb{B}) + \epsilon_0 \dot{\mathbb{E}} \tag{5}$$

$$\frac{\partial \eta}{\partial t} + \nabla \cdot \mathbb{J}_{tot} = 0 \tag{6}$$

$$\vec{f} \equiv \text{Lorentz force-density} = \eta \mathbb{E} + \mathbb{J}_{tot} \times \mathbb{B} \tag{7}$$

η is the electric charge density. Equation 5 expresses the total current as the sum of a convection current due to the bodily convection of free charge by the fluid, a conduction current created by the electric field $\overset{*}{\mathbb{E}} = \mathbb{E} + \mathbb{V} \times \mathbb{B}$ seen by the moving fluid, and the displacement current. We shall shortly show that the convective and displacement currents are, in hydromagnetics, completely negligible in comparison with the conduction current, and that the electric part, $\eta \mathbb{E}$, of the Lorentz force is negligible in comparison with the magnetic part. Thus we shall finally use the quasi-static form of the Maxwell equations, with a purely magnetic body force.

(b) Navier-Stokes and continuity equations:

We assume that our medium is a "Stokes fluid", with zero bulk viscosity. Then, defining

$$\mathbb{F} \equiv \rho \frac{D\mathbb{V}}{Dt} + \rho\nu\left(\nabla\times\nabla\times\mathbb{V} - \frac{4}{3}\nabla\nabla\cdot\mathbb{V}\right) + \nabla P$$

$$\left(\frac{D}{Dt} \equiv \frac{\partial}{\partial t} + \mathbb{V}\cdot\nabla\right) \tag{8}$$

the equation of motion of the fluid is

$$\mathbb{F} = \vec{f} \tag{9}$$

In addition, of course, we have the continuity equation

$$\frac{D\rho}{Dt} + \rho\nabla\cdot\mathbb{V} = 0 \tag{10}$$

$\rho\nu$ is the shear-viscosity coefficient, ν being the kinematical viscosity coefficient. Equation 9 equates the Lorentz ponderomotive force to the sum of pressure, inertial, and viscous forces.

(c) Energy equation and the equations of state:

The equation of energy balance is

$$\rho \frac{Du}{Dt} = -p\nabla\cdot\mathbb{V} + \Phi_v + \nabla\cdot(\tau\nabla T) + \mathbb{J}_{tot}\cdot(\mathbb{E} + \mathbb{V}\times\mathbb{B}) \tag{11}$$

where

u = specific internal energy of the fluid

τ = heat conduction coefficient

T = temperature

$$\Phi_v = \rho\nu\left[2\sum_{i=1}^{3}\left(\frac{\partial v_i}{\partial x_i}\right)^2 + \sum_{i<j}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right)^2\right]$$

the viscous dissipation term, and the meanings of the individual terms in equation 11 are clear.

The thermal equation of state is

$$p = p(\rho, T) \tag{12}$$

and the 'caloric' equation of state is

$$u = u(\rho, T) \tag{13}$$

Before we proceed to combine and manipulate the foregoing equations, we digress to discuss the orders of magnitude of certain important quantities and the conditions for pronounced hydromagnetic coupling effects. This information is readily developed by order-of-magnitude estimates from the equations.

3. Orders of Magnitude; Hydromagnetic Approximations.*

Let V denote a representative velocity, L a representative length within which a specified field changes by a significant fraction of itself, τ a representative time for the same thing, etc. The $(\sim)$ sign shall denote rough equality in the order-of-magnitude sense.

As we have seen, we have

$$\frac{\nabla \times \mathbb{B}}{\mu_0} = \mathbb{J}_{tot} = \mathbb{J}_{cond} + \mathbb{J}_{ind} + \mathbb{J}_{conv} + \mathbb{J}_{disp}$$

where

$$\mathbb{J}_{cond} = \sigma \mathbb{E} \quad \text{, the conduction current}$$

$$\mathbb{J}_{ind} = \sigma \mathbb{V} \times \mathbb{B} \quad \text{, the "induction current"}$$

$$\mathbb{J}_{conv} = \eta \mathbb{V} = \mathbb{V}\, \nabla \cdot \mathbb{E} \quad \text{, the convection current}$$

$$\mathbb{J}_{disp} = \epsilon_0 \dot{\mathbb{E}} \quad \text{, the displacement current}$$

For periodic processes,

$$\frac{\mathbb{J}_{disp}}{\mathbb{J}_{ind}} \simeq \frac{\epsilon_0 \omega}{\sigma} = \gamma \tag{12}$$

*Our discussion in this section is largely based on those given by W. M. Elsasser, (6 and 12).

a quantity well-known in the theory of electromagnetic waves in rigid conductors. Now, in hydromagnetics we are concerned with the motion of highly conducting fluids, moving with long periods (limited by the inertia of the fluid as a whole). The conductivities rarely fall below metallic order. A lower limit for σ is surely $\approx 10^7$ mhos/meter, the conductivity of iron. An upper limit for ω is surely, say, 10^4 radians/second.* Even with these liberal limits γ is minute, of the order of 10^{-13}. So the displacement current may always be neglected in hydromagnetics.

Now,

$$\eta = \epsilon_0 \nabla \cdot \mathbb{E} \sim \frac{\epsilon_0 E}{L}$$

Therefore

$$\frac{J_{conv.}}{J_{cond.}} \sim \frac{\eta V}{\sigma E} \sim \frac{\epsilon_0 V}{L\sigma} \sim \gamma \tag{13}$$

So $\mathbb{J}_{conv.}$ is enormously smaller than $\mathbb{J}_{cond.}$ and we may likewise neglect the convection current against the conduction current. We see then that, to an excellent approximation,

$$\frac{\nabla \times \mathbb{B}}{\mu_0} = \mathbb{J} = \sigma \left(\mathbb{E} + \mathbb{V} \times \mathbb{B}\right) \tag{14}$$

Let us now compare the "induction current" to the total current

$$\frac{J_{ind}}{J_{tot.}} \sim \frac{\sigma V B}{(B/\mu_0 L)} = \mu \sigma L V \equiv R_m \tag{15}$$

where the useful dimensionless number R_m is often called the "magnetic Reynolds number", for formal reasons to be mentioned shortly.

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*In cosmic hydromagnetics the conductivities are considerably higher than the value cited, and the frequencies much lower.

Let us clarify the physical meaning of this quantity.

For free decay of currents in a rigid conductor we have the well known eddy-current equation

$$\frac{\partial \mathbf{B}}{\partial t} = \frac{1}{\mu_0 \sigma} \nabla^2 \mathbf{B} \tag{16}$$

Then

$$\frac{B}{\tau_{decay}} \sim \frac{1}{\mu_0 \sigma}\left(\frac{B}{L^2}\right), \qquad \text{or} \qquad \tau_{decay} \sim \mu_0 \sigma L^2 \tag{17}$$

This describes the diffusive decay (or penetration) of magnetic fields in conductors, which proceeds slowly when the conductivity is high or the dimensions of the field large; it is of course the basis of the skin effect . Relation 17 states that the time necessary for a magnetic field to penetrate to a depth L in a rigid conductor is of order $\mu_0 \sigma L^2$. Roughly, then, we may think of the field as moving through the conductor with a "velocity" $\frac{1}{\mu_0 \sigma L}$.

Now, the periods of the fluid's motion are of the order of

$$\tau_{fluid} = \frac{L}{V}$$

Therefore

$$R_m \equiv \mu_0 \sigma L V \sim \frac{\tau_{decay}}{\tau_{fluid}} \tag{18}$$

We see that a large R_m means that the changes of the field due to its convection by the fluid are large compared to the changes caused by resistive damping of the field, in the same amount of time.

To consider the matter in another way, we first derive the generalization of equation 16 for a fluid conductor. From equation 14, in our hydromagnetic approximation, $\mathbf{E} = \frac{\nabla \times \mathbf{B}}{\mu_0 \sigma} - \mathbf{V} \times \mathbf{B}$. Using this in equation 1, together with the identity

$$\nabla \times \nabla \times \mathbf{B} = -\nabla^2 \mathbf{B} \qquad \text{(since } \nabla \cdot \mathbf{B} = 0\text{)}$$

yields

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{V} \times \mathbf{B}) + \frac{1}{\mu_0 \sigma} \nabla^2 \mathbf{B} \tag{19}$$

This relation, which is referred to throughout this thesis as the "equation of induction", is the companion to the Navier-Stokes equation of motion in all hydromagnetic problems.

The equation of induction, clearly, describes the bodily convection of the magnetic field by the fluid motion, proceeding apace with its diffusive damping by Joule heat losses* -- these two effects being described respectively by the first and second terms on the right side of equation 19. The ratio of these two terms is, in order of magnitude,

$$\frac{\dot{B}_{conv}}{\dot{B}_{diff.}} \sim \frac{(VB/L)}{(B/\mu_0 \sigma L^2)} = \mu_0 \sigma L V = R_m \tag{20}$$

We see that is a measure of the <u>strength of coupling</u> of the velocity field to the magnetic-induction field. If R_m is <u>large</u>, the convective effects dominate the diffusive effects and the field can be significantly changed by the fluid motion before being appreciably damped. <u>Clearly, $R_m \gg 1$ is a necessary condition for appreciable hydromagnetic coupling effects</u>. For the fluid motions in the earth's core, $R_m \sim 10^3$, and it is is larger by many powers of ten for phenomena on a cosmic scale.

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*As a rough picture we may think of a partial entrainment of the magnetic field lines by the moving fluid, with a relative slip and damping effect from the resistivity. Such a picture is useful and exact for the case of a perfectly-conducting fluid, but for finite conductivity it is apparently not a precise notion, though often an aid to the imagination.

Under the condition of large R_m we see from equation 15 that both the 'induction current' $\sigma \mathbb{V}\times \mathbb{B}$ and the 'conduction current' $\sigma \mathbb{E}$ are large compared to the total current. Therefore, for large R_m, $\mathbb{E} \simeq -\mathbb{V}\times\mathbb{B}$ to a very good approximation. Now the condition $\mathbb{E} = -\mathbb{V}\times\mathbb{B}$ is precisely the definition of infinite or "perfect" conductivity. So a fluid with large R_m behaves like a perfectly conducting fluid. That is, the larger the scale of a hydromagnetic disturbance, the larger the conductivity of the fluid and the larger its velocity, the more nearly does it approximate to a perfectly-conducting fluid. This is one of the reasons supporting the use of the infinite-conductivity idealization, especially in cosmic physics. This idealization, however, must be handled with great care; not only does it often make ambiguous the proper fitting of boundary conditions, but in other respects as well it represents a singular limiting case, as we shall see elsewhere in this thesis.

For a perfectly-conducting fluid equation 19 reduces to

$$\frac{\partial \mathbb{B}}{\partial t} = \nabla \times \left(\mathbb{V} \times \mathbb{B}\right) \tag{19a}$$

and this equation is readily integrated to yield the result that

$$\frac{D}{Dt}\iint \mathbb{B}\cdot d\vec{S} = 0 \qquad (\text{for } R_m = \infty) \tag{21}$$

where the integral is taken over any closed surface whose boundary moves with the fluid everywhere. This clearly expresses the feature already mentioned, that in this limiting case the magnetic field is "locked" to the fluid motion.

In fluid dynamics the important dimensionless Reynolds' number is defined as

$$\mathcal{R} \equiv \frac{VL}{\nu} \tag{22}$$

It is the ratio of the sizes of the inertial terms to the viscous frictional terms in the Navier-Stokes equation. It is well known that if $\mathcal{R} \gg 1$ for a flow, the situation is unstable with regard to the development of a turbulent regime out of a laminar one. Now, in cosmical physics the Reynolds number is probably always very large. Therefore we must expect that turbulent motion is the rule in cosmic hydromagnetics and hydrodynamics, and we must frankly face the likelihood that laminar flows represent at best only a crude approximation in these cases. In all probability the scope of applicability of laminar flow theories is no greater than, say, in the field of meteorology. The development of a good theory of hydromagnetic turbulence is therefore of great importance for the astrophysical applications of our subject. Efforts on this formidable task have already been initiated by Chandrasekhar.

In analogy with the role of the kinematic viscosity in the Navier-Stokes equation, we may attach the term "magnetic viscosity" to the quantity

$$\lambda \equiv \frac{1}{\mu_0 \sigma} \tag{23}$$

λ then has the same formal role in the induction equation (equation 19) as ν has in the Navier-Stokes equation. Furthermore, with definition 23 we have, from equation 15 that

$$\mathcal{R}_m = \frac{VL}{\lambda} \tag{24}$$

in analogy with equation 22; for this reason $\mathcal{R}_m$ was termed the magnetic Reynolds' number. It is by no means to be implied, however, that this analogy is a very close one physically.

The ratio of the rates of local heat generation by viscosity to that by resistivity is easily expressed. This ratio is of the order of magnitude

$$\frac{w_\nu}{w_\sigma} \sim \frac{(\rho\nu)\frac{V^2}{L^2}}{\frac{1}{\sigma}\left(\frac{B}{\mu_0 L}\right)^2} = (\mu_0\sigma\nu)\left(\mu_0\rho\frac{V^2}{B^2}\right) \tag{25}$$

Now, for reasonably pronounced hydromagnetic coupling, the magnetic pressure $\frac{B^2}{2\mu}$ must be comparable with ρV^2, the dynamic pressure of the fluid. Therefore

$$\rho V^2 \approx \frac{B^2}{\mu_0} \tag{26}$$

so that equation 25 may be written

$$\frac{w_\nu}{w_\sigma} \sim \mu_0\sigma\nu = \frac{\nu}{\lambda} \tag{26}$$

Equation 26 is but one condition for pronounced hydromagnetic coupling; as we have seen in the considerations following equation 20, we must also have $\mathcal{R}_m \equiv \mu_0\sigma L V \gg 1$. Combining this condition with equation 26 yields

$$BL\sigma\sqrt{\frac{\mu_0}{\rho}} \gg 1 \tag{28}$$

as a necessary condition for appreciable hydromagnetic effects. Thus, we require sufficiently high conductivity, small enough density, large enough wave disturbances and strong enough fields. These requirements create some of the experimental difficulties mentioned at the beginning of this chapter.

Let us now compare the relative importance of electrostatic to magnetic ponderomotive forces on the moving fluid. This ratio is

$$\frac{f_e}{f_m} = \frac{|\epsilon_0 \nabla \cdot \mathbb{E}|}{\left|\frac{\nabla \times \mathbb{B}}{\mu_0} \times \mathbb{B}\right|} \sim \frac{\epsilon_0 E^2/L}{B^2/\mu_0 L} = \frac{1}{c^2}\frac{E^2}{B^2} \tag{29}$$

where $c = \frac{1}{\sqrt{\epsilon_0 \mu_0}}$ is the vacuum velocity of light. But, if R_m is not $\lll 1$, we may estimate

$$E \sim \frac{BL}{\tau} \sim BV$$

Therefore

$$\frac{f_e}{f_m} \sim \frac{V^2}{c^2} \lll 1, \qquad (30)^*$$

since we deal with completely non-relativistic fluid velocities throughout. So, unless R_m is minute, one may neglect the electrostatic body force in comparison with the magnetic one.

With the excellent approximation of a purely magnetic Lorentz force we may, from equations 8, 9, and 14, with the use of the identity

$$\nabla \times \nabla \times \mathbb{V} = \nabla \nabla \cdot \mathbb{V} - \nabla^2 \mathbb{V}$$

write the hydromagnetic Navier-Stokes equation explicitly as

$$\frac{\partial \mathbb{V}}{\partial t} + (\mathbb{V} \cdot \nabla)\mathbb{V} = \frac{(\nabla \times \mathbb{B}) \times \mathbb{B}}{\mu_0 \rho} - \frac{\nabla p}{\rho} + \nu\left(\nabla^2 \mathbb{V} + \frac{1}{3}\nabla \nabla \cdot \mathbb{V}\right) \tag{31}$$

where any external body-force is to be added on the right side. Equation 31 together with the induction equation

$$\frac{\partial \mathbb{B}}{\partial t} = \nabla \times (\mathbb{V} \times \mathbb{B}) + \frac{1}{\mu_0 \sigma} \nabla^2 \mathbb{B} \tag{19}$$

are the basic equations of hydromagnetics.

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*We note, incidentally, that V^2/c^2 is also the ratio of electric to magnetic energy-densities.

4. The Linearized Equations of Hydromagnetics.

(a) The linearization process.

Summarizing the results of sections 2 and 3, we see that the solution of a hydromagnetic problem, with the very good approximations already indicated, involves the simultaneous treatment of equations 31 (the equation of motion), 19)the equation of induction), 10 (the equation of continuity), 11 (the equation of energy balance), 13 (the thermal equation of state), and 12 (the "caloric" equation of state), together with appropriate boundary conditions on the magnetic field, the velocity field, and the accompanying electric field.* Since this array of relations is completely intractable save possibly for certain one-dimensional problems, we are led to consider simpler expedients for certain kinds of situations.

As in ordinary hydrodynamics, it seems that we may gain much useful information by studying flows which may be considered as small perturbations about a known one; in particular, small perturbations about a quiescent state of a uniform fluid. Our object will then be to obtain "linearized" equations describing the "small" deviations of the various dependent variables from their "unperturbed" values.

In hydromagnetics, an unperturbed state cannot be one of vanishing magnetic field, for then, since the magnetic field enters quadratically in the equation of motion and bilinearly with the velocity in the equation of induction, there would be no first order effects

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*The electric field must in principle be taken account of even though, inside the fluid, it may be expressed in terms of the velocity and magnetic fields. However, it seems (analogously to the discussion of the electric field in and about a wire carrying a steady current) that the matching of $\mathbb{E}$ across the boundaries will result merely in a surface-charge. This charge, because of the largeness of σ, will be very small, just as in the case of the wire.

involving the magnetic field. So we are forced to consider an unperturbed state in which there is already a "large" magnetic field permeating the medium. In nearly all studies in hydromagnetics, from sheer mathematical necessity, this zero-order magnetic field is taken as constant in time and uniform in direction; external currents are necessary, of course, to maintain it. Admittedly, this necessity greatly limits the range of problems to which such linearized equations are applicable. If the large field varies slowly enough, with respect to the scale of the fluid motions, we may still hope to use the uniform-field approximation. More elaborate perturbation methods, similar to the "WKB approximation" may also be used in such cases, starting with the uniform-field case.

(b) Linearized velocity-equations for a polytropic fluid.

Returning to equations 10, 11, 12, 13, 19, and 31 we proceed now to linearize them throughout, writing new equations for small departures of all our dependent variables from an unperturbed state of a uniform isotropic fluid at rest in a large constant magnetic field $\mathbb{B}_0$.

We put $\mathbb{B}=\mathbb{B}_0+\mathbb{b}^{(1)}$, $\mathbb{V}=\mathbb{V}^{(1)}$, $\rho=\rho_0+\rho^{(1)}$, $T=T_0+T^{(1)}$, $u=u_0+u^{(1)}$, $p=p_0+p^{(1)}$ where the superscripted symbols are 'first order' quantities. The velocity $\mathbb{V}$ and all derivatives of first-order quantities are also first-order quantities. Quantities of second- and higher order are neglected throughout. We shall omit the superscripts, as it will be clear which our first-order terms are. The dotted equality sign $(\doteq)$ shall be understood to mean "equality to within second- and higher-order quantities".

Taking the curl of equation 1, and using equation 14 and the vector identity $\nabla\times\nabla\times(\ )\equiv\nabla\nabla\cdot(\ )-\nabla^2(\ )$ yields

$$\nabla\nabla\cdot\mathbb{E}-\nabla^2\mathbb{E}+\mu_0\frac{\partial \mathbb{J}}{\partial t}=0 \tag{32}$$

Also, since $\mathbb{E} \doteq \frac{\nabla \times \mathbb{b}}{\mu_0 \sigma} - \mathbb{V} \times \mathbb{B}$, equation 32 becomes

$$-\nabla \nabla \cdot (\mathbb{V} \times \mathbb{B}_0) - \nabla^2 \mathbb{E} + \mu_0 \frac{\partial \dot{\mathbb{J}}}{\partial t} \doteq 0 \tag{33}$$

when a second-order term $\nabla \nabla \cdot (\mathbb{V} \times \mathbb{b})$ has been dropped. Next, upon post-multiplying equation 33 with the constant vector $\mathbb{B}_0$ in the vector cross-product, and noting that $(\nabla^2 \mathbb{E}) \times \mathbb{B}_0 \equiv \nabla^2 (\mathbb{E} \times \mathbb{B}_0)$ (since ∇^2 is a scalar operator), we obtain

$$-\left[\nabla \nabla \cdot (\mathbb{V} \times \mathbb{B}_0)\right] \times \mathbb{B}_0 - \nabla^2 (\mathbb{E} \times \mathbb{B}_0) + \mu_0 \frac{\partial}{\partial t} (\dot{\mathbb{J}} \times \mathbb{B}_0) = 0 \tag{34}$$

Now, since $\vec{f} = \dot{\mathbb{J}} \times \mathbb{B}$ to a very close approximation, as we have seen, we have

$$\mathbb{E} \times \mathbb{B}_0 \doteq \left(\frac{\dot{\mathbb{J}}}{\sigma} - \mathbb{V} \times \mathbb{B}_0 \right) \times \mathbb{B}_0 = \frac{\mathbb{F}}{\sigma} - (\mathbb{V} \times \mathbb{B}_0) \times \mathbb{B}_0$$

where, from the linearized approximation to equation 8 we have

$$\mathbb{F} \doteq \rho_0 \frac{\partial \mathbb{V}}{\partial t} + \rho_0 \nu \left(\nabla \times \nabla \times \mathbb{V} - \frac{4}{3} \nabla \nabla \cdot \mathbb{V} \right) + \nabla P \tag{35}$$

Therefore, equation 34 becomes

$$-\nabla \nabla \cdot (\mathbb{V} \times \mathbb{B}_0) - \frac{1}{\sigma} \nabla^2 \mathbb{F} + \nabla^2 \left[(\mathbb{V} \times \mathbb{B}_0) \times \mathbb{B}_0 \right] + \mu_0 \frac{\partial \mathbb{F}}{\partial t} = 0$$

But, in this last equation, identically

$$\nabla^2 \left[(\mathbb{V} \times \mathbb{B}_0) \times \mathbb{B}_0 \right] \equiv \left[\nabla^2 (\mathbb{V} \times \mathbb{B}_0) \right] \times \mathbb{B}_0 \equiv \left[\nabla \nabla \cdot (\mathbb{V} \times \mathbb{B}_0) \right] \times \mathbb{B}_0 - \left[\nabla \times \nabla \times (\mathbb{V} \times \mathbb{B}_0) \right] \times \mathbb{B}_0$$

and so finally we obtain the linearized equation

$$-\left[\nabla \times \nabla \times (\mathbb{V} \times \mathbb{B}_0) \right] \times \mathbb{B}_0 + \mu_0 \frac{\partial \mathbb{F}}{\partial t} \doteq \frac{1}{\sigma} \nabla^2 \mathbb{F} \tag{36}$$

where the linearized form of $\mathbb{F}$ is given by equation 35. Let us now write out the explicit forms this takes for compressible and for incompressible fluids; in so doing we shall, for the case of monochromatic time-dependence, convert equation 36 into an equation in the velocity alone.

The linearized form of the continuity equation 10 is

$$\rho_o \nabla\cdot \mathbb{V} \doteq -\frac{\partial \rho}{\partial t} \tag{37}$$

Now, if all quantities have the time-dependence $e^{i\omega t}$ then, from equation 37,

$$\rho = \frac{\rho_o}{i\omega} \nabla\cdot \mathbb{V} \tag{38}$$

If we choose to neglect the energy balance equation, and thereby take no account of heat conduction effects, we may assume that our medium is a so-called "polytropic fluid", with equation 12 replaced by

$$p = p(\rho) \tag{39}$$

(For a general fluid this assumes, in effect, either isothermal or adiabatic compressions.) Then

$$\nabla p \doteq \left(\frac{\partial p}{\partial \rho}\right)_o \nabla \rho = V_s^2 \nabla \rho \tag{40}$$

where V_S *is the isothermal speed of sound in the fluid.* Thereupon, for monochromatic time-dependence, we obtain, with the use of equations 38 and 40, the linearized equation for the velocity field, namely

$$-\left[\nabla\times\nabla\times(\mathbb{V}\times\mathbb{B}_o)\right]\times\mathbb{B}_o - \mu_o\rho_o\omega^2\mathbb{V} + i\omega\mu_o\rho_o\nu\left[\nabla\times\nabla\times\mathbb{V} - \frac{4}{3}\nabla\nabla\cdot\mathbb{V}\right] + \mu_o\rho_o V_s^2\nabla\nabla\cdot\mathbb{V}$$
$$\doteq \frac{\rho_o}{\sigma}\left[i\omega\nabla^2\mathbb{V} + \nu\nabla^2\left\{\nabla\times\nabla\times\mathbb{V} - \frac{4}{3}\nabla\nabla\cdot\mathbb{V}\right\}\right] + \frac{\rho_o V_s^2}{i\omega\sigma}\nabla^2(\nabla\nabla\cdot\mathbb{V}) \tag{41}$$

This fourth-order equation becomes of second order when we put $\sigma = \infty$. In Chapter II of this thesis we shall be concerned with the case $\nu = 0$, $\sigma = \infty$, for which equation 41 becomes

$$-\omega^2\rho_o\mathbb{V} + \rho_o V_s^2\nabla\nabla\cdot\mathbb{V} - \frac{\left[\nabla\times\nabla\times(\mathbb{V}\times\mathbb{B}_o)\right]\times\mathbb{B}_o}{\mu_o} \doteq 0 \tag{42}$$

Actually, equation 42 is needlessly restrictive; with the assumptions $\nu = 0$, $\sigma = \infty$ one may easily derive the equation

$$\rho_0 \frac{\partial^2 \mathbb{V}}{\partial t^2} + \rho_0 V_s^2 \nabla \nabla \cdot \mathbb{V} - \left[\frac{\nabla \times \nabla \times (\mathbb{V} \times \mathbb{B}_0)}{\mu_0} \right] \times \mathbb{B}_0 = 0 \tag{42'}$$

which is studied in Chapter II. Even this dissipationless case, because of the anisotropic third-term, is a rather difficult one, as we shall see.

Another case of interest is that studied by Alfvén in his basic paper, namely the case of a perfectly-conducting non-viscous incompressible fluid. For this case

$$\mathbb{F} \doteq \rho_0 \frac{\partial \mathbb{V}}{\partial t} + \nabla p$$

and since $\nabla \cdot \mathbb{V} = 0$, equation 36 becomes

$$\left[\nabla \times \nabla \times (\mathbb{V} \times \mathbb{B}_0) \right] \times \mathbb{B}_0 = \mu_0 \rho_0 \frac{\partial^2 \mathbb{V}}{\partial t^2} + \mu_0 \nabla \left(\frac{\partial p}{\partial t} \right) \tag{43}$$

Putting $\mathbb{B}_0 = \hat{e}_z B_0$ equation 43 is seen to become

$$\frac{\partial^2 \mathbb{V}}{\partial z^2} - \frac{1}{V_A^2} \frac{\partial^2 \mathbb{V}}{\partial t^2} = \nabla \left[\frac{\mu_0}{B_0^2} \frac{\partial p}{\partial t} + \frac{\partial v_z}{\partial z} \right] \tag{44}$$

where $V_A = \frac{B_0}{\sqrt{\mu_0 \rho_0}}$ is the so-called Alfven velocity. Taking the divergence yields

$$\nabla^2 \left[\frac{\mu_0}{B_0^2} \left(\frac{\partial p}{\partial t} \right) + \frac{\partial v_z}{\partial z} \right] = 0$$

Now, (considering the case of an unbounded fluid), a harmonic function which is everywhere non-singular must be a constant. Since $p \equiv v_z \equiv 0$ in the undisturbed fluid at infinity, this constant must be zero. Therefore,

$$\frac{\partial p}{\partial t} = - \frac{B_0^2}{\mu_0} \frac{\partial v_z}{\partial z}, \tag{45}$$

giving the pressure once v_z is known. Substituting equation 45 into equation 44 yields a one-dimensional wave equation for the propagation of the transverse "Alfven waves", namely

$$\frac{\partial^2 \mathbb{V}}{\partial z^2} - \frac{1}{V_A^2}\frac{\partial^2 \mathbb{V}}{\partial t^2} = 0 \tag{46}$$

It may be mentioned that a very similar derivation gives the more general linearized equation

$$\left(\frac{\partial}{\partial t} - \frac{\nabla^2}{\mu_0 \sigma}\right)\left(\frac{\partial}{\partial t} - \nu \nabla^2\right)\mathbb{V} = V_A^2 \frac{\partial^2 \mathbb{V}}{\partial z^2} \tag{46'}$$

for small motions of a viscous, finitely conducting incompressible fluid in a field $\mathbb{B}_0 = \hat{e}_z B_0$. Moreover, equation 46' is exact, for all amplitudes, when $\mathbb{V}$ lies everywhere orthogonal to $\mathbb{B}_0$.

The linearized equation 36 was derived and discussed by Baños (13, 17) for the case $\nu = 0$; our derivation is an abridged version of his. Baños (15) has studied the plane- and cylindrical-wave solutions of equation 36 and has shown that the incompressible fluid sustain two modes -- one accompanied by pressure fluctuations ("p-modes" in Baños' nomenclature) and the other ("v-modes") devoid of them; the compressible fluid, on the other hand, sustains the same "v-modes" as the incompressible one, and in addition <u>two</u> shear-compression "p-modes", which behave respectively like a modified sound wave and like a modified Alfven wave.* This author has also (13) made it highly plausible that for validity of equation 36 we need only require that $|\mathbb{V}| \ll V_A$.

*In the case of a finitely-conducting <u>viscous</u> compressible fluid, with neglect of the heat-conductivity, <u>describable</u> by equation 41, there is an additional pair of damped shear waves, making five modes altogether. The effects of heat-conductivity introduce yet another mode, giving two shear modes and four shear-compression modes. These matters will not be further pursued here, however.

In Chapter II we shall discuss the generation of waves by a point source in a compressible dissipationless fluid, and shall be concerned with the last-mentioned three wave types.

c. Inclusion of the equation of energy balance.

Logically, for the discussion of flows in a compressible fluid in the presence of dissipation, either viscous or ohmic, we should include the equation of energy balance along with the equation of momentum balance already discussed. Not to do so is to neglect the effect of heat conductivity and the resulting changes in the velocity field via the thermal equation of state. In acoustics, for example, it has long been known that the effects of heat conductivity can be as important as those of viscosity.

In published work on hydromagnetics, with but one or two exceptions*, the equation of energy-balance has not been taken into acount. It may be of interest, however, to show how its inclusion within the linearized scheme discussed in section b is straightforward, and leads to a pair of coupled equations for the velocity and temperature fields; plane-wave equations for these are derivable by the same procedure used by Baños (17) for the polytropic fluid.

Let us return to equations 12 and 13, to linearize them and to eliminate the pressure, density and internal-energy variables. (Compare reference 14.)

$$\dot{u} = \left(\frac{\partial u}{\partial \rho}\right)_T \dot{\rho} + \left(\frac{\partial u}{\partial T}\right)_\rho \dot{T} \quad \text{and} \quad \nabla p = \left(\frac{\partial p}{\partial \rho}\right)_T \nabla \rho + \left(\frac{\partial p}{\partial T}\right)_\rho \dot{T}$$

*These are studies on the structure of 1-dimensional hydromagnetic shock waves, for which the energy-balance equation is of prime importance, because of the violent compression.

In linearized form, the partial derivatives will refer to the 'unperturbed' state of the fluid and will be constants. These derivatives may, by the use of familiar thermodynamic relations, be reduced to familiar quantities, as follows. Let

$V_s \equiv$ Isothermal sound velocity

$\gamma \equiv c_p/c_v$, the ratio of specific heats

$\alpha = -\frac{1}{\rho_0}\left(\frac{\partial \rho}{\partial T}\right)_p$ = Thermal expansion coefficient

Then $\left(\frac{\partial u}{\partial T}\right)_\rho = c_v$, $\left(\frac{\partial p}{\partial T}\right)_\rho = \rho_0(\gamma-1)\frac{c_v}{\alpha T_0}$, and therefore

$$\dot{u} = -p_0\left(\frac{\partial u}{\partial \rho}\right)_T \nabla\cdot \mathbb{V} + c_v \dot{T} \tag{47}$$

Now, the linearized form of the energy-balance equation 11 is

$$\rho\dot{u} \doteq -p_0 \nabla\cdot\mathbb{V} + \tau\nabla^2 T \tag{48}$$

the viscous dissipation and Joule-heat terms being of second-order. (p_0 is the unperturbed pressure.) Inserting equation 47 into equation 48 yields

$$\frac{\tau}{\rho_0 c_v}\nabla^2 T - \dot{T} = \left[\frac{p_0}{\rho_0 c_v} - \frac{\rho_0}{c_v}\left(\frac{\partial u}{\partial \rho}\right)_T\right]\nabla\cdot\mathbb{V} \tag{49}$$

Now, from thermodynamics,

$$\rho_0^2\left(\frac{\partial u}{\partial \rho}\right)_T = p_0 - T_0\left(\frac{\partial p}{\partial T}\right)_\rho = p_0 - \rho_0(\gamma-1)\frac{c_v}{\alpha} \tag{50}$$

(T_0 is the unperturbed temperature.) Hence

$$\frac{\tau}{\rho_0 c_v}\nabla^2 T - \frac{\partial T}{\partial t} \doteq \frac{\gamma-1}{\alpha}\nabla\cdot\mathbb{V} \tag{51}$$

showing clearly the coupling between the velocity and temperature fields.

To include energy-balance in linearized approximation one must solve equation 51 simultaneously with equation 36, $\mathbb{F}$ being defined by equation 35. Now, however, instead of equation 40, we have

$$\nabla p \doteq \left(\frac{\partial p}{\partial \rho}\right)_0 \nabla \rho + \left(\frac{\partial p}{\partial T}\right)_0 \nabla T$$

which for monochromatic time-dependence $e^{i\omega t}$ becomes

$$\nabla p = -\frac{\rho_0 V_s^2}{i\omega} \nabla \nabla \cdot \mathbb{V} + \frac{\rho_0(\gamma - 1)}{\alpha T_0} c_v \nabla \dot{T}$$

To include energy-balance together with momentum-balance in a consistent linear approximation, we must therefore handle the simultaneous equations

$$\left[\nabla \times \nabla \times (\mathbb{V} \times \mathbb{B}_0)\right] \times \mathbb{B}_0 + i\omega\mu_0 \mathbb{F} \doteq -\frac{1}{6} \nabla^2 \mathbb{F} \tag{36}$$

and

$$\frac{\tau}{\rho_0 c_v} \nabla^2 T - \frac{\partial T}{\partial t} \doteq \frac{\gamma - 1}{\alpha} \nabla \cdot \mathbb{V} \tag{51}$$

where

$$\mathbb{F} \equiv i\omega\rho_0 \mathbb{V} + \rho_0 \nu\left(\nabla \times \nabla \times \mathbb{V} - \frac{4}{3} \nabla \nabla \cdot \mathbb{V}\right) - \frac{\rho_0 V_s^2}{i\omega} \nabla \nabla \cdot \mathbb{V} + \frac{\rho_0(\gamma - 1)}{\alpha T_0} c_v \nabla \dot{T} \tag{52}$$

II. RADIATION OF HYDROMAGNETIC WAVES.

1. Introduction.

We consider here the generation of hydromagnetic waves by the action of applied body forces in an unbounded dissipationless compressible fluid permeated by a large, constant, uniform magnetic field. The basic problem considered is that of the wave motion produced by a body force applied at a point in the medium. The disturbances are to be small enough so that the linearized wave equation applies, which means that the velocity must everywhere be small compared to the Alfven velocity $V_A = B_0/\sqrt{\mu_0 \rho_0}$. This leads to a Green's function type solution which may then be used to study the radiation from extended sources.

The exact integration of our vector wave equation in a practically usable form is probably out of the question. The present work is for the most part addressed to obtaining asymptotic results, giving the radiated amplitude in the far-zone region, at distances for which it diminishes inversely with the distance from the source. (One mode, however, is discussed exactly, for all distances.)

Expressions are derived, asymptotically exact at large distances for a sinusoidally oscillating source, from which one may numerically compute the amplitude, phase, and polarization of the velocity field* in the wave zone. Formulae are also derived for the power radiated per unit solid angle per unit frequency, in any direction, for each of the three modes involved. Detailed geometrical interpretations are

*The accompanying induced magnetic field is of course immediately derivable from this, by use of the equation of induction.

given for various points in the analysis.

The three modes, one of which is purely transverse and the others direction-dependent mixtures of longitudinal and transverse waves, behave in quite different fashions. Two of these propagate sharp wave fronts from pulses, but the third does not, namely, Huyghens' Principle* does not hold for it. This last mode, most prominent when the sound velocity and the Alfven velocity are not very different, is characterized by disconnected cusp-shaped closed disturbance fronts, not surrounding the disturbance source, and propagating away from it parallel and anti-parallel to the magnetic field, while growing uniformly within a limiting cone whose vertex angle depends on the ratio of sound velocity to Alfvén velocity.

The other two modes obey Huyghen's Principle.* One of these, the purely transverse Alfvén mode, polarized perpendicularly to the magnetic field, shows a peculiar behavior in that a point source nevertheless sets up disturbances everywhere along planes perpendicular to the magnetic field; these plane wave fronts propagate along and anti-parallel to the magnetic field with the Alfvén velocity. The intensity along such a plane has an azimuthal angular dependence with respect to the transverse radius. Just as with the heat-conduction equation (though for entirely different reasons), some disturbance appears instantaneously at all points of a transverse plane through the source, whenever this undergoes a change. This physical anomaly is, as we

---

*This term is used in a variety of senses in the literature. We say that a wave motion proceeding from a point source propagates according to Huyghens' Principle if a) the amplitude at any space-time point depends only on what the source was doing at a single previous instant, and b) a pulse source, with time dependence $\delta(t-t_0)$, creates a pulse wave front (see ref. 20, pp. 169-170).

shall see, largely due to the approximation of infinite conductivity, and illustrates again that results based on this common idealization must be examined with care. The second mode obeying Huyghens' Principle, a mixed longitudinal-transverse wave, propagates outward from the source with expanding ovaloid disturbance fronts enclosing the cusp shaped disturbances previously mentioned. This mode becomes a pure sound wave as the magnetic field vanishes.

Typical polar plots of the wave-fronts are given, on common scales. An exact solution, valid for all distances, is given for the Alfvén mode. This, by itself, is the exact solution of our problem for an incompressible fluid. The other two modes, to which most of this work is addressed, are studied by an asymptotic integration procedure utilizing approximate inversion of Fourier transforms by the stationary-phase method. This procedure is of wide applicability and should enable the study of wave propagation problems in quite general *uniform anisotropic media*.

The stationary phase approximation, because it is non-uniform, renders certain parts of our analysis somewhat inconclusive. With an aperiodic source some odd features appear in the solution for one of the modes, which anomalies are partly traceable to the breakdown of the stationary-phase approximation when there are Fourier components present in the source time-dependence of such low frequency that the field-point does not lie in the wave-zone for these frequencies. This has to do with the order in which integrations are performed in evaluating inverse transforms.

The validity of Huyghens' Principle for wave motions described by a hyperbolic set of equations is of course a very exceptional circumstance and so the occurrence of a mode not obeying it need occasion no surprise. We may make a tentative plausible re-interpretation of the peculiar properties predicted for this mode by the stationary-phase approximation. A full investigation of these points, involving a detailed study of the "domain of dependence" for the hyperbolic system of equations involved here, involves elaborate considerations in recently developed theory for linear hyperbolic equations of order higher than the second, and is not entered into here. (See, however, ref. 15 .)

It is shown in the appendix that the disturbance fronts derived in the explicit solutions for our three modes are also the characteristic surfaces of our set of hyperbolic equations, thus identifying these as the only possible true wave fronts (loci upon which the amplitude and/or its derivatives may be discontinuous).

The driving source treated in the mathematical analysis here -- a body force applied at a point -- may seem somewhat artificial to realize in concrete terms, especially for a compressible fluid. Presumably, however, in principle at least, the results given here are applicable to cover the study of the hydromagnetic radiation from quite general disturbing sources in such a medium, whatever their energy supply may be. One needs only to specify the equivalent body forces in the regions where driving agencies are active, these body forces being expressed (necessarily in a rather complicated way) in terms of certain space and time derivatives of the fluid velocity which is to be specified in the source region, and then to use an appropriate superposition of the point source solutions given here. In the case of excitation by heat liberation within the fluid we shall see that, for "weak" heat sources, permitting use of a linearized equation of state, one may quite simply express the driving forces in terms of the gradient of the rate of heat generation per unit volume.

2. The Plane Wave Solutions.

In commencing the analysis, we first consider the various modes of vibration of our medium. We have small oscillations in a perfectly conducting, non-viscous compressible fluid, permeated by a constant uniform magnetic field $\mathbb{B}_o = \hat{e}_z B_o$. As we have seen (equation 42 in the Introduction), the linearized equation describing free waves is

$$\rho_o \ddot{\mathbb{V}} - \rho_o V_s^2 \nabla\nabla\cdot\mathbb{V} + \frac{\mathbb{B}_o}{\mu_o} \times \left[\nabla\times\nabla\times\left(\mathbb{V}\times\mathbb{B}_o\right)\right] = 0 \qquad (1)$$

where ρ_o is the fluid density and V_s the sound velocity. This linearized equation is valid for $V \ll V_A = \dfrac{B_o}{\sqrt{\mu_o \rho_o}}$.

We wish first to examine some of the characteristics of plane waves in this medium, and then the shapes of the wave fronts caused by a localized pulse disturbance. Seeking plane wave solutions, put

$$\mathbb{V}(\mathbb{r}, t) = e^{i(\mathbb{k}\cdot\mathbb{r} - \omega t)}\, \hat{\mathbb{V}} \qquad (2)$$

A slight reduction yields

$$\rho_o \omega^2 \hat{\mathbb{V}} + \rho_o V_s^2 (\mathbb{k}\cdot\hat{\mathbb{V}})\mathbb{k} - \frac{(\mathbb{B}_o\cdot\hat{\mathbb{V}})(\mathbb{B}_o\cdot\mathbb{k})}{\mu_o}\mathbb{k} + \frac{\mathbb{B}_o^2}{\mu_o}(\mathbb{k}\cdot\hat{\mathbb{V}})\mathbb{k} + \frac{(\mathbb{B}_o\cdot\mathbb{k})^2}{\mu_o}\hat{\mathbb{V}} - \frac{(\mathbb{B}_o\cdot\mathbb{k})(\hat{\mathbb{V}}\cdot\mathbb{k})}{\mu_o}\mathbb{B}_o = 0 \qquad (3)$$

Let θ denote the angle between $\mathbb{B}_o$ and $\mathbb{k}$.

(a) For $\hat{\mathbb{V}}\cdot\mathbb{B} = \hat{\mathbb{V}}\cdot\mathbb{k} = 0$ equation 3 yields

$$\left[\frac{(\mathbb{B}_o\cdot\mathbb{k})^2}{\mu} - \omega^2\rho_o\right]\hat{\mathbb{V}} = 0 \qquad \text{or} \qquad \rho_o\omega^2 = \frac{B_o^2 k^2}{\mu}\cos^2\theta \qquad (4)$$

These are plane Alfvén waves, purely transverse. When we consider the excitation problem we shall see that this mode, alone, is a purely <u>transverse</u> wave whose generation and propagation is best treated separately from the other modes.

(b) If $\mathbb{V}$ is in the $\mathbb{B}_o$, $\mathbb{K}$ plane, $\mathbb{V} = a\mathbb{K} + b\,\mathbb{B}_o$, then equation 3 gives

$$\left(-\rho_o\omega^2 + \rho_o V_s^2 k^2 + \frac{B_o^2 k^2}{\mu}\right)a + \rho_o V_s^2 (\mathbb{B}_o \cdot \mathbb{K})\, b = 0$$

and

$$\rho_o\omega^2 b + a k^2 (\mathbb{B}_o \cdot \mathbb{K}) = 0$$

from which we obtain

$$\omega^2\left[-\rho_o\omega^2 + \rho V^2 k^2 + \frac{\mathbb{B}_o^2 k^2}{\mu}\right] - V_s^2 k^2 \frac{(\mathbb{B}_o\cdot\mathbb{K})^2}{\mu} = 0 \tag{5}$$

or, after slight reduction,

$$\omega^4 - \left(V_s^2 + V_A^2\right)\omega^2 k^2 + V_s^2 V_A^2 k^4 \cos^2\theta = 0 \tag{5'}$$

from which

$$\frac{2\omega^2}{k^2} = \left(V_s^2 + V_A^2\right) \pm \sqrt{\left(V_s^2 + V_A^2\right)^2 - 4\, V_s^2 V_A^2 \cos^2\theta}$$

Putting

$$\frac{V_s}{V_A} = e^{\lambda} \tag{6}$$

and

$$\varkappa = \operatorname{sech}\lambda = \frac{2 V_A V_s}{V_A^2 + V_s^2} \tag{7}$$

and

$$F_{\pm}(\theta) = \sqrt{V_s V_A \cosh\lambda}\ \sqrt{1 \pm \sqrt{1 - \varkappa^2\cos^2\theta}} = \sqrt{\frac{V_s^2 + V_A^2}{2}}\sqrt{1 \pm \sqrt{1 - \varkappa^2 \cos^2\theta}} \tag{8}$$

we have

$$\omega^2 = k^2 F_{\pm}^2(\theta) \tag{9}$$

or

$$\mathbb{K} = \pm \frac{\omega}{F(\theta)} \hat{e}_k \tag{10}$$

for F_+ or F_-, where $\hat{e}_{\mathbb{K}}$ is a unit vector making an angle θ with $\mathbb{B}_o$.

Typical polar plots of $\frac{1}{\omega}\sqrt{\frac{V_S^2 + V_A^2}{2}}\, \mathbb{K}$ against θ are shown in figures 1 through 3. We see that these plots are in general 3-sheeted surfaces, except when $\lambda = 0$, when they coalesce. These surfaces describe the behavior of a pair of "magneto-acoustic" wave modes which we shall denote as the " F_+ " and " F_- " modes. Both are mixtures of transverse and longitudinal waves, the percentages varying with θ.

For propagation along the magnetic field ($\theta = 0$), and for this direction alone, there occur purely longitudinal (sound) waves with $\omega^2 = V_s^2 k^2$. For both $F_{\pm}$ modes the phase velocity is independent of k (though depending on the direction of $\mathbb{K}$), and so the lossless medium, though anisotropic, is non-dispersive, in the usual sense of that term. Plane waves of arbitrary form can be propagated without change of shape.

The phase velocity (along $\mathbb{K}$) is

$$\mathbb{V}_{ph} = \frac{\omega}{k} \hat{e}_k = \hat{e}_k F_{\pm}(\theta) \tag{11}$$

while the group velocity is

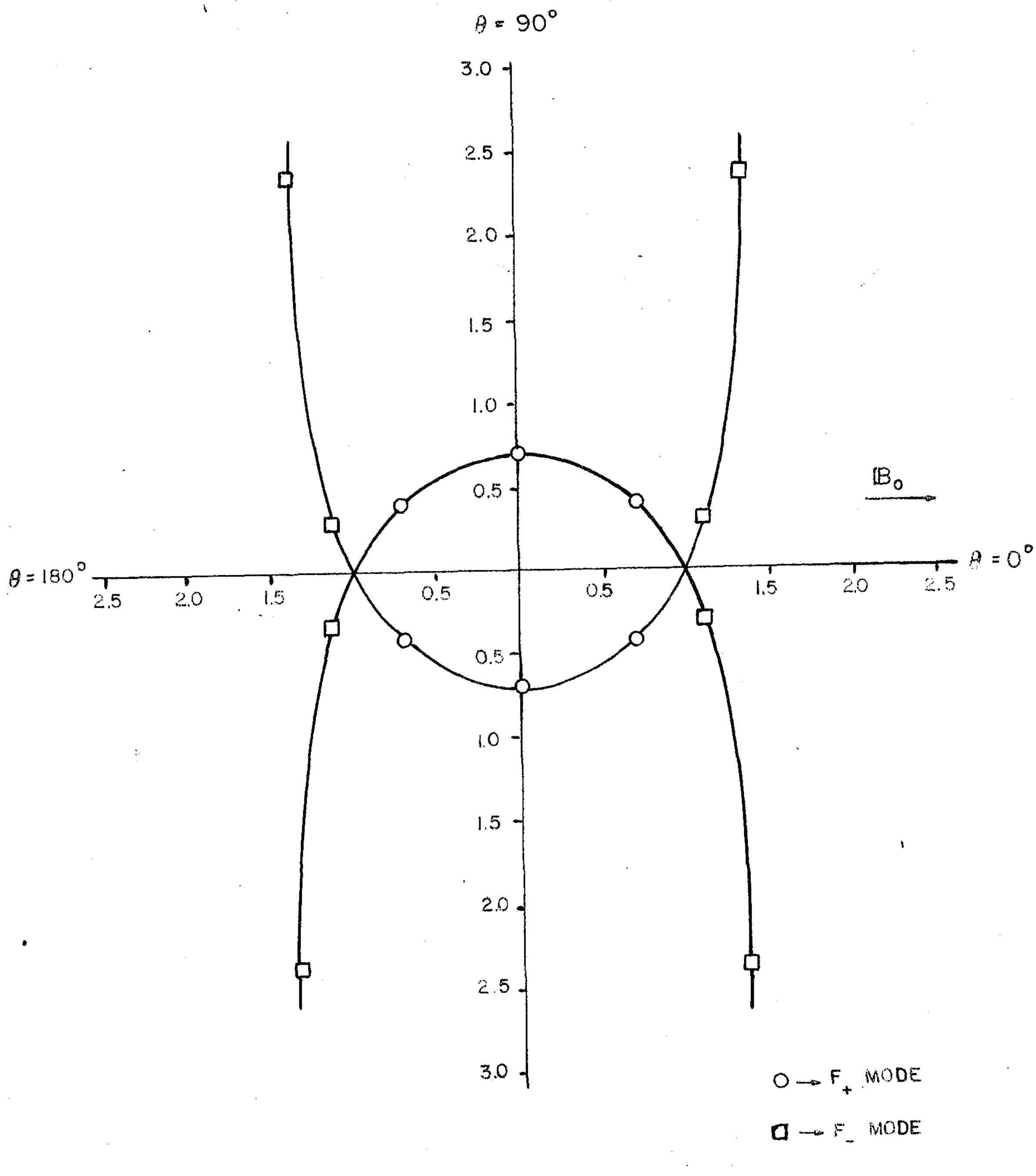


FIGURE I

PLOT OF $k^* \equiv \frac{1}{\omega}\sqrt{\frac{V_S^{\,2} + V_A^{\,2}}{2}}\; k$ VS. θ FOR $\lambda = 0$ $(V_A / V_S = 1)$

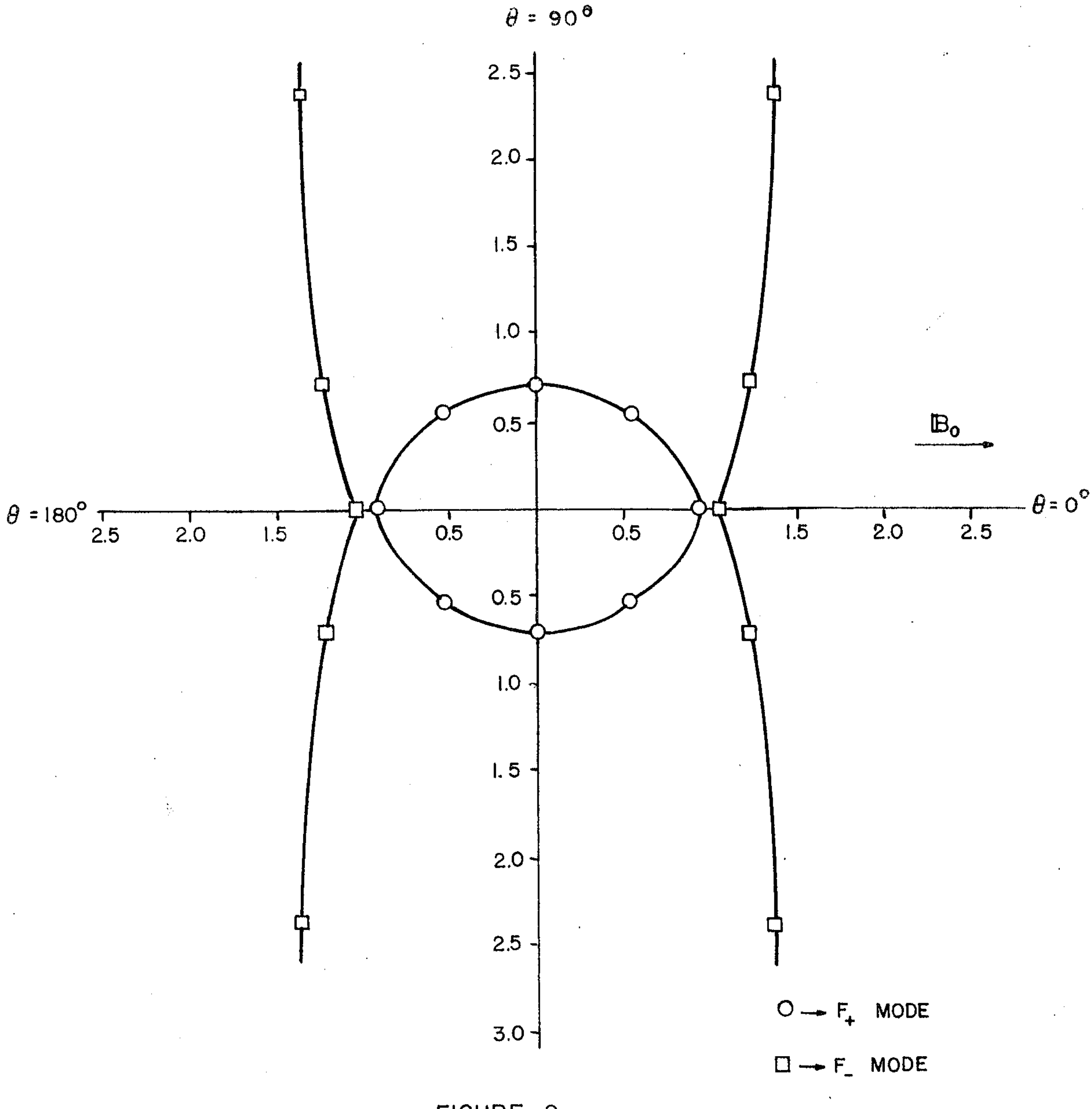


FIGURE 2

PLOT OF $k^* \equiv \frac{1}{\omega}\sqrt{\frac{V_S{}^2 + V_A{}^2}{2}}\, k$ VS. θ FOR $\lambda = \pm 0.1 (V_A / V_S = 1.105$ OR $0.9050)$

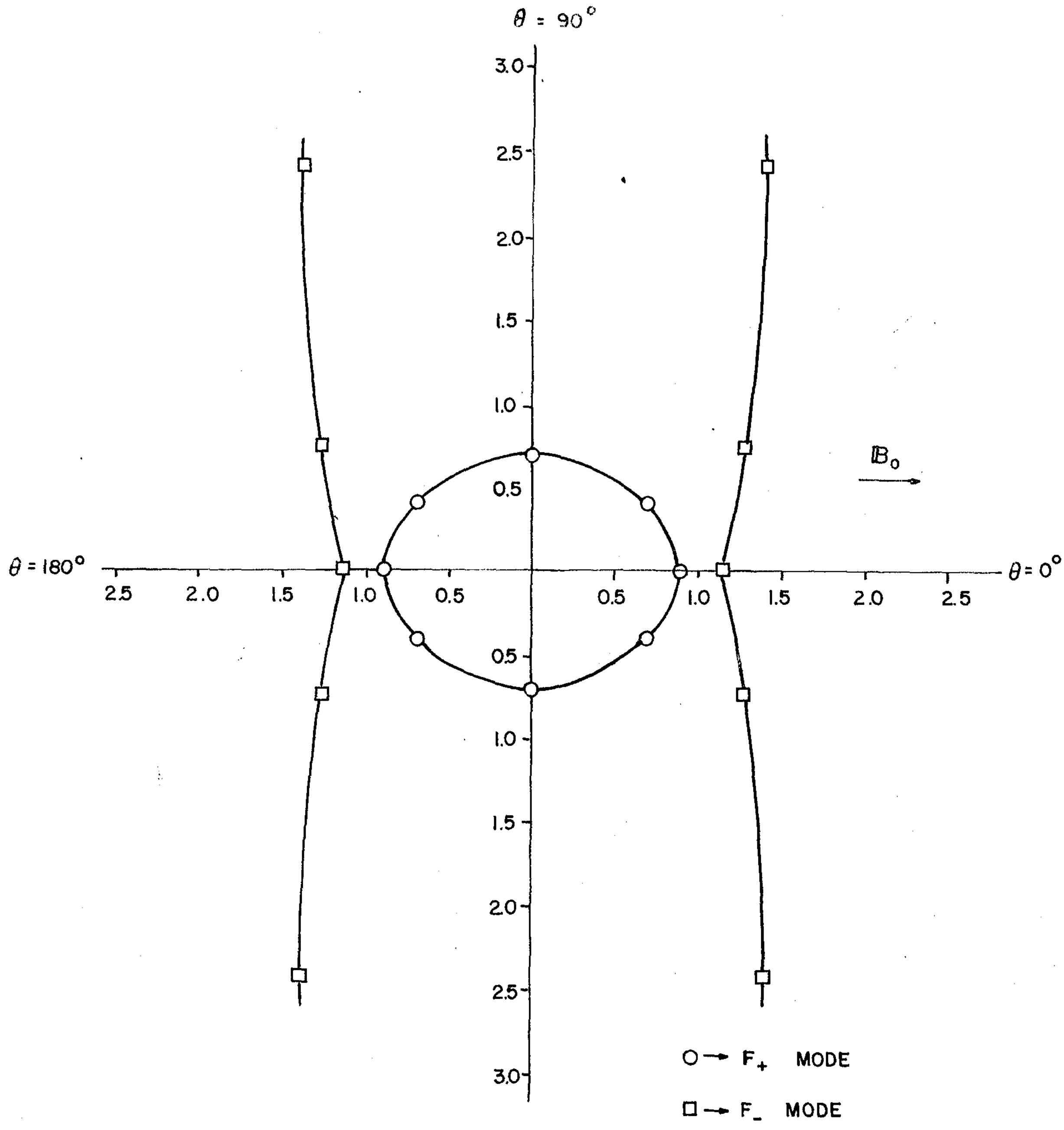


FIGURE 3

PLOT OF $k^* \equiv \frac{1}{\omega}\sqrt{\frac{V_S^2 + V_A^2}{2}}\; k$ VS. θ FOR $\lambda = \pm 0.25$ ($V_A/V_S = 1.284$ OR 0.7788)

$$\mathbb{V}_g = \nabla_{\mathbb{k}} \omega = \hat{e}_x \frac{\partial \omega}{\partial k_x} + \hat{e}_y \frac{\partial \omega}{\partial k_y} + \hat{e}_z \frac{\partial \omega}{\partial k_z} \tag{12}$$

and these are in general unequal, both in magnitude and direction, even though the medium is "non-dispersive".

The F_+ and F_- modes, whose velocity fields are accompanied by pressure fluctuations, are the " P -modes" of Baños (13). The third, purely transverse mode (see equation 3) is the "Alfvén mode" of Banos, and its velocity field is devoid of pressure fluctuations. This mode, which also exists in an incompressible fluid, is the type of hydromagnetic wave discussed by Alfvén in his basic paper (2). The dispersion relation for this mode is

$$\omega^2 = V_A^2 \kappa^2 \cos^2\theta \tag{13}$$

For this mode, the plot of $\frac{V_A}{\omega}\mathbb{k}$ against θ is a pair of planes orthogonal to $\mathbb{B}_0$ and at unit distance from the origin. The group velocity of this mode is

$$\mathbb{V}_g = \pm V_A \hat{e}_z \tag{14}$$

3. The Wave Fronts.

Next, it is of interest to study the form of the phase fronts (the equi-phase surfaces) emanating from a point source, still considering sinusoidal time-dependence. Since, as we have seen, the medium is non-dispersive, we might also expect that these phase fronts will also define true wave fronts, on which the amplitude can have discontinuous derivatives or itself be discontinuous. This is explicitly verified for the $F_\pm$ modes in the Appendix, where it is shown that these two equi-phase surfaces are also characteristic surfaces of our equations.*

Very simple considerations will reveal into what regions energy will spread from a concentrated pulse source (and an interesting new type of wave-front, as well). It will be the object of later more elaborate considerations to study the distribution of amplitude, polarization and energy flux in the far-zone.

To study the equi-phase surfaces emanating from a monochromatic point source, we must find the envelope of the equi-phase planes

$$\mathbb{k}\cdot\mathbb{r} = const.$$

With $\mathbb{B}_0$ as polar axis, let $\mathbb{k}$ have polar angles θ, ϕ and let $\mathbb{r}$ (the radius vector to any field point) have polar angles $\psi\ h$. (We suppose a fixed reference system in which to measure these angles.) Then, we vary $\mathbb{k}$ over all directions and find the envelope of the two-parameter family of equi-phase planes. Now, in the subsequent application of the stationary-phase method we shall also be interested in stationary values of the phase-factor $\mathbb{k}\cdot\mathbb{r}$, namely, values of $\mathbb{k}$ about which, for any given $\mathbb{r}$, this expression is stationary. The two questions are very closely related and the following analysis answers them both.

Using equation 10 we find the equation of a plane of constant phase to be

$$\frac{r}{F(\theta)}\left[\cos\theta\cos\psi + \sin\theta\sin\psi\cos(\phi - h)\right] = \omega t = const. \tag{15}$$

Also, the partial derivative of the left-hand side with respect to θ must vanish for all θ, ψ, h, which gives

*Curiously, the third (Alfvén) characteristic is not identical with the envelope of equi-phase planes for that mode; this envelope consists of just a single point.

$$\sin\theta \sin\psi \sin(\phi - h) = 0$$

for all θ, ψ , thereby determining that

$$\phi - h = 0, \quad \pi$$

and hence, from equation 15,

$$\frac{r}{F(\theta)} \cos(\theta \pm \psi) = const. \tag{16}$$

where the (+) sign goes with $\phi - h = \pi$ (F_- case) and the (-) sign goes with $\phi - h = 0$ (F_+ case). Finally, the partial derivative of equation 15 with respect to θ must vanish for all θ, ψ, h, which gives

$$\tan(\theta \pm \psi) = \frac{-F'(\theta)}{F(\theta)} \tag{17}$$

So far, in equation 15 to 17, "F" denotes either F_+ or F_- (the Alfvén case being trivial). θ and ψ are taken as positive colatitude angles.

We want now to remove the remaining sign ambiguity; to find a unique pairing of $(\theta \pm \psi)$ with F_+ , F_- . In equation 16 we may take the constant > 0 without loss of generality; this amounts merely to stipulating $r > 0$ in the first quadrant for $\theta - \psi$. Let us solve the equation

$$\tan(\theta - \beta) = \frac{-F'}{F} \tag{17'}$$

where

$$\cos(\theta - \beta) > 0 \qquad \text{(which follows from equation 16)}$$

If $\beta > 0$, $\phi - h = 0$ and $\psi = \beta$. If $\beta < 0$, $\phi - h = \pi$ and $\psi = -\beta = |\beta|$. We know that $F_+' > 0$ and $F_-' < 0$. Hence, from equation 17', for F_+ we get $\theta - \beta < 0$, guaranteeing $\beta > 0$. So, for the F_+ mode we require $\theta - h = 0$ and $\beta = \psi$, and use the (-) sign $(\theta - \psi)$ in equations 16 and 17. For F_- , equation 17' gives $\theta - \beta > 0$, which says nothing. If we can show that $\tan\theta < \tan(\theta-\beta)$ for F_- , this will establish $\beta < 0$ for F_- . Now,

$$\frac{-F_-'}{F_-} = \tan(\theta-\beta) = \frac{k^2 \sin\theta\cos\theta}{2\sqrt{1-k^2\cos^2\theta}\left[1-\sqrt{1-k^2\cos^2\theta}\right]}$$

To show $\tan\theta < \tan(\theta-\beta)$ we need merely show that

$$\sqrt{1-k^2\cos^2\theta}\left(1-\sqrt{1-k^2\cos^2\theta}\right) < \frac{\cos^2\theta}{2}$$

which is easily demonstrated. Consequently, $\beta > 0$ for the F_+ mode, and $\beta < 0$ for the F_- mode.

Assembling these results, we have the following:

For the F_+ mode,

$$\tan(\theta-\psi) = -\frac{F_+'(\theta)}{F_+(\theta)} \tag{18a}$$

and

$$\frac{r}{F_+(\theta)}\cos(\theta-\psi) = t \left(= \frac{\mathbb{k}\cdot\mathbb{r}}{\omega}\right) \tag{19a}$$

while, at a stationary point,

$$\phi - h = 0 \tag{20a}$$

For the F_- mode,

$$\tan(\theta+\psi) = -\frac{F_-'(\theta)}{F_-(\theta)} \tag{18b}$$

and

$$\frac{r}{F_-(\theta)}\cos(\theta-\psi) = t \left(= \frac{\mathbb{k}\cdot\mathbb{r}}{\omega}\right) \tag{19b}$$

while, at a stationary point,

$$\phi - h = \pi \tag{20b}$$

Equations 18 and 20 establish, for both F_+ and F_- modes, an association between directions in physical space (ψ, h) and directions in $\mathbb{k}$ -space (θ, ϕ), which will play a role later in our use of the stationary-phase approximation. On the other hand, equations 18 and 19 together define, for both modes, the parametric equations (in terms of the $\mathbb{k}$ -space angle coordinate θ) of a family of curves $\mathbb{r} = \mathbb{r}(\psi)$, which are sections, by planes through the source and parallel to $\mathbb{B}_0$, of a family of equiphase surfaces, (envelopes of planes $\mathbb{k}\cdot\mathbb{r}$ = constant), axially symmetric about the polar axis. From equation 18 and equation 19,

$$r = t\sqrt{F_{\pm}^{2}(\theta) + F_{\pm}'^{2}(\theta)} \tag{21}$$

Clearly this is the equation of the equiphase surface $\Phi = 0$, in time t.*

We shall see later that, for θ and ψ connected by equation 18, $\sqrt{F^2+F'^2}$ is the speed of a wave front (e.g. from a pulse disturbance) in the direction ψ, so that equation 21 is also the equation of the trace of such a wave front at time t after its initiation. For the Alfvén mode we shall see that the analog of equation 21 is the pair of planes

$$z = \pm V_A t \tag{22}$$

In figures 1 through 3, the dimensionless wave number

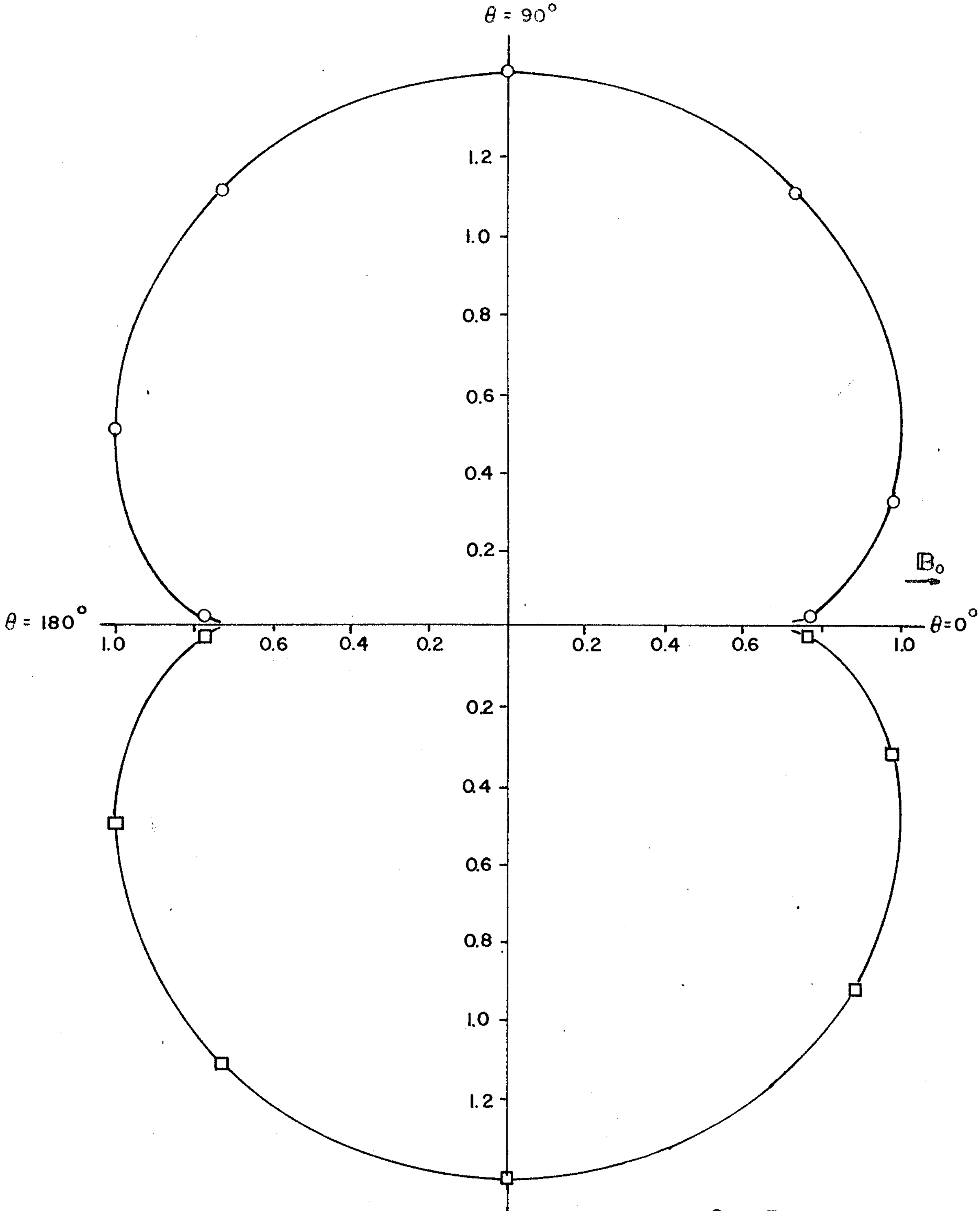
$$k^* = \frac{1}{\omega}\sqrt{\frac{V_s^2 + V_A^2}{2}}\, k$$

is plotted against θ for the F_+ and F_- modes, for λ = 0, $\pm$ 0.1, $\pm$ 0.25. In figures 4, 5, 6, the dimensionless quantity

$$r^* = \sqrt{\frac{2}{V_s^2 + V_A^2}}\,\sqrt{F^2 + F'^2}$$

is plotted against ψ for the same set of values of λ. The correspondence between θ and ψ is graphically depicted in figures 7, 8, 9. The ovaloid surfaces are for the F_+ mode, which radiates in all directions. The disjoint tri-cusped surfaces are for the F_- mode, which radiates only within a double-cone whose angular width, $2\psi_{max.}$, is determined by V_s/V_A. We can readily ascertain from the equation following equation 5' that, if $V_A < V_s$, the plane wave front for the Alfvén mode intersects the F_- tri-cusped surface at $\theta = 0$ and π on its caps, whereas if $V_A > V_s$, the Alfvén wave front intersects the F_+ ovaloid at $\theta = 0$ and π. (See figure 10).

*We define $\Phi = \mathbb{k}\cdot\mathbb{r} - \omega t$ as the phase of a plane wave.



○ → F_+ MODE

□ → F_- MODE

FIGURE 4

PLOT OF $r^* = \sqrt{\frac{2}{V_S^2+V_A^2}}\sqrt{F^2+F'^2}$ VS ψ FOR $\lambda = 0\,(V_A/V_S = 1)$
FOR THE F_+ AND F_- MODES

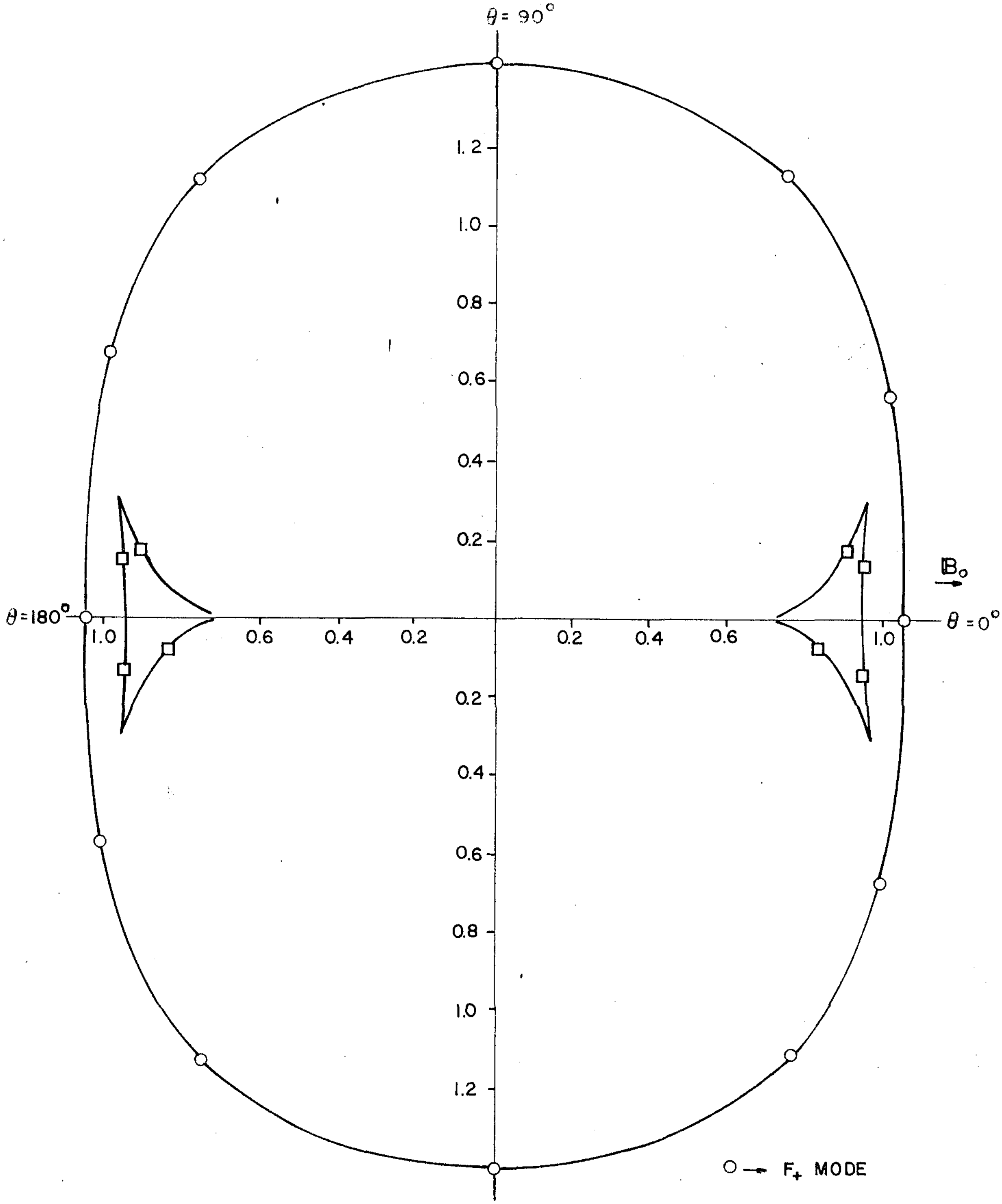


FIGURE 5

PLOT OF $r^* = \sqrt{\frac{2}{V_S^2 + V_A^2}} \sqrt{F^2 + F'^2}$ VS. ψ FOR $\lambda = \pm 0.1$ ($V_A/V_S = 1.105$ OR 0.9050)
FOR THE F_+ AND F_- MODES

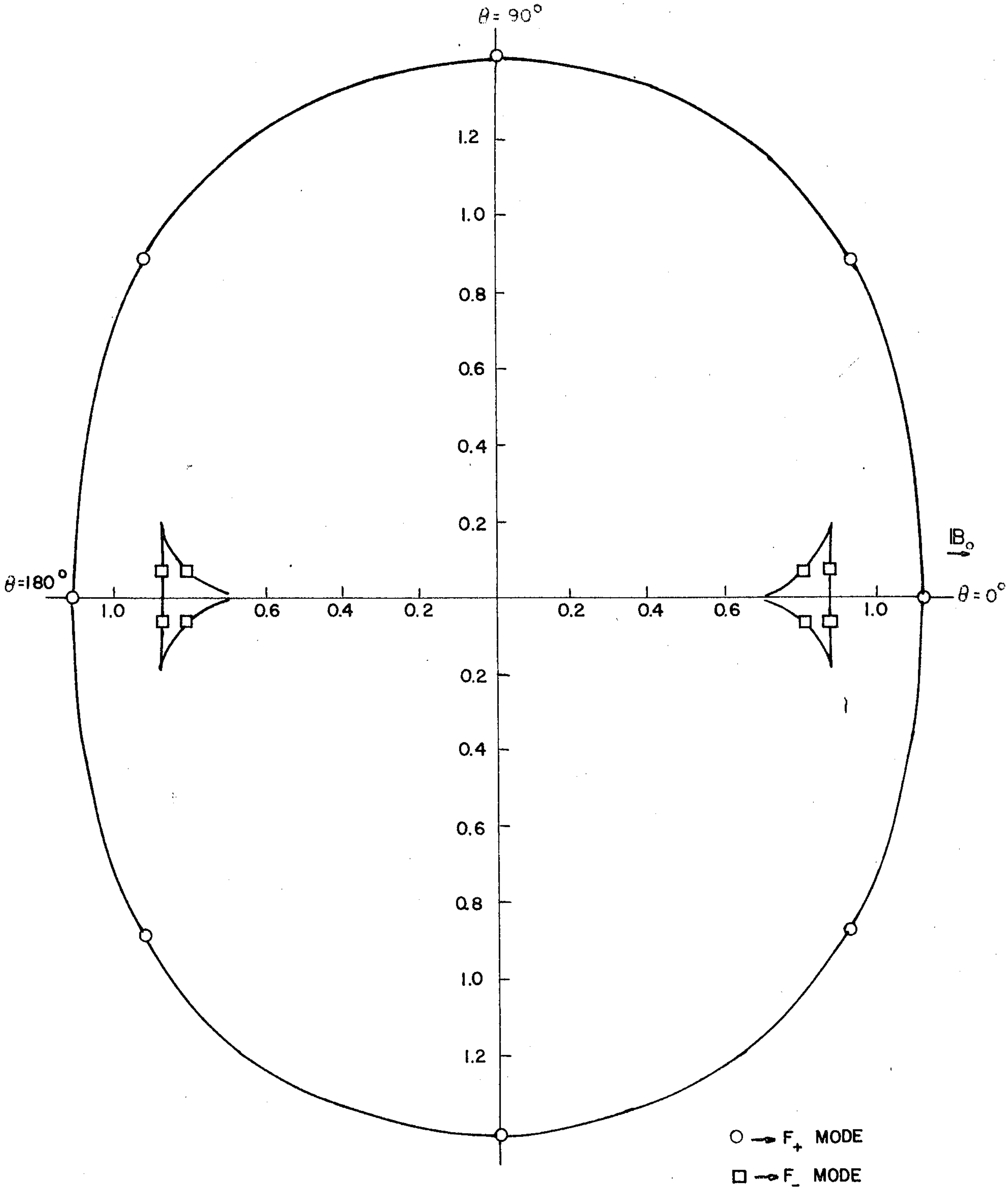


FIGURE 6

PLOT OF $r^* = \sqrt{\frac{2}{V_S^2 + V_A^2}}\sqrt{F^2 + F'^2}$ VS. ψ FOR $\lambda = \pm 0.25$ ($V_A/V_S = 1.284$ OR 0.7788) FOR THE F_+ AND F_- MODES

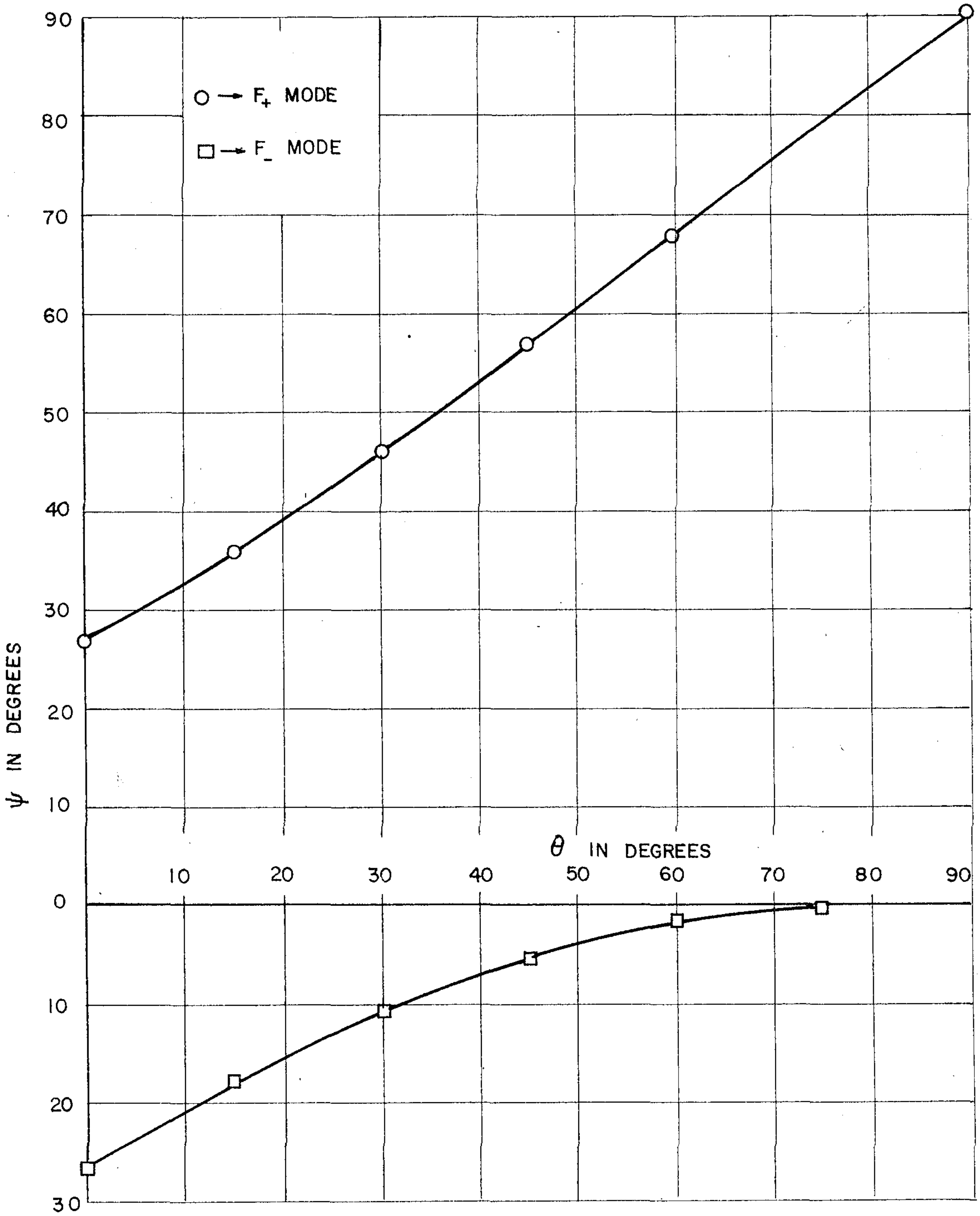


FIGURE 7

ψ VS. θ FOR F_+ AND F_- MODES, WITH $\lambda = 0$ $(V_A / V_S = 1)$

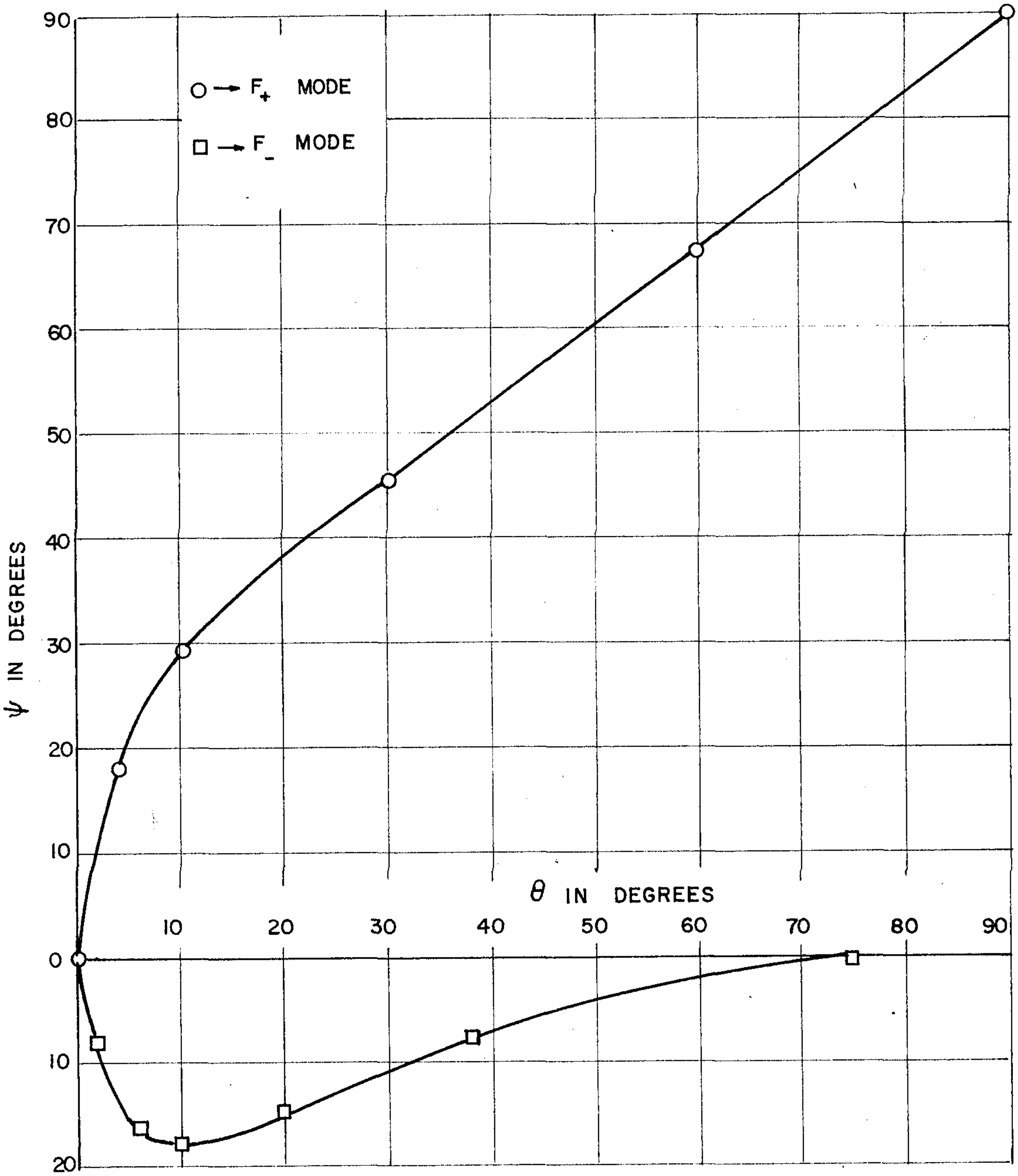


FIGURE 8

ψ VS. θ FOR F_+ AND F_- MODES, WITH $\lambda = \pm 0.1$ (V_A/V_S =1.105 OR 0.9050)

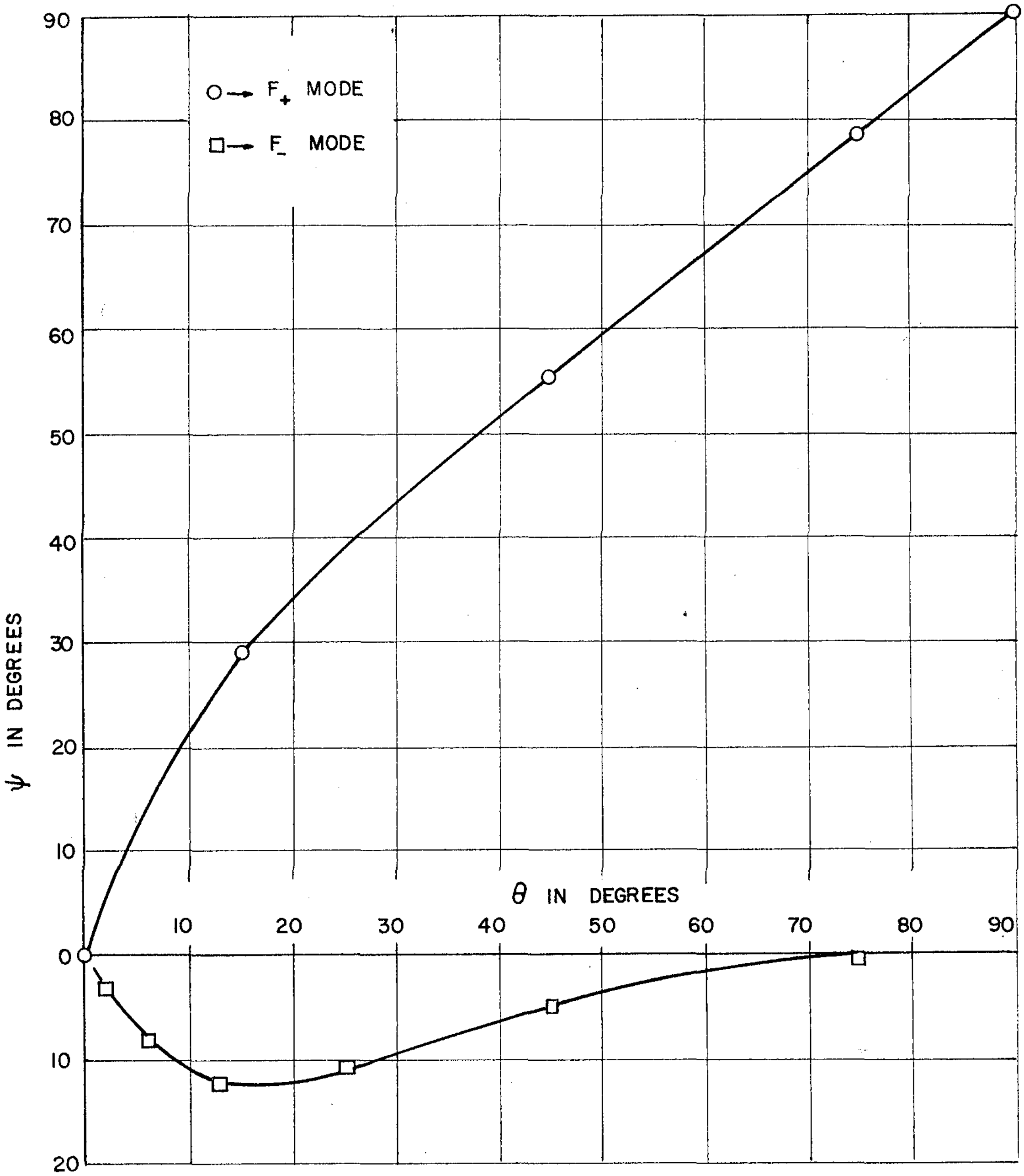


FIGURE 9

ψ VS. θ FOR F_+ AND F_- MODES, WITH $\lambda = \pm 0.25 (V_A/V_S = 1.284$ OR $0.7788)$

For a pulse point disturbance at the origin, we now have the following simple partial picture of the way in which the energy subsequently spreads. The Alfvén mode carries off its energy in a pair of plane disturbance points orthogonal to the magnetic field. The faster of the shear-compression modes (F_+ mode) radiates out as an expanding ovaloid, flattened in the direction of the magnetic field. Within the ovaloid there moves outwards, parallel and anti-parallel to the field, a pair of tri-cusped closed wave fronts, remaining always touching a double cone whose angular aperture is determined by V_S/V_A alone. These disjoint fronts are most prominent when V_S/V_A is nearly unity. As $V_S/V_A \to 1$ the F_+ and F_- modes coalesce into a wave front in the shape of an ovaloid pinched in at its minor axis. As $V_S/V_A \to \infty$ or 0 the critical cone angle $2\psi_{max.}$ tends monotonically to zero.

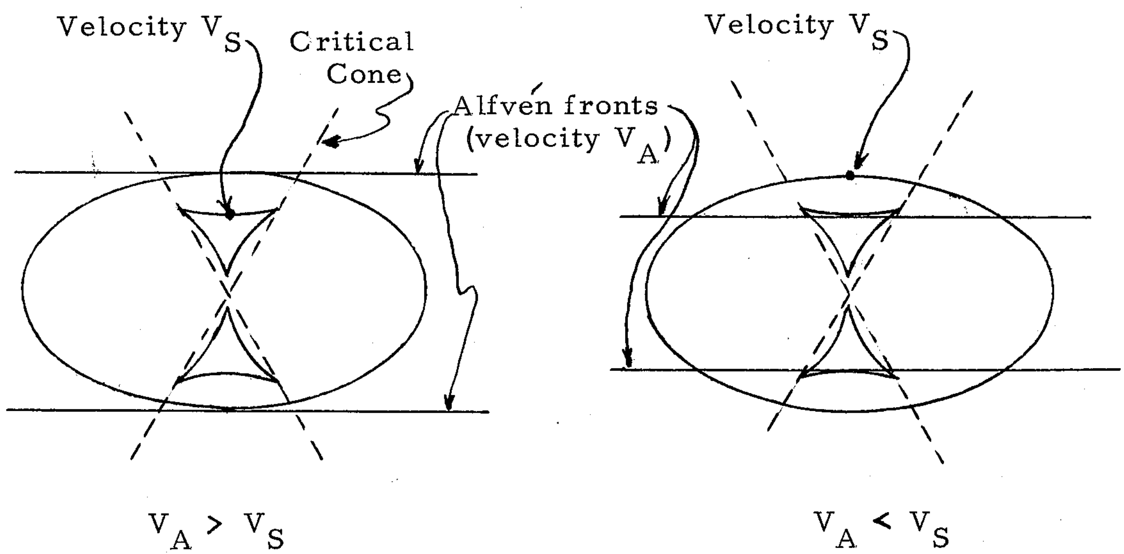


Figure 10 - showing the relative positions of $F_\pm$ fronts and Alfvén fronts for $V_A \gtrless V_S$.

Our analysis does not entirely settle the question of the distribution of energy within the ovaloid front. There can be no amplitude ahead of either Alfvén front, outside the ovaloid. There can be no amplitude outside of the ovaloid front and ahead of the Alfvén front.* However, as already stated, Huyghens' Principle does not hold for part of the front associated with the F_- mode. Presumably, then, there is some disturbance everywhere inside the critical cone and within the ovaloid front, at all distances from the source. Further, there will be a discontinuity in the amplitude across each of these surfaces. Further remarks about the wave fronts and some geometrical interpretations are given later in the text, and in the Appendix.

4. Fourier Analysis of the Inhomogeneous Wave Equation.

We consider again our linearized vector wave equation, this time with a forcing term. We have then

$$\rho_o \frac{\partial^2 \mathbb{V}}{\partial t^2} - \rho_o V_s^2 \nabla\nabla\cdot\mathbb{V} + \frac{\mathbb{B}_o}{\mu} \times \left[\nabla\times\nabla\times(\mathbb{V}\times\mathbb{B}_o)\right] = \mathbb{Q}(\mathbb{r}, t) \tag{23}$$

where $\mathbb{Q}(\mathbb{r},t)$ is the time derivative of the applied body force density at $\mathbb{r}$, t. Attempts were made to treat this as a vector equation, perhaps along lines similar to those used in the integration of the ordinary vector wave equation (16, Chapter 16), but the anisotropic term (the third term on the left-hand side of equation 23) seems to block such an approach. Baños (13 and 17) has constructed complete sets of plane and cylindrical wave solutions of equation 1 in terms of solutions of the scalar Helmholtz equation and it was attempted to use these in the integration of equation 23, by constructing a tensor Green's function from them. However, it

*These two statements hold for all distances, being based on the fact that these surfaces are characteristic surfaces, as shown in the Appendix.

is by no means evident how to find a "vector Green's Theorem" for the operator in equation 23, and, also, the divergenceless and irrotational parts of $\mathbb{V}$ are by no means simply related to the divergenceless and irrotational parts of $\mathbb{Q}$. Also, it seems impossible to find equations for these parts of $\mathbb{V}$, by themselves. Consequently, the only feasible approach has seemed to be via the decomposition of equation 23 into a system of equations, and solution of this system by Fourier analysis. To this we now proceed.

We take the 4-dimensional Fourier transform of equation 23, putting

$$\mathbb{V}(\mathbb{r}, t) = \frac{1}{(2\pi)^2}\iint d^3\mathbb{K}\, d\omega\, e^{i(\mathbb{K}\cdot\mathbb{r}-\omega t)}\, \vec{v}(\mathbb{K}, \omega) \tag{24}$$

$$\mathbb{Q}(\mathbb{r}, t) = \frac{1}{(2\pi)^2}\iint d^3\mathbb{K}\, d\omega\, e^{i(\mathbb{K}\cdot\mathbb{r}-\omega t)}\, \vec{q}(\mathbb{K}, \omega) \tag{25}$$

of which the inverses are

$$\vec{v}(\mathbb{K}, \omega) = \frac{1}{(2\pi)^2}\iint d^3\mathbb{r}\, dt\, e^{-i(\mathbb{K}\cdot\mathbb{r}-\omega t)}\, \mathbb{V}(\mathbb{r}, t) \tag{24'}$$

and

$$\vec{q}(\mathbb{K}, \omega) = \frac{1}{(2\pi)^2}\iint d^3\mathbb{r}\, dt\, e^{-i(\mathbb{K}\cdot\mathbb{r}-\omega t)}\, \mathbb{Q}(\mathbb{r}, t) \tag{25'}$$

With the use of equations 24 and 25, equation 23 becomes

$$\rho_0\omega^2\vec{v}(\mathbb{K}, \omega) + \rho_0 V_s^2(\mathbb{K}\cdot\vec{v})\mathbb{K} - \frac{\mathbb{B}_0 \times}{\mu_0}\left[\mathbb{K}\times\left(\mathbb{K}\times\left[\vec{v}\times\mathbb{B}\right]\right)\right] = \vec{q}(\mathbb{K}, \omega) \tag{26}$$

Throughout most of this work, and unless otherwise indicated, our source will be the point source

$$\mathbb{Q}(\mathbb{r}, t) = \delta(\mathbb{r})\,\mathbb{Q}(t) = \delta(\mathbb{r})\,\mathcal{F}(t)\,\vec{Q} \tag{27}$$

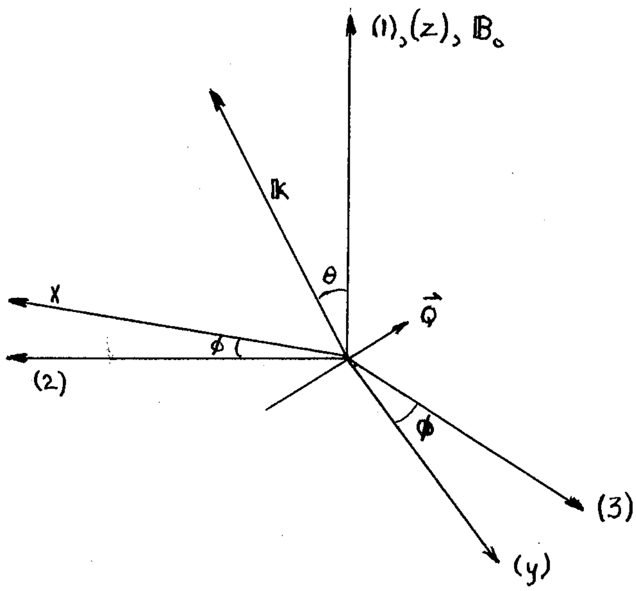
(where $\vec{Q}$ is a const. vector) so that, in this case, from equation 25'

$$q(\mathbb{K}, \omega) = \frac{1}{(2\pi)^{3/2}}\,\vec{Q}\, f(\omega) = \vec{q}(\omega) \tag{28}$$

where

$$f(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \mathcal{F}(t)\, e^{i\omega t}\, dt$$

We resolve our vector Fourier transforms for the moment along the axes of a mobile coordinate system defined as follows. Introduce the unit vectors $\hat{e}_1$ parallel to $\mathbb{B}_0$, $\hat{e}_2$ in the $\mathbb{B}_0$, $\mathbb{k}$ plane but orthogonal to $\mathbb{B}_0$, and $\hat{e}_3 = \hat{e}_1 \times \hat{e}_2$. This mobile coordinate system is referred to a set of fixed Cartesian axes $x\,y\,z$ whose z-axis coincides with $\hat{e}_1$ and with respect to which $\mathbb{k}$ has polar angles θ, ϕ and $\mathbb{r}$ (the vector from the source to the field point) has polar angles ψ, h as shown below.



Axis (2) lies always in the plane, so that $\mathbb{k} = \hat{e}_1 \kappa_1 + \hat{e}_2 \kappa_2$
Axes $x\,y$ and (2), (3) lie in a common plane perpendicular to $\mathbb{B}_0$.
The source $\mathbb{Q}$ has polar angles θ_q, ϕ_q.

Figure 11

Clearly, then

$$\left.\begin{aligned} V_x^{(total)} &= V_2\cos\phi - V_3\sin\phi = V_x^{(\pm)} + V_x^{(A)} \\ V_y^{(total)} &= -V_2\sin\phi + V_3\cos\phi = V_y^{(\pm)} + V_y^{(A)} \\ V_z^{(total)} &= V_3 \end{aligned}\right\}$$

where the superscript $(\pm)$ denotes the part due to the $F\pm$ modes, and the superscript (A) denotes the part due to the Alfvén mode. (29)

Resolving equation 26 along the mobile axes yields

$$\hat{e}_1 \text{ -component : } -\rho_o\omega^2 v_1 + \rho_o V_S^2 \kappa_1(\kappa_1 v_1 + \kappa_2 v_2) = q_1 \tag{30}$$

$$\hat{e}_2 \text{ -component : } -\rho_o\omega^2 v_2 + \rho_o V_S^2 \kappa_2(\kappa_1 v_1 + \kappa_2 v_2) + \frac{B_o^2}{\mu}(\kappa_1^2 + \kappa_2^2) = q_2 \tag{31}$$

$$\hat{e}_3 \text{ -component : } -\rho_o\omega^2 v_3 + \kappa^2 \frac{B_o^2}{\mu} v_3 = q_3 \tag{32}$$

From equation 29 thru 32 we see that we may treat separately the amplitude due to the F_+ and F_- modes (described by equations 30 and 31), and that due to the Alfvén mode, described by equation 32, for which $\vec{V}_z = 0$ and $\nabla\cdot\vec{V}=0$. It will be convenient to obtain the total amplitude by solving equations 30, 31 together, separately solving equation 32, and adding the results. We commence by treating the $F_\pm$ modes, leaving the Alfvén mode for a separate treatment later.

Equations 30 and 31 may be written in matrix form as

$$\begin{pmatrix} -\rho_o\omega^2 + \rho_o V_S^2 \kappa_1^2 & \rho_o V_S^2 \kappa_1\kappa_2 \\ \rho_o V_S^2 \kappa_1\kappa_2 & -\rho_o\omega^2 + \rho_o V_S^2 \kappa_2^2 + \frac{B_o^2}{\mu}\kappa^2 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} \tag{33}$$

Introducing

$$V_A = \frac{B_o}{\sqrt{\mu_o\rho_o}}, \qquad \kappa_1 = \kappa\cos\theta, \quad \text{and} \quad \kappa_2 = \kappa\sin\theta$$

the determinant of this matrix may be written as

$$\begin{aligned} D &= \rho_o^2\left[\omega^4 - \omega^2\kappa^2(V_S^2 + V_A^2) + V_S^2 V_A^2 \kappa^4\cos^2\theta\right] \\ &= \rho_o^2\left(\omega^2 - \kappa^2 F_+^2(\theta)\right)\left(\omega^2 - \kappa^2 F_-^2(\theta)\right) \end{aligned} \tag{34}$$

Then

$$\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \frac{\rho_o\kappa^2}{D}\begin{pmatrix} V_A^2 + V_S^2\sin^2\theta - \frac{\omega^2}{\kappa^2} & -V_S^2\sin\theta\cos\theta \\ -V_A^2\sin\theta\cos\theta & V_S^2\sin^2\theta - \frac{\omega^2}{\kappa^2} \end{pmatrix}\begin{pmatrix} q_1 \\ q_2 \end{pmatrix} \tag{35}$$

But

$$\frac{\rho_0 K^2}{D} = \frac{1}{\rho_0 (F_+^2 - F_-^2)} \left[\frac{1}{\omega^2 - K^2 F_+^2} - \frac{1}{\omega^2 - K^2 F_-^2} \right]$$

Then

$$\begin{pmatrix} v_1 \\ \\ v_2 \end{pmatrix} = \frac{1}{\rho_0 (F_+^2 - F_-^2)} \left[\frac{1}{\omega^2 - K^2 F_+^2} - \frac{1}{\omega^2 - K^2 F_-^2} \right] \begin{pmatrix} V_A^2 + V_S^2 \sin^2\theta - \frac{\omega^2}{K^2} & -V_S^2 \sin\theta\cos\theta \\ \\ -V_S^2 \sin\theta\cos\theta & V_S^2 \sin^2\theta - \frac{\omega^2}{K^2} \end{pmatrix} \begin{pmatrix} q_1 \\ \\ q_2 \end{pmatrix}$$

$$= \frac{1}{\rho_0 (F_+^2 - F_-^2)} \left\{ \left[\frac{1}{\omega^2 - K^2 F_+^2} - \frac{1}{\omega^2 - K^2 F_-^2} \right] \begin{pmatrix} V_A^2 + V_S^2 \sin^2\theta & -V_S^2 \sin\theta\cos\theta \\ \\ -V_S^2 \sin\theta\cos\theta & V_S^2 \sin^2\theta \end{pmatrix} - \underset{\sim}{I} \frac{\omega^2}{K^2} \left[\frac{1}{\omega^2 - K^2 F_+^2} - \frac{1}{\omega^2 - K^2 F_-^2} \right] \right\} \begin{pmatrix} q_1 \\ \\ q_2 \end{pmatrix}$$

$$= \frac{1}{\rho_0 (F_+^2 - F_-^2)} \left\{ \left[\frac{1}{\omega^2 - K^2 F_+^2} - \frac{1}{\omega^2 - K^2 F_-^2} \right] \begin{pmatrix} V_A^2 + V_S^2 \sin^2\theta & -V_S^2 \sin\theta\cos\theta \\ \\ -V_S^2 \sin\theta\cos\theta & V_S^2 \sin^2\theta \end{pmatrix} - \underset{\sim}{I} \left[\frac{F_+^2}{\omega^2 - K^2 F_+^2} - \frac{F_-^2}{\omega^2 - K^2 F_-^2} \right] \right\} \begin{pmatrix} q_1 \\ \\ q_2 \end{pmatrix}$$

where $\underset{\sim}{I}$ is the unit matrix $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

Then, defining

$$\underset{\sim}{M}_{\pm}(\theta) \equiv \begin{pmatrix} V_A^2 + V_S^2 \sin^2\theta - F_\pm^2 & -V_S^2 \sin\theta\cos\theta \\ \\ -V_S^2 \sin\theta\cos\theta & V_S^2 \sin^2\theta - F_\pm^2 \end{pmatrix} \tag{36}$$

we have, in an obvious notation,

$$\vec{v}(\mathbb{K}, \omega) = \frac{1}{\rho_0 (F_+^2 - F_-^2)} \left[\frac{\underset{\sim}{M}_+}{\omega^2 - K^2 F_+^2} - \frac{\underset{\sim}{M}_-}{\omega^2 - K^2 F_-^2} \right] \cdot \vec{q}(\mathbb{K}, \omega)$$

$$= \frac{1}{\rho_0 (F_+^2 - F_-^2)} \left[\frac{1}{2\omega F_+} \left(\frac{\underset{\sim}{M}_+}{K + \frac{\omega}{F_+}} - \frac{\underset{\sim}{M}_+}{K - \frac{\omega}{F_+}} \right) - \frac{1}{2\omega F_-} \left(\frac{\underset{\sim}{M}_-}{K + \frac{\omega}{F_-}} - \frac{\underset{\sim}{M}_-}{K - \frac{\omega}{F_-}} \right) \cdot \vec{q}(\mathbb{K}, \omega) \right. \tag{37}$$

Our mobile coordinate system has aided in a very convenient decomposition of the amplitude into 'modes', but, before we can transform equation 37 back into physical space we must express it in terms of fixed axes.

In equation 29, picking out the first terms on the right hand sides (which refer to that part of the total amplitude due to the F_+ modes), we have

$$\begin{pmatrix} V_x \\ V_y \\ V_z \end{pmatrix} = \begin{pmatrix} 0 & \cos\phi \\ 0 & -\sin\phi \\ 1 & 0 \end{pmatrix} \begin{pmatrix} V_1 \\ V_2 \end{pmatrix}$$

and

$$\begin{pmatrix} q_1 \\ q_2 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ \cos\phi & \sin\phi & 0 \end{pmatrix} \begin{pmatrix} q_x \\ q_y \\ q_z \end{pmatrix}$$

Consequently, from equation 37, we may define

$$\underset{\sim}{H}_{\pm}(\theta,\phi) \equiv \begin{pmatrix} 0 & \cos\phi \\ 0 & -\sin\phi \\ 0 & 0 \end{pmatrix} \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix}_{\pm} \begin{pmatrix} 0 & 0 & 1 \\ \cos\phi & -\sin\phi & 0 \end{pmatrix} = \begin{pmatrix} M_{22}\cos^2\phi & M_{22}\cos\phi\sin\phi & M_{21}\cos\phi \\ M_{22}\cos\phi\sin\phi & M_{22}\sin^2\phi & M_{21}\sin\phi \\ M_{12}\cos\phi & M_{12}\sin\phi & M_{11} \end{pmatrix}_{\pm}$$

$$= \begin{pmatrix} (V_S^2\sin^2\theta - F_\pm^2)\cos^2\phi & (F_\pm^2 - V_S^2\sin^2\phi)\sin\phi\cos\phi & -V_S^2\sin\theta\cos\theta\cos\phi \\ (F_\pm^2 - V_S^2\sin^2\theta)\sin\phi\cos\phi & (V_S^2\sin^2\theta - F_\pm^2)\sin^2\phi & V_S^2\sin\theta\cos\theta\sin\phi \\ -V_S^2\sin\theta\cos\theta\cos\phi & V_S^2\sin\theta\cos\theta\sin\phi & V_A^2 + V_S^2\sin^2\theta - F_\pm^2 \end{pmatrix}$$

so that

$$\vec{v}(\mathbb{K},\omega) = \frac{1}{\rho_0(F_+^2 - F_-^2)}\left[\frac{1}{2\omega F_+}\left(\frac{\underset{\sim}{H}_+}{\kappa + \frac{\omega}{F_+}} - \frac{\underset{\sim}{H}_+}{\kappa - \frac{\omega}{F_+}}\right) - \frac{1}{2\omega F_-}\left(\frac{\underset{\sim}{H}_-}{\kappa + \frac{\omega}{F_-}} - \frac{\underset{\sim}{H}_-}{\kappa - \frac{\omega}{F_-}}\right)\right] \cdot \vec{q}(\mathbb{K},\omega)$$

$$= \vec{a}_+ - \vec{a}_- \quad \text{, say} \tag{39}$$

so,

$$\mathbb{V}_{\pm}(\mathbb{r},t) = \frac{1}{4\pi^2}\int_{-\infty}^{\infty} d\omega \int_{(\text{Hemisphere})} d\Omega \int_{-\infty}^{\infty} \kappa^2 d\kappa \, (\vec{a}_+ - \vec{a}_-)\, e^{i(\mathbb{K}\cdot\mathbb{r} - \omega t)} \quad \text{(Exact formal solution)} \tag{40}$$

where we have introduced polar coordinates in $\mathbb{K}$ -space and we integrate first radially on κ , then over angles, and finally over ω . We adopt the convention here of carrying the angular integration over a <u>hemisphere</u> $(\cos\angle(\mathbb{K}\cdot\mathbb{r}) \geq 0)$, while the radius κ is allowed to assume both signs.

An exact evaluation of equation 40 would certainly be very difficult. Here, we limit ourselves to seeking only the <u>asymptotic</u> form of equation

40, for large r , which makes things much simpler. We shall find that the dominant terms fall off as $\frac{1}{r}$, and we shall discard higher reciprocal powers of the radius. We now proceed step by step to an asymptotic evaluation of equation 40, where the stationary-phase approximation will be used for doing the angular part of the integral.

5. Radial Part of the Inversion Integral. Selection of Outgoing Waves.

Consider the radial integrals in $\mathbb{K}$ -space

$$\vec{I}_{\pm}(\Omega, r, \omega) \equiv \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \kappa^2 \vec{a}_{\pm}(\kappa, \Omega, \omega) e^{i\kappa\mu r} d\kappa \tag{41}$$

where $\mu = \cos\angle(\mathbb{K}, \mathbb{r})$ and Ω denotes the angular coordinates (θ, ϕ) of $\mathbb{K}$. These integrals are to be interpreted in the sense of appropriate principal values, as we whall see presently. Substituting equation 39, equation 41 becomes

$$\vec{I}(\Omega, r, \omega) = \frac{1}{\sqrt{2\pi}} \frac{1}{\rho_0(F_+^2 - F_-^2)} \frac{\underset{\sim}{H}_{\pm} \cdot q(\mathbb{K}, \omega)}{2\omega F_{\pm}} \left\{ \int_{-\infty}^{\infty} \frac{\kappa^2 e^{i\kappa\mu r} d\kappa}{\kappa + \frac{\omega}{F_{\pm}}} - \int_{-\infty}^{\infty} \frac{\kappa^2 e^{i\kappa\mu r} d\kappa}{\kappa - \frac{\omega}{F_{\pm}}} \right\} \tag{42}$$

To see how to handle the poles, we go back to equation 23 and introduce a small amount of dissipation, which will displace the poles off the real axis. Now equation 23 states (interpreting $\mathbb{Q}$ as the time derivative of a body force) that

$$\overrightarrow{\text{Force}} = \rho_0 \dot{\mathbb{V}} + \text{gradient terms}.$$

The simplest sort of dissipation to introduce is a resistance to motion proportional to the velocity, so that

$$\overrightarrow{\text{Force}} = \rho_0 \left(\dot{\mathbb{V}} + 2\delta \cdot \mathbb{V} \right) + \text{gradient terms}.$$

Hence

$$\mathbb{Q} = \rho_0 \left(\ddot{\mathbb{V}} + 2\delta \cdot \dot{\mathbb{V}} \right) \quad + \text{ gradient terms, so that}$$

$$\vec{q} = \rho_0 \left[(-i\omega)^2 + 2\delta \cdot (-i\omega) \right] \vec{v} + \ldots\ldots$$

$$= -\rho_0 \left(\omega + i\delta \right)^2 \vec{v} + \ldots\ldots + \text{ gradient terms (on neglecting}$$

a term in δ^2). We see that the effect of a slight amount of dissipation is to replace ω by $\omega + i\delta$, which displaces the poles on the real axis at $\kappa = \pm \frac{\omega}{F}$ to $\pm \frac{\omega + i\delta}{F}$. Thus the indented contour of integration for equation 42 is as shown in fig. 12. So we have now determined how to evaluate the principal values of our radial integrals.

By our convention, stated earlier, of carrying the angular part of the $\mathbb{K}$-space integrations only over a hemisphere (while κ takes on both signs), we may take this hemisphere as the one for which $\mu = \cos \angle(\mathbb{k} \cdot \mathbb{r}) > 0$ and thus close our rectilinear contour by a large semicircle in the upper half κ-plane. Then it is always just the second integral on the right side of equation 42 which contributes. Thereupon ω has always the same sign as κ, and so we see that our choice of contour shown in fig. 12 has assured us of purely outgoing waves, which is required. (This would also be the case if we chose the hemisphere $\mu < 0$).

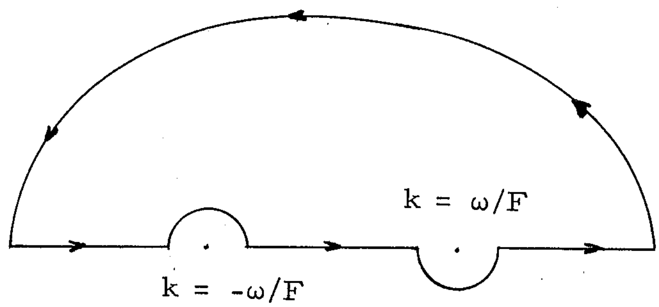


Figure 12

We may now immediately evaluate equation 42 to obtain

$$\vec{I}(\Omega, r; \omega) = \frac{1}{\sqrt{2\pi}} \frac{i\pi}{2\rho_0(F_+^2 - F_-^2)} \frac{\underset{\sim}{H}_\pm \cdot \vec{q}(\mathbb{K}, \omega)}{F_\pm^3} \omega e^{i\frac{\omega\mu r}{F_\pm}} \tag{43}$$

6. Association of Directions in $\mathbb{K}$ -space and Physical Space. The Critical Cone.

For large $\frac{\omega r}{F_\pm}$, the phase factor in $\vec{I}(\Omega, r; \omega)$ varies rapidly with angle, so the angular integral may be asymptotically evaluated by the method of stationary phase. In this procedure we must know the stationary points of the phase factor $\frac{\omega\mu r}{F_\pm}$, that is, the values θ_0, ϕ_0 about which this expression is stationary under small variations of θ, ϕ. We have already answered this question in the discussion leading up to equations 18 and 20. For $\mathbb{K}$ restricted to end on one of the surfaces $K = \frac{\omega}{F_\pm(\theta)}$, which is always required,

$$\mathbb{K} \cdot \mathbb{r} \equiv \frac{\omega\mu r}{F_\pm}$$

and in the earlier discussion we have seen that, if $\mathbb{r}$ has the polar angles ψ, h, the angular coordinates θ_0, ϕ_0 of $\mathbb{K}$ for stationary $\frac{\mu r}{F}$ are given by

$$\tan(\theta_0 \mp \psi) = -\frac{F_\pm'(\theta_0)}{F_\pm(\theta_0)} \tag{44}$$

and

$$\phi_0 = h \; (F_+ \text{ case}) \qquad \phi_0 = \pi - h \; (F_- \text{ case}) \tag{45}$$

where θ and ψ are always measured positively from the direction of $\mathbb{B}_0$. In this way, with a given direction $\mathbb{r}(\psi, h)$ in the physical space is associated, via equations 44 and 45, one or more directions $\mathbb{K}(\theta_0, \phi_0)$ in $\mathbb{K}$ -space, such that the dominant contribution to the radiation in the far-zone field in the direction ψ, h comes from those portions of the $F_\pm$ surfaces in the immediate neighborhood of θ_o, ϕ_o. It is further easily seen (as shown in the next section) that, at these stationary points, the normal to the $F_\pm$ surface is parallel to $\mathbb{r}$. For the surface there are two such points or none, depending on whether ψ is less or greater than a critical value given in terms of V_A/V_S; for the F_+ surface there is always just one. The contributions of all contributing points must be summed, to obtain the field strength.

It is easily shown that this connection between directions in $\mathbb{K}$-space and in physical space has also the property that the group velocity

$$\mathbb{V}_g = \nabla_{\mathbb{K}} \omega = \hat{e}_\kappa F(\theta_o) + \hat{e}_\theta F'(\theta_o) \tag{46}$$

for a train of plane waves with wave numbers around $\mathbb{K}$, is in the direction of $\mathbb{r}(\psi)$, where ψ is the associated colatitude angle in physical space, and has the magnitude $\frac{F_\pm}{\mu}$, the speed of propagation of a $F_\pm$ phase or wave front in the direction ψ. A further property of this connection between $\mathbb{K}$-space and physical space is that, at a stationary point θ_o, ϕ_o, the normal to a $F_\pm$ surface is in the direction of the associated ray $\mathbb{r}(\psi)$ in physical space. This is made clear by the following construction, which illustrates the statement for the F_- mode.

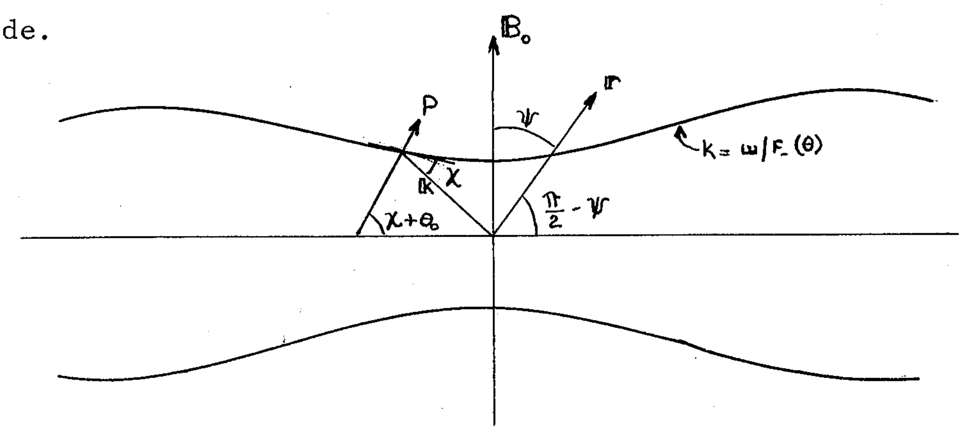


Figure 13

In the above figure, which illustrates the association of directions for the F_- mode*, P is one of the stationary points on the F_- surface which contributes to the radiation at large distances in the direction $\mathbf{r}(\psi)$ as we have already seen, $\phi - h = \pi$ here so that $\mathbf{k}$ lies in the plane of $\mathbf{B}_o$ and $\mathbf{r}$. Further,

$$\mathbf{k}\cdot\mathbf{r} = \frac{\omega\mu r}{F_-(\theta_o)} = \omega r \frac{\cos(\theta_o+\psi)}{F_-(\theta_o)}$$

and

$$\tan(\theta_o+\psi) = -\frac{F_-'(\theta_o)}{F_-(\theta_o)}$$

The radius $K = \overline{OP}$ is proportional to $\dfrac{1}{F_-(\theta)}$. Then, by geometry,

$$\cos\chi = \frac{1}{K}\frac{\partial K}{\partial\theta} = \tan(\theta_o+\psi)$$

so that

$$\theta_o+\psi = \frac{\pi}{2} - \chi$$

establishing that $\overline{QP}$ is parallel to $\mathbf{r}$. So the normal to an F_- surface at a stationary point lies in the associated direction in physical space.

We will now see that there is a critical angle ψ_{max}, (depending on V_S/V_A alone) such that, for $\psi > \psi_{max}$ there is no contribution from the F_- surface (no stationary point on it), meaning that the F_- mode radiates only into a limited cone in physical space.... a "critical cone" whose half-angle is ψ_{max}. This radiation is characterized by the disjoint wave-fronts already discussed, which remain always touching this cone as they expand.

The critical cone angle ψ_{max}, being an extremum, is defined from

$$\frac{\partial\psi}{\partial\theta} = 0 \tag{47}$$

---

* A similar construction can obviously be made for the F_+ mode.

Since

$$\psi = \theta_0 + \tan^{-1}\frac{F'(\theta_0)}{F(\theta_0)} \tag{48}$$

the condition becomes

$$\frac{F}{F^2+F'^2}\left(F''+F\right) = 0 \tag{49}$$

In expression 49 the first factor is everywhere positive, for both modes. Equation 49 has no solution for the F_+ mode, for which

$$F_+'' + F_+(\theta) > 0$$

but it does possess a solution for the F_- mode, for which $(F_-'' + F_-)$ can change sign. Therefore the critical cone-half-angle ψ_{max} is defined by equation 48, together with the condition

$$F_-''(\theta_m) + F_-(\theta_m) = 0 \tag{50}$$

It is easily shown that equation 50 is just the condition defining the inflection point on the F_- surface. We see that the F_+ mode, whose associated surface has no inflection point, radiates in all directions, as stated. Clearly, then, there is always just one stationary point on the F_+ surface, and zero or two stationary points on the F_- surface, according as $\psi > \psi_{max}$ or $\psi < \psi_{max}$. It is easily shown from equations 50, 8, and 48 that

$$\text{as} \quad \frac{V_A}{V_S} \to 0 \quad \text{or} \quad \infty\,, \qquad \psi_m \to 0$$

$$\text{as} \quad \frac{V_A}{V_S} \to 1\,, \qquad \psi_m \to 45^\circ$$

7. The Angular Integral. Stationary Phase Approximation.

We return now to equation 40 and carry out the angular integral in $\mathbb{K}$-space by use of the stationary-phase method. It is well known that any integral of the form

$$J(m) = \int_a^b g(x)\, e^{imf(x)}\, dx \tag{51}$$

with a, b, m, f, g real, f and g sufficiently regular, and f possessed of a single stationary point x_o, for which $f'(x) = 0$, in the interval, has the asymptotic form

$$J(m) \sim \sqrt{\frac{2\pi}{|m f''(x_o)|}}\; g(x_o) \exp\left[im f(x_o) + i\frac{\pi}{4}\,\mathrm{sgn}\left(m f''(x_o)\right)\right] + O\left(\frac{1}{m}\right) \tag{52}$$

This is easily derived from the lowest order expansion

$$f(x) = f(x_o) + \frac{(x-x_o)^2}{2} f''(x_o) + \cdots\cdots$$

In our case the phase factor $\mu r/F$ is a function of the two polar angles of $\mathbb{K}$ and we have a double integral. The lowest order expansion of the phase factor is now given from

$$\frac{\mu(\theta,\phi)}{F(\theta)} = \left(\frac{\mu}{F}\right)_o + \frac{1}{2}\left[\frac{\mu_{\phi\phi}}{F}\right]_o (\phi-\phi_o)^2 + \frac{1}{2}\left[\left(\frac{\mu}{F}\right)_{\theta\theta}\right]_o (\theta-\theta_o)^2$$

the mixed 2nd partial derivative vanishing at a stationary point. Let the monochromic intensity at r contributed by a single stationary point be $\vec{I}_\pm(\omega, \mathbb{r})$. This is given by a double angular integral which is just a product of one-dimensional integrals and we obtain

$$\vec{I}_\pm(\omega,\mathbb{r}) = \frac{1}{2\pi}\int d\phi \int \sin\theta\, d\theta\, \vec{I}_\pm(\Omega, r; \omega) \sim \vec{I}_\pm(\Omega_o, \omega, r)\cdot \mathfrak{G}(\Omega_o) \tag{53}$$

where

$$\mathfrak{G}(\Omega_o) = \frac{\sin\theta_o}{r}\,\frac{1}{\sqrt{\left|\left\{\omega\left(\frac{\mu}{F}\right)_{\theta\theta}\right\}_o\right|\left|\left(\frac{\omega\mu_{\phi\phi}}{F}\right)_o\right|}}\,\epsilon \qquad \text{where} \qquad \epsilon = \begin{Bmatrix} i \\ 1 \\ -i \end{Bmatrix} \tag{54}$$

according as $\left\{\omega\left(\frac{\mu}{F}\right)_{\theta\theta}\right\}_o$ and $\left(\frac{\omega\mu_{\phi\phi}}{F}\right)_o$ are $\begin{Bmatrix} \text{both positive} \\ \text{of opposite signs} \\ \text{both negative} \end{Bmatrix}$

Now, we easily compute that

$$\left[\left(\frac{\mu}{F_\pm}\right)_{\theta\theta}\right]_o = -\frac{F_\pm + F_\pm''}{F_\pm^2}\cos(\theta_o \mp \psi_o) = -\frac{F_\pm + F_\pm''}{F_\pm\sqrt{F_\pm^2 + F_\pm'^2}} \tag{55}$$

and

$$\left[\frac{\mu_{\phi\phi}}{F_\pm}\right]_o = \mp\frac{\sin\theta_o \sin\psi_o}{F_\pm} \tag{56}$$

Now, both θ_0 and ψ_0 lie between 0 and $\pi/2$. Therefore

$$\left[\frac{\mu_{\phi\phi}}{F}\right]_0 \begin{matrix} <0 & \text{for } F_+ \text{ mode} \\ >0 & \text{for } F_- \text{ mode} \end{matrix} \quad \text{everywhere}$$

Further,

$$F_+ + F_+''(\theta) < 0 \qquad \text{everywhere}$$

$$F_- + F_-''(\theta) > 0 \qquad \text{for } \theta > \theta_m = \text{polar angle of inflection point}$$

$$F_- + F_-''(\theta) < 0 \qquad \text{for } \theta < \theta_m$$

We must now proceed carefully to check the behavior of $\vec{I}_\pm(\theta)$ for positive and negative frequencies.

For $\omega > 0$ $\left[\left(\frac{\omega\mu}{F}\right)_{\theta\theta}\right]_0 > 0$ on F_- surface for $\theta < \theta_m$

$\left[\left(\frac{\omega\mu}{F}\right)_{\theta\theta}\right]_0 < 0$ on F_- surface for $\theta > \theta_m$, and everywhere on F_+ surface

$\left(\frac{\omega\mu_{\phi\phi}}{F}\right)_0$ <0 everywhere on F_+ surface

>0 everywhere on F_- surface

Hence, for $\omega > 0$

$\epsilon = i$ for the F_- surface at $\theta < \theta_m$

$\epsilon = 1$ for the F_- surface at $\theta > \theta_m$

$\epsilon = -i$ everywhere on the F_+ surface

Similarly, for $\omega < 0$

$\left[\left(\frac{\omega\mu}{F}\right)_{\theta\theta}\right]_0 < 0$ on F_- surface at $\theta < \theta_m$

$\left[\left(\frac{\omega\mu}{F}\right)_{\theta\theta}\right]_0 > 0$ on F_- surface for $\theta > \theta_m$, and everywhere on F_+ surface

$\left[\frac{\omega\mu_{\phi\phi}}{F}\right]_0$ >0 everywhere on F_+ surface

<0 everywhere on F_- surface

Hence, for $\omega < 0$

$\epsilon = -i$ for the F_- surface at $\theta < \theta_m$

$\epsilon = 1$ for the F_- surface at $\theta > \theta_m$

$\epsilon = i$ everywhere on the F_+ surface

Using these results in conjunction with equations 28, 43, 53, 55, and 56 yields

$$\vec{I}_{\pm}(\omega,r) = \sqrt{\frac{\pi}{2}}\,\frac{\sin\theta_0}{2r}\cdot\frac{1}{\rho_0\left(F_+^2-F_-^2\right)}\,\frac{1}{\sqrt{\frac{(F+F'')\sin\theta_0\sin\psi_0}{\sqrt{F^2+F'^2}}}}\,\frac{\underset{\sim}{H}_{\pm}(\theta_0,\phi_0)\cdot\vec{q}(\omega)}{F^2(\theta_0)}\,e^{i\omega\left(\frac{\mu}{F_0}\right)r}\begin{cases}1\\ i\,\mathrm{sgn}\,\omega\\ -1\end{cases} \tag{57}$$

for contributions from stationary points $\begin{array}{l}\theta<\theta_m \text{ on } F_- \text{ surface}\\ \theta>\theta_m \text{ on } F_- \text{ surface}\\ \text{all } \theta \text{ on } F_+ \text{ surface}\end{array}$, respectively, where F may be F_+ or F_- as desired, except as indicated, and all angular variables are evaluated at a stationary point.

Now (see figure 11), for the point source (equation 28),

$$\begin{aligned} q_1(\omega) &= q_z(\omega) = Q_z\frac{f(\omega)}{(2\pi)^{3/2}}\\ q_2(\omega) &= q_x(\omega)\cos\phi + q_y(\omega)\sin\phi = \left(Q_x\cos\phi - Q_y\sin\phi\right)\frac{f(\omega)}{(2\pi)^{3/2}} \end{aligned} \tag{58}$$

and, for the $F_\pm$ modes, $q_3(\omega)=0$

It is convenient to orient the fixed axes so that $Q_y=0$. Then, defining

$$K_{\pm}(\theta_0,\psi_0) = \frac{\sin\theta_0}{8\pi\rho_0\left(F_+^2-F_-^2\right)F^2\sqrt{\left|\frac{F+F''}{\sqrt{F^2+F'^2}}\sin\theta_0\sin\psi_0\right|}}\,\underset{\sim}{H}_{\pm}(\theta_0,\phi_0) \tag{59}$$

where the argument of the F_- functions is θ_0 and $\underset{\sim}{H}_{\pm}(\theta,\phi)$ is given by equation 38, we have, for the monochromatic intensity at long distances in the direction $\mathbb{r}(\psi,h)$:

$$\vec{I}^{(1)}_{-}(\omega;\mathbb{r}) \sim \frac{e^{i\omega\left(\frac{\mu}{F_0}\right)r}}{r}\,f(\omega)\,\underset{\sim}{K}_{-}\left(\theta_0(\psi_0),\pi-h\right)\begin{Bmatrix}Q_z\\ -Q_x\cosh\\ 0\end{Bmatrix} \tag{60a}$$

for contributions from $\theta<\theta_m$ on F_- surface

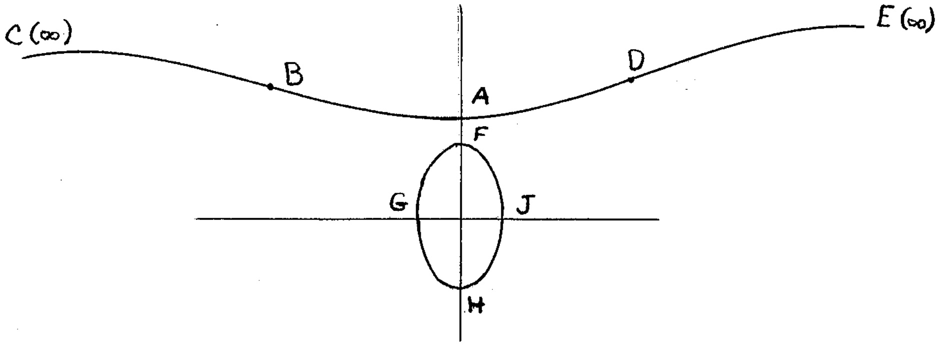
$$\vec{I}_{+}(\omega;\mathbb{r}) \sim -\frac{e^{i\omega\left(\frac{\mu}{F}\right)r}}{r}\,f(\omega)\,\underset{\sim}{K}_{+}\left(\theta_0(\psi_0),h\right)\begin{Bmatrix}Q_z\\ Q_x\cosh\\ 0\end{Bmatrix} \tag{60b}$$

for contributions from <u>any</u> part of the F_+ surface

$$I^{(2)}_{-}(\omega;\mathbb{r}) \sim \frac{e^{i\omega\left(\frac{\mu}{F}\right)r}}{r}\,i\,f(\omega)\,\mathrm{sgn}(\omega)\,\underset{\sim}{K}\left(\theta_0(\psi_0),h\right)\begin{Bmatrix}Q_z\\ -Q_x\cosh\\ 0\end{Bmatrix} \tag{60c}$$

for contributions from $\theta>\theta_m$ on F_- surface and where $\sim$ means "asymptotically approaches, for large r " and $\theta_0\equiv\theta_0(\psi_0)$.

The correspondence between points on the $F_\pm$ mode wave-fronts and the contributing stationary points on the $F_\pm$ surfaces are as shown below.



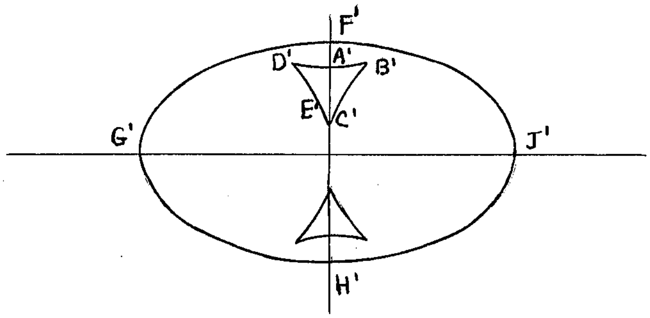


Figure 14 -- showing corresponding points on " F surfaces" and wave fronts.

(The correspondence for the lower parts of the F-surfaces is found by symmetry.)

We note from equations 60 that the tri-cusped phase surfaces, despite the fact that they were derived as envelopes of equi-phase planes, are not single equi-phase surfaces, but pairs of these in contact, there being a 90° phase jump at the corners, corresponding to $\theta = \theta_m$, where the Gaussian curvature of the F surface changes sign. Also, because of the peculiar $(i \,\mathrm{sgn}\,\omega)$ factor in equation 60c, there is a 180° phase jump in going from negative to positive frequencies in that expression. This phase shift (whose discontinuous nature is a property of the stationary phase approximation) will shortly lead us to an interesting consequence--the invalidity of Huyghens' Principle for part of one of the wave fronts.

Further, there is a 180° phase difference between the ovaloid phase front and the cap of the tri-cusped phase fronts lying within it. As $V_S/V_A \to 1$, the caps approach the adjacent portion of the ovaloid front and coalesce, with destructive interference, producing the dumb-bell-shaped radiation pattern shown in fig. 4.

Equations 60a - c give, for a given frequency component of a concentrated driving force, the contributions to the far-zone velocity field $\mathbb{V}(\mathbf{r}, t)$ from the various possible stationary points on the F_+, F_- surfaces. To obtain the total field from these two modes one must sum over these stationary points corresponding to the observation-direction of interest. Our results are the leading terms in asymptotic expansions in $\frac{1}{r}$, the rest of the field falling off with higher inverse powers of r provided that certain conditions are met.

The conditions for expressions 60 to give the dominant contributions for large r are that, in the Taylor expansion of $\omega r\left(\frac{\mu}{F}\right)$ about a stationary point, the third-order terms shall be small compared with the second-order terms even when the latter are a moderate multiple of 2π. These conditions require (see, e.g. reference 18 , pp. 395-396)

$$\left.\begin{aligned} \omega r\left[\left(\frac{\mu}{F}\right)_{\theta\theta\theta}\right]_0 &\ll \left[\omega r\left(\frac{\mu}{F}\right)_{\theta\theta}\right]_0^{3/2} \\ \omega r\left[\frac{\mu_{\phi\phi\phi}}{F}\right]_0 &\ll \left[\omega r\,\frac{\mu_{\phi\phi}}{F}\right]_0^{3/2} \end{aligned}\right\}\text{for } F_+ \text{ and } F_- \qquad (61)$$

The derivatives are all bounded except in the neighborhood of the inflection point on the E surface, where $\left(\frac{\mu}{F}\right)_{\theta\theta}$ vanishes. Over the whole radiation pattern except for the E mode very near the critical cone $\psi = \psi_m$ (corresponding to the invlection point), the condition for the validity of the preceding inequalities is

$$\frac{\omega r}{F} \gg 1 \tag{62}$$

which states that, for a given observation distance r, ω shall be large enough so that the observation point lies in the wave zone for the source. This condition is assumed in most of the present work, and our stationary-phase amplitude formulae give the dominant contributions only when it is fulfilled. This condition creates some limitations in our discussion of aperiodic disturbances, as we shall shortly see. Equations 60 become invalid in the close neighborhoods of the cusps touching the critical cone, at which angles equation 59 becomes singular. The amplitude formulae break down completely at these points, giving an infinite amplitude. This is a frequently met property of stationary-phase approximations, occurring notably in various diffraction problems such as the theory of the rainbow, diffraction fields near caustics, etc. We could obtain expressions valid near the cusps by carrying the approximation to one higher order, obtaining amplitude formulae differing from equations 60 in that the θ-dependence would involve the so-called "Airy integral". This will not be done here, but no difficulty is involved.

The extension of these results, by superposition, to expressions for the far-zone radiation from a finite distribution of applied body forces, in terms of the spectral analysis $\mathbb{F}_\omega(r')$ of these forces, can be written down at once from equations 60, in exactly the same way in which the far-zone radiation formulae for extended current distributions in electrodynamics are derived from the point source solutions.

8. Concentrated Driving Force with Aperiodic Time Variation.

We come now to the question of a driving source with arbitrary time dependence. To find the far-zone response, we perform the final ω -integration on equations 60 obtaining the velocity from

$$\mathbb{V}(\mathbb{r}, t) = \int_{-\infty}^{\infty} e^{-i\omega t} \vec{I}_{\pm}(\omega, \mathbb{r})\, d\omega \tag{63}$$

For cases 60a and 60b we obtain the "retarded potential"formulae

$$\mathbb{V}_{-}^{(1)}(\mathbb{r}, t) = \underset{\sim}{K}_{-}\left(\theta_0(\varphi_0),\ \pi - h\right)\cdot\begin{pmatrix} Q_z \\ -Q_x \cosh \\ 0 \end{pmatrix} \mathcal{F}\left[t - \left(\frac{\mu}{F}\right)_0 r\right] \tag{64a}$$

and

$$\mathbb{V}(\mathbb{r}, t) = -\underset{\sim}{K}_{+}\left(\theta_0(\varphi_0),\ \pi - h\right)\cdot\begin{pmatrix} Q_z \\ Q_x \cosh \\ 0 \end{pmatrix} \mathcal{F}\left[t - \left(\frac{\mu}{F}\right)_0 r\right] \tag{64b}$$

indicating that apparently the F_+ mode wave front and the caps of the wave fronts (portions D'A'B' and the corresponding lower part, in figure 14) propagate according to Huyghens' Principle, i.e. a pulse source gives a pulse wave-front shape, and there are no precursors or "post cursors". This is not true, however, for case 60c, on account of the $(i \operatorname{sgn} \omega)$factor, arising from the negative Gaussian curvature of the corresponding portion of the F_- surfaces.

We may easily evaluate

$$P\int_{-\infty}^{\infty} e^{-i\omega t} \operatorname{sgn} d\omega = \frac{1}{\pi t} \tag{65}$$

which gives us, from case 60c the formula

$$\mathbb{V}_{-}^{(2)}(\mathbb{r}, t) = \underset{\sim}{K}_{-}\left(\theta_0(\varphi_0),\ \pi - h\right)\begin{pmatrix} Q_z \\ -Q_x \cosh \\ 0 \end{pmatrix} \mathcal{F}^{+}\left(t - \left(\frac{\mu}{F}\right)_0 r\right) \tag{64c}$$

where

$$\overset{+}{\mathcal{F}}(\tau) \doteq \frac{1}{\pi} P\int_{-\infty}^{\infty} \frac{\mathcal{F}(\tau)}{t-\tau}\, d\tau \tag{66}$$

(principal value)

where we have used equation 65 and the Convolution Theorem on equation 60c. In other words, the back portions (portions D'E'B', in figure 14) of the $\underline{E}$ wave front do not, apparently, propagate according to Huyghens' Principle. We have a "retarded potential" expression for this part of the $\underline{E}$ front, but it is given in terms of an "effective source" (equation 66), which depends on the entire history of the actual source. This is most assuredly not correct, and is traceable to the breakdown of the stationary phase approximation. This approximation, we recall, is, at a given observation point, accurate only for frequencies sufficiently high so that this observation point lies in the wave-zone of the source. However, in the ω -integration, we carried the integral over the entire infinite range of all positive and negative frequencies, traversing the region where the stationary-phase approximation is inaccurate. This reflects itself in the discontinuous phase-jump of 180° in passing through zero frequency.

The mathematics is, presumably, inaccurately expressing the fact that this part of the wave front does, indeed, not propagate "sharply", and that the amplitude on it, at a given instant, depends on what the source was doing over a (presumably finite) interval of time. This is just saying that the energy from the point source gets to a given point of the retrograde portion of the $\underline{E}$ front by different "routes", taking different "travel times" through the anisotropic medium in so doing.

If we evaluate equation 66 for $\mathcal{F}(t) = \delta(t)$, corresponding to a driving force with time dependence $\delta(t)$, we obtain

$$\mathcal{F}^{+}(\tau) = \frac{1}{\pi} P \int_{-\infty}^{\infty} \frac{\dot{\delta}(t)}{t-\tau} = -\frac{1}{\pi t^2}$$

so that equation 64c would involve the time dependence

$$\frac{1}{\left(t - \left(\frac{\mu}{F}\right)_0 r\right)^2}$$

giving a wave front with precursor and postcursor.* Since the delta function is very "rich" in low frequencies, we may expect the response to be rather poorly reproduced by the stationary-phase approximation.

The discontinuous behavior of the phase at zero frequency, as given by our approximation, seems to be largely responsible for the exaggeratedly anomalous behavior we obtain for this disturbance front. Giving an electric circuit theory analogy we are, so to speak, passing a signal through a "filter" with a non-physical, discontinuous phase shift. It is well known that non-physical "premonitory" responses then result.

Suppose we arbitrarily replace our discontinuous phase shift by one which passes smoothly thru a 180° phase alteration in a finite frequency interval about $\omega = 0$. Specifically, let us arbitrarily replace

$$i \operatorname{sgn} \omega \qquad \text{by} \qquad \frac{2i}{\pi} \tan^{-1} \frac{\omega}{\omega_0}$$

It is an easy exercise in contour integration to show that (in the sense of a Cauchy limit),

$$\frac{2i}{\pi} \int_{-\infty}^{\infty} \tan^{-1} \frac{\omega}{\omega_0}\, e^{i\omega t} = \frac{2}{t}\, e^{-\omega_0 |t|}$$

On doing the integral $\frac{2i}{\pi} \int_{-\infty}^{\infty} e^{-i\omega t} \tan^{-1} \frac{\omega}{\omega_0} f(\omega)\, d\omega$ we then obtain an "effective source"

*The time derivative of the delta function occurs because $Q = \frac{\partial \mathbb{F}}{\partial t}$.

$$\overset{+}{\mathcal{F}}(\tau) = 2P\int_{-\infty}^{\infty} \frac{\mathcal{F}(\tau)\, e^{-\omega_0 |t-\tau|}}{|t-\tau|}\, d\tau \qquad (67)$$

in place of equation 66, showing that smoothing out the phase jump diminishes heavily the weighting of the past and future of the source. Further information relevant to finding what an exact solution would give, in place of equation 66, could be found by studying the domain of dependence for the solutions of our wave equation. Equation 66 should give a reasonably accurate representation of the amplitude when $\mathcal{F}(t)$ is not "too rich" in frequencies so low that criterion 62 fails to be satisfied.

9. The Alfvén Mode.

We come now to the problem of finding the part of the total velocity field which is due to the Alfven mode. As already indicated after equation 29, we can give a separate treatment to this partial amplitude. Our solution here will be valid for all distances and, by itself, is the exact solution for the waves excited in an incompressible perfectly conducting fluid.

Referring to the Alfvén amplitude in equation 29, but suppressing the superscript, we have

$$v_x = v_3 \sin\phi \;; \qquad v_y = v_3 \cos\phi \qquad (68)$$

while (see figure 11),

$$q_3 = -q_x \sin\phi + q_y \cos\phi \qquad (69)$$

We may write equation 32 as

$$\left(k_z^2 - \frac{\omega^2}{V_A^2}\right) v_3 = \frac{q_3}{\rho_0} \qquad (70)$$

where v_3 and q_3 are functions of k_z, $k_\perp$ and ω.

Now, for our point source, as we saw in equation 28

$$\vec{q}(\omega) = \frac{1}{(2\pi)^{3/2}} \vec{Q} f(\omega)$$

so that

$$q_3 = \frac{f(\omega)}{(2\pi)^{3/2}} \left[Q_y \cos\phi - Q_x \sin\phi \right] \tag{71}$$

As before, we choose the fixed axes so that $Q_y = 0$. Then we have

$$\left(K_z^2 - \frac{\omega^2}{V_A^2}\right) V_x = -\frac{q}{\rho_0} \sin\phi = \frac{f(\omega)}{(2\pi)^{3/2}} \frac{Q_x \sin^2\phi}{\rho_0} \tag{72}$$

and

$$\left(K^2 - \frac{\omega^2}{V_A^2}\right) V_y = \frac{q_3}{\rho_0} \cos\phi = -\frac{f(\omega)}{(2\pi)^{3/2}} Q_x \sin\phi \cos\phi \tag{73}$$

Introduce the two-dimensional transforms $\tilde{V}_x(K_x, K_y; z, t)$, $\tilde{V}_y(K_x, K_y; z, t)$

such that

$$\tilde{V}_x(K_x, K_y; z, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{i(K_z z - \omega t)} V_x(\mathbb{K}, \omega)\, dK_z\, d\omega$$

$$\tilde{V}_y(K_x, K_y; z, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{i(K_z z - \omega t)} V_y(\mathbb{K}, \omega)\, dK_z\, d\omega \tag{74}$$

so that

$$V_x(x, y, z, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{i(K_x x + K_y y)} \tilde{V}_x(K_x, K_y; z, t)\, dK_x\, dK_y$$

and

$$V_y(x, y, z, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{i(K_x x + K_y y)} \tilde{V}_y(K_x, K_y; z, t)\, dK_x\, dK_y \tag{74'}$$

Equations 72 and 73 are then equivalent to

$$\left(\frac{1}{V_A^2}\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial z^2}\right) \tilde{V}_x = \frac{\mathcal{F}(t)}{2\pi\rho_0} \delta(z) Q_x \sin^2\phi \tag{75}$$

and

$$\left(\frac{1}{V_A^2}\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial z^2}\right) \tilde{V}_y = \frac{-\mathcal{F}(t)}{2\pi\rho_0} \delta(z) Q_x \sin\phi\cos\phi \tag{76}$$

Now,

$$\sin\phi = \frac{K_y}{\sqrt{K_x^2 + K_y^2}} = \frac{K_y}{K_\perp} \qquad \cos\phi = \frac{K_x}{K_\perp}$$

Therefore, we have the following 1-dimensional wave equations for the components of the Alfvén amplitude

$$\left(\frac{\partial^2}{\partial t^2} - V_A^{\,2}\frac{\partial^2}{\partial z^2}\right)V_x = R(x,y)\,\delta(z)\,\mathcal{F}(t)\,Q_x \tag{77}$$

and

$$\left(\frac{\partial^2}{\partial t^2} - V_A^{\,2}\frac{\partial^2}{\partial z^2}\right)V_y = S(x,y)\,\delta(z)\,\mathcal{F}(t)\,Q_x \tag{78}$$

where

$$R(x,y) = \frac{1}{(2\pi)^2}\iint dk_x\,dk_y\,\frac{k_y k_y}{k_\perp^2}\,e^{i(xk_x+yk_y)} = \frac{-1}{(2\pi)^2}\left(\frac{\partial^2}{\partial y^2}\iint\frac{dk_x\,dk_y}{k_\perp^2}\,e^{i\vec{k}_\perp\cdot\vec{\rho}}\right) \tag{79}$$

and

$$S(x,y) = -\frac{1}{(2\pi)^2}\iint dk_x\,dk_y\,\frac{k_x k_y}{k_\perp^2}\,e^{i(xk_x+yk_y)} = \frac{1}{(2\pi)^2}\left(\frac{\partial^2}{\partial x\,\partial y}\iint\frac{dk_x\,dk_y}{k_\perp^2}\,e^{i\vec{k}\cdot\vec{\rho}}\right) \tag{80}$$

where $\vec{\rho} = x\hat{e}_x + y\hat{e}_y$ and $\vec{k}_\perp = k_x\hat{e}_x + k_y\hat{e}_y$.

We recognize that

$$-\frac{1}{(2\pi)^2}\iint\frac{dk_x\,dk_y}{k_\perp^2}\,e^{i\vec{k}_\perp\cdot\vec{\rho}} = \frac{1}{2\pi}\log\rho \quad \text{, except for an infinite constant} \tag{81}$$

the left hand side being just a formal Fourier integral representation for the fundamental solution of the two-dimensional Laplace equation, appearing on the right hand side. Therefore

$$R(x,y) = 2\pi\,\frac{y^2-x^2}{\rho^4} \tag{82}$$

and

$$S(x,y) = -2\pi\,\frac{2xy}{\rho^4} \tag{83}$$

The well known solutions to equations 77 and 78 are

$$V_x(z,t) = \frac{\pi Q_x}{V_A}\,\frac{y^2-x^2}{\rho^4}\,\Phi\!\left(t-\frac{|z|}{V_A}\right) \tag{84}$$

and

$$V_y(z,t) = -\frac{\pi Q_x}{V_A}\,\frac{xy}{\rho^4}\,\Phi\!\left(t-\frac{|z|}{V_A}\right) \tag{85}$$

where

$$\Phi(\tau) = \int_{-\infty}^{\tau}\mathcal{F}(t)\,dt = F_x(\tau) \tag{86}$$

the x -component of the actual body force, since $\mathbb{Q} = \dot{\mathbb{F}}$ as we recall. So the Alfvén amplitude, unlike waves on a string, propagates according to Huyghens' Principle, parallel and anti-parallel to $\mathbb{B}_0$.

We notice at once a very peculiar feature of the solution. Even though the actual source is concentrated at a point, yet the "effective source" for the Alfven mode part of the amplitude is distributed over the entire plane $z = 0$!*,** This anomaly is, however, of purely formal origin, and is easily explained physically as a consequence of the idealizations made in setting up our wave equation ... chiefly, the assumption of infinite conductivity and, much less importantly, the neglect of displacement current.

For the Alfvén mode, $V_z = b_z = 0$. Since $\nabla \cdot \mathbb{B} = 0$, we have

$$\frac{\partial b_x}{\partial x} + \frac{\partial b_y}{\partial y} = 0$$

Because of the perfect conductivity, our idealized fluid is "locked" to the magnetic field lines. The field, in virtue of the preceding equation, must respond instantaneously, everywhere on an infinite transverse plane through the source, whenever this undergoes a change. The fluid, being locked to the field, is thereby set into transverse motion everywhere on this plane, and this plane disturbance-configuration then propagates parallel and anti-parallel to $\mathbb{B}_0$. In this mode the medium responds to a sudden change of the transverse component of the source as a two-dimensional incompressible fluid would respond to the sudden creation of a dipole within it. This feature is a good illustration of the non-physical anomalies which may be expected to follow from the assumption of perfect conductivity.

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*This "effective source" may easily be shown directly to be the divergenceless part of $\mathbb{Q}_\perp(\mathbb{r}, t)$, the transverse component of $\mathbb{Q}(\mathbb{r}, t)$.

**The "infinite transverse propagation speed" for this mode may seem the more surprising in view of the fact that the Alfven phase velocity $U_A = \frac{\omega}{K} = V_A \cos\theta$ goes to zero at $\theta = 90°$.

Much the same situation would occur in the excitation of disturbances in a finitely conducting incompressible fluid, although the magnetic field could then "move" with respect to the fluid and would readjust itself at a finite rate and one would have diffusion of eddy currents in the medium. The same anomaly as just discussed would appear, because the incompressibility would give rise to infinite transverse propagation speed.

It would be of some interest to study our problem in a finitely conducting compressible fluid. There would then be no infinite spreading velocity. The mathematical difficulties would, however, be much increased.

10. Angular Distribution of Spectral Intensity in the Radiation Field.

Let us consider a finite-sized driving source region, with an applied body force density $\mathbb{F}(\mathbf{r}, t)$ distributed over it. We shall discuss here the power radiated per unit frequency, and its angular distribution. The rate at which energy is being applied by the body forces, at time t, is

$$\frac{dW}{dt} = \int d^3 \mathbf{r}\; \mathbb{F}(\mathbf{r}, t) \cdot \mathbb{V}(\mathbf{r}, t) \qquad (86)$$

with $\mathbb{F}$ and $\mathbb{V}$ real. Introducing the Fourier analyses

$$\mathbb{F}(\mathbf{r}, t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \mathbb{F}_\omega(\mathbf{r})\, e^{i\omega t}\, d\omega$$

and

$$\mathbb{V}(\mathbf{r}, t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \mathbb{V}_\omega(\mathbf{r})\, e^{i\omega t}\, d\omega$$

with $\mathbb{F}_{-\omega} = \mathbb{F}_\omega^*$; $\mathbb{V}_{-\omega} = \mathbb{V}_\omega^*$ (reality conditions), we may write equation 86 as

$$\frac{dW}{dt} = \frac{1}{2\pi} \int d^3\mathbf{r} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} d\omega\, d\omega' \Big[\mathbb{F}_\omega(\mathbf{r}) \cdot \mathbb{V}_{\omega'}(\mathbf{r})\, e^{-i(\omega+\omega')t} + \mathbb{F}_\omega^*(\mathbf{r}) \cdot \mathbb{V}_{\omega'}^*(\mathbf{r})\, e^{i(\omega+\omega')t} + \mathbb{F}_\omega(\mathbf{r}) \cdot \mathbb{V}_{\omega'}^*(\mathbf{r})\, e^{-i(\omega-\omega')t} + \mathbb{F}_\omega^*(\mathbf{r}) \cdot \mathbb{V}_{\omega'}(\mathbf{r})\, e^{i(\omega-\omega')t} \Big]$$

Let the driving forces be active over some finite period of time. Then the total energy radiated during their action is

$$W=\int_{-\infty}^{\infty}\frac{dW}{dt}\,dt=\int d^3\mathbf{r}\int_{0}^{\infty}\left[\mathbf{F}_\omega(\mathbf{r})\cdot\mathbf{V}_\omega^*(\mathbf{r})+\mathbf{F}_\omega^*(\mathbf{r})\cdot\mathbf{V}_\omega(\mathbf{r})\right]d\omega$$
$$=2\mathrm{Re}\int_0^\infty d\omega\int d^3\mathbf{r}\cdot\mathbf{F}_\omega(\mathbf{r})\cdot\mathbf{V}_\omega^*(\mathbf{r}) \tag{87}$$

where we have used the reality condition and the identity $\frac{1}{2\pi}\int e^{\pm i\xi x}dx=\delta(\xi)$

Now, by Parseval's theorem for Fourier transforms,

$$\int d^3\mathbf{r}\;\mathbf{F}_\omega(\mathbf{r})\cdot\mathbf{V}_\omega^*(\mathbf{r})=\int d^3\mathbf{k}\;\vec{f}(\mathbf{k},\omega)\cdot\vec{v}^*(\mathbf{k},\omega) \tag{88}$$

where
$$\vec{f}(\mathbf{k},\omega)=\int\mathbf{F}_\omega(\mathbf{r})e^{i\mathbf{k}\cdot\mathbf{r}}d^3\mathbf{r}=\text{4-dimensional transform of }\mathbf{F}(\mathbf{r},t)$$

and similarly for $\vec{v}(\mathbf{k},\omega)$. But as we saw following equation 23, $\frac{\partial F}{\partial t}=\mathbf{Q}$ the vector used in our work as the driving term. So

$$\vec{f}(\mathbf{k},\omega)=\frac{1}{i\omega}\vec{q}(\mathbf{k},\omega)$$

and hence

$$W=2\mathrm{Re}\int_0^\infty\frac{d\omega}{i\omega}\int d^3\mathbf{k}\cdot\vec{q}(\mathbf{k},\omega)\cdot\vec{v}(\mathbf{k},\omega) \tag{89}$$

Now, in our previously-used notation,

$$\vec{q}(\mathbf{k},\omega)\cdot\vec{v}^*(\mathbf{k},\omega)=\vec{q}(\mathbf{k},\omega)\cdot\vec{v}^*_{(\pm)}(\mathbf{k},\omega)+\vec{q}(\mathbf{k},\omega)\cdot\vec{v}_{(A)}(\mathbf{k},\omega)$$

so that we may treat separately the power radiated into the $F_\pm$ modes and into the Alfvén mode, adding these to obtain the total. We look first at the radiation into the $F_\pm$ modes.

Using equations 39 and 89 we have, in an obvious notation

$$W_\pm=2\mathrm{Re}\int_0^\infty\frac{d\omega}{i\omega}\int d^3\mathbf{k}\;\vec{q}_\pm(\mathbf{k},\omega)\cdot\vec{v}_\pm^*(\mathbf{k},\omega)$$

The total radiated energy is then

$$W=2\mathrm{Re}\int_0^\infty\frac{d\omega}{i\omega}d^3\mathbf{k}\left[\frac{1}{2\omega F_+}\left(\frac{N_+}{k+\frac{\omega}{F_+}}-\frac{N_+}{k-\frac{\omega}{F_+}}\right)-\frac{1}{2\omega F_-}\left(\frac{N_-}{k+\frac{\omega}{F_-}}-\frac{N_-}{k-\frac{\omega}{F_-}}\right)\right] \tag{90}$$

where

$$N_\pm(k,\Omega,\omega)\equiv\left(q_x^*,\ q_y^*,\ q_z^*\right)\cdot\underset{\sim}{H}_\pm\cdot\begin{pmatrix}q_x\\q_y\\q_z\end{pmatrix} \tag{91}$$

(matrix multiplication) is a purely real function. Now, expression 90 is only given a definite meaning by inserting a small amount of dissipation and then passing to the dissipationless limit, as before. Without this, the formal value of quantity 89 would be zero, since the quantity in the square bracket would be purely real. We do not use quite the same device as before, however, for $N_\pm$ is not an analytic function of K, involving $|q_x|^2$ etc., as it does.* Using the same "pole displacement" as in equation 42, we note that

$$Re\left[\frac{1}{i}\frac{1}{K\pm(K_0+i\epsilon)}\right] = \mp\frac{\epsilon}{\epsilon^2+(K\pm K_0)^2} \rightarrow \mp\pi\delta(k-k_0) \quad as \quad \epsilon\to 0 \tag{92}$$

since one representation of the Dirac δ -function is

$$\delta(x) = \frac{1}{\pi}\lim_{\epsilon\to 0}\frac{\epsilon}{\epsilon^2+x^2} \tag{93}$$

Using equation 93 in equation 90, we may write, upon introducing polar coordinates in $\mathbb{K}$ -space,

$$W = 2\pi\int_0^\infty\frac{d\omega}{\omega}\int d\Omega\int_{-\infty}^{\infty}\kappa^2 d\kappa\left[\frac{1}{2\omega F_-}\left\{\delta\left(\kappa+\frac{\omega}{F_-}\right)+\delta\left(\kappa-\frac{\omega}{F_-}\right)\right\}N_-(\kappa,\Omega,\omega)\right]$$
$$-2\pi\int_0^\infty\frac{d\omega}{\omega}\int d\Omega\int_{-\infty}^{\infty}\kappa^2 d\kappa\left[\frac{1}{2\omega F_+}\left\{\delta\left(\kappa+\frac{\omega}{F_+}\right)+\delta\left(\kappa-\frac{\omega}{F_+}\right)\right\}N_+(\kappa,\Omega,\omega)\right] \tag{94}$$

where, as in equation 40, our radial integration extends over an infinite range and our angular integration is over a hemisphere only. Doing the radial integration in equation 94 gives

$$W = 2\pi\int_0^\infty d\omega\int d\Omega\left[\frac{N_-\left(\frac{\omega}{F_-},\Omega,\omega\right)}{F_-^3} - \frac{N_+\left(\frac{\omega}{F_+},\Omega,\omega\right)}{F_+^3}\right] \tag{95}$$

where our angular integration is now carried over all directions in $\mathbb{K}$-space. Now, note that, once k has been eliminated above, $N_\pm$ is an even function of ω (as follows from its definition, equation 91). (Before elimination of k one could only say that $N_\pm(\kappa,\Omega,\omega)=N_\pm(-\kappa,-\Omega,-\omega)$)

* The device of using a small amount of dissipation assures us of purely outgoing waves, and that there are no sources of energy except where the applied forces act, instead of a mixture of ingoing and outgoing waves, and energy sources at infinity.

Equation 95 is a formal expression, in terms of quite definite quantities, for the energy radiated during the action of the driving forces. Its spectral resolution is evident. We now appeal to our previously established asymptotic connection between directions in $\mathbb{K}$ -space and in physical space to extract directly from equation 96 an expression for the angular distribution of the spectral intensity. To a given solid angle in $\mathbb{K}$ -space corresponds a definite solid angle in physical space, given in terms of equations 18 and 20, as we have seen, $\psi = \tan^{-1}\frac{F_{\pm}'(\theta)}{F_{\pm}(\theta)} \mp \theta$. The ratio of corresponding solid angles in physical space and in $\mathbb{K}$ -space is clearly

$$\frac{\sin\psi}{\sin\theta} = \frac{d\psi}{d\theta} \tag{96}$$

An easy calculation then gives

$$R(\psi) = \frac{\sin\psi}{\sin\theta}\frac{d\psi}{d\theta} = \frac{F_{\pm}(F_{\pm}'' + F_{\pm})}{(F^2 + F'^2)^{3/2}}\left(F_{\pm} + F_{\pm}'\cot\theta\right) \tag{97}$$

The angular distribution of radiated spectral intensity is then

$$2\pi R(\psi)\left[\frac{N_-\left(\frac{\omega}{F_-}, \Omega_0, \omega\right)}{F_-^3} - \frac{N_+\left(\frac{\omega}{F_+}, \Omega_0, \omega\right)}{F_+^3}\right] \tag{98}$$

where the polar angles in $\mathbb{K}$ -space are expressed in terms of the polar angles of the direction of interest, by means of equations 18 and 20.

11. Power Radiated into the Alfvén Mode.

From first principles, we now give a separate treatment to the Alfvén mode, and derive a formal expression for the energy radiated into this mode, and its spectral distribution. This energy is radiated entirely parallel and anti-parallel to $\mathbb{B}_0$, as we have seen.

As we have already seen, the 4-dimensional transform of the Alfvén mode part of the total amplitude is

$$\vec{V}^{(A)} = \hat{e}_x V_x + \hat{e}_y V_y = -e_x V_3 \sin\phi + e_y V_3 \cos\phi$$

From equation 32,

$$V_3 = \frac{1}{B_0} \frac{q_3}{k_z^2 - \frac{\omega^2}{V_A^2}} = \frac{1}{B_0^2} \frac{-q_x \sin\phi + q_y \cos\phi}{k_z^2 - \frac{\omega^2}{V_A^2}}$$

As before, take the fixed axes so $Q_y = 0$. Then (for use in equation 89), introducing polar coordinates for $\mathbb{K}$,

$$\vec{q}\,(\mathbb{K},\omega)\cdot \vec{V}_A^{\,*}(\mathbb{K},\omega) = \frac{1}{B_0^2} \frac{|q_x|^2 \sin^2\phi}{\kappa^2\cos^2\theta - \frac{\omega^2}{V_A^2}} \tag{99}$$

Now, the energy radiated into this mode is

$$W_{(A)} = 2\,\mathrm{Re}\int \frac{d\omega}{i\omega}\int d^3\mathbb{K}\; \vec{q}\,(\mathbb{K},\omega)\cdot \vec{V}_{(A)}^{\,*}(\mathbb{K},\omega)$$

as before. Defining $F_A(\theta) = V_A \cos\theta$, in analogy to $F_\pm(\theta)$ for the shear-compression modes, we have

$$\frac{1}{\kappa^2\cos^2\theta - \frac{\omega^2}{V_A^2}} = \frac{1}{2\omega F_A(\theta)}\left(\frac{1}{\kappa - \frac{\omega}{F_A}} - \frac{1}{\kappa + \frac{\omega}{F_A}}\right)$$

Then

$$W_{(A)} = \mathrm{Re}\,\frac{1}{i}\int_0^\infty \frac{d\omega}{\omega^2}\int \frac{\sin^2\phi\, d\Omega}{F_A(\theta)}\int_{-\infty}^{\infty} \kappa^2 |q(\kappa,\Omega,\omega)|^2 \left\{\frac{1}{\kappa - \frac{\omega}{F_A}} - \frac{1}{\kappa + \frac{\omega}{F_A}}\right\} d^3\mathbb{K} \tag{101}$$

Proceeding similarly to the treatment following equation 89, we re-write this as

$$W_{(A)} = \frac{2\pi}{B_0^2}\int_0^\infty d\omega \int \frac{d\Omega\,\sin^2\phi}{F_A^3(\theta)}\, |q_x(\kappa,\Omega,\omega)|^2 \tag{101}$$

There seems to be a delicate question of convergence involved here, for we cannot rely on the θ-dependence of $|q_x|^2$ to alleviate what would otherwise be a divergent θ-integral of the form

$$\int \frac{\sin\theta}{\cos^3\theta}\, d\theta \qquad \text{(if } |q_x| \text{ were independent of } \theta \text{, say).}$$

Such difficulties do not occur in the treatment of the F_+ mode.*

12. Thermal Excitation of Hydromagnetic Waves.

As a simple illustration of a concrete mechanism for the excitation of hydromagnetic radiation, let us consider our fluid to be a perfect gas (whose heat conductivity is neglected), and within which by some means, heat is being added in amount $\Lambda(r, t)$ unit volume and time. Let its specific internal energy, pressure, and density be E, p, ρ. We consider only a "weak" heat source. We have the equation of energy balance

$$\rho\left[\frac{DE}{Dt} + p\,\frac{D}{D}\,\frac{1}{\rho}\right] = \Lambda \tag{102}$$

where the indicated derivatives are convective derivatives in a co-moving coordinate system locally.

$$E = c_v T = \frac{c_v}{R}\frac{p}{\rho} = \frac{1}{\gamma-1}\frac{p}{\rho}$$

We easily derive the form in p, ρ variables, namely

$$\frac{p}{\gamma-1}\cdot\frac{D}{Dt}\log\left(\frac{p}{\rho^\gamma}\right) = \Lambda \tag{103}$$

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*In an investigation of the total energy radiated by turbulence into hydromagnetic waves, R. M. Kulsrud (19) has found similar divergent expressions for the radiation into the Alfven mode and what corresponds to our E mode, and estimated the order of magnitude of these by rather ad hoc heuristic arguments. The subject of this paper has some points of contact with the work in this chapter.

The root of the difficulty is that, for the E and Alfvén modes, the phase velocities vanish at $\theta = 90^\circ$. This is a consequence of the idealizations made in setting up the linearized wave equation. It would seem, however, that our expressions for the angular distribution of the radiated power should be accurate, except for θ near 90° (ψ near 0°) in the F_- mode.

Linearizing this for small variations of p, ρ about p_0, ρ_0, our energy equation becomes

$$\frac{\dot{p}}{p_0} - \gamma \frac{\dot{\rho}}{\rho_0} = \Omega \equiv \frac{\Lambda}{\rho_0 c_v T_0} \tag{104}$$

The perfect-gas equation of state,

$$p = \rho R' T \qquad (R' = \text{gas constant per gram})$$

becomes

$$\left(\frac{p}{p_0} - 1\right) = \left(\frac{\rho}{\rho_0} - 1\right) + \left(\frac{T}{T_0} - 1\right) \tag{105}$$

when linearized. Elimination of variables between the above, the linearized continuity equation $\rho_0 \nabla\cdot\mathbb{V} = -\dot{\rho}$, the induction equation $\dot{\mathbb{b}} = \nabla\times(\mathbb{V} + \mathbb{B}_0)$ and the momentum equation $\rho_0 \dot{\mathbb{V}} = -\nabla p(\rho) + (\nabla\times\mathbb{b})\times\mathbb{B}_0$ leads easily to the wave equation

$$\rho_0 \ddot{\mathbb{V}} - \rho_0 V_s^2 \nabla\nabla\cdot\mathbb{V} - \left[\nabla\times\nabla\times(\mathbb{V}\times\mathbb{B}_0)\right]\times\mathbb{B}_0 = -p_0 \nabla\Omega \tag{106}$$

We thereby identify $-p_0\nabla\Omega = \frac{-p_0}{\rho_0 c_v T}\nabla\Lambda(\mathbf{r},t)$ as our effective driving term, $\mathbb{Q}(\mathbf{r},t)$.

APPENDIX

The Characteristic Surfaces of the Hydromagnetic Wave Equation.

We show briefly here that the characteristic surfaces of our wave equation are three in number; the ovaloid and tri-cusped surfaces previously derived as envelopes of equi-phase planes, for the F_+, F_- modes respectively, and planes* orthogonal to $\mathbb{B}_0$ for the Alfvén mode. True wave fronts (discontinuity loci for the velocity) must lie on these expanding surfaces, thus supporting our explicit analytical solutions for the amplitude. The underlying mathematical theory is not discussed. (But see reference 20, especially pp. 376-378 and 455-465.) Our wave equation in vector form is

$$\rho_0 \ddot{\mathbb{V}} - \rho_0 V_S^2 \nabla\nabla\cdot\mathbb{V} + \mathbb{B}_0 \times \left[\nabla\times\nabla\times(\mathbb{V}\times\mathbb{B}_0)\right] = \mathbb{Q}(\mathbb{r}, t)$$

Written in component form, it becomes the system of equations

$$-\frac{\partial^2 V_x}{\partial t^2} + a\frac{\partial^2 V_x}{\partial x^2} + b\frac{\partial^2 V_x}{\partial z^2} + a\frac{\partial^2 V_y}{\partial x \partial y} + (a-b)\frac{\partial^2 V_z}{\partial x \partial z} = -Q_1 \qquad (107a)$$

$$-\frac{\partial^2 V_y}{\partial t^2} + a\frac{\partial^2 V_y}{\partial y^2} + b\frac{\partial^2 V_y}{\partial z^2} + a\frac{\partial^2 V_x}{\partial x \partial y} + (a-b)\frac{\partial^2 V_z}{\partial y \partial z} = -Q_2 \qquad (107b)$$

$$-\frac{\partial^2 V_z}{\partial t^2} + (a-b)\frac{\partial^2 V_z}{\partial z^2} + (a-b)\frac{\partial^2 V_x}{\partial x \partial z} + (a-b)\frac{\partial^2 V_y}{\partial y \partial z} = -Q_3 \qquad (107c)$$

where $a = V_A^2 + V_S^2$ and $b = V_A^2$.

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*One can easily show that, for the Alfven mode, the envelope of the equi-phase planes degenerates to a pair of points at $\theta = 0$ and $\theta = 180°$ on the F_+ surface if $V_S > V_A$ and on the F_- surface if $V_S < V_A$. It is thus reassuring to see that the corresponding characteristic surface is nevertheless a pair of planes orthogonal to $\mathbb{B}$, in agreement with our direct solution for the Alfvén-mode part of the amplitude.

The differential equation for a characteristic manifold $w(x,y,z,t)=0$ is

$$\begin{vmatrix} a\xi^2+b\zeta^2-\tau^2 & a\xi\eta & (a-b)\xi\zeta \\ a\xi\eta & a\eta^2+b\zeta^2-\tau^2 & (a-b)\eta\zeta \\ (a-b)\xi\zeta & (a-b)\eta\zeta & (a-b)\zeta^2-\tau^2 \end{vmatrix} = 0 \qquad (108)$$

where $\xi \equiv \frac{\partial w}{\partial x}$, $\eta \equiv \frac{\partial w}{\partial y}$, $\zeta \equiv \frac{\partial w}{\partial z}$, $\tau \equiv \frac{\partial w}{\partial t}$. Expanded and rearranged, this may be written

$$\left(\tau^2 - V_A^2\zeta^2\right)\left[\left(\xi^2+\eta^2+\zeta^2\right)\left(\{V_A^2+V_s^2\}\tau^2 - V_A^2V_s^2\zeta^2\right)-\tau^4\right] = 0 \qquad (109)$$

To find the characteristic surfaces* (expanding surfaces on which discontinuities in the solution or its derivatives may exist) we put

$$w = t - \pi(x, y, z) \qquad (110)$$

(now the equation $w=0$ will define the possible wave fronts), and we have $\tau = 1$, $\xi = -\frac{\partial \pi}{\partial x}$, $\eta = -\frac{\partial \pi}{\partial y}$, $\zeta = -\frac{\partial \pi}{\partial z}$. Equation 109 is then exactly equivalent to the pair of equations

$$1 - V_A^2\left(\frac{\partial \pi}{\partial z}\right)^2 = 0 \qquad (110a)$$

and

$$\left[\left(\frac{\partial \pi}{\partial x}\right)^2 + \left(\frac{\partial \pi}{\partial y}\right)^2 + \left(\frac{\partial \pi}{\partial z}\right)^2\right]\left[\left(V_A^2+V_s^2\right) - V_A^2V_s^2\left(\frac{\partial \pi}{\partial z}\right)^2\right] - 1 = 0 \qquad (110b)$$

To solve equation 110b, we first seek a so called "complete solution" in the form

$$\pi_2(x, y, z) = \kappa_1 x + \kappa_2 y + \kappa z \qquad (111)$$

where $\kappa_1, \kappa_2, \kappa_3$ are parameters subject only to the simple constraint

*Technically "ray-conoid" or a section of the "characteristic manifold" by a plane t = const.

$$\left(\varkappa_1^2 + \varkappa_2^2 + \varkappa_3^2\right)\left[\left(V_A^2 + V_S^2\right) - V_A^2 V_S^2 \varkappa_3^2\right] - 1 = 0$$

eqn.111 is a 2-parameter family of planes. Any two parameters will do, so we are at liberty to put

$$\varkappa_1 = \varkappa \sin\Theta \cos\Phi$$

$$\varkappa_2 = \varkappa \sin\Theta \sin\Phi$$

$$\varkappa_3 = \varkappa \cos\Theta$$

(Θ, Φ need not be identified with angles in space.)

whereupon we now require

$$\left(V_A^2 + V_S^2\right)\varkappa^2 - V_A^2 V_S^2 \varkappa^4 \cos^2\Theta - 1 = 0 \tag{112}$$

which is of exactly the same form as equation 5' with $\omega = 0$, so that

$$\varkappa = \frac{1}{F_\pm(\Theta)} \tag{113}$$

with $F_\pm(\Theta)$ defined in exactly the same way as $F_\pm(\theta)$ is defined ... by equation 8. Then

$$\pi_2(x, y, z) = \varkappa r\left[\cos\Theta\cos\psi + \sin\Theta\sin\psi\cos(\phi - h)\right] \tag{114}$$

To find the equations of the characteristic surfaces we need only find the envelope of the planes $\pi_2 = t$. This problem is almost exactly the same as that solved in equations 15 ff., and culminates in the result that the characteristic surfaces for the $F_\pm$ modes are surfaces rotationally symmetric about $\psi = 0$ such that their cross-sections in any meridian plane are given by

$$r = \sqrt{F_\pm^2(\psi) + F_\pm'^2(\psi)}\; t \tag{115}$$

The Alfvén characteristic surfaces are immediately derived from equation 110a as the planes

$$z = \pm V_A \cdot t \tag{116}$$

III. SOME TWO-DIMENSIONAL HYDROMAGNETIC STEADY FLOWS OF AN INCOMPRESSIBLE FLUID.

1. Introduction.

The published literature on hydromagnetics, with but one very recent notable paper (24), does not contain a single discussion of a solution of the combined equations of motion and induction. All the work on the extremely difficult and basic problem of hydromagnetic dynamo action has been based upon the induction equation alone -- assuming a steady velocity field fulfilling certain geometrical boundary conditions (e.g. containment within a sphere), and studying its effect on a prescribed initial magnetic field, seeking asymptotic solutions. Thus, in these studies, the back-reaction of the magnetic field on the flow was nowhere taken into account. Therefore it has seemed worthwhile to see if one might develop some information on this point by studying some very primitive examples.

In this chapter we give the results of an exploratory effort to find and study some solutions of the exact nonlinear coupled equations of hydromagnetics, under simplifying assumptions stringent enough to reduce the analytical problems to at least partially manageable proportions and to permit the obtaining of some exact solutions.

The objects of this study were to formulate the equations for some specially chosen flows of simple geometry, to discuss some of their general properties, to ascertain whether one could find means of generating fairly wide classes of solutions (perhaps along lines analogous to some of the procedures used in ordinary hydrodynamics), and to seek from the special solutions obtained some further clues about the properties of these equations.

To achieve the greatest analytical simplification, we select for consideration two classes of two dimensional flows of simple symmetry types. Restriction to two dimensions, together with the assumption of incompressibility, permits formulation of the equations of motion and induction in scalar form by the introduction of stream functions for the interacting divergenceless fields $\mathbb{V}$ and $\mathbb{B}/\sqrt{\mu_0 \rho_0}$ (which have the same dimensions).

The idealized physical situation considered to begin with is that of a viscous incompressible conducting fluid filling all space, and with the fluid velocity, magnetic induction and current having prescribed, very special, two-dimensional symmetry. The fluid is thus acted upon by internal pressure forces, viscous forces, and magnetic ponderomotive forces due to the currents, which are themselves maintained by the fluid motion. We may thus speak descriptively of a kind of <u>fluid dynamo.</u>

The fluid velocity is assumed to lie everywhere parallel to a given plane ("planar geometry"), or in axially-symmetric meridian planes ("axially-symmetric geometry"), and it is stipulated that there is no variation of any quantity in a direction orthogonal to such a plane.

In flows of "Type I" symmetry, $\mathbb{V}$ and $\mathbb{B}$ lie, at each point, in one of these planes, while $\mathbb{j}$ is perpendicular thereto; in "Type II" flows $\mathbb{j}$ and $\mathbb{V}$ lie everywhere in such planes, with $\mathbb{B}$ perpendicular to them. Thus, in both cases, the magnetic ponderomotive force lies everywhere in a flow plane.

The equations of motion and induction are written down for both types of flow, for both planar and axially-symmetric geometry. Almost all of our discussion is limited to the steady-flow situation, and we do not investigate the development of this state from a time-dependent initial state.

Also, it turns out that, at least for finite conductivity, there are no physically acceptable steady solutions of the axially-symmetric cases, as is shown by general theorems due to Cowling and Chandrasekhar. As far as steady flows are concerned, then, only the planar cases are of interest to us. Some general features of both types of planar steady flows are discussed, without solution; explicit solutions are given only for Type I . . . the Type II flows being similar in essentials to the plane rotational flows of ordinary hydrodynamics, as we shall see. Thus our discussion is carried out in order of diminishing generality, commencing with the formulation of some general time-dependent equations and culminating in the derivation of a number of explicit special solutions for some Type I planar flows of a non-viscous fluid.

It has not proven possible to find wide classes of solutions of our equations, even in the simplest cases. There is no analogy with the extensively-developed two-dimensional potential-flow theory of ordinary hydrodynamics; the simplest cases of interest are more complicated than the two-dimensional rotational flows of fluid mechanics; even in that subject only a small number of solutions are known.*

The solutions given for type I flows are all obtained by the same

*The hydromagnetic flows are in general rotational because the magnetic forces impart vorticity to the fluid.

simple expedient, by means of which solutions for $\mathbb{V}$ and $\mathbb{B}$ are simultaneously determined, for the case where the fluid fills all space. We shall note some quite peculiar features of these flows.

A planar flow solution for an unbounded fluid body may be used as an approximate solution for a bounded cylindrical fluid body. If we examine the distribution of pressure, fluid velocity, electric current and electric and magnetic fields yielded by a solution for an infinite fluid, we may, it seems, determine an environment for a bounded fluid body which will be approximately equivalent to the surrounding moving fluid. This will necessitate supplying a rigid cylindrical container on whose surface are distributed appropriate active fluid sources ("pumps"), fluid sinks, suitable sources of electromotive force, surface currents and charges and, in general, supplying appropriately chosen return paths for electric current.

None of the special solutions obtained here may be usefully applied as descriptions of real physical systems of interest. They do, however, illustrate features which must be taken account of in the much more complicated "realistic" problems.

2. Derivation of the Scalar Equations.

As we have seen in Chapter I, the basic equations of hydromagnetics for a uniform, incompressible, viscous conducting fluid are

$$\frac{\partial \mathbb{V}}{\partial t}+(\mathbb{V}\cdot\nabla)\mathbb{V} = \frac{(\nabla\times\mathbb{B})\times\mathbb{B}}{\mu_0\rho}+\nu\nabla^2\mathbb{V}-\frac{1}{\rho}\nabla p \quad \text{(equation of motion)} \tag{1}$$

and

$$\frac{\partial \mathbb{B}}{\partial t} = \nabla\times(\mathbb{V}\times\mathbb{B})+\lambda\nabla^2\mathbb{B} \quad \text{(equation of induction)} \tag{2}$$

where ν is the kinematic viscosity and $\lambda = \frac{1}{\mu_0\sigma}$ is the "magnetic viscosity".

To eliminate the pressure-gradient term we operate through on equation 1 with the curl operator. Using the vector identity

$$(\mathbb{V}\cdot\nabla)\mathbb{V} = \nabla\left(\frac{V^2}{2}\right) - \mathbb{V}\times(\nabla\times\mathbb{V})$$

we obtain

$$\frac{\partial}{\partial t}(\nabla\times\mathbb{V}) = \nabla\times\left[\mathbb{V}\times(\nabla\times\mathbb{V})\right] + \nabla\times\left[\frac{(\nabla\times\mathbb{B})\times\mathbb{B}}{\mu_0\rho}\right] + \nu\nabla\times(\nabla^2\mathbb{V}) \qquad (3)$$

Equations 2 and 3 may be taken as the basic equations. Any solution of equations 1 and 2 is of course a solution of equations 2 and 3. Further, it is easy to show that any solution of equation 3 for which ∇p is integrable to a single-valued function $p(r)$ is also a solution of equation 1. So equations 2 and 3, together with the requirement of a single-valued pressure, may be taken as a starting point.*

Let us describe our flows in an orthogonal curvilinear coordinate system with coordinate surfaces u^1 = const., u^2 = const., u^3 = const., with unit vectors $\hat{e}_1$, $\hat{e}_2$, $\hat{e}_3$ normal to the respective surfaces, and where the line-element is

$$ds^2 = h_1^2(du^1)^2 + h_2^2(du^2)^2 + h_3^2(du^3)^2$$

The orientation of axes is such that $\hat{e}_1\times\hat{e}_2=\hat{e}_3$, etc.

For planar geometry, u^1, u^2, u^3 will be a set of rectangular Cartesian axes, with $u^3 = z$. For axially symmetric geometry, u^1 and u^2 will be arbitrary orthogonal curvilinear coordinates in a meridian plane, $u^3 = \phi$ the azimuth angle of such a plane, while $h_1 = h_1(u^1, u^2)$, $h_2 = h_2(u^1, u^2)$

*It may happen, as in the case of the special solution discussed on page 114 that p is not single valued in the region occupied by the fluid. We may still use such a solution by inserting a suitable partition such that p becomes single valued in the divided region. The partition has then the significance of an "actuator surface", supplying momentum to the fluid, and across which the pressure is discontinuous.

and $h_3 = \rho(u^1, u^2)$, the radius from the symmetry axis.

We now proceed to write the scalar equivalents of equations 2 and 3 for flows of both symmetry types.

Type I flows. The velocity field $\mathbb{V}$ and the magnetic induction field $\mathbb{B}$ lie everywhere in surfaces u^3 = const., and everything is independent of u^3. $\mathbb{V}$ and $\frac{\mathbb{B}}{\sqrt{\mu_0 \rho}}$, being two-dimensional divergenceless vector fields, may be derived from scalar stream-functions $\phi(u^1, u^2, t)$ and $\psi(u^1, u^2, t)$, respectively. Consider, e.g. the velocity field $\mathbb{V}$. Since $V_3 = 0$ we have

$$\nabla \cdot \mathbb{V} = \frac{1}{h_1 h_2 h_3}\left[\frac{\partial}{\partial u^1}(h_2 h_3 V_1) + \frac{\partial}{\partial u^2}(h_3 h_1 V_2)\right] = 0$$

This condition may always be identically satisfied by constructing, for any $\phi(u^1, u^2, t)$.

$$V_1 = -\frac{1}{h_2 h_3}\frac{\partial \phi}{\partial u^2}$$

$$V_2 = \frac{1}{h_1 h_3}\frac{\partial \phi}{\partial u^1}$$

so that

$$\mathbb{V} = \frac{\hat{e}_3}{h_3} \times \nabla\phi = -\nabla \times \left(\frac{\hat{e}_3}{h_3}\phi\right) \tag{4}$$

Then we find that

$$\nabla \times \mathbb{V} = \hat{e}_3 h_3 D\{\phi\} \tag{5}$$

$$\mathbb{V} \times (\nabla \times \mathbb{V}) = D\{\phi\} \nabla\phi \tag{6}$$

$$\nabla \times \nabla \times \mathbb{V} = -\frac{\hat{e}_3}{h_3} \times \nabla F\{\phi\} \tag{7}$$

$$\nabla \times \left[\nabla \times \nabla \times \mathbb{V}\right] = -\frac{\hat{e}_3}{h_1 h_2} G(\phi) \tag{8}$$

$$\nabla \times \left[\mathbb{V} \times (\nabla \times \mathbb{V})\right] = \frac{\hat{e}_3}{h_1 h_2}\frac{\partial(D\{\phi\}, \phi)}{\partial(u^1, u^2)} \tag{9}$$

where $\frac{\partial(D, \phi)}{\partial(u^1, u^2)}$ is the Jacobian, and where

$$D\{\phi\} \equiv \frac{1}{h_1 h_2 h_3}\left[\frac{\partial}{\partial u^1}\left(\frac{h_2}{h_1 h_3}\frac{\partial \phi}{\partial u^1}\right) + \frac{\partial}{\partial u^2}\left(\frac{h_1}{h_2 h_3}\frac{\partial \phi}{\partial u^2}\right)\right] \tag{10}$$

$$F\{\phi\} \equiv h_3^2 \, D\{\phi\}$$

$$G\{\phi\} \equiv \frac{\partial}{\partial u^1}\left[\frac{h_2}{h_1 h_3}\frac{\partial}{\partial u^1} h_3^2 \, D\{\phi\}\right] + \frac{\partial}{\partial u^2}\left[\frac{h_1}{h_2 h_3}\frac{\partial}{\partial u^2}\left(h_3^2 \, D\{\phi\}\right)\right] \tag{11}$$

In the planar geometry, $D\{\phi\} = F\{\phi\} = \nabla^2\phi$ and $G\{\phi\} = \nabla^4\phi$. $(h_3 = 1)$

We may similarly derive the magnetic-induction field from a stream function $\psi = \psi(u^1, u^2, t)$. Put

$$\frac{\mathbb{B}}{\sqrt{\mu_0 \rho}} = \frac{\hat{e}_3}{h_3} \times \nabla\psi \tag{12}$$

Then

$$\frac{\mathbb{V}\times\mathbb{B}}{\sqrt{\mu_0\rho}} = \frac{1}{h_3^2}\nabla\phi\times\nabla\psi = \frac{-\hat{e}_3}{h_1 h_2 h_3^2}\,\frac{\partial(\phi,\psi)}{\partial(u^1,u^2)} \tag{13}$$

and

$$\nabla\times\left(\frac{\mathbb{V}\times\mathbb{B}}{\sqrt{\mu_0\rho}}\right) = \frac{-\hat{e}_3}{h_3}\times\nabla\left[\frac{1}{h_1 h_2 h_3}\,\frac{\partial(\phi,\psi)}{\partial(u^1,u^2)}\right] \tag{14}$$

Equation 3 then becomes

$$\hat{e}_3 h_3 \dot{D}\{\phi\} = \frac{\hat{e}_3}{h_1 h_2}\frac{\partial(D\{\phi\},\phi)}{\partial(u^1,u^2)} - \frac{\hat{e}_3}{h_1 h_2}\frac{\partial(D\{\psi\},\psi)}{\partial(u^1,u^2)} + \frac{\hat{e}_3}{h_1 h_2}\nu G\{\psi\}$$

which shows that such a motion with "Type I" symmetry continues to evolve in time with this symmetry, and gives the scalar equation of motion

$$h_1 h_2 h_3 \, \dot{D}\{\phi\} = \frac{\partial(D\{\phi\},\phi)}{\partial(u^1,u^2)} - \frac{\partial(D\{\psi\},\psi)}{\partial(u^1,u^2)} + \nu G(\psi) \tag{15}$$

Likewise, equation 2 becomes

$$\frac{\sqrt{\mu\rho}}{h_3}\,\hat{e}_3\times\nabla\left[\dot{\psi} + \frac{1}{h_1 h_2 h_3}\frac{\partial(\phi,\psi)}{\partial(u^1,u^2)} - \lambda F(\psi)\right] = 0$$

giving the scalar equation of induction

$$\dot{\psi} + \frac{1}{h_1 h_2 h_3}\cdot\frac{\partial(\phi,\psi)}{\partial(u^1,u^2)} - \lambda F(\psi) = \kappa = \text{const.} \tag{16}$$

which is to be paired with equation 15.

In a similar way, we set up the scalar equations for the Type II case.

Type II flows. The velocity $\mathbb{V}$ and the electric current density lie everywhere in a surface u^3 = constant, while $\mathbb{B} = \hat{e}_3 B(u^1, u^2, t)$ lies everywhere perpendicular to $\mathbb{J}$ and $\mathbb{V}$. For this case, if we put

$$\frac{\mathbb{B}}{\sqrt{\mu\rho}} = -\frac{\psi(u^1, u^2, t)}{h_3}\hat{e}_3 \tag{17}$$

we find

$$\nabla \times \frac{\mathbb{B}}{\sqrt{\mu_0\rho}} = \frac{\hat{e}_3}{h_3} \times \nabla\psi \tag{18}$$

(i.e. ψ is a stream function for $\mathbb{J}$ in case II)

$$\begin{aligned}\frac{(\nabla\times\mathbb{B})\times\mathbb{B}}{\mu_0\rho} &= -\frac{1}{h_3^2}\psi\nabla\psi \\ &= \frac{-1}{2h_3^2}\nabla(\psi^2)\end{aligned} \tag{19}$$

$$\nabla \times \frac{\nabla\times\mathbb{B}}{\sqrt{\mu\rho}} = h_3 D\{\psi\}\hat{e}_3 = \frac{F\{\psi\}}{h_3}\hat{e}_3$$

$$\frac{\nabla\times[(\nabla\times\mathbb{B})\times\mathbb{B}]}{\mu\rho} = \frac{-\hat{e}_3}{h_1 h_2}\frac{\partial\left(\frac{\psi}{h_3^2}, \psi\right)}{\partial(u^1, u^2)} \tag{20}$$

$$\frac{\mathbb{V}\times\mathbb{B}}{\sqrt{\mu\rho}} = -\frac{1}{h_3^2}\psi\nabla\phi \tag{21}$$

$$\nabla \times \frac{(\mathbb{V}\times\mathbb{B})}{\sqrt{\mu_0\rho}} = -\frac{\hat{e}_3}{h_1 h_2}\frac{\partial\left(\frac{\psi}{h_3^2}, \psi\right)}{\partial(u^1, u^2)} \tag{22}$$

The resulting equations of motion and induction for Type II flows are, respectively

$$h_1 h_2 h_3 D\{\phi\} = \frac{\partial(D\{\phi\}, \phi)}{\partial(u^1, u^2)} - \frac{\partial\left(\frac{\psi}{h_3^2}, \psi\right)}{\partial(u^1, u^2)} + \nu G\{\phi\} \tag{23}$$

and

$$\dot{\psi} + \lambda F\{\psi\} - \frac{h_3}{h_1 h_2}\frac{\partial(\phi, \psi/h_3^2)}{\partial(u^1, u^2)} = 0 \tag{24}$$

The equations that have been derived are the general time-dependent equations of motion and induction for the two flow types. Hence-forth, however, we consider mainly steady flows, which must satisfy equations 15 and 16 or 23 and 24, with the time derivatives set equal to zero. Before discussing the special solutions that have been obtained, we note various general features which are ascertainable without solution.

3. Some General Features of the Steady Flows.

(a) For planar Type I steady flows the equations of motion and induction are, in rectangular coordinates

$$\frac{\partial(\nabla^2\phi,\phi)}{\partial(x,y)} - \frac{\partial(\nabla^2\psi,\psi)}{\partial x,y} + \nu\nabla^4\psi = 0 \tag{25}$$

and

$$\frac{\partial(\phi,\psi)}{\partial(x,y)} - \lambda\nabla^2\psi = \kappa = const. \tag{26}$$

Equation 25 is a generalization of the well-known stream-function equation for the steady rotational flow of an incompressible fluid in hydrodynamics. Equation 26, multiplied by $\hat{e}_z$, may be seen to be an integration of the static Maxwell equation $\nabla\times\mathbb{E}=0$. For in this case the terms on the left-hand side may be identified with the aid of equations 5 and 13 as

$$\frac{\mathbb{V}\times\mathbb{B} - \lambda\nabla\times\mathbb{B}}{\sqrt{\mu_0\rho}} = -\frac{1}{\sqrt{\mu_0\rho}}\left(\frac{\mathbb{J}}{\sigma} - \mathbb{V}\times\mathbb{B}\right) = -\frac{\mathbb{E}}{\sqrt{\mu_0\rho}}$$

That is, if $\mathbb{E}=\mathbb{E}(x,y)$ then the integral of $\nabla\times\mathbb{E}=0$ is just $\mathbb{E}(x,y)=$ const., which is the content of equation 26. Thus, the electric field, seen in the "fixed" coordinate system with respect to which $\mathbb{V}$ and $\mathbb{B}$ are measured, is constant and directed normally to the flow plane, for stationary flows with Type I symmetry. The current $\mathbb{J}=\frac{\nabla\times\mathbb{B}}{\mu}$ is also always normal to these planes, though not necessarily in the same sense everywhere, being a function of the transverse coordinates. There are changes at infinity creating the constant field, but no volume charge density. This current in the z-direction is a circumstance which causes difficulty in the fitting of appropriate boundary conditions for a bounded fluid body, because the necessary return currents must be taken account of. For Type I planar steady flow of a perfectly conducting fluid ($\lambda = 0$), we have

$$\nabla \times \mathbb{B} = -\mathbb{E} = \varkappa \hat{e}_z = \text{const.} \tag{28}$$

This feature leads to some peculiar properties for this case, as we shall see (compare (i)).

(b) For steady Type II planar flows the equations of motion and induction are

$$\frac{\partial(\nabla^2\phi, \phi)}{\partial(x, y)} - \nu \nabla^4 \phi = 0 \tag{29}$$

and

$$\frac{\partial(\phi, \psi)}{\partial(x, y)} - \lambda \nabla^2 \psi = 0 \tag{30}$$

The velocity stream-function ϕ, being uncoupled to ψ in equation 29, is determined exactly as in the ordinary hydrodynamics of plane rotational flows. The magnetic ponderomotive force does not appear in equation 29 because, for these flows, it is irrotational, and so affects only the pressure distribution; the fluid streamlines are just as in the absence of magnetic effects. One readily finds that

$$\nabla\left(\frac{p}{\rho} + \frac{V^2 + \psi}{2}\right) = \nabla^2\phi \, \nabla\phi - \nu \hat{e}_z \times \nabla\nabla^2\psi \tag{31}$$

For any ϕ determined from equation 29 the corresponding ψ is determined by solving what is then a linear equation, equation 30. A number of solutions of equation 29 are known (see, e.g. reference 22 pp. 492-97) and it is quite possibly feasible to obtain some corresponding solutions of equation 25, obtaining the accompanying magnetic fields.

In these cases, a current $\mathbb{j} = \sqrt{\frac{\rho}{\mu_0}}\, \hat{e}_z \times \nabla\psi$ flows transverse to the magnetic field. The same difficulty with return currents already mentioned occurs here also, unless solutions of equation 30 are sought with the boundary condition $\mathbb{j} \cdot \hat{n} = \sqrt{\frac{\rho}{\mu_0}} \frac{\partial \psi}{\partial s} = 0$ on the transverse boundary, where the derivative is taken along the boundary. Thus, to have no normal current at a transverse boundary, the magnetic field must be constant on it. The electric current is accompanied by a volume charge density

$$\nabla \cdot \mathbb{E} = -\nabla \cdot (\mathbb{V} \times \mathbb{B}) = \sqrt{\mu_0 \rho}\, \left[\psi \nabla^2 \phi + \nabla\phi \cdot \nabla\psi \right] \tag{32}$$

The accompanying electric ponderomotive force $E \nabla \cdot E$ is completely negligible in comparison with the magnetic ponderomotive force, as we saw in Chapter I, and was neglected from the outset in the equation of motion.

In the case of infinite conductivity, $(\lambda = 0)$, equation 30 requires that $\psi = g(\phi)$, where g is an arbitrary function. Thus, in this case, the contour lines of constant $\mathbb{B}$ coincide with the fluid streamlines. This can be physically inferred from the incompressibility condition and the fact that the magnetic field lines are, for $\lambda = 0$, bodily convected by the fluid.

(c) Since

$$\nabla \left(\frac{p}{\rho} + \frac{V^2}{2} \right) = \mathbb{V} \times (\nabla \times \mathbb{V}) + \frac{(\nabla \times \mathbb{B}) \times \mathbb{B}}{\mu_0 \rho}$$

then, upon integrating both sides between two points on any streamline, we observe that the integral vanishes if $\mathbb{B}$ is constant on a streamline, as is the case for planar Type II steady flows of a perfectly conducting fluid, or for those Type I flows of a perfectly conducting fluid with $\mathbb{V}$ parallel to $\mathbb{B}$ (i.e. $\lambda = 0$ $\varkappa = 0$). Hence in these cases the "weak form" of Bernoulli's theorem holds ... $p + \frac{\rho V^2}{2}$ is constant along a streamline. (Had we included a conservative external force such as gravity, with potential Ω, then $p + \frac{\rho V^2}{2} + \Omega$ would be constant on a streamline).

(d) It may be parenthetically noted that, for any 3-dimensional steady flow of a perfectly-conducting incompressible fluid, a corresponding hydromagnetic flow may be trivially found by superposing a magnetic field everywhere in constant proportion to the velocity. For we have then

$$(\nabla \times \mathbb{V}) \times \mathbb{V} - \frac{(\nabla \times \mathbb{B}) \times \mathbb{B}}{\mu_0 \rho} = \nu \nabla^2 \mathbb{V} - \frac{1}{\rho} \nabla \left(p + \rho \frac{V^2}{2}\right)$$

and

$$\nabla \times (\mathbb{V} \times \mathbb{B}) = 0$$

The last equation is identically satisfied by putting $\frac{\mathbb{B}}{\sqrt{\mu_0 \rho}} = A \mathbb{V}$, where is a constant. Then the first equation becomes

$$(1 - A^2)(\nabla \times \mathbb{V}) \times \mathbb{V} = \nu \nabla^2 \mathbb{V} - \frac{1}{\rho} \nabla \left(p + \rho \frac{V^2}{2}\right)$$

defining an ordinary hydrodynamic flow with altered pressure and viscosity coefficient. If a magnetic field is thus superposed on a potential flow, it remains a potential flow and there is no electric current in the fluid. In the potential flow case, of course, the "strong form" of Bernoulli's theorem holds; $p + \rho \frac{V^2}{2}$ is a constant.

(e) There are two dissipation parameters, λ and ν, in our equations. When, to lighten analytical difficulties, approximations are made in which one or both of these are put equal to zero (and there are many cases in which these would seem to be good approximations, physically), we must take cognizance of a phenomenon well known and much studied in hydrodynamics -- namely, the non-uniform behavior of the solutions in the limit in which various coefficients in the equations of motion approach zero. Consider, as a simple typical illustration, equation 29, the stream-function equivalent of the Navier-Stokes equation:

$$\frac{\partial(\nabla^2\phi, \phi)}{\partial(x, y)} + \nu \nabla^4 \phi = 0 \tag{29}$$

If ν is put equal to zero, equation 29 implies that

$$\nabla^2\phi + f(\phi) = 0 \tag{33}$$

with f arbitrary thus far ... an equation which has quite different physical content and mathematical properties from equation 29. Equation 29 describes the diffusion and transport of vorticity, whereas equation 33 states that the vorticity is constant on each streamline.

Consider the solutions of these equations inside some region in which there are fixed boundaries: Equation 29 presumably has a unique solution subject to the boundary conditions ϕ = const. and $\nabla\phi = 0$ on fixed boundaries, with the behavior of ϕ at infinity also required.* The solutions of equation 33, on the other hand, are by no means uniquely fixed by boundary conditions alone, but require much more information, equivalent to specifying

*There exists no mathematical proof establishing the existence of a unique solution of equation 29 for these boundary conditions; they are assumed on physical grounds, and no case is known to provide a counter-example.

the vorticity on each streamline. This amounts to specifying f . Now, the manifold of solutions of equation 33 contains in it the limit of the (unique) solution of equation 29 as $\nu \to 0$, but this limit cannot be gotten, in any obvious way, by putting $\nu = 0$ before solving.* We may expect this situation for passage to both limits, $\lambda \to 0$ and $\nu \to 0$ in our equations.

Consider, for example, equation 26 for $\varkappa = 0$. When $\lambda \neq 0$ it is one of a pair of coupled differential equations; when $\lambda = 0$ it becomes $\psi = f(\psi)$, so that the pair is equivalent to one equation, equation 25, in which an undetermined function appears, and which surely requires more stringent conditions for its unique solubility.

As another example, we shall shortly see that the axially-symmetric steady flows are ruled out by a general theorem (Cowling-Chandrasekhar) which, however, would not go through on the basis of the infinite-conductivity approximation.

Clearly the province of hydromagnetics, being like hydrodynamics "doubled up", so to speak, involves these questions of non-uniform limits in even more subtle ways than does the latter subject.

(f) It is quite clear that the planar flows for which some explicit solutions are given here are not applicable even as approximate descriptions of realistic physical problems; they are illustrative only. Nevertheless it is of some interest to inquire to what extent they are com-

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*It happens that this limiting solution can be found from equation 29, when the flow is such that there is at most one set of closed nested streamlines. Feynman and Lagerstrom have shown that the limiting solution is got by putting $f(\psi)$ = constant and solving equation 29 with the usual boundary conditions used for it. The value of the constant may be found by the use of boundary-layer theory. (Private communication from Professor P. A. Lagerstrom.)

patible with physically required or physically useful boundary conditions, i.e. to inquire whether they are, at least approximately, physically self-consistent. We shall now see that, for a variety of reasons, they are at best only approximately consistent with necessary boundary conditions.

From the beginning, to achieve maximum simplicity of description the symmetries of our flows were prescribed. This is already because of the simultaneous occurrence of $\mathbb{V}$, $\mathbb{B}$, $\mathbb{j}$ and $\mathbb{E}$, much more of a restriction than the analogous thing is in hydrodynamics. Then, from analytical necessity, solutions were sought only for the case of an unbounded fluid body, without reference to asymptotic behavior or the specification of singularities. It would have been much more satisfactory, (were this feasible), to look for solutions dying out suitably at infinity -- or, in the case of a bounded fluid body, to have imposed at least the boundary conditions of vanishing normal component of fluid velocity and electric current, say by expanding ϕ and ψ in a set of base functions satisfying the boundary conditions.

The solutions we obtain have in general non-vanishing j_n and V_n necessitating return paths for the current outside the moving fluid body, and fluid sources and sinks on the boundaries.* The external currents will perturb the interior fields. Rigorously, this alone makes the solutions self-inconsistent for finite fluid bodies, but one could presumably arrange dimensions so that this perturbation would be small over most of the moving fluid. Fluid sources and sinks could be arranged to be consistent with conservation of fluid if they merely re-

*For Type II planar flows solutions with vanishing normal current may exist -- if there are solutions of equation 30, for the ϕ of interest, with the boundary condition $\frac{\partial \psi}{\partial s} = 0$, as we saw in section 3(b).

circulate the fluid outside the boundaries but, since a conducting fluid tends to convect magnetic flux, it seems impossible to imagine such a fluid source or sink in which perturbing electric currents do not also flow. Again, one could presumably render the perturbation small over much of the fluid body, with suitable geometry. Also, of course, the presence even of current-free matter outside the fluid body will perturb the unbounded space solutions, via the accompanying electric field extending outside of the fluid.*

Therefore, when we speak of planar flows in bounded fluid bodies it is to be understood as an approximation of the kind roughly indicated above. It is clear, incidentally, that for such an approximation, the fluid body must be of cylindrical shape, with its extension in the z-direction much greater than its transverse dimensions. Otherwise the currents flowing in the fluid, with the types of symmetry assumed, would not generate a magnetic field approximately consistent with these symmetries.

(g) The currents and electric fields accompanying our flows are in general such that $\mathbb{j} \cdot \mathbb{E}$ is non-vanishing.** This means that the motion is in general accompanied by the conversion of mechanical energy into heat and electrical energy. Therefore, except in special cases, even for a non-dissipative fluid, the motion must be driven by active fluid sources, i.e. "pumps", rendering electrical energy available as output, or vice versa, driven by sources of e.m.f. and rendering mechanical potential energy available at the boundaries. Thus we have

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*Matching the electric field across the boundary will require a surface charge on the boundary, as has already been stated.

**For Type I flows this is always true, even if the fluid is assued non-viscous and perfectly conducting. The statement fails to hold only for Type II flows of a perfectly conducting fluid, for which $\mathbb{j} \cdot \mathbb{E} = 0$.

a sort of fluid "dynamo".

To see this more clearly, let us examine the overall energy balance. Starting with the momentum-balance equation

$$\mathbb{F} \equiv \rho\left[\frac{\partial \mathbb{V}}{\partial t} + (\mathbb{V}\cdot\nabla)\,\mathbb{V}\right] + \nabla p = \mathbb{j} \times \mathbb{B} + \rho\nu\,\nabla^2\mathbb{V} \tag{34}$$

we have

$$\mathbb{V}\cdot\mathbb{F} = \frac{\partial}{\partial t}\left(\tfrac{1}{2}\rho V^2\right) + \nabla\cdot\left(\tfrac{1}{2}\rho V^2\mathbb{V}\right) + \nabla\cdot(p\,\mathbb{V})$$
$$= -\,\mathbb{j}\cdot(\mathbb{V}\times\mathbb{B}) + \rho\nu\,\mathbb{V}\cdot\nabla^2\mathbb{V}$$

Since $\mathbb{V}\times\mathbb{B} = \frac{\mathbb{j}}{\sigma} - \mathbb{E}$ we obtain

$$\frac{\partial}{\partial t}\left(\tfrac{1}{2}\rho V^2\right) + \nabla\cdot\left[\left(p + \tfrac{1}{2}\rho V^2\right)\mathbb{V}\right] \equiv \mathbb{V}\cdot\nabla p + \frac{D}{Dt}\left(\tfrac{1}{2}\rho V^2\right) = \frac{-j^2}{\sigma} + \mathbb{j}\cdot\mathbb{E} + \rho\nu\,\mathbb{V}\cdot\nabla^2\mathbb{V}$$

or

$$\mathbb{V}\cdot\nabla p = \mathbb{j}\cdot\mathbb{E} - \frac{j^2}{\sigma} - \rho\nu\,\mathbb{V}\nabla^2\mathbb{V} - \frac{D}{Dt}\left(\tfrac{1}{2}\rho V^2\right) \tag{35}$$

where $\frac{D}{Dt}$ is the convective derivative.

Equation 35 has a clear meaning, expressing $\mathbb{V}\cdot\nabla p$, the rate of working of the pressure forces acting on a fluid element, in terms of the rate of working of the current against the electric field, the Joule heat loss and the rate of change of the kinetic energy of the moving fluid element. In integral form equation 35 may be written, for steady state conditions, as

$$\iint\left(\tfrac{1}{2}\rho V^2 + p\right)\mathbb{V}\cdot\hat{n}\,dS = -\iiint\left(\mathbb{j}\cdot\mathbb{E} - \frac{j^2}{\sigma} + \rho\nu\,\mathbb{V}\cdot\nabla^2\mathbb{V}\right)d\tau \tag{36}$$

or, alternatively

$$\iint\left(\tfrac{1}{2}\rho V^2 + p\right)\mathbb{V}\cdot\hat{n}\,dS = -\iint\frac{(\mathbb{E}\times\mathbb{B})}{\mu_0}\cdot\hat{n}\,dS + \iiint\left(\rho\nu\,\mathbb{V}\cdot\nabla^2\mathbb{V} - \frac{j^2}{\sigma}\right)d\tau \tag{37}$$

expressing the balancing of the work done by the "kinetic pressure" at the boundaries against the energy transformed into heat by viscosity and resistivity, and the electrical energy made available by the current flowing out of the fluid.

(h) The property $\mathbb{V}\times\mathbb{B}=\varkappa\hat{e}_z$ for planar Type I steady flows of a perfectly-conducting fluid implies that one cannot find such solutions with $\varkappa\neq 0$ if the fluid is viscous and there are fixed boundaries present, or if the velocity is required to die out at infinity. Neither $\mathbb{V}$ nor $\mathbb{B}$ can approach zero anywhere unless the other becomes unbounded.

The same property also implies that one cannot have Type I planar motion of any perfectly-conducting fluid, with $\varkappa\neq 0$, in a simply-connected finite region free from sources or sinks, for there would then be at least one stagnation point, which is impossible.* The same feature is illustrated in the special solutions derived later, and may be more comprehensively stated in the form of a simple theorem as follows.

(i) For any steady Type I planar flow of a perfectly conducting fluid, with $\varkappa\neq 0$, there is a simple symmetrical relation between the velocity and magnetic stream functions, which is a consequence of the assumptions of infinite conductivity, incompressibility, and the Type I symmetry. Namely, if there exist closed contour lines $\psi=c$ (closed flux lines), then within such a curve ψ may be taken as any multiple of the area enclosed by itself. Further, the corresponding ϕ has a singularity (infinity) within the closed contour, corresponding to a source or sink of fluid, and is accordingly multi-valued in such a

*To have a steady flow within closed boundaries requires a driving agency within the fluid. One will obtain a single-valued pressure only upon making a cut in the region, the cut having the significance of an 'actuator plate' to supply mechanical energy to the fluid. This is illustrated in the flow discussed on page 114 .

way that, on each traversal of a closed ψ -curve, ϕ increases by a constant amount. In these statements, ϕ may be everywhere replaced by ψ and vice versa. Thus, closed magnetic flux lines imply a source or sink of fluid.

This result is easily proved formally as follows. If $\psi = c$ is a closed curve, so that there is a nested family of closed ψ contours within it, then either $\nabla\psi = 0$ at some interior point or ψ becomes unbounded at some interior point. The latter cannot happen, as we can see from the meaning of a stream function for a field in terms of the flux of that field. Now, equation 26, for $\lambda = 0$ states that

$$\nabla\phi \times \nabla\psi \equiv \nabla \times (\phi \nabla \psi) = \varkappa \hat{e}_z \tag{38}$$

So, at that point, within $\psi = c$, where $\nabla\psi$ vanishes, the corresponding ϕ must become unbounded.

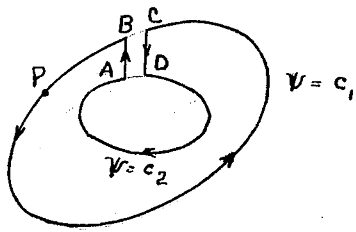

Figure 15

Let $\psi = c_1$, $\psi = c_2$ be a pair of closed ψ-contours, one lying within the other as in figure 15 and draw the cut-lines $\overline{AB}$, $\overline{CD}$ along segments of contour lines ψ = const. connecting these points. That is, a point such as P is specified by $\psi = c_1$ or c_2 and the value of ψ there.

Integrating both sides of equation 38 over the area between $\psi = c_1$ and $\psi = c_2$ with the use of Green's theorem, we obtain

$$\varkappa \iint_{(S)} \hat{e}_z \, dS = \varkappa A \oint_{(c)} \phi \nabla \psi \, d\vec{l} = \oint \phi \, d\psi$$

where A is the area contained between $\psi = c_1$ and $\psi = c_2$, and, in the line integral, we are integrating along contours of constant and along the cut-lines shown. $d\psi = 0$ along the inner and outer loops, so that

$$\varkappa A = \int_{\psi = c_1}^{\psi = c_2} (\phi_A - \phi_D) \, d\psi$$

where ψ_S and ψ_D are the values of ϕ along the outgoing and ingoing cut-lines respectively. The left-hand side here is $\neq 0$ so ϕ cannot be single-valued. But $\nabla\phi$, which determines $\mathbb{V}$, must be single-valued. Hence ϕ_A and ϕ_D must differ by some constant, γ, the amount by which ϕ increases on completing one circuit of a ψ - curve. Thus $\varkappa A = \gamma(c_2 - c_1)$. So, for any closed contour $\psi = c$ we have $A(c)$ = (area within $\psi = c$) = γc +const., and we may take ψ as any constant times this enclosed area. In virtue of the symmetry of our equations in ϕ and ψ these symbols may be everywhere interchanged in the foregoing argument.

These statements are really intuitively evident. They follow from the induction equation alone, and are consequences of $\mathbb{V} \times \mathbb{B}$ = const. and the stationary nature of the flow. This is illustrated in fig. 16.

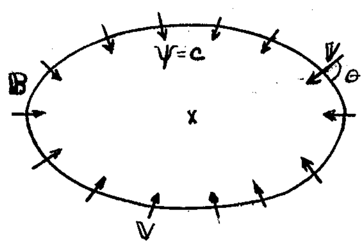

Figure 16

The presence of a source or sink of fluid within the closed curve is geometrically clear from the condition $VB\sin\theta$=const. over the closed $\mathbb{B}$ line shown; and the statement $\psi \propto$ (area enclosed) is clear if ψ = constant represents a fluid streamline, expressing the requirement that, to be stationary, the configuration must be carried continuously into itself as the $\mathbb{B}$ -lines are convected along bodily by the fluid motion (because of the infinite conductivity).

(j) It is of some interest, though apparently not a practical means of achieving new solutions, to note that, for planar Type I steady flows of a dissipationless fluid, the equations of motion and induction may be formally combined into a single equation. By a trivial scale change of coordinates, we may write equations 25 and 26 as

$$\frac{\partial(\nabla^2\phi, \phi)}{\partial(x, y)} - \frac{\partial(\nabla^2\psi, \psi)}{\partial(x, y)} = 0 \tag{25'}$$

and

$$\frac{\partial(\phi, \psi)}{\partial(x, y)} = 1 \tag{26'}$$

where $\lambda = 0$.

One may now "solve" equation 26', obtaining ϕ and ψ in terms of a "generating function", and then use equation 25' to obtain a differential equation for this function. The procedure may be sketched as follows (compare reference 20 , pp. 49-50). Introduce new variables α, β by putting

$$\begin{aligned} x &= x(\alpha, \beta) \qquad & \phi &= \phi(\alpha, \beta) \\ y &= y(\alpha, \beta) \qquad & \psi &= \psi(\alpha, \beta) \end{aligned} \tag{38}$$

where $x(\alpha, \beta)$ and $y(\alpha, \beta)$ are arbitrary differentiable functions for the moment. Using the multiplication theorem for Jacobians and multiplying equation 21' by $\dfrac{\partial(x,y)}{\partial(\alpha,\beta)}$, we may write

$$\frac{\partial(\phi,\psi)}{\partial(\alpha,\beta)} = \frac{\partial(x,y)}{\partial(\alpha,\beta)} \tag{40}$$

as exact equivalent for equation 26'.

Now, whatever $x(\alpha,\beta)$ and $y(\alpha,\beta)$ may be, solutions of equation 40 go over into solutions of equation 26' by use of equation 39. So we are at liberty to impose the further conditions that

$$\begin{aligned} x(\alpha,\beta) + \phi(\alpha,\beta) &= 2\alpha \\ y(\alpha,\beta) + \psi(\alpha,\beta) &= 2\beta \end{aligned} \tag{41}$$

thus specializing the transformations of equation 39 somewhat. Relations 31 are, however, identically satisfied by putting

$$\begin{aligned} x(\alpha,\beta) &= \alpha + P(\alpha,\beta) \qquad & y(\alpha,\beta) &= \beta + Q(\alpha,\beta) \\ \phi(\alpha,\beta) &= \alpha - P(\alpha,\beta) \qquad & \psi(\alpha,\beta) &= \beta - Q(\alpha,\beta) \end{aligned} \tag{42}$$

where $P(\alpha,\beta)$ and $Q(\alpha,\beta)$ are, as far as equations 42 are concerned, arbitrary functions, but must be particularized so that equation 40 is satisfied.

Using equation 42 we compute

$$\frac{\partial(x,y)}{\partial(\alpha,\beta)} = 1 + \frac{\partial P}{\partial \alpha} + \frac{\partial Q}{\partial \beta} + \frac{\partial P}{\partial \alpha}\frac{\partial Q}{\partial \beta} - \frac{\partial P}{\partial \alpha}\frac{\partial Q}{\partial \beta}$$

and

$$\frac{\partial(\phi,\psi)}{\partial(\alpha,\beta)} = 1 - \frac{\partial P}{\partial \alpha} - \frac{\partial Q}{\partial \beta} + \frac{\partial P}{\partial \alpha}\frac{\partial Q}{\partial \beta} - \frac{\partial P}{\partial \alpha}\frac{\partial Q}{\partial \beta}$$

Consequently, to satisfy equation 40 we require that

$$\frac{\partial P}{\partial \alpha}, + \frac{\partial Q}{\partial \beta} = 0$$

which is satisfied identically by putting

$$P(\alpha, \beta) = \frac{\partial W(\alpha, \beta)}{\partial \beta} \quad ; \qquad Q(\alpha, \beta) = -\frac{\partial W(\alpha, \beta)}{\partial \alpha}$$

where $W(\alpha, \beta)$ is an arbitrary function of these variables.

So we generate an implicit solution of equation 40 by the relations

$$\begin{aligned} x &= \alpha + \frac{\partial W}{\partial \beta} & \qquad \phi &= \alpha - \frac{\partial W}{\partial \beta} \\ y &= \beta - \frac{\partial W}{\partial \alpha} & \qquad \psi &= \beta + \frac{\partial W}{\partial \alpha} \end{aligned} \tag{43}$$

where $W(\alpha, \beta)$ must satisfy the condition that the Jacobians be non-zero, namely

$$D = \frac{\partial(x, y)}{\partial(\alpha, \beta)} \equiv \frac{\partial(\phi, \psi)}{\partial(\alpha, \beta)} \equiv 1 + \frac{\partial^2 W}{\partial \alpha^2}\frac{\partial^2 W}{\partial \beta^2} - \left(\frac{\partial^2 W}{\partial \alpha \partial \beta}\right)^2 \neq 0 \tag{44}$$

but is otherwise arbitrary.

We may now formally insert equations 43 into the equation of motion (equation 25'), obtaining a single (albeit very complicated) non-linear equation in W alone, any solution of which would give, via equations 43, a solution of the coupled equations 25' and 26'. This procedure will be recognized as quite similar to the finding of solutions of the Hamilton canonical equations of mechanics by means of a contact transformation defined through a generating function.

(k) Let us at this point consider, briefly, the flows with axially symmetric geometry. As was already mentioned, the steady flows of this kind are probably of no interest, for reasons now to be set forth.

In 1934, Cowling (23) gave a very simple and most elegant proof that, when the magnetic field and the fluid motions are confined to the meridian planes of an axially symmetric configuration, no stationary hydromagnetic flow is possible. Implicit in the proof are the further, most natural, assumptions that no singularities are to be allowed in either field and that there is no magnetic flux at infinity.

This firmly established theorem has played a central role for over two decades in the attempts to construct mechanisms for the operation of a fluid dynamo. Type I axially symmetric steady flows are thus impossible in realistic mechanisms, but Type II flows do not conflict with Cowling's result. Very recently, however, by an ingenious generalization of Cowling's arguments, Backus and Chandrasekhar (24)* have demonstrated the stronger theorem that, provided the magnetic field is smooth enough to be three times continuously differentiable, <u>no</u> stationary axially symmetric hydromagnetic flow is possible. That is, the assumption that $\mathbb{V}$ and $\mathbb{B}$ lie in meridian planes is not needed.

On the basis of this result it would seem that there is probably no point in seeking stationary solutions for axially symmetric flows of either type. The generalization given by Backus and Chandrasekhar follows the general lines of Cowling's argument, and it seems worth while to reproduce here the essence of Cowling's original argument.

The conclusion is drawn entirely from the equation of induction, without reference to the equation of motion, as follows. Introduce cylindrical coordinates r, ϕ, z. The current is given by $\mathbb{j} = \sigma(\mathbb{E} + \mathbb{V} \times \mathbb{B})$. Now, $\mathbb{E}$ cannot have a ϕ-component, for if it did we could not simultaneously have axial symmetry and $\nabla \times \mathbb{E} = 0$. But $\mathbb{j} = \frac{1}{\mu} \nabla \times \mathbb{B}$ and $\mathbb{V} \times \mathbb{B}$ are azimuthally directed, so that $\mathbb{E} = 0$. Since $\nabla \cdot (\rho \mathbb{V}) = 0 = \nabla \cdot \mathbb{B}$ we may introduce a velocity stream function $\Phi(r, z)$ and a magnetic stream function $\Psi(r, z)$ such that

*Reference 24 came to the attention of the author only very shortly before the conclusion of the present work.

$$V_z = \frac{1}{2\pi}\frac{\partial \Phi}{\partial r}, \qquad V_r = -\frac{1}{2\pi r}\frac{\partial \Phi}{\partial z}$$

$$\frac{B_z}{\mu_0} = \frac{1}{2\pi}\frac{\partial \Psi}{\partial r}, \qquad \frac{B_r}{\mu_0} = -\frac{1}{2\pi r}\frac{\partial \Psi}{\partial z} \tag{45}$$

These stream functions also denote the total flux of the corresponding fields through a circle, normal to the z axis, and passing through (r, z) .

Assume now that there is no net flux of $\mathbb{B}$ through such a plane in the limit as $r \to \infty$, i.e. that the electric currents lie in a finite region. Then $\Psi = 0$ for $r = 0$ and for $r = \infty$ and so, in each meridional half-plane, Ψ must have a maximum at some 'critical point' at which

$$\frac{\partial \Psi}{\partial r} = \frac{\partial \Psi}{\partial z} \qquad \frac{\partial^2 \Psi}{\partial r^2} + \frac{\partial^2 \Psi}{\partial z^2} \neq 0 \tag{46}$$

Now, since $\mathbb{E} = 0$, $\dot{\mathbb{J}} = \sigma \mathbb{V} \times \mathbb{B}$. We have also that

$$\dot{\mathbb{J}} = \nabla \times \frac{\mathbb{B}}{\mu} = -\frac{1}{2\pi}\left(\frac{\partial^2 \Psi}{\partial z^2} + \frac{\partial^2 \Psi}{\partial r^2} + \frac{1}{r}\frac{\partial \Psi}{\partial z}\right) \tag{47}$$

and

$$\mathbb{V} \times \mathbb{B} = \frac{\mu_0}{4\pi^2 \rho}\left(\frac{\partial \Phi}{\partial z}\frac{\partial \Psi}{\partial r} - \frac{\partial \Phi}{\partial r}\frac{\partial \Psi}{\partial z}\right) \tag{48}$$

So we require

$$\left(\frac{\partial^2 \Psi}{\partial z^2} + \frac{\partial^2 \Psi}{\partial r^2} - \frac{1}{r}\frac{\partial \Psi}{\partial r}\right) = \frac{\mu_0 \sigma}{2\pi \rho}\left(\frac{\partial \Phi}{\partial z}\frac{\partial \Psi}{\partial r} - \frac{\partial \Phi}{\partial r}\frac{\partial \Psi}{\partial z}\right) \tag{49}$$

But equation 48 is inconsistent with equations 46 at the critical point, where $\mathbb{B}$ vanishes. Physically, the current cannot support the magnetic field in the neighborhood of the critical point. We note that this argument depends on σ being finite; were one to start with the infinite conductivity idealization the theorem would obviously not go through.

It has been stated in the literature (see, e.g. reference 9) that steady hydromagnetic flows are impossible in two dimensions, including the case of planar motion. This is considered to follow from the same considerations that enter into Cowling's theorem, or at least to be plausible on the basis of that result. Yet it seems by no means evident how to prove Cowling's theorem directly for this case, and one may well doubt whether the statement is correct. The examples explicitly given in this study cannot, however, be considered as counter examples because, rigorously, they refer only to an unbounded fluid.

4. Solution of the Steady-Flow Equations.

We come now to the question of solving our equations of motion and induction. Unfortunately, as far as analytical solutions are concerned, very little can be done, with the possible exception of planar Type II flow, for which no solutions were attempted. Exact solutions of any degree of generality are totally out of the question; the analytical problems lie beyond the range of available mathematical methods. Even perturbation procedures, iterating back and forth between solutions of the equations of motion and induction, which can in principle be set up in several ways, are very difficult of execution, although this question must be answered with reference to specific problems with specific boundaries, etc.

After a variety of attempts at seeking solutions (special "ansatzes", "similarity solutions", power series solutions, reduction to a single equation), one very simple procedure yielded some solutions for Type I flow. This procedure is as follows.

If we are considering the motion in an orthogonal coordinate system u^1, u^2, pick one of the stream functions, say ϕ, to be a function of only one of the coordinates, say u^1. Substituting $\phi(u^1)$ into the equation of induction we may often perform a formal integration, obtaining the corresponding $\psi(u^1, u^2)$ in terms of $\phi'(u^1)$ and a function $f(u^2)$, to be determined along with $\phi(u^1)$. The necessary derivatives are then computed and substituted into the equation of motion. If one is lucky (and this seems to depend on the coordinate system in a way hard to judge in advance) one can then split off a pair of non-linear ordinary differential equations from which f and ϕ may be determined, one at a time. In most cases, the equations for f and ϕ will be coupled. We now illustrate this procedure for planar Type I motions.

a. Planar Type I Steady Flows of a Dissipationless Fluid.

The equations of motion and induction are

$$\frac{\partial(\nabla^2\phi, \phi)}{\partial(x, y)} = \frac{\partial(\nabla\psi, \psi)}{\partial(x, y)} \tag{50}$$

and

$$\frac{\partial(\phi, \psi)}{\partial(x, y)} = K \tag{51}$$

Put $\phi = \phi(x)$. Then, from equation 50,

$$\psi(x, y) = \frac{Ky}{\phi'(x)} + f(x) \tag{52}$$

where $f(x)$ is to be determined. Accordingly $\nabla^2\phi = \phi''(x)$ and $\nabla^2\psi = -\frac{K}{\phi'^2}\phi''y + f''(x) + \frac{2K}{\phi'^3}\phi''^2$. Inserting these in equation 50 yields, after reduction,

$$\frac{Ky}{\phi'^2}\left[4\phi''^3 - 5\phi'\phi''\phi''' + \phi'^2\phi^{(iv)}\right] - \phi'^2 f''' + \left[2\phi''^2 - \phi'\phi'''\right]f' = 0$$

Clearly, we may split off from this a pair of equations, which will permit determination of ϕ and f in sequence. Equating separately to zero the coefficient of y and the sum of the other terms gives

$$4\phi''^3 - 5\phi'\phi''\phi''' + \phi'^2\phi^{(IV)} = 0 \tag{53}$$

and

$$\phi'^2 f''' + \left[\phi'\phi''' - 2\phi''^2\right] f' = 0 \tag{54}$$

Putting $\phi' = e^u$ converts these respectively into

$$u''' - 2u'u'' = 0 \tag{55}$$

and

$$f''' + \left[u'' - u'^2\right] f' = 0 \tag{56}$$

Equation 55 integrates immediately to yield

$$u'' = u'^2 - a^2 \tag{57}$$

Putting $u' = \zeta$ gives $\dfrac{d\zeta}{\zeta^2 - a^2} = dx$; $\zeta = -a\coth(ax+b)$, so that $u = -\log\sinh(ax+b) + \text{const.}$ Then $\phi' = e^u = A\,\mathrm{csch}(ax+b)$ and

$$\phi(x) = A\int \frac{dx}{\mathrm{csch}(ax+b)} = A_1 \log\left|\tanh\tfrac{1}{2}(ax+b)\right|$$

For this solution, $u'^2 - u'' = a^2$. Then, from equation 47, $f''' - a^2 f' = 0$ so that $f(x) = B_1\sinh(ax+d)$. In this way we obtain our first solution

$$\begin{aligned} \phi(x) &= A_1 \log\tanh\left|\tfrac{1}{2}(ax+b)\right| \\ \psi(x,y) &= \frac{Ky}{A_1 a}\sinh(ax+b) + B_1\sinh(ax+d) \end{aligned} \tag{58}$$

Correspondingly,

$$\begin{aligned} &V_x = 0\,, && V_y = aA_1\,\mathrm{csch}(ax+b) \\ &\frac{B_x}{\sqrt{\mu_0\rho}} = -\frac{K}{A_1 a}\sinh(ax+b), && B_y = \frac{Ky}{A_1}\cosh(ax+b) + aB_1\cosh(ax+d) \end{aligned} \tag{59}$$

$$\sqrt{\frac{\mu_0}{\rho}}\,\mathbf{j} = \hat{e}_z\nabla^2\psi = \hat{e}_z\left(\frac{aKy}{A_1}\sinh(ax+b) + a^2B_1\sinh(ax+d)\right)$$

The velocity is a parallel flow field in the y -direction, with a velocity gradient in the x -direction. The faster the fluid motion at a point, the more nearly parallel $\mathbb{V}$ and $\mathbb{B}$ are. Since $(\mathbb{V}\cdot\nabla)\mathbb{V}=0$ for a parallel flow, we have $\nabla p = \frac{(\nabla\times\mathbb{B})\times\mathbb{B}}{\mu} = \mathbb{J}\times\mathbb{B}$ and $p(x,y) = \frac{\alpha K^2 y^2}{2A_1^2}\sinh^2(\alpha x+b) + \frac{\alpha B_1 K y}{A_1}\sinh(\alpha x+b)\sinh(\alpha x+d) + \frac{\alpha^3 B_1^2}{2}\sinh^2(\alpha x+d) + \text{const.}$

That part of the pressure which is independent of K is perpendicular to the velocity. "Dynamo action", necessitating a component of ∇p parallel to $\mathbb{V}$, occurs when $K\neq 0$.

Similarly, taking $i\alpha$ for the previous "a" in equation 48 we obtain a second solution

$$\phi(x) = A_2 \log\left|\tan\tfrac{1}{2}(\alpha x+b)\right|$$
$$\psi(x,y) = \frac{Ky}{A_1\alpha}\sin(\alpha x+b) + B_2\sin(\alpha x+d) \tag{60}$$

Correspondingly,

$$V_x = 0 \quad , \qquad V_y = \alpha A_2 \csc(\alpha x+b)$$
$$\frac{B_x}{\sqrt{\mu_0\rho}} = -\frac{K}{A_2\alpha}\sin(\alpha x+b) \;; \qquad B_y = \frac{Ky}{A_2}\cos(\alpha x+b) + \alpha B_2\cos(\alpha x+d) \tag{61}$$

Taking "α"$=0$ in equation 48 gives $u''-u'^2=1$. Solving anew by the same procedure, $\zeta=\frac{du}{dx}=-\frac{\alpha}{\alpha x+b}$, so $u=\log\frac{A}{\alpha x+b}$, and $\phi'=e^u=\frac{A}{\alpha x+b}$.

Then

$$\phi(x) = A_3\log|(\alpha x+b)|$$

Correspondingly, $u'^2-u''=0$, so $f'''(x)=0$, $\quad f(x)=gx^2+hx$

We thus have a third solution

$$\phi = A_3\log|(\alpha x+b)|$$
$$\psi = \frac{Ky(\alpha x+b)}{A_3\alpha} + gx^2+hx \tag{62}$$

for which

$$V_x = 0 \quad , \qquad V_y = \frac{\alpha A_3}{\alpha x + b}$$

$$\frac{B_x}{\sqrt{\mu_0 \rho}} = -\frac{\kappa}{A_3 \alpha}(\alpha x + b) \qquad B_y = \frac{\kappa y}{A_3} + 2gx + h \tag{63}$$

In this case we have a uniform current, given by $\sqrt{\frac{\mu_0}{\rho}}\,\mathring{j} = \hat{e}_z \nabla^2 \psi = 2g$

We obtain another solution by putting $u' = \alpha$ in equation 48, identically satisfying it. Then $u(x) = \alpha x + b$ and so we find the solution

$$\begin{aligned} \phi(x) &= A_4 e^{\alpha x} \\ \psi(x, y) &= \frac{\kappa y}{A_4 \alpha} e^{-\alpha x} + B_4 e^{\alpha x} \end{aligned} \tag{64}$$

for which

$$\begin{aligned} V_x &= 0 \quad , & V_y &= \alpha A_4 e^{\alpha x} \\ \frac{B_x}{\sqrt{\mu_0 \rho}} &= \frac{-\kappa}{A_4 \alpha} e^{-\alpha x} \quad , & \frac{B_y}{\sqrt{\mu_0 \rho}} &= -\frac{\kappa y}{A_4} e^{-\alpha x} + \alpha B_4 e^{\alpha x} \end{aligned} \tag{65}$$

These four solutions each involve four essential constants, the other two relating to placement of the origin.

Finally, an obvious fifth solution is

$$\begin{aligned} \phi(x) &= A_5 x \\ \psi(x, y) &= \frac{\kappa y}{A_5} + f(x) \end{aligned} \tag{66}$$

with $f(x)$ an arbitrary function. From all these solutions, of course, new ones may be trivially found by interchanging ϕ and ψ.

The foregoing are all exact solutions, everywhere, of the equations of motion and induction for Type I planar flows of a dissipationless fluid. They were derived on the assumption of an unbounded fluid, but yet the first four must necessarily become infinite in unbounded space. Rigorously, then, they do not fulfill obvious physical requirements. Further, it is by no means clear what sorts of conditions should be used to determine the constants and thus specify a unique solution.

Suppose that, in the approximate sense discussed in section 3f, we seek to use the above solutions to describe a steady flow within a region bounded by insulating non-magnetic planes $x = x_1$, $x = x_2$ and $y = y_1$, $y = y_2$, the extension of this region in the z-direction being large compared with its lateral dimensions.*

The foregoing solutions describe flows with velocity field in the y-direction and a velocity gradient in the x-direction. Clearly, in each case, a distribution of fluid sources is required on the boundary $y = y_1$, say, and a distribution of fluid sinks is required at $y = y_2$. The necessary fluid boundary conditions (vanishing normal component of fluid velocity at a rigid boundary) are automatically satisfied on $x = x_1$, $x = x_2$. On the approximation of vacuum surroundings, then, as far as ordinary electromagnetic and hydrodynamical boundary conditions are concerned, there are no further conditions to be imposed on our solutions to determine the arbitrary constants. What further conditions are necessary is by no means clear. One can impose, in various ways, conditions which will specify a unique flow in each case, but our small collection of particular solutions is too limited to give any strong clues about this. It seems clear, however, that -- just as in the perfect-fluid rotational flows of ordinary hydrodynamics -- a unique solution is not fixed by boundary conditions alone.

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*With vacuum outside the fluid body, simple arguments show that, with a prescribed 'interior' solution for $\mathbb{V}$, $\mathbb{B}$ (and hence $\mathbb{E}$) satisfaction of the necessary boundary conditions for $\mathbb{B}$ and $\mathbb{E}$ will require a surface charge layer and a surface current.

A simple solution in plane polar coordinates (r, θ) is readily derivable as follows.

The equations of motion and induction are

$$\frac{\partial(\nabla^2\phi, \phi)}{\partial(r, \theta)} = \frac{\partial(\nabla^2\psi, \psi)}{\partial(r, \theta)} \tag{67}$$

and

$$\frac{\partial(\phi, \psi)}{\partial(r, \theta)} = \kappa r \tag{68}$$

Put $\psi = \frac{Ar^2}{2}$. Then equation 68 yields $\phi(r,\theta) = -\frac{\kappa\theta}{A} + f(r)$. Inserting this into equation 67 gives $\frac{\partial\phi}{\partial\theta}\frac{\partial}{\partial r}(\nabla^2\phi) = 0$, or $\nabla^2\phi = 2\beta = const.$ Hence $f(r) = \alpha \log r + \frac{\beta r}{2}$. Our solution is then

$$\begin{aligned} \psi(r) &= \frac{Ar^2}{2} \\ \phi(r, \theta) &= -\frac{\kappa\theta}{A} + \alpha \log r + \beta\frac{r^2}{2} \end{aligned} \tag{69}$$

This solution prescribes a uniform current density in the z-direction, with $\sqrt{\frac{\mu}{\rho}}\,\mathring{j} = \hat{e}_z \nabla^2\psi = 2A\hat{e}_z$, and a fluid source or sink of strength $Q = \frac{2\pi\kappa}{A}$ centered at the origin. The fluid streamlines $\phi = const.$ spiral outward or inward, depending on the sign of κ, while the magnetic field is purely azimuthal. We have

$$\begin{aligned} \frac{B_\theta}{\sqrt{\mu_0\rho}} &= Ar & B_r &= 0 \\ V_\theta &= \frac{\alpha}{r} + \beta r & V_r &= \frac{\kappa}{Ar} \end{aligned} \tag{70}$$

The pressure gradient is $\nabla p = \frac{(\nabla\times B)\times B}{\mu_0\rho} - (V\cdot\nabla)V$. Using the identity

$$(V\cdot\nabla)V = \hat{e}_r\left[(V\cdot\nabla)v_r - \frac{v_\theta^2}{r}\right] + \hat{e}_\theta\left[(V\cdot\nabla)v_\theta + \frac{v_r v_\theta}{r}\right]$$

we find

$$\frac{\nabla p}{\rho} = \hat{e}_r\left[\left(\alpha^2 + \frac{\kappa^2}{A^2}\right)\frac{1}{r^3} + (\beta^2 - 2A^2)r + \frac{2\alpha\beta}{r}\right] - \frac{2\beta\kappa}{Ar}\hat{e}_\theta ,$$

so that

$$p(r,\theta) = 2\alpha\beta \log r + \left(\frac{\beta^2}{2} - A^2\right) r^2 - \frac{1}{2}\left(\alpha^2 + \frac{\kappa^2}{A^2}\right)\frac{1}{r^2} - \frac{2\beta\kappa}{A}\theta + \text{const.} \quad (71)$$

We see immediately that the pressure function is not single-valued in any circular region about the source. In accordance with the remarks in the footnote on page 87, in order to interpret this solution we must insert, say radially outward from the origin a "cut" or "actuator plate", across which the pressure everywhere undergoes a jump of amount $\frac{2\beta\kappa}{A}$. This can hardly be realized in concrete physical form but can exist in principle. Part of the pumping action, keeping the fluid in steady motion against magnetic braking forces, occurs across this cut line -- the rest occurs at the fluid source. The energy fed into the fluid at these places appears in the form of electrical energy available at the ends of the fluid cylinder. The energy balance given in equation 36 may readily be explicitly verified, taking the 'boundaries' as a pair of concentric circles and the ('two-sided') cut-line. Had we allowed for a driving agency from the beginning by including a rotational external driving force in the equation of motion, we would have found a different solution for which, however, the artifice of the 'cut' would not be necessary.

If we take as bounding region the cut line and a pair of concentric cylinders, one of which is a source and the other a sink, our remarks following equation 66 hold here as well. It is open to conjecture which are the proper auxiliary conditions, going with equations 67 and 68 to determine a unique solution.

b. Inclusion of Dissipation in the Fluid

Let us examine what equations result and what solutions are obtainable by the same simple method in seeking exact solutions for planar Type I flow with a viscous, finitely-conducting fluid. Our equations are

$$\frac{\partial(\nabla^2\phi, \phi)}{\partial(x, y)} - \frac{\partial(\nabla^2\psi, \psi)}{\partial(x, y)} + \nu \nabla^4\phi = 0 \tag{73}$$

and

$$\frac{\partial(\phi, \psi)}{\partial(x, y)} - \lambda \nabla^2\psi = K \tag{74}$$

It is hopeless to seek such solutions if λ, ν, K are all non-zero. Let us, however, look at the cases where $K=0$ and $\nu=0$, respectively, and examine the equations encountered in seeking solutions by our simple method.

(i) $\underline{K=0}$, $\underline{\nu\neq 0}$ (viscous Type I flow in cases where $\mathbb{J}=\sigma(\mathbb{V}\times\mathbb{B})$; $\mathbb{E}=0$). Putting $\psi=\psi(x)$ gives, after considerable computation,*

$$\phi(x, y) = f(x) - \lambda y W(x) \tag{75}$$

where
$$\begin{aligned} W(x) &\equiv \frac{d}{dx}\log\psi'(x) \\ WW''' - W'W'^2 &= \frac{\nu}{\lambda}W^{(iv)} \\ -f'''W + f'W'' + \frac{\nu}{\lambda}f^{(iv)} &= 0 \end{aligned} \tag{76}$$

Putting $Z(x)\equiv f'(x)$ gives, then

$$\begin{aligned} \frac{d}{dx}\left(\frac{W''}{W}\right) &= \frac{\nu}{\lambda}\left(\frac{Z'''}{Z^2}\right) \\ ZW'' - WZ'' &= -\frac{\nu}{\lambda}Z''' \end{aligned} \tag{77}$$

which seem intractable unless $\nu=0$.

(ii) $\underline{\nu=0}$, $\underline{\lambda\neq 0}$ (Type I planar flow of an inviscid resistive fluid). Putting $\psi=\psi(x)$ we obtain, from equation 66

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*We could not proceed here by choosing ϕ dependent on one coordinate; it can only be ψ .

$$\phi(x, y) = f(x) - y\left(\frac{\kappa}{\psi'} + \lambda\frac{d}{dx}\log\psi'\right)$$

From equation 65 we may then split off the equations

$$\frac{\kappa^2}{\psi'^3}\psi^{iv} - \frac{5\kappa^2}{\psi'^4}\psi''\psi''' + \frac{4\kappa^2\psi''^3}{\psi'^5} - \frac{\lambda\kappa}{\psi'}\frac{d^4}{dx^4}\log\psi' + \frac{\lambda\kappa}{\psi'^3}\psi''\psi''' -$$
$$- \frac{6\lambda\kappa\psi''^2\psi'''}{\psi'^4} + \frac{6\lambda\kappa}{\psi'^5}\psi''^4 - \lambda^2\frac{\psi''}{\psi'}\frac{d^4}{dx^4}\log\psi' - \frac{\lambda\kappa}{\psi'^2}\frac{d^2}{dx^2}\log\psi' +$$
$$+ 2\lambda\kappa\frac{\psi''^2}{\psi'^3}\frac{d^2}{dx^2}\log\psi' + \lambda^2\left(\frac{d^2}{dx^2}\log\psi'\right)\left(\frac{d^3}{dx^3}\log\psi'\right) -$$
$$- \frac{\lambda\kappa}{\psi'^2}\psi''\frac{d^3}{dx^3}\log\psi' = 0 \qquad (78)$$

and

$$\frac{f'''}{f'}\left[\frac{\kappa}{\psi'} + \lambda\frac{\psi''}{\psi'}\right] + \left[\frac{\kappa\psi''}{\psi'^2} - \frac{2\kappa\psi''^2}{\psi'^3} - \lambda\frac{d^3}{dx^3}\log\psi'\right] = 0 \qquad (79)$$

Examination of equations 76-77 and 78-79 shows that we can at best treat only the case $\underline{\kappa = 0}$ and $\underline{\nu = 0}$*. (Type I planar flow of an inviscid resistive fluid in the special case where $\mathbb{E} = 0$.) For this special case the equations become, upon putting $W(x) \equiv \frac{d}{dx}\log\psi'(x)$

$$WW''' = W'W'' \qquad (80)$$

and

$$f'''W = f'W'' \qquad (81)$$

which are immediately soluble. We obtain the following solutions

$$\psi'(x) = A\exp\left(c_1x^2 + c_2x\right)$$
$$\phi(x, y) = \left(c_3e^{mx} + c_4e^{-mx}\right) - \lambda my\left(c_1e^{mx} - c_2e^{-mx}\right) \qquad (82)$$

for which

$$V_x = m\lambda\left(c_1e^{mx} - c_2e^{-mx}\right) \qquad \frac{B_y}{\sqrt{\mu\rho}} = A\exp\left[c_1e^{mx} + c_2e^{-mx}\right]$$

$$V_y = m\left[c_3e^{mx} - c_4e^{-mx} - \lambda my\left(c_1e^{mx} + c_2e^{-mx}\right)\right] \qquad B_x = 0 \qquad (83)$$

---

*Since $\lambda \neq 0$, the stipulation $\kappa = 0$ does not mean that $\mathbb{V}$ and $\mathbb{B}$ are parallel.

Also,

$$\psi'(x) = A \exp\left(c_1 \cos mx + c_2 \sin mx\right)$$
$$\phi(x,y) = c_3 \cos mx + c_4 \sin mx - \lambda m y\left(c_2 \cos mx - c_1 \sin mx\right) \tag{84}$$

for which

$$V_x = m\lambda\left(c_2 \cos mx - c_1 \sin mx\right), \quad B_x = 0, \quad \frac{B_y}{\sqrt{\mu\rho}} = A \exp\left(c_1 \cos mx + c_2 \sin mx\right)$$
$$V_y = m\left(c_4 \cos mx - c_3 \sin mx\right) + \lambda m^2 y\left(c_2 \sin mx + c_1 \cos mx\right) \tag{85}$$

and finally,

$$\psi'(x) = A \exp\left(c_1 x^2 + c_2 x\right)$$
$$\phi(x,y) = c_3 x^2 + c_4 x - \lambda y\left(2c_1 x + c_2\right) \tag{86}$$

for which

$$V_x = 2c_1 x + c_2 \qquad V_y = -2c_3 x + 2c_1 \lambda y + c_4$$
$$B_x = 0 \qquad \frac{B_y}{\sqrt{\mu\rho}} = A \exp\left(c_1 x^2 + c_2 x\right) \tag{87}$$

Again, these solutions can at best be used to describe flows in bounded regions, because they are unbounded in infinite space. It is not clear what restrictions on the choice of coefficients are physically necessary to allow for the damping effects of resistivity. If we seek a solution in plane polar coordinates for the same case we start from the equations

$$\frac{\partial(\nabla^2\phi, \phi)}{\partial(x, y)} = \frac{\partial(\nabla^2\psi, \psi)}{\partial(x, y)} \quad \text{and} \quad \frac{\partial(\phi, \psi)}{\partial(x,y)} = \lambda r \nabla^2\psi \tag{89}$$

Choose $\psi = \psi(r)$ (azimuthal $\mathbb{B}$ -lines). After some manipulation we split off the ordinary equations

$$\left(r\alpha''\right)' - \frac{\alpha'}{\alpha}\left(r\alpha'\right)' = \frac{\alpha'}{r}$$

and

$$r^2 f''' + r f'' - r f - r^2 \frac{\alpha'' f'}{\alpha} - r \frac{f'^2}{\alpha} = 0 \tag{90}$$

where $\alpha(r) \equiv 1 + r\frac{d}{dr}\log\psi'$ and $\phi(r,\theta) \equiv f(r) - \frac{\lambda\theta}{\psi'}\left[\psi' + r\psi''\right]$

$f = f(r)$

And these, too, seem hard to manage analytically.

IV. HYDROMAGNETIC SURFACE WAVES.

In this chapter we consider the case of a very deep 'lake' of dissipationless incompressible conducting fluid, permeated by a constant uniform magnetic field and situated in a uniform gravitational field. We investigate whether there exist characteristic small-amplitude gravity surface-waves different from those known in hydrodynamics..

By the term "surface wave" we mean a solution of the hydromagnetic equations with amplitude dying out to zero at great depths, $z \to -\infty$, and increasing with z so that they would diverge for an unbounded fluid, thus making the surface a necessary feature. By "characteristic waves" we mean disturbances with a characteristic dispersion law for their Fourier components, and so described by a characteristic differential equation. We shall see that, if the magnetic field has a component perpendicular to the undisturbed surface, there are no surface waves at all, the energy of any initial disturbance being transmitted with undiminished intensity downward along the magnetic field. If the field is tangential to the undisturbed surface, one may set up, in many arbitrary ways, disturbances localized near the surface, but there are no characteristic surface waves.

Let the free boundary of the undisturbed fluid be the plane $z = 0$, with $z < 0$ in the fluid below and $z > 0$ in the vacuum above. Two cases will be considered in succession -- the case where the constant field $\mathbf{B}_0$ is oblique to the undisturbed surface, and the case where it is tangential thereto.

(1). Oblique Magnetic Field

Let $\mathbf{b}^+$ and $\mathbf{b}^-$ be the induced magnetic fields in the vacuum space and in the fluid, respectively. Let the form of the disturbed surface be $z = \zeta(x, y, t)$. Define $\Pi \equiv p + U$, where p is the total pressure at point, and $U = \rho g z$ where ρ is the fluid density and g the acceleration of gravity. The boundary conditions at the free surface are then the following:

$$\mathbf{b}^{+}(x, y, \zeta, t) = \mathbf{b}^{-}(x, y, \zeta, t) \qquad (1)$$

(There can be no surface current, for otherwise the surface layer would have unbounded acceleration.)

$$V_z(x, y, \zeta, t) = \frac{D}{Dt}\zeta(x, y, t) \qquad (2)$$

(The fluid has no normal relative velocity with respect to the instantaneous surface.)

$$\Pi(x, y, \zeta, t) = \rho g \zeta(x, y, t) \qquad (3)$$

(Condition of zero pressure at the instantaneous surface, following from equation 1 and the fact that $p = 0$ for $z > 0$.)

Neglecting second-order quantities in the wave amplitudes, conditions, equations 1 to 3 can be replaced by the linearized approximations

$$\mathbf{b}^{+}(x, y, 0, t) = \mathbf{b}^{-}(x, y, 0, t) \qquad (1')$$

$$v_z(x, y, 0, t) = \frac{\partial \zeta}{\partial t} \qquad (2')$$

$$\Pi(x, y, 0, t) = \rho g \zeta \qquad (3')$$

Further necessary relations are

$$\rho_0 \frac{\partial \mathbf{V}}{\partial t} = \frac{(\nabla \times \mathbf{b}^{-})}{\mu_0} \times \mathbf{B}_0 - \nabla \Pi = \frac{(\mathbf{B}_0 \cdot \nabla)\mathbf{b}^{-}}{\mu_0} - \nabla\left(\Pi + \frac{\mathbf{B}_0 \cdot \mathbf{b}^{-}}{\mu_0}\right) \qquad (4)$$

(Linearized equation of motion, with gravity body-force added)

$$\nabla \cdot \mathbb{b} = 0 \qquad \text{(Maxwell equation)} \tag{5}$$

$$\nabla \cdot \mathbb{V} = 0 \qquad \text{(Incompressibility assumption)} \tag{6}$$

$$\frac{\partial \mathbb{b}}{\partial t} = \nabla \times (\mathbb{V} \times \mathbb{B}_0) = (\mathbb{B}_0 \cdot \nabla) \mathbb{V} \qquad \text{(Equation of induction)} \tag{7}$$

$$\nabla \times \mathbb{b}^+ = \nabla^2 \mathbb{b}^+ = 0 \qquad \text{(No currents in vacuum space above the fluid)} \tag{8}$$

Combining equations 2' and 3' yields

$$\dot{\Pi}(x, y, 0, t) = \rho g v_z (x, y, 0, t) \tag{9}$$

From equation 1' and the fact that the only possible discontinuities can occur just across the plane $z = 0$ it follows that $\frac{\partial b_x}{\partial x}$ and $\frac{\partial b_y}{\partial y}$ are continuous across $z = 0$. Therefore, in virtue of equation 5 we have

$$\left.\frac{\partial b_z^+}{\partial z}\right|_{z=0} = \left.\frac{\partial b_z^-}{\partial z}\right|_{z=0} \tag{10}$$

Taking the divergence of equation 4 yields

$$\nabla^2 \left(\Pi + \frac{\mathbb{B}_0 \cdot \mathbb{b}^-}{\mu_0} \right) = 0 \tag{11}$$

Combining equation 4 and equation 7 gives

$$\left[\frac{\partial^2}{\partial t^2} - \frac{(\mathbb{B}_0 \cdot \nabla)^2}{\mu_0 \rho} \right] \mathbb{b}^- = -(\mathbb{B}_0 \cdot \nabla) \nabla W \tag{12}$$

and where $W = \frac{\Pi}{\rho} + \frac{\mathbb{B}_0 \cdot \mathbb{b}}{\mu_0 \rho}$

$$\left[\frac{\partial^2}{\partial t^2} - \frac{(\mathbb{B}_0 \cdot \nabla)^2}{\mu_0 \rho} \right] \mathbb{V} = -\frac{\partial}{\partial t} \nabla W \tag{13}$$

From equations 12 and 13 we immediately obtain

$$\left[\frac{\partial^2}{\partial t^2} - \frac{(\mathbb{B}_0 \cdot \nabla)^2}{\mu_0 \rho} \right] (\nabla \times \mathbb{b}^-) = 0 \tag{14}$$

and

$$\left[\frac{\partial^2}{\partial t^2} - \frac{(\mathbb{B}_0 \cdot \nabla)^2}{\mu_0 \rho} \right] (\nabla \times \mathbb{V}) = 0 \tag{15}$$

Equations 14 and 15 express the transport of vorticity and electric current into the fluid in the direction of $\mathbb{B}_o$, with no damping, at speed V_A. Therefore, a surface wave (with all wave quantities dying out to zero deep in the fluid) could only exist, if at all, provided $\nabla^2 \bar{\mathbb{b}} = \nabla^2 \bar{\mathbb{V}} \equiv 0$. That is, such a wave would have to be irrotational and with no accompanying electric current, i.e. no magnetic forces. Now, if $\nabla^2 \bar{\mathbb{b}} = 0$ then of course $\nabla^2 \bar{\mathbb{b}}_z = 0$ as well. But, from equations 1 and 10, both b_z and $\frac{\partial b_z}{\partial z}$ are continuous across $z = 0$. Therefore, $\nabla^2 b_z = 0$ implies $b_z = 0$.* Thereupon, from equation 7, $(\mathbb{B}_o \cdot \nabla) v_z \equiv 0$, so v_z cannot vary in the direction of $\mathbb{B}$. By the definition of a surface wave, we must then require $v_z \equiv 0$ so that, by equation 2, the surface cannot deform itself. We could go on to show that all the wave quantities vanish identically, but this is already quite clear. Thus, if $\mathbb{B}_{oz} \neq 0$, not even ordinary irrotational gravity waves can exist.

2. Tangential Magnetic Field.

We consider now the case where $\mathbb{B}_o$ lies in the plane of the undisturbed surface. Our procedure will be to obtain general solutions for $\mathbb{V}$ and $\bar{\mathbb{b}}$ from equations 12 and 13 and then to impose the necessary boundary conditions. It will turn out that this does not suffice to particularize $\mathbb{V}$ and $\mathbb{b}$ enough to determine any new characteristic wave types. Ordinary irrotational gravity waves, with rectilinear wavefronts parallel to $\mathbb{B}_o$, are possible in this case.

Put $\mathbb{B}_o = B_o \hat{e}_x$, with the same orientation of axes as in 1. Now, it is well known that a particular solution of the equation

$$\left(\frac{\partial^2}{\partial t^2} - V_A^2 \frac{\partial^2}{\partial x^2}\right) F = W(x, y, z, t) \tag{16}$$

*Formally, we could have b_z = const., but this could not be associated with a wave motion.

is given by the 'retarded potential' expression

$$F(x, y, z, t) = \frac{1}{2V_A} \int_{-\infty}^{t} d\tau \int_{x-V_A(t-\tau)}^{x+V_A(t-\tau)} dx' \, W(x', y, z, \tau) \tag{17}$$

Now, for any W whatever, the corresponding F given by equation 17, when substituted into equations 12 and 13, yields

$$\left(\frac{\partial^2}{\partial t^2} - \frac{B_o^2}{\mu_o \rho} \frac{\partial^2}{\partial x^2}\right)\left(\mathbb{b} + B_o \frac{\partial}{\partial x} \nabla F\right) = 0 \tag{18}$$

and

$$\left(\frac{\partial^2}{\partial t^2} - \frac{B_o^2}{\mu_o \rho} \frac{\partial^2}{\partial x^2}\right)\left(\mathbb{V} + \frac{\partial}{\partial t} \nabla F\right) = 0 \tag{19}$$

Now, if $W = \frac{\pi}{\rho} + \frac{\mathbb{B}_o \cdot \mathbb{b}}{\mu \rho}$, then, to satisfy equation 11 we must require $\nabla^2 W = 0$. However, it is easily verified by a simple change of variables in equation 17 that, if $\nabla^2 W = 0$ then $\nabla^2 F = 0$ as well.

As a result, the general solutions of equations 12 and 13 may be written

$$\mathbb{b} = -B_o \frac{\partial}{\partial x} \nabla F + \vec{f}_1 (x - V_A t, y, z) + \vec{f}_2 (x + V_A t, y, z) \tag{20}$$

and

$$\mathbb{V} = -\frac{\partial}{\partial t} \nabla F + \vec{f}_3 (x - V_A t, y, z) + \vec{f}_4 (x + V_A t, y, z) \tag{21}$$

where F is any harmonic function whatever and the vector functions $\vec{f}_i$ are (free-wave) solutions of the homogeneous 1-dimensional wave equation. Clearly, the first terms in equation 20 and equation 21 are the irrotational parts of these fields.

We shall now inquire, for any prescribed harmonic function F to what extent necessary further conditions (boundary conditions and the necessity of satisfying our full complement of equations) particularize the f_i . Substitution of equations 20 and 21 into equation 7 yields

$$V_A \vec{f}_1' + B_0 \vec{f}_3' = V_A \vec{f}_2' - B_0 \vec{f}_4' \tag{22}$$

where the primes denote differentiation of an $\vec{f}_i$ with respect to its first argument $x \pm V_A t$. Now, the left-hand side of equation 22 is independent of $x + V_A t$, while its right-hand side is independent of $x - V_A t$. So both sides must be at most functions of y, z only.* However, <u>wave</u> motions will be unaffected if we drop such time-independent terms and write

$$V_A \vec{f}_1 + B_0 \vec{f}_3 = 0 = V_A \vec{f}_2 - B_0 \vec{f}_4 \tag{22'}$$

so that

$$\vec{f}_3 = -\frac{1}{\sqrt{\mu_0 \rho}} \vec{f}_1 \qquad \text{and} \qquad \vec{f}_4 = \frac{1}{\sqrt{\mu_0 \rho}} \vec{f}_2 \tag{23}$$

Putting equation 20 into equation 5, we require

$$\nabla \cdot \vec{f}_1 + \nabla \cdot \vec{f}_2 = 0 \qquad \text{since} \qquad \nabla^2 W = 0 \tag{24}$$

Putting equation 21 into equation 6, we require (again using $\nabla^2 W = 0$) that $\nabla \cdot \vec{f}_3 + \nabla \cdot \vec{f}_4 = 0$, which, with the use of equation 23 requires

$$\nabla \cdot \vec{f}_1 - \nabla \cdot \vec{f}_2 = 0 \tag{25}$$

So to satisfy equations 5 and 6 we need only require

$$\nabla \cdot \vec{f}_1 = \nabla \cdot \vec{f}_2 = 0 \tag{26}$$

Now, with equations 20 and 21 we find that

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*The only alternative, putting $V_A \vec{f}_1 + B_0 \vec{f}_3 = (x - V_A t)\vec{\Phi}_1(y, z) + \vec{\Phi}_2(y, z)$ and $V_A \vec{f}_2 - B_0 \vec{f}_4 = (x + V_A t)\vec{\chi}_1(y, z) + \vec{\chi}_2(y, z)$ does not give a bounded amplitude.

$$\frac{\mathbb{B}_0 \cdot \mathbb{b}^-}{\mu_0} = -\frac{B_0^2}{\mu_0}\frac{\partial^2}{\partial x^2}F + \mathbb{B}_0 \cdot (\vec{f}_1 + \vec{f}_2)$$

and

$$\Pi = p + U = \rho W - \frac{\mathbb{B}_0 \cdot \mathbb{b}}{\mu_0} = \rho\left(\frac{\partial^2}{\partial t^2} - V_A \frac{\partial^2}{\partial x^2}\right)F - \frac{\mathbb{B}_0 \cdot \mathbb{b}}{\mu_0}$$
$$= \rho \frac{\partial^2 F}{\partial t^2} - \mathbb{B}_0 \cdot (\vec{f}_1 + \vec{f}_2)$$

Substituting this expression for Π into equation 9, with the use of equation 21, we find another condition on the $\vec{f}_1, \vec{f}_2$, namely

$$\left[\rho \frac{\partial^3 F}{\partial t^3} - B_0(f_{1x}' + f_{2x}') + \rho g \frac{\partial^2 F}{\partial t \partial z} - \rho g (f_{1z} + f_{2z})\right]_{z=0} = 0 \qquad (27)$$

The only remaining required condition on the $\vec{f}_i$, for any given F, is the matching of $\mathbb{b}^+$ to $\mathbb{b}^-$ at the surface $z=0$, with $\nabla^2 \mathbb{b}^+ = 0$ for $z>0$. Subject to this last condition, together with equations 26 and 27, a solution of our entire array of equations is given by equations 20 and 21. However, with one 'volume condition' $(\nabla \cdot \vec{f}_1 = \nabla \cdot \vec{f}_2 = 0)$ and two surface conditions, there are not enough restrictions to determine the form of the $\vec{f}_i$ or to determine a relation between $\mathbb{K}$ and ω for their Fourier components.

If we put $\vec{f}_1 = \vec{f}_2 = 0$ then, from equation 20, $\mathbb{b}^- = -\mathbb{B}_0 \frac{\partial}{\partial x} \nabla F$ so that $\nabla^2 \mathbb{b}^- = 0$ since F is a harmonic function. Since there is no current outside, $\nabla^2 \mathbb{b}^+ = 0$ as well so that the only physically acceptable solution is $\mathbb{b} = 0$. In particular, $0 = b_z^- = -B_0 \frac{\partial^2 F}{\partial x \partial z}$, so we see that $\frac{\partial F}{\partial z}$ is independent of x, so that v_z and Π are independent of x. Also, $\frac{\partial F}{\partial x}$ is independent of z, which is not allowable for a surface-wave solution unless $\frac{\partial F}{\partial x} = 0$, or $F = F(y, z, t)$. This, however, gives $\mathbb{b}^- \equiv 0$, $v_x \equiv 0$, and v_y independent of x. If we take for F a wave solution with $\frac{\partial F}{\partial x} = 0 = \nabla^2 F$, namely $F = F_0 e^{[i(ky - \omega t) + kz]}$ then $v_z = -\frac{\partial^2 F}{\partial z \partial t} = i\omega k F$ and $\dot{\Pi} = \rho \frac{\partial^3 F}{\partial t^3} = i\omega^3 \rho F$. Thereupon, evaluating these expressions at $z=0$, equation 9 gives the characteristic dispersion-law for ordinary gravity waves, namely $\omega^2 = gk$. The only characteristic surface waves existing in the presence of a tangential magnetic field are thus irrotational gravity waves of the ordinary type, running with their crests and troughs parallel to $\mathbb{B}_0$. One may, of course set up, in many arbitrary ways, disturbances localized near the surface, but we have shown that there are no new characteristic wave types here.

The same procedure which was used here, based on the use of equations 20 and 21, may be employed to find the wave-types which exist for a finite-depth fluid, bounded by a rigid bottom at $z = -h$ and by a free surface at $z = \zeta$. In this problem, for oblique $\mathbb{B}_0$, the boundary conditions at these two surfaces determine a discrete spectrum of waves. These are standing waves in the z-direction, while running along the surface. The details are quite involved, however.

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