Spectral Analysis of Julia Sets

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Abstract

We investigate different measures defined geometrically or dynamically on polynomial Julia sets and their scaling properties. Our main concern is the relationship between harmonic and Hausdorff measures.

We prove that the fine structure of harmonic measure at the more exposed points of an arbitrary polynomial Julia set is regular, and *dimension spectra* or *pressure* for the corresponding (negative) values of parameter are real-analytic. However, there is a precisely described class of polynomials , where a set of preperiodic critical points can generate a unique very exposed tip, which manifests in the *phase transition* for some kinds of spectra.

For parabolic and subhyperbolic polynomials, and also semihyperbolic quadratics we analyze the spectra for the positive values of parameter, establishing the extent of their regularity.

Results are proved through spectral analysis of the transfer (Perron-Frobenius-Ruelle) operator.

Contents

List of Figures

Chapter 1

Introduction

The main topic of this dissertation lies in the study of the dynamics of polynomial iterations and the geometry of corresponding Julia sets. We investigate different measures defined geometrically or dynamically on Julia sets and their scaling properties. This is done via describing a collection of characteristics, which we will call *spectra.*

Holomorphic dynamics. Considering a holomorphic dynamical system, such as Julia set, one notices that dynamics yields self-similarity. Hence geometry is "homogeneous" in different places, and the logical way to describe it is to count "how often" certain behavior occurs in different scales. It is also logical to expect that all geometrical objects have also dynamical/ergodic meaning, and the same set of parameters will describe both geometry and dynamical properties.

As usual in dynamics, one finds that hyperbolic Julia sets with expanding dynamics are easier to study and have very nice properties.

Non-hyperbolic dynamics is more difficult to understand, but still some remnants of expanding can be observed, hence one can expect to find difficult, but interesting behavior.

Complex analysis. Approaching the problem of describing the geometry of domains and their boundaries in the complex plane, one notices that many questions can be reduced to understanding the structure of harmonic measure. However, harmonic measure will have good scaling properties only for self-similar sets, so one can obtain more interesting results in such a case. On the other hand, the extremal behavior of harmonic measure can be approximated on self-similar fractals, which also motivates studying of their properties.

1. Overview and general discussion

Julia sets represent a class of dynamical systems which are defined seemingly easily but can exhibit very difficult properties. They were studied, starting with the works of L. Böttcher, P. Fatou, and G. Julia throughout this century, and very intensively over the last two decades. Partially this interest was evoked by beautiful computer pictures, which showed difficult "fractal" structure and resemblance to many physical phenomena arising in nature.

For a rational function F one can define the Julia set $J_F \subset \hat{\mathbb{C}}$ as the complement to the set of points in whose neighborhoods iterates of F form a normal family:

$$J_F \ := \ \hat{\mathbb{C}} \setminus \left\{ z : \ \exists U \ni z, \ \{F^n|_U\} \text{ is normal family} \right\} ,$$

another possible definition is the closure of all repelling periodic points:

$$J_F \ := \ \mathrm{Clos} \ \left\{ z : \ F^n(z) = z, \ |(F^n)'(z)| > 1 \text{ for some } n \right\} .$$

We mainly will be interested in the case when F is a polynomial, then J_F coincides with the boundary of domain of attraction to infinity

$$A(\infty) \ := \ \left\{ z : \ F^n(z) \to \infty \text{ as } n \to \infty \right\} ,$$

which is fully invariant under the action of F. The nicest class of Julia sets are the so called hyperbolic ones, for which dynamics F on the Julia set is expanding, i.e. $|(F^n)'(z)| > CQ^n$, $C > 0$, $Q > 1$ for any $z \in J_F$ and $n \in \mathbb{Z}_+$. For these definitions and basic properties of the Julia sets see monograph **[DH]** of A. Douady and J. Hubbard; books **[Be]** by A. Beardon, **[Mi1]** by J. Milnor and **[CG]** by L. Carleson and T. Gamelin, the latter follows a more analytical approach. The expository paper **[ELy]** of A. Eremenko and M. Lyubich gives a good presentation of their ergodic properties.

1a. Measures and transfer operator

Trying to understand the properties of a Julia set, one can start looking at the different measures (defined geometrically or dynamically), and at their behavior under dynamics.

1.1. Entropy and Balanced Measures

Balanced measures. A first logical class of measures to look at are the invariant ones. However, this class is too broad; so we can start with considering the so called *balanced* measures, where the mass is uniformly distributed among the preimages. Namely

$$\deg F \cdot \mu(A) = \mu(F(A)), \quad \text{if } F \text{ is injective on } A,$$

i.e. Jacobian of μ is equal to $\deg F$. H. Brolin in **[Bro]** has proved existence of a balanced measure for polynomials and has shown that it coincides with the equilibrium (harmonic) measure from potential theory, which is well-defined since capacity of the Julia set is equal to 1. He has also shown that it has strong mixing property, namely

$$\lim_{n\to\infty} \int h \cdot g(F^n)\, d\mu \;=\; \int h\, d\mu \cdot \int g\, d\mu \, .$$

Uniqueness of the balanced measure was established later by A. Freire, A. Lopes, and R. Mañé in **[FLM]**, **[Mañ1]**; also in **[BGH]** by M. F. Barnsley, J. S. Geronimo, and A. N. Harrington. See the latter paper, **[BH]**, and **[Lo2]** for the properties of potential, generated by the balanced (equilibrium) measure and its connections to Pade approximations and theory of moments.

Another way to view the balanced measure is to notice that preimages $F^{-n}z$ as n tends to infinity have uniform distribution with respect to it, hence one can construct the balanced measure as a weak limit of the sums

$$\mathrm{w}-\lim_{n\to\infty} \quad d^{-n} \sum_{y\in F^{-n}z} \delta_y \, .$$

Here δ_y denotes the δ-measure supported at point y. This process can be viewed as considering operator L^* acting on measures:

$$d\,(L^*\nu)\,(y) \;:=\; \frac{1}{\deg F}\, d\nu\,(Fy) \;\; ,$$

and analyzing its iterates. In such a way M. Lyubich has constructed balanced measures for rational functions (see **[Ly2-3]**).

One can play with these sums (or operator) in a different way, putting a unit mass at some point, $1/d$ masses at its preimages, $1/d^2$ at their preimages, etc., and taking a weak limit of measures obtained at the n-th step, properly normalized. A similar approach to another measure will appear to be useful later.

Harmonic measure and symbolic dynamics. One would like to have a good model of the dynamics on the Julia set. If it is connected and locally connected, then (via the Riemann uniformization map) dynamics $z \mapsto z^d$ ($d := \deg F$) on the unit circle gives us a topological model (modulo some lamination). Hence it is logical to expect that it will also be a proper metric model.

A more sophisticated way, which also works in a non-connected case, is symbolic dynamics, which models the dynamics F on the Julia set by equal-weighted one-sided Bernoulli shift on $\mathbb{Z}_d^\infty$. It was first introduced for Julia sets by M. Jakobson and J. Guckenheimer (see **[J1-4]** and **[Gu]**) and used extensively later. Bernoulli shift indeed appears to be a proper metric model: R. Mañé has shown in **[Mañ2]** that for some iterate N, (F^N, μ) is equivalent to the equal-weighted one-sided Bernoulli shift on d^N symbols.

Length plays an important role in the dynamics on the unit circle, particularly it is balanced and maximizes the entropy (with a value of $\log d$). The same is true for its analogue – d-adic measure on $\mathbb{Z}_d^\infty$.

In the connected case mapping the length from the unit circle we obtain harmonic measure on the Julia set. As was mentioned, H. Brolin proved that it coincides with the balanced measure, moreover A. Lopes has shown that this property characterizes the polynomials among the rational functions (see [**Lo1**] and also paper of M. Lyubich and A. Volberg [**LV1-2**]). Harmonic measure is a very important tool in understanding the structure of some set. Hence the unique balanced measure, except for dynamical, plays quite an important geometric role.

Harmonic measure also admits a probabilistic approach, since it measures the probability of a set being hit by a Brownian motion, which is invariant under the conformal maps. Hence it naturally fits in the framework of holomorphic dynamics. This connection of probability to dynamics hasn't remained unnoticed: some of the mentioned theorems were proved in this way by S. P. Lalley in [**La**].

Entropy. Another logical thing to expect is that this measure will maximize the entropy. Indeed, this was proved by M. Lyubich (see [**Ly1-3**]): particularly, there exists a unique invariant measure maximizing the entropy, it coincides with balanced measure and has entropy $\log \deg F$, which is the topological entropy of this dynamical system. The latter equality was conjectured earlier by R. Bowen in [**Bo1**], and partial results in establishing it were obtained by M. Jakobson, J. Guckenheimer, M. Misiurewicz, and F. Przytycki (see **[J1-3]**, **[Gu]**, **[MP]**).

Moreover, this balanced measure has very nice ergodic properties: in the case of totally disconnected Julia set it is Gibbs (see e.g. **[C]** and **[MV]** for more general setting of conformal Cantor sets), for hyperbolic polynomials the mixing is exponentially fast.

1.2. Geometric Measures

One knows that self-similar sets (e.g. Cantor sets, snowflakes, etc. – see **[Fa]**) usually carry some geometric measure – like Hausdorff or Minkowski – since they have nice scaling properties. It is logical to expect the same from the Julia set.

Since the Julia set has more complicated origins of self-similarity than, say, an affine cantor set, it is hard to evaluate (even estimate) its Hausdorff dimension and understand geometrically what will be the proper Hausdorff measure. One has to approach this question dynamically, as was first done by D. Sullivan. If we assume that the Julia set has Hausdorff dimension t and, moreover $0 < \mathcal{H}_t(J_F) < \infty$, then the Jacobian of the Hausdorff measure $\mathcal{H}_t$ (or normalized $\nu := \mathcal{H}_t/\mathcal{H}_t(J_F)$) will be equal to $|F'|^t$:

$$\int_A |F'|^t \, d\nu = \nu(F(A)), \quad \text{if } F \text{ is injective on } A .$$

Therefore one can look for a probability measure on J_F with such property, and hope that it will be a t-dimensional Hausdorff or some other geometric measure. We will call such measures t-conformal.

Patterson-Sullivan construction. A constructive method to find conformal measure, comes from the theory of Kleinian groups, where it was introduced by S. J. Patterson [**Pa1**] and then used by D. Sullivan [**Su1,3**] (see also exposition of related results in [**Pa2**]). It is pretty much the same as the described method for constructing a balanced measure: we fix t, pick some point z outside the Julia set, and put a unit mass on it. To "make" this measure conformal, we have to put appropriate masses on preimages of z, then on their preimages, etc. and consider a weak limit. Roughly speaking, we take

$$\nu_n := \sum_{k=0}^{n} \sum_{y \in F^{-k} z} \left| \left(F^k\right)' (y) \right|^{-t} \delta_y ,$$

and set ν to be the weak limit of the normalized measures $\nu_n / \mathrm{Var}(\nu_n)$. In practice the process is more difficult: one has to chose t depending on the convergence of the sum above, and sometimes "correct" the terms in it to make the weak limit conformal. D. Sullivan has shown that for any rational map there exists t for which we can construct a conformal measure in such a way.

Particularly, there exists a δ-conformal measure for

$$\delta := \inf \left\{ t : \sum_{k=0}^{\infty} \sum_{y \in F^{-k}z} \left| \left(F^k\right)'(y) \right|^{-t} < \infty \right\} ,$$

for some (any) z outside J_F. The latter series is an analogue of the Poincaré series for a Kleinian group.

Geometric properties of conformal measures. Unfortunately it is difficult to deduce much about the conformal measure from the construction or conformal property itself in the general case. There might not be unique δ with existing conformal measure, hypothetically conformal measure for given δ might not be unique, it might be atomic and does not have much of a geometric meaning.

First results were obtain by D. Sullivan, **[Su2]**, in hyperbolic case, following the work **[Bo2]** of R. Bowen on the dimension of quasicircles (see about Bowen's formula later). Since every small ball is mapped eventually by some iterate of F to large scale with bounded distortion, and after some work one obtains that

$$\nu(B_r) \asymp r^{\delta} ,$$

for any ball B_r of radius r centered on the Julia set. Thereafter ν is (up to a constant) equal to a δ-dimensional Hausdorff measure, δ being the Hausdorff dimension of J_F. In this case there is a unique conformal measure for a unique exponent.

In the general case, results are harder and many things are still unknown. If we set δ to be a minimum exponent with an existing conformal measure, it is unknown whether δ is always equal to Hausdorff, Minkowski or hyperbolic dimension of J_F. The latter dimension is defined as a supremum of the Hausdorff dimensions of hyperbolic subsets of J_F, where dynamics is expanding.

However, some partial results are known. M. Denker and M. Urbański have shown in **[DU5]** that the following numbers coincide: 1) minimal zero of the pressure function, 2) supremum of Hausdorff dimensions of ergodic invariant measures with positive entropy (*dynamical dimension*), and 3) the minimal exponent δ' for which a measure conformal except on some finite set exists, coincide. Moreover, for many rational maps δ' and the minimal exponent δ for conformal measures are the same (some kind of "expanding" on critical orbits is sufficient).

If there are no recurrent critical points in the Julia set (but we allow parabolic points), the situation is even nicer – M. Urbański has shown in **[U]** (partial results were obtained earlier by him and M. Denker in **[DU2-4,7]**) that the exponent δ will coincide with Hausdorff and packing dimensions of the Julia set and the δ-conformal measure will be equal to normalized δ-dimensional Hausdorff or/and packing measure. The latter is determined by the existence of parabolic points, which can produce interesting dichotomies for the possible properties of conformal and invariant measures, see **[ADU]**.

It remains to mention that work **[Sh]** of M. Shishikura implies that for topologically generic quadratics with parameter in the boundary of Mandelbrot set all mentioned dimensions will be equal to 2. However, we don't know what will be the 2-conformal measure in this case if the Julia set has zero area.

1.3. Transfer Operator

It is plausible to have another approach, maybe not working for a general Julia set, but giving more properties of a conformal measure. First we introduce a notion of (λ, t)-conformal measure, i.e. such measure ν that

$$\lambda \int_A |F'|^t \, d\nu \;=\; \nu\left(F\left(A\right)\right), \quad \text{if } \; F \; \text{ is injective on } \; A \; .$$

Note that it generalizes two previous definitions: t-conformal measure is $(1, t)$-conformal, whereas balanced is $(d, 0)$-conformal.

This notion is very closely related to one of a transfer (*Perron-Frobenius-Ruelle*) operator. *Transfer operator* with weight ϕ is defined by

$$\left(L_\phi f\right)(z) \;:=\; \sum_{y \in F^{-1} z} \phi(y) f(y) \; ,$$

in a proper functional space. Choice of it is very important and will be discussed later.

The main role is played by the family L_t of transfer operators with weights $\phi_t := |F'|^{-t}$. If the Julia set does not contain critical points, L_t acts on the space of functions, continuous on J_F. Hence its formal adjoint acts on the space of Borel measures supported on J_F:

$$d\left(L_t^*\nu\right)(y) \;=\; |F'(y)|^{-t}\, d\nu(Fy)\ .$$

It is easy to see that a measure ν is (λ, t)-conformal if and only if it is an eigenmeasure of L_t^* with eigenvalue λ. Moreover, if L_t has an eigenfunction f with the same eigenvalue, the measure $f\nu$ is an invariant measure equivalent to ν.

Pressure. For hyperbolic polynomial let $r_F(t)$ denote the spectral radius of the transfer operator L_t in a proper space (e.g. $C(J_F)$). We define the *pressure* by $P_F(t) = \log r_F(t)$. As we shall see, by carrying out the spectral analysis of the transfer operator, one can establish nice properties of pressure and connect it to other characteristics of the Julia set.

For the Julia sets, pressure was first used by D. Ruelle in **[Ru2]** to establish a conjecture of D. Sullivan that HDim J_F depends real analytically on a hyperbolic rational function F. Particularly, he applied the machinery of thermodynamic formalism (see **[Ru1,4]**) and showed that when the Julia set J_F of a rational function F is hyperbolic, the pressure function $P_F(t)$ is real analytic as a function of t and F. The remaining ingredient of his proof was

Bowen's formula. In the hyperbolic case the only zero of $P_F(t)$ is the Hausdorff dimension HDim J_F of the Julia set – the intuitive reason is that conformal measure for $t = \text{HDim } J_F$ should coincide with Hausdorff in the same dimension and hence $\lambda_t = 1$ and $P_F(t) = 0$. This formula was noticed by R. Bowen for quasicircles, **[Bo2]**; D. Ruelle established it for hyperbolic Julia sets in **[Ru2]** and **[Ru3]** (see also **[DS]** for another proof), it is equivalent to the mentioned result **[Su2]** of D. Sullivan on conformal measures.

Spectral analysis of the transfer operator. In the hyperbolic case the map F is expanding, so the transfer operator makes "smooth" functions even "smoother" which implies its quasicompactness in the spaces of smooth functions (e.g. Hölder continuous – see later), and $r_F(t)$ is a simple isolated eigenvalue with eigenfunction f_t and eigenmeasure ν_t:

$$L_t\, f_t \;=\; r_F(t)\, f_t\ , \quad L_t^*\, \nu_t \;=\; r_F(t)\, \nu_t\ .$$

Operator L_t depends real analytically on t, thus by perturbation theory $P_F(t)$ is real analytic as a function of t. Actually, in "infinitely smooth" spaces (e.g. functions real analytic in the neighborhood of the Julia set) the transfer operator behaves even nicer: D. Ruelle proved that it is nuclear in the sense of A. Grothendieck (**[Gr1]**), see **[Ru2]**.

Eigenfunction f_t and eigenmeasure ν_t also depend real-analytically (in a proper sense) on t and play an important role: ν_t is (r,t)-conformal (e.g. for $t=0$ it is balanced and for $t = \text{HDim } J_F$ is equal to the normalized t-dimensional Hausdorff measure), $f_t\nu_t$ gives us an equivalent invariant measure.

Quasicompactness. Another method of making spectral analysis and establishing quasicompactness is due to C. T. Ionescu-Tulcea and G. Marinescu (see **[IM]** and also the paper **[N]** of R. Nussbaum): the "smoothing" property of the transfer operator leads us to the inequality of the type

$$\|L^n f\|_{\text{smooth space}} \lesssim (\lambda-\varepsilon)^n \|f\|_{\text{smooth space}} + \lambda^n \|f\|_{\text{usual space}} ,$$

which "pushes" the essential spectral radius in the "smooth space" down to $(\lambda-\varepsilon)$ and makes operator L quasicompact.

By quasicompactness we mean that the essential spectral radius $r_{ess}(L)$ is strictly less than the spectral radius $r(L)$ of the transfer operator L. Therefore the spectrum outside of the disk of radius $r_{ess}(L)$ consists of finite number of isolated eigenvalues. We are interested in the main eigenvalue $r(L)$, determining the pressure; under nice circumstances it appears to be an isolated eigenvalue of multiplicity 1, which provides good properties of pressure.

Zeta-function. To prove the real analyticity of $P_F(t)$ in F, D. Ruelle considered another approach, which gives the same pressure: he defined

$P_F(t)$ as the inverse to the pole of the dynamical ζ-function

$$\zeta(u) \;=\; \zeta_{F,t}(u) \;:=\; \exp \sum_{n=1}^{\infty} \frac{u^n}{n} \sum_{F^n z = z} \left|(F^n)'(z)\right|^{-t} ,$$

with the smallest modulus. Real analyticity follows then from nice spectral properties of the transfer operator and general theory of Fredholm determinants (see **[Gr2]**), to which ζ-function is closely connected.

Variational principle. Variational approach works in the general setting too. For $\phi \in C(J_F)$, the *pressure* is defined as

$$P(\phi) \;:=\; \sup \left(h_\mu + \int \phi d\mu \right) ,$$

where the supremum is taken over all probability measures μ on J_F invariant under F. In the hyperbolic case, this variational problem has a unique solution for smooth ϕ and the pressure function is $P_F(t) = P(-t \log|F'|)$. Moreover, the maximizing measure for $\phi = -t\log|F'|$ is equal to $f_t \nu_t$.

On computer experiments. Nice spectral properties of the transfer operator imply that, iterating it, we converge to the main eigenfunction. Particularly, in the hyperbolic case, the *partition function* Z_n satisfies

$$Z_n(z) \;:=\; (L^n 1)(z) \;\asymp\; \lambda^n ,$$

giving us an opportunity to estimate the main eigenvalue numerically (with an error of $\mathrm{const}^{1/n} - 1 \asymp 1/n$, if we compute L^n by taking the preimages of z under F^{-n}).

Hence whenever the transfer operator behaves "nicely," particularly in the hyperbolic case, one can make rigorous computer estimates of all the parameters involved: spectra, dimensions, etc. – see **[STV]** and **[V]** by G. Servizi, G. Turchetti, S. Vaienti. Moreover, reasonable algorithms should converge exponentially fast, depending on the "expansion" constants for the dynamics. See, e.g., paper **[G]** of L. Garnett for computations of Hausdorff dimension for hyperbolic quadratics $z^2 + c$ with small c. Roughly speaking, she evaluated zero of the pressure via considering the finite rank approximations to the transfer operator and computing their spectral radii.

Unfortunately, for most interesting cases with low hyperbolicity, the exponent is close to 1 and can have a nasty constant in front of it, spoiling the situation. So, with the present abilities of computers, it still seems favorable to use "unrigorous" methods (e.g. Monte-Carlo), speeding up the computations: consult the paper **[BZ]** of O. Bodart and M. Zinsmeister.

1.4. Non-hyperbolic situation

The key to analysis of spectra and properties of J_F lies in the spectral analysis of the transfer operator. If the dynamics is "expanding" in some sense, the transfer operator has a nice spectrum (in a proper space), its eigenvalues and eigenfunctions (eigenmeasures of its adjoint) behave "nicely" and all mentioned objects are "nicely" connected and have "good" properties. The main problem is hence to introduce a proper notion of "expanding," and analyze the spectrum of transfer operator in a proper space – the rest will follow.

In a non-hyperbolic situation there are many difficulties, since the dynamics is not expanding in the usual sense. Moreover, some of the previously mentioned definitions must be modified in this case (e.g. if there are critical points on the Julia set, the transfer operator does not act on $C(J_F)$ for $t > 0$) and a priori they can lead to different objects.

Since the first work of Ruelle, some progress has been made for the case of expansive maps, which corresponds to the Julia sets with parabolic points, see e.g. **[HR]**, **[Ru7]**, **[ADU]** or **[DU2,3,6,7]**.

In the case of an arbitrary rational function M. Denker, M. Urbański, and F. Przytycki (see **[DU1]**, **[Pr1]**, **[DPU]**) showed that transfer operator L_ψ with a Hölder continuous, positive ψ is almost periodic on the space of Hölder continuous functions (under an additional assumption ($\diamondsuit$) -see the Subsection 2.3). This is a weaker property than quasicompactness, and if there are critical points on the Julia set, $L_t = L_{|F'|^{-t}}$ for $t \neq 0$ does not satisfy their assumptions. Recently N. Haydn has proved a stronger theorem, establishing the quasicompactness of the transfer operator in this case – see **[Ha]**.

An interesting partial case consists of polynomials $z^2 - c$ with real

$c < -2$. Then the Julia set is a Cantor subset of the real line, and the potential $|F'(z)|^{-t}$, restricted to it, is equal to a holomorphic function $F'(z)^{-t}$. This was used by A. Eremenko, G. Levin, M. Sodin, and P. Yuditskiĭ in **[ELS]** and **[LSY1-2]** to thoroughly investigate the spectrum of the transfer operator with $t = 2$.

Substantial progress has been made on similar questions in one-dimensional dynamics, mainly using bounded variation spaces, see, e.g., **[HK1-2]**, **[Pol]**, **[Ru5]**, **[Ry]** (see book **[Ru8]** of D. Ruelle for exposition of this subject). There is no good analogue of the bounded variation space for the complex plane; BV on the interval is invariant under monotone transformations, while in the complex plane we can only hope to find a space invariant under conformal maps, and still we will have some problems with critical points. Nevertheless, considering the space BV_2 of functions whose second partial derivatives are measures, D. Ruelle **[Ru6]** proved the quasicompactness of L_ψ with $\psi \in BV_2$ assuming certain conditions. His theorem implies some partial results for the negative spectrum.

Another way is in using Markov extensions, introduced by F. Hofbauer in **[Ho]** and developed by G. Keller in **[K]**. This technique is very much related to the "jump transformation," when instead of given dynamics for every point one considers a proper iterate, carrying enough expansion (such method usually works in expansive case – see, e.g.

[**ADU**]). The idea is to do the same thing, but "separating" different iterates – we make a "tower" of countably many copies of the original dynamical system: the point goes up, till it accumulates enough expansion, then returns to the first floor. Hofbauer towers were used very successfully in analyzing maps of the interval, see papers [**BK**], [**KN**], [**HK3-4**], and thesis [**Bru**] of H. Bruin, which contains very instructive exposition of the tower construction. We will use Markov extensions build on the Yoccoz puzzle to analyze the non-recurrent quadratics.

Relating the pressure to the ζ-function is more difficult (see, e.g., [**Ru5-6**]), consult [**Ba**] and [**Ru8**] for the account of results in one-dimensional dynamics. For non-hyperbolic Julia sets, we are aware only of the work [**Hi2**] of A. Hinkkanen, who deduced an explicit formula for the ζ-function with constant weights (i.e. $t = 0$), which appeared to be always rational.

1b. Fine structure of measures

There are a few other approaches of analytical and geometrical nature, rather than dynamical, which lead to the same objects.

1.5. Complex analysis

Harmonic measure. One of the most important notions in the study of the subsets of complex plane (more precisely, boundaries of domains) is harmonic measure. For a given domain (and a point inside) one can define harmonic measure as an equilibrium distribution for logarithmic potential, probability of being hit by Brownian motion, or as an image of the length on the unit circle under the Riemann uniformization map for simply connected domains. There are other definitions, generating more applications of this notion; we refer the reader to the monograph **[GM]** of J. Garnett and D. Marshall. The choice of the point is not important for the local behavior, so we will consider the harmonic measure on the boundary of the basin of attraction to infinity with respect to infinity, which we will denote by ω.

In the past decade there was a lot of progress in understanding geometric behavior and fine structure of harmonic measure, see exposition and references in the lectures **[Mak2]** of N. Makarov. Of particular interest to us are the papers **[CJ]**, **[JM]**, and **[CM]** of L. Carleson, P. Jones, and N. Makarov, where they obtained bounds on the extremal behavior of harmonic measure. These papers show that it can be approximated on self-similar sets, which have good scaling properties and hence nice fine structure of harmonic measure. One should note, that many of its properties were first (and easier) observed and proved for self-similar sets: e.g. the dimension of harmonic measure on the boundary of any simply connected domain by Makarov's theorem is equal to 1 ([**Mak1**]); the proof for domains of attraction to infinity (for connected hyperbolic polynomial Julia sets) is easier and has more intuitive reasons, than in general case – see paper [**Man**] of A. Manning. This provides an additional motivation to investigate harmonic measure on self-similar fractals, since still there are many open questions about its behavior in general case.

Distortion of the Riemann map. Having a simply connected domain, one can learn about the geometry of the boundary by measuring the distortion of the Riemann uniformization map. One possible way is to consider the rate of growth of the integral means of its derivative, this makes even more sense if domain has self-similar boundary. Particularly, for a connected polynomial Julia set we can define the *conformal spectrum* in the following way:

$$\beta_F(t) \;:=\; \limsup_{r\to 1+} \frac{\log \int_{|z|=r} |\varphi'|^t}{\log \frac{1}{r-1}},\ t \in \mathbb{R}\ ,$$

where φ is the Riemann map from the outside of the unit disk to the basin of attraction to ∞. Considering the conjugation of F outside J_F with $z \mapsto z^d$ outside the unit disk, given by properly normalized φ, it is easy to see that in hyperbolic case $P_F(t) = (\beta_F(t) - t + 1) \cdot \log(\deg F)$.

There are also ways to extend this definition to the multiply connected case, e.g. considering mean values of distance to the boundary (or modulus of gradient of the Green's function), taken to some power, along the Green's lines.

Harmonic Measure on Julia Sets. Many geometric properties of Julia sets are closely related to the behavior of harmonic measure and different spectra under discussion:

The dichotomy of A. Zdunik (see [**Zd**], for hyperbolic Julia sets it was proved by F. Przytycki, M. Urbański, and A. Zdunik in [**PUZ**]) for the boundary of attractive basin to be either an analytic curve or have Hausdorff dimension strictly more than 1, is proved using LIL (Law of Iterated Logarithm) principle for harmonic measure. Her theorem is in fact stronger, and has the following meaning in terms of conformal β-spectrum: either the boundary is an analytic curve (then $\beta \equiv 0$), or the second derivative $\beta''(0)$ is strictly positive.

The Pommerenke-Yoccoz-Levin-Petersen inequality, which estimates the multipliers of periodic points in connected Julia sets (see [**Pom**], [**ELe1-2**], [**Le**], [**Pe**]), is related to the A. Beurling's estimate for the possible concentration of harmonic measure at a point when we know how "twisted" is our domain.

By [**GS**] and [**CJY**] we know that Collet-Eckmann condition (i.e. exponential expansion on critical orbits) forces Fatou components to be Hölder, particularly semihyperbolic polynomials have Fatou components even John domains. Hölder property is closely related to spectra and their regularity: simply connected domain is Hölder if and only if for the conformal spectrum $\lim_{t\to+\infty}\beta(t)/t < 1$ (equivalently if and only if $P_F(t) < 0$ for large t), the value of the limit depending on the Hölder exponent. In this case, by **[Mak2]**, the only root of the equation $\beta(t) = t - 1$ is the Minkowski dimension of the boundary (an analogue of the R. Bowen's formula in non-dynamical context).

We want also to note that many things known about the geometric regularity of the Julia sets imply some nice properties of harmonic measure. R. Mañé and L. F. da Rocha have shown in **[MR]** (see also **[Hi1]**) that Julia sets of rational functions are uniformly perfect, this property ensures some kind of regularity usually needed for the study of harmonic measure on disconnected sets.

On rational functions and conformal Cantor sets. We will mainly discuss the harmonic measure on the basin of attraction to infinity for polynomial Julia sets, which coincides with the balanced measure of maximal entropy. In the general case one has to consider, instead of a pressure as function of t, $P = P(t\log|F'|)$, a pressure of two parameters: $P = P(t\log|F'| + s\log J_\omega)$, taking into account the non-constant Jacobian J_ω of the harmonic measure.

However, many of our constructions do apply to conformal Cantor sets, or (super)attractive/parabolic basins of attraction for rational functions. It appears that the situation is still good enough to establish the regularity of the fine structure, though some other things, like rigidity properties, may "go wrong" (see e.g. **[C]**, **[MV]**, **[LV1-2]** for questions concerning conformal Cantor sets).

1.6. Multifractal Analysis – Chaotic Sets

Multifractal analysis is an intensively developing subject on the border between mathematics and physics. It was introduced by T. Halsey, M. Jensen, L. Kadanoff, I. Procaccia, and B. Shraiman in a physical paper **[HJKPS]**, where they tried to understand and describe scaling laws of physical measures on different fractals of physical nature (strange attractors, stochastic fractals like DLA, etc.). They considered the *dimension spectrum* of those measures – a continuum of parameters characterizing the size of the set where certain power law applies to the mass concentration.

Since then there appeared a number of papers with rigorous multifractal analysis of different dynamically defined measures, see e.g. **[PW]** for references. For a physical approach to multifractal analysis and its mathematical counterpart, consult **[BS]**, **[Fa]**, **[Fe]**, **[Mand]**, and **[T2]**. It is natural to perform (and expect to perform well) multifractal analysis for self-similar sets and for measures behaving nicely under rescaling. Generally it is done for expanding dynamical systems (– hyperbolic Julia sets) and for Gibbs measures.

It appears, that the fine structure of harmonic measure is best described via its multifractal analysis, see **[Mak2]**. A. Lopes (**[Lo3]**) made multifractal analysis of equilibrium (harmonic) measure for hyperbolic Julia sets, when all reasonable definitions of spectra work and lead to the same objects. He also pointed out in **[Lo5-6]** some irregularities (to be discussed later), which may occur in subhyperbolic situation. Also see papers of Y. Pesin and H. Weiss **[PW]**, **[P]**, who, in particular, analyze equilibrium measures with Hölder potentials for conformal expanding case.

Dimension spectra. Dimension spectra will estimate the size of the set where our measure has certain power law behavior. It makes sense to consider two kinds of dimension spectra, of "Minkowski" and "Hausdorff" types. Roughly speaking, we define (for rigorous definitions see **[Mak2]**) *Hausdorff dimension spectrum* as

$$\tilde{f}(\alpha) \ := \ \mathrm{HDim}\,\{z:\ \omega B_{z,\delta} \approx \delta^{\alpha}\,,\ \delta \to 0\}\ .$$

To define *box dimension spectrum*, we not just change the Hausdorff dimension to box-counting, but also "flip" limits in the definitions of spectrum and dimension:

$$f(\alpha) \ := \ \lim_{\delta\to 0} \mathrm{BDim}_{\delta}\,\{z:\ \omega B_{z,\delta} \approx \delta^{\alpha}\}\ .$$

Here BDim_{δ} is the box-counting dimension, estimated with disks of radii δ:

$$\mathrm{BDim}_\delta E \;:=\; \inf\left\{p:\ \exists \text{ disjoint } \{B_{z,\delta}\},\ z\in E \text{ with } \sum_B \delta^p \asymp 1\right\}.$$

Of course, in the general situation there will be many points, where measure behaves differently at different scales, so we will have to add lim sup's and lim inf's to our definitions.

By analyzing the multifractal structure of some measure we mean investigating relationship of dimension spectra with other objects and proving some kind of its regularity – like the possibility of taking real limits (instead of upper or lower) in the definitions.

Packing and covering spectra. Sometimes it is more convenient to play with disks, having certain concentration of harmonic measure, in a different fashion, analogous to considering *grand ensemble* in statistical mechanics (cf. [**Ru1**]). Resulting spectra appear (under nice circumstances) to be connected via Legendre transform to dimension spectra and behave more like pressure in dynamical situations. In fact, pressure evaluates the same quantities with Lyapunov exponents instead of rescaling coefficients for measures; in dynamical context those are closely related, as was noticed by J.-P. Eckmann and I. Procaccia in [**EP**] (see also [**FPT**], [**ST**], and [**T1**]).

We define the *packing spectrum* $\pi(t)$ as

$$\sup\left\{q:\ \forall\delta>0\ \exists\ \delta-\text{packing } \{B\} \text{ with } \sum \delta(B)^t\omega(B)^q \geq 1\right\},$$

where $\delta(B)$ is the diameter of disk B. *Covering spectrum* $c(t)$ is defined similarly, as

$$\inf\left\{q:\ \forall\delta>0\ \exists\ \delta-\text{cover }\{B\}\text{ with }\sum\delta(B)^t\omega(B)^q\le 1\right\}.$$

In nice situations (e.g. hyperbolic Julia sets, there, in fact, also $f\equiv\tilde f$ and $\pi\equiv c$), spectra and dimension spectra are related via Legendre-type transforms (see Figure 1):

$$\begin{aligned}
\pi(t) &= \sup_{\alpha>0}\frac{f(\alpha)-t}{\alpha}, \quad & f(\alpha) &= \inf_t\,(t+\alpha\pi(t)),\\
c(t) &= \sup_{\alpha>0}\frac{\tilde f(\alpha)-t}{\alpha}, \quad & \tilde f(\alpha) &= \inf_t\,(t+\alpha c(t)).
\end{aligned}$$

For the properties in general case, see **[Mak2]**.

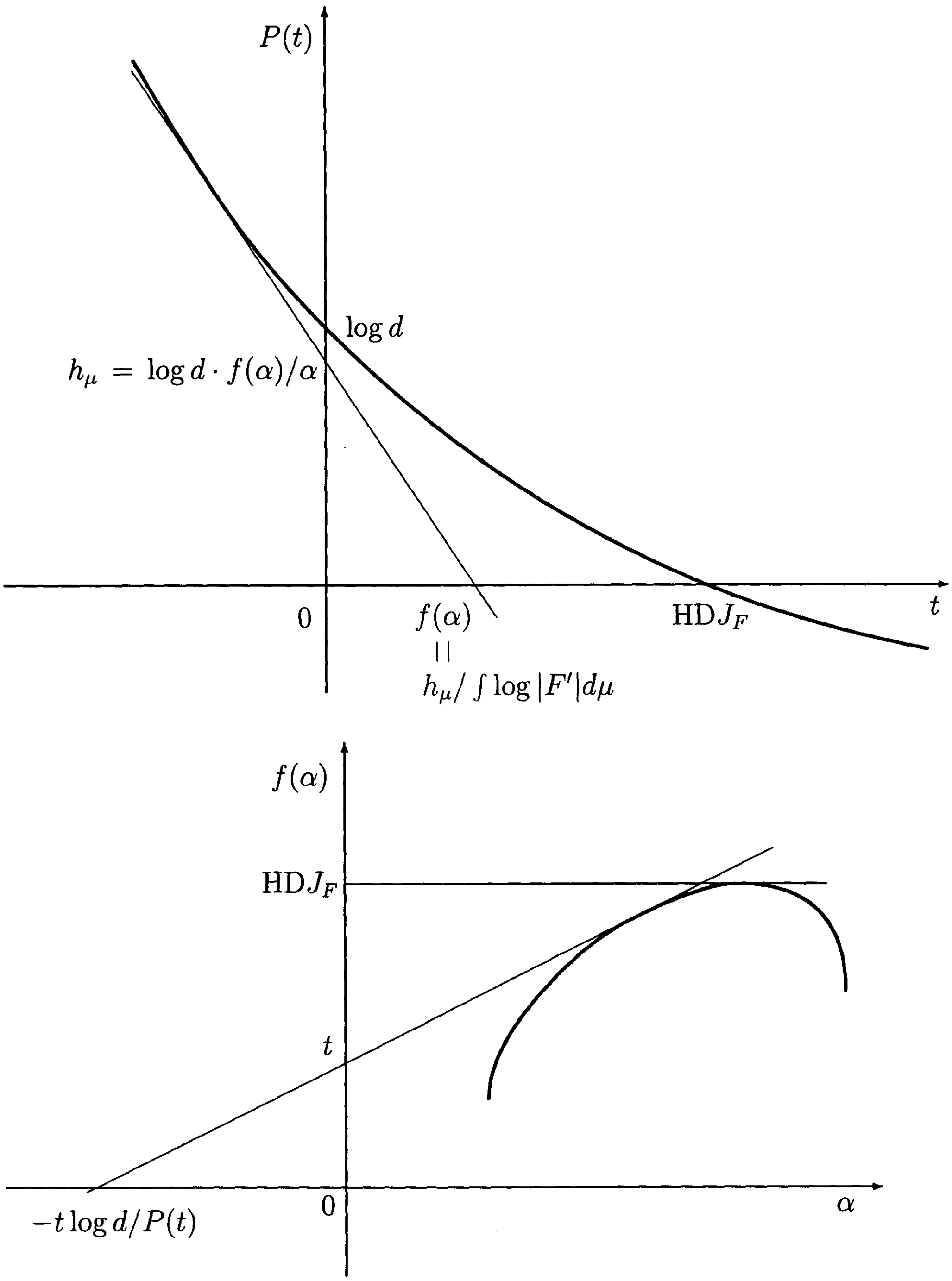

Figure 1. Spectra of hyperbolic polynomials

2. Results

Summing it up, one observes that spectra for negative values of t (or small α) describe the geometry of the set at "more exposed" points – tips – with higher concentration of harmonic measure. The more negative is t, the higher the concentration. On the other hand, spectra for positive t (big α) describe the geometry of "fjords," where the harmonic measure is low, but the set is "more dense" (here the Hausdorff dimension "lives").

A physical interpretation of this is discussed by T. Bohr, P. Cvitanović, and M. H. Jensen in [**BCJ**], where they suggest that spectra for small α should be robust under small parameter perturbation, when large α's are noisy and poorly convergent – the fjords are screened, and this can manifest in a "phase transition" at the Hausdorff dimension.

We prove nice behavior of the spectra for negative values of t for *any* polynomial Julia set. All of them coincide, nicely converge and are real analytic except for the rare class of polynomials with very specific combinatorics of the critical orbits, which causes their Julia sets to have points with unique geometry, unrepeated elsewhere. In some cases this phenomenon can cause the "Hausdorff" and "Minkowski" spectra to be different, with latter having a "*phase transition.*" The reason is that "Hausdorff" definitions neglect the input from individual, not repeated patterns, while "Minkowski" take them into account.

For positive spectra the situation is indeed more difficult – lack of expansion has more impact: when the critical point belongs to the Julia set the weight $|F'(z)|^{-t}$ is not even bounded, and we have troubles defining the pressure. However, using different methods, we were able to work out three particular cases.

For parabolic Julia sets (dynamics is not expanding, but is expansive, and there are no critical points on the Julia set) we prove nice behavior of spectra up to the value $t = \text{HDim } J_F$, where the phase transition happens. In this case it is caused not by a unique pattern, but by infinitely duplicated cusp at the parabolic point. Then work **[ADU]** of J. Aaronson, M. Denker, and M. Urbański implies an intriguing dichotomy for discontinuity of the derivative at the phase transition point and the value of HDim J_F.

For subhyperbolic Julia sets (M. Misiurewicz - W. Thurston sets where there can be preperiodic critical points and the dynamics is expanding with respect to a metric with finite number of singularities) we were able to transfer the problem via the Riemann uniformization map to the unit circle, where preperiodicity of the critical point allows to multiply the weight function by a proper homology, thus getting rid of its singularities. Resulting pressure behaves nicely and has proper connections to other spectra.

In a more difficult semihyperbolic case (M. Misiurewicz's Julia sets where dynamics is expanding with respect to some singular metric – see paper **[CJY]** of L. Carleson, P. Jones, and J.-C. Yoccoz for other equivalent definitions and proof of the John property for their Fatou components) we constructed, for non-recurrent quadratics, Markov extension (analogous to F. Hofbauer tower) on the J.-C. Yoccoz puzzle (see J. Hubbard's **[Hu]** and J. Milnor's **[Mi2]** expositions of the J.-C. Yoccoz's results). Dynamics on this tower is expanding, hence spectra for tower behave nicely and have good relation to some spectra for the original dynamical system.

2.1. Negative spectra

Establishing the quasicompactness of the transfer operator on Sobolev spaces (see Chapter 2 for precise formulation), we prove the following results, which almost completes the analysis of the negative spectra.

Theorem A. **A1.** *For any polynomial F and negative t either*

(i) *$P_F(t)$ is real analytic on $(-\infty, 0)$, or*

(ii) *there exists a "phase transition" point* $t_0 < 0$ *such that*

$$\begin{cases} P_F(t) & \text{is real analytic on } [t_0, 0)\ , \text{ and} \\ P_F(t) & = \ -P_{-\infty} \cdot t \ \text{ on } \ (-\infty, t_0]\ . \end{cases}$$

In fact $P_F(t) = \max(-P_{-\infty} \cdot t, \tilde{P}_F(t))$ *for some* $\tilde{P}_F(t)$ *real analytic on* $(-\infty, 0)$.

A2. *In the case (i) all mentioned definitions of pressure give the same function* $P_F(t)$. *In the case (ii) function* $P_F(t)$ *corresponds to "Box" definitions, when "hidden" spectrum* $\tilde{P}_F(t)$ *– to "Hausdorff." All definitions converge nicely, i.e. one can take* lim *instead of* lim sup, *etc.*

A3. *For the occurrence of a "phase transition" it is necessary that either*

(iia) *F is conjugate to a Chebyshev polynomial (then J_F is an interval), or*

(iib) *there are a fixed point a, $Fa = a$, and a positive number ε such that*

$$F^n b = b\,, \quad b \neq a \quad \Rightarrow \quad \mu(b) \ \leq \mu(a) \ - \ \varepsilon\ ,$$

where $\mu(b) \ := \left|(F^n)'(b)\right|^{\frac{1}{n}}$ *denotes the multiplier of a periodic point b. This implies that* $a \in J_F$, $P_{-\infty} = \log \mu(a)$ *and*

$$F^{-1}a \setminus \{a\} \ \subset \ \text{Crit}\, F \quad (= \quad \textit{zeroes of } \ F'\ )\ . \qquad (\clubsuit)$$

A4. *For quadratic polynomials (iib) cannot happen (for combinatorial reasons), and for cubics it only happens for some polynomials with disconnected Julia set. However, there is a degree 4 subhyperbolic polynomial with connected Julia set for which it occurs.*

A5. *The pressure function depends continuously on F as a function in the space $C^\infty(-\infty, 0)$, or $C^\infty\left((-\infty, t_0 - \varepsilon) \cup (t_0 + \varepsilon, 0)\right)$ if the phase transition occurs.*

Parts A1 and A2 state that the distribution of the harmonic measure at the "more exposed" points is very regular, and multifractal analysis does apply. Part A3 and A4 show that the phase transition is a rather rare phenomenon. The Julia set J_F itself does not depend continuously on F, nevertheless A5 shows that "more exposed" parts of the Julia set are "rigid" and depend continuously on the dynamics.

The geometrical meaning of the phase transition is that the tip at point a is unique and "significantly more exposed" than any other point of the Julia set. Thus for $t > t_0$ the spectrum is determined by the combined influence of many similar parts, but for $t \geq$ the input from the "very exposed" tip at point "a" "screens" the rest of the real-analytic spectrum (which, nevertheless, continues to exist and can be calculated as Hausdorff spectrum).

The "thermodynamical" meaning of the phase transition is that for each $t > t_0$ we have a unique equilibrium state, supported on the whole

Julia set. For $t = t_0$ we obtain another equilibrium state, given by a δ-measure at the point a, which dominates for $t < t_0$. However, the original equilibrium state "continues" to exist being hidden. For Chebyshev quadratic ($F(z) = z^2 - 2$ and Julia set is an interval) this phenomenon was observed by E. Ott, W. Withers,and J. A. Yorke in [**OWY**] and A. Lopes in [**Lo5,6**]. Also in the latter paper maps having the so called gap (see [**Lo5**] and compare to the maps falling into the case (ii) of Theorem A) are discussed, including Chebyshev polynomial and the Lattes example ($F(z) = ((z-2)/z)^2$, Julia set is the whole sphere).

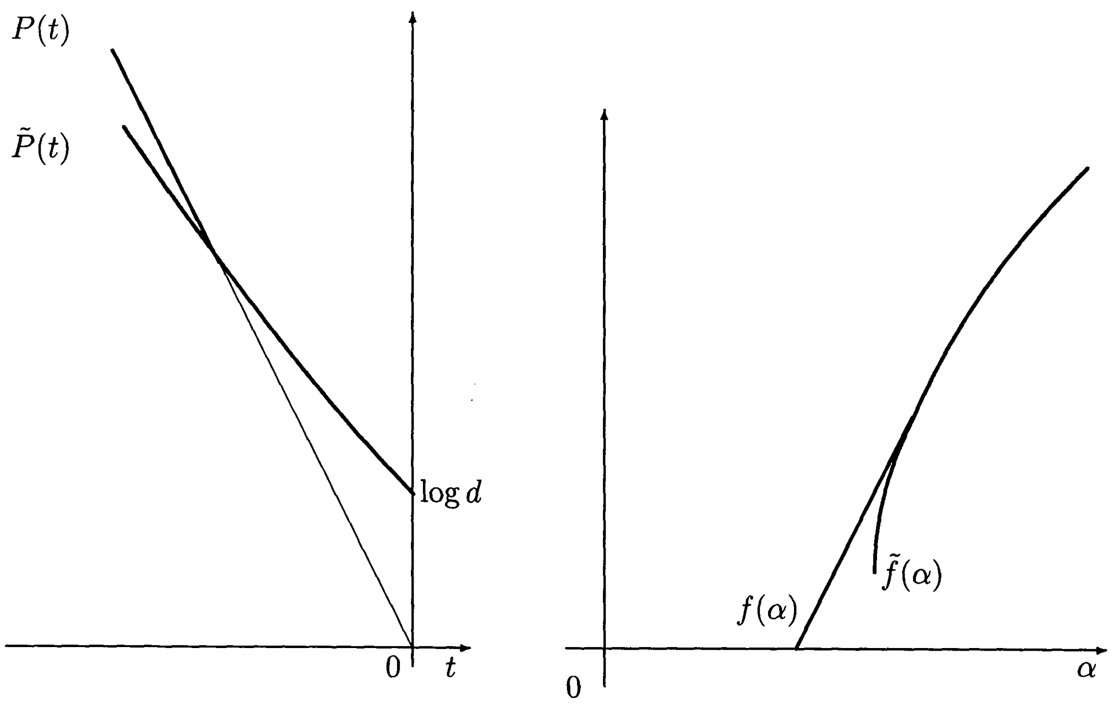

Figure 2. Phase transition in negative spectra

In other words, the transfer operator is always quasicompact, and the maximal eigenvalue is isolated, but for t_0 two eigenvalues (one generated by a nice measure, and another – by a δ-measure at the tip) "cross" and the maximal eigenvalue has multiplicity 2. Note that this differs from the phase transition in positive spectrum (e.g. for parabolics), or one described by M. Feigenbaum, I. Procaccia, and T. Tél in **[FPT]**, when the essential spectral radius reaches the maximal eigenvalue. In our case the transfer operator stays quasicompact and hence spectra behave nicely.

2.2. Positive spectrum

Parabolic case. Assume that all critical points are attracted parabolic or (super) attractive cycles. Establishing the quasicompactness of transfer operator for the parameter values $t < \mathrm{HDim}\ J_F$, we prove the following

Theorem B. B1. *For a parabolic polynomial F, the function $P_F(t)$ is real analytic on $[0, \mathrm{HDim}\ J_F)$ and $P_F(t) = 0,\ t \in [\mathrm{HDim}\ J_F, +\infty)$.*
B2. *The derivative of $P_F(t)$ is discontinuous at the point* $\mathrm{HDim}\ J_F$ *if and only if*

$$\mathrm{HDim}\ J_F > 2 - \frac{2}{p+1}, \qquad (\spadesuit)$$

where p is the maximal number of petals at the parabolic points.

The reason for the phase transition is that, while for $t < \delta$ transfer operator is quasicompact and conformal measure is "nice", for $t > \delta$ the situation changes, conformal measure becomes atomic – supported on the preimages of the parabolic cycle. Another interpretation is that main input in the spectra for $t >$ HDim J_F comes from the infinitely many times duplicated cusp at the parabolic point.

The dichotomy comes from the work [**ADU**] of J. Aaronson, M. Denker, and M. Urbański on the existence of the invariant measures equivalent to δ-conformal.

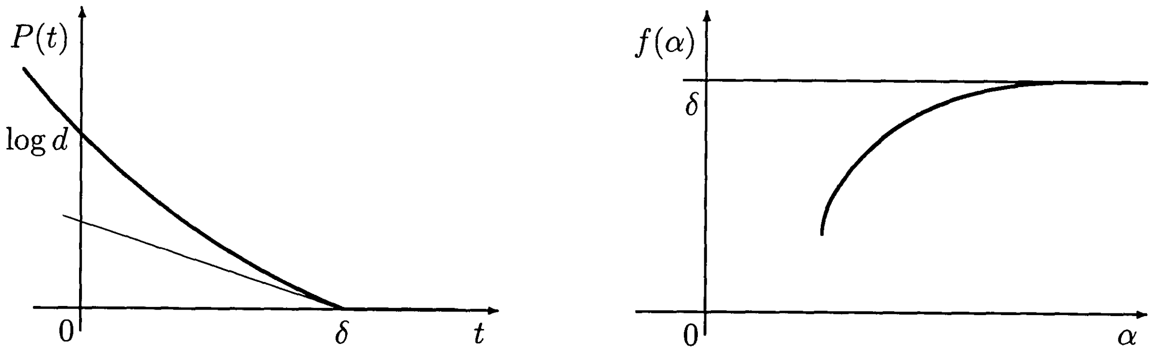

Figure 3. Spectra of parabolics with $\mathrm{HD}J_f > 2 - \frac{2}{p+1}$

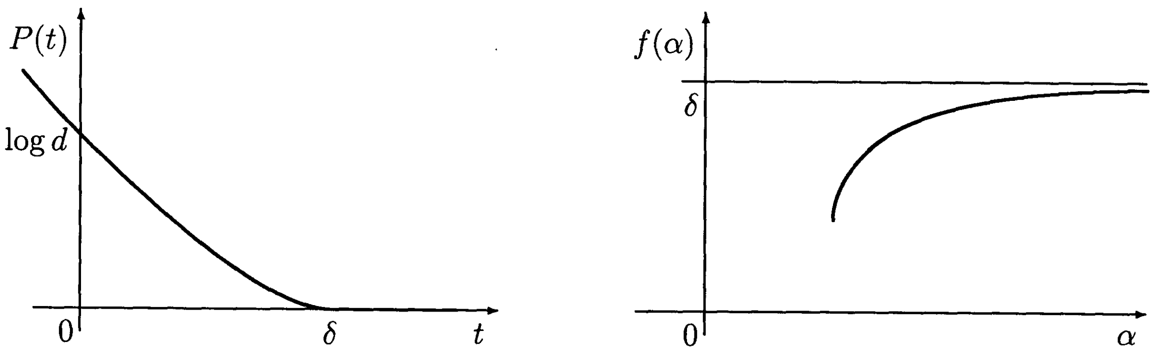

Figure 4. Spectra of parabolics with $\mathrm{HD}J_f \leq 2 - \frac{2}{p+1}$

The inequality (♠) is true for $z^2 + 1/4$ (the "cauliflower" Julia set – indeed, its dimension is greater than 1), and hence there is a discontinuity of the derivative at the phase transition point. We do not know which case of the dichotomy occurs even for other parabolic quadratics; and we were not able to prove the discontinuity of the derivative for $z^2+1/4$ directly. Note that computer experiments in **[BZ]** suggest that for $z^2 - 3/4$ the inequality (♠) fails.

Note that as an application of the quasicompactness result we obtain an improved estimate on the radius of essential spectrum of the transfer operator (in hyperbolic or parabolic case), which depends only on t, $P(t)$, and $P(\pm\infty)$ (see the Remark 1.7 in the corresponding Chapter).

Subhyperbolic case. Assume that all critical points in the Julia set are strictly preperiodic. Transferring the problem to the unit circle, we prove for the critically finite polynomials the following

Theorem C. *For a subhyperbolic polynomial F with connected Julia set, the function $P_F(t)$ is real analytic on $[0, +\infty)$.*

Semihyperbolic case. For semihyperbolic quadratic polynomials (i.e. those with non-recurrent critical point) we build a Markov extension of the original system (analogous to the Hofbauer tower), taking the Yoccoz puzzle as a base. Then we establish the quasicompactness of the corresponding transfer operator, and prove the following

Theorem D. *For a non-recurrent quadratic polynomial $F(z) = z^2 + c$ either*

(i) $P_F(t)$ *is real analytic on* $[0, +\infty)$, *or*

(ii) *A phase transition occurs: there exists* $t_0 > \mathrm{HDim}\, J_F$ *such that*

$$\begin{cases} P_F(t) & \text{is real analytic on } [0, t_0), \text{ and} \\ P_F(t) & = \; -\frac{1}{2}\, t \, \log \left(\liminf_{n\to\infty} |(F^n)'(c)|^{1/n} \right) \text{ on } [t_0, +\infty) \; . \end{cases}$$

We do not know whether case (ii) can occur. By the work **[CJY]** of L. Carleson, P. Jones and J.-C. Yoccoz the polynomials under consideration have domain of attraction to infinity $A(\infty)$ John, thus one can speculate that spectrum should be "nice" and (ii) impossible. On the other hand, by **[GS]**, Collet-Eckmann polynomials have Hölder $A(\infty)$, so $\beta(t) < 0$ for large t and one can expect to have (ii) for Collet-Eckmann recurrent polynomials.

2.3. Transfer operator on Sobolev spaces

We will apply the methods used for the analysis of negative spectrum to investigating transfer operators with general Sobolev weights. Considering functional spaces on the complex sphere, or some open set Ω with

$$F^{-1}\Omega \subset \Omega \; ,$$

taken with spherical metric, we define transfer operator on the space of continuous functions by

$$Lf(z) := \sum_{y \in F^{-1}z} f(y)\, g(y) ,$$

where preimages y of z are counted with multiplicities. Note that

$$L^n f(z) := \sum_{y \in F^{-n}y} f(y)\, g_n(y) ,$$

where $g_n(y) = g(y)g(Fy)\ldots g(F^{n-1}(y))$. Denote by λ the spectral radius:

$$\lambda := \lim_{n\to\infty} \|L^n\|_\infty^{1/n} .$$

We prove the following theorem, establishing the quasicompactness of L:

Theorem E. *Suppose that*

$$\lambda > (\sup g_n)^{1/n} , \qquad (\diamondsuit)$$

for some (any) large n. Then for $p > 2$ sufficiently close to 2 operator L acts on $W_{1,p}$ and

$$r_{ess}(L, W_{1,p}) < r(L, W_{1,p}) = \lambda ,$$

where r and r_{ess} are spectral and essential spectral radii of L as an operator on $W_{1,p}$. Moreover, λ is an isolated eigenvalue of multiplicity one.

This theorem is analogous to the results [**DU1**], [**Pr1**], [**DPU**] of M. Denker, M. Urbański, and F. Przytycki for the transfer operator on the space of Hölder continuous functions They have established the (weaker) property of almost periodicity under the assumption ($\diamondsuit$). Recently N. Haydn has extended their results, proving the quasicompactness (see [**Ha**]).

2.4. Open problems

Negative spectrum. Some work still remains to be done for negative values of t. If we establish a connection between the spectrum and the ζ-function (the similar problem is stated in [**Ru6**]), we might be able to learn more about the dependence of $P_F(t)$ on F and get some necessary and sufficient conditions for the phase transition to occur. Particularly it is interesting whether or not (iib) is sufficient for the phase transition (it is so in degrees 2 and 3).

Positive spectrum. A very optimistic statement is

Conjecture. *For any polynomial F either*

(I) *$P_F(t)$ is real analytic on $[0, +\infty)$, or*

(II) *$P_F(t)$ is real analytic on $[0, t_{ii})$ and $P_F(t) = P_+ t,\ t \in [t_0, +\infty)$, or*

(III) *$P_F(t)$ is real analytic on $[0, t_{iii})$ and $P_F(t) = 0,\ t \in [t_0, +\infty)$.*

A possible step in this direction is to establish

Conjecture. *For any polynomial F, $P_F(t)$ is real analytic in some neighborhood of* 0.

Work **[Zd]** of A. Zdunik implies that for any polynomial (except ones with Julia set being an interval or a circle) the pressure at zero has strictly positive second derivative, which advances us in that direction.

Next step will be to find some necessary and (or) sufficient conditions for (I), (II), (III) to hold: in terms of the orbits of the critical points of F or geometry of the Julia set J_F. There are some euristic reasons to expect that case of John domain of attraction to infinity (i.e. semi-hyperbolic polynomials – see **[CJY]**) corresponds to (I), while Hölder (but not John, e.g. Collet-Eckmann polynomials fall into this category, see **[GS]**) case – to (II).

If the phase transition occurs, we need to analyze its nature. One example is the question about the meaning of the first zero of the pressure function (the "phase transition point" in case (III)). It is interesting to compare it with such parameters as the dynamical, Hausdorff and hyperbolic dimensions of J_F. There is some hope that in cases (I), (II) all of these dimensions will coincide. On the other hand, the geometry of the Julia sets with the phase transition (case (III)) is likely to be "bad" and it might happen that the dimensions above will differ.

Dependence on F. The logical statement to prove is that $P_F(t)$ depends continuously on F whenever it is positive. As in the case of the negative spectrum, it might follow from the "sufficiently good" spectral analysis of the transfer operator. If one wants to prove some stronger results, like analyticity, he will probably need to study ζ-function.

2.5. About methods and organization

The general scheme of the proofs is to find a proper notion of expanding, establish the quasicompactness of the transfer operator, and deduce the results about the spectra. The thesis is organized as three (independent) chapters, devoted to negative spectrum, parabolic and semihyperbolic polynomials. Some arguments are similar in different cases, but still they have (sometimes substantial) differences, so we repeat them for the sake of completeness.

The Chapter 2 is devoted to the analysis of negative spectra for arbitrary polynomials (Theorem A). In the arbitrary case F is not expanding and L_t does not act on the space of Hölder continuous functions. This problem for $t < 0$ is solved by considering the Sobolev space, where bad behavior of F near critical points is compensated by the zeroes of the weight function $|F'|^{-t}$. Then we apply the same technique to the study of transfer operators with arbitrary Sobolev weights, establishing the Theorem E.

In the Chapter 3 we consider the parabolic case (Theorem B), when dynamics is not expanding but expansive, and careful computation shows that non-expanding branches of F^{-n} do not contribute much to the transfer operator for $t < \mathrm{HDim}\ J_F$, hence spectra are good for the corresponding parameters.

For other Julia sets and $t > 0$ the situation is more complicated, since the function $|F'|^{-t}$ is unbounded and thus L_t does not preserve the space of bounded functions.

In the subhyperbolic case we "cheat" by multiplying the weight by a precisely defined homology and cancelling its singularities. Then the problem is transferred via the Riemann uniformization map to the unit circle, where the corresponding dynamics is expanding. The pressure for the new operator gives the same spectra. The corresponding result (Theorem C) is proved in the paper **[MS]** by N. Makarov and the author. We will also rely heavily on this paper during our analysis of the phase transition phenomenon in the Chapter 2 (negative spectrum).

In the Chapter 4 we work (establishing Theorem D) with non-recurrent quadratics, building a Markov extension of original system – an analogue of a Hofbauer tower in one-dimensional dynamics, which gives the desired expansion.

Chapter 2

Negative spectra

In this chapter we will consider the spectra for negative values of parameter t, in which case transfer operator preserves the space of continuous functions. In the first Section we prove Ionescu-Tulcea and Marinescu inequality and establish the quasicompactness of transfer operator in the Sobolev space, while the second is devoted to the analyticity of spectra and their properties, which result in the following

Theorem A. **A1.** *For any polynomial F and negative t either*

(i) *$P_F(t)$ is real analytic on $(-\infty, 0)$, or*

(ii) *there exists a "phase transition" point $t_0 < 0$ such that*

$$\begin{cases} P_F(t) & \text{is real analytic on } [t_0, 0)\ , \text{ and} \\ P_F(t) & = \ -P_{-\infty} \cdot t \ \text{ on } \ (-\infty, t_0]\ . \end{cases}$$

In fact $P_F(t) = \max(-P_{-\infty} \cdot t, \tilde{P}_F(t))$ for some $\tilde{P}_F(t)$ real analytic on $(-\infty, 0)$.

A2. *In the case (i) all mentioned definitions of pressure give the same function $P_F(t)$. In the case (ii) function $P_F(t)$ corresponds to "Box" definitions, when "hidden" spectrum $\tilde{P}_F(t)$ – to "Hausdorff." All definitions converge nicely, i.e. one can take* lim *instead of* lim sup, *etc.*

A3. *For the occurrence of a "phase transition" it is necessary that either*

(iia) *F is conjugate to a Chebyshev polynomial (then J_F is an interval),* or

(iib) *there are a fixed point a, $Fa = a$, and a positive number ε such that*

$$F^n b = b\,, \quad b \neq a \quad \Rightarrow \quad \mu(b) \; \leq \mu(a) \, - \, \varepsilon \; ,$$

where $\mu(b) \; := \left|(F^n)'\,(b)\right|^{\frac{1}{n}}$ *denotes the multiplier of a periodic point b. This implies that $a \in J_F$, $P_{-\infty} \; = \; \log \mu(a)$ and*

$$F^{-1}a \setminus \{a\} \;\subset\; \text{Crit } F \quad (= \quad \text{zeroes of } \; F' \;) \; . \qquad (\clubsuit)$$

A4. *For quadratic polynomials (iib) cannot happen (for combinatorial reasons), and for cubics it only happens for some polynomials with disconnected Julia set. However, there is a degree 4 subhyperbolic polynomial with connected Julia set for which it occurs.*

A5. *The pressure function depends continuously on F as a function in the space $C^\infty(-\infty, 0)$, or $C^\infty\left((-\infty, t_0 - \varepsilon) \cup (t_0 + \varepsilon, 0)\right)$ if the phase transition occurs.*

Finally, in the last Section we apply the developed technique to the study of transfer operators with arbitrary Sobolev weights. This results in the Theorem E (see the Subsection 3.1 for more details).

1. Quasicompactness

1.1. Notation

Let Ω be a big disk compactly containing J_F such that

$$F^{-1}\Omega \subset \Omega \, ,$$

and orbits of the critical points do not intersect $\partial\Omega$.

For $t \le 0$ we define operator L_t on $C\left(\bar{\Omega}\right)$ by

$$Lf(z) \; := \; \sum_{y \in F^{-1}z} f(y) \left|F'(y)\right|^{-t} \, ,$$

where preimages y of z are counted with multiplicities. Note that

$$L^n f(z) \; := \; \sum_{y \in F^{-n}(z)} f(y) \left|F_n'(y)\right|^{-t}$$

(here and later we write $F_n = F^n$).

Denote

$$s(t) \; := \; \log_d \, r\left(L_t, C\left(\bar{\Omega}\right)\right) \quad \text{(spectral radius)} \, .$$

It will be shown later that for connected J_F

$$s(t) \;=\; 1 - t + \beta(t) \,.$$

We will also consider this operator in various Sobolev spaces $W_{1,p}(\Omega)$. Observe that for $p > 2$, $W_{1,p}(\Omega) \subset C\left(\bar{\Omega}\right)$ (see **[Zi]** for this and other properties of Sobolev spaces).

1.2. Quasicompactness theorem

Theorem. *For any negative t for $p > 2$ sufficiently close to 2 operator L_t acts on $W_{1,p}(\Omega)$ and*

$$r_{ess}\left(L_t, W_{1,p}(\Omega)\right) \;<\; r\left(L_t, W_{1,p}(\Omega)\right) \;=\; d^{s(t)} \,,$$

where r and r_{ess} are spectral and essential spectral radii of L_t as an operator on $W_{1,p}(\Omega)$.

1.3. Sobolev spaces

We consider Sobolev space $W_{1,p}(\Omega)$ with the norm

$$\|f\|_{1,p} \;:=\; \|f\|_p \;+\; \|\partial f\|_p \,.$$

If $p > 2$, then

$$W_{1,p}(\Omega) \;\subset\; C\left(\bar{\Omega}\right) \,, \quad \text{and} \quad \|f\|_\infty \;\lesssim\; \|f\|_p \,.$$

Moreover, $W_{1,p}$-functions are Hölder continuous: $W_{1,p} \subset H\ddot{o}l_{1-\frac{2}{p}}$.

More precisely, if $|z-y| = \delta$, then

$$|f(z) - f(y)| \lesssim \delta^{1-\frac{2}{p}} \|\partial f\|_{L^p(B(z,\delta))} .$$

For the proofs of these results see W.Ziemer "Weakly Differentiable Functions", particularly [**Zi**, 2.4.4].

Lemma. *If $t < -2(1-\frac{2}{p})$, then $L_t W_{1,p} \subset W_{1,p}$.*

Proof. It is sufficient to estimate

$$\int_\Omega \left| \partial_{F(y)} \left(|F'|^{-t} f \right)(y) \right|^p dm_2(F(y))$$

by const $\|f\|_{1,p}$. Clearly

$$\ldots = \int_\Omega \left| \partial \left(|F'|^{-t} f \right)(y) \right|^p |F'(y)|^{2-p} dm_2 y \leq I + II .$$

Here

$$\begin{aligned} I &= \int |\partial f|^p |F'|^{-tp} |F'|^{2-p} dm_2 \\ &< \text{const} \int |\partial f|^p , \end{aligned}$$

since $t < -2(1-\frac{2}{p})$ implies $-tp+2-p > 0$ and $\sup_\Omega |F'|^{-tp+2-p} < \infty$. The second term can be estimated by

$$\begin{aligned} II &= \int |f|^p \left(\partial \left(|F'|^{-t} \right) \right)^p |F'|^{2-p} dm_2 \\ &\leq \|f\|_\infty^p \int \left(\partial \left(|F'|^{-t} \right) \right)^p |F'|^{2-p} dm_2 , \end{aligned}$$

so we just need to prove the convergence of the last integral which is non-trivial only at the critical points of F.

Consider some critical point (say zero), let F' have singularity (zero) of order k:

$$F' \asymp z^k .$$

Then

$$\begin{aligned} |F'|^{-t} &\asymp |z|^{-kt} \\ \partial\left(|F'|^{-t}\right) &\lesssim |z|^{-1-kt} \\ \left(\partial\left(|F'|^{-t}\right)\right)^p |F'|^{2-p} &\lesssim |z|^{-p(1+kt)+k(2-p)} . \end{aligned}$$

Since $k \geq 1$, $t < -(1-\frac{2}{p})(1+\frac{1}{k})$ thus

$$-p(1+kt)+k(2-p) > -2 ,$$

which implies the convergence and hence the desired estimate. ■

1.4. Integral means $(t \leq 0)$

Lemma. *For any (all) $z \in \partial\Omega$*

$$\|L_t^n\|_\infty \asymp L_t^n 1\,(z) .$$

In J_F is connected, then

$$\|L_t^n\|_\infty \asymp d^{n(1-t)} \int_{|z|=1+d^{-n}} |\varphi'|^t \qquad (\overset{e^{o(n)}}{\asymp} d^{n(1-t+\beta(t))}) .$$

Remark. Lemma implies that $\beta(t)$ is well-defined and

$$r\left(L_t, C\left(\bar{\Omega}\right)\right) \;=\; d^{s(t)} \;=\; d^{(1-t+\beta(t))} \;.$$

Proof. Clearly $\|L_t^n\|_\infty \;=\; \|L_t^n 1\|_\infty$. The function $z \mapsto L_t^n 1\,(z)$ is subharmonic. Therefore

$$\|L_t^n 1\|_\infty \;=\; \sup_{\partial\Omega} L_t^n 1 \;.$$

Fix $z_0 \in \partial\Omega$. For any $z \in \partial\Omega$ we can choose a domain $\Psi \ni z, z_0$ without forward iterates of critical points. Then any branch of F^{-n} is conformal on Ψ, so

$$|F_n'(y)| \;\asymp\; |F_n'(y_0)|$$

where y, y_0 are images of z, z_0 under the same branch of F^{-n}. We can estimate the distortion by a constant independent of z, hence

$$\sup_{\partial\Omega} L_t^n 1 \;\asymp\; L_t^n 1\,(z_0) \;\asymp\; L_t^n 1\,(z) \;,$$

which implies the first statement of the Lemma.

Note that similar estimates (and $F^{-n}\,\Omega \searrow K_F \;=\;$ filled Julia set) imply that

$$|F_n'\,(y_0)| \;\geq\; 1 \quad \text{for} \quad y_0 \;\in\; F^{-n} z_0 \;\text{ and large }\; n \;,$$

thus for $t < u < 0$

$$L_t^n 1\,(z_0) \;\gtrsim\; L_u^n 1\,(z_0) \;\gtrsim\; d^n \;,$$

and

$$s(t) \ \geq \ s(u) \ \geq \ 1 \ .$$

Moreover area estimate shows that

$$\sum_{y \in F^{-n}(z_0)} |F_n'(y)|^{-2} \ \lesssim \ 1 \ ,$$

implying that for fixed $d > q > 1$ for large n most of the $y \in F^{-n} z_0$ satisfy $|F_n'(y)| \ > \ q^n$. Therefore for $n \gg 1$

$$\begin{aligned} \frac{d}{dt} L_t^n 1(z_0) \ &= \ \sum_{y \in F^{-n}(z_0)} - \log |F_n'(y)| \ \cdot |F_n'(y)|^{-t} \\ &\leq \ -\frac{1}{2} \ n \ \log q \ \sum_{y \in F^{-n}(z_0)} |F_n'(y)|^{-t} \\ &= \ -\frac{1}{2} \ n \ \log q \ L_t^n 1(z_0) \ . \end{aligned}$$

So for

$$\phi_n(t) \ := \ \frac{1}{n} \ \log_d \ (L_t^n 1(z_0))$$

we have $\phi_n(t) \ < \ -\frac{1}{2} \log q \ < \ 0$ which implies that $s(t)$ is strictly decreasing because $\phi_n \underset{n \to \infty}{\longrightarrow} s$.

To prove the second part note that without loss of generality φ conjugates F with dynamics $T : \ z \mapsto z^d$ on $\mathbb{D}_-$:

$$F \circ \varphi \ = \ \varphi \circ T \ .$$

Differentiating the identity $F^n \circ \varphi \ = \ \varphi \circ T^n$, we obtain

$$F_n' \circ \varphi \ \cdot \ \varphi' \ = \ \varphi' \circ T^n \ \cdot \ T^{n\prime} \ .$$

Applying this equality to the preimages $\zeta \in T^{-n}$ of some fixed point $\xi \in \mathbb{D}_-$ we notice that the right side is $\asymp d^n$, thus (taking to power t)

$$d^{-nt}\left|\varphi'(\zeta)\right|^t \asymp \left|F_n'(y)\right|^{-t},$$

where $y = \varphi(\zeta)$ is a corresponding preimage of $z = \varphi(\xi)$ under F^n.

The points ζ are equidistributed on the circle

$$|x| = r_n = |\xi|^{\frac{1}{d^n}},$$

with $(r_n - 1) \asymp \frac{1}{d^n}$. Therefore summing over all $\zeta \in T^{-n}\xi$ ($y \in F^{-n}z$), we have

$$\begin{aligned} L^n 1(z) &= \sum_{y \in F^{-n}(z)} \left|F_n'(y)\right|^{-t} \\ &\asymp d^{-nt} \sum_{\zeta \in T^{-n}\xi} \left|\varphi'(\zeta)\right|^t \\ &\asymp d^{n(1-t)} \int_{|x|=r_n} \left|\varphi'\right|^t, \end{aligned}$$

which together with the first statement completes the proof. ■

1.5. Ionescu-Tulcea and Marinescu inequality

Lemma. *Suppose $t < -2(1-\frac{2}{p})$, so L_t acts on $W_{1,p}(\Omega)$. Then*

$$\|L_t^n f\|_{1,p} \lesssim d^{nw(p,t)+o(n)}\|f\|_{1,p} + C_n\|f\|_\infty,$$

with

$$w(p,t) = \frac{1}{p'}\, s\left(p'\left(1+t-\frac{2}{p}\right)\right) .$$

Proof. Clearly

$$\begin{aligned}\|\partial\,(L^n f)\|_p \le& \left\|\left(\sum_{y\in F^{-n}(z)} \partial_z f(y)\,|F_n'(y)|^{-t}\right)(z)\right\|_p \\ &+ \left\|\left(\sum_{y\in F^{-n}(z)} f(y)\,\partial_z |F_n'(y)|^{-t}\right)(z)\right\|_p =: I + II .\end{aligned}$$

We can estimate the right-hand side by

$$\begin{aligned} I^p &\le \int_\Omega \left[\sum_{y\in F^{-n}(z)} |\partial_y f(y)|\,|F_n'(y)|^{-(1+t)}\right]^p dm_2(z) \\ &= \int_\Omega \left[\sum |\partial f||F_n'|^{-\frac{2}{p}}\,|F_n'|^{\frac{2}{p}-1-t}\right]^p dm_2(z) \\ &\underset{*}{\le} \int_\Omega \left[\sum_{y\in F^{-n}(z)} |\partial f|^p |F_n'|^{-2}(y)\right] \cdot \\ &\qquad \cdot \left[\sum_{y\in F^{-n}(z)} |F_n'(y)|^{p'\left(\frac{2}{p}-1-t\right)}\right]^{\frac{p}{p'}} dm_2(z) \\ &\le \int_\Omega \sum_{y\in F^{-n}(z)} |\partial f|^p |F_n'|^{-2}(y)\, dm_2(z) \cdot \\ &\qquad \cdot \left[\sup_{z\in\Omega} \sum_{y\in F^{-n}(z)} |F_n'(y)|^{p'\left(\frac{2}{p}-1-t\right)}\right]^{\frac{p}{p'}} \\ &= \|\partial f\|_p^p \left\|L^n_{p'(1+t-\frac{2}{p})}\right\|_\infty^{\frac{p}{p'}} \end{aligned}$$

(in (*) we used Hölder inequality: $(\sum a_i b_i)^p \;\le\; \sum a_i^p \left(\sum b_i^{p'}\right)^{\frac{p}{p'}}$),

and

$$
\begin{aligned}
II^p &= \int_\Omega \left(\sum_{y \in F^{-n}(z)} f(y)\, \partial_y |F_n'(y)|^{-t} |F_n'(y)|^{-1} \right)^p dm_2(z) \\
&\le \|f\|_\infty^p \int_\Omega \left(\partial_y |F_n'(y)|^{-t} |F_n'(y)|^{-1} \right)^p dm_2(z) \;=:\; \|f\|_\infty^p\, C_n^p \,.
\end{aligned}
$$

Here C_n is finite since we assumed $t < -2(1-\frac{2}{p})$ – use the same estimate as in 1.3. It remains to notice that (see 1.4)

$$
\begin{aligned}
\left\| L^n_{p'(1+t-\frac{2}{p})} \right\|_\infty^{\frac{1}{p'}} &\overset{e^{o(n)}}{\asymp} d^{n \frac{1}{p'} s\left(p'(1+t-\frac{2}{p})\right)} \\
&= d^{nw(p,t)} \,.
\end{aligned}
$$

■

1.6. Finite rank approximation

Lemma. *For any* $\varepsilon > 0$ *there exists a finite rank operator* M *in* $W_{1,p}(\Omega)$ *such that*

$$
\begin{aligned}
\|M\|_{1,p} &\le \text{abs. const}\,, \\
\|(I-M)\,f\|_\infty &\le \varepsilon\, \|f\|_{1,p}\,.
\end{aligned}
$$

Proof. Cover Ω with a grid of triangles ϕ of size $\delta \ll 1$. Define Mf as a continuous function satisfying

$$
Mf \;=\; \begin{cases} f \;\text{ at all vertices}\,, \\ \text{is linear in each triangle } \phi\,. \end{cases}
$$

For a fixed ϕ, we have

$$|\partial (Mf)| \lesssim \frac{1}{\delta} \|f - f(center)\|_{L^\infty(\phi)}$$

and by the Hölder continuity (see 1.3) this is

$$\ldots \lesssim \delta^{-\frac{2}{p}} \left(\int_\phi |\partial f|^p \right)^{\frac{1}{p}} .$$

Taking to the power p and integrating over Ω (i.e. summing up over all ϕ's) we have

$$\int_\Omega |\partial (Mf)|^p \lesssim \int_\Omega |\partial f|^p ,$$

which proves the first inequality.

The second one follows from the Hölder continuity. ■

1.7. Analysis of $w(p,t)$

We show here that for any negative $t < -2\left(1 - \frac{2}{p}\right)$ with $p > 2$ function $w(p,t)$ satisfies

$$w(p,t) < s(t) .$$

Clearly $s(t)$ satisfies following two properties:

$$s(kt) \leq k\, s(t) \text{ for } k > 1 \text{ and}$$

$$s(t) \text{ is decreasing .}$$

Proof. In fact, $d^{Ns(t)} \overset{e^{o(n)}}{\asymp} \left\|L_t^N 1\right\|_\infty$ so obvious

$$L_{kt}^N 1 \ \le\ \left(L_t^N 1\right)^k$$

implies the first property.

For the second see Proof of the Lemma 1.4. There we proved even that $s(t)$ is strictly decreasing but we use the simpler property for the sake of 2.7. ∎

Now we can write

$$\begin{aligned} w(p,t) &= \frac{1}{p'}\, s\left(p'\left(t+1-\frac{2}{p}\right)\right) \\ &\le \frac{1}{p'}\, s\left(p'\, t\right) \\ &\le s\left(t\right) . \end{aligned}$$

If the second inequality is strict, we are done. Otherwise for any $k \in [1, p']$

$$k\, s(t) \ =\ \frac{k}{p'}\, s\left(p't\right) \ \le\ s(kt) \ \le\ k\, s(t) \ ,$$

hence $s(kt) = k\, s(t)$ and $s(t)$ is strictly decreasing on $[p't, t]$. This implies that the first inequality is strict and again we are done. ∎

1.8. Quasicompactness

Claim. *Suppose $t < -2(1-\frac{2}{p})$ and $p > 2$. Then*

$$r(L_t, W_{1,p}(\Omega)) \ =\ d^{s(t)} \ ,$$

and

$$r_{ess}(L_t, W_{1,p}(\Omega)) \;<\; r(L_t, W_{1,p}(\Omega)) \;.$$

Remark. Clearly $p \in (2, \frac{4}{t+2})$ for $t \in (-2, 0)$ and any $p > 2$ for $t \leq -2$ satisfies the condition of this Claim and therefore it completes the Proof of our Theorem.

Proof. First recall that

$$r\left(L_t, C\left(\bar{\Omega}\right)\right) \;=\; d^{s(t)} \;.$$

Also

$$\|L_t^n\|_\infty \asymp \|L_t^n 1\|_\infty \lesssim \|L_t^n 1\|_{1,p} \lesssim \|L_t^n\|_{1,p} \;,$$

and $d^{s(t)} \;\leq\; r(L_t, W_{1,p}(\Omega))$.

To prove the converse, by the Ionescu-Tulcea and Marinescu inequality choose $\varepsilon > 0$ and N such that

$$\left\|L^N f\right\|_{1,p} \;\leq\; d^{N(s(t)-\varepsilon)}\, \|f\|_{1,p} \;+\; K\, \|f\|_\infty \;.$$

Define inductively an increasing sequence $\{M_k\}$ by

$$M_0 \;:=\; 1 \;, \qquad M_k \;:=\; \max\left\{M_{k-1}\, d^{Ns(t)},\; \left\|L^{Nk}\right\|_\infty\right\} \;.$$

Then

$$M_k \;=\; d^{Nks(t)+o(k)} \;\geq\; \left\|L^{Nk}\right\|_\infty \;, \qquad M_{k+1} \,/\, M_k \;\geq\; d^{Ns(t)} \;;$$

and we can choose a constant C satisfying

$$C\,d^{N(s(t)-\varepsilon)} + 1 \;\le\; C\,d^{Ns(t)} \quad \text{and} \quad K \;\le\; C\,M_1 \;.$$

By induction,

$$\begin{aligned} \left\|L^{Nk} f\right\|_{1,p} &\le\; d^{Nk(s(t)-\varepsilon)}\, \|f\|_{1,p} \;+\; C\,M_k\, \|f\|_\infty \\ &\lesssim\; d^{Nks(t)}\, \|f\|_{1,p} \;, \end{aligned}$$

therefore

$$r(L_t, W_{1,p}(\Omega)) \;\le\; d^{s(t)} \;,$$

which implies the desired inequality and thus the first statement.

To prove the second statement choose q small such that

$$q\left(1 + \|M_\varepsilon\|_{1,p}\right) \;<\; \frac{1}{3}$$

for any approximation operator M_ε. Since $w(p,t) < s(t)$, we have

$$\left\|L^N f\right\|_{1,p} \;\le\; q\,d^{Ns(t)}\, \|f\|_{1,p} \;+\; K\,\|f\|_\infty$$

for some N and K. Take $\varepsilon < \frac{1}{3K} d^{Ns(t)}$, and define $M := M_\varepsilon$. Then

$$\begin{aligned} \left\|L^N (f - Mf)\right\|_{1,p} &\le\; q\,d^{Ns(t)}\, \|f - Mf\|_{1,p} \;+\; K\,\|f - Mf\|_\infty \\ &\le\; d^{Ns(t)}\, q \left(1 + \|M\|_{1,p}\right) \|f\|_{1,p} \;+\; K\,\varepsilon\, \|f\|_{1,p} \\ &\le\; \frac{2}{3}\, d^{Ns(t)}\, \|f\|_{1,p} \;, \end{aligned}$$

and

$$r_{ess}\left(L_t, W_{1,p}(\Omega)\right) \leq \left(\frac{2}{3}\right)^{\frac{1}{N}} d^{s(t)} < d^{s(t)} .$$

■

2. Analyticity

2.1. Remark on the proofs

We will prove Theorem A1 in the Subsections 2.3 - 2.8 below, last two of which are also devoted to the analysis of the phase transition phenomenon (Theorem A3). Multifractal analysis (Theorem A2) is completed in the Subsections 2.2 and 2.9. In the remaining part of this Subsection we prove Theorem A4-5.

Proof of Theorem A4. Discussion in [**MS**, 6.2] shows that for combinatorial reasons (♣) cannot happen for quadratic polynomials. Comparison of multipliers in [**MS**, 6.5] for cubic polynomials satisfying (♣) implies that iib) cannot be true for F with connected J_F. See [**MS**, 6.6] for an example of a degree 4 subhyperbolic polynomial with a phase transition. ■

Corollary. *If F is quadratic or cubic polynomial, it has a phase transition if and only if (iia) or (iib) is true.*

Proof of Theorem A5. Discussion below shows that the main eigenvalue of the transfer operator is isolated and of multiplicity one for all $t \in (-\infty, 0)$, or for $t \in (-\infty, t_0 - \varepsilon) \cup (t_0 + \varepsilon, 0)$, if the phase transition occurs. Using perturbation theory for transfer operators with weights $|F'|^{-(t+is)}$, we conclude that for specified values of t pressure can be extended to a holomorphic function of $t + is$ in a thin strip $-\epsilon < s < \epsilon$. By the perturbation theory again, this extended pressure depends continuously on F, implying Theorem A5 via Cauchy formula. ■

2.2. Variational principle

For $t < 0$ define *pressure* by

$$P(t) \ := \ P\left(-t \log |F'|\right) \ = \ \sup \left(h_\mu \ - \ t \int \log |F'| d\mu \right) \ ,$$

where supremum is taken over all probability measures μ on J_F invariant under F. We also define $\tilde{P}(t)$ as the same supremum, taken over all probability non-atomic measures (or equivalently, with positive entropy).

Lemma. $s(t) \ = \ P(t) \,/\, \log d$ *and* $\tilde{s}(t) \ = \ \tilde{P}(t) \,/\, \log d$ *.*

Proof. Equivalent property $r(L_t) \ = \ \exp\left(P\left(t\right)\right)$ is mentioned in **[Ru6,** 6.3], where he refers to [**Pr**]. The same method works for the second formula – see section 2.7 for the definition of $\tilde{s}(t)$. ∎

2.3. Eigenvalues and multiplicity

Fix t and denote

$$\lambda = \lambda_t := r\left(L_t, C\left(\bar{\Omega}\right)\right) .$$

Lemma. $\ker\left(L_t - \lambda\right) \neq \{0\}$.

Proof. See [**Ru6**, Theorem 2.2].

Idea of the proof: since $r_{ess}\left(L_t, W_{1,p}(\Omega)\right) < r\left(L_t, W_{1,p}(\Omega)\right) = \lambda$, we have some eigenvalues λ_j of absolute value λ, and a decomposition

$$1 = X_0 + \sum X_j ,$$

where X_0 is the subspace corresponding to the rest of the spectrum.

Then

$$0 \leq L_t^n 1 = L_t^n X_0 + \sum L_t^n X_j .$$

Here $L_t^n X_0 = o\left(\lambda^n\right)$ and $L_t^n X_j = \lambda_j^n X_j$, implying that one of the λ_j's is positive and hence is equal to $\lambda = r\left(L_t, C\left(\bar{\Omega}\right)\right)$. ∎

Remark. This reasoning also implies that $L_t^n 1 \lesssim \lambda^n$ and therefore for $z \in \partial\Omega$ we have $L_t^n 1 \asymp \lambda^n$.

2.4. Conformal measures

Let L_t^* denote the adjoint of $L_t : C(J_F) \circlearrowleft$.

Lemma. *There exists probability measure ν on J_F such that*

$$L_t^* \nu = \lambda \nu .$$

Proof. Fix a point $z \in \partial\Omega$ and denote

$$\mu_n := \lambda^{-n} (L_t^*)^n \delta_z .$$

Clearly $L_t^* \mu_n = \lambda \mu_{n+1}$. Since

$$d L_t^* \nu(z) = |F'(z)|^{-t} d\nu(Fz) ,$$

we have a following formula for μ_n:

$$\mu_n = \lambda^{-n} \sum_{y \in F^{-n}(z)} |F'(y)|^{-t} \delta_y ,$$

and $\mathrm{Var}(\mu_n) = \lambda^{-n} L_t^n 1(z) \asymp 1$.

Set

$$\nu_n := \sum_{j=0}^{n} \mu_n .$$

Then $\mathrm{Var}(\nu_n) \asymp n$ and

$$\mathrm{Var}(L_t^* \nu_n - \lambda \nu_n) = \mathrm{Var}(\lambda \mu_{n+1} - \lambda \mu_0) \lesssim 1 .$$

Following Patterson and Sullivan choose a subsequence of normalized measures $\nu_n \,/\, \mathrm{Var}\,(\nu_n)$ weakly converging to some measure ν. Clearly ν is probability measure supported on J_F. Since

$$\mathrm{Var}\,(L_t^* \nu_n \;-\; \lambda \nu_n) \;/\; \mathrm{Var}\,(\nu_n) \underset{n \to \infty}{\longrightarrow} 0 \;,$$

measure ν satisfies $L_t^* \nu \;=\; \lambda \nu$. ■

Lemma. *Suppose ν is a measure from the previous Lemma. Then* $\mathrm{supp}\nu \;=\; J_F$ *unless F satisfies iia) or* (♣).

Proof. Clearly ν satisfies

$$\lambda\, d\nu(z) \;=\; d\, L_t^*\, \nu(z) \;=\; |F'(z)|^{-t}\, d\nu(Fz) \;,$$

hence

$$z \in \mathrm{supp}\nu \;\Rightarrow\; Fz \in \mathrm{supp}\nu \quad \text{and}$$

$$Fz \in \mathrm{supp}\nu \;, \quad z \notin \mathrm{Crit}\; F \Rightarrow\; z \in \mathrm{supp}\nu \;.$$

Then the same reasoning as in [**MS**, 6.1] proves this Lemma.

Main idea: preimages of any point $z \in J_F$ are dense in J_F, so if $z \in \mathrm{supp}\nu$ does not belong to the forward orbits of critical points, $\mathrm{supp}\nu \;=\; J_F$ and we are done. The only cases when ν can be supported only on the forward orbits of critical points are iia) and (♣). ■

2.5. Jordan cells

Example. Operator

$$Q_t \; := \; \begin{pmatrix} o & 1 \\ x^2 & 0 \end{pmatrix} \; ,$$

acting on $\mathbb{R}^2$ depends real analytically on t. But $\sigma(Q_t) = \{\pm t\}$ and therefore $r(Q_t) = |t|$ is not a real analytic function.

Lemma. *Suppose there exists an eigenmeasure ν, $L^* \nu = \lambda \nu$ with* $\operatorname{supp}\nu = J_F$. *Then* dim ker $(L_t - \lambda)^2 = 1$, *i.e. λ is a simple eigenvalue.*

Proof. First we will prove the following

Sublemma. *For $f \in W_{1,p}(\Omega)$*

$$\begin{cases} L_t f = \lambda f \\ f|_{J_F} = 0 \end{cases} \quad \textit{implies} \quad f = 0 \, .$$

Proof of Sublemma. For any $z \in \Omega$ distortion estimates in 1.4 imply

$$\begin{aligned} \left|F_n'(y)\right|^{-1} &\asymp d^{-n} \left|\varphi'(\zeta)\right| \asymp d^{-n} \frac{\text{dist }(y, J_F)}{\text{dist }(\zeta, \partial\mathbb{D}_-)} \\ &\asymp d^{-n} \frac{\text{dist }(y, J_F)}{d^{-n}} = \text{dist }(y, J_F) \; , \end{aligned}$$

where notation is taken from 1.4. Using $W_{1,p}(\Omega) \subset H\ddot{o}l_\alpha$ with small $\alpha < 1 - \frac{2}{p}$ we obtain

$$
\begin{aligned}
|f(z)| &= \left|\lambda^{-n} L_t^n f(z)\right| \\
&= \left|\lambda^{-n} \sum_{y \in F^{-n}(z)} |F_n'(y)|^{-t} f(y)\right| \\
&\lesssim \lambda^{-n} \sum_{y \in F^{-n}(z)} |F_n'(y)|^{-t} \text{ dist } (y, J_F)^\alpha \\
&\lesssim \lambda^{-n} \sum_{y \in F^{-n}(z)} |F_n'(y)|^{-t-\alpha} \\
&\lesssim d^{n(s(t-\alpha)-s(t))} \underset{n\to\infty}{\longrightarrow} 0 ,
\end{aligned}
$$

because $s(t)$ is strictly decreasing by 1.4. Consequently $f = 0$ and the Sublemma is proved. ■

If we consider only functions in $C(J_F)$, the same reasoning as in [**MS**, 3.6] gives us that dim ker $(L_t - \lambda) = 1$, and then Sublemma implies this for $f \in W_{1,p}(\Omega)$.

Also [**MS**, §3.6] shows that we have an eigenfunction f non-negative on J_F:

$$(L_t - \lambda)\, f = 0 .$$

If λ is not a simple eigenvalue, then there exists function h such that

$$(L_t - \lambda)\, h = f .$$

Therefore

$$\begin{aligned}\langle f, \nu\rangle &= \langle Lh, \nu\rangle - \langle \lambda h, \nu\rangle \\ &= \langle h, L^*\nu\rangle - \lambda\, \langle h, \nu\rangle = 0 .\end{aligned}$$

Thus ν-a.e. we have $f = 0$, hence everywhere on J_F and (by the Sublemma) in Ω. ∎

2.6. Analyticity

Lemmas 1.8 and 2.5 show that if for some t' there exists eigenmeasure ν, $L^*_{t'}\,\nu = \lambda_{t'}\,\nu$ with supp$\nu = J_F$, then for $p > 2$ close to 2 number $d^{s(t)}$ is a simple isolated eigenvalue of $L_t : W_{1,p}(\Omega) \circlearrowleft$. Clearly L_t depends real analytically on t therefore $s(t)$ and hence $\beta(t)$ is real analytic in the neighborhood of t' (see [**MS**, 4.1]).

So 2.5 implies that $s(t)$ is real analytic on $(-\infty, 0)$, unless iia) or (♣) happens.

In the case iia) J_F is an interval, without loss of generality

$$J_F = [-2, 2]\ , \quad \varphi(z) = z + \frac{1}{z}\ ,$$

and simple calculation shows that

$$\beta(t) = \max\,(-1 - t, 0)\ .$$

It remains to deal with the case (♣) when for some t' there is no eigenmeasure supported on the whole J_F. Then analysis in [**MS**, 6] implies that in fact $\nu_{t'} = \delta_a$ and $\lambda_{t'} = \mu(a)^{-t}$ (or equivalently $s(t') = -s_- t'$ with $s_- = \log_d \mu(a)$). Set

$$t_0 := \sup\{t : \nu_t = \delta_a\} = \sup\{t : s(t) = -s_- t\} .$$

By the proof of 1.4 $s(t) \geq 1$, thus $t_0 < 0$.

For $t \in (t_0, 0)$ by the discussion above $s(t)$ is real analytic on $(t_0, 0)$. Since

$$L_t^* \delta_a = \mu(a)^{-t} \delta_t = d^{-s_- t} \delta_a ,$$

we have $s(t) \geq -s_- t$. On the other hand by 1.7 for $t \leq t_0$

$$s(t) \leq \frac{t}{t_0} s(t_0) = s_- t ,$$

which implies that $s(t) = -s_- t$ for $t \in (-\infty, t_0]$.

Remembering the relation $\beta(t) = s(t) + t - 1$ we notice that to prove the main Theorem it remains to show that $s(t)$ can be extended from $[t_0, 0)$ to a real analytic function on $(-\infty, 0)$, and occurrence of the phase transition together with (♣) implies condition iib).

2.7. Hidden spectrum

We want to analyze the case when F satisfies (♣) and a phase transition occurs.

Let $\kappa := \min\{\text{multiplicity } c \in F^{-1}(a)\}$. We will consider functions

$$G_{\alpha,t} := |F'|^{-t} \frac{H_{\alpha,t}}{H_{\alpha,t} \circ F} ,$$
$$H_{\alpha,t}(z) := |z-a|^{-\alpha t} ,$$

and associated transfer operator

$$L_{\alpha,t} f(z) := \sum_{y \in F^{-n}(z)} G_{\alpha,t}(y)\, f(y) .$$

For all $\alpha < \frac{\kappa}{\kappa+1} =: \alpha_{max}$ function $G_{\alpha,t}$ belongs to $W_{1,p}(\Omega)$ for $p > 2$ close to 2 and has zeroes of multiplicity $\geq -tk_{min}$ with

$$k_{min} := \min\left(1, \kappa - \alpha\,(\kappa+1)\right) > 0$$

at the critical points of F. Also denote

$$k_{max} := \max\{\text{multiplicity } c \in \operatorname{Crit} F\} .$$

Set

$$r_\alpha(t) := r\left(L_{\alpha,t}, C\left(\bar{\Omega}\right)\right) ,$$
$$s_\alpha(t) := \log_d r_\alpha(t) .$$

We can repeat all previous arguments for the operator $L_{\alpha,t}$ with $\alpha \in [0, \alpha_{max})$ (or we can use results of the Section 3).

A few keypoints:

Obviously

$$\left(G_{\alpha,t}\right)_n := \prod_{j=0}^{n-1} G_{\alpha,t} \circ F^j = |F'|^{-t} \frac{H_{\alpha,t}}{H_{\alpha,t} \circ F^n} , \quad \text{and}$$
$$\left(L^n_{\alpha,t} f\right)(z) = \frac{1}{H(z)} \left(L^n_{\alpha,t} H f\right)(z) ,$$

this relation helps to reformulate properties of $L_{\alpha,t}$ in terms of L_t. It means that considering $L_{\alpha,t}$ we in fact restrict L_t to co-dimension one subspace $W_{1,p}(\Omega) \cap \{f : f(a) = 0\}$ of $W_{1,p}(\Omega)$, "forgetting" the eigenvalue of L_t which gives us the phase transition.

Then note that the same reasoning as in 1.4 shows that for $0 \le \alpha \le \frac{\kappa}{\kappa+1}$ and $z \in \partial\Omega$

$$\begin{aligned} r_\alpha(t)^n &\overset{e^{o(n)}}{\asymp} \left\| L^n_{\alpha,t} \right\|_\infty \\ &\asymp L^n_{\alpha,t} 1\,(z) \\ &= \frac{1}{H_{\alpha,t}(z)}\ (L^n_t H_{\alpha,t})\,(z) \\ &\asymp (L^n_t H_{\alpha,t})\,(z)\ . \end{aligned}$$

Since $G_{\alpha,t}$ has smaller orders of zeroes, to make $L_{\alpha,t}$ act on $W_{1,p}(\Omega)$ we need a stronger condition, particularly

$$-p\ (1 + k_{min} t)\ +\ k_{max}\ (2 - p)\ >\ -2$$

will be sufficient. But this inequality is equivalent to

$$p\ (k_{max} + 1 + k_{min} t)\ <\ 2\ (k_{max} + 1)\ ,$$

which is clearly true for $p > 2$ close to 2.

An analog of $w(p,t) < s(t)$ which we need for the proof of Ionescu-

Tulcea and Marinescu inequality is

$$\frac{1}{p'}\, s_{\alpha'}\left(p'\left(t-\frac{2}{p}+1\right)\right) \;<\; s_\alpha(t) \quad \text{with}$$
$$\alpha' \;=\; \alpha\,\frac{t}{\left(t-\frac{2}{p}+1\right)} \;>\; \alpha\;.$$

This can be proved following 1.7, using the inequality $s_{\alpha'}(t) \leq s_\alpha(t)$, which is true since

$$\begin{aligned} r_{\alpha'}(t)^n &\overset{e^{o(n)}}{\asymp} L^n_{\alpha',t}1(z) \\ &= L^n_{\alpha,t}\, H_{\alpha'-\alpha}\,(z) \\ &\lesssim L^n_{\alpha,t}1(z) \overset{e^{o(n)}}{\asymp} r_\alpha(t)^n \end{aligned}$$

for fixed $z \in \partial\Omega$.

Like in 2.6 we obtained that $s(t)$ is real analytic whenever $s(t) > -s_- t$, here we conclude in the same way that whenever

$$s_\alpha(t) \;>\; -s_{\alpha,-}t \quad \text{with} \quad s_{\alpha,-} \;:=\; \frac{1}{-t}\log_d G_{\alpha,t}(a) \;=\; (1-\alpha)\; s_- \;,$$

all our theory is applicable to $L_{\alpha,t}$, hence the function $s_\alpha(t)$ is real analytic and adjoint operator has eigenmeasure

$$L^*_{\alpha,t}\,\nu_{\alpha,t} \;=\; r_\alpha(t)\,\nu_{\alpha,t}\;,$$

with support dense in J_F.

Then for any $\alpha' \in [\alpha, \alpha_{max}]$ we have $H := H_{\alpha'-\alpha,t} \in W_{1,p}(\Omega)$ for $p > 2$ close to 2 and

$$\left\langle L^n_{\alpha,t} H, \nu_{\alpha,t} \right\rangle = r_\alpha(t)^n \, \langle H, \nu_{\alpha,t} \rangle \gtrsim r_\alpha(t)^n ,$$

implying for fixed $z \in \partial\Omega$ that

$$\left(L^n_{\alpha,t} H \right)(z) \asymp r_\alpha(t)^n .$$

Thus (fix $z \in \partial\Omega$)

$$\begin{aligned} r^n_{\alpha'}(t) &\overset{e^{o(n)}}{\asymp} \left(L^n_{\alpha',t} 1 \right)(z) \\ &= (H(z))^{-1} \left(L^n_{\alpha,t} H \right)(z) \\ &\asymp \left(L^n_{\alpha,t} H \right)(z) \asymp r_\alpha(t)^n . \end{aligned}$$

This leads us to

Claim. *Let $\alpha \in [0, \alpha_{max})$ and*

$$t_\alpha := \inf \left\{ t : s_\alpha(t) > -s_{\alpha,-t} \right\} .$$

Then $s_\alpha(t)$ is real analytic on the interval $(t_\alpha, 0)$ and is equal (on this interval) to $s_{\alpha'}(t)$ for any $\alpha' \in [\alpha, \alpha_{max}]$.

Set $\tilde{s}(t) := s_{\alpha_{max}}(t)$. Noticing that $s_0(t) = s(t)$, we conclude that

1) $\tilde{s}(t) = s(t)$ for $t \in [t_0, 0)$.

2) $\tilde{s}(t)$ is real analytic whenever $\tilde{s}(t) > -s_{\alpha_{max},-t}$.

So it remains to prove the following

Lemma. *For any t the condition $\tilde{s}(t) > -s_{\alpha_{max},-t}$ holds, i.e.*

$$r_{\alpha_{max}}(t) \ > \ G_{\alpha_{max},t}(a) \ .$$

Proof. Throughout this proof for simplicity we denote $L := L_{\alpha_{max},t}$ and $G := G_{\alpha_{max},t}$.

Function G (unlike $G_{\alpha,t}$ for smaller α) does not have zeroes at all critical points of F, moreover for any $z \in \Omega$ at least one of its preimages is not a zero of G. Therefore we can consider the following operator on probability measures on J_F:

$$\nu \ \mapsto \ \mathrm{Var}\ (L^*\nu)^{-1} \ L^*\nu \ ,$$

and by the Schauder theorem it fixes some measure which is an eigenmeasure of L^*:

$$L^*\nu \ = \ \lambda\,\nu \ .$$

Clearly $\lambda \leq r_{\alpha_{max}}(t)$.

Consider some neighborhood of the point a which is mapped by F onto itself. Take small disk U inside it such that it's preimages U_n under the branches of F^{-n} preserving a are disjoint.

Reasoning from [**MS**, 6.1] shows that $\mathrm{supp}\nu = J_F$ thus $\nu(U) > 0$. Since ν is an eigenmeasure of L^* and simple distortion estimates imply

$$G_n|_{U_n} \ \asymp \ G_n(a) \ ,$$

we have

$$\begin{aligned}\nu(U_n) &= \lambda^{-n} \int_{U_n} G_n(z)\, d\nu(F^n z) \\ &\asymp \lambda^{-n}\, G_n(a)\, \nu(U) \gtrsim \left(\frac{G(a)}{\lambda}\right)^n .\end{aligned}$$

Therefore

$$\sum_{n=1}^{\infty} \left(\frac{G(a)}{\lambda}\right)^n \lesssim \sum_{n=1}^{\infty} \nu\,(U_n) < \nu(\Omega) = 1 ,$$

which implies $G(a) < \lambda$ and hence the desired inequality. ■

2.8. Phase transition and multipliers

We want to prove that the occurrence of a phase transition together with (♣) implies

$$\mu(a) > \mu := \sup\{\mu(b : b \neq \{a\})\} .$$

Suppose it is not true and a phase transition occurs, though $\mu(a) \le \mu$.

For any cycle $b = \{b_1, ..b_n\} \neq \{a\}$ of order n (i.e. $F^n b_1 = b_1$) and for $N = kn$ we have

$$\begin{aligned} d^{N\tilde{s}(t)+o(N)} &\gtrsim \left(L_{0,t}^N 1\right)(b_1) \\ &\ge (G_{0,t})_N\,(b_1) \\ &= |F_N'(b_1)|^{-t} \\ &= \mu(b)^{-tN} . \end{aligned}$$

Thus

$$\tilde{s}(t) \geq -t \log_d \mu \geq -t \log_d \mu(a) = -ts_- \; .$$

Together with real analyticity of $\tilde{s}(t)$ this implies

$$\tilde{s}(t) > -ts_- \; .$$

But if a phase transition occurs at a point t_0 then

$$\tilde{s}(t_0) = s(t_0) = -t_0 s_- \; ,$$

and this condition is false.

We got a contradiction therefore iib) follows from the phase transition together with (♣). ■

2.9. Multifractal analysis

For simplicity we will work out the case of connected Julia set. We already established the equality $s(t) = \beta(t) - t + 1$, and by **[Mak2]** this implies

$$\sup_{\alpha>0} \frac{f(\alpha)-t}{\alpha} = \pi(t) = s(t) = \frac{1}{\log d} P(t) = \beta(t) - t + 1 \; .$$

Moreover, since we proved nice convergence of the limits for β it will imply the same for $\pi(t)$ and $f(\alpha)$.

It remains to analyze the Hausdorff spectra. Following the Subsection 2.7 we notice that for some (any) $z \in A(\infty)$ we have

$$d^{\tilde{s}(t)} \asymp (L_t^n H_{\alpha_{max},t}) \; ,$$

and following 1.4 this implies

$$d^{\tilde{s}(t)} \asymp d^{n(1-t)} \int_{|z|=1+d^{-n}} |\varphi'|^t \, |\varphi - a|^\kappa ,$$

where $\kappa := -\alpha_{max} t > 0$.

Hence the spectrum $\tilde{s}(t)$ represents the β-spectrum with the neutralized input from the point a. Particularly, that implies that if $\omega' := \omega \llcorner U$ is the restriction of ω to some open set U not containing a, then

$$c_{\omega'}(t) \le \tilde{s}(t) .$$

But (unlike the packing spectrum) the covering spectrum has "Hausdorff" properties, particularly if $\omega = \sum_j \omega_j$, then $c_\omega(t) = \sup_j c_{\omega_j}(t)$. Therefore we deduce that

$$c(t) \le \tilde{s}(t) .$$

To establish the inverse inequality it is sufficient to consider the Hausdorff dimension spectrum $\tilde{f}(\alpha)$ and prove (cf. [**Mak2**, 2.2]) that for $\alpha_t := -1/\tilde{s}'(t)$ we have the inequality

$$\tilde{f}(\alpha_t) \ge \inf_\tau (\tau + \alpha_t \tilde{s}(\tau)) = t + \alpha_t \tilde{s}(t) = t - \frac{\tilde{s}(t)}{\tilde{s}'(t)} .$$

By the Subsection 2.2 for any t there exists a probability measure $\mu = \mu_t := f_t \nu_t$ with positive entropy such that

$$\begin{aligned} \log d \cdot \tilde{s}(t) &= \tilde{P}(t) = h_\mu - t \int \log |F'| d\mu \\ &= \sup_m \left(h_m - t \int \log |F'| dm \right) . \end{aligned}$$

From the variational principle and convexity of $\tilde{s}(t)$ we also deduce that for any $\tau < 0$

$$\log d \cdot \tilde{s}(\tau) \;\geq\; h_\mu - \tau \int \log |F'| d\mu \ ,$$

hence, by the smoothness of $\tilde{s}(\tau)$,

$$\log d \cdot \tilde{s}'(t) \;=\; -\int \log |F'| d\mu \ ,$$

and also

$$\log d \cdot \tilde{s}(t) \;=\; h_\mu \,+\, t \log d \cdot \tilde{s}'(t) \ .$$

Then, by Pesin's theory (see, e.g., result **[Mañ3]** of R. Mañe) for μ-almost every z the local behavior of ω can be evaluated as

$$\begin{aligned}
\lim_{\delta\to 0} \frac{\log \omega B_\delta}{\log \delta} \;&=\; \lim_{\delta\to 0} \frac{\log \omega B_\delta}{\log \mu B_\delta} \lim_{\delta\to 0} \frac{\log \mu B_\delta}{\log \delta} \\
&=\; h_\omega \ (h_\mu)^{-1} \ h_\mu \left(\int \log |F'| d\mu \right)^{-1} \\
&=\; \log \ (-\log d \cdot \tilde{s}'(t))^{-1} \;=\; -1/\tilde{s}'(t) \;=\; \alpha_t \ ,
\end{aligned}$$

which together with the estimate of dimension (see **[Mañ3]** and **[Y]**)

$$\begin{aligned}
\mathrm{HDim}\ \mu = \; & h_\mu \left(\int \log |F'| d\mu \right)^{-1} \\
=\; & (\log d \cdot \tilde{s}(t) \,-\, t \log d \cdot \tilde{s}'(t)) \ (-\log d \cdot \tilde{s}'(t))^{-1} \\
=\; & t \,-\, \frac{\tilde{s}(t)}{\tilde{s}'(t)} \ ,
\end{aligned}$$

gives the desired inequality $\tilde{f}(\alpha_t) \geq t - \tilde{s}(t)/\tilde{s}'(t)$ and completes the proof. ∎

3. Transfer operator on Sobolev spaces

3.1. Main theorem

We will consider functional spaces on the complex sphere, or some open set Ω with

$$F^{-1}\Omega \subset \Omega \, ,$$

taken with spherical metric.

Transfer operator on the space of continuous functions is defined by

$$Lf(z) := \sum_{y \in F^{-1}z} f(y)\, g(y) \, ,$$

where preimages y of z are counted with multiplicities. Note that

$$L^n f(z) := \sum_{y \in F^{-n}(z)} f(y)\, g_n(y) \, ,$$

where $g_n(y) = g(y)g(Fy)\ldots g(F^{n-1}(y))$. Denote by λ the spectral radius:

$$\lambda := \lim_{n\to\infty} \|L^n\|_\infty^{1/n} \, .$$

We assume that the weight g belongs to Sobolev space $W_{1,p}$ for some $p > 2$ close to 2, is non-negative and vanishes exactly at the critical points of the dynamics F. Equivalently, we can assume that $g = h \cdot |F'|^{\tau}$ for small $\tau > 0$ and g, h, $|F'|^{\tau} \in W_{1,p}$ for some (any) number $p > 2$ sufficiently close to 2. Note that $g_n = h_n |F_n'|^{\tau}$.

We will assume that an additional condition is satisfied, namely

$$\lambda > (\sup g_n)^{1/n} , \qquad (\diamondsuit)$$

for some (any large) n, equivalently

$$\text{const } q^n \lambda^n > \sup g_n ,$$

for any n and a constant $q < 1$.

Theorem E. *Under the assumption* $(\diamondsuit)$ *for* $p > 2$ *sufficiently close to 2 operator* L *acts on* $W_{1,p}$ *and*

$$r_{ess}(L, W_{1,p}) < r(L, W_{1,p}) = \lambda ,$$

where r *and* r_{ess} *are spectral and essential spectral radii of* L *as an operator on* $W_{1,p}$. *Moreover,* λ *is an isolated eigenvalue of multiplicity one.*

Proof. After some technical lemmas contained in the Subsection 3.2 we will establish Ionescu-Tulcea and Marinescu inequality in the Subsection 3.3. As in the Subsection 1.8, together with finite rank approximation it will imply quasicompactness of the transfer operator.

Following the Subsection 2.3, we obtain that λ is an isolated eigenvalue of L, and also an eigenvalue of the adjoint operator L^* with some eigenmeasure ν:

$$L^* \nu \ = \ \lambda \nu \ .$$

It remains to prove that $\mathrm{supp}\nu \ = \ J_F$, and then the argument of the Subsection 2.5 will yield that multiplicity of λ is one. But considerations from [**MS**, 6] show that if this is not the case, then measure ν must be supported on a single fixed point or an orbit of order 2 (the latter might happen only for Chebyshev polynomials). Let a be this point (or one of these two points), then for even n we have $F_n(a) = a$ and hence

$$g_n(a)\, \nu(\{a\}) \ = \ ((L^*)^n\, \nu)\, (\{a\}) \ = \ \lambda^n\, \nu(\{a\}) \ ,$$

which contradicts the condition $(\diamondsuit)$. ∎

3.2. Technical lemmas

Lemma. *There exists $q_1 < 1$ such that for small $\epsilon > 0$ we have*

$$\sup \left(g_n\, |F_n'|^{-\epsilon} \right) \ \leq \ \mathrm{const}\ (q_1)^n\ \lambda^n \ .$$

Proof. If we take ϵ to be small enough so that $q \ < \ (q\lambda / \sup h)^{\epsilon / tau}$, then there exists $q_1 < 1$ such that

$$q^{\frac{\tau-\epsilon}{\tau}}\, \lambda^{\frac{\tau-\epsilon}{\tau}}\ (\sup h)^{\frac{\epsilon}{\tau}} \ < \ q_1\, \lambda \ .$$

Therefore, noting that $\sup h_n \leq (\sup h)^n$ we can write

$$\begin{aligned}
\sup\left(g_n \left|F_n'\right|^{-\epsilon}\right) &= \sup\left((h_n)^{\frac{\epsilon}{\tau}} (h_n)^{\frac{\epsilon}{\tau}} \left|F_n'\right|^{\tau-\epsilon}\right) \\
&\leq (\sup h_n)^{\frac{\epsilon}{\tau}} (\sup g_n)^{\frac{\tau-\epsilon}{\tau}} \\
&\leq (\sup h)^{n\frac{\epsilon}{\tau}} (\text{const } q^n \lambda^n)^{\frac{\tau-\epsilon}{\tau}}
\end{aligned}$$

$$< \text{ const } (q_1)^n \lambda^n ,$$

proving the desired estimate. ■

Lemma. *If $2 < p < \frac{2}{1-\tau}$ and g, h, $\in W_{1,p}$, then $LW_{1,p} \subset W_{1,p}$.*

Proof. As before, it is sufficient to estimate

$$\int \left|\partial_{F(y)} (gf)(y)\right|^p dm_2 (F(y))$$

by const $\|f\|_{1,p}$. Clearly

$$\ldots = \int_\Omega |\partial (gf)(y)|^p \left|F'(y)\right|^{2-p} dm_2 y \leq I + II .$$

Here

$$\begin{aligned}
I &= \int |\partial f|^p (g)^p \left|F'\right|^{2-p} dm_2 \\
&< \text{ const } \int |\partial f|^p ,
\end{aligned}$$

and since we consider $2 < p < \frac{2}{1-\tau}$, then $\tau p + 2 - p > 0$ and

$$\sup (g)^p \left|F'\right|^{2-p} = \sup (h)^p \left|F'\right|^{\tau p + 2 - p} < \infty .$$

The second term can be estimated by

$$
\begin{aligned}
II &= \int |f|^p (\partial g)^p |F'|^{2-p} \, dm_2 \\
&\leq \int |f|^p (\partial h)^p |F'|^{\tau p+2-p} \, dm_2 \\
&\qquad + \int |f|^p (h)^p \left(\partial \left(|F'|^\tau\right)\right)^p |F'|^{2-p} \, dm_2 \\
&\leq \|f\|_\infty^p \left(\sup |F'|^{\tau p+2-p} \int (\partial h)^p \, dm_2 \right. \\
&\qquad \left. + \sup (h)^p \int \left(\partial \left(|F'|^\tau\right)\right)^p |F'|^{\tau p+2-p} \, dm_2 \right),
\end{aligned}
$$

so it remains to show the convergence of the last integral, which, as before, is implied by the condition $p < \frac{2}{1-\tau}$. ■

3.3. Ionescu-Tulcea and Marinescu inequality

Lemma. *For $p > 2$ close to 2, L acts on $W_{1,p}$ and*

$$
\|L_t^n f\|_{1,p} \leq \text{const } q^n \lambda^n \|f\|_{1,p} + C_n \|f\|_\infty ,
$$

for some $q < 1$ and constant C_n.

Proof. Clearly

$$
\begin{aligned}
\|\partial (L^n f)\|_p \leq & \left\| \left(\sum_{y \in F^{-n}(z)} \partial_z f(y) \cdot g_n(y) \right) (z) \right\|_p \\
& + \left\| \left(\sum_{y \in F^{-n}(z)} f(y) \, \partial_z g_n(y) \right) (z) \right\|_p =: I + II .
\end{aligned}
$$

Choosing a small ϵ, we can estimate the right hand side by

$$
\begin{aligned}
I^p &\leq \int \left(\sum_{y \in F^{-n}(z)} |\partial_y f(y)| \, g_n(y) \, |F_n'(y)|^{-1} \right)^p dm_2(z) \\
&= \int \left(\sum |\partial f| |F_n'|^{-\frac{2}{p}} \, g_n \, |F_n'|^{\frac{2}{p}-1} \right)^p dm_2(z) \\
&\leq \int \left(\sum |\partial f| |F_n'|^{-\frac{2}{p}} \, \text{const} \, (q_1)^n \, \lambda^n \, |F_n'|^{\epsilon} \, |F_n'|^{\frac{2}{p}-1} \right)^p dm_2(z) \\
&\leq \text{const} \, (q_1)^{pn} \, \lambda^{pn} \, \sup \left(|F_n'|^{p(\epsilon + \frac{2}{p} - 1)} \right) \\
&\qquad \cdot \int \sum |\partial f|^p \, |F_n'|^{-2} \, dm_2(z) \\
&\leq \text{const} \, (q_1)^{pn} \, \lambda^{pn} \, \|\partial f\|_p^p \, ,
\end{aligned}
$$

if we assume that $(p\epsilon + 2 - p) > 0$, i.e. p is sufficiently close to 2. Following the previous Subsection we obtain

$$
II^p \leq \ldots \leq \|f\|_\infty^p \, C_n^p \, ,
$$

which completes the proof. ∎

Chapter 3
Parabolic case

In this chapter we will be concerned with parabolic polynomials, i.e. those where all critical points are attracted either to (super)attractive cycles or parabolic cycles.

In the first section we will establish the quasicompactness of the transfer operator, and in the second apply it to the analyticity of spectrum, proving the following

Theorem B. **B1.** *For a parabolic polynomial F, the function $P_F(t)$ is real analytic on* $[0, \mathrm{HDim}\ J_F)$ *and* $P_F(t) = 0,\ t \in [\mathrm{HDim}\ J_F, +\infty)$.
B2. *The derivative of $P_F(t)$ is discontinuous at the point* HDim J_F *if and only if*

$$\mathrm{HDim}\ J_F \ > \ 2 \ - \ \frac{2}{p+1}\ , \tag{$\spadesuit$}$$

where p is the maximal number of petals at the parabolic points.

Dynamics in the parabolic case is not expanding, but expansive;

namely, any two points eventually go far apart. Such polynomials are known to have locally connected if connected Julia sets (see [**DH**]), we will also use some metric properties established in [**ADU**].

1. Transfer operator

1.1. Preliminaries

Definition. For $X \subset \mathbb{C}$ we define the space $\mathrm{Lip}(X)$ as a subspace of $C(X)$ with the following norm

$$\|f\|_{\mathrm{Lip}(X)} := \|f\|_{\infty} + \mathrm{Lip}_X(f) \quad , \quad \text{where}$$

$$\mathrm{Lip}_X(f) := \sup_{x,y\in X} \frac{|f(x)-f(y)|}{|x-y|} .$$

We denote $\mathrm{Lip} := \mathrm{Lip}_{J_F}$.

Proposition. *There exists compact U with piecewise smooth boundary such that*

I. $J_F \subset U$, $0 \notin U$.

II. $F^{-1}U \subset U$, *each point $z \in U$ has such small neighborhood V in U that*

$$\sup \operatorname{diam} \big(\textit{component of connectivity of } F^{-k}V\big) \to 0 , \quad k \to \infty .$$

III. *There exists constant C such that any $x, y \in U$ can be joined inside U by the arc of length less than $C|x-y|$.*

Remark. The third property implies that for any $\delta > 0$, $f \in C(U)$

$$\mathrm{Lip}_U(f) \quad < \quad C \sup_{|x-y|<\delta} \frac{|f(x)-f(y)|}{|x-y|} .$$

Idea of the Proof. As in [**DH**, X.2.4], one can take some neighborhood of the Julia set and then cut out a few angles, inside which critical points approach parabolic cycles. ∎

1.2. Transfer operator

For $z \in U$ and $t \in \mathbb{R}$ we define

$$g(z) \;=\; g_t(z) \;:=\; |F'(z)|^{-t} ,$$

$$g_n(z) \;=\; g_{t,n}(z) \;:=\; \prod_{j=0}^{n-1} g_t\left(F^j z\right) .$$

Clearly for bounded subsets $X \subset \mathbb{C}$ for any n $g_n \in \mathrm{Lip}(X)$. This allows us to consider the *transfer operator* on the $\mathrm{Lip}(U)$:

$$Lf(z) \;=\; L_t f(z) \;:=\; \sum_{y \in F^{-1}z} f(y)\, g_t(y) ,$$

where preimages y of z are counted with multiplicities. Note that

$$L_t^n f(z) \;:=\; \sum_{y \in T^{-n}x} f(y)\, g_{t,n}(y) .$$

Definition. For $t \in \mathbb{R}$ and positive integer n we define the *partition function*

$$Z_n(z) \;=\; Z_{t,n}(z) \;:=\; \sum_{y \in F^{-n}(z)} g_{t,n}(y)\;, \quad z \in U\;.$$

Denote also

$$\lambda_{\pm} \;:=\; \limsup_{n\to\infty} \sqrt[n]{\sup g_{\pm 1,n}}\;.$$

Theorem. *Fix $t \in \mathbb{R}$.*

I. *There exist a probability measure $\nu := \nu_t$ on J_F, and a positive number $\lambda := \lambda_t$ such that*

$$L^*\nu = \lambda\nu, \quad \mathrm{supp}\nu = J_F\;, \tag{1}$$

and for any positive integer n and $z \in U$

$$Z_n(z) \overset{e^{o(n)}}{\asymp} \lambda^n\;, \tag{2}$$

thus λ is the spectral radius of the operator L_g in $C(J_F)$.

II. *If*

$$\begin{array}{c} t \;<\; 0\;, \quad or \\ \lambda \;>\; \lambda_+^t \;\; \text{for } t \geq 0\;, \end{array} \tag{3}$$

then λ is the spectral radius and an isolated eigenvalue of multiplicity one of the operator $L_t : \mathrm{Lip}(U) \to \mathrm{Lip}(U)$. Furthermore for any positive integer n and $z \in J_F$

$$Z_n(z) \asymp \lambda^n\;. \tag{4}$$

The first part of the theorem is proved in the sections 1.3–1.4 below, and the second in the sections 1.5–1.9. Throughout these sections we often omit t, denoting ν_t by ν, λ_t by λ, etc.

1.3. Existence of conformal measure

Clearly, for $\mu \in \mathrm{Prob}(J_F)$, $L^*\mu$ is a positive non-zero measure. Thus we can consider the following operator on Prob:

$$P: \ \mu \ \mapsto \ \frac{L^*\mu}{\mathrm{Var}\, L^*\mu} \ .$$

It fixes some measure ν which is the desired one. Condition $\mathrm{supp}\nu = J_F$ holds since $z \in \mathrm{supp}\nu$ implies $F^{-n}(z) \subset \mathrm{supp}\nu$ and preimages of any point $z \in J_F$ are dense in J_F.

1.4. Partition function

Lemma. *For any positive integer n and $x, y \in U$*

$$Z_n(x) \ \overset{e^{o(n)}}{\asymp} \ Z_n(y) \ .$$

Proof. Since U is compact it is sufficient to prove this for $x, y \in V$, where V is some small connected subset of U. Denote by V_j^k components of $F^{-k}V$. By the Proposition 1.1. we know that

$$\sup_j \mathrm{diam}\ V_j^k \ = \ o(1), \quad k \to \infty \ ,$$

thus

$$\omega_k := \sup_j \frac{\sup_{V_j^k} g}{\inf_{V_j^k} g} = e^{o(n)} ,$$

because g is separated from zero on U. Therefore

$$\frac{\sup_{V_j^n} g_n}{\inf_{V_j^n} g_n} \leq \prod_{k=0}^{n-1} \omega_k = e^{o(n)} .$$

This implies our Lemma. ∎

Remark. We also proved that for x, y in one component of connectivity of $F^{-n}V$

$$g_n(x) \overset{e^{o(n)}}{\asymp} g_n(y) .$$

Now we can prove (2). In fact for any $x \in J_F$ we have

$$\begin{aligned} \|L^n\|_{C(J_F)} &= \sup_y Z_n(y) \\ &\overset{e^{o(n)}}{\asymp} Z_n(x) \\ &\overset{e^{o(n)}}{\asymp} \int Z_n(y) d\nu(y) \\ &= \langle L^n 1, \nu \rangle \\ &= \left\langle 1, (Ł^*)^n \nu \right\rangle = \lambda^n . \end{aligned}$$

So (2) holds and the spectral radius of L_g in $C(J_F)$ is equal to

$$\lim_{n\to\infty} \sqrt[n]{\|L_g^n\|_{C(J_F)}} = \lambda .$$

∎

We turn now to the proof of the second part of Theorem 1.1.

1.5. Prevalence of the expanding branches

Lemma. *Fix $Q > 1$. For $z \in U$ denote*

$$\mathcal{N}_n(z) \;:=\; \left\{ y \in F^{-n}(z),\ \left|(F^n)'(y)\right| < Q^n \right\} \;.$$

Then

$$\#\mathcal{N}_n(z) \;\le\; e^{o(n)} Q^{2n} \;.$$

Also

$$\inf_{z \in U} \left|(F^n)'(z)\right| \;\ge\; e^{-o(n)} \;.$$

Proof. Take V to be a neighborhood of z in U. Then

$$\mathrm{Area}(U) \;\ge\; \mathrm{Area}(\cup F^{-n} V) \;\asymp\; \mathrm{Area}(V)\ Z_{2,n}(z) \;,$$

hence by the first part of the Theorem 1.2

$$\begin{aligned} 1 &\overset{e^{o(n)}}{\gtrsim} Z_{2,n}(z) \\ &= \sum_{y \in F^{-n}(z)} \left|(F^n)'(y)\right|^{-2} \;\ge\; \#\mathcal{N}_n(z)\, Q^{-2n} \;, \end{aligned}$$

which implies

$$\#\mathcal{N}_n(z) \;\le\; e^{o(n)} Q^{2n} \;.$$

The second statement can be proved in the same way. ∎

1.7. Ionescu-Tulcea and Marinescu inequality

Suppose the condition (3) is satisfied. Normalize $g(z)$ by setting

$$h(z) := g(z)/\lambda .$$

Denote

$$L := L_h = \frac{1}{\lambda} L_g ,$$

then

$$\|L^n\|_{C(J_F)} = \frac{1}{\lambda^n} \|L_g^n\|_{C(J_F)} \overset{e^{o(n)}}{\asymp} 1 .$$

Lemma. *In the above notation there exist* $0 < q < 1$ *and* $M_n = e^{o(n)}$ *such that*

$$\|L^n f\|_{\mathrm{Lip}} \leq \text{const}\, q^n \|f\|_{\mathrm{Lip}} + M_n \|f\|_\infty . \tag{5}$$

Remark. Note that in our construction, we obtain

$$q \approx e^{o(1)} \left((Q^{2+t}/\lambda) + Q^{-n} \right)^{1/n} .$$

Choosing Q properly and taking large n, we can obtain some value of q, depending only on λ and t. This method, applied in parabolic or hyperbolic case, will give an estimate on the radius of essential spectrum of the transfer operator L_t, in terms of t, $P(t)$, and $P(\pm\infty)$ only.

However, this does not provide us with an estimate for the absolute value of the second eigenvalue.

Proof. First we want to estimate

$$D = D_n :=$$

$$:= \#\mathcal{N}_n(z) \inf_{z\in U}{}^{-1}\left|(F^n)'(z)\right| \sup_{y\in\mathcal{N}_n(z)} h_n(y) + \|L^n 1\|_\infty Q^{-n} .$$

If $t \geq 0$ then for $h(z)$

$$\limsup_{n\to\infty} \sqrt[n]{\|h_n\|_\infty} =: p = \lambda_+^t/\lambda < 1 ,$$

and

$$\sup_{y\in\mathcal{N}_n(z)} h_n(y) \leq \|h_n\|_\infty \leq e^{o(n)} p^n .$$

Choose $Q > 1$ such that $pQ^2 < 1$. Then

$$D \leq e^{o(n)} \left(\left(pQ^2\right)^n + Q^{-n} \right) .$$

If $t < 0$ then

$$\sup_{y\in\mathcal{N}_n(z)} h_N(y) = \sup_{y\in\mathcal{N}_n(z)} \left|(F^n)'(y)\right| / \lambda^n$$
$$\leq Q^{tn}/\lambda^n .$$

Clearly Lemma 1.5 implies that $\lambda > 1$, so we can choose $Q > 1$ such that inequality $Q^{2+t}/\lambda < 1$ holds. Then

$$D \leq e^{o(n)} \left(\left(Q^{2+t}/\lambda\right)^n + Q^{-n} \right) .$$

In any case we can pick large n (denote it by N) such that

$$2CD \ < \ q \ < \ 1$$

for some q (C is the constant from the Proposition 1.1).

For $f \in \mathrm{Lip}$ consider such continuation $\tilde{f} \in C(U)$ that

$$\mathrm{Lip}_U\left(\tilde{f}\right) \ \le \ \mathrm{Lip}(f)\ , \quad \left\|\tilde{f}\right\|_{C(U)} \ \le \ \|f\|_\infty\ , \tag{6}$$

for example it is true for

$$\tilde{f}\,(z) \ := \ \min\left\{ \ -\|f\|_\infty\ , \ \sup_{x\in J_F}\ \{\,f(x) \ - \ |x-z|\,\mathrm{Lip}(f)\,\} \ \right\} \ .$$

Now if points $z, z' \in U$ are sufficiently close we have

$$\begin{aligned}
&\frac{\left|\left(L^N\tilde{f}\right)(z) - \left(L^N\tilde{f}\right)(z')\right|}{|z-z'|} \\
&\le \ \sum_{y\in F^{-n}(z)} \frac{|h_N(y)-h_N(y')|}{|z-z'|}\left|\tilde{f}\,(y)\right| + \sum_{y\in F^{-n}(z)} \frac{\left|\tilde{f}(y)-\tilde{f}(y')\right|}{|z-z'|} h_N(y') \\
&= \quad A \quad + \quad B\ ,
\end{aligned}$$

where we denote by y' the preimage of z' obtained by the branch of F^{-n} for which $F^{-n}(z) = y$. Using (6) we can estimate A by

$$\begin{aligned}
A \ &\le \ \sum_{y\in F^{-n}(z)} \|h_N\|_{\mathrm{Lip}(U)}\left\|\tilde{f}\right\|_{C(U)} \frac{|y-y'|}{|z-z'|} \\
&\le \ 2^N\,\|h_N\|_{\mathrm{Lip}(U)}\,\|f\|_\infty\, e^{o(n)} \quad =: \ A'\,\|f\|_\infty\ ,
\end{aligned}$$

and B (again, we suppose z and z' are sufficiently close) by

$$
\begin{aligned}
B &= \sum_{y\in\mathcal{N}_N(z)} \cdots \quad + \sum_{y\in T^{-N}(z)\setminus\mathcal{N}_N(z)} \cdots \\
&\le \#\mathcal{N}_N(z) \sup_{|x-y|<\delta} \frac{\left|\tilde f(x)-\tilde f(y)\right|}{|x-y|} \sup_{y\in\mathcal{N}_N(z)} \frac{|y-y'|}{|z-z'|} \sup_{y\in\mathcal{N}_n(z)} h_N(y) \\
&\quad + \sup_{|x-y|<\delta} \frac{\left|\tilde f(x)-\tilde f(y)\right|}{|x-y|} \sup_{y\in F^{-N}(z)\setminus\mathcal{N}_N(z)} \frac{|y-y'|}{|z-z'|} \left(L^N 1\right)(z') \\
&\le \left(\#\mathcal{N}_n(z) \inf_{z\in U}{}^{-1} \left|(F^n)'(z)\right| \sup_{y\in\mathcal{N}_n(z)} h_N(y) \; + \; \|L^n 1\|_\infty Q^{-n} \right) \cdot \\
&\qquad\qquad \cdot\, 2 \sup_{|x-y|<\delta} \frac{\left|\tilde f(x)-\tilde f(y)\right|}{|x-y|} \\
&\le 2\,D\,C \left\|\tilde f\right\|_{\mathrm{Lip}(U)} \quad \le \quad 2\,\frac{q}{2C}\,C\,\|f\|_{\mathrm{Lip}} \quad = \quad q\,\|f\|_{\mathrm{Lip}}
\end{aligned}
$$

Therefore

$$
\left\|L^N f\right\|_{\mathrm{Lip}} \le q\,\|f\|_{\mathrm{Lip}} + M\,\|f\|_\infty \tag{7}
$$

with

$$
M = \left\|L^N\right\|_{C(J_F)} + A' .
$$

Now by induction we have

$$
\left\|L^{Nk} f\right\|_{\mathrm{Lip}} \le q^k\,\|f\|_{\mathrm{Lip}} + \frac{C_k M}{1-q}\,\|f\|_\infty
$$

with $C_k \uparrow \infty$,

$$
\left\|L^{Nk}\right\|_{C(J_F)} \le C_k = e^{o(k)} .
$$

This implies (5). ∎

1.8. Quasicompactness

A Banach space linear operator L is called *quasicompact* if

$$\|L^N - K\| < 1$$

for some compact operator K and some integer N. An equivalent statement is that the image $[L^N]$ of L^N in the *Calkin algebra* (= bounded operators modulo compact) has norm strictly less than one. This implies that the spectral radius of $[L]$ in the Calkin algebra is strictly less than one, and therefore there is $r < 1$ such that the part of the spectrum of L lying outside of the disk $\{|z| \leq r\}$ consists of a finite number of eigenvalues that all have finite geometric multiplicity.

Lemma. *Any operator L on* Lip *satisfying* (5) *is quasicompact.*

Proof. Cover J by some equilateral triangle B. By the Whitney extension theorem there exists bounded operator

$$W : \operatorname{Lip}(J_F) \to \operatorname{Lip}(B) ,$$

with $(Wf)|_{J_F} = J_F$.

Set $K := \|W\|_{\text{Lip}} \geq 1$ and fix N such that

$$\|L^N f\|_{\text{Lip}} \leq \frac{1}{100K} \|f\|_{\text{Lip}} + M \|f\|_{\infty}, \quad f \in \text{Lip} .$$

Also fix small ε (what we need is $MK\varepsilon \leq \frac{1}{100}$) and take partition of B into equilateral triangles with sides less then ε. Define an operator $\tilde{P}$ on $\mathrm{Lip}(B)$ by the requirement

$$\begin{cases} \tilde{P}f \;=\; f \quad \text{on the vertices of our small triangles,} \\ \tilde{P}f(x) \text{ is a linear function of } \operatorname{Re} x\,, \operatorname{Im} x \text{ inside any of them.} \end{cases}$$

Then $\dim \tilde{P}(\mathrm{Lip}(B)) < \infty$, and $\left\|\tilde{P}\right\|_{\mathrm{Lip}(B)} < 2$.

Set now $P := \tilde{P}W$, $P : \mathrm{Lip} \to \mathrm{Lip}$. It is clear that $\dim P\,(\mathrm{Lip}) < \infty$ thus to prove that L is quasicompact it remains to check that

$$\left\|L^N - L^N P\right\| \;<\; 1.$$

Since $\|P\|_{\mathrm{Lip}} \leq \left\|\tilde{P}\right\|_{\mathrm{Lip}} \|W\|_{\mathrm{Lip}} \leq 2K$, and

$$\begin{aligned} \|f - Pf\|_\infty \;&\leq\; \left\|Wf - \tilde{P}Wf\right\|_{C(B)} \;\leq\; \varepsilon \left\|Wf - \tilde{P}Wf\right\|_{\mathrm{Lip}(B)} \\ &\leq\; 4\varepsilon \,\|Wf\|_{\mathrm{Lip}(B)} \;\leq\; 4K\varepsilon\,\|f\|_{\mathrm{Lip}}\,, \end{aligned}$$

we have

$$\begin{aligned} \left\|L^N\,(f - Pf)\right\|_{\mathrm{Lip}} &\leq \frac{1}{100K}\|f - Pf\|_{\mathrm{Lip}} + M\|f - Pf\|_\infty \\ &\leq \frac{1}{100K}(1 + 2K)\|f\|_{\mathrm{Lip}} \;+\; 4MK\varepsilon\|f\|_{\mathrm{Lip}} \\ &\leq \frac{1}{2}\|f\|_{\mathrm{Lip}}\,. \end{aligned}$$

■

Returning to the proof of the second part of Theorem 1.2, we see that the last two lemmas imply that 1 is the spectral radius and an isolated eigenvalue of $L =:$ Lip $\circlearrowleft$, therefore λ is the spectral radius and an isolated eigenvalue of L_g : Lip $\circlearrowleft$. It remains to show that

$$\dim \bigcap_{n\geq 1} \ker(L_g - \lambda)^n \ = \ 1 \ .$$

1.9. Multiplicity

In this section we denote by L the unnormalized operator L_g. First we prove

$$\dim \ker(L - \lambda) \ = \ 1. \tag{8}$$

Observe that if f satisfies $Lf = \lambda f$ with $\lambda > 0$ then $L|f| = \lambda|f|$. Indeed,

$$\begin{aligned}\langle \lambda|f|, \nu\rangle = \langle |Lf|, \nu\rangle \leq \langle L|f|, \nu\rangle \\ = \langle |f|, L^*\nu\rangle = \langle |f|, \lambda\nu\rangle\end{aligned}$$

and $L|f| = \lambda|f|$ ν-a.e. and hence everywhere because supp$\nu = J_F$. Next observe that if f is a real-valued eigenfunction, $Lf = \lambda f$, then either $f \geq 0$ or $f \leq 0$. Otherwise, we can find a nontrivial eigenfunction $\tilde{f} \geq 0$ (e.g. $\tilde{f} = |f| - f$) which is zero at some point. The equation

$$\lambda^n \tilde{f}(z) \ = \ \sum_{y\in F^{-n}(z)} g_n(y)\tilde{f}(y)$$

then implies that $\tilde{f}$ must vanish on a dense set, hence everywhere.

Note that we also proved that eigenfunction f is strictly positive. Thus equality

$$\lambda^n f(z) = L^n f(z) = \sum_{y \in F^{-n}(z)} g_n(y) f(y)$$

implies (4): in fact, for any n and $z \in J_F$

$$\lambda^n \asymp \sum_{y \in F^{-n}(z)} g_n(y) = Z_n(z) .$$

Therefore we also have

$$\|L^n\|_{C(J_F)} = \sup_{z \in J_F} Z_n(z) \asymp \lambda^n . \tag{9}$$

Suppose now that we have two real functions f_1, f_2 satisfying

$$Lf_j = \lambda f_j, \ f_j \geq 0 \ \ (j = 1, 2).$$

Normalizing them by the condition

$$\int f_j d\nu = 1,$$

we have

$$\int |f_1 - f_2| d\nu = \left| \int (f_1 - f_2) d\nu \right| = 0,$$

and $f_1 = f_2$. This proves (8).

It remains to show that

$$\ker(L - \lambda)^2 = \ker(L - \lambda).$$

Suppose this is not true. Then there are $f \neq 0$ and h such that

$$Lf = \lambda f, \ Lh = \lambda h + cf, \quad (c \neq 0).$$

By induction, we have

$$L^n h \ = \ \lambda^n \left(h + nc\lambda^{-1} f\right)$$

which contradicts the relation (9). ■

2. Analyticity

2.1. Analyticity

Define $P(t) \ := \ \log \lambda_t$. Area estimate in the Subsection 1.5 shows that $P(t) \ \leq \ 0$ for $t \geq 2$, hence $\lambda_+ \leq 1$. On the other hand, existence of the parabolic cycle with multiplier 1 gives us $\lambda_+ \geq 1$. Thus $\lambda_+ = 1$, and we can reformulate Theorem 1.2 in the following form:

As long as $P(t)$ is positive, λ_t is the spectral radius and an isolated eigenvalue of multiplicity one of the operator $L_t : \mathrm{Lip}(U) \rightarrow \mathrm{Lip}(U)$.

Let δ be the first root of $P(t)$. Then λ_t for $t < \delta$ is an isolated eigenvalue of multiplicity one of the operator L_t which depends real-analytically on t, hence, by perturbation theory, the pressure $P(t)$ is real analytical on this interval. On the other hand, $P(t)$ is decreasing and non-negative, thus $P(t) = 0$ for $t \geq \delta$.

To complete the proof of the Theorem B it remains to analyze the phase transition at the point δ.

2.2. Phase transition

By the paper **[DU2]** the Bowen's formula holds for parabolic maps, hence $\delta = \mathrm{HDim}\ J_F$. By **[ADU]** (see especially Theorem 9.9) there is a δ-conformal measure $\nu = \nu_\delta$ (which is either Hausdorff or Packing, depending on the dimension), and it admits an equivalent (σ-finite) invariant measure μ, which is finite if and only if the inequality ($\spadesuit$) holds.

Now, if ($\spadesuit$) holds, such a measure μ is finite, thus (being equivalent to the conformal measure ν) it maximizes the expression in the variational principle:

$$P(\delta) \;=\; 0 \;=\; h_\mu \,-\, \delta \int \log|F'|d\mu \ ,$$

and has a positive entropy, hence (by the variational principle)

$$P(t) \;\geq\; h_\mu \,-\, t \int \log|F'|d\mu \;=\; (\delta - t) \int \log|F'|d\mu \ .$$

The last integral is positive (it is equal to h_μ/δ), therefore $P'(\delta-) < 0 = P'(\delta+)$ and the derivative of the pressure is discontinuous at δ.

On the other hand, suppose that $P'(\delta-) < 0$. Then for each $t < \delta$ there is an invariant measure μ_t (equivalent to the (λ_t, t)-conformal measure ν_t: particularly, $\mu_t = f_t \nu_t$, where f_t is the eigenfunction of the transfer operator), for which $\int \log |F'| d\mu_t > -P'(\delta-) > 0$. Considering the limit of f_t as $t \to \delta$ (and normalizing them properly) it is easy to construct a finite invariant measure μ equivalent to $\nu = \nu_\delta$ and therefore deduce the inequality ($\spadesuit$).

Chapter 4

Semihyperbolic case

In this chapter we restrict ourselves to the study of non-recurrent quadratic polynomial $F(z) = z^2 + c$:

$$0 \notin \omega 0 := \cup_{k \geq 1} F^k 0 \ , \quad c \in J_F \ .$$

We prove the following

Theorem D. *For a non-recurrent quadratic polynomial $F(z) = z^2 + c$ either*

(i) *$P_F(t)$ is real analytic on $[0, +\infty)$, or*

(ii) *A phase transition occurs: there exists $t_0 >$ HDim J_F such that*

$$\begin{cases} P_F(t) & \text{is real analytic on } [0, t_0), \text{ and} \\ P_F(t) & = -\frac{1}{2} t \log \left(\liminf_{n \to \infty} |(F^n)'(c)|^{1/n} \right) \text{ on } [t_0, +\infty) \ . \end{cases}$$

The chapter is organized as follows: in the Section 1 we introduce Hofbauer tower, changing the dynamics and forcing expansion. In the

Section 2 we prove quasicompactness of the transfer operator, and in the Section 3 investigate its eigenvalues and their connection to spectrum,

1. Yoccoz puzzle and Hofbauer tower

1.1. Preliminaries

Define

$$s(t) \;:=\; \limsup_{n\to\infty} \frac{1}{n} \log_2 \sum_{y\in F^{-n}(z)} \left|(F^n y)'\right|^{-t} ,$$

for some (any – there is no difference, see Proof of the Lemma) $z \in A(\infty)$.

Lemma. *For any* $z \in A(\infty)$

$$\sum_{y\in F^{-n}(z)} \left|(F^n y)'\right|^{-t} \;\asymp\; 2^{n(1-t)} \int_{|z|=1+2^{-n}} |\varphi'|^t .$$

Therefore

$$s(t) \;=\; 1 - t + \beta(t) .$$

Proof. Without loss of generality φ conjugates F with dynamics $T : z \mapsto z^2$ on $\mathbb{D}_-$:

$$F \circ \varphi \;=\; \varphi \circ T .$$

Differentiating the identity $F^n \circ \varphi = \varphi \circ T^n$, we obtain

$$F_n' \circ \varphi \cdot \varphi' = \varphi' \circ T^n \cdot T^{n\prime} .$$

Applying this equality to the preimages $\zeta \in T^{-n}$ of some fixed point $\xi \in \mathbb{D}_-$ we notice that the right side is $\asymp 2^n$, thus (taking to power t)

$$2^{-nt} \left|\varphi'(\zeta)\right|^t \asymp \left|F_n'(y)\right|^{-t} ,$$

where $y = \varphi(\zeta)$ is a corresponding preimage of $z = \varphi(\xi)$ under F^n.

The points ζ are equidistributed on the circle

$$|x| = r_n = |\xi|^{\frac{1}{2^n}} ,$$

with $(r_n - 1) \asymp \frac{1}{2^n}$. Therefore summing over all $\zeta \in T^{-n}\xi$ ($y \in F^{-n}z$), we have

$$\begin{aligned} \sum_{y \in F^{-n}(z)} \left|(F^n y)'\right|^{-t} &\asymp 2^{-nt} \sum_{\zeta \in T^{-n}\xi} \left|\varphi'(\zeta)\right|^t \\ &\asymp 2^{n(1-t)} \int_{|x|=r_n} |\varphi'|^t , \end{aligned}$$

which completes the proof. ■

1.2. Tower construction

For non-recurrent polynomials dynamics on the orbit of the critical point is expanding, i.e

$$Q_c := \liminf_{n\to\infty} \left|F^{n\prime}(c)\right|^{\frac{1}{n}} > 1 .$$

Consider Yoccoz puzzle (see **[Mi2]** and **[Hu]** for the construction and its properties) and denote by $P_n(0)$ the puzzle-piece of depth $n + n_0$ containing 0 (in this case 0 cannot lie on the boundary of a puzzle-piece). Since for such Julia sets diameter of puzzle-pieces tends to 0 when depth increases, we can choose n_0 in such a way that $P_0(0)$ is disjoint from $\omega 0$.

Denote by Ω formal disjoint union of all puzzle pieces of depth $n_0 + n_1$, n_1 to be chosen later. Our tower will be a union of floors equivalent to subsets of Ω:

$$\mathcal{T} := \cup_{k\geq 1} \mathcal{F}_k = \cup_{k\geq 1} \left(\mathcal{F}_k{}', k\right) ,$$

where $\mathcal{F}_k{}' \subset \Omega$. Define floors by

$$\mathcal{F}_k{}' = \begin{cases} \Omega\,, \ k = 1 \\ P_{n_1}(0)\,, \ k = 2 \ldots n_1 \\ P_k(0)\,, \ k = n_1 \ldots \end{cases} .$$

1.3. Metrics on the tower

We define metrics on the tower by setting

$$\rho\left((z_1, k_1), (z_2, k_2)\right) = \begin{cases} Q_\rho{}^{-(k_1-1)} \left|z_1 - z_2\right|, \ k_1 = k_2 > 1 \\ \left|z_1 - z_2\right|, \ k_j = 1, \ z_j \text{ are in} \\ \qquad \text{the same puzzle-piece} \\ \infty \quad \text{otherwise} \ , \end{cases}$$

where Q_ρ is some constant satisfying $Q_c{}^{\frac{1}{2}} > Q_\rho$, to be chosen later. To simplify notation we will write $|x - y|$ instead of $\rho(x, y)$ for $x, y \in \mathcal{T}$.

1.4. Dynamics on the tower

Set

$$\mathcal{U}_k{}' := \mathcal{F}_{k+1}{}' , \quad \mathcal{D}_k{}' := \mathcal{F}_k{}' \setminus \mathcal{U}_k{}' ,$$

and

$$\mathcal{U}_k := \left(\mathcal{U}_k{}', k\right) , \quad \mathcal{D}_k := \left(\mathcal{D}_k{}', k\right) .$$

Define $\tilde{F}$ by

$$\tilde{F} : (z,k) \mapsto \begin{cases} (z, k+1) , & z \in \mathcal{U}_k{}' \\ \left(F^k z, 1\right) , & z \in \mathcal{D}_k{}' \end{cases} .$$

Note that $\tilde{F}$ is undefined for some points on the first level of the tower, however transfer operator and its' formal adjoint are still well-defined (we adopt an agreement that sum of empty set of terms is zero). To simplify the formulas we will write F_n in place of n-th iterate $\tilde{F}^n$.

1.5. Properties of the tower

Proposition.

i] *Dynamics $\tilde{F}$ maps 1-to-1*

$$\mathcal{U}_k \ \text{to} \ \mathcal{F}_{k+1} ,$$

$$\mathcal{D}_k, \ k \geq 2 \ \text{ to } \ \text{a union of some components of } \mathcal{F}_1 .$$

ii] *If x, y are in one component, i.e. $|x - y| < \infty$, then they have preimages in the same components.*

iii]

$$\begin{aligned}\left|\tilde{F}'z\right| &= Q_\rho, \quad z \in \mathcal{U}_k \\ \left|\tilde{F}'z\right| &\asymp {Q_\rho}^{-(k-1)} \left|{F^k}'c\right|^{\frac{1}{2}} \gtrsim {Q_c}^{\frac{k}{2}} {Q_\rho}^{-k}, \quad z \in \mathcal{D}_k\end{aligned}$$

iv] *By the choice of n_1 we can make $\tilde{F}$ expanding, i.e. for some C and $Q > 1$*

$$|F_n'| > CQ^n .$$

v] *For some $Q > 1$ for any n and C we can choose T in such a way that $y \in \mathcal{F}_k$, $k \geq T$ which after less than n iterations goes down*

$$|F_n'(y)| > C\, Q^k .$$

Proof.

i],ii]. Follow directly from the construction. The property ii] means that we can estimate $|\nabla\left(Lf(z)\right)|$ through estimating $|\nabla f(y)|$ for $y \in F^{-1}$.

iii]. First note that

$$\delta_k := \text{diam}\left(F\left(P_k(0)\right)\right) \asymp |F_k'(c)|^{-1} .$$

Take $y \in P_k \setminus P_{k+1} = \mathcal{D}_k$, $k \geq n_1$, from the distortion theorems

$$\begin{aligned}
\left|F^{k'}(y)\right| &= |F'(y)|\left|F^{(k-1)'}F(y)\right| \\
&\asymp |y|\left|F^{(k-1)'}(c)\right| \\
&\asymp \text{diam}\,(F\,(P_k(0)))^{\frac{1}{2}}\,\left|F^{(k-1)'}(c)\right| \\
&\asymp \left|F^{(k-1)'}(c)\right|^{-\frac{1}{2}+1} = \left|F^{(k-1)'}(c)\right|^{\frac{1}{2}} .
\end{aligned}$$

Therefore

$$\begin{aligned}
\left|\tilde{F}'\,(y,k)\right| &= {Q_\rho}^{-(k-1)}\left|F^{k'}(y)\right| \\
&\asymp {Q_\rho}^{-(k-1)}\left|F^{(k-1)'}(c)\right|^{\frac{1}{2}} \gtrsim {Q_\rho}^{-k}\,{Q_c}^{-\frac{k}{2}} .
\end{aligned}$$

iv],v]. Follow from $Q_\rho < {Q_c}^{\frac{1}{2}}$.

■

2. Analysis of the transfer operator

2.1. Transfer operator on the tower

We will consider two function spaces on the tower: $C\,(\mathcal{T})$, $\text{Lip}\,(\mathcal{T})$ with norms

$$\begin{aligned}
\|f\|_\infty &= \|f\|_{C(\mathcal{T})} := \sup_k \|f\|_{\infty,\mathcal{F}_k} \\
\|f\|_{Lip} &= \|f\|_{\text{Lip}(\mathcal{T})} := \sup_k \|f\|_{Lip,\mathcal{F}_k}
\end{aligned}$$

For $t \in \mathbb{R}$ we define operator L_t on $C(\mathcal{T})$ by

$$Lf(z) := \sum_{y \in \tilde{F}^{-1}z} f(y) \left|\tilde{F}'(y)\right|^{-t},$$

note that

$$L^n f(z) := \sum_{y \in \tilde{F}^{-n}(z)} f(y) \left|F_n'(y)\right|^{-t}.$$

Denote by λ_t the spectral radius $r(L_t, C(\mathcal{T}))$.

Lemma. *L_t is bounded operator on $C(\mathcal{T})$, and* $\lambda_t \geq 2^{s(t)}$.

Proof. The boundness of Lf follows from the property v] of the tower.

To prove the inequality take some point $z \in A(\infty)$ such that

$$z \in \mathcal{F}_{n_1}{}' \setminus \mathcal{F}_{n_1+1}{}'.$$

For such z and $n \geq n_1$ all points $y \in F^{-n}z$ satisfy $\tilde{F^n}(y,1) = (z,1)$ and $\tilde{F^n}'(y,1) = F^{n\prime}y$. Thus by Lemma 1.1 for $n \geq n_1$

$$\begin{aligned} \|L_t^n\|_\infty \geq \left|(L_t^n 1)(z,1)\right| &= \sum_{y' \in \tilde{F}^{-n}} \left|F_n' y'\right|^{-t} \\ &= \sum_{y \in F^{-n}} \left|F^{n\prime} y\right|^{-t} \asymp 2^{s(t)} . \end{aligned}$$

■

2.2. Ionescu-Tulcea and Marinescu inequality

Lemma. *Suppose* $\lambda_t > Q_\rho{}^{-t}$. *Then for* $n \gg 1$ *and* $T \gg n$ *we have*

$$\|L^n f\|_{\mathrm{Lip}} \le \frac{1}{5}\lambda_t^n \,\mathrm{Lip}\,(f) + \mathrm{const}\sup_{k<T}\|f\|_{\infty,\mathcal{F}_k} + \frac{1}{5}\lambda_t^n \sup_{k\ge T}\|f\|_{\infty,\mathcal{F}_k}\ .$$

Proof.

For $z \in \mathcal{F}_k$, $k < n$ we will use the estimate (since all functions are Lipschitz the gradients are defined almost everywhere, and later we will be able to consider an essential supremum)

$$\begin{aligned}
|\nabla L^n f(z)| &= \left|\sum_{y\in\tilde{F}^{-n}(z)} \nabla_z\left(f(y)\,|F_n'(y)|^{-t}\right)\right| \\
&\le \sum_{y\in\tilde{F}^{-n}(z)} |\nabla f(y)|\,|F_n'(y)|^{-t-1} \\
&\qquad + \sum_{y\in\tilde{F}^{-n}(z)} |f(y)|\,|F_n'(y)|^{-1}\left|\nabla\left(|F_n'(y)^{-t}|\right)\right| \\
&\le \sup_{y\in\tilde{F}^{-n}(z)} |\nabla f(y)|\,|(L_t^n 1)(z)|\,\sup_y |F_n' y|^{-1} \\
&\qquad + C\sum_{y\in\tilde{F}^{-n}(z)} |f(y)||F_n'(y)|^{-t} \\
&\le \sup_k \mathrm{Lip}\,(f,\mathcal{F}_k)\ Q^{-n}\,\lambda_t^n\, e^{o(n)} + C\,|L^n f(z)|\ ,
\end{aligned}$$

where C is an absolute constant.

Note that $z \in \mathcal{F}_k$, $k \geq n$ has only one preimage $y \in \mathcal{F}_{k-n}$ and in its neighborhood F_n increases all distances exactly by a multiplier of ${Q_\rho}^n$, thus

$$|\nabla L^n f(z)| \;=\; |\nabla f(y)|\, {Q_\rho}^{-n(t+1)}\;.$$

Using the latter inequality for $z \in \mathcal{F}_k$, $k \geq n$ we obtain that for $n \gg 1$ for sufficiently large $T \gg n$

$$\begin{aligned}
\sup_k \operatorname{Lip}(L^n f) &= \sup_k \ \text{ess-sup}_{\mathcal{F}_k} \nabla L^n f \\
&= \sup \left\{ \text{ess-sup}_{\mathcal{F}_k, k\geq n} \nabla L^n f\,,\ \text{ess-sup}_{\mathcal{F}_k, k<n} \nabla L^n f \right\} \\
&\leq \sup \Big\{ \sup_k \operatorname{Lip}(f, \mathcal{F}_k)\ {Q_\rho}^{-n(t+1)}\,, \\
&\quad \sup_k \operatorname{Lip}(f, \mathcal{F}_k)\ Q^{-n}\, \lambda_t^n\, e^{o(n)} \;+\; C \sup_{k<n} \|L^n f\|_{\infty, \mathcal{F}_k} \Big\} \\
&\leq \frac{1}{5}\, \lambda_t^n\ \operatorname{Lip}(f) \;+\; C\ \|L^n f\|_\infty\,.
\end{aligned}$$

Now estimate in the same way $|f(z)|$.

Since $\lambda_t > {Q_\rho}^{-t}$, for large n we have ${Q_\rho}^{-tn} < \frac{1}{5(C+1)}\lambda_t^n$. Then for $z \in \mathcal{F}_k$, $k \geq n$ we can estimate

$$|L^n f(z)| \;=\; {Q_\rho}^{-tn}\, |f(y)| \;<\; \frac{1}{5\,(C+1)} \lambda_t^n\ |f(y)|\,,$$

where $y \in \mathcal{F}_{n-k}$ is the only preimage of z.

For $z \in \mathcal{F}_k$, $k < n$ number of preimages of z on each level of the tower is bounded by some constant K (depending only on n). We can choose T in such a way that for some $q < 1$ and any $y \in \mathcal{F}_k$, $k \geq T$ which after less then n iterations goes down

$$|F_n'(y)|^{-t} \, (C+1) \, K \; < \; \frac{1}{5} \, \lambda_t^n \, \frac{1}{1-q} \, q^k \, .$$

Then

$$\begin{aligned}
|L^n f(z)| \; &= \; \left| \sum_{y \in \tilde{F}^{-n}(z)} f(y) \, |F_n'(y)|^{-t} \right| \\
&\leq \; \sum_{y \in \mathcal{F}_k, k<T} |F_n'(y)|^{-t} |f(y)| \; + \; \sum_{y \in \mathcal{F}_k, k \geq T} |F_n'(y)|^{-t} |f(y)| \\
&\leq \; \text{const} \; \sup_{k<T} \|f\|_{\infty, \mathcal{F}_k} \\
&\quad + \; \sum_{k \geq T} \|f\|_{\infty, \mathcal{F}_k} \, K \, \frac{1}{5} \, \lambda_t^n \, \frac{1}{1-q} q^k \, ((C+1) \, K)^{-1} \\
&\leq \; \text{const} \; \sup_{k<T} \|f\|_{\infty, \mathcal{F}_k} \; + \; \sup_{k \geq T} \|f\|_{\infty, \mathcal{F}_k} \, \frac{1}{5} \, \lambda_t^n \, (C+1)^{-1} \, .
\end{aligned}$$

Together these estimates give us

$$\begin{aligned}
\|L^n f\|_\infty \; &= \; \sup_k \|L^n f\|_{\infty, \mathcal{F}_k} \\
&\leq \; \sup_{k<n} \|L^n f\|_{\infty, \mathcal{F}_k} \; + \; \sup_{k \geq n} \|L^n f\|_{\infty, \mathcal{F}_k} \\
&\leq \; \sup \Big\{ \text{const} \; \sup_{k<T} \|f\|_{\infty, \mathcal{F}_k} \\
&\qquad\qquad + \; \sup_{k \geq T} \|f\|_{\infty, \mathcal{F}_k} \, \frac{1}{5} \, \lambda_t^n \, (C+1)^{-1} \; , \\
&\qquad\qquad \frac{1}{5\,(C+1)} \lambda_t^n \sup_{k \geq 1} \|L^n f\|_{\infty, \mathcal{F}_k} \Big\} \\
&\leq \; \text{const} \; \sup_{k<T} \|f\|_{\infty, \mathcal{F}_k} \; + \; \frac{1}{5\,(C+1)} \, \lambda_t^n \; \sup_{k \geq T} \|f\|_{\infty, \mathcal{F}_k} \, .
\end{aligned}$$

Therefore

$$
\begin{aligned}
\|L^n f\|_\infty &= \operatorname{Lip}(L^n f) + \|L^n f\|_\infty \\
&\leq \frac{1}{5}\,\lambda_t^n \,\sup_k \operatorname{Lip}(f) + (C+1)\,\|L^n f\|_\infty \\
&\leq \frac{1}{5}\,\lambda_t^n \,\sup_k \operatorname{Lip}(f) \\
&\qquad + \text{const} \sum_{k<T} \|f\|_{\infty,\mathcal{F}_k} + \frac{1}{5}\,\lambda_t^n \sum_{k\geq T} \|f\|_{\infty,\mathcal{F}_k} \ .
\end{aligned}
$$

■

2.3. Finite rank approximation

Lemma. *For any integer T and $\delta > 0$ there exists a finite rank operator M in* Lip $(\mathcal{T})$ *such that*

$$
\begin{aligned}
\|M\|_{\mathrm{Lip}} &\leq 1 \ , \\
\|(I-M)\,f\|_{\infty,\mathcal{F}_k} &\leq \delta\ \|f\|_{\mathrm{Lip},\mathcal{F}_k} \quad , \quad k<T \ , \\
Mf(z) &= 0 \ , \quad k \geq T \ .
\end{aligned}
$$

Proof. Cover $\mathcal{F}_k$, $k < T$ with a grid of triangles ϕ of size $\delta \ll 1$. Define Mf as a continuous function vanishing on $\mathcal{F}_k$, $k \geq T$ and satisfying

$$
Mf = \begin{cases} f \ \text{at all vertices} \\ \text{is linear in each triangle } \phi \ , \end{cases}
$$

on $\mathcal{F}_k$, $k < T$. The properties are obvious. ■

2.4. Quasicompactness

Claim. *If* $\lambda_t > {Q_\rho}^{-t}$ *then*

$$r(L_t, \mathrm{Lip}\,(\mathcal{T})) \;=\; \lambda_t \;,$$

and

$$r_{ess}(L_t, \mathrm{Lip}\,(\mathcal{T})) \;<\; r(L_t, \mathrm{Lip}\,(\mathcal{T})) \;.$$

Proof.

First recall that

$$r\,(L_t, C\,(\mathcal{T})) \;=\; \lambda_t \;.$$

Also

$$\|L_t^n\|_\infty \asymp \|L_t^n 1\|_\infty \lesssim \|L_t^n 1\|_{\mathrm{Lip}} \leq \|L_t^n\|_{\mathrm{Lip}} \;,$$

and $\lambda_t \leq r(L_t, \mathrm{Lip}\,(\mathcal{T}))$.

To prove the converse, by the Ionescu-Tulcea and Marinescu inequality choose $\varepsilon > 0$ and N such that

$$\left\|L^N f\right\|_{\mathrm{Lip}} \;\leq\; \lambda_t^{N-\varepsilon}\,\|f\|_{\mathrm{Lip}} \;+\; K\,\|f\|_\infty \;.$$

Define inductively an increasing sequence $\{M_k\}$ by

$$M_0 \;:=\; 1 \;, \qquad M_k \;:=\; \max\left\{M_{k-1}\,\lambda_t^N,\, \left\|L^{Nk}\right\|_\infty\right\} \;.$$

Then

$$M_k \;=\; \lambda_t^{Nk+o(k)} \;\geq\; \left\|L^{Nk}\right\|_\infty , \quad M_{k+1} \,/\, M_k \;\geq\; \lambda_t^N \;;$$

and we can choose a constant C satisfying

$$C\,\lambda_t^{N-\varepsilon} + 1 \;\leq\; C\,\lambda_t^N \;\; \text{and} \;\; K \;\leq\; C\,M_1 \;.$$

By induction,

$$\begin{aligned} \left\|L^{Nk} f\right\|_{\text{Lip}} \;&\leq\; \lambda_t^{Nk-\varepsilon k}\,\|f\|_{\text{Lip}} \;+\; C\,M_k\,\|f\|_\infty \\ &\lesssim\; \lambda_t^{Nk}\,\|f\|_{\text{Lip}} \;, \end{aligned}$$

therefore

$$r(L_t, \operatorname{Lip}(\mathcal{T})) \;\leq\; \lambda_t \;,$$

which implies the desired inequality and thus the first statement.

To prove the second statement choose n and T such that

$$\begin{aligned} \|L^n f\|_{\text{Lip}} \;\leq\; &\frac{1}{5}\,\lambda_t^n\,\|f\|_{\text{Lip}} \\ &+\; \text{const}\sum_{k<T}\|f\|_{\infty,\mathcal{F}_k} \;+\; \frac{1}{5}\,\lambda_t^n\sum_{k\geq T}\|f\|_{\infty,\mathcal{F}_k} \;, \end{aligned}$$

and a finite rank approximation M for $\delta < \frac{1}{5\,\text{const}}\,\lambda_t^n$.

Then

$$\begin{aligned}
\|L^n(f - Mf)\|_{\mathrm{Lip}} \;\le\; & \frac{1}{5}\,\lambda_t^n\,\|f - Mf\|_{\mathrm{Lip}} \\
& + \text{const} \sum_{k<T} \|f - Mf\|_{\infty,\mathcal{F}_k} \\
& + \frac{1}{5}\,\lambda_t^n \sum_{k\ge T} \|f - Mf\|_{\infty,\mathcal{F}_k} \\
\le\; & \frac{2}{5}\,\lambda_t^n\,\|f\|_{\mathrm{Lip}} \\
& + \text{const}\;\delta \sum_{k<T} \|f\|_{\mathrm{Lip},\mathcal{F}_k} + \frac{1}{5} \sum_{k\ge T} \|f\|_{\infty,\mathcal{F}_k} \\
\le\; & \frac{4}{5}\,\lambda_t^n\,\|f\|_{\mathrm{Lip}}
\end{aligned}$$

and

$$r_{ess}(L_t, \mathrm{Lip}(\mathcal{T})) \;\le\; \left(\frac{4}{5}\right)^{\frac{1}{n}} \lambda_t \;<\; \lambda_t\,.$$

■

3. Analyticity of spectrum

In the subsections 3.1 – 3.3 we will assume that $\lambda_t > {Q_\rho}^{-t}$, i.e. Claim 2.4 holds.

3.1. Eigenvalues and multiplicity

Fix t and denote $\lambda = \lambda_t := r(L_t, C(\mathcal{T}))$.

Lemma. $\ker(L_t - \lambda) \neq \{0\}$. *Moreover there is a strictly positive eigenfunction* $f = f_t$.

Proof. See [**Ru6**, Theorem 2.2].

Idea of the proof: since $r_{ess}(L_t, \mathrm{Lip}(\mathcal{T})) < r(L_t, \mathrm{Lip}(\mathcal{T})) = \lambda$, we have some eigenvalues λ_j of absolute value λ, and a decomposition

$$1 = X_0 + \sum X_j ,$$

where X_0 is the subspace corresponding to the rest of the spectrum.

Then

$$0 \le L_t^n 1 = L_t^n X_0 + \sum L_t^n X_j .$$

Here $L_t^n X_0 = o(\lambda^n)$ and $L_t^n X_j = \lambda_j^n X_j$, implying that one of the λ_j's is positive and hence is equal to $\lambda = r(L_t, C(\mathcal{T}))$.

Also it follows that there is non-negative (and hence strictly positive: if $f(z) = 0$ then $f(y) = 0$ for $F_n y = z$, from continuity $f \equiv 0$ on J_F and by Sublemma everywhere) eigenfunction f with eigenvalue λ. ∎

Sublemma. *For* $f \in \mathrm{Lip}(\mathcal{T})$

$$\begin{cases} L_t f = \lambda f \\ f|_{\tilde{J}_F} = 0 \end{cases} \quad \textit{implies} \quad f = 0 .$$

Proof of Sublemma. For fixed $z \in \mathcal{T}$ and $y \in \tilde{F}^{-n} z$ distortion estimates imply

$$|F_n'(y)|^{-1} \asymp \mathrm{dist}\left(y, \tilde{J}_F\right) .$$

Therefore

$$\begin{aligned}
|f(z)| &= \left|\lambda^{-n}\, L_t^n f(z)\right| \\
&= \left|\lambda^{-n} \sum_{y\in\tilde{F}^{-n}(z)} |F_n'(y)|^{-t} f(y)\right| \\
&\lesssim \lambda^{-n} \sum_{y\in\tilde{F}^{-n}(z)} |F_n'(y)|^{-t} \ \mathrm{dist}\left(y, \tilde{J}_F\right) \\
&\lesssim \lambda^{-n} \sum_{y\in\tilde{F}^{-n}(z)} |F_n'(y)|^{-t-1} \\
&\lesssim \lambda^{-n} \frac{1}{CQ^n} \sum_{y\in\tilde{F}^{-n}(z)} |F_n'(y)|^{-t} \lesssim e^{o(n)} \,/\, Q^{-n} \underset{n\to\infty}{\longrightarrow} 0\,.
\end{aligned}$$

Consequently $f = 0$ and Sublemma is proved. ■

3.2. Conformal measures

Consider the formal adjoint L^* to L, acting on the space of measures on $\mathcal{T}$:

$$d\,(L^*\,\mu)\,(y) \;:=\; \left|\tilde{F}'y\right|^{-t}\, d\,\mu\,(Fy)\ .$$

Lemma. *There is a probability measure $\nu = \nu_t$ supported on $\mathcal{T}$ with $L^*\nu = \lambda\nu$.*

Proof. Clearly

$$(L^{*n}\mu)\,(\mathcal{T}) \;=\; \langle 1, L^{*n}\mu\rangle \;=\; \langle L^n 1, \mu\rangle \;\asymp\; \lambda^n\; \mu\,(\mathcal{T})\ .$$

Take some positive measure μ supported on the first floor of the tower and define

$$\mu_n' := \lambda^{-n} L^{*n} \mu ,$$

$$\mu_n := \frac{1}{n} \sum_{k=1}^{n} \mu_k' .$$

Then

$$\mathrm{Var}\,(\mu_n) \asymp 1 , \quad \mathrm{Var}\,(L^* \mu_n - \mu_n) \asymp \frac{1}{n} ,$$

thus to find ν it is sufficient to prove that some subsequence $\{\mu_n\}$ has a weak limit. The only trouble is that $\mathcal{T}$ is not compact.

But one can easily check by induction that for small $Q > 1$ and large positive C

$$\mathrm{Var}\,(\mu_n', \mathcal{F}_k) < C\, Q^{-n} ,$$

which is sufficient. ∎

Remark. Denote $\tilde{J}_F := \cup_k \{(z,k) \in \mathcal{F}_k : z \in J_F\}$. Then $\tilde{J}_F$ is invariant under $\tilde{F}$ and $\mathrm{supp}\nu \supset \tilde{J}_F$.

3.3. Pressure and spectrum

Lemma. *We have* $\lambda = 2^{s(t)}$.

Proof. We know that $\lambda \geq 2^{s(t)} \geq {Q_c}^{-\frac{t}{2}}$. Assume the claim of the Lemma is not true. Then $\lambda > 2^{s(t)}$. Chose such λ_1 and Q_ρ that

$$\lambda > {Q_\rho}^{-t} > \lambda_1\, 2^{s(t)} .$$

By the definition of $s(t)$ the series

$$\sum_{n=0}^{\infty} \sum_{y \in F^{-n}0} (\lambda_1)^{-n} \left|(F^n y)'\right|^{-t} ,$$

is convergent, hence we can define a finite measure ν_0 on J_F by

$$\nu_0 := \sum_{n=0}^{\infty} \sum_{y \in F^{-n}0} (\lambda_1)^{-n} \left|(F^n y)'\right|^{-t} \delta_y .$$

Note that $d\nu_0(Fy) = \lambda_1 \left|F'y\right|^{-t} d\nu_0(y)$.

Now transfer this measure to the tower by setting

$$d\,\nu_1(z,k) := \lambda_1^n \, {Q_\rho}^{nt} \, d\,\nu_0(z) ,$$

we obtain a finite measure again (since $\lambda_1^n \, QM^{nt} < 1$). It is easy to check that the mentioned property of ν_0 implies

$$d\nu_1(\tilde{F}y) = \left|\tilde{F}'y\right|^{-t} d\,\nu_1(y) ,$$

and hence $L^*\nu_1 = \lambda_1 \, \nu_1$. Therefore for the positive eigenfunction f we can write

$$\lambda^n \langle f, \nu_1 \rangle = \langle L^n f, \nu_1 \rangle = \langle f, {L^*}^n \nu_1 \rangle = \lambda_1^n \langle f, \nu_1 \rangle .$$

Since f is strictly positive this implies $\lambda = \lambda_1$, and we obtain a contradiction. ∎

Claim. *We have* dim ker $(L_t - \lambda)^2 = 1$, *i.e.* λ *is a simple eigenvalue.*

Proof. If we consider only functions in $C\left(\tilde{J}_F\right)$, the same reasoning as in [**MS**, §3.6] gives us that dim ker $(L_t - \lambda) = 1$, and then Sublemma implies this for $f \in \mathrm{Lip}\,(\mathcal{T})$.

Lemma 3.1 shows that we have a non-negative eigenfunction f:

$$(L_t - \lambda)\, f = 0\,.$$

If λ is not a simple eigenvalue, then there exists function h such that

$$(L_t - \lambda)\, h = f\,.$$

Therefore

$$\begin{aligned} \langle f, \nu\rangle &= \langle Lh, \nu\rangle - \langle \lambda h, \nu\rangle \\ &= \langle h, L^*\nu\rangle - \lambda\,\langle h, \nu\rangle = 0\,. \end{aligned}$$

Thus ν-a.e. we have $f = 0$, hence everywhere on $\tilde{J}_F$ and (by the Sublemma) in $\mathcal{T}$. ∎

3.4. Analyticity

Claims 2.4 and 3.3 show that if for some t' we can choose Q_ρ in such a way that $\lambda_t > {Q_\rho}^{-t} > {Q_c}^{-\frac{t}{2}}$ then $2^{s(t)}$ is a simple isolated eigenvalue of L_t : Lip $(\mathcal{T})$ ↺. Clearly it implies that L_t depends real analytically on t therefore $s(t)$ and hence $\beta(t)$ is real analytic in the neighborhood of t' (see [**MS**, 4.1]).

By Lemma 2.1 $\lambda_t \geq 2^{s(t)}$, thus inequality $2^{s(t)} > {Q_c}^{-\frac{t}{2}}$ is sufficient to find Q_ρ satisfying our conditions. On the other hand we always have $2^{s(t)} \geq {Q_c}^{-\frac{t}{2}}$ thus to prove the main Theorem we only have to check $2^{s(t)} \geq {Q_c}^{-\frac{t}{2}}$.

First note that

$$\delta_n := \text{diam}\left(F\left(P_n(0)\right)\right) \asymp \left|F_n'(c)\right|^{-1}.$$

We can choose a point $z \in A(\infty)$ in each part of Ω, so that for any n one of them has a preimage $y \in P_n \setminus P_{n+1}$ under the F^{-n}. From the distortion theorems

$$\begin{aligned}
\left|F^{n\prime}(y)\right| &= \left|F'(y)\right| \left|F^{(n-1)\prime}F(y)\right| \\
&\asymp |y| \left|F^{(n-1)\prime}(c)\right| \\
&\asymp \text{diam}\left(F\left(P_n(0)\right)\right)^{\frac{1}{2}} \left|F^{(n-1)\prime}(c)\right| \\
&\asymp \left|F^{(n-1)\prime}(c)\right|^{-\frac{1}{2}+1} = \left|F^{(n-1)\prime}(c)\right|^{\frac{1}{2}}.
\end{aligned}$$

Then for the corresponding z:

$$\sum_{y \in \tilde{F}^{-n}(z)} \left|F^{n\prime}y\right|^{-t} \geq \left|F^{n\prime}(y)\right|^{-t} \asymp \left|F^{(n-1)\prime}(c)\right|^{-\frac{t}{2}},$$

and taking $\limsup$ as $n \to \infty$ we obtain the desired inequality. ■

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