NON-STATIONARY LATTICE THEORY

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Part I

APPLICATION OF PRANDTL'S ACCELERATION POTENTIAL TO LATTICE THEORY

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APPLICATION OF PRANDTL'S ACCELERATION POTENTIAL TO LATTICE THEORY

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I. Introduction

The lattice theory has important applications, especially to the design of airscrews and turbine blades. The problem of a stationary lattice in a uniform flow has been treated by several authors using different methods. Kármán and Burgers used the method of conformal transformation (Ref. 5, p. 91), while Pistolesi used the method of vortex distribution (Ref. 4). This article deals with the combined use of conformal transformation and acceleration potential for the solution of the problem.

In the following treatment, the basic idea of the acceleration potential, as it was introduced by L. Prandtl, is first reviewed, and then shown how it can be applied to the solution of steady-state two-dimensional problems. The method is then applied to the determination of the lift and moment of a stationary lattice in a uniform flow. A new expression for the moment is obtained by this method.

II. The Acceleration Potential

The equation of motion for a non-viscous incompressible fluid, if the body force is neglected, is given by

$$\frac{d\bar{q}}{dt} = -\frac{1}{\rho}\nabla p \qquad (1)$$

where $\bar{q}$, p and ρ are the velocity, pressure and density of the fluid respectively. For the acceleration vector $\frac{d\bar{q}}{dt}$, a scalar function φ can be defined such that

$$\nabla\varphi = \frac{d\bar{q}}{dt} \tag{2}$$

Then Eq. (1) becomes

$$\rho\nabla\varphi = -\nabla p \tag{3}$$

where φ is called the acceleration potential (Prandtl, ref.2). In applications, both φ and p can be taken as zero at infinity, then Eq. (3) gives

$$\rho\varphi = -p \tag{4}$$

Hence there exists a very simple relation between the pressure field and the acceleration potential.

In the so-called linearized theory where the perturbation velocity $\bar{q}'$ of the fluid is assumed to be everywhere small as compared with the uniform velocity $\bar{U}$ in the x-direction, so that

$$\bar{q} = \bar{U} + \bar{q}'$$

Then, by neglecting higher order terms, Eq. (2) may be written as

$$\nabla\varphi = \frac{\partial\bar{q}'}{\partial t} + U\frac{\partial\bar{q}'}{\partial x} \tag{5}$$

The equation of continuity in this case is

$$\nabla\cdot\bar{q}' = 0 \tag{6}$$

Taking the divergence on both sides of Eq. (5) gives

$$\nabla^2\varphi = 0 \tag{7}$$

Hence the acceleration potential satisfies the Laplace's differential equation for the linearized theory.

III. The Method of Acceleration Potential

The method of solving two-dimensional problems by the use of acceleration potential is to find a complex potential function.

$$F(x+iy) = \varphi(x,y) + i\,\psi(x,y) \tag{8}$$

to satisfy the boundary conditions. In Eq. (8), φ is the acceleration potential and ψ is the conjugate function of φ, The two functions φ and ψ satisfy the so-called Cauchy-Riemann relations:

$$\left.\begin{aligned} \frac{\partial \varphi}{\partial x} &= \frac{\partial \psi}{\partial y} \\ \frac{\partial \varphi}{\partial y} &= -\frac{\partial \psi}{\partial x} \end{aligned}\right\} \tag{9}$$

which give the accelerations of the fluid in the x- and y-direction respectively. It is evident from Eq.(9) that both φ and ψ satisfy the Laplace's differential equation.

To obtain the boundary condition for the complex function (8), consider a stationary thin airfoil. The equation of the airfoil can be represented by

$$y = y(x) \tag{10}$$

Then the vertical velocity of the fluid adjacent to the airfoil will be

$$\frac{v}{U} = \frac{dy}{dx} \tag{11}$$

From Eq.(9), for steady state:

$$-\frac{d\psi}{dx} = U\frac{dv}{dx} \tag{12}$$

Integrating both sides of Eq.(12) with respect to x and assuming that both ψ and v are zero at infinity gives, on the boundary:

$$\psi = -Uv = -U^2\frac{dy}{dx} \tag{13}$$

The restriction now imposed on the complex function (8) is that the imaginary part must satisfy the boundary condition (13). In addition, the function must also satisfy the condition at $z = \pm\infty$ which will be considered later.

IV. The Stationary Lattice

The above method is now applied to the problem of a stationary lattice in a uniform flow field, as shown in Fig.1, where the uniform velocity U is at a small inclination α to the lattice. In treating this problem, the conformal representation of the lattice will be first considered.

1. Conformal Representation of the Lattice.

Using the idea of "hydrodynamic analogy" (see Ref. 5, p. 84 and p. 92), a mapping function between a unit circle and a non-staggered lattice composed of straight airfoils (see Fig.1) can be constructed as follows:

$$z = \frac{h}{2\pi}\left[\operatorname{Log}\frac{\zeta+k}{\zeta-k} + \operatorname{Log}\frac{\zeta+\frac{1}{k}}{\zeta-\frac{1}{k}}\right] \qquad 0 < k < 1 \tag{14}$$

where k is the parameter of conformal transformation. In the expression (14), it is to be noted that for $k < |\zeta| < \frac{1}{k}$, the term

$\frac{\zeta+k}{\zeta-k}$ is positive, while $\frac{\zeta+\frac{1}{k}}{\zeta-\frac{1}{k}}$ is negative, the negative behavior of the latter term can be eliminated by writing (14), with the introduction of an unimportant additive constant, as:

$$z = \frac{h}{2\pi}\left[\operatorname{Log}\frac{\zeta+k}{\zeta-k} + \operatorname{Log}\frac{\frac{1}{k}+\zeta}{\frac{1}{k}-\zeta}\right] \qquad 0<k<1 \tag{15}$$

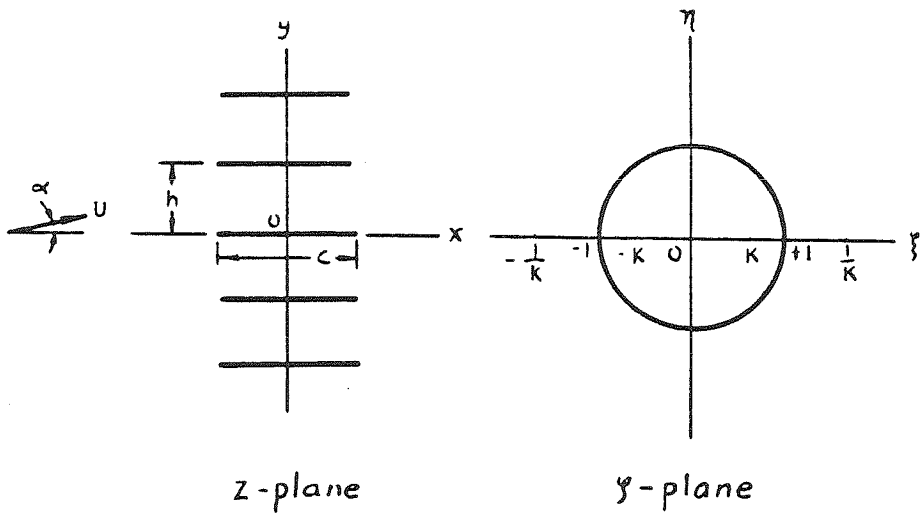

Fig. 1

On the unit circle $\zeta = e^{i\theta}$, the transformation equation (15), owing to the many-valued nature of the logarithmic function, * gives

$$\left.\begin{aligned} x &= \frac{h}{2\pi}\log\frac{1+2k\cos\theta+k^2}{1-2k\cos\theta+k^2} \\ y &= nh \qquad \text{n is any integer.} \end{aligned}\right\} \tag{16}$$

* Here the value of $2n\pi$ has been added only to the imaginary part of Log $\frac{\frac{1}{k}+\zeta}{\frac{1}{k}-\zeta}$, for the reason that only the exterior region of the unit circle $|\zeta|=1$ is to be considered.

which are the equations of the boundaries of the non-staggered lattice composed of straight airfoils. The chord of each airfoil can be easily found from the first equation of (16) by calculating the maximum and minimum x-coordinates and by subtracting one from the other, there results:

$$c = \frac{2h}{\pi} \log \frac{1+k}{1-k} \tag{17}$$

which may be expressed as

$$k = \tanh \frac{\pi c}{4h} \tag{18}$$

For very small k, Eq.(17) can be approximated, by taking only the first term of the series expansion, as:

$$c = \frac{4hk}{\pi} \tag{19}$$

It is to be observed that in the expression (19), when k becomes infinitely small, h will become infinitely large, if c is fixed. The product hk, however, will remain to be finite and is equal to $\frac{\pi c}{4}$. Hence it appears that the first-term approximation for very small k is really the limiting case $k \to 0$ which corresponds to the case of a single flat plate.

2. The Complex Potential Function

The complex potential function for the flow past the non-staggered lattice will be of the form:

$$\varphi + i\psi = Uz(1 - i\alpha) + \varphi' + i\psi' \tag{20}$$

where $\varphi' + i\psi'$ represents the disturbance function; α is the angle between the direction of the uniform velocity U and the chord line of each airfoil of the lattice, and this angle is assumed to be small. The conditions which determine the function (19) are the following:

(a) On the boundary, $\frac{dy}{dx}=0$, $\psi=0$, according to Eq.(13).

$$(\psi')_{|\zeta|=1} = U^2\alpha$$

(b) Condition at $z=\pm\infty$, when expressed in terms of ζ, will be

$$(\varphi'+i\psi')_{\zeta=\pm\frac{1}{k}} = \text{finite} \rightarrow 0, \text{ as } k \rightarrow 0$$

The meaning of the second condition is that the disturbance function $\varphi'+i\psi'$ has to be finite at the finite points of the ζ-plane; but as $k\rightarrow 0$, these points approach to infinity, and the problem reduces to that of a single flat plate, accordingly $\varphi'+i\psi'$ has to vanish.

The complex potential function which satisfies the conditions (a) and (b) is found to be

$$\varphi+i\psi = U^2(1-i\alpha) + \frac{2iU^2\alpha}{1+k^2}\,\frac{1+k^2\zeta}{1+\zeta} \tag{21}$$

On the unit circle $\zeta=e^{i\theta}$, the real part of (21) is

$$\varphi_b = U^2 + \frac{1-k^2}{1+k^2}\,U^2\alpha\,\frac{1-\cos\theta}{\sin\theta} \tag{22}$$

where the subscript b denotes the value of the quantity on the boundary.

3. Determination of the Lift and Moment

The lift is calculated by

$$L = \oint p_b\,dx \tag{23}$$

By substituting (4) and the series expansion of x from (16), there is obtained:

$$L = \int_{-\pi}^{\pi} (-\rho \varphi_b)\left(-\frac{2h}{\pi} \sum_{n=1,3\cdots}^{\infty} k^n \sin n\theta\right) d\theta \tag{24}$$

Eq.(24), after substituting φ_b from (22), becomes

$$L = \frac{4\rho h}{\pi} \frac{1-k^2}{1+k^2} U^2 \alpha \int_0^{\pi} \left(\frac{1-\cos\theta}{\sin\theta}\right) \sum_{n=1,3\cdots}^{\infty} k^n \sin n\theta \, d\theta \tag{25}$$

To evaluate the integral in Eq.(25), expand $\frac{\sin n\theta}{\sin\theta}$ into the following series:

$$\frac{\sin n\theta}{\sin\theta} = (-1)^{\frac{n-1}{2}} \left\{ 1 - \frac{n^2-1^2}{\lfloor 2} \cos^2\theta + \frac{(n^2-1^2)(n^2-3^2)}{\lfloor 4} \cos^4\theta - \frac{(n^2-1^2)(n^2-3^2)(n^2-5^2)}{\lfloor 6} \cos^6\theta + \cdots + (-1)^{\frac{n-1}{2}} (2\cos\theta)^{n-1} \right\}$$

where n is odd.

Then integrating gives

$$\int_0^{\pi} (1-\cos\theta) \sum_{n=1,3\cdot}^{\infty} k^n \frac{\sin n\theta}{\sin\theta} \, d\theta = \sum_{n=1,3\cdots}^{\infty} k^n (-1)^{\frac{n-1}{2}} \pi U_n \tag{26}$$

where $$U_n = 1 - \frac{n^2-1^2}{\lfloor 2} \cdot \frac{1}{2} + \frac{(n^2-1^2)(n^2-3^2)}{\lfloor 4} \cdot \frac{3}{4} \cdot \frac{1}{2} - \frac{(n^2-1^2)(n^2-3^2)(n^2-5^2)}{\lfloor 6} \cdot \frac{5}{6} \cdot \frac{3}{4} \cdot \frac{1}{2} + \cdots$$ to n terms.

Substituting (26) into (25), the result is

$$L = 4\rho h \frac{1-k^2}{1+k^2} U^2 \alpha \sum_{n=1,3\cdots}^{\infty} k^n (-1)^{\frac{n-1}{2}} U_n$$

which can be simplified to

$$L = \frac{4\rho h k}{1+k^2} U^2 \alpha \tag{27}$$

If the lift is expressed in the form:

$$L = C_L \frac{\rho}{2} U^2 c \tag{28}$$

Then the lift coefficient is given by

$$C_L = 8 \frac{h}{c} \frac{k}{1+k^2} \alpha \tag{29}$$

which, in view of (18), may be expressed as

$$C_L = 4 \frac{h}{c} \alpha \tanh \frac{\pi c}{2h} \tag{30}$$

The moment is calculated by

$$M = \oint p_b \, x \, dx \tag{31}$$

By substituting (4) and the series expansion of x from (16), there is obtained:

$$M = \int_{-\pi}^{\pi} (-\rho \varphi_b) \left\{ -\frac{4h^2}{\pi^2} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{k^{2(m+n-1)}}{2m-1} \cos(2m-1)\theta \sin(2n-1)\theta \right\} d\theta \tag{32}$$

Using the identity:

$$\cos(2m-1)\theta \; \sin(2n-1)\theta = \tfrac{1}{2}\left[-\sin 2(m-n)\theta + \sin 2(m+n-1)\theta\right]$$

the expression (32) can be written in the form:

$$M = \frac{2\rho h^2}{\pi^2} \int_{-\pi}^{\pi} \varphi_b \sum_{m=2,4..}^{\infty} C_m \sin m\theta \, d\theta \tag{33}$$

where $$C_m = \sum_{n=2,4..}^{m} \frac{k^m}{n-1} + \sum_{n=2,4..}^{\infty} \left(\frac{k^{m+2n-2}}{n-1} - \frac{k^{m+2n-2}}{m+n-1} \right)$$

Eq.(33), after introducing φ_b from (22), becomes

$$M = \frac{4\rho h^2}{\pi^2} \frac{1-k^2}{1+k^2} U^2 \alpha \int_0^{\pi} (1-\cos\theta) \sum_{m=2,4..}^{\infty} C_m \frac{\sin m\theta}{\sin\theta} \, d\theta \tag{34}$$

The integral in Eq.(34) can be evaluated by expanding $\frac{\sin m\theta}{\sin\theta}$ into the following series:

$$\frac{\sin m\theta}{\sin\theta} = (-1)^{\frac{m}{2}+1}\left\{ m\cos\theta - \frac{m(m^2-2^2)}{\lfloor 3}\cos^3\theta + \frac{m(m^2-2^2)(m^2-4^2)}{\lfloor 5}\cos^5\theta - \cdots + (-1)^{\frac{m}{2}+1}(2\cos\theta)^{m-1}\right\}$$

where m is even.

Then integrating gives

$$\int_0^{\pi}(1-\cos\theta)\sum_{m=2,4\cdots}^{\infty} C_m \frac{\sin m\theta}{\sin\theta}\,d\theta = -\sum_{m=2,4\cdot}^{\infty} C_m (-1)^{\frac{m}{2}+1}\pi V_m \tag{35}$$

where $$V_m = m\cdot\frac{1}{2} - \frac{m(m^2-2^2)}{\lfloor 3}\cdot\frac{3}{4}\cdot\frac{1}{2} + \frac{m(m^2-2^2)(m^2-4^2)}{\lfloor 5}\cdot\frac{5}{6}\cdot\frac{3}{4}\cdot\frac{1}{2} - \cdots$$ to m terms.

Substituting (35) into (34), the result is

$$M = -\frac{4\rho h^2}{\pi}\frac{1-k^2}{1+k^2}U^2\alpha\sum_{m=2,4\cdots}^{\infty} C_m(-1)^{\frac{m}{2}+1}V_m$$

which can be simplified to

$$M = -\frac{2\rho h^2 U^2\alpha}{\pi}\log\frac{1+k^2}{1-k^2} \tag{36}$$

If the moment is expressed in the form:

$$M = C_M\frac{\rho}{2}U^2c^2 \tag{37}$$

Then the moment coefficient is given by

$$C_M = -\frac{4h^2}{\pi c^2}\alpha\log\frac{1+k^2}{1-k^2} \tag{38}$$

which, in view of (18), may be expressed as

$$C_M = -\frac{4h^2}{\pi c^2}\alpha\log\cosh\frac{\pi c}{2h} \tag{39}$$

It is interesting to note that in the limiting case $k \rightarrow 0$, which corresponds to the case of a single flat plate; the expressions (29) and (38), in view of (19), are reduced respectively to

$$C_L = 2\pi\alpha \tag{40}$$

$$C_M = -\frac{\pi}{2}\alpha \tag{41}$$

which are the well-known results for a flat-plate airfoil.

V. Results

The expressions for the lift and moment are

$$L = C_L \frac{\rho}{2} U^2 c \tag{28}$$

$$M = C_M \frac{\rho}{2} U^2 c^2 \tag{37}$$

where C_L and C_M are given by Eqs.(30) and (39)

$$C_L = 4\frac{h}{c}\alpha \tanh\frac{\pi c}{2h} \tag{30}$$

$$C_M = -\frac{4h^2}{\pi c^2}\alpha \log\cosh\frac{\pi c}{2h} \tag{39}$$

It is better to re-write Eqs.(30) and (39) as

$$C_L = 2\pi\sigma\alpha \tag{42}$$

$$C_M = -\frac{\pi}{2}\lambda\alpha \tag{43}$$

where σ and λ are interference factors which are given by

$$\sigma = \frac{2h}{\pi c}\tanh\frac{\pi c}{2h} \tag{44}$$

$$\lambda = \frac{8h^2}{\pi^2 c^2} \log \cosh \frac{\pi c}{2h} \tag{45}$$

The relations (44) and (45) may be represented in graphical forms by plotting σ and λ against the chord-gap ratio $\frac{c}{h}$, as shown in Fig.2. From the figure, it is seen that as the chord-gap ratio $\frac{c}{h}$ increases, the values of σ and λ decrease, that is, the interference effect increases.

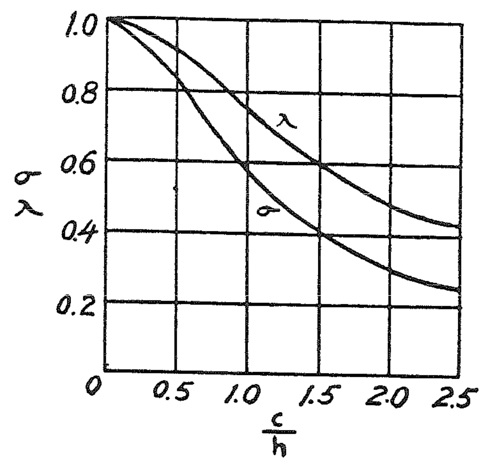

Fig.2

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Part II

NON-STATIONARY LATTICE THEORY

Contents

List of Symbols

h = gap between airfoils of the lattice

c = chord of each airfoil of the lattice

k = parameter of conformal transformation

$$k = \tanh \frac{\pi c}{4h} \qquad 0<k<1$$

b = $\frac{1}{2}(k+\frac{1}{k})$

r = $s+\sqrt{s^2-1}$

s = $-b \tanh\frac{\pi x}{h} \qquad x < -\frac{c}{2}$

i = imaginary unit, used in connection with the complex potential function

j = imaginary unit, used in connection with the harmonic functions of time

y_o = amplitude of the oscillation of each airfoil of the lattice

t = time, also used as the variable of integration

v = velocity

a = acceleration

U = uniform velocity in the x-direction

L = $L_o e^{j\omega t}$ = lift

M = $M_o e^{j\omega t}$ = moment

L_s and M_s = quasi-steady lift and moment respectively

A = a function, but not of x, as given by Eq.(68)

P and Q = functions of k and s, as defined by Eqs.(54) and (51) respectively

G_1 and G_2 = functions of b and $j\gamma$, as defined by Eqs.(74) and (76) respectively

R_a and I_a = real and imaginery parts of $\dfrac{G_1}{G_1 + G_2}$

E and F = functions of b and $j\gamma$, as defined by Eqs.(80) and (81) respectively

ρ = density of fluid

φ = acceleration potential, also used as the phase angle

ψ = conjugate function of

ω = circular frequency of the oscillation

$\sigma = \dfrac{1}{2\mu} \tanh 2\mu$

$\lambda = \dfrac{1}{2\mu^2} \log\cosh 2\mu$

$\mu = \dfrac{\pi c}{4h}$

$\nu = \dfrac{\omega c}{2U}$ = reduced frequency

$\gamma = \dfrac{\omega h}{2\pi U} = \dfrac{\nu}{4\mu}$

Note: The subscript o has the meaning, such as

$$\varphi = \varphi_0 e^{j\omega t} \qquad \psi = \psi_0 e^{j\omega t}$$

The subscript b denotes the value of a quantity on the boundary.

NON-STATIONARY LATTICE THEORY

A. H. Shieh

I. Introduction

The problem of non-stationary lattice composed of straight airfoils in a uniform flow is treated by the combined use of conformal transformation and acceleration potential. The given conditions in the physical z-plane are first transformed to those in the ζ-plane (the transformed plane). Then the complex potential function is expressed directly in the ζ-plane in terms of the boundary conditions in this plane. This procedure is quite similar to the expressing of the complex velocity directly in terms of ζ (see Ref. 3), and it greatly simplifies the solution of certain airfoil problems.

In the following treatment, the method of acceleration potential used to solve two-dimensional problems is first introduced. The method is then applied to the determination of the lift and moment of a non-stationary lattice in a uniform flow. The results obtained involve Gamma-functions together with hypergeometric functions of the complex variable. For numerical calculation, the expressions for the lift and moment are reduced to simpler forms (without approximation). The calculated results are represented graphically by curves with different values of chord-gap ratio $\frac{c}{h}$ and the reduced frequency ν. It is found that in order to keep the amplitudes of the periodic lift and moment at their lowest values, the optimum conditions are: $\frac{c}{h}$ 1.5 and $\gamma = 0.5$. Following this, the physical interpretation of the lift and moment equations is given. The equations for calculating the amplitudes of the periodic lift and moment at any values of $\frac{c}{h}$ and γ are finally summarized.

II. The Method of Acceleration Potential

In the steady flow of a non-viscous incompressible fluid, if the body force is neglected, it can be shown that there exists a very simple relation between the pressure field and the acceleration potential

$$\rho \varphi = -p \tag{1}$$

where p and ρ are the pressure and density of the fluid respectively, and φ is the acceleration potential. In the so-called linearized theory where the perturbation velocity of the fluid is assumed to be everywhere small as compared with the uniform velocity U , the acceleration potential satisfies the Laplace's differential equation, $\nabla^2 \varphi = 0$.

The method of acceleration potential used to solve two - dimensional problems is to find a complex potential function

$$F(x + iy) = \varphi(x , y) + i \psi(x , y) \tag{2}$$

to satisfy the boundary condition. In Eq.(2), φ is the acceleration potential and ψ is the conjugate function of φ. The two functions φ and ψ satisfy the so-called Cauchy — Riemann relations:

$$\left.\begin{aligned} \frac{\partial \varphi}{\partial x} &= \frac{\partial \psi}{\partial y} \\ \frac{\partial \varphi}{\partial y} &= -\frac{\partial \psi}{\partial x} \end{aligned}\right\} \tag{3}$$

which give the accelerations in the x- and y-direction respectively. It is evident from (3) that both φ and ψ satisfy the

Laplace's differential equation.

To obtain the boundary condition for the complex function (2), consider a non-stationary thin airfoil. The equation of the airfoil may be represented by

$$y = y(x, t) \tag{4}$$

then the vertical velocity and vertical acceleration of the fluid adjacent to the airfoil will be

$$v_b = \frac{\partial y}{\partial t} + U \frac{\partial y}{\partial x} \tag{5}$$

$$a_b = \frac{\partial v_b}{\partial t} + U \frac{\partial v_b}{\partial x} \tag{6}$$

respectively, where the subscript b denotes the value on the boundary. The restrictions imposed on the complex function (2) may now be stated as follows: The complex function (2) must satisfy the following three conditions:

(a) On the boundary,

$$-\frac{\partial \psi_b}{\partial x} = a_b \tag{7}$$

(b) In the fluid,

$$-\frac{\partial \psi}{\partial x} = \frac{\partial v}{\partial t} + U \frac{\partial v}{\partial x} \tag{8}$$

where v is the velocity of the fluid which has to satisfy the condition of continuity at the edge of the airfoil. This corresponds to the so-called "Kutta-Joukowski condition" in the classical theory that the velocity is to be finite at the trailing edge.

(c) The condition at $z = \pm\infty$, which will be considered later.

III. The Non-Stationary Lattice

The above method is now applied to the problem of a non-stationary lattice in a uniform flow field, as shown in Fig.1, where the fluid is flowing at a uniform velocity U to the lattice. In treating this problem, the conformal representation of the lattice will be first considered.

1. Conformal Representation of the Lattice

The equation for the conformal transformation of the non-staggered lattice composed of straight airfoils to a unit circle (see Fig.1) may be expressed as

$$z = \frac{h}{2\pi}\left[\operatorname{Log}\frac{\zeta + k}{\zeta - k} + \operatorname{Log}\frac{\frac{1}{k}+\zeta}{\frac{1}{k}-\zeta}\right] \qquad 0 < k < 1 \tag{9}$$

On the unit circle, $\zeta = e^{i\theta}$, the transformation equation (9), owing to the many-valued nature of the logarithmic function,* gives

$$\left.\begin{aligned} x &= \frac{h}{2\pi}\log\frac{1 + 2k\cos\theta + k^2}{1 - 2k\cos\theta + k^2} \\ y &= nh \qquad \text{n is any integer} \end{aligned}\right\} \tag{10}$$

which are the equations of the boundaries of the non-staggered lattice composed of straight airfoils. The chord of each airfoil can be easily found from the first equation of (10)

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* Here the value of $2n\pi$ has been added only to the imaginary part of $\operatorname{Log}\frac{\frac{1}{k}+\zeta}{\frac{1}{k}-\zeta}$, for the reason that only the exterior region of the unit circle $|\zeta| = 1$ is to be considered.

by calculating the maximum and minimum x-coordinates and by subtracting one from the other. There results:

$$c = \frac{2h}{\pi} \log \frac{1+k}{1-k} \tag{11}$$

which may be expressed as

$$k = \tanh \frac{\pi c}{4h} \tag{12}$$

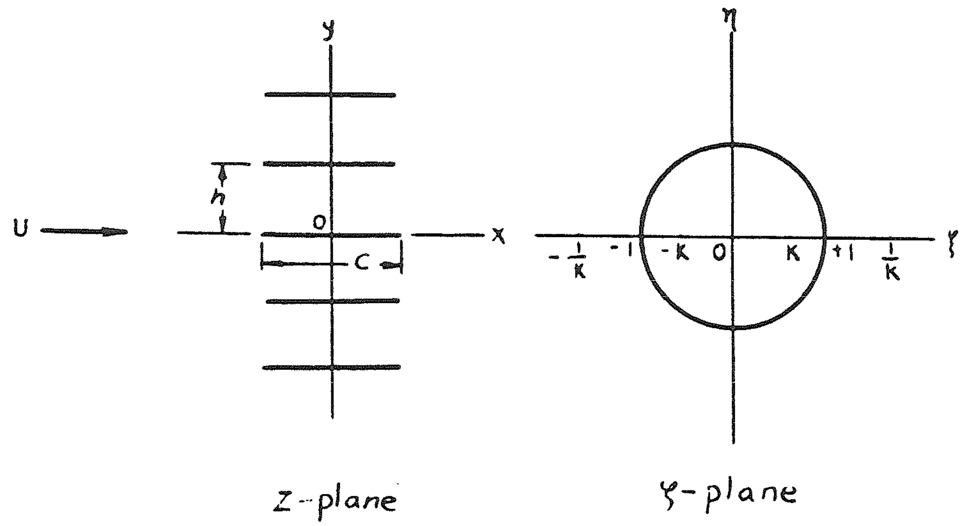

Fig. 1

On solving Eq.(10) for ζ , there is obtained

$$\zeta = b \tanh\frac{\pi z}{h} \pm \sqrt{b^2 \tanh^2\left(\frac{\pi z}{h}\right) - 1} \tag{13}$$

where $$b = \tfrac{1}{2}\left(k + \frac{1}{k}\right) \tag{14}$$

It is to be observed that the inverse function (13) is a two-valued function of z . The sign before the radical must be chosen carefully. In later developments, the region outside the unit circle $|\zeta| = 1$ and along the negative real axis will be considered. For this case, $\zeta = \rho^{i\pi}$, the proper expression to be used is

$$-r = b\tanh\frac{\pi x}{h} - \sqrt{b^2\tanh^2\left(\frac{\pi x}{h}\right) - 1} \qquad x < -\frac{c}{2} \tag{15}$$

2. The Complex Potential Function

Let the oscillation of each airfoil of the lattice be a harmonic function of time; thus

$$y = y_o e^{j\omega t} \tag{16}$$

where y_o and ω are the amplitude and circular frequency of the oscillation respectively. Then the velocity and acceleration of the fluid adjacent to each airfoil will be

$$v_b = j\omega y_o e^{j\omega t} \tag{17}$$

$$a_b = -\omega^2 y_o e^{j\omega t} \tag{18}$$

Substituting (18) into (7) and integrating gives, on the lattice boundary, the following expression for ψ_b:

$$\psi_b = (\omega^2 y_o x + A)e^{j\omega t} \tag{19}$$

where A is a function which will be determined later.

Let $$\psi_b = \psi_{ob}\, e^{j\omega t}$$

Then Eq.(19) gives

$$\psi_{ob} = \omega^2 y_o x + A \tag{20}$$

Introducing the series expansion of x from (10) into (20), the expression becomes

$$\psi_{ab} = \frac{2h}{\pi}\omega^2 y_0 \sum_{n=1}^{\infty} \frac{k^{2n-1}}{2n-1}\cos(2n-1)\theta + A \tag{21}$$

which gives the condition on the boundary of the unit circle in the ζ-plane. The more complicated problem of fluid flow in the physical z-plane has now been reduced to a much simpler one in the ζ-plane with the condition (21) together with the condition at $\zeta = \pm\frac{1}{k}$ which corresponds to $z = \pm\infty$ according to the transformation equation (9).

The next step is to find a complex potential function of the form:

$$\varphi_0 + i\psi_0 = \frac{2h}{\pi}\omega^2 y_0 \sum_{n=1}^{\infty} \frac{k^{2n-1}}{2n-1}\cos(2n-1)\theta + A(\varphi_1 + i\psi_1) \tag{22}$$

where $\varphi_1 + i\psi_1$ represents the disturbance function which has to satisfy the following two conditions:

$$\left.\begin{aligned} &(a) \qquad (\psi_1)_{|\zeta|=1} = 1, \qquad \text{according to Eq.(21)} \\ &(b) \qquad (\varphi_1 + i\psi_1)_{\zeta=\pm\frac{1}{k}} = \text{finite} \rightarrow 0, \text{ as } k \rightarrow 0 \end{aligned}\right\} \tag{23}$$

The meaning of the second condition is that the disturbance function $\varphi_1 + i\psi_1$ has to be finite at the finite points of the ζ-plane; but as $k \rightarrow 0$, these points approach to infinity, and the problem reduces to that of a single flat plate; accordingly the disturbance function $\varphi_1 + i\psi_1$ has to vanish. One more remark which should be made here is that the mere points $\zeta = \pm\infty$ on the ζ-plane have nothing to do with the boundary-condition consideration, as there are no corresponding points on the physical z-plane.

The complex potential function which satisfies the conditions (23) is found to be

$$\varphi_0 + i\psi_0 = \frac{2h}{\pi}\,\omega^2 y_0 \sum_{n=1}^{\infty} \frac{k^{2n-1}}{2n-1}\,\frac{i}{\zeta^{2n-1}} + \frac{2iA}{1+k^2}\,\frac{1+k^2\zeta}{1+\zeta} \qquad (24)$$

On the unit circle $\zeta = e^{i\theta}$, the real part of (24) is given by

$$\varphi_{0b} = \frac{2h}{\pi}\,\omega^2 y_0 \sum_{n=1}^{\infty} \frac{k^{2n-1}}{2n-1}\,\sin(2n-1)\theta + A\,\frac{1-k^2}{1+k^2}\,\frac{1-\cos\theta}{\sin\theta}. \qquad (25)$$

Before going to the determination of the function A, it is expedient to consider the lift and moment first.

3. Determination of the Lift and Moment

The lift is of the form:

$$L = L_0 e^{j\omega t} \qquad (26)$$

where L_0 is given by

$$L_0 = \oint p_{0b}\,dx \qquad (27)$$

By substituting (1) and the series expansion of x from (10), there is obtained:

$$L_0 = \int_{-\pi}^{\pi} (-\rho\,\varphi_{0b})\left(-\frac{2h}{\pi}\sum_{n=1}^{\infty} k^{2n-1}\ \sin(2n-1)\theta\right)d\theta \qquad (28)$$

where φ_{0b} is given by Eq.(25). Since the integrand is an even function; Eq.(28), after substituting φ_{0b} from (25), becomes

$$\left.\begin{aligned} L_\theta = {} & \frac{8\rho h^2}{\pi^2}\omega^2 y_0 \int_0^\pi \sum_{m=1}^\infty \sum_{n=1}^\infty \frac{k^{2(m+n-1)}}{2m-1} \sin(2m-1)\theta \sin(2n-1)\theta \, d\theta \\ & + \frac{4\rho h}{\pi} A \frac{1-k^2}{1+k^2} \int_0^\pi \frac{1-\cos\theta}{\sin\theta} \sum_{n=1,3\cdot}^\infty k^n \sin n\theta \, d\theta \end{aligned}\right\} \qquad (29)$$

The first integral in (29) is readily evaluated:

$$\left.\begin{aligned} & \int_0^\pi \sum_{m=1}^\infty \sum_{n=1}^\infty \frac{k^{2(m+n-1)}}{2m-1} \sin(2m-1)\theta \sin(2n-1)\theta \, d\theta \\ & = \frac{\pi}{2} \sum_{n=1}^\infty \frac{k^{2(2n-1)}}{2n-1} = \frac{\pi}{4} \log \frac{1+k^2}{1-k^2} \end{aligned}\right\} \qquad (30)$$

To evaluate the second integral in (29), expand $\frac{\sin n\theta}{\sin\theta}$ into the following series:

$$\begin{aligned} \frac{\sin n\theta}{\sin\theta} = (-1)^{\frac{n-1}{2}} \Big\{ & 1 - \frac{n^2-1^2}{\underline{|2}} \cos^2\theta + \frac{(n^2-1^2)(n^2-3^2)}{\underline{|4}} \cos^4\theta \\ & - \frac{(n^2-1^2)(n^2-3^2)(n^2-5^2)}{\underline{|6}} \cos^6\theta + \cdots + (-1)^{\frac{n-1}{2}} (2\cos\theta)^{n-1} \Big\} \end{aligned}$$

when n is odd.

Then integrating gives

$$\int_0^\pi (1-\cos\theta) \sum_{n=1,3\cdots}^\infty k^n \frac{\sin n\theta}{\sin\theta} \, d\theta = \sum_{n=1,3\cdots}^\infty k^n (-1)^{\frac{n-1}{2}} \pi U_n \qquad (31)$$

where

$$\begin{aligned} U_n = {} & 1 - \frac{n^2-1^2}{\underline{|2}} \cdot \frac{1}{2} + \frac{(n^2-1^2)(n^2-3^2)}{\underline{|4}} \cdot \frac{3}{4} \cdot \frac{1}{2} \\ & - \frac{(n^2-1^2)(n^2-3^2)(n^2-5^2)}{\underline{|6}} \cdot \frac{5}{6} \cdot \frac{3}{4} \cdot \frac{1}{2} + \cdots \quad \text{to n terms.} \end{aligned}$$

Substituting (30) and (31) into (29), the result is

$$L_0 = \frac{2\rho h}{\pi}\omega^2 y_0 \log\frac{1+k^2}{1-k^2} + 4\rho h A \frac{1-k^2}{1+k^2}\sum_{n=1,3\cdots}^{\infty} k^n (-1)^{\frac{n-1}{2}} U_n$$

which can be simplified to

$$L_0 = \frac{2\rho h}{\pi}\omega^2 y_0 \log\frac{1+k^2}{1-k^2} + \frac{4\rho h k A}{1+k^2} \tag{32}$$

which, in view of (12), may be expressed as

$$L_0 = \frac{2\rho h}{\pi}\omega^2 y_0 \log\cosh\frac{\pi c}{2h} + 2\rho h A \tanh\frac{\pi c}{2h} \tag{33}$$

where A remains to be determined.

The moment is also of the form:

$$M = M_0 e^{j\omega t} \tag{34}$$

where M_0 is given by

$$M_0 = \oint p_{ob}\, x\, dx \tag{35}$$

By substituting (1) and the series expansion of x from (10) into (35), there is obtained:

$$M_0 = \int_{-\pi}^{\pi}(-\rho\varphi_{ob})\left(-\frac{4h^2}{\pi^2}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{k^{2(m+n-1)}}{2m-1}\cos(2m-1)\theta\,\sin(2n-1)\theta\right)d\theta \tag{36}$$

Using the identity:

$$\cos(2m-1)\theta\,\sin(2n-1)\theta = \tfrac{1}{2}\left[-\sin 2(m-n)\theta + \sin 2(m+n-1)\theta\right]$$

the expression (36) can be written in the form:

$$M_0 = \frac{2\rho h^2}{\pi^2}\int_{-\pi}^{\pi}\varphi_{ob}\sum_{m=1}^{\infty} C_{2m}\sin 2m\theta\, d\theta \tag{37}$$

where

$$C_{2m} = \sum_{n=1}^{m}\frac{k^{2m}}{2n-1} - \sum_{n=1}^{\infty}\left(\frac{k^{2(m+2n-1)}}{2m+2n-1} - \frac{k^{2(m+2n-1)}}{2n-1}\right)$$

and y_{ob} is given by Eq. (25). Since the integrand is an even function, Eq. (37), after introducing y_{ob} from (25), becomes

$$\left.\begin{aligned} M_0 = \frac{8\rho h^3}{\pi^3}\omega^2 y_0 \int_0^{\pi} \sum_{n=1}^{\infty} \frac{k^{2n-1}}{2n-1} \sin(2n-1)\theta \sum_{m=1}^{\infty} C_{2m} \sin 2m\theta \, d\theta \\ + \frac{4\rho h^2}{\pi^2} A \frac{1-k^2}{1+k^2} \int_0^{\pi} (1-\cos\theta) \sum_{m=2,4\cdots}^{\infty} C_m \frac{\sin m\theta}{\sin\theta} \, d\theta \end{aligned}\right\} \qquad (38)$$

where $$C_m = \sum_{n=2,4\cdots}^{m} \frac{k^m}{n-1} + \sum_{n=2,4\cdot}^{\infty} \left(\frac{k^{m+2n-2}}{n-1} - \frac{k^{m+2n-2}}{m+n-1} \right)$$

The first integral in (38) is zero because of the orthogonality of the trigonometric functions. The second integral in (38) can be evaluated by expanding $\frac{\sin m\theta}{\sin\theta}$ into the following series:

$$\frac{\sin m\theta}{\sin\theta} = (-1)^{\frac{m}{2}+1} \left\{ m\cos\theta - \frac{m(m^2-2^2)}{3!} \cos^3\theta + \frac{m(m^2-2^2)(m^2-4^2)}{5!} \cos^5\theta - \cdots + (-1)^{\frac{m}{2}+1} (2\cos\theta)^{m-1} \right\}$$

when m is even.

Then integrating gives

$$\int_0^{\pi} (1-\cos\theta) \sum_{m=2,4\cdots}^{\infty} C_m \frac{\sin m\theta}{\sin\theta} \, d\theta = -\sum_{m=2,4\cdots}^{\infty} C_m (-1)^{\frac{m}{2}+1} \pi V_m \qquad (39)$$

where
$$V_m = m \cdot \frac{1}{2} - \frac{m(m^2-2^2)}{3!} \cdot \frac{3}{4} \cdot \frac{1}{2} + \frac{m(m^2-2^2)(m^2-4^2)}{5!} \cdot \frac{5}{6} \cdot \frac{3}{4} \cdot \frac{1}{2} - \cdots \quad \text{to m terms.}$$

Substituting (39) into (38), the result is

$$M_0 = -\frac{4\rho h^2}{\pi} A \frac{1-k^2}{1+k^2} \sum_{m=2,4\cdots}^{\infty} C_m (-1)^{\frac{m}{2}+1} V_m$$

which can be simplified to

$$M_0 = -\frac{2\rho h^2}{\pi} A \log \frac{1+k^2}{1-k^2} \tag{40}$$

which, in view of (12), may be expressed as

$$M_0 = -\frac{2\rho h^2}{\pi} A \log \cosh \frac{\pi c}{2h} \tag{41}$$

where A remains to be determined.

4. Determination of the Function A.

The function A will be determined by the condition (8). By substituting

$$\psi = \psi_0 e^{j\omega t} \qquad v = v_0 e^{j\omega t}$$

into (8) and after cancelling out the term $e^{j\omega t}$, there is obtained:

$$-\frac{d\psi_0}{dx} = j\omega v_0 + U\frac{dv_0}{dx} \tag{42}$$

or

$$e^{\frac{j\omega x}{U}}\left(\frac{dv_0}{dx} + \frac{j\omega}{U}v_0\right) = -\frac{1}{U} e^{\frac{j\omega x}{U}} \frac{d\psi_0}{dx}$$

Integrating the above by using the conditions:

$$v_0 = \text{finite} \qquad \text{at} \quad x = -\infty$$

$$v_0 = j\omega y_0 \qquad \text{at} \quad x = -\frac{c}{2}$$

one obtains:

$$-j\omega y_0 e^{-\frac{j\omega c}{2U}} = \frac{1}{U}\int_{-\infty}^{-\frac{c}{2}} e^{\frac{j\omega x}{U}} \frac{d\psi_0}{dx}\,dx \tag{43}$$

where ψ_0 is given by Eq. (24)

Write $\qquad \psi_0 = A\psi_1 + \psi_2$

Then Eq. (43) becomes

$$-j\omega y_0 U e^{-\frac{j\omega c}{2U}} = \int_{-\infty}^{-\frac{c}{2}} e^{\frac{j\omega x}{U}} \left(A \frac{d\psi_1}{dx} + \frac{d\psi_2}{dx} \right) dx \tag{44}$$

where

$$\psi_1 = I_m \frac{2i}{1+k^2} \frac{1+k^2\zeta}{1+\zeta} \tag{45}$$

$$\psi_2 = \frac{2h}{\pi} \omega^2 y_0 \, I_m \sum_{n=1}^{\infty} \frac{k^{2n-1}}{2n-1} \frac{i}{\zeta^{2n-1}} \tag{46}$$

Notice that there is a singularity at $\zeta = -1$ for ψ_1 and in order to make the first integral in (48) integrable, it is convenient to integrate this by parts, using the following conditions:

ψ_1 = finite at $x = -\infty$

$\psi_1 = 1$ at $x = -\frac{c}{2}$

Thus

$$\int_{-\infty}^{-\frac{c}{2}} e^{\frac{j\omega x}{U}} \frac{d\psi_1}{dx} dx = e^{-\frac{j\omega c}{2U}} - \frac{j\omega}{U} \int_{-\infty}^{-\frac{c}{2}} e^{\frac{j\omega x}{U}} \psi_1 \, dx \tag{47}$$

By substituting (47) into (44) and after transposing, there is obtained:

$$A = \frac{-j\omega y_0 U e^{-\frac{j\omega c}{2U}} - \int_{-\infty}^{-\frac{c}{2}} e^{\frac{j\omega x}{U}} \frac{d\psi_2}{dx} dx}{e^{-\frac{j\omega c}{2U}} - \frac{j\omega}{U} \int_{-\infty}^{-\frac{c}{2}} e^{\frac{j\omega x}{U}} \psi_1 \, dx} \tag{48}$$

The integrals in Eq. (48) will now be considered. On the negative real axis, $\zeta = r\,e^{i\pi}$, Eq. (45) gives

$$\psi_1 = \frac{2}{1+k^2}\,\frac{1-k^2 r}{1-r} \tag{49}$$

Now, from Eq. (15)

$$r = s + \sqrt{s^2 - 1}$$

where

$$s = -b \tanh \frac{\pi x}{h} \qquad x < -\frac{c}{2} \tag{50}$$

After some calculation, ψ_1 can be expressed as

$$\psi_1 = 1 - Q$$

where

$$Q = \frac{1-k^2}{1+k^2}\,\frac{s}{\sqrt{s^2-1}} + \frac{1-k^2}{1+k^2}\,\frac{1}{\sqrt{s^2-1}} \tag{51}$$

Then

$$\int_{-\infty}^{-\frac{c}{2}} e^{\frac{j\omega x}{U}} \psi_1 \, dx = \frac{U}{j\omega} e^{-\frac{j\omega c}{2U}} - \int_{-\infty}^{-\frac{c}{2}} e^{\frac{j\omega x}{U}} Q \, dx \tag{52}$$

Similarly, from Eq.(46), with $\gamma = r e^{i\pi}$

$$\psi_2 = -\frac{2h}{\pi} \omega^2 y_0 \sum_{n=1}^{\infty} \frac{k^{2n-1}}{2n-1}\,\frac{1}{r^{2n-1}} \tag{53}$$

$$\frac{d\psi_2}{dx} = \frac{d\psi_2}{dr}\,\frac{dr}{ds}\,\frac{ds}{dx} = \omega^2 y_0 (1 - P)$$

where

$$P = \frac{1-k^2}{1+k^2}\,\frac{s}{\sqrt{s^2-1}} \tag{54}$$

Then

$$\int_{-\infty}^{-\frac{c}{2}} e^{\frac{j\omega x}{U}} \frac{d\psi_2}{dx} \, dx = -j\omega y_0 U e^{-\frac{j\omega c}{2U}} - \omega^2 y_0 \int_{-\infty}^{-\frac{c}{2}} e^{\frac{j\omega x}{U}} P \, dx \tag{55}$$

Substituting (52) and (55) into (48) gives

$$A = \frac{\omega y_0 U}{j} \frac{\int_{-\infty}^{-\frac{c}{2}} e^{\frac{j\omega x}{U}} P\, dx}{\int_{-\infty}^{-\frac{c}{2}} e^{\frac{j\omega x}{U}} Q\, dx} \tag{56}$$

Using the following substitutions:

$$\left.\begin{aligned} s &= -b \tanh\frac{\pi x}{h} \\ x &= -\frac{h}{2\pi}\log\frac{1+\frac{s}{b}}{1-\frac{s}{b}} \\ dx &= -\frac{h}{\pi b}\frac{ds}{1-\frac{s^2}{b^2}} \\ x &= -\frac{c}{2} \qquad s = 1 \\ x &= -\infty \qquad s = b \end{aligned}\right\} \tag{57}$$

Then Eq.(56) becomes

$$A = \frac{\omega y_0 U}{j} \frac{\int_1^b e^{-\frac{j\omega}{U}\frac{h}{2\pi}\log\frac{1+\frac{s}{b}}{1-\frac{s}{b}}} \frac{P}{1-\frac{s^2}{b^2}}\, ds}{\int_1^b e^{-\frac{j\omega}{U}\frac{h}{2\pi}\log\frac{1+\frac{s}{b}}{1-\frac{s}{b}}} \frac{Q}{1-\frac{s^2}{b^2}}\, ds} \tag{58}$$

where P and Q are given by Eqs. (54) and (51) respectively. Before proceeding further, it is advisable here to examine the limiting case $k \to 0$.

5. The Limiting Case $k \to 0$

Consider the limiting case $k \to 0$ which corresponds to a single flat plate.

For $k \ll 1$, Eq.(11) can be approximated, by taking only the first term of the series expansion, as

$$\underset{\sim}{c} = \frac{4hk}{\pi} \tag{59}$$

and for $\frac{s}{b} \ll 1$.

$$\log \frac{1+\frac{s}{b}}{1-\frac{s}{b}} = \frac{2s}{b} \tag{60}$$

where $$b = \tfrac{1}{2}\left(k + \frac{1}{k}\right)$$

Substituting (60) into (58) gives

$$A = \frac{\omega y_0 U}{j} \frac{\int_1^b e^{-\frac{j\omega}{U}\frac{2hk}{\pi(1+k^2)}s} \frac{P}{1-\frac{s^2}{b^2}}\, ds}{\int_1^b e^{-\frac{j\omega}{U}\frac{2hk}{\pi(1+k^2)}s} \frac{Q}{1-\frac{s^2}{b^2}}\, ds} \tag{61}$$

For $k \to 0$, $b \to \infty$, Eq.(61), in virtue of (59), becomes

$$A = -j\omega y_0 U \frac{\int_1^\infty e^{-\frac{j\omega c}{2U}s} \frac{s}{\sqrt{s^2-1}}\, ds}{\int_1^\infty e^{-\frac{j\omega c}{2U}s} \sqrt{\frac{s+1}{s-1}}\, ds} \tag{62}$$

which can be expressed in terms of the modified Bessel functions of the second kind, thus

$$A = -j\omega y_0 U \frac{K_1(j\nu)}{K_0(j\nu) + K_1(j\nu)} \tag{63}$$

where $$\nu = \frac{\omega c}{2U}$$

ν is called the reduced frequency.

For $k \to 0$, the lift and moment expressions (33) and (41) become respectively

$$L_o = \frac{\pi \rho c^2}{4} \omega^2 y_o + \pi \rho c A \tag{64}$$

$$M_o = -\frac{\pi \rho c^2}{4} A \tag{65}$$

The results obtained, as given by Eqs. (63), (64) and (65) check with those obtained by Kármán and Sears (Ref.6).

6. Evaluation of the Integrals

The integrals in Eq.(58) will now be evaluated. Using the notations:

$$\mu = \frac{c}{4h} \qquad \nu = \frac{\omega c}{2U} \qquad \gamma = \frac{\nu}{4\mu} \tag{66}$$

Then Eq.(61) becomes

$$A = \frac{\omega y_o U}{j} \frac{\int_1^b \left(\frac{b+s}{b-s}\right)^{-j\gamma} \frac{P}{b^2-s^2} ds}{\int_1^b \left(\frac{b+s}{b-s}\right)^{-j\gamma} \frac{Q}{b^2-s^2} ds} \tag{67}$$

By substituting (51) and (54) into (67), one obtains:

$$A = -j\omega y_o U \frac{G_1}{G_1 + G_2} \tag{68}$$

where

$$G_1 = \int_1^b \left(\frac{b+s}{b-s}\right)^{-j\gamma} \frac{1}{b^2-s^2} \frac{s}{\sqrt{s^2-1}} ds \tag{69}$$

$$G_2 = \int_1^b \left(\frac{b+s}{b-s}\right)^{-j\gamma} \frac{1}{b^2-s^2} \frac{1}{\sqrt{s^2-1}} ds \tag{70}$$

These integrals (69) and (70) can be evaluated by means of the following substitutions:

$$\left.\begin{aligned}
&\frac{b - s}{b - s} = \frac{1}{t}\ \frac{b-1}{b-1} \\
&s = \frac{b(b+1)-b(b-1)t}{b+1+(b-1)t} \\
&ds = \frac{-2b(b^2-1)dt}{[b+1+(b-1)t]^2} \\
&s = b \qquad t = 0 \\
&s = 1 \qquad t = 1
\end{aligned}\right\} \tag{71}$$

First, consider the integral (69). By substituting (71) into (69) and after reducing, there is obtained:

$$G_1 = \frac{1}{2\sqrt{b^2-1}}\left(\frac{b-1}{b+1}\right)^{j\gamma}\int_0^1 t^{-1+j\gamma}\frac{1-\frac{b-1}{b+1}t}{\sqrt{(1-t)\left[1-\left(\frac{b-1}{b+1}\right)^2 t\right]}}\,dt \tag{72}$$

Introducing the series expansion

$$\left[1-\left(\frac{b-1}{b+1}\right)^2 t\right]^{-\frac{1}{2}} = \sum_{n=0}^{\infty}\frac{1\cdot 3\cdots(2n-1)}{2\cdot 4\cdots 2n}\left(\frac{b-1}{b+1}\right)^{2n} t^n$$

into (75), then

$$\left.\begin{aligned}
G_1 = {} & \frac{1}{2\sqrt{b^2-1}}\left(\frac{b-1}{b+1}\right)^{j\gamma}\sum_{n=0}^{\infty}\frac{1\cdot 3\cdots(2n-1)}{2\cdot 4\cdots 2n}\left(\frac{b-1}{b+1}\right)^{2n}\int_0^1\frac{t^{n-1+j\gamma}}{\sqrt{1-t}}\,dt \\
& -\frac{1}{2\sqrt{b^2-1}}\left(\frac{b-1}{b+1}\right)^{1+j\gamma}\sum_{n=0}^{\infty}\frac{1\cdot 3\cdots(2n-1)}{2\cdot 4\cdots 2n}\left(\frac{b-1}{b+1}\right)^{2n}\int_0^1\frac{t^{n+j\gamma}}{\sqrt{1-t}}\,dt
\end{aligned}\right\} \tag{73}$$

Using the formulas:

$$\int_0^1 \frac{t^{m+j\gamma}}{\sqrt{1-t}}\,dt = \frac{\Gamma(\frac{1}{2})\,\Gamma(m+1+j\gamma)}{\Gamma(m+\frac{3}{2}+j\gamma)}$$

$$\frac{1\cdot 3\cdots(2n-1)}{2\cdot 4\cdots 2n}\,\Gamma(\tfrac{1}{2}) = \frac{\Gamma(n+\frac{1}{2})}{\lfloor n}$$

Eq.(73) can thus be expressed in terms of the Gamma-functions.

$$\left.\begin{aligned} G_1 = {} & \frac{1}{2\sqrt{b^2-1}}\left(\frac{b-1}{b+1}\right)^{j\gamma}\sum_{n=0}^{\infty}\frac{\Gamma(n+\frac{1}{2})\,\Gamma(n+j\gamma)}{\lfloor n\;\Gamma(n+\frac{1}{2}+j\gamma)}\left(\frac{b-1}{b+1}\right)^{2n} \\ & -\frac{1}{2\sqrt{b^2-1}}\left(\frac{b-1}{b+1}\right)^{1+j\gamma}\sum_{n=0}^{\infty}\frac{\Gamma(n+\frac{1}{2})\,\Gamma(n+1+j\gamma)}{\lfloor n\;\Gamma(n+\frac{3}{2}+j\gamma)}\left(\frac{b-1}{b+1}\right)^{2n} \end{aligned}\right\} \qquad (74)$$

This can also be expressed in terms of the hypergeometric functions (see Ref. 7, p.288).

$$\left.\begin{aligned} G_1 = {} & \frac{1}{2\sqrt{b^2-1}}\left(\frac{b-1}{b+1}\right)^{j\gamma} B\left(\tfrac{1}{2}, j\gamma\right) F\left(\tfrac{1}{2}, j\gamma; \tfrac{1}{2}+j\gamma; \left(\tfrac{b-1}{b+1}\right)^2\right) \\ & -\frac{1}{2\sqrt{b^2-1}}\left(\frac{b-1}{b+1}\right)^{1+j\gamma} B\left(\tfrac{1}{2}, 1+j\gamma\right) F\left(\tfrac{1}{2}, 1+j\gamma; \tfrac{3}{2}+j\gamma; \left(\tfrac{b-1}{b+1}\right)^2\right) \end{aligned}\right\} \qquad (75)$$

where

$$B\left(\tfrac{1}{2}, j\gamma\right) = \frac{\Gamma(\frac{1}{2})\,\Gamma(j\gamma)}{\Gamma(\frac{1}{2}+j\gamma)}$$

The integral (70) can be evaluated in a similar manner by using the substitutions (71). The result is

$$\left.\begin{aligned} G_2 = {} & \frac{1}{2b\sqrt{b^2-1}}\left(\frac{b-1}{b+1}\right)^{j\gamma}\sum_{n=0}^{\infty}\frac{\Gamma(n+\frac{1}{2})\,\Gamma(n+j\gamma)}{\lfloor n\;\Gamma(n+\frac{1}{2}+j\gamma)}\left(\frac{b-1}{b+1}\right)^{2n} \\ & +\frac{1}{2b\sqrt{b^2-1}}\left(\frac{b-1}{b+1}\right)^{1+j\gamma}\sum_{n=0}^{\infty}\frac{\Gamma(n+\frac{1}{2})\,\Gamma(n+1+j\gamma)}{\lfloor n\;\Gamma(n+\frac{3}{2}+j\gamma)}\left(\frac{b-1}{b+1}\right)^{2n} \end{aligned}\right\} \qquad (76)$$

or expressed as:

$$\left.\begin{aligned} G_2 = {} & \frac{1}{2b\sqrt{b^2-1}}\left(\frac{b-1}{b+1}\right)^{j\gamma} B\left(\tfrac{1}{2}, j\gamma\right) F\left(\tfrac{1}{2}, j\gamma; \tfrac{1}{2}+j\gamma; \left(\frac{b-1}{b+1}\right)^2\right) \\ & + \frac{1}{2b\sqrt{b^2-1}}\left(\frac{b-1}{b+1}\right)^{1+j\gamma} B\left(\tfrac{1}{2}, 1+j\gamma\right) F\left(\tfrac{1}{2}, 1+j\gamma; \tfrac{3}{2}+j\gamma; \left(\frac{b-1}{b+1}\right)^2\right) \end{aligned}\right\} \quad (77)$$

7. Formulas for the Lift and Moment

The formulas for the lift and moment are now summarized below:

$$L = L_o e^{j\omega t} \quad (26)$$

$$M = M_o e^{j\omega t} \quad (34)$$

$$L_o = \frac{2\rho h^2}{\pi}\omega^2 y_o \log \cosh 2\mu + 2\rho h A \tanh 2\mu \quad (33)$$

$$M_o = -\frac{2\rho h}{\pi} A \log\cosh 2\mu \quad (41)$$

$$A = -j\omega y_o U \frac{G_1}{G_1 + G_2} \quad (68)$$

where G_1 and G_2 are given by (74) and (76), or by (75) and (77) respectively.

IV. Numerical Calculation

Let

$$R_a + j\, I_a = \frac{G_1}{G_1 + G_2} \tag{78}$$

Substituting (74) and (76) into (78) and after cancelling out the term

$$\frac{1}{2\sqrt{b^2-1}} \left(\frac{b-1}{b+1}\right)^{j\gamma} \frac{\Gamma(\frac{1}{2})\,\Gamma(j\gamma)}{\Gamma(\frac{1}{2}+j\gamma)}$$

gives the following expression:

$$R_a + j\, I_a = \frac{1+E-F}{1+E-F-\frac{1}{b}(1+E+F)} \tag{79}$$

where

$$E = \sum_{m=0}^{\infty} \left(\frac{b-1}{b+1}\right)^{2(m+1)} \prod_{n=0}^{m} \frac{(n+\frac{1}{2})(n+j\gamma)}{(n+1)(n+\frac{1}{2}+j\gamma)} \tag{80}$$

$$F = \sum_{m=0}^{\infty} \frac{m+1}{m+\frac{1}{2}} \left(\frac{b-1}{b+1}\right)^{2m+1} \prod_{n=0}^{m} \frac{(n+\frac{1}{2})(n+j\gamma)}{(n+1)(n+\frac{1}{2}+j\gamma)} \tag{81}$$

The next step is to substitute $R_a + j\, I_a$ into the lift and moment equations and then to express them in non-dimensional forms. This is done as follows: By substituting (68), (78) into (33, (43) and then into (26), (34), one obtains:

$$\left.\begin{aligned} L = \Big[&\frac{2\rho h^2}{\pi} \omega^2 y_0 \log\cosh 2\mu \\ &- 2\rho h U j\omega y_0 (R_a + jI_a) \tanh 2\mu \Big] e^{j\omega t} \end{aligned}\right\} \tag{82}$$

$$M = \left[\frac{2\rho h^2 U}{\pi} j\omega y_0 (R_a + jI_a) \log\cosh 2\mu \right] e^{j\omega t} \tag{83}$$

In the stationary lattice theory, the lift and moment have been shown to be

$$L_s = 2\rho h U\alpha \tanh 2\mu \tag{84}$$

$$M_s = -\frac{2\rho h^2}{\pi} U^2\alpha \log\cosh 2\mu \tag{85}$$

Now replace $U\alpha$ by $-j\omega y_o e^{j\omega t}$ in the above equations *

$$L_s = -2\rho h U j\omega y_o e^{j\omega t} \tanh 2\mu \tag{86}$$

$$M_s = \frac{2\rho h^2}{\pi} U j\omega y_o e^{j\omega t} \log\cosh 2\mu \tag{87}$$

where L_s and M_s may be called the quasi-steady lift and moment respectively.

Introducing the notations:

$$\sigma = \frac{1}{2\mu} \tanh 2\mu \tag{88}$$

$$\lambda = \frac{1}{2\mu^2} \log\cosh 2\mu \tag{89}$$

which have been called the interference factors in the stationary lattice theory.

Then Eqs.(82) and (83) can be expressed in the following non-dimensional forms:

---

* Here the negative sign means that when $v_b = j\omega y_o e^{j\omega t}$ is in the upward direction, the perturbation velocity $U\alpha$ is downward.

$$\frac{L}{L_s} = R_a + j\left(I_a + \frac{1}{2}\frac{\lambda}{\sigma}\nu\right) = \left|\frac{L}{L_s}\right| e^{j\varphi_L} \tag{90}$$

$$\frac{M}{M_s} = R_a + j\, I_a = \left|\frac{M}{M_s}\right| e^{j\varphi_M} \tag{91}$$

where

$$\left|\frac{L}{L_s}\right| = \sqrt{R_a^2 + \left(I_a + \frac{1}{2}\frac{\lambda}{\sigma}\nu\right)^2} \tag{92}$$

$$\frac{M}{M_s} = \sqrt{R_a^2 + I_a^2} \tag{93}$$

$$\varphi_L = \tan^{-1} \frac{I_a + \frac{1}{2}\frac{\lambda}{\sigma}\nu}{R_a} \tag{94}$$

$$\varphi_M = \tan^{-1} \frac{I_a}{R_a} \tag{95}$$

φ_L and φ_M are the phase angles relative to the directions of L_s and M_s respectively.

The calculation were performed at various values $\frac{c}{h}$ and ν.

The results of calculation are plotted in Figs. 2-5.

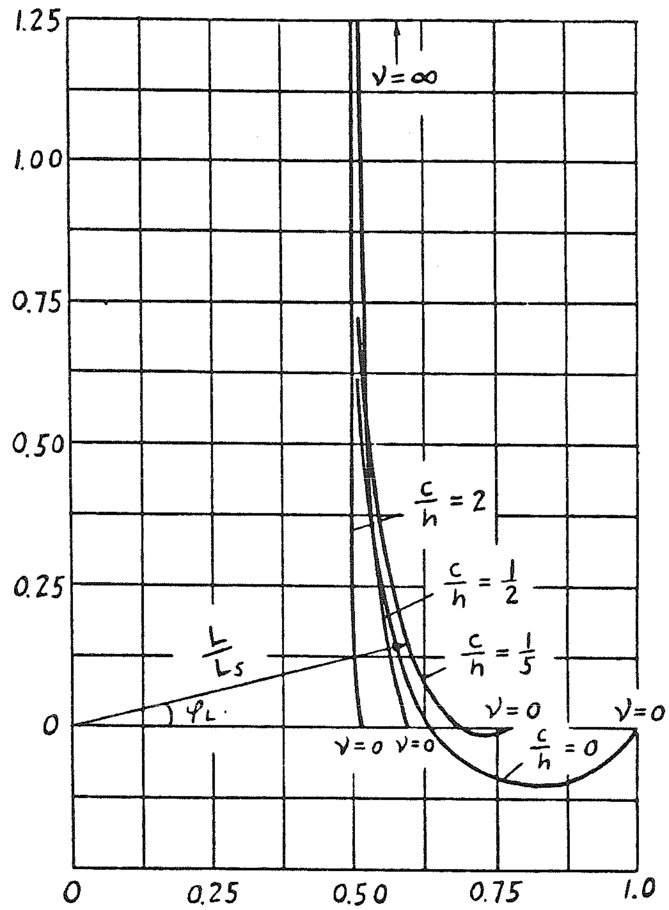
1.25
1.00
0.75
0.50
0.25
0
0
0.25
0.50
0.75
1.0
ν=∞
c/h = 2
c/h = 1/2
c/h = 1/5
c/h = 0
L/Ls
φL
ν=0
ν=0
ν=0
ν=0

Fig. 2

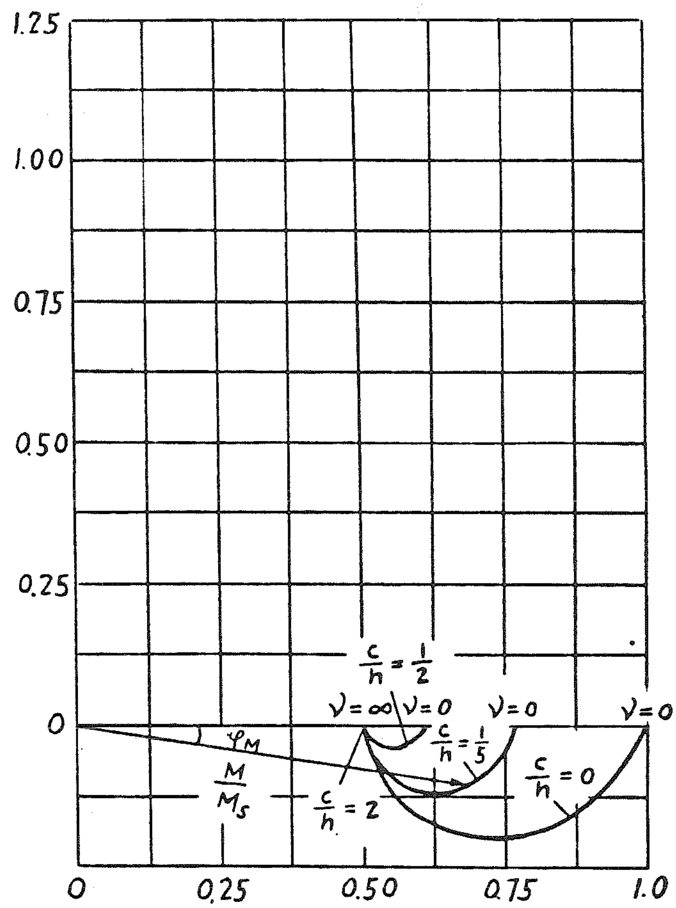
1.25
1.00
0.75
0.50
0.25
0
$\frac{c}{h} = \frac{1}{2}$
$\nu = \infty$
$\nu = 0$
$\nu = 0$
$\nu = 0$
φ_M
$\frac{c}{h} = \frac{1}{5}$
$\frac{c}{h} = 0$
$\frac{M}{M_s}$
$\frac{c}{h} = 2$
0
0.25
0.50
0.75
1.0

Fig. 3

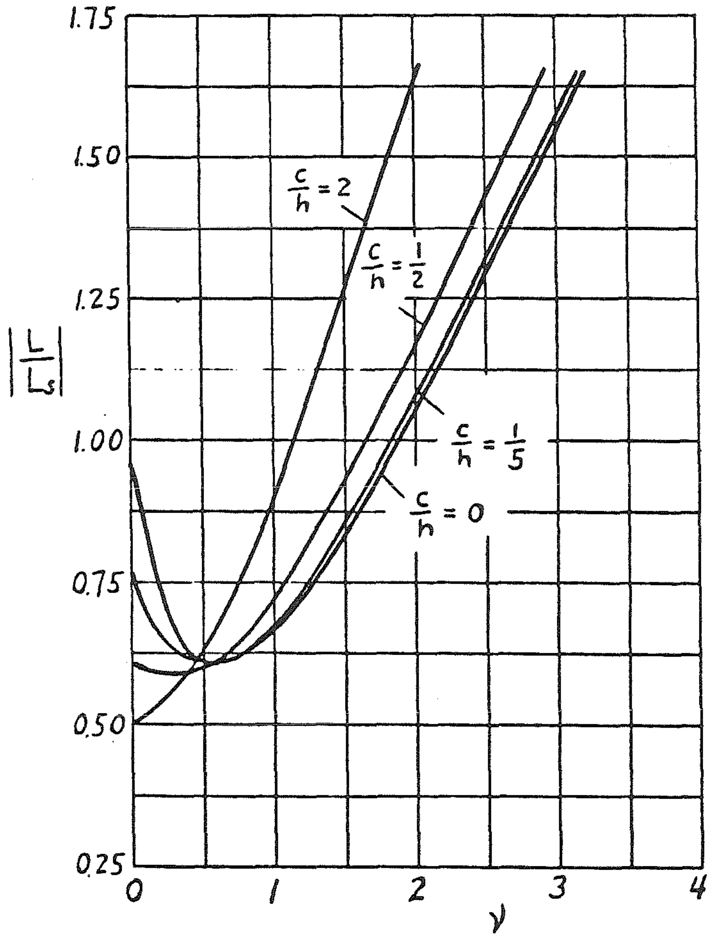
1.75
1.50
1.25
1.00
0.75
0.50
0.25
$\left|\frac{L}{L_s}\right|$
0
1
2
3
4
ν
$\frac{c}{h} = 2$
$\frac{c}{h} = \frac{1}{2}$
$\frac{c}{h} = \frac{1}{5}$
$\frac{c}{h} = 0$

Fig. 4

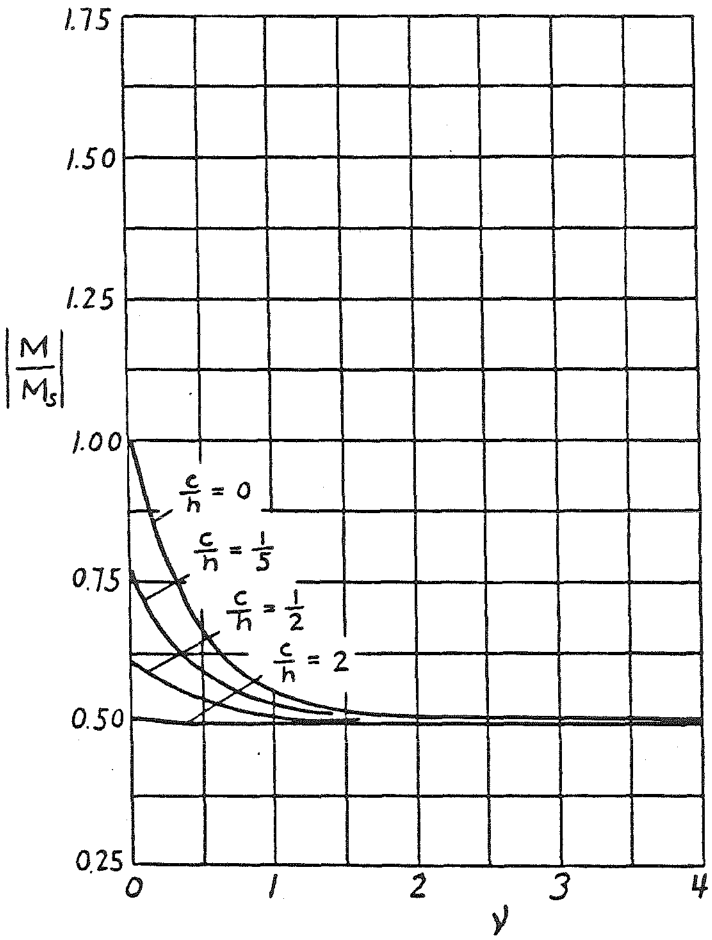
1.75
1.50
1.25
$\left|\frac{M}{M_s}\right|$
1.00
0.75
0.50
0.25
$\frac{c}{h} = 0$
$\frac{c}{h} = \frac{1}{5}$
$\frac{c}{h} = \frac{1}{2}$
$\frac{c}{h} = 2$
0
1
2
3
4
y

Fig. 5

V. Discussion of Results

Since complex quantities may be represented graphically by points in a plane or by vectors drawn from the origin to these points, in Figs. 2 and 3 are plotted the vector diagrams which give the complete information about the vectors $\frac{L}{L_s}$ and $\frac{M}{M_s}$ with their varying magnitudes $\left|\frac{L}{L_s}\right|$ and $\left|\frac{M}{M_s}\right|$ and varying phase angles φ_L and φ_M at different values of $\frac{c}{h}$ and ν. In Figs. 4 and 5 are plotted the diagrams which give the values of $\left|\frac{L}{L_s}\right|$ and $\left|\frac{M}{M_s}\right|$ at different values of $\frac{c}{h}$ and ν. From Fig.5, it is seen that for $\frac{c}{h} = 1.5$, the value of $\left|\frac{M}{M_s}\right|$ remains practically constant at the lowest value 0.5 for all values of ν; while from Fig. 4, it is seen that for $\nu = 0.5$, the value of $\left|\frac{L}{L_s}\right|$ remains practically constant at the approximate lowest value 0.60 for all values of $\frac{c}{h}$.

For discussion of the above results, write Eqs. (86) and (87) in the following forms:

$$L_s = |L_s| \; e^{j(\omega t + \varphi_{Ls})} \tag{96}$$

$$M_s = |M_s| \; e^{j(\omega t + \varphi_{LM})} \tag{97}$$

where

$$|L_s| = \pi \rho U^2 c \cdot \frac{y_0}{c} \cdot 2\tau\nu \tag{98}$$

$$|M_s| = \pi \rho U^2 c^2 \cdot \frac{y_0}{c} \cdot \frac{1}{2} \lambda \nu \tag{99}$$

$$\varphi_{Ls} = \tan^{-1} - \infty = -\frac{\pi}{2} \tag{100}$$

$$\varphi_{LM} = \tan^{-1} \infty = \frac{\pi}{2} \tag{101}$$

Then Eqs. (90) and (91) can be written as

$$\mathfrak{L} = \left|\frac{\mathfrak{L}}{\mathfrak{L}_s}\right| \left|\mathfrak{L}_s\right| \; e^{j\left(\omega t - \frac{\pi}{2} + \varphi_L\right)} \tag{102}$$

$$M = \left|\frac{M}{M_s}\right| \left|M_s\right| \; e^{j\left(\omega t + \frac{\pi}{2} + \varphi_M\right)} \tag{103}$$

It has been seen that for a fixed value of $\frac{c}{h} = 1.5$, the lowest values of $\left|\frac{\mathfrak{L}}{\mathfrak{L}_s}\right|$ and $\left|\frac{M}{M_s}\right|$ occur at a low value of $\nu = 0.5$. At this low value of ν , it is evident from Eqs. (98) and (99) that for $\frac{c}{h}$ fixed, the values of $\left|\mathfrak{L}_s\right|$ and $\left|M_s\right|$ will also be low. Thus it may be concluded that the lowest amplitudes of $\mathfrak{L}$ and M occur approximately at $\frac{c}{h} = 1.5$ and $\nu = 0.5$. In practical applications, the amplitudes of the periodic lift and moment $\mathfrak{L}$ and M should be kept at their lowest values. Hence

$$\frac{c}{h} = 1.5 \qquad \nu = 0.5 \tag{104}$$

may be considered as the optimum conditions.

At these conditions,

$$\left|\frac{\mathfrak{L}}{\mathfrak{L}_s}\right| = 0.60 \tag{105}$$

$$\left|\frac{M}{M_s}\right| = 0.50 \tag{106}$$

$$\left|\mathfrak{L}_s\right| = 0.415\pi\rho U^2 c \cdot \frac{y_0}{c} \tag{107}$$

$$\left|M_s\right| = 0.150\pi\rho U^2 c^2 \cdot \frac{y_0}{c} \tag{108}$$

The next discussion is to give the physical meaning to the two terms in Eq.(82) In Eq.(82), the lift may be considered as consisting of two parts:

$$L_1 = \left(\frac{2\rho h^2}{\pi}\,\omega^2 y_o \log\cosh 2\mu\right)e^{j\omega t} \tag{109}$$

$$L_2 = -\left[2\rho h j\omega y_o (R_a + j\,I_a)\ \tanh 2\mu\right]e^{j\omega t} \tag{110}$$

It should be noted here that the two terms L_1 and L_2 correspond to the two terms in the complex potential expression (24). The physical meaning of L_1 and L_2 may now be stated as follows: the lift L_1 is produced due to the reaction of the accelerated masses, while the lift L_2 is produced due to the circulation about each airfoil of the lattice. The lift L_1 which may be called the apparent-mass lift, acts through the center of each airfoil, while the lift L_2 produces the moment about the center of each airfoil, and its line of action can be obtained from Eqs. (83) and (110).

$$x = \frac{M}{L_2} = -\frac{1}{4}\,\frac{\lambda}{\sigma}\,c \tag{111}$$

For $k \rightarrow 0$, $$x = \frac{M}{L_2} = -\frac{1}{4}\,c \tag{112}$$

This result checks again with that obtained by Kármán and Sears (Ref.6).

VI. Summary of Equations

The equations for calculating the amplitudes of the periodic lift and moment at any values of $\frac{c}{h}$ and ν are now summarized below:

$$L = \left|\frac{L}{L_s}\right| \, |L_s| \; e^{j(\omega t - \frac{\pi}{2} + \varphi_L)} \tag{102}$$

$$M = \left|\frac{M}{M_s}\right| \, |M_s| \; e^{j(\omega t + \frac{\pi}{2} + \varphi_M)} \tag{103}$$

$$\left|\frac{L}{L_s}\right| = \sqrt{R_a^2 + (I_a + \frac{1}{2}\frac{\lambda}{\sigma}\nu)^2} \tag{92}$$

$$\left|\frac{M}{M_s}\right| = \sqrt{R_a^2 + I_a^2} \tag{93}$$

$$|L_s| = \pi \rho U^2 c \cdot \frac{y_0}{c} \cdot 2\sigma\nu \tag{98}$$

$$|M_s| = \pi \rho U^2 c^2 \cdot \frac{y_0}{c} \cdot \frac{1}{2}\lambda\nu \tag{99}$$

$$R_a + j\, I_a = \frac{1 + E - F}{1 + E - F - \frac{1}{b}(1 + E + F)} \tag{79}$$

$$E = \sum_{m=0}^{\infty} \left(\frac{b-1}{b+1}\right)^{2(m+1)} \prod_{n=0}^{m} \frac{(n+\frac{1}{2})(n+j\gamma)}{(n+1)(n+\frac{1}{2}+j\gamma)} \tag{80}$$

$$F = \sum_{m=0}^{\infty} \frac{m+1}{m+\frac{1}{2}} \left(\frac{b-1}{b+1}\right)^{2m+1} \prod_{n=0}^{m} \frac{(n+\frac{1}{2})(n+j\gamma)}{(n+1)(n+\frac{1}{2}+j\gamma)} \tag{81}$$

$$\frac{b-1}{b+1} = \left(\frac{1-k}{1+k}\right)^2 \tag{14}$$

$$k = \tanh \mu \tag{12}$$

$$\sigma = \frac{1}{2\mu} \tanh 2\mu \tag{88}$$

$$\lambda = \frac{1}{2\mu^2} \log \cosh 2\mu \tag{89}$$

$$\mu = \frac{\pi c}{4h} \qquad \nu = \frac{\omega c}{2U} \qquad \gamma = \frac{\nu}{4\mu} \tag{66}$$

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