I. THE ANALYSIS OF THE REWETTING OF A VERTICAL SLAB USING A WIENER-HOPF TECHNIQUE

II. ASYMPTOTIC EXPANSIONS OF INTEGRALS WITH THREE COALESCING SADDLE POINTS

Thesis by

James R. Mueller

In Partial Fulfillment of the Requirements
for the Degree of
Doctor of Philosophy

California Institute of Technology
Pasadena, California
1982

ACKNOWLEDGEMENTS

The acknowledgement of support and gratitude in a dissertation is perhaps the most onerous portion to write. Over the many years of my education, it is difficult to collectively identify those individuals and events which have lead me down the path I have chosen. Certain individuals stand out clearly in the forefront, though.

First and foremost, I wish to express my deepest thanks to my advisor, Professor Donald S. Cohen, for his invaluable help, strong encouragement, and infinite patience (yes, *infinite*) in the development of this thesis.

I wish also to thank my fellow graduate students and friends at Caltech for many stimulating discussions both academic and nonacademic. My research at the Institute was supported in the form of Graduate Teaching Assistantships and Graduate Research Assistantships. Special thanks also go to Vivian Davies for a superb typing job.

I would also like to express my gratitude to my family for their continuing support and encouragement during my studies. Finally, I wish to dedicate this thesis to my aunt and uncle, Mr. and Mrs. H. E. Ott. Without their love and kindness, this thesis would never have been possible.

ABSTRACT

Part I

A problem which arises in the analysis of the emergency core cooling system of a nuclear reactor is the rewetting of the fuel rods following a loss of coolant accident. Due to the high initial temperatures in the rods, the emergency coolant initially flashes to steam on contact, effectively insulating the rods from the coolant. It is observed experimentally, however, that a constant velocity traveling quench front is set up on the surface of the rod, moving from the cold to the hot end.

We approximate the rod by an infinite two-dimensional slab with adiabatic boundary conditions ahead of the quench front, and a constant heat transfer coefficient behind in the wet region. The temperature at the front is found using Fourier transforms and an exact Wierner-Hopf Factorization. Using a reversion of series, the dimensionless velocity of the quench front (Peclet number) for a small dimensionless heat transfer coefficient A (Biot number) is found approximately to lowest order in A. This approximate quench front velocity is found to be in agreement with the known front velocity for a one-dimensional slab.

Part II

Contour integrals of the form $I(\lambda;\underline{\alpha}) = \int_{C_z} g(z)\exp[\lambda w(z;\underline{\alpha})]dz$ are considered for a large parameter λ. In problems of interest, the exponent w is assumed to have simple saddle points at $z = \alpha_i$, i=1,2,3 which are allowed to coalesce, forming a single saddle of order three. Using a conformal map, the integral I is shown to be asymptotically equivalent to the study of the canonical integral $J(\lambda;\gamma) = \int_{C_t} \exp\left[\lambda\left(\frac{-t^4}{4} + \frac{\gamma^2}{2}t^2\right)\right]dt$, which has simple saddles at $t = 0, \pm \gamma$. By applying the method of steepest descent, the complete asymptotic behavior of $J(\lambda;\gamma)$ is obtained for $\lambda \to \infty$, uniformly as $\gamma \to 0$.

TABLE OF CONTENTS

PART I

THE ANALYSIS OF THE REWETTING OF A VERTICAL SLAB USING A WIENER-HOPF TECHNIQUE

I. INTRODUCTION

During the past few years, especially since the Three Mile Island accident, the design safety of nuclear power plants has come under increasing scrutiny. A nuclear reactor is a very complex device, at the heart of which is the core assembly containing the fuel elements. In round numbers, a typical reactor core contains approximately 100 tons of enriched uranium oxide, arranged in tens of thousands of thin cylindrical fuel rods. This core is enclosed in a steel pressure vessel a few meters in diameter and about 10 meters high, with walls about 30 centimeters thick. It is cooled by two or more independent water loops, which transfer the heat to steam turbines. The approximate cost of the entire system is $1 billion.

One of the technical problems associated with the safety of the nuclear fission reactors is that of modeling the effectiveness of the emergency core cooling system (ECCS). After a loss of coolant accident (LOCA), the temperature in the cladding of the fuel rod elements in the core increases due to the decay heat from fission products and heat energy stored in the rods prior to the accident. This decay heat is far from negligible [1]. Approximately 7% of the total energy output of an operating reactor is from fission product decay. Thus, in a typical reactor operating at a total power level of 3000 MW, 210 MW come from fission decay. Following a shutdown, the decay heat drops very rapidly: after one minute it drops from 210 MW to 120 MW; after one hour it is down to 30 MW, though still large by any standard. Unless this substantial "afterheat" is removed, the cladding material (a zirconium alloy) will begin to oxidize; ultimately the core itself will melt, with the possible consequence of releasing radioactive products into the environment.

Fortunately, this "China Syndrome" scenario can be precluded. Following the LOCA, water is forced into the core by the ECCS either by spraying from the top (for boiling water reactors) or via reflooding from the bottom (for pressurized water reactors). Cooling an overheated fuel rod requires that the liquid water be in contact with the surface of the rod. However, as a result of the high temperatures attained in the cladding prior to the arrival of the emergency coolant, the water does not initially wet the surface of the clad. Instead, it instantly flashes to steam, forming a thin film on the surface of the cladding. This Leidenfrost layer effectively prevents cooling of the fuel rod elements due to the poor heat conductivity of steam.

The surface of the cladding does cool, however, though the mechanisms are not entirely understood at present. It has been proposed in the engineering literature [2], [3], based on some empirical results, that the flow of heat along the temperature gradient from the hot part to the cooler part of the surface provides enough cooling to establish a traveling quench front moving from the cold to the hot end of the rod. Behind the quench front, the water is able to rewet the surface (maintain contact) and provide a high heat transfer normal to the surface. Ahead of the quench front the steam insulates the cladding and provides a low heat transfer normal to the surface.

The quench front itself is characterized by violent nucleate boiling (or sputtering) at the leading edge of the film. The position of the quench front is determined by a critical temperature T_0, called the sputtering temperature, which is the temperature on the surface of the cladding at the position of the front. That is, the surface of the cladding is at temperature $T < T_0$ behind the quench front, and at $T > T_0$ ahead of the front. This situation is illustrated in Figure 1-1 for the case of top spray.

Many experiments have been performed on the subjects of quenching and rewetting. A review of the results can be found in [4]. It was found experimentally that the rewetting velocity was nearly constant with time, and that it was a function of coolant flow rate, coolant temperature, and pressure. Clearly, the important quantity of physical importance is the determination of the quench front velocity, for this ultimately determines the rate at which heat is transferred to the coolant. The solution of the temperature distribution within the rod is of secondary importance. In our analysis, we will exploit the observation that the quench front velocity is constant by finding traveling wave solutions, which in turn will be used to find the velocity of the quench front.

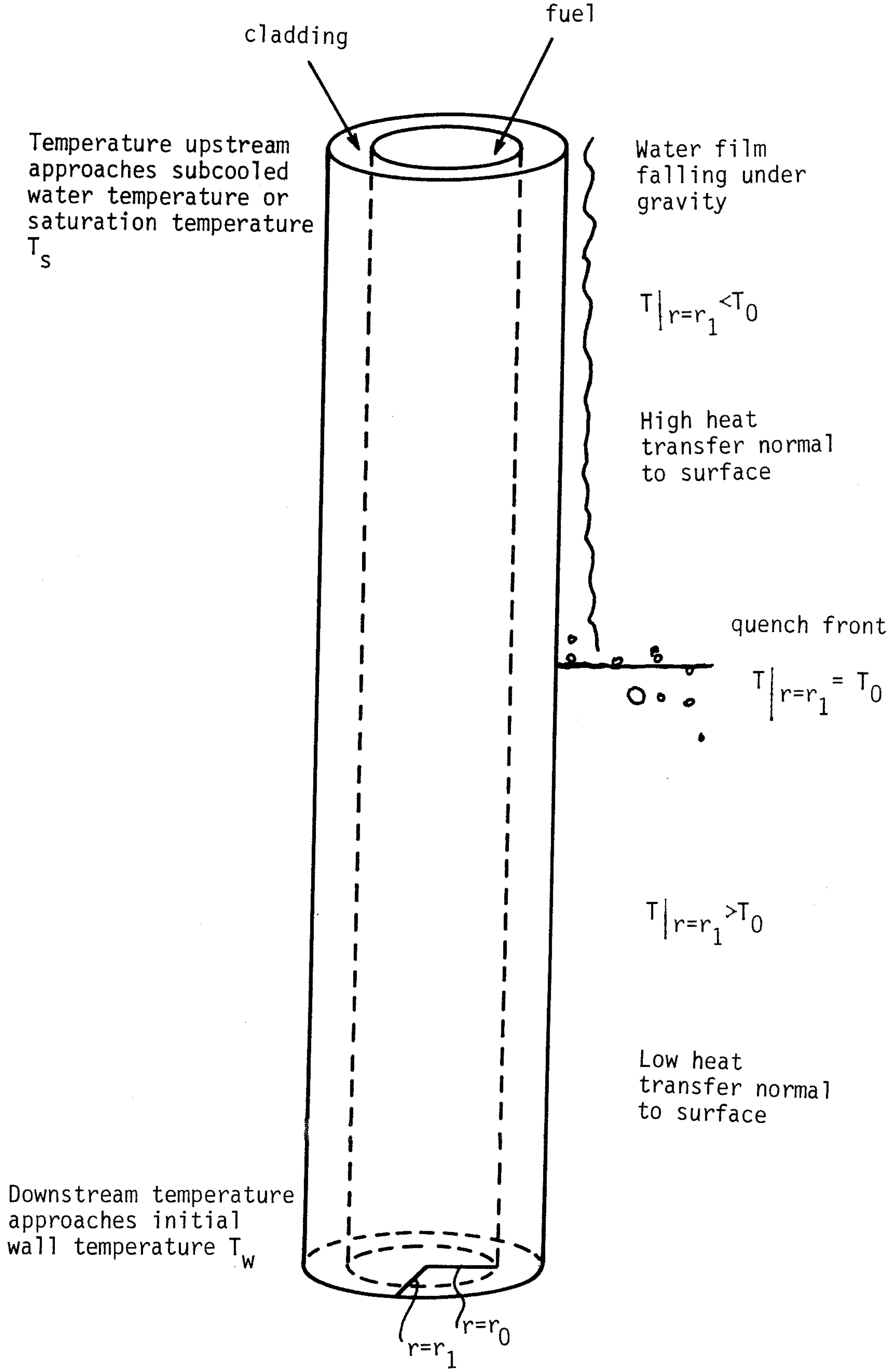

cladding
fuel
Temperature upstream approaches subcooled water temperature or saturation temperature T_s
Water film falling under gravity
$T|_{r=r_1} < T_0$
High heat transfer normal to surface
quench front
$T|_{r=r_1} = T_0$
$T|_{r=r_1} > T_0$
Low heat transfer normal to surface
Downstream temperature approaches initial wall temperature T_w
$r=r_0$
$r=r_1$

Figure 1-1

II. FORMULATION OF THE PROBLEM

In this section, we formulate the dimensionless version of the model equations which will then be solved using Fourier transforms and a Wiener-Hopf technique.

Approximating a single fuel rod by an infinitely long cylinder, we can write the heat equation in the cladding

$$\frac{\rho c}{k}\frac{\partial T}{\partial t} = \frac{\partial^2 T}{\partial x^2} + \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) \qquad \begin{matrix} r_0 < r < r_1 \\ -\infty < x < \infty \\ t > 0 \end{matrix}$$

with a given initial temperature distribution

$$T(x,r,0) \text{ given.}$$

Assume at the outer radius $r = r_1$ that due to top spray or reflooding

$$k\frac{\partial T}{\partial r} = \begin{cases} hf(T-T_s) , & \text{if } T < T_0 \text{ at } r = r_1 , \\ 0 , & \text{if } T > T_0 \text{ at } r = r_1 . \end{cases}$$

At the inner radius $r = r_0$, assume that there is no normal heat transfer, i.e.,

$$\frac{\partial T}{\partial r} = 0 \text{ at } r = r_0 .$$

Finally, at the ends of the rod

$$T(-\infty,r,t) = T_s \qquad \text{(cool end)}$$

$$T(+\infty,r,t) = T_w \qquad \text{(hot end)} .$$

In the above equations, the density is ρ, specific heat c, thermal conductivity k, heat transfer coefficient h, sputtering temperature T_0, saturation temperature T_s, and wall temperature T_w, all constant. The heat transfer function is $f(T-T_s)$, which we will assume to be linear (Newton cooling).

Before proceeding further, the boundary conditions on the inner and outer radii of the cladding deserve further comment. Though we have chosen the "standard" ones, several alternate boundary conditions have been considered in the literature in varying degrees of detail.

In particular, at the outer radius $r = r_1$ we have assumed adiabatic conditions with a zero heat transfer coefficient in the dry region ahead of the quench front. Other models exist [5] in which the downstream heat transfer coefficient is nonzero, though smaller than the upstream coefficient. However, these models have only a limited range of application and do not agree well with experimental results.

At the inner radius $r = r_0$ we have assumed that the cladding is completely insulated. This cannot be entirely correct in that it ignores the effect of the fission decay heat altogether. The rationale in ignoring the decay heat is that the heat flux across the inner radius $r = r_0$ due to fission decay is small compared to the very large values (10-100 MW m^{-2}) reached in the quenching process itself. In [6] an approximate integral technique is used to obtain approximate solutions in the case of small internal heat generation.

In the equations and boundary conditions above, we next approximate the cylindrical geometry by a two-dimensional slab. If we nondimensionalize the equations for a nondimensional temperature $u = u(x,y,t)$,

$$\left.\begin{aligned}
&\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}, \quad -\infty < x < \infty, \quad 0 < y < 1, \quad t > 0 \\
&u(x,y,0) \text{ given} \\
&\left.\frac{\partial u}{\partial y}\right|_{y=0} = 0 \\
&\left.\frac{\partial u}{\partial y}\right|_{y=1} = \begin{cases} -Au\big|_{y=1} & , \; 0 < u < u_c \\ 0 & , \; u_c < u < 1 \end{cases} \\
&u(-\infty,y,t) = 0 \quad , \quad u(+\infty,y,t) = 1
\end{aligned}\right\} \qquad (2.1)$$

Here A is the dimensionless heat transfer coefficient or Biot number and is given by

$$A = \frac{h}{k}(r_1 - r_0) \; .$$

The dimensionless sputtering temperature u_c is

$$u_c = \frac{T_0 - T_s}{T_w - T_s} \; .$$

Both A and u_c are known constants.

Next, assume traveling wave solutions of the form

$$u(x,y,t) = u(x',y)$$

where $x' = x-pt$, and where p is the (unknown) dimensionless velocity of the quench front or Peclet number. Note that by changing to a moving coordinate system, we fix the location of the quench front at $x' = 0$, $y = 1$. Finally, in order to simplify the subsequent analysis, let $x'' = x'p$; then dropping the primes we have

$$\left.\begin{array}{ll}
p^2 \dfrac{\partial^2 u}{\partial x^2} + p^2 \dfrac{\partial u}{\partial x} + \dfrac{\partial^2 u}{\partial y^2} = 0 & \begin{array}{c} -\infty < x < \infty \\ 0 < y < 1 \end{array} \\
\left.\dfrac{\partial u}{\partial y}\right|_{y=0} = 0 & \\
\left.\dfrac{\partial u}{\partial y}\right|_{y=1} = \begin{cases} -Au|_{y=1} \ , & 0 < u < u_c \ , \text{ or } \quad x < 0 \\ 0 \ , & u_c < u < 1 \ , \text{ or } \quad x > 0 \end{cases} & \\
u(-\infty,y) = 0 & \\
u(+\infty,y) = 1 & \\
u(0,1) = u_c & \\
p \ \text{ unknown .} &
\end{array}\right\} \qquad (2.2)$$

That this problem even has a solution has been investigated in a forthcoming publication by B. Nicolaenko and B. Wendroff [7]. Briefly, they found that there exists a unique pair (p,u) (depending on A and u_c) with $p > 0$, $u(x,y) > 0$. In addition, $u_y(x,y) > 0$ for $y \neq 0$ and $y \neq 1$, and $u_x(x,y) > 0$.

Numerical solutions of (2.2) have been obtained in [8] using an eigenfunction expansion technique, by finite differences in [9], and by finite differences using isotherms as a coordinate in [10]. For the case where the two-dimensional problem is reduced to one dimension by integrating the y-dependence, numerical solutions have been obtained for small values of A [11], [12] and for large values of A in [13].

Asymptotic solutions using a Wierner-Hopf technique with an approximate kernel substitution have been examined in [14] for large and small Biot numbers, and an empirical formula for the continuous range of Biot numbers has been obtained from this work in [15].

In this thesis we shall obtain asymptotic solutions for the quench front velocity using a Wierner-Hopf technique and the exact kernel factorization.

To solve (2.2), we would like to apply a Fourier transform in x. Unfortunately, the boundary condition at $x = +\infty$ is inappropriate for this, especially in light of the subsequent Wierner-Hopf analysis. However, the trivial change in the dependent variable

$$v(x,y) = 1 - u(x,y)$$

will markedly simplify the development. With this change (2.2) becomes

$$\left.\begin{aligned}
&\frac{\partial^2 v}{\partial x^2} + \frac{\partial v}{\partial x} + \frac{1}{p^2}\frac{\partial^2 v}{\partial y^2} = 0 \\
&\left.\frac{\partial v}{\partial y}\right|_{y=0} = 0 \\
&\left.\frac{\partial v}{\partial y}\right|_{y=1} = \begin{cases} -A\left(v\big|_{y=1} - 1\right), & x < 0 \\ 0, & x > 0 \end{cases} \\
&v(-\infty,y) = 1 \\
&v(+\infty,y) = 0 \\
&v(0,1) = 1 - u_c \equiv v_c \,.
\end{aligned}\right\} \qquad (2.3)$$

If we define the Fourier transform pair

$$V(\xi,y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} v(x,y)e^{ix\xi}dx$$

$$v(x,y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} V(\xi,y)e^{-ix\xi}d\xi \ , \qquad (2.4)$$

the differential equation in (2.3) transforms to

$$\frac{d^2V}{dy^2} - p^2(\xi^2+i\xi)\ V(\xi,y) = 0$$

with solutions

$$V(\xi y) = C(\xi;p)\cosh\gamma y + B(\xi;p)\sinh\gamma y \ . \qquad (2.5)$$

In (2.5) $\gamma \equiv p\sqrt{\xi^2 + i\xi}$, where we consider the positive branch of the square root for definiteness.

If we routinely transform the boundary conditions on the sides of the slab, we reach an impasse, since $v(x,1)e^{ix\xi}$ must be integrated on the half-line $x < 0$, and this is unknown. This leads us to define

$$V_+(\xi,y) = \frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} v(x,y)e^{i\xi x}dx$$

$$V_-(\xi,y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{0} v(x,y)e^{i\xi x}dx \qquad (2.6)$$

so that

$$V(\xi,y) = V_+(\xi,y) + V_-(\xi,y). \tag{2.7}$$

In general, $V_+(\xi,y)$ defines an analytiic function of the complex variable ξ in some upper half-plane $Im(\xi) > \tau_-$. Likewise, $V_-(\xi,y)$ defines a function analytic in a lower half-plane $Im(\xi) < \tau_+$, where $\tau_- < \tau_+$, so that the inversion integral in (2.4) may be deformed anywhere in the overlap strip $\tau_- < Im(\xi) < \tau_+$.

Next, since

$$\frac{dV(\xi,y)}{dy} = \frac{dV_-(\xi,y)}{dy} + \frac{dV_+(\xi,y)}{dy} = V_-'(\xi,y) + V_+'(\xi,y) \ ,$$

the boundary conditions on the sides of the slab imply that

$$\left.\begin{aligned} & V_-'(\xi,0) = V_+'(\xi,0) = V'(\xi,0) = 0 \\ & V_+'(\xi,1) = 0 \\ & V_-'(\xi,1) = -AV_-(\xi,1) + \frac{A}{\sqrt{2\pi}} \int_{-\infty}^{0} e^{i\xi x} \, dx \\ & \qquad = -A\left[V_-(\xi,1) + g(\xi)\right] \\ & \qquad \text{where } g(\xi) = \frac{i}{\sqrt{2\pi}\xi} , \text{ provided } Im(\xi) < 0 . \end{aligned}\right\} \tag{2.8}$$

Applying (2.7 and (2.8) to (2.5), we find that

$$B(\xi;p) = 0$$

$$C(\xi;p) = \frac{-A\left[V_-(\xi,1)+g(\xi)\right]}{\gamma \sinh\gamma} \ , \quad \text{so that}$$

$$V(\xi,1) = V_+(\xi,1) + V_-(\xi,1) = \frac{-A}{\gamma}\left[V_-(\xi,1) + g(\xi)\right]\coth\gamma \ .$$

Rearranging this last equation slightly yields

$$\left.\begin{aligned} &V_+(\xi,1) + K(\xi)V_-(\xi,1) + N(\xi) = 0 \\ &\text{where} \\ &\qquad K(\xi) = 1 + \frac{A\coth\gamma}{\gamma} \\ &\qquad N(\xi) = \frac{A\coth\gamma}{\gamma}\, g(\xi) \end{aligned}\right\} \tag{2.9}$$

The functions $K(\xi)$ and $N(\xi)$ are known and will be shown to be analytic and nonzero in the strip $\tau_- < \text{Im}(\xi) < \tau_+$. Thus, (2.9) is a single equation in the unknowns $V_+(\xi,1)$ and $V_-(\xi,1)$, valid in the strip. We will solve it for $V_+(\xi,1)$ and $V_-(\xi,1)$ using a Wiener-Hopf factorization technique.

The fundamental step in the Wiener-Hopf procedure for the solution of this equation is to factor the expression $K(\xi)$ such that

$$K(\xi) = \frac{K_-(\xi)}{K_+(\xi)} ,$$

where $K_-(\xi)$ is analytic and nonzero in $\text{Im}(\xi) < \tau_+$ and where $K_+(\xi)$ is analytic and nonzero in $\text{Im}(\xi) > \tau_-$. This will be accomplished in Section III. The product factorization above allows us to rewrite (2.9) as

$$K_+(\xi)V_+(\xi,1) + K_-(\xi)V_-(\xi,1) + K_+(\xi)\, N(\xi) = 0 . \tag{2.10}$$

Next, we factor $K_+(\xi)\, N(\xi)$ as a sum

$$K_+(\xi)\, N(\xi) = R_+(\xi) + R_-(\xi) ,$$

using the same ± conventions as above. The sum factorization will be obtained in Section IV.

With this result we can rearrange (2.10), defining the function $E(\xi)$ as

$$K_+(\xi)V_+(\xi,1) + R_+(\xi) = -K_-(\xi)V_-(\xi,1)-R_-(\xi) \equiv E(\xi) \ . \tag{2.11}$$

$E(\xi)$ is originally defined only in the strip $\tau_- < \text{Im}(\xi) < \tau_+$. However, since the first part of (2.11) is analytic for $\text{Im}(\xi) > \tau_-$ and the second part is analytic for $\text{Im}(\xi) < \tau_+$, we are able to analytically continue $E(\xi)$ to the entire plane so that $E(\xi)$ is an entire function.

Finally, using asymptotic properties of the functions $K_+(\xi)$, $V_+(\xi,1)$, $R_+(\xi)$, $K_-(\xi)$, $V_-(\xi,1)$, and $R_-(\xi)$ as $|\xi| \to \infty$ in their respective half-planes, we will show in Section V that $E(\xi)$ tends to zero as ξ tends to infinity. By Liouville's theorem then, $E(\xi) \equiv 0$ allowing us to solve for $V_+(\xi,1)$ and $V_-(\xi,1)$ explicitly as

$$V_+(\xi,1) = \frac{-R_+(\xi)}{K_+(\xi)} \ , \quad \text{and} \quad V_-(\xi,1) = \frac{-R_-(\xi)}{K_-(\xi)} \ .$$

This allows us to invert the transform, solving for v.

Before moving to the sum and product factorizations, we close this section with the determination of the regions of analyticity of $V_+(\xi,1)$ and $V_-(\xi,1)$. In order to do so, we need to consider the growth of $v(x,y)$ as $x \to \pm \infty$ in $0 < y < 1$.

For $x > 0$, the characteristic functions of (2.3) are

$$\exp\left[\left(-\frac{1}{2}-\frac{1}{2}\sqrt{1+\frac{4n^2\pi^2}{p^2}}\right)x\right]\cos(n\pi y)\ ; \qquad n = 0,1,2,\ \ldots\ .$$

The complete solution will be a sum of homogeneous solutions of the form

$$v(x,y) = \sum_{n=0}^{\infty} A_n \exp\left[\left(-\frac{1}{2}-\frac{1}{2}\sqrt{1+\frac{4n^2\pi^2}{p^2}}\right)x\right]\cos(n\pi y)\ ,$$

where the A_n's are unknown expansion coefficients. Therefore, for x large and positive, $v(x,y) = O(e^{-x})$. Referring to the definition of $V_+(\xi,y)$ in (2.6), we conclude that $V_+(\xi,1)$ is analytic in $\mathrm{Im}(\xi) > -1$.

In the half-strip $x < 0$, $0 < y < 1$, we use a slightly different technique. Since $v(x,y)$ is bounded for all $x < 0$, we obtain using (2.6) the (crude) estimate that $V_-(\xi,1)$ is analytic in $\mathrm{Im}(\xi) < 0$. Note that this correlates well with the observation that $g(\xi)$ in (2.8) is defined for $\mathrm{Im}(\xi) < 0$.

III. PRODUCT FACTORIZATION

In this section, we exhibit the explicit product factorization that allows us to write

$$K(\xi) \equiv 1 + \frac{A \coth\gamma}{\gamma} = \frac{K_-(\xi)}{K_+(\xi)} \quad , \tag{3.1}$$

where $K_-(\xi)$ is analytic and free of zeros in the lower half-plane $\mathrm{Im}(\xi) < \tau_+$, and where $K_+(\xi)$ is analytic and free of zeros in the upper half-plane $\mathrm{Im}(\xi) > \tau_-$.

The procedure we will follow in obtaining a product factorization of $K(\xi)$ is well known [16] and relies on the Weierstrass Factor Theorem [17]. Briefly, it states that if $K(\xi)$ is entire and can be expanded in an infinite product, and if $K(\xi)$ is even, then we may write

$$K(\xi) = K(0) \prod_{n=1}^{\infty} \left\{1 - \left(\frac{\xi}{\xi_n}\right)^2 \right\}$$

where $\xi = \pm\, \xi_n$ are the (simple) zeros of $K(\xi)$. This allows us to decompose $K(\xi)$ as a product of the form

$$K(\xi) = K_+(\xi)K_-(\xi) \quad , \text{ where}$$

$$K_\pm(\xi) = \{K(0)\}^{1/2} e^{\mp\chi(\xi)} \prod_{n=1}^{\infty} \left\{1 \pm \frac{\xi}{\xi_n}\right\} \exp\left(\mp \frac{\xi}{\xi_n}\right) . \tag{3.2}$$

In (3.2) the exponential factors $\exp(\mp\, \xi/\xi_n)$ are inserted into the infinite products to ensure absolute convergence in their respective half-planes. The function $\chi(\xi)$ also deserves comment. For the moment, $\chi(\xi)$ is any arbitrary entire function, and can be chosen later so that the factors $K_+(\xi)$ and $K_-(\xi)$ have algebraic behavior as ξ tends to infinity.

For (3.1), we need to factor $K(\xi)$ as a quotient. Some preliminary manipulation is required first, however. At first glance we seem doomed to failure since $K(\xi)$ is not an even function of ξ. However, if we let

$$\xi = \alpha - i/2 \quad , \tag{3.3}$$

then $\xi^2 + i\xi = \alpha^2 + \frac{1}{4}$ which is even, so that $K = K(\alpha)$ is an even function of the variable α.

In order to facilitate further analysis, we will consider K, K_+, and K_- as functions of the complex variable α and perform the factorizations in the complex α-plane. We note in passing that as functions of α, $V_+(\alpha,1)$ is analytic in $\mathrm{Im}(\alpha) > -\frac{1}{2}$ and $V_-(\alpha,1)$ is analytic in $\mathrm{Im}(\alpha) < +\frac{1}{2}$, using the results obtained at the end of Section II. Next, for the function $K(\alpha)$ we notice that the branch point singularities of $\gamma = p\sqrt{\alpha^2+1/4}$ are removable, so that $K(\alpha)$ is meromorphic. Therefore, if we rewrite $K(\alpha)$ as the quotient of two entire functions $L(\alpha)$ and $M(\alpha)$ such that

$$K(\alpha) = \frac{L(\alpha)}{M(\alpha)}$$

where

$$L(\alpha) = \gamma\sinh\gamma + A\cosh\gamma$$

$$M(\alpha) = \gamma\sinh\gamma \quad , \qquad \text{then}$$

we may factor $L(\alpha)$ and $M(\alpha)$ individually.

The expression $L(\alpha)$ has simple imaginary zeros at $\gamma = \pm i\rho_n$ satisfying $\tan\rho_n = A/\rho_n$. Therefore $L(\alpha)$ may be written in factor form as

$$L(\alpha) = A \prod_{n=1}^{\infty} \left(1 + \frac{\gamma^2}{\rho_n^2}\right)$$

$$= A\left\{ \prod_{n=1}^{\infty} \left[\left(1+\frac{p^2}{4\rho_n^2}\right)^{1/2} + \frac{i\alpha p}{\rho_n}\right]\exp\left(\frac{-i\alpha p}{\rho_n}\right)\right\} \left\{ \prod_{n=1}^{\infty} \left[1+\frac{p^2}{4\rho_n^2}\right)^{1/2} - \frac{i\alpha p}{\rho_n}\right]\exp\left(\frac{+i\alpha p}{\rho_n}\right)\right\}$$

Hence $L(\alpha) = L_-\alpha/L_+(\alpha)$ where

$$L_-(\alpha) = A^{1/2}\, e^{-\chi(\alpha)} \prod_{n=1}^{\infty} \left[\left(1 + \frac{p^2}{4\rho_n^2}\right)^{1/2} + \frac{i\alpha p}{\rho_n}\right] \exp\left(\frac{-i\alpha p}{\rho_n}\right)$$

and (3.4)

$$\frac{1}{L_+(\alpha)} = A^{1/2}\, e^{+\chi(\alpha)} \prod_{n=1}^{\infty} \left[\left(1+\frac{p^2}{4\rho_n^2}\right)^{1/2} - \frac{i\alpha p}{\rho_n}\right] \exp\left(\frac{+i\alpha p}{\rho_n}\right) .$$

Now $L_-(\alpha)$ is analytic and free of zeros in the lower half-plane $\mathrm{Im}(\alpha) < \left(\frac{\rho_1^2}{p^2} + \frac{1}{4}\right)^{1/2}$ (or for sure $\mathrm{Im}(\alpha) < \frac{1}{2}$). Likewise $1/L_+(\alpha)$ is analytic and free of zeros in the upper half-plane $\mathrm{Im}(\alpha) > -\left(\frac{\rho_1^2}{p^2} + \frac{1}{4}\right)^{1/2}$ (or for sure $\mathrm{Im}(\alpha) > -\frac{1}{2}$). We also note that for n sufficiently large and positive

$$\rho_n = (n-1)\pi + \frac{A}{(n-1)\pi} + 0\left(\frac{A^2}{n^3}\right)$$

Next, consider $M(\alpha) = \gamma\sinh\gamma$. As it stands, we cannot factor $M(\alpha)$ directly since $\gamma = 0$ is a double zero. However, if we consider instead $\sinh\gamma/\gamma$ (with a removable singularity at $\gamma = 0$) we may factor it as

$$\frac{\sinh\gamma}{\gamma} = \prod_{n=1}^{\infty} \left(1 + \frac{\gamma^2}{n^2\pi^2}\right)$$

$$= \left\{ \prod_{n=1}^{\infty} \left[\left(1+\frac{p^2}{4n^2\pi^2}\right)^{1/2} + \frac{i\alpha p}{n\pi}\right]\exp\left(\frac{-i\alpha p}{n\pi}\right)\right\} \left\{ \prod_{n=1}^{\infty} \left[\left(1+\frac{p^2}{4n^2\pi^2}\right)^{1/2} - \frac{i\alpha p}{n\pi}\right]\exp\left(\frac{+i\alpha p}{n\pi}\right)\right\} .$$

Then, using $M(\alpha) = \gamma^2 \frac{\sinh\gamma}{\gamma}$, we find that

$$M(\alpha) = \frac{M_-(\alpha)}{M_+(\alpha)} , \text{ where}$$

$$M_-(\alpha) = p\left(\tfrac{1}{2}+i\alpha\right) \prod_{n=1}^{\infty} \left[\left(1 + \frac{p^2}{4n^2\pi^2}\right)^{1/2} + \frac{i\alpha p}{n\pi}\right] \exp\left(\frac{-i\alpha p}{n\pi}\right)$$

and (3.5)

$$\frac{1}{M_+(\alpha)} = p\left(\tfrac{1}{2}-i\alpha\right) \prod_{n=1}^{\infty} \left[\left(1 + \frac{p^2}{4n^2\pi^2}\right)^{1/2} - \frac{i\alpha p}{n\pi}\right] \exp\left(\frac{+i\alpha p}{n\pi}\right)$$

Here $M_-(\alpha)$ is analytic and free of zeros in $\mathrm{Im}(\alpha) < \frac{1}{2}$ and $\frac{1}{M_+(\alpha)}$ is analytic and free of zeros in $\mathrm{Im}(\alpha) > -\frac{1}{2}$.

Finally, since $K(\alpha) = \frac{L(\alpha)}{M(\alpha)}$, we obtain from (3.4) and (3.5) that

$$K_-(\alpha) = \frac{L_-(\alpha)}{M_-(\alpha)} = \frac{A^{1/2}\rho^{-\chi(\alpha)} \prod_{n=1}^{\infty} \left[\left(1 + \frac{p^2}{4\rho_n^2}\right)^{1/2} + \frac{i\alpha p}{\rho_n}\right] \exp\left(\frac{-i\alpha p}{\rho_n}\right)}{p\left(\tfrac{1}{2}+i\alpha\right) \prod_{n=1}^{\infty} \left[\left(1+ \frac{p^2}{4n^2\pi^2}\right)^{1/2} + \frac{i\alpha p}{n\pi}\right] \exp\left(\frac{-i\alpha p}{n\pi}\right)} \qquad (3.6)$$

is analytic and free of zeros in $\mathrm{Im}(\alpha) > -\frac{1}{2}$, and

$$\frac{1}{K_+(\alpha)} = \frac{M_+(\alpha)}{L_+(\alpha)} = \frac{A^{1/2} e^{+\chi(\alpha)} \prod_{n=1}^{\infty} \left[\left(1 + \frac{p^2}{4\rho_n^2}\right)^{1/2} - \frac{i\alpha p}{\rho_n}\right] \exp\left(\frac{i\alpha p}{\rho_n}\right)}{p\left(\frac{1}{2} - i\alpha\right) \prod_{n=1}^{\infty} \left[\left(1 + \frac{p^2}{4n^2\pi^2}\right)^{1/2} - \frac{i\alpha p}{n\pi}\right] \exp\left(\frac{i\alpha p}{n\pi}\right)} \tag{3.7}$$

is analytic and free of zeros in $\mathrm{Im}(\alpha) > -\frac{1}{2}$. This completes the factorization of $K(\alpha) = \frac{K_-(\alpha)}{K_+(\alpha)}$. In closing this section, we will now obtain the asymptotic behaviors of $K_+(\alpha)$ and $K_-(\alpha)$ as α tends to infinity in their respective half-planes. This information will be of further use in Sections IV and V. Recall first that for n large and positive that $\rho_n = (n-1)\pi + O(\frac{1}{n})$. This suggests writing the infinite product expansions in (3.6) and (3.7) in a slightly different form, say for $K_-(\alpha)$ as

$$K_-(\alpha) = \frac{A^{1/2} e^{-\chi(\alpha)}}{\left(\frac{p}{2} + ip\alpha\right)} \left\{\left(1 + \frac{p^2}{4\rho_1^2}\right)^{1/2} + \frac{i\alpha p}{\rho_1}\right\} \exp\left(\frac{-i\alpha p}{\rho_1}\right) \times$$

$$\frac{\prod_{n=2}^{\infty} \left[\left(1 + \frac{p^2}{4\rho_n^2}\right)^{1/2} + \frac{i\alpha p}{\rho_n}\right] \exp\left(\frac{-i\alpha p}{\rho_n}\right)}{\prod_{n=1}^{\infty} \left[\left(1 + \frac{p^2}{4n^2\pi^2}\right)^{1/2} + \frac{i\alpha p}{n\pi}\right] \exp\left(\frac{-i\alpha p}{n\pi}\right)} .$$

Now for $|\alpha|$ sufficiently large, we may neglect the terms $p^2/4\rho_n^2$ and $p^2/4n^2\pi^2$ relative to unity. Hence, since $\rho_n \sim (n-1)\pi$ as $n \to \infty$, we find that $K_-(\alpha)$ is of the order

$$\frac{e^{-\chi(\alpha)}\ \alpha\exp\left(\frac{-i\alpha p}{\rho_1}\right)}{\alpha}\qquad\frac{\prod_{n=1}^{\infty}\left[1+\frac{i\alpha p}{n\pi}\right]\exp\left(\frac{-i\alpha p}{n\pi}\right)}{\prod_{n=1}^{\infty}\left[1+\frac{i\alpha p}{n\pi}\right]\exp\left(\frac{-i\alpha p}{n\pi}\right)}$$

$$= O\left(\exp(-\chi(\alpha) - \frac{i\alpha p}{\rho_1}\right) \quad \text{as } |\alpha| \to \infty \text{ in } \mathrm{Im}(\alpha) < \frac{1}{2}\ .$$

Likewise, it can be shown that

$$\frac{1}{K_+(\alpha)} = O\left(\exp\left(\chi(\alpha) + \frac{i\alpha p}{\rho_1}\right)\right) \text{ as } |\alpha| \to \infty \text{ in } \mathrm{Im}(\alpha) > -\frac{1}{2}\ .$$

Until this point, the function $\chi(\alpha)$ has been arbitrary, though entire; we can now fix the value of $\chi(a)$ by requiring that $K_-(\alpha)$ and $K_+(\alpha)$ have algebraic behavior as $|\alpha| \to \infty$ in their respective half-planes. Clearly, we set

$$\chi(\alpha) \equiv \frac{-i\alpha p}{\rho_1} \quad , \tag{3.8}$$

so that

$$K_-(\alpha) = O(1) \text{ as } |\alpha| \to \infty \text{ in } \mathrm{Im}\alpha) < \frac{1}{2}$$

$$K_+(\alpha) = O(1) \text{ as } |\alpha| \to \infty \text{ in } \mathrm{Im}(\alpha) > -\frac{1}{2}\ . \tag{3.9}$$

Note also that

$$K(\alpha) = O(1) \text{ as } |\alpha| \to \infty \text{ in } -\frac{1}{2} < \mathrm{Im}(\alpha) < \frac{1}{2}\ . \tag{3.10}$$

IV. SUM DECOMPOSITION

The topic of this section is the sum factorization of the expression

$$R(\xi) \equiv K_+(\xi) \frac{A\coth\gamma}{\gamma} g(\xi) = R_+(\xi) + R_-(\xi)$$

where

$$g(\xi) = \frac{i}{\sqrt{2\pi}\ \xi}, \quad \mathrm{Im}(\xi) < 0 .$$

In terms of the new independent variable α, this can be rewritten,

$$R(\alpha) = \frac{Ai\ K_+(\alpha)\ \coth\gamma}{\sqrt{2\pi}\ \gamma\ (\alpha - \frac{i}{2})} \quad . \tag{4.1}$$

We will factor this expression into a sum $R_+(\alpha) + R_-(\alpha)$ where $R_+(\alpha)$ is analytic in the upper half-plane $\mathrm{Im}(\alpha) > -\frac{1}{2}$, and $R_-(\alpha)$ is analytic in the lower half-plane $\mathrm{Im}(\alpha) < +\frac{1}{2}$.

As in the case of the product factorization, the general procedure for sum decomposition is well known; several examples are given in [16] and [18]. The appropriate sum factorization theorem states that if (i) $R(\alpha)$ is analytic in $\tau_- < \tau < \tau_+$ $(\alpha = \sigma + i\tau)$

(ii) $|R(\alpha)| < |\sigma|^{-k}$; k = constant > 0 as $|\sigma| \to \infty$ in $\tau_- < \tau < \tau_+$

then

$$\left.\begin{aligned} &R(\alpha) = R_+(\alpha) + R_-(\alpha) \quad \text{where} \\ &R_+(\alpha) = \frac{1}{2\pi i}\int_{-\infty+ic}^{\infty+ic} \frac{R(\zeta)}{\zeta-\alpha}\,d\zeta, \text{ analytic for } \tau > \tau_- \\ &R_-(\alpha) = \frac{-1}{2\pi i}\int_{-\infty+id}^{\infty+id} \frac{R(\zeta)}{\zeta-\alpha}\,d\zeta, \text{ analytic for } \tau < \tau_+ \\ &\text{where} \quad \tau_- < c < \tau < d < \tau_+ . \end{aligned}\right\} \tag{4.2}$$

Schematically the situation is illustrated in Figure 4-1 in the complex ζ-plane, where for our purposes we have $\tau_- = -\frac{1}{2}$ and $\tau_+ = \frac{1}{2}$.

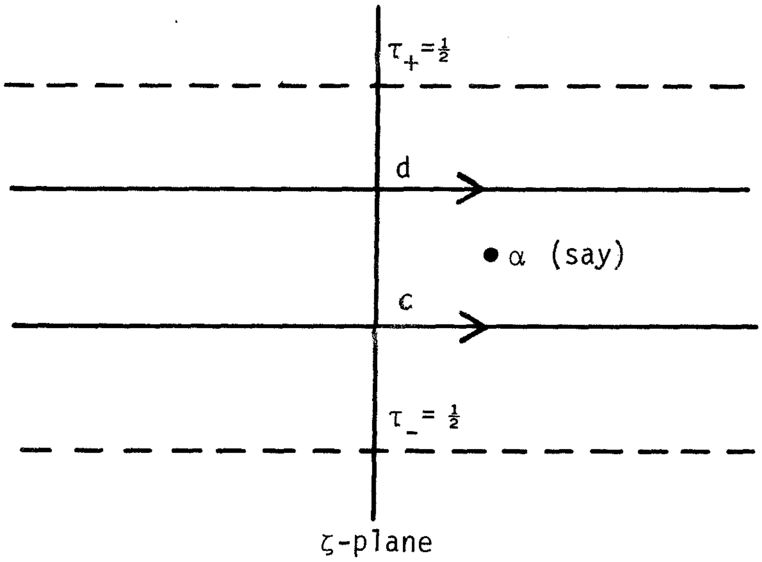


Figure 4-1

In the strip $-\frac{1}{2} < \tau < +\frac{1}{2}$, $R(\alpha)$ is analytic and $\gamma \sim p\alpha$ as $|\alpha| \to \infty$ so that $\coth\gamma = O(1)$ as $|\alpha| \to \infty$ there. Also, since $K_+(\alpha) = O(1)$ as $|\alpha| \to \infty$ in $\tau > -\frac{1}{2}$, we conclude that

$$R(\alpha) = O\left(\frac{1}{\alpha^2}\right) = O\left(\frac{1}{\sigma^2}\right) \text{ as } |\sigma| \to \infty \text{ in } -\frac{1}{2} < \tau < +\frac{1}{2} .$$

Therefore, the hypotheses of (4.2) are satisfied.

To evaluate the integrals in (4.2) it would seem expedient to close the contours in the upper half-plane $\text{Im}(\zeta) > -\frac{1}{2}$, since $K_+(\zeta)$ is analytic there. However, the integrand $\frac{R(\zeta)}{\zeta-\alpha}$ also has a double pole at $\zeta = \frac{i}{2}$, and the complications from calculating the residue at that point preclude closing the contours from above as a viable choice. In order to close the contours in the lower half-plane though, we need an alternate expression for $K_+(\zeta)$since the infinite products of that function only converge for $\text{Im}(\zeta) > -\frac{1}{2}$. Recalling that

$$K_+(\zeta) = \frac{K-(\zeta)}{K(\zeta)} = \frac{K_-(\zeta)}{1 + \frac{A\coth\gamma}{\gamma}} ,$$

we can rewrite the integrand as

$$\frac{R(\zeta)}{\zeta-\alpha} = \frac{AiK_-(\zeta)\cosh\gamma}{\sqrt{2\pi}(\zeta-\alpha)(\zeta-\frac{i}{2})(\gamma\sinh\gamma + A\cosh\gamma)}$$

$$= \frac{AiK_-(\zeta)\coth\gamma}{\sqrt{2\pi}\,(\zeta-\alpha)(\zeta-\frac{i}{2})(\gamma + A\coth\gamma)} .$$

Now, in $\mathrm{Im}(\zeta) < \frac{1}{2}$, we see that the integrand has a simple pole at $\zeta = \alpha$, and an infinite sequence of simple poles at $\zeta = -i\sqrt{\frac{\rho_k^2}{p^2} + \frac{1}{4}} \equiv -i\omega_k$, $k = 1,2,3,\ldots$, where $\cot\rho_k = \frac{\rho_k}{A}$. If we consider the infinite sequence of rectangular contours C_k with lower corners at $\pm \frac{\pi}{p}\left(k + \frac{1}{2}\right) - i\,\frac{\pi}{p}\left(k + \frac{1}{2}\right)$, we find that $\coth\gamma = O(1)$ on C_k so that the integrand is $O\left(\frac{1}{\zeta^3}\right)$ on C_k. Hence $\left|\int_{C_k}\right|$ as $k \to \infty$ and the integrals for $R_+(\alpha)$ and $R_-(\alpha)$ are reduced to a sum of residues. After some fairly routine calculations we obtain that

$$\left.\begin{aligned}
R_+(\alpha) &= \frac{-Ai}{\sqrt{2\pi}\,p^2} \sum_{k=1}^{\infty} \frac{\rho_k^2\, K_-(-i\omega_k)}{\omega_k\left(\omega_k + \frac{1}{2}\right)\left(\rho_k^2 + A^2 + A\right)\left(\alpha + i\omega_k\right)} \\
R_-(\alpha) &= R(\alpha) + \frac{Ai}{\sqrt{2\pi}\,p^2} \sum_{k=1}^{\infty} \frac{\rho_k^2\, K_-(-i\omega_k)}{\omega_k\left(\omega_k + \frac{1}{2}\right)\left(\rho_k^2 + A^2 + A\right)\left(\alpha + i\omega_k\right)} \\
&= \frac{AiK_-(\alpha)\cosh\gamma}{\sqrt{2\pi}\left(\alpha - \frac{i}{2}\right)\left(\gamma\sinh\gamma + A\cosh\gamma\right)} \\
&\quad + \frac{Ai}{\sqrt{2\pi}\,p^2} \sum_{k=1}^{\infty} \frac{\rho_k^2 K_-(-i\omega_k)}{\omega_k\left(\omega_k + \frac{1}{2}\right)\left(\rho_k^2 + A^2 + A\right)\left(\alpha + i\omega_k\right)} \\
\text{where} \quad \omega_k &= \sqrt{\frac{\rho_k^2}{p^2} + \frac{1}{4}}, \quad k = 1,2,\ldots \\
\text{and where} \quad \cot\rho_k &= \frac{\rho_k}{A}\,.
\end{aligned}\right\} \tag{4.3}$$

In the sum decomposition above, $R_+(\alpha)$ is analytic in $\mathrm{Im}(\alpha) > -\frac{1}{2}$, and $R_-(\alpha)$ is analytic in $\mathrm{Im}(\alpha) < +\frac{1}{2}$. Also, since $K_-(-i\omega_k) = O(1)$ for k large, we see that the general term in the series is $O(1/k^3)$ so that convergence is absolute. Finally, we close with the observation that

$$R_+(\alpha) = O\left(\frac{1}{\alpha}\right) \quad \text{as} \quad |\alpha| \to \infty \quad \text{in} \quad \text{Im}(\alpha) > -\frac{1}{2}$$

$$R_-(\alpha) = O\left(\frac{1}{\alpha}\right) \quad \text{as} \quad |\alpha| \to \infty \quad \text{in} \quad \text{Im}(\alpha) < \frac{1}{2} \; . \tag{4.4}$$

V. INVERSION OF THE TRANSFORM

A. ANALYTIC CONTINUATION

We are now in a position to determine the integral function $E(\alpha)$ from equation (2.11), which in terms of the variable α can be rewritten

$$E(\alpha) = K_+(\alpha)V_+(\alpha,1) + R_+(\alpha) = -K_-(\alpha)V_-(\alpha,1) - R_-(\alpha) \; . \qquad (5.1)$$

We recall from (3.9) that

$$K_-(\alpha) = O(1) \quad \text{as} \quad |\alpha| \to \infty \quad \text{in} \quad \text{Im}(\alpha) < \frac{1}{2}$$

$$K_+(\alpha) = O(1) \quad \text{as} \quad |\alpha| \to \infty \quad \text{in} \quad \text{Im}(\alpha) > -\frac{1}{2} \, ,$$

and from (4.4) that (5.2)

$$R_+(\alpha) = O(\frac{1}{\alpha}) \quad \text{as} \quad |\alpha| \to \infty \quad \text{in} \quad \text{Im}(\alpha) > -\frac{1}{2}$$

$$R_-(\alpha) = O(\frac{1}{\alpha}) \quad \text{as} \quad |\alpha| \to \infty \quad \text{in} \quad \text{Im}(\alpha) < \frac{1}{2} \; .$$

In order to determine the asymptotic behavior of $V_+(\alpha,1)$ and $V_-(\alpha,1)$ in their respective half-planes, recall from (2.3) that $v(x,1) \to v_c$ = constant as $x \to 0^{\pm}$ on $y = 1$, so that

$$V_+(\alpha,1) = \frac{1}{\sqrt{2\pi}} \int_0^\infty e^{i\alpha x} e^{x/2} v(x,1)dx \sim \frac{\text{const}}{\alpha} \text{ as } |\alpha| \to \infty \text{ in } \text{Im}(\alpha) > -\frac{1}{2} \, ,$$

and likewise,

$$V_-(\alpha,1) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{0} e^{i\alpha x} e^{x/2} v(x,1)dx \sim \frac{\text{const}}{\alpha} \text{ as } |\alpha| \to \infty \text{ in } \mathrm{Im}(\alpha) < -\frac{1}{2} , \tag{5.3}$$

Using (5.1)-(5.3) we see that when $\mathrm{Im}(\alpha) < \frac{1}{2}$, $E(\alpha)$ goes to zero as $|\alpha| \to \infty$. Similarly, when $\mathrm{Im}(\alpha) > -\frac{1}{2}$, $E(\alpha)$ behaves as $O(1/\alpha)$ as $|\alpha| \to \infty$. Thus, since $E(\alpha)$ is an entire function, we therefore conclude that $E(\alpha) \equiv 0$. We have finally then

$$V_+(\alpha,1) = \frac{-R_+(\alpha)}{K_+(\alpha)} = \frac{Ai}{\sqrt{2\pi}\; p^2 K_+(\alpha)} \sum_{k=1}^{\infty} \frac{\rho_k^2 K_-(-i\omega_k)}{\omega_k(\omega_k + \frac{1}{2})(\rho_k^2 + A^2 + A)(\alpha + i\omega_k)}$$

and

$$V_-(\alpha,1) = \frac{-R_-(\alpha)}{K_-(\alpha)} = \frac{-Ai\cosh\gamma}{\sqrt{2\pi}\; (\alpha - \frac{i}{2})(\gamma\sinh\gamma + A\cosh\gamma)}$$

$$- \frac{Ai}{\sqrt{2\pi}\; p^2 K_-(\alpha)} \sum_{k=1}^{\infty} \frac{\rho_k^2 K_-(-i\omega_k)}{\omega_k(\omega_k + \frac{1}{2})(\rho_k^2 + A^2 + A)(\alpha + i\omega_k)} . \tag{5.4}$$

B. INVERSION OF THE TRANSFORM

In this section we will invert the transform and obtain a series representation for the spattering temperature v_c on the surface of the slab $y = 1$ at $x = 0$.

Now, using (2.5) and the results of (2.8), we find that in terms of α,

$$V(\alpha,y) = C(\alpha;p) \cosh(\gamma y) ,$$

where $C(\alpha;p) = \dfrac{[V_+(\alpha,1) + V_-(\alpha,1)]}{\cosh\gamma}$.

Upon inversion using (2.4),

$$v(x,y) = \frac{e^{-x/2}}{\sqrt{2\pi}} \int_{-\infty+i\tau}^{\infty+i\tau} \left[\frac{V_+(\alpha,1) + V_-(\alpha,1)}{\cosh\gamma} \right] \cosh(\gamma y) e^{-i\alpha x} d\alpha \ ,$$

where $-\frac{1}{2} < \tau < +\frac{1}{2}$.

As mentioned previously, the actual temperature distribution within the solid is of less importance than the velocity of the quench front itself. Thus, focusing on the surface of the slab $y = 1$ we find as expected that

$$v(x,1) = \frac{e^{-x/2}}{\sqrt{2\pi}} \int_{-\infty+i\tau}^{\infty+i\tau} \left[V_+(\alpha,1) + V_-(\alpha,1)\right] e^{-i\alpha x} d\alpha \ , \tag{5.5}$$

where $V_+(\alpha,1)$ and $V_-(\alpha,1)$ are given by (5.4). In the integral (5.5) we close the contour above or below depending on whether $x < 0$ or $x > 0$. We now consider the two cases separately.

<u>Case I:</u> $x < 0$ and $x = 0$

Here we consider the contour integral

$$\frac{e^{-x/2}}{\sqrt{2\pi}} \int_{\Gamma} \left[V_+(\alpha,1) + V_-(\alpha,1)\right] e^{-i\alpha x} d\alpha \ ,$$

where Γ is the contour obtained from (5.5), closing in the upper half-plane $\mathrm{Im}(\alpha) > -\frac{1}{2}$. Note first of all that $V_+(\alpha,1) = -R_+(\alpha)/K_+(\alpha)$ is analytic in $\mathrm{Im}(\alpha) > -\frac{1}{2}$, and hence will contribute nothing to the integral. Next, since

$$K_-(\alpha) = K_+(\alpha)K(\alpha) = K_+(\alpha)\left[1 + \frac{A\coth\gamma}{\gamma}\right] \ ,$$

the term $V_-(\alpha,1)$ may be rewritten using (5.4) as

$$V_-(\alpha,1) = \frac{-R_-(\alpha)}{K_-(\alpha)} = \frac{-Ai\cosh\gamma}{\sqrt{2\pi}\,(\alpha-\frac{i}{2})(\gamma\sinh\gamma + A\cosh\gamma)}$$

$$-\frac{Ai\gamma\sinh\gamma}{\sqrt{2\pi}\,p^2K_+(\alpha)(\gamma\sinh\gamma + A\cosh\gamma)}\sum_{k=1}^{\infty}\frac{\rho_k^2\,K_-(-i\omega_k)}{\omega_k(\omega_k+\frac{1}{2})(\rho_k^2+A^2+A)(\alpha+i\omega_k)}\,, \qquad (5.6)$$

in order to simplify the calculation of residues. In the first term of (5.6) we have in the upper half-plane a simple pole at $\alpha = i/2$ and an infinite sequence of simple poles at $\alpha = i\sqrt{\frac{\rho_n^2}{p^2}+\frac{1}{4}} \equiv i\omega_n$, $n = 1,2,3,\cdots$ corresponding to the zeros of $\gamma\sinh\gamma + A\cosh\gamma$ at $\gamma = +i\rho_n$ satisfying $\cot\rho_n = \rho_n/A$. The second term of (5.6) only has simple poles at $\alpha = i\omega_n$, since $K_+(\alpha)$ is nonzero and analytic, and since the infinite series is analytic in $\mathrm{Im}(\alpha) > -\frac{1}{2}$. If we consider the sequence of rectangular contours C_n with upper corners at $\frac{\pi}{p}(n+\frac{1}{2})(\pm 1+i)$, we find that $\coth\gamma = O(1)$ on C_n; in a fashion similar to our experiences in Section IV, $|\int_{C_n}| \to 0$ as $n \to \infty$. Hence, the evaluation of (5.5) can be obtained from the sum of residues of terms of (5.6). In standard fashion, we find that for $x < 0$

$$v(x,1) = 1 - \frac{A}{p^2}\sum_{n=1}\frac{\rho_n^2\exp[(\omega_n-\frac{1}{2})x]}{\omega_n(\omega_n-\frac{1}{2})(\rho_n^2+A^2+A)}$$

$$+\frac{A^2}{p^4}\sum_{n=1}^{\infty}\frac{\rho_n^2\exp[(\omega_n-\frac{1}{2})x]}{\omega_nK_+(i\omega_n)(\rho_n^2+A^2+A)}\sum_{k=1}^{\infty}\frac{\rho_k^2\,K_-(-i\omega_k)}{\omega_k(\omega_k+\frac{1}{2})(\rho_k^2+A^2+A)(\omega_n+\omega_k)}\,,$$

where $$\omega_k = \sqrt{\frac{\rho_k^2}{p^2}+\frac{1}{4}}\;; \quad k = 1,2\,\cdots,$$

satisfying $$\cot\rho_k = \rho_k/A\,. \qquad (5.7)$$

In the first series above, the general term is $O(1/n^2)$ for n large. The terms in the iterated series behave as $O(1/n^2)$ and $O(1/k^3)$, so that the series in (5.7) will then converge uniformly and absolutely for $x \leq 0$. Also, as $x \to -\infty$, we see that $v(x,1) \to 1$ in agreement with the boundary condition from (2.3). We note in passing that when $x = 0$, the temperature $v(0,1) = v_c$ is the (known) sputtering temperature. We shall make use of this observation in the next section to obtain an approximate value of the quench front velocity p, when $p << 1$.

Case II: $x > 0$

In this case we close the contour in the lower half-plane $\mathrm{Im}(\alpha) < \frac{1}{2}$. The only part of the integrand which contributes to the contour integral is $V_+(\alpha,1)$ which can be rewritten

$$V_+(\alpha,1) = \frac{-R_+(\alpha)}{K_+(\alpha)}$$

$$= \frac{Ai(\gamma\sinh\gamma + A\cosh\gamma)}{\sqrt{2\pi}\, p^2 K_-(\alpha)\, \gamma\sinh\gamma} \sum_{k=1}^{\infty} \frac{\rho_k^2\, K_-(-i\omega_k)}{\omega_k(\omega_k+\frac{1}{2})(\rho_k^2+A^2+A)(\alpha+i\omega_k)} \, . \qquad (5.8)$$

Note first of all that the simple poles in the infinite series at $\alpha = -i\omega_k$ are removable, since these are precisely the zeros of the factor $\gamma\sinh\gamma + A\cosh\gamma$ in the numerator. Otherwise, the only other singularities in the lower half-plane are at $\alpha = -i/2$ and $\alpha = -i\sigma_n$, where $\sigma_n \equiv \sqrt{\frac{n^2\pi^2}{p^2} + \frac{1}{4}}$, $n = 1,2,\ldots$. Therefore, for $x > 0$,

$$v(x,1) = \frac{A^2}{p^4}\left\{ \frac{e^{-x}}{K_-(-i/2)} \sum_{k=1}^{\infty} \frac{\rho_k^2 K_-(-i\omega_k)}{\omega_k(\rho_k^2+A^2+A)(\omega_k{}^2-\frac{1}{4})} \right.$$

$$\left. + \sum_{n=1}^{\infty} \frac{\exp[(-\sigma_n-\frac{1}{2})x]}{\sigma_n K_-(-i\sigma_n)} \sum_{k=1}^{\infty} \frac{\rho_k^2 K_-(-i\omega_k)}{\omega_k(\rho_k^2+A^2+A)(\omega_k+\frac{1}{2})(\omega_k-\sigma_n)} \right\} \qquad (5.9)$$

For x large, and positive, we see that $v(x,1) = O(e^{-x})$, in agreement with our asymptotic analysis from Section II. However, there may be some convergence problems with the iterated series in (5.9). In particular, when $k = n+1$, the term $\omega_k-\sigma_n$ is then $O(1/n^2)$, and series may fail to converge at all. For this reason, we shall use the series (5.7) instead in finding the velocity of the quench front.

VI. REVERSION OF THE SERIES. THE APPROXIMATE VELOCITY OF THE QUENCH FRONT. DISCUSSION.

At this point we are ready to find the velocity p of the quench front. On the surface of the slab at $x = 0$, $y = 1$, the temperature $v(0,1) = v_c$ is the sputtering temperature, a known constant. From (5.7) we see that

$$v_c = 1 - \frac{A}{p^2} \sum_{n=1}^{\infty} \frac{\rho_n^2}{\omega_n\left(\omega_n - \frac{1}{2}\right)\left(\rho_n^2+A^2+A\right)}$$

$$+ \frac{A^2}{p^4} \sum_{n=1}^{\infty} \frac{\rho_n^2}{\omega_n K_+(i\omega_n)\left(\rho_n^2+A^2+A\right)} \sum_{k=1}^{\infty} \frac{\rho_k^2\, K_-(-i\omega_k)}{\omega_k\left(\omega_k+\frac{1}{2}\right)\left(\rho_k^2+A^2+A\right)\left(\omega_n+\omega_k\right)} ,$$

where $$\omega_k = \sqrt{\frac{\rho_k^2}{p^2} + \frac{1}{4}} , \quad k = 1,2, \ldots , \tag{6.1}$$

satisfying $$\cot\rho_k = \rho_k/A .$$

It should be emphasized that the results of (6.1) are exact within the approximations of our model. In principle, we would like to revert the series above and solve for $p = p(A,v_c,\rho_k)$. Unfortunately, given the complicated nature of (6.1), especially due to the infinite products $K_+(i\omega_n)$ and $K_-(-i\omega_k)$, such a task seems hopeless. Instead, we opt for an approximation to (6.1) valid when both the quench front velocity p and the Biot number A are small.

It is reasonable to assume that both p and A are small together since the mechanism responsible for the existence of the quench front is the temperature gradient between the hot and cold portions of the slab;

when the heat transfer coefficient A is small, the temperature gradient is small, and so the velocity of the quench front should also be small.

When the two-dimensional problem we have considered is reduced to one dimension by integrating out the y-dependence of the differential equations, the velocity of the quench front [11], [12] is given exactly by

$$p = \sqrt{\frac{A}{v_c}}\,(1-v_c)\ . \tag{6.2}$$

The approximate solution proposed for the two-dimensionnal problem should compare closely with the one-dimensional solution (6.2), since an assumption in that model is that the temperature variations in y are small, and this is true only for a small heat transfer coefficient.

For our problem, we will show that the solution of (6.1) for p is given by (6.2) accurate up to terms of O(A).

To begin, we recall that the ρ_k are roots of the transcendental equation $\cot\rho_k = \rho_k/A$. It can easily be shown that for A sufficiently small,

$$\rho_1 = \sqrt{A}\,\left(1 - \frac{A}{6} + O(A^2)\right)\ . \tag{6.3}$$

We also recall that for n large,

$$\rho_n = (n-1)\pi + \frac{A}{(n-1)\pi} + O\left(\frac{A^2}{n^3}\right)\ . \tag{6.4}$$

Actually, this later result is also the expansion for ρ_n when A is small, and is reasonably accurate even for n = 2. In [19], the roots of $\cot\rho_k = \rho_k/A$ are tabulated and are repeated here in Table 6-1 for different values of A.

A	ρ_2	ρ_3	ρ_4
0	π = 3.14159	2π = 6.28319	3π = 9.42478
0.05	3.15743	6.29113	9.43008
0.15	3.18860	6.30696	9.44067
0.25	3.21910	6.32270	9.45122
0.50	3.29231	6.36162	9.47749

Table 6-1

Next, we make use of these results in simplifying the infinite products in K_+ and K_-. If we recall from (3.6) and (3.7) the definitions of K_+ and K_-, it is clear that $K_-(-\alpha) = 1/K_+(\alpha)$. Thus, we need only consider the form of K_+, which can be written using (3.8) as

$$K(i\omega_k) = \frac{(\frac{p}{2} + p\omega_k) \prod_{n=1}^{\infty} [(1+ \frac{p^2}{4n^2\pi^2})^{\frac{1}{2}} + \frac{p\omega_k}{n\pi}]\exp(\frac{-p\omega_k}{n\pi})}{\sqrt{A}\,\exp(\frac{p\omega_k}{\rho_1}) \prod_{n=1}^{\infty} [(1 + \frac{p^2}{4\rho_n^2})^{\frac{1}{2}} + \frac{p\omega_k}{\rho_n}]\,\exp(\frac{-p\omega_k}{\rho_n})}$$

$$= \frac{(\frac{p}{2} + p\omega_k)}{\sqrt{A}[(1+\frac{p^2}{4\rho_1^2})^{\frac{1}{2}} + \frac{p\omega_k}{\rho_1}]} \times \frac{\prod_{n=1}^{\infty} [(1+ \frac{p^2}{4n^2\pi^2})^{\frac{1}{2}} + \frac{p\omega_k}{n\pi}]\exp(\frac{-p\omega_k}{n\pi})}{\prod_{n=2}^{\infty} [(1 + \frac{p^2}{4\rho_n^2})^{\frac{1}{2}} + \frac{p\omega_k}{\rho_n}]\exp(\frac{-p\omega_k}{\rho_n})} \tag{6.5}$$

where $\omega_k = \sqrt{\frac{\rho_k^2}{p^2} + \frac{1}{4}}$, $k = 1,2, \ldots$

satisfying $\cot\rho_k = \rho_k/A$.

Using the behavior of the ρ_k's from (6.3) and (6.4), it can be shown that the quotient of infinite products in (6.5) behaves as

$$\frac{\prod_{n=1}^{\infty}\left[\left(1+\frac{p^2}{4n^2\pi^2}\right)^{\frac{1}{2}}+\frac{p\omega_k}{n\pi}\right]\exp\left(\frac{-p\omega_k}{n\pi}\right)}{\prod_{n=2}^{\infty}\left[\left(1+\frac{p^2}{4\rho_n^2}\right)^{\frac{1}{2}}+\frac{p\omega_k}{\rho_n}\right]\exp\left(\frac{-p\omega_k}{\rho_n}\right)} = \begin{cases} 1+O(A^2) & \text{if} \quad k=1 \\ 1+O(A) & \text{if} \quad k\geq 2 \, . \end{cases}$$

Hence, after some simplification we can write

$$K_+(i\omega_k) = \frac{\rho_1\left(\frac{p}{2}+p\omega_k\right)}{\sqrt{A}\left[\left(1+\frac{p^2}{4\rho_1^2}\right)^{\frac{1}{2}}+\frac{p\omega_k}{\rho_1}\right]} + O(A) \, , \quad k \geq 1 \, . \tag{6.6}$$

Armed with this result, we can approximate (6.1) by truncating each series at the first term. It can be shown that such a seemingly crude approximation introduces errors of O(A) at most so that

$$v_c = 1 - \frac{\delta}{(p\omega_1)(p\omega_1-p/2)} + \frac{2A\,\delta^2}{\rho_1^2(p\omega_1)(p/2+p\omega_1)^3} + O(A) \, ,$$

where

$$\delta = \frac{A\rho_1^2}{\rho_1^2 + A^2 + A} \, . \tag{6.7}$$

The validity of this truncation is further reinforced when we consider that the series in (6.1) converge rapidly as $\left(\frac{1}{n^2}\right)$ or $\left(\frac{1}{k^3}\right)$ depending on the appropriate summation index.

We next note that under the approximation (6.3),

$$\delta = \frac{A}{2}\left[1 - \frac{2}{3}A + O(A^2)\right] ,$$

so that (6.7) becomes

$$v_c = 1 - \frac{A/2}{(p\omega_1)(p\omega_1 - p/2)} + \frac{A^2/2}{(p\omega_1)(p/2 + p\omega_1)^3} + O(A) , \qquad (6.8)$$

where $$p\omega_1 = \sqrt{\rho_1^2 + \frac{p^2}{4}} .$$

As a check of our approximations, we note that if $p = 0$, then v_c should equal the (modified) upstream temperature of 1 from (2.3). In this case (6.8) reduces to

$$v_c = 1 + O(A) ,$$

as expected.

Finally, if $p << 1$ it can be shown that

$$P = \sqrt{\frac{A}{v_c}}\,(1 - v_c) + O(A) . \qquad (6.9)$$

As pointed out earlier, the leading term in this expansion is the exact solution for a one-dimensional slab.

It is also worth comparing our results with those obtained in [14] using an approximate kernel factorization and a nonrigorous asymptotic analysis. Again, we find exact agreement for the velocity of the quench front to lowest order, pointing to the validity of the choice of the approximate kernel in that analysis.

REFERENCES

[1] T. H. Moss, D. L. Sills, eds., "The Three Mile Island Nuclear Accident: Lessons and Implications," Annals of the New York Academy of Sciences 365 (1981).

[2] R. B. Duffey, D. T. C. Porthouse, "The Physics of Rewetting in Water Reactor Emergency Core Cooling," Nuc. Eng. Des. 25 (1973), 379-394.

[3] T. S. Thompson, "On the Process of Rewetting a Hot Surface by a Falling Liquid Film," Nuc. Eng. Des. 31 (1974), 234-245.

[4] D. Butterworth, R. G. Owen, "The Quenching of Hot Surfaces by Top and Bottom Flooding - a Review," AERE Report R7992, AERE Harwell, Oxfordshire, England (March 1975).

[5] J. J. Carbajo, A. D. Siegel, "Review and Comparison Among the Different Models for Rewetting in LWR's," Nuc. Eng. Des. 58 (1980), 33-44.

[6] L. S. Yao, "Rewetting of a Vertical Surface with Internal Heat Generation, Proc. Solar and Nuclear Heat Transfer, AIChE Symposium 73 (164) (1976), 46-50.

[7] cited in: H. T. Laguer, B. Wendroff, "Bounds for the Model Quench Front," LA-UR-80-727, Los Alamos Scientific Laboratory (1980).

[8] M. W. E. Coney, "Calculations on the Rewetting of Hot Surfaces," Nuc. Eng. Des. 31 (1974), 246-259.

[9] J. G. M. Andersen, P. Hansen, "Two-Dimensional Heat Conduction in Rewetting Phenomenon," Report No. NORHAV-D-6, Danish Atomic Energy Commission Research Establishment, Risø, Denmark (1974).

[10] D. Durack, B. Wendroff, "Computing a Two-Dimensional Quench Front," Nucl. Sci. Eng. 64 (1977), 187-191.

[11] R. Semeria, B. Martinet, "Colefaction Spots on a Heating Wall: Temperature Distribution and Resorption," Proc. Inst. Mech. Eng. 180 (1965), 192-205.

[12] A. Yamanouchi, "Effect of Core Spray Cooling in Transient State After Loss of Coolant Accident," J. Nucl. Sci. Tech. 5 (1968), 547-558.

[13] B. Wendroff, "A Surface Approximation for Quench Front Motion," LA-UR-77-2565, Los Alamos Scientific Laboratory (1977).

[14] C. L. Tien, L. S. Yao, "Analysis of Conduction-Controlled Rewetting of a Vertical Surface," J. Heat Transfer 97 (1975), 161-165.

[15] S. S. Dua, C. L. Tien, "A Generalized Two-Parameter Relation for Conduction-Controlled Rewetting of a Hot Vertical Surface," Int. J. Heat Mass Transfer 20 (1977), 174-176.

[16] B. Noble, Methods Base on the Wiener-Hopf Technique for the Solution of Partial Differential Equations, Pergamon Press (1958).

[17] E. C. Titchmarsh, Theory of Functions, 2nd Ed., Oxford University Press (1939).

[18] G. F. Carrier, M. Krook, C. E. Pearson, Functions of a Complex Variable, McGraw-Hill (1966).

[19] M. Abramowitz, I. A. Stegun, ed., Handbook of Mathematical Functions, Dover (1972).

PART II

ASYMPTOTIC EXPANSIONS OF INTEGRALS

WITH

THREE COALESCING SADDLE POINTS

I. INTRODUCTION

A great deal of research has been devoted to problems of wave propagation in dispersive media in which the dispersive waves are defined by integrals of the form

$$I(x,t) = \int_\Gamma g(k) \exp[i(kx-\omega t)]dk \quad , \tag{1.1}$$

where $\omega = \omega(k)$ is called the dispersion relation. In simple cases, the asymptotic behavior of (1.1) for large t, say, can easily be evaluated by the method of stationary phase. In this situation the dominant behavior of the integral is governed by the critical points $k = k_i$ satisfying

$$\frac{d\omega}{dk} = \frac{x}{t} \tag{1.2}$$

Recently, an interest has arisen in integrals such as (1.1) containing multiple critical points that are allowed to coalesce forming a single critical point of higher order; specifically, the case of three simple critical points coalescing to form a third order critical point. Physically, such problems can arise in the interaction of surface and internal waves [1], wave trains in stratified shear flows [2], and in the field structure of a cylindrical electromagnetic wave train in the neighborhood of a line focus [3]. Using standard asymptotic techniques such as the methods of stationary phase or steepest descent, the asymptotic behavior of the integral for distinct critical points or for coalesced critical points is well known. What has not been considered previously, however, is the uniform asymptotic behavior governing the coalescence of the three critical points into one.

From another viewpoint, we can examine this problem by allowing the group velocity $d\omega/dk$ in (1.2) to depend on some parameter θ as in

$$\frac{d\omega}{dk} = C_g(k;\theta) \quad , \tag{1.3}$$

where a typical C_g is sketched in Figure 1-1 for varying θ. We see that

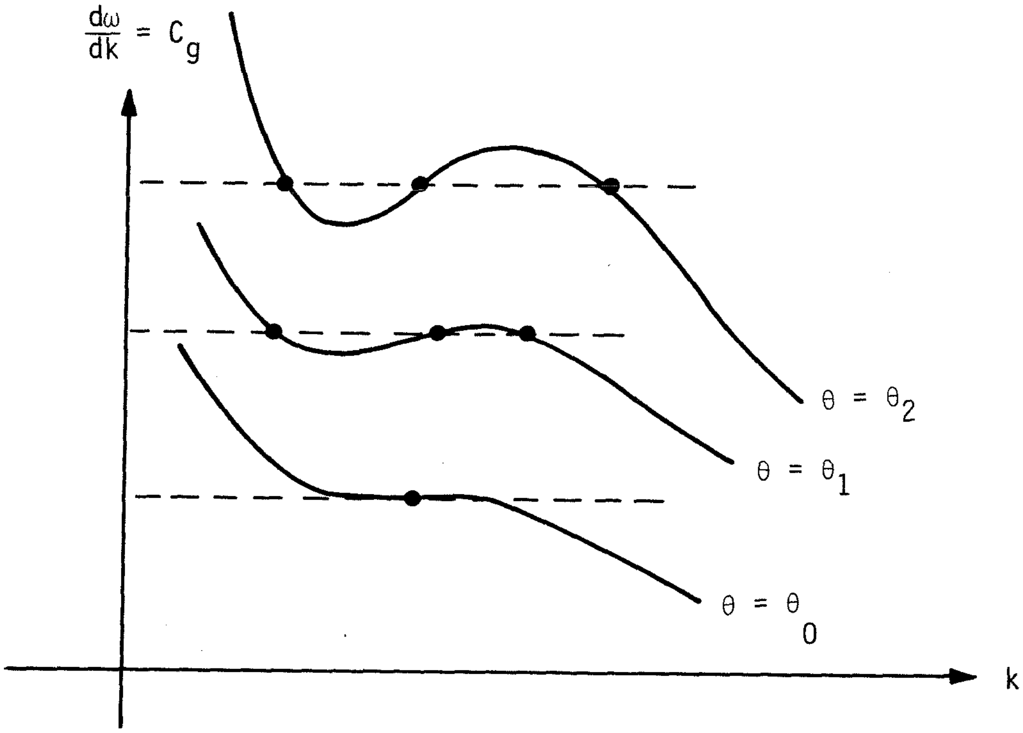


Figure 1-1

as $\theta \to \theta_0$, the distinct critical points coalesce. In terms of I, where t is the large parameter, we seek an asymptotic expansion of I as $t \to \infty$, uniformly as $\theta \to \theta_0$.

Mathematically, integrals with two coalescing saddle points have been examined extensively in the literature. The fundamental papers in this area, due to Chester, Friedman, and Ursell [4], and later Friedman

[5], express the uniform asymptotic behavior in terms of Airy functions and their derivatives. Some results in the more general problem of p coalescing critical points ($p \geq 2$) are presented in [6] by Bleistein in terms of "generalized" Airy functions. In all cases, the critical step in the asymptotic analysis is the transformation of the given integral (such as (1.1)) into a simpler "canonical" integral that exhibits the same properties. For two coalescing saddle points, the canonical integral is the Airy Integral; hence the resulting solutions are related to Airy Functions.

The situation involving three coalescing critical points is not nearly as well formulated as the case for two saddle points, probably due in part to its more infrequent occurrence in nature, but also because the solutions are not expressible in terms of known simple functions. Pearcey [3] and Hughes [1] consider the canonical integrals (Pearcey Functions?)

$$I(x,y) = \int_{-\infty}^{\infty} \exp\left[i\left(\frac{1}{4} u^4 - \frac{1}{2} xu^2 - yu\right)\right]du \tag{1.4}$$

and show that the three critical points present for $y^2 < \frac{4}{27} x^3$ coalesce on the cusped curve $y^2 = \frac{4}{27} x^3$. Using the method of stationary phase, they obtain the leading (one term) behavior on and off the cusp of coalescence.

Our approach to this problem is to adopt a much simpler canonical integral than (1.4), namely the contour integral

$$J(\lambda;\gamma) = \int_{C_t} \exp\left[\lambda\left(\frac{-t^4}{4} + \frac{\gamma^2}{2} t^2\right)\right]dt \;, \tag{1.5}$$

for some appropriately chosen contour C_t. This form has the immediate advantage over (1.4) of having explicitly factorable saddle points at $t = 0, \pm \gamma$. As γ tends to zero, the three saddle points coalesce to the origin. We find the complete asymptotic expansion of (1.5) for $\lambda \to \infty$ uniformly as $\gamma \to 0$ using the method of steepest descent. Much of the notation and formalism of our analysis has been borrowed from Bleistein and Handelsman [7] in their expository discussion of the case of two coalescing saddle points.

II. FORMULATION OF THE PROBLEM

In this section, we consider contour integrals of the form

$$I(\lambda;\underline{\alpha}) = \int_{C_z} g(z)\ \exp[\lambda w(z;\underline{\alpha})]dz\ , \qquad (2.1)$$

with parameters $\lambda \varepsilon \mathbb{R}$ and $\underline{\alpha}$, where $\underline{\alpha} = (\alpha_1,\alpha_2,\alpha_3)$, for $\alpha_i\ \varepsilon\ \mathbb{C}$. We shall assume that $g(z)$ and $w(z;\underline{\alpha})$ are analytic functions of z in a simply connected domain D which contains the contour C_z and the points $z = \alpha_1$, $z = \alpha_2$ and $z = \alpha_3$. We further assume that the exponent $w(z;\underline{\alpha})$ has simple saddle points at $z = \alpha_1$, $z = \alpha_2$, $z = \alpha_3$ provided that $\alpha_1 \neq \alpha_2$, $\alpha_2 \neq \alpha_3$, $\alpha_1 \neq \alpha_3$. Therefore,

$$\frac{\partial w}{\partial z}(\alpha_i;\underline{\alpha}) = 0$$

and $\qquad (2.2)$

$$\frac{\partial^2 w}{\partial z^2}(\alpha_i;\underline{\alpha}) \neq 0 \qquad ; \qquad i = 1,2,3$$

provided that the α_i are distinct. To simplify the analysis, we shall assume that all other saddle points of the exponent w lie in the valleys of w. The contour C_z may be of infinite extent, but must approach infinity in the valleys.

We next suppose that the three saddle points are free to move in D and in particular that they can coalesce there, forming a saddle of order 3. Thus, when $\alpha_1 = \alpha_2 = \alpha_3$,

$$w_z(\alpha_i;\underline{\alpha}) = w_{zz}(\alpha_i;\underline{\alpha}) = w_{zzz}(\alpha_i;\underline{\alpha}) = 0$$

and (2.3)

$$w_{zzzz}(\alpha_i;\underline{\alpha}) \neq 0 \ ; \quad i = 1,2,3 \ .$$

Our objective is to find an asymptotic expansion of $I(\lambda;\underline{\alpha})$ as $\lambda \to \infty$ which is valid uniformly as the α_i coalesce in D. In what follows, we shall adopt much of the nomenclature of Bleistein and Handelsman in [7]. Briefly, the idea is to introduce a new variable of integration, say t, which simplifies the exponent $w(z;\underline{\alpha})$ while retaining all of the basic properties in (2.2) and (2.3). Thus, the transformation $z = z(t)$ should be a one-to-one conformal map of the domain D containing all the saddle points of interest onto a domain D_1 in the complex t-plane while being as simple as possible. Under this change, the exponent w becomes

$$w(z;\underline{\alpha}) = w\big(z(t);\underline{\alpha}\big) \equiv \phi(t;\underline{\alpha}) \ .$$

In order to satisfy (2.2) and (2.3), the simplest form for ϕ would be a polynomial of degree 4. One possible choice for ϕ would be

$$w\big(z(t);\underline{\alpha}\big) = \frac{-t^4}{4} - \frac{\gamma(c-1)}{3}\, t^3 + \frac{\gamma^2 c}{2}\, t^2 + \beta \ , \qquad (2.4)$$

for some constants γ,c, and β. In this representation, the critical points are located at $t = \gamma$, $t = 0$, and $t = -\gamma c$ for $\gamma \neq 0$. As $\gamma \to 0$, the saddle points coalesce at the origin of the t-plane. In actuality, however, we may choose an even simpler form for the exponent w by setting c=1 in (2.4), yet still retaining the salient characteristics of (2.2) and (2.3) above. We have then that

$$w(z(t);\underline{\alpha}) \equiv \phi(t;\gamma) = \frac{-t^4}{4} + \frac{\gamma^2}{2}t^2 + \beta . \qquad (2.5)$$

From another viewpoint, the choice of the "canonical" polynomial (2.5) is (conformally) equivalent to any other polynomial having three distinct saddle points in that we can always find a bilinear transformation, conformal in the extended plane, which maps three distinct points into three distinct points.

In order that the transformation $z = z(t)$ is a conformal map of D onto D_1, we must require that dz/dt is finite and nonzero for all t in D_1 and all z in D. By differentiating (2.5) with respect to t, we find that

$$\dot{z}(t) = \frac{dz}{dt} = \frac{-t^3 + \gamma^2 t}{w_z(z;\underline{\alpha})} . \qquad (2.6)$$

Clearly, the only difficulties in this expression arise when $t = 0, \pm \gamma$ and when $w_z = 0$ at one of the saddle points $z = \alpha_i$, $i = 1,2,3$. Thus, at the outset, we need to impose the condition that saddle points in the t-plane correspond to saddle points in the z-plane, say

$$\begin{aligned} t &= \gamma & &\Longleftrightarrow & z &= \alpha_1 \\ t &= 0 & &\Longleftrightarrow & z &= \alpha_2 \\ t &- -\gamma & &\Longleftrightarrow & z &= \alpha_3 . \end{aligned} \qquad (2.7)$$

Hence we obtain the expressions

$$\begin{aligned} \beta &= w(\alpha_2;\underline{\alpha}) \\ \gamma^4 &= 4[w(\alpha_1;\underline{\alpha}) - w(\alpha_2;\underline{\alpha})] \\ &= 4[w(\alpha_3;\underline{\alpha}) - w(\alpha_2;\underline{\alpha})] \end{aligned} \qquad (2.8)$$

It should be noted at this point that (2.8) does not determine γ uniquely. In particular, we shall need to specify the appropriate branch of the fourth root, a consideration to be addressed later in this section.

With the assignment of saddle points in (2.7), the expression for $\dot{z}(t)$ in (2.6) is indeterminate at any one of the critical points. Applying L'Hospital's rule, we find that

$$\left(\dot{z}(t)\right)^2\Big|_{\substack{t\,=\,\pm\,\gamma,0\\ z\,=\,\alpha_i}} = \frac{-3t^2+\gamma^2}{w_{zz}(z;\underline{\alpha})}\Big|_{\substack{t\,=\,\pm\,\gamma,0\\ z\,=\,\alpha_i}} \tag{2.9}$$

which is finite and nonzero provided that $\gamma \neq 0$. If $\gamma = 0$ (corresponding to the coalescence of the saddle points), then we must apply L'Hospital's rule twice more so that

$$\left(\dot{z}(t)\right)^4\Big|_{\substack{\gamma=0=t\\ z=\alpha_1=\alpha_2=\alpha_3}} = \frac{-6}{w_{zzzz}(z;\underline{\alpha})}\Big|_{\substack{\gamma=0=t\\ z=\alpha_1=\alpha_2=\alpha_3}} \tag{2.10}$$

which is finite and nonzero.

Thus, under the transformation (2.5) we can rewrite (2.1) as

$$I(\lambda;\underline{\alpha}) = \int_{C_t} G_0(t;\underline{\alpha})\, \exp\left[\lambda\phi(t;\gamma)\right]dt \ , \tag{2.11}$$

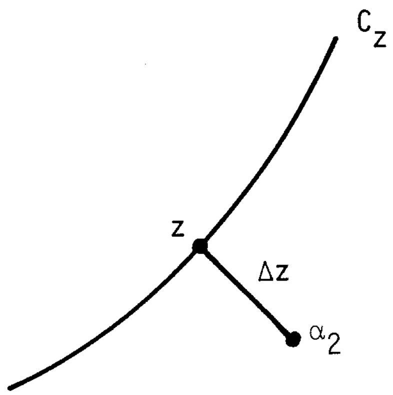
where $$G_0(t;\underline{\alpha}) = g\left(z(t)\right)\frac{dz}{dt}$$

and where C_t is the image of C_z in the t-plane. In performing the asymptotic expansion of (2.11) we will need to deform the contour C_t to the steepest descent paths through the saddle points. If C_t is of infinite extent, it must approach infinity in the valleys of the exponent. from (2.5), we see that $\exp[\lambda\phi(t;\gamma)]$ vanishes as $|t| \to \infty$, independent of γ in any one of the four sectors

$$\frac{\pi}{2}\left(n - \frac{1}{4}\right) < \arg(t) < \frac{\pi}{2}\left(n + \frac{1}{4}\right) , \quad n = 0,1,2,3 .$$

By hypothesis, C_t must begin and end in these sectors.

At this juncture we are prepared to return to the unanswered question of a branch assignment for γ in equation (2.8). In a related note, we have seen that a necessary condition for the conformality of the transformation $z = z(t)$ is that γ and β are defined by (2.7) and (2.8). Whether these conditions are sufficient for a conformal map to exist is unknown. In the analysis of two coalescing saddle points, Chester, Friedman and Ursell [4] have shown that the canonical polynomial ϕ (in this case of degree 3) and the conditions corresponding to (2.7) and (2.8) have just one branch which defines a conformal map of the domain D containing the saddle points and the contour C_z. When there are more than two critical points, Bleistein [6] has shown that in the neighborhood of each point in the z-plane, the transformation $z = z(t)$ of the exponent w is locally (1-1) analytic in all variables. Following this work, we will assume that where required, the transformation $z = z(t)$ is globally (1-1) analytic. Next, in order to resolve the ambiguity surrounding the branch choice for γ in (2.8), consider the transformation $z = z(t)$ which maps the contour C_z in the z-plane into, say, C_t in the t-plane as shown in Figure 2-1. Examining the transformation in the neighborhood of the origin of the t-plane, we may define $\Delta t = t$ to be the image of $\Delta z = z - \alpha_2$, where t is an arbitrary point on C_t and z its preimage on C_z. For the above choice of C_t we see



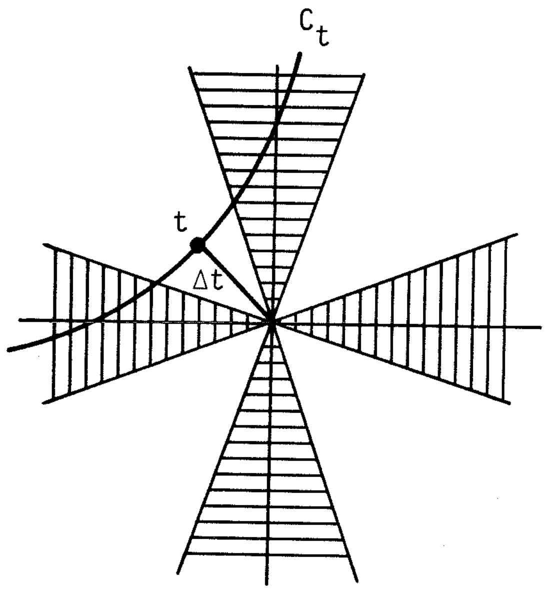


Figure 2-1. The contour C_z and its image C_t. The shaded regions in (b) denote the sectors $\frac{\pi}{2}(n - \frac{1}{4}) < \arg(t) < \frac{\pi}{2}(n + \frac{1}{4})$, n = 0,1,2,3.

that

$$\frac{\pi}{2} < \arg(\Delta t) < \pi \tag{2.12}$$

Since we know that approximately

$$\dot{z}(t)\Big|_{t=0} \simeq \frac{\Delta z}{\Delta t} \quad ,$$

we see using (2.9) that

$$(\Delta z)^2 \simeq \frac{\gamma^2(\Delta t)^2}{w_{zz}(\alpha_2;\underline{\alpha})} \; . \tag{2.13}$$

Taking the argument of both sides of (2.13) yields

$$\arg(\Delta t)^2 \simeq \arg\left[\frac{(\Delta z)^2 w_{zz}(\alpha_2;\underline{\alpha})}{\gamma^2}\right] \; .$$

Finally, from (2.12) we find that

$$\arg(\Delta z) + \frac{1}{2}\arg\left[w_{zz}(\alpha_2;\underline{\alpha})\right] - \pi < \arg(\gamma) < \arg(\Delta z) + \frac{1}{2}\arg\left[w_{zz}(\alpha_2;\underline{\alpha})\right] - \frac{\pi}{2} \; , \tag{2.14}$$

thus restricting γ to a sector in the complex plane having an angle of $\pi/2$, and fixing the branch of the fourth root. Clearly, we could have chosen for C_t any of the other three contours ending in neighboring shaded sectors in the t-plane of Figure 2-1.

Returning to equation(2-11), our next task is to expand the amplitude term $G_0(t;\underline{\alpha})$ to simplify the derivation of the uniform asymptotic expansion. To this end, we set

$$G_0(t;\underline{\alpha}) = g_0 + g_1 t + g_2 t^2 + \left[t^3 - \gamma^2 t\right] Q_0\,(t;\underline{\alpha}) \; , \tag{2.15}$$

where g_0, g_1, and g_2 are constants and where $Q_0(t;\underline{\alpha})$ is by hypothesis an analytic function of t in D_1. With Q_0 regular, it should be noted that the last term in (2.15) vanishes at the critical points $t = 0, \pm\gamma$. Because of this, the contribution of the final term to the asymptotic behavior as $\lambda \to \infty$ is lessened. We will exploit this observation later in obtaining the complete asymptotic expansion for the integral.

If we set $t = 0, \pm \gamma$ in (2.15) we find that

$$g_0 = G_0(0;\underline{\alpha})$$

$$g_1 = \frac{G_0(\gamma;\underline{\alpha}) - G_0(-\gamma;\underline{\alpha})}{2\gamma} \tag{2.16}$$

$$g_2 = \frac{G_0(\gamma;\underline{\alpha}) + G_0(-\gamma;\underline{\alpha}) - 2G_0(0;\underline{\alpha})}{2\gamma^2} \quad .$$

The expressions for g_1 amd g_2 have removable singularities for $\gamma = 0$. Using L'Hospital's rule,

$$\lim_{\gamma\to 0} g_1 = \frac{dG_0(0;\underline{\alpha})}{dt}$$

$$\lim_{\gamma\to 0} g_2 = \frac{d^2G_0(0;\underline{\alpha})}{dt^2} \quad . \tag{2.17}$$

With g_0, g_1, and g_2 from above, we can solve for Q_0 as

$$Q_0(t;\underline{\alpha}) = \frac{G_0(t;\underline{\alpha}) - g_0 - g_1 t - g_2 t^2}{t^3 - \gamma^2 t} \quad , \tag{2.18}$$

where, in this equation, the right hand side is indeterminate at $t=0, \pm \gamma$. As usual, appealing to L'Hospital's rule, we obtain that as $t \to 0$

$$\lim_{t\to 0} Q_0(t;\underline{\alpha}) = \frac{G_0(\gamma;\underline{\alpha}) - G_0(-\gamma;\underline{\alpha}) - 2\gamma\dot{G}_0(0;\underline{\alpha})}{2\gamma^3} \quad , \tag{2.19}$$

which is finite provided $\gamma \neq 0$. When $\gamma \to 0$, (2.19) becomes indeterminate so that

$$\lim_{\gamma\to 0}\ \lim_{t\to 0}\ Q_0(t;\underline{\alpha}) = \frac{\dddot{G}_0\,(0;\underline{\alpha})}{6} \tag{2.20}$$

Likewise, as $t \to \pm\gamma$,

$$\lim_{t\to\pm\gamma}\ Q_0(t;\underline{\alpha}) = \frac{\dot{G}_0(\pm\gamma;\underline{\alpha}) - g_1 \mp 2\gamma g_2}{2\gamma^2}\ , \tag{2.21}$$

which is also indeterminate as $\gamma \to 0$. As before, we find that

$$\lim_{\gamma\to 0}\ \lim_{t\to\pm\gamma}\ Q_0(t;\underline{\alpha}) = \frac{\dddot{G}_0(0;\underline{\alpha})}{6} \tag{2.22}$$

Finally, if we insert (2.15) into (2.11) using (2.5) we conclude that

$$I(\lambda;\underline{\alpha}) = e^{\lambda\beta}\int_{C_t} (g_0+g_1t+g_2t^2)\ \exp\{\lambda(\frac{-t^4}{4} + \frac{\gamma^2}{2}\ t^2)\}dt + R_0(\lambda;\underline{\alpha})\ ,$$

where

$$R_0(\lambda;\underline{\alpha}) = e^{\lambda\beta}\int_{C_t} (t^3-\gamma^2t)Q_0(t;\underline{\alpha})\ \exp\{\lambda(\frac{-t^4}{4} + \frac{\gamma^2t^2}{2})\}dt\ . \tag{2.23}$$

If the asymptotic properties of the integrals in (2.23) were known for $\lambda \to \infty$ and $\gamma \to 0$, our work would be complete. Unfortunately, they are not, nor do the integrals seem to be expressible in terms of "known" simple functions. Therefore, our next undertaking, in Section III, is a digression on the asymptotic expansion of integrals of the form

$$J(\lambda;\gamma) = \int_{C_t} \exp\{\lambda(\frac{-t^4}{4} + \frac{\gamma^2}{2}\ t^2)dt \tag{2.24}$$

as $\lambda \to \infty$ for $\gamma \to 0$. In point, the expressions will be simplified if instead we consider

$$H(\zeta) = \int_{C_t} \exp(\zeta t^2 - \frac{t^4}{4}) dt \ , \tag{2.25}$$

where J and H are related by

$$J(\lambda;\gamma) = \frac{1}{\lambda^{1/4}} H(\frac{\gamma^2 \lambda^{1/2}}{2}) \ . \tag{2.26}$$

Following this, in Section IV, we will obtain an asymptotic expansion for $I(\lambda;\underline{\alpha})$ in (2.23) in terms of H and its derivatives.

III. ASYMPTOTIC BEHAVIOR OF THE CANONICAL INTEGRAL

The topic of this section is the asymptotic analysis of the integral

$$J(\lambda;\gamma) = \int_{C_t} \exp\left[\lambda\left(\frac{-t^4}{4} + \frac{\gamma^2}{2} t^2\right)\right]dt \;. \tag{3.1}$$

In (3.1) we will examine two different regimes for the parameters λ and γ. In the first case we shall examine the behavior of (3.1) for λ large and positive and γ nonzero. In the second we will consider the behavior for λ large and positive and γ small or zero.

A. BEHAVIOR AS $\lambda \to \infty$ FOR $\gamma \neq 0$.

In order to facilitate an analysis using the method of steepest descent, we let $t = \gamma t'$ in (3.1). After dropping primes,

$$J(\lambda;\gamma) = \gamma \int_{C_t} \exp\left(k\, g(t)\right)dt \tag{3.2}$$

where $$g(t) = \frac{t^2}{2} - \frac{t^4}{4}$$

and where $$k = \lambda\gamma^4 \;. \tag{3.3}$$

The central idea in the method of steepest descent is to deform the original contour C_t to the steepest path of descent through the appropriate saddle points of the exponent $g(t)$. On the steepest paths, $\mathrm{Im}\left(g(t)\right)$ is constant, and in particular on the steepest path of descent $\mathrm{Re}\left(g(t)\right)$ is decreasing away from the saddle points. For our problem, the saddle points are the solutions of

$$g'(t) = 0 ,$$

which are (3.4)

$$t = 0, \pm 1 .$$

We find that

$$g(0) = 0$$

$$g(\pm 1) = \frac{1}{4} \qquad (3.5)$$

so that if $t = x + iy$, the steepest paths are given by

$$\text{Im}\left[g(t)\right] = xy\left(1 - x^2 + y^2\right) = 0 . \qquad (3.6)$$

as sketched in Figure 3-1.

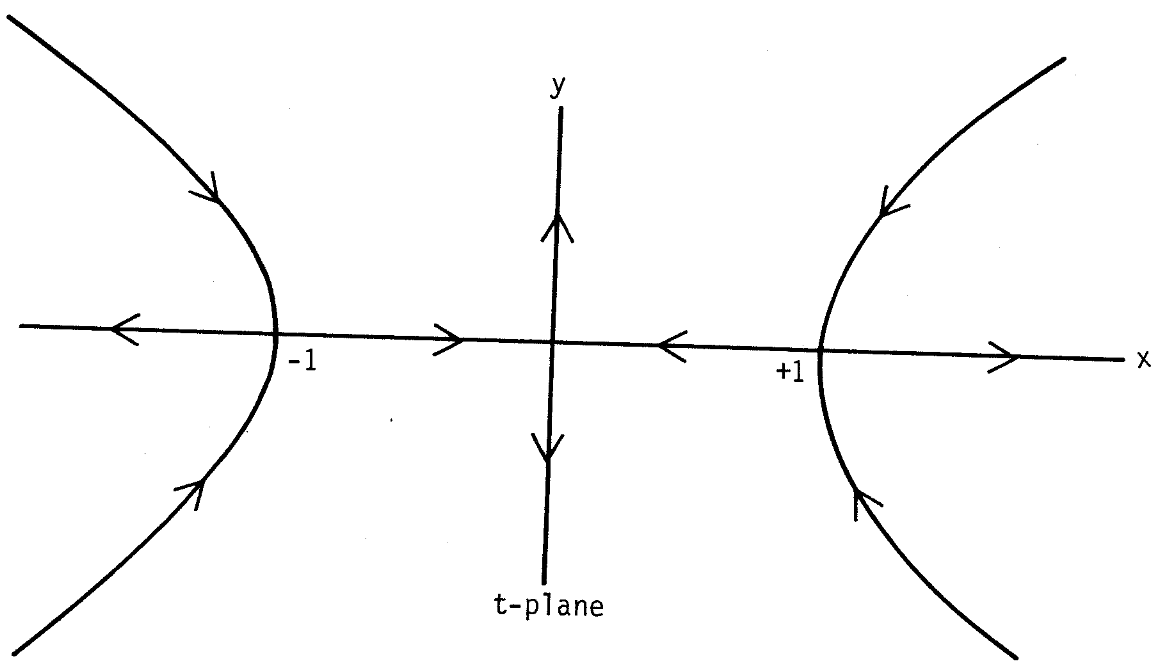


Figure 3-1. Steepest paths $x = 0$, $y = 0$, and $x^2 - y^2 = 1$. Arrows indicate direction of decreasing $\text{Re}\left[g(t)\right]$.

In order to complete the picture of the global topography, we also need to find the level curves $\text{Re}\left[g(t)\right] = \text{constant}$ through each of the saddle points. The level curve through the saddle at the origin is given by $\mathrm{Re}\big(g(t)\big) = \mathrm{Re}\big(g(0)\big) = 0$, or

$$y^2 = 3x^2 - 1 + \sqrt{8x^4 - 4x^2 + 1}\ . \tag{3.7}$$

Likewise, the level curves through $t = \pm 1$ are given by $\mathrm{Re}\big(g(t)\big) = \frac{1}{4}$, or

$$y^2 = 3x^2 - 1 - 2\sqrt{2x^4 - x^2}\ . \tag{3.8}$$

Geometrically, the level curves give the boundaries between the hills and valleys of the saddle points. The configuration for our problem is sketched roughly in Figure 3-2.

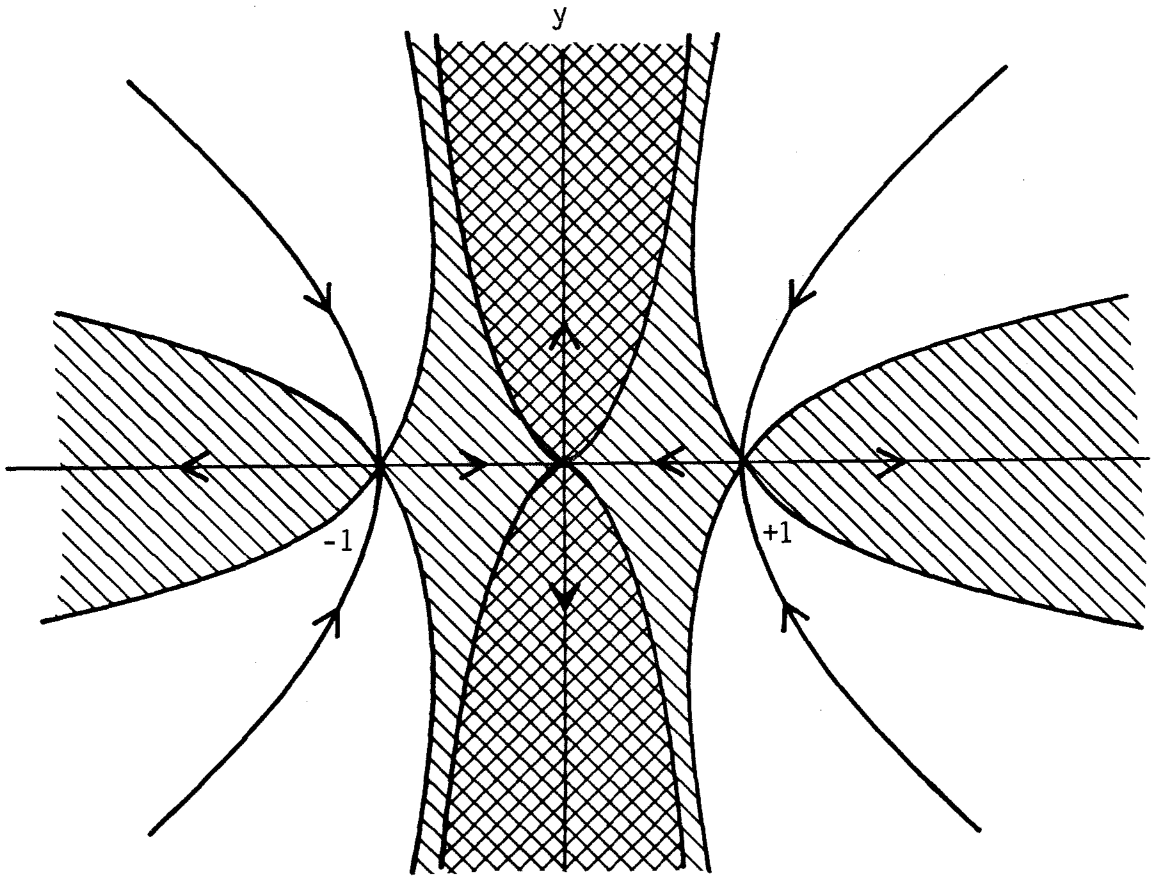


Figure 3-2. Level curves and steepest paths for g(t).

 valleys of t = 0 valleys of t = ± 1

We note immediately that the saddle point $t = 0$ lies entirely in the valleys of the saddles at $t = \pm 1$. This implies that for any steepest path contour through the origin and either of the saddles at $t = \pm 1$, the contribution from the saddle at the origin will be exponentially smaller and hence asymptotically negligible compared with the contributions from $t = \pm 1$. Since the topography is symmetric with respect to the y-axis, the above analysis suggests that for the deformed contours we consider one contour involving only the saddle at the origin, and another contour involving any combination of the other saddle points. For definiteness, let C_1 denote the conttour from left to right along the Re(t)-axis, and $J_1(\lambda;\gamma)$ the corresponding integral. Likewise, we let C_2 denote the contour from bottom to top along the Im(t)-axis passing only through the saddle at the origin, and $J_2(\lambda;\gamma)$ the integral involving C_2.

For the path along the Re(t)-axis we let

$$g(t) - g(\pm 1) = -s \ , \text{ where } s \in \mathbb{R} \ , \tag{3.9}$$

and non-negative on the path of integration. Thus

$$J_1(\lambda;\gamma) \sim \gamma e^{k/4} \int_{C_1} e^{-ks} \left(\frac{dt}{ds}\right) ds \ , \tag{3.10}$$

which is a Laplace-type integral. The next step in the procedure is to find $(\frac{dt}{ds})$, and in particular, to expand $\frac{dt}{ds}$ in the vicinity of the saddle points (i.e., where $s = 0$) since that is where the major contribution to the integral occurs. Using (3.9) we have that

$$(t^2-1)^2 = 4s \ ,$$

so that referring to Figure 3-3,

$$\left.\begin{aligned}
\frac{dt}{ds} &= \frac{1}{2\sqrt{s}\,(1 + 2\sqrt{s})^{1/2}} &\quad \text{on} \quad & 0_1A_1 \\
\frac{dt}{ds} &= \frac{-1}{2\sqrt{s}\,(1 + 2\sqrt{s})^{1/2}} &\quad \text{on} \quad & 0_{-1}A_{-1} \\
\frac{dt}{ds} &= \frac{-1}{2\sqrt{s}\,(1 - 2\sqrt{s})^{1/2}} &\quad \text{on} \quad & 0_1B_1 \\
\frac{dt}{ds} &= \frac{1}{2\sqrt{s}\,(1 - 2\sqrt{s})^{1/2}} &\quad \text{on} \quad & 0_{-1}B_{-1}
\end{aligned}\right\} \tag{3.11}$$

If we expand the expressions in parentheses in each of the denominators of (3.11) in a series involving powers of $\sqrt{s}$, the integral in (3.10 can be evaluated using Watson's Lemma.

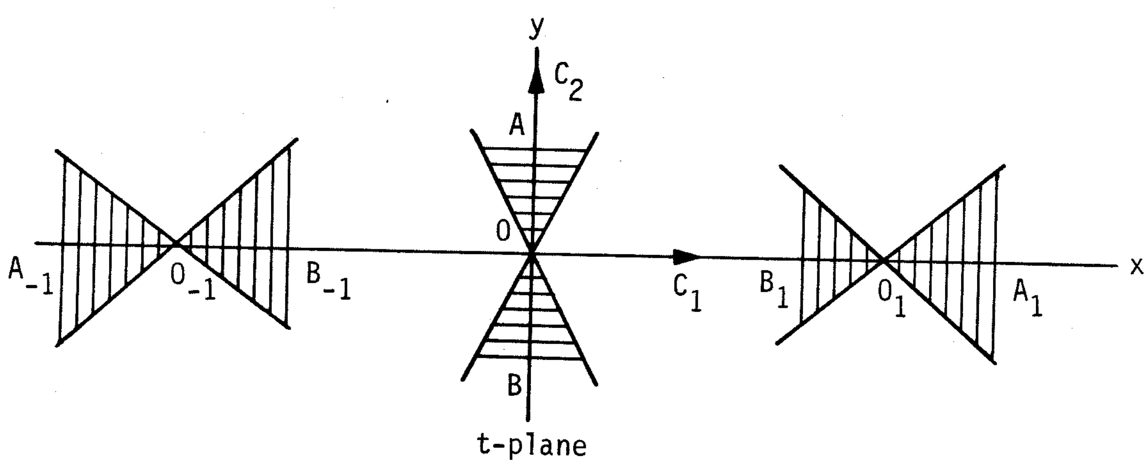


Figure 3-3. Valleys are shaded.

We find after some fairly routine calculations that

$$J_1(\lambda;\gamma) \sim 2\gamma \exp\left(\frac{k}{4}\right) \sum_{n=0}^{\infty} \frac{\left\{\Gamma\left(2n+\frac{1}{2}\right)\right\}^2 \; 2^{2n}}{(2n)! \; \Gamma\frac{1}{2} \; k^{2n+1/2}} \tag{3.12}$$

as $k \to \infty$ in $\frac{-\pi}{2} < \arg(k) < \frac{\pi}{2}$.

For the second integral, this time along the path C_2, we let

$$g(t) = -s \; , \quad \text{where} \quad s \in \mathbb{R} \tag{3.13}$$

since $g(0) = 0$. As before, the variable s is non-negative and decreasing away from the saddle point at $t = 0$ on C_2. As in the previous case

$$J_2(\lambda;\gamma) \sim \gamma \int_{C_2} e^{-ks}\left(\frac{dt}{ds}\right)ds \; . \tag{3.14}$$

Here, the expression for $\frac{dt}{ds}$ is somewhat more complicated. Referring to Figure 3-3,

$$\frac{dt}{ds} = \frac{1}{2\sqrt{s}} \; e^{+i\pi/2}\left(1 + (1+4s)^{1/2}\right)^{1/2} (1+4s)^{-1/2} \quad \text{on OA}$$

$$\frac{dt}{ds} = \frac{1}{2\sqrt{s}} \; e^{-i\pi/2}\left(1 + (1+4s)^{1/2}\right)^{1/2} (1+4s)^{-1/2} \quad \text{on OB} \tag{3.15}$$

When we expand for s small, there does not seem to be a simple formula for the expansion coefficients. Instead, we have that

$$\frac{dt}{ds} = \frac{1}{2} \; e^{\pm i\pi/2} \sum_{n=0}^{\infty} C_n \; s^{n-1/2} \tag{3.16}$$

where in this expression

$$\left.\begin{aligned}
C_n &= \sum_{j=0}^{n} r_j\, h_{n-j} \\
\text{and where } \left(1 + (1+4s)^{1/2}\right)^{1/2} &= \sum_{j=0}^{\infty} r_j\, s^j \\
&= \sqrt{2}\ \left\{1 + \frac{s}{2} + \sum_{j=2}^{\infty} \frac{\frac{1}{2}\left(\frac{1}{2}-(j+1)\right)\left(\frac{1}{2}-(j+2)\right)\cdots\left(\frac{1}{2}-(2j-1)\right)}{j!}\, s^j\right\} \\
\text{and } (1+4s)^{-1/2} &= \sum_{j=0}^{\infty} h_j\, s^j \\
&= 1 + \sum_{j=1}^{\infty} \frac{(-1)^j\, 2^j\, (1\cdot 3\cdot 5\cdots(2j-1))}{j!}\, s^j
\end{aligned}\right\} \quad (3.17)$$

The first few coefficients are given explicitly by

$$C_0 = \sqrt{2} \qquad C_1 = \frac{-3}{\sqrt{2}\ 1!}$$

$$C_2 = \frac{35}{2\sqrt{2}\ 2!} \qquad C_3 = \frac{-693}{4\sqrt{2}\ 3!}$$

$$C_4 = \frac{20635}{8\sqrt{2}\ 4!} \quad .$$

Finally, if we substitute (3.16) into (3.14) we find that

$$J_2(\lambda;\gamma) \sim \gamma i \sum_{n=0}^{\infty} C_n \frac{\Gamma\left(n+\frac{1}{2}\right)}{k^{n+1/2}} \ , \qquad (3.18)$$

$$\text{as } k \to \infty \text{ in } \frac{-\pi}{2} < \arg(k) < \frac{\pi}{2}$$

where the C_n are given by (3.17). We close this portion of the asymptotic analysis by noting that in terms of the original variables, the leading term behavior of (3.12) and (3.18) is given by

$$J_1(\lambda;\gamma) \sim \frac{2\exp\left(\frac{\lambda\gamma^4}{4}\right)\,\Gamma\left(\frac{1}{2}\right)}{\gamma\lambda^{1/2}} \quad \text{as } \lambda \to \infty$$

$$J_2(\lambda;\gamma) \sim \frac{\sqrt{2}i\;\Gamma\left(\frac{1}{2}\right)}{\gamma\lambda^{1/2}} \quad \text{as } \lambda \to \infty\,. \tag{3.19}$$

B. BEHAVIOR AS $\lambda \to \infty$ FOR $\gamma \to 0$.

In this case, for γ small, we begin our analysis by converting the integral in (3.1) into a differential equation. Letting $t = \frac{t'}{\lambda^{1/4}}$ in (3.1), we find after dropping primes that

$$J(\lambda;\gamma) = \frac{1}{\lambda^{1/4}}\,H(\zeta)$$

where $$H(\zeta) = \int_{C_t} \exp\left(\zeta t^2 - \frac{t^4}{4}\right)dt\,,$$

and where $$\zeta = \frac{\gamma^2\lambda^{1/2}}{2}\,. \tag{3.20}$$

We can convert the integral for $H(\zeta)$ in (3.20) into a differential equation in the independent variable ζ by integrating by parts as

$$H(\zeta) = t\exp\left(\zeta t^2 - \frac{t^4}{4}\right)\Big|_{C_t} - \int_{C_t}\left(2\zeta t^2 - t^4\right)\exp\left(\zeta t^2 - \frac{t^4}{4}\right)dt\,.$$

The boundary term vanishes on C_t, leaving

$$H''(\zeta) - 2\zeta H'(\zeta) - H(\zeta) = 0 \qquad (3.21)$$

where the primes denote differentiation with respect to the argument. The behavior of equation (3.21) near $\zeta=0$ can be found easily since the origin is an ordinary point of the differential equation. It can be shown that

$$H(\zeta) = d_0 \sum_{n=0}^{\infty} \frac{2^{2n}\,\Gamma\left(n+\frac{1}{4}\right)}{(2n)!\;\Gamma\left(\frac{1}{4}\right)}\,\zeta^{2n}$$

$$+ d_1 \sum_{n=0}^{\infty} \frac{2^{2n}\,\Gamma\left(n+\frac{3}{4}\right)}{(2n+1)!\;\Gamma\left(\frac{3}{4}\right)}\,\zeta^{2n+1} \quad , \qquad (3.22)$$

for some constants d_0 and d_1 to be determined. The series in (3.22) have an infinite radius of convergence since every finite point is ordinary. We note that given a contour C_t, the integral solution for $H(\zeta)$ in (3.20) is unique, whereas the series solution (3.22) is determined only within multiplicative constants d_0 and d_1. In order to fix d_0 and d_1, we merely evaluate (3.20) and (3.22) explicitly when $\zeta = 0$. If we adopt the convention that $H_1(\zeta)$ and $H_2(\zeta)$ are the solutions involving the contours C_1 and C_2 respectively from Section III-A, we find from (3.20) that

$$
\left.
\begin{aligned}
H_1(0) &= \int_{C_1} e^{-t^4/4}\,dt = \frac{\sqrt{2}}{2}\,\Gamma\left(\frac{1}{4}\right) \\
H_2(0) &= \int_{C_2} e^{-t^4/4}\,dt = \frac{\sqrt{2}}{2}i\,\Gamma\left(\frac{1}{4}\right) \\
H_1'(0) &= \int_{C_1} t^2 e^{-t^4/4}\,dt = \sqrt{2}\,\Gamma\left(\frac{3}{4}\right) \\
H_2'(0) &= \int_{C_2} t^2\, e^{-t^4/4}\,dt = -\sqrt{2}\,i\,\Gamma\left(\frac{3}{4}\right) \quad .
\end{aligned}
\right\} \tag{3.23}
$$

Finally, solving for d_0 and d_1, we have that

$$
\begin{aligned}
H_1(\zeta) &= \frac{\sqrt{2}}{2} \sum_{n=0}^{\infty} \frac{2^{2n}\Gamma\left(n+\frac{1}{4}\right)}{(2n)!}\,\zeta^{2n} + \frac{\sqrt{2}}{2} \sum_{n=0}^{\infty} \frac{2^{2n+1}\Gamma\left(n+\frac{3}{4}\right)}{(2n+1)!}\,\zeta^{2n+1} \\
H_2(\zeta) &= \frac{\sqrt{2}}{2}i \sum_{n=0}^{\infty} \frac{2^{2n}\Gamma\left(n+\frac{1}{4}\right)}{(2n)!}\,\zeta^{2n} - \frac{\sqrt{2}}{2}i \sum_{n=0}^{\infty} \frac{2^{2n+1}\Gamma\left(n+\frac{3}{4}\right)}{(2n+1)!}\,\zeta^{2n+1} \quad .
\end{aligned} \tag{3.24}
$$

It should be mentioned that in addition to the solutions $H_1(\zeta)$ and $H_2(\zeta)$, it can easily be shown that $H_1(-\zeta)$ and $H_2(-\zeta)$ also solve (3.21). Since only any two solutions are linearly independent, there must be a functional relationship connecting the others. In fact, it is that

$$H_2(\zeta) = i\,H_1(-\zeta)$$

and

$$H_2(-\zeta) = i\,H_1(\zeta) \quad . \tag{3.25}$$

These linear functional relations can be used to extend the range of ζ in the asymptotic behavior of the canonical integral for large arguments in Part A.

Converting back to the original variables, the behavior when $\gamma = 0$ is given by

$$J_1(\lambda;0) = \frac{1}{\lambda^{1/4}} \frac{\sqrt{2}}{2} \Gamma\left(\frac{1}{4}\right)$$

$$J_2(\lambda;0) = \frac{1}{\lambda^{1/4}} \frac{\sqrt{2}}{2} i \; \Gamma\left(\frac{1}{4}\right) \tag{3.26}$$

We close this section by finding the leading term behavior of $H(\zeta)$ in (3.21) near $\zeta = \infty$. Clearly, this will serve to check our results from Part A of this section. The point at infinity is an irregular singular point of the differential equation; asymptotically, the solutions can be shown to behave like

$$H(\zeta) \sim \frac{e^{\zeta^2}}{\zeta^{1/2}} \qquad \text{as} \qquad \zeta \to \infty$$

and

$$H(\zeta) \sim \frac{1}{\zeta^{1/2}} \qquad \text{where} \qquad \zeta = \frac{\gamma^2 \lambda^{1/2}}{2} \; , \tag{3.27}$$

modulo some multiplicative constants. We note that in terms of the original variables, the results of (3.27) are in qualitative agreement with the leading term behavior in (3.19) using the method of steepest descent.

IV. THE COMPLETE ASYMPTOTIC EXPANSION

At this point we are prepared to return to equation (2.23), armed with the knowledge of the asymptotic expansions of the canonical integrals, either $J_{1,2}$ or $H_{1,2}$ from the preceeding section.

We note first of all, due to symmetry in the integrand on the contours C_1 or C_2, that the term involving g_1 cancels in (2.23), leaving

$$I_{1,2}(\lambda;\underline{\alpha}) \sim e^{\lambda\beta} \int_{C_{1,2}} (g_0+g_2t^2) \exp[\lambda(-\frac{t^4}{4} + \frac{\gamma^2}{2} t^2)]dt + R_{0_{1,2}}(\lambda;\underline{\alpha}) \tag{4.1}$$

where $$R_{0_{1,2}}(\lambda;\underline{\alpha}) \sim e^{\lambda\beta} \int_{C_{1,2}} (t^3-\gamma^2 t)Q_0(t;\underline{\alpha}) \exp[\lambda(\frac{-t^4}{4} + \frac{\gamma^2}{2} t^2)]dt \ ,$$

and where I_1 or I_2 denote the integral $I(\lambda;\underline{\alpha})$ deformed from C_t to either of the contours C_1 or C_2 respectively. In terms of H_1 and H_2, (4.1) can be rewritten

$$I_{1,2}(\lambda;\underline{\alpha}) \sim e^{\lambda\beta}[\frac{g_0}{\lambda^{1/4}} H_{1,2} (\frac{\gamma^2\lambda^{1/2}}{2}) + \frac{g_2}{\lambda^{3/4}} H'_{1,2}(\frac{\gamma^2\lambda^{1/2}}{2})] + R_{0_{1,2}}(\lambda;\underline{\alpha})$$

where we recall $$H_{1,2}(\zeta) = \int_{C_{1,2}} \exp[\zeta t^2 - \frac{t^4}{4}]dt \ . \tag{4.2}$$

In R_0 we integrate by parts and find that due to vanishing boundary terms

$$R_{0_{1,2}}(\lambda;\underline{\alpha}) \sim \frac{1}{\lambda} e^{\lambda\beta} \int_{C_{1,2}} G_1 t;\underline{\alpha}) \exp[\lambda(\frac{-t^4}{4} + \frac{\gamma^2}{2} t^2)]dt \ ,$$

where $$G_1(t;\underline{\alpha}) = \frac{d}{dt} Q_0(t;\underline{\alpha}) \ . \tag{4.3}$$

The integral in (4.3) is of the same form as that in (2.11) except that it is multiplied by $1/\lambda$ and hence is of lower asymptotic order. If we continue to integrate by parts (N + 1) times we obtain that

$$I_{1,2}(\lambda;\underline{\alpha}) \sim e^{\lambda\beta}\left[\frac{H_{1,2}\left(\frac{\gamma^2\lambda^{\frac{1}{2}}}{2}\right)}{\lambda^{1/4}}\sum_{n=0}^{N}\frac{g_{4n}}{\lambda^n} + \frac{H'_{1,2}\left(\frac{\gamma^2\lambda^{\frac{1}{2}}}{2}\right)}{\lambda^{3/4}}\sum_{n=0}^{N}\frac{g_{4n+2}}{\lambda^n}\right] + R_N(\lambda;\underline{\alpha}) \ ,$$

where $$R_N(\lambda;\underline{\alpha}) \sim \frac{1}{\lambda^{N+1}} e^{\lambda\beta}\int_{C_{1,2}} G_{N+1}(t;\underline{\alpha}) \exp\left[\lambda\left(\frac{-t^4}{4}+\frac{\gamma^2}{2}t^2\right)\right]dt \ . \tag{4.4}$$

In this expansion, the coefficients g_n are given recursively by the formulae

$$\left.\begin{aligned}
g_{4n} &= G_n(0;\underline{\alpha}) \\
g_{4n+1} &= \frac{G_n(\gamma;\underline{\alpha}) - G_n(-\gamma;\underline{\alpha})}{2\gamma} \\
g_{4n+2} &= \frac{G_n(\gamma;\underline{\alpha}) + G_n(-\gamma;\underline{\alpha}) - 2G_n(0;\underline{\alpha})}{2\gamma^2} \\
G_n(t;\underline{\alpha}) &= g_{4n} + g_{4n+1}t + g_{4n+2}t^2 + (t^3-\gamma^2 t)Q_n(t;\underline{\alpha}) \\
\text{where} \qquad G_{n+1}(t;\underline{\alpha}) &= \frac{d}{dt}Q_n(t;\underline{\alpha}) \ .
\end{aligned}\right\} \tag{4.5}$$

Formally, the procedure of repeated integration by parts can be continued indefinitely to yield the complete asymptotic expansion of $I_{1,2}(\lambda;\underline{\alpha})$.

In an effort to consolidate the relevant equations, we now list the asymptotic expansions of $H_{1,2}(\zeta)$ from the preceding section. From (3.24) we have directly that for ζ small,

$$H_1(\zeta) = \frac{\sqrt{2}}{2} \sum_{n=0}^{\infty} \frac{2^{2n}\,\Gamma\left(n+\frac{1}{4}\right)}{(2n)!} \zeta^{2n} + \frac{\sqrt{2}}{2} \sum_{n=0}^{\infty} \frac{2^{2n+1}\,\Gamma\left(n+\frac{3}{4}\right)}{(2n+1)!} \zeta^{2n+1}$$

$$H_2(\zeta) = \frac{\sqrt{2}}{2} i \sum_{n=0}^{\infty} \frac{2^{2n}\,\Gamma\left(n+\frac{1}{4}\right)}{(2n)!} \zeta^{2n} - \frac{\sqrt{2}}{2} i \sum_{n=0}^{\infty} \frac{2^{2n+1}\,\Gamma\left(n+\frac{3}{4}\right)}{(2n+1)!} \zeta^{2n+1} \qquad (4.6)$$

Next, when $\zeta \to \infty$ we have from (3.12) and (3.18) using (3.20) that

$$H_1(\zeta) \sim \frac{\sqrt{2}}{\zeta^{1/2}} \exp(\zeta^2) \sum_{n=0}^{\infty} \frac{\left[\Gamma\left(2n+\frac{1}{2}\right)\right]^2}{(2n)!\,2^{2n}\,\Gamma\left(\frac{1}{2}\right)\,\zeta^{4n}}$$

$$H_2(\zeta) \sim \frac{\sqrt{2}\ i}{2\zeta^{1/2}} \sum_{n=0}^{\infty} C_n \frac{\Gamma\left(n+\frac{1}{2}\right)}{2^{2n}\ \zeta^{2n}} \ , \quad \text{in} \quad |\arg(\zeta)| < \frac{\pi}{4} \ . \qquad (4.7)$$

The functional relations

$$H_2(\zeta) = i\ H_1(-\zeta)$$

$$H_2(-\zeta) = i\ H_1(\zeta) \qquad (4.8)$$

also allow us to extend the validity of the expansions in (4.7) to other regions of the complex ζ-plane.

The expansion (4.4) for $I_{1,2}(\lambda;\underline{\alpha})$ involves the functions $H_{1,2}(\zeta)$ whose asymptotic properties are given above in (4.6) and (4.7). The advantage of such an expansion in terms of H is that a single formula (namely (4.4)) yields the smooth transition as $\gamma \to 0$ when $\lambda \to \infty$. That is to say, we know that for distinct simple saddle points (i.e., when $\gamma \neq 0$) the algebraic order of $I_{1,2}$ should be $O(\frac{1}{\lambda^{1/2}})$. Likewise, when all three saddle points coalesce to form a third order saddle ($\gamma = 0$) the algebraic order is $O(\frac{1}{\lambda^{1/4}})$. That this same behavior is exhibited by (4.4) can easily be shown by examining the leading order behavior of $H_{1,2}$. After some manipulation we find the asymptotic formulae

$$I_1(\lambda;\underline{\alpha}) \sim \frac{2\Gamma(\frac{1}{2})\ \exp[\lambda(\beta + \frac{\gamma^4}{4})]}{\lambda^{1/2}} \left(\frac{g_0}{\gamma} + g_2\gamma\right)$$

$$I_2(\lambda;\underline{\alpha}) \sim \frac{\sqrt{2}i\ \Gamma(\frac{1}{2})e^{\lambda\beta}g_0}{\lambda^{1/2}\ \gamma}\ , \quad \text{as } \lambda \to \infty \text{ for } \gamma \neq 0 \tag{4.9}$$

and

$$I_1(\lambda;\underline{\alpha}) \sim \frac{\sqrt{2}\ \Gamma(\frac{1}{4})e^{\lambda\beta}g_0}{2\ \lambda^{1/4}}$$

$$I_2(\lambda;\underline{\alpha}) \sim \frac{\sqrt{2}i\ \Gamma(\frac{1}{4})e^{\lambda\beta}g_0}{2\ \lambda^{1/4}}\ , \quad \text{as } \lambda \to \infty,, \text{ for } \gamma = 0 \tag{4.10}$$

in agreement with the heuristic results in the preceeding discussion.

REFERENCES

[1] B. A. Hughes, "On the Interaction of Surface and Internal Gravity Waves: Uniformly Valid Solution by Extended Stationary Phase," J. Fluid Mech. 74, Part 4 (1976), 667-683.

[2] A. K. Liu, D. J. Benney, "The Evolution of Nonlinear Wave Trains in Stratified Shear Flows," Studies in Appl. Math 64 (1981), 247-269.

[3] T. Pearcey, "The Structure of an Electromagnetic Field in the Neighborhood of a Cusp of a Caustic," Phil. Mag. 37 (1946), 311-317.

[4] C. Chester, B. Friedman, F. Ursell, "An Extension of the Method of Steepest Descents," Proc. Camb. Phil. Soc. 53 (1957), 599-611.

[5] B. Friedman, "Stationary Phase with Neighboring Critical Points, SIAM Review 7 (1959), 280-289.

[6] N. Bleistein, "Uniform Asymptotic Expansions of Integrals with Many Nearby Stationary Points and Algebraic Singularities," J. Math. Mech. 17 (1967), 533-560.

[7] N. Bleistein, R. Handelsman, Asymptotic Expansions of Integrals, Holt, Rinehart and Winston (1975).