BOUNDARY CURRENT EFFECTS IN MAGNETOHYDRODYNAMICS WITH ANISOTROPIC CONDUCTIVITY

Thesis by

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ABSTRACT

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A theoretical investigation is conducted to determine the effects of currents flowing through a boundary into the magnetohydrodynamic flow of an inviscid, incompressible fluid with anisotropic conductivity. The particular arrangement of an externally applied magnetic field parallel to the velocity field is investigated for two flow geometries; (i) semi-infinite flow over a conducting flat wall, and (ii) channel flow between a conducting lower wall and an insulating upper wall. In both cases the applied boundary currents are assumed to be sinusoidal in shape and flow into the fluid normal to the boundary.

A small perturbation analysis is used to linearize the macroscopic steady flow equations of a fully ionized gas. A Cartesian coordinate system is adopted in which the x-axis is in the flow direction and the y-axis is normal to the conducting wall. The problem is considered two dimensional from the standpoint that the perturbed quantities are independent of the z-coordinate although the z-components are, in general, non-zero. The general solution to the linearized equations is obtained for case (i). Because of the complexity of this solution, it is studied in detail only in the limits

of small and large magnetic Reynold's number. Solutions for case (ii) are obtained in the limits of small and large magnetic Reynold's number by applying the limiting procedure to the linearized equations before solving them.

In the limit of small magnetic Reynold's number for both cases (i) and (ii), the magnetic and velocity field vectors are composed of an irrotational part and a rotational part. The irrotational portion always remains in the x-y plane. However, the rotational portion and, hence, the currents lie in a plane which is rotated about the x-axis; the angle between this plane and the x-y plane being strongly dependent upon the degree of anisotropy in the fluid's electrical conductivity. The currents in the fluid form symmetric loops closing at the conducting boundary. Anisotropic effects on the magnitude of the magnetic and velocity field components and the currents are generally moderate except near the conducting wall. At this wall the x and z current components can become quite large for strong anisotropic conductivity. Both the irrotational and rotational portions of the velocity field vector behave in a manner analogous to ordinary incompressible flow with the applied sinusoidal boundary current in the flat wall replaced by a solid sinusoidal wall.

In the limit of large magnetic Reynold's number for both cases (i) and (ii), anisotropic effects are absent to the order of the inverse square root of the magnetic Reynold's number. In addition, the currents and field perturbations are found to be confined to a thin magnetic boundary layer near the conducting wall. The currents lie entirely in the x-y plane and again form loops closing at the conducting boundary, but are steeply inclined toward the x-axis. The x-component of the current flowing in the fluid is found to be larger than the applied boundary current by a factor of the square root of the magnetic Reynold's number.

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LIST OF SYMBOLS

a	$\left(\frac{\lvert k \rvert}{2\lambda}\right)^{1/2}$
A, A_1 to A_6	complex constants
b	channel height
B, B_1 to B_4	complex constants
c	exponent
C_1 to C_6	complex constants
D, D_1 to D_4	complex constants
$-e$	electron charge
$\vec{E}$	electric field vector
f	function of y
g	function of y
G_1 to G_6	complex constants
$\vec{h}$	magnetic field perturbation vector
$\vec{H}$	magnetic field vector
i	$(-1)^{1/2}$
$\vec{i}$	unit vector
$\vec{j}$	current density vector
J	amplitude of boundary current
k	$4\pi\sigma\mu U_0\left(\frac{m^2-1}{m^2}\right)$
K, K_1 to K_6	complex constants
m	Alfvén number
m_e	mass of an electron
M	complex constant

n	number of electrons or ions per unit volume
p	pressure
$\vec{q}$	fluid velocity vector
Q_1 to Q_6	complex constants
R_1, R_2	complex constants
Rm	magnetic Reynold's number
$\vec{u}$	fluid velocity perturbation vector
U_o	unperturbed fluid velocity
x, y, z	Cartesian coordinates
α, β	exponents
γ	$(1+\omega_o^2\tau^2)^{1/2}$
ϵ	amplitude of wavy wall
ξ	coordinate
η	auxiliary function
θ	auxiliary function
λ	2π/wavelength
μ	magnetic permeability
$\vec{\xi}$	$\nabla \times \vec{H}$
ρ	fluid density
ρ_e	charge density
σ	scalar electric conductivity
τ	mean time between collisions of electrons with ions
ϕ	auxiliary function
ψ	stream function

ω	electron cyclotron frequency
$\vec{\Omega}$	vorticity vector

SUBSCRIPTS

x, y, z, ρ	components of a vector quantity
0	unperturbed quantity
1	irrotational vector quantity
2	rotational vector quantity
p	pressure
w	wall

I. INTRODUCTION

In the magnetohydrodynamic flow of an ionized gas, the interaction of the magnetic field with the charged particles in the fluid coupled with the collisions between particles give rise to Hall currents. These currents are responsible for what is commonly called anisotropic conductivity of the fluid. Derivations of the flow equations of a fully ionized gas, including Hall currents, are given by Cowling (1), Spitzer (2), and Delcroix (3).

If currents are introduced into the fluid through a boundary, they also will interact with the fluid and the magnetic field. It is the effects produced by these boundary currents in a fluid of anisotropic conductivity that is the subject of this thesis. In particular, a theoretical investigation is conducted for an externally applied magnetic field parallel to the velocity field for two flow geometries; (i) semi-infinite flow over a conducting flat wall, and (ii) channel flow between a conducting lower wall and an insulating upper wall. In both cases the boundary currents are assumed sinusoidal in shape and flow into the fluid normal to the boundary.

Linearization of the macroscopic steady flow equations of a fully ionized gas is accomplished using the small perturbation method in a manner identical to that of Sonnerup (4). A Cartesian coordinate system is adopted in which the x-axis is in the flow direction and the y-axis is normal to the conducting wall. The problem is considered two dimensional from the standpoint that the perturbed quantities are independent of the z-coordinate although the z-components are, in general, non-zero. The general solution to the linearized equations is obtained for the semi-infinite flow case. This solution is studied in detail in the limits of small and large magnetic Reynold's number. In the channel flow case, solutions are obtained in the limits of small and large magnetic Reynold's number by applying the limiting procedure to the linearized flow equations first.

II. ANISOTROPIC CONDUCTIVITY

The basic equation which describes the effects of electrical conductivity in a conducting fluid is the generalized form of Ohm's law. When a conducting fluid flows in the presence of an electric field, Ohm's law relates the current density to the electric field; the constant of proportionality being the scalar conductivity. However, if a magnetic field is also present, a Hall current is introduced due to the interaction of the magnetic field with the charged particles in the fluid and collisions between particles. It is this Hall current which causes the anisotropic conductivity effects in magnetohydrodynamic flows.

The generalized form of Ohm's law is derived in references (1), (2), and (3) for a fully ionized gas consisting of two components (ions and electrons) and where electrical neutrality is maintained overall. Assuming (i) that the velocity of the electrons relative to the ions is small, (ii) the mass velocity of the gas is the velocity of the ions (since the mass ratio of ions to electrons is large), (iii) no fluid stresses exist (only hydrodynamic pressure), and (iv) on the average the electrons lose their entire momentum in each collision with ions, then the steady state form of Ohm's law in a coordinate system moving with the mass velocity of the gas is

$$\vec{j}-\left(\frac{\omega\tau}{H}\right)\vec{j}\times\vec{H}=\sigma\left[\vec{E}+\mu\vec{q}\times\vec{H}+\left(\frac{1}{me}\right)\nabla p_e\right] \qquad (2.1)$$

where electromagnetic units have been used and

$\vec{j}$ = current density vector

$\vec{H}$ = magnetic field vector

$\vec{E}$ = electric field vector

$\vec{q}$ = fluid velocity vector

σ = scalar electric conductivity

μ = magnetic permeability (in emu, μ = 1 in free space)

n = number of electrons (or ions) per unit volume

-e = electron charge

p_e = electron partial pressure

ω = cyclotron frequency of electrons

τ = mean time between collisions of electrons with ions

Expressions for the electron cyclotron frequency and scalar conductivity, respectively, are

$$\omega = \frac{e\mu H}{m_e} \tag{2.2}$$

$$\sigma = \frac{ne^2\tau}{m_e} \tag{2.3}$$

where m_e is the electron mass.

The form of Ohm's law most suitable for the analysis to be conducted is that of equation (2.1). However, some additional remarks on the nature of anisotropic conductivity will be presented for completeness. Equation (2.1) can be written as

$$\vec{j} = \sigma\vec{E}' + \left(\frac{\omega\tau}{H}\right)\vec{j}\times\vec{H} \tag{2.4}$$

where the electric field $\vec{E}'$ is the sum of the electric field $(\vec{E}+\mu\vec{q}\times\vec{H})$ seen by an observer moving with the gas velocity $\vec{q}$

and the electric field $(\nabla p_e / ne)$ produced by the pressure distribution in the gas. The term $\left[\left(\frac{\omega\tau}{H}\right) \vec{j} \times \vec{H} \right]$ is the Hall current arising from the particle drift across the electric field due to the presence of the magnetic field. The Hall current is small for dense gases and weak magnetic fields since many collisions take place during the time needed for an electron to make a complete turn around a magnetic field line. It is large for rareified gases and/or strong magnetic fields since the electrons can spiral freely between collisions. It is this latter case that is of interest herein since it may cause strong anisotropy in the electric conductivity.

The anisotropic effect of the Hall current is most easily seen by solving for the current density vector. Assuming that the magnetic field is in the x-direction in a right-handed Cartesian coordinate system, the current density may be found by crossing equation (2.4) with $\vec{H}$, using vector identities, and substituting for $(\vec{j} \times \vec{H})$ in equation (2.4). The result is

$$
\begin{aligned}
j_x &= \sigma E_x' \\
j_y &= \frac{\sigma}{1+\omega^2\tau^2}\left(E_y' - \omega\tau E_z' \right) \\
j_z &= \frac{\sigma}{1+\omega^2\tau^2}\left(\omega\tau E_y' + E_z' \right)
\end{aligned}
\qquad (2.5)
$$

It is easily seen that the current density depends strongly on the factor $\omega\tau$, which is the ratio of the electron cyclotron frequency to the electron collision frequency with ions. When this factor is small, the Hall current is small and Ohm's law reduces to the isotropic form:

$$\vec{j} = \sigma \vec{E}' \tag{2.6}$$

The anisotropic form of Ohm's law, equation (2.5), can be put into a matrix form which possesses tensor properties under orthogonal transformations; hence the term "tensor conductivity" which is sometimes used synonymously with anisotropic conductivity. This matrix representation is

$$\begin{pmatrix} j_x \\ j_y \\ j_z \end{pmatrix} = \frac{\sigma}{1+\omega^2\tau^2} \begin{pmatrix} 1+\omega^2\tau^2 & 0 & 0 \\ 0 & 1 & -\omega\tau \\ 0 & \omega\tau & 1 \end{pmatrix} \begin{pmatrix} E'_x \\ E'_y \\ E'_z \end{pmatrix} \tag{2.7}$$

III. THE GOVERNING EQUATIONS

The equations governing magnetohydrodynamics are the appropriate forms of Ohm's law, Maxwell's equations, and the equations of fluid mechanics including all necessary electromagnetic terms. Because of the complexity of the equations resulting from combining the above sets of equations in their general form, several simplifying assumptions will now be made. First, only fully ionized, electrically neutral, non-viscous gases will be considered so that Ohm's law can be expressed in the form of equation (2.1) for steady state conditions. In electromagnetic units, the steady state Maxwell's equations for a moving non-polarized medium are

$$\begin{aligned} \nabla\cdot\vec{E} &= 4\pi c^2 \rho_e \\ \nabla\cdot\vec{H} &= 0 \\ \nabla\times\vec{E} &= 0 \\ \nabla\times\vec{H} &= 4\pi\vec{j} \end{aligned} \tag{3.1}$$

where convective currents and the vacuum displacement current have been neglected, c is the speed of light in a vacuum and ρ_e is the electric charge density.

As a further simplifying assumption, only incompressible flows will be considered. The continuity of mass equation then becomes

$$\nabla\cdot\vec{q} = 0 \tag{3.2}$$

The momentum equations for steady state, incompressible, non-viscous flows are

$$\rho \vec{q} \cdot \nabla \vec{q} = -\nabla p + \mu \vec{j} \times \vec{H} \tag{3.3}$$

where ρ is the fluid mass density and p is the pressure.

Ohm's law and the momentum equations will now be linearized by the introduction of small perturbations in the magnetic and velocity fields as follows:

$$\begin{aligned} \vec{H} &= \vec{H}_o + \vec{h} \\ \vec{q} &= \vec{U}_o + \vec{u} \end{aligned} \tag{3.4}$$

The unperturbed magnetic and velocity field vectors $\vec{H}_o$ and $\vec{U}_o$ are assumed to be constant in space and time while the perturbation vectors $\vec{h}$ and $\vec{u}$ are small in comparison to the unperturbed vectors.

$$\begin{aligned} |h| &\ll |H_o| \\ |u| &\ll |U_o| \end{aligned} \tag{3.5}$$

Taking the curl of Ohm's law, equation (2.1), in order to eliminate the pressure, using Maxwell's equations and some vector identities, and introducing equations (3.4) produces the following linearized form of Ohm's law upon neglect of higher order terms.

$$\begin{aligned} &\vec{H}_o \cdot \nabla \vec{u} - \vec{U}_o \cdot \nabla \vec{h} \\ &\qquad = -\frac{1}{4\pi\sigma\mu}\left[\nabla^2 \vec{h} - \left(\frac{\omega_o \tau}{H_o}\right)\vec{H}_o \cdot \nabla(\nabla \times \vec{h})\right] \end{aligned} \tag{3.6}$$

where $\omega_o = \dfrac{e\mu H_o}{m_e}$ (3.7)

Similarly, the momentum equation can be linearized to the following form.

$$\vec{U}_o \cdot \nabla \vec{u} - \frac{\mu}{4\pi\rho} \vec{H}_o \cdot \nabla \vec{h} = -\nabla\left(\frac{p}{\rho} + \frac{\mu}{4\pi\rho} \vec{H}_o \cdot \vec{h}\right) \tag{3.8}$$

A more useful form of the momentum equation is obtained by taking the curl of equation (3.8).

$$\vec{U}_o \cdot \nabla \vec{\Omega} = \frac{\mu}{4\pi\rho} \vec{H}_o \cdot \nabla \vec{\xi} \tag{3.9}$$

where
$$\vec{\Omega} \equiv \nabla \times \vec{u}$$
$$\vec{\xi} \equiv \nabla \times \vec{h} \tag{3.10}$$

It should be noted that so far no conditions have been imposed on the orientation of the magnetic and velocity fields. In the case to be considered here, the magnetic and velocity fields are parallel and will be assumed to be in the x-direction. Inserting this information into equations (3.6) and (3.9) results in the following forms of Ohm's law and the momentum equations for parallel fields.

$$H_o \frac{\partial \vec{u}}{\partial x} - U_o \frac{\partial \vec{h}}{\partial x} = -\frac{1}{4\pi\sigma\mu}\left(\nabla^2 \vec{h} - \omega_o \tau \frac{\partial \vec{\xi}}{\partial x}\right) \tag{3.11}$$

and

$$U_o \frac{\partial \vec{\Omega}}{\partial x} = \frac{\mu H_o}{4\pi\rho} \frac{\partial \vec{\xi}}{\partial x} \tag{3.12}$$

Equation (3.12) can be integrated to give

$$\vec{u} = \left(\frac{U_o}{m^2 H_o}\right) \vec{h} + U_o \nabla\phi \tag{3.13}$$

where Φ is a function satisfying Laplace's equation and m is the Alfvén number; i.e., the ratio of the fluid speed to the speed of ordinary Alfvén waves. The Alfvén number is given by the formula

$$m = \left(\frac{4\pi \rho U_0^2}{\mu H_0^2} \right)^{1/2} \tag{3.14}$$

Taking the curl of equation (3.11) and substituting equation (3.12) into the result gives

$$k \frac{\partial \vec{\xi}}{\partial x} = \nabla^2 \vec{\xi} - \omega_0 \tau \left(\nabla \times \frac{\partial \vec{\xi}}{\partial x} \right) \tag{3.15}$$

where k is defined as

$$k \equiv 4\pi \sigma \mu U_0 \left(\frac{m^2 - 1}{m^2} \right) \tag{3.16}$$

The assumption will now be made that the problem under consideration is two dimensional in the sense that the solution depends on only one other coordinate in addition to the x-coordinate. Choosing the solution to be independent of the z-coordinate, this assumption takes the form

$$\frac{\partial}{\partial z} = 0 \tag{3.17}$$

Although no variation of the solution with the z-coordinate is allowed under this assumption, the z-components of the solution can be non-zero functions of the x and y-coordinates. On the basis of this assumption, equation (3.15) can be reduced to the following two scalar equations.

$$\nabla^2 h_z = k \frac{\partial h_z}{\partial x} + \omega_0 \tau \frac{\partial \xi_z}{\partial x} \tag{3.18}$$

$$\nabla^2 \xi_z = k \frac{\partial \xi_z}{\partial x} - \omega_0 \tau \frac{\partial}{\partial x} \nabla^2 h_z \tag{3.19}$$

Combined with the appropriate boundary conditions, equations (3.18), (3.19), (3.13), and (3.11) allow the magnetic and velocity field perturbations to be found. Using the last of Maxwell's equations (3.1) determines the currents flowing in the fluid and use of Ohm's law (2.1) allows determination of the electric field. Once the electric field is known, the first of Maxwell's equations (3.1) determines the electric charge density distribution. Finally, the pressure field can be determined by substituting the velocity field given by equation (3.13) into the momentum equation (3.8) and integrating. The result is

$$p - p_0 = -\rho U_0 u_x \tag{3.20}$$

where p_0 is the unperturbed pressure and u_x is the x-component of the perturbation velocity. Similarly, the pressure coefficient is

$$C_p = -\frac{2 u_x}{U_0} \tag{3.21}$$

Although only the magnetic and velocity field perturbations and the currents flowing in the fluid will be determined in the subsequent analysis, the above equations allow any quantity of interest to be evaluated.

IV. SEMI-INFINITE FLOW OVER A FLAT WALL WITH SINUSOIDAL BOUNDARY CURRENTS

The equations just derived will now be used to investigate the effects of boundary currents in the magnetohydrodynamic flow of an anisotropically conducting fluid. In particular, flow of semi-infinite extent over a flat conducting wall through which sinusoidal currents are induced into the fluid will be considered here. A general solution will be obtained valid for arbitrary magnetic Reynold's number and $\omega_o \tau$. Approximate solutions will then be obtained in the limits of small and large magnetic Reynold's number.

A. GENERAL SOLUTION

One of the simplest and, yet, one of the most useful boundary current distributions that can be specified is one having a sinusoidal profile. Solutions for boundary currents of any arbitrary profile of interest can, in principle, be constructed from the sinusoidal boundary current solution using Fourier superposition. Therefore, the effects of sinusoidally distributed boundary currents will be investigated here.

The geometrical arrangement to be considered is as shown in Figure 1. The unperturbed magnetic and velocity fields are parallel and in the x-direction. The flat wall is considered to be of infinite extent in the x and z-directions and, for the purposes of this investigation, can be considered infinitely thick. It is supposed that the wall is a non-magnetic conductor which has the same magnetic permeability as the fluid. In addition, it is assumed that no electric fields exist in the wall parallel to the boundary. Thus, the only non-zero current component in the wall is in the y-direction. It is further assumed that this current flows undistorted and undiminished through the wall. This requires that the current throughout the wall be equal to the current at the boundary.

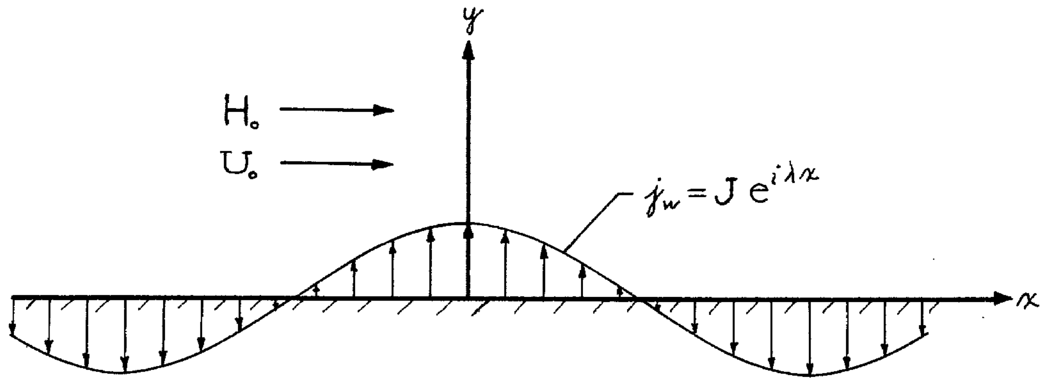


FIGURE 1

GEOMETRICAL ARRANGEMENT OF THE SEMI-INFINITE FLOW CASE

Using complex notation, the boundary current induced into the fluid through the conducting wall is specified to be

$$j_w = J e^{i\lambda x} \tag{4.1}$$

where J is the amplitude of the boundary current and $(2\pi/\lambda)$ is equal to the wave length of the current distribution. In order to obtain solutions valid within the small perturbation approach used to linearize the equations governing this problem, it is necessary to restrict the magnitude of the boundary current amplitude according to following inequality.

$$J \ll \lambda H_o \tag{4.2}$$

The electromagnetic fields in the fluid and the conducting wall are coupled at the boundary. Because of this coupling, it is necessary, in effect, to solve for the fields in the wall as well as in the fluid. The boundary conditions that will be applied are as follows:

(i) the normal component of the current density vector is continuous across the boundary

(ii) no fluid flows through the surface of the wall

(iii) the components of the magnetic field perturbation vector are continuous across the boundary

(iv) both the magnetic field and velocity field perturbations vanish as y approaches positive infinity

(v) the x and y-components of the magnetic field perturbation vector vanish as y approaches negative infinity while the z-component remains finite.

Condition (iii) is equivalent to specifying that no surface currents exist at the boundary. In condition (v), the z-component of the magnetic field perturbation is not allowed to vanish since its derivative specifies the current flowing in the wall.

From Maxwell's equations (3.1) we have

$$\begin{aligned} \nabla\cdot\vec{h} &= 0 \\ \nabla\times\vec{h} &= 4\pi\vec{j} \end{aligned} \qquad (4.3)$$

Since the only current in the wall is in the y-direction and is equal to the value at the boundary given by equation (4.1), equations (4.3) reduce to the following four scalar equations.

$$\frac{\partial h_x}{\partial x} + \frac{\partial h_y}{\partial y} = 0 \tag{4.4}$$

$$\frac{\partial h_z}{\partial y} = 0 \tag{4.5}$$

$$\frac{\partial h_z}{\partial x} = -4\pi J e^{i\lambda x} \tag{4.6}$$

$$\frac{\partial h_y}{\partial x} - \frac{\partial h_x}{\partial y} = 0 \tag{4.7}$$

A stream-type function Ψ will now be introduced which is defined by the following equations.

$$h_x = -\frac{\partial \Psi}{\partial y}$$
$$h_y = \frac{\partial \Psi}{\partial x} \tag{4.8}$$

Equations (4.8) automatically satisfy equation (4.4) and when introduced into equation (4.7) result in the following form of Laplace's equation.

$$\frac{\partial^2 \Psi}{\partial x^2} + \frac{\partial^2 \Psi}{\partial y^2} = 0 \tag{4.9}$$

This equation is easily solved by separation of variables to give, using boundary condition (v),

$$\psi = -\left(\frac{iA}{\lambda}\right) e^{i\lambda x + \lambda y} \tag{4.10}$$

where A is a complex constant to be determined. Thus, except for the constant A, the x and y-components of the magnetic field in the wall are determined. The z-component is easily found from equations (4.5) and (4.6) with the result that the magnetic field perturbations in the wall are

$$\begin{aligned} h_x &= iA\, e^{i\lambda x + \lambda y} \\ h_y &= A\, e^{i\lambda x + \lambda y} \\ h_z &= \frac{4\pi i J}{\lambda}\, e^{i\lambda x} \end{aligned} \tag{4.11}$$

The magnetic field perturbations in the fluid are determined using equations (3.18) and (3.19). By eliminating ε_z from these equations, the following fourth order partial differential equation for h_z is obtained.

$$\gamma^2 \frac{\partial^4 h_z}{\partial x^4} + (\gamma^2+1)\frac{\partial^4 h_z}{\partial x^2 \partial y^2} + \frac{\partial^4 h_z}{\partial y^4} - 2k\left(\frac{\partial^3 h_z}{\partial x^3} + \frac{\partial^3 h_z}{\partial x \partial y^2}\right) + k^2 \frac{\partial^2 h_z}{\partial x^2} = 0 \tag{4.12}$$

where γ is defined as

$$\gamma \equiv \left(1 + \omega_0^2 \tau^2 \right)^{1/2} \tag{4.13}$$

If a particular solution of the form

$$h_z = K\, e^{i\lambda x - cy} \tag{4.14}$$

is substituted into equation (4.12) and the coefficients of the exponential in the resulting algebraic equation are equated, it is found that equation (4.14) is a possible solution if c takes on any of the four values given below.

$$c = \pm \lambda \left[1 + \frac{1}{2}\omega_0^2\tau^2 + \frac{ik}{\lambda} \pm \frac{1}{2}\omega_0^2\tau^2 \left(1 + \frac{4ik}{\lambda \omega_0^2 \tau^2} \right)^{1/2} \right]^{1/2} \tag{4.15}$$

Since by boundary condition (iv) the magnetic field perturbations must vanish as y approaches infinity, the negative values of c outside the outer square root are not valid for the physical problem at hand. The remaining two values of c are physically allowable and, for convenience will be denoted by α and β as follows:

$$\alpha \equiv \lambda \left[1 + \frac{1}{2}\omega_0^2\tau^2 + \frac{ik}{\lambda} + \frac{1}{2}\omega_0^2\tau^2 \left(1 + \frac{4ik}{\lambda \omega_0^2 \tau^2} \right)^{1/2} \right]^{1/2}$$

$$(4.16)$$

$$\beta \equiv \lambda \left[1 + \frac{1}{2}\omega_0^2\tau^2 + \frac{ik}{\lambda} - \frac{1}{2}\omega_0^2\tau^2 \left(1 + \frac{4ik}{\lambda \omega_0^2 \tau^2} \right)^{1/2} \right]^{1/2}$$

Thus, it has been determined that the z-component of the magnetic field perturbation has a particular solution of the form

$$h_z = e^{i\lambda x}\left(C_1 e^{-\alpha y} + C_2 e^{-\beta y}\right) \tag{4.17}$$

Since h_z and ξ_z are related by equations (3.18) and (3.19), ξ_z must also have the same form as equation (4.17). By definition

$$\xi_z = \frac{\partial h_y}{\partial x} - \frac{\partial h_x}{\partial y} \tag{4.18}$$

and, therefore, h_x and h_y must also have the form of equation (4.17). Thus, the particular solution for the perturbed magnetic field in the fluid is

$$\begin{aligned} h_x &= e^{i\lambda x}\left(C_3 e^{-\alpha y} + C_4 e^{-\beta y}\right) \\ h_y &= e^{i\lambda x}\left(C_5 e^{-\alpha y} + C_6 e^{-\beta y}\right) \\ h_z &= e^{i\lambda x}\left(C_1 e^{-\alpha y} + C_2 e^{-\beta y}\right) \end{aligned} \tag{4.19}$$

Since $\vec{h}$ must be divergence free, the following relations between constants must hold.

$$\begin{aligned} C_5 &= \frac{i\lambda}{\alpha} C_3 \\ C_6 &= \frac{i\lambda}{\beta} C_4 \end{aligned} \tag{4.20}$$

The homogeneous solution in the fluid corresponding to the magnetic field being both divergence free and curl free is easily seen to be

$$h_x = B e^{i\lambda x - \lambda y}$$

$$h_y = i B e^{i\lambda x - \lambda y} \tag{4.21}$$

$$h_z = 0$$

where B is a complex constant and boundary condition (iv) has again been used. Since the equations are linear, the general solution is obtained by combining the homogeneous and particular solutions. Thus in the fluid the magnetic field perturbations are given by

$$h_x = e^{i\lambda x}\left(B e^{-\lambda y} + C_3 e^{-\alpha y} + C_4 e^{-\beta y}\right)$$

$$h_y = i e^{i\lambda x}\left(B e^{-\lambda y} + \frac{\lambda}{\alpha} C_3 e^{-\alpha y} + \frac{\lambda}{\beta} C_4 e^{-\beta y}\right) \tag{4.22}$$

$$h_z = e^{i\lambda x}\left(C_1 e^{-\alpha y} + C_2 e^{-\beta y}\right)$$

Since the components of $\vec{h}$ are continuous across the boundary by boundary condition (iii), the components of equations (4.11) and (4.22) may be equated at the boundary. This results in the following relations between coefficients.

$$C_2 = \frac{4\pi i J}{\lambda} - C_1$$

$$C_3 = \frac{\alpha}{\lambda(\alpha-\beta)}\left[\beta(B+iA) - \lambda(B-iA)\right] \tag{4.23}$$

$$C_4 = -\frac{\beta}{\lambda(\alpha-\beta)}\left[\alpha(B+iA) - \lambda(B-iA)\right]$$

It will be noted that boundary condition (i) is automatically satisfied by the first of equations (4.23). If equations (4.18), (4.22), and (4.23) are substituted into equation (3.18), the following two equations emerge by equating the coefficients of like exponentials.

$$\frac{(\alpha^2 - i\lambda k - \lambda^2)\, C_1}{\left[\beta(B+iA) - \lambda(B-iA)\right]} = \frac{i\omega_0 \tau(\alpha^2-\lambda^2)}{(\alpha-\beta)} \tag{4.24}$$

$$\frac{(\beta^2 - i\lambda k - \lambda^2)\, C_2}{\left[\alpha(B+iA) - \lambda(B-iA)\right]} = \frac{i\omega_0 \tau(\beta^2-\lambda^2)}{(\alpha-\beta)} \tag{4.25}$$

Similarly, by substitution into equation (3.19) the following two equations are obtained.

$$(\alpha^2 - i\lambda k - \lambda^2) = -\frac{i\omega_0 \tau \lambda^2(\alpha-\beta)\, C_1}{\left[\beta(B+iA) - \lambda(B-iA)\right]} \tag{4.26}$$

$$(\beta^2 - i\lambda k - \lambda^2) = -\frac{i\omega_0 \tau \lambda^2(\alpha-\beta)\, C_2}{\left[\alpha(B+iA) - \lambda(B-iA)\right]} \tag{4.27}$$

Taking the ratio of equations (4.24) and (4.26) results in

$$\beta(B+iA)-\lambda(B-iA)=\pm\frac{i\lambda(\alpha-\beta)C_1}{(\alpha^2-\lambda^2)^{1/2}} \tag{4.28}$$

Likewise, the ratio of equations (4.25) and 4.27) produces

$$\alpha(B+iA)-\lambda(B-iA)=\pm\frac{i\lambda(\alpha-\beta)C_2}{(\beta^2-\lambda^2)^{1/2}} \tag{4.29}$$

The ± signs occur since only the squares of the quantities on the left hand side of equations (4.28) and (4.29) can be determined. By substitution of equation (4.28) back into either equation (4.24) or equation (4.26) and examining the limiting cases in which $|k/\lambda|$ is allowed to approach zero or infinity, it is found that only the negative sign is applicable in equation (4.28). Using a similar procedure, it is also found that only the negative sign is applicable for equation (4.29). Substitution of these results into equations (4.23) produces the following expressions for the complex constants C_2, C_3 and C_4.

$$\begin{aligned} C_2 &= \frac{4\pi i J}{\lambda} - C_1 \\ C_3 &= -\frac{i\alpha C_1}{(\alpha^2-\lambda^2)^{1/2}} \\ C_4 &= \frac{i\beta C_2}{(\beta^2-\lambda^2)^{1/2}} \end{aligned} \tag{4.30}$$

Solving for the complex constant B from equations (4.28) and (4.29), using the negative signs, gives

$$B = \frac{i}{2}\left[\left(\frac{\alpha+\lambda}{\alpha-\lambda}\right)^{1/2} C_1 - \left(\frac{\beta+\lambda}{\beta-\lambda}\right)^{1/2} C_2\right] \tag{4.31}$$

Since C_2 is given by equation (4.23) in terms of C_1, only the constant C_1 remains to be determined for the complete solution of the magnetic field in the fluid. This last undetermined constant will be obtained by equating coefficients of like exponentials in the linearized form of Ohm's law (3.11). However, it is necessary to obtain a solution for the velocity field first. The velocity perturbation vector is given by equation (3.13).

$$\vec{u} = \left(\frac{U_o}{m^2 H_o}\right)\vec{h} + U_o \nabla\phi \tag{3.13}$$

Since $\nabla^2\phi = 0$ and by boundary condition (iv) the velocity perturbations must vanish as y approaches infinity, the proper expression for ϕ is easily seen to be

$$\phi = D e^{i\lambda x - \lambda y} \tag{4.32}$$

By boundary condition (ii) the y-component of the velocity must be zero at wall. Using this condition, the complex constant D is found to be

$$D = \frac{i}{\lambda m^2 H_o}\left(B + \frac{\lambda}{\alpha} C_3 + \frac{\lambda}{\beta} C_4\right) \tag{4.33}$$

The y-component of Ohm's law (3.11), after substitution for the velocities using (3.13), is

$$4\pi\sigma\mu U_0\left[\left(\frac{m^2-1}{m^2}\right)\frac{\partial h_y}{\partial x} - H_0\frac{\partial^2\phi}{\partial x\partial y}\right] = \nabla^2 h_y + \omega_0\tau\frac{\partial^2 h_y}{\partial x^2} \qquad (4.34)$$

Substituting equations (4.22) and (4.32) into equation (4.34) and equating coefficients of $e^{-\lambda y}$ leads to the result that

$$\left(\frac{m^2-1}{m^2}\right)B = i\lambda H_0 D \qquad (4.35)$$

Using equations (4.30), (4.31), and (4.33) to solve for the final undetermined constant produces the following result.

$$C_1 = \frac{4\pi i J}{\lambda}\,\frac{(\beta+\lambda)m^2 - 2\lambda}{\left[(\beta+\lambda)m^2 - 2\lambda\right] + \left(\frac{\beta^2-\lambda^2}{\alpha^2-\lambda^2}\right)^{1/2}\left[(\alpha+\lambda)m^2 - 2\lambda\right]} \qquad (4.36)$$

Thus the exact solution for the perturbations of the magnetic and velocity fields has been completely determined. The detemination of any other quantities of interest, such as the currents flowing in the fluid, can be accomplished as outlined previously on page 11.

The pertinent equations for the magnetic and velocity field perturbations are the following: (4.22), (4.16), (4.36), (4.30), (4.31), (3.13), (4.32), and (4.33). These equations form a very formidable set with the basic character of the solution being obscured by the complexity of the expressions. In the limits of small and large magnetic Reynold's number considerably simplified approximate solutions can be obtained and the features in these limits demonstrated. These approximate solutions are presented below.

B. SMALL MAGNETIC REYNOLD'S NUMBER APPROXIMATION

For the problem under consideration, the magnetic Reynold's number is

$$R_m = \frac{4\pi\sigma\mu U_o}{\lambda} \tag{4.37}$$

It will be noticed that the magnetic Reynold's number is closely related to k as previously defined in equation (3.16). The relationship between the magnetic Reynold's number and k is

$$R_m = \left(\frac{m^2-1}{m^2}\right)\frac{k}{\lambda} \tag{4.38}$$

In order to expand the exponents α and β given by equations (4.16) in the limit of small magnetic Reynold's number, it is necessary to assume that

$$\left|\frac{k}{\lambda}\right| \ll 1 \tag{4.39}$$

With this assumption, the various constants in the general solution are found to approach the following values.

$$\begin{aligned}
\alpha &= \gamma\lambda \\
\beta &= \lambda \\
C_1 &= \frac{4\pi i J}{\lambda} \\
C_2 &= 0 \\
C_3 &= \frac{4\pi\gamma J}{\lambda\omega_o\tau} \\
C_4 &= 0 \\
B &= -\frac{2\pi J}{\lambda}\left(\frac{\gamma+1}{\gamma-1}\right)^{1/2} \\
D &= -\frac{2\pi i(\gamma-1)J}{m^2 H_o \lambda^2 \omega_o \tau}
\end{aligned} \tag{4.40}$$

Substituting these values into equations (4.22) results in the following representation for the magnetic field perturbations in the limit of small magnetic Reynold's number.

$$h_x = \frac{4\pi J}{\lambda \omega_o \tau} e^{i\lambda x}\left[\gamma e^{-\gamma\lambda y} - \frac{1}{2}(\gamma+1)e^{-\lambda y}\right]$$

$$h_y = \frac{4\pi i J}{\lambda \omega_o \tau} e^{i\lambda x}\left[e^{-\gamma\lambda y} - \frac{1}{2}(\gamma+1)e^{-\lambda y}\right] \tag{4.41}$$

$$h_z = \frac{4\pi i J}{\lambda} e^{i\lambda x - \gamma\lambda y}$$

Using equations (3.13), (4.32), and (4.33) determines the velocity field perturbations in the limit of small magnetic Reynold's number. They are

$$u_x = \frac{4\pi J U_o}{m^2 H_o \lambda \omega_o \tau} e^{i\lambda x}\left(\gamma e^{-\gamma\lambda y} - e^{-\lambda y}\right)$$

$$u_y = \frac{4\pi i J U_o}{m^2 H_o \lambda \omega_o \tau} e^{i\lambda x}\left(e^{-\gamma\lambda y} - e^{-\lambda y}\right) \tag{4.42}$$

$$u_z = \frac{4\pi i J U_o}{m^2 H_o \lambda} e^{i\lambda x - \gamma\lambda y}$$

Now, using Maxwell's equation $\nabla\times\vec{H}=4\pi\vec{j}$, the currents flowing in the fluid become

$$j_x = -i\gamma J\, e^{i\lambda x - \gamma\lambda y}$$

$$j_y = J\, e^{i\lambda x - \gamma\lambda y} \tag{4.43}$$

$$j_z = \omega_o \tau J\, e^{i\lambda x - \gamma\lambda y}$$

Eliminating $\omega_o\tau$ in favor of it's equivalent γ and taking the real part of the complex expressions, the vector form of the magnetic field becomes

$$\begin{aligned}\left(\frac{\lambda}{4\pi J}\right)\vec{h} = &-\frac{1}{2}\left(\frac{\gamma+1}{\gamma-1}\right)^{1/2} e^{-\lambda y}\left(\cos\lambda x\,\vec{i}_x - \sin\lambda x\,\vec{i}_y\right)\\ &+\frac{\gamma}{(\gamma^2-1)^{1/2}}\, e^{-\gamma\lambda y}\left(\cos\lambda x\,\vec{i}_x - \frac{1}{\gamma}\sin\lambda x\,\vec{i}_y\right.\\ &\left. -\frac{(\gamma^2-1)^{1/2}}{\gamma}\sin\lambda x\,\vec{i}_z\right)\end{aligned} \tag{4.44}$$

Similarly, the velocity field vector is

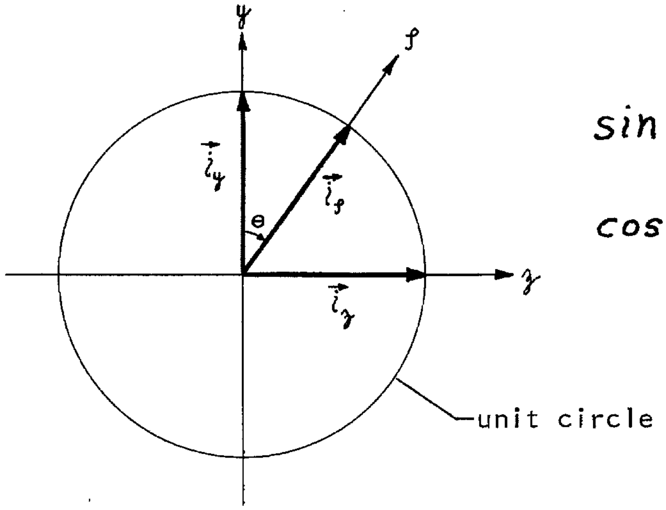
$$\begin{aligned}\left(\frac{m^2 H_o\lambda}{4\pi J}\right)\frac{\vec{u}}{U_o} = &-\frac{1}{(\gamma^2-1)^{1/2}}\, e^{-\lambda y}\left(\cos\lambda x\,\vec{i}_x - \sin\lambda x\,\vec{i}_y\right)\\ &+\frac{\gamma}{(\gamma^2-1)^{1/2}}\, e^{-\gamma\lambda y}\left(\cos\lambda x\,\vec{i}_x - \frac{1}{\gamma}\sin\lambda x\,\vec{i}_y\right.\\ &\left. -\frac{(\gamma^2-1)^{1/2}}{\gamma}\sin\lambda x\,\vec{i}_z\right)\end{aligned} \tag{4.45}$$

and the current density vector is

$$\begin{aligned}\frac{\vec{j}}{J} = \gamma\, e^{-\gamma\lambda y}&\left(\sin\lambda x\,\vec{i}_x + \frac{1}{\gamma}\cos\lambda x\,\vec{i}_y\right.\\ &\left. +\frac{(\gamma^2-1)^{1/2}}{\gamma}\cos\lambda x\,\vec{i}_z\right)\end{aligned} \tag{4.46}$$

where $\vec{i}_x$, $\vec{i}_y$, and $\vec{i}_z$ are the unit vectors in the x, y, and z-directions respectively. These expressions can be simplified by defining a new unit vector $\vec{i}_\zeta$ in the y-z plane (see Figure 2) as

$$\vec{i}_\zeta \equiv \frac{1}{\gamma}\left[\,\vec{i}_y + (\gamma^2-1)^{1/2}\,\vec{i}_z\,\right] \tag{4.47}$$



$$\sin\theta = \frac{(\gamma^2-1)^{1/2}}{\gamma}$$

$$\cos\theta = \frac{1}{\gamma}$$

FIGURE 2

DEFINITION OF THE ζ-AXIS

Denoting the direction of $\vec{i}_\zeta$ by ζ, where the magnitude of ζ is equal to γy, the magnetic and velocity field vectors can be written as the sum of two similar vectors;

$$\vec{h} = \vec{h}_1 + \vec{h}_2 \tag{4.48}$$

where $$\left(\frac{\lambda}{4\pi J}\right)\vec{h}_1 = -\frac{1}{2}\left(\frac{\gamma+1}{\gamma-1}\right)^{1/2} e^{-\lambda y}\left(\cos\lambda x\,\vec{i}_x - \sin\lambda x\,\vec{i}_y\right) \tag{4.49}$$

$$\left(\frac{\lambda}{4\pi J}\right)\vec{h}_2 = \frac{\gamma}{(\gamma^2-1)^{1/2}}\, e^{-\lambda\zeta}\left(\cos\lambda x\,\vec{i}_x - \sin\lambda x\,\vec{i}_\zeta\right) \tag{4.50}$$

and $\vec{u} = \vec{u}_1 + \vec{u}_2$ (4.51)

where $\left(\frac{m^2 H_0 \lambda}{4\pi J}\right) \frac{\vec{u}_1}{U_0} = -\frac{1}{(\gamma^2-1)^{1/2}} e^{-\lambda y}\left(\cos \lambda x \, \vec{i}_x - \sin \lambda x \, \vec{i}_y\right)$ (4.52)

$$\left(\frac{m^2 H_0 \lambda}{4\pi J}\right) \frac{\vec{u}_2}{U_0} = \frac{\gamma}{(\gamma^2-1)^{1/2}} e^{-\lambda \xi}\left(\cos \lambda x \, \vec{i}_x - \sin \lambda x \, \vec{i}_\xi\right) \qquad (4.53)$$

It will be noted that $\vec{h}_1$ and $\vec{u}_1$ are in the x-y plane while $\vec{h}_2$ and $\vec{u}_2$ are in the x-ξ plane. The currents are entirely in the x-ξ plane and are given by

$$\frac{\vec{j}}{J} = \gamma e^{-\lambda \xi}\left(\sin \lambda x \, \vec{i}_x + \cos \lambda x \, \vec{i}_\xi\right) \qquad (4.54)$$

The two vector $\vec{h}_1$ and $\vec{u}_1$ are irrotational and so contribute nothing to the currents and vorticity, respectively. The currents flowing in the fluid are due entirely to the $\vec{h}_2$ portion of the magnetic field perturbation and the vorticity present in the fluid is, likewise, due entirely to the $\vec{u}_2$ portion of the velocity field perturbation.

The expressions for the magnetic field, velocity field, and currents do not involve the magnetic Reynold's number explicitly and, therefore, do not depend upon the absolute magnitude of the magnetic Reynold's number. It is only required that the magnetic Reynold's number be small enough for the inequality (4.39) to hold. As might be expected in this limiting case, the magnetic field and currents do not depend upon the unperturbed velocity U_0 and, therefore, behave as though the fluid is stationary.

Figures 3, 4, and 5 show the variation of the magnetic field, velocity field, and current components as a function of λy with $\omega_o \tau$ as a parameter. Note that different scales have been used in Figures 3 and 4 for the three components. These figures show that near the wall the x and y-components of the magnetic and velocity fields are considerably smaller than the corresponding component in the z-direction while the current components are all of comparible magnitude for $\omega_o \tau$ of order one. In the limit of isotropic conductivity ($\omega_o \tau = 0$ or $\gamma = 1$), the x and y-components of the magnetic and velocity fields disappear as does the z-component of the current. In the opposite limit of strong anisotropy, the x and z-components of the current near the wall are considerably greater than the induced boundary current. In all cases the dependence on λy is very strong and, for all practical purposes, the boundary current effects are limited to values of λy less than about three.

In the limit of small magnetic Reynold's number, the current lines lie entirely in the x-ξ plane as shown in Figure 6. Each current line is symmetric about the ξ-axis and forms a closed loop with the wall. The current lines, when referred to the x-ξ plane, are functions of λx and $\lambda \xi$ only and are independent of the various parameters involved; i.e., U_o, H_o, $\omega_o \tau$, and J. The only dependence on a parameter occurs in the orientation of the x-ξ plane. Since the tangent of the angle which the x-ξ plane makes with the x-y plane is equal to $\omega_o \tau$, the unperturbed magnetic field serves to orient the x-ξ plane through its effect on the electron cyclotron frequency.

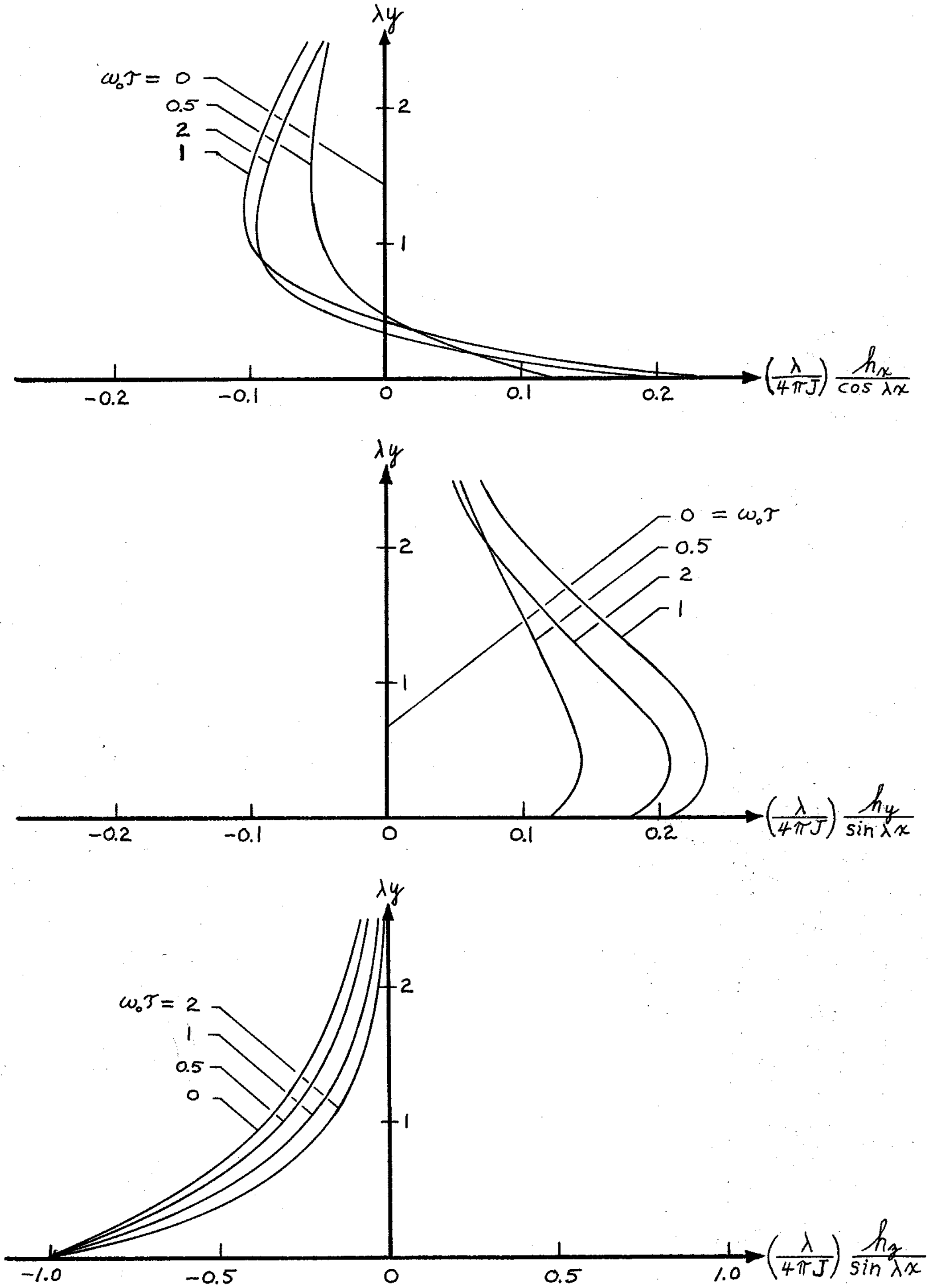


FIGURE 3

MAGNETIC FIELD COMPONENTS IN THE SEMI-INFINITE FLOW CASE FOR SMALL MAGNETIC REYNOLD'S NUMBER

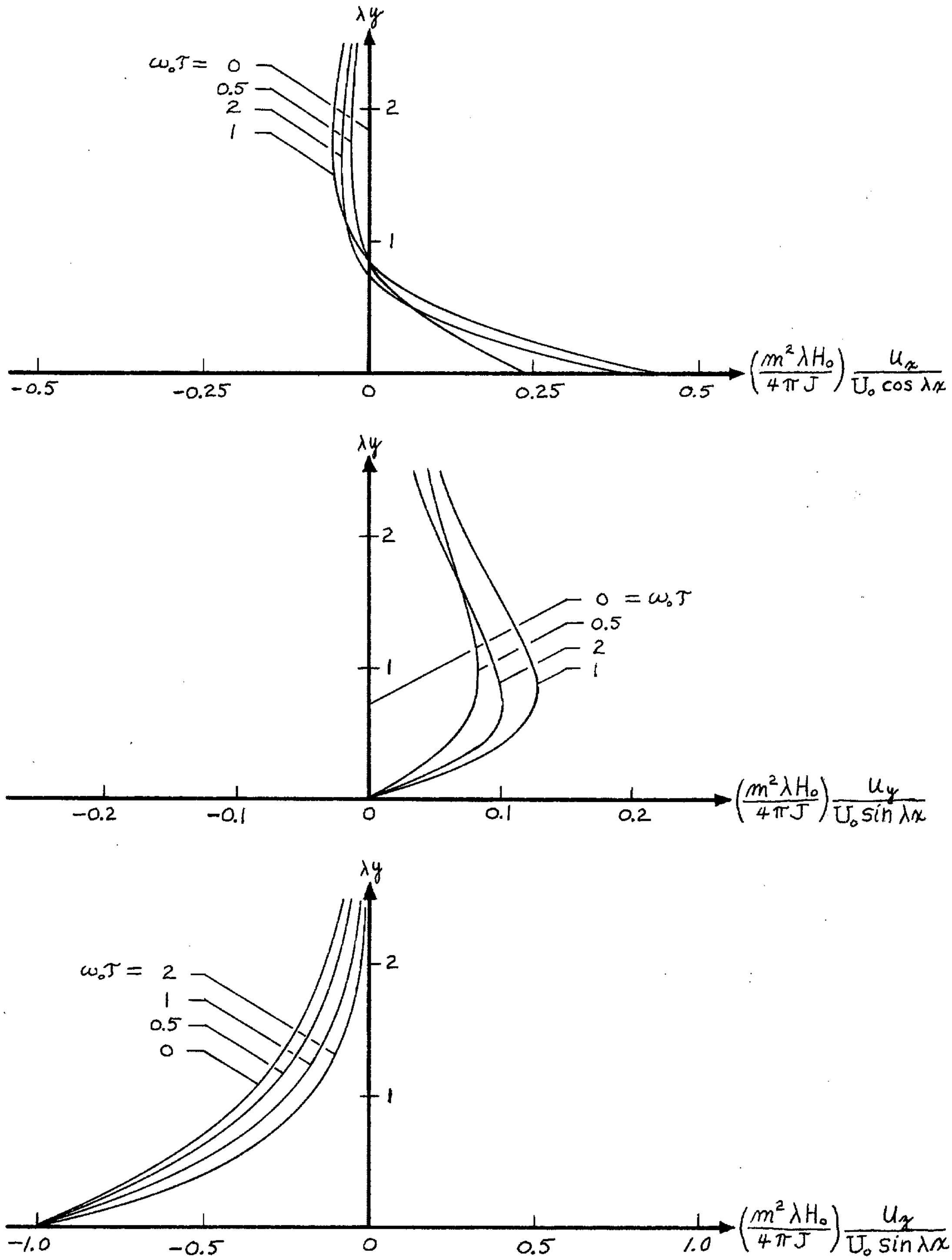


FIGURE 4

VELOCITY COMPONENTS IN THE SEMI-INFINITE FLOW CASE
FOR SMALL MAGNETIC REYNOLD'S NUMBER

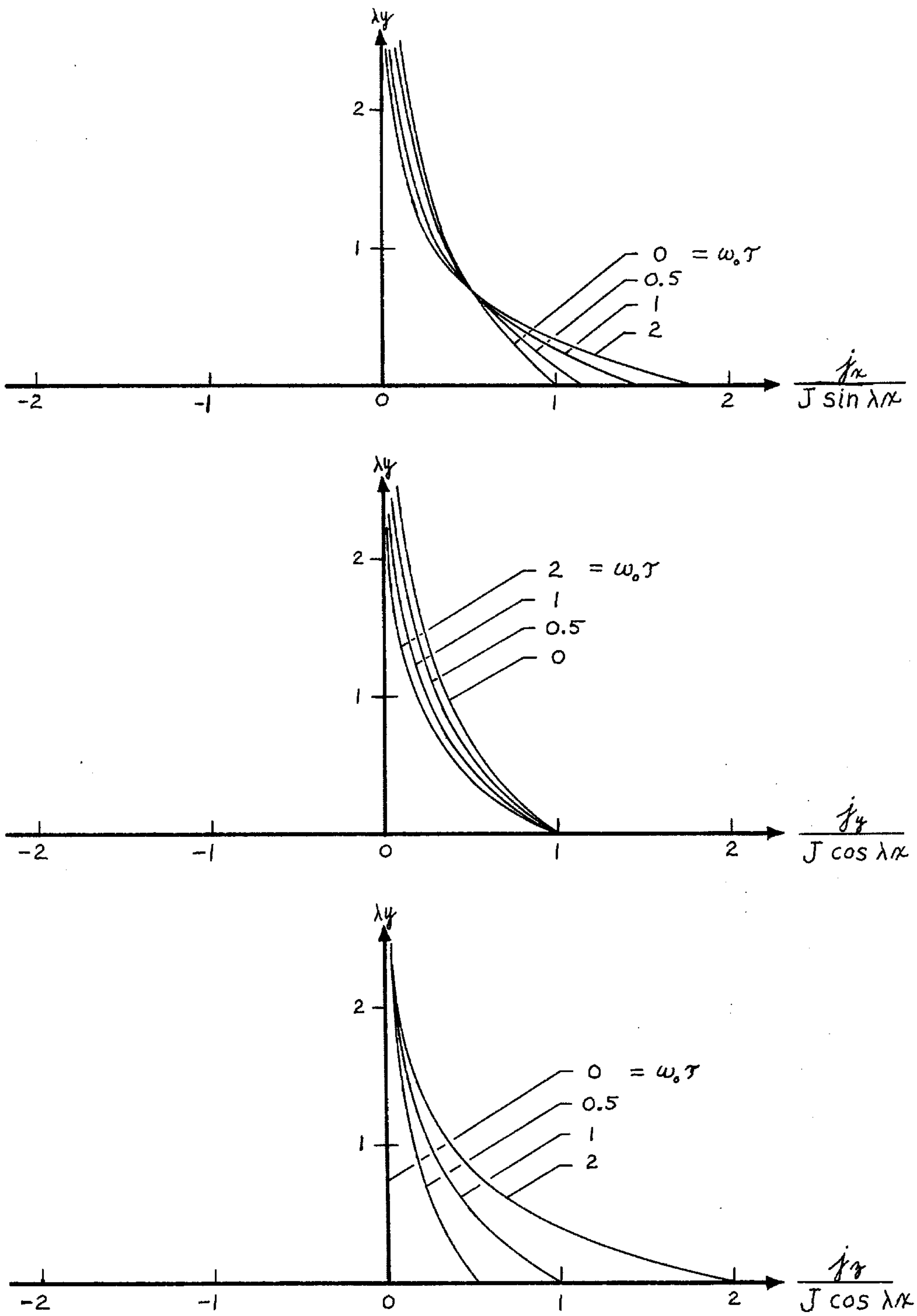


FIGURE 5

CURRENT COMPONENTS IN THE SEMI-INFINITE FLOW CASE
FOR SMALL MAGNETIC REYNOLD'S NUMBER

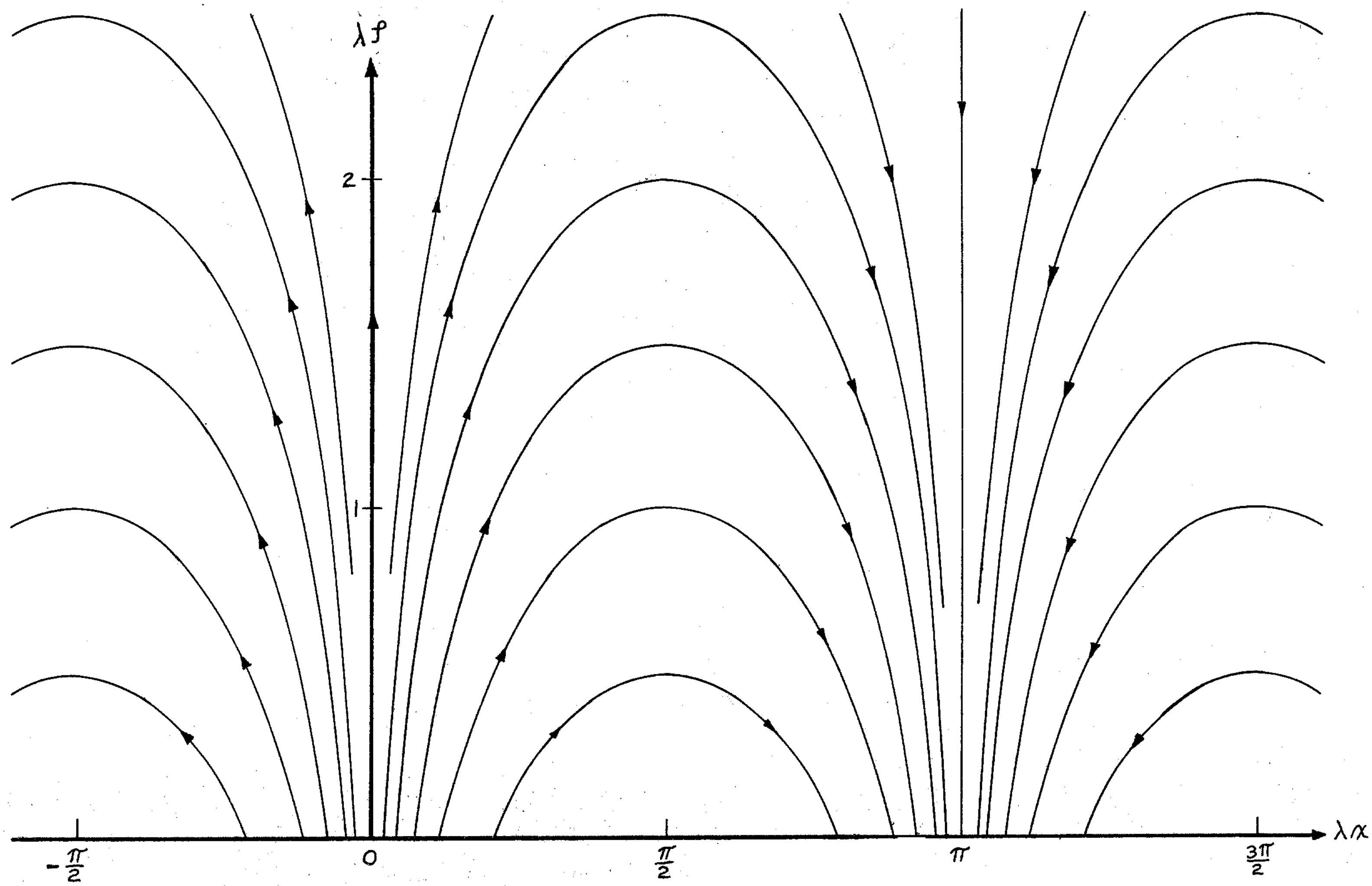


FIGURE 6

CURRENT LINES IN THE SEMI-INFINITE FLOW CASE
FOR SMALL MAGNETIC REYNOLD'S NUMBER

An interesting analogy can be made between the velocity perturbations calculated here and those due to ordinary incompressible flow over a wavy wall. If the wavy wall is expressed as $y = \epsilon \cos \lambda x$, the ordinary incompressible flow velocity perturbation vector is

$$\vec{u} = \epsilon \lambda U_o \, e^{-\lambda y} \left(\cos \lambda x \, \vec{i}_x - \sin \lambda x \, \vec{i}_y \right) \tag{4.55}$$

By comparing equation (4.55) to equations (4.52) and (4.53), it can be seen that, in the limit of small magnetic Reynold's number, the velocity perturbations obtained with sinusoidal boundary currents in an anisotropically conducting fluid are of the same form as the velocity perturbations due to ordinary incompressible flow over a wavy wall. For the irrotational portion $\vec{u}_1$ given by equation (4.52), the equivalent wave height is given by

$$\epsilon = - \frac{4\pi J}{m^2 H_o \lambda^2 (\gamma^2 - 1)^{1/2}} \tag{4.56}$$

while for the rotational part $\vec{u}_2$ given by equation (5.16), the equivalent wave height is given by

$$\epsilon = \frac{4\pi \gamma J}{m^2 H_o \lambda^2 (\gamma^2 - 1)^{1/2}} \tag{4.57}$$

Thus, since the solution for incompressible flow over a wavy wall is well known, the hydrodynamic effects of the problem under consideration are also well known by analogy.

C. LARGE MAGNETIC REYNOLD'S NUMBER APPROXIMATION

In order to expand the exponents α and β in the limit of large magnetic Reynold's number, it is necessary to assume that

$$\left|\frac{k}{\lambda}\right| \gg 1 \tag{4.58}$$

and, in addition, that $\omega_0 \tau$ is of the order of unity. It is now convenient to make the following definition.

$$a \equiv \left(\frac{|k|}{2\lambda}\right)^{1/2} = \left(\frac{1}{2} R_m \left|\frac{m^2-1}{m^2}\right|\right)^{1/2} \tag{4.59}$$

With this definition and the above assumptions, the various constants in the general solution are found to approach the following values.

$$\begin{aligned}
\alpha &= (1 \pm i)\lambda a \\
\beta &= (1 \pm i)\lambda a \\
C_1 &= \frac{2\pi i J}{\lambda} \\
C_2 &= \frac{2\pi i J}{\lambda} \\
C_3 &= \frac{2\pi J}{\lambda} \\
C_4 &= -\frac{2\pi J}{\lambda} \\
B &= 0 \\
D &= 0
\end{aligned} \tag{4.60}$$

Where the symbol $(1 \pm i)$ appears, the positive sign is to be taken when $k > 0$ (corresponding to superalfvénic flow) and the negative sign when $k < 0$ (corresponding to subalfénic flow). Substitution of the above constants into equations (4.22) results in the following representation for the magnetic field perturbations in the limit of large magnetic Reynold's number.

$$h_x = 0$$

$$h_y = 0 \tag{4.61}$$

$$h_z = \frac{4\pi i J}{\lambda} e^{i\lambda x - (1 \pm i)\lambda a y}$$

Since D = 0, the velocity field in the limit of large magnetic Reynold's number is strictly proportional to the magnetic field, and is given by

$$\vec{u} = \left(\frac{U_0}{m^2 H_0}\right) \vec{h} \tag{4.62}$$

The currents flowing in the fluid in this limit are

$$j_x = -i(1 \pm i) a J e^{i\lambda x - (1 \pm i)\lambda a y}$$

$$j_y = J e^{i\lambda x - (1 \pm i)\lambda a y} \tag{4.63}$$

$$j_z = 0$$

The unit vectors in the x, y, and z-directions will again be denoted by $\vec{i}_x$, $\vec{i}_y$, and $\vec{i}_z$ respectively. Taking the real part of the above complex expressions, the vector forms of the magnetic and velocity field perturbations and the currents flowing in the fluid, for $k>0$, are

$$\vec{h} = -\frac{4\pi J}{\lambda}\, e^{-\lambda a y} \sin\lambda(x-ay)\, \vec{i}_z$$

$$\vec{u} = -\frac{4\pi J U_0}{m^2 H_0 \lambda}\, e^{-\lambda a y} \sin\lambda(x-ay)\, \vec{i}_z \tag{4.64}$$

$$\vec{j} = J e^{-\lambda a y}\Big\{a\big[\cos\lambda(x-ay) + \sin\lambda(x-ay)\big]\vec{i}_x + \cos\lambda(x-ay)\,\vec{i}_y\Big\}$$

and for $k<0$ are

$$\vec{h} = -\frac{4\pi J}{\lambda}\, e^{-\lambda a y} \sin\lambda(x+ay)\, \vec{i}_z$$

$$\vec{u} = -\frac{4\pi J U_0}{m^2 H_0 \lambda}\, e^{-\lambda a y} \sin\lambda(x+ay)\, \vec{i}_z \tag{4.65}$$

$$\vec{j} = -J e^{-\lambda a y}\Big\{a\big[\cos\lambda(x+ay) - \sin\lambda(x+ay)\big]\vec{i}_x - \cos\lambda(x+ay)\,\vec{i}_y\Big\}$$

The solutions obtained are independent of $\omega_0\tau$ and, hence, anisotropic effects are absent to the order of the approximation made. Small anisotropic effects of the order of $R_m^{-1/2}$ do occur but will not be considered here since they are far overshadowed by the isotropic solution.

Opposite to the case of small magnetic Reynold's number, all quantities involved depend strongly on the actual magnitude of the magnetic Reynold's number, which is contained in the parameter a. Although the solution in the limit of large magnetic Reynold's number is independent of the anisotropy of the fluid's conductivity, the unperturbed magnetic field does affect the solution through the Alfvén number, which is also contained in the parameter a.

Figures 7, 8, and 9 show the variation of the magnetic field, velocity field, and current components with $\lambda a y$ for various values of λx. Note that the ordinates of all three figures are magnified by the square root of the magnetic Reynold's number while in Figure 8 the abscissa is reduced by the same factor. It is easily seen from these figures that the boundary current effects are essentially limited to values of $\lambda a y$ less than about three. Since for large magnetic Reynold's numbers a is much greater than unity, λy must be much less than unity for currents to flow in the fluid. Thus, the dominating characteristic of this solution is the existence of a thin magnetic boundary layer close to the wall within which the currents are confined. Also indicated in these figures are some of the wave characteristics, $\lambda(x-ay)=$ constant.

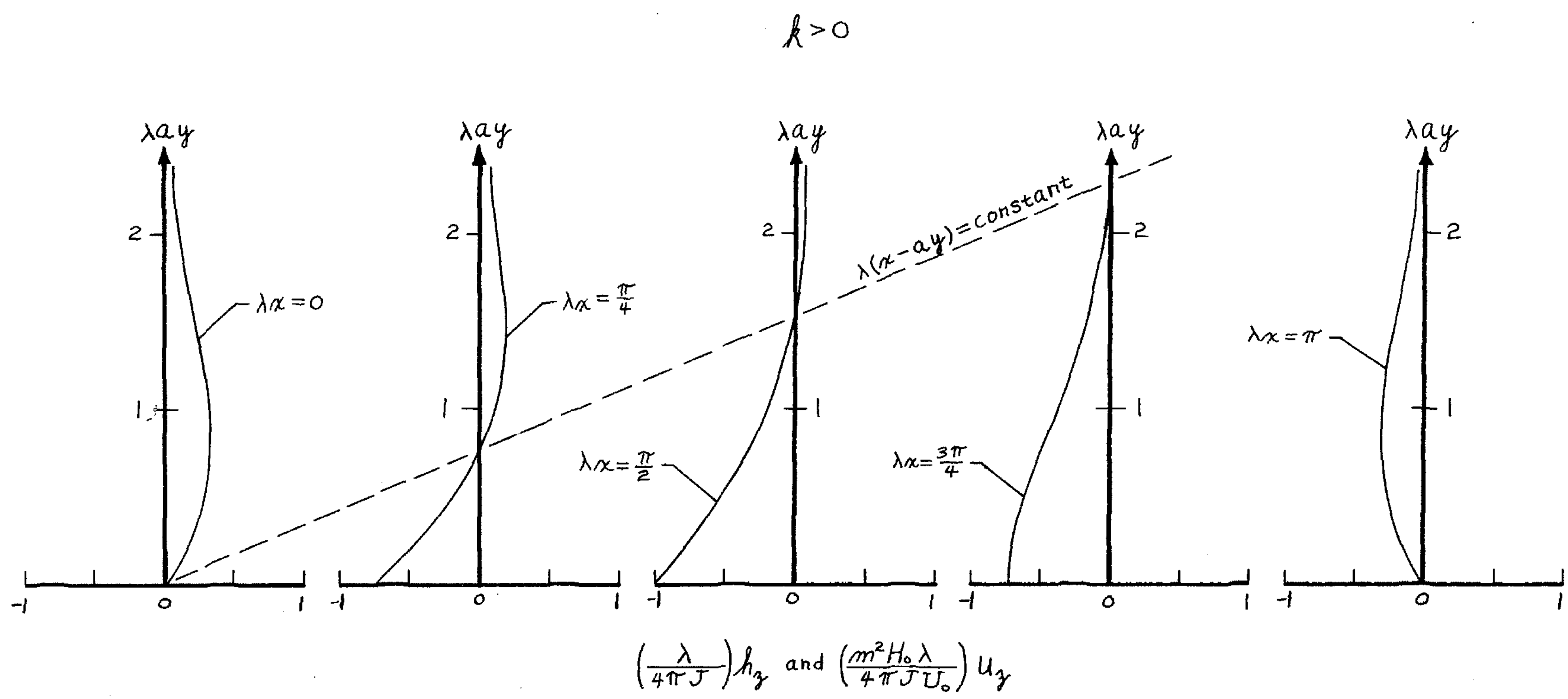


FIGURE 7

MAGNETIC FIELD AND VELOCITY COMPONENTS IN THE SEMI-INFINITE FLOW CASE FOR LARGE MAGNETIC REYNOLD'S NUMBER

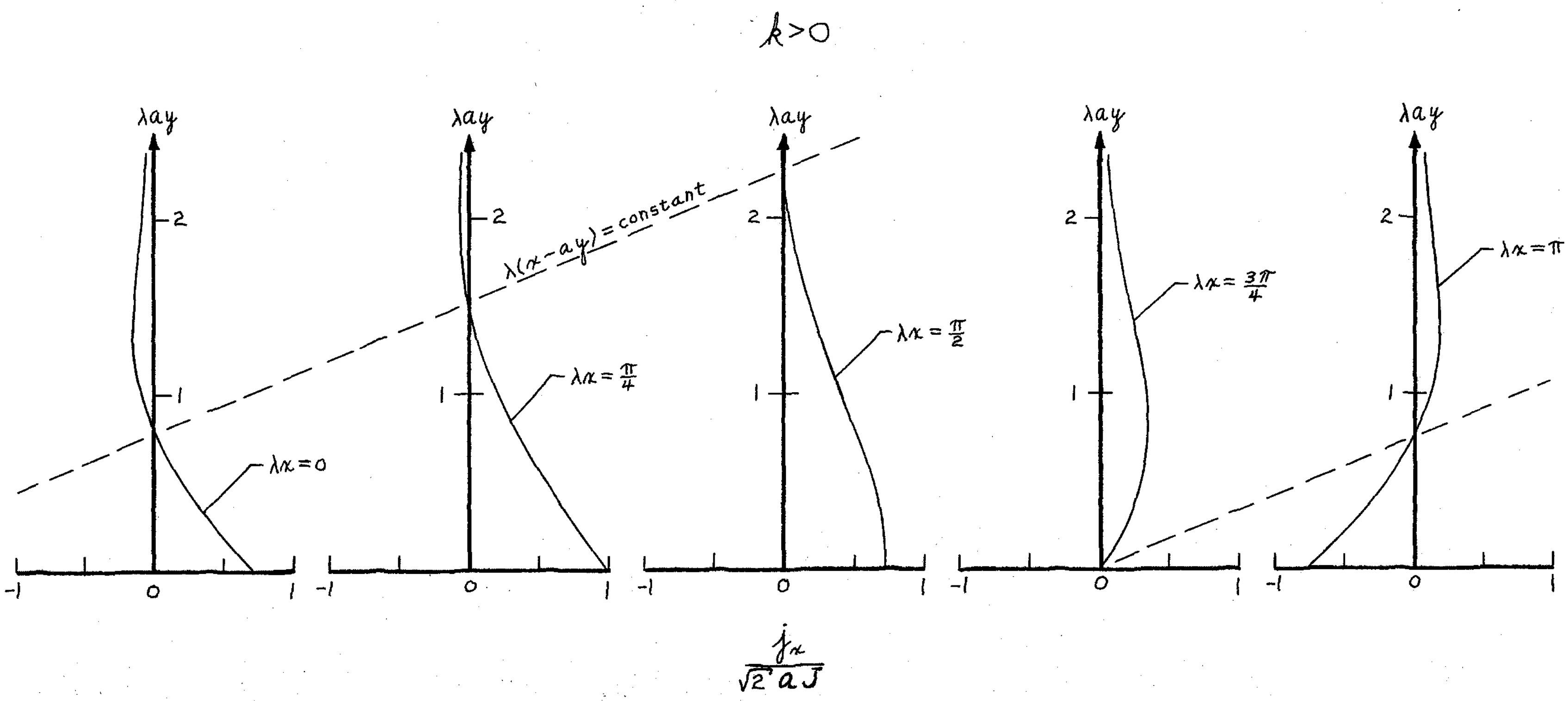


FIGURE 8

THE X-CURRENT COMPONENT IN THE SEMI-INFINITE FLOW CASE FOR LARGE MAGNETIC REYNOLD'S NUMBER

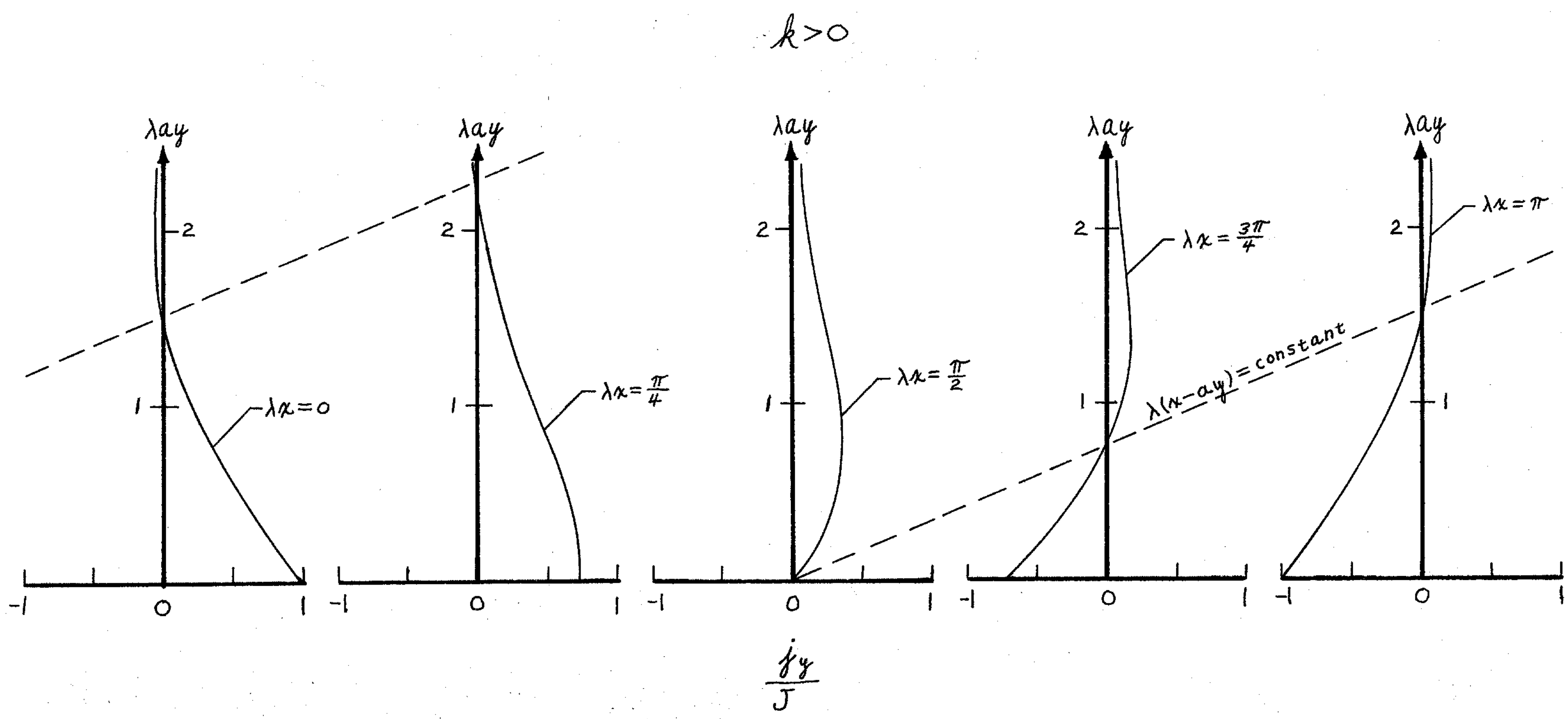


FIGURE 9

THE Y-CURRENT COMPONENT IN THE SEMI-INFINITE FLOW CASE
FOR LARGE MAGNETIC REYNOLD'S NUMBER

The current lines in the x-y plane are shown in Figure 10. As in Figures 7 through 9, the ordinate is magnified by the square root of the magnetic Reynold's number. The current lines shown are functions of λx and $\lambda a y$ only and are independent of all other parameters involved. As in the small magnetic Reynold's number case, the currents form closed loops with the wall but are no longer symmetrical. In the physical plane, the current loops shown would be inclined very sharply toward the x-axis; the degree of inclination depending upon the square root of the magnetic Reynold's number.

Because of the interesting analogy that was made between the velocity solutions obtained in the limit of small magnetic Reynold's number and ordinary incompressible flow over a wavy wall, it seems reasonable that a similar analogy with ordinary supersonic flow over a wavy wall might be made in the present case of large magnetic Reynold's number. Although the velocity obtained in this latter case has wave type solutions characteristic of ordinary supersonic flows, it is exponentially damped in y and the presence of only a z-component makes an analogy impossible.

It is interesting to note that the x-component of the current flowing in the fluid is a factor a larger than the y-component of the current, even though it is the y-component at the boundary which is causing the perturbations in the flow. Thus, this magnetohydrodynamic flow acts as a very effective current amplifier; an increase of δ in the applied boundary current produces an increase of $a\delta$ in the x-component of the current in the fluid at the wall. Because the

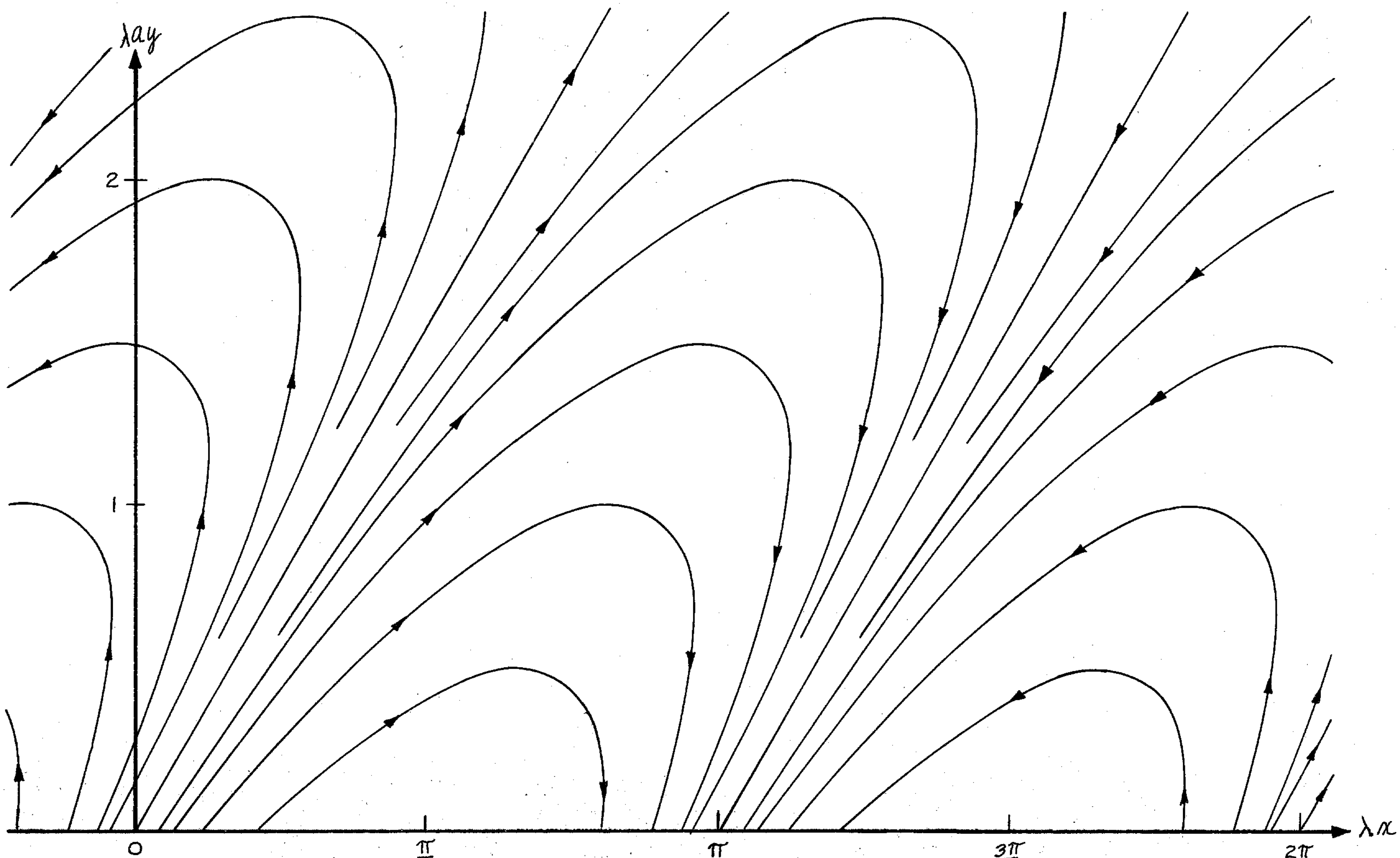


FIGURE 10

CURRENT LINES IN THE SEMI-INFINITE FLOW CASE
FOR LARGE MAGNETIC REYNOLD'S NUMBER

factor a is proportional to the square root of the magnetic Reynold's number, this amplifying effect can be very significant. Since the currents are very strongly damped in the y-direction, it is at the boundary itself where the largest currents are produced. Referring back to the case of small magnetic Reynold's number, equations (4.43) show a similar amplification takes place. There, however, it is the anisotropy of the fluid's conductivity that produces an amplification of both the x and z-components.

V. CHANNEL FLOW WITH SINUSOIDAL BOUNDARY CURRENTS IN ONE WALL

The channel flow problem can be solved in general form using the same procedures used in the semi-infinite flow case. However, the solution of the channel flow problem is much more complex because of the large increase in the number of constants which must be evaluated using the appropriate boundary conditions. Since detailed examination of the semi-infinite flow problem has been restricted to the limiting cases of small and large magnetic Reynold's number, the approximate forms of the governing equations for these two limiting cases are solved for the channel flow problem instead of obtaining the general solution. Before proceeding to solve these limiting cases, a definitive statement of the problem and the applicable boundary conditions will be given.

The geometrical arrangement of the channel flow problem being considered is as shown in Figure 11. Once again the unperturbed magnetic and velocity fields are parallel and in the x-direction. However, the fluid now flows between two parallel flat walls which are a distance b apart. For the purposes of this investigation both walls are considered to be of infinite extent in the x and z-directions and, in addition, infinitely thick. The lower wall is assumed to be a non-magnetic conductor having the same permeability as the fluid and, as in the semi-infinite flow case, the only non-zero current component in this wall is in the y-direction with a magnitude equal to the current at the boundary. It is supposed that the upper wall is a non-magnetic insulator in which no currents exist.

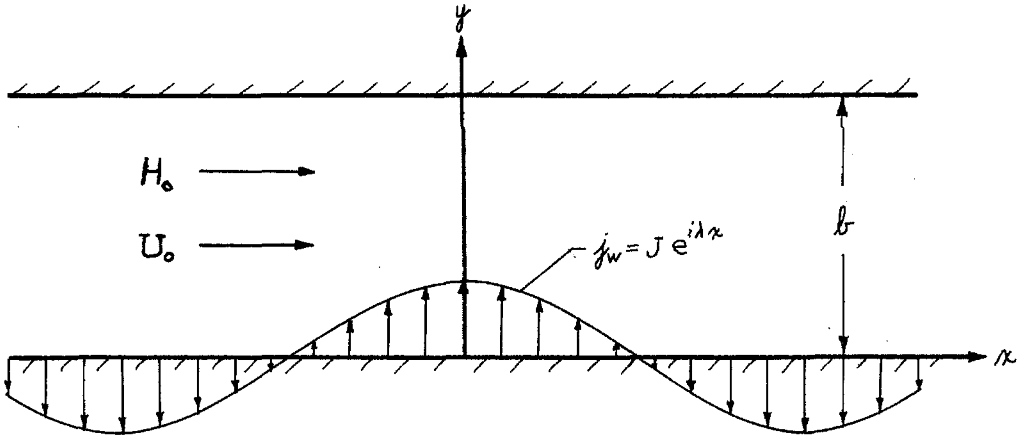


FIGURE II

GEOMETRICAL ARRANGEMENT OF THE CHANNEL FLOW CASE

The boundary current induced into the fluid through the conducting wall is again specified to be sinusoidal in profile and, using complex notation, is given by

$$j_w = J e^{i\lambda x} \tag{5.1}$$

where $J \ll \lambda H_o$ for valid solutions to be obtained within the small perturbation approach being used. The boundary conditions that will be applied, which are very similar to those used in the semi-infinite flow case, are as follows:

(i) the normal component of the current density vector is continuous across the boundaries of both walls

(ii) no fluid flows through the surface of either wall

(iii) the components of the magnetic field perturbation vector are continuous across the boundaries of both walls

(iv) the magnetic field perturbation components in the upper wall vanish as y approaches positive infinity

(v) the x and y-components of the magnetic field perturbation vector in the lower wall vanish as y approaches negative infinity while the z-component remains finite.

Condition (iii) is equivalent to specifying that no surface currents exist at either boundary while, once again, the z-component of the magnetic field perturbation in the conducting wall is not allowed to vanish by condition (v) since its derivative specifies the current flowing in that wall.

Note that this problem reduces to the semi-infinite flow problem as the distance between the walls is allowed to become very large. It will be seen later that the solutions obtained for the channel flow problem do indeed reduce to the semi-infinite flow solutions as b approaches infinity in the limit. Also note that by simple superposition of the solution to the problem considered here and the solution to the same problem with the positions of the conducting and insulating walls reversed results in the solution to the channel flow problem with two conducting walls.

As a final preparatory step before proceeding with the actual solution of the limiting cases of the channel flow problem, the applicable governing equations will be put into non-dimensional form. The dimensionless variables to be used are

$$x^* \equiv \lambda x$$

$$y^* \equiv \lambda y \qquad (5.2)$$

$$h^* \equiv \frac{h}{H_o}$$

$$\xi^* \equiv \frac{\xi}{\lambda H_o}$$

where subscripts have been omitted. Using equation (3.13) to eliminate the velocity terms from Ohm's law (3.11) and substituting the above dimensionless variables results in the following dimensionless form of Ohm's law.

$$\frac{k}{\lambda}\frac{\partial \vec{h}^*}{\partial x^*} - \left(\frac{m^2}{m^2-1}\right)\frac{k}{\lambda}\frac{\partial}{\partial x^*}\nabla\phi = \frac{\partial^2 \vec{h}^*}{\partial x^{*2}} + \frac{\partial^2 \vec{h}^*}{\partial y^{*2}} - \omega_o\tau\frac{\partial \vec{\xi}^*}{\partial x^*} \qquad (5.3)$$

Similarly, the dimensionless forms of equations (3.18) and (3.19) are

$$\frac{\partial^2 h_z^*}{\partial x^{*2}} + \frac{\partial^2 h_z^*}{\partial y^{*2}} = \frac{k}{\lambda}\frac{\partial h_z^*}{\partial x^*} + \omega_o\tau\frac{\partial \xi_z^*}{\partial x^*} \qquad (5.4)$$

$$\frac{\partial^2 \xi_z^*}{\partial x^{*2}} + \frac{\partial^2 \xi_z^*}{\partial y^{*2}} = \frac{k}{\lambda}\frac{\partial \xi_z^*}{\partial x^*} - \omega_o\tau\left(\frac{\partial^3 h_z^*}{\partial x^{*3}} + \frac{\partial^3 h_z^*}{\partial x^*\partial y^{*2}}\right) \qquad (5.5)$$

A. SMALL MAGNETIC REYNOLD'S NUMBER APPROXIMATION

In the limit of small magnetic Reynold's number, it will be assumed that the inequality

$$\left| \frac{k}{\lambda} \right| \ll 1 \tag{5.6}$$

is valid where k is related to the magnetic Reynold's number by equation (4.38).

$$R_m = \left(\frac{m^2}{m^2 - 1} \right) \frac{k}{\lambda} \tag{4.38}$$

Since $\frac{\partial h_z^*}{\partial x^*}$ and $\frac{\partial^2 h_z^*}{\partial x^{*2}}$ are of the same order of magnitude, the first term on the right of equation (5.4) is negligible compared with the first term on the left because of the inequality (5.6). Similarly, the first term on the right of equation (5.5) can be neglected in comparison with the first term on the left. For moderate values of $\omega_0 \tau$ the remaining terms are of comparable magnitude and, in terms of dimensional variables, the equations to be solved are

$$\nabla^2 h_z = \omega_0 \tau \frac{\partial \xi_z}{\partial x} \tag{5.7}$$

$$\nabla^2 \xi_z = - \omega_0 \tau \frac{\partial}{\partial x} \nabla^2 h_z \tag{5.8}$$

A stream type function Ψ will now be introduced by the following defining equations which identically satisfy Maxwell's equation $\nabla \cdot \vec{h} = 0$.

$$h_x = -\frac{\partial \Psi}{\partial y}$$
$$h_y = \frac{\partial \Psi}{\partial x} \tag{5.9}$$

Using equations (5.9) and the definition of $\mathcal{E}_z$ in equations (5.7) and (5.8) results in the following single equation for the stream function Ψ.

$$\nabla^2 \left(\frac{\partial^2 \Psi}{\partial x^2} + \frac{1}{\gamma^2} \frac{\partial^2 \Psi}{\partial y^2} \right) = 0 \tag{5.10}$$

This equation can be integrated immediately to give

$$\frac{\partial^2 \Psi}{\partial x^2} + \frac{1}{\gamma^2} \frac{\partial^2 \Psi}{\partial y^2} = \Theta \tag{5.11}$$

where Θ is a function defined by

$$\nabla^2 \Theta = 0 \tag{5.12}$$

The solution of equation (5.12) is easily found by separation of variables to be

$$\Theta = e^{i\lambda x} \left(A_1' e^{-\lambda y} + A_2' e^{\lambda y} \right) \tag{5.13}$$

where A_1' and A_2' are complex constants. The particular solution of equation (5.11) is then found to be

$$\Psi = e^{i\lambda x} \left(A_1 e^{-\lambda y} + A_2 e^{\lambda y} \right) \tag{5.14}$$

and it is obvious that the homogeneous solution is given by

$$\psi = e^{i\lambda x}\left(A_3 e^{-\gamma\lambda y} + A_4 e^{\gamma\lambda y}\right) \tag{5.15}$$

Thus, by adding the particular and homogeneous solutions, the general solution for the stream function is found to be

$$\psi = e^{i\lambda x}\left(A_1 e^{-\lambda y} + A_2 e^{\lambda y} + A_3 e^{-\gamma\lambda y} + A_4 e^{\gamma\lambda y}\right) \tag{5.16}$$

Thus, the x and y-components of the magnetic field perturbation vector can be determined by substitution of equation (5.16) into equations (5.9).

The z-component is obtained from equation (5.7) by noting that

$$\nabla^2 h_z = \omega_o \tau \frac{\partial \xi_z}{\partial x} = \omega_o \tau \frac{\partial}{\partial x} \nabla^2 \psi = \omega_o \tau \nabla^2 h_y \tag{5.17}$$

This equation integrates immediately to give

$$h_z = \omega_o \tau h_y + \eta \tag{5.18}$$

where η is a function defined by

$$\nabla^2 \eta = 0 \tag{5.19}$$

The function η has the same form as θ given by equation (5.13). Thus, the form of all components of the magnetic field perturbation vector in the fluid have been determined and are given by

$$h_x = \lambda e^{i\lambda x}\left(A_1 e^{-\lambda y} - A_2 e^{\lambda y} + \gamma A_3 e^{-\gamma\lambda y} - \gamma A_4 e^{\gamma\lambda y}\right)$$

$$h_y = i\lambda e^{i\lambda x}\left(A_1 e^{-\lambda y} + A_2 e^{\lambda y} + A_3 e^{-\gamma\lambda y} + A_4 e^{\gamma\lambda y}\right) \qquad (5.20)$$

$$h_z = i\omega_0 \tau\lambda e^{i\lambda x}\left(A_5 e^{-\lambda y} + A_6 e^{\lambda y} + A_3 e^{-\gamma\lambda y} + A_4 e^{\gamma\lambda y}\right)$$

Inside the lower wall, the magnetic field again has the form of equation (4.11).

$$h_x = i B_1 e^{i\lambda x + \lambda y}$$

$$h_y = B_1 e^{i\lambda x + \lambda y} \qquad (5.21)$$

$$h_z = \frac{4\pi i J}{\lambda} e^{i\lambda x}$$

In the upper wall where no currents exist, the magnetic field is both divergence free and curl free. The form of the magnetic field in this wall is

$$h_x = B_2 e^{i\lambda x - \lambda(y-b)}$$

$$h_y = i B_2 e^{i\lambda x - \lambda(y-b)} \qquad (5.22)$$

$$h_z = 0$$

Note that boundary condition (i) is automatically satisfied by equations (5.21) and (5.22); i.e., the normal current at the lower boundary is the induced boundary current j_w and at the upper boundary no current flows through the surface of the wall.

Since the components of the magnetic field are continuous at both boundaries by boundary condition (iii), equations (5.20) can be equated to equations (5.21) at the lower boundary and to equations (5.22) at the upper boundary. In this manner, the constants A_1, A_2, A_5, and A_6 are related to A_3 and A_4. At this point it is convenient to introduce the following definitions for some repeatedly occurring combinations of terms.

$$
\begin{aligned}
Q_1 &\equiv 1 - e^{-(\gamma-1)\lambda b} \\
Q_2 &\equiv 1 - e^{(\gamma-1)\lambda b} \\
Q_3 &\equiv 1 - e^{-(\gamma+1)\lambda b} \\
Q_4 &\equiv 1 - e^{(\gamma+1)\lambda b} \\
Q_5 &\equiv 1 - e^{-2\lambda b} \\
Q_6 &\equiv 1 - e^{2\lambda b}
\end{aligned}
\tag{5.23}
$$

Using this notation, equating components of the magnetic field perturbations at the two boundaries results in

$$A_1 = -\tfrac{1}{2}(\gamma+1)A_3 + \tfrac{1}{2}(\gamma-1)A_4$$

$$A_2 = \tfrac{1}{2}(\gamma-1)(1-Q_3)A_3 - \tfrac{1}{2}(\gamma+1)(1-Q_2)A_4$$

$$A_5 = \frac{4\pi J}{\lambda^2\omega_0\tau Q_5} - \left(\frac{Q_3}{Q_5}\right)A_3 - \left(\frac{Q_2}{Q_5}\right)A_4 \tag{5.24}$$

$$A_6 = \frac{4\pi J}{\lambda^2\omega_0\tau Q_6} - \left(\frac{Q_1}{Q_6}\right)A_3 - \left(\frac{Q_4}{Q_6}\right)A_4$$

In the limit of small magnetic Reynold's number as defined by the inequality (5.6), the left hand side of the dimensionless form of Ohm's law (5.3) can be neglected in comparison to the right hand side. Thus, to the order of the approximation made in obtaining equations (5.20) for the magnetic field perturbations, Ohm's Law is

$$\nabla^2 \vec{h} - \omega_0 \tau \frac{\partial \vec{\xi}}{\partial x} = 0 \tag{5.25}$$

Substituting the expressions already obtained for the magnetic field perturbations into the above form of Ohm's law and equating coefficients of like exponentials results in the determination of the remaining constants A_3 and A_4.

$$A_3 = \frac{4\pi J}{\lambda^2\omega_0\tau}\left(\frac{Q_2 - Q_4}{Q_1 Q_2 - Q_3 Q_4}\right) \tag{5.26}$$

$$A_4 = \frac{4\pi J}{\lambda^2\omega_0\tau}\left(\frac{Q_1 - Q_3}{Q_1 Q_2 - Q_3 Q_4}\right)$$

Evaluating all the constants A_1 through A_6 and substituting into equation (5.20) results in the following expressions for the magnetic field components.

$$h_x = \frac{4\pi J}{\lambda\omega_0 \mathcal{T}} e^{i\lambda x}\left(\gamma K_1 e^{-\gamma\lambda y} - \gamma K_2 e^{\gamma\lambda y} + K_3 e^{-\lambda y} - K_4 e^{\lambda y}\right)$$

$$h_y = \frac{4\pi i J}{\lambda\omega_0 \mathcal{T}} e^{i\lambda x}\left(K_1 e^{-\gamma\lambda y} + K_2 e^{\gamma\lambda y} + K_3 e^{-\lambda y} + K_4 e^{\lambda y}\right) \qquad (5.27)$$

$$h_z = \frac{4\pi i J}{\lambda} e^{i\lambda x}\left(K_1 e^{-\gamma\lambda y} + K_2 e^{\gamma\lambda y}\right)$$

where the constants K_1 through K_4 are defined as follows:

$$K_1 \equiv \frac{Q_2 - Q_4}{Q_1 Q_2 - Q_3 Q_4}$$

$$K_2 \equiv \frac{Q_1 - Q_3}{Q_1 Q_2 - Q_3 Q_4}$$

$$K_3 \equiv \frac{(\gamma-1)(Q_1 - Q_3) - (\gamma+1)(Q_2 - Q_4)}{2\,(Q_1 Q_2 - Q_3 Q_4)} \qquad (5.28)$$

$$K_4 \equiv \frac{\gamma Q_5}{Q_1 Q_2 - Q_3 Q_4}$$

The velocity field perturbations are again determined from equation (3.13), which is

$$\vec{u} = \left(\frac{U_0}{m^2 H_0}\right)\vec{h} + U_0 \nabla\phi \qquad (3.13)$$

where $\nabla^2\phi=0$. Using separation of variables, ϕ is easily found to be

$$\phi = e^{i\lambda x}\left(D_1 e^{-\lambda y} + D_2 e^{\lambda y}\right) \tag{5.29}$$

The y-component of the velocity now becomes

$$\begin{aligned} u_y = \frac{4\pi i J U_0}{m^2 H_0 \lambda \omega_0 \mathcal{T}} e^{i\lambda x}\Big[& K_1 e^{-\gamma\lambda y} + K_2 e^{\gamma\lambda y} \\ & + \left(K_3 + \frac{i m^2 H_0 \lambda^2 \omega_0 \mathcal{T}}{4\pi J} D_1\right) e^{-\lambda y} \\ & + \left(K_4 - \frac{i m^2 H_0 \lambda^2 \omega_0 \mathcal{T}}{4\pi J} D_2\right) e^{\lambda y} \Big] \end{aligned} \tag{5.30}$$

Since no fluid flows through the surface of the walls, u_y vanishes at both walls. Using these conditions results in the following relations for D_1 and D_2.

$$\begin{aligned} D_1 &= \frac{4\pi i J}{m^2 H_0 \lambda^2 \omega_0 \mathcal{T}} \left[\left(\frac{Q_3}{Q_5}\right) K_1 + \left(\frac{Q_2}{Q_5}\right) K_2 + K_3\right] \\ D_2 &= -\frac{4\pi i J}{m^2 H_0 \lambda^2 \omega_0 \mathcal{T}} \left[\left(\frac{Q_1}{Q_6}\right) K_1 + \left(\frac{Q_4}{Q_6}\right) K_2 + K_4\right] \end{aligned} \tag{5.31}$$

Combining terms gives the following result for the velocity field perturbations.

$$\begin{aligned} u_x &= \frac{4\pi J U_0}{m^2 H_0 \lambda \omega_0 \mathcal{T}} e^{i\lambda x}\Big[\gamma K_1 e^{-\gamma\lambda y} - \gamma K_2 e^{\gamma\lambda y} - \frac{1}{Q_5} e^{-\lambda y} + \frac{1}{Q_6} e^{\lambda y}\Big] \\ u_y &= \frac{4\pi i J U_0}{m^2 H_0 \lambda \omega_0 \mathcal{T}} e^{i\lambda x}\Big[K_1 e^{-\gamma\lambda y} + K_2 e^{\gamma\lambda y} - \frac{1}{Q_5} e^{-\lambda y} - \frac{1}{Q_6} e^{\lambda y}\Big] \\ u_z &= \frac{4\pi i J U_0}{m^2 H_0 \lambda} e^{i\lambda x}\Big[K_1 e^{-\gamma\lambda y} + K_2 e^{\gamma\lambda y}\Big] \end{aligned} \tag{5.32}$$

The currents flowing in the fluid are determined from Maxwell's equation $\nabla\times\vec{H} = 4\pi\vec{j}$. They are

$$j_x = -i\gamma J e^{i\lambda x}\left(K_1 e^{-\gamma\lambda y} - K_2 e^{\gamma\lambda y}\right)$$

$$j_y = J e^{i\lambda x}\left(K_1 e^{-\gamma\lambda y} + K_2 e^{\gamma\lambda y}\right) \tag{5.33}$$

$$j_z = \omega_0\tau J e^{i\lambda x}\left(K_1 e^{-\gamma\lambda y} + K_2 e^{\gamma\lambda y}\right)$$

Once again the perturbation vectors can be put into a simplified form by the introduction of the unit vector $\vec{i}_f$ in the y-z plane (see Figure 2 on page 27). This unit vector is defined by equation (4.47) as

$$\vec{i}_f \equiv \frac{1}{\gamma}\left[\vec{i}_y + (\gamma^2-1)^{1/2}\,\vec{i}_z\right] \tag{4.47}$$

Denoting the direction of $\vec{i}_f$ by f, where the magnitude of f is equal to γy, the magnetic and velocity field vectors can again be written as the sum of two similar vectors. Taking the real part of the complex expressions and eliminating $\omega_0\tau$ in favor of its equivalent γ, these vectors are

$$\vec{h} = \vec{h}_1 + \vec{h}_2 \tag{5.34}$$

where $$\left(\frac{\lambda}{4\pi J}\right)\vec{h}_1 = \frac{1}{(\gamma^2-1)^{1/2}}\Big[\left(K_3 e^{-\lambda y} - K_4 e^{\lambda y}\right)\cos\lambda x\,\vec{i}_x - \left(K_3 e^{-\lambda y} + K_4 e^{\lambda y}\right)\sin\lambda x\,\vec{i}_y\Big] \tag{5.35}$$

$$\left(\frac{\lambda}{4\pi J}\right)\vec{h}_2 = \frac{\gamma}{(\gamma^2-1)^{1/2}}\Big[\left(K_1 e^{-\lambda\zeta} - K_2 e^{\lambda\zeta}\right)\cos\lambda x\ \vec{i}_x - \left(K_1 e^{-\lambda\zeta} + K_2 e^{\lambda\zeta}\right)\sin\lambda x\ \vec{i}_\zeta\Big] \tag{5.36}$$

and $\vec{u} = \vec{u}_1 + \vec{u}_2$ (5.37)

where
$$\left(\frac{m^2 H_0 \lambda}{4\pi J}\right)\frac{\vec{u}_1}{U_0} = -\frac{1}{(\gamma^2-1)^{1/2}}\Big[\left(\frac{e^{-\lambda y}}{Q_5} - \frac{e^{\lambda y}}{Q_6}\right)\cos\lambda x\ \vec{i}_x - \left(\frac{e^{-\lambda y}}{Q_5} + \frac{e^{\lambda y}}{Q_6}\right)\sin\lambda x\ \vec{i}_y\Big] \tag{5.38}$$

$$\left(\frac{m^2 H_0 \lambda}{4\pi J}\right)\frac{\vec{u}_2}{U_0} = \frac{\gamma}{(\gamma^2-1)^{1/2}}\Big[\left(K_1 e^{-\lambda\zeta} - K_2 e^{\lambda\zeta}\right)\cos\lambda x\ \vec{i}_x - \left(K_1 e^{-\lambda\zeta} + K_2 e^{\lambda\zeta}\right)\sin\lambda x\ \vec{i}_\zeta\Big] \tag{5.39}$$

As in the semi-infinite flow case, the vectors $\vec{h}_1$ and $\vec{u}_1$ are in the x-y plane while the vectors $\vec{h}_2$ and $\vec{u}_2$ are in the x-ζ plane. In addition the vectors $\vec{h}_1$ and $\vec{u}_1$ are once more irrotational and so do not contribute to the currents and vorticity, respectively. The currents, which are due to $\vec{h}_2$ only, are vectorially represented as

$$\frac{\vec{j}}{J} = \gamma\Big[\left(K_1 e^{-\lambda\zeta} - K_2 e^{\lambda\zeta}\right)\sin\lambda x\ \vec{i}_x + \left(K_1 e^{-\lambda\zeta} + K_2 e^{\lambda\zeta}\right)\cos\lambda x\ \vec{i}_\zeta\Big] \tag{5.40}$$

As can be seen, the currents are contained entirely in the x-ζ plane.

The variation of the coefficients K_1, K_2, K_3, K_4, $1/Q_5$ and $1/Q_6$ with λb and $\omega_0\tau$ is shown in Figure 12. It is easily seen in this figure that as λb approaches infinity, K_1, $-K_3$, and $1/Q_5$ approach unity while K_2, K_4, and $1/Q_6$ approach zero. These are just the values required for the channel flow solution to reduce to the semi-infinite flow solution. Thus, as required, the channel flow solution obtained coincides with the semi-infinite flow solution as the upper wall moves to infinity.

Figures 13, 14, and 15 show the variation of the magnetic field, velocity field, and current components as a function of (y/b) for two values of λb (0.25 and 1.0) and $\omega_0\tau = 1$. The current lines in the x-ξ plane are shown in Figure 16 for $\lambda b = 0.5$ and $\omega_0\tau = 1$. Each current line again forms a closed loop with the wall symmetric about the ξ-axis and, as would be expected, resembles a compressed version of the current lines in the semi-infinite flow case.

Generally speaking, most of the characteristics of the semi-infinite flow case are also applicable to the channel flow case presently being considered. Therefore, they will not be repeated here and only the effects produced by the insulating wall will be discussed.

Because of the rapid rate at which the coefficients shown in Figure 12 approach their limiting values as λb increases, it might be thought that the effect of the insulating wall is essentially limited to values of λb less than unity. That this is not the case is amply demonstrated in Figures 13, 14, and 15. In these figures, it can be

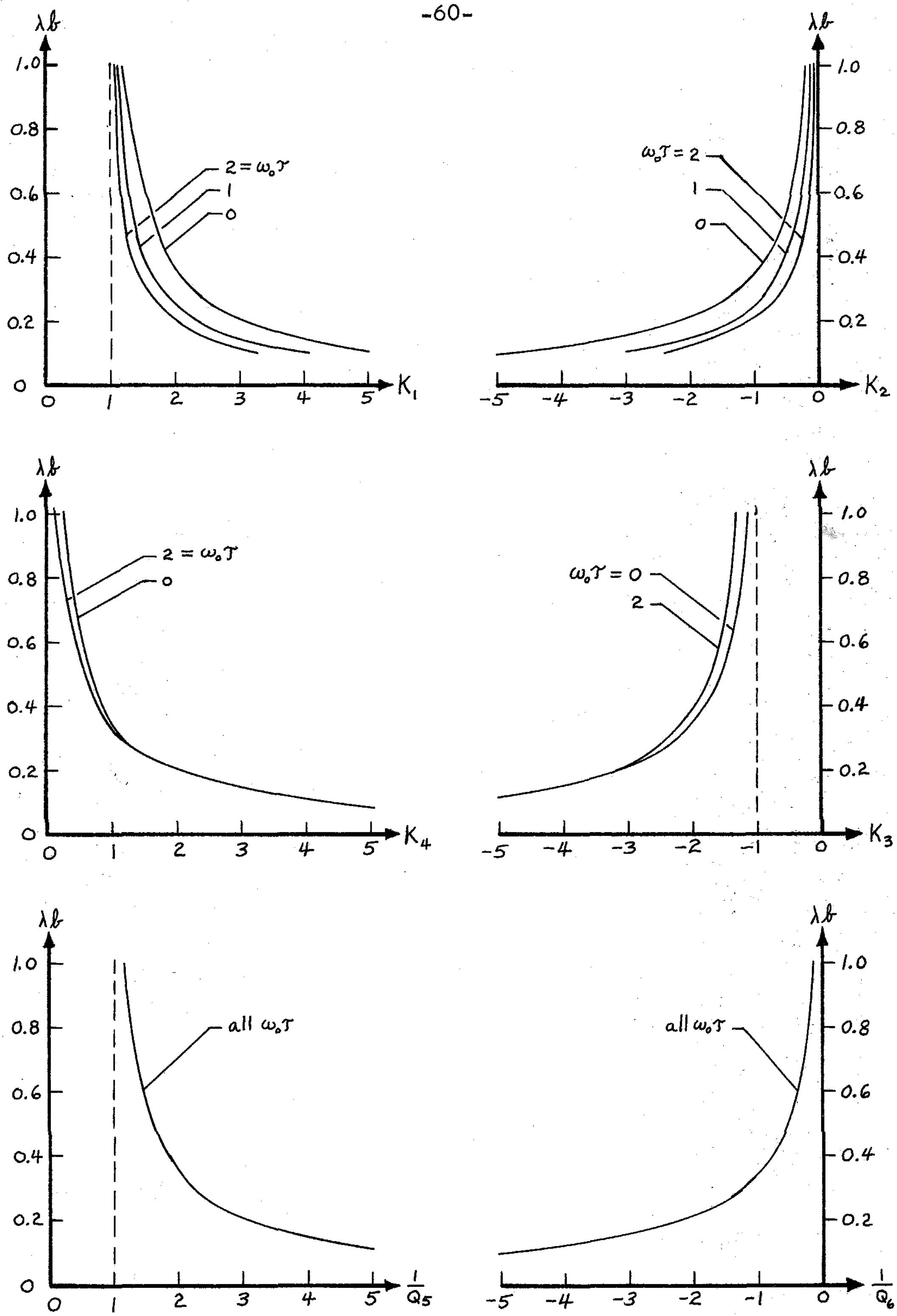


FIGURE 12

CHANNEL FLOW COEFFICIENTS FOR SMALL MAGNETIC REYNOLD'S NUMBER

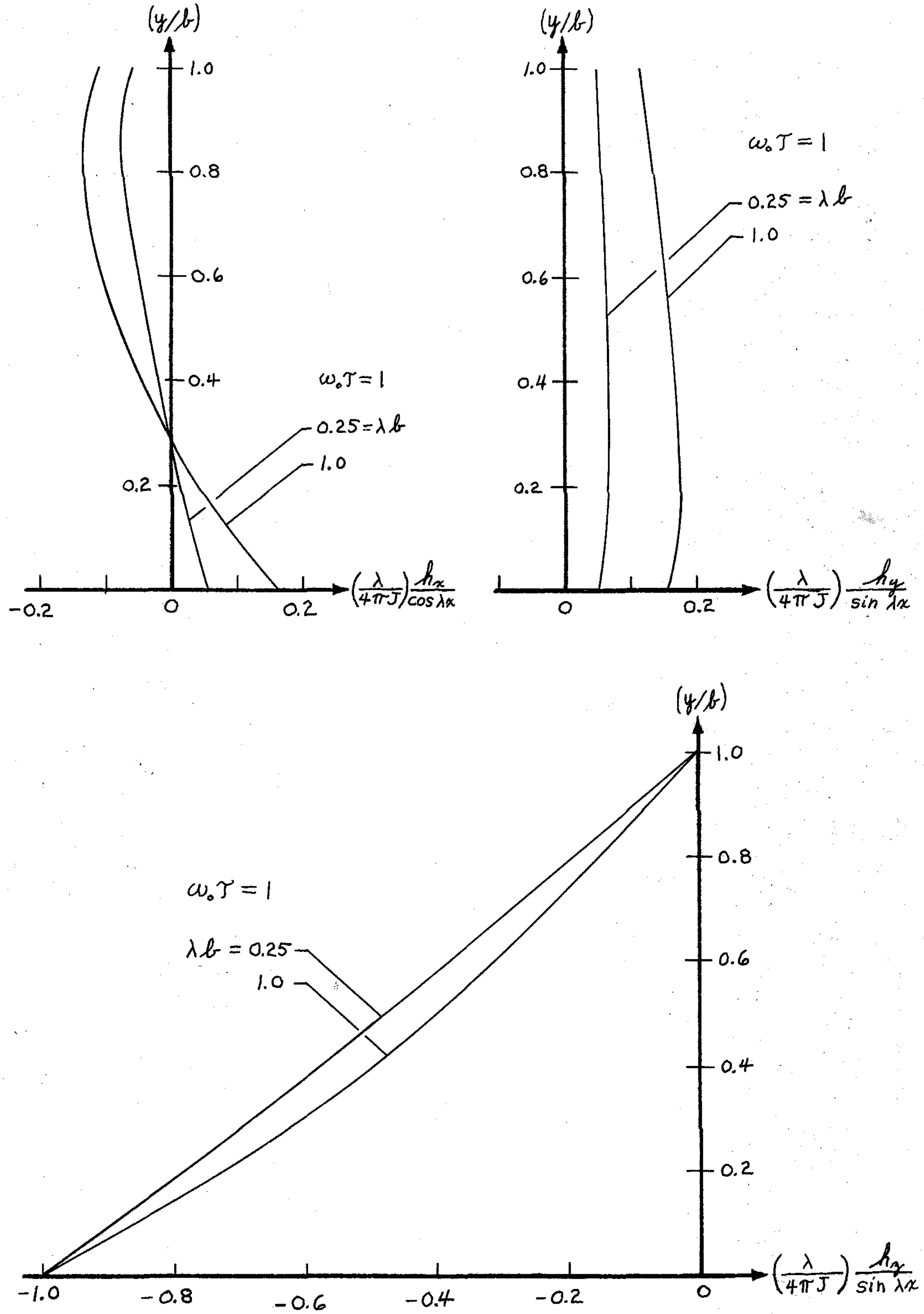


FIGURE 13

MAGNETIC FIELD COMPONENTS IN THE CHANNEL FLOW CASE
FOR SMALL MAGNETIC REYNOLD'S NUMBER

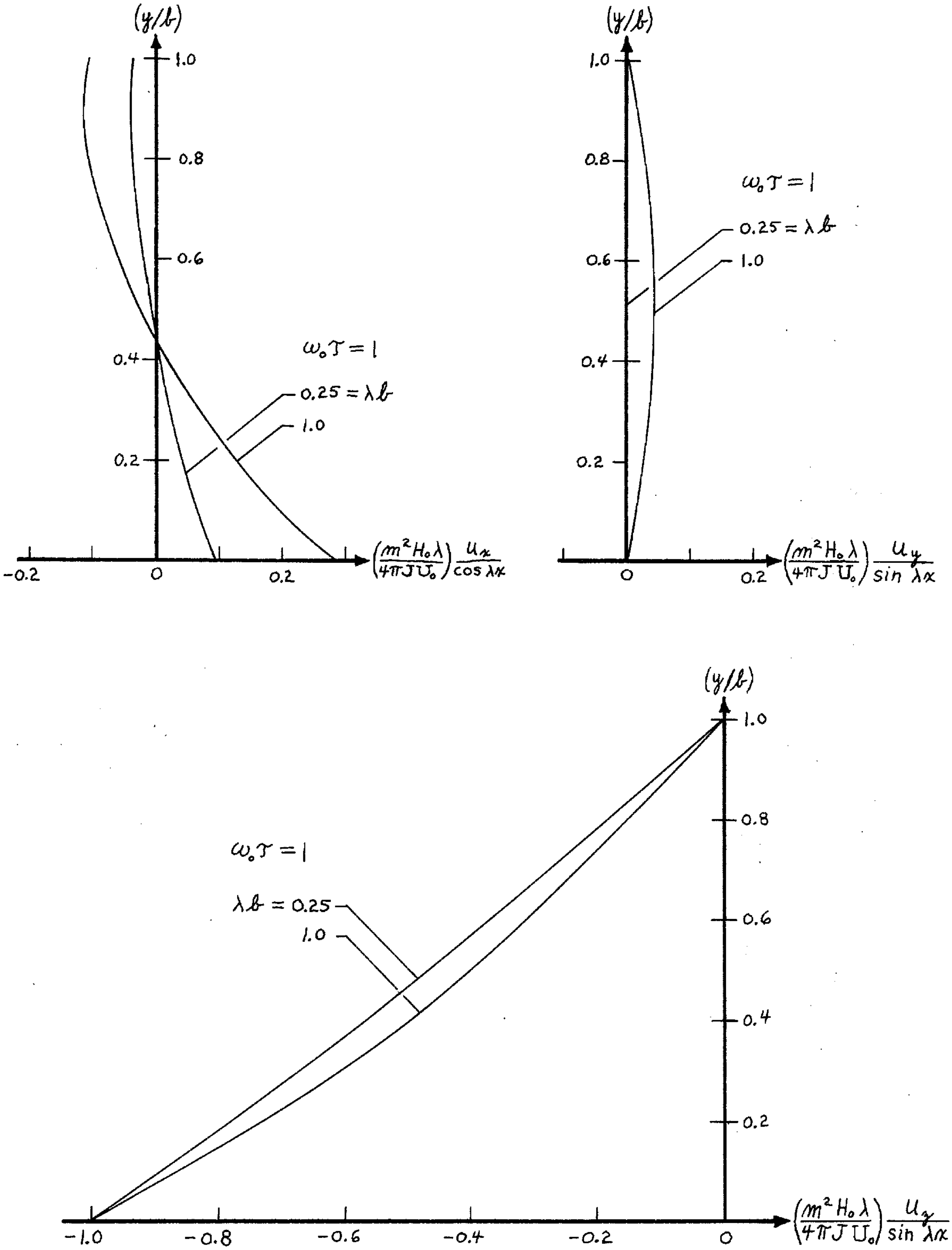


FIGURE 14

VELOCITY COMPONENTS IN THE CHANNEL FLOW CASE
FOR SMALL MAGNETIC REYNOLD'S NUMBER

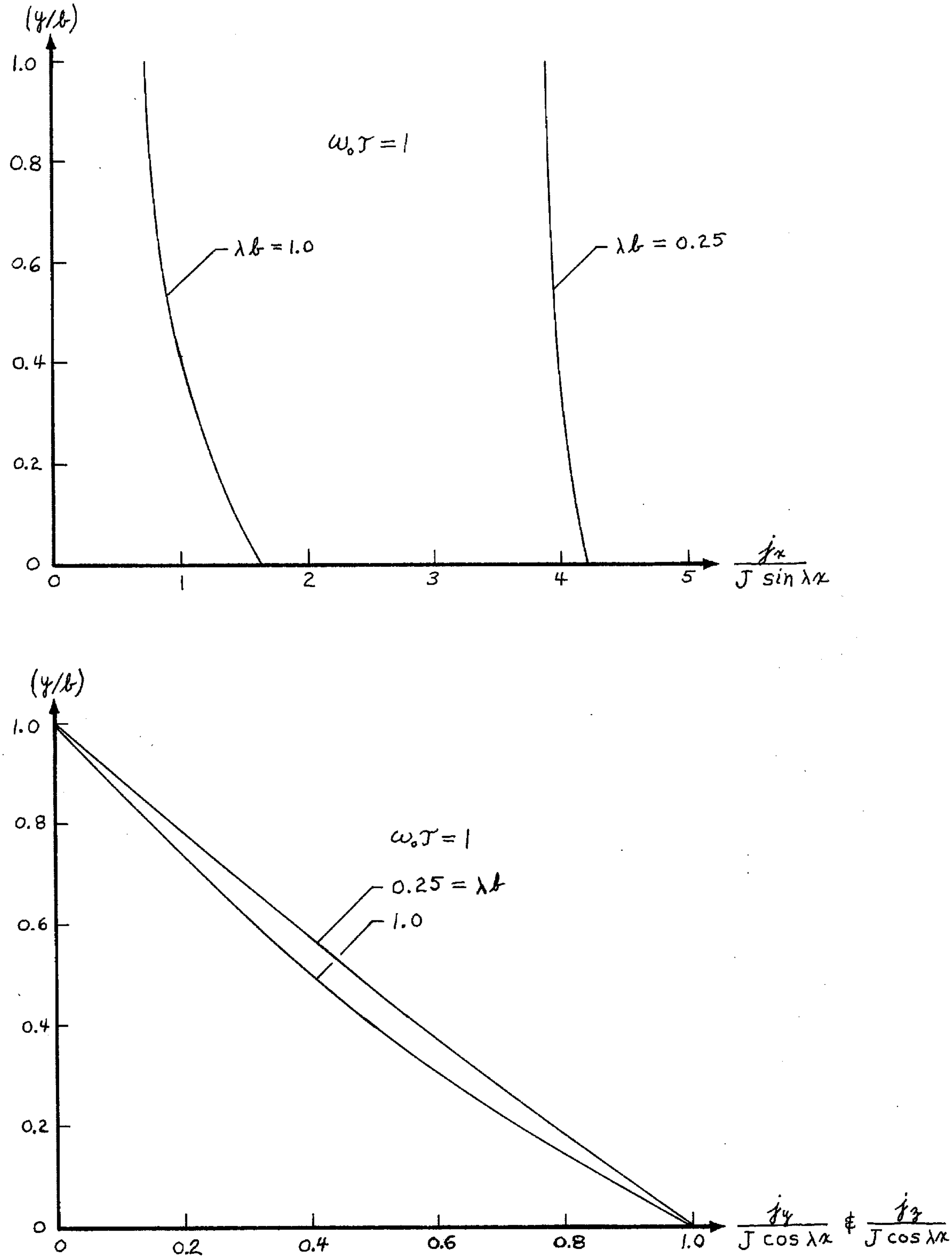


FIGURE 15

CURRENT COMPONENTS IN THE CHANNEL FLOW CASE
FOR SMALL MAGNETIC REYNOLD'S NUMBER

$\omega_0 \tau = 1$

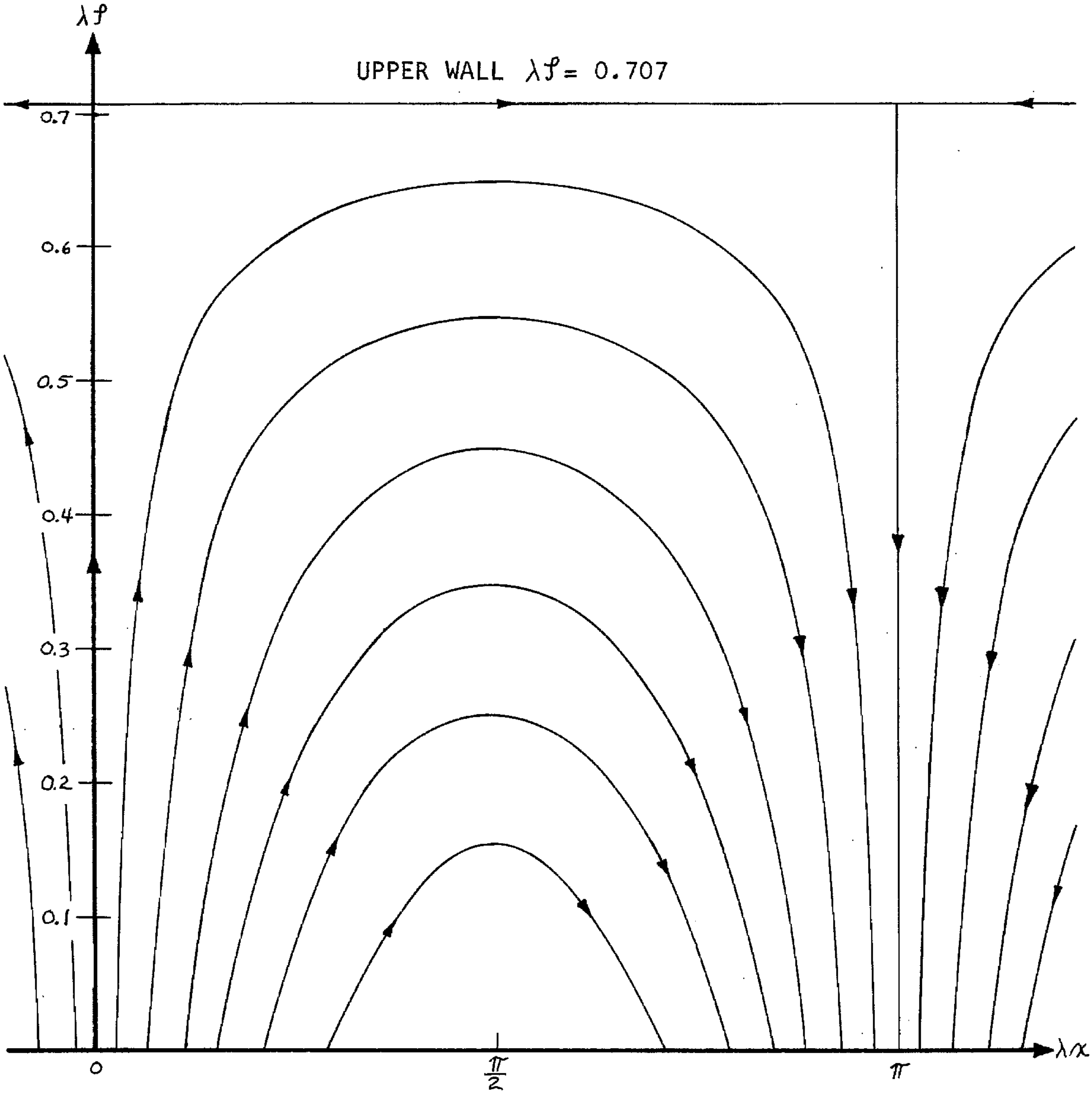


FIGURE 16

CURRENT LINES IN THE CHANNEL FLOW CASE
FOR SMALL MAGNETIC REYNOLD'S NUMBER

seen that for λb of unity, the components do not approach very closely the values of the semi-infinite flow case as shown in Figures 3, 4, and 5. In actuality, the influence of the insulating wall is appreciable until the walls are far enough apart that the perturbations die out before reaching the upper wall. This occurs at a value of λb of approximately three as can be seen in Figures 3, 4, and 5. Thus, the influence of the insulating wall extends to values of λb of approximately three even though the coefficients shown in Figure 12 approach their limiting values at about unity. The reason this occurs is due to the fact that in the component equations the coefficients, which decay exponentially with λb, are multiplied by exponentials in λy; some of which can grow rather than decay. These growing exponentials tend to retard the decay of the component with λy.

At the other end of the scale, Figure 12 shows that all the coefficients approach infinity as λb approaches zero. Since the approach to this problem has been that of a small perturbation analysis, the distance between the walls cannot be so small that the perturbed quantities become appreciable in comparison to the unperturbed values. It appears that for moderate values of $\omega_0 \tau$, the lower limit on the distance between the walls is a value of λb of approximately one tenth.

Several interesting features are produced by the presence of the insulating wall. First, as the channel height decreases and the effect of the insulating wall becomes appreciable, the x and y-components of the velocity steadily decrease in magnitude. In fact, the y-component effectively disappears for λb less than unity. The z-component, on the other hand, changes from an exponential decay when the channel height is large to a very nearly linear velocity profile across the height of the channel when λb is less than unity. These features are easily seen in Figure 14. Generally speaking, the effect of the insulating wall on the magnetic field perturbations, as shown in Figure 13, is the same as those just described for the velocity field perturbations.

Perhaps the most interesting features produced by the presence of the insulating wall occur in the currents flowing in the fluid. Decreasing the channel height causes the y and z-components to change from an exponential decay, corresponding to no effect produced by the insulating wall (as in the semi-infinite flow case shown in Figure 5, page 32), to a very nearly linear decay across the channel height when λb is less than unity (see Figure 15). In addition, two significant effects occur simultaneously in the x-component of the current as the channel height becomes small. First, the strong exponential decay present when the insulating wall is far away changes to a very moderate decay when λb approaches unity and, as λb decreases further, the magnitude of the x-component becomes virtually constant across the height of the channel. Second, the magnitude of x-component of the current density increases rapidly as λb decreases below a value of about one. This increase in magnitude is inversely proportional to the channel height since the total current flowing through any y-z plane must be conserved.

B. LARGE MAGNETIC REYNOLD'S NUMBER APPROXIMATION

It will be assumed that, in the limit of large magnetic Reynold's number, the inequality

$$\left|\frac{k}{\lambda}\right| \gg 1 \tag{5.41}$$

is valid and, in addition, that $\omega_0\tau$ is of the order of unity. This inequality will now be used to reduce the dimensionless equations (5.4) and (5.5) to the forms appropriate for this limiting case.

Since $\frac{\partial h_z^*}{\partial x^*}$ and $\frac{\partial^2 h_z^*}{\partial x^{*2}}$ are of the same order of magnitude, the first term on the left of equation (5.4) can be neglected in comparison with the first term on the right by virtue of the inequality (5.41). Similarly, because $\frac{\partial \xi_z^*}{\partial x^*}$ and $\frac{\partial^2 \xi_z^*}{\partial x^{*2}}$ are of the same order, the first term on the left of equation (5.5) can be neglected in comparison with the first term on the right. Substituting the resulting expression for $\frac{\partial \xi_z^*}{\partial x^*}$ obtained from equation (5.4) into the reduced form of equation (5.5) results in

$$\frac{\partial^2 \xi_z^*}{\partial y^{*2}} + \omega_0\tau\frac{\partial}{\partial x^*}\left[\frac{\partial^2 h_z^*}{\partial x^{*2}} + \left(\frac{k}{\lambda\omega_0\tau}\right)^2 h_z\right]$$
$$= -\omega_0\tau\frac{\partial^2}{\partial y^{*2}}\left[\frac{\partial h_z^*}{\partial x^*} - \left(\frac{k}{\lambda\omega_0^2\tau^2}\right)h_z^*\right] \tag{5.42}$$

Now, since $\omega_0\tau$ is of order one and h_z^*, $\frac{\partial h_z^*}{\partial x^*}$, and $\frac{\partial^2 h_z^*}{\partial x^{*2}}$ are all of the same order, the first term inside both square brackets can be neglected in comparison with the second term. Thus, in the limit of large magnetic Reynold's number the governing equations, in dimensional form, become

$$\left(\frac{\partial^2}{\partial y^2} - k\frac{\partial}{\partial x}\right) h_z = \omega_0 \tau \frac{\partial \xi_z}{\partial x} \tag{5.43}$$

$$\left(\frac{\partial^2}{\partial y^2} - k\frac{\partial}{\partial x}\right) \xi_z = 0 \tag{5.44}$$

Equation (5.44) is easily solved by separation of variables. The result is

$$\xi_z = e^{i\lambda x}\left[G_1 e^{-(1\pm i)\lambda a y} + G_2 e^{(1\pm i)\lambda a y}\right] \tag{5.45}$$

where G_1 and G_2 are complex constants to be determined and a is as previously defined by equation (4.59).

$$a \equiv \left(\frac{|k|}{2\lambda}\right)^{1/2} = \left(\frac{1}{2} R_m \left|\frac{m^2-1}{m^2}\right|\right)^{1/2} \tag{4.59}$$

The positive sign in the symbol $(1 \pm i)$ is to be taken when $k>0$ (superalfvénic flow) and the negative sign when $k<0$ (subalfvénic flow).

Substituting the expression for ξ_z given by equation (5.45) into equation (5.43) produces the following equation for h_z.

$$\frac{\partial^2 h_z}{\partial y^2} - k\frac{\partial h_z}{\partial x} = i\lambda\omega_0\tau e^{i\lambda x}\left[G_1 e^{-(1\pm i)\lambda a y} + G_2 e^{(1\pm i)\lambda a y}\right] \tag{5.46}$$

The homogeneous solution to this equation has the same form as equation (5.45)

$$h_z = e^{i\lambda x}\left[G_3 e^{-(1\pm i)\lambda a y} + G_4 e^{(1\pm i)\lambda a y} \right] \tag{5.47}$$

For the particular solution, assume h_z is of the form

$$h_z = e^{i\lambda x} f(y) \tag{5.48}$$

where f(y) is a function of y only.

Substituting this form of h_z into equation (5.46) results in the following total differential equation for f(y).

$$\begin{aligned} &\frac{d^2 f}{dy^2} - i\lambda k f \\ &\qquad = i\lambda \omega_o \tau \left[G_1 e^{-(1\pm i)\lambda a y} + G_2 e^{(1\pm i)\lambda a y} \right] \end{aligned} \tag{5.49}$$

The particular solution to this equation is easily found to be

$$f(y) = -\frac{i\omega_o \tau y}{2(1\pm i)a}\left[G_1 e^{-(1\pm i)\lambda a y} - G_2 e^{(1\pm i)\lambda a y} \right] \tag{5.50}$$

Thus, combining the homogeneous and particular solutions, the z-component of the magnetic field perturbation is found to be

$$\begin{aligned} h_z = e^{i\lambda x}\Big[&(G_3 - M G_1 y)\, e^{-(1\pm i)\lambda a y} \\ &+ (G_4 + M G_2 y)\, e^{(1\pm i)\lambda a y} \Big] \end{aligned} \tag{5.51}$$

where the complex constant M is defined as

$$M \equiv \frac{i\omega_o \tau}{2(1\pm i)a} \tag{5.52}$$

In order to determine the x and y-components of the magnetic field, a stream type function Ψ will be used once again. This function automatically satisfies Maxwell's equation $\nabla\cdot\vec{H}=0$ and is defined by

$$h_x = -\frac{\partial \Psi}{\partial y}$$
$$h_y = \frac{\partial \Psi}{\partial x} \tag{5.53}$$

By definition

$$\xi_z = \frac{\partial h_y}{\partial x} - \frac{\partial h_x}{\partial y} \tag{5.54}$$

Introducing equations (5.45) and (5.53) into equation (5.54) results in the following equation for the stream function Ψ.

$$\nabla^2 \Psi = e^{i\lambda x}\left[G_1 e^{-(1\pm i)\lambda a y} + G_2 e^{(1\pm i)\lambda a y}\right] \tag{5.55}$$

The homogeneous solution of this equation is easily found to be

$$\Psi = e^{i\lambda x}\left(G_5' e^{-\lambda y} + G_6' e^{\lambda y}\right) \tag{5.56}$$

The particular solution will be assumed to be of the form

$$\Psi = e^{i\lambda x} g(y) \tag{5.57}$$

where g (y) is a function of y only. Substitution of equation (5.57) into equation (5.55) results in the following total differential equation for g(y).

$$\frac{d^2 g}{dy^2} - \lambda^2 g = G_1 e^{-(1\pm i)\lambda a y} + G_2 e^{(1\pm i)\lambda a y} \tag{5.58}$$

This equation is easily solved to give

$$g(y) = \frac{1}{(1\pm i)^2 \lambda^2 a^2} \left(G_1 e^{-(1\pm i)\lambda a y} + G_2 e^{(1\pm i)\lambda a y} \right) \tag{5.59}$$

where the fact that $a^2 \gg 1$ has been used. Thus, the stream function Ψ, upon combination of the homogeneous and particular solutions, is

$$\Psi = \frac{1}{(1\pm i)^2 \lambda^2 a^2} e^{i\lambda x} \Big[G_1 e^{-(1\pm i)\lambda a y} + G_2 e^{(1\pm i)\lambda a y} + G_5 e^{-\lambda y} + G_6 e^{\lambda y} \Big] \tag{5.60}$$

where the primed constants of equation (5.56) have been replaced by equivalent unprimed constants for convenience.

Using equations (5.53), the magnetic field perturbations in the limit of large magnetic Reynold's number become

$$h_x = \frac{1}{(1\pm i)^2 \lambda a^2} e^{i\lambda x} \Big[(1\pm i) a G_1 e^{-(1\pm i)\lambda a y} - (1\pm i) a G_2 e^{(1\pm i)\lambda a y} + G_5 e^{-\lambda y} - G_6 e^{\lambda y} \Big] \tag{5.61a}$$

$$h_y = \frac{i}{(1\pm i)^2 \lambda a^2} e^{i\lambda x} \Big[G_1 e^{-(1\pm i)\lambda a y} + G_2 e^{(1\pm i)\lambda a y} + G_5 e^{-\lambda y} + G_6 e^{\lambda y} \Big] \tag{5.61b}$$

$$h_z = \frac{1}{2(1\pm i)a} e^{i\lambda x}\Big\{\big[2(1\pm i)aG_3 - i\omega_0 \tau G_1 y\big] e^{-(1\pm i)\lambda a y} \qquad (5.61c)$$

$$+ \big[2(1\pm i)aG_4 + i\omega_0 \tau G_2 y\big] e^{(1\pm i)\lambda a y}\Big\}$$

The magnetic field perturbations in the lower wall have the same form as in the previous cases considered, which is

$$h_x = i B_3 e^{i\lambda x + \lambda y}$$

$$h_y = B_3 e^{i\lambda x + \lambda y} \qquad (5.62)$$

$$h_z = \frac{4\pi i J}{\lambda} e^{i\lambda x}$$

As in the small magnetic Reynold's number case, the magnetic field perturbations in the upper wall are

$$h_x = B_4 e^{i\lambda x - \lambda(y-b)}$$

$$h_y = i B_4 e^{i\lambda x - \lambda(y-b)} \qquad (5.63)$$

$$h_z = 0$$

Since the magnetic field components are continuous across both boundaries by boundary condition (iii), equations (5.61) can be equated to equations (5.62) at the lower boundary and to equations (5.63) at the upper boundary. The resulting relations between coefficients are

$$G_3 = \frac{4\pi i J}{\lambda}\left(\frac{R_1}{R_1 - R_2}\right) - \frac{i\omega_0 \tau b}{2(1\pm i)a}\left(\frac{G_1 R_2 - G_2 R_1}{R_1 - R_2}\right)$$

$$G_4 = -\frac{4\pi i J}{\lambda}\left(\frac{R_2}{R_1 - R_2}\right) + \frac{i\omega_0 \tau b}{2(1\pm i)a}\left(\frac{G_1 R_2 - G_2 R_1}{R_1 - R_2}\right) \tag{5.64}$$

$$G_5 = \frac{1}{2}(1\pm i)a\,(G_2 - G_1)$$

$$G_6 = \frac{1}{2}(1\pm i)a\, e^{-\lambda b}\left(\frac{G_1 R_2 - G_2 R_1}{R_1 - R_2}\right)$$

where, for convenience, the complex constants R_1 and R_2, which are defined as follows, have been introduced.

$$R_1 \equiv e^{(1\pm i)\lambda a b}$$

$$R_2 \equiv e^{-(1\pm i)\lambda a b} \tag{5.65}$$

The remaining undetermined constants G_1 and G_2 are found by solving for the velocity field perturbations using equation (3.13) and introducing the magnetic and velocity field expressions into the form of Ohm's law appropriate for the large magnetic Reynold's number approximation. The coefficients of like exponentials are then equated and the undetermined constants G_1 and G_2 obtained.

Since the function ϕ in the defining equation for the velocity perturbations, equation (3.13), satisfies Laplace's equation, it has the form

$$\phi = e^{i\lambda x}\left(D_3 e^{-\lambda y} + D_4 e^{\lambda y}\right) \tag{5.66}$$

Thus, the y-component of the velocity becomes

$$\begin{aligned} u_y = \frac{i\, U_o\, e^{i\lambda x}}{m^2 H_o (1\pm i)^2 \lambda a^2}\Big[& G_1 e^{-(1\pm i)\lambda a y} + G_2 e^{(1\pm i)\lambda a y} \\ & + \left(G_5 + i m^2 H_o D_3\right) e^{-\lambda y} \\ & + \left(G_6 - i m^2 H_o D_4\right) e^{\lambda y} \Big] \end{aligned} \tag{5.67}$$

Equating u_y to zero at both boundaries according to boundary condition (ii) results in the following expressions for D_3 and D_4.

$$\begin{aligned} D_3 &= \frac{i}{2m^2 H_o (1\pm i)^2 \lambda^2 a^2}\left[\left(\frac{e^{\lambda b} - R_2}{\sinh \lambda b}\right) G_1 + \left(\frac{e^{\lambda b} - R_1}{\sinh \lambda b}\right) G_2 + 2G_5\right] \\ D_4 &= \frac{i}{2m^2 H_o (1\pm i)^2 \lambda^2 a^2}\left[\left(\frac{e^{-\lambda b} - R_2}{\sinh \lambda b}\right) G_1 + \left(\frac{e^{-\lambda b} - R_1}{\sinh \lambda b}\right) G_2 - 2G_6\right] \end{aligned} \tag{5.68}$$

In the limit of large magnetic Reynold's number, the only simplification that can be made in the dimensionless form of Ohm's law (5.3) is to neglect the first term on the right hand side of the equation in comparison to the first term on the left hand side. Making this simplification, the dimensional form of Ohm's law becomes

$$k\frac{\partial \vec{h}}{\partial x} - \left(\frac{m^2}{m^2-1}\right) k \frac{\partial}{\partial x} \nabla \phi = \frac{\partial^2 \vec{h}}{\partial y^2} - \omega_0 r \frac{\partial \vec{\xi}}{\partial x} \tag{5.69}$$

Substitution of the expressions for the magnetic field perturbations into the z-component of the above equation and equating coefficients of like exponentials leads to the result that

$$\begin{aligned} G_1 &= G_2 = 0 \\ G_3 &= \frac{4\pi i J}{\lambda}\left(\frac{R_1}{R_1 - R_2}\right) \\ G_4 &= -\frac{4\pi i J}{\lambda}\left(\frac{R_2}{R_1 - R_2}\right) \\ G_5 &= G_6 = 0 \\ D_3 &= D_4 = 0 \end{aligned} \tag{5.70}$$

where it has been assumed that

$$\frac{\lambda y}{a} \ll 1 \tag{5.71}$$

Thus, the channel flow magnetic field perturbations in the limit of large magnetic Reynold's number are

$$h_x = 0$$

$$h_y = 0 \tag{5.72}$$

$$h_z = \frac{4\pi i J}{\lambda}\, e^{i\lambda x}\left[\left(\frac{R_1}{R_1-R_2}\right) e^{-(1\pm i)\lambda a y} - \left(\frac{R_2}{R_1-R_2}\right) e^{(1\pm i)\lambda a y}\right]$$

while the velocity field perturbations are

$$u_x = 0$$

$$u_y = 0 \tag{5.73}$$

$$u_z = \frac{4\pi i J U_0}{m^2 H_0 \lambda}\, e^{i\lambda x}\left[\left(\frac{R_1}{R_1-R_2}\right) e^{-(1\pm i)\lambda a y} - \left(\frac{R_2}{R_1-R_2}\right) e^{(1\pm i)\lambda a y}\right]$$

The current flowing in the fluid is determined from Maxwell's equation $\nabla \times \vec{H} = 4\pi \vec{j}$. The current components are

$$j_x = -i(1\pm i) a J\, e^{i\lambda x}\left[\left(\frac{R_1}{R_1-R_2}\right) e^{-(1\pm i)\lambda a y} + \left(\frac{R_2}{R_1-R_2}\right) e^{(1\pm i)\lambda a y}\right]$$

$$j_y = J\, e^{i\lambda x}\left[\left(\frac{R_1}{R_1-R_2}\right) e^{-(1\pm i)\lambda a y} - \left(\frac{R_2}{R_1-R_2}\right) e^{(1\pm i)\lambda a y}\right] \tag{5.74}$$

$$j_z = 0$$

The real part of the complex coefficients in the above equations will be denoted by the following expressions.

$$K_5 \equiv Re\left(\frac{R_1}{R_1-R_2}\right) = \frac{1}{1-e^{-2\lambda a b}}$$
$$K_6 \equiv Re\left(\frac{-R_2}{R_1-R_2}\right) = \frac{1}{1-e^{2\lambda a b}} \tag{5.75}$$

Taking the real part of equations (5.72), (5.73), and (5.74), the vector form of the magnetic and velocity perturbations and the currents flowing in the fluid, for $k > 0$, become

$$\vec{h} = -\frac{4\pi J}{\lambda}\left(K_5 e^{-\lambda a y} + K_6 e^{\lambda a y}\right) \sin\lambda(x-ay)\ \vec{i}_z$$
$$\vec{u} = -\frac{4\pi J U_0}{m^2 H_0 \lambda}\left(K_5 e^{-\lambda a y} + K_6 e^{\lambda a y}\right) \sin\lambda(x-ay)\ \vec{i}_z$$
$$\vec{j} = J\Big\{a\left[K_5 e^{-\lambda a y} - K_6 e^{\lambda a y}\right]\left[\cos\lambda(x-ay) + \sin\lambda(x-ay)\right]\vec{i}_x \tag{5.76}$$
$$+\left(K_5 e^{-\lambda a y} + K_6 e^{\lambda a y}\right)\cos\lambda(x-ay)\ \vec{i}_y\Big\}$$

and, for $k < 0$, are

$$\vec{h} = -\frac{4\pi J}{\lambda}\left(K_5 e^{-\lambda a y} + K_6 e^{\lambda a y}\right) \sin\lambda(x+ay)\ \vec{i}_z$$
$$\vec{u} = -\frac{4\pi J U_0}{m^2 H_0 \lambda}\left(K_5 e^{-\lambda a y} + K_6 e^{\lambda a y}\right) \sin\lambda(x+ay)\ \vec{i}_z \tag{5.77}$$
$$\vec{j} = J\Big\{a\left[K_5 e^{-\lambda a y} - K_6 e^{\lambda a y}\right]\left[\sin\lambda(x+ay) - \cos\lambda(x+ay)\right]\vec{i}_x$$
$$+\left(K_5 e^{-\lambda a y} + K_6 e^{\lambda a y}\right)\cos\lambda(x+ay)\ \vec{i}_y\Big\}$$

The variation of the coefficients K_5 and K_6 with λab is shown in Figure 17. It is easily seen that as λab becomes large, the coefficients asymptotically approach the values required for the channel flow solution to reduce to the semi-infinite flow solution; namely, K_5 approaches unity and K_6 approaches zero. Thus, as required, the channel flow solution obtained coincides with the semi-infinite flow solution as the upper wall moves to infinity.

Since the ordinate of Figure 17 is magnified by the factor a, the actual channel height required for the influence of the upper wall to be appreciable is quite small. Once again, the solution in the limit of large magnetic Reynold's number is independent of the anisotropy of the fluid conductivity but does depend strongly on the actual magnitude of the magnetic Reynold's number.

Figures 18, 19, and 20 show the variation of the magnetic field, velocity field, and current components as a function of (y/b) for two values of λab (0.4 and 1.0). Note that the abscissa of Figure 19 is reduced by a factor a. Generally speaking, most of the characteristics of the semi-infinite flow case are also applicable to the channel flow case being considered here.

Inspection of Figures 18 and 20 show that the z-components of the magnetic field and velocity and the y-component of the current are not appreciably affected by varying the channel height. The x-component of the current, however, is significantly affected by the variation of the channel height as can be seen in Figure 19. Its behavior is very similar to that of the small magnetic Reynold's

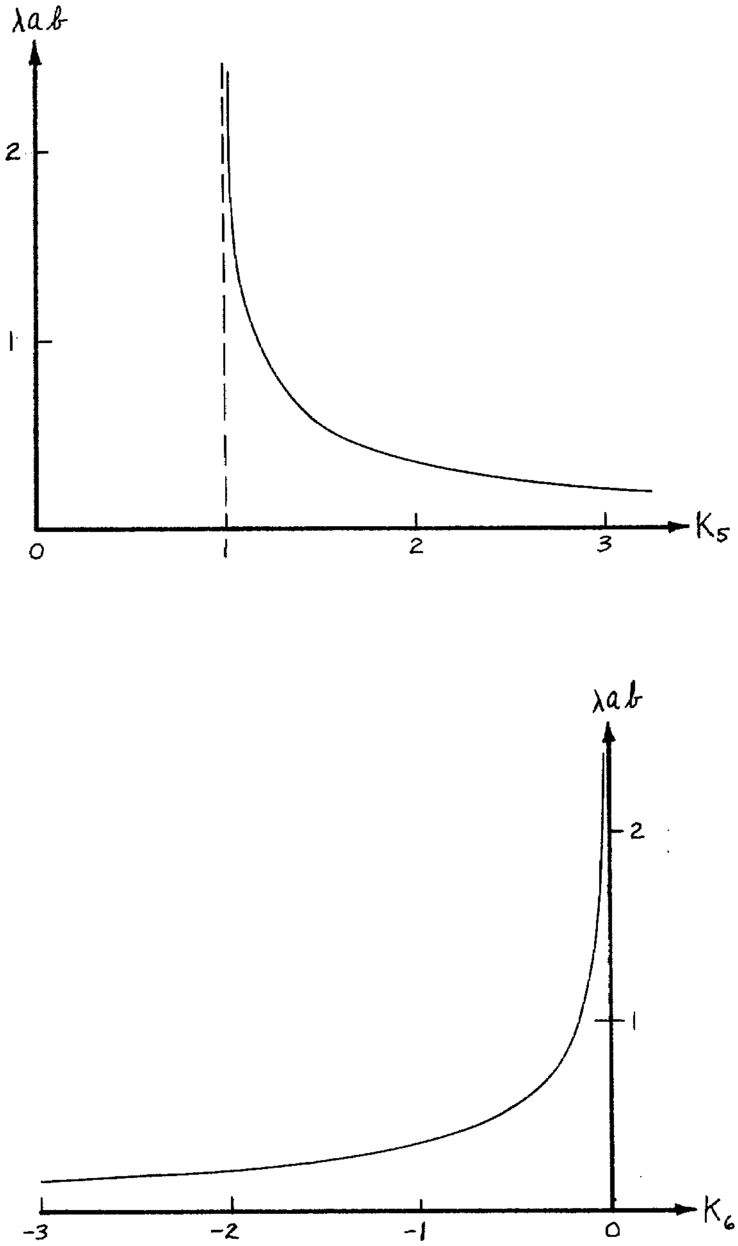


FIGURE 17

CHANNEL FLOW COEFFICIENTS

FOR LARGE MAGNETIC REYNOLD'S NUMBER

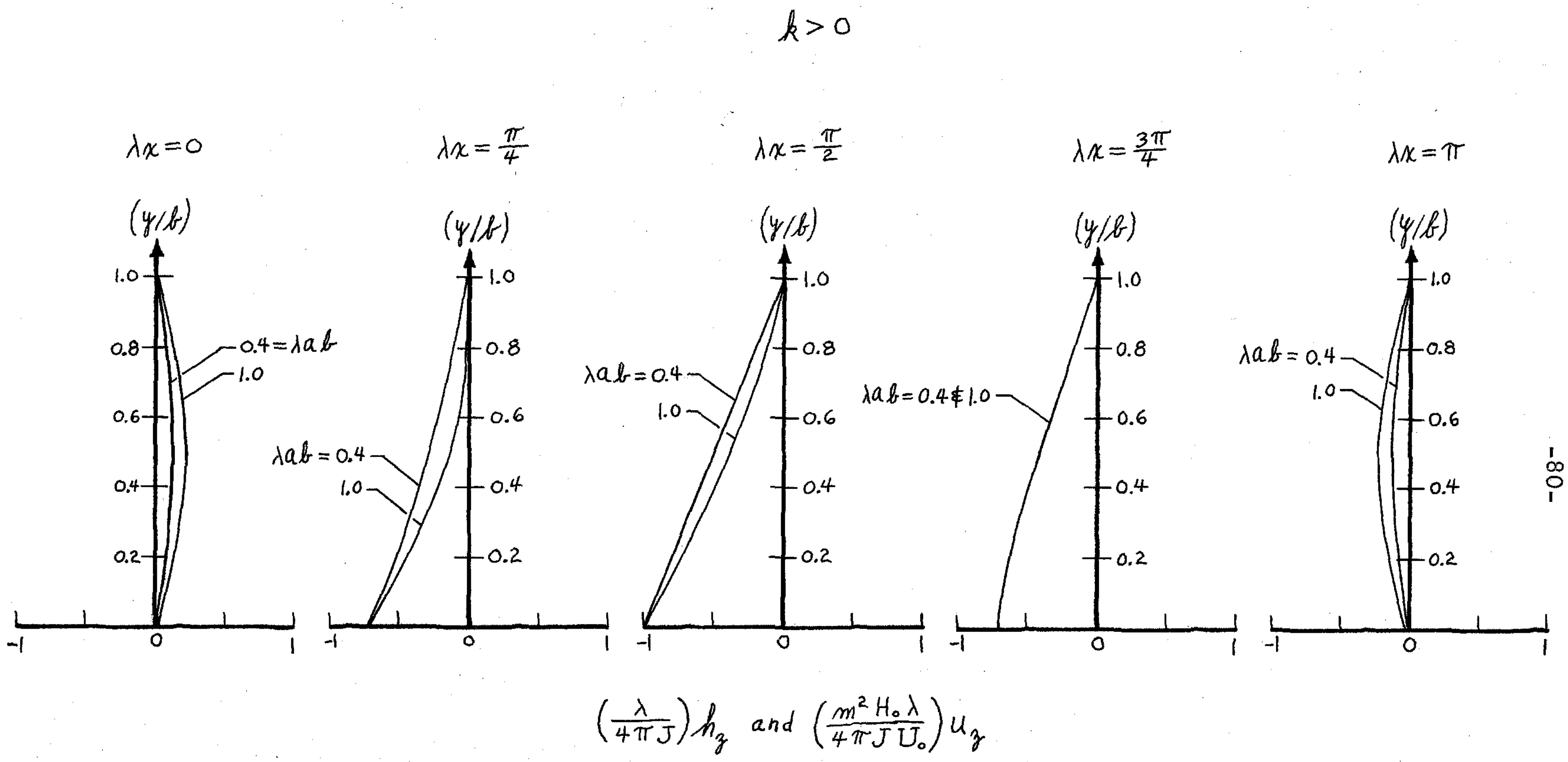


FIGURE 18

MAGNETIC FIELD AND VELOCITY COMPONENTS IN THE CHANNEL FLOW CASE FOR LARGE MAGNETIC REYNOLD'S NUMBER

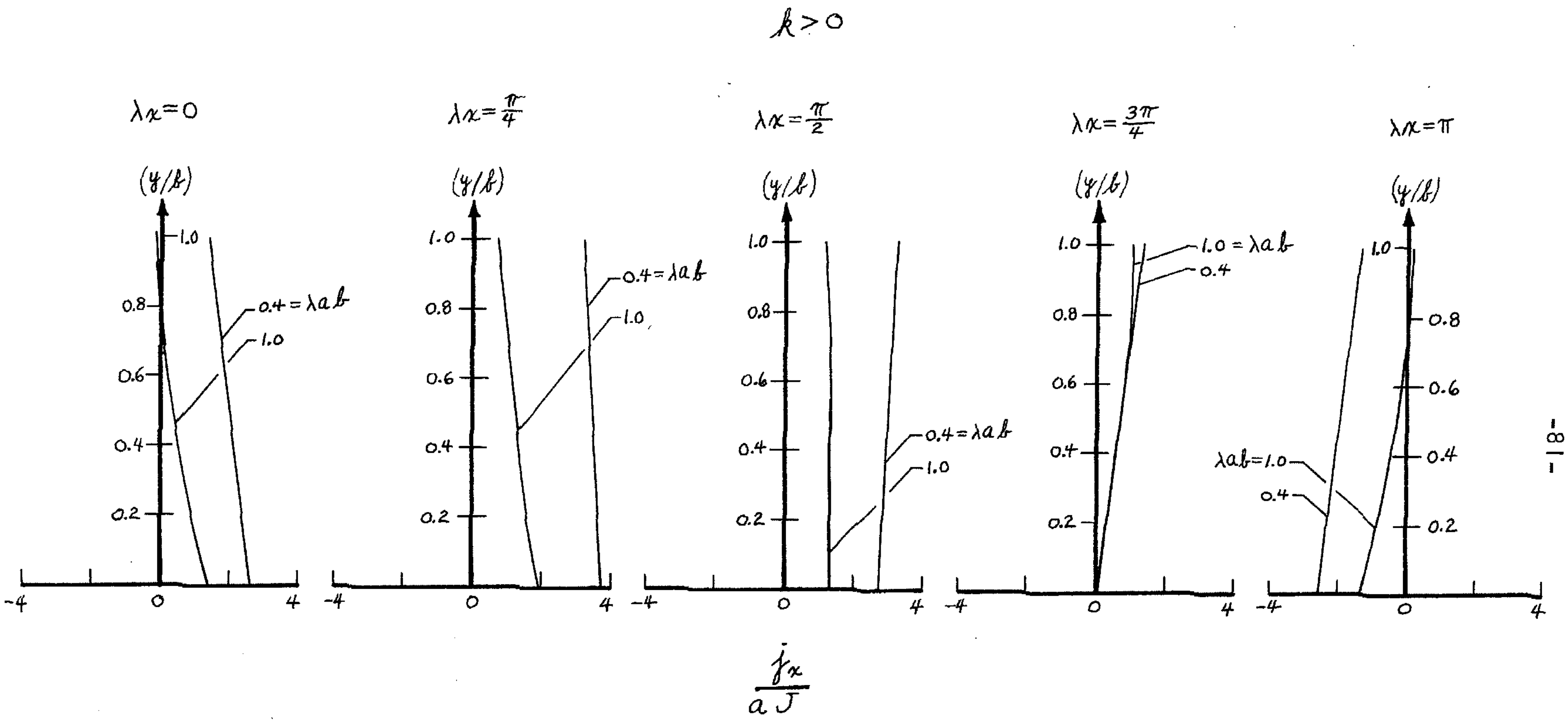


FIGURE 19

THE X-CURRENT COMPONENT IN THE CHANNEL FLOW CASE
FOR LARGE MAGNETIC REYNOLD'S NUMBER

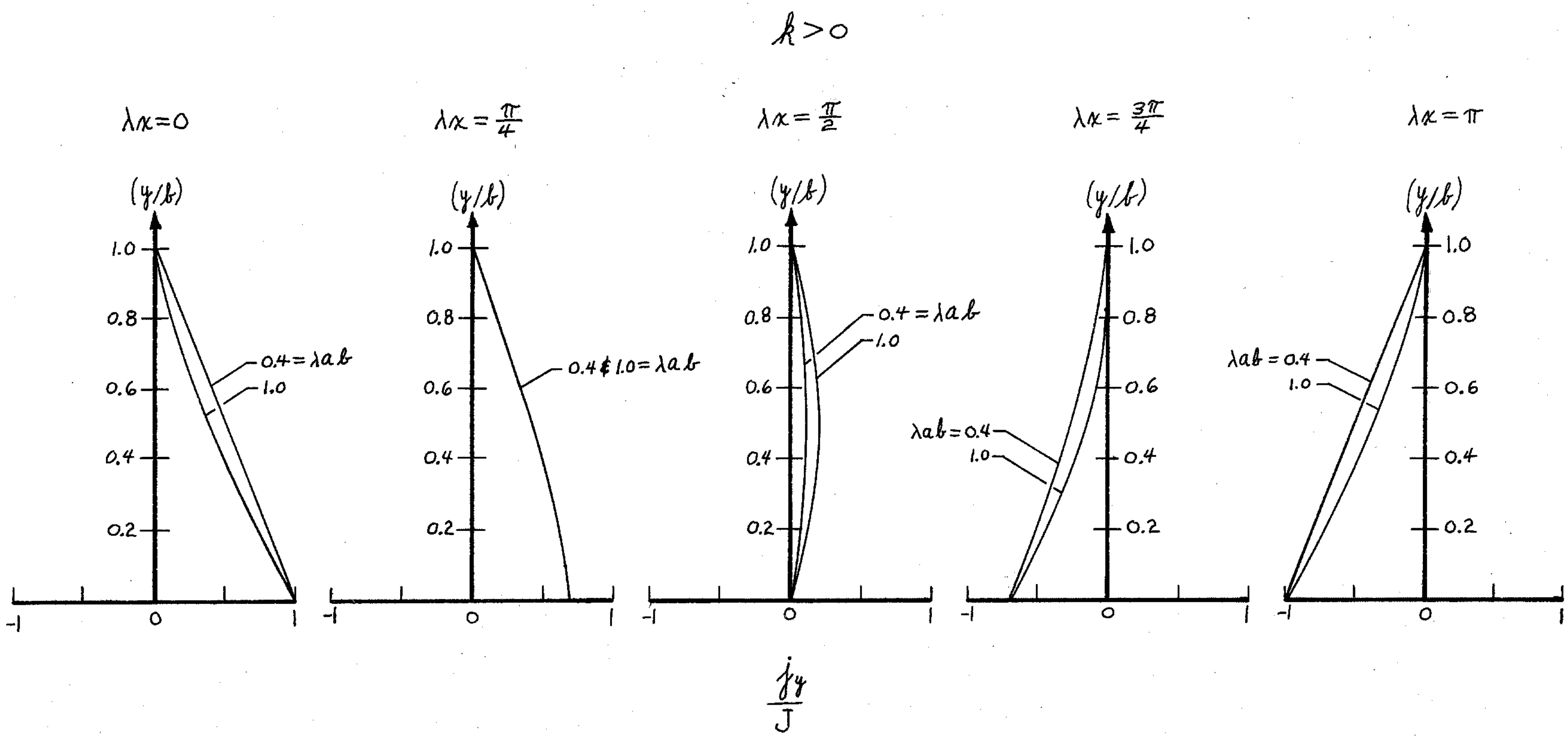


FIGURE 20

THE Y-CURRENT COMPONENT IN THE CHANNEL FLOW CASE
FOR LARGE MAGNETIC REYNOLD'S NUMBER

number case just considered. The strong exponential decay present when the insulating wall is far away changes to a very moderate, almost linear decrease as λab approaches unity and, as λab decreases further, the magnitude of the x-component of the current density becomes very nearly constant across the height of the channel.

VI. CONCLUDING REMARKS

In the limit of small magnetic Reynold's number for both the semi-infinite flow and channel flow cases, the magnetic and velocity field vectors are composed of an irrotational and a rotational part. The irrotational portion always remains in the x-y plane. However, the rotational portion and, hence, the currents lie in a plane which is rotated about the x-axis; the angle between this plane and the x-y plane being strongly dependent upon the degree of anisotropy in the fluid's electrical conductivity. This rotational effect could be used to determine the degree of anisotropy of a conducting fluid experimentally.

Anisotropic effects on the magnitude of the magnetic and velocity field components and the currents are generally moderate except near the conducting wall. At this wall the x and z-components of the current can become quite large for strong anisotropic conductivity. In the channel flow case, the x-component of the current becomes large as the channel height becomes small for all degrees of anisotropy. In both the semi-infinite and channel flow cases, the currents in the fluid form symmetric loops closing at the conducting boundary.

Both the irrotational and rotational portions of the velocity field vector behave in a manner analogous to ordinary incompressible flow with the applied sinusoidal boundary current in the flat wall replaced by a solid sinusoidal wall. Since the hydrodynamic effects of ordinary fluid flow over a wavy wall are well known, the hydrodynamic effects of a conducting fluid over a flat wall with sinusoidal boundary currents are also well known by analogy.

In the limit of large magnetic Reynold's number for both the semi-infinite flow and channel flow cases, anisotropic effects are absent to the order of the inverse square root of the magnetic Reynold's number. In addition, the currents and field perturbations are confined to a thin magnetic boundary layer near the conducting wall. The currents lie entirely in the x-y plane and again form loops closing at the conducting boundary, but are steeply inclined toward the x-axis. At the conducting wall, the x-component of the current in the fluid is larger than the applied boundary current by a factor of the square root of the magnetic Reynold's number. Thus, in the limit of large magnetic Reynold's number, the magnetohydrodynamic flow considered acts as a current amplifier; a small boundary current controls a large current in the fluid.

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