QUARKS, GLUONS, BAGS, AND HADRONS

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ABSTRACT

We consider two semiclassical hadron models, the MIT bag model and a generalization, the confined free quark model. The low momentum transfer matrix elements of some of the S-wave hadrons are derived and compared with experiment in both models, and we conclude that the constraints imposed on the latter model by the MIT model are largely justified by experiment. We then generalize the MIT model to include lowest order gluon exchange spin-spin forces between quarks and remove some degeneracies of the zeroth order MIT model. Finally we predict the masses and magnetic moments of some of the hypothetical charmed hadrons.

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I. Introduction

Recently there has been a great deal of interest in semi-classical models of hadron structure which impose quark confinement ab initio, and which are simple enough to allow a large class of hadron interactions to be treated numerically[1-11]. The most successful of these models in terms of agreement with experiment is the MIT bag model, which was introduced by A. Chodos, B. Jaffe, K. Johnson, C. Thorn, and V. Weisskopf in 1974[1]. The initial form of this model was a "bag" of massless quarks confined to a spherical region by (1) the addition of a constant "bag strength" to the Lagrangian density, and (2) the constraint that the quark fields vanished outside the bag. The imposition of Poincaré invariance required step (1) to render the model nontrivial. The generalization to massive quarks was then carried out concurrently by several groups in 1975[3,4,8], and the MIT group introduced corrections due to zero point fluctuations and quark-gluon interactions[3].

In this thesis we introduce a generalization of the MIT model which abandons Poincaré invariance and hence the power to predict hadron spectra. This allows us to investigate a larger model parameter space in treating weak and electromagnetic matrix elements, which serves as a check on the more restrictive bag model parameters. We then generalize the bag model to massive quarks (which is now a well known result) and obtain a value for the strange quark mass. This allows us to predict a number of SU(6) violations due to this mass, some of which are well-satisfied. Finally we consider gluon effects in a fashion which is quite different from the approach of the MIT group and estimate the size of some mass splittings of S-wave hadrons that result from the lowest order quark-gluon interaction.

II. Explicit CFQM (Confined Free Quark Model) Wave Functions

In much of the following we shall obtain explicit numbers for quark model calculations using the simplest possible assumption for confined quark wave functions; the quarks are assumed to be confined (by an unspecified mechanism) to a spherical region of radius a and are free within that region.

The first-quantized quark wavefunctions satisfy

$$\begin{aligned} (\not{\partial} - m_q)\psi_q &= 0 \qquad\qquad r \leq a \\ \psi_q &= 0 \qquad\qquad r > a \end{aligned} \tag{1}$$

and the normalization integral for a 1-quark wavefunction is

$$\int_{r \leq a} d^3x\, \psi_q^\dagger \psi_q = 1 \tag{2}$$

the solutions of equation (1) with definite J^2, J_z and parity are well known;[12]

$$\begin{aligned} \psi_{jm}^{(+)}(x) &= \sqrt{4\pi}\, \alpha_j^{+} \begin{bmatrix} j_{j-\frac{1}{2}}(kr)\, \varphi_{jm}^{(+)}(\Omega) \\ ik\, j_{j+\frac{1}{2}}(kr)\, \varphi_{jm}^{(-)}(\Omega) \end{bmatrix} e^{-i\omega t} \qquad P = (-)^{j-\frac{1}{2}} \\ \psi_{jm}^{(-)}(x) &= \sqrt{4\pi}\, \alpha_j^{-} \begin{bmatrix} i\, j_{j+\frac{1}{2}}(kr)\, \varphi_{jm}^{(-)}(\Omega) \\ k\, j_{j-\frac{1}{2}}(kr)\, \varphi_{jm}^{(+)}(\Omega) \end{bmatrix} e^{-i\omega t} \qquad P = (-)^{j+\frac{1}{2}} \end{aligned} \tag{3a,b}$$

where $\omega = \sqrt{k^2+m^2}$, $\not{k} \equiv \sqrt{\frac{\omega-m}{\omega+m}}$, and the $\{\varphi_{jm}^{(\pm)}(\Omega)\}$ are spinor spherical harmonics[12],

$$\varphi_{jm}^{(+)} = \frac{1}{\sqrt{2j}}\begin{bmatrix}\sqrt{j+m}\; Y_{j-\frac{1}{2}}^{m-\frac{1}{2}} \\ \sqrt{j-m}\; Y_{j-\frac{1}{2}}^{m+\frac{1}{2}}\end{bmatrix} \qquad \varphi_{jm}^{(-)} = \frac{1}{\sqrt{2(j+1)}}\begin{bmatrix}\sqrt{j-m+1}\; Y_{j+\frac{1}{2}}^{m-\frac{1}{2}} \\ -\sqrt{j+m+1}\; Y_{j+\frac{1}{2}}^{m+\frac{1}{2}}\end{bmatrix} \tag{4}$$

A special case of interest is the lowest positive parity mode, $J^P = \frac{1}{2}^+$;

$$\psi_{\frac{1}{2}\uparrow}^{(+)} = \alpha_{\frac{1}{2}+}\begin{bmatrix} j_0(kr) \\ 0 \\ i\not{k} j_1(kr)\begin{bmatrix}\cos\theta \\ \sin\theta e^{i\varphi}\end{bmatrix}\end{bmatrix} e^{-i\omega t}$$
$$\psi_{\frac{1}{2}\downarrow}^{(+)} = \alpha_{\frac{1}{2}+}\begin{bmatrix} 0 \\ j_0(kr) \\ i\not{k} j_1(kr)\begin{bmatrix}\sin\theta e^{-i\varphi} \\ -\cos\theta\end{bmatrix}\end{bmatrix} e^{-i\omega t} \tag{5a,b}$$

The normalization integral (2) determines $\alpha_{j\pm}(a,m,k)$;

$$|\alpha_{j\pm}|^2 = \frac{x_0^3}{4\pi a^3}\left[I(j\mp\tfrac{1}{2},x_0) + \not{k}^2 I(j\pm\tfrac{1}{2},x_0)\right]^{-1} \tag{6}$$

where $x_0 \equiv ka$ and the function I is the integral

$$I(\ell,x_0) \equiv \int_0^{x_0} \eta^2 j_\ell(\eta)^2 d\eta \tag{7}$$

for the special case $J^P = \frac{1}{2}^+$ the normalization integral gives explicitly

$$|\alpha_{\frac{1}{2}+}|^2 = \frac{x_0^2(1+\frac{\mu}{\epsilon})}{4\pi}\left[1 - \frac{\mu}{\epsilon} j_0(2x_0) - (1-\frac{\mu}{\epsilon}) j_0(x_0)^2\right]^{-1} a^{-3} \tag{8}$$

where $\mu \equiv ma$, $\epsilon \equiv \omega a$. For each $J^P = \frac{1}{2}^\pm, \frac{3}{2}^\pm, \ldots$ the normalization (6) and the explicit spinors (3) determine a three parameter quark wavefunction $\psi^\pm_{jm}(x_0(j)^\pm, a, m)$ where $x_0(j)^\pm = k_j^\pm a$ and a is the well radius. If we wish we may take $m = 0$, since most quark model results are insensitive to small changes ($\lesssim 50$ MeV) in m.

For reference purposes we now work out the matrix elements of various Dirac matrices between quark states. For the lowest mode ($\frac{1}{2}^+$) we may determine these with some trivial algebra;

$$\langle\bar{\uparrow}|\uparrow\rangle = |\alpha_+|^2 \left\{ j_0(kr)^2 - \bar{k}^2 j_1(kr)^2 \right\} \tag{9,a}$$

$$\langle\bar{\uparrow}|\downarrow\rangle = \langle\bar{\downarrow}|\uparrow\rangle = 0 \tag{9,b}$$

$$\langle\bar{\uparrow}|\gamma_5|\uparrow\rangle = 2i|\alpha_+|^2 \bar{k}\, j_0(kr) j_1(kr) \cos\theta \tag{9,c}$$

$$\langle\bar{\downarrow}|\gamma_5|\uparrow\rangle = 2i|\alpha_+|^2 \bar{k}\, j_0(kr) j_1(kr) \sin\theta\, e^{i\varphi} = -\langle\bar{\uparrow}|\gamma_5|\downarrow\rangle^* \tag{9,d}$$

$$\langle\bar{\uparrow}|\gamma_0|\uparrow\rangle = \langle\uparrow|\uparrow\rangle = \langle\downarrow|\downarrow\rangle = |\alpha_+|^2 \left\{ j_0(kr)^2 + \bar{k}^2 j_1(kr)^2 \right\} \tag{9,e}$$

$$\langle\bar{\uparrow}|\gamma_0|\downarrow\rangle = \langle\bar{\downarrow}|\gamma_0|\uparrow\rangle = 0 \tag{9,f}$$

$$\begin{aligned} \langle\bar{\uparrow}|\vec{\gamma}|\uparrow\rangle &= -4\sqrt{\tfrac{2\pi}{3}}\, i|\alpha_+|^2 \bar{k}\, j_0(kr) j_1(kr)\, \vec{Y}_{110}(\Omega) \\ &= -\langle\bar{\downarrow}|\vec{\gamma}|\downarrow\rangle \qquad\qquad \vec{Y}_{110} = i\sqrt{\tfrac{3}{8\pi}} \sin\theta\, \hat{\varphi} \end{aligned} \tag{9,g}$$

$$\langle \bar{\uparrow} | \vec{\gamma} | \downarrow \rangle = -8\sqrt{\frac{\pi}{3}}\, i|\kappa_{+}^{2}\, k\, j_0(kr)\, j_1(kr)\, \vec{Y}_{11-1}(\Omega) \tag{9,h}$$

$$\vec{Y}_{11-1} = \sqrt{\frac{3}{16\pi}}\, e^{-i\varphi}\left(\hat{\theta} - i\cos\theta\,\hat{\varphi}\right)$$

$$\langle \bar{\downarrow} | \vec{\gamma} | \uparrow \rangle = \langle \bar{\uparrow} | \vec{\gamma} | \downarrow \rangle^{*} \tag{9,i}$$

Now we are prepared to treat the electromagnetic and weak current matrix elements between our CFQM quark states.

A. Electromagnetic Interactions

The electromagnetic interaction of hadrons and photons is of the form

$$\mathcal{L}_I = J_\mu^{em} A_\mu \tag{10}$$

where the hadron electromagnetic current J_μ^{em} is assumed to be a sum of minimal-coupling type quark currents;

$$J_\mu^{em} = \sum_{quarks} :j_\mu^{(q)}: = \sum_q e_q :\bar{\psi}_q \gamma_\mu \psi_q: \tag{11}$$

This form of $j_\mu^{(q)}$ is appropriate to pointlike quarks with no anomalous magnetic moment term . The directly measurable electromagnetic matrix element of individual hadrons are

(1) diagonal matrix elements:

electric and magnetic form factors $G_E(q^2)$, $G_M(q^2)$

These may be interpreted nonrelativistically as Fourier transforms of the charge and magnetization densities of the hadron in question . In particular their values at $q^2=0$ are the electric charge and magnetic dipole moment of the hadron.

(2) off diagonal matrix elements:

electric and magnetic multipole transition moments

These are the matrix elements of the E.M. current J_μ^{em} between two different hadrons. The transition moments we shall calculate are for magnetic dipole transitions with $J^P = 1^+$, which are the best known experimentally.

Diagonal Matrix Elements

First we consider the diagonal matrix elements $\langle \text{hadron} | J_\mu^{em} | \text{hadron} \rangle$. The magnetic dipole moment $\vec{\mu}$ of a hadron is defined as its interaction with a static magnetic field;

$$H_I = -\vec{\mu} \cdot \vec{B} \tag{12}$$

In the Coulomb gauge $A_0 = 0$, $\vec{A} = -\frac{1}{2}\vec{B} \times \vec{r}$, so we find from $\mathcal{L}_I$;

$$H_I = -\mathcal{L}_I = -\frac{1}{2}(\vec{r} \times \vec{J}^{em}) \cdot \vec{B} \tag{13}$$

so $\quad \vec{\mu}_{had.} = \frac{1}{2} \vec{r} \times \vec{J}^{em}_{had.}$

and the total magnetic moment operator is $\vec{\mu} = \int d^3x \, \vec{\mu}(\vec{x})$,

$$\vec{\mu}_q = \frac{e_q}{2} \int d^3x \, \vec{r} \times (\bar{\psi}_q \vec{\gamma} \psi_q) \tag{14}$$

so

$$\vec{\mu}_{hadron} = \langle \text{hadron} | \int d^3x \sum_q \frac{1}{2} e_q \, \vec{r} \times (\bar{\psi}_q \vec{\gamma} \psi_q) | \text{hadron} \rangle \tag{15}$$

The $\vec{\mu}$ operator is diagonal in quark space so the total moment is just the weighted sum of individual quark moments. The required integral for free quark spherical wavefunctions gives the result;

$$\vec{\mu}_\pm(J_z) = J_z \frac{e k a f^{(j)}(x_0)}{x_0}\left[\frac{j+\frac{1}{2}}{j(j+1)}\right]\left[I(j\mp\tfrac{1}{2},x_0)+k^2 I(j\pm\tfrac{1}{2},x_0)\right]^{-1}\hat{z} \qquad (16)$$

where $f^{(j)}(x_0) \equiv \int_0^{x_0}\eta^3 j_{j-\frac{1}{2}}(\eta)\, j_{j+\frac{1}{2}}(\eta)\,d\eta$ for $|jm(\pm)\rangle$ modes

and $I(\ell,x_0)\equiv\int_0^{x_0}\eta^2 j_\ell(\eta)^2\, d\eta$

A special case that is easily solvable is $J^P=\frac{1}{2}^\pm$, the lowest modes;

$$\mu(\tfrac{1}{2}^+\uparrow) = \frac{eka}{6x_0}\left\{x_0(2+\cos 2x_0)-\tfrac{3}{2}\sin 2x_0\right\}\left[I(0,x_0)+k^2 I(1,x_0)\right]^{-1} \qquad (17)$$

$$\mu(\tfrac{1}{2}^-\uparrow) = \text{"}\ \{\qquad\text{"}\qquad\}\left[I(1,x_0)+k^2 I(0,x_0)\right]^{-1}$$

where $I(0,x_0)=\frac{x_0}{2}\left[1-j_0(2x_0)\right]$ $\qquad I(1,x_0)=\frac{x_0}{2}\left[1+j_0(2x_0)-2j_0(x_0)^2\right]$

We note that $[\mu]=\ell$ as required by dimensional analysis: for $m_q=0$, $\mu\propto a$. To illustrate the dependence of the quark magnetic moment on the parameters (x_0 , m_q) we have explicitly computed and plotted $\frac{2\mu}{ea}(x_0, m_q=0)$ in Fig. A and $\frac{2\mu}{ea}(x_0=2, m_q, a=1.4\ \text{fm})$ in Fig B.

The first graph also indicates the experimental value of μ(proton), which gives us an approximate value for a of $\sim$1.5 fm assuming $x_0\sim 2$. The fit to the remaining baryons will be discussed below.

The second graph shows the fairly strong dependence of $\mu(\frac{1}{2}^+)$ on the quark mass; a value of $m_q = 300\,\mathrm{MeV}$ cuts the quark moment at $m_q = 0$ by $\sim 50\%$. We shall see that this effect is observable in the $\Lambda(1115)$ baryon moment.

To obtain the hadron moments we form the matrix element $\langle \mathrm{hadron} | \vec{\mu} | \mathrm{hadron} \rangle$, which gives $\mu(\mathrm{hadron})$ as a sum over quark moments. Naive quark model calculations set $\mu_u = -2\mu_d = -2\mu_s$ proportional to the quark charges, but this approximation is unjustified in view of the strong m_q dependence of μ and the large mass of the strange quark. Keeping the moments separate, we obtain for the octet baryons;

$$\mu_P = \tfrac{1}{3}(4\mu_u - \mu_d) \qquad \mu_N = \tfrac{1}{3}(4\mu_d - \mu_u) \qquad \mu_\Lambda = \mu_s \tag{18,1-h}$$

$$\mu_{\Sigma^+} = \tfrac{1}{3}(4\mu_u - \mu_s) \qquad \mu_{\Sigma^0} = \tfrac{1}{3}(2\mu_u + 2\mu_d - \mu_s) \qquad \mu_{\Sigma^-} = \tfrac{1}{3}(4\mu_d - \mu_s)$$

$$\mu_{\Xi^0} = \tfrac{1}{3}(4\mu_s - \mu_u) \qquad \mu_{\Xi^-} = \tfrac{1}{3}(4\mu_s - \mu_d)$$

Clearly the strongest breaking of the SU(3) moment ratios due to $m_s \gg m_{u,d}$ will be observable in the Λ and the Ξ's.

To make some specific predictions we assume $a = 1.4\,\mathrm{fm}$, $m_u = m_d = 0$, $x_0 = 2$, and calculate the ratios μ (octet baryon)/μ_P as a function of m_s. The result of this calculation, together with the experimental data, is shown in Fig. C. The deviation of the well-known Σ^- and Λ

moments from the SU(6) results are correctly given to 1 std. deviation by $m_s \sim 200$ Mev, though the splittings shown are rather sensitive to the choice of χ_0; a value of $\chi_0 = 2.5$ changes the strange quark mass required to fit the μ_Λ from ~150 Mev to ~250 Mev. We also note that the predicted Ξ^- moment deviates further from the experimental result as we increase m_s. In view of the large error bars and the qualitatively correct SU(6) breaking of the Σ^- and Λ it is possible that the Ξ^- experiment was in error.

Now we consider diagonal matrix elements between hadrons with different momenta (form factors). In view of the lack of data for metastable baryons we shall consider only the nucleon form factors.

In eP elastic scattering the most general form of the proton vertex consistent with current conservation and Lorentz invariance is

$$e\Gamma_\mu^P = e\left[F_1(q^2)\gamma_\mu + \frac{i\kappa_P}{2M_P} F_2(q^2)\sigma_{\mu\nu} q_\nu \right] \tag{19}$$

$\kappa_P \equiv g_P - 1$

This gives the invariant amplitude

$$\mathcal{M} = e^2 \bar{u}_e(e_f)\gamma_\mu u_e(e_i)\, \bar{u}_P(p_f)\, \Gamma_\mu^P(p_f - p_i)\, u_P(p_i) \tag{20}$$

which, on squaring and summing over spins, leads to the well known Rosenbluth cross section[13]. The relation of the form factors F_1, F_2 to the electric and magnetic form factors which we shall calculate is

$$\begin{aligned} G_E(q^2) &= F_1(q^2) + \frac{q^2 \kappa_P}{(2M_P)^2} F_2(q^2) \\ G_M(q^2) &= F_1(q^2) + \kappa_P F_2(q^2) \end{aligned} \tag{21}$$

As previously stated, $G_E(0) = 1$ and $G_M(0) = \frac{\mu_p}{e} = \frac{g_p}{2M_p}$, the gyromagnetic ratio. The form factors are related to $\rho(\vec{x})$ and $\vec{j}(\vec{x})$ by[14]

$$G_E(q^2) = \left\{ \sum_q \int d^3x \, \rho_q(\vec{x}) e^{-i\vec{q}\cdot\vec{x}} \right\} / e_{proton} \tag{22}$$
$$G_M(q^2) = \left\{ \frac{1}{2|\vec{q}|^2} \sum_q (\vec{\sigma} \times \vec{\nabla}) \cdot \vec{j}_q(\vec{x}) e^{-i\vec{q}\cdot\vec{x}} \right\} / e_{proton}$$

where the charge and current densities are

$$\rho_q(\vec{x}) = \sum_q e_q \psi_q^\dagger \psi_q$$
$$\vec{j}_q(\vec{x}) = \sum_q e_q \bar{\psi}_q \vec{\gamma} \psi_q$$

First consider the electric form factors; for quarks in the $\frac{1}{2}^+$ mode we find for a single quark

$$G_E(q^2) = \frac{4\pi \alpha_+^2 a^3}{x_0^3} \int_0^{x_0} y^2 j_0\left(\frac{qa}{x_0} y\right) \left[j_0(y)^2 + k^2 j_1(y)^2 \right] dy \tag{23}$$

and for a hadron we have

$$G_E(q^2) = \left\{ \sum_q \int d^3x \, e^{-i\vec{q}\cdot\vec{x}} \langle \text{hadron} | \, e_q \psi_q^\dagger \psi_q \, | \text{hadron} \rangle \right\} / e_{proton} \tag{24}$$

For large q^2 the proton will not recoil coherently, so the G_E we calculate above cannot be compared with the experimentally measured form factor above $|q^2| \gtrsim 1\,\text{GeV}^2$. Since the nucleon form factors are crudely of the form $\frac{G(q^2)}{G(0)} \sim \left(1 - \frac{q^2}{.71}\right)^{-2}$ for $q^2 < 0$ we can see that only the first two terms in the expansion in q^2 are important for $q^2 \lesssim (.3\,\text{GeV})^2$. The first two terms are

$$G_E(q^2) = 1 - \frac{1}{6} r_Q^2 |q^2| + \ldots \qquad (25a,b)$$

where

$$r_Q^2 = \int d^3x \, r^2 \rho(r) =$$

(r.m.s. charge radius)^2.

The $G_E(q^2)$ integral may be carried out explicitly for $\frac{1}{2}^+$ quarks in the limit $q^2 \mapsto 0$ – the result is

$$r_Q^2 = \frac{4\pi \alpha_+^2 a^5}{x_0^5} \left[A(x_0) + k^2 B(x_0) \right] \qquad (26)$$

where

$$A(x_0) = \frac{x_0^3}{6} - \frac{x_0 \cos 2x_0}{4} - \frac{1}{4}\left(x_0^2 - \frac{1}{2}\right) \sin 2x_0$$

$$B(x_0) = \frac{x_0^3}{6} + \frac{x_0}{2} + \frac{3x_0 \cos 2x_0}{4} + \frac{1}{4}\left(x_0^2 - \frac{5}{2}\right) \sin 2x_0$$

For the proton we know experimentally that $r_Q(P) = .89 \pm .03$ fm.[15] If we assume massless quarks this gives a region in the (a, x_0) parameter space which is shown in Fig. D. Clearly, fitting the experimental $r_Q^2(P)$ does not constrain x_0 but instead gives us approximate limits on the proton radius $a \simeq 1.2 \pm .1$ fm. In this figure we also indicate the curve $a(x_0)$ determined by imposing the physical value of the proton gyromagnetic ratio $g_p = 2.793$.[16] The curves are somewhat inconsistent, but the best agreement is obtained by choosing a, x_0 to lie in the region $1.2\text{fm} \lesssim a \lesssim 1.4\text{fm}$, $2.0 \lesssim x_0 \lesssim 2.6$.

For q^2 larger than .25 Gev^2 we can't neglect terms of $O(q^4)$ in evaluating $G_E(q^2)$ and so we evaluate numerically the integral (24). The form factor $G_E(q^2)$ resulting from the choices $x_o=2$ (a=.75, 1., 1.25 fm) and $x_o=2.5$ (a=.75, 1., 1.25 fm) are shown in Figs. E and F respectively. It is apparent that for $|q| \gtrsim .4$ Gev the free quark model form factors are unsatisfactory - the experimental proton $G_E(q2)$, taken from Price et al [15], indicate that the real quark wavefunctions have much larger contributions from spatial frequencies $\lesssim$ $(.5\ \mathrm{fm})^{-1}$ than do the free quark wavefunctions. In a phrase, the free quark wavefunctions vary too slowly.

One might speculate that a moderate (~100 Mev) quark mass could sharpen the quark wavefunctions near their origin sufficiently to improve the medium q^2 behavior of the proton $G_E(q^2)$ form factor. This effect is shown to be insignificant in Fig. G, where we plot $G_E(q^2)$ for $x_o=2.$, a=1.25 fm, and $m_q=0$ and 500 Mev.

Now we consider the magnetic form factor. The messy integral given previously is derived from the result given by Sachs[14], which is equivalent to the operator equation

$$\int d^3x\, \vec{j}(\vec{x})\, e^{i\vec{q}\cdot\vec{x}} = ie\,\vec{\sigma}\times\vec{q}\; G_M(q^2) \tag{27}$$

where the two dimensional space on which the $\vec{\sigma}$ and $\vec{j}$ operate is the $j=\frac{1}{2}$ baryon spin space. For a spin $\uparrow$ baryon we have

$$\vec{V}(\vec{q}) \equiv \int d^3x\, \langle B,\uparrow|\,\vec{j}(\vec{x})\,|B,\uparrow\rangle\, e^{i\vec{q}\cdot\vec{x}} = ie\,\{-\sin\varphi_q, \cos\varphi_q, 0\}\,|\vec{q}|\sin\theta_q\, G_M(q^2) \tag{28}$$

in cartesian components.

This equation only holds if $V(q) = V(q)\,\hat{\varphi}_q$, so we have

$$\int d^3x\ \hat{\varphi}_q \cdot \langle B,\uparrow|\vec{j}(\vec{x})|B,\uparrow\rangle\, e^{-i\vec{q}\cdot\vec{x}} = ie\sin\theta_q\, |\vec{q}|\, G_M(q^2) \tag{29}$$

As an explicit example we consider a spin $\uparrow$ proton with CFQM quark wavefunctions. From the SU(6) decomposition of $|P,\uparrow\rangle$ we have

$$\begin{aligned}\langle P,\uparrow|\vec{j}(\vec{x})|P,\uparrow\rangle &= \tfrac{1}{3}\left\{5\langle u\uparrow|\vec{j}|u\uparrow\rangle + \langle u\downarrow|\vec{j}|u\downarrow\rangle + \langle d\uparrow|\vec{j}|d\uparrow\rangle + 2\langle d\downarrow|\vec{j}|d\downarrow\rangle\right\}\\ &= \tfrac{1}{3}\left\{4\langle u\uparrow|\vec{j}|u\uparrow\rangle - \langle d\uparrow|\vec{j}|d\uparrow\rangle\right\}\end{aligned} \tag{30}$$

One may evaluate the quark matrix elements explicitly to find

$$\langle q\uparrow|\vec{j}|q\uparrow\rangle = 2e_q k\alpha_+^2 j_0(kr) j_1(kr)\sin\theta\,\hat{\varphi} \equiv e_q f(kr)\sin\theta\,\hat{\varphi} \tag{31}$$

Then we have, assuming the u and d quark wavefunctions are the same,

$$\langle P,\uparrow|\vec{j}|P,\uparrow\rangle = e f(kr)\sin\theta\,\hat{\varphi} \tag{32}$$

In spherical harmonics and spherical components this is

$$\begin{aligned}\langle P,\uparrow|\vec{j}|P,\uparrow\rangle &= \sqrt{\tfrac{4\pi}{3}}\, ie f(kr)\left\{-Y_{11}\hat{e}_- + Y_{1-1}\hat{e}_+\right\}\\ &= -2\sqrt{\tfrac{2\pi}{3}}\, ie f(kr)\,\vec{Y}_{110}(\Omega)\end{aligned} \tag{33}$$

where

$$\hat{e}_\pm = \frac{1}{\sqrt{2}}\left(\mp\hat{x} - i\hat{y}\right)$$

We may now Fourier transform this vector;

$$\int d^3x \, \langle P,\uparrow | \vec{j}(\vec{x}) | P,\uparrow \rangle \, e^{i\vec{q}\cdot\vec{x}} = 4\pi\sqrt{\tfrac{4\pi}{3}} \, ie \sum_{\ell m} i^\ell \int r^2 dr \, f(kr) \, j_\ell(qr) \int d\Omega \cdot Y_{\ell m}(\Omega_q) Y^*_{\ell m}(\Omega) \{-Y_{11}\hat{e}_- + Y_{1-1}\hat{e}_+\}$$

$$= \frac{(4\pi)^{3/2}}{\sqrt{3}} \, ie \int r^2 f(kr) j_1(qr) \, dr \, \sqrt{\tfrac{3}{4\pi}} \sin\theta_q \, \hat{\varphi}_q \qquad (34)$$

Comparison of this result with (27) gives

$$G_M(q^2) = \frac{4\pi}{q} \int r^2 f(kr) j_1(qr) \, dr$$

where we have implicitly replaced $|\vec{q}|$ by q. Recalling the specific form for $f(kr)$ we found in (31) this becomes

$$G_M(q^2) = \frac{8\pi k d_+^2}{q} \int_0^a r^2 j_0(kr) j_1(kr) j_1(qr) \, dr$$

$$= \frac{8\pi k d_+^2 a^4}{qa \, x_0^3} \int_0^{x_0} \eta^2 j_0(\eta) j_1(\eta) j_1\!\left(\frac{qa}{x_0}\eta\right) d\eta \qquad (35)$$

We may check this result by taking the small qa limit;

$$G_M(q^2) \simeq \frac{8\pi k d_+^2 a^4}{3 x_0^4} \int_0^{x_0} \eta^3 j_0(\eta) j_1(\eta) \, d\eta = \frac{\mu_P}{e}$$

For massless quarks we may simplify the normalization of the integral in (35) slightly;

$$G_M(q^2) = \frac{2x_0}{q\,(x_0^2 - \sin^2 x_0)} \int_0^{x_0} \eta^2 j_0(\eta) j_1(\eta) j_1\!\left(\frac{qa}{x_0}\eta\right) d\eta$$

To compare this result with experiment we fix $a(x_o)$ such that $G_M(o) = \frac{\mu_p}{e}$ (expt.) which means that we stay on the curve shown in Fig. D. Choosing other values of a, x_o means that our curve starts at $G_M(o) \neq \frac{\mu_p}{e}$ (expt), though the functional dependence on q^2 will be qualitatively the same as the constrained curves. We have previously discussed $\mu_p(a, x_o)$, so we lose nothing by fixing $G_M(o)$ at a convenient value. The resulting $G_M(q^2)$ for various x_o and the experimental data are shown in Fig. H. If we introduce a quark mass of 500 Mev and calculate the effect on $G_M(q^2)$ as we did previously with $G_E(q^2)$ we find that the scale of the magnetic moment decreases significantly but that the explicit q^2 dependence does not appreciably change. (Fig. I). The same qualitative q^2 behavior with respect to m was seen in $G_E(q^2)$.

In conclusion, we find that our model predictions for both nucleon form factors fall too rapidly with q^2 for $|q^2| \gtrsim .25$ Gev^2, though they are not unreasonable for $|q^2| \lesssim .25$ Gev^2 and spacelike.

Off-Diagonal Electromagnetic Matrix Elements

Now we consider off-diagonal matrix elements of the electromagnetic current $\langle \text{hadron}' | J_\mu^{em} | \text{hadron} \rangle$ (36)

which to lowest order are responsible for transitions between two different baryons or mesons with the emission (or absorption) of a single photon. Expanding the photon field in $J_\mu^{em} A_\mu$ in states of definite J^P we find that the photon can have the following J^P;

$J = 1, 2, \ldots \qquad P = (-)^J \qquad$ electric multipole (2^j-pole) transitions

$J = 1, 2, \ldots \qquad P = (-)^{J+1} \qquad$ magnetic 2^j-pole transitions

The resulting selection rules on allowed J^P, $J'^{P'}$ for each case are obvious.

In the long wavelength approximation ($kr \ll 1$) the partial rate for 2 -pole transitions is of the form ;

$$\Gamma\,(2^{\ell}\text{ pole electric}) \sim E_\ell\, \alpha\, (ka)^{2\ell}\, k$$
$$\Gamma\,(2^{\ell}\text{ pole magnetic}) \sim M_\ell\, \alpha\, (ka)^{2\ell}\, k \qquad (37)$$

where a is the approximate extent of the hadron, k is the photon energy, and E_ℓ and M_ℓ are functions of ℓ which decrease rapidly with increasing

ℓ. To get an idea of the rates we expect for each multipole we can plug in some typical numbers for a (1.2 fm) and k_γ (300 Mev) in a table reproduced in Ref.17 with the result:

Order of magnitude widths for multipole transitions; $k_\gamma =$ 300 Mev, a=1.2 fm, Γ in Mev, for $B^* \mapsto B\gamma$ transitions.

multipole	electric	magnetic (assuming $g \cong 3$)
$\ell = 1$	1^-; 2 Mev	1^+; .5 Mev
$\ell = 2$	2^+; .1 Mev	2^-; .05 Mev
$\ell = 3$	3^-; .01 Mev	3^+; .002 Mev

Since these numbers are derived assuming only the first term in the photon radial wavefunction $j_\ell(kr) \sim \frac{(kr)^{2\ell+1}}{(2\ell+1)!!}$ is important, they are clearly only very crude guesses for our case with ka $\sim$ 2. Leaving out the higher terms of order $(kr)^{\ell+2}$ as we have done overestimates the size of the interaction $j_\mu(x) A_\mu(x)$ for $r \sim a$, especially for large ℓ (assuming the quark e.m. current j_μ doesn't vary rapidly with r). We shall consider the effect of using the correct photon and quark wavefunctions to calculate the transition rates later.

From the table above it is clear that electric dipole radiative transitions are the most important when they are not forbidden by a selection rule. The available data, however, are concerned with transitions between the lowest lying baryon octet and decimet ($B_{10}(\frac{3}{2}^+) \mapsto B_8(\frac{1}{2}^+)\gamma$) and the vector and pseudoscalar meson nonets ($M_{8+1}(1^-) \mapsto M_{8+1}(0^-)\gamma$). Clearly only (+) parity transitions are allowed, so electric dipole transitions are excluded; the dominant matrix element for the observed photon emissions are magnetic dipole

(1^+) transitions, with perhaps some contribution from the electric quadrupole (2^+) term.* In the long wavelength approximation the transition rate for the decay $A \mapsto B\gamma(1^+)$ where A is unpolarized is

$$\Gamma\left(\text{hadron A} \longmapsto \text{hadron B}\gamma(1^+)\right) = \sum_f \frac{4}{3}|\vec{\mu}_{fi}|^2 k^3_{\gamma,f} \qquad (38)$$

$\sum_f$ means sum over $|\text{hadron B}\rangle$ polarizations

$\vec{\mu}_{fi}$ is the transition magnetic moment, i.e. the expectation value of the magnetic moment operator $\vec{\mu}$ between the two hadron states;

$$\vec{\mu}_{fi} = \langle \text{hadron A}_i | \sum_q \frac{1}{2}\int d^3x\, e_q \vec{r} \times \bar{\psi}_q \vec{\gamma} \psi_q \; | \text{hadron B}_f \rangle \qquad (39)$$

Assuming that we may neglect recoil effects this is just an integral over quark wavefunctions at rest. We may now consider specific reactions.

(1) $\Delta^+ \longmapsto P\gamma$

The experimental value for the partial width of this reaction is[18] .65 ± .04 MeV.

First we shall calculate the partial width using the naive SU(6) quark model, then we shall do the same calculation using explicit quark wavefunctions.

* If the decay is caused by a single quark $\mapsto$ quark transition in $\frac{1}{2}^+$ modes we of course have $\Delta\ell = 1$ only and expect pure magnetic dipole transitions.

Assume the Δ^+ has $j_z = 3/2$, then the P has $j_z = \frac{1}{2}$ and the γ $j_z = 1$. The transition moment between these states is

$$\vec{\mu}_{\frac{3}{2}\frac{1}{2}} = \langle \Delta^+, \tfrac{3}{2} | \vec{\mu} | P, \tfrac{1}{2} \rangle \tag{40}$$

$$|\Delta^+, \tfrac{3}{2}\rangle = \frac{1}{\sqrt{3}} \left\{ |u\uparrow u\uparrow d\uparrow\rangle + \text{et. cyc.} \right\} \tag{41}$$

$$|P, \tfrac{1}{2}\rangle = \frac{1}{\sqrt{18}} \left\{ 2\left(|u\uparrow d\downarrow u\uparrow\rangle + \text{et. cyc.}\right) - \left(|u\uparrow u\downarrow d\uparrow\rangle + \text{all perms.}\right) \right. \tag{42}$$

so $$\vec{\mu}_{\frac{3}{2}\frac{1}{2}} = \sqrt{\tfrac{2}{3}} \left\{ \langle d\uparrow | \vec{\mu} | d\downarrow\rangle - \langle u\uparrow | \vec{\mu} | u\downarrow\rangle \right\} \tag{43}$$

assuming $$_{\Delta}\langle q | q' \rangle_P = \delta(q, q') \tag{44}$$

This is not a trivial assumption, since quark wavefunctions for hadrons of different radii (in the free quark model, for example) or of different momenta will not necessarily be orthonormal. Our assumption $_{\Delta}\langle q | q'\rangle_P = \delta(q,q')$ here implies that we neglect recoil and that the quark wavefunctions in the $|\Delta\rangle$ and $|P\rangle$ are the same.

In the naive quark model the quark wavefunctions are just two component Pauli spinors; $u(\uparrow) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $u(\downarrow) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, and the magnetic moment operator is taken to be [19]

$$\vec{\mu}_q = \frac{e_q}{e} \mu \vec{\sigma}_q \tag{45}$$

where μ is an undetermined scale. (Note that we found previously* that the assumption $|\vec{\mu}_q| \propto e_q$ made here is unable to account for the observed strange baryon magnetic moments). With this ansatz we find

$$\vec{\mu}_{\frac{3}{2}\frac{1}{2}} = -\sqrt{\tfrac{2}{3}}\,\mu\,\{1,-i,0\} \tag{46}$$

Then
$$\Gamma(\Delta^+_{\frac{3}{2}} \mapsto P_{\frac{1}{2}}\gamma) = \frac{16}{9}\mu^2 k_\gamma^3 \tag{47}$$

with $k_\gamma = \left[\dfrac{M_\Delta^2 - M_P^2}{2M_\Delta}\right]$

μ may be related to the proton moment by the simple observation of (18a)

$$\mu_P = \langle P,\tfrac{1}{2}|\,\mu_3\,|P,\tfrac{1}{2}\rangle = \tfrac{1}{3}(4\mu_u - \mu_d) \quad ; \text{ using (45),}$$

$$= \tfrac{4}{3}\left(\tfrac{2}{3}\mu\right) - \tfrac{1}{3}\left(-\tfrac{1}{3}\mu\right) = \mu$$

So we predict for the M1 decay rate of $\Delta^+_{3/2} \longmapsto P_{1/2}\gamma$,

$$\Gamma_{3/2\,1/2} = \frac{16}{9}\mu_P^2 k_\gamma^3 \tag{46}$$

Physical numbers are M = 1211. Mev. M_P = 939. Mev, k = 241. Mev,

* See pp.10, equations (18 a-h) and Fig. C.

$\mu_p = 2.793 \frac{e}{2M_p}$ so we obtain (with $e^2 = \alpha$)

$$\Gamma_{\frac{3}{2}\frac{1}{2}}(\Delta^+ \mapsto p\gamma) = 402 \text{ Kev} \tag{47}$$

We may obtain the other matrix elements using the Wigner-Eckart theorem, once we show that the magnetic moment operator $\vec{\mu}$ transforms as an irreducible vector under the rotation group:

An irreducible tensor $T_m^{(\ell)}$ with respect to SO(3) satisfies the commutation relations

$$\begin{aligned} [J_\pm, T_m^{(\ell)}] &= [\ell(\ell+1) - m(m\pm 1)]^{1/2} T_{m\pm 1}^{(\ell)} \\ [J_z, T_m^{(\ell)}] &= m T_m^{(\ell)} \end{aligned} \tag{48}$$

If these are satisfied the matrix elements of the irreducible tensor operator $T^{(\ell)}$ between any two basis vectors of irreducible representations of the rotation group will be of the form

$$\langle jm | T_\mu^{(\ell)} | j'm' \rangle = \frac{\langle j \| T^{(\ell)} \| j' \rangle}{\sqrt{2j+1}} \langle \ell\mu ; j'm' | jm \rangle \tag{49}$$

where $\langle j \| T^{(\ell)} \| j' \rangle$ is the reduced matrix element of T.
Recall the definition of $\vec{\mu}$:

$$\mu_i = \frac{1}{2} \epsilon_{ijk} x_j J_k \tag{50}$$

First we infinitesimally rotate $\vec{j}$;
under an infinitesimal rotation about the z-axis,

$$\psi'(\vec{x}') \simeq \psi(\vec{x}) + \frac{i}{2}\omega\sigma_{12}\psi(\vec{x}) \tag{51}$$

$$\vec{j}'(\vec{x}') \simeq e\psi^{\dagger}(\vec{x})\left[1-\frac{i}{2}\omega\sigma_{12}\right]\vec{\alpha}\left[1+\frac{i}{2}\omega\sigma_{12}\right]\psi(\vec{x})$$

$$= \vec{j}(\vec{x}) + \omega\left(j_2, -j_1, 0\right) \tag{52}$$

so

$$R_\omega j_i R_{\omega^{-1}} \simeq (j_1+\omega j_2, j_2-\omega j_1, j_3)_i \sim j_i - i\omega \mathcal{D}^{(1)}_{ji}(J_z) j_j \tag{53}$$

which is equivalent to (48).

This must be satisfied if $\vec{j}$ transforms as an irreducible vector under the rotation group. ($D^{(1)}(w)$ is the 3-dimensional matrix rep. of a rotation about the z-axis and $D^{(1)}(J_z)$ is the 3-dimensional rep. of J_z. If we change the components of j to the spherical components $j_\pm$, $j_o = j_3$, we find that our infinitesimal rotation becomes

$\delta j_\pm = \pm i\omega j_\pm$, $\delta j_0 = 0$ This means the matrix representing J_z in the basis we have chosen is

$$\mathcal{D}^{(1)}(J_z) = \begin{bmatrix} 1 & & \\ & 0 & \\ & & -1 \end{bmatrix} \tag{55}$$

a little more work gives us the matrix representatives of J_x and J_y:

$$\mathcal{D}^{(1)}(J_x) = \frac{1}{\sqrt{2}}\begin{bmatrix} 0 & 1 & \\ 1 & 0 & 1 \\ & 1 & 0 \end{bmatrix} \qquad \mathcal{D}^{(1)}(J_y) = \frac{1}{\sqrt{2}}\begin{bmatrix} 0 & -i & \\ i & 0 & -i \\ & i & 0 \end{bmatrix} \tag{54,c}$$

These are the familiar spin-1 reps. of the rotation group generators, so the spherical components of $\vec{J}$ indeed transform as an irreducible spin-1 tensor under the rotation group.

It is trivial to show that the components of μ transform similarly, so we now consider how $\vec{\mu} = \frac{1}{2}\vec{r}\times\vec{j}$ transform under an infinitesimal rotation; for a rotation about the z-axis,

$$\delta\mu_1 = \tfrac{1}{2}(\delta y\, j_0 + y\,\delta j_0 - \delta z\, j_2 - z\,\delta j_2) = \tfrac{\omega}{2}(-x j_0 + z j_1) = \omega\mu_2$$

$$\delta\mu_2 = -\omega\mu_1 \qquad\qquad \delta\mu_3 = 0 \tag{56}$$

These are just the transformation laws we found for j, so by analogy we can form a spin-1 irreducible tensor operator out of $\vec{\mu}$;

$$\vec{\mu}^{(1)} = \begin{bmatrix} -\frac{1}{\sqrt{2}}(\mu_1 + i\mu_2) \\ \mu_3 \\ \frac{1}{\sqrt{2}}(\mu_1 - i\mu_2) \end{bmatrix} = \begin{bmatrix} \mu_+ \\ \mu_0 \\ \mu_- \end{bmatrix} \tag{57}$$

Having demonstrated this well known fact we return to the problem of calculating the matrix elements of $\vec{\mu}$ between all possible polarization states of $|\Delta^+\rangle$ and $|P\rangle$

The Wigner-Eckart theorem (49) tells us that all these matrix elements may be determined from a single reduced matrix element;

$$\langle \Delta^+, m \,|\, \mu_\nu^{(1)} \,|\, P, m' \rangle = \frac{1}{2} \langle \tfrac{3}{2} \| \mu^{(1)} \| \tfrac{1}{2} \rangle \langle \tfrac{1}{2} m'; 1 \nu \,|\, \tfrac{3}{2} m \rangle \tag{58}$$

one previously determined matrix element (46) gives us

$$\langle \tfrac{3}{2} \| \mu^{(1)} \| \tfrac{1}{2} \rangle = \frac{4}{\sqrt{3}} \mu_p \tag{59}$$

so the remaining matrix elements are

$$\langle \Delta^+, m \,|\, \mu_\nu^{(1)} \,|\, P, m' \rangle = \frac{2}{\sqrt{3}} \mu_p \langle \tfrac{1}{2} m'; 1 \nu \,|\, \tfrac{3}{2} m \rangle \tag{60}$$

explicitly this gives

$$\begin{aligned}
\langle \tfrac{1}{2} | \mu_\nu^{(1)} | \tfrac{1}{2} \rangle &= \delta_{\nu 0} \frac{2\sqrt{2}}{3} \mu_p \\
\langle \tfrac{1}{2} | \mu_\nu^{(1)} | -\tfrac{1}{2} \rangle &= \delta_{\nu 1} \frac{2}{3} \mu_p \\
\langle -\tfrac{1}{2} | \mu_\nu^{(1)} | \tfrac{1}{2} \rangle &= \delta_{\nu -1} \frac{2}{3} \mu_p \\
\langle -\tfrac{1}{2} | \mu_\nu^{(1)} | -\tfrac{1}{2} \rangle &= \delta_{\nu 0} \frac{2\sqrt{2}}{3} \mu_p \\
\langle -\tfrac{3}{2} | \mu_\nu^{(1)} | -\tfrac{1}{2} \rangle &= \delta_{\nu -1} \frac{2}{\sqrt{3}} \mu_p
\end{aligned} \tag{61, a-e}$$

We may now return to (46) and calculate the naive SU(6) quark model rate for unpolarized $\Delta^+ \mapsto P\gamma$,

$$\Gamma = \frac{4}{3}\frac{\mu_p^2}{(2\cdot\frac{3}{2}+1)} k_\gamma^3 \left\{2\cdot\frac{4}{3}+2\cdot\frac{8}{9}+2\cdot\frac{4}{9}\right\} = \frac{16}{9}\mu_p^2 k_\gamma^3 = 402\ \mathrm{Kev} \tag{62}$$

which is just the rate for polarized $\Delta^+_{\frac{3}{2}} \mapsto P_{\frac{1}{2}}\gamma$ decay. This teaches us that we need not do the sum over all polarization states to obtain the decay rate, since the isotropy of space insures that a particle with spin up decays just as fast as one "standing on his head."

Now we calculate the $\Delta^+ \mapsto P\gamma$ lifetime using the free quark model wavefunctions. If we again assume that the initial $|\Delta^+\rangle$ has $j_z = 3/2$ we find for the transition moment

$$\vec{\mu}_{\frac{3}{2}\frac{1}{2}} = \frac{1}{2}\int d^3x\, \vec{r} \times \sum_q e_q \langle \Delta^+, \tfrac{3}{2} | \bar{\psi}_q \vec{\gamma} \psi_q | P, \tfrac{1}{2} \rangle \tag{63}$$

Inserting explicit $\frac{1}{2}^+$ wavefunctions and assuming that the quark masses and hadron model radii are all equal gives the result

$$\vec{\mu}_{\frac{3}{2},\frac{1}{2}} = -\sqrt{\frac{2}{3}}\int d^3x \left\{ {}_\Delta\langle u\uparrow | \overline{\mu(x)} | u\downarrow \rangle_P - {}_\Delta\langle d\uparrow | \overline{\mu(x)} | d\downarrow \rangle_P \right\} \tag{64}$$

If the quark wavefunctions are the same in both cases, $|\rangle_p = |\rangle_\Delta$, this gives

$$\vec{\mu}_{\frac{3}{2}\frac{1}{2}} = -\sqrt{\tfrac{2}{3}}\,\tfrac{4\pi}{3}\, e\,\tilde{I}(a,x_0)\,\{-i,-1,0\}$$ for the Cartesian comps. of $\vec{\mu}$, or (65)

$$\mu^{(1)}_{\frac{3}{2}\frac{1}{2}} = -\frac{8\pi i}{3\sqrt{3}}\, e\,\tilde{I}(a,x_0)\{1,0,0\}$$ for the spherical comps. of $\vec{\mu}^{(1)}$

where

$$\tilde{I}(a,x_0) \equiv \frac{2\alpha_+^2 k a^4}{x_0^4}\int_0^{x_0} y^3 j_0(y)\, j_1(y)\, dy. \tag{66}$$

This gives for the width $\Delta \mapsto P\gamma$ (38)

$$\Gamma_{\Delta^+ \mapsto p\gamma} = \frac{1024\pi^2}{81}\, k_\gamma^3\, \frac{(\alpha_+^2 a^3)^2 k^2 a^2 e^2}{x_0^8}\left[\int_0^{x_0} y^3 j_0(y)\, j_1(y)\, dy\right]^2 \tag{67}$$

For massless quarks $k=1$, $\alpha_+^2 a^3 = \frac{x_0^2}{4\pi}\left[1 - j_0(x_0)^2\right]^{-1}$, and the above result simplifies to

$$\Gamma_{\Delta^+ \mapsto p\gamma} = \left(\frac{2}{9}\right)^2 \alpha\, k_\gamma^3 a^2 \left[\frac{x_0(2+\cos 2x_0) - \frac{3}{2}\sin 2x_0}{x_0^2 - \sin^2 x_0}\right]^2 \tag{68}$$

Since the same integral appeared in the magnetic moment calculation, we may once again write $\Gamma(\Delta^+ \mapsto p\gamma)$ in terms of $\mu_p^2 k_\gamma^3$, where the μ_p is now the value calculated from the free quark wavefunctions. (The naive SU(6) calculation takes μ_p as input.)

The result is

$$\Gamma(\Delta^+ \mapsto p\gamma) = \frac{16}{9} \mu_p^2 k_\gamma^3 \tag{69}$$

which is exactly the SU(6) rate as $\Gamma(\mu_p, k_\gamma)$. Carrying out the above calculation with massive quarks gives the same result. The fact that these calculations all give the above result is not surprising, since we are calculating the transition magnetic moment between $|\Delta^+\rangle$ and $|P\rangle$ assuming that the up(down) quark wavefunctions in the $|\Delta^+\rangle$ and $|P\rangle$ are identical. This assumption implies $|\vec{\mu}_{\Delta P}| \propto |\vec{\mu}_p|$ where the constant of proportionality is an SU(6) Clebsch-Gordon coefficient that doesn't depend on the structure of the quark wavefunctions.

Experimentally we know

$$\frac{\Gamma(\Delta^+ \mapsto p\gamma)}{\frac{16}{9} \mu_p^2 k_\gamma^3} = \frac{9 M_p^2 \,\Gamma(\Delta^+ \mapsto p\gamma)}{4 g_p^2 \,\alpha\, k_\gamma^3} \simeq 1.44 \tag{70}$$

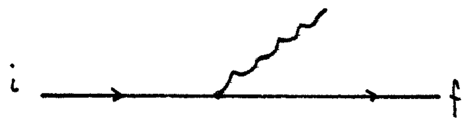
instead of 1., with an uncertainty of about 10% due to uncertainty in the Δ^+ total width. This test of the SU(6) quark model is independent of the assumed quark wave functions, so one or more of our other assumptions must be in significant error. These assumptions are:

1) long wavelength approximation for the emitted photon
2) neglect of recoil effects
3) equivalence of $|u\rangle_\Delta, |u\rangle_p$; $|d\rangle_\Delta$, and $|d\rangle_p$ quark wavefunctions
4) neglect of whatever amplitude the $|P\rangle$ and $|\Delta^+\rangle$ have to be in exotic (other than $|qqq\rangle$) states.

First we shall see how serious the long wavelength approximation was by recalculating the decay rate without assuming $k_\gamma a \ll 1$. This assumption is especially suspect because $k_\gamma a \sim 1.5$, although the decreasing behavior of $\frac{j_1(x)}{x}$ as x increases suggests that the long wavelength approx. led us to overestimate $\Gamma(\Delta^+ \mapsto P\gamma)$. This implies that $\frac{\Gamma}{\frac{16}{9}\mu_p^2 k_\gamma^3}$ will be <u>less</u> than one when done correctly, giving even poorer agreement with the experimental value of ~ 1.44. We shall see that this premonition is indeed correct.

The $\mathcal{S}$-matrix element for single photon emission from a spin-$\frac{1}{2}$ fermion is [20]

$$\mathcal{S}_{fi} = i\int d^4x\, \mathcal{L}_I(x) = -i\sqrt{4\pi\alpha}\int d^4x\, \bar{\psi}_f(x)\, \not{A}(x)\, \psi_i(x) \; * \qquad (71)$$



* We note that two conventions for e are in common usage; "rationalized", $e^2/4\pi = \alpha$ (e.g. refs. 12,19) and "unrationalized", $e^2 = \alpha$ (e.g. refs. 16,18). Using α directly avoids this ambiguity.

The photon wavefunction A_μ is a plane wave normalized to $\frac{1 \text{ photon}}{\text{unit volume}}$,

$$A_\mu(x) = \frac{\epsilon_\mu}{\sqrt{2k_\gamma V}}\left(e^{ik_\gamma \cdot x} + e^{-ik_\gamma \cdot x}\right) \tag{72}$$

where the polarization vector satisfies $\epsilon_\mu k_\mu = 0, \quad \epsilon^2 = -1.$ Without loss of generality we may choose the two independent polarization vectors to be totally transverse;

$$\epsilon_\mu^{(1)} = (0, \vec{\epsilon}^{(1)}) \quad , \epsilon_\mu^{(2)} = (0, \vec{\epsilon}^{(2)}) \quad ; \ \vec{\epsilon}^{(1)} \perp \vec{\epsilon}^{(2)} \perp \hat{k}_\gamma \tag{73}$$

The transition current $e\bar{\psi}_f \vec{\gamma} \psi_i = \vec{J}_{fi}$ is easily evaluated using the CFQM $\frac{1}{2}^+$ quark wavefunctions (5); for the calculation of the rate $\Delta^+_{\frac{3}{2}} \mapsto P_{\frac{1}{2}}\gamma$ we require only the quark spin-flip transition current

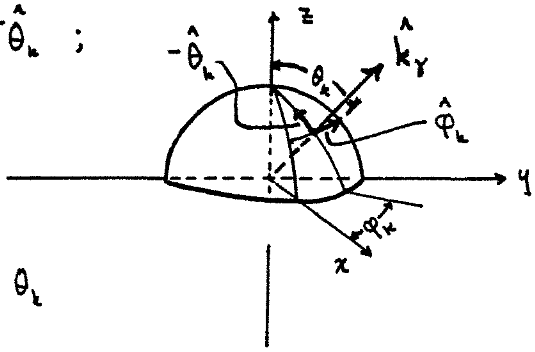
$$\vec{J}_{\downarrow\uparrow}(\vec{x}) = \sqrt{4\pi\alpha}\, \bar{\psi}_{f\downarrow} \vec{\gamma} \psi_{i\uparrow} = 2\sqrt{4\pi\alpha}\, i\, \alpha_+^2 k\, j_0(kr) j_1(kr) \{\cos\theta, i\cos\theta, -\sin\theta e^{i\varphi}\} \tag{74}$$

The outgoing photon has $\hat{k}_\gamma = \hat{r}$ so we may choose the photon polarization vectors to be $\hat{e}_1 = \hat{\varphi}_k \ , \hat{e}_2 = \hat{\theta}_k$;

so

$$\hat{\epsilon}^{(1)} = -\hat{x}\sin\varphi_k + \hat{y}\cos\varphi_k$$

$$\hat{\epsilon}^{(2)} = -\hat{x}\cos\theta_k\cos\varphi_k - \hat{y}\cos\theta_k\sin\varphi_k + \hat{z}\sin\theta_k$$



(75)

and the projection of the transition current along each of these is

$$\vec{j}\cdot\hat{\epsilon}^{(1)} = -\sqrt{\frac{4\pi}{3}}\,g(r)\,Y_{10}(\Omega)\,e^{i\varphi_k}$$

$$\vec{j}\cdot\hat{\epsilon}^{(2)} = -i\sqrt{\frac{4\pi}{3}}\,g(r)\left\{Y_{10}(\Omega)\cos\theta_k e^{i\varphi_k} - \sqrt{2}\,Y_{11}(\Omega)\sin\theta_k\right\} \tag{76}$$

where $g(r) = 2\sqrt{4\pi d}\,\alpha_+^2\,k\,j_0(kr)\,j_1(kr)$

Expanding the plane wave $e^{-i\vec{k}_\gamma\cdot\vec{x}} = 4\pi\sum_{lm}(-i)^l\,Y_{lm}(\Omega_{k_\gamma})\,Y_{lm}^*(\Omega)\,j_l(k_\gamma r)$ and inserting it in the S-matrix (71), assuming $\omega_f < \omega_i$ gives us the result

$$S_{fi}^{(\lambda)} = \frac{8\pi^2\delta(\omega_f+\omega-\omega_i)}{\sqrt{2k_\gamma V}}\int d^3x\;\vec{j}_{fi}\cdot\hat{\epsilon}^{(\lambda)}\left\{Y_{10}(\Omega_{k_\gamma})Y_{10}(\Omega)^* + Y_{11}(\Omega_{k_\gamma})Y_{11}(\Omega)^*\right\} j_1(k_\gamma r) \tag{77}$$

Other photon angular momenta clearly don't contribute, as one sees from (76). Explicitly doing the integral for each polarization state gives

$$S_{fi}^{(1)} = -\frac{8\pi^2}{\sqrt{2k_\gamma V}}\int_0^a r^2 g(r)\,j_1(k_\gamma r)\,dr\,\cos\theta_k\,e^{i\varphi_k}\,\delta(\omega_f+\omega-\omega_i) \equiv -\frac{\Lambda 2\pi}{\sqrt{V}}\cos\theta_k\,e^{i\varphi_k}\,\delta(\;)$$

$$S_{fi}^{(2)} = -\frac{i\Lambda}{\sqrt{V}}\,e^{i\varphi_k}\,2\pi\,\delta(\omega_f+\omega-\omega_i)$$

where

$$\Lambda = \frac{4\pi}{\sqrt{2k_\gamma}}\int_0^a r^2 g(r)\,j_1(k_\gamma r)\,dr$$

The rate to shoot off a photon of each polarization is then as usual

$$\Gamma^{(\lambda)} = V \cdot \underbrace{\frac{|S_{fi}^{(\lambda)}|^2}{VT}}_{\text{rate/vol}} \cdot \underbrace{\delta_{fi}}_{\text{fermion phase space}} \cdot \underbrace{V\frac{d^3k_\gamma}{(2\pi)^3}}_{\text{photon phase space}} \tag{78}$$

Doing the substitutions for S_{fi} gives

$$\Gamma^{(1)} = \frac{k_\gamma^2 \Lambda^2}{3\pi}$$

Recall $\Lambda = \frac{4\pi}{\sqrt{2k_\gamma}} \int_0^a r^2 g(r) j_1(k_\gamma r)\, dr$

so $\Gamma^{(1)} = \frac{8\pi}{3} k_\gamma \left[\int_0^a r^2 g(r) j_1(k_\gamma r)\, dr \right]^2$ (79)

$\hat{\varphi}_k$ polarized photon emission width

Similarly we may obtain the width for photon emission with $\hat{\epsilon}^{(2)} = \hat{\theta}_k$;

$$\Gamma^{(2)} = 3\Gamma^{(1)} \tag{80}$$

So the total width for $q\uparrow \mapsto q\downarrow \gamma$ is

$$\Gamma = \frac{32\pi}{3} k_\gamma \left[\int_0^a r^2 g(r) j_1(k_\gamma r)\, dr \right]^2 \tag{81}$$

In the long wavelength approximation this gives

$$\Gamma \simeq \frac{32\pi}{27} k_\gamma^3 \left[\int_0^a r^3 g(r)\, dr \right]^2 \tag{82}$$

We can easily check this unlikely looking result - we know the long wavelength M1 dipole transition rate from (38)

$$\Gamma = \frac{4}{3} |\vec{\mu}_{fi}|^2 k_\gamma^3$$

and we know the transition magnetic moment from (50)

$$\vec{\mu}_{fi} = \frac{1}{2} \int d^3x\, \vec{r} \times \vec{J}_{fi}$$

where $\vec{J}_{fi}$ is given by (74). A little work gives us

$$\vec{\mu}_{fi} = \frac{2\sqrt{\pi}}{3} \int_0^a r^3 g(r)\, dr \quad \{1, i, 0\} \tag{83}$$

in cartesian components, so

$$|\vec{\mu}_{fi}|^2 = \frac{8\pi}{9} \left[\int_0^a r^3 g(r)\, dr \right]^2 \tag{84}$$

and finally

$$\Gamma = \frac{32\pi}{27} k_\gamma^3 \left[\int_0^a r^3 g(r)\, dr \right]^2 \tag{85}$$

which is exactly the long wavelength limit found in (82).

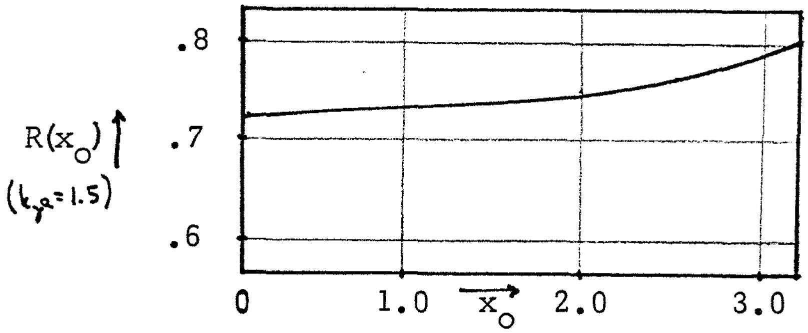
We are primarily interested in the importance of the long wavelength assumption, so we just look at the ratio of the widths Γ and Γ_0 calculated with and without $k_\gamma r \ll 1$ assumed;

$$R \equiv \frac{\Gamma(k_\gamma r)}{\Gamma_0(k_\gamma r \ll 1)} = \frac{9x_0^2}{(k_\gamma a)^2}\left[\frac{\int_0^{x_0} \eta^2 j_0(\eta) j_1(\eta) j_1\left(\frac{k_\gamma a}{x_0}\eta\right) d\eta}{\int_0^{x_0} \eta^3 j_0(\eta) j_1(\eta) d\eta}\right]^2 \tag{86}$$

For a specific case we again consider $\Delta^+ \mapsto p\gamma$, with $k_\gamma a \sim 1.5$, which gives

$$R(x_0) \simeq 4x_0^2\left[\frac{\int_0^{x_0} \eta^2 j_0(\eta) j_1(\eta) j_1\left(\frac{3\eta}{2x_0}\right) d\eta}{\int_0^{x_0} \eta^3 j_0(\eta) j_1(\eta) d\eta}\right]^2 \tag{87}$$

Evaluating this function for $0 < x_0 \leq 3$ and $k_\gamma a = 1.5$, we find the following behavior;



So we find that not making the long wavelength approximation leads to a lower prediction of the width for $\Delta^+ \mapsto p\gamma$;

$$\Gamma(\Delta^+ \mapsto p\gamma) = R(x_0)\,\Gamma(\Delta^+ \mapsto p\gamma \text{ assuming } k_\gamma r \ll 1)$$

with $R(x_0) \sim .75-.8$ for reasonable values of x_0 (2-3). The correction factor $R(x_0)$ is model dependent, so we can no longer make a wave-function independent statement analogous to (69), though the slow dependence of $R(x_0)$ on x_0 tells us that the short wavelength correction to (69) will be about a 20-30% decrease unless the wavefunctions are radically different than the CFQM functions.

Our result for the CFQM functions is then

$$\Gamma(\Delta^+ \mapsto p\gamma) = R(x_0)\,\frac{16}{9}\,\mu_p^2 k_\gamma^3 \tag{88}$$

or

$$\frac{9M_p^2\,\Gamma(\Delta^+ \mapsto p\gamma)}{4g_p^2\alpha\, k_\gamma^3} = R(x_0) \sim .75-.8$$

which is worse than the long wavelength result of 1. (Recall that the experimental value is 1.4-1.5.) Clearly we have found an effect that can't be neglected, but we have also found that including it makes our agreement with experiment poorer.

Now we consider some meson magnetic dipole decays. The available data are concerned with the decays of the low lying $\ell = 0$ vector nonet $(\rho, \varphi, \omega, K^*)$ into the pseudoscalar nonet (π, η, η', K) - as of 1975 the available data are: [21-25]

reaction	Γ (kev)	k_γ (Mev)
$\rho^- \mapsto \pi^- \gamma$	35 ± 10	372.
$K^{*+} \mapsto K^+ \gamma$	< 80	309.
$K^{*0} \mapsto K^0 \gamma$	75 ± 35	307.
$\omega \mapsto \pi^0 \gamma$	870 ± 80	380.
$\omega \mapsto \eta \gamma$	< 50	200.
$\varphi \mapsto \pi^0 \gamma$	5.7 ± 2.1	501.
$\varphi \mapsto \eta \gamma$	65 ± 15	362.
$\eta' \mapsto \rho \gamma$	< 270	170.
$\eta' \mapsto \omega \gamma$	< 80	158.

First we shall calculate the rates for $\omega \mapsto \pi^0 \gamma$, and $\rho^- \mapsto \pi^- \gamma$ using the naive quark model just to be certain that we understand how to do meson decays. As with $\Delta^+ \mapsto P\gamma$ we choose a particular polarization channel $(1^-)_{+1} \longmapsto (0^-)_0 (\gamma)_{+1}$ and use the long wavelength approximation;

$$\Gamma_{fi} = \sum_f \frac{4}{3} |\vec{\mu}_{fi}|^2 k_\gamma^3 \tag{38}$$

where

$$\vec{\mu}_{fi} = \langle 1^-, j_z = +1 | \vec{\mu} | 0^- \rangle \tag{89}$$

is the usual transition moment. The naive SU(6) quark decompositions for the mesons we are considering are:

$$|\omega\rangle^{*}_{+1} = \frac{1}{\sqrt{2}}\left(|u\uparrow\bar{u}\uparrow\rangle + |d\uparrow\bar{d}\uparrow\rangle\right) \qquad |\rho^{-}\rangle_{+1} = |u\uparrow\bar{d}\uparrow\rangle \tag{90}$$

$$|\pi^{0}\rangle = \frac{1}{2}\left(|u\uparrow\bar{u}\downarrow\rangle - |u\downarrow\bar{u}\uparrow\rangle - |d\uparrow\bar{d}\downarrow\rangle + |d\downarrow\bar{d}\uparrow\rangle\right)$$

$$|\pi^{-}\rangle = \frac{1}{\sqrt{2}}\left(-|d\downarrow\bar{u}\uparrow\rangle + |d\uparrow\bar{u}\downarrow\rangle\right) \tag{91}$$

for the ω we have

$$\vec{\mu}_{\omega\pi^0} = \langle\omega|\vec{\mu}|\pi^0\rangle = \frac{-1}{2\sqrt{2}}\left({}_{\omega}\langle u\uparrow|\vec{\mu}|u\downarrow\rangle_{\pi} - {}_{\omega}\langle\bar{u}\uparrow|\vec{\mu}|\bar{u}\downarrow\rangle_{\pi} - {}_{\omega}\langle d\uparrow|\vec{\mu}|d\downarrow\rangle_{\pi} + {}_{\omega}\langle\bar{d}\uparrow|\vec{\mu}|\bar{d}\downarrow\rangle_{\pi}\right) \tag{92}$$

Since our theory is charge conjugation invariant we can treat the antiquark matrix elements as quark matrix elements with $e_{\bar{q}} = -e_q$;

$$\vec{\mu}_{\omega\pi^0} = \frac{-1}{\sqrt{2}}\left({}_{\omega}\langle u\uparrow|\vec{\mu}|u\downarrow\rangle_{\pi} - {}_{\omega}\langle d\uparrow|\vec{\mu}|d\downarrow\rangle_{\pi}\right) \tag{93}$$

or, assuming the u and d quarks have identical wavefunctions,

$$\vec{\mu}_{\omega\pi^0} = \frac{-e}{\sqrt{2}}\ {}_{\omega}\langle q\uparrow|\frac{\vec{\mu}}{e_q}|q\downarrow\rangle_{\pi} \tag{74}$$

* We assume ω-φ mixing such that the φ is pure $s\bar{s}$.

For naive SU(6) with $\vec{\mu} = \frac{e_q}{e}\mu\vec{\sigma_q}$ this gives

$$\vec{\mu}_{\omega\pi^0} = \frac{\mu}{\sqrt{2}}\{1,-i,0\} \quad , \quad |\vec{\mu}_{\omega\pi^0}|^2 = \mu^2 \tag{95}$$

so

$$\Gamma(\omega \mapsto \pi^0\gamma) = \frac{\alpha g_P^2}{3}\frac{k_\gamma^3}{M_P^2} = \frac{4}{3}k_\gamma^3\mu_P^2 \tag{96}$$

The last line assumes the quark wavefunctions are the same in both the nucleons (where we found $\mu = \mu_P$) and in the mesons - we will of course drop this assumption and use a meson μ for these calculations if necessary and just say that meson quarks look different than baryon quarks.

Plugging in numbers, we find (with $\mu = \mu_P$)

$$\Gamma(\omega \mapsto \pi^0\gamma) = 1190 \text{ Kev} \tag{97}$$

Compared with the experimental $\Gamma = 870 \pm 80$ Kev this result is not bad, although it neglects recoil effects, which can't be much larger than for $\omega \mapsto \pi^0\gamma$. For the present we shall continue neglecting recoil to see how well the other reactions fit into the nonrelativistic SU(6) prediction (they deviate substantially from nonrel. SU(6)).

Afterward we shall put in an estimate of recoil effects and see how strongly it affects the SU(6) pattern of mesonic magnetic dipole decay rates. Now we merely state the results for the other two reactions neglecting recoil.

$$|\vec{\mu}_{\rho^-_{1^+}\pi^-}|^2 = \frac{1}{9}\mu^2 \qquad (98)$$

$$|\vec{\mu}_{K^{*0}_{1^+}K^0}|^2 = \frac{4}{9}\mu^2$$

$$\Gamma(\rho^- \mapsto \pi^-\gamma) = \frac{4}{27}\mu^2 k_\gamma^3 = \frac{\alpha}{27} g_\rho^2 \frac{k_\gamma^3}{M_\rho^2} = 123 \text{ Kev} \qquad (99)$$

$$\Gamma(K^{*0} \mapsto K^0\gamma) = \frac{16}{27}\mu^2 k_\gamma^3 = 277 \text{ Kev}$$

which we compare to the experimental values

$$\Gamma(\rho^- \mapsto \pi^-\gamma) = 35 \pm 10 \text{ Kev} \qquad (100)$$

$$\Gamma(K^{*0} \mapsto K^0\gamma) = 75 \pm 35 \text{ Kev}$$

These are both a factor of three low. A priori we could claim that the discrepancies are due to recoil or short photon wavelength effects, but we must also explain why the $\omega \mapsto \pi^0\gamma$ result was in fairly good agreement with experiment. The failure of the ρ calculation and the success of the ω are a great surprise to anyone who believes (1) the SU(6) quark decomposition of these states is correct and (2) the quark wavefunctions in the ω and ρ are similar. We shall se that there are reasons to believe that the $\rho^- \mapsto \pi^- \gamma$ expt. is in error.

We stated previously that the scale of the magnetic moment operator $\vec{\mu}$ might be changed from the μ_p we found for baryons; to test our result independently of μ and complications from the strange quark mass we look at $\frac{\Gamma(\rho^- \mapsto \pi^- \gamma)}{\Gamma(\omega \mapsto \pi^0 \gamma)}$;

$$\frac{\Gamma(\rho^- \mapsto \pi^- \gamma)}{\Gamma(\omega \mapsto \pi^0 \gamma)} = \frac{1}{9} \frac{(m_\rho^2 - m_{\pi^-}^2) m_\omega}{(m_\omega^2 - m_{\pi^0}^2) m_{\rho^-}} = .103 \tag{101}$$

Experimentally $\frac{\Gamma(\rho^-)}{\Gamma(\omega)}$ = .040±.012, so we still have a prediction from naive SU(6) which is 5 standard deviations from the experimental result. For those who distrust the ρ experiment we note that the $K^{*0} \mapsto K^0 \gamma$ experiment, with a similar deviation from the ω rate and SU(6), was done by a different group. We have yet to see how recoil corrections affect these rates, but it is clear that the experiments are inconsistent with recoil-neglecting SU(6).

The last reaction for which there are good data is $\varphi \mapsto \eta \gamma$, so we treat it also; there is some η-η' mixing,

$$\begin{aligned} |\eta'\rangle &= \cos\theta_s |\underset{\sim}{1}\rangle - \sin\theta_s |\underset{\sim}{8}\rangle \\ |\eta\rangle &= \sin\theta_s |\underset{\sim}{1}\rangle + \cos\theta_s |\underset{\sim}{8}\rangle \end{aligned} \tag{102}$$

with $\theta_s \sim -10^{\circ}$. In view of the crudeness of the results for ρ and K^{*o} decay we don't expect a small θ_s to matter much, but we include it anyway. As above we work out $\vec{\mu}_{fi}$;

$$\langle \varphi_{1+} | \vec{\mu} | \underset{\sim}{1} \rangle = \langle s\uparrow\bar{s}\uparrow | \vec{\mu} \frac{1}{\sqrt{6}} \{ |u\uparrow\bar{u}\downarrow\rangle + \ldots + |s\uparrow\bar{s}\downarrow\rangle - |s\downarrow\bar{s}\uparrow\rangle \}$$

$$= \frac{1}{3}\sqrt{\frac{2}{3}}\,\mu\,\{1,-i,0\} \tag{103}$$

$$\langle \varphi_{1+} | \vec{\mu} | \underset{\sim}{8} \rangle = \langle s\uparrow\bar{s}\uparrow | \vec{\mu} \frac{1}{\sqrt{3}} \{ |s\uparrow\bar{s}\downarrow\rangle - |s\downarrow\bar{s}\uparrow\rangle + \ldots \}$$

$$= \frac{2}{3\sqrt{3}}\,\mu\,\{1,-i,0\} \tag{104}$$

so we find

$$\langle \varphi_{1+} | \vec{\mu} | \eta' \rangle = \frac{1}{3}\sqrt{\frac{2}{3}}\,\mu\,(\cos\theta_s - \sqrt{2}\sin\theta_s)\,\{1,-i,0\} \tag{105}$$

$$\langle \varphi_{1+} | \vec{\mu} | \eta \rangle = \frac{1}{3}\sqrt{\frac{2}{3}}\,\mu\,(\sin\theta_s + \sqrt{2}\cos\theta_s)\,\{1,-i,0\}$$

and for the widths we find

$$\Gamma(\varphi \mapsto \eta'\gamma) = \frac{4\alpha}{81}\left(\frac{g_\rho}{M_\rho}\right)^2 (\cos\theta_s - \sqrt{2}\sin\theta_s)^2 k_\gamma^3 \tag{106}$$

$$\Gamma(\varphi \mapsto \eta\gamma) = \frac{4\alpha}{81}\left(\frac{g_\rho}{M_\rho}\right)^2 (\sin\theta_s + \sqrt{2}\cos\theta_s)^2 k_\gamma^3 = 303 \text{ Kev}\Big|_{\theta_s=0}$$

Comparing this with the experimental value $\Gamma_{\varphi\mapsto\eta\gamma} = 65\pm15\,\mathrm{Kev}$ we see that this reaction too is inconsistent with $\Gamma_{\omega\mapsto\pi^0\gamma}$ and the assumption of SU(6) symmetry neglecting recoil.

Are these rates particularly sensitive to the assumed quark wave functions? As with the $\Delta^+\mapsto p\gamma$ decay we answer this question by replacing the two component naive SU(6) Pauli spinors by the four component CFQM Dirac spinors with x_0 and a left as free parameters.

Again assuming that u and d quarks have the same wavefunction leads to the ω result

$$\vec{\mu}_{\omega\pi^0} = \frac{-e}{\sqrt{2}}\ {}_{\omega}\langle q\uparrow|\frac{\vec{\mu}}{e_q}|q\downarrow\rangle_{\pi^0} \tag{107}$$

where the quark spinors are given by (5a, b) and the magnetic moment operator is (39). For massless quarks the matrix element in (107) is

$$\langle q\uparrow|\frac{\vec{\mu}}{e_q}|q\downarrow\rangle = \frac{4\pi}{3}J(x_0)a\,\{i,-1,0\} \tag{108}$$

where

$$J(x_0) = \frac{2d_+^2a^3}{x_0^4}\int_0^{x_0} \eta^3 j_0(\eta)j_1(\eta)\,d\eta \tag{109}$$

so

$$|\vec{\mu}_{\omega\pi^0}|^2 = \frac{16\pi^2 d}{9}J(x_0)^2a^2 \tag{110}$$

and the decay rate in the long wavelength approximation is

$$\Gamma(\omega \mapsto \pi^0\gamma) = \frac{64\pi^2\alpha}{27} J(x_0)^2 a^2 k_\gamma^3$$

We know the integral in (109) from its previous occurrence in $\Delta^+ \mapsto P\gamma$ decay (68);

$$\int_0^{x_0} \eta^3 j_0(\eta) j_1(\eta)\, d\eta = \frac{1}{4}\left[x_0(2+\cos 2x_0) - \frac{3}{2}\sin 2x_0 \right]$$

so we may evaluate (110) explicitly;

$$\Gamma(\omega \mapsto \pi^0\gamma) = \frac{\alpha a^2}{27}\left[\frac{x_0(2+\cos 2x_0) - \frac{3}{2}\sin 2x_0}{x_0^2 - \sin^2 x_0}\right]^2 k_\gamma^3 \qquad (111)$$

To relate this width to μ_p as does the naive $SU(6)$ quark model requires the unreasonable assumptions x_0(mesons)=x_0(baryons), a (mesons) = a (baryons). If we do make these assumptions we should get the naive SU(6) result; from (17) and (18, a) we have

$$\mu_p = \frac{ea}{6}\left[\frac{x_0(2+\cos 2x_0) - \frac{3}{2}\sin 2x_0}{x_0^2 - \sin^2 x_0}\right] \qquad (112)$$

so we verify that the above assumptions cast (111) in the form (96):

$$\Gamma(\omega \mapsto \pi^0\gamma) = \frac{4}{3}\mu_p^2 k_\gamma^3 \qquad \text{(not true in general)}$$

More generally the meson parameters will be different from the baryon parameters. The question we want to ask is how rapidly (111) varies as a function of x_o (and a, which is obvious). To illustrate this variation we give $\Gamma_{calc}(\omega\mapsto\pi^o\gamma; x_o, a)$ for various values of a and x_o;

$\Gamma_{calc}^{CFQM}(\omega\mapsto\pi^o\gamma; x_o, a)$ in Kev   expt Γ = 870$\pm$80 kev

naive SU(6)Γ= 1190 kev

x_o	a=1.0fm	1.2fm	1.4fm
1.5	412	594	808
2.0	556	801	1090
2.5	563	810	1102

Table B

$\omega\mapsto\pi^o\gamma$ width in the long wavelength approximation neglecting π^o recoil

For x_o in the range we found to be reasonable for baryon models ($2 \lesssim x_o \lesssim 2.5$) we see that almost all the dependence of Γ is on a; the width is very insensitive to x_o. (The scale of the numbers in Table B cannot be taken seriously in view of the two approximations made in calculating the width, long photon wavelength and no π^o recoil.)

We considered the effect of not assuming long photons on the width calculation in the discussion of $\Delta^+\mapsto P\gamma$ where we found that the SU(6) result was suppressed by a factor $R(x_o, k_\gamma a)$ which for the CFQM is given by (87). For $\omega \mapsto \pi^0\gamma$ we have k_γ =380 Mev, so $k_\gamma a \sim 2.5$, and the suppression factor may be evaluated numerically to give $R(x_0=2, k_\rho) = .426$.

As we have seen it is possible to scale down the naive quark model $1^- \mapsto 0^- \gamma$ width by a factor of 2 by including the effects of short photons or reduced meson radius, so the "anomalously small" widths of $K^{*0} \mapsto K^0\gamma$ and so forth are not hard to account for. The paradox which we can't explain at this level is

$$\left.\frac{\Gamma(\rho^- \mapsto \pi^- \gamma)}{\Gamma(\omega \mapsto \pi^0 \gamma)}\right|_{\text{expt.}} = .040 \pm .012 \tag{113}$$

The near equality of the ρ and ω masses leads us to believe they are similar internally, differing principally in isospin. The photons in these reactions are very close in energy, so π recoil and short γ effects should be approximately equal. These assertions and the single quark spin flip model of these decays leads us to relate the Γ ratio to an SU(6) coefficient, which gives

$$\frac{\Gamma(\rho^- \mapsto \pi^- \gamma)}{\Gamma(\omega \mapsto \pi^0 \gamma)} \simeq \frac{1}{9} \tag{114}$$

One possible explanation for this measured deviation from the SU(6) prediction is that the large admixture of $|q\bar{q}A\rangle^*$ in the standard quark model $|q\bar{q}\rangle$ states is responsible for a modification of the SU(6) transition moment ratios. This is of course a difficult effect to estimate, although with a number of approximations we may see how large we

* By A we mean a neutral colored vector gluon.

expect this effect to be. If we jump ahead to the bag model and consider mixing between the $|q\bar{q}\rangle$ and lowest lying $|q\bar{q}A\rangle$ states through the $g\bar{\psi}\frac{\lambda^a}{2}\gamma_\mu\psi A^a$ Yukawa part of the quark-gluon lagrangian, we may estimate the amount of mixing by fitting the $\pi\rho$ splitting as a function of g. Then we may work out the matrix elements $\langle 1^-|\vec{\mu}|0^-\rangle$ with no free parameters. This messy (and inconsistent, since we find large mixing angles which imply that we probably have significant mixing with other configurations) calculation is deferred to Appendix D; the result we find is that $|\vec{\mu}_{\omega\pi}|^2 = 9\,|\vec{\mu}_{\rho\pi}|^2$. The mixing angles are the same for the $(\pi\rho)$ and $(\pi\omega)$ in this approximation, $m_\rho = m_\omega \gg m_\pi$, and the angles cancel out of the ratio of the moments, changing only the overall scale. Presumably mixing with gluons will correct ratios of transition moments of strange and nonstrange mesons, but we haven't treated this problem. The upshot of all this is that we can find no effect which modifies the SU(6) prediction of $\Gamma(\omega\mapsto\pi^0\gamma)/\Gamma(\rho^-\mapsto\pi^-\gamma)\sim 9$, so we tenatively claim that the $\rho^-\mapsto\pi^-\gamma$ experiment is in error. This is a difficult experiment involving ρ^- production in a nuclear Coulomb field by a π^- beam, which has amplitudes to go through a photon or through strong ρ^- production within the nucleus. In the above experiment both effects are sizable, and the relative phase of the two must be known in order to find the size of the electromagnetic amplitude. In doing this the authors of the experiment find two solutions, $\Gamma = 80 \pm 10$ Kev[23] and $\Gamma = 35 \pm 10$ Kev, the former of which (which is consistent with SU(6) and the $\omega\mapsto\pi^0\gamma$ rate) they reject. We suggest that the former solution is correct, simply because we can find no mechanism which allows the lower rate. Of course there remains apparent SU(6) violation in the rates $K^{*0} \mapsto K^0\gamma$ and $\varphi \mapsto \eta\gamma$ which has given credence to the first such reported violation and has led to a flock of theoretical papers on the form of the SU(6) violation (which incidently are unable to consistently explain all four of the measured rates given above with the exception of one model with four free parameters). These papers have in common two essential ingredients; (1) acceptance of the $\rho^- \mapsto \pi^-\gamma$ experiment of Gobbi et al,[23] which we question, and (2) neglect of recoil effects. Now we consider how bad the latter approximation is. It is a straightforward but tedious exercise to compute the change in the quark overlap integrals once the π is boosted, but lack of time prevents this exercise from being carried out with the CFQM wavefunctions. Instead we merely quote the result of doing this in the relativistic harmonic oscillator quark model of Feynman, Kislinger, and Ravndal [27]. These fellows find that including quark recoil in photoelectric meson decays modifies the nonrelativistic SU(6) quark model prediction as

$$\Gamma(1^- \mapsto 0^-\gamma\,;\,\text{recoiling } 0^-) = \left[\frac{2m(1^-)}{m(1^-)+m(0^-)}\right]^3 \Gamma(1^- \mapsto 0^-\gamma\,;\,\text{no recoil}) \quad (115)$$

In the limit $m(1^-) \gg m(0^-)$ (which is not unreasonable for some of the decays) we find that the predicted rate is enhanced by a factor of 8 over the nonrel. SU(6) prediction. In addition, if there is a large spread in the range of $m(1^-)/m(0^-)$ in the measured decays (which there is), we expect the recoil effects to significantly modify the pattern of decay rates predicted by SU(6). For convenience we start with the rate known with the greatest certainty, $\Gamma_{\omega\mapsto\pi^0\gamma} = 870 \pm 80$ Kev, and use the above recoil factor to predict the other decay rates with SU(6) and with/without recoil included. The results we find are;

Mesonic M1 Decay Rates With FKR Recoil Factor

reaction	$\lvert\vec{\mu}_{fi}\rvert^2/\lvert\vec{\mu}_{\omega\pi^0}\rvert^2$	nonrel. SU(6)	SU(6) with recoil	expt.	
$\omega\mapsto\pi^0\gamma$	1	input	input	870±80	
$\rho^-\mapsto\pi^-\gamma$	$\frac{1}{9}$	91±8.3	88±8.1	35±10	believed incorrect
$K^{*0}\mapsto K^0\gamma$	$\frac{4}{9}$	200±19	87±8.0	75±35	
$\varphi\mapsto\eta\gamma$	$\frac{8}{27}f(\theta_s)$	160±15	73±6.7	65±15	

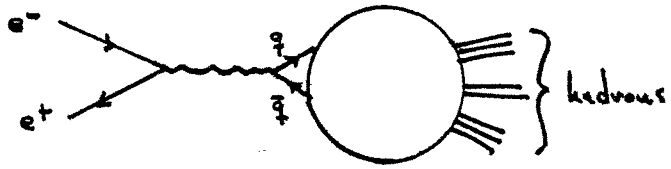
(we take $\theta_s = -10°$). $f(\theta_s) = 1 + \sqrt{2}\sin\theta_s\cos\theta_s - \frac{1}{2}\sin^2\theta_s$

Table C

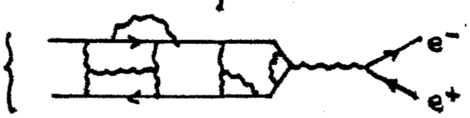
There are a number of other effects such as suppression of the strange quark moment which further modify these predictions by up to 40% for some cases, which will be treated in a letter on M1 mesonic decays,[26] together with predictions for other rates which are experimentally known only as upper limits at present. The essential point of our argument is seen above, however; the FKR recoil factor brings the $\omega\pi^0$, $K^{*0}K^0$, and $\varphi\eta$ transition moments into agreement with the experimental results to within experimental error. Only the $\rho^-\pi^-$ moment breaks the pattern, and we hold that this experiment is wrong. It follows that the theoretical work on SU(6) violation[28-30] is based on an invalid interpretation of experiment (recoil neglect) and is thus simply ill-motivated. An interesting calculation would be to explicitly work out the recoil effects in the CFQM and Bag models to see how model dependent these effects really are.

The decay 1^- meson $\mapsto \ell^+\ell^-$

The production of the heavy ψ-particles in e^+e^- colliding beam experiments has given rise to a great deal of interest in the process $e^+e^- \mapsto$ hadrons. The largest contributions to this must go through a virtual photon:



At current energies we expect diagrams of this form to make the only important contribution. We wish to avoid the complication of recoil effects in the hadron production so we study instead the inverse reaction, in particular the decay of a vector meson to e^+e^-;



glue exchange indicated schematically

This process is equivalent to the well-known problem $\mu^+\mu^- \mapsto e^+e^-$ except for the gluon effects, if we assume that quarks are Dirac fermions. The lowest order diagram for $\mu^+\mu^- \mapsto e^+e^-$ is trivial to evaluate, with the result

$$\sigma_{tot}(s) = \frac{4\pi\alpha^2}{3s}\left[\frac{1-\frac{4m_e^2}{s}}{1-\frac{4m_\mu^2}{s}}\right]^{1/2}\left(1+\frac{2}{s}(m_e^2+m_\mu^2)+\frac{4m_e^2m_\mu^2}{s^2}\right) \simeq \frac{4\pi\alpha^2}{3s}\,\frac{1+\frac{2m_\mu^2}{s}}{\left[1-\frac{4m_\mu^2}{s}\right]^{1/2}} \quad \begin{matrix} s \gg m_e \\ m_\mu \gg m_e \end{matrix} \tag{116}$$

Suppose we had confined the initial $\mu^+\mu^-$ pair to a volume V and calculated the rate for the reaction, keeping V finite. We then find, assuming $m_\mu \gg m_e$,

$$\Gamma(\mu^+\mu^- \mapsto e^+e^-) = \frac{8\pi\alpha^2}{3sV}\left(1+\frac{2m_\mu^2}{s}\right) \sim \frac{\pi\alpha^2}{m_\mu^2 V}\bigg|_{s\sim 4m_\mu^2} \tag{117,8}$$

We may connect this to the order of magnitude $1^- \mapsto e^+e^-$ rate by a simple argument;

the s of $V \mapsto e^+e^-$ is clearly $s = m_V^2$, and we simply call the muons quarks in $\mu^+\mu^- \mapsto e^+e^-$, which gives a rate

$$\Gamma(V \mapsto e^+e^-) = \frac{2\alpha^2}{m_V^2 a^3}\left(1 + \frac{2m_q^2}{m_V^2}\right)\left\langle\frac{e_q}{e}\right\rangle^2 \tag{119}$$

with two limiting cases for very light or heavy quarks;

$$\Gamma(V \mapsto e^+e^-) \sim \begin{cases} \dfrac{2\alpha^2}{m_V^2 a^3}\left\langle\dfrac{e_q}{e}\right\rangle^2 & m_q \ll m_V \\ \dfrac{3\alpha^2}{4m_q^2 a^3}\left\langle\dfrac{e_q}{e}\right\rangle^2 & m_q \sim \dfrac{m_V}{2} \end{cases} \tag{120a,b}$$

where $\left\langle\frac{e_q}{e}\right\rangle$ is the mean quark charge weighted by the amplitude to find each type of $(q\bar{q})$ pair in the meson.

To see whether or not this order of magnitude estimate is correct we look at some specific decays:[31,32]

Reaction	Rate (Kev)
$\rho^0 \mapsto e^+e^-$	$6.32 \pm .68$
$\rho^0 \mapsto \mu^+\mu^-$	9.89 ± 1.77
$\omega^0 \mapsto e^+e^-$	$.758 \pm .173$
$\varphi \mapsto e^+e^-$	$1.33 \pm .10$
$\varphi \mapsto \mu^+\mu^-$	$1.04 \pm .15$
$\psi \mapsto e^+e^-$	$4.8 \pm .6$

(121a,...,h)

$$\psi \mapsto \mu^+\mu^- \quad = \quad 4.8 \pm .6 \text{ Kev}$$
$$\psi' \mapsto e^+e^- \quad = \quad 2.2 \pm .6 \text{Kev}$$

First consider the $\rho^0 \to e^+e^-$ decay, assuming $m_q \ll m_\rho$.

The rate is proportional to the sum of the $u\bar{u}$ and $d\bar{d}$ annihilation amplitudes squared;

$$\Gamma(\rho^0 \mapsto e^+e^-) \sim \frac{2\alpha^2}{m_\rho^2 a^3}\left[\frac{1}{\sqrt{2}}\left(\frac{2}{3}\right) - \frac{1}{\sqrt{2}}\left(-\frac{1}{3}\right)\right]^2 = \frac{\alpha^2}{m_\rho^2 a^3} \tag{123}$$

With $a^{-1} \sim 159$ Mev.* and $m_v = m_\rho = 770$ Mev, we find

$$\Gamma(\rho^0 \mapsto e^+e^-) \sim .36 \text{ Kev} \tag{124}$$

which is in error by a factor of $\sim$ 20. In the conventional phenomenology of $v \to e^+e^-$ an arbitrary factor is introduced at this point to correct the scale of these rates, for which there is some justification ; the rate is proportional to (the amplitude to find the quark and antiquark at zero separation)^2 , which we have no way of knowing without solving a model of quark dynamics explicitly for $|\psi_{q\bar{q}}(0)|^2$. Our incorrect result above merely indicates that $\frac{|\psi(0)|^2}{\langle|\psi(x)|^2\rangle} \gg 1$ in the ρ-meson. We shall see what the bag model gives for this correction factor later, but for the moment we shall assume $|\psi(0)|^2/\langle|\psi(x)|^2\rangle \equiv R$ is a universal constant for the light vector mesons (which implies that this ratio is spin-independent), and concern ourselves only with ratios of rates, i.e. $\Gamma(V_1 \mapsto e^+e^-)/\Gamma(V_2 \mapsto e^+e^-)$. First we consider the cases for which we expect $m_q \ll a^{-1}$;

$$\left.\frac{\Gamma(\rho_0^0 \mapsto e^+e^-)}{\Gamma(\omega \mapsto e^+e^-)}\right|_{\text{theoretical}} = \frac{\frac{1}{2}\left(\frac{2}{3}+\frac{1}{3}\right)^2}{\frac{1}{2}\left(\frac{2}{3}-\frac{1}{3}\right)^2} = 9 \tag{125}$$

* This follows from the zeroth order bag model with $B_0^{1/4} = 120$ Mev.

Experimentally $\dfrac{\Gamma(\rho^\circ \mapsto e^+e^-)}{\Gamma(\omega \mapsto e^+e^-)} = 8.34 \pm 2.10.$ (126)

This case is in good agreement. Now we consider the effect of an appreciable lepton mass ($\mu^+\mu^-$) on this ratio.

If we don't neglect m_ℓ we find;

$$\frac{\Gamma(\rho^\circ,\omega \mapsto \mu^+\mu^-)}{\Gamma(\rho^\circ,\omega \mapsto e^+e^-)} = \left[1 - \frac{4m_\mu^2}{m_v^2}\right]^{1/2}\left[1 + 2\frac{m_\mu^2}{m_v^2}\right] \qquad m_q \ll m_\mu, m_v$$

$$\simeq 1 - 6\left(\frac{m_\mu}{m_v}\right)^4 \qquad \frac{m_\mu}{m_v} \ll 1 \tag{127}$$

here $m_v \sim .78$ Gev and $m_\mu \sim .1$ Gev so we find

$$\Gamma(\rho^\circ,\omega \mapsto \mu^+\mu^-)/\Gamma(\rho^\circ,\omega \mapsto e^+e^-) \simeq .998 \tag{128}$$

which is unity to the accuracy of the experimentally measured rates;

$$\left.\frac{\Gamma(\rho^\circ \mapsto \mu^+\mu^-)}{\Gamma(\rho^\circ \mapsto e^+e^-)}\right|_{expt.} = 1.56 \pm .33 \tag{129}$$

$$\left.\frac{\Gamma(\omega \mapsto \mu^+\mu^-)}{\Gamma(\omega \mapsto e^+e^-)}\right|_{expt} < 2.5 \tag{130}$$

For the $\rho^\circ \mapsto \ell^+\ell^-$ the increased lepton mass seems experimentally to increase the rate, although the effect is not large compared to the experimental errors. The $\omega \mapsto \mu^+\mu^-$ decay is unfortunately known only as an upper limit.

To treat the $\varphi \mapsto \ell^+\ell^-$ decays with our model, we take m_s = 270 Mev as found previously from the decimet baryons,[8] assuming $m_u = m_d = 0$.

We have an unknown parameter, the correction factor $|\psi(0)|^2/\langle|\psi(x)|^2\rangle$ which we expect to be different for the strange quark, due to its mass, although we have no way of estimating it here. If we simply assume this correction factor is the same as it is in $\rho^\circ \mapsto e^+e^-$ we predict the ratio $\frac{\Gamma(\varphi \mapsto e^+e^-)}{\Gamma(\rho^\circ \mapsto e^+e^-)}$ as

$$\frac{\Gamma(\varphi \mapsto e^+e^-)}{\Gamma(\rho^\circ \mapsto e^+e^-)} = \frac{2}{9}\left(\frac{m_\rho}{m_\varphi}\right)^2\left(\frac{a_\rho}{a_\varphi}\right)^3\left(1+2\left(\frac{m_s}{m_\varphi}\right)^2\right) = .108 \tag{131}$$

where we have taken $\frac{a_\rho}{a_\varphi}$ = 1.022 which results from the bag model with $B_o^{\frac{1}{4}}$ = 120 Mev, m_s= 270 Mev (which is only a 7% correction to the simpler assumption $a_\rho = a_\varphi$)

From the experimental rates, we find

$$\frac{\Gamma(\varphi \mapsto e^+e^-)}{\Gamma(\rho^\circ \mapsto e^+e^-)} = .210 \pm .028 \tag{132}$$

which is significantly larger than the rate we predict assuming $|\psi(0)|^2/\langle|\psi|^2\rangle$ is the same in the ρ° and φ. Explicitly, if we assume that the change in the rate from our prediction is due to changes in $|\psi_v(0)|^2/\langle|\psi_v(x)|^2\rangle \equiv R_v$ we find

$$\left.\frac{R(\omega)}{R(\rho^\circ)}\right|_{expt.} = .927 \pm .233 \tag{133}$$

$$\left.\frac{R(\varphi)}{R(\rho^\circ)}\right|_{expt.} = 1.94 \pm .259 \tag{134}$$

We may also calculate the ratio of correction factors $R(\varphi)/R(\rho^\circ)$ by comparing the rates to $\mu^+\mu^-$ for these two mesons. There we find

$$\frac{\Gamma(\varphi \mapsto \mu^+\mu^-)}{\Gamma(\rho^\circ \mapsto \mu^+\mu^-)} = \frac{R(\varphi)}{R(\rho^\circ)} \frac{2\, m_\rho^2 a_\rho^3 \left[1+2\frac{m_\mu^2+m_s^2}{m_\varphi^2}+4\frac{m_\mu^2 m_s^2}{m_\varphi^4}\right]}{9\, m_\varphi^2 a_\varphi^3 \left[1+2\frac{m_\mu^2}{m_\rho^2}\right]} \left\{\frac{1-4\frac{m_\mu^2}{m_\varphi^2}}{\left(1-4\frac{m_s^2}{m_\varphi^2}\right)\left(1-4\frac{m_\mu^2}{m_\rho^2}\right)}\right\}^{1/2} \tag{135}$$

$$= .1824 \frac{R(\varphi)}{R(\rho^\circ)} \tag{136}$$

which is experimentally

$$\frac{\Gamma(\varphi \mapsto \mu^+\mu^-)}{\Gamma(\rho^\circ \mapsto \mu^+\mu^-)} = .105 \pm .0242 \tag{137}$$

so the $\mu^+\mu^-$ decay gives the ratio of correction factors as

$$\left.\frac{R(\varphi)}{R(\rho^\circ)}\right|_{expt} = .576 \pm .133 \tag{138}$$

which is not consistent with the result 1.94 $\pm$ .259 found in comparing the e^+e^- decay rates of the φ and ρ^0. We are not able to explain the difference in the two results with this simple model.

Another prediction made by our model is $\Gamma(\varphi \mapsto \mu^+\mu^-)/\Gamma(\varphi \mapsto e^+e^-)$ which is predicted to be

$$\frac{\Gamma(\varphi \mapsto \mu^+\mu^-)}{\Gamma(\varphi \mapsto e^+e^-)} = \frac{\sqrt{1-4\frac{m_\mu^2}{m_\varphi^2}}\left[1+2\frac{m_\mu^2+m_s^2}{m_\varphi^2}+4\frac{m_\mu^2 m_s^2}{m_\varphi^4}\right]}{\left(1+2\frac{m_s^2}{m_\varphi^2}\right)} = .999 \qquad (139)$$

which is essentially unity.

The experimental result is

$$\left.\frac{\Gamma(\varphi \mapsto \mu^+\mu^-)}{\Gamma(\varphi \mapsto e^+e^-)}\right|_{expt} = .78 \pm .13 \qquad (140)$$

As with the ρ^0 we find a branching ratio two standard deviations from unity, although here it deviates in the opposite direction. Our simple models give this ratio as close to unity for both cases, so we don't understand the apparent deviation.

Finally, we consider the very heavy ψ particles. Here we have $m_c \sim 1\,\text{Gev} \gg a^{-1} \sim .3\,\text{Gev}$, so we predict a rate

$$\Gamma(\psi \mapsto e^+e^-) \simeq \Gamma(\psi \mapsto \mu^+\mu^-) \simeq \frac{4\alpha^2}{3m_\psi^2 a_\psi^3} R(\psi) = .165\, R(\psi) \;\; \text{Kev} \qquad (141)$$

where we have used a = .7 fm as will be found in section IV when we treat glue effects in the vector and pseudoscalar mesons; the correction factor R(ψ) due to peaking of the amplitude to find the c and $\bar{c}$ quarks at zero separation has been left free. Experimentally, we know $\Gamma(\psi \mapsto e^+e^-) = 4.8 \pm .6$ Kev so we find for R(ψ)

$$R(\psi) = 29.1 \pm 3.6 \qquad (142)$$

and essentially the same result for $\psi \mapsto \mu^+\mu^-$. For the ψ' we predict

$$\Gamma(\psi' \mapsto e^+e^-) \simeq \Gamma(\psi' \mapsto \mu^+\mu^-) \simeq .070\, R(\psi') \;\; \text{Kev} \qquad (143)$$

where we take $\frac{a_{\psi'}}{a_\psi} = 1.19$, as suggested by fitting the ψ and ψ' masses in the zeroth order bag model. For the ψ' we have experimentally $\Gamma(\psi' \to e^+e^-) = 2.2 \pm 0.6$ Kev, which gives

$$R(\psi') = 36.8 \pm 8.6 \,, \tag{144}$$

which is somewhat larger than the ψ result we found previously. We note that the amplitude to find these heavy quarks at the same point is of the same order of magnitude as the amplitude to find the light and strange quarks at the same point. Unfortunately we have no model of quark-quark interactions which we may use here to calculate how large R should be and how strongly it should depend on the quark mass.

If we use the bag model quark spinors instead of assuming that the quarks are uniformly distributed, we will of course find that $R(v) \equiv |\psi_v(0)|^2 / \langle |\psi_v(x)|^2 \rangle \neq 1$ although this peaking effect is much less than the experimental values of $R(v) \sim 20\text{-}40$ for conventional and charmed vector mesons. For the sake of completeness we shall explicitly work out the correction factor R predicted by the bag model for the ψ and ρ^0.[33]

Our two quark wavefunction for the $\psi(3095)$ is of the form

$$\psi(\vec{r}_1,\vec{r}_2) = \frac{1}{\sqrt{2}}\left(u_q(\vec{r}_1)\, v_{\bar{q}}(\vec{r}_2) + u_{\bar{q}}(\vec{r}_1)\, v_q(\vec{r}_2) \right) \tag{145}$$

where u and v are the spinors corresponding to the quark states in the ψ. For the ψ both q and $\bar{q}$ are in the ground state, so we have

$$\psi(\vec{r}_1,\vec{r}_2) = \frac{1}{\sqrt{2}}\left(u_q(\vec{r}_1)\, u_{\bar{q}}(\vec{r}_2) + u_{\bar{q}}(\vec{r}_1)\, u_q(\vec{r}_2) \right) \tag{146}$$

our previous assumption that the quarks were uniformly distributed gives $u^\dagger u(r) = \theta(a-r)\, u_0^\dagger u_0$ where u_0 is a constant spinor. We know that the [annihilation amplitude]^2 is proportional to the [amplitude to find q and $\bar{q}$ at the same point]^2, which is

$$|A|^2 \sim \frac{\int d^3r \; |\psi(\vec{r},\vec{r})|^2}{\iint d^3r_1\, d^3r_2 \; |\psi(\vec{r}_1,\vec{r}_2)|^2} \tag{147}$$

The ratio of this (amplitude)^2 to the case of uniformly distributed quarks is

$$\frac{|A|^2}{|A_0|^2} = \frac{V\int d^3r\, |\psi(\vec{r},\vec{r})|^2}{\iint d^3r_1 d^3r_2\, |\psi(\vec{r}_1,\vec{r}_2)|^2} \equiv R \tag{148}$$

We shall compensate for the neglect of $\vec{x}$ dependence of the quark wave functions in the rate for $V\mapsto e^+e^-$ by multiplying the uniform quark rate by this factor.

For the case of the $\psi(3095)$, the above attenuation factor becomes

$$R(\psi) = \frac{V\int d^3x\, |u(\vec{x})|^4}{[\int d^3x\, |u(\vec{x})|^2]^2} \tag{149}$$

or, specializing further to the bag model

$$R(\psi) = \frac{x^3}{3}\frac{\int_0^x y^2 f(k,y)^2 dy}{\{\int_0^x y^2 f(k,y)\, dy\}^2} \tag{150}$$

where $\quad f(k,y) = j_0(y)^2 + k^2 j_1(y)^2$

Another effect we have neglected is that the physical $|\psi\rangle$ is not pure $|c\bar{c}\rangle$, but has an amplitude to be $|c\bar{c}\rangle, |c\bar{c}A\rangle, |c\bar{c}AA\rangle, |c\bar{c}q\bar{q}\rangle, \cdots,$ of which only the $|c\bar{c}\rangle$ configuration contributes to $\psi\mapsto e^+e^-$ to lowest order. If we parameterize the $|\psi\rangle$ as $|\psi\rangle = \cos\theta_\psi |c\bar{c}\rangle + \sin\theta_\psi |\text{others}\rangle$ then the rate for $\psi\mapsto e^+e^-$ will be suppressed by $|\langle c\bar{c}|\psi\rangle|^2 = \cos^2\theta_\psi$ from our calculated rate for a pure $|c\bar{c}\rangle\ \psi$. Assembling all these factors we obtain a rate for $\psi\mapsto e^+e^-$ which is

$$\Gamma(\psi\mapsto e^+e^-) = \underbrace{\frac{3\alpha_c^2}{m_\psi^2 a_\psi^3}}_{\Gamma_0}\underbrace{\left[\frac{V\int d^3x\, |u(\vec{x})|^4}{\{\int d^3x\, |u(\vec{x})|^2\}^2}\right]}_{R(\psi)} \cos^2\theta_\psi \tag{151}$$

An estimate of the mixing angle θ_ψ (given in section IV) gives $\theta_\psi \sim 27°$, so $\cos^2\theta_\psi \sim .8$. Since m_ψ is fixed (and hence m_c (a)), our only independent free parameter here is a.

If we use the parameters which follow from fitting the ψ and ψ' masses ($m_c = 1.1$ Gev , a = .644 fm) we obtain a rate

$$\Gamma(\psi\mapsto e^+e^-) \simeq .84 \text{ Kev} \tag{152}$$

which is a factor of 5 below the experimental rate of 4.8 ± 0.6 Kev.

Now we consider the effect of our quark wavefunction and mixing angle corrections on the rates we predicted earlier for the light vector mesons. As with the $\psi \mapsto e^+e^-$ we predict a corrected rate

$$\Gamma(V \mapsto e^+e^-) \simeq \frac{3\alpha^2}{m_V^2 a^3} \left\langle \frac{e_q}{e} \right\rangle^2 R(V) \cos^2\theta_V \tag{153}$$

where the correction factor with bag model wavefunctions is (assuming massless quarks),

$$R(V) = \frac{x_0^3}{3} \frac{\left\{\int_0^{x_0} y^2 [j_0(y)^2 + j_1(y)^2]\, dy\right\}^2}{\left\{\int_0^{x_0} y^2 [j_0(y)^2 + j_1(y)^2]\, dy\right\}^2} = 1.070 \Big|_{x_0 = 2.043} \tag{154}$$

Since the experimental rate for $\rho^0 \longmapsto e^+e^-$ gives $R(\rho^0) \sim 20$, it is clear that a calculation of the rate for $V \mapsto \ell^+\ell^-$ which neglects the short-range quark-quark interaction in obtaining $|\psi(0)|^2$ will give a result which is an order of magnitude too low.

Weak Interactions

We shall consider here only the low energy semileptonic baryon decays of the form

$$B' \mapsto B \ell \bar{\nu}_\ell$$

which we assume are described adequately for our purposes by the current-current interaction

$$\mathcal{L}_I = \frac{G}{\sqrt{2}} J_\mu^\dagger J_\mu$$

where

$$J_\mu \equiv J_\mu^{weak} = J_\mu^{weak}(\text{hadrons}) + J_\mu^{weak}(\text{leptons}) \tag{155}$$

and

$$J_\mu^{had.} = \bar{\psi}_u \gamma_\mu (1-\gamma_5)\left[\psi_d \cos\theta_c + \psi_s \sin\theta_c\right] + h.c.$$

$$J_\mu^{lept.} = \bar{\psi}_\ell \gamma_\mu (1-\gamma_5)\psi_{\bar{\nu}_\ell} + h.c. \qquad \theta_c \equiv \text{Cabibbo angle}$$

For simplicity we shall neglect the effect of recoil of the final baryon B, which is probably justified only for neutron β-decay. The hadronic weak current for $B' \mapsto B$ is

$$J_\mu^{had.}(B' \mapsto B) = \langle B' | \gamma_0 \gamma_\mu (1-\gamma_5)\left(|u\rangle\langle d| \cos\theta_c + |u\rangle\langle s| \sin\theta_c \right) | B \rangle + h.c.$$

In the nonrelativistic limit we shall see that only one component of γ_μ has a nonvanishing contribution to the spin-flip and nonflip transition amplitudes. By comparing the numerical coefficient of this component for plane-wave quark wave functions and for the explicit CFQM spinors we shall see what effect our modified wave functions have on the usual SU(6) predictions for g_A and g_V.

First we do neutron β-decay with plane waves (SU(6)) and with CFQM spinors. Consider the no spin-flip contribution to the nucleonic weak current:

$$J_\mu = \langle P,\uparrow | \gamma_0 \left\{ \gamma_\mu (1-\gamma_5) |u\rangle\langle d| \cos\theta_c \right\} | N,\uparrow \rangle + h.c. \tag{156}$$

We now decompose this into quarks and look at the vector and axial vector currents individually. If we approximate the real quark spinors with rest

Dirac spinors we find for these two currents

$$V_\mu(N\mapsto P) \equiv v_\mu(N\mapsto P)\, g_V \cos\theta_c = \delta_{\mu 0}\cos\theta_c \tag{157}$$

$$a_\mu(N\mapsto P) \equiv -a_\mu(N\mapsto P)\, g_A \cos\theta_c = \delta_{\mu 3}\langle P,\uparrow|\sigma_z|u\rangle\langle d|N,\uparrow\rangle\cos\theta_c = \tfrac{5}{3}\delta_{\mu 3}\cos\theta_c \tag{158}$$

where v_μ and a_μ are V_μ and a_μ normalized to unity. We note that only a single component of J_μ survives nonrelativistically in each case, as promised. We shall use these equations (with $\sin\theta_c$ for strangeness changing decays) as our defining equations for the vector and axial vector weak transition moments g_A and g_V. Clearly for nonrelativistic SU(6) we have (for $N\mapsto P\ell\bar{\nu}_\ell$)

$$g_V = 1 \quad , \quad g_A = -\frac{5}{3} \quad , \text{so} \quad \left.\frac{g_A}{g_V}\right|_{N\mapsto P} = -\frac{5}{3} \tag{159}$$

To obtain $\frac{g_A}{g_V}(N\mapsto P)$ with CFQM spinors we simply insert the explicit coordinate dependent spinors in the expansion for J_μ in terms of quarks and do the overlap integral; the result is

$$V_\mu(N\mapsto P) = \delta_{\mu 0}\cos\theta_c \quad , \quad \text{so} \quad g_V(N\mapsto P) = 1 \tag{160}$$

which is obvious if the neutron and proton quark wave functions are identical, since the integral is just the normalization integral $\int d^3x\, \psi_q^\dagger \psi_q = 1$.

Similarly we find

$$a_\nu(N\mapsto P) = \delta_{\nu 3}\, f(\mu, x_0)\, \frac{5}{3}\cos\theta_c \tag{161}$$

where

$$f(\mu,x_0) = \frac{4\pi\alpha_+^2 a^3}{x_0^3}\int_0^{x_0} \eta^2\left\{ j_0(\eta)^2 - \frac{k^2}{3} j_1(\eta)^2 \right\} d\eta$$

is a wave function dependent correction to the axial-vector moment.

Thus we find

$$\left.\frac{g_A}{g_V}\right|_{\substack{N\mapsto P\\ \text{CFQM}}} = -\frac{5}{3} f(\mu, x_0) \tag{162}$$

This correction function $f(\mu,x_0)$ is explicitly

$$f(\mu,x_0) = \frac{I(0,x_0) - \frac{k^2}{3} I(1,x_0)}{I(0,x_0) + k^2 I(1,x_0)} \tag{163}$$

where

$$I(\ell,x_0) \equiv \int_0^{x_0} y^2 j_\ell(y)^2 dy \tag{164}$$

The numerical value of $\frac{g_A}{g_V}(N \mapsto P)$ as a function of χ_0 and m_q for a particular a (1.4 fm) is shown explicitly in Fig. K, and the constraint $\mu(x_0)$ required to reproduce the experimental value $\left.\frac{g_A}{g_V}\right|_{N\mapsto Pe^-\bar{\nu}_e} = -1.25\pm.01$ [34] is given in Fig. L. It is clear that a χ_0 of 2.0 - 2.5 as suggested by the proton r_Q and gyromagnetic ratio leads to a quark mass on the order of 100 Mev. . This is consistent because r_Q and g_p drop by only ~6% and ~13% respectively when we increase m_q from 0 to 100 MeV, so we can increase a by ~10% and get r_Q, g_P, and $\left.\frac{g_A}{g_V}\right|_{N\mapsto P}$ all correct to within about 10%. We hasten to point out that $\frac{g_A}{g_V}$ is quite sensitive to variation of other parameters than the quark mass. As an example we show the variation of $\frac{g_A}{g_V}$ in the nucleon as a function of a vector potential well (as well as a scalar, which is a mass) in Fig. M, and it is clear that a vector well of a depth comparable to the hypothetical quark mass would also give $\left.\frac{g_A}{g_V}\right|_{N\mapsto P}$ correctly. Our model is simply not realistic enough to determine the light quark masses to any accuracy by fitting the experimental nucleon parameters, since we are not able to rule out effects such as the effective vector potential (glue, ρ -mesons?) alluded to above. For this reason we shall usually assume that the light quark masses are zero unless we are considering the effect of small deviations in $m_{u,d}$ on observable hadron properties (for example, iso multiplet splittings).

Now we consider the strangeness changing semileptonic decays. First we will look in some detail at the best known decay $\Lambda \mapsto Pe^-\bar{\nu}_e$. As previously we are interested in seeing the effect of our CFQM quark wavefunctions on the nonrelativistic SU(6) predictions g_A and g_V. We construct the hadronic weak current

$$J_\mu^{had.}(\Lambda \mapsto P) = \langle P|\gamma_0\gamma_\mu(1-\gamma_5)|u\rangle \langle s| \sin\theta_c|\Lambda\rangle + h.c. \tag{165}$$

Here we note that the quark projection operator $|u\rangle\langle s|$ suffers from an ambiguity, as the Λ quark wavefunctions will not in general be the same as the P quark wavefunctions. In practice for S-wave baryons the difference (from different hadron model radii) will be small, so the effect of assuming that $|u\rangle\langle s|$ means $|u\rangle_\Lambda\langle s|_\Lambda$ rather than forcing all our hadron models to have the same radii is not significant.

Once again we neglect recoil and first approximate the quarks by rest Dirac spinors, which gives for the vector and axial-vector currents

$$\begin{aligned} V_\mu^{had}(\Lambda\uparrow \mapsto P\uparrow) &= \delta_{\mu 0}\langle P,\uparrow|\gamma_0^2|u\rangle_\Lambda\langle s|_\Lambda|\Lambda,\uparrow\rangle\sin\theta_c \\ &= \sqrt{\tfrac{3}{2}}\,\delta_{\mu 0}\langle P,\uparrow|P,\uparrow\rangle_\Lambda\sin\theta_c \simeq \sqrt{\tfrac{3}{2}}\,\delta_{\mu 0}\sin\theta_c \qquad \text{so } g_V\Big|_{\substack{SU(6)\\ \Lambda\mapsto P}} = \sqrt{\tfrac{3}{2}} \end{aligned} \tag{166}$$

where $|u\rangle_\Lambda$ is an up quark spinor with the same wavefunction as the Λ s-quark. (They are identical in nonrelativistic SU(6) in any case.)

Similarly

$$a_\mu^{had}(\Lambda\uparrow \mapsto P\uparrow) = -\delta_{\mu 3}\langle P,\uparrow|\gamma_0\gamma_3\gamma_5|u\rangle_\Lambda\langle s|_\Lambda|\Lambda,\uparrow\rangle\sin\theta_c = -\delta_{\mu 3}\sqrt{\tfrac{3}{2}}\sin\theta_c \tag{167}$$

$$\text{so } g_A\Big|_{\substack{SU(6)\\ \Lambda\mapsto P}} = -\sqrt{\tfrac{3}{2}}$$

which gives the SU(6) result

$$\frac{g_A}{g_V}\Big|_{\substack{SU(6)\\ \Lambda\mapsto P}} = -1 \tag{168}$$

The experimental result is $\frac{g_A}{g_V}\Big|_{\Lambda\mapsto Pe^-\bar{\nu}_e}^{35} = -.64 \pm .07$, which is 5 standard deviations from the SU(6) prediction. In neutron β-decay we found that the quark model prediction was a factor of 1.33 larger than experiment and that the CFQM wavefunctions could be chosen to give $\left.\frac{g_A}{g_V}\right|_{N\mapsto P}$ correctly with massless quarks by taking $\chi_0 \sim 1.6$. For Λ β-decay we again expect that realistic quark wavefunctions will give a more reasonable value for $\frac{g_A}{g_V}$, since the SU(6) prediction for $\left.\frac{g_A}{g_V}\right|_{\Lambda\mapsto P}$ is again too large (this time by ~50%).

Previously we assumed that the neutron and proton had the same radius; to keep the number of parameters down to a manageable size we assume the same a for the P and Λ. We further assume massless light quarks and a universal mode number $\chi = ka$ for all quarks (the latter assumption is the most serious, and the bag model in particular does not satisfy it). After all these assumptions we have only the two parameters $\mu_s = m_s a$ and χ which $\left. g_A/g_V \right|_{\Lambda\mapsto P}$ depends on. The insertion of the CFQM wavefunctions in the expressions for the vector and axial-vector currents then proceeds exactly as with the decay $N \mapsto P e^- \bar{\nu}_e$, and we find

$$\left.\frac{g_A}{g_V}\right|_{\substack{\Lambda\mapsto P \\ CFQM}} = -f_2(\mu_s, \chi) \tag{169}$$

where $F_2(\mu_s,\chi)$ is given by

$$f_2(\mu_s, \chi) = \frac{\int d^3x\, \psi^\dagger(m=0)\gamma_0\gamma_3\gamma_5\psi(m_s)}{\int d^3x\, \psi^\dagger(m=0)\psi(m_s)} = \frac{I(0,\chi) - \frac{k_s}{3} I(1,\chi)}{I(0,\chi) + k\, I(1,\chi)} \tag{170}$$

where

$$k_s = \left[\frac{\sqrt{\mu_s^2+\chi^2} - \mu_s}{\sqrt{\mu_s^2+\chi^2} + \mu_s}\right]^{1/2}, \quad \chi_0 = \chi_s = \chi$$

Had we made the more realistic choice $\chi_0 \neq \chi_s$ which gives three free parameters we would have found instead

$$f_2(\mu, \chi_0, \chi_s) = \frac{I(0,\chi_0,\chi_s) - \frac{k_s}{3} I(1,\chi_0,\chi_s)}{I(0,\chi_0,\chi_s) + k_s\, I(1,\chi_0,\chi_s)} \tag{171}$$

where $I(\ell, x_o, x_s) \equiv \int_0^{x_o} y^2 j_\ell(y) j_\ell(\frac{x_s}{x} y)$. To make the problem easily visualizable, however, we assume $x_o = x_s \equiv x$ and use the earlier form for $f_2(\mu_s, x)$. A plot of the $\left.\frac{g_A}{g_V}\right|_{\Lambda \mapsto p}$ we predict in this fashion for $\Lambda \mapsto p e^- \bar{\nu}_e$ for various values of x and μ_s is given in Fig. IV, together with the region in x, μ_s space determined by the experimental value $\left.\frac{g_A}{g_V}\right|_{\Lambda \mapsto p e^- \bar{\nu}_e} = -.64 \pm .07$. We note that the dependence of $\frac{g_A}{g_V}$ on μ_s is rather slow, so we can't expect to estimate the strange quark mass with any certainty from this experiment alone. We do learn (once again) that x is in the range $2 \lesssim x \lesssim 3$ for the strange quark as well as the nonstrange quarks, which correlates well with the magnetic moment and charge radius results. The remainder of the baryon semileptonic decays may be treated similarly, but we shall defer these calculations to the section on the bag model, which will give us a value for m_s to use in our expression for the weak current matrix elements.

III. Review of the zeroth order bag model

Now that we are familiar with some of the properties of a hadron model consisting of free quarks confined to a spherical region, we may begin our study of the MIT bag model. This model, which was introduced by Chodos et al.[1] in 1974, assumes that quarks are free massless pointlike Dirac fermions constrained to lie within a sphere of radius a. The confinement mechanism is the addition of a constant term B_0, the bag strength, to the lagrangian density within the quark sphere only. When one imposes the boundary conditions (1) no component of the quark electric current density normal to the bag surface, and (2) conservation of the Poincaré charges P_μ and $M_{\mu\nu}$, one finds that the radius and energy of the rest bag are completely determined by the bag strength parameter $B_0^{1/4}$(Mev). In their next paper Chodos et al.[2] fitted this parameter to the mean non-strange S-wave baryon mass of ~ 1180 Mev, which led them to the value $B_0^{1/4} = 120$ Mev. This massless quark model predicts a proton gyromagnetic ratio $g_p = 2.64$ (compared to 2.793 experimentally), a nucleon $g_A/g_V = -1.09$ (experimentally - 1.25 ± .01), and a proton rms charge radius $r_Q = 1.00$ fm (experimentally .89 ± .03 fm. for small q^2). This phenomenological success in explaining nucleon properties which depend primarily on the quark wave functions inside the nucleon led to the generalization of the bag model to bags containing massive quarks[3,4,8] and later gluons and excited quark states.[3] (The latter problems are very difficult in that the two boundary conditions one imposes on the bag surface are not satisfied on a sphere in general when the bag contains vector gluons or non S-wave quarks*.) The massive quark bag gives predictions of the pattern of SU(6) symmetry breaking of the octet and decimet baryon masses, magnetic moments, axial vector charges, and rms charge radii, although only the first three are currently accessible to experiment. The decimet masses are fit rather well by $(B_0^{1/4}, m_s) = (124\ \text{Mev}, 270\ \text{Mev})$,[8] and with this value for the strange quark mass one obtains explicit values for the magnetic moments and g_A/g_V of the strange octet baryons, some of which are known experimentally. The moments, with the exception of the Ξ^- moment, agree rather well, although the $\Sigma^- \mapsto N$ g_A/g_V is broken from the SU(6) result in the wrong direction.[8]

One of the worst results of the bag model is the degeneracy of states differing only in the quark spin orientation, that is the neglect of the strong spin-spin interaction, which is presumably due to vector gluon exchange. Neglect of this effect predicts degeneracy between the $[\pi\rho]$, $[P\Delta]$, and so forth. The result of including gluon effects, which we cannot do consistently in a spherical bag, will be considered later.

Details of the zeroth order (spherical) bag model

Here we shall be rather schematic, due to the large number of papers which treat the zeroth order bag model in great detail. We assume that the strong interaction lagrangian is of the form

---

*With the exception of $\begin{bmatrix} \ell=1 \\ \ell=0 \end{bmatrix}$ free quark spinors.

$$\mathcal{L}_{strong} = \underbrace{\sum_q \bar{\psi}_q (\not{\partial} - m_q) \psi_q}_{\mathcal{L}_0(\psi)} + \mathcal{L}_{0_{glue}} + \mathcal{L}_I (\psi_q, \text{glue}) \tag{172}$$

and further that the effect of $\mathcal{L}_I$ is to strongly bind the quarks and gluons to a spherical region of radius a within which its effects may be neglected:

$$\mathcal{L}_{strong} \sim \begin{cases} \mathcal{L}_0(\psi) + \mathcal{L}_{0\,glue} & r < a \\ 0 & r > a \end{cases} \tag{173}$$

What conditions must we impose to neglect the region r > a and still have current conservation and Poincaré invariance? First, we must impose that the normal component of the quark current is zero:

$$\eta_\mu \bar{\psi}_q \gamma_\mu \psi_q = 0 \Big|_S \tag{174}$$

where η_μ is the 4-normal, which is at rest $\eta_\mu = (0, \hat{r})$. Second, we must impose conservation of P_μ and $M_{\mu\nu}$. To illustrate how this works we just consider P_μ:

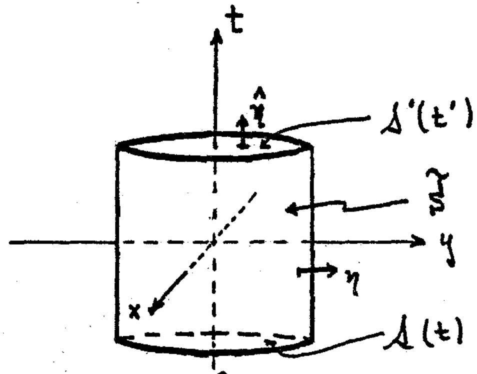
$$P_\mu = \int d\Sigma_\nu T_{\mu\nu} \qquad \text{where } d\Sigma_\nu = \hat{\eta}_\nu d^3x \tag{175}$$

and $\hat{\eta}_\nu$ is a timelike normal.

Taking $\hat{\eta}_\nu = \hat{e}_0$ we have

$$P_\mu(t) = \int d^3x \, T_{\mu 0}(\vec{x}, t) \tag{176}$$

We must impose $P_\mu(t) = P_\mu(t')$ to have a Poincaré invariant theory. Now we consider a 4-dimensional surface integral with ends that are 3-integrals over the bag volume:



$$\int_{\mathcal{S}} d\Sigma_\nu T_{\mu\nu} + \underbrace{\int_{s'} T_{\mu 0}(t')\, d^3x}_{P_\mu(t')} - \underbrace{\int_{s} T_{\mu 0}(t)\, d^3x}_{P_\mu(t)} = 0 \tag{177}$$

by Gauss's law. Since we want P_μ to be constant we must force the first integral to be zero, and it is easy to see that $T_{\mu\nu}\eta_\nu$ must be zero everywhere on the bag surface. Thus we are led to the second boundary condition

$$T_{\mu\nu}\eta_\nu = 0 \Big|_{\mathcal{S}} \tag{178}$$

It is easy to show that the first boundary condition $\bar{\psi}\not{\eta}\psi\big|_{\mathcal{S}} = 0$ is obtained from the linear conditions $i\not{\eta}\psi = \pm\psi\big|_{\mathcal{S}}$[8]. With our conventions the (+) lends to a (+) parity ground state of lower energy than the lowest (-) energy state, so we take the boundary condition $i\not{\eta}\psi = \psi\big|_{\mathcal{S}}$ as physical. (The opposite result follows from the (-) choice.) For a massless Dirac field $T_{\mu\nu}$ is (after some partial integration),

$$T_{\mu\nu} = -\frac{i}{2}\left[\bar{\psi}\gamma_\nu\psi_{,\mu} - \bar{\psi}_{,\mu}\gamma_\nu\psi\right] \tag{179}$$

so

$$T_{\mu\nu}\eta_\nu = -\frac{i}{2}\left[\bar{\psi}\not{\eta}\psi_{,\mu} - \bar{\psi}_{,\mu}\not{\eta}\psi\right] \tag{180}$$

Applying the linear b.c. $i\not{\eta}\psi = \psi$ to this gives

$$T_{\mu\nu}\eta_\nu\Big|_{\mathcal{S}} = \frac{1}{2}\left(\bar{\psi}_{,\mu}\psi + \bar{\psi}\psi_{,\mu}\right) = \frac{1}{2}\left(\bar{\psi}\psi\right)_{,\mu} \tag{181}$$

Thus the general second boundary condition for a free fermion and free gluon bag model is

$$\frac{1}{2}\partial_\mu(\bar\psi\psi) + T_{\mu\nu}(\text{glue})\eta_\nu = 0\Big|_{\mathfrak{F}} \tag{182}$$

Now we have to make some assumption about the form of $T_{\mu\nu}$ (glue). Neglecting it completely doesn't work, because we then find

$$\partial_\mu(\bar\psi\psi) = 0\Big|_{\mathfrak{F}} \tag{183}$$

which cannot be satisfied for free fermions confined to a finite sphere. The next simplest assumption is to just assume that the glue contributes a constant to the lagrangian,[1] which we call $(-B_0)$, so as to give a (+) contribution to the hamiltonian. This gives a $T_{\mu\nu}$

$$T_{\mu\nu}(\text{glue}) = B_0 g_{\mu\nu} \tag{184}$$

and the second boundary condition becomes[1]

$$\partial_\mu(\bar\psi\psi) + 2B_0\eta_\mu = 0\Big|_{\mathfrak{F}} \tag{185}$$

This and the linear condition $i\not\eta\psi = \psi\big|_{\mathfrak{F}}$ are the two boundary conditions one imposes in the zeroth order bag model, and which determine the surface $\mathfrak{F}$. For free quarks in the lowest mode with $j^P = 1/2^+$, $\bar\psi\psi$ is a function of r only, and we have

$$\hat r_\mu \delta_{\mu r}\frac{d}{dr}(\bar\psi\psi) + 2B_0\eta_\mu = 0\Big|_{\mathfrak{F}} \tag{186}$$

This can only be satisfied for $\eta_\mu = (0,\hat r)$, so the surface must be a sphere, with the two boundary conditions

$$i\not\eta\psi = \psi\Big|_{r=a} \qquad\qquad \frac{d}{dr}(\bar\psi\psi) = -2B_0\Big|_{r=a} \tag{187a,b}$$

Inside the sphere ψ must satisfy the free Dirac equation:

$$\not\partial\psi = 0 \qquad r < a \tag{188}$$

For a given mode these three equations together with the requirement that the fermion spinor be normalized to one particle in the sphere,

$$\int d^3x\, \psi^\dagger\psi = 1 \tag{189}$$

completely determine the bag model wavefunctions as a function of the single parameter B. If we have more than one quark field $\{\psi_q\}$ these equations generalize to

$$\begin{gathered} i\gamma\psi_q = \psi_q\Big|_{r=a}, \quad \not\partial\psi_q = 0, \quad \int d^3x\,\psi_q^\dagger\psi_q = 1 \quad \forall q \\ \sum_q \frac{d}{dr}\left(\bar\psi_q\psi_q\right) = -2B_0\Big|_{r=a} \end{gathered} \tag{190}$$

It is easy to see that the set of quark wavefunctions allowed in this massless quark spherical bag model is a subset of the free quark model introduced previously

	Free dimensional params.	Free dimless params	Wave eq. for ψ's
CFQM	a, $\{m_q\}$	$x_0 \equiv ka$, mode numbers of ψ_q	$(\not\partial - m_q)\psi_q = 0 \quad r<a$
Bag Model	B_0 (or a) (these we related by the quadratic b.c.)	mode numbers of ψ_q (x_0 determined by lin. b.c.)	$\not\partial\psi_q = 0 \quad r<a$
Massive Quark* Bag Model	B_0(or a), $\{m_q\}$ (see above)	mode numbers of ψ_q (see above)	$(\not\partial - m_q)\psi_q = 0 \quad r<$
Massive Quark* Bag Model with Yang-Mills gluons	B_0(or a), $\{m_q\}$ (see above)	mode numbers of ψ_q, A (see above)	$(\not\partial - m_q)\psi_q = \text{sources}$, $\Box^2 A^a_\mu = j^a_\mu$

*We include these generalizations for comparison although they have not yet been treated.

Since the first two bag models are specializations of the CFQM, most of the results we found for the CFQM can be used to obtain the predictions of the first two spherical bag models. There are only two problems. First, we must do some trivial algebra with the CFQM spinors to obtain the constraints on the $\{\psi_q\}$ imposed by the two boundary conditions. Second, we note that there is a fundamental difference between the CFQM and bag model, in that the bag model imposes Poincaré invariance as an additional constraint. This means that all the Poincaré charges (energy, $\vec{p}$, $M_{\mu\nu}$) are well determined in the bag model, so we can obtain predictions for the hadron spectrum. In the CFQM, confinement is imposed by an unspecified external force (~ infinitely high potential well extending to infinity), so we can't say what the energy of the entire system is. In this sense the bag model is a clear improvement over the CFQM.

Now we recall the solutions of $(\not{p}-m)\psi = 0$, $r < a$ we found with definite total J that satisfy the spherical boundary conditions* and see what the two boundary conditions require:

$$\psi_{\frac{1}{2}^+,\uparrow}(\vec{x},t) = \alpha_+ \begin{bmatrix} j_0(kr) \\ 0 \\ ik\, j_1(kr)\begin{bmatrix} \cos\theta \\ \sin\theta e^{i\varphi} \end{bmatrix} \end{bmatrix} e^{-i\omega t} \qquad k \equiv \sqrt{\frac{\omega-m}{\omega+m}}$$

$$\psi_{\frac{1}{2}^+,\downarrow}(\vec{x},t) = \alpha_+ \begin{bmatrix} 0 \\ j_0(kr) \\ ik\, j_1(kr)\begin{bmatrix} \sin\theta e^{-i\varphi} \\ -\cos\theta \end{bmatrix} \end{bmatrix} e^{-i\omega t} \tag{191}$$

*Only $J^P = 1/2^{\pm}$ work here; states with higher J don't satisfy the second boundary condition on a sphere.

$$\psi_{\frac{1}{2}^-,\uparrow}(\vec{x},t) = \alpha_- \begin{bmatrix} \frac{i}{k} j_1(kr) \begin{bmatrix} \cos\theta \\ \sin\theta e^{i\varphi} \end{bmatrix} \\ j_0(kr) \\ 0 \end{bmatrix} e^{-i\omega t}$$

$$\psi_{\frac{1}{2}^-,\downarrow}(\vec{x},t) = \alpha_- \begin{bmatrix} \frac{i}{k} j_1(kr) \begin{bmatrix} \sin\theta e^{-i\varphi} \\ -\cos\theta \end{bmatrix} \\ 0 \\ j_0(kr) \end{bmatrix} e^{-i\omega t}$$

The condition $i\gamma\psi = \psi\big|_{r=a}$ gives

$$\tan \chi_q = \frac{\chi_q}{1 - \mu_q \mp \sqrt{\chi_q^2 + \mu_q^2}} \quad (\pm \text{ parity}) \tag{192}$$

where $\chi_q = k_q a$, $\mu_q = m_q a$. As claimed, with a as our independent dimensional parameter (and $\{m_q\}$ fixed) the "shape parameters" $\{\chi_q\}$ must satisfy an eigenvalue equation, rather than being free as in the CFQM. (Since this boundary condition is imposed to keep $n_r j_r = 0\big|_{r=a}$ it might be argued that it is necessary in the CFQM too, but we left it out there simply because it is so restrictive.) For the massless quark case we have

$$\tan \chi_q = \frac{\chi_q}{1 \mp \chi_q} \quad (\pm \text{ parity}) \tag{193}$$

which we can solve numerically for each successive radial excitation:

N	χ(+ par.) $\equiv$ ka	χ(− par.) $\equiv$ ka
1	2.043	3.812
2	5.396	7.002
⋮	⋮	⋮

For $m_q \neq 0$ we have a trancendental equation to solve which depends on the parameter $\mu_q = m_q a$, $0 \leq \mu_q < \infty$, but the solutions turn out to depend rather weakly on μ:

N	$\chi_+(\mu=0)$	$\chi_+(\mu=\infty)$	$\chi_-(\mu=0)$	$\chi_-(\mu=\infty)$
1	2.043	π	3.812	4.49
2	5.396	2π	7.002	7.73
⋮	⋮	$N\pi$	⋮	⋮

In general one may show that the n^{th} radial excitation mode numbers $\chi^{(\pm n)}(\mu)$ ($\pm$ parity) are within the bounds

$$\begin{aligned} (n-\tfrac{1}{2})\pi &< \chi^{(n)}(\mu) \leq n\pi = \chi^{(n)}(\infty) \\ n\pi &< \chi^{(-n)}(\mu) < (n+\tfrac{1}{2})\pi \end{aligned} \tag{194}$$

Thus imposing the first boundary condition (187a) has given us a mass-momentum constraint of the form $ka = \chi^{(\pm n)}(ma)$. A graph of the first 4 functions $\chi^{(\pm n)}(\mu)$ is given in Fig. 8.

Now we impose the second (quadratic) boundary condition (187b) which gives Poincaré invariance:

$$\sum_q \frac{d}{dr}(\bar{\Psi}\Psi) = -2B_0 \Big|_{r=a}$$

Plugging in the explicit massive quark spinors gives

$$2\pi B_0 a^4 = \sum_q 4\pi d_{q\pm}^2 a^3 j_0(x_q)^2(\epsilon_q \mp 1) \quad (\pm \text{ parity}) \tag{195}$$

where $\epsilon = \omega a = \sqrt{x^2+\mu^2}$. This is simply a constraint relating the radius a and the bag strength B_0 for given quark masses $\{m_q\}$, which tells us that we need only one of these two (a, B_0) dimensional parameters;

the other is then determined by the second boundary condition. In practice we shall usually eliminate a and use B_0 as our model defining parameter, implicitly hoping that it is somehow fundamental to have a universal confining pressure. B_0 has units of $(\text{energy})^4$, so we shall usually quote $B_0^{1/4}$(Mev).

The final constraint is normalization, $\int \psi_q^\dagger \psi_q \, d^3x = 1 \;\forall q$, which we also had to impose for the CFQM. This integral determines our normalization constants for the two $j^P = 1/2^\pm$ cases as

$$\alpha_\pm = x \left[\frac{1 \pm \frac{\mu}{\epsilon}}{4\pi} \right]^{1/2} \left(1 \mp \frac{\mu}{\epsilon} j_0(x) - \left(1 \mp \frac{\mu}{\epsilon}\right) j_0(x)^2 \right)^{-1/2} \tag{196}$$

The equations (190) with the Dirac equation generalized to massive quarks completely determine the bag model for a given number of quarks and their mode numbers, masses, and the bag pressure $B_0^{1/4}$. Now we are ready to compare our results with real hadron properties. Most of these calculations were carried out with the CFQ model (which has quark wavefunctions of which the bag model wavefunctions are a subset) and exhibited as a function of $x \equiv ka$ and a(fm). For the nucleons with $m_q \lesssim a^{-1}$ we may save some work by simply working out the location of the bag model $\psi(B_0^{1/4}, m_q; \vec{x})$ in the x, a plane. We assume a constant $m_q = m$ for the quark masses. The location of the bag model solutions is shown in Fig. ? .

We hasten to point out that most of the earlier CFQM calculations were done assuming $m_q = 0$, so we can read off bag model hadron properties from the earlier work for $m_q = 0$ only. For example, for a $B_0^{1/4} = 150$ Mev, $m_q = 0$ proton we find $M_{\text{proton}} = 1.47$ Gev, a = 1.10 fm, $x_0 = 2.04$, and from Figs. A, D in the CFQM section we may estimate $r_Q \sim .80$ fm, $g_p \sim 2.1$. The correct numbers are a = 1.099 fm, $\chi_0 = 2.043$, $r_Q = .801$ fm, and $g_p = 2.114$.

Now that we have our normalized spinors we wish to gain some intuition regarding their quark momentum spectrum. A Fourier decomposition of the spinors into plane wave spinors of all momenta $\vec{k}$ gives the following result for $m_q = 0$: Our spinors are a superposition of plane waves with all directions of momenta $\hat{p}$ equally likely, with the amplitude to find a given scale of momentum $p = |\vec{p}|$ a delta function for $|\vec{p}| = \omega = x_o/a$ as is required by the constraint that the quarks lie on their mass shell. For $m_q \neq 0$ we find a similiar result; in this case the quarks satisfy the free massive Dirac equation, so the momentum spectrum is proportional to $\delta(|\vec{p}|^2 + m^2 - \omega^2)$.

The first point of phenomenological interest is the hadron spectrum predicted by the bag model. Since the CFQM assumed an unspecified confinement mechanism we were not able to obtain hadron masses. Bag model confinement is due to the B_0 term in the lagrangian, however, so we may trivially evaluate H to obtain

$$H = \int T_{oo}\, d^3x = \int_{r<a} d^3x \left(\sum_q (\omega_q \psi_q^\dagger \psi_q) + B_o \right) \tag{197}$$

Taking the matrix element of this operator between hadron states gives us the mass of the hadron. For example, for a hypothetical massless quark bag at rest we find

$$M_q = \langle q | H | q \rangle = \omega_q + \frac{4\pi}{3} B_o a^3 \tag{198}$$

The second boundary condition (187b) here becomes

$$2\pi B_0 a^4 = \frac{x^2(x-1)j_0(x)^2}{1-j_0(x)^2} \tag{199}$$

so we find

$$M_q = a^{-1}\left\{ x + \frac{2x^2(x-1)j_0(x)^2}{3(1-j_0(x)^2)} \right\} = \frac{4x}{3a} \tag{200}$$

where we have used the first boundary condition to simplify this result. Similarly for a bag containing n massless quarks we find

$$M_{nq} = n^{3/4} M_q \tag{201}$$

This is one of the well known problems of the zeroth order bag model; a bag containing six quarks (an exotic) weighs less than two baryons with the same bag strength. If this were true all nuclei would be strongly unstable and would coalesce into giant 3N-quark hadrons. At least one model of gluon effects corrects this difficulty[3] and gives a deuteron that weighs less than a 6-quark exotic by ~ 300 Mev, so this problem is not to be taken seriously. For the most part we will simply ignore the exotic n-quark states, assuming that they are actually above the non-exotic masses and hence are very broad objects.

Now we shall specialize to S-wave baryons, since the meson spectrum is so strongly broken by gluon effects. If we assume that the light quarks are massless we find for the non-strange S-wave baryon (P,Δ) mass

$$M_{P,\Delta} = \left[\frac{2^{10}\pi x^3}{3}\right]^{1/4} B_0^{1/4} = 9.778\, B_0^{1/4} \Big|_{2.043=x} \tag{202}$$

The units of B_0 are clearly $(\text{Mev})^4$. In their first phenomenological bag model paper (which treats non-strange baryons only) Chodos et al.[2] have chosen the mean multiplicity weighted [PΔ] mass of 1180 Mev as a determination of B_0, which gives $B_0^{1/4} \sim 120$ Mev. Their hope is that the bag strength $B_0^{1/4}$ is a fundamental constant for a large class of hadrons, so that one may fix $B_0^{1/4}$ and determine other parameters such as the strange quark mass and the medium-q^2 quark-gluon coupling constant. We shall see that this hope is justified in the zeroth order model for S-wave baryons and perhaps P-wave baryons, although certainly not for charmed mesons. The non-charmed S-wave mesons have a number of difficulties due to large gluon effects, although they too require a bag strength on the order of the 120 Mev we use for baryons.

To treat the strange baryons we need to know the strange quark mass. We obtain a value for m_s by assuming $m_u = m_d = 0$ as usual and fixing $B_0^{1/4} = 124$ Mev to give the Δ mass as 1211 Mev, following which we vary m_s to give a best fit to the Σ^*, Ξ^*, and Ω^- masses. The bag mass with massive quarks is a simple generalization of the previous result (198),

$$M = \left[\frac{4\pi}{3} B_0 a^4 + \sum_q \epsilon_q \right] a^{-1} \tag{203}$$

where $a\,(B_0, \{\chi_q, \mu_q\})$ is determined by the infamous second boundary condition (187b). This fit to the decimet masses is shown in Fig. Q, and we obtain from it a value of 270 ± 20 Mev.[8] Independent of this work the MIT group obtained a value for m_s (in a bag model including gluons and zero-point energy shift effects) of 279 Mev[3] which is consistent with ours.

With the value $m_s = 270 \pm 20$ MeV we may now see how well the octet masses come out. For the octet we lower $B_0^{1/4}$ to 96 MeV in order to correctly give the proton mass, and the Σ, Λ, and Ξ masses then come out with no free parameters.[8] This bag model has no dependence on spin symmetry or isospin, so the Σ and Λ are degenerate;

decimet: $B_{\underset{\sim}{10}} = (124\text{MeV})^4$		octet: $B_{\underset{\sim}{8}} = (96\text{MeV})$	
resonance	predicted mass (MeV)	resonance	predicted mass (MeV)
Δ(1211)	1212 (fixes B_{10})	N(940)	939 (fixes B_8)
Σ^*(1385)	1377 ± 15	Λ(1115)	1110 ± 16
Ξ^*(1530)	1539 ± 29	Σ(1193)	1110 ± 16
Ω(1672)	1703 ± 43	Ξ(1317)	1282 ± 30

The S-wave meson masses are too badly broken by gluon effects to be treated similarly. A simple example of this is seen in comparing the π and the ρ. If we assume a single bag strength for all S-wave mesons we find $m_\pi = m_\rho$ while experimentally $m_\rho \sim 5.5\, m_\pi$, which makes any attempt to learn about these mesons from only their masses and the free quark bag model predictions rather futile. In the baryons we observe a similar effect; the state with more $(\uparrow\uparrow)$ quark pairs is experimentally the more massive of the states predicted to be degenerate in the bag model, as for example $m_\Delta = 1211$ MeV , $m_p = 939$ MeV. In the baryons, however, this effect is not as large as in the mesons and there is some hope that it may be treated as a perturbation. We shall look at the meson masses more seriously when we consider gluon effects, but for the moment we merely observe that a $q\bar{q}$ bag with $B_0^{1/4} = 120$ MeV has a mass $m_{q\bar{q}}$ = 865 MeV, which is not far from the ρ mass m_ρ = 770 MeV.

For the remainder of the S-wave baryon calculations here we assume m_s = 270 MeV, $m_u = m_d = 0$, and $B_0^{1/4}$ = 120 MeV. With these numbers we may predict the host of baryon properties (magnetic moments, electromagnetic form factors, weak and electromagnetic transition moments, and $1^- \to \ell^+\ell^-$ decay rates) that we derived for the CFQM model. To keep the list finite, however, we stop with the octet baryon magnetic moments (which we can trust due to the lack of recoil effects) and g_A/g_V for semileptonic decays. The numbers we find are[8] ;

Magnetic moments

resonance	quark moment composition	$\frac{\mu(\text{hadron})}{\mu(\text{proton})}$: bag model	nonrel SU(6)	expt
P(938)	$\frac{1}{3}(4\mu_u-\mu_d)$	$\equiv 1(g_p=2.643)$	$\equiv 1(g_p=2.793)$	$\equiv 1$
N(939)	$\frac{1}{3}(4\mu_d-\mu_u)$	- 2/3 (0%)	- 2/3	- .6849
Λ(1115)	μ_s	- .233±.006 (-9%)	- 1/3	- .24±.02
Σ^+(1189)	$\frac{1}{3}(4\mu_u-\mu_s)$	.959±.003 (-1%)	1	.927±.165
Σ^0(1193)	$\frac{1}{3}(2\mu_u+2\mu_d-\mu_s)$	.298±.002 (-3%)	1/3	-
Σ^-(1198)	$\frac{1}{3}(4\mu_d-\mu_s)$	- .363±.002 (+1%)	- 1/3	- .53±.13
Ξ^0(1314)	$\frac{1}{3}(4\mu_s-\mu_u)$	- .527±.009 (-5%)	- 2/3	-
Ξ^-(1321)	$\frac{1}{3}(4\mu_s-\mu_d)$	- .200±.008 (-13%)	- 1/3	- .69±.27

Table D

Weak transition moments[8]

decay	g_V	$g_A(SU(6))/g_V$	$g_A(Cab.)/g_V$	$g_A(bag)/g_V$	$g_A/g_V(expt.)$
$N \mapsto P e^- \bar{\nu}_e$	1	-1.67	-1.26	-1.09	-1.250±.009
$\Sigma^- \mapsto \Sigma^0 e^- \bar{\nu}_e$	$-\sqrt{2}$	- .67	- .41	- .44	---
$\Sigma^\pm \mapsto \Lambda e^\pm \overset{(-)}{\nu}_e$	0	g_A= -.816	g_A= -.69	g_A= -.53	g_A= -.62±.03
$\Xi^- \mapsto \Xi^0 e^- \bar{\nu}_e$	- 1	.33	.44	.22	---
$\Lambda \mapsto P e^- \bar{\nu}_e$	$\sqrt{3/2}$	-1.00	- .69	- .72	- .653±.054
$\Sigma^- \mapsto N e^- \bar{\nu}_e$	1	.33	.44	.24	.435±.035
$\Xi^- \mapsto \Lambda e^- \bar{\nu}_e$	$\sqrt{3/2}$	- .33	- .13	- .24	---
$\Xi^- \mapsto \Sigma^0 e^- \bar{\nu}_e$	$\sqrt{1/2}$	-1.67	-1.26	-1.19	---

Table E

In the magnetic moments we quote $\frac{\mu_B}{\mu_P}$ for m_s = 270 ± 20 MeV with a bag strength of $B_0^{1/4}$ = 120 MeV, which gives the scale g_p = 2.643 (experimentally g_p = 2.793). Since there is some variation of μ_B/μ_P with the somewhat arbitrary parameter $B_0^{1/4}$, we also quote the change in this ratio when we use $B_0^{1/4}$ = 96 MeV which correctly gives the proton mass. Simple dimensional arguments tell us that the scale changes to g_p = 3.304, so the 120 MeV numbers are presumably more reliable for magnetic moments.

As with the CFQM magnetic moments we find that the strange quark's mass reduces its contribution to the baryon magnetic moment, which is seen in the fall of the Λ moment from the SU(6) value $-\frac{1}{3}\mu_p$ to $-.23\,\mu_P$ as we increase m_s from 0 to 270 MeV. This compares quite well with the experimental result $(-.24 \pm .02)\mu_p$. Similarly we expect the Σ^- moment to be larger than the SU(6) prediction, which is confirmed experimentally. The only serious discrepancy is the Ξ^- moment, which we previously claimed to be a possible experimental error.

The behavior of μ_B/μ_P as a function of m_s is shown explicitly in Fig. R .

The semileptonic weak g_A/g_V coefficients we derive with m_s = 270 MeV do not come out as well as the magnetic moments. The $\Sigma \mapsto \Lambda$ and $\Lambda \mapsto P$ decays agree fairly well with experiment, though the $\Sigma \mapsto N$ decay prediction breaks SU(6) in the wrong direction. For comparison we include the predictions of the Cabibbo octet model of weak decays with the currently accepted coefficients (D,F) = (.85, .41)[34]. Any SU(6) quark theory with only quark currents in its lagrangian which neglects recoil will give F/D = 2/3 for these decays, with only the scale arbitrary. Comparison with the F,D above clearly tells us that any such quark theory will have trouble fitting these coefficients, since they give F/D = .48. The nonrelativistic SU(6) model is equivalent to the Cabibbo theory with D = 1 (hence F = 2/3), while the bag model with m_s = 270 MeV gives D = .65 for S = 0 and D = .72 for S = 1 decays. It would be interesting to see whether or not the inclusion of recoil effects significantly changes the erroneous F/D = 2/3 constraint forced on us by a recoilless quark model. Another possible source of significant correction to g_A/g_V is the mixing of $|qqq\rangle$ states with $|qqqA\rangle$ states through the strong interaction lagrangian, which can presumably be obtained by fitting the $[P\Delta]$ mass difference. This mixing will also modify the magnetic moments we predicted previously.

Now we abandon the more serious topics for a short while and mention the results we find for some random classical electromagnetic calculations. The first question we consider is whether or not the mean isomultiplet splitting we see in the baryon octet (about -5 Mev/e) has anything to do with classical electromagnetism. One would naively expect that this splitting is due to $(m_d - m_u)$ rather than to the classical electromagnetic field energy. In the bag model we can explicitly check this hypothesis. We proceed as follows; (1) construct the electric current j_μ for each quark within a baryon, (2) solve for A_μ inside the baryon (vector spherical harmonics simplify the math considerably), (3) obtain the classical e.m. self energy as $E = \int d^3x\, j \cdot A$. Typical energy shifts we find in this fashion for $B_0^{1/4} = 120$ Mev are on the order of .5 Mev, an order of magnitude too small to account for isomultiplet splittings. We are thus led to the suspicion that $(m_d - m_u)$ ~ 10 Mev (assuming m_d, $m_u \ll m_s$) which gives the desired 5 Mev splitting. The second calculation we perform is to obtain the charge density within each octet baryon, which we previously did implicitly in finding the electric and magnetic form factors for nucleons. At that time we did not treat the strange baryons, and our results there are rather amusing. The heavy strange quark mass confines that negatively charged quark to a smaller mean radius than the u and d quarks, so that for hadrons containing both u and s quarks the charge distribution may actually change sign. In Fig. 5 we explicitly show the charge distribution we find for the proton and the lambda with $B_0^{1/4} = 120$ Mev and $m_s = 270$ Mev. The Λ with its heavy strange quark is negative on the inside and positive on the outside, rather like an exotic piece of confection. Such predictions are of course impossible to check without data on the Λ electric form factor, although we note that the bag model does rather well with Λ weak and electromagnetic properties for which data exist. The proton electromagnetic form factors, however, are quite well known, and we can certainly use the small q^2 behavior of these form factors (i.e., g_p and r_Q) as independent tests of the validity of the bag model parameter $B_0^{1/4} = 120$ Mev that Chodos et al. suggest. Fitting the experimental proton electric form factor[15] gives a value of $r_Q = .89 \pm .03$ fermis. If we assume massless light quarks we may read off the limits for $B_0^{1/4}$ that fitting r_Q imposes simply by staring at Figs. D and P. The result is $B_0^{1/4} = 135 \pm 5$ Mev, which compares well with the 120 Mev value suggested by the mean nonstrange baryon mass of 1180 Mev, but not so well with the 96 Mev required to give $M_p = 939$ Mev. (The 135 ± 5 Mev value we find for $B_0^{1/4}$ gives a proton mass of 1320 ± 50 Mev.) Since we have not yet estimated the downward shift in the proton energy due to the spin-spin force (which we naively expect to be on the order of the $[P\Delta]$ mass difference), we can only say that this result is superficially reasonable. The question of the size of the gluon shift of the proton energy will be considered in the section on glue effects.

Previously we noted that the large splitting of the π and ρ mesons which the bag model predicts to be degenerate made the fitting of these masses a very ambiguous exercise, although use of the usual proton bag strength of 120 Mev gives a meson mass on the order of the

ρ mass ($m_{q\bar{q}}$ = 865 Mev). Fortunately, however, data on r_Q of the pion are available from a $\pi^- e^-$ (hydrogen-electron) experiment recently done at Dubna[36]. The value of r_Q quoted by this JINR-IHEP-UCLA collaboration is .78 $^{+.09}_{-.10}$ fermis. Since we do not expect r_Q to be particularly sensitive to the admixture of $|q\bar{q}A''\rangle$ and $|q\bar{q}\rangle$ states in the pion (unlike m_π), we may obtain an estimate of the mesonic bag strength $B_0^{1/4}$ merely by fitting this experimental $r_Q(\pi)$ in the zeroth order bag model. This is shown in Fig. I, where we find $B_0^{1/4}(\pi) = 140\ ^{+20}_{-15}$ Mev. Comparison of this $B_0^{1/4}$ with the 135 ± 5 Mev obtained by fitting $r_Q(P)$ gives some credence to the hope that $B_0^{1/4}$ is a constant for the S-wave hadrons. The pion mass predicted by this range of $B_0^{1/4}$ is enormous; $m_\pi = 1010\ ^{+145}_{-110}$ Mev. If this model is correct (and if the experimental r_Q quoted above is correct) we expect that the quark-gluon coupling will shift this zeroth order pion bag model mass downward by about 1 Gev.

V. Glue Mixing Effects

We have seen that the hadron structure assumed in the bag model, i.e. free quarks confined to a spherical region by a constant pressure, reproduces to lowest order some well known hadron properties. In particular the S-wave baryon properties (magnetic moments, transition magnetic moments, nucleon charge radii, and axial vector weak transition moments) come out surprisingly well for such a simple model.

There remain very strong forces in hadrons that are not included in this "zeroth order" bag model. The strongest is evidently the spin-spin force responsible for the splittings $\pi\rho$, K^0K^{0*}, $P\Delta$, and numerous others. In addition there is the force responsible for the large $\pi\eta\eta'$ splittings, which is presumably distinct from the spin-spin force.

The most popular model for the origin of the spin-spin force at present is the well known colored quark - Yang-Mills SU(3) vector gluon theory, which is described by the lagrangian

$$\mathcal{L} = \sum_q \left\{ \bar{\psi}_q (\not\partial - m_q)\psi - g\left(\bar{\psi}_q \gamma_\mu \frac{\lambda^a}{2} \psi_q\right) A^a_\mu \right\} - \frac{1}{4} G^{*a}_{\mu\nu} G^a_{\mu\nu} \tag{204}$$

$$G^a_{\mu\nu} = A^a_{\nu,\mu} - A^a_{\mu,\nu} + g c_{abc} A^b_\mu A^c_\nu \tag{205}$$

$\{c_{abc}\}$ are the SU(3) structure coefficients and $\{\lambda^a\}$ are the $\underset{\sim}{3}$ representation of the SU(3) generators.

Attempts to include the effects of such an interaction between the quarks in extended quark models have thus far followed two lines. The first is to assume the quarks are nonrelativistic and to use the well known Breit effective hamiltonian for single massless vector exchange as one does with positronium, which was carried out by De Rujula, Georgi, and Glashow[37]. Their phenomonological paper uses no explicit quark wavefunctions and is instead content to just estimate the size of mass splittings and deviations from naïve SU(6) electromagnetic properties of baryons, given various terms in the Breit hamiltonian. They find that the S-wave meson and baryon splittings and the P-wave baryon splittings can successfully be explained in this fashion, although each multiplet must be fit separately (they have no model for quark excited states). Another interesting speculation advanced in this paper is that the $\eta\eta'$ splitting is due to mixing with a two-gluon intermediate state, which they cannot evaluate with any certainty. As is pointed out by the authors of the second approach, quark dynamics are certainly relativistic and the assumption of the Breit hamiltonian is of questionable validity. The alternate method of including gluon effects, due to the MIT group [3], is to treat the octet of colored gluons as classical fields excited by the color currents of the quarks $j_\mu^a = \sum_q \bar{\psi}_q \gamma_\mu \frac{\lambda^a}{2} \psi_q$, exactly as one would do in classical electromagnetism. Since this group is working with the bag model they find it necessary to arbitrarily "toss out" the part of the field A_μ^a that doesn't satisfy the linear boundary condition for a spherical hadron [3]. This ad hoc procedure is said to not significantly alter their results. With this prescription a "mixed bag" model of hadrons is obtained that depends on four parameters; $\alpha_c = \frac{g^2}{4\pi}$, Z_0 (finite part of zero point energy $\cdot R_{hadron}$), B_0, and m_s. The fit obtained to the S-wave baryons and mesons is quite good, although the π comes out ~300 MeV and the η and η' remain a problem. The η, η' masses are speculatively attributed to mixing with a two gluon state, as DeRújula et. al. suggest. Problems with this approach are that fitting baryon masses with the gluon interaction requires a smaller baryon, so the scale of the magnetic moments falls by about 30% from the reasonable zeroth order bag value. Similarly the nucleon and pion r_Q fall by about 30%, leaving them in quite significant disagreement with the experimental result. Yet another problem is that this approach apparently does not give similarly reasonable values for the P-wave baryon masses.[38]

We would like to include gluon effects in the bag model in order to explain the observed spin-dependent mass splittings, and we shall consider an approach which treats quarks and gluons symmetrically. The scheme we use is as follows; we assume that a meson is $|m\rangle = a|q\bar{q}\rangle + b|q\bar{q}A\rangle + c|q\bar{q}AA\rangle + \ldots$ and truncate the Hilbert space at $|q\bar{q}A\rangle$ so our mesons have amplitudes to be the normal $|q\bar{q}\rangle$ states with some admixture of the exotic $|q\bar{q}A\rangle$ states. The confining effect of many-gluon exchange which is mocked by $\mathcal{B}_0$ is assumed to be the same for all such states. This is still an infinite dimensional Hilbert space, since we have states with all possible quark and gluon angular momenta and all radial excitations of those states. To make this calculation tractable we consider a further truncated Hilbert space consisting only of the non-exotic SU(6) quark state and the lowest lying quark-gluon state. Tossing out all the higher states in our Hilbert space of course makes this a shadow of the original problem, although it will hopefully illustrate the salient features of the analogous mixed bag model problem with a more complete Hilbert space.

First we develop the bag model with vector constituents, as we need to know the quark-gluon wave functions in order to construct our perturbation Hamiltonian in the truncated Hilbert space. The boundary conditions imposed on a confined vector Yang-Mills field as obtained by Chodos et. al.[3] are

$$\eta_\mu G^a_{\mu\nu} = 0 \Big|_S \qquad \frac{1}{4} G^{*a}_{\mu\nu} G^a_{\mu\nu} = -\mathcal{B}_0 \Big|_S \tag{206}$$

Since we are considering perturbations about $g=0$ for the medium g^2 gluon effects, we impose instead the boundary conditions

$$\eta_\mu F^a_{\mu\nu} = 0 \Big|_S \qquad \frac{1}{4} F^{*a}_{\mu\nu} F_{\mu\nu} = -\mathcal{B}_0 \Big|_S \tag{207}$$

$$F^a_{\mu\nu} \equiv A^a_{\nu,\mu} - A^a_{\mu,\nu} \tag{208}$$

The low q^2 gluons, represented by B_μ, are of course left turned on always. We are to consider only lowest order effects in g, so we obtain the quark-gluon bag states assuming initially no interaction between the quarks and gluons, i.e. $g=0$. This means that the gluons are just an octet of colored photons with no self-coupling, so we may treat the equivalent problem of a bag with quarks and (confined) photons as constituents.

We choose our gauge such that $A_0=0$, so the Lorentz condition $A_{\mu,\mu}=0$ implies $\vec{\nabla}\cdot\vec{A}=0$*. Combining the gluon spin and orbital angular momentum to form a state with definite j, j_z gives us vector spherical harmonics for the gluon wave-functions;

$$\vec{A}_{jm} = \sum_l f_l(r)\,\vec{Y}_{jlm}(\Omega) \tag{209}$$

where $\vec{Y}_{jlm} = \sum_s \langle jm \mid ls, 1\, m-s\rangle Y_{ls}(\Omega)\,\hat{e}_{m-s}$ and $\hat{e}_i = \begin{bmatrix} -\frac{1}{\sqrt{2}}(\hat{x}+i\hat{y}) \\ \hat{z} \\ \frac{1}{\sqrt{2}}(\hat{x}-i\hat{y}) \end{bmatrix}$ (210)

Imposing $\vec{\nabla}\cdot\vec{A}=0$ and $(\nabla^2+\omega^2)\vec{A}=0$ gives two sets of solutions for given j;

$$\vec{A}^E_{jm} = \left\{ j_{j-1}(\omega r)\vec{Y}_{jj-1m} - \sqrt{\frac{j}{j+1}}\, j_{j+1}(\omega r)\vec{Y}_{jj+1m} \right\} e^{-i\omega t} \tag{211}$$

electric multipoles $P(\vec{H}) = (-)^j$

$$\vec{A}^M_{jm} = j_j(\omega r)\,\vec{Y}_{jjm}\, e^{-i\omega t} \tag{212}$$

magnetic multipoles $P(\vec{H}) = (-)^{j+1}$

The linear boundary condition for a bag at rest gives the two conditions

$$\hat{\eta}\cdot\vec{A}\Big|_S = 0 = \hat{\eta}\times(\vec{\nabla}\times\vec{A})\Big|_S \tag{213}$$

*The instantaneous Coulomb interaction which we ignore gives no contribution, as the $|q\bar{q}(1)\rangle$ state has zero color charge density.

For each set of multipoles this implies, for a spherical bag,

$$A^{E}_{jm}: \qquad j_{j-1}(\omega a) + j_{j+1}(\omega a) = 0 = j_j(\omega a) \qquad j \geq 1 \tag{214}$$

$$A^{M}_{jm}: \qquad j_{j-1}(\omega a) = \frac{j}{j+1}\, j_{j+1}(\omega a) \qquad j \geq 1 \tag{215}$$

Since a is fixed by B_0, these are eigenvalue equations for ωa for electric and magnetic multipole gluons respectively. (The two constraints in (214) are equivalent.)

Now we must satisfy the quadratic boundary condition in (206), or more properly its generalization to the mixed bag problem:

$$-\frac{1}{2} G^{*}_{\mu\nu} G_{\mu\nu}\Big|_{S} + \eta\cdot\partial(\bar{\Psi}\Psi)\Big|_{S} = 2B_0 \tag{216}$$

Setting g = 0 and again assuming a spherical bag, this gives

$$\left\{ \omega^2 |\vec{A}|^2 - |\vec{B}|^2 \right\}\Big|_{S} - \frac{d}{dr}(\bar{\Psi}\Psi)\Big|_{S} = 2B_0 \tag{217}$$

If we take a particular magnetic multipole field for the gluon, $a_j j_j(\omega r)\vec{Y}_{jjm}$, the second boundary condition term in brackets on the left hand side of (217) becomes

$$\{\ \} = a_j^2\omega^2 \Big[j_j(\omega a)^2 |\vec{Y}_{jjm}|^2 - \frac{j+1}{2j+1} j_{j-1}(\omega a)^2 |\vec{Y}_{jj-1m}|^2 - \frac{j}{2j+1} j_{j+1}(\omega a)^2 |\vec{Y}_{jj+1m}|^2 + \frac{\sqrt{j(j+1)}}{2j+1} j_{j-1}(\omega a) j_{j+1}(\omega a) \left(\vec{Y}^{*}_{jj-1m}\cdot\vec{Y}_{jj+1m} + c.c. \right) \Big] \tag{218}$$

which has explicit dependence on the solid angle. We thus reach the well known result that it is inconsistent with the two boundary conditions to put constituents with orbital angular momentum in a spherical bag in general. Rather than concern ourselves with the very complicated problem of nonspherical perturbations of the bag model [39] we simply allow the bag strength variation as a function of the solid angle as is necessary to satisfy the linear and quadratic boundary conditions on a spherical surface. Furthur, the average pressure over the surface $\langle B(\Omega)\rangle$ we assume to be the universal bag constant B_0. This ad hoc prescription is necessary to make even this simple model solvable. Having made this assumption the second b.c. becomes, for a magnetic j-pole gluon,

$$\frac{|a_j|^2\omega^2}{4\pi}\left[j_j(\omega a)^2 - \frac{j+1}{2j+1}j_{j-1}(\omega a)^2 - \frac{j}{2j+1}j_{j+1}(\omega a)^2\right] - \frac{d}{dr}\langle\bar\Psi\Psi\rangle_\Omega = 2B_0\Big|_{\mathcal{S}} \qquad (219)$$

or, using the first boundary condition,

$$\frac{|a_j|^2\omega^2}{4\pi}\left[j_j(\omega a)^2 - \frac{j+1}{j}j_{j-1}(\omega a)^2\right] - \frac{d}{dr}\langle\bar\Psi\Psi\rangle_\Omega = 2B_0\Big|_{\mathcal{S}} \qquad (220)$$

The coefficient a_j is determined by the normalization condition of N gluons per bag;

$$N_\omega = \frac{1}{2}\int d^3x\,(|\vec{E}|^2+|\vec{H}|^2) = \frac{|a_j^{(N)}|^2}{2\omega}\int_0^{\omega a}\eta^2\left\{j_j(\eta)^2 + \frac{j+1}{2j+1}j_{j-1}(\eta)^2 + \frac{j}{2j+1}j_{j+1}(\eta)^2\right\}d\eta \qquad (221)$$

$$\equiv \frac{|a_j|^2}{2\omega}I^{(j)}(\omega a)$$

$$a_j^{(N)} = \frac{x_j}{a}\sqrt{\frac{2N}{I^{(j)}(x_j)}} \qquad (222)$$

where $x_j = \omega_j a$ is the gluon mode number, which is a solution of the first boundary condition (215);

$$j_{j-1}(x_j) = \frac{j}{j+1}j_{j+1}(x_j)$$

The lowest lying gluon state is the $j^P = 1^+$ magnetic dipole gluon, for which we have

$$j_0(x_1) = \frac{1}{2}j_2(x_1) \qquad x_1 = 2.744,\ldots \qquad (223)$$

The integral $I^{(1)}(x_1=2.744) = 2.277$ so the normalization of the gluon field is

$$a_1 = 2.572a^{-1} \qquad (224)$$

$$\vec{A}^{M}_{1m} = 2.572\,a^{-1} j_1(2.744\, r/a)\,\vec{Y}_{11m}\, e^{-i\omega t} \qquad \omega = 2.744\,a^{-1} \tag{225}$$

Since the massless gluons do not carry a scale of length the gluon term in the second boundary condition (216) must by dimensional arguments be of the form

$$-\tfrac{1}{2} G^{*}_{\mu\nu} G_{\mu\nu} \Big|_{S} = Z[\ell, x_\ell]\, a^{-4} \tag{226}$$

where the numerical coefficient Z is in general

$$Z[\ell, x_\ell] = \frac{x_\ell^{4}}{2\pi I^{(\ell)}(x_\ell)} \left(j_\ell(x_\ell)^2 - \frac{\ell+1}{\ell} j_{\ell-1}(x_\ell)^2 \right) \tag{227}$$

This result means that from the point of view of the quarks in the bag (given our assumptions about the boundary conditions and with $g_{qqA} = 0$) the only effect of putting a gluon in the bag is to replace the bag strength with a lower effective bag strength

$$B_{\text{eff}} = B_0 - \tfrac{1}{2} Z[\ell, x_\ell]\, a^{-4} \tag{228}$$

for our lowest 1^+ gluon we find $Z[1, 2.744] = .4368$. To quote some numbers we compare the qqq baryon bag with a $qqqA$ configuration with A in the lowest 1^+ mode (massless quarks);

	qqq	$qqqA(1^+)$
E(Mev)	1173	1549
a(fm)	1.37	1.51
$B^{1/4}_{\text{eff}}$(Mev)	120	109.4

So the no gluon and lowest one gluon baryon states are separated by about 300 MeV. If the qqA coupling is not small we shall have sizable mixing between glueless and glue containing states. This is not at all surprising, since it is the $q\bar{q}A$ coupling that we expect to separate the masses of the $\pi\rho$, $P\Delta$, and so forth. As an example the mixing of the usual quark states and exotic quark-and-glue states we shall consider the $\pi\rho$ system.

We shall work only to lowest order in g so we take only the Yukawa term in the Yang-Mills glue and colored quark lagrangian (204);

$$\mathcal{L} = \sum_q \bar{\psi}_q(\not{p} - m_q)\psi_q - g\,\bar{\psi}_q \gamma_\mu \lambda^a \psi_q A^a_\mu + \mathcal{L}_0(A^a_\mu) \qquad (229)$$

It is easy to see that the gA^3 term in the lagrangian (204) will not in itself give rise to a $\pi\rho$ splitting, so we neglect this term.

To lowest order in g with these approximations our mesons are of the form

$$|\pi\rangle = |q\bar{q}\rangle_\pi + \mathcal{O}(g)\,|q\bar{q}A\rangle_\pi$$

$$|\rho\rangle = |q\bar{q}\rangle_\rho + \mathcal{O}(g)\,|q\bar{q}A\rangle_\rho \qquad (230)$$

where the $\{|q\bar{q}\rangle\}$ are the conventional quark model states and the $\{|q\bar{q}A\rangle_{\pi,\rho}\}$ are quark and gluon states having the same quantum numbers as the π and ρ respectively. We consider mixing with the lowest gluon mode only, which is a magnetic dipole gluon $(J^P = 1^+)$. There is an electric dipole gluon state which lies very close in energy to the M1 gluon, but this state will notcombine with the ground state of the $q\bar{q}$ system to form a negative parity state without relative orbital angular momentum. For this reason it is not unreasonable to consider only the M1 gluon state as a first approximation.

Now we consider which of the possible $|q\bar{q}A(M1)\rangle$ ground states we expect to mix with the $|q\bar{q}\rangle_\pi$ and the $|q\bar{q}\rangle_\rho$ by determining the allowed spins and parities;

		A: $J^P = 1^+$
$q\bar{q}$:	$J^P = 0^-$	$\|q\bar{q}A(1^-)\rangle$
	$J^P = 1^-$	$\|q\bar{q}A(0^-, 1^-, 2^-)\rangle$

There is only one candidate state to mix with the $|q\bar{q}\rangle_\pi$, a $J^P = 0^-$ state in which the quark-antiquark pair is in an $s_{q\bar{q}}=1$ state. We have two states to mix with the $|q\bar{q}\rangle_\rho$, with $s_{q\bar{q}}=0$ and $s_{q\bar{q}}=1$ respectively. To find the $\pi\rho$ masses predicted by our model as a function of $\mathcal{B}_0^{1/4}$ and g, we need only find the matrix elements of the Yukawa part of the quark-gluon interaction hamiltonian,

$$H_I = -g \sum_q \int d^3x\, \bar{\psi}_q \vec{\gamma} \lambda_a \psi_q \cdot \vec{A}^a \tag{231}$$

following which we may diagonalize the perturbed hamiltonian matrix. This will tell us the energy eigenvalues $E\pi$, $E\rho$, and the mixing angles between the $|q\bar{q}\rangle$ and $|q\bar{q}A\rangle$ substates. The principal difficulty in this calculation is the technical problem of explicitly constructing the color singlet $|q\bar{q}A\rangle$ states and obtaining the matrix element of H_I , as we have three quark colors, eight gluon colors, and eight groups of terms in the interaction H_I . There is also some subtlety in dealing with the combinatorics of colored antiquarks. First, we name our quark colors;

$$|\text{quark } \underset{\sim}{3} \text{ color}\rangle = \begin{bmatrix} m & & c \\ & & \\ & y & \end{bmatrix} \qquad \begin{matrix} c \doteq \text{cyan} \\ m = \text{magenta} \\ y = \text{yellow} \end{matrix} \tag{232}$$

The m,c quarks form a color $SU(2)$ isodoublet, with $I_z|c\rangle = +\frac{1}{2}|c\rangle$.
These correspond to the d, u of quark flavor. The antiquarks transform according to the conjugate representation $D^*(\underset{\sim}{3})$ of the color $SU(3)$ group, which implies that the three dimensional color generator matrices for the antiquarks are the negative transposes of the color quark generators. By merely rearranging the antiquark basis states and introducing some phases we may construct a basis of colored antiquarks for which the conventional λ matrices act as color generators. Explicitly, this is

$$|\text{antiquark } \bar{\underset{\sim}{3}} \text{ color}\rangle = \begin{bmatrix} & -\bar{y} & \\ & & \\ \bar{c} & & -\bar{m} \end{bmatrix} \tag{233}$$

We may combine these states to form overall $|q\bar{q}\rangle$ color octet and singlet states. In terms of explicit colors, these states are;

Color octets:

$$
\begin{array}{ccccc}
 & -|m\bar{y}\rangle & & -|c\bar{y}\rangle & \\
 & & \frac{1}{\sqrt{6}}\left(|c\bar{c}\rangle + |m\bar{m}\rangle - 2|y\bar{y}\rangle\right) & & \\
|m\bar{c}\rangle & & \frac{1}{\sqrt{2}}\left(|c\bar{c}\rangle - |m\bar{m}\rangle\right) & & -|c\bar{m}\rangle \\
 & |y\bar{c}\rangle & & -|y\bar{m}\rangle &
\end{array}
\tag{234}
$$

Color singlet;

$$\frac{1}{\sqrt{3}}\left(|c\bar{c}\rangle + |m\bar{m}\rangle + |y\bar{y}\rangle\right) \tag{235}$$

The color octet states are arranged to have the same " c - m isospin " and " y strangeness " as the corresponding well known baryon octet states. We also have the color octet of gluons

$$
\begin{array}{ccccc}
 & |A_N\rangle & & |A_P\rangle & \\
 & & |A_\Lambda\rangle & & \\
|A_{\Sigma^-}\rangle & & |A_{\Sigma^0}\rangle & & |A_{\Sigma^+}\rangle \\
 & |A_{\Xi^-}\rangle & & |A_{\Xi^0}\rangle &
\end{array}
\tag{236}
$$

The solution to the problem of constructing a color singlet $|q\bar{q}\rangle$ state is given above. To construct a color singlet $|q\bar{q}A\rangle$ state, however, we must vectorially combine the color octet $|q\bar{q}\rangle$ and $|A\rangle$ basis states, using the $\langle \underline{1} | \underline{8}\,\underline{8}\rangle$ SU(3) Clebsch-Gordon coefficients. To within overall normalization, the color singlet combination is;[45]

$$|q\bar{q}A(\underline{1})\rangle = |Q_N\rangle|A_{\Xi^0}\rangle + |Q_\Lambda\rangle|A_\Lambda\rangle + (\text{all neutrals}) - |Q_P\rangle|A_{\Xi^-}\rangle - (\text{all charged}) \tag{237}$$

where $\quad |Q_N\rangle \equiv -|m\bar{y}\rangle$

In view of the large number of terms we shall deal with it is probably wiser to deal with cartesian labels 1,...,8 for the color state vectors than the particle labels p,...., . The relation between these different bases is obvious up to overall phases, e.g.

$$|A(\Sigma^+)\rangle = \pm \frac{1}{\sqrt{2}}\left(|A(1)\rangle + i\,|A(2)\rangle\right) \tag{238}$$

It is necessary for our purposes to have correct relative phases between the states in our matrix element. We may obtain a consistent set of states by adopting the phase conventions of de Swart[46], which requires that the matrix elements of the " isospin " operators and K_+ (which raises both " isospin " and " hypercharge ") all be positive if nonzero. We may also relate the " particle " and cartesian components of the λ matrices of color SU(3) by requiring that their commutators with $\vec{I}, K_+$ give the same linear combination of λ matrices as the linear combination of particle states we obtain by operating on the corresponding initial state with $\vec{I}$ and K_+. For example,

$$K_+|\Xi^-\rangle = \frac{1}{\sqrt{2}}|\Sigma^0\rangle + \sqrt{\frac{3}{2}}\,|\Lambda\rangle \tag{239}$$

implies

$$\left[\mathcal{O}^{(3)}(K_+), \lambda_{\Xi^-}\right] = \frac{1}{\sqrt{2}}\left(\lambda_{\Sigma^0} + \sqrt{3}\,\lambda_\Lambda\right) \tag{240}$$

We fix the phase of the λ s by choosing

$$\mathcal{O}^{(3)}(K_+) = \frac{1}{2}\left(\lambda_4 + i\lambda_5\right) \tag{241}$$

in the conventional representation of the λ matrices.[47] The resulting λ " particle " matrices are

$$\begin{array}{ccccc} & -\frac{\lambda_6+i\lambda_7}{2} & & -\frac{\lambda_4+i\lambda_5}{2} & \\ \frac{\lambda_1-i\lambda_2}{2} & & \frac{\lambda_8}{\sqrt{2}} \quad \frac{\lambda_3}{\sqrt{2}} & & -\frac{\lambda_1+i\lambda_2}{2} \\ & \frac{\lambda_4-i\lambda_5}{2} & & -\frac{\lambda_6-i\lambda_7}{2} & \end{array} \tag{242}$$

We may check these phases by forming the dot product $\lambda(1)^a\lambda(2)^a$ in terms of particle states as given previously;

$$\lambda(1)^a\lambda(2)^a = \lambda(1)^N\lambda(2)^{\Xi^0} + \ldots - \lambda(1)^P\lambda(2)^{\Xi^-} - \ldots$$

$$= (-)^2\frac{\lambda(1)_6 + i\lambda(1)_7}{2}\,\frac{\lambda(2)_6 - i\lambda(2)_7}{2} + \ldots - (-)\frac{\lambda(1)_4 + i\lambda(1)_5}{2}\,\frac{\lambda(2)_4 - i\lambda(2)_5}{2} - \ldots$$

$$= \frac{1}{2}\left[\lambda(1)_6\lambda(2)_6 + \lambda(1)_7\lambda(2)_7 + \ldots + \lambda(1)_4\lambda(2)_4 + \lambda(1)_5\lambda(2)_5 + \ldots\right] \tag{243}$$

This, to an irrelevant factor of 2, is exactly the cartesian scalar product which we expect. The value of this exercise is that we may now take the " particle " $|q\bar{q}\rangle$ bilinear color octet states and recast them as the cartesian components $|Q_i\rangle$, as this problem is isomorphic to the problem of relating the λ " particle " and cartesian components. Explicitly we find

$$|Q_1\rangle = \frac{1}{\sqrt{2}}(|m\bar{c}\rangle + |c\bar{m}\rangle) \qquad |Q_2\rangle = \frac{i}{\sqrt{2}}(|m\bar{c}\rangle - |c\bar{m}\rangle) \qquad |Q_3\rangle = \frac{1}{\sqrt{2}}(|c\bar{c}\rangle - |m\bar{m}\rangle)$$

$$|Q_4\rangle = \frac{1}{\sqrt{2}}(|y\bar{c}\rangle + |c\bar{y}\rangle) \qquad |Q_5\rangle = \frac{i}{\sqrt{2}}(|y\bar{c}\rangle - |c\bar{y}\rangle) \qquad |Q_6\rangle = \frac{1}{\sqrt{2}}(|m\bar{y}\rangle + |y\bar{m}\rangle) \tag{244}$$

$$|Q_7\rangle = -\frac{i}{\sqrt{2}}(|m\bar{y}\rangle - |y\bar{m}\rangle) \qquad |Q_8\rangle = \frac{1}{\sqrt{6}}(|c\bar{c}\rangle + |m\bar{m}\rangle - 2|y\bar{y}\rangle)$$

This increase in complexity is compensated for by the fact that our color singlet state may now be written as

$$|q\bar{q}A(\underline{1})\rangle = \frac{1}{\sqrt{8}}\left[\,|Q_1\rangle|A_1\rangle + \ldots + |Q_8\rangle|A_8\rangle\,\right] \tag{245}$$

To complete the construction of our basis states we must now bring in the quark and gluon spins. Our convention for antiquarks is that $|\bar{q}\uparrow\rangle$ has $j_z = \frac{+1}{2}$, and under charge conjugation

$$C|q\uparrow\rangle = |\overline{q\downarrow}\rangle = |\bar{q}\uparrow\rangle$$

$$C|q\downarrow\rangle = |\overline{q\uparrow}\rangle = -|\bar{q}\downarrow\rangle \tag{246}$$

Now we consider the $|q\bar{q}\rangle$ π and ρ states. The full state is simply a direct product of the color and spin substates, so we find

$$|q\bar{q}\,(\text{color } \underline{1}, J\,J_z = 1\,1)\rangle_\rho = \frac{1}{\sqrt{3}}\left(|c\uparrow\bar{c}\uparrow\rangle + |m\uparrow\bar{m}\uparrow\rangle + |y\uparrow\bar{y}\uparrow\rangle\right)$$

$$|q\bar{q}\rangle_\pi = \frac{1}{\sqrt{6}}\left(|c\uparrow\bar{c}\downarrow\rangle - |c\downarrow\bar{c}\uparrow\rangle + |m\uparrow\bar{m}\downarrow\rangle - \ldots\right) \tag{247}$$

Similarly we may combine the spins and colors in the $|q\bar{q}A\rangle$ states to obtain J,J_z=1,1 and 0,0 states to mix with the ρ and π respectively. There, we have

$$|q\bar{q}A\rangle_{\rho\ \text{spins}} = \begin{cases} \frac{1}{2}\left[(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle)|A(+)\rangle - \sqrt{2}\,|\uparrow\uparrow\rangle|A(0)\rangle\right] & J_{q\bar{q}}=1 \\ \frac{1}{\sqrt{2}}\left(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle\right)|A(+)\rangle & J_{q\bar{q}}=0 \end{cases} \tag{248}$$

$$|q\bar{q}A\rangle_{\pi\ \text{spins}} = \frac{1}{\sqrt{3}}\left[|\downarrow\downarrow\rangle|A(+)\rangle - \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle)|A(0)\rangle + |\uparrow\uparrow\rangle|A(-)\rangle\right] \tag{249}$$

where $|A(+)\rangle$, $|A(0)\rangle$, and $|A(-)\rangle$ are gluon states with J_z= 1,0,and -1 respectively.
The full $|q\bar{q}A\rangle$ states are

$$|q\bar{q}A\rangle_{\rho} = \begin{cases} \sum_{i=1}^{8} \frac{1}{2\sqrt{8}}\left[(|Q_i\uparrow\downarrow\rangle + |Q_i\downarrow\uparrow\rangle)|A_i(+)\rangle - \sqrt{2}\,|Q_i\uparrow\uparrow\rangle|A_i(0)\rangle\right] & J_{q\bar{q}}=1 \\ \sum_{i=1}^{8} \frac{1}{4}\left[|Q_i\uparrow\downarrow\rangle - |Q_i\downarrow\uparrow\rangle\right]|A_i(+)\rangle & J_{q\bar{q}}=0 \end{cases} \tag{250}$$

and

$$|q\bar{q}A\rangle_{\pi} = \sum_{i=1}^{8} \frac{1}{2\sqrt{6}}\left[|Q_i\uparrow\uparrow\rangle|A_i(-)\rangle - \frac{1}{\sqrt{2}}(|Q_i\uparrow\downarrow\rangle + |Q_i\downarrow\uparrow\rangle)|A_i(0)\rangle + |Q_i\downarrow\downarrow\rangle|A_i(+)\rangle\right] \tag{251}$$

The additional flavor degree of freedom is a trivial modification of this result. Now we consider the matrix elements of our interaction hamiltonian (231) between these $|q\bar{q}\rangle$ and $|q\bar{q}A\rangle$ states. In terms of cartesian components this hamiltonian is

$$H_I = \sum_{i=1}^{8} \int d^3x\ -g\,\bar{\psi}\vec{\gamma}\lambda^i\,\psi\cdot\vec{A}^i = -g\int d^3x \sum_{i=1}^{8} \bar{\psi}\lambda^i\left[\gamma_z A^i(0) - \gamma_+ A^i(-) - \gamma_- A^i(+)\right]\psi \tag{252}$$

where the spherical $\vec{\gamma}$ and $\vec{A}$ components are

$$V_\pm = \mp\frac{1}{\sqrt{2}}\left[V_x \pm iV_y\right] \tag{253}$$

As our cartesian color components do not have a preferred direction, the full matrix element of H_I will simply be the number of cartesian colors (8) times the matrix element of any one of these. It is at this stage that we see the advantage of using this color bases.

Before we evaluate the H_I matrix elements explicitly, we should note whether or not we expect any of them to be zero. The only nontrivial quark and gluon quantum number which we haven't explicitly conserved is charge conjugation, which is of course conserved by the strong interaction. Our matrix element and the properties of the individual components under charge conjugation are;

$$\mathcal{M}_{fi} = \underbrace{\langle q\bar{q}(\underset{\sim}{8})|\langle A(\underset{\sim}{8})|}_{\eta_{c'}}\ \underbrace{\mathrm{J}\cdot A}_{(+)}\ \underbrace{\underbrace{|q\bar{q}(\underset{\sim}{1})\rangle}_{\eta_c}\ \underbrace{|0_A(\underset{\sim}{1})\rangle}_{(+)}}_{\eta_c} \tag{254}$$

where η_c and $\eta_{c'}$ are phases. ($\eta_c = +1,-1$ for the π and ρ respectively.) Obviously we require that the $|q\bar{q}A\rangle$ and $|q\bar{q}\rangle$ states have the same C-parity. Under C, most of the $|q\bar{q}\rangle$ substates change their color magnetic quantum numbers in a complicated fashion, but if we restrict our attention to the $3(\Sigma^0)$ or $8(\Lambda)$ components we see only a change in phase;

$$|q\bar{q}A\rangle = c_1|Q_p\rangle|A_{\Xi^-}\rangle + \ldots + c_8|Q_\Lambda\rangle|A_\Lambda\rangle + c_3|Q_{\Sigma^0}\rangle|A_{\Sigma^0}\rangle \tag{255}$$

$$\mathcal{C}|q\bar{q}A\rangle = c_1\eta_{Q_p}|Q_{\Xi^-}\rangle(-|A_p\rangle) + \ldots + c_8\eta_{Q_\Lambda}|Q_\Lambda\rangle(-|A_\Lambda\rangle) - c_3\eta_{Q_{\Sigma^0}}|Q_{\Sigma^0}\rangle|A_{\Sigma^0}\rangle \tag{256}$$

Our $|q\bar{q}A\rangle$ states are $\mathcal{C}$-eigenstates, so the phases $\eta_{Q_p},\ldots,\eta_{Q_\Lambda},\eta_{Q_{\Sigma^0}}$ must all be the same phase η_Q;

$$\eta_Q = -\eta_{q\bar{q}A} \tag{257}$$

For a color singlet $|q\bar{q}\rangle$ pair $\eta_{q\bar{q}} = (-)^{\ell+s}$, where s and ℓ are the combined $|q\bar{q}\rangle$ spin and relative orbital angular momentum respectively. The $|q\bar{q}\rangle$ pair in the $|q\bar{q}A\rangle$ state has the same $\mathcal{C}$-parity as a $|q\bar{q}\rangle$ pair in a color singlet;

$$\eta_Q = (-)^{\ell+s} = -\eta_{q\bar{q}A} \tag{258}$$

Thus, we find

$$\mathcal{C}|\underbrace{q\bar{q}}_{s,\ell}A\rangle = (-)^{\ell+s+1}|q\bar{q}A\rangle \tag{259}$$

As we have no relative orbital angular momentum between the quarks in our states, we simply have a charge conjugation phase of $(-)^{S_{q\bar{q}(A)}+1}$. The initial $|q\bar{q}\rangle$ state must have the same C-parity as the $|q\bar{q}A\rangle$ state, since our strong Yukawa term $\vec{j}\cdot\vec{A}$ doesn't change the C-parity of the states it operates on. The initial state has C-parity $(-)^{S_{q\bar{q}}}$, so this is equivalent to the statement

$$\langle q\bar{q}A|g\bar{\psi}\vec{\gamma}\lambda^a\psi\cdot\vec{A}^a|q\bar{q}\rangle = 0 \tag{260}$$

unless $S_{q\bar{q}(A)} = S_{q\bar{q}} \pm 1 \quad (\ell = 0)$

This has an important consequence. One state we were to mix with the $|q\bar{q}\rangle_\rho$ has $S_{q\bar{q}(A)} = 1$, so the matrix element

$$\underset{(S_{q\bar{q}(A)}=1)}{{}_\rho\langle q\bar{q}A|H_I|q\bar{q}\rangle_\rho} = 0 \tag{261}$$

In truncating our Hilbert space to the lowest lying quark and gluon states for which the matrix element of the interaction hamiltonian is nonzero, we have reduced the problem of computing the πρ splitting to a model problem in which we are simply diagonalizing a 2×2 hamiltonian for each of the two mesons. Before we proceed, however, we should consider the question of whether or not we expect the model to give results of increased accuracy when we add more of the excited quark and gluon states to our small Hilbert space. Although the only way to actually answer this question is to explicitly bring in these excited states, we can naively say that the model will not be greatly improved for two reasons. First, the bag model has many more excited states than we expect in a real physical quark and gluon theory, as a result of not removing the CM motion of the constituents. (We discuss this problem in Appendix B .) Second, we have no reason to believe that our free, on mass shell bag model states bare more than a passing resemblance to the physical meson state vectors, so including more of these states need not give a more accurate picture of the real mesons.

Now we return to our 2x2 $\pi\rho$ mixing problem. First, we note that our C-parity argument predicts that one of the ρ meson $\langle q\bar{q}A|H_I|q\bar{q}\rangle$ matrix elements must be zero ;

$$_{\rho}\langle q\bar{q}(s_{q\bar{q}}=1)A|H_I|q\bar{q}\rangle_{\rho} = 0 \tag{262}$$

This zero we shall check to insure the correctness of our C-parity argument. We have previously constructed the full $|q\bar{q}\rangle$ and $|q\bar{q}A\rangle$ ρ meson basis states in (247) and (250) respectively. We expect to find that the hamiltonian (252) between (247) and the (250) $s_{qq} = 1$ state gives zero. Before we proceed, we may simplify our calculation somewhat further by introducing a shorthand notation for our matrix elements. In a bag model without color but otherwise identical to the model we have been considering, we define

$$\langle q\uparrow|\langle A(0)|\int \bar{\psi}\vec{\gamma}\psi\cdot\vec{A}\, d^3x\, |q\uparrow\rangle \equiv \xi \tag{263}$$

This allows us to write a colorless hamiltonian as a combination of projection operators;

$$\begin{aligned} H_I(\text{colorless}) &= -g\int d^3x\, \bar{\psi}\vec{\gamma}\psi\cdot\vec{A} = -g\xi\left[|q\uparrow\rangle\langle q\uparrow| - |q\downarrow\rangle\langle q\downarrow|\right]|A(0)\rangle\langle 0| + \ldots + h.c. \\ &= -g\xi\Big\{|A(0)\rangle\langle 0|\left[|q\uparrow\rangle\langle q\uparrow| - |q\downarrow\rangle\langle q\downarrow| - |\bar{q}\uparrow\rangle\langle\bar{q}\uparrow| + |\bar{q}\downarrow\rangle\langle\bar{q}\downarrow|\right] \\ &\quad + \sqrt{2}\Big(-|A(+)\rangle\langle 0|\left[|q\downarrow\rangle\langle q\uparrow| - |\bar{q}\downarrow\rangle\langle\bar{q}\uparrow|\right] + |A(-)\rangle\langle 0|\left[|q\uparrow\rangle\langle q\downarrow| - |\bar{q}\uparrow\rangle\langle\bar{q}\downarrow|\right]\Big)\Big\} + h.c. \end{aligned} \tag{264}$$

$$\begin{aligned} = -g\xi\Big\{&\Pi_{A(0)}\left(\Pi_{q(0)} + \Pi_{\bar{q}(0)}\right) - \sqrt{2}\,\Pi_{A(+)}\left(\Pi_{q(-)} + \Pi_{\bar{q}(-)}\right) \\ &+ \sqrt{2}\,\Pi_{A(-)}\left(\Pi_{q(+)} + \Pi_{\bar{q}(+)}\right)\Big\} + h.c. \end{aligned} \tag{265}$$

Our SU(3) color hamiltonian (252) is simply a sum of eight such sets of projection operators, one for each cartesian component of the color octet gluon field;

$$H_I(\text{SU(3) color}) = -g\xi\sum_{i=1}^{8}\left[\Pi_{A_i(0)}\left(\Pi_{q_i(0)} + \Pi_{\bar{q}_i(0)}\right) - \ldots\right] + h.c. \tag{266}$$

Where the colored quark and gluon projection operators are

$$\Pi_{A_i(0)} \equiv |A_i(0)\rangle\langle 0| \tag{267}$$

$$\Pi_{q_i(0)} \equiv \lambda_i\left[|q\uparrow\rangle\langle q\uparrow| - |q\downarrow\rangle\langle q\downarrow|\right] \tag{268}$$

$$\Pi_{\bar q_i(0)} \equiv \lambda_i\left[|\bar q\uparrow\rangle\langle \bar q\uparrow| - |\bar q\downarrow\rangle\langle \bar q\downarrow|\right] \tag{269}$$

We may further write the λ matrices as quark color projection operators,

$$\lambda_1 = \left[|c\rangle\langle m| + |m\rangle\langle c| - |\bar c\rangle\langle \bar m| - |\bar m\rangle\langle \bar c|\right], \ldots \tag{270}$$

which allows us to write the $\{\Pi_{q_i}\}$ as operators like

$$\Pi_{q,(0)} = |c\uparrow\rangle\langle m\uparrow| + |m\uparrow\rangle\langle c\uparrow| - |c\downarrow\rangle\langle m\downarrow| - |m\downarrow\rangle\langle c\downarrow| \tag{271}$$

and

$$\Pi_{\bar q,(0)} = -|\bar c\uparrow\rangle\langle \bar m\uparrow| - |\bar m\uparrow\rangle\langle \bar c\uparrow| + |\bar c\downarrow\rangle\langle \bar m\downarrow| + |\bar m\downarrow\rangle\langle \bar c\downarrow| \tag{272}$$

Now we evaluate the ρ-meson H_I matrix elements. First, we have

$$_\rho\langle q\bar q(s=0)A|H_I|q\bar q\rangle_\rho = -g\xi\sum_{i=1}^{8}\langle q\bar q(s=0)A|\Pi_{A_i(+)}^{(-\sqrt2)}(\Pi_{q_i(-)} + \Pi_{\bar q_i(-)})|q\bar q\rangle_\rho = 8g\xi\langle -|i=1|-\rangle$$

$$= \sqrt2\,\frac{2g\xi}{\sqrt3}\left(\langle Q_1\uparrow\downarrow| - \langle Q_1\downarrow\uparrow|\right)\left(\Pi_{q,(-)} + \Pi_{\bar q,(-)}\right)\left(|c\uparrow\bar c\uparrow\rangle + |m\uparrow\bar m\uparrow\rangle + |y\uparrow\bar y\uparrow\rangle\right)\underbrace{\langle A_1(+)|\Pi_{A_1(+)}|0\rangle}_{1} \tag{273}$$

We note

$$\Pi_{q,(-)} = |c\downarrow\rangle\langle m\uparrow| + |m\downarrow\rangle\langle c\uparrow| \qquad -\Pi_{\bar q,(-)} = |\bar c\downarrow\rangle\langle \bar m\uparrow| + |\bar m\downarrow\rangle\langle \bar c\uparrow| \tag{274}$$

and
$$|Q_1\rangle = \frac{1}{\sqrt2}\left(|m\bar c\rangle + |c\bar m\rangle\right) \tag{275}$$

so we find

$$\langle H_I\rangle = \frac{2}{\sqrt3}g\xi\left[\langle m\uparrow\bar c\downarrow| - \langle m\downarrow\bar c\uparrow| + \langle c\uparrow\bar m\downarrow| - \langle c\downarrow\bar m\uparrow|\right]\left[|c\downarrow\rangle\langle m\uparrow| + |m\downarrow\rangle\langle c\uparrow|\right.$$
$$\left. - |\bar c\downarrow\rangle\langle \bar m\uparrow| - |\bar m\downarrow\rangle\langle \bar c\uparrow|\right]\left[|c\uparrow\bar c\uparrow\rangle + |m\uparrow\bar m\uparrow\rangle + |y\uparrow\bar y\uparrow\rangle\right] = -\frac{8}{\sqrt3}g\xi \neq 0 \tag{276}$$

which is exactly as expected by our C-parity argument. Now we evaluate the ρ meson matrix element which must be zero by C-parity conservation ;

$$
\begin{aligned}
{}_{\rho}\langle q\bar{q}(s=1)A|H_I|q\bar{q}\rangle_{\rho} &= -8g\xi\,{}_{\rho}\langle q\bar{q}A|\left[\Pi_{A,(0)}(\Pi_{q,(0)}+\Pi_{\bar{q},(0)})-\sqrt{2}\,\Pi_{A,(+)}(\Pi_{q,(-)}\right.\\
&\left.+\Pi_{\bar{q},(-)})\right]|q\bar{q}\rangle_{\rho} = \sqrt{\tfrac{2}{3}}\,g\xi\Big[-\sqrt{2}\underbrace{\langle A,(+)|\Pi_{A,(+)}|0\rangle}_{1}(\langle Q,\uparrow\downarrow|+\langle Q,\downarrow\uparrow|)(\Pi_{q,(-)}\\
&+\Pi_{\bar{q},(-)})-\sqrt{2}\underbrace{\langle A,(0)|\Pi_{A,(0)}|0\rangle}_{1}\langle Q,\uparrow\uparrow|(\Pi_{q,(0)}+\Pi_{\bar{q},(0)})\Big]\left[|c\uparrow\bar{c}\uparrow\rangle+|m\uparrow\bar{m}\uparrow\rangle+|y\uparrow\bar{q}\uparrow\rangle\right]
\end{aligned}
\tag{277}
$$

The quark matrix elements are

$$
\begin{aligned}
&(\langle Q,\uparrow\downarrow|+\langle Q,\downarrow\uparrow|)(\Pi_{q,(-)}+\Pi_{\bar{q},(-)})(|c\uparrow\bar{c}\uparrow\rangle+|m\uparrow\bar{m}\uparrow\rangle+|y\uparrow\bar{q}\uparrow\rangle)=0\\
&\langle Q,\uparrow\uparrow|(\Pi_{q,(0)}+\Pi_{\bar{q},(0)})(|c\uparrow\bar{c}\uparrow\rangle+|m\uparrow\bar{m}\uparrow\rangle+|y\uparrow\bar{q}\uparrow\rangle)=0
\end{aligned}
\tag{278}
$$

So we find for our full matrix element

$$
{}_{\rho}\langle q\bar{q}(s=1)A|H_I|q\bar{q}\rangle_{\rho} = 0 \tag{279}
$$

as we expect for states of opposite C-parity.

To obtain a number for the size of this perturbation we simply evaluate ξ as defined in (263) for quarks and gluons in the lowest bag model modes. Using equations (9g), (212), (215), (225), we find

$$
\xi = \int \langle q\uparrow|\bar{\psi}\vec{\gamma}\psi|q\uparrow\rangle\cdot\langle A(0)|\vec{A}|0\rangle\, d^3x \tag{280}
$$

$$
\begin{aligned}
&= \int d^3x\left[-i\,4\sqrt{\tfrac{2\pi}{3}}\,k\alpha_+^2 j_0(kr)j_1(kr)\overrightarrow{Y_{110}(\Omega)}\right]\cdot\left[a_1 j_1(\omega r)\overrightarrow{Y_{110}(\Omega)}\right]\\
&= -\frac{i}{a}\,4\sqrt{\frac{2\pi}{3}}\,\frac{(\alpha_+^2a^3)(a_1a)}{\chi_0^3}\,k\int_0^{\chi_0}\eta^2 j_0(\eta)j_1(\eta)j_1\!\left(\frac{x_1}{\chi_0}\eta\right)d\eta
\end{aligned}
\tag{281}
$$

For massless quarks we have $k = 1$, $\chi_0 = 2.043$, and doing this integral with x_1 (gluon) = 2.744 gives the result

$$
\xi = -i\,\frac{c_0}{a} \tag{282}
$$

where $c_0 = .196$ and a is the radius of either the $|q\bar{q}\rangle$ or $|q\bar{q}A\rangle$ state. (We have assumed the difference in the radii of the two states is small). The parameters we find for the unperturbed bag model states with $B_0^{1/4} = 120$ Mev are;

	a(fermis)	E(Mev)
$\lvert q\bar{q}\rangle$	1.242	866
$\lvert q\bar{q}A\rangle$	1.412	1273

If we take $a \sim 1.33$ fermis, we have

$$\xi \simeq -i\,29.3 \text{ Mev} \tag{283}$$

so our hamiltonian matrix element is numerically

$$_\rho\langle q\bar{q}(s_{q\bar{q}}=0)A|H_I|q\bar{q}\rangle_\rho = i\,135.\,g \text{ Mev} \tag{284}$$

For a strong coupling of $\alpha_s \equiv \frac{g^2}{4\pi} = 1$, this gives an off diagonal matrix element of

$$\langle q\bar{q}A|H_I|q\bar{q}\rangle = i\,480. \text{ Mev} \tag{285}$$

Which is indeed of the correct order of magnitude if we are to account for the $\pi\rho$ splitting in this fashion.

Now we consider the problem of mixing the $|q\bar{q}\rangle_\pi$ state (247) with the exotic $|q\bar{q}A\rangle_\pi$ state (251) through the interaction hamiltonian (266). We form the matrix element of H_I between the $|q\bar{q}\rangle_\pi$ state exactly as we did for the ρ meson above, and we find the result

$$_\pi\langle q\bar{q}A|H_I|q\bar{q}\rangle_\pi = 8\xi g \tag{286}$$

We recall that the energy scale ξ is defined in (263) and is given numerically in (283).

At this point we note that a great simplification is possible in dealing with the color combinatorics of our meson configuration mixing problem. The matrix elements of the hamiltonian we must evaluate here are all of the form

$$\underbrace{\langle q\bar{q}(\underset{\sim}{8})|\otimes\langle A(\underset{\sim}{8})|}_{\underset{\sim}{1}}\,\vec{J}^a\cdot\vec{A}^a\,|q\bar{q}(\underset{\sim}{1})\rangle \tag{287}$$

where the numbers refer to the color SU(3) transformation properties of the state vectors. As the color SU(3) assignments of these states do not change in the ρ to the π problem, these matrix elements must be proportional to the matrix elements of state vectors with the same spins in an Abelian (no color degree of freedom) theory. Thus,

$$\langle q\bar{q}(\underline{8})| \otimes \langle A(\underline{8})| \vec{J}^a \cdot \vec{A}^a |q\bar{q}(\underline{1})\rangle = N \langle q\bar{q}|\langle A| \vec{J}\cdot\vec{A}\, |q\bar{q}\rangle \tag{288}$$

with $\quad \vec{J} = g\bar{\psi}\vec{\gamma}\psi \qquad \vec{J}^a = g\bar{\psi}\vec{\gamma}\psi\lambda^a$ (289)

We have of course assumed that the corresponding spin states are identical. The numerical value of N we find by doing the Abelian version of the ρ mixing problem. Our states are

$$|q\bar{q}\rangle_\rho = |q\uparrow\bar{q}\uparrow\rangle \tag{290}$$

$$|q\bar{q}(s_{q\bar{q}}=1)\, A\rangle = \frac{1}{\sqrt{2}}\left[\, |q\uparrow\bar{q}\uparrow\rangle|A(0)\rangle - \frac{1}{\sqrt{2}}\left(|q\uparrow\bar{q}\downarrow\rangle + |q\downarrow\bar{q}\uparrow\rangle\right)|A(+)\rangle\right] \tag{291}$$

and the colorless hamiltonian is given in (264). The matrix element we find is

$$_\rho\langle q\bar{q}(s_H=0)A|H_I(\text{colorless})|q\bar{q}\rangle_\rho = -2\xi g \tag{292}$$

We then obtain the ratio N of the color SU(3) and colorless matrix elements by comparison with (276);

$$N = \frac{\left(-\frac{8}{\sqrt{3}}\xi\right)}{(-2\xi)} = \frac{4}{\sqrt{3}} \tag{293}$$

Proceeding similarly, we may show that the corresponding π-meson matrix element in the colorless theory and the new equivalent colorless interaction are

$$_\pi\langle q\bar{q}A|H_I(\text{colorless})|q\bar{q}\rangle_\pi = \frac{8\xi g}{(4/\sqrt{3})} = 2\sqrt{3}\,\xi g \tag{294}$$

$$\mathcal{H}_{I\,\text{equiv.}}(\text{colorless}) = -f\bar{\psi}\vec{\gamma}\psi\cdot\vec{A} = -\frac{4}{\sqrt{3}}\, g\bar{\psi}\vec{\gamma}\psi\cdot\vec{A} \tag{295}$$

This is of course only true for matrix elements of H_I betreen $|q\bar{q}\rangle$ and $|q\bar{q}A\rangle$ states with the color assignments which we have considered here. Presumably similar relations may be derived relating other color SU(3) matrix elements to corresponding Abelian matrix elements in terms of SU(3) Racah coefficients. A knowledge of these $\pi\rho$ coefficients would be a useful tool in calculations of this nature. As a final aside on this general topic, we note that the interaction strength ratio N we would have found for color SU(3) quarks and gluons in our meson problem is

$$N[\text{color } SU(n)] = \sqrt{6\left(1-\frac{1}{3n}\right)} \tag{296}$$

This tells us that the size of the H_I matrix element we find for color SU(3) is already $\sqrt{8/9}$ of the matrix element we find for mesons in a model with an infinite number of colors. (Of course we don't suggest that n is greater than three physically, as we would then have trouble explaining $|qqq\rangle$ baryons as color singlets.)

Now we return to the $\pi\rho$ mass problem. Rather then use the parameters given on p.103 for the bag model states with an assumed value of the bag strength, we shall instead fit the two free parameters in our model ($B_0^{1/4}$ or a^{-1} and g) to the experimental masses. From the parameters given on p.103, we may write the π and ρ hamiltonian matrices for an arbitrary bag strength in an equivalent theory as

$$H_\pi = \begin{bmatrix} E_1 & -g\xi_a^* \\ -g\xi_a & E_2 \end{bmatrix} \qquad H_\rho = \begin{bmatrix} E_1 & \frac{g\xi_a^*}{\sqrt{3}} \\ \frac{g\xi_a}{\sqrt{3}} & E_2 \end{bmatrix} \tag{297}$$

where E_1 and E_2 are the unperturbed $E_{q\bar{q}}$ and $E_{q\bar{q}A}$ state energies, respectively. As the only independent scale of energy in this problem is a^{-1} or $B_0^{1/4}$, we may factor this scale out of the hamiltonians ;

$$H_\pi = \begin{bmatrix} n_1 & -8ic_0 g \\ 8ic_0 g & n_2 \end{bmatrix} a^{-1} \qquad H_\rho = \begin{bmatrix} n_1 & \frac{8ic_0 g}{\sqrt{3}} \\ -\frac{8ic_0 g}{\sqrt{3}} & n_2 \end{bmatrix} a^{-1} \tag{298}$$

We find the lowest energy eigenvalues of these matrices, which we identify with the physical π and ρ mesons respectively, by introducing a rotation between the $|q\bar{q}\rangle$ and $|q\bar{q}A\rangle$ states;

$$\begin{aligned} |\rho\rangle &= \cos\theta_\rho |q\bar{q}\rangle_\rho + \sin\theta_\rho |q\bar{q}A\rangle_\rho \\ |\pi\rangle &= \cos\theta_\pi |q\bar{q}\rangle_\pi + \sin\theta_\pi |q\bar{q}A\rangle_\pi \end{aligned} \tag{299}$$

where the angles θ_ρ, θ_π are chosen to diagonalize the hamiltonians H_ρ and H_π. The energy eigenvalues we find are

$$\begin{aligned} E_\rho &= \left\{ \frac{n_1+n_2}{2} - \left[\left(\frac{n_1-n_2}{2}\right)^2 + \frac{64}{3} c_0^2 g^2 \right]^{1/2} \right\} a^{-1} \\ E_\pi &= \left\{ \frac{n_1+n_2}{2} - \left[\left(\frac{n_1-n_2}{2}\right)^2 + 64 c_0^2 g^2 \right]^{1/2} \right\} a^{-1} \end{aligned} \tag{300}$$

The unperturbed bag model state parameters n_1 and n_2 we may write down from $E_{q\bar{q}}$, $E_{q\bar{q}A}$, $a_{q\bar{q}}$, and $a_{q\bar{q}A}$, and we know c_0 from (282);

$$n_1 = 5.45 \qquad n_2 = 9.11 \qquad c_0 = .196 \tag{301}$$

Now we simply fit $E_\rho = .77$ Gev and $E_\pi = .14$ Gev by adjusting a^{-1} and $\alpha_s = \frac{g^2}{4\pi}$. The values we find for these parameters are

$$a = .801 \text{ fm} \qquad \alpha_s = 1.35 \qquad (\text{ and hence } B_c^{1/4} = 126 \text{ Mev.}) \tag{302}$$

If we recall that the color Yang-Mills coupling g and the equivalent Abelian coupling f are related by $f = \frac{4}{\sqrt{3}} g$, we see that our color model at this level looks like an Abelian coupling of strength

$$\alpha = \frac{f^2}{4\pi} = \frac{16}{3} \alpha_s = 9.71 \tag{303}$$

Thus, a moderately strong color SU(3) coupling leads to mass splittings which would require a very strong coupling to reproduce in an Abelian theory.

One way in which we may check the consistency of our model to some extent is by noting the size of the mixing angles θ_π and θ_ρ defined in (299). If these angles are near $\frac{\pi}{4}$, we have driven our system into " saturation ", and we expect that significant fractions of the physical meson states are the heavier excited bag model states which we have left out of our model. The mixing angles we find here are

$$\tan\theta_\pi = -\frac{\sqrt{1+3\tilde{g}^2}-1}{\sqrt{3}\,\tilde{g}}$$
$$\tan\theta_\rho = \frac{\sqrt{1+\tilde{g}^2}-1}{\tilde{g}} \qquad (304)$$

where we have introduced

$$\tilde{g} = \frac{|\xi| f}{E_2-E_1} \equiv \frac{2|\xi| f}{\Delta} = \frac{16 c_0 g}{\sqrt{3}(n_2-n_1)} \qquad (305)$$

For the numbers given above we find

$$\tilde{g} = 2.04 \qquad (306)$$

which gives numerical values for the mixing angles of

$$\theta_\pi = -37.1^\circ \qquad \theta_\rho = 31.9^\circ \qquad (307)$$

In terms of probabilities, this implies that the $|\pi\rangle$ is ~36% $|q\bar{q}A\rangle_\pi$ and the $|\rho\rangle$ is ~28% $|q\bar{q}A\rangle_\rho$. Clearly, we have driven our model near saturation, so it seems likely that other excited states contribute significantly to the physical π and ρ meson states. The general features of the mixing, however, should not look particularly different than the model we have sketched here, although many more states are involved.

The masses and mixing angles we found in this π, ρ model as a function of f are shown explicitly in Figures V and W. To find the numbers given in these figures we approached the problem slightly differently, using $a = \frac{1}{2}(a_{q\bar{q}} + a_{q\bar{q}A})$ as the scale of length through which we define n_1, n_2, and so forth (297,8). This approach leads to parameters

$$f = 8.9 \quad (\alpha_s = 1.18) \qquad\qquad B_0^{1/4} = 193.5 \text{ MeV} \tag{308}$$

which we shall use for the remaining light meson calculation.

Now that we have approximate values for $\mathcal{B}_0^{1/4}$ and f to fit the masses of the π and ρ it will be of interest to see whether or not the value of m_s required to fit the K mass is near the 270 MeV value obtained in fitting the decimet in the lowest order bag model.

The $K(495)$ and $K^*(892)$ we may treat similarly to the π and ρ considered previously, except that we must include the effect of the strange quark mass on the $q\bar{q}A$ overlap integrals.

We expand the physical $|K^+\rangle$ as

$$|K^+\rangle = \cos\theta_k |K_0^+\rangle + \sin\theta_k |K_0^{*+}A\rangle \qquad (+\ldots \text{ neglected}) \tag{309}$$

where

$$\begin{aligned} |K_0^+\rangle &= \tfrac{1}{\sqrt{2}}\left(|u\uparrow\bar{s}\downarrow\rangle - |u\downarrow\bar{s}\uparrow\rangle\right) \\ |K_0^{*+}A\rangle &= \tfrac{1}{\sqrt{3}}\left(|K_0^{*+}(+)\rangle|A(-)\rangle - |K_0^{*+}(0)\rangle|A(0)\rangle + |K_0^{*+}(-)\rangle|A(+)\rangle\right) \end{aligned} \tag{310}$$

so as previously we find quark currents

$$\begin{aligned} \langle K_0^+|\vec{j}\,|K_0^{*+}(+)\rangle &= \tfrac{1}{\sqrt{2}}\left(\langle u\downarrow|\vec{j}\,|u\uparrow\rangle + \langle s\downarrow|\vec{j}\,|s\uparrow\rangle\right) \\ \langle K_0^+|\vec{j}\,|K_0^{*+}(0)\rangle &= -\langle u\uparrow|\vec{j}\,|u\uparrow\rangle - \langle s\uparrow|\vec{j}\,|s\uparrow\rangle \\ \langle K_0^+|\vec{j}\,|K_0^{*+}(-)\rangle &= -\tfrac{1}{\sqrt{2}}\left(\langle u\uparrow|\vec{j}\,|u\downarrow\rangle + \langle s\uparrow|\vec{j}\,|s\downarrow\rangle\right) \end{aligned} \tag{311}$$

Now we find for the matrix element

$$\varepsilon = -\langle K_0^+|\, f\sum_q(\bar{\psi}_q\vec{\gamma}\psi_q)\cdot\vec{A}\,|K_0^{*+}A\rangle = \sqrt{3}\,f\left(\xi(m_u) + \xi(m_s)\right) \tag{312}$$

where ξ is defined in the $|\pi_0\rangle, |\rho_0 A\rangle$ mixing problem;

$$\xi = \frac{4\sqrt{\frac{2\pi}{3}}\,(a_1 a)(\alpha_+^2 a^3)\,k}{x_0^3 a}\underbrace{\int_0^{x_0} y^2 j_0(y)\, j_1(y)\, j_1\!\left(\tfrac{x_1}{x_0}y\right)dy}_{I(x_0,x_1)} \tag{313}$$

Using the bag model parameter $\mathcal{B}_0^{1/4} = 193.5$ MeV as with the π and ρ gives the unperturbed $|K_o^+\rangle$ and $|K_o^{+*}A\rangle$ the following properties (for $m_s = 100, 200$, and 300 Mev);

	K_o^+: m_s=.1Gev	.2 Gev	.3 Gev	$K_o^{+*}A$: m_s=.1Gev	.2 Gev	.3 Gev
Energy (Mev)	1447	1504	1566	2104	2161	2225
Radius (fm)	.770	.768	.766	.875	.874	.872
x(s quark)	2.207	2.335	2.437	2.227	2.366	2.473
$\alpha_+^2 a^3$(s quark)	.490	.570	.646	.501	.591	.676
k (fm^{-1} s quark)	2.87	3.04	3.18	2.54	2.71	2.84

For simplicity we take the K_o^+ quark wavefunctions to compute the overlap integrals $I(x, x_1)$ in $\xi(m_q)$ defined above. This will lead to an overall error of ~5%, but the same procedure was used in computing the $\pi\rho$ splitting. The result for each case is

$$I(2.207) = .3260 \qquad I(2.335) = .3641 \qquad I(2.437) = .3918$$

so the overlap $\xi(m_q)$ is (314)

$$2\xi(100\,\text{Mev}) = .371\ a^{-1} \qquad 2\xi(200\,\text{Mev}) = .350\ a^{-1} \qquad 2\xi(300\,\text{Mev}) = .329\ a^{-1}$$
$$= 95.1\ \text{Mev} \qquad\qquad = 89.9\ \text{Mev} \qquad\qquad = 84.7\ \text{Mev}$$

We see that the effect of a $m_s \sim 300$ Mev is a suppression of the $q\bar{q}A$ overlap integral by ~10%. As with the π we may now obtain the K energy and mixing angle with the $K_o^* A$ configuration as a function of f;

$$E_K = \Sigma - \Delta\sqrt{1+3\tilde{q}^2} \qquad \Sigma = \tfrac{1}{2}(E_{K_o^*A} + E_{K_o})\ ,\ \Delta = \tfrac{1}{2}(E_{K_o^*A} - E_{K_o})$$
$$\tan\theta_K = \frac{1-\sqrt{1+3\tilde{q}^2}}{\sqrt{3}\,\tilde{q}} \qquad \tilde{\xi} = \tfrac{1}{2}\left(\xi(0) + \xi(m_s)\right) \qquad \tilde{q} = \frac{2f\tilde{\xi}}{\Delta} \tag{315}$$

A plot of the K and K* masses we find as a function of m_s is given in Fig. X . It is clear that the model is too crude to allow an accurate determination of m_s using the π, ρ values of $B_o^{1/4}$ and f, since fitting m_K requires $m_s \sim 370$ Mev and m_{K^*} requires 150 MeV. The best fit to both masses is given by $m_s = 270$ Mev , which gives $m_K = 405$ Mev, $m_{K^*} = 980$ Mev . The fact that this is the same value of m_s which we obtained by fitting the baryon decimet in the glueless bag model is of course coincidental, although it is reassuring that both approaches give similar values for m_s. The K and K^* mixing angles for $f \simeq 10$ do not deviate from the π and ρ mixing angles by more than a degree, so they are not quoted here.

A topic of current interest which we are now able to consider is the mass of the pseudoscalar analogue of the $\psi(3105)$ vector meson. We shall use values of $B_0^{1/4}$ and m_c which correctly give the ψ and ψ' masses, $B_0^{1/4} = 215$ Mev and $m_c = 1.1$ Gev, as is shown in Figure V. Having fit the $\psi(3105)$ exactly we shall obtain a value for M_{χ_c} as a function of $\alpha_s = \frac{g^2}{4\pi}$ which we found to be $\alpha_s \sim 1.3$ in fitting the $\pi\rho$ splitting. (The χ_c is the pseudoscalar S-wave $c\bar{c}$ meson.)

First consider the $\psi(3105)$; we assume mixing between the states $|\psi_0\rangle = |c\uparrow\bar{c}\uparrow\rangle$ and the $|\chi_{c_0} A\rangle$ state

$$|\chi_{c_0} A\rangle = \frac{1}{\sqrt{2}}\left(|c\uparrow\bar{c}\downarrow\rangle - |c\downarrow\bar{c}\uparrow\rangle\right)|A(+)\rangle \tag{316}$$

through the interaction Hamiltonian (in the Coulomb gauge),

$$H_I = \sum_q -f(\bar{\psi}_q \vec{\gamma}\, \psi_q)\cdot \vec{A} \tag{317}$$

which gives the off-diagonal matrix element

$$\varepsilon = \langle \psi_0 | H_I | \chi_{c_0} A\rangle = -\sqrt{2}\, f \int d^3x \,(\bar{c}\uparrow \vec{\gamma}\, c\downarrow)\cdot \vec{A} \tag{318}$$

Using the explicit form for the quark current (9g,h) we find

$$\epsilon = 2f\xi(m_c) = f a_1 \alpha_+^2\, 8\sqrt{\frac{2\pi}{3}}\, k \int_0^a r^2 j_0(kr)\, j_1(kr)\, j_1\!\left(\frac{x_g}{a} r\right) dr \tag{319}$$

in terms of which the ψ energy and mixing angle are

$$E_\psi = \Sigma - \sqrt{\Delta^2 + \xi^2} \qquad \tan\theta_\psi = \frac{\sqrt{1+\tilde{\xi}^{-2}} - 1}{\tilde{\xi}} \quad \left(\tilde{\xi} = \frac{2f\xi}{\Delta}\right) \tag{320}$$

where $\Sigma = \frac{1}{2}(E_{\psi_0} + E_{\chi_{c_0}A})$ and $\Delta = \frac{1}{2}(E_{\chi_{c_0}A} - E_{\psi_0})$ as with the ρ. We have estimated the $B_0^{1/4}$ and m_c necessary to correctly give the ψ and ψ' masses, neglecting glue effects, as $B_0^{1/4} = 215$ Mev, $m_c = 1.1$ Gev . For simplicity we shall use these parameters and calculate m_{χ_c}/m_ψ rather than worry about getting m_ψ to agree exactly with experiment. The parameters we find for the unperturbed ψ_0 and $\chi_{c_0}A$ states are;

	ψ_0	$\chi_{c0}A$	mean
Energy (Mev)	3090	3860	——
Radius (fm)	.644	.750	.697
x_0	2.77	2.81	2.79
k	.341	.305	.323
$\alpha_+^2 a^3$	1.001	1.058	1.030

$a_1 a = 2.572$ (1^+ gluons)
$x_g = 2.744$

Since the radii and quark parameters of these two states are similar we take the mean values of $k, \alpha_+^2 a^3$, x_0 and a in our estimate of the $q\bar{q}A$ overlap integrals. With the values given above we find

$$\varepsilon = 2\mathcal{F}\xi(1.1\ \mathrm{Gev}) = f\,\frac{(a_1 a)(\alpha_+^2 a^3)\,8\sqrt{\frac{2\pi}{3}}\,k}{x_0^3 a}\int_0^{x_0} \eta^2 j_0(\eta)\, j_1(\eta)\, j_1\!\left(\frac{x_g}{x_0}\eta\right) d\eta \tag{321}$$

$$= .210 f a^{-1} = 59.5\, f\ \mathrm{Mev}$$

so our numerical prediction for $E_\psi(f)$ is

$$E_\psi(f) = 3.48 - \left[.148 + .00354 f^2\right]^{1/2}\ \mathrm{Gev} \tag{322}$$

Now we consider the effect of glue on the χ_c. We expand this state as

$$|\chi_c\rangle = \cos\theta_\chi\,|\chi_{c_0}\rangle + \sin\theta_\chi\,|\psi_0 A\rangle \quad (+\ldots\ \text{neglected}) \tag{323}$$

where

$$|\chi_{c_0}\rangle = \frac{1}{\sqrt{2}}\left(|c\uparrow\bar{c}\downarrow\rangle - |c\downarrow\bar{c}\uparrow\rangle\right)$$

$$|\psi_0 A\rangle = \frac{1}{\sqrt{3}}\left(|\psi_0(+)\rangle|A(-)\rangle - |\psi_0(0)\rangle|A(0)\rangle + |\psi_0(-)\rangle|A(+)\rangle\right) \tag{324}$$

so the matrix element of the hamiltonian is

$$\varepsilon = -f\langle \chi_{c_0} | \sum_q (\bar{\psi}_q \vec{\gamma} \psi_q)\cdot\vec{A} | \psi_0 A\rangle = -\sqrt{\tfrac{2}{3}}\, f \Big\{ (\overline{c\downarrow}\,\vec{\gamma}\, c\uparrow)\cdot\overrightarrow{A(-)} + \sqrt{2}\,(\overline{c\uparrow}\,\vec{\gamma}\, c\uparrow)\cdot\overrightarrow{A(0)} - (\overline{c\uparrow}\,\vec{\gamma}\, c\downarrow)\cdot\overrightarrow{A(+)}\Big\} \tag{325}$$

which eventually becomes

$$\begin{aligned}\varepsilon &= -2\sqrt{3}\, \xi(1.1\ \text{Gev}) \\ &= -.1031\, f\ \text{Gev}\end{aligned} \tag{326}$$

This predicts for the pseudoscalar energy $E_{\chi_c}(f)$

$$E_{\chi_c}(f) = 3.48 - [.148 + .0106 f^2]^{1/2}\ \text{Gev} \tag{327}$$

The scale of the masses is somewhat arbitrary, so we consider only the predicted ratio $E_{\chi_c}(f)/E_\psi(f)$, which for our values of $B_0^{1/4}$ and f comes out

$$\frac{E_{\chi_c}(f)}{E_\psi(f)} = \frac{1-[.0122+8.77\cdot 10^{-4} f^2]^{1/2}}{1-[.0122+2.92\cdot 10^{-4} f^2]^{1/2}} = \frac{1-[.0122+.0591\alpha_s]^{1/2}}{1-[.0122+.0197\alpha_s]^{1/2}} < 1 \quad \text{as expected} \tag{328}$$

If we use the value of f we found in fitting the $\pi\rho$ splitting f = 8.9 corresponding to $\alpha_s = 1.18$, we find

$$\frac{E_{\chi_c}(f=8.9)}{E_\psi(f=8.9)} = .880 \tag{329}$$

If we now impose $E_\psi = 3.09\ \text{Gev}$ we find $E_{\chi_c} = 2.72\ \text{Gev}$. This result compares favorably with the recently reported evidence for a pseudoscalar $c\bar{c}$ state with $E_{c\bar{c}} \sim 2.8\ \text{Gev}$[41]. The fact that our simple prescription gives the $c\bar{c}$ $1^- - 0^-$ mass difference this well with no free parameters gives support to the hope that the most important part of the quark-gluon interaction in determining meson mass splittings is the Yukawa term. For comparison of our value of α_s with others a plot of $E_{\chi_c}(\alpha_s)/E_\psi(\alpha_s)$ is given in Fig. Z . We also give the mass ratios

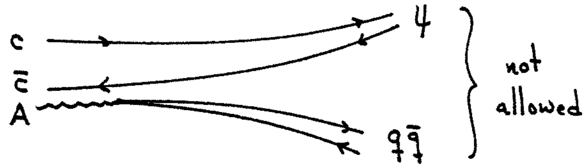
$E_{\chi_c A}(\alpha_s)/E_\psi(\alpha_s)$, $E_{\psi A}(\alpha_s)/E_\psi(\alpha_s)$, and the mixing angles θ_ψ, θ_{χ_c} which are numerically

$$\frac{E_{\chi A}}{E_\psi} = \frac{1+\sqrt{.0122+.0591\alpha_s}}{1-\sqrt{.0122+.0197\alpha_s}} \qquad \tan\theta_\psi = \frac{\sqrt{1+1.61\alpha_s}-1}{1.27\sqrt{\alpha_s}} \tag{330}$$

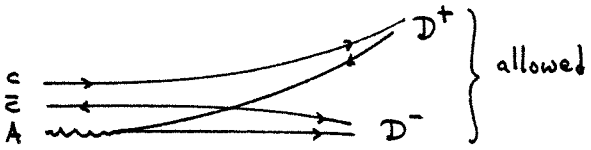
$$\frac{E_{\psi A}}{E_\psi} = \frac{1+\sqrt{.0122+.0197\alpha_s}}{1-\sqrt{.0122+.0197\alpha_s}} \qquad \tan\theta_{\chi_c} = \frac{1-\sqrt{1+4.83\alpha_s}}{2.20\sqrt{\alpha_s}}$$

We note that our mixing angles for $\alpha_s = 1.18$ are $\theta_\psi = 27°$ and $\theta_{\chi_c} = -34°$, which are small enough to make our naïve mixing scheme credible. This was not the case with the $\pi\rho$ splitting where the mixing angles approached $\pm 45°$, although the value of α_s we found there works reasonably well in giving the $\chi_c \psi$ splitting.

An interesting result of the $\chi_c \psi$ problem is that the $|\chi_c A\rangle$ and $|\psi A\rangle$ vector and pseudoscalar states are not shifted very far from their initial energies, so these states (with admixtures of $q^2\bar{q}^2$ and $q\bar{q}A^2, \ldots$ which we have neglected) presumably exist in the general neighborhood of the energies we estimate here, $m_{\chi A} \sim 4.5$ Gev and $m_{\psi A} \sim 4.9$ Gev. The 1^- $|\chi A\rangle$ state is produced in e^+e^- annihilation as is the $|\psi\rangle$, and its principal decay mechanism must be gluon annihilation to produce a quark pair. The quarks made by the gluon form a color octet, so each must accompany one of the c quarks to decay to color singlet mesons;



$$|\chi A\rangle \not\longmapsto |\psi\rangle|\rho^0, \pi^0, \eta, \ldots\rangle$$



$$|\chi A\rangle \longmapsto |D^+\rangle|D^-\rangle,\ |D^0\rangle|\overline{D^0}\rangle,$$
$$|F^+\rangle|F^-\rangle \text{ if massive enough}$$

The signature of these largely $|c\bar{c}A\rangle$ vector mesons will be (1) strong decays into $(c\bar{q})(\bar{c}q)$ rather than $(c\bar{c})(q\bar{q})$ for the reasons given above, (2) suppression of the rate to e^+e^- by $\tan^2\theta_\psi$, because only the $|c\bar{c}\rangle$ component may annihilate into a photon, and presumably (3) the existence of a 0^- $|c\bar{c}A\rangle$ state ~ 500 Mev above the vector state which will also decay into $D^+D^-, \ldots$, rather than $\chi_c\pi, \ldots$.

A possible candidate for the $|c\bar{c}A\rangle$ vector is the recently reported $\psi(4414)$[42], which has a partial width to e^+e^- of $.44 \pm .14$ Kev. If we extrapolate the rate for $\psi \mapsto e^+e^-$ of 4.8 ± 0.6 KeV to a meson with m = 4.414 GeV and assume the enhancement factor at zero separation is unchanged (see the section on $1^- \mapsto e^+e^-$), we would predict a $\psi_{4.4} \mapsto e^+e^-$ rate

$$\Gamma(\psi(4.4) \mapsto e^+e^-) = 2.4 \pm .3 \quad \text{Kev} \tag{331}$$

extrapolating from the $\psi'(3695)$ rate of $2.2 \pm .3$ KeV gives

$$\Gamma(\psi(4.4) \mapsto e^+e^-) = 1.5 \pm .2 \quad \text{Kev} \tag{332}$$

These rates are too large by factors of 5.5 and 3.5 respectively. If we pretend that the $\psi(4.4)$ is the largely $|c\bar{c}A\rangle$ state and use the value of $\theta_\psi \sim 27°$ we found in the section on $c\bar{c} \leftrightarrow c\bar{c}A$ mixing, we would predict a rate to e^+e^- which is suppressed by $\tan^2\theta_\psi \sim .26$;

$$\Gamma(c\bar{c}A(4414) \mapsto e^+e^-) = \begin{cases} .62 \pm .08 \text{ Kev (from } \psi) \\ .39 \pm .05 \text{ Kev (from } \psi') \end{cases} \tag{333}$$

Both estimates overlap the experimental result of $.44 \pm .14$ Kev.

This simple estimate of the rate to e^+e^- is certainly not a proof that the $\psi(4.4)$ is largely $|c\bar{c}A\rangle$. More convincing evidence would be the discovery that the πK mass distribution in $\psi(4.4)$ decays peaks at about 1.87 Gev (mass of the D's) as would be expected from the predominance of decays like

$$\psi(4.4) \longmapsto D^+ D^-, \quad D^+ \to K^0\pi^+, \quad D^- \to \overline{K^0}\pi^-$$

$$\psi(4.4) \longmapsto D^0 \overline{D^0}, \quad D^0 \to K^0\pi^0, \quad \overline{D^0} \to \overline{K^0}\pi^0$$

Other decays are suppressed by $\tan\theta_{Cab.}$, $\tan^2\theta_{\psi}$, or the larger mass of the strange-charmed mesons .

Thus far in treating the charmed particles we have assumed the parameters $B_0^{1/4} = 215$ Mev and $m_c = 1.1$ Gev , which result from fitting the $\psi(3105)$ and $\psi'(3695)$ masses with the lowest order bag model (neglecting gluon effects). The two constraint curves $B_0^{1/4}(m_c)$ that result from taking the $\psi(3105)$ to be two lowest mode quarks and the $\psi'(3695)$ to be one lowest mode quark and one first-radially-excited mode quark are shown in Fig. U ; clearly we have a solution of these two constraints at $B_0^{1/4} = 215$ Mev and $m_c = 1.1$ Gev. The failure of attempts in the literature[6] to correctly find the ψ and ψ' masses is due to the assumption that $B_0^{1/4} \sim 140$ Mev (motivated by the light hadrons) is a universal constraint applicable to the $c\bar{c}$ mesons as well. We find that this assumption is invalid and that the ψ and ψ' are somewhat smaller than conventional hadrons. To be more explicit we find for r_Q for each of these with the above parameters $r_Q(\psi) = .396$ fm , $r_Q(\psi') = .459$ fm , which we may compare with the measured π charge radius $r_Q(\pi) = .78^{+.09}_{-.10}$ fm[36].

A question which rises immediately is the spectrum of charmed mesons other than the ψ and ψ' which are predicted by the bag model with these parameters. We display here only those levels composed of $j = 1/2$ quarks, i.e. those states which rigorously satisfy the bag model constraints. There will of course be other nonspherical states whose energies could be estimated in a spherical bag approximation, but lack of time excludes their treatment here. The $c\bar{q}$ mesons (with q a light $u, d,$ or s quark) we treat by arbitrarily fixing the mass of the lowest states (D's) at 1.9 Gev . This mass is chosen to give the ψ' stability $(m_D > 1.85$ Gev$)$ and to allow the ~50 MeV wide bump seen in e^+e^- annihilation at ~3.95 GeV to decay strongly to $D\bar{D}$ $(m_D \lesssim 1.95$ Gev$)$. This fixes the bag strength for the $c\bar{q}$ mesons at $B_0^{1/4} = 148$ Mev , and the energies we predict for the lowest lying states, neglecting gluon effects, are given in Table F. The mass we use for the $c\bar{u}$ (D) pseudoscalar meson in this table was guessed in April 1976. The D resonance has since been found ($m_{D^0} = 1.864 \pm .0054$ Gev) together with a candidate first excited state ($m_{D^{*0}} = 2.004 \pm .0036$ Gev)[48]. We note that scaling down all the $c\bar{u}$ and $c\bar{s}$ masses in Table F by $\frac{1864}{1900}$ predicts a first excited state at $M_{D^{*}}^{0}[c(1P)\bar{u}(1S)] = 2.002$ Gev, in close agreement with the observed mass. By analogy with these successes, we expect to find the elusive F pseudoscalar meson at $M_F = 2.017$ Gev, and $M_{F^*} = 2.157$.

We note in passing that in comparison with the situation of the light mesons, the charmed ($c\bar{c}$) meson spectrum is a more tractable laboratory in which to study the mixing of pure quark and exotic gluon states. The reason for this assertion is that our off diagonal Yukawa matrix element $\langle q\bar{q}A | \vec{j}\cdot\vec{A} | q\bar{q}\rangle$ which mixed the $|q\bar{q}\rangle$ and $|q\bar{q}A\rangle$ states in the ρ meson is essentially a number times $k g a^{-1}$ for quarks and gluons in the lowest mode, where $k \equiv [(\omega_q - m_q)/(\omega_q + m_q)]^{1/2} \sim 1$ for the π and ρ. For charmed quarks we typically have $k \sim .3$, which implies that the amplitude to find a $|c\bar{c}A\rangle$ state in a mostly $|c\bar{c}\rangle$ meson will be about $\frac{1}{3}$ the corresponding $|q\bar{q}A\rangle$ meson admixture in the light mesons. Thus, the charmed mesons should not only lie closer to the energies predicted by the bag model than do the corresponding light hadrons, they should be less contaminated with Hilbert space vectors containing different constituents as well.

Finally we might also predict the masses of the exotic $|(q\bar{q})^n\rangle$, $|(qqq)^n\rangle$, $|A^n\rangle$,... states in the bag model, but we are limited by not knowing the value of the bag model parameter $B_0^{1/4}$ to use for these states. The range of values we have used previously in fitting hadron masses ($B_0^{1/4} = 96$ to 215 Mev) is too large to allow us to make reasonably precise predictions for these exotic states.

state	$\vert c\bar{u}\rangle$	$\vert c\bar{s}\rangle$	$\vert c\bar{c}\rangle$
$\vert(1S)^2\rangle$	1900 (input) [0^-]	2055 [0^-]	3090 (input) [1^-]
$\vert(1P)(1S)\rangle$	2040	2197	3352
$\vert(1S)(1P)\rangle$	2246	2320	—
$\vert(1P)^2\rangle$	2366	2442	3585
$\vert(2S)(1S)\rangle$	2238	2397	3692 (input) [1^-]
$\vert(1S)(2S)\rangle$	2531	2597	—
$\vert(2S)(1P)\rangle$	2544	2622	3901
$\vert(1P)(2S)\rangle$	2639	2706	—

...

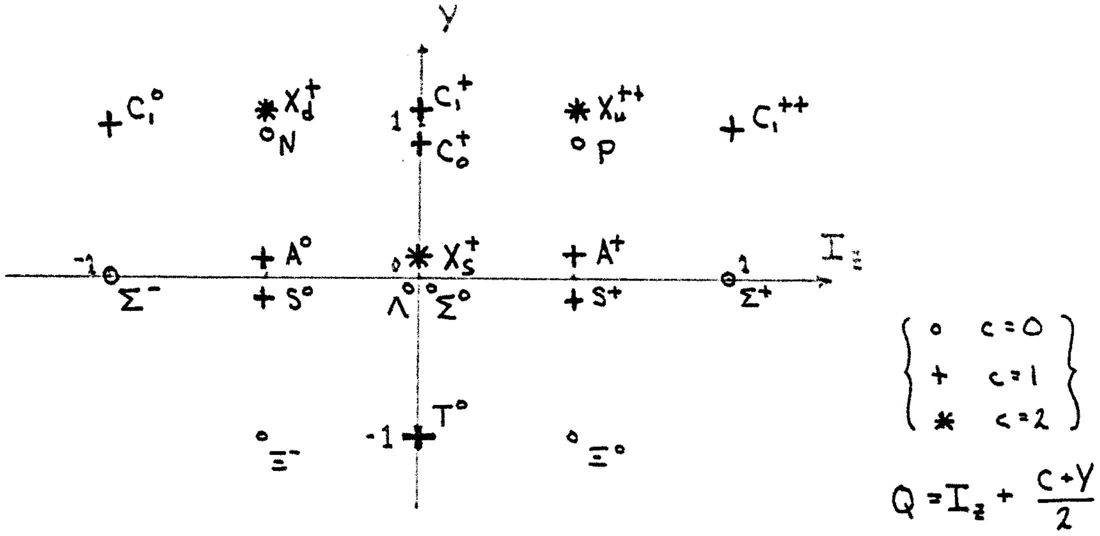
Table F

Energies of $c\bar{q}$ and $c\bar{c}$ mesons in the bag model (MeV)

(See text pp.118-119.)

A final topic in the world of charmed particles which we shall treat is the magnetic moments of the S-wave charmed baryons. Excepting the group theory of SU(8) necessary to derive the quark decomposition of each of these baryons, as is summarized in Appendix C, this is a trivial generalization of the previous CFQM and Bag model sections on baryon magnetic moments.

The baryons in the $J^P = \frac{1}{2}^+$ $\underset{\sim}{20}$ representation of SU(8) have been dubbed[43] as follows;



In terms of the quark magnetic moments μ_u, μ_d, μ_s, and μ_c we find for each charmed baryon the magnetic moment (with $\mu = c_1\mu_u + c_2\mu_d + c_3\mu_s + c_4\mu_c$);

Baryon	c_1	c_2	c_3	c_4	$\frac{\mu(\text{bag model})}{\mu_B(\text{proton})}$
C_1^{++}	$\frac{4}{3}$	0	0	$-\frac{1}{3}$	1.538
C_1^{+}	$\frac{2}{3}$	$\frac{2}{3}$	0	$-\frac{1}{3}$	.262
C_1^{0}	0	$\frac{4}{3}$	0	$-\frac{1}{3}$	−1.014
C_0^{+}	0	0	0	1	.489
A^{+}	0	0	0	1	.483

A^0	0 0 0 1	.488
S^+	$\frac{2}{3}$ $\frac{2}{3}$ 0 $-\frac{1}{3}$	.260
S^0	0 $\frac{2}{3}$ $\frac{2}{3}$ $-\frac{1}{3}$	-.878
T^0	0 0 $\frac{4}{3}$ $-\frac{1}{3}$	-.817
X_u^{++}	$-\frac{1}{3}$ 0 0 $\frac{4}{3}$	.237
X_d^+	0 $-\frac{1}{3}$ 0 $\frac{4}{3}$	.850
X_s^+	0 0 $-\frac{1}{3}$ $\frac{4}{3}$	.804

To obtain a bag model prediction for these moments we have taken $m_{u,d} = 0$ $m_s = .27$ Gev, and $m_c = 1.1$ Gev as found in the zeroth order bag model previously. The bag strength we take to be $B_0^{1/4} = 160$ Mev, which gives a lowest charmed baryon mass of 2.40 Gev These predictions differ from SU(8) in that the scale of the strange and especially the charmed quark moments are suppressed relative to the nonstrange quark moments by their large masses, and with the bag model we have a model of this suppression with no free parameters.
With the above choices for $\{m_q\}$ and $B_0^{1/4}$ we find that $\frac{\mu_c}{\mu_u}$ is suppressed by up to 80% from the naive SU(8) ratio of unity, so our predictions show substantial deviation from SU(8). It is of course not likely that these moments will be measured in the near future. The transition moments between neutral charmed baryons such as $A^0 \leftrightarrow S^0$ may well be known soon, however, as the magnetic dipole decay of the heavier of these into the lighter will be the principal decay mode unless they are very close in energy.
These matrix elements will also show considerable suppression of the naive SU(8) prediction of the charmed quark contribution to the total magnetic dipole decay rate.

Conclusion

We have treated some of the weak and electromagnetic properties of strongly interacting particles in two models which impose quark confinement as an initial constraint. The first model, which is a generalized bag model without Poincaré invariance, allows us to determine reasonable values for the quark wave function parameters $\chi_0 = ka$ (mode number), a (hadron model radius), and $\{m_i\}$ (quark masses) from a number of experimentally measured weak and electromagnetic matrix elements. The values we find for these parameters are generally in good agreement with the values we are forced to take by the requirement of Poincaré invariance in the second (bag) model. This lends credence to the reasonableness of the low momentum transfer matrix elements given by the bag model which do not involve quark production or annihilation. For a process involving quark-antiquark annihilation we find that these models are incomplete, however, and that a sharp peaking of the amplitude to find a $q\bar{q}$ pair at zero separation inside a meson exists which is not allowed for in either model. Finally we consider gluon effects in the hadronic mass spectrum and show that a relatively small color SU(3) Yang-Mills coupling gives rise to large quark-gluon configuration mixing which lifts the degeneracies observed in the S-wave meson states in the zeroth order bag model.

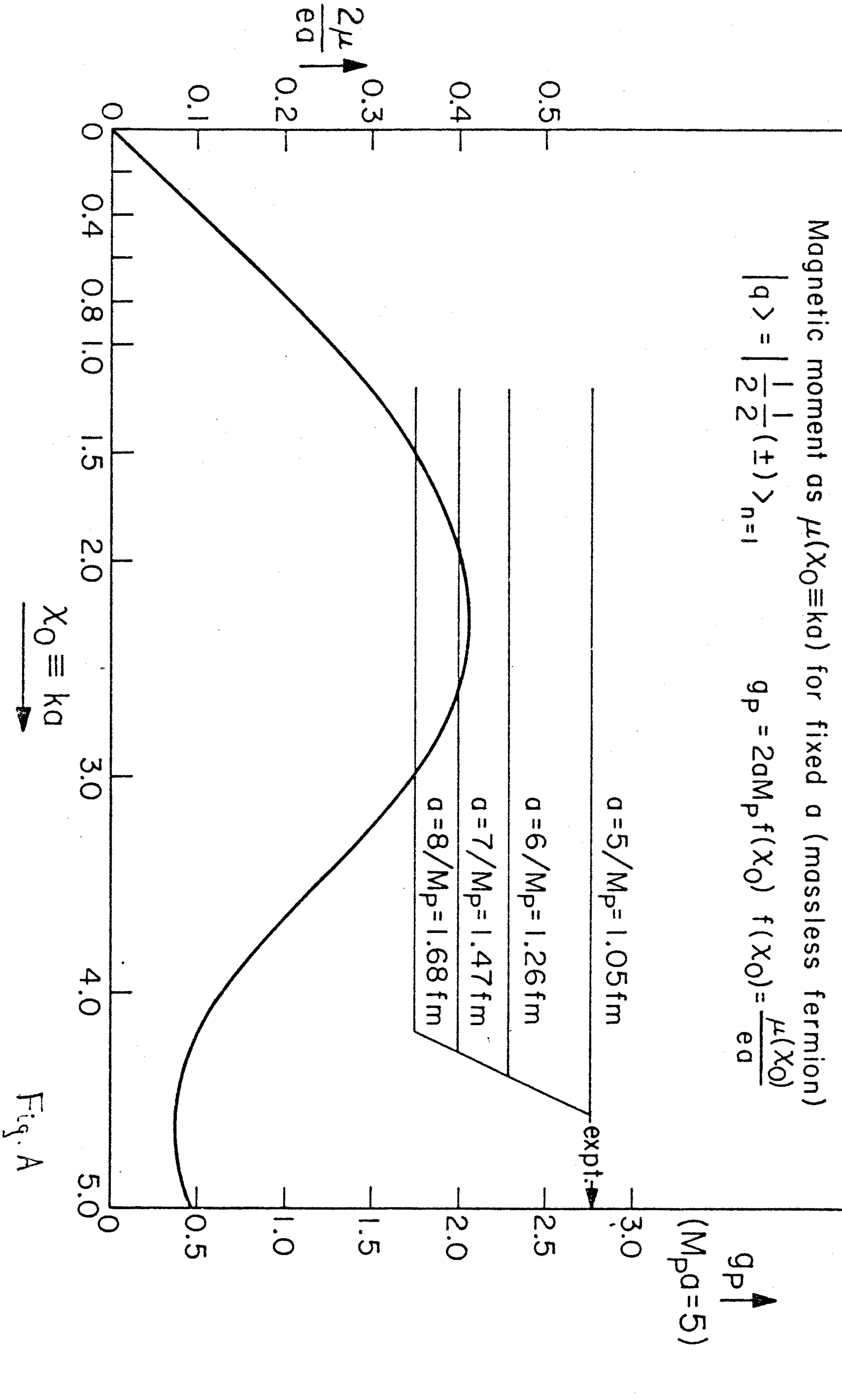
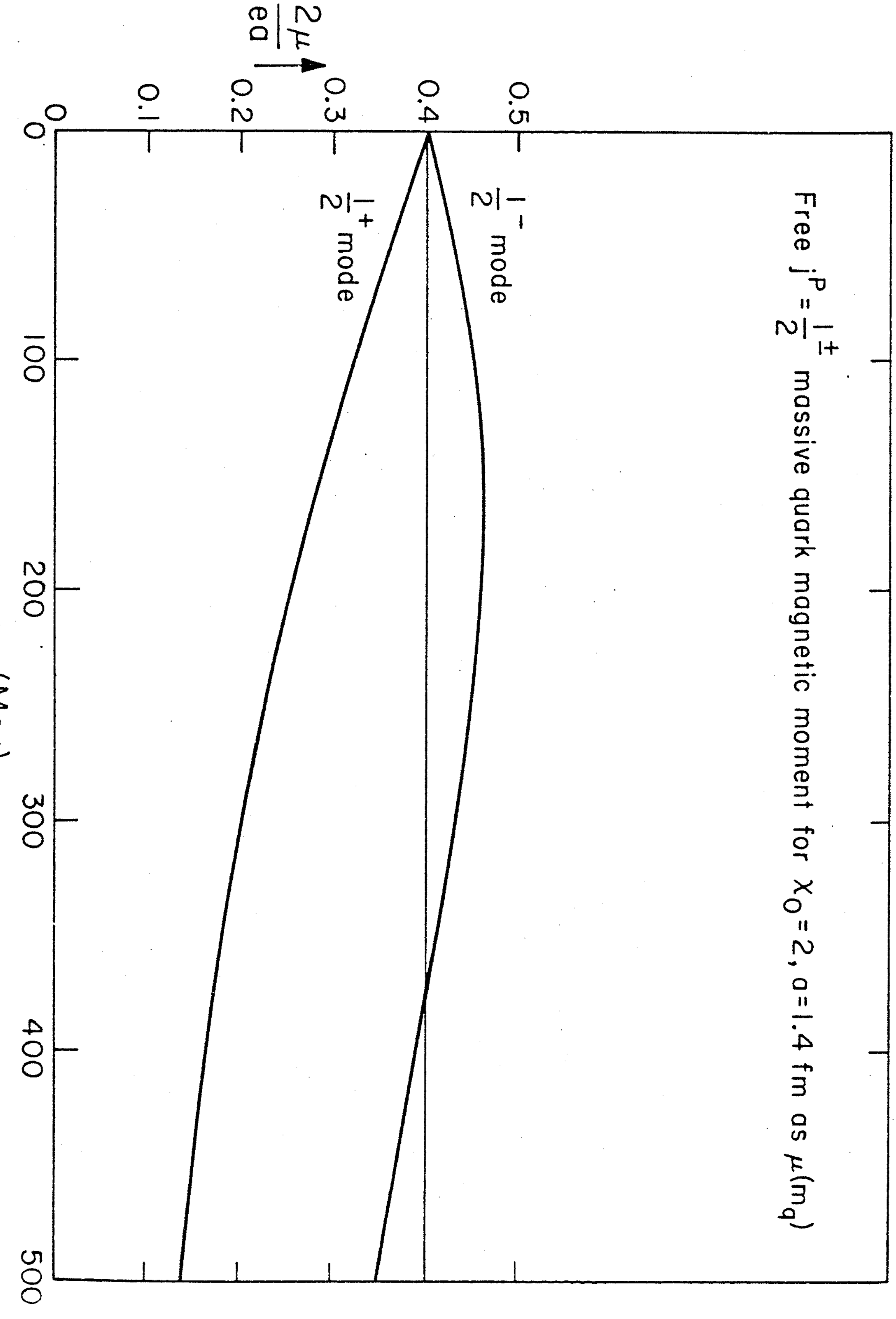

Magnetic moment as $\mu(\chi_0 \equiv ka)$ for fixed a (massless fermion)
$|q\rangle = \left|\frac{1}{2}\frac{1}{2}(\pm)\right\rangle_{n=1}$
$g_p = 2aM_p f(\chi_0)$ $f(\chi_0) = \frac{\mu(\chi_0)}{ea}$
$a = 5/M_p = 1.05$ fm
$a = 6/M_p = 1.26$ fm
$a = 7/M_p = 1.47$ fm
$a = 8/M_p = 1.68$ fm
expt.
$\frac{2\mu}{ea}$
0
0.1
0.2
0.3
0.4
0.5
$\chi_0 \equiv ka$
0
0.4
0.8
1.0
1.5
2.0
3.0
4.0
5.0
g_p
$(M_p a = 5)$
0
0.5
1.0
1.5
2.0
2.5
3.0

Fig. A



Free $j^P = \frac{1}{2}^{\pm}$ massive quark magnetic moment for $x_0 = 2$, $a = 1.4$ fm as $\mu(m_q)$
$\frac{2\mu}{ea}$
0
0.1
0.2
0.3
0.4
0.5
$\frac{1}{2}^-$ mode
$\frac{1}{2}^+$ mode
0
100
200
300
400
500
m_q(Mev)

Fig. B

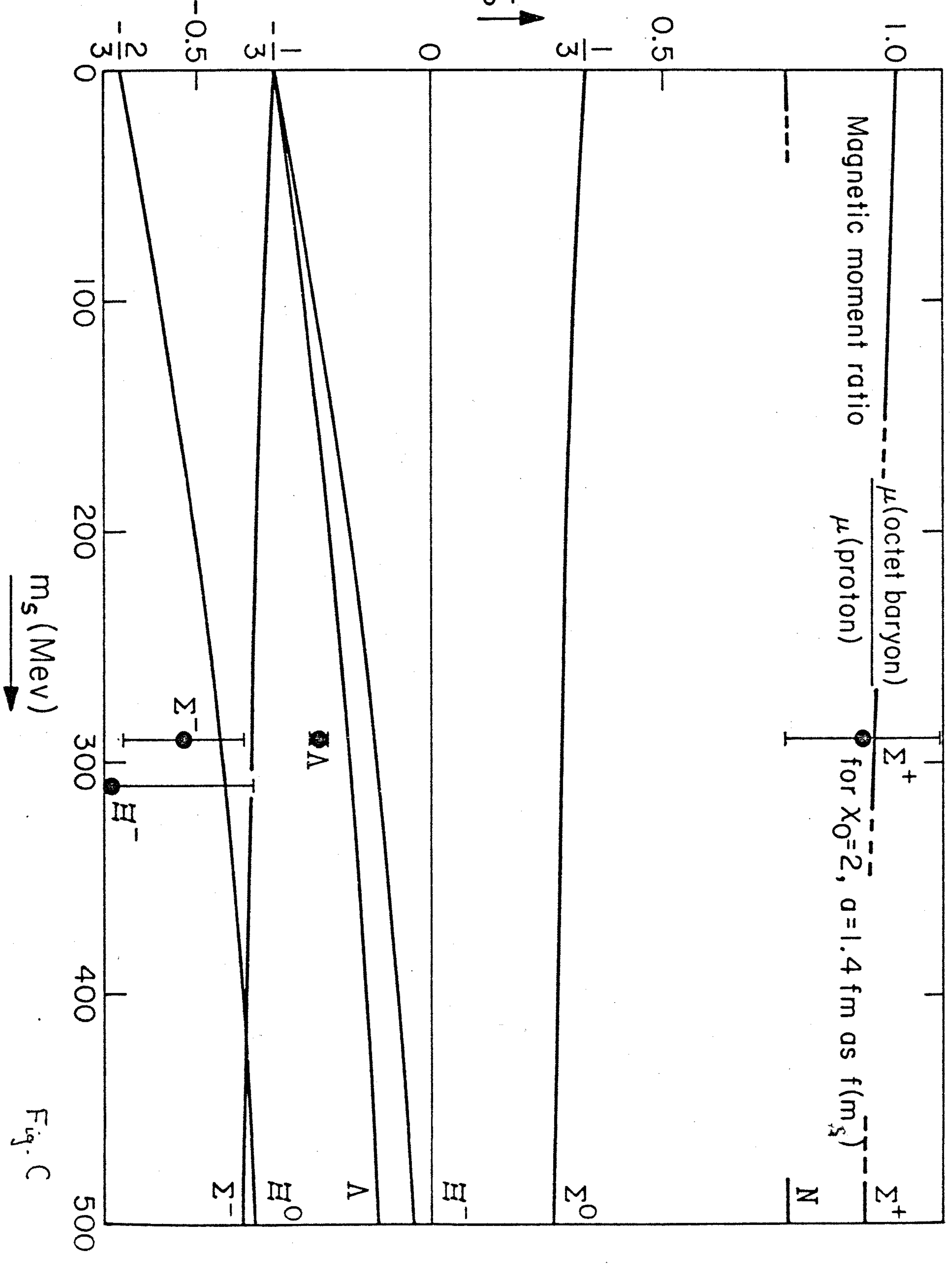
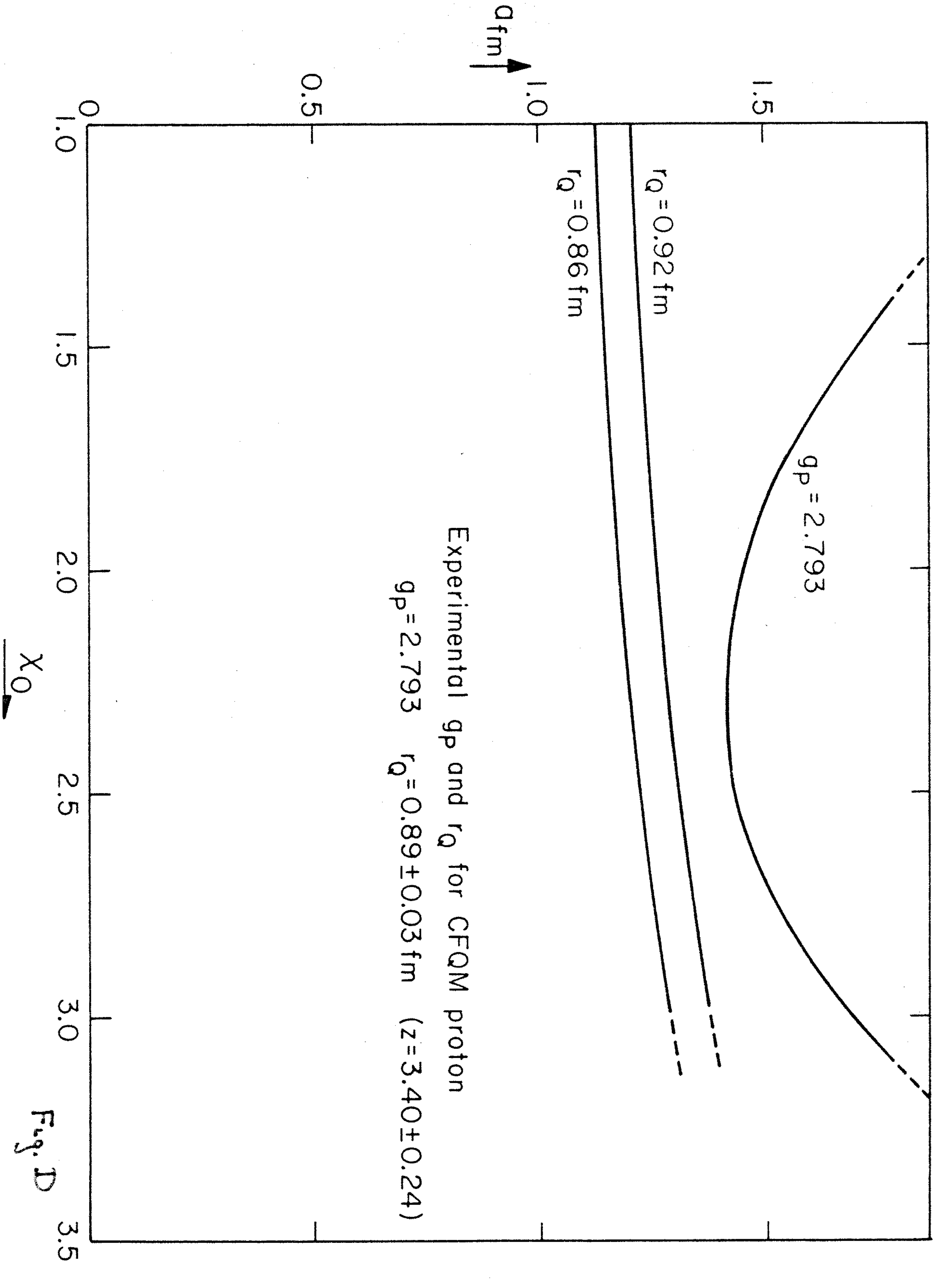

Magnetic moment ratio
$\frac{\mu(\text{octet baryon})}{\mu(\text{proton})}$ for $X_0=2$, $a=1.4$ fm as $f(m_s)$
$\frac{\mu}{\mu_p}$
1.0
0.5
$\frac{1}{3}$
0
$-\frac{1}{3}$
-0.5
$-\frac{2}{3}$
0
100
200
300
400
500
m_s (Mev)
Σ^+
N
Σ^0
Ξ^-
Λ
Ξ^0
Σ^-

Fig. C



a_{fm}
0
0.5
1.0
1.5
1.0
1.5
2.0
2.5
3.0
3.5
χ_0
$r_Q = 0.92$ fm
$r_Q = 0.86$ fm
$g_P = 2.793$
Experimental g_P and r_Q for CFQM proton
$g_P = 2.793$ $r_Q = 0.89 \pm 0.03$ fm $(z = 3.40 \pm 0.24)$

Fig. D

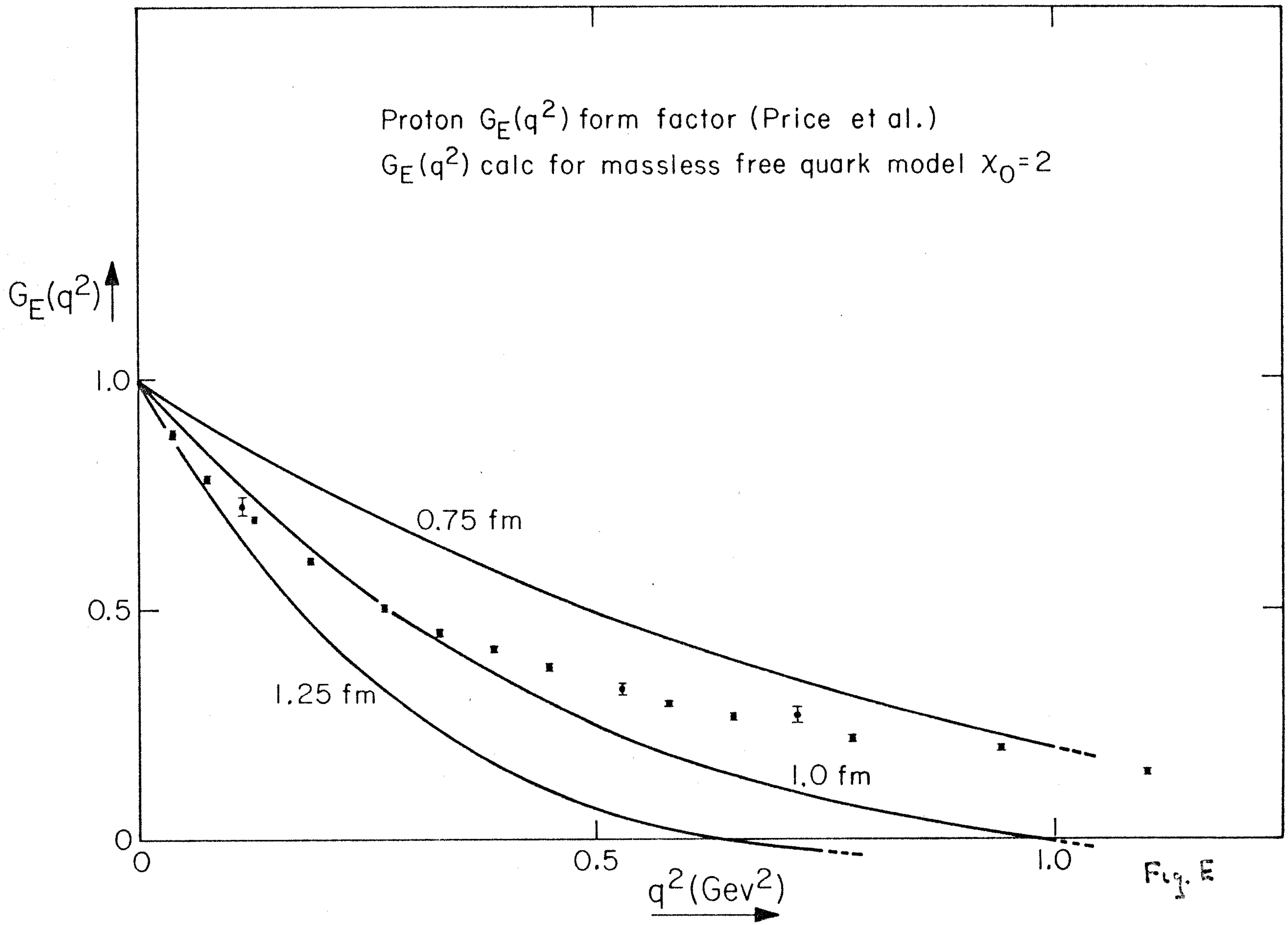

Proton $G_E(q^2)$ form factor (Price et al.)
$G_E(q^2)$ calc for massless free quark model $\chi_0 = 2$
$G_E(q^2)$
1.0
0.5
0
0.75 fm
1.0 fm
1.25 fm
0
0.5
1.0
q^2 (Gev^2)

Fig. E

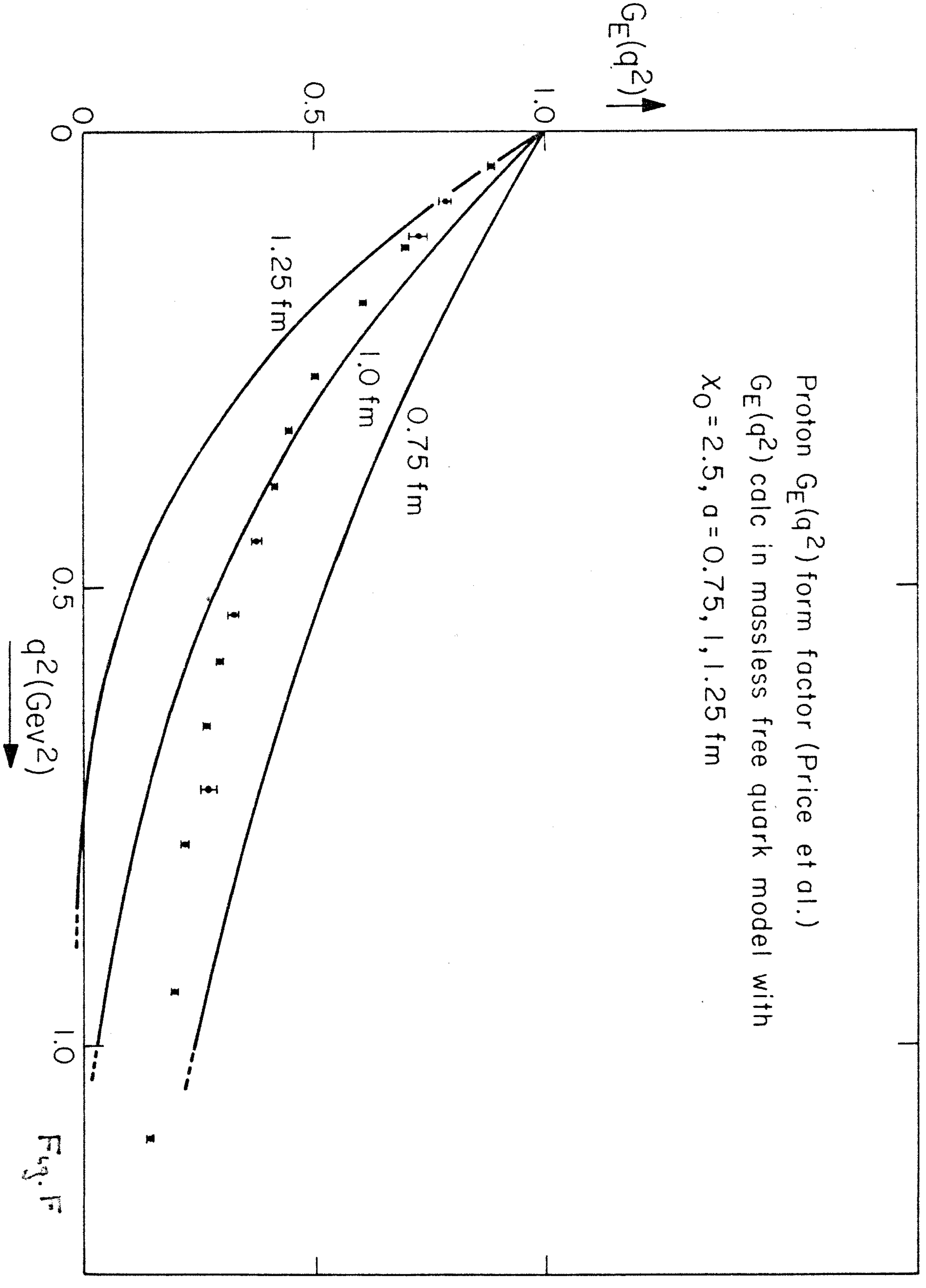

Proton $G_E(q^2)$ form factor (Price et al.)
$G_E(q^2)$ calc in massless free quark model with
$X_0 = 2.5$, $a = 0.75, 1, 1.25$ fm
$G_E(q^2)$
1.0
0.5
0
0.5
1.0
q^2(Gev^2)
0.75 fm
1.0 fm
1.25 fm

Fig. F

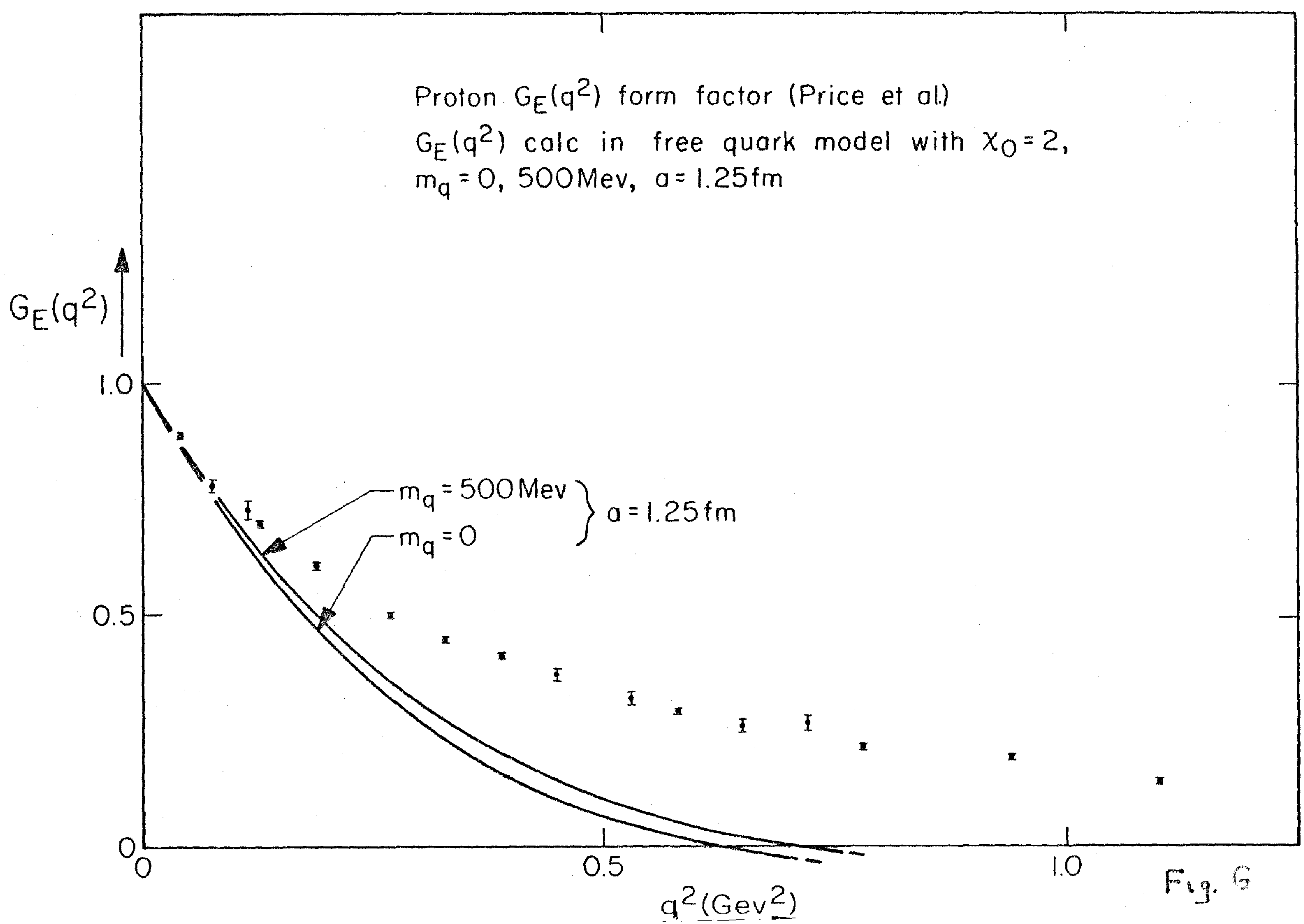

Proton $G_E(q^2)$ form factor (Price et al.)
$G_E(q^2)$ calc in free quark model with $\chi_0 = 2$,
$m_q = 0$, 500Mev, $a = 1.25$fm
$G_E(q^2)$
1.0
0.5
0
0
0.5
1.0
$m_q = 500$Mev
$m_q = 0$
$a = 1.25$fm
q^2(Gev^2)

Fig. G

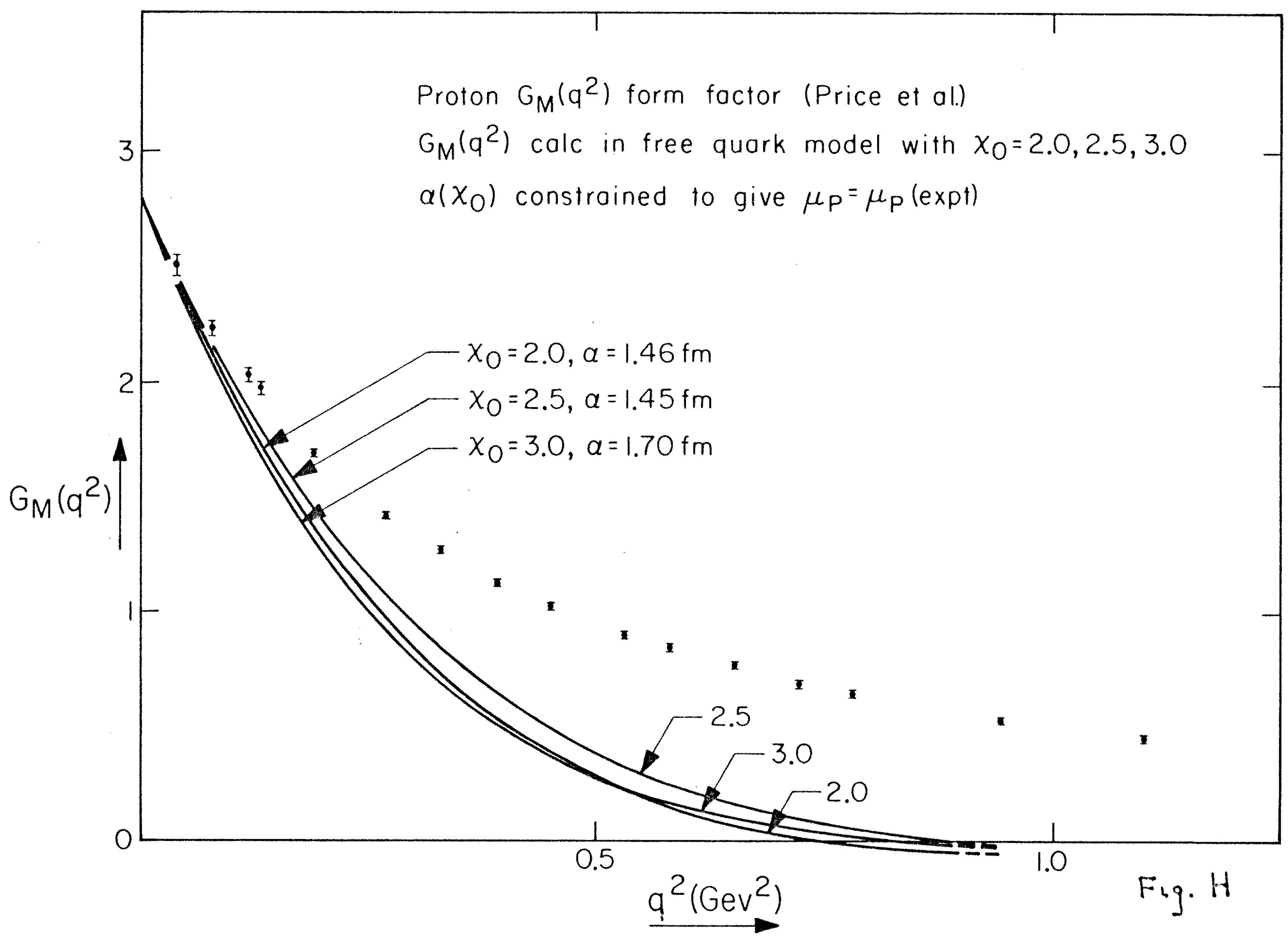
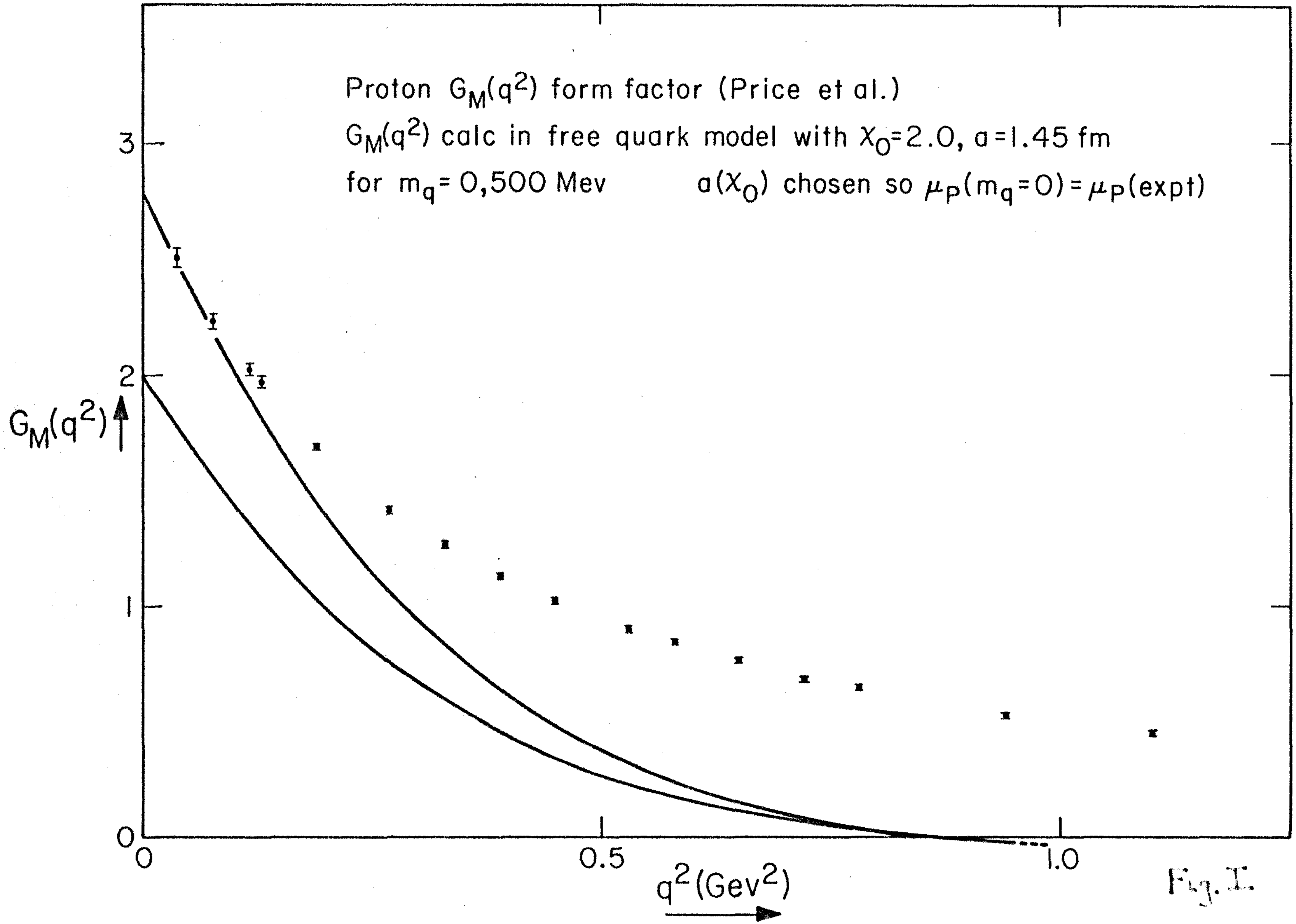
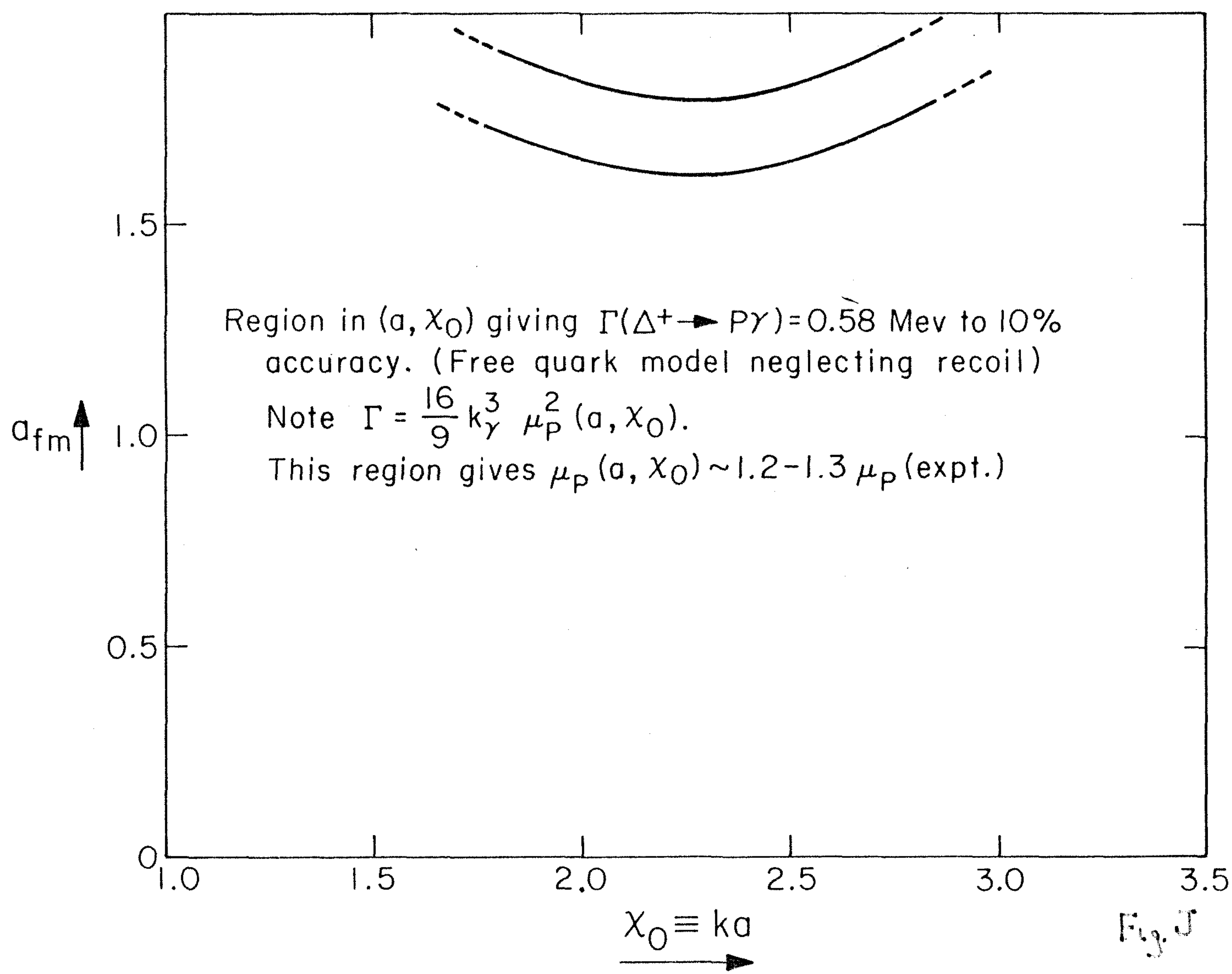

Proton $G_M(q^2)$ form factor (Price et al.)
$G_M(q^2)$ calc in free quark model with χ_0 = 2.0, 2.5, 3.0
$\alpha(\chi_0)$ constrained to give $\mu_P = \mu_P$(expt)
$\chi_0 = 2.0$, $\alpha = 1.46$ fm
$\chi_0 = 2.5$, $\alpha = 1.45$ fm
$\chi_0 = 3.0$, $\alpha = 1.70$ fm
2.5
3.0
2.0
$G_M(q^2)$
3
2
1
0
0.5
1.0
q^2(Gev^2)

Fig. H



Proton $G_M(q^2)$ form factor (Price et al.)
$G_M(q^2)$ calc in free quark model with $X_0=2.0$, $a=1.45$ fm
for $m_q= 0,500$ Mev
$a(X_0)$ chosen so $\mu_P(m_q=0)=\mu_P(\text{expt})$
$G_M(q^2)$
0
1
2
3
0
0.5
1.0
$q^2(\text{Gev}^2)$

Fig. I.



a_{fm}
1.5
1.0
0.5
0
1.0
1.5
2.0
2.5
3.0
3.5
$\chi_0 \equiv ka$
Region in (a, χ_0) giving $\Gamma(\Delta^+ \rightarrow P\gamma) = 0.58$ Mev to 10% accuracy. (Free quark model neglecting recoil)
Note $\Gamma = \frac{16}{9} k_\gamma^3 \ \mu_P^2 (a, \chi_0)$.
This region gives $\mu_P(a, \chi_0) \sim 1.2 - 1.3\ \mu_P$(expt.)

Fig. J

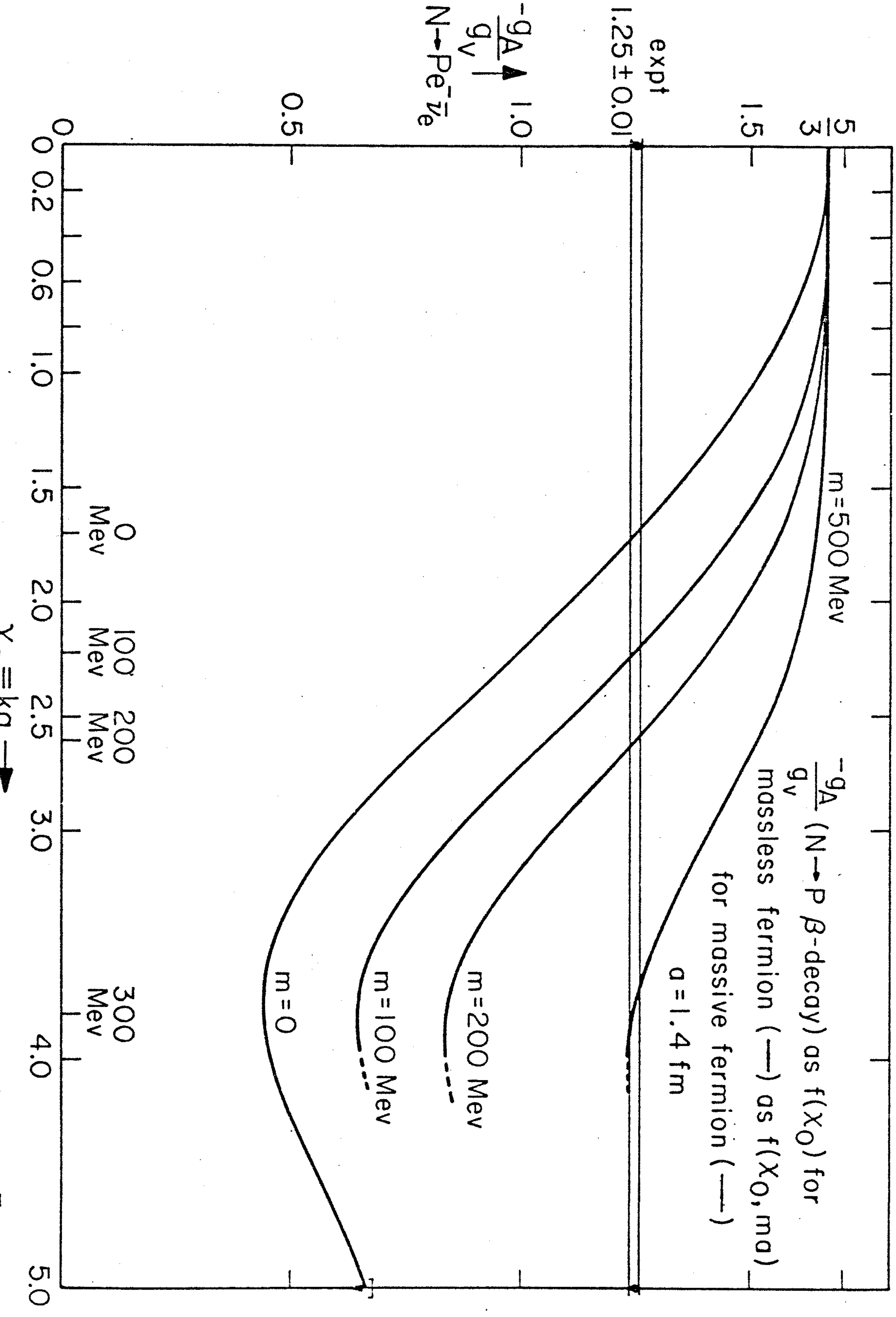

$\frac{-g_A}{g_V}$ (N→P β-decay) as f(χ_0) for
massless fermion (—) as f(χ_0, ma)
for massive fermion (—)
a = 1.4 fm
m = 500 Mev
m = 200 Mev
m = 100 Mev
m = 0
$\frac{5}{3}$
1.5
expt
1.25 ± 0.01
$\frac{-g_A}{g_V}$
N→P$e^-\bar{\nu}_e$
1.0
0.5
0
0 0.2 0.6 1.0 1.5 2.0 2.5 3.0 4.0 5.0
0 Mev
100 Mev
200 Mev
300 Mev
$\chi_0 \equiv ka$

Fig K

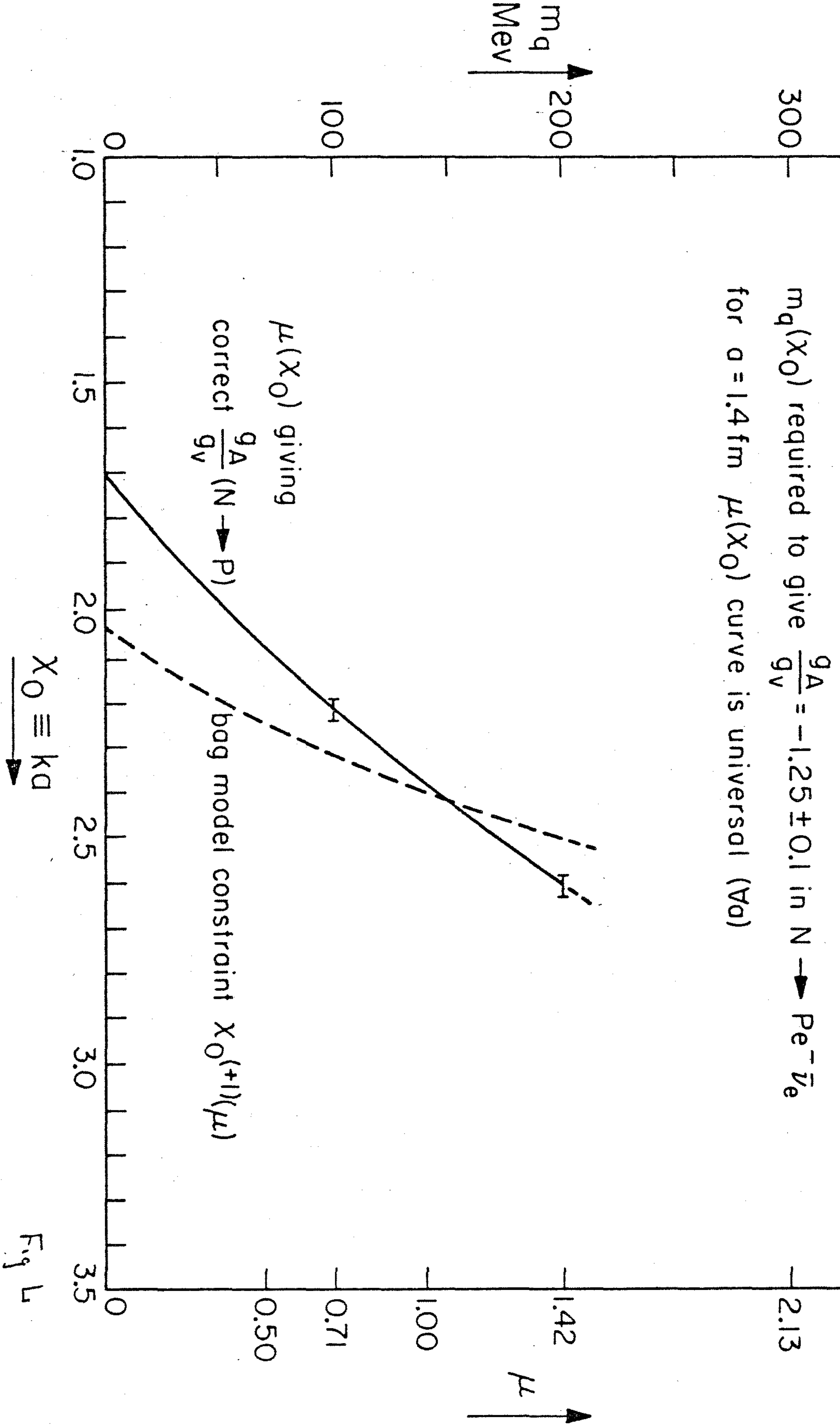
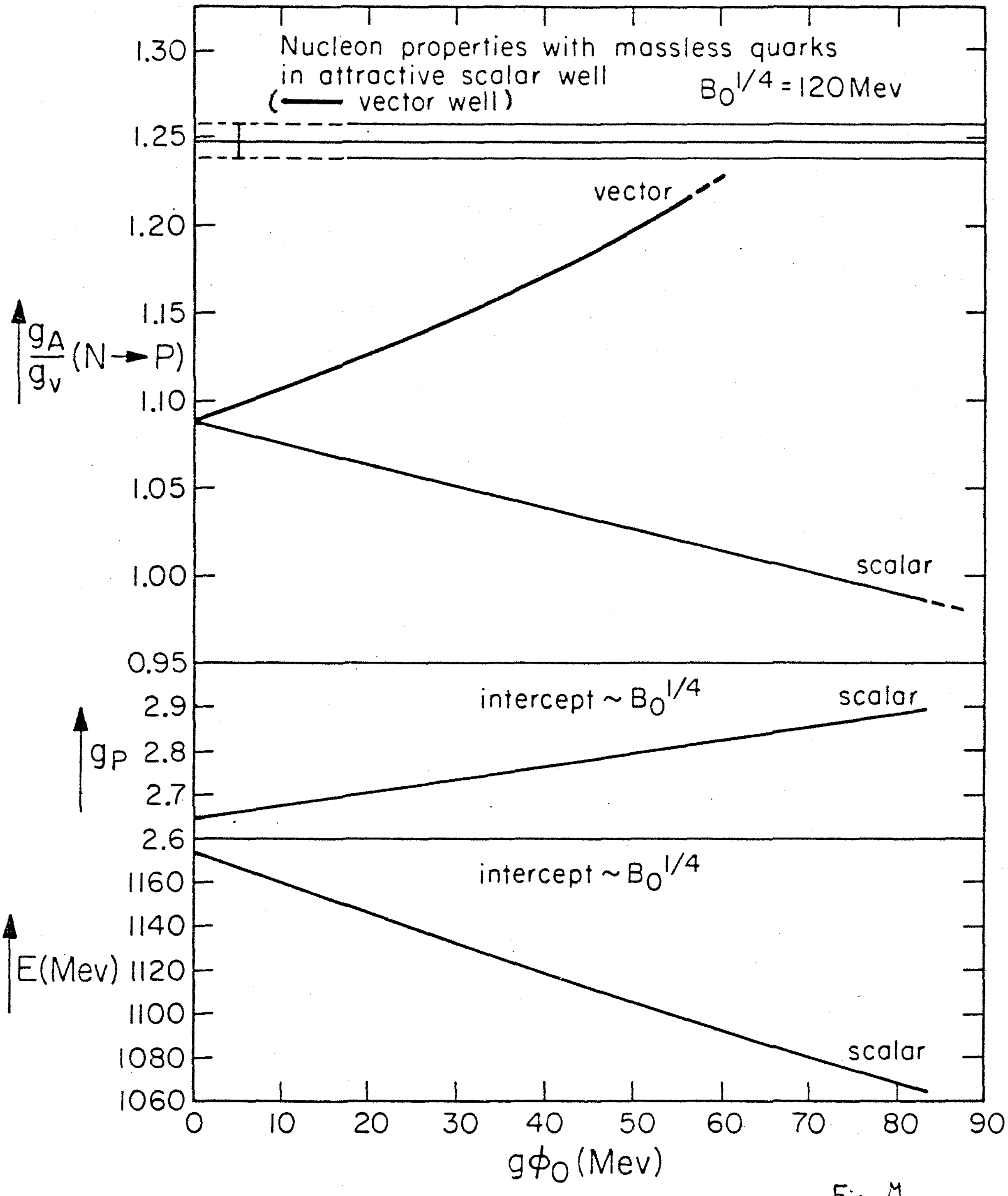
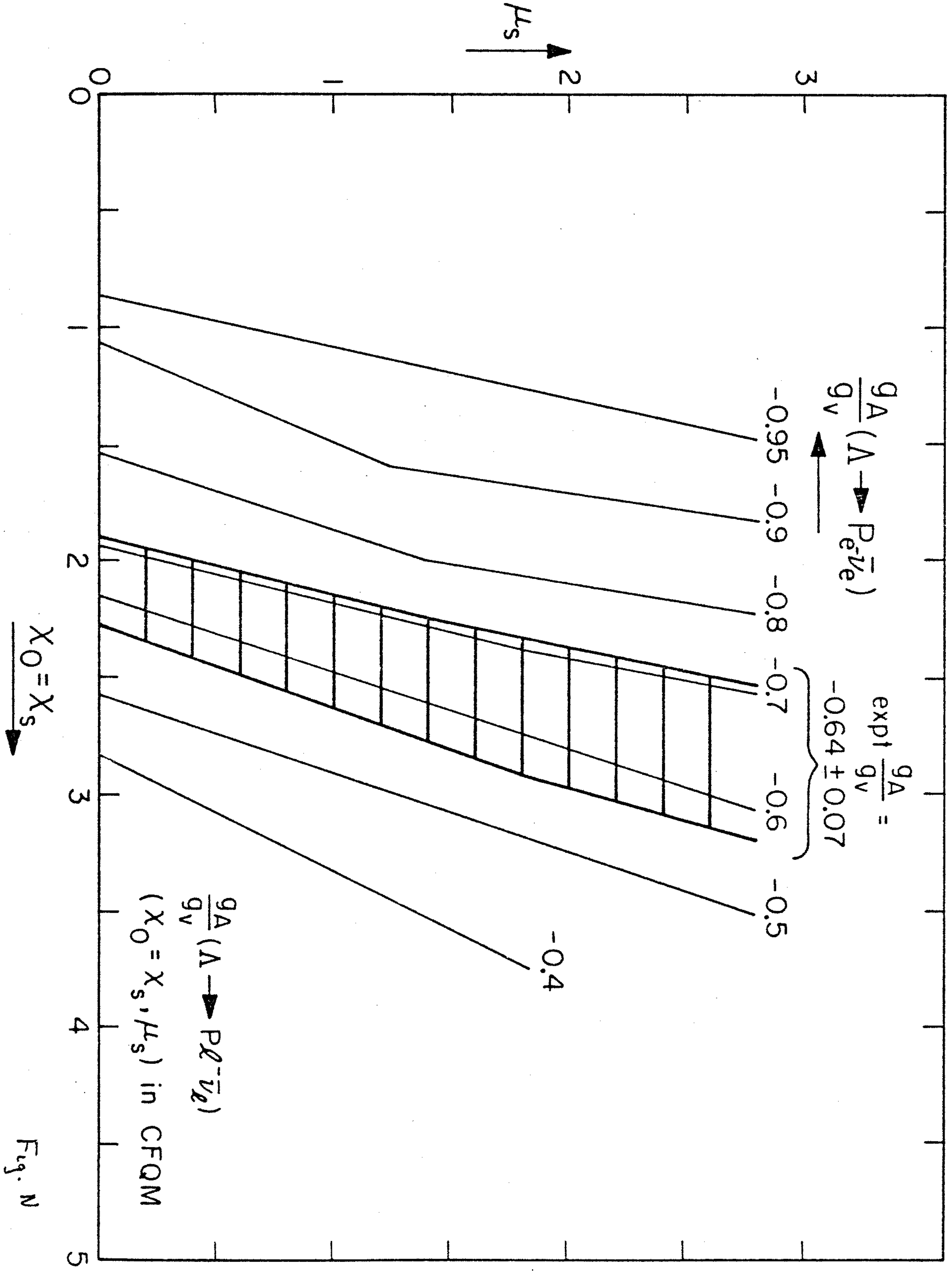

$m_q(X_0)$ required to give $\frac{g_A}{g_V} = -1.25 \pm 0.1$ in $N \to Pe^- \bar{\nu}_e$
for $a = 1.4$ fm $\mu(X_0)$ curve is universal ($\forall a$)
$\mu(X_0)$ giving
correct $\frac{g_A}{g_V}$ $(N \to P)$
bag model constraint $X_0^{(+1)}(\mu)$
m_q
Mev
0
100
200
300
1.0
1.5
2.0
2.5
3.0
3.5
$X_0 \equiv ka$
0
0.50
0.71
1.00
1.42
2.13
μ

Fig L



Nucleon properties with massless quarks
in attractive scalar well
(⟵ vector well)
$B_0^{1/4} = 120$ Mev
vector
scalar
$\frac{g_A}{g_V}(N \to P)$
1.30
1.25
1.20
1.15
1.10
1.05
1.00
0.95
intercept ~ $B_0^{1/4}$
scalar
g_P
2.9
2.8
2.7
2.6
intercept ~ $B_0^{1/4}$
scalar
E(Mev)
1160
1140
1120
1100
1080
1060
0 10 20 30 40 50 60 70 80 90
$g\phi_0$ (Mev)

Fig. M



μs
0
1
2
3
4
5
gA/gV (Λ → P e- ν̄e)
-0.95
-0.9
-0.8
-0.7
-0.6
-0.5
-0.4
expt gA/gV = -0.64 ± 0.07
gA/gV (Λ → P ℓ- ν̄ℓ)
(χO = χs, μs) in CFQM
χO = χs

Fig. N

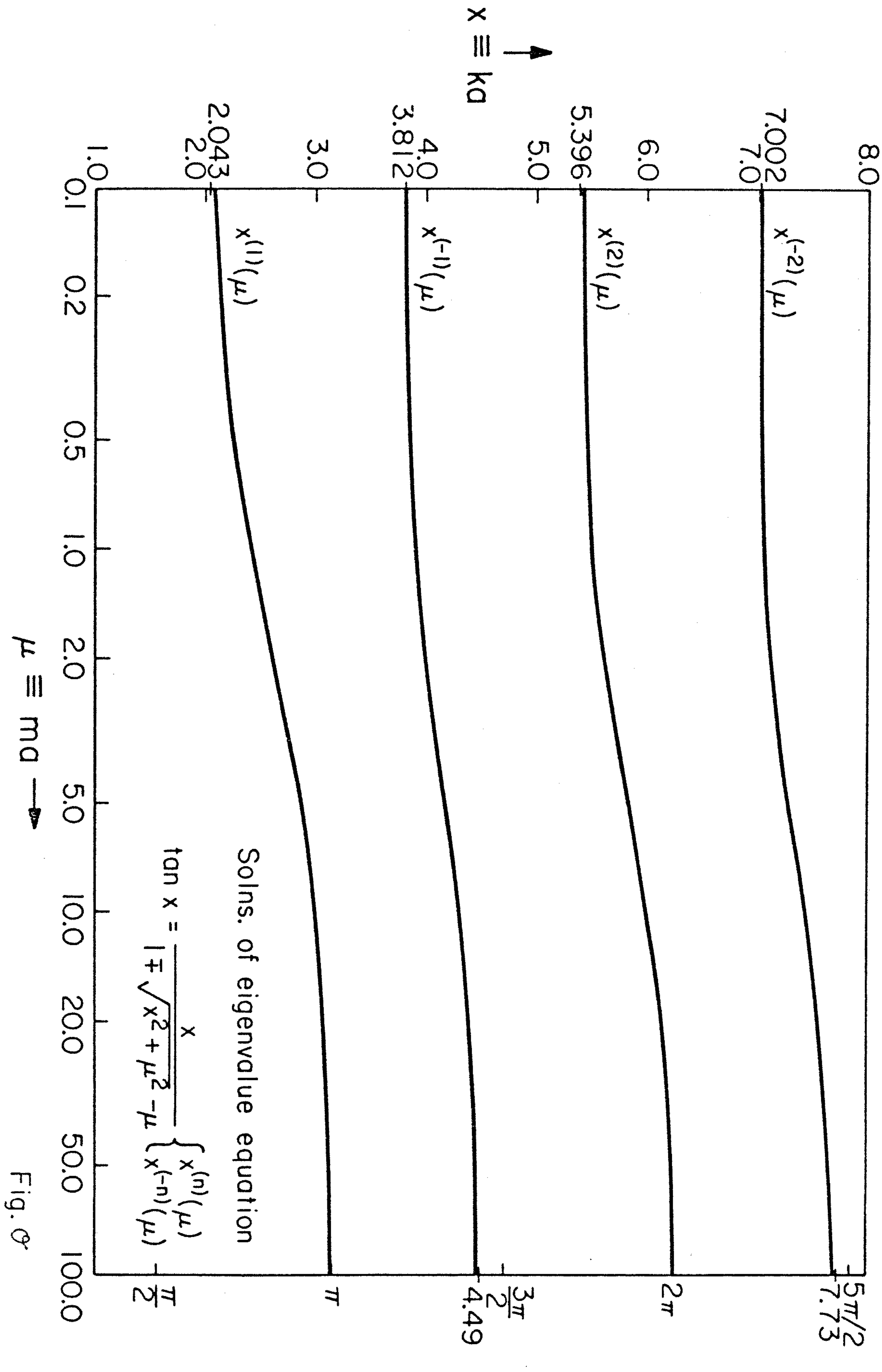
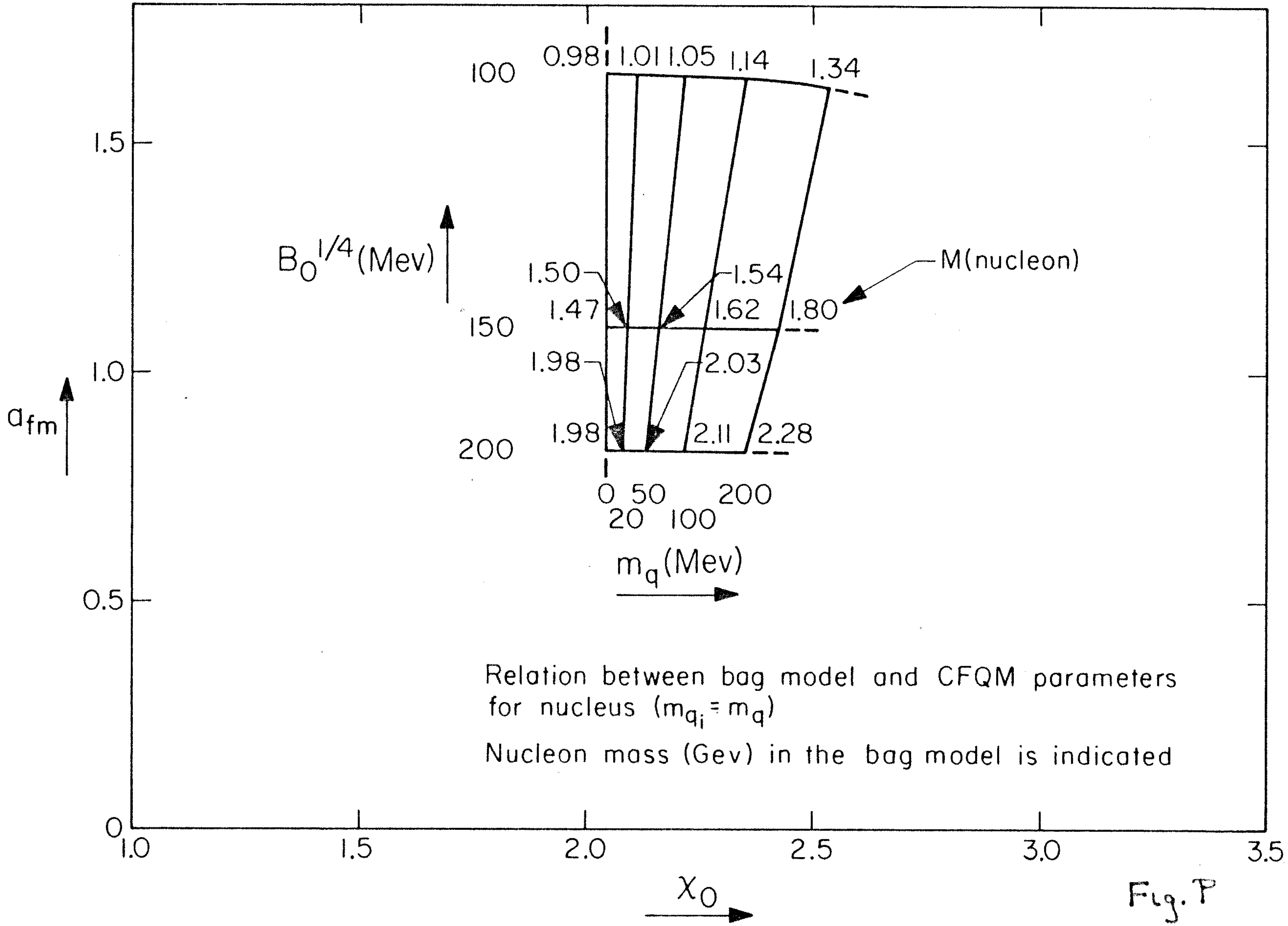

x ≡ ka
μ ≡ ma
8.0
7.002
7.0
6.0
5.396
5.0
4.0
3.812
3.0
2.043
2.0
1.0
0.1
0.2
0.5
1.0
2.0
5.0
10.0
20.0
50.0
100.0
$x^{(-2)}(\mu)$
$x^{(2)}(\mu)$
$x^{(-1)}(\mu)$
$x^{(1)}(\mu)$
$5\pi/2$
7.73
2π
$\frac{3\pi}{2}$
4.49
π
$\frac{\pi}{2}$
Solns. of eigenvalue equation
$\tan x = \frac{x}{1 \mp \sqrt{x^2+\mu^2} - \mu} \begin{cases} x^{(n)}(\mu) \\ x^{(-n)}(\mu) \end{cases}$

Fig. 8



Relation between bag model and CFQM parameters for nucleus ($m_{q_i} = m_q$)

Nucleon mass (Gev) in the bag model is indicated

Fig. P

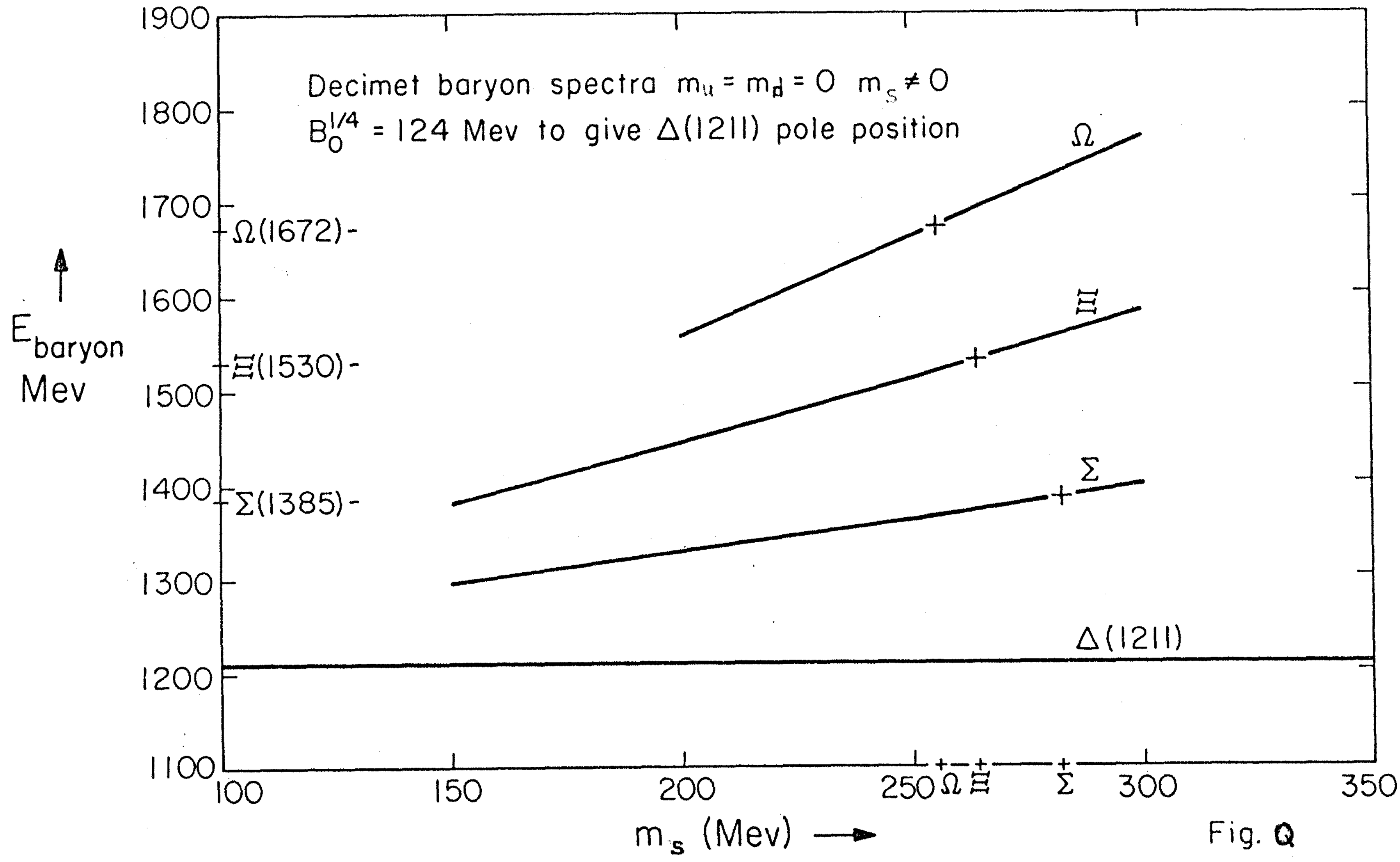
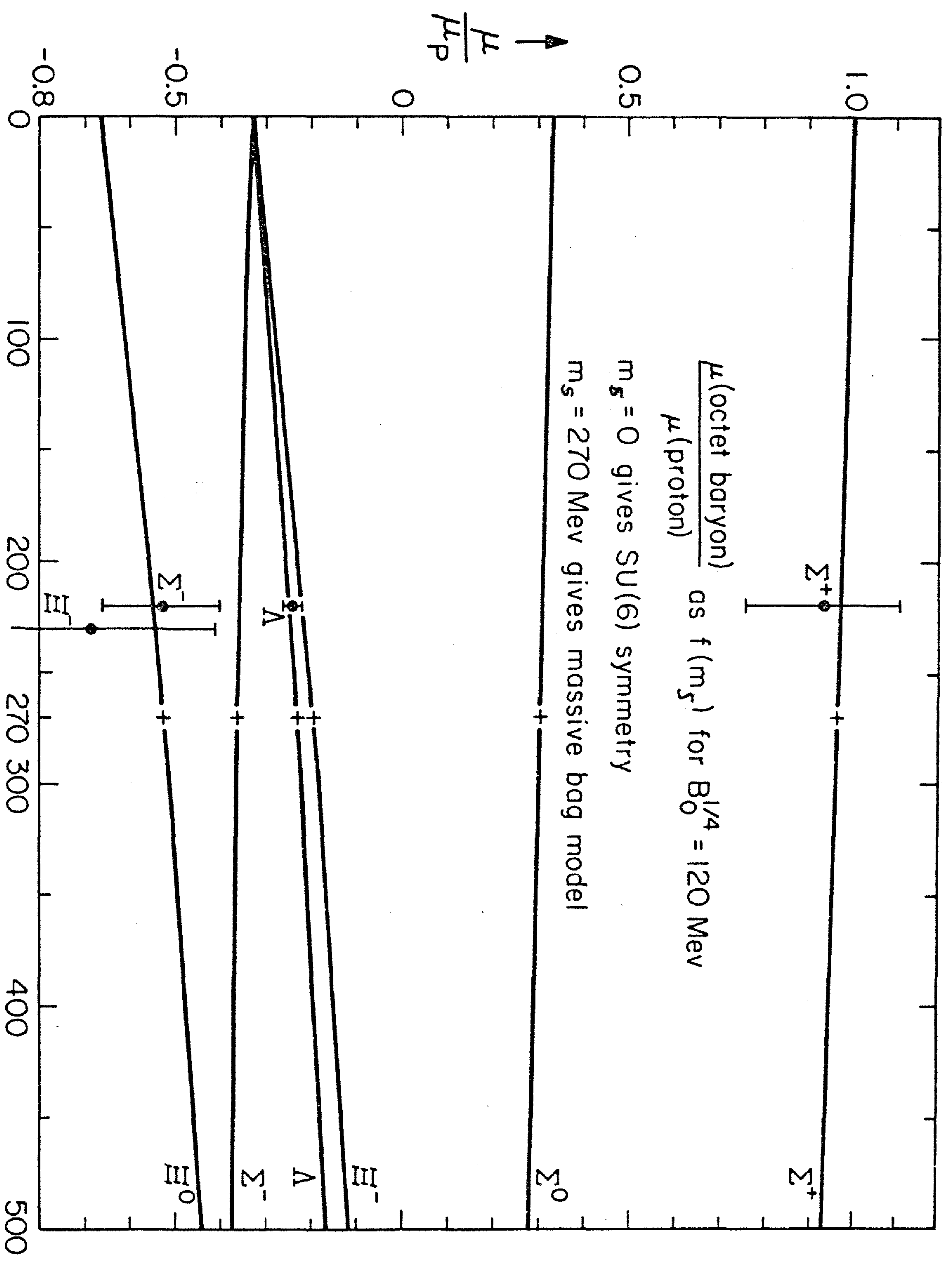

Decimet baryon spectra $m_u = m_d = 0$ $m_s \neq 0$
$B_0^{1/4} = 124$ Mev to give $\Delta(1211)$ pole position
E_{baryon} Mev
1900
1800
1700
1600
1500
1400
1300
1200
1100
$\Omega(1672)$
$\Xi(1530)$
$\Sigma(1385)$
Ω
Ξ
Σ
$\Delta(1211)$
100
150
200
250
300
350
Ω Ξ Σ
m_s (Mev)

Fig. Q



$\frac{\mu}{\mu_p}$
1.0
0.5
0
-0.5
-0.8
0
100
200
270
300
400
500
m_s (Mev)
$\frac{\mu(\text{octet baryon})}{\mu(\text{proton})}$ as $f(m_s)$ for $B_0^{1/4}$ = 120 Mev
m_s = 0 gives SU(6) symmetry
m_s = 270 Mev gives massive bag model
Σ^+
Σ^0
Ξ^-
Λ
Σ^-
Ξ^0

Fig. R

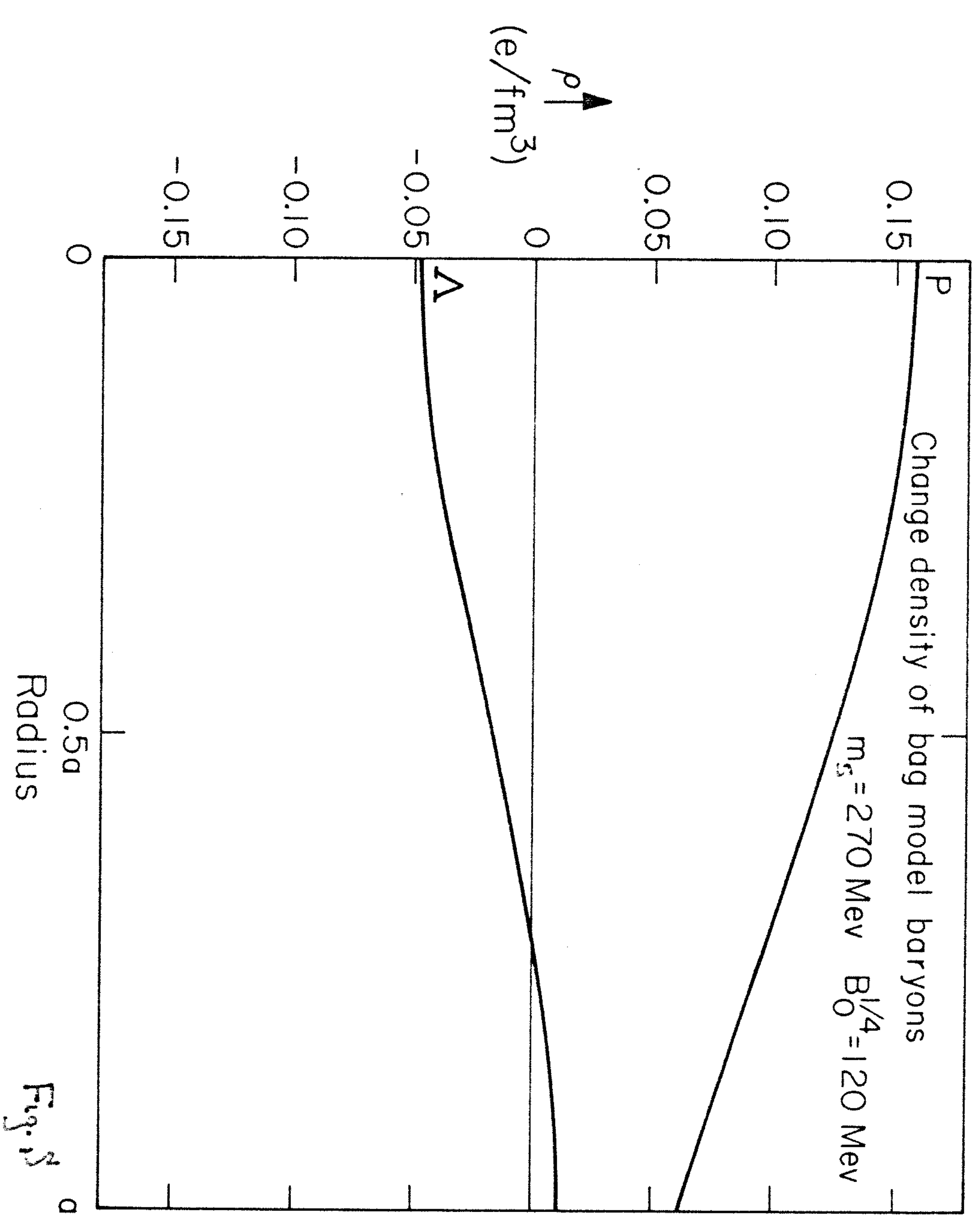
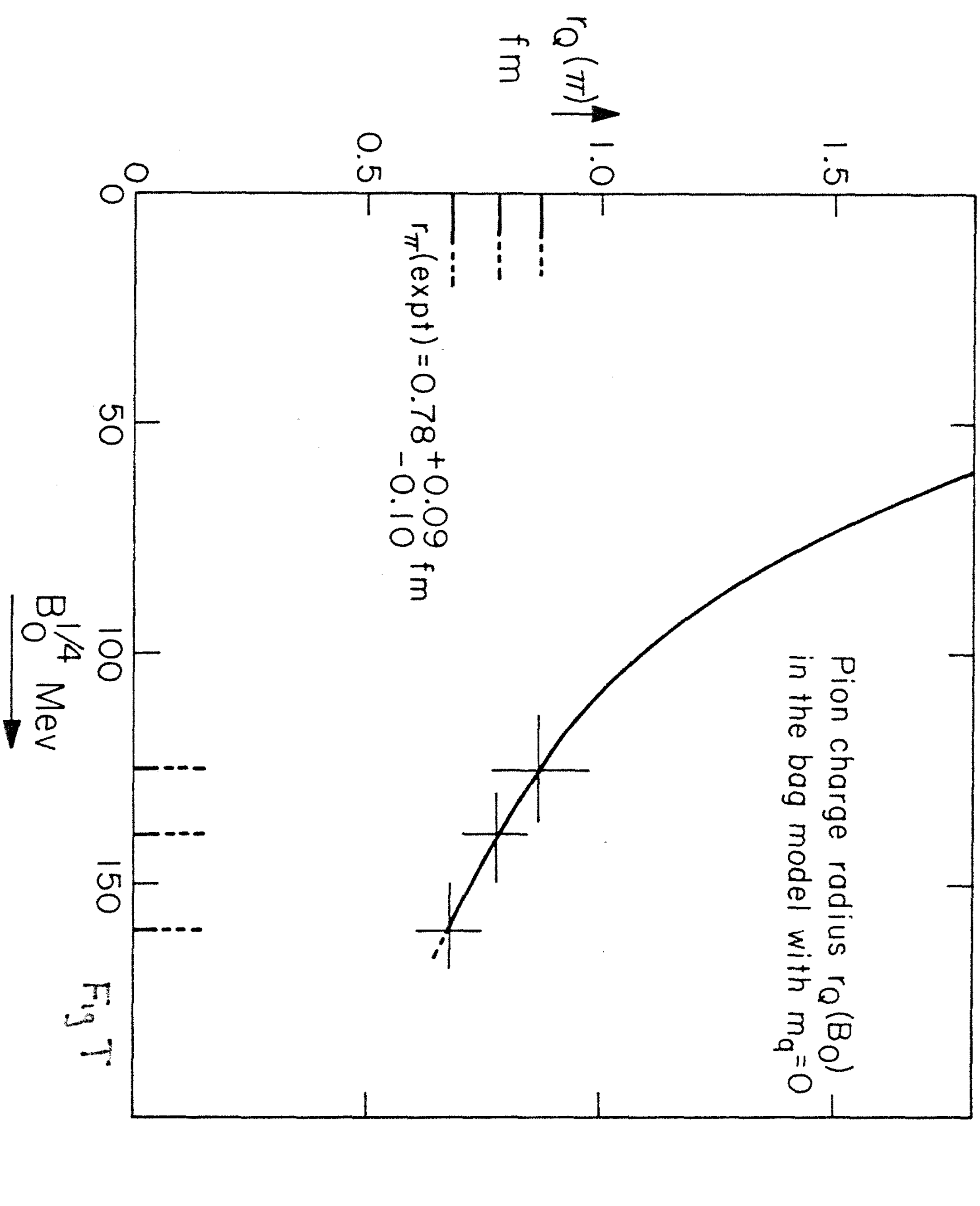
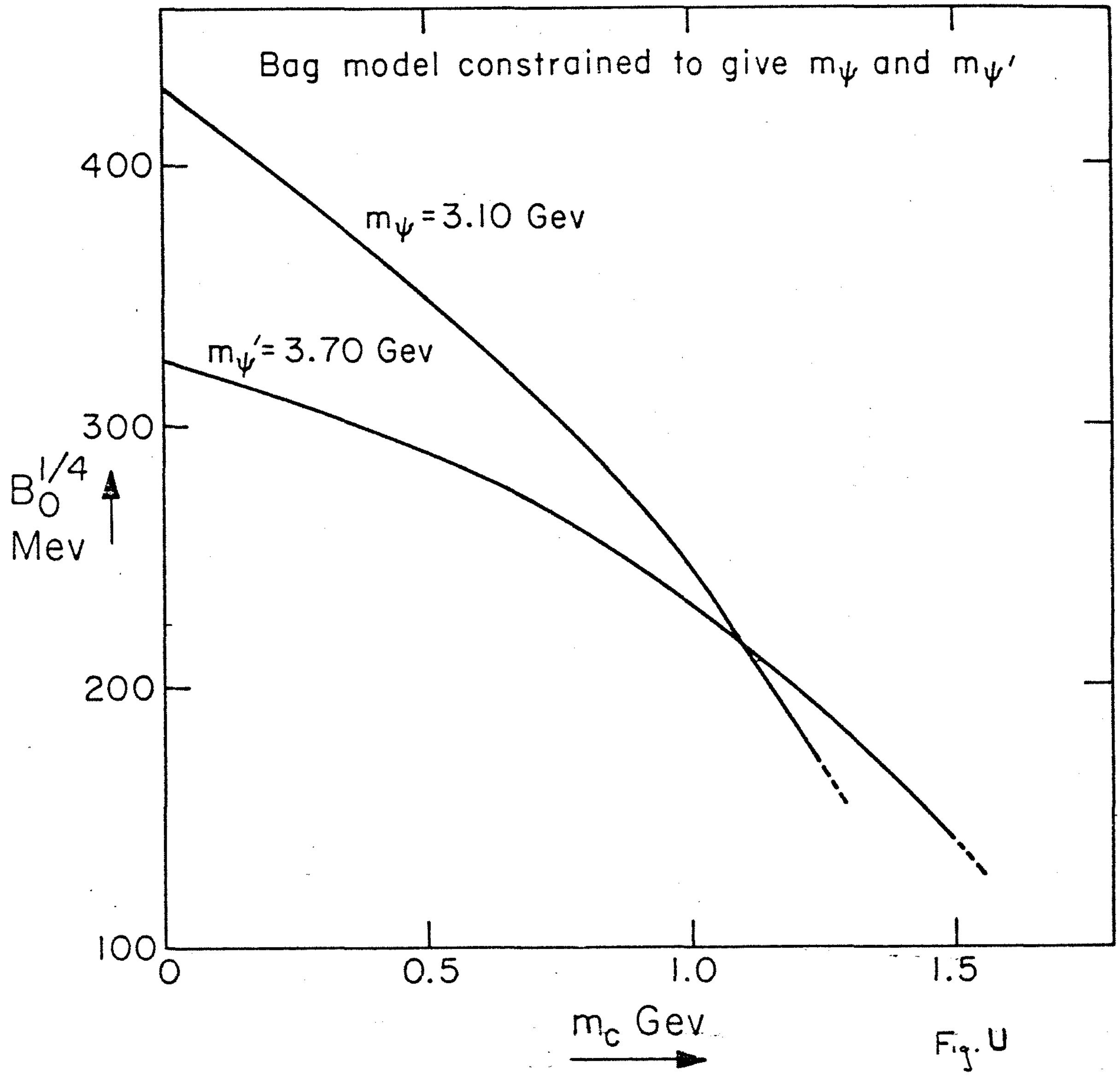

ρ
(e/fm^3)
0.15
0.10
0.05
0
−0.05
−0.10
−0.15
P
Λ
Change density of bag model baryons
m_s=270 Mev $B_0^{1/4}$=120 Mev
0
0.5a
a
Radius

Fig. 12



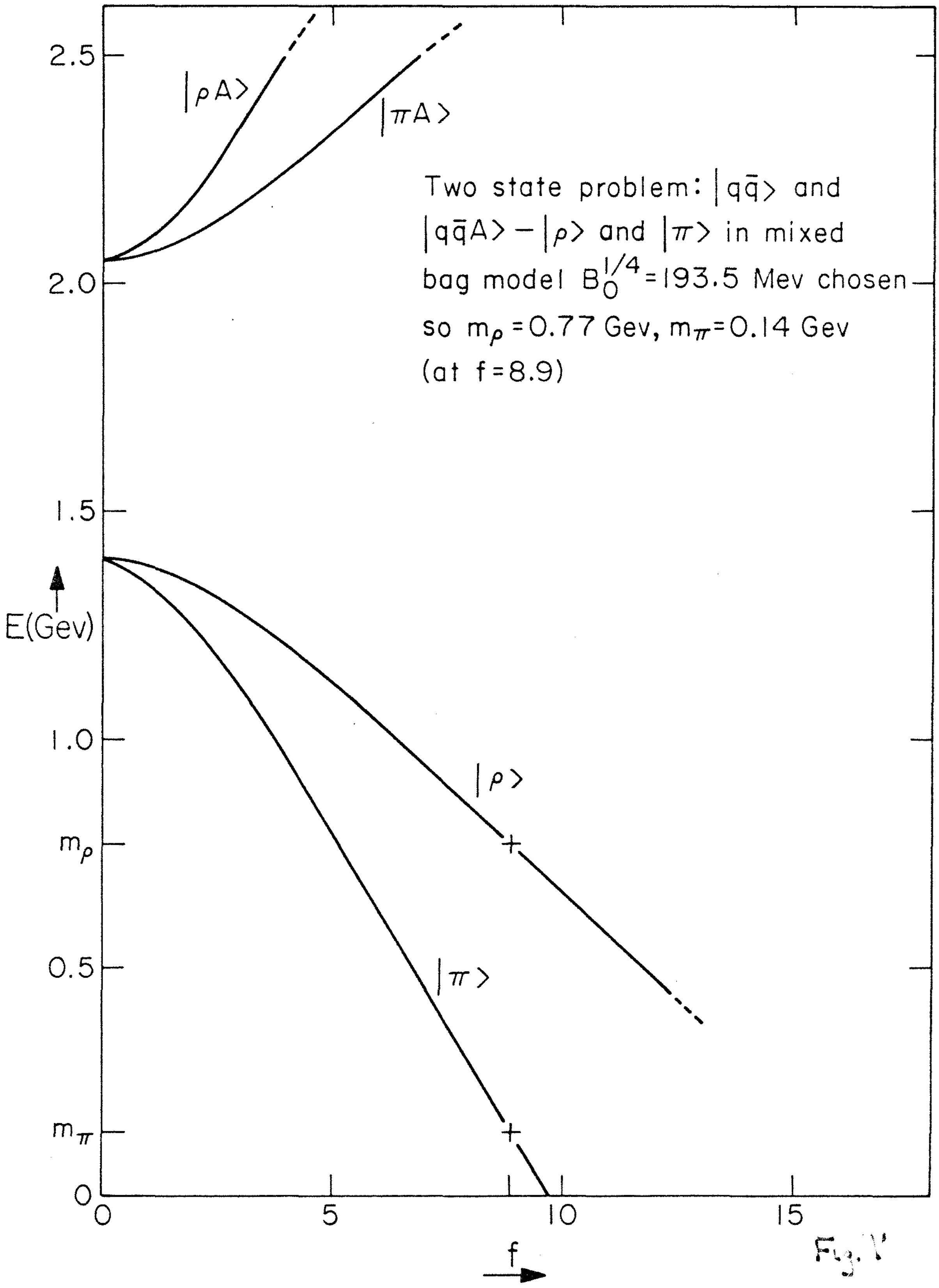
Pion charge radius $r_Q(B_0)$ in the bag model with $m_q=0$
$r_Q(\pi)$ fm
1.5
1.0
0.5
0
$r_\pi(\text{expt})=0.78^{+0.09}_{-0.10}$ fm
0
50
100
150
$B_0^{1/4}$ Mev

Fig 7



Bag model constrained to give m_ψ and $m_{\psi'}$
$m_\psi = 3.10$ Gev
$m_{\psi'} = 3.70$ Gev
400
300
200
100
$B_0^{1/4}$ Mev
0
0.5
1.0
1.5
m_c Gev

Fig. U



2.5
2.0
1.5
1.0
0.5
0
m_ρ
m_π
E(Gev)
0
5
10
15
f
$|\rho A\rangle$
$|\pi A\rangle$
Two state problem: $|q\bar{q}\rangle$ and $|q\bar{q}A\rangle$ — $|\rho\rangle$ and $|\pi\rangle$ in mixed bag model $B_0^{1/4}$=193.5 Mev chosen so m_ρ=0.77 Gev, m_π=0.14 Gev (at f=8.9)
$|\rho\rangle$
$|\pi\rangle$

Fig. V

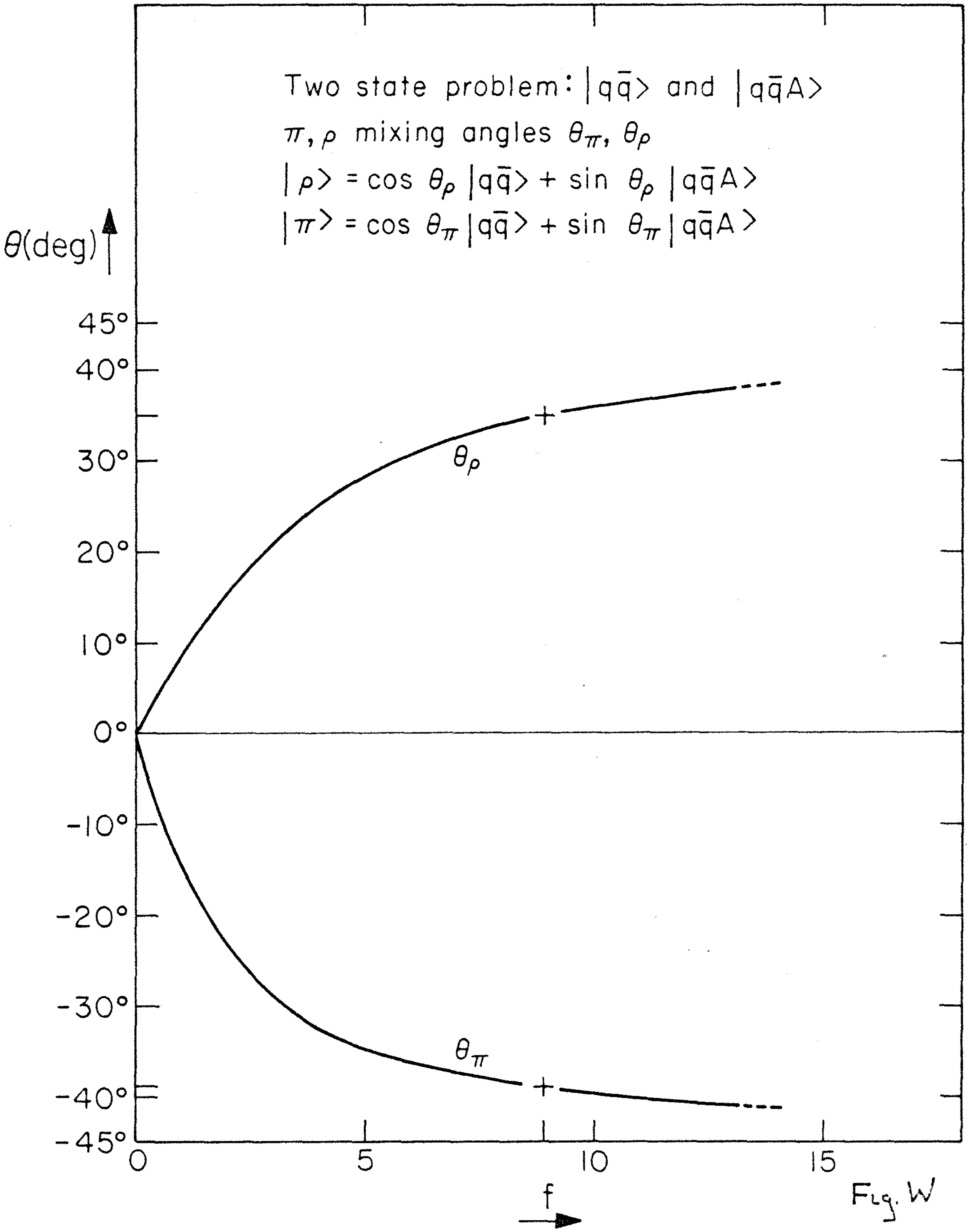
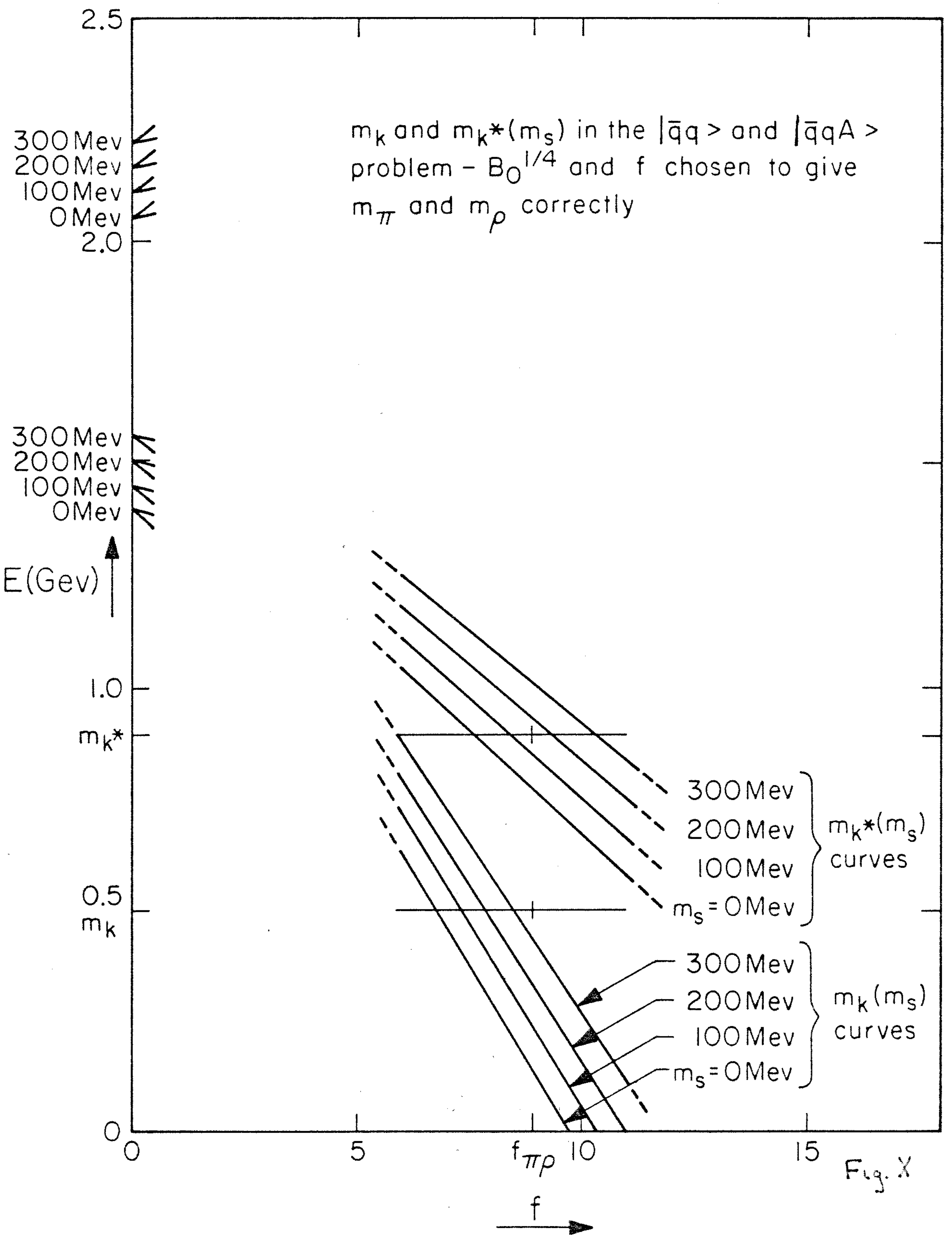

Two state problem: $|q\bar{q}\rangle$ and $|q\bar{q}A\rangle$
π, ρ mixing angles θ_π, θ_ρ
$|\rho\rangle = \cos\theta_\rho |q\bar{q}\rangle + \sin\theta_\rho |q\bar{q}A\rangle$
$|\pi\rangle = \cos\theta_\pi |q\bar{q}\rangle + \sin\theta_\pi |q\bar{q}A\rangle$
θ(deg)
45°
40°
30°
20°
10°
0°
-10°
-20°
-30°
-40°
-45°
θ_ρ
θ_π
0
5
10
15
f

Fig. W



m_K and $m_{K^*}(m_s)$ in the $|\bar{q}q\rangle$ and $|\bar{q}qA\rangle$ problem - $B_0^{1/4}$ and f chosen to give m_π and m_ρ correctly
2.5
300 Mev
200 Mev
100 Mev
0 Mev
2.0
300 Mev
200 Mev
100 Mev
0 Mev
E(Gev)
1.0
m_{K^*}
0.5
m_K
0
0
5
$f_{\pi\rho}$
10
15
f
300 Mev
200 Mev
100 Mev
m_s = 0 Mev
$m_{K^*}(m_s)$ curves
300 Mev
200 Mev
100 Mev
m_s = 0 Mev
$m_K(m_s)$ curves

Fig. X

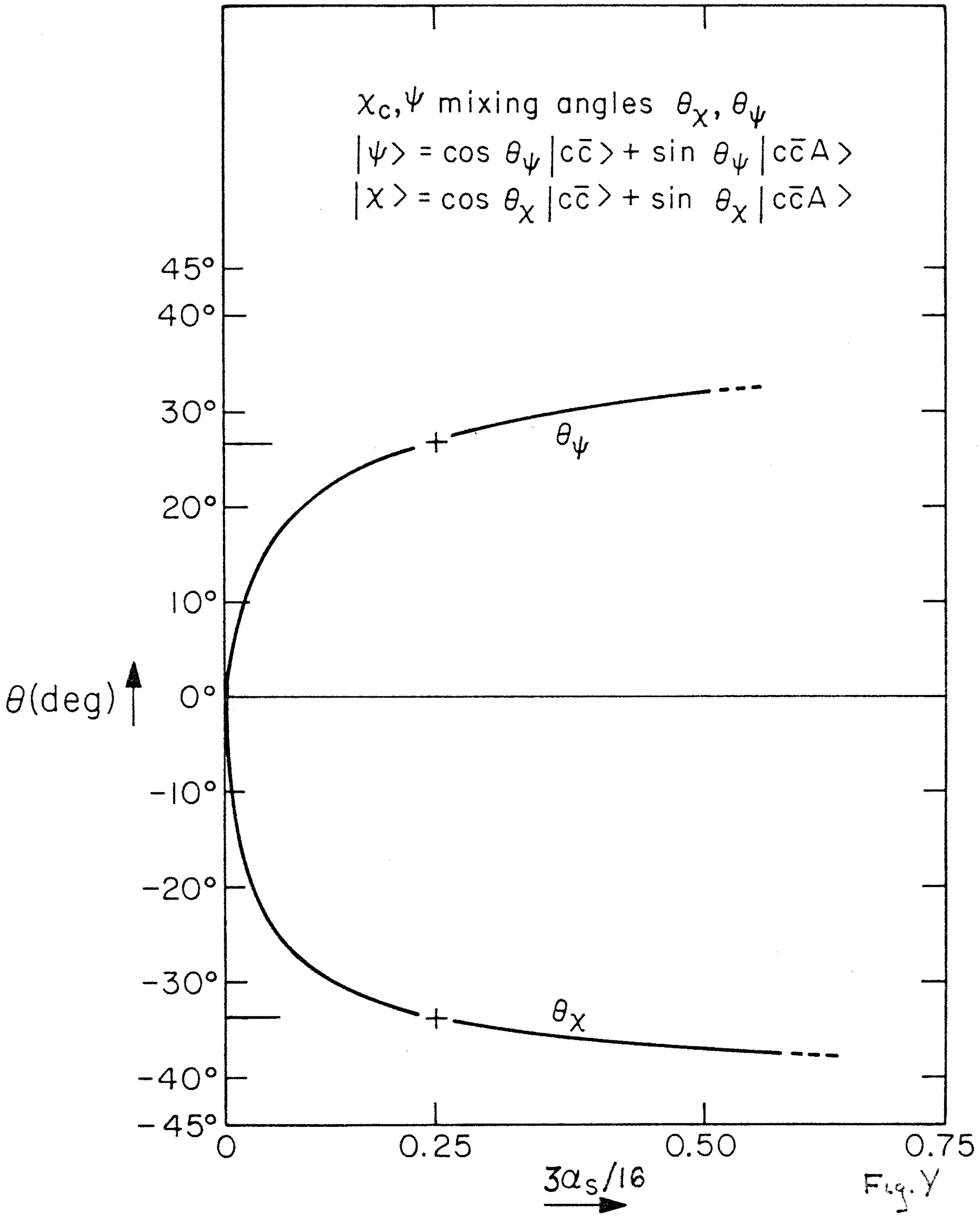

χ_c, ψ mixing angles θ_χ, θ_ψ
$|\psi\rangle = \cos\theta_\psi |c\bar{c}\rangle + \sin\theta_\psi |c\bar{c}A\rangle$
$|\chi\rangle = \cos\theta_\chi |c\bar{c}\rangle + \sin\theta_\chi |c\bar{c}A\rangle$
45°
40°
30°
20°
10°
0°
-10°
-20°
-30°
-40°
-45°
θ(deg)
θ_ψ
θ_χ
0
0.25
0.50
0.75
$3\alpha_s/16$

Fig. Y

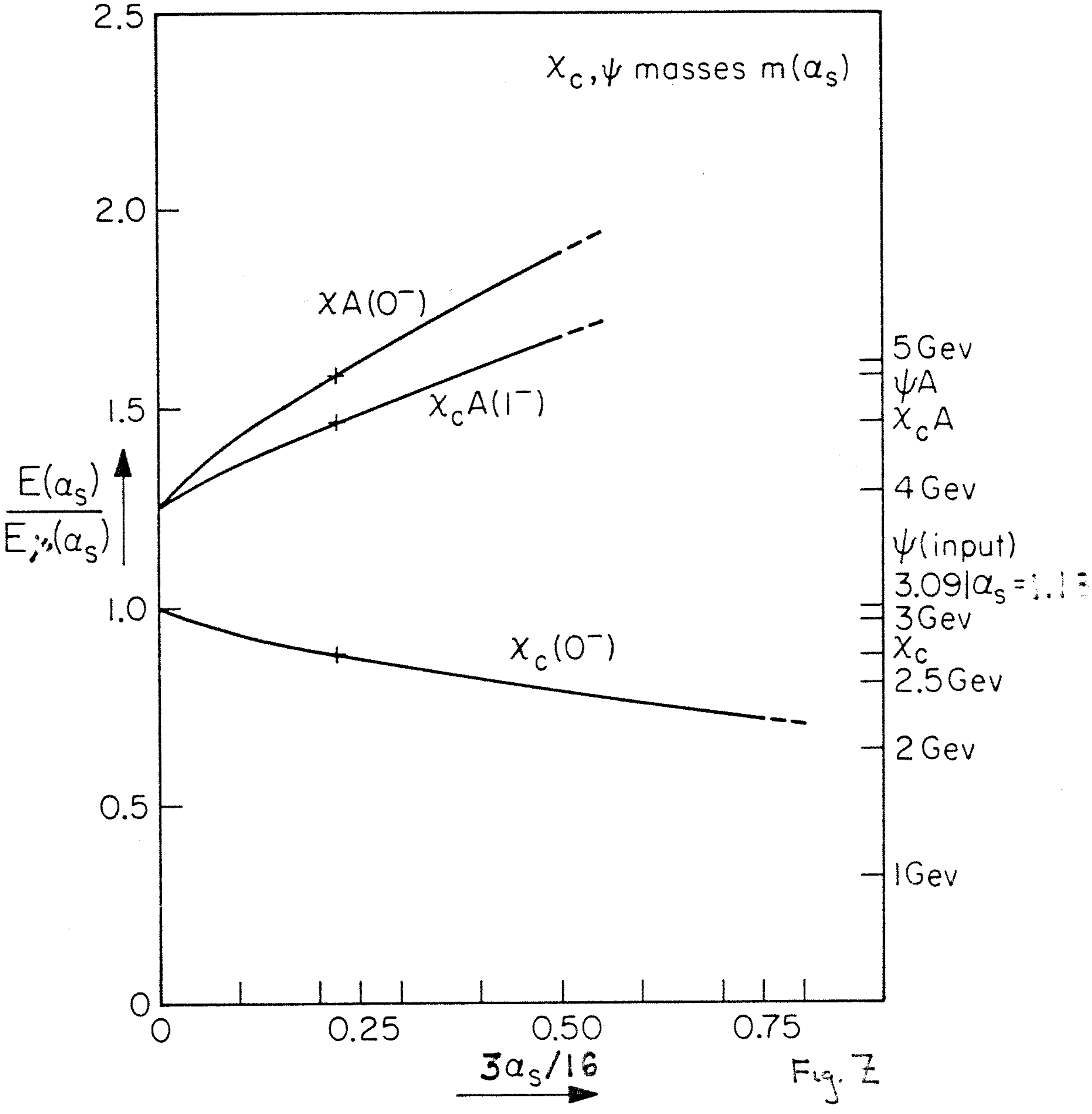

χ_c, ψ masses $m(\alpha_s)$
$\chi A(0^-)$
$\chi_c A(1^-)$
$\chi_c(0^-)$
$\frac{E(\alpha_s)}{E_{\psi}(\alpha_s)}$
$3\alpha_s/16$
2.5
2.0
1.5
1.0
0.5
0
0
0.25
0.50
0.75
5 Gev
ψA
$\chi_c A$
4 Gev
ψ (input)
3.09 | α_s =
3 Gev
χ_c
2.5 Gev
2 Gev
1 Gev

Fig. 7

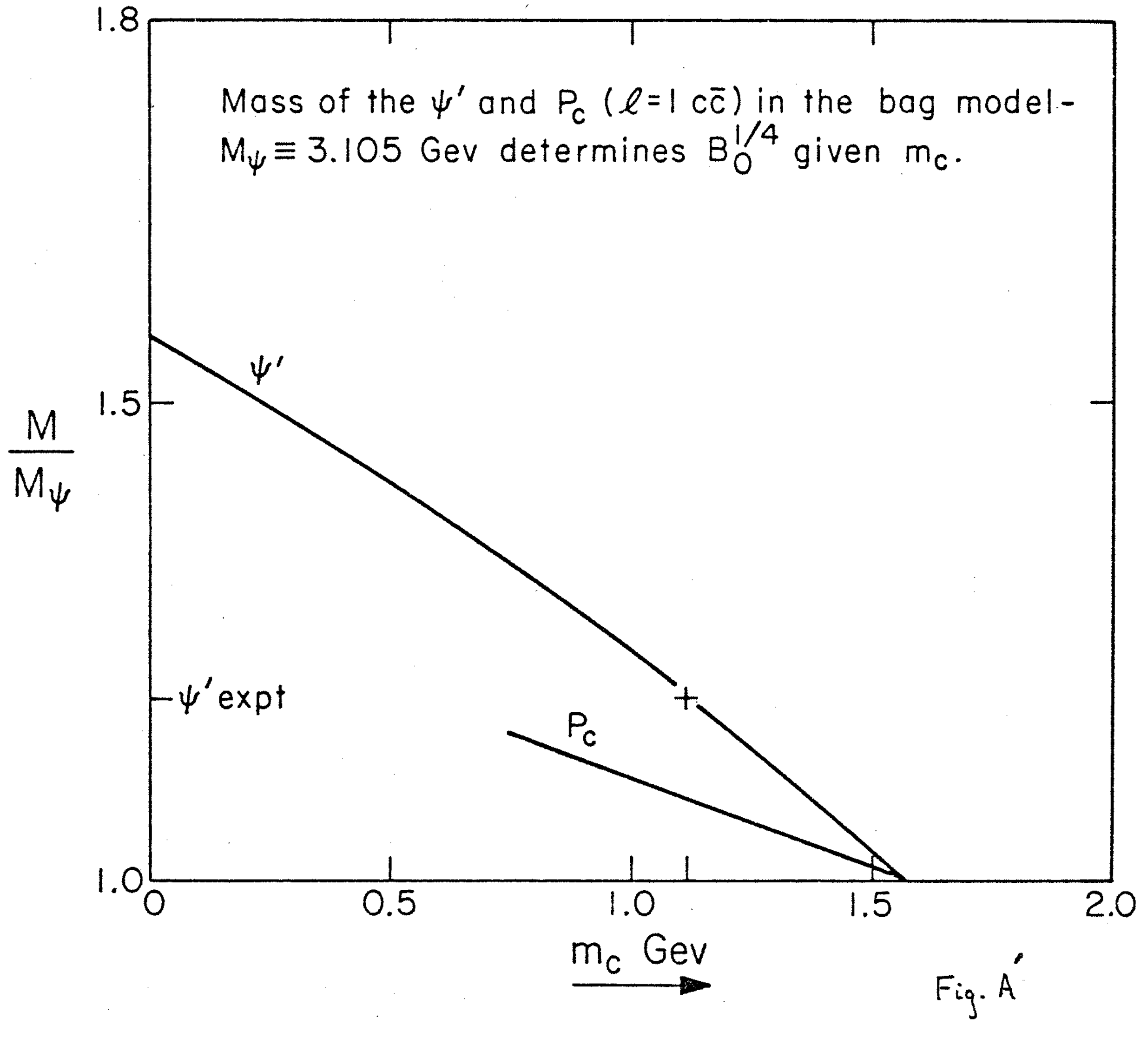

Mass of the ψ′ and P_c (ℓ=1 $c\bar{c}$) in the bag model-
$M_\psi \equiv$ 3.105 Gev determines $B_0^{1/4}$ given m_c.
1.8
1.5
1.0
$\frac{M}{M_\psi}$
ψ′
ψ′ expt
P_c
0
0.5
1.0
1.5
2.0
m_c Gev

Fig. A′

APPENDIX A CONVENTIONS AND INTEGRALS

We use the conventions of Bjorken and Drell[44] for Dirac matrices, spinors, and the metric ;

$$g_{\mu\nu} = \begin{bmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{bmatrix} \qquad \vec{\gamma} = \begin{bmatrix} & \vec{\sigma} \\ -\vec{\sigma} & \end{bmatrix} \qquad \gamma_0 = \begin{bmatrix} I & \\ & -I \end{bmatrix}$$

$$\gamma_5 = i\gamma_0\gamma_1\gamma_2\gamma_3 = \begin{bmatrix} & I \\ I & \end{bmatrix} \qquad \sigma_{\mu\nu} = \frac{i}{2}\left[\gamma_\mu, \gamma_\nu\right] \tag{A1}$$

We also note that some of the integrals left as implicit functions in the text may be carried out explicitly in terms of special functions[45] ;

$$I(\ell, x_0) \equiv \int_0^{x_0} \eta^2 j_\ell(\eta)^2 d\eta = \frac{x_0^3}{3}\left[j_\ell(x_0)^2 - j_{\ell-1}(x_0)\, j_{\ell+1}(x_0) \right] \tag{A2}$$

$$f^{(j)}(x_0) \equiv \int_0^{x_0} \eta^3 j_{j-\frac{1}{2}}(\eta)\, j_{j+\frac{1}{2}}(\eta) = \frac{x_0^2}{\pi}\left[j\, J_j(x_0)^2 - (j+1)\, J_{j-1}(x_0)\, J_{j+1}(x_0) \right] \tag{A3}$$

APPENDIX B BAG MODEL EXCITED STATES

The problems of combining angular momenta to form excited hadron states in the bag model and in free space are similar but not identical. The bag model tends to have more possible states for a given set of basis states, due to the fact that the relative center of mass motion of the constituents has not been removed. Here we shall illustrate the differences between the two cases and obtain a suitable set of labels for the bag model states.

First, consider the problem of constructing a state of total angular momentum J from two particles with spins s_1 and s_2. Normally we assume that physics is translationally invariant, so we may remove the location of the center of mass of the two particles from the description of their relative motion. The assumption of invariance of the problem with respect to rotations about the center of mass leads to the conservation of the relative angular momentum of the two particles, which we may combine with the two intrinsic angular momenta $s_{1,2}$ to obtain a state of total angular momentum J and Z-projection $J_z|\rangle = M|\rangle$;

$$|JM;\ell_{12},s_1,s_2\rangle \tag{B1}$$

Independent sets of states are obtained by first combining two of the three available angular momenta ℓ_{12}, s_1, and s_2 to make an intermediate angular momentum j, and then combining the third angular momentum. For example, suppose we combine the two spins s_1 and s_2 to make an intermediate angular momentum j;

$$\vec{J} = \vec{\ell}_{12} + \vec{s}_1 + \vec{s}_2 = \vec{\ell}_{12} + \vec{\mathcal{S}} \tag{B2}$$

We may simultaneously diagonalize $\vec{\ell}_{12}^{\,2}$ and $\vec{\mathcal{S}}^{\,2}$, which gives us states which are eigenstates of $\vec{J}^2, J_z, \vec{\ell}_{12}^{\,2}, \vec{\mathcal{S}}^{\,2}, \vec{s}_1^{\,2}$, and $\vec{s}_2^{\,2}$;

$$|JM;\ell_{12},\mathcal{S},s_1,s_2\rangle = \sum_{m_{12}\,\mu\,\mu_1\,\mu_2} \begin{pmatrix} J & \ell_{12} & \mathcal{S} \\ M & m_{12} & \mu \end{pmatrix} \begin{pmatrix} \mathcal{S} & s_1 & s_2 \\ \mu & \mu_1 & \mu_2 \end{pmatrix} |\ell_{12}\, m_{12}\rangle \cdot |s_1\mu_1\rangle |s_2\mu_2\rangle \tag{B3}$$

Alternatively we could have combined $\vec{\ell}_{12}$ with one of the spins to make a different intermediate angular momentum j; these states are related to the above $|JM\rangle$ states by Racah coefficients.

In a cavity, combining the angular momenta of two particles proceeds differently. There is no translational invariance, so we may not remove the center of mass motion of the two particles from the problem. The physics of a single particle in the cavity is globally invariant only with respect to rotations about the center of the cavity, so we have a conserved angular momentum for a particle only about that point. Each particle has such an angular momentum; we may combine this orbital angular momentum $\vec{\ell}_1$ with the particle's intrinsic spin $\vec{s}_1$ to make a total angular momentum for that particle.

$$\vec{J}_1 = \vec{\ell}_1 + \vec{s}_1 \tag{B4}$$

The one particle states with definite $\vec{J}_1^{\,2}$ are obtained by adding the spin and orbital angular momenta $\vec{s}_1$ and $\vec{\ell}_1$ about the center of the cavity;

$$|j_1 m_1\rangle = \sum_{\mu} \begin{pmatrix} j_1 & \ell_1 & s_1 \\ m_1 & \mu & m_1-\mu \end{pmatrix} |\ell_1\mu\rangle |s_1\, m_1-\mu\rangle \tag{B5}$$

These states have diagonal $\vec{j}_1^{\,2}$, $\vec{\ell}_1^{\,2}$, $\vec{s}_1^{\,2}$, and j_{1z}.

The one-quark solutions of the free Dirac equation which we discussed previously, the CFQM wavefunctions, are superpositions of these one particle state vectors with j_1 and s_1 fixed, summed over ℓ_1. (The free Dirac hamiltonian mixes states with different ℓ_1,

$$|j_1 m_1\rangle_{\substack{\text{Dirac}\\ \text{E eigenstate}}} = \sum_{\ell_1=|j_1-1/2|}^{|j_1+1/2|} c_{\ell_1} \sum_{\mu} \begin{pmatrix} j_1 & \ell_1 & \frac{1}{2} \\ m_1 & \mu & m_1-\mu \end{pmatrix} |\ell_1 \mu\rangle |\tfrac{1}{2}\, m_1-\mu\rangle \tag{B6}$$

so $\vec{\ell}_1^{\,2}$ is not diagonal for quark state vectors which are energy eigenstates). In constructing a many quark state in a cavity the most natural basis is one in which the quarks are eigenstates of the free Dirac hamiltonian, so we simply combine the above one-quark states to form a state of total angular momentum $\vec{J}$;

$$\vec{J} = \vec{j}_1 + \vec{j}_2 + \ldots \tag{B7}$$

The orbital angular momenta ℓ_1, ℓ_2,... have been summed over implicitly in constructing simultaneous eigenstates of $\vec{j}_1^{\,2}$, $\vec{j}_2^{\,2}$,... and the free Dirac hamiltonian. There are two independent ways of combining spin and orbital angular momenta to form an energy eigenstate $(\ell = j \pm \frac{1}{2})$, so we require an additional label to identify these cavity states. In the nonrelativistic limit ($m_{quark} \mapsto \infty$) only one of the orbital angular momenta contributes to the state vector, so we may identify it by this ℓ_1 value. Alternately we may distinguish the two states with identical s_1 and j_1 by their spatial parities, which are opposite. For example, with $s_1=j_1=\frac{1}{2}$ we have

$$|j_1=1/2\,,\ S\ \text{wave}\rangle = |j_1=1/2\,,P=(+)\rangle$$
$$|j_1=1/2\,,\ P\ \text{wave}\rangle = |j_1=1/2\,,P=(-)\rangle \tag{B8}$$

In combining many quark state vectors to form excited hadron cavity states we must of course specify these labels as well as the quark angular momenta.

To construct an n-gluon state we proceed similarly and combine one-gluon state vectors which are eigenstates of the free gluon hamiltonian. The individual spins and angular momenta in the cavity must sum to a total angular momentum J;

$$\vec{J} = \vec{\ell}_1+\vec{s}_1+\vec{\ell}_2+\vec{s}_2+\ldots+\vec{\ell}_n+\vec{s}_n \tag{B9}$$

Unlike the free quark hamiltonian, the free gluon hamiltonian does not mix one-particle states with different orbital angular momenta. This means that we may diagonalize all the $\{\vec{\ell}_i^{\,2}\}$ and $\{\vec{s}_i^{\,2}\}$ simultaneously, and we could combine the individual angular momenta in any way we wish without creating undue complexity.* In particular we choose to imitate the quark case and combine the $\vec{l}_i$ and $\vec{s}_i$ to form one-particle total angular momenta $\vec{j}_i^{\,2}$ (also diagonal), which we then combine to form the total J ;

$$\vec{J} = \vec{j}_1+\vec{j}_2+\ldots+\vec{j}_n \tag{B10}$$
$$\vec{j}_i = \vec{\ell}_i+\vec{s}_i \tag{B11}$$

One important result of working in a cavity is that we have more excited states than in the free particle case, as a result of not re-

* We note, however, that the transversality constraint $A_{\mu,\mu}=0$ mixes $\ell = j\pm 1$ states for electric multipole gluons.

moving the center of mass motion. A simple example of this is the construction of a two quark state with distinguishable quarks. Suppose that we wish to construct a $J^P = 0^+$ state. Outside the cavity we can do this in only one way; we combine the two spins to make a state with $s^P = 1^-$, to which we add a unit of relative orbital angular momentum $\ell_{qq}^P = 1^-$ to make $J^P = 0^+ \subset 1^- \otimes 1^-$. In the cavity, however, each quark has an orbital angular momentum, and we have two independent $J^P = 0^+$ states;

$$\begin{aligned} |0^+\rangle_1 &= |j_1 = 1/2, P_1 = (+)\rangle \otimes |j_2 = 1/2, P_2 = (-)\rangle \\ |0^+\rangle_2 &= |j_1 = 1/2, P_1 = (-)\rangle \otimes |j_2 = 1/2, P_2 = (+)\rangle \end{aligned} \tag{B12}$$

These states will not in general be degenerate in energy. However, all $|J^P\rangle$ states which we may construct from

$$|j_1 = 1/2, P_1 = (+)\rangle |j_2 = 1/2, P_2 = (-)\rangle \tag{B13}$$

will be degenerate in the zeroth order bag model, as will all such states we may construct from the second set. As a shorthand notation for all such states, we simply write the degree of radial excitation (1,2,...) and the orbital angular momentum carried by each quark in the nonrelativistic limit. Thus, we have

$$|0^+\rangle_1 \subset |(1S)(1P)\rangle \tag{B14}$$

where $|(1S)(1P)\rangle$ is an abbreviation for a degenerate set of states with negative parity and spins j=2,1,0. This is the notation which we use to label the charmed meson bag model energy levels in Table F.

As a last observation, we note that these additional excited states of the bag model relative to the free space problem make any comparison of the bag states with experimentally measured resonance masses an ambiguous exercise.

APPENDIX C SU(4) $\underline{20}$ $1/2^+$ BARYONS

When one generalizes the SU(6) quark model to SU(8) by introducing the two charmed quark states $|c\uparrow\rangle$ and $|c\downarrow\rangle$, one finds that the $\underline{56}$-dimensional irreducible representation of SU(6) spanned by the light S-wave baryons is contained in a $\underline{120}$ in SU(8). This $\underline{120}$ contains a $J^P = 1/2^+$ SU(4) $\underline{20}$ and a $3/2^+$ SU(4) $\underline{20}'$. The quark decomposition of the charmed baryons in the $1/2^+$ $\underline{20}$ part of the SU(8) $\underline{120}$ we have derived and reproduced below. The correct relative phases of these states have not been determined, as they were not required for our purposes. The notation used for the baryons is that of Ref. 43, and the $1/2^+$ $\underline{20}$ weight diagram is displayed in the text, p.121.

<u>Baryon</u> <u>Quark Decomposition</u>

e.g. $|P,\uparrow\rangle = \frac{1}{\sqrt{18}}\left\{2\left(|u\uparrow u\uparrow d\downarrow\rangle + \text{et. cyc.}\right) - \left(|u\uparrow u\downarrow d\uparrow\rangle + \text{all perms } u\uparrow u\downarrow d\uparrow\right)\right\}$

$$|C_1^{++},\uparrow\rangle = |P,\uparrow\rangle\,(d\mapsto c)$$

$$|C_1^{+},\uparrow\rangle = \frac{1}{6}\left\{2\left(|u\uparrow d\uparrow c\downarrow\rangle + \text{all perms}\right) - \left(|u\uparrow d\downarrow c\uparrow\rangle + \text{a.p.}\right) - \left(|u\downarrow d\uparrow c\uparrow\rangle + \text{a.p.}\right)\right\}$$

$$|C_1^{0},\uparrow\rangle = |N,\uparrow\rangle\,(u\mapsto c)$$

$$|C_0^{+},\uparrow\rangle = \frac{1}{\sqrt{12}}\left\{\left(|u\uparrow d\uparrow c\downarrow\rangle + \text{a.p.}\right) - \left(|u\downarrow d\uparrow c\uparrow\rangle + \text{a.p.}\right)\right\} = |\Lambda,\uparrow\rangle\,(s\mapsto c)$$

$$|X_u^{++},\uparrow\rangle = |N,\uparrow\rangle\,(d^2\mapsto c^2)$$

$$|X_d^{+},\uparrow\rangle = |N,\uparrow\rangle \begin{pmatrix} d^2\mapsto c^2 \\ u\mapsto d \end{pmatrix}$$

$$|X_s^{+},\uparrow\rangle = |N,\uparrow\rangle \begin{pmatrix} d^2\mapsto c^2 \\ u\mapsto s \end{pmatrix}$$

$$|A^0,\uparrow\rangle = |C_0^+,\uparrow\rangle\,(u\mapsto s)\ ;\quad |A^+,\uparrow\rangle = |C_0^+,\uparrow\rangle \begin{pmatrix} u\mapsto s \\ d\mapsto u \end{pmatrix}\ ;\quad |T^0,\uparrow\rangle = |P,\uparrow\rangle \begin{pmatrix} u^2\mapsto s^2 \\ d\mapsto c \end{pmatrix}$$

$$|S^+,\uparrow\rangle = \frac{1}{6}\left\{2\left(|u\uparrow s\uparrow c\downarrow\rangle + \text{a.p.}\right) - \left(|u\uparrow s\downarrow c\uparrow + \text{a.p.}\right) - \left(|u\downarrow s\uparrow c\uparrow\rangle + \text{a.p.}\right)\right\}$$

$$|S^0,\uparrow\rangle = |S^+,\uparrow\rangle\,(u\mapsto d) \qquad \text{(C1 a-k)}$$

APPENDIX D GLUONIC M1 TRANSITION MOMENT CORRECTION

Mixing of the naive SU(6) quark states normally given in hadron decompositions with states containing vector gluons modifies many of the classical SU(6) predictions. Here we consider the effect of mixing the $|q\bar{q}>$ and lowest lying $|q\bar{q}A>$ states as discussed in section IV of the text on the $\rho\pi$ and $\omega\pi$ transition moments.

These moments are defined as

$$\begin{aligned} \vec{\mu}_{\rho\pi} &= <\rho|\vec{\mu}|\pi> \\ \vec{\mu}_{\omega\pi} &= <\omega|\vec{\mu}|\pi> \end{aligned} \tag{D1}$$

where

$$\vec{\mu} = \sum_{q,\lambda} \mu \, (e_q/e) \, \vec{\sigma}_q \; |q\lambda><q\lambda| \tag{D2}$$

is the magnetic moment operator. We assume a universal scale μ for the light quark moments.

Mixing the $|q\bar{q}>$ and $|q\bar{q}A>$ gives a generalized meson decomposition of the form

$$\begin{aligned} |\rho> &= \cos\theta_\rho \; |q\bar{q}> + \sin\theta_\rho \; |q\bar{q}A> \\ |\omega> &= \cos\theta_\omega \; |q\bar{q}> + \sin\theta_\omega \; |q\bar{q}A> \end{aligned} \tag{D3}$$

and similarly for the remaining mesons.

Since the mixing angles $\{\theta_{mes}\}$ are charge independent, the relative sizes of the matrix elements $<\pi^{+0-}|\vec{\mu}|\rho^{+0-}>$ will be the same as in the $|q\bar{q}>$-only meson decomposition. This implies that the reduced matrix elements of the I = 0 and I = 1 parts of the magnetic moment operator will change by at most a common scale, and the well known SU(6) results for the ratios of moments will be unchanged. In particular we may show that

$$\begin{aligned} \frac{|<\omega|\vec{\mu}|\pi^0>|^2}{|<\rho^-|\vec{\mu}|\pi^->|^2} &= 9\,\frac{(\cos\theta_\pi \cos\theta_\omega - \sin\theta_\pi \sin\theta_\omega/\sqrt{3})^2}{(\cos\theta_\pi \cos\theta_\rho - \sin\theta_\pi \sin\theta_\rho/\sqrt{3})^2} \\ &= 9 \text{ if } \theta_\omega = \theta_\rho \end{aligned} \tag{D4}$$

We will find $\theta_\omega = \theta_\rho$ if the $q\bar{q}$, $q\bar{q}A$ mixing hamiltonian is isospin independent. As the mixing is due to the strong interaction we expect that this is indeed the case, neglecting small electromagnetic effects.

The overall scale of these moments changes as

$$\frac{|< v|\vec{\mu}|\pi >|^2_{\text{mixed}}}{|< v|\vec{\mu}|\pi >|^2_{\text{naive SU(6)}}} = (\cos\theta_\pi \cos\theta_{\rho,\omega} - \sin\theta_\pi \sin\theta_{\rho,\omega}/\sqrt{3})^2 \tag{D5}$$

In section IV our model of this mixing gives $\theta_{\rho,\omega} \sim 37^0$, $\theta_\pi \sim -39^0$, which gives a suppression of the moments of

$$\frac{|< \vec{\mu}_{v\pi}>|^2_{\text{mixed}}}{|< \vec{\mu}_{v\pi}>|^2_{\text{naive SU(6)}}} \sim .71 \tag{D6}$$

We thus find a suppression of the absolute rates for M1 vector meson decays of about 30 % due to mixing of the $q\bar{q}$ and $q\bar{q}A$ exotic states. This effect will be much larger between states which do not have mixing angles of opposite sign as defined here.

APPENDIX E GLUONIC ΔP MASS SPLITTING

In section IV we developed the concept of gluon constituents in the bag model and used a truncated Hilbert space to estimate the strong Yukawa coupling required to give the S-wave mesons the observed spin-spin splitting.

In this appendix we shall treat the P and Δ baryon masses using the same approximations in order to estimate the amount of exotic mixing in the baryons and to compare our result with the well known ΔP splitting of ~ 300 Mev.

As previously we consider mixing only between the lowest lying qqqA exotic states with magnetic dipole gluons and the usual SU(6) qqq baryon states. The states which we expect to mix, taking $s_z(P) = 1/2$ and $s_z(\Delta) = 3/2$, are

Proton Basis States	Delta Basis States
$\vert P_0,\uparrow\rangle_{J^P=\frac{1}{2}^+} = \vert 1\rangle$	$\vert \Delta_0, 3/2\rangle_{\frac{3}{2}^+} = \vert 4\rangle$
$\vert P_0 A,\uparrow\rangle_{\frac{1}{2}^+} = \vert 2\rangle$	$\vert \Delta_0 A, 3/2\rangle_{\frac{3}{2}^+} = \vert 5\rangle$
$\vert \Delta_0 A,\uparrow\rangle_{\frac{1}{2}^+} = \vert 3\rangle$	$\vert P_0 A, 3/2\rangle_{\frac{3}{2}^+} = \vert 6\rangle$

(E1)

We assume only the Yukawa part of the quark-gluon strong interaction hamiltonian (204) is important in these medium-q^2 splittings. Between two baryon color singlet states the Yukawa term gives a matrix element of the form

$$\langle qqq, \text{color } \underset{\sim}{1} | H_I | qqq, \text{color } \underset{\sim}{1}\rangle = \langle qqq\, \underset{\sim}{1} | \sum_{q',a} g \int d^3x\, \bar{\psi}_{q'} \gamma_\mu \frac{\lambda^a}{2} \psi_{q'} A^a_\mu | qqq\, \underset{\sim}{1}\rangle \tag{E2}$$

which is the matrix element we find for a Yukawa quark-gluon theory with a different coupling constant κ and without a color degree of freedom,

$$\langle qqq, \text{color } \underline{1} | H_I | qqq, \text{color } \underline{1} \rangle = \langle qqq | \sum_{q'} \kappa \int d^3x\, \bar{\psi}_{q'} \gamma_\mu \psi_{q'} A_\mu | qqq \rangle \tag{E3}$$

where $|\kappa| = g\sqrt{8/3}$ for this case.[40] We work in a gauge in which $A_0(x) = 0$, so we have in our effective colorless theory the matrix element

$$\langle \, | H_I | \, \rangle = -\langle \, | \sum_{q'} \kappa \int d^3x\, \bar{\psi}_{q'} \vec{\gamma} \psi_{q'} \cdot \vec{A} | \, \rangle = -\langle \, | \kappa \sum_{q'} \vec{J}_{q'}(x) \cdot \vec{A}(x) | \, \rangle \tag{E4}$$

To determine the mixing of the basis states in the physical proton and delta we need only diagonalize the hamiltonian matrix of this operator between the two sets of basis states given above. First we note that this operator is a flavor isospin singlet in the limit $m_u = m_d$, so the matrix elements $\langle P_0 | H_I | \Delta_0 A \rangle$ and $\langle \Delta_0 | H_I | P_0 A \rangle$ must vanish identically in the limit of isomultiplet degeneracy and negligible electromagnetic effects. We expand the physical states as linear combinations of the states which are mixed strongly by H_I;

$$\begin{aligned} |P, \uparrow \rangle &= \cos\beta\, |1\rangle + \sin\beta\, |2\rangle \\ |\Delta, 3/2 \rangle &= \cos\gamma\, |4\rangle + \sin\gamma\, |5\rangle \end{aligned} \tag{E5}$$

We call the nonzero matrix elements

$$A = \langle 2 | H_I | 1 \rangle \qquad\qquad E = \langle 5 | H_I | 4 \rangle \tag{E6}$$

in terms of which we may find the mixing angles β, γ and the physical state energies E_P, E_Δ, E_{PA}, $E_{\Delta A}$. These angles and energies are

$$\tan\beta = -\frac{2A^*}{(E_2 - E_1)\left[1 + \sqrt{1 + \frac{4|A|^2}{(E_2 - E_1)^2}}\right]} \qquad E_{\substack{P \\ (PA)}} = \frac{1}{2}\left[E_2 + E_1 \underset{(+)}{-} (E_2 - E_1)\sqrt{1 + \frac{4|A|^2}{(E_2 - E_1)^2}}\right] \tag{E7}$$

where $E_{1,2}$ are the energies $E_{P_0, P_0 A}$ of the basis states before the mixing hamiltonian H_I is turned on. The results for the Δ are given by the substitutions $\beta \to \gamma$, $A \to B$, and $P \to \Delta$. The initial energies are unchanged, as $E_1 = E_4$ and $E_2 = E_5$. Now we need only determine the matrix elements and initial energies. For the proton we have

$$|1\rangle = \frac{1}{\sqrt{18}}\big(2|u\uparrow d\downarrow u\uparrow\rangle + 2|u\uparrow u\uparrow d\downarrow\rangle + 2|d\downarrow u\uparrow u\uparrow\rangle - |u\uparrow u\downarrow d\uparrow\rangle - |u\uparrow d\uparrow u\downarrow\rangle - |u\downarrow d\uparrow u\uparrow\rangle$$
$$- |u\downarrow d\uparrow u\uparrow\rangle - |d\uparrow u\downarrow u\uparrow\rangle - |d\uparrow u\uparrow u\downarrow\rangle - |u\downarrow u\uparrow d\uparrow\rangle\big) = |P_0, \uparrow\rangle$$

$$|2\rangle = \sqrt{\tfrac{2}{3}}\,|A(+)\rangle|P_0,\downarrow\rangle - \sqrt{\tfrac{1}{3}}\,|A(0)\rangle|P_0,\uparrow\rangle = |P_0 A, \uparrow\rangle \qquad \text{(E8)}$$

where
$$|P_0,\downarrow\rangle = -\frac{1}{\sqrt{18}}\big(2|u\downarrow d\uparrow u\downarrow\rangle + \ldots - |u\uparrow u\downarrow d\downarrow\rangle - \ldots\big)$$

$$|A(+)\rangle = |\,1\text{ gluon}, j^P = 1^+, j_z = +1\rangle \qquad |A(0)\rangle = |\,1\text{ gluon}, j^P = 1^+, j_z = 0\rangle$$

so the matrix element is

$$A = \langle 2|H_I|1\rangle = -\kappa\int d^3x\left[\sqrt{\tfrac{2}{3}}\langle A(+)|\vec{A}|0\rangle\cdot\langle P_0,\downarrow|\sum_q \vec{j}_q|P_0,\uparrow\rangle - \sqrt{\tfrac{1}{3}}\langle A(0)|\vec{A}|0\rangle \cdot\langle P_0,\uparrow|\sum_q \vec{j}_q|P_0,\uparrow\rangle\right] \qquad \text{(E9)}$$

We assume the basis states all have approximately the same radii, which gives for the quark currents and gluon fields

$$\langle q\downarrow|\vec{j}|q\uparrow\rangle = i g(r)\overrightarrow{Y_{111}(\Omega)} \qquad \langle q\uparrow|\vec{j}|q\uparrow\rangle = -\frac{i}{\sqrt{2}}g(r)\overrightarrow{Y_{110}(\Omega)}$$
$$\overrightarrow{A_{(+)}}(x)^* = i a_1 j_1(\omega r)\overrightarrow{Y_{111}}^* \qquad \overrightarrow{A_{(0)}}(x)^* = i a_1 j_1(\omega r)\overrightarrow{Y_{110}}^* \qquad \text{(E10)}$$

where g(r) is defined in (231) and the gluon field parameters are defined in (209-218). This gives for the H_I matrix element

$$A = \sqrt{3}\,\kappa\xi$$

where
$$\xi = \frac{4\sqrt{\frac{2\pi}{3}}\,k\,(\alpha_+^2 a^3)(a_1 a)}{x_0^3}\left[\int_0^{x_0} y^2 j_0(y) j_1(y) j_1\!\left(\frac{\omega a}{x_0}y\right)dy\right]a^{-1} \qquad \text{(E11)}$$

For massless quarks and a magnetic dipole gluon in the lowest bag model modes this is simply a constant times κa^{-1};

$$\kappa\,\xi = c_0 \kappa a^{-1} \qquad \text{(E12)}$$

Evaluating this constant numerically gives $c_0 = .3929/2$.

In an $SU(3)_{color}$ theory we have $|g^2| = |3K^2/8|$ for this baryon problem, which gives the proton energy and mixing angle β as functions of the strong coupling constant $\alpha_s = g^2/4\pi$;

$$\tan\beta = \frac{16 c_0 g}{3(E_2-E_1)a\left[1+\sqrt{1+\frac{8g^2c_0^2}{(E_2-E_1)^2a^2}}\right]} \qquad E_p = \frac{1}{2}\left[E_2+E_1-(E_2-E_1)\sqrt{1+\frac{8g^2c_0^2}{(E_2-E_1)^2a^2}}\right] \tag{E13}$$

Similarly we may treat the Δ-ΔA mixing problem. There we find the result $B = -\sqrt{5}\,A$, which gives for the corresponding Δ parameters;

$$\tan\gamma = -\frac{16\sqrt{5}\,c_0 g}{3(E_2-E_1)a\left[1+\sqrt{1+\frac{40g^2c_0^2}{(E_2-E_1)^2a^2}}\right]} \qquad E_\Delta = \frac{1}{2}\left[E_2+E_1-(E_2-E_1)\sqrt{1+\frac{40g^2c_0^2}{(E_2-E_1)^2a^2}}\right] \tag{E14}$$

The ΔP mass splitting we find is

$$E_\Delta - E_p = -\frac{1}{2}(E_2-E_1)\left[\sqrt{1+\frac{160\pi c_0^2\alpha_s}{(E_2-E_1)^2a^2}} - \sqrt{1+\frac{32\pi c_0^2\alpha_s}{(E_2-E_1)^2a^2}}\right] \tag{E15}$$

which is less then zero, contrary to reality. If we use the numerical results we found for E_1, E_2, and a for the basis states with $B_0^{1/4} = 120$ Mev (pp. 92) we find

$$\left(\alpha_s = \frac{8\hat{\alpha}_s}{3}\right)$$

$$\tan\beta = \frac{2.71\sqrt{\hat{\alpha}_s}}{1+\sqrt{1+5.52\hat{\alpha}_s}} \qquad E_p = 1.36 - .188\sqrt{1+5.52\hat{\alpha}_s} \ \text{Gev}$$

$$\tan\gamma = -\frac{6.06\sqrt{\hat{\alpha}_s}}{1+\sqrt{1+27.6\hat{\alpha}_s}} \qquad E_\Delta = 1.36 - .188\sqrt{1+27.6\hat{\alpha}_s} \ \text{Gev} \tag{E16}$$

For the value $\alpha_s = 1.18$ found in considering meson spin-spin splittings, we obtain the numerical results;

$$\beta = 32^\circ, \quad E_p = 1.01 \text{ Gev} \qquad \gamma = -41^\circ, \quad E_\Delta = .677 \text{ Gev} \tag{E17}$$

Although we obtain qualitatively the correct splitting, it has the wrong sign. There are two plausible explanations for this invalid result; (1) other states contribute significantly to the physical proton and delta, which reverses the sign of the splitting, or (2) the bag model wavefunctions for the $|qqq\rangle$ and $|qqqA\rangle$ states without the center of mass motion removed simply give matrix elements unlike the physical state matrix elements. The fact that the classical gluon field model of the MIT group[3] gives this splitting reasonably well seems to imply that (1) is correct, as the classical result should follow in the limit of taking a very large number of gluonic states in computing the energy shifts.

As we have not found the set of basis states which gives the ΔP mass splitting reasonably well, we can't expect to obtain valid estimates of gluonic corrections to the classical SU(6) predictions of baryon properties. This very interesting problem requires a more complete model of the baryon, i.e. a larger initial Hilbert space of quark and gluon basis states. An attempt to carry out this extension in the bag model would be flawed by the presence of many more basis states than would be found in the quark-gluon theory itself, due to the existence of the bag as an extra degree of freedom. For this reason enlarging the Hilbert space in the bag model would probably not lead to a useful description of the physical baryon states.

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