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LAMINAR BOUNDARY LAYER STABILITY IN FREE CONVECTION

II

LAMINAR FREE CONVECTION WITH VARIABLE FLUID PROPERTIES

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ABSTRACT

I. LAMINAR BOUNDARY LAYER STABILITY IN FREE CONVECTION

The stability with respect to transition to turbulent flow of laminar free convection flow along a semi-infinite flat plate was studied analytically by the method of small oscillations. Six procedures of varying degrees of mathematical complexity for solving approximately the stability problem were developed according to different assumptions regarding the presence or absence of heat conduction and viscous and body forces. The procedure which considers that only inertial, pressure, and viscous forces control the stability was applied to the case of free convection of air along a vertical plate. It was found that the analytic predictions agreed qualitatively but not quantitatively with experimental observations made by other investigators of the appearance of instability in such a flow.

II. LAMINAR FREE CONVECTION WITH VARIABLE FLUID PROPERTIES

Heat transfer by laminar free convection along a semi-infinite flat plate was analyzed with the assumption that the density, specific heat, viscosity, and thermal conductivity of the fluid are functions of the temperature. An approximate method for obtaining heat-transfer rates was developed and was applied to cases of heating and cooling oils with large changes in their Prandtl numbers. By comparison, it was found that earlier and more general analyses based on essentially constant-property assumptions give good results in the variable-property case when they are based on the properties of the fluid at the plate surface.

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LAMINAR BOUNDARY LAYER STABILITY IN FREE CONVECTION

A

SUMMARY

It has been observed in some free convection flows that the flow is laminar near its origin and that it becomes turbulent after proceeding some distance in its course. This work is an investigation by analytic means of the initial stage of transition from the laminar to the turbulent situation for such a free convection flow. More specifically, it is a study of the instability of a given laminar free convection flow with respect to small disturbances, which, according to the present theories of the origin of turbulence, are amplified until the flow becomes turbulent. The free convection situation chosen for study is the two-dimensional flow existing in the neighborhood of a semi-infinite flat plate either inclined or parallel to the body force field which produces the flow. The plate is kept isothermal and at a temperature either greater or less, depending on the orientation of the body force field and the plate, than that of the unaffected, stationary fluid far from the plate so that the flow proceeds along the plate away from its leading edge.

With very general assumptions regarding the fluid properties, disturbance equations for the analysis of the laminar flow stability were derived by the method of two-dimensional small oscillations. This often-used method of stability analysis consists essentially of assuming that small wave-like disturbances of velocity and fluid properties occur in the flow and then of attempting to find whether these disturbances are amplified or damped. Throughout the derivations it was assumed that the Reynolds number of the basic laminar flow is large and that the product of the fluid coefficient of thermal expansion and the temperature difference between the plate and the fluid far from it is small. The

further assumptions that the mean flow is parallel and that temperature differences are so small that the only effect of variable fluid properties is the appearance of a body force term in the momentum equation were used to simplify the disturbance equations to forms for which it was feasible to attempt to construct approximate solutions.

Six methods for approximately solving the stability problem using these stability equations were developed for the cases of both an inclined and a vertical plate. These methods are differentiated by the simplifying assumptions regarding the fluid and flow properties made in developing them. The most complex of these methods takes into account inertial, pressure, viscous, and body forces as well as the effect of a finite fluid thermal conductivity. The least complex method considers that only inertial and pressure forces determine the stability or instability of the flow. All of the remaining possible simplifications were included in the other four methods. Unfortunately, employing any of these methods except the two simplest ones would necessitate a very great amount of computation. The amount of calculation anticipated is so great that the use of automatic, high-speed computing devices would be a necessity.

The two simplest methods were developed with the assumptions that only inertial and pressure forces in one case and only inertial, pressure, and viscous forces in the other case determine the stability of the flow. Both these methods are sufficiently simple to admit their application with the use of no more than a desk calculator as a computational aid, but the amount of calculation that they require is nevertheless apt to be quite tedious. These cases reduce simply to the problems of solving the stability problem for a nonviscous, incompressible flow in the one case and for a viscous, incompressible flow in the other case.

Both these problems have been studied extensively in the past in treating the stability of other velocity profiles, and existing techniques were appropriately applied to the present situation with the free convection velocity profile.

The stability of laminar free convection flow along a vertical, semi-infinite flat plate as a function of the disturbance wave number α* and the mean-flow Reynolds number Re** was investigated by using the method in which only inertial, pressure, and viscous forces are considered. This was done for two velocity profiles corresponding to free convection in a fluid with a Prandtl number of 0.72 so that a comparison with experimental observations of instability in air, which has a Prandtl number of this value, could be made. (The Prandtl number is the principal parameter that determines the shapes of the velocity and temperature profiles in free convection.) One of the velocity profiles was a cubic polynomial approximation, and the other was the "exact" profile obtained by another investigator in solving numerically the flow equations which describe the laminar free convection situation. An "indifference" curve or curve dividing the $\alpha - Re$ plane into stable and unstable regions was obtained for the cubic polynomial profile, and

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* The wave number α of the disturbance is equal to $\frac{2\pi\delta}{\Lambda}$, in which δ is the boundary layer thickness defined to be $\frac{1}{U_m}\int_0^\infty \bar{U}dy$ and Λ is the wave length of the disturbance. $\bar{U}$ is the mean-flow velocity component parallel to the plate, U_m is the maximum value of $\bar{U}$ in the boundary layer at given distance from the edge of the plate, and y is distance measured normal to the plate surface from that surface.

** The mean-flow Reynolds number Re is defined to be $\frac{U_m \delta}{\nu_0}$, in which ν_0 is the kinematic viscosity of the fluid far from the plate. Re is a monotonically increasing function of the distance from the edge of the plate because U_m and δ both depend in this way on the distance from the edge of the plate.

a significant portion of such a curve was secured for the exact profile. An examination of the mathematical procedures necessary to obtain these curves led the author to believe that the curve for the cubic polynomial profile represents a valid solution of the assumed problem but that the portion obtained of the curve for the exact profile may be considerably displaced from its true position. Both these curves agree qualitatively with observations of the appearance of instability in free convection in that they predict that a minimum Reynolds number exists below which the flow is stable for oscillations of all wave numbers. However, the curve for the cubic polynomial profile indicates that the instability waves which initially* appear in the flow do so for values of the Reynolds number, the wave number, and the phase velocity or velocity of propagation of the disturbance approximately 29, 1.9, and 0.49, respectively, times the values of these parameters which were observed. The extreme discrepancy between the predicted and observed values of the minimum Reynolds numbers for instability is believed to be largely the result of poor approximation by the cubic polynomial to the true profile. The indifference curve for the exact profile gives values of the Reynolds and wave numbers and the disturbance phase velocity at the first appearance of instability which are 1.19, 7.0, and 0.96 times the observed values. In this case the agreement for the Reynolds numbers and the phase velocities was considered to be satisfactory, but the discrepancy between the values of the wave numbers is far larger than what it should be if the analysis were valid and the observations were accurate. It was suspected that a more extensive mathematical treatment would reduce this dis-

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* That is, at the lowest value of the Reynolds number.

crepancy, although it might increase the discrepancy between the values of the Reynolds number.

The indifference curves for the two velocity profiles along with points representing the observed first appearance of instability are plotted in Figures 7 and 8.

B

INTRODUCTION

B1 The General Problem of Laminar Flow Instability

In the science of fluid mechanics the problems of turbulent flow and of the transition of laminar flow to turbulent flow are among those which are most difficult and yet which present themselves in many situations of importance to the worker in this field. Probably the most familiar example of a situation in which one desires information regarding the second of these two problems, transition from laminar to turbulent flow, is the case of boundary layer flow on an airplane wing. In this instance the higher rate of momentum transport in turbulent flow increases the profile drag over the laminar case and can in specific situations prevent flow separation. This thesis is concerned with the stability, or ability to remain laminar rather than become turbulent, of a class of laminar flows. The class of laminar flows considered is the class associated with the two-dimensional free convection of air near an inclined* or vertical* semi-infinite flat plate, which process will be described in detail later in this introductory section. Instability of this class of flows has not been studied analytically previously, and only a relatively small amount of experimental investigation has been done in the field. An analytic investigation for the purpose of adding to the present body of knowledge concerning laminar flow stability was considered to be desirable. Such an investigation may also be considered to be one of the

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* The plate is said to be inclined if the vector which represents the body force producing the flow makes a finite angle with the plate surface. It is described as being vertical when the body force vector is parallel to the plate surface.

first steps in learning how to control whether any particular free convection flow of practical importance will be laminar or turbulent. Since heat-transfer coefficients differ greatly between the laminar and turbulent regimes, the ability to govern which of these types of flow is present would be important in applications such as nuclear reactors in which heat transfer by free convection is significant.

The classical experiments of Osborne Reynolds regarding transition from laminar to turbulent flow in a conduit of circular cross-section indicated that the formation of turbulence is the result of an instability in the laminar flow. This instability causes any flow disturbances such as are caused by roughness of a bounding surface to be amplified until the random, unsteady fluctuations of turbulence become established. By the same reasoning a flow which is laminar must damp out any disturbances which appear in it. The task of the worker treating the problem analytically is to predict mathematically whether a flow is stable or unstable.

B2 Historical Survey of Analytic Laminar Flow Stability Investigations

Here only a brief synopsis of those analytic studies of laminar flow stability which are most pertinent to the present work will be given. The general field of laminar flow stability problems is a rather broad one, and the material presented here comprises a relatively small fraction of the total amount of work which has been done, although the basic analyses are reported.

In 1895 Reynolds(1) examined the problem of the stability of a simple type of laminar flow through a consideration of the rate of change of the mean kinetic energy of small superposed disturbances such as appear in turbulent flow. The essential idea was to find conditions under which this kinetic energy would either be dissipated by viscosity or be increased by the transfer of energy from the mean motion to the disturbance motion. When this method was adapted to the case of cylindrical Poiseuille flow the result was the determination of a minimum Reynolds number below which turbulent motion could not be sustained. This value of this Reynolds number was much lower than experimental values for transition from laminar to turbulent flow, and the theory thus was confirmed to the extent that its predictions were not found to be in conflict with observed phenomena. Unfortunately, this approach of treating the kinetic energy of disturbance motions has apparently been too complicated in application to have yielded much more in the way of interesting results.

Before Reynolds had published his energy method for treating the problem, Rayleigh(2) had in 1880 introduced a technique which is fundamental in almost every present-day study and which was used in all the following contributions discussed here. This is the method of small oscillations in which sinusoidal disturbance waves are assumed that proceed in the direction of the mean laminar flow and are either amplified or damped exponentially with time. Actually, this method is not as restricted as it might first appear because the sinusoidal disturbance waves can be considered to be Fourier components of disturbance waves of arbitrary form. That laminar flows on the verge of transition first indicate their instability by developing disturbance waves progressing in the direction of the mean flow was observed by Reynolds in his experiments on transition in a circular tube. Ordinarily this method is applied to plane, parallel flows, i.e., two-dimensional flows with parallel streamlines; and it results in reducing the number of independent variables in the problem to one, the space variable measured perpendicular to the direction of the mean flow. Also, the problem is linearized by assuming that the disturbances are sufficiently small to justify the neglect of terms of order higher than the first in the disturbance quantities. Even in the case that the disturbances are amplified, this assumption that they are very small is valid if one considers that they have just appeared and have not had sufficient time to become of appreciable size compared with corresponding mean-flow quantities. This method of small oscillations was applied by Rayleigh and others to flow stability problems in which viscosity was neglected, but little success was had by them. An important contribution by Rayleigh, however, was the deduction of two important theorems concerned with the relation of the phase velocity of the disturbance wave to the mean-flow velocity profile when viscosity is neglected.

An equation for the study of the stability of a laminar, viscous, incompressible, parallel flow with respect to three-dimensional disturbances was derived by Kelvin[3] in 1887. It is essentially what is now called the Orr-Sommerfeld equation, except for the fact that Kelvin's equation was formulated for three-dimensional disturbances with the dependent variable related to a disturbance velocity component, while the Orr-Sommerfeld equation is restricted to two-dimensional disturbances with the dependent variable related to a disturbance stream function. Kelvin's equation was studied by Orr[4], who apparently has been given credit for deriving it. Sommerfeld[5] in 1908 set the equation in its present form for two-dimensional disturbances in terms of a dependent variable related to a disturbance stream function.

Asymptotic methods for solving the Orr-Sommerfeld equation when the Reynolds number of the mean flow is large were developed by Heisenberg(6) in 1924. He applied these methods in a study of plane Poiseuille flow in an attempt to classify combinations of disturbance wave length and mean-flow Reynolds number as stable or unstable, but limitations in the methods prevented him from completing this classification. Nevertheless, his method of obtaining solutions with the Reynolds number in the equation infinite, his noting of the great importance of the curvature of the mean-flow velocity profile, and his determination of the region of validity of the asymptotic solutions in the complex plane of the independent variable have been particularly useful to later investigators.

Tollmien(7) in 1929 applied the Orr-Sommerfeld equation to the Blasius boundary layer and was able to find the relation between disturbance wave length and mean-flow Reynolds number for the case of neutral, i.e., neither amplified nor damped, oscillations. The results of his analysis were confirmed beautifully by the experimental investigations of Schubauer and Skramstad(8) in 1943.

An investigation of the effect of stratification of density upon the stability of the Blasius boundary layer was made by Schlichting(9) in 1935. The fluid density was assumed to vary across the boundary layer, and a body force field such as a gravitational field acting on each fluid particle in proportion to its mass was assumed to be present and to be directed perpendicular to the plate. Although Schlichting's stability equation was not derived in the most general way possible, it bears a strong resemblance to corresponding ones developed in the present work, as noted in Appendix 2. His stability equation was the Orr-Sommerfeld equation with small additional terms, and his method of finding solutions was essentially that of Tollmien with a perturbation process added.

In 1945 Lin[10] published an important, comprehensive paper containing the results of an examination of the incompressible, two-dimensional, parallel-flow, stability problem. He utilized the best methods of both Heisenberg and Tollmien for obtaining approximate solutions of the Orr-Sommerfeld equation. His procedure and modifications of it form the basis of the methods used for approximating solutions for the stability equations of the present work.

The problem of the stability of a compressible, viscous, boundary layer flow was investigated by Lees and Lin[11] in 1947. Disturbance equations for two-dimensional, parallel flow were derived with the inclusion of the energy equation, which was not considered in the previously mentioned work of Schlichting. In the present problem dealing with free convection, the methods of derivation and solution of the equations that were applied by Lees and Lin are quite useful, although they were principally interested in cases for which the Mach number is finite and the Froude number* is infinite. For at least the case of air, free convection flow along a flat plate undergoes transition at negligible Mach numbers. When the Froude number is finite there is added to the free convection disturbance equation corresponding to the Orr-Sommerfeld equation a term which provides coupling with a disturbance energy equation. The importance of this term will be apparent when the reader

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* The Froude number appears in nondimensionalized boundary layer equations when there are present both a variation in the fluid density and a body force such as gravity.

arrives at Appendix 4, in which methods of solving the equations are developed.

B3 Laminar Free Convection Along an Inclined or Vertical Semi-Infinite Flat Plate

Free convection is the process of energy transport by a fluid in motion when the motion is the result of the interaction of a variable fluid density distribution with a body force field such as gravity. Every-day examples of the process are observed in the rising of smoke from a burning cigarette or the upward flow of heated air near a steam radiator. In these situations the fluid in motion has a density lower than that of the surrounding fluid, and the result is an upward flow due essentially to the fact that the downward gravitational force acting on an element of heated fluid is less than the upward buoyant force of the static pressure field, which is determined by the hydrostatic relation in the unheated surrounding fluid. Similarly, the flow of air down a mountain side on a clear night is a process of free convection. In this case the mountain radiates heat into space until its surface temperature drops below that of the ambient atmosphere. The air immediately adjacent to the surface of the mountain is cooled and flows downward because of its increased density compared with that of the surrounding atmosphere in which the hydrostatic relation determines the variation of pressure with height.

In order to investigate free convection analytically it is necessary to solve simultaneously the continuity, momentum, and energy equations of fluid mechanics as well as the equation of state of the fluid being convected. This is much more difficult in general than dealing with forced convection problems, in the usual cases of which the fluid properties are taken to be constant and the velocity field is taken to be predetermined.

In forced convection the problem of solving a heat-transfer problem becomes simply that of solving the energy equation with appropriate boundary conditions. Since in free convection the velocity field is dependent upon the fluid density field and this latter field is dependent principally upon the temperature field, all the equations have to be solved simultaneously. Because of this complication, free convection problems involving only the simplest types of configurations have been examined analytically. The first configuration to be so examined in detail was a semi-infinite flat plate parallel to the body force field, the flow being considered to be two-dimensional. The plate is kept isothermally at a temperature different from that of the fluid far away. This plate temperature is chosen to be higher or lower than that of the surrounding or ambient fluid, depending on the sense of the body force field, in order that flow will start at the leading edge of the plate and proceed away from the leading edge along the plate. The general shapes of the velocity and temperature profiles at a given distance from the plate edge are indicated in Figure 3. Polhausen(12) found similarity relations for the velocity and temperature profiles for this configuration and solved the problem for a fluid Prandtl number of 0.733 to correspond to that of air. Schmidt and Beckmann(12) measured temperature and velocity profiles for this case of a semi-infinite vertical flat plate in air and found very good agreement with the results of Polhausen's analysis. Schuh(13) in addition solved Polhausen's equations with Prandtl numbers of 10, 100, and 1000; and Ostrach(14) obtained accurate solutions of the same equations with the use of an electronic digital computer for these values of the Prandtl number as well as for the values 0.01, 0.72, 0.733, 1, and 2.

Heat-transfer coefficients agreeing quite well over a wide range of Prandtl numbers with those predicted by solving the Polhausen equations were obtained by Squire[15], who applied integral relations to assumed simple polynomial velocity and temperature profiles.

These analyses of laminar flow along a vertical, semi-infinite flat plate can be extended to the case of an inclined plate if the body force producing the flow is taken to be that component of the total body force vector that is parallel to the plate. Although the component of the body force field normal to the plate may have a tendency to separate the flow on one side of the plate, it can be shown by a consideration of the momentum equation that it should have no effect while the flow remains attached and laminar.

B4 Instability and Transition of Laminar Free Convection Along a Semi-Infinite Flat Plate

It has been established by observation that free convection flow along a semi-infinite flat plate is initially laminar near the leading edge but becomes turbulent after it has progressed some distance along the plate. For a fluid with a given Prandtl number, the Grashof number Gr_x[*] based on distance from the leading edge of the plate is the main parameter describing the stability of the flow according to observation and according to analysis as well. (The Grashof number is related to the

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[*] Gr_x is defined to be $\frac{|a_1| g \varepsilon x^3}{\nu_0^2}$, g being the magnitude of the body force vector, a_1 being the cosine of the angle between the body force vector and the plate surface, ε being the product $\beta_0 \Delta T$ in which β_0 is the coefficient of thermal expansion of the fluid far from the plate and ΔT is the absolute value of the temperature difference between the plate and the fluid far from it, x being the distance from the leading edge of the plate, and ν_0 being the kinematic viscosity of the fluid far from the plate.

Reynolds number used in the analysis in that Gr_x is proportional to $(Re)^4$.) For heat-transfer purposes, transition from laminar to turbulent flow is considered to occur at a value of Gr_x of about 10^9 [16]. According to the laminar flow instability theory of the origin of turbulence, instability should appear first at somewhat lower values of Gr_x in order that disturbances carried along with the flow be amplified in the process of transition until the turbulent regime is established. To the knowledge of the author, the observations reported by Eckert and Soehnghen[17] are the only experimental investigations of laminar flow instability in free convection along a flat plate. They found that with air as the convecting fluid, instability waves traveling with the mean flow first appeared at a Gr_x of 4×10^8. These initial waves were of a length equal to 3.1 times the distance measured normal to the plate from the plate surface to the outer point in the flow at which the velocity was 0.01 of its maximum. Motion pictures studies indicated that the phase velocity of these waves was 0.73 times the maximum velocity of the flow at the particular values of Gr_x and distance from the plate leading edge at which they were first observed. It is important to note that these initial waves were considered to be two-dimensional or, equivalently, that they propagated essentially parallel to the plate surface in the direction of the mean flow. This is in accord with the fact that for incompressible flow it has been shown analytically by Squire[18] that such two-dimensional waves, rather than "three-dimensional" ones which propagate parallel to the plate but at an angle with the mean flow velocity, are the first to be unstable.

In Figures 7 and 8, points which represent Eckert and Soehnghen's observed incipient instability waves are plotted in terms of the nondimensional disturbance wave number α and the mean-flow Reynolds number

Re. The values of α and Re for the experimental observations in the two figures are slightly different because the nondimensionalizing process depends on the shapes of the velocity profiles, which are different for the two cases.

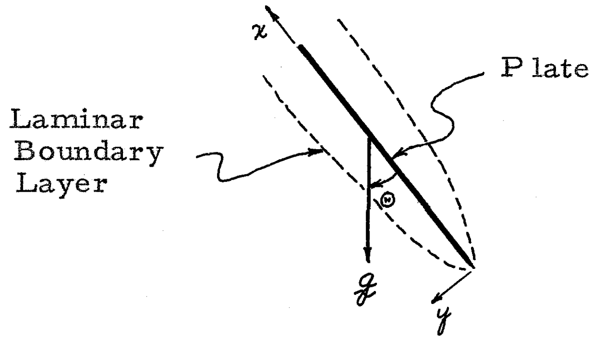
B5 The Following Sections

Section C contains a general description of the application of the method of small oscillations to the study of the stability of laminar free convection flow along an inclined or vertical flat plate. A discussion and conclusions concerning six approximate procedures developed for the application of this method with various simplifying assumptions are also contained in this section. The development of these six procedures is presented in Appendix 4. With the exception of the simplest, they were developed for the purpose of classifying combinations of the Reynolds number Re and the wave number α as being stable or unstable.

Section D describes the application of one of the methods developed in Appendix 4 to the case of air convecting about a vertical plate. Gravitational forces were neglected for simplification. This implies, as is indicated in the appendix, that the thermal conductivity of the fluid is irrelevant to the problem. An indifference curve or plot in the α-Re plane representing neutrally stable oscillations was obtained for the cubic polynomial velocity profile used by Squire[15] in his free convection analysis, and a significant portion of such a curve was secured for the profile obtained by Ostrach[14] in his solution of Polhausen's equations with a Prandtl number corresponding to that of air. The indifference curves are compared with each other and with the observations of instability in laminar free convection in air reported in Reference 17, and discrepancies are discussed thoroughly.

C

APPLICATION OF THE METHOD OF SMALL OSCILLATIONS TO THE STUDY OF THE STABILITY OF LAMINAR FREE CONVECTION ALONG A FLAT PLATE



The physical configuration considered for two-dimensional free convection along a semi-infinite flat plate is that shown above. If the plate is heated above the temperature of the surrounding fluid, the body force g will cause a flow of the boundary layer type as indicated by the dotted lines to develop on both sides of the plate. (If the plate were cooled below the temperature of the surrounding fluid, the component of g parallel to the plate would have to be directed away from the leading edge in order for the flow to proceed away from the leading edge along the plate as it is considered to do.) The notational convention employed is such that for the case of the figure shown the angle Θ measured as indicated between g and the plate is positive when the component of g perpendicular to the plate is directed away from the surface on the side along which the flow is being studied. For the study of flow along the bottom surface of the plate in the diagram, Θ is positive; if flow along the top surface were to be treated, the angle would be taken to be negative.

As mentioned in Section B2, the method of small oscillations for the investigation of laminar flow stability consists essentially of perturbing the flow and finding whether the perturbations amplify or die away in time. If they amplify, the conclusion is that the flow is unstable and that turbulence could be expected to develop; if they die away, the flow should be stable. In the process of applying this method of small oscillations one assumes that small disturbances of velocity, pressure, temperature, and density and other fluid properties occur in the form of waves traveling parallel to the plate in the direction of the flow. The Fourier components of these wave-type disturbances are placed in the equations which describe the free convection flow. That is, a disturbance is assumed in the form of a sinusoidal wave having an arbitrary wave length and phase velocity which are to be determined by the flow. The processes of placing these sinusoidal disturbances into the flow equations and appropriately simplifying them are described in detail in Appendix 2. As is indicated in the following divisions of the present Section C, when the Prandtl number of the mean flow has been chosen, the problem becomes resolved to the task of determining the relations among the wave number α of the disturbance, the phase velocity c of the disturbance, and the Reynolds number Re of the mean flow at a given rate of disturbance amplification or damping.

C1 The Disturbance Differential Equations

For the case of laminar free convection along an inclined or vertical flat plate the simplified disturbance equations according to the method of small oscillations are

$$(\bar{u}-c)(\varphi''-\alpha^2\varphi)-\bar{u}''\varphi+\frac{\varepsilon}{\alpha F}\{i a_1 s'-\alpha a_2 s\}+\frac{i}{\alpha Re}\{\varphi^{IV}-2\alpha^2\varphi''+\alpha^4\varphi\}=0 \tag{1}$$

and

$$(\bar{u}-c)s-\bar{\theta}'\varphi+\frac{i}{\sigma_0\alpha Re}\{s''-\alpha^2 s\}=0. \tag{2}$$

The derivation of these equations is presented in detail in Appendix 2. $\bar{u}$ is the nondimensional mean-flow velocity component parallel to the wall equal to the local dimensional mean velocity component divided by the maximum dimensional velocity U_m in the boundary layer, and $\bar{\theta}$ is the nondimensional mean fluid temperature equal to the difference between the local mean temperature of the fluid and the temperature of the fluid far from the plate, divided by the absolute value of the difference between the temperatures of the plate and the fluid far from it. Both $\bar{u}$ and $\bar{\theta}$ are taken to be functions of only a co-ordinate η measured normal to the plate. φ and s are functions similarly nondimenional-ized associated with a disturbance stream function Ψ and a temperature perturbation θ such that $\Psi(\xi,\eta,\tau) = \varphi(\eta)\, e^{i\alpha(\xi - c\tau)}$ and $\theta(\xi,\eta,\tau) = s(\eta)\, e^{i\alpha(\xi - c\tau)}$. ξ is the value of a co-ordinate measured along the plate from its leading edge and nondimensionalized by dividing by the boundary layer thickness δ, η is a similarly nondimensionalized co-ordinate measured norm-al to the plate from its surface, τ is time made nondimensional by divid-ing by $\frac{\delta}{U_m}$, c is the nondimensional phase velocity or velocity of prop-agation of the disturbance oscillation, and α is the wave number of the oscillation equal to $\frac{2\pi\delta}{\Lambda}$, Λ being the wave length. δ is the bound-ary layer thickness defined previously. ε in equation 1 is equal to $\beta_0 \Delta T$, in which β_0 is the coefficient of thermal expansion of the fluid far from the plate and ΔT is the absolute value of the difference between the plate temperature and temperature of the fluid far from the plate. F is a Froude number defined to be $\frac{U_m^2}{g\delta}$, g being the magni-tude of the body force vector, a_1 is the cosine of the angle Θ between the body force vector and the plate surface, and a_2 is the sine of that angle. Re is a Reynolds number defined to be $\frac{U_m \delta}{\nu_0}$, with ν_0 the kinematic viscosity of the fluid far from the plate; and σ_0 is the Prandtl number of the fluid far from the plate. A prime (') denotes differentiation with respect to η.

Equation 1 was obtained through eliminating the pressure terms between the two components of the momentum equation by cross-differentiation and subtraction, and in this and various other forms it will be called the "combined momentum equation". Equation 2 is simply the flow energy equation. In both of the equations the disturbance terms have been linearized as previously mentioned, and steady-state terms have been cancelled. As shown in Appendix 2, these simplified forms of the equations hold strictly only for the case of parallel flow with all fluid properties other than the density taken to be constant. The effect of density variation appears only in the presence of the term $\frac{\varepsilon}{\alpha F}\left\{ i a_1 S' - \alpha a_2 S \right\}$ which couples the two equations together. That the simplifications made in deriving these forms are reasonable is demonstrated in Appendix 2.

An important fact for the free convection flow considered is that the Froude number F and the Reynolds number Re are related, along with the parameter ε. In Appendix 3 it is shown that one can write

$$\frac{\varepsilon}{F} = \frac{f}{Re},$$

in which, for a given value of Θ, f is a dimensionless function of the velocity profile shape. When the difference between the temperature of the plate and of the fluid far from it is small, this profile shape depends only on the Prandtl number σ_0 of the convecting fluid so that f is ultimately a function of the Prandtl number if Θ is fixed. Figure 2 is a plot of $|a_1| f$ as a function of σ_0, values of $|a_1| f$ being obtained according to Appendix 3 from information contained in Reference 14.

Equation 1 with $\frac{\varepsilon}{F}$ replaced by $\frac{f}{\sqrt{Re}}$ is

$$(\bar{u}-c)(\varphi''-\alpha^2\varphi) - \bar{u}''\varphi + \frac{f}{\alpha\sqrt{Re}}\{i a_1 s' - \alpha a_2 s\} + \frac{i}{\alpha Re}\{\varphi^{IV} - 2\alpha^2\varphi'' + \alpha^4\varphi\} = 0. \qquad (1a)$$

One can see that when the value of αRe is specified the size of the term which couples this equation with the energy equation 2 is proportional to f . The significance of this appears in the development in Appendix 4 of approximate methods for solving the stability problem.

C2 Boundary Conditions

In general, six homogeneous boundary conditions must be specified in order to make the problem determinate. These boundary conditions are derived from a consideration of restrictions on the velocity and temperature disturbances which are imposed far from the plate and at the plate surface. The requirement far from the plate is that the disturbances die away; that is, one specifies that the disturbances associated with incipient turbulence originate in the layer in which there is flow. At the plate surface the requirements on the disturbance velocity are that its component v normal to the surface must vanish and, if the fluid viscosity is considered, i. e., if the term $\frac{i}{\alpha Re}\{\varphi^{IV} - 2\alpha^2\varphi'' + \alpha^4\varphi\}$ in equation 1a is retained, that the component u parallel to the wall must vanish as well. Since the disturbance stream function $\psi(\xi, \eta, \tau)$ has been assumed to be of the form $\varphi(\eta)e^{i\alpha(\xi - c\tau)}$, the requirement that u or $\frac{\partial\psi}{\partial\eta}$ vanish at the plate is $\varphi'(0) = 0$; and the requirement that v or $-\frac{\partial\psi}{\partial\xi}$ disappear there is $\varphi(0) = 0$. The requirements which should be specified on the temperature disturbance at the wall are more difficult to formulate from physical reasoning because of the interaction between the temperature disturbances in the fluid and heat transfer in the plate. An analysis taking into account the heat transfer process in the plate would really be necessary for determining the exact boundary condition to impose on the temperature disturbance at the plate surface. The results of this analysis as well as the analysis itself would probably be quite complex; and only the two conditions that the plate is isothermal or that it is adiabatic with respect to the temperature disturbances are considered in Appendix 4, where methods of approximately solving the stability problem with various simplifications are developed. An isothermal plate surface requires that $s(0) = 0$; if the plate surface is adiabatic with respect to the temperature disturbances, $s'(0) = 0$.

In Appendix 4 various simplifications are made in equations 1a and 2 to correspond to different simplifications regarding the fluid properties and flow characteristics. When s and its derivatives are eliminated between equations 1a and 2 and certain simplifications are made, the order of the resulting equation in φ and its derivatives is reduced compared with the order it would have without the simplifications. The result is that the number of boundary conditions which can be satisfied is decreased. This reduction in the number of boundary conditions is done in accordance with the simplifying assumptions made concerning the flow. For instance, taking the flow to be nonviscous implies that the boundary condition requiring the disappearance of the velocity component parallel to the plate at its surface should be dropped.

C3 The Boundary-Value Problem

The general procedure for obtaining information about the stability of the flow by using equations 1a and 2 in conjunction with the boundary conditions discussed is described in this section.

If one eliminates s and its derivatives between equations 1a and

2, a sixth-order equation in φ will result. (A procedure for doing this is described in Appendix 4.) Because this equation is linear and homogeneous, its general solution φ will be capable of expression as a linear combination of six linearly independent solutions φ_j as

$$\varphi = \sum_{j=1}^{6} C_j \, \varphi_j(\sigma_0, c, \alpha, \alpha Re). \tag{3}$$

As indicated, each of the solutions φ_j will depend on the parameters σ_0, c, α, and αRe which appear in the original pair of equations. One method of obtaining an expression for s analogous to that for φ in equation 3 begins by securing an expression for s which is homogeneous and linear in φ and its derivatives. This expression can be gotten by proceeding as indicated in Part 4. 3. 3 of Appendix 4. One then substitutes for φ according to equation 3 to obtain

$$s = \sum_{j=1}^{6} C_j \, s_j(\sigma_0, c, \alpha, \alpha Re), \tag{4}$$

the S_j's being defined to be the coefficients of the C_j's.

Six homogeneous linear boundary conditions of the types previously discussed can be stated in the form of six linear equations in φ, s, and their derivatives. These six equations are written symbolically as

$$\mathcal{J}_{\iota}(\varphi, s) = 0. \quad {}_{\iota = 1, 2, \ldots 6} \tag{5}$$

If one substitutes the expressions for φ and s defined by equations 3 and 4 into the set of equations 5, the result is a set of six simultaneous linear equations in the C_j's. For non-trivial solutions C_j of this set of equations to exist, the determinant $D(\sigma_0, c, \alpha, \alpha Re)$ of the coefficients of the C_j's must disappear, or

$$D(\sigma_0, c, \alpha, \alpha Re) = 0. \tag{6}$$

It is of importance that the expression $D(\sigma_0, c, \alpha, \alpha Re)$ is the sum of a real and an imaginary part, each of which must separately be equal to zero according to the preceding equation. For a given fluid, σ_0 is specified. Also, the imaginary part of c is determined when a rate of amplification or damping of the disturbances is chosen, since the disturbances are proportional to $e^{i\alpha(\xi - c\tau)}$. The only free parameters, then, which appear in equation 6 are $\mathcal{Rl}(c)$, α, and αRe. The final step in solving the boundary-value problem can be taken to be that of choosing values of $\mathcal{Rl}(c)$ and finding corresponding pairs of values of α and αRe simultaneously satisfying the equations

$$\mathcal{Rl}\{D(\sigma_0, c, \alpha, \alpha Re)\} = 0 \tag{6a}$$

and

$$\mathcal{Im}\{D(\sigma_0, c, \alpha, \alpha Re)\} = 0. \tag{6b}$$

Enough pairs of values of α and αRe are to be found to determine a curve in the α-Re plane for the particular value of $\mathcal{Im}(c)$ chosen. Usually one investigates the case of neutral stability in which the disturbances are neither amplified nor damped and therefore takes $\mathcal{Im}(c) = 0$. In this case the curve in the α-Re plane divides pairs of values of α and Re into those associated with stable or damped oscillations and those associated with unstable or amplified oscillations. Such a curve is called an "indifference" or "neutral stability" curve.

Although this procedure is perhaps one of the neater ways in which the general process of solving the boundary-value problem can be described there are modifications of it which should be more useful in practice. In the methods of approximate solution of the boundary-value problems developed in Appendix 4, the formidable task of dealing with a sixth-order determinant is avoided; three is the highest order of the determinants considered which correspond to $D(\sigma_0, c, \alpha, \alpha Re)$.

C4 Simplified Methods for Solving the Free Convection Stability Problem

The general method outlined in Sections C1 through C3 for studying analytically the stability of free convection flow is in reality extremely difficult to apply. Although the overall procedure is quite straightforward, obtaining exact expressions for φ and s and solving the corresponding boundary-condition equation 6 would be impossible in practice. Accordingly, approximate methods for solving the stability problem based on the assumption that the Reynolds number is large were developed as shown in Appendix 4.

Six of these methods, which are of varying degrees of mathematical complexity, were derived. These methods differ basically from one another according to the consideration or neglect of heat conduction and inertial, pressure, viscous, and body forces. In each of the six methods inertial and pressure forces are considered, while heat conduction and viscous and body forces are included or not included in the various cases in order to yield a group of methods covering all possible significant combinations of these factors. Only inertial and pressure forces are considered in the simplest of these methods, while the most complicated involves the consideration of inertial, pressure, viscous, and body forces as well as heat conduction. Each of the six methods was derived for the case of an inclined plate, and the modifications necessary when the plate is vertical were determined.

C4.1 Conclusions Regarding the Methods Developed for Solving Approximately the Free Convection Stability Problem

After deriving and examining these methods, the author has arrived at the following conclusions concerning treating the free convection stability problem analytically:

I. The method of two-dimensional small oscillations can be used to develop techniques for analyzing the stability with respect to turbulent transition of high-Reynolds-number, laminar, free convection flows about a vertical or inclined semi-infinite flat plate.

II. In general, both the disturbance energy equation and the disturbance combined momentum equation, which is obtained by eliminating the pressure terms between the two components of the momentum equation, must be considered.

III. When the plate is inclined only slightly from the vertical, the effects upon stability of inclination are opposite in character when the component of the body force is directed away from the plate surface compared with the effects when the component is directed toward the plate surface.

IV. For a vertical plate, the stability of the flow is independent of whether the plate is heated or cooled with respect to the surrounding fluid if the velocity and temperature profiles have the same shapes in the two cases. If the plate is inclined, the stability could be different for heating and cooling even if the body force vector makes the same angle with a perpendicular to the plate surface in the two cases.

V. Important reductions in the mathematical complexity of the problem can be made by treating only the combined momentum equation as well as by taking the Prandtl number to be either very large or very small when both the combined momentum and the energy equations are considered.

VI. With the possible exception of the cases in which only the combined momentum equation is considered, the mathematical complexity of the methods of solution which have been developed is so great that the use of electronic computing facilities is anticipated to be necessary for their successful application.

C4. 2 Discussion of Conclusions

Conclusion I follows simply from the facts that six different schemes for investigating analytically the stability of a laminar free convection flow have been developed and that these six schemes are based on differential equations derived by the method of small oscillations. Obtaining indifference curves by these six methods and by experiment for comparison would have to be done in order to determine fully the validity of the methods. The validity of one of these methods was partially checked as indicated in Section D by comparing indifference curves obtained with its use with the first instability observed interferometrically in the free convection of air along a flat plate. The general results were that the qualitative agreement between the analysis and the experiments was satisfactory but that the quantitative agreement was poor. Computational limitations affecting the analysis prevent these results from applying directly to the method as developed in Appendix 4, however.

The second conclusion is merely a statement of the fact that both the combined momentum equation 1 or 1a and the energy equation 2 appear when the method of small oscillations is applied to the general two-dimensional flow equations with simplifications as indicated in Appendix 2. The various simplifying assumptions made in arriving at these disturbance equations are described and justified in the same appendix.

The third conclusion is a consequence of the fact that linear terms in a_2 , which is the sine of the angle of inclination Θ of the plate with the vertical, appear in the terms of the boundary-condition equations when the effect of the body force on stability is taken into account. Because linear rather than higher-order terms in a_2 are most important when a_2 is small, the effects of a finite value of a_2 on any parameter which depends on it through a boundary-condition equation will reverse when a_2 changes sign.

When a_2 is positive, the component of the body force normal to the plate is directed away from the plate surface; and when a_2 is negative, this component is directed toward the plate surface. Thus the conclusion follows.

An important parameter which inclination of the plate may affect is the lowest value of the Reynolds number for which instability appears. Schlichting[9] found analytically for the Blasius boundary layer that the minimum Reynolds number for instability was increased when the boundary layer was "stably" stratified with respect to a body force field normal to the bounding surface and a fluid density that varied in the direction normal to the surface. That is, if the body force field and fluid density variation acted to restore a disturbed fluid particle to its original position, the minimum Reynolds number for instability was increased. When the flow was taken to be "unstably" stratified, i.e., when the body force field and the fluid density variation were taken to act so that a disturbed particle would be moved farther from its original position, the minimum Reynolds number for instability decreased. One might expect to find similar results in the free convection case when the plate is inclined, since inclining the plate by introducing a component of the body force field normal to the plate has the effect of causing the flow to be either stably or unstably stratified, depending on which side of the plate is considered. However, the great dissimilarity between the Blasius velocity profile and the typical free convection velocity profile prevents one from being at all sure that the stability would be affected similarly for the two cases.

Conclusion V follows primarily from a comparison of the coupled, viscous, heat-conducting case of Part 4. 2. 4 of Appendix 4 with the uncoupled cases and with the coupled cases for which the Prandtl number was taken to be either very large or very small. The coupled, viscous, heat-conducting case is the only one in which three rather than one or two boundary conditions are applied at the wall. This additional boundary condition necessitates the use of two solutions valid near the inner critical point η_{c_1} in addition to a combination of two nonviscous solutions, while the other, simplified cases require at most only one solution in addition to the nonviscous ones. The boundary-condition equation is correspondingly increased in complexity.

The truth of the sixth conclusion should be apparent after the six approximate methods of solution of the stability problem which are developed in Parts 4. 1 and 4. 2 of Appendix 4 have been examined. Experience in using a desk calculator to compute indifference curves for the uncoupled, viscous case as described in Section D is another reason for the author's arriving at this conclusion that the use of elaborate computing facilities would be necessary in applying the more complex methods.

D

INVESTIGATION WITH NEGLECT OF TEMPERATURE DISTURBANCE EFFECTS OF THE STABILITY OF LAMINAR FREE CONVECTION IN A VISCOUS FLUID WITH A PRANDTL NUMBER OF 0.72

D1 The General Method

In Appendix 4, approximate methods for solving the neutral stability problem are presented for six cases with different assumptions regarding the effects of viscosity, heat conduction, and the body force. Of these six methods, only the two for which the effects of the body force on stability are neglected are sufficiently simple to be treated without the use of extensive, high-speed computing facilities. These two are the uncoupled*, nonviscous case and the uncoupled, viscous case. The uncoupled, viscous case was investigated for the situation in which the fluid has a Prandtl number of 0.72. This value of the Prandtl number was chosen principally because it is that of air, for which some information[17] concerning the appearance of instability is available. Also, using this value of the Prandtl number is advantageous when making an analysis in which the effect of coupling between the combined momentum and energy equations is neglected. This is because the term f, which depends on the Prandtl number and is a factor of the expression $\frac{f}{\alpha\sqrt{Re}}\{ia_1 S' - \alpha a_2 S\}$ in the combined momentum equation which provides coupling with the energy equation, has its minimum near this value of the Prandtl number.

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* A case is said to be "uncoupled" if only the combined momentum equation 1a rather than both it and the energy equation 2 are considered in the analysis. When the body force is neglected the term in the combined momentum equation which couples it to the energy equation disappears.

Thus, one should expect that if coupling can be neglected in any range of values of the Prandtl number, it can be neglected for a value of 0.72, provided that the product αRe is sufficiently large.

Part 4.1.2 of Appendix 4 is the development of an approximate method for solving the uncoupled, viscous case. This method involves solving the Orr-Sommerfeld equation,

$$(\bar{u}-c)(\varphi''-\alpha^2\varphi) - \bar{u}''\varphi + \frac{i}{\alpha Re}\{\varphi^{IV} - 2\alpha^2\varphi'' + \alpha^4\varphi\} = 0, \tag{4-3}$$

$\bar{u}$ being specified by the free convection velocity profile and φ being required to meet the boundary conditions

$$\left.\begin{aligned} \varphi(0) &= 0, \\ \varphi'(0) &= 0, \end{aligned}\right\} \tag{4-11a}$$

and

$$\left.\begin{aligned} \varphi(\infty) &= 0, \\ \varphi'(\infty) &= 0. \end{aligned}\right\} \tag{4-11b}$$

These boundary conditions are implied by the requirements that the components of the disturbance velocity both normal and parallel to the plate disappear at the plate surface and far away from it. The resulting boundary-condition equation is

$$\frac{-\{\varphi_1(0) - \alpha c^2 \varphi_2(0)\}}{\eta_{c_1}\{\varphi_1'(0) - \alpha c^2 \varphi_2'(0)\}} = \frac{\Phi_3(\zeta_0)}{\zeta_0 \Phi_3'(\zeta_0)}, \tag{4-32}$$

which defines a relation for neutral disturbance oscillations among the Reynolds number Re of the mean flow, the phase velocity c, and the wave number α. In this equation φ_1 and φ_2 are the solutions of the nonviscous Orr-Sommerfeld equation which are given by equations 4-6 and 4-7 of Part 4.1.1.3 of Appendix 4, and Φ_3 is one of the viscous solutions defined by equations 4-18 of Part 4.1.2.3 of the same appendix.

η_{c_1} is the inner critical point or value of η for which $\bar{u}$ is equal to c. ζ_o is $-\eta_{c_1}(\bar{u}'_{c_1}\alpha Re)^{\frac{1}{3}}$, $\bar{u}'_{c_1}$ being the value of $\bar{u}'$ at η_{c_1}. For convenience the left-hand side of the equation will be designated by $\mathcal{E}(\alpha,c)$, and the right-hand side by $\mathcal{L}(\zeta_o)$ so that the relation becomes

$$\mathcal{E}(\alpha,c) = \mathcal{L}(\zeta_o). \tag{7}$$

The first step in solving this boundary-condition equation is to plot $\mathcal{L}(\zeta_o)$ as a function of ζ_o in the complex plane. Then a value of c is chosen, and $\mathcal{E}(\alpha,c)$ is plotted as a function of α for this value of c. Intersections, if any, of the curves representing $\mathcal{L}(\zeta_o)$ and $\mathcal{E}(\alpha,c)$ determine combinations of values of α, c, and ζ_o for which neutral oscillations can exist. Since $\zeta_o = -\eta_{c_1}(\bar{u}'_{c_1}\alpha Re)^{\frac{1}{3}}$ and $\bar{u}'_{c_1}$ and η_{c_1} are fixed when c is chosen, these sets of values of α, c, and ζ_o can be used to find pairs of values of α and αRe which specify points on an indifference curve in the α-Re plane.

D2 The Velocity Profiles Treated

D2. 1 The Cubic Polynomial Profile

The first of the two velocity profiles examined for stability was the cubic polynomial approximation to the free convection velocity profile which was employed by Squire(15) in his integral method of treating laminar free convection along a vertical, semi-infinite flat plate. In terms of the present dimensionless distance η from the surface this profile is represented analytically by

$$\bar{u} = \left\{ \begin{array}{ll} \frac{243}{64}\eta\left(1-\frac{9}{16}\eta\right)^2, & 0 \le \eta \le \frac{16}{9} = b \\ 0, \quad \eta \ge \frac{16}{9} & , \end{array} \right\} \tag{8}$$

in which b is the value of η at the edge of the boundary layer. The profile is plotted in Figure 9 in terms of the free convection dimensionless similarity velocity $\frac{\bar{U}x}{2\nu_0 (Gr_x)^{\frac{1}{2}}}$ and similarity distance $\left(\frac{Gr_x}{4}\right)^{\frac{1}{4}} \frac{y}{x}$ from the plate surface. One can ascertain from either this plot or the analytic representation that the profile satisfies the requirements that $\bar{u}$ be zero at both the plate surface and the edge of the boundary layer; also, $\bar{u}'$ disappears at the edge of the boundary layer as it should.

With the use of the results of Squire's laminar analysis and the formulae of Appendix 3 one can establish the following formulas which hold for free convection in a fluid which has a Prandtl number σ_0 and in which the velocity profile is that given by equation 8:

$$\left.\begin{aligned} f &= \frac{6.38}{|a_1|}\left(1 + \frac{20}{21\sigma_0}\right) \\ \frac{\delta}{x} &= \frac{2.21\left(\sigma_0 + \frac{20}{21}\right)^{\frac{1}{4}}}{\sigma_0^{\frac{1}{2}}(Gr_x)^{\frac{1}{4}}} \\ Re &= \frac{1.69\,(Gr_x)^{\frac{1}{4}}}{\sigma_0^{\frac{1}{2}}\left(\sigma_0 + \frac{20}{21}\right)^{\frac{1}{4}}} \end{aligned}\right\} \quad (9)$$

Here f is a factor of the term which couples the combined momentum and energy equations, and $\frac{\delta}{x}$ is the ratio of the boundary layer thickness to the distance from the edge of the plate. Setting $\sigma_0 = 0.72$ reduces these relations to

$$\left.\begin{aligned} f &= \frac{14.8}{|a_1|}, \\ \frac{\delta}{x} &= \frac{2.96}{(Gr_x)^{\frac{1}{4}}}, \\ \text{and} \quad Re &= 1.75\,(Gr_x)^{\frac{1}{4}}. \end{aligned}\right\} \quad (10)$$

D2. 2 <u>The Exact Profile</u>

The velocity profile termed the "exact" profile is that obtained by the numerical solution of Polhausen's equations describing laminar free convection along a vertical, semi-infinite flat plate of a fluid with a

Prandtl number of 0.72. These equations were derived on the assumptions that the fluid density varies linearly with the temperature and that the other fluid properties are constant. The particular solutions of the equations that were employed are those tabulated in Reference 14. For comparison with the cubic polynomial profile, the velocity profile obtained from these solutions is plotted in Figure 9 in terms of the previously mentioned free convection similarity variables.

By using information contained in Reference 14 and the formulas of Appendix 3 one can determine that for the exact profile

$$\left.\begin{aligned} f &= \frac{16.92}{|a_1|}, \\ \frac{\delta}{x} &= \frac{3.055}{(Gr_x)^{\frac{1}{4}}}, \\ \text{and} \quad Re &= 1.686\,(Gr_x)^{\frac{1}{4}}. \end{aligned}\right\} \quad (11)$$

D3 Solution of the Boundary-Condition Equation $\mathcal{E}(\alpha, c) = \mathcal{L}(\zeta_0)$

As indicated in Section D1, this equation is solved by plotting curves representing its two sides in the complex plane and noting the values of α, c, and ζ_0 at points of intersection of the curves. The processes of obtaining values of $\mathcal{L}(\zeta_0)$ and $\mathcal{E}(\alpha, c)$ are described in Sections D3.1 and D3.2.

D3.1 Determination of $\mathcal{L}(\zeta_0)$

In Reference 18 the real and imaginary parts of $\int_{\infty}^{\zeta_0} d\Omega_1 \int_{\infty}^{\Omega_1} d\Omega_2 \Omega_2^{\frac{1}{2}} H_{\frac{1}{3}}^{(1)}\left\{\frac{2}{3}(i\Omega_2)^{\frac{3}{2}}\right\}$ or $\Phi_3(\zeta_0)$ and $\int_{\infty}^{\zeta_0} d\Omega\, \Omega^{\frac{1}{2}} H_{\frac{1}{3}}^{(1)}\left\{\frac{2}{3}(i\Omega)^{\frac{3}{2}}\right\}$ or $\Phi_3'(\zeta_0)$ are tabulated for values of ζ_0 differing by 0.5 over the range $-8 \leq \zeta_0 \leq 8$. Additional values of $\Phi_3(\zeta_0)$ and $\Phi_3'(\zeta_0)$ were calculated in order to reduce to 0.1 the size of the interval between points at which the functions were known. This was done by performing single and double numerical integrations of $\zeta^{\frac{1}{2}} H_{\frac{1}{3}}^{(1)}\left\{\frac{2}{3}(i\zeta)^{\frac{3}{2}}\right\}$, values of this function being found by the use of Reference 19. The process of interpolation was completed by drawing curves of the real and imaginary parts of $\Phi_3(\zeta_0)$ and $\Phi_3'(\zeta_0)$. With the aid of these, $\mathcal{X}(\zeta_0)$, which is $\frac{\Phi_3(\zeta_0)}{\zeta_0 \Phi_3'(\zeta_0)}$ by definition, was determined and plotted in the complex plane for use in solving the boundary-condition equation

$$\mathcal{E}(\alpha, c) = \mathcal{X}(\zeta_0). \tag{7}$$

Since the viscous solution $\Phi_3(\zeta)$ is developed on the assumption that $(\bar{u}-c) = \bar{u}'_{c_1}(\eta-\eta_{c_1})$, there is some question concerning the validity of using $\Phi_3(\zeta)$ as a solution when c is quite large. This is because the velocity profile between η_{c_1} and the plate, where the boundary conditions are applied, is not actually linear but curved. In order to ascertain how serious this shortcoming of the solution $\Phi_3(\zeta)$ actually is, a calculation of the expression corresponding to $\mathcal{X}(\zeta_0)$ was made with the use of a viscous solution in which the effect of curvature of the velocity profile was considered. This viscous solution was one of those developed by Tollmien in Reference 20. Unfortunately, it requires extensive numerical integration for its application. In the case of the cubic polynomial profile and for a value of c of 0.40 and a value of ζ_0 of - 2.66, the values of $\mathcal{X}(\zeta_0)$ and the corresponding expression for Tollmien's improved viscous solution were practically identical. With larger values of c such as appear in the solution of the boundary-condition equation for the exact profile, the discrepancy between the expressions for the two solutions can be expected to be larger; but the poor accuracy with which $\mathcal{E}(\alpha,c)$ can be calculated for this case is more serious in its effect on the determination of the indifference curve. The problems attendant to calculating $\mathcal{E}(\alpha,c)$ for the exact profile are discussed in the following Section D3. 2.

D3. 2 Determination of $\mathcal{E}(\alpha, c)$

D3. 2. 1 Simplifications

$\mathcal{E}(\alpha, c)$ is defined to be $\dfrac{-\{\varphi_1(0) - \alpha c^2 \varphi_2(0)\}}{\eta_{c_1}\{\varphi_1'(0) - \alpha c^2 \varphi_2'(0)\}}$. Equations 4-6 of Appendix 4 defining φ_1 and φ_2 state that

$$\varphi_1 = (\bar{u} - c) \sum_{m=0}^{\infty} \alpha^{2m} \mathcal{Q}_m(\eta)$$

and

$$\varphi_2 = (\bar{u} - c) \sum_{m=0}^{\infty} \alpha^{2m} \mathcal{G}_m(\eta) \tag{4-6}$$

so that one can write

$$\mathcal{E}(\alpha, c) = \frac{c}{\eta_{c_1}} \left\{ \frac{\sum_{m=0}^{\infty} \alpha^{2m} Q_m - c^2 \sum_{m=0}^{\infty} \alpha^{2m+1} G_m}{\bar{u}_p' \left[\sum_{m=0}^{\infty} \alpha^{2m} Q_m - c^2 \sum_{m=0}^{\infty} \alpha^{2m+1} G_m \right] - c \left[\sum_{m=0}^{\infty} \alpha^{2m} Q_m' - c^2 \sum_{m=0}^{\infty} \alpha^{2m+1} G_m' \right]} \right\}, \tag{12}$$

in which $\bar{u}_p' = \left.\dfrac{d\bar{u}}{d\eta}\right|_{\eta=0}$, $Q_m = \mathcal{Q}_m(0)$, $G_m = \mathcal{G}_m(0)$, $Q_m' = \left.\dfrac{d\mathcal{Q}_m(\eta)}{d\eta}\right|_{\eta=0}$, and $G_m' = \left.\dfrac{d\mathcal{G}_m(\eta)}{d\eta}\right|_{\eta=0}$. According to equations 4-7 of Appendix 4, the following relations hold:

$$\left.\begin{aligned}
\mathcal{Q}_0 &= 1 \\
\mathcal{G}_0 &= \int_b^{\eta} d\Omega (\bar{u} - c)^{-2} \\
\underset{m \geq 1}{\mathcal{Q}_m} &= \int_b^{\eta} d\Omega_1 (\bar{u} - c)^{-2} \int_b^{\Omega_1} d\Omega_2 (\bar{u} - c)^2 \mathcal{Q}_{m-1} \\
\underset{m \geq 1}{\mathcal{G}_m} &= \int_b^{\eta} d\Omega_1 (\bar{u} - c)^{-2} \int_b^{\Omega_1} d\Omega_2 (\bar{u} - c)^2 \mathcal{G}_{m-1}
\end{aligned}\right\} \tag{4-7}$$

One can see that the calculation of the coefficients of α^{2m} and α^{2m+1} in equation 12 becomes excessively complicated very rapidly with increasing values of m so that a stringent limit is placed on the number of terms which can be utilized in any determination of values of $\mathcal{E}(\alpha, c)$. In the present work only the terms Q_0, Q_1, G_0, Q_0', Q_1', G_0', and G_1' are retained. These are, according to the definitions 4-7,

$$\left.\begin{aligned}
Q_0 &= 1, \\
Q_1 &= \int_b^0 d\Omega_1 (\bar{u}-c)^{-2} \int_b^{\Omega_1} d\Omega_2 (\bar{u}-c)^2; \\
G_0 &= \int_b^0 d\Omega\, (\bar{u}-c)^{-2}; \\
Q_0' &= 1, \\
Q_1' &= \frac{1}{c^2}\int_b^0 d\Omega\, (\bar{u}-c)^2, \\
G_0' &= \frac{1}{c^2},
\end{aligned}\right\} \qquad (13)$$

and

$$G_1' = \frac{1}{c^2}\int_b^0 d\Omega_1 (\bar{u}-c)^2 \int_b^{\Omega_1} d\Omega_2 (\bar{u}-c)^{-2} = G_0 Q_1' - \frac{1}{c^2} Q_1 .$$

Neglecting other terms in equation 12 gives

$$\mathcal{E}(\alpha,c) \doteq \frac{c}{\eta_{c_1}}\left\{\frac{1+\alpha^2 Q_1 - \alpha c^2 G_0}{\bar{u}_p'[1+\alpha^2 Q_1 - \alpha c^2 G_0] \quad - \alpha c[\alpha Q_1'(1-\alpha c^2 G_0) + \alpha^2 Q_1 - 1]}\right\}. \qquad (14)$$

Here the symbol $\doteq$ indicates approximate equality. The validity of this equation depends upon the relative sizes of the terms which are retained and those which have been neglected; it will be best then when α is small, since the higher-order terms in α have been neglected. Only the integrals for G_0, Q_1, and Q_1' must be determined; but finding them is quite a task, as is explained in the next two parts.

One might ask why in approximately calculating the nonviscous solutions and their derivatives appearing in $\mathcal{E}(\alpha,c)$ that certain transformations of the series in α^2 used by Lin were not employed. Lin's transformations result in more rapidly converging series when the velocity profile is that of forced convection, but it was found by the author that they are inapplicable to the case of a profile with two critical points such as the general free convection profile.

D3. 2. 2 Calculation of G_0, Q_1', and Q_1 for the Cubic Polynomial Profile and Validity of the Approximation to $\mathcal{E}(\alpha, c)$ for this Profile

Calculating G_0 is simply an exercise in the integration of the reciprocal of a sixth-order polynomial. It is convenient to write $(\bar{u}-c)$ as $\left(\frac{9}{16}\right)^2 \frac{243}{64}(\eta-E_1)(\eta-E_2)(\eta-E_3)$, in which E_1 and E_2 are η_{c_1} and η_{c_2}, and E_3 is the third zero of the polynomial $\frac{243}{64}\eta\left(1-\frac{9}{16}\eta\right)^2 - c$. With this convention $(\bar{u}-c)^{-2}$ can be written as the sum of three terms of the form

$$\frac{A_j\eta + B_j}{(\eta - E_j)^2}. \tag{15}$$

The signs of the imaginary parts of the logarithmic terms in the integrals of these expressions are determined by the requirements

$$\frac{-7\pi}{6} < \arg(\eta - \eta_{c_1}) < \frac{\pi}{6} \tag{4-26}$$

and

$$-\frac{\pi}{6} < \arg(\eta - \eta_{c_2}) < \frac{7\pi}{6}. \tag{4-29}$$

Determining Q_1' involves the simple operation of integrating a sixth-order polynomial. Finding Q_1 requires the integration of products of terms of the form 15 with powers of η running from 0 to 7. In Q_1 the signs of imaginary parts of logarithmic terms are also determined by the requirements 4-26 and 4-29. Although the operations of finding the integrals are straightforward, the algebraic complication is extreme, particularly for Q_1. Seventy-one successive computations are necessary to find G_0 after c has been chosen and E_1, E_2, and E_3 have been secured by solving a cubic equation. Approximately 300 additional operations are necessary to secure Q_1.

The question may arise of why G_1 was not chosen as one of the coefficients of α^{2m} to be retained in the expression for $\mathcal{E}(\alpha,c)$ along with G_0, Q_0, Q_1, G_0', G_1', Q_0', and Q_1', since G_1 and G_1' are both coefficients of α^3. That is, all coefficients of α^0, α^1, and α^2 have been retained; but of the coefficients of α^3, G_1 and G_1', only G_1' has been kept. This inconsistency is the consequence of an excessive algebraic complexity which was found to be a result of performing the integrations necessary to determine G_1. This complexity was so great that the author decided that the slight additional accuracy in determining $\mathcal{E}(\alpha,c)$ that would be gotten from including G_1 would not justify the inordinate amount of calculation that would have to be done. To secure a value of G_1, several times the amount of computation necessary to obtain Q_1, G_0, and Q_1' would have had to be performed.

Although the approximate expression 14 for $\mathcal{E}(\alpha,c)$ utilizes only the first few terms of infinite series, it is believed that with its use fairly accurate values for $\mathcal{E}(\alpha,c)$ were obtained in this case of the cubic polynomial profile. From an examination of equation 14 one finds that when α approaches ∞, the approximate expression for $\mathcal{E}(\alpha,c)$ should vanish, since α appears to the third power in the denominator of the fraction and to only the second power in the numerator. Therefore, one should suspect that the accuracy with which $\mathcal{E}(\alpha,c)$ is represented is poor when α is so large that $\mathcal{E}(\alpha,c)$ approaches 0 unless $\mathcal{E}(\alpha,c)$ lies close to 0 for even moderate values of α. The converse, i.e., that when $\mathcal{E}(\alpha,c)$ is not close to 0 the accuracy is good, is not necessarily true; but when $\mathcal{E}(\alpha,c)$ is plotted in the complex plane, its tendency to turn into the origin as α increases from 0 to ∞ can be used to estimate the accuracy with which the function is represented. The representation is absolutely correct when $\alpha=0$; as α increases the representation decreases in validity. It was found that the plots shown in Figure 5 of $\mathcal{E}(\alpha,c)$ for this cubic polynomial profile with different values of c did not turn in toward the origin until α was made many times the size of the values for which the curves of $\mathcal{E}(\alpha,c)$ intersected those of $\mathcal{L}(\mathfrak{Z}_0)$. It is therefore believed that the representation of $\mathcal{E}(\alpha,c)$ within the important range of values of α is fairly good, although there is no way of knowing so positively unless one could somehow determine the function exactly for comparison by obtaining all of the terms of the series.

D3. 2. 3 Calculation of G_0, Q_1', and Q_1 for the Exact Profile and Validity of the Approximation to $\mathcal{E}(\alpha,c)$ for this Profile

Although the integration for this case was actually performed in terms of a tabulated function F', which is proportional to the mean velocity $\bar{u}$, of Reference 14 rather than in terms of $\bar{u}$, the process will be described here using $\bar{u}$ for clarity.

Because $(\bar{u}-c)^2$ has no singularities for $0 \leq \eta \leq b$, the trapezoidal rule of numerical integration was used over this entire range to compute Q_1'. On the other hand, the singularities of $(\bar{u}-c)^{-2}$ at the critical points η_{c_1} and η_{c_2} make imperative the employment of analytic integration for determining G_0 and Q_1 in the neighborhoods of these points. If $(\bar{u}-c)^{-2}$ is expanded in a Laurent series, the following relation, which is useful in the neighborhood of a critical point η_{c_j}, can be written:

$$(\bar{u}-c)^{-2} = \frac{1}{(\bar{u}_{c_j}')^2}(\eta-\eta_{c_j})^{-2} - \frac{\bar{u}_{c_j}''}{(\bar{u}_{c_j}')^3}(\eta-\eta_{c_j})^{-1} + \left\{\frac{3}{4}\frac{(\bar{u}_{c_j}'')^2}{(\bar{u}_{c_j}')^4} - \frac{1}{3}\frac{\bar{u}_{c_j}'''}{(\bar{u}_{c_j}')^3}\right\} + O\{(\eta-\eta_{c_j})\} \qquad (16)$$

The various derivatives $\bar{u}_{c_j}^{(n)}$ can be obtained by using the tabulated functions of Reference 14 and the laminar flow equations contained therein. One must require, of course, that $|\eta-\eta_{c_j}|$ be less than the distance from η_{c_j} to the nearest other singularity of the function so that the series will converge. Integrating, one gets the relation

$$\int_{\eta_{c_j}+{}_j\ell_1}^{\eta_{c_j}-{}_j\ell_2} d\Omega(\bar{u}-c)^{-2} = \frac{1}{(\bar{u}'_{c_j})^2}\left\{\frac{1}{{}_j\ell_2}+\frac{1}{{}_j\ell_1}\right\} - \frac{\bar{u}''_{c_j}}{(\bar{u}'_{c_j})^3}\log\left(-\frac{{}_j\ell_2}{{}_j\ell_1}\right)$$

$$-\left\{\frac{3}{4}\frac{(\bar{u}''_{c_j})^2}{(\bar{u}'_{c_j})^4}-\frac{1}{3}\frac{\bar{u}'''_{c_j}}{(\bar{u}'_{c_j})^3}\right\}\left\{{}_j\ell_2+{}_j\ell_1\right\}+O\left\{{}_j\ell_1^2,{}_j\ell_2^2\right\}. \tag{17}$$

In this equation ${}_j\ell_1$ and ${}_j\ell_2$ are two small positive quantities the sizes of which are chosen to obtain a favorable compromise between the inaccuracy of numerical integration near η_{c_j} and the poor representation of the integrand by the first few terms of the series far from η_{c_j}. The sign of the imaginary part of $\log\left(-\frac{{}_j\ell_2}{{}_j\ell_1}\right)$ is decided by the appropriate one of the stipulations 4-26 and 4-29 of Appendix 4. With the use of this analytic means of integrating around the singular points, the following approximations are taken to be valid:

$$G_0 \doteq \int_b^{\eta_{c_2}+{}_2\ell_1} d\Omega(\bar{u}-c)^{-2} + \int_{\eta_{c_2}+{}_2\ell_1}^{\eta_{c_2}-{}_2\ell_2} d\Omega(\bar{u}-c)^{-2} + \int_{\eta_{c_2}-{}_2\ell_2}^{\eta_{c_1}+{}_1\ell_1} d\Omega(\bar{u}-c)^{-2} + \int_{\eta_{c_1}+{}_1\ell_1}^{\eta_{c_1}-{}_1\ell_2} d\Omega(\bar{u}-c)^{-2} + \int_{\eta_{c_1}-{}_1\ell_2}^{0} d\Omega(\bar{u}-c)^{-2} \tag{18}$$

$$Q_1 \doteq \int_b^{\eta_{c_2}+{}_2\ell_1} d\Omega_1(\bar{u}-c)^{-2}\int_b^{\Omega_1} d\Omega_2(\bar{u}-c)^2 + \left\{\int_{\eta_{c_2}+{}_2\ell_1}^{\eta_{c_2}-{}_2\ell_2} d\Omega(\bar{u}-c)^{-2}\right\}\left\{\int_b^{\eta_{c_2}} d\Omega(\bar{u}-c)^2\right\} + \int_{\eta_{c_2}-{}_2\ell_2}^{\eta_{c_1}+{}_1\ell_1} d\Omega_1(\bar{u}-c)^{-2}\int_b^{\Omega_1} d\Omega_2(\bar{u}-c)^2$$

$$+\left\{\int_{\eta_{c_1}+{}_1\ell_1}^{\eta_{c_1}-{}_1\ell_2} d\Omega(\bar{u}-c)^{-2}\right\}\left\{\int_b^{\eta_{c_1}} d\Omega(\bar{u}-c)^2\right\} + \int_{\eta_{c_1}-{}_1\ell_2}^{0} d\Omega_1(\bar{u}-c)^{-2}\int_{\eta_{c_2}+{}_2\ell_1}^{\Omega_1} d\Omega_2(\bar{u}-c)^2 \tag{19}$$

Of the integrals appearing in these equations, $\int_{\eta_{c_2}+{}_2\ell_1}^{\eta_{c_2}-{}_2\ell_2} d\Omega(\bar{u}-c)^{-2}$ and $\int_{\eta_{c_1}+{}_1\ell_1}^{\eta_{c_1}-{}_1\ell_2} d\Omega(\bar{u}-c)^{-2}$ are obtained analytically as indicated by equation 17, all definite and indefinite integrals having $(\bar{u}-c)^2$ as integrands are secured by the trapezoidal rule of numerical integration, and the remaining integrals are found numerically according to Simpson's rule where it is applicable.

In summing the terms on the right-hand sides of equations 18 and 19, it was found that the real part of each final expression is a small difference of large numbers. On account of this, small errors in integration can cause large percentage differences between the computed and true values of G_0 and Q_1; and considerably more scatter of points in the α-Re plane should be expected for this case of the exact profile than for that of the cubic polynomial profile. The scatter of these points in Figure 8 is apparently the result of this sensitivity of G_0 and Q_1 to these errors in integration. For each of the critical velocities 0.40, 0.50 and 0.65, the original points of which seemed to have a particularly bad scatter, a second point was computed with different choices of the values of the ${}_j\ell_n$'s. The fairly large differences in position between the members of pairs of points for $c = 0.50$ and $c = 0.65$ is indicative that the cause of the scatter is inaccuracy of integration. Taking smaller intervals for the numerical integration and more terms in the series for the analytic integration should increase the accuracy of the determination of $\mathcal{E}(\alpha, c)$. However, the work required to increase the accuracy of the numerical integration was not considered to be justified because $\mathcal{E}(\alpha, c)$ for the exact profile appears to be rather poorly approximated by the right-hand side of equation 14.

The indifference curve was drawn as in the figure for two reasons: each of the pairs of points for $c = 0.50$ and $c = 0.65$ was far from coincident, and no other indifference curve known to the author has a peculiar bend in its corresponding region.

As was done for the case of the cubic polynomial profile, the validity with which $\mathcal{E}(\alpha, c)$ was represented within a given range of values of α was estimated by noting the tendency of the curves of

$\mathcal{E}(\alpha, c)$ in the complex plane to turn toward the origin. As can be seen in Figure 6, the curves of $\mathcal{E}(\alpha, c)$ used in solving graphically the equation $\mathcal{E}(\alpha, c) = \mathcal{Z}(\zeta_0)$ turn toward the origin very appreciably with increasing values of α after they intersect the curve of $\mathcal{Z}(\zeta_0)$ once. Because the curves of $\mathcal{E}(\alpha, c)$ turn toward the origin so rapidly, only the indifference point corresponding to what should have been the smaller value of α could be plotted. Points defining the upper branch of the indifference curve were thus not found, and it was sketched to conform roughly to the upper branch of the indifference curve for the cubic polynomial profile.

In addition, it is quite possible that the representation of $\mathcal{E}(\alpha, c)$ for values of α corresponding to the intersections of the curves of $\mathcal{E}(\alpha, c)$ with the curve of $\mathcal{Z}(\zeta_0)$ which were plotted may not be very good. This is because the values of α at these intersections are already quite large, and dropping the terms which have been neglected and which have higher powers of α as factors may have introduced serious errors. There is in all probability serious error in the representation of $\mathcal{E}(\alpha, c)$ for $c = 0.70$ at the intersection with the curve for $\mathcal{Z}(\zeta_0)$, since α has a value between 5 and 6 there, which is very large for terms having higher powers of α as factors to be neglected.

D4 Values of Re, c, and α at the First Experimentally Determined Appearance of Instability

Through the utilization of interferometric techniques, Eckert and Soehnghen(17) observed that instability waves in the laminar free convection of air along a heated, vertical, semi-infinite flat plate first appeared at a value of Gr_x, the Grashof number based on distance from the leading edge of the plate, of 4×10^8. These investigators also reported that these waves progressed in the direction of the flow with a phase velocity 0.73 times the maximum velocity in the boundary layer where they were observed and that the length of the waves was 3.1 times the distance from the plate surface to the outer point in the flow at which the velocity was 0.01 of its maximum. Because no measurements of velocity were mentioned in Eckert and Soehnghen's paper, it was assumed that the maximum mean-flow velocity and the characteristic distance from the plate to the outer point at which the flow was 0.01 of its maximum were computed from exact solutions of the Polhausen equations for laminar free convection, which have been found to correspond closely with experimental observations by other investigators[12].

Eckert and Soehnghen's observations of Grashof number, phase velocity, and wave length for the initial instability were reduced to the present parameters of Reynolds number, dimensionless phase velocity, and wave number for the purpose of allowing some comparison to be made between theory and experiment regarding stable and unstable regions of the α-Re plane and the phase velocity of the waves of incipient instability. To reduce the value of the Grashof number to a Reynolds number, the relation

$$Re = 1.75\,(Gr_x)^{\frac{1}{4}} \tag{20}$$

of equations 10 for the cubic polynomial profile and the relation

$$Re = 1.686\,(Gr_x)^{\frac{1}{4}} \tag{21}$$

of equations 11 for the exact profile were employed. The values of this Reynolds number were found to be 248 referred to the cubic polynomial profile and 238 referred to the exact profile. These values as well as the values of the phase velocity and wave number are indicated in Figures 7 and 8.

The definition of the Reynolds number, equation 20 and the relation

$$\frac{\delta}{x} = \frac{2.96}{(Gr_x)^{\frac{1}{4}}} \tag{22}$$

of equations 10 for the cubic polynomial profile, and equation 21 and the relation

$$\frac{\delta}{x} = \frac{3.055}{(Gr_x)^{\frac{1}{4}}} \tag{23}$$

of equations 11 for the exact profile are sufficient to enable one to ascertain that the maximum velocity for the cubic polynomial profile is 1.07 times the maximum velocity for the exact profile. Using this fact, one can determine that the dimensionless phase velocities c referred to the cubic polynomial and exact profiles are respectively 0.68 and 0.73. As mentioned previously, it is assumed that Eckert and Soehnghen's value of the ratio of the disturbance phase velocity to the maximum mean-flow velocity in the boundary layer is based on the maximum mean-flow velocity of the exact profile.

In order to find the values of α , the wave number, corresponding to the initial experimental appearance of instability, one first defines $\delta_{.01}$ to be the distance from the plate surface to the outer point in the flow at which the mean velocity is 0.01 of its maximum value. Through the employment of a tabulation for the exact profile of the dimensionless similarity velocity $\frac{\bar{U}x}{2\nu_0 (Gr_x)^{\frac{1}{2}}}$ given in Reference 14 as a function of the dimensionless similarity distance $\left(\frac{Gr_x}{4}\right)^{\frac{1}{4}} \frac{y}{x}$ from the plate surface, one can find that

$$\delta_{.01} = 5.56\, x \left(\frac{4}{Gr_x}\right)^{\frac{1}{4}}. \tag{24}$$

The wave number α is defined by the equation

$$\alpha = \frac{2\pi\delta}{\Lambda}, \tag{25}$$

Λ being the wave length of the disturbance. According to the observations of Eckert and Soehnghen,

$$\Lambda = 3.1\ \delta_{.01} \tag{26}$$

initially. Utilizing equations 24, 25, and 26 along with equation 22 for the cubic polynomial profile or equation 23 for the exact profile enables one to determine that the values of α for the first appearance of instability are 0.76 referred to the cubic polynomial profile and 0.79 referred to the exact profile.

D5 Discussion of Results

The general result of this analytic investigation of stability in free convection along a semi-infinite flat plate is that the predictions of the analysis agree qualitatively, but not quantitatively, with the experimental observations of earlier investigators. Both the analysis and the experiments indicate that the flow is stable near the leading edge of the plate and that it becomes unstable after it proceeds some distance along the plate, but there is particularly poor agreement between prediction and observation concerning the wave number of the unstable oscillations that are first to appear.

The specific results of the analysis are difficult to analyze because it is impossible to differentiate satisfactorily among the effects of mathematical inadequacies, velocity profile shape, and neglect of coupling between the combined momentum and energy equations. Few definite statements regarding the findings can be made on account of this difficulty of separating these effects. However, a thorough analysis of the findings is desirable as a guide to future investigators who may attempt to use the results or to pursue the problem further, although this analysis must of necessity be of a speculative rather than conclusive nature.

D5. 1 Mathematical Validity of the Indifference Curves

The process of constructing an indifference curve for this case in which inertial, pressure, and viscous forces are considered and coupling between the combined momentum and energy equations is neglected is described in Part 4. 1. 2 of Appendix 4. The procedure requires that approximate solutions of the Orr-Sommerfeld equation be developed. Two general assumptions must be satisfied for these approximate solutions to represent the corresponding exact solutions well for the purposes of the stability analysis.

The first of these assumptions is that the product αRe of the wave and Reynolds numbers is large. One of the reasons that this is assumed is that terms of order $\frac{1}{\alpha Re}$ are considered to be neglectable when the "nonviscous" approximate solutions of the equation are developed. A severer restriction is placed on the size of αRe by dropping terms of order $\frac{1}{(\alpha Re)^{\frac{1}{3}}}$ in the construction of the group of "viscous" solutions of which $\Phi_3(\zeta)$ is a member. To determine the actual magnitude of the error introduced by neglecting terms having $\frac{1}{\alpha Re}$ and $\frac{1}{(\alpha Re)^{\frac{1}{3}}}$ as factors, one would have to construct for comparison the more accurate solutions obtained by including these higher-order terms as well as the terms of order zero in $\frac{1}{\alpha Re}$ and $\frac{1}{(\alpha Re)^{\frac{1}{3}}}$ that were employed. Doing so would involve dealing with what was considered to be an inordinate amount of mathematical complexity for the information that would be gained, and it was thus not attempted.

Nevertheless, some estimate of an upper bound for the magnitude of the negected terms can be secured by computing the sizes of $\frac{1}{\alpha Re}$ and $\frac{1}{(\alpha Re)^{\frac{1}{3}}}$ at the points on the indifference curves for the two velocity profiles which correspond to the smallest values of αRe on these two curves. Neglecting terms of order $\frac{1}{\alpha Re}$ in developing the nonviscous solutions was in all probability done with impunity because the maximum value of $\frac{1}{\alpha Re}$ for the case of the cubic polynomial is about 0.0001 and it is roughly 0.001 for the exact profile. For the cubic polynomial profile the maximum value of $\frac{1}{(\alpha Re)^{\frac{1}{3}}}$ is approximately 0.05, and it is about 0.1 for the exact profile. Although these estimates of the magnitude of the maximum corrections to the viscous solution are considerably larger than the corresponding corrections to the nonviscous solutions, it appears improbable that serious errors have been introduced by considering only the terms of zeroth order in $\frac{1}{(\alpha Re)^{\frac{1}{3}}}$ during the development of the viscous solution.

The second general assumption that was made in constructing the approximate solutions of the Orr-Sommerfeld equation is that the velocity profile is essentially linear between the plate surface and the inner critical point η_{c_1} at which the disturbance phase velocity is equal to the velocity of the laminar flow. This assumption is made in the development of the viscous solution $\Phi_3(\zeta)$. Although neither the cubic polynomial profile nor the exact profile is strictly linear in this range, it is believed that only minor errors result from this assumption. The basis of this belief is the close check mentioned in Section D3. 1 between the values of the function $\mathcal{L}(\zeta_0)$ of the boundary-condition equation

$$\mathcal{E}(\alpha, c) = \mathcal{L}(\zeta_0) \tag{7}$$

as computed both by using $\Phi_3(\zeta)$ and by employing a much more complicated viscous solution for which the nonlinearity of the velocity profile was taken into account. While this check was made for only one value of ζ_0 and one value of the disturbance phase velocity c, it was sufficiently representative to suggest strongly that the assumption of linearity has only minor adverse effects on the validity of the final indifference curves.

Restrictions on the size of the wave number α are not explicitly required in the development of approximate solutions of the Orr-Sommerfeld equation in Appendix 4, but they are necessary on account of computational limitations if one is to obtain valid indifference curves. The nonviscous solutions φ_1 and φ_2 that satisfy the equation

$$(\bar{u}-c)(\varphi''-\alpha^2\varphi)-\bar{u}''\varphi=0, \qquad (4\text{-}4)$$

which is obtained by setting $\frac{1}{\alpha Re}=0$ in the Orr-Sommerfeld equation 4-3, are developed as infinite series in α^2 in the forms

$$\left.\begin{aligned} \varphi_1 &= (\bar{u}-c)\sum_{m=0}^{\infty}\alpha^{2m}\mathcal{L}_m(\eta) \\ \text{and} \qquad \varphi_2 &= (\bar{u}-c)\sum_{m=0}^{\infty}\alpha^{2m}\mathcal{J}_m(\eta), \end{aligned}\right\} \qquad (4\text{-}6)$$

the functions $\mathcal{L}_m(\eta)$ and $\mathcal{J}_m(\eta)$ being defined by equations 4-7. As Lin[10] noted, these solutions are entire functions of α^2 except when η has values which are singular points of equation 4-4. That is, the solutions are regular in the entire finite α^2 plane; and this implies that the series in α^2 for φ_1, φ_2, and their first derivatives converge at the plate surface where the boundary conditions are applied. Thus one could ideally represent φ_1, φ_2, and their derivatives in the boundary-condition equation to any degree of accuracy desired by taking a sufficient number of terms in the series. Practically, however, the complexity of calculating the coefficients of α^{2m} increases so rapidly as m increases that taking more than the first term or two in the series would have required an amount of computation that would have been prohibitive with the available facilities. The retention and neglect of the terms in α^{2m} for the present case is described in Section D3.2.1.

An actual determination of the effect on the positions of the indifference curves which result from neglecting the higher-order terms in α^{2m} for φ_1, φ_2, and their derivatives cannot be made because exact solutions for which all the terms in α^{2m} are included are not available for comparison. However, some very inexact inferences regarding the validity of these curves can be made by roughly estimating the accuracy with which the nonviscous term $\mathcal{E}(\alpha, c)$ of the boundary-condition equation 7 is known for various values of α. This was done by examining the behavior for increasing values of α of the curves which represent the function at various constant values of c in the complex plane. These curves for the two velocity profiles are plotted in Figures 5 and 6, and Sections D3.2.2 and D3.2.3 contain discussions of the examinations. The results of the inspections were that the representation of $\mathcal{E}(\alpha, c)$ was considered to be fairly good in the range of values of α relevant to the determination of the indifference curve for the cubic polynomial profile but that it was not thought to be very good for the exact profile. In fact, this poor representation of $\mathcal{E}(\alpha, c)$ for the exact profile is responsible for the failure to determine the upper branch of the indifference curve in this case, and the points corresponding to the higher values of c for the lower branch that was determined are of doubtful validity. The validity of the point for a value of c of 0.70 is particularly suspect.

Because so few terms of the series for φ_1, φ_2, and their derivatives were retained, estimating the magnitude of the neglected terms by examining the sizes of the last few retained terms was not considered to be satisfactory, although doing so was attempted.

As implied previously, predicting the appearance of the indifference curves if higher-order terms in α^2 were to be used in the nonviscous solutions can be done only speculatively. It is not expected that the curve for the cubic polynomial profile would be moved much since it is believed that the nonviscous solutions are already fairly well represented for this case. For the case of the exact profile it is thought that considerable change in the position of the curve could easily follow from the use of higher-order approximations because it is suspected that the nonviscous solutions are quite poorly represented at present. Just how the curve would change its position cannot be predicted exactly, but it could very easily drop to lower values of α. The reason for supposing that this would happen is that it is suspected that neglecting the higher-order terms in α^2 has caused points representing values of $\mathcal{E}(\alpha, c)$ in the complex plane to be displaced toward the right of their correct positions. A comparison between the curves of this function as represented in Figures 5 and 6 for the cubic polynomial and exact profiles is the basis of this suspicion. Since at a given value of c, α increases as one proceeds toward the left along one of these curves, a displacement of points toward the right would result in values of α larger than the correct ones at the intersections of the curves of $\mathcal{E}(\alpha, c)$ and $\mathcal{Z}(\zeta_0)$ in the solution of the boundary-condition equation. Thus if the points representing $\mathcal{E}(\alpha, c)$ were returned to their correct positions, the values of α at the intersections would be lower than those at present. Also, if the intersections for given values of c were to remain at approximately the same values of ζ_o on the curve of $\mathcal{Z}(\zeta_o)$, the indifference curve would be shifted to larger values of Re. This is true because the equation defining ζ_o, $\zeta_o = -\eta_{c_1}(\bar{u}'_{c_1}\alpha Re)^{\frac{1}{3}}$, requires that Re increases if α decreases when all other terms remain unchanged.

The scatter of points defining the indifference curve for the exact profile is apparently due to inaccuracies in the numerical and analytic integration which was performed to determine the coefficients of powers of α^2 in the nonviscous solutions and their derivatives, as explained in the discussion of Section D3. 2. 3. Although this scatter is objectionable, it should be of secondary importance in affecting the indifference curves compared with neglecting the higher powers of α^2 in these nonviscous solutions and their derivatives.

In summary, it can be stated concerning the mathematical validity of the process of solving this uncoupled, viscous case that the indifference curve for the cubic polynomial profile is considered to be quite sound but that the branch obtained of the curve for the exact profile is of doubtful reliability.

D5. 2 Comparison of the Indifference Curves for the Cubic Polynomial and Exact Velocity Profiles

The indifference curves of Figures 7 and 8 for the cubic polynomial and exact velocity profiles lie in considerably different positions in the α-Re plane. The lower branch of the curve for the exact profile lies at higher values of the wave number than almost the entire curve for the cubic polynomial profile, and instability for the exact profile is indicated to appear first at a Reynolds number only one twenty-fifth of the value of this parameter at its first appearance for the cubic profile. Determining in just what proportions mathematical limitations on the validity of the analyses and inherent differences in the stability characteristics of the two profiles are responsible for the discrepancy between these two curves of neutral stability is impossible without representations of the function $\mathcal{E}(\alpha, c)$ for the exact profile which are much better than those that were used. It is not difficult, however, to understand why both of these factors should be responsible for the differences between the curves.

The preceding Section D5. 1 contains a discussion of the possibility that because the function $\mathcal{E}(\alpha, c)$ was inaccurately determined, the lower branch of the indifference curve for the exact profile should actually lie at a position in the α-Re plane considerably different from its present position; and reasons are presented there for suspecting that its true position is below and to the right of this present position. Changing the position of the curve in this manner would diminish the discrepancy between it and the curve for the cubic polynomial profile, but it is doubted that more than a fraction of the difference between the Reynolds numbers at which the two curves indicate that instability should first appear could be resolved in this way.

The other factor, inherently different stability characteristics of the two velocity profiles, is probably primarily responsible for the discrepancy between the two indifference curves. This statement is made because of the great effects on indifference curves that slight changes in profile shapes were found to produce by investigators studying the stability of the forced convection boundary layer profile in the presence of pressure gradients. Reynolds numbers for the first appearance of instability differing by a factor of 20 and corresponding wave numbers differing by a factor of 2.5 were computed for velocity profiles of only mildly different shapes according to information given in Reference 21. That differences of similar magnitudes between the values of these parameters should be present for the cubic polynomial and exact free convection profiles due to inherent differences in the stability characteristics of the two profiles is therefore quite reasonable.

D5.3 Comparison of the Indifference Curves with the Observed First Appearance of Instability Waves

The positions of the points in Figures 7 and 8 which represent the experimentally observed first appearance of instability waves were determined from information given in Reference 17 that was secured by studying the flow with the use of a Zehnder-Mach interferometer. A consideration of the interferometric process of observing the waves indicated that they must have been definitely two-dimensional; that is, they must have proceeded with no more than very little deviation from the direction of the mean flow rather than at an appreciable angle with respect to it. As explained in Section D4, the experimental values of the Reynolds number, the wave number, and the phase velocity given on the figures for the two profiles are slightly different; and these differences are the results of using somewhat different values of the characteristic velocities and lengths in the two cases.

Comparing the experimental point with the indifference curve for the cubic polynomial profile in Figure 7 indicates that instability is predicted to appear at values of Re, α, and c approximately

29, 1.9, and 0.49, respectively, times the observed values of these parameters. For the case of the exact profile, the predictions of the indifference curve of Figure 8 are that instability appears at values of Re, α, and c approximately 1.19, 7.0, and 0.96 times the observed values. The reasons for these discrepancies can be grouped into two classes: those that cause the observed physical situation and that assumed in the analysis to differ, and those that result from inadequacies in the mathematical treatment of the assumed situation.

The first reason considered in the class that keeps the observed and assumed situations from being equivalent is poor approximation to the actual velocity profile by the assumed profiles. As mentioned in Section D5.2, the stability characteristics of forced convection boundary layers are very sensitive to the profile shape, and the very poor correlation in free convection between the lowest Reynolds numbers for instability from experiment and from the indifference curve for the cubic polynomial profile could easily be largely the result of inherently different stability characteristics of the experimental and assumed profiles. The good agreement between the experimental values of the minimum Reynolds number for instability and the disturbance phase velocity on the one hand and the values of the parameters predicted by the analysis of the exact profile on the other hand helps to substantiate this view, but there is poor agreement between the observed and predicted values of the wave number in this case. Also, this indifference curve for the exact profile is subject to some change in its position that could result from an improved mathematical treatment. Nevertheless, it is believed that this curve as it stands is a considerably better description of the stability of the actual flow than is the curve for the cubic polynomial profile.

The second factor that adds to the difference between the observed and assumed situations is the neglect of coupling between the combined momentum and energy equations. From the complete combined momentum equation,

$$(\bar{u}-c)(\varphi''-\alpha^2\varphi)-\bar{u}''\varphi+\frac{f}{\alpha^{\nu}Re}\{ia_1S'-\alpha a_2S\}+\frac{i}{\alpha Re}\{\varphi^{IV}-2\alpha^2\varphi''+\alpha^4\varphi\}=0, \tag{1a}$$

the coupling term $\frac{f}{\alpha^{\nu}Re}\{ia_1S'-\alpha a_2S\}$ has been dropped to produce the Orr-Sommerfeld equation, upon which the stability analysis of the uncoupled, viscous case is based. For a value of f corresponding to a Prandtl number of the convecting fluid of 0.72 and values of α and Re equal to those at the observed first appearance of instability, the combination $\frac{f}{\alpha^{\nu}Re}$ has a magnitude of about 0.09. The maximum sizes of this combination for values of α and Re specified by the indifference curves for each of the two profiles are smaller. As shown in Appendix 5, the effect of an appreciable coupling term should be felt wholly in the nonviscous solutions for very small values of the combination $\frac{f}{\alpha^{\nu}Re}$. The nonviscous solutions with coupling included differ from their "uncoupled" counterparts in the presence of factors of order $e^{\frac{f}{\alpha^{\nu}Re}}$ and terms of order $\frac{f}{\alpha^{\nu}Re}$. It is thus estimated that an indifference curve obtained with the coupled nonviscous solutions would not differ greatly from one obtained with the uncoupled nonviscous solutions as long as $\frac{f}{\alpha^{\nu}Re}$ were equal to or smaller than its value at the first observed appearance of instability. Possibly one might find at the same value of c differences in the values of α and Re for the two curves on the order of one-quarter or one-third of their values on the curve for the uncoupled case.

Third in this group of factors is the neglect of the effect of the variable mean temperature in the inertial terms of the combined momentum equation. The authors of Reference 17 reported that mean temperature variations across the boundary layer up to about 8 per cent of the ambient air temperature were present during their observations of the appearance of instability waves in laminar free convection along a heated plate. It was brought to the attention of the author by Professor Lester Lees that heating the surface of a flat plate in laminar forced convection at an appreciable Mach number has a destabilizing effect on the boundary layer through the variation in mean fluid density in the inertial terms of the disturbance equation, and it is possible that such an effect was present during the experimental observations of instability in free convection which are reported in Reference 17. The magnitude of this effect is impossible to estimate with accuracy, but it is very doubtful that it alone could be responsible for the great discrepancy between the minimum Reynolds number for instability predicted for the cubic polynomial profile and that observed.

A final item in the lack of correspondence between the observed and assumed flows is the presence of effects due to the lateral edges of the plate in the experimental flows. These effects are quite possibly of minor importance, but they could alter the stability characteristics of the flow from what they would be if it were strictly two-dimensional. Without further information it is impossible to say just how the stability would be changed by the presence of these lateral edges, although an introduction by them of disturbances of finite amplitude into the flow might initiate transition at a value of the Reynolds number lower than that at which it could be expected to start in the undisturbed flow.

There might also be some effect due to differences between the velocity profile near the edges of the plate and what it would be in strictly two-dimensional flow.

The only inadequacy in the analytic treatment of the assumed situation which is apt to be of importance in affecting the agreement between experiment and analysis is the poor degree of accuracy with which the function $\mathcal{E}(\alpha, c)$ is represented for the exact profile. As mentioned in Section D5. 1, it is suspected that a more accurate determination of the function would result in shifting the lower branch of the indifference curve to smaller values of α and that it might also be shifted to higher values of Re . If the curve were to move to lower values of α , the agreement for this parameter with observation for the first appearance of instability would be improved; but moving it to higher values of Re would make the agreement for this other parameter poorer.

D6 Suggestions for Further Analytic Work on the Problem of Laminar Boundary Layer Stability in the Free Convection of Air

Two additional studies in connection with the general analytic problem of stability in the laminar free convection of air are suggested for an investigator who has the necessary computing facilities at his disposal. Both these would require extensive numerical computation.

The first of these is to obtain the entire indifference curve for the exact free convection velocity profile with the assumption that the combined momentum and energy equations are uncoupled, which is the same assumption employed in obtaining the lower half of the indifference curve of Figure 8. The use of an electronic digital computer should allow one to obtain an indifference curve of much greater mathematical validity than the one of Figure 8. This is because the computational limitations on the accuracy of the nonviscous solutions of the disturbance equation should be eased considerably with the availability of a device for very rapidly integrating numerically. Although with the use of a computer it is anticipated that algebraic complexity would still be a serious problem, it should be feasible to work with several more of the terms in powers of α^2 in the representations of the nonviscous solutions and thus increase very appreciably the validity of these representations, particularly for larger values of α. Smoothing the velocity profile and using small intervals in numerical integration should hold to an acceptable amount any scatter of points defining the indifference curve in the $\alpha - Re$ plane such as is present in Figure 8.

The second suggested investigation is the determination of an indifference curve with the effect of coupling between the combined momentum and energy equations taken into account. In this case the exact velocity and temperature profiles should be used so that comparisons with both experimental results and the indifference curve obtained with neglect of the coupling could be made. Also, a sufficient number of terms in powers of α^2 should be taken in the nonviscous solutions for them to be valid over the range of values of α which is of interest. The procedure for the coupled, viscous, non-heat-conducting case outlined in Part 4.2.2 of Appendix 4 should be followed for this investigation, since it is the simplest of those methods of treatment which have been developed for taking the coupling into account. It differs in application from the uncoupled, viscous case already considered only in that the nonviscous solutions of the disturbance equation are modified to take the coupling into account. As explained in Part 4.2.2 of Appendix 4, one can expect that solving the boundary-condition equation in this coupled case would be much more laborious than in the uncoupled case. The situation in which the plate is vertical should be attacked before any attempts are made to investigate the effects of inclination of the plate.

APPENDIX 1

NOTATION*

1.1 Latin Letters

a_1 cos Θ.

a_2 sin Θ.

b Value of η at edge of boundary layer chosen so that $\bar{u}, \bar{\theta} = 0$ for $\eta \geq b$.

c Dimensionless complex disturbance propagation or phase velocity equal to $\frac{\mathcal{K}}{U_m}$, in which $\mathcal{K}$ is the complex disturbance propagation velocity parallel to the plate.

c_p Specific heat of fluid at constant pressure, assumed to be a function of temperature only.

F Froude number equal to $\frac{U_m^2}{g\delta}$.

f Dimensionless function of σ_0 determined from mean-flow boundary layer relations as indicated in Appendix 3 so that $f = \frac{\delta}{x} Re$.

f Dimensionless function of σ_0 determined from mean-flow boundary layer relations as indicated in Appendix 3 so that $\mathit{f} = \frac{\varepsilon}{F} Re$.

g Magnitude of body force per unit mass.

Gr_x Grashof number based on ambient fluid properties and distance from leading edge of plate. It is equal to $\frac{|a_1| g \varepsilon x^3}{\nu_0^2}$.

h Enthalpy of fluid per unit mass, assumed to be a function of temperature only. Zero enthalpy is taken at $T = T_0$ so that $h = \int_{T_0}^{T} c_p(\Omega)\, d\Omega$.

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* A fluid property with the subscript $_p$ represents the property at the plate surface. Similarly, the subscript $_0$ refers to the ambient condition in which the fluid is unaffected by the presence of the plate. Also, a symbol over which a bar is placed denotes a mean or laminar flow variable, while a star * signifies that the symbol represents a small fluctuating quantity associated with incipient turbulence.

$H_j^{(\ell)}$	Hankel function of the ℓ^{th} kind of order j.
k	Thermal conductivity of fluid, assumed to be a function of temperature only.
L	Characteristic length equal to distance from leading edge of plate to point of application of equations (a constant).
p	Pressure or negative of average of x and y-components of normal stress in fluid.
$q, 1+\bar{q}$	Dimensionless specific heats at constant pressure equal, respectively, to $\frac{C_p^*}{C_{p_0}}, \frac{\bar{C}_p}{C_{p_0}}$.
$r, 1+\bar{r}$	Dimensionless mass densities of fluid equal, respectively, to $\frac{\rho^*}{\rho_0}, \frac{\bar{\rho}}{\rho_0}$.
Re	Reynolds number equal to $\frac{U_m \delta}{\nu_0}$.
s	Dimensionless function of η defined by the equation $\theta(\xi,\eta,\tau) = s(\eta)\, e^{i\alpha(\xi - c\tau)}$.
T	Temperature of fluid.
T_0	Temperature of ambient fluid (fluid unaffected by plate).
T_p	Temperature of plate.
ΔT	Temperature difference defined to be $\lvert T_p - T_0 \rvert$.
t	Time.
U	x-component of fluid velocity.
U_m	Maximum value of $\bar{U}$ in the boundary layer at a given distance from the leading edge of the plate.
$u, \bar{u}$	Dimensionless velocity components equal, respectively, to $\frac{U^*}{U_m}, \frac{\bar{U}}{U_m}$.
$\bar{u}_f$	Value of $\bar{u}$ corresponding to the value of η at which $\bar{u}''=0$.
$\bar{u}_j^{(n)}$	Value of the n^{th} derivative of $\bar{u}$ at the critical point η_{c_j}.
V	y - component of fluid velocity.
$v, \bar{v}$	Dimensionless velocity components equal, respectively, to $\frac{V^*}{U_m}, \frac{L}{\delta}\frac{\bar{V}}{U_m}$.

x	Cartesian co-ordinate representing distance from leading edge of plate measured along an axis parallel to the plate.
x_1	Dimensionless Cartesian co-ordinate equal to $\frac{x}{L}$.
y	Cartesian co-ordinate representing distance from plate surface measured along an axis perpendicular to the plate surface.

1.2 Greek Letters

α	Dimensionless disturbance wave number equal to $\frac{2\pi\delta}{\Lambda}$.
β	Coefficient of thermal expansion of fluid equal to $-\frac{1}{\rho}\left(\frac{\partial\rho}{\partial T}\right)_p$.
γ	Dimensionless parameter equal to $\frac{(\Delta T)^2}{2\varepsilon^2\rho_0}\left(\frac{\partial^2\rho}{\partial T^2}\right)_{B,P}\Big\rvert_{\rho=\rho_0}$.
δ	Boundary layer thickness defined to be $\frac{1}{\bar{U}_m}\int_0^\infty \bar{U}\,dy$.
ε	Dimensionless parameter equal to $\beta_0\Delta T$ or $-\frac{\Delta T}{\rho_0}\left(\frac{\partial\rho}{\partial T}\right)_P\Big\rvert_{\rho=\rho_0}$
ζ	Independent variable defined to be $(\bar{U}'_{c_1}\alpha Re)^{\frac{1}{3}}(\eta-\eta_{c_1})$ in Parts 4.1.2, 4.2.2, and 4.2.4 of Appendix 4; or $(\bar{U}'_{c_1}\sigma_0\alpha Re)^{\frac{1}{3}}(\eta-\eta_{c_1})$ in Part 4.2.3 of Appendix 4.
ζ_0	Value of ζ for $\eta=0$.
η	Dimensionless Cartesian co-ordinate equal to $\frac{y}{\delta}$.
η_{c_1}	Inner critical point or smaller value of η for which $\bar{u}=c$.
η_{c_2}	Outer critical point or larger value of η for which $\bar{u}=c$.
Θ	Angle between body force vector and plate surface.
$\theta, \bar{\theta}$	Dimensionless temperatures equal, respectively, to $\frac{T^*}{\Delta T}, \frac{\bar{T}-T_0}{\Delta T}$.
$\kappa, 1+\bar{\kappa}$	Dimensionless thermal conductivities equal, respectively, to $\frac{k^*}{k_0}, \frac{\bar{k}}{k_0}$.
Λ	Disturbance wave length.
λ_1	Parameter equal to $\left(\frac{\partial\rho}{\partial p}\right)_T\Big\rvert_{\rho=\rho_0}$.
λ_2	Parameter equal to $\Delta T\left(\frac{\partial^2\rho}{\partial p\,\partial T}\right)_{B,T}\Big\rvert_{\rho=\rho_0}$.

μ	Dynamic viscosity of fluid, assumed to be a function of temperature only.
ν	Kinematic viscosity of fluid, assumed to be a function of temperature only.
ξ	Dimensionless Cartesian co-ordinate equal to $\frac{x}{\delta}$.
$\pi, \bar{\pi}$	Dimensionless pressures equal, respectively, to $\frac{p^*}{\rho_0 U_m^2}$, $\frac{\bar{p}}{\rho_0 U_m^2}$.
ρ	Mass density of fluid.
σ	Prandtl number of fluid equal to $\frac{c_p \mu}{k}$.
τ	Dimensionless time variable equal to $\frac{U_m}{\delta} t$.
φ	Dimensionless function of η defined by the equation $\psi(\xi, \eta, \tau) = \varphi(\eta) e^{i\alpha(\xi - c\tau)}$.
ψ	Dimensionless disturbance stream function.
$\Omega, \Omega_j, j=1,2,..$	Dummy variables of integration.
$\omega, 1+\bar{\omega}$	Dimensionless dynamic viscosities equal, respectively, to $\frac{\mu^*}{\mu_0}, \frac{\bar{\mu}}{\mu_0}$.

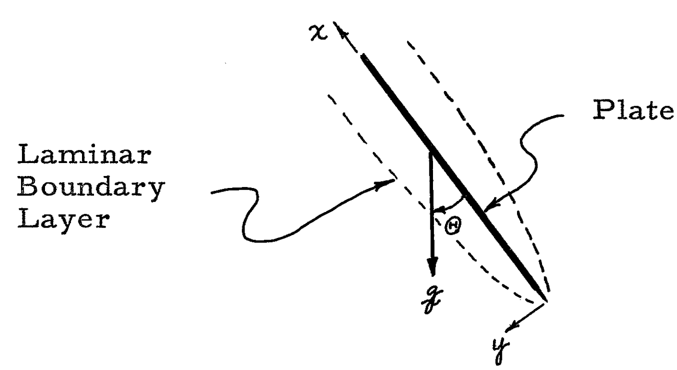
APPENDIX 2

DERIVATION OF DISTURBANCE EQUATIONS UNDER GENERAL ASSUMPTIONS

2.1 Fluid Properties

For generality, the assumption is made that c_p , the specific heat at constant pressure, μ , the dynamic viscosity, and k , the thermal conductivity, are functions of temperature only, in both steady and non-steady flow. On the other hand, under nonsteady-flow conditions the mass density ρ is taken to be a function of both disturbance pressure and disturbance temperature, although it is considered to depend on temperature alone in the steady-flow situation.

2.2 Diagram of Physical Configuration and Co-ordinate System



2.3 Derivation of Disturbance Equations

The basic equations describing two-dimensional nonsteady flow are as follows:

Continuity equation:

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho U)}{\partial x} + \frac{\partial(\rho V)}{\partial y} = 0 \tag{2-1}$$

x -momentum equation:

$$\begin{aligned}\rho\left\{\frac{\partial U}{\partial t} + U\frac{\partial U}{\partial x} + V\frac{\partial U}{\partial y}\right\} = -\frac{\partial p}{\partial x} - \rho a_1 g - \frac{2}{3}\frac{\partial}{\partial x}\left\{\mu\left(\frac{\partial U}{\partial x} + \frac{\partial V}{\partial y}\right)\right\} \\ + 2\frac{\partial}{\partial x}\left\{\mu\frac{\partial U}{\partial x}\right\} + \frac{\partial}{\partial y}\left\{\mu\left(\frac{\partial U}{\partial y} + \frac{\partial V}{\partial x}\right)\right\}\end{aligned} \tag{2-2}$$

y -momentum equation:

$$\begin{aligned}\rho\left\{\frac{\partial V}{\partial t} + U\frac{\partial V}{\partial x} + V\frac{\partial V}{\partial y}\right\} = -\frac{\partial p}{\partial y} + \rho a_2 g - \frac{2}{3}\frac{\partial}{\partial y}\left\{\mu\left(\frac{\partial U}{\partial x} + \frac{\partial V}{\partial y}\right)\right\} \\ + 2\frac{\partial}{\partial y}\left\{\mu\frac{\partial V}{\partial y}\right\} + \frac{\partial}{\partial x}\left\{\mu\left(\frac{\partial U}{\partial y} + \frac{\partial V}{\partial x}\right)\right\}\end{aligned} \tag{2-3}$$

Energy equation:

$$\begin{aligned}\rho\left\{\frac{\partial h}{\partial t} + U\frac{\partial h}{\partial x} + V\frac{\partial h}{\partial y}\right\} = \frac{\partial}{\partial x}\left\{k\frac{\partial T}{\partial x}\right\} + \frac{\partial}{\partial y}\left\{k\frac{\partial T}{\partial y}\right\} + \frac{\partial p}{\partial t} + U\frac{\partial p}{\partial x} + V\frac{\partial p}{\partial y} \\ + \mu\left\{-\frac{2}{3}\left(\frac{\partial U}{\partial x} + \frac{\partial V}{\partial y}\right)^2 + 2\left(\frac{\partial U}{\partial x}\right)^2 + 2\left(\frac{\partial V}{\partial y}\right)^2 + \left(\frac{\partial U}{\partial y} + \frac{\partial V}{\partial x}\right)^2\right\}\end{aligned} \tag{2-4}$$

State equation:

$$\rho = \rho(p, T) \tag{2-5}$$

In these equations U is the component of velocity in the x-direction, V is the component in the y-direction, p is the pressure considered to be the negative of the average of the x and y normal stress components, and h is the enthalpy per unit mass. a_1 is equal to $\cos\Theta$ and a_2 is equal to $\sin\Theta$, Θ being the angle as indicated between the body force vector, which has a magnitude of g, and the plate. ρ, μ, and k are the mass density, dynamic viscosity, and thermal conductivity, respectively.

No z-momentum equation is shown because the assumption is made that the disturbances along with the basic or steady-state flow are two-dimensional. Squire[22] showed that the minimum Reynolds number for instability of a parallel, viscous, constant-density flow is smaller for two-dimensional disturbances than for three-dimensional ones. Although in compressible flow at a Mach number greater than unity the opposite can be true, the present study is restricted to a consideration of two-dimensional disturbances for simplicity.

Each flow variable in the preceding equations will now be written as the sum of a steady-state part depending upon x and y only and a fluctuating part depending upon time t as well as x and y. The steady-state part is determined by the basic laminar flow, and the fluctuating part is considered to be due to some small disturbance which will be either amplified or damped. For example, $U(x,y,t) = \bar{U}(x,y) + U^*(x,y,t)$ will be assumed. The fluctuating part can be taken to be arbitrarily small compared with the steady-state part, since the study is restricted to a consideration of the stability of the flow with respect to infinitesimal disturbances. If it is noted that the steady-state quantities by themselves satisfy the flow equations and if the equations are linearized in terms of the disturbance quantities, the results are

$$\frac{\partial \rho^*}{\partial t} + \frac{\partial \bar{\rho}}{\partial x} U^* + \bar{\rho}\frac{\partial U^*}{\partial x} + \frac{\partial \bar{U}}{\partial x}\rho^* + \bar{U}\frac{\partial \rho^*}{\partial x} + \frac{\partial \bar{\rho}}{\partial y} V^* + \bar{\rho}\frac{\partial V^*}{\partial y} + \frac{\partial \bar{V}}{\partial y}\rho^* + \bar{V}\frac{\partial \rho^*}{\partial y} = 0 \tag{2-1a}$$

$$\rho^*\left\{\bar{U}\frac{\partial \bar{U}}{\partial x} + \bar{V}\frac{\partial \bar{U}}{\partial y}\right\} + \bar{\rho}\left\{\frac{\partial U^*}{\partial t} + U^*\frac{\partial \bar{U}}{\partial x} + \bar{U}\frac{\partial U^*}{\partial x} + V^*\frac{\partial \bar{U}}{\partial y} + \bar{V}\frac{\partial U^*}{\partial y}\right\}$$
$$= -\frac{\partial p^*}{\partial x} - \rho^* a_1 g - \frac{2}{3}\left\{\bar{\mu}\frac{\partial}{\partial x}\left(\frac{\partial U^*}{\partial x} + \frac{\partial V^*}{\partial y}\right) + \frac{\partial \bar{\mu}}{\partial x}\left(\frac{\partial U^*}{\partial x} + \frac{\partial V^*}{\partial y}\right) + \mu^*\frac{\partial}{\partial x}\left(\frac{\partial \bar{U}}{\partial x} + \frac{\partial \bar{V}}{\partial y}\right)\right.$$
$$\left.+ \frac{\partial \mu^*}{\partial x}\left(\frac{\partial \bar{U}}{\partial x} + \frac{\partial \bar{V}}{\partial y}\right)\right\} + 2\left\{\bar{\mu}\frac{\partial^2 U^*}{\partial x^2} + \frac{\partial \bar{\mu}}{\partial x}\frac{\partial U^*}{\partial x} + \mu^*\frac{\partial^2 \bar{U}}{\partial x^2} + \frac{\partial \mu^*}{\partial x}\frac{\partial \bar{U}}{\partial x}\right\} + \left\{\bar{\mu}\frac{\partial}{\partial y}\left(\frac{\partial U^*}{\partial y} + \frac{\partial V^*}{\partial x}\right)\right.$$

$$+\frac{\partial\bar{\mu}}{\partial y}\left(\frac{\partial U^*}{\partial y}+\frac{\partial V^*}{\partial x}\right)+\mu^*\frac{\partial}{\partial y}\left(\frac{\partial \bar{U}}{\partial y}+\frac{\partial \bar{V}}{\partial x}\right)+\frac{\partial \mu^*}{\partial y}\left(\frac{\partial \bar{U}}{\partial y}+\frac{\partial \bar{V}}{\partial x}\right)\Bigg\}, \tag{2-2a}$$

$$\begin{aligned}
&\rho^*\left\{\bar{U}\frac{\partial \bar{V}}{\partial x}+\bar{V}\frac{\partial \bar{V}}{\partial y}\right\}+\bar{\rho}\left\{\frac{\partial V^*}{\partial t}+U^*\frac{\partial \bar{V}}{\partial x}+\bar{U}\frac{\partial V^*}{\partial x}+V^*\frac{\partial \bar{V}}{\partial y}+\bar{V}\frac{\partial V^*}{\partial y}\right\}\\
&=-\frac{\partial p^*}{\partial y}+\rho^* a_2 g-\frac{2}{3}\left\{\bar{\mu}\frac{\partial}{\partial y}\left(\frac{\partial U^*}{\partial x}+\frac{\partial V^*}{\partial y}\right)+\frac{\partial \bar{\mu}}{\partial y}\left(\frac{\partial U^*}{\partial x}+\frac{\partial V^*}{\partial y}\right)+\mu^*\frac{\partial}{\partial y}\left(\frac{\partial \bar{U}}{\partial x}+\frac{\partial \bar{V}}{\partial y}\right)\right.\\
&\left.+\frac{\partial \mu^*}{\partial y}\left(\frac{\partial \bar{U}}{\partial x}+\frac{\partial \bar{V}}{\partial y}\right)\right\}+2\left\{\bar{\mu}\frac{\partial^2 V^*}{\partial y^2}+\frac{\partial \bar{\mu}}{\partial y}\frac{\partial V^*}{\partial y}+\mu^*\frac{\partial^2 \bar{V}}{\partial y^2}+\frac{\partial \mu^*}{\partial y}\frac{\partial \bar{V}}{\partial y}\right\}+\left\{\bar{\mu}\frac{\partial}{\partial x}\left(\frac{\partial U^*}{\partial y}+\frac{\partial V^*}{\partial x}\right)\right.\\
&\left.+\frac{\partial \bar{\mu}}{\partial x}\left(\frac{\partial U^*}{\partial y}+\frac{\partial V^*}{\partial x}\right)+\mu^*\frac{\partial}{\partial x}\left(\frac{\partial \bar{U}}{\partial y}+\frac{\partial \bar{V}}{\partial x}\right)+\frac{\partial \mu^*}{\partial x}\left(\frac{\partial \bar{U}}{\partial y}+\frac{\partial \bar{V}}{\partial x}\right)\right\},
\end{aligned} \tag{2-3a}$$

$$\begin{aligned}
&\rho^*\bar{C}_p\left\{\bar{U}\frac{\partial \bar{T}}{\partial x}+\bar{V}\frac{\partial \bar{T}}{\partial y}\right\}+\bar{\rho}\,C_p^*\left\{\bar{U}\frac{\partial \bar{T}}{\partial x}+\bar{V}\frac{\partial \bar{T}}{\partial y}\right\}+\bar{\rho}\,\bar{C}_p\left\{\frac{\partial T^*}{\partial t}+U^*\frac{\partial \bar{T}}{\partial x}+\bar{U}\frac{\partial T^*}{\partial x}+V^*\frac{\partial \bar{T}}{\partial y}\right.\\
&\left.+\bar{V}\frac{\partial T^*}{\partial y}\right\}=\left\{\bar{k}\frac{\partial^2 T^*}{\partial x^2}+\frac{\partial \bar{k}}{\partial x}\frac{\partial T^*}{\partial x}+k^*\frac{\partial^2 \bar{T}}{\partial x^2}+\frac{\partial k^*}{\partial x}\frac{\partial \bar{T}}{\partial x}+\bar{k}\frac{\partial^2 T^*}{\partial y^2}+\frac{\partial \bar{k}}{\partial y}\frac{\partial T^*}{\partial y}+k^*\frac{\partial^2 \bar{T}}{\partial y^2}+\frac{\partial k^*}{\partial y}\frac{\partial \bar{T}}{\partial y}\right\}\\
&+\left\{\frac{\partial p^*}{\partial t}+\bar{U}\frac{\partial p^*}{\partial x}+U^*\frac{\partial \bar{p}}{\partial x}+\bar{V}\frac{\partial p^*}{\partial y}+V^*\frac{\partial \bar{p}}{\partial y}\right\}+\bar{\mu}\left\{-\frac{4}{3}\left(\frac{\partial \bar{U}}{\partial x}\frac{\partial U^*}{\partial x}+\frac{\partial \bar{U}}{\partial x}\frac{\partial V^*}{\partial y}+\frac{\partial U^*}{\partial x}\frac{\partial \bar{V}}{\partial y}\right.\right.\\
&\left.\left.+\frac{\partial V^*}{\partial y}\frac{\partial \bar{V}}{\partial y}\right)+4\left(\frac{\partial \bar{U}}{\partial x}\frac{\partial U^*}{\partial x}+\frac{\partial \bar{V}}{\partial y}\frac{\partial V^*}{\partial y}\right)+2\left(\frac{\partial \bar{U}}{\partial y}\frac{\partial U^*}{\partial y}+\frac{\partial \bar{U}}{\partial y}\frac{\partial V^*}{\partial x}+\frac{\partial U^*}{\partial y}\frac{\partial \bar{V}}{\partial x}+\frac{\partial V^*}{\partial x}\frac{\partial \bar{V}}{\partial x}\right)\right\}\\
&+\mu^*\left\{-\frac{2}{3}\left(\frac{\partial \bar{U}}{\partial x}+\frac{\partial \bar{V}}{\partial y}\right)^2+2\left(\frac{\partial \bar{U}}{\partial x}\right)^2+2\left(\frac{\partial \bar{V}}{\partial y}\right)^2+\left(\frac{\partial \bar{U}}{\partial y}+\frac{\partial \bar{V}}{\partial x}\right)^2\right\},
\end{aligned} \tag{2-4a}$$

and
$$\rho^* = T^*\left(\frac{\partial \rho}{\partial T}\right)_p\Bigg|_{\rho=\bar{\rho}} + p^*\left(\frac{\partial \rho}{\partial p}\right)_T\Bigg|_{\rho=\bar{\rho}}. \tag{2-5a}$$

In order to determine which of their terms are largest, these equations are rewritten in dimensionless form:

$$\left\{\frac{\partial r}{\partial \tau}+\bar{u}\frac{\partial r}{\partial \xi}+(1+r)\left(\frac{\partial u}{\partial \xi}+\frac{\partial v}{\partial \eta}\right)+\frac{\partial \bar{r}}{\partial \eta}v\right\}+\frac{f}{Re}\left\{\frac{\partial \bar{r}}{\partial x_1}u+\bar{v}\frac{\partial r}{\partial \eta}+\left(\frac{\partial \bar{u}}{\partial x_1}+\frac{\partial \bar{v}}{\partial \eta}\right)r\right\}=0 \tag{2-1b}$$

$$\left\{(1+\bar{r})\left(\frac{\partial u}{\partial \tau}+\bar{u}\frac{\partial u}{\partial \xi}+\frac{\partial \bar{u}}{\partial \eta}v\right)+\frac{\partial \Pi}{\partial \xi}\right\}+\frac{a_1 r}{F}+\frac{1}{Re}\left\{\frac{2}{3}(1+\bar{\omega})\frac{\partial}{\partial \xi}\left(\frac{\partial u}{\partial \xi}+\frac{\partial v}{\partial \eta}\right)\right.$$

$$-2(1+\bar\omega)\frac{\partial^2 u}{\partial\xi^2} - \frac{\partial\bar\omega}{\partial\eta}\left(\frac{\partial u}{\partial\eta}+\frac{\partial v}{\partial\xi}\right) - (1+\bar\omega)\frac{\partial}{\partial\eta}\left(\frac{\partial u}{\partial\eta}+\frac{\partial v}{\partial\xi}\right) - \frac{\partial\bar U}{\partial\eta}\frac{\partial\omega}{\partial\eta} - \frac{\partial^2\bar U}{\partial\eta^2}\omega$$

$$+f\Big[\left(\bar U\frac{\partial\bar U}{\partial x_1}+\bar V\frac{\partial\bar U}{\partial\eta}\right)r + (1+\bar r)\left(\frac{\partial\bar U}{\partial x_1}u+\bar V\frac{\partial u}{\partial\eta}\right)\Big]\Big\} + \frac{f}{R_e^2}\Big\{\frac{2}{3}\Big[\frac{\partial\bar\omega}{\partial x_1}\left(\frac{\partial u}{\partial\xi}+\frac{\partial v}{\partial\eta}\right)$$

$$+\left(\frac{\partial\bar U}{\partial x_1}+\frac{\partial\bar V}{\partial\eta}\right)\frac{\partial\omega}{\partial\xi}\Big] - 2\left[\frac{\partial\bar\omega}{\partial x_1}\frac{\partial u}{\partial\xi}+\frac{\partial\bar U}{\partial x_1}\frac{\partial\omega}{\partial\xi}\right]\Big\} + \frac{f^2}{R_e^3}\Big\{\frac{2}{3}\left[\frac{\partial}{\partial x_1}\left(\frac{\partial\bar U}{\partial x_1}+\frac{\partial\bar V}{\partial\eta}\right)\right]\omega$$

$$-2\frac{\partial^2\bar U}{\partial x_1^2}\omega - \frac{\partial\bar V}{\partial x_1}\frac{\partial\omega}{\partial\eta} - \frac{\partial^2\bar V}{\partial\eta\,\partial x_1}\omega\Big\} = 0 \qquad (2\text{-}2b)$$

$$\Big\{(1+\bar r)\left(\frac{\partial v}{\partial\tau}+\bar U\frac{\partial v}{\partial\xi}\right)+\frac{\partial\pi}{\partial\eta}\Big\} - \frac{a_2 r}{F} + \frac{1}{R_e}\Big\{\frac{2}{3}\Big[\frac{\partial\bar\omega}{\partial\eta}\left(\frac{\partial u}{\partial\xi}+\frac{\partial v}{\partial\eta}\right) + (1+\bar\omega)\frac{\partial}{\partial\eta}\left(\frac{\partial u}{\partial\xi}+\frac{\partial v}{\partial\eta}\right)\Big]$$

$$-2\left[\frac{\partial\bar\omega}{\partial\eta}\frac{\partial v}{\partial\eta}+(1+\bar\omega)\frac{\partial^2 v}{\partial\eta^2}\right] - (1+\bar\omega)\frac{\partial}{\partial\xi}\left(\frac{\partial u}{\partial\eta}+\frac{\partial v}{\partial\xi}\right) - \frac{\partial\bar U}{\partial\eta}\frac{\partial\omega}{\partial\xi} + f\Big[(1+\bar r)\left(\frac{\partial\bar V}{\partial\eta}v+\bar V\frac{\partial v}{\partial\eta}\right)\Big]\Big\}$$

$$+\frac{f}{R_e^2}\Big\{\frac{2}{3}\left[\frac{\partial}{\partial\eta}\left(\frac{\partial\bar U}{\partial x_1}+\frac{\partial\bar V}{\partial\eta}\right)\right]\omega + \frac{2}{3}\left(\frac{\partial\bar U}{\partial x_1}+\frac{\partial\bar V}{\partial\eta}\right)\frac{\partial\omega}{\partial\eta} - 2\left(\frac{\partial\bar V}{\partial\eta}\frac{\partial\omega}{\partial\eta}+\frac{\partial^2\bar V}{\partial\eta^2}\omega\right)$$

$$-\frac{\partial\bar\omega}{\partial x_1}\left(\frac{\partial u}{\partial\eta}+\frac{\partial v}{\partial\xi}\right) - \frac{\partial^2\bar U}{\partial x_1\partial\eta}\omega + f\Big[\left(\bar U\frac{\partial\bar V}{\partial x_1}+\bar V\frac{\partial\bar V}{\partial\eta}\right)r + (1+\bar r)\frac{\partial\bar V}{\partial x_1}u\Big]\Big\}$$

$$+\frac{f^2}{R_e^3}\Big\{-\frac{\partial\bar V}{\partial x_1}\frac{\partial\omega}{\partial\xi}\Big\} + \frac{f^3}{R_e^4}\Big\{-\frac{\partial^2\bar V}{\partial x_1^2}\omega\Big\} = 0 \qquad (2\text{-}3b)$$

$$\Big\{(1+\bar r)(1+\bar q)\left(\frac{\partial\theta}{\partial\tau}+\bar U\frac{\partial\theta}{\partial\xi}+\frac{\partial\bar\theta}{\partial\eta}v\right)\Big\} + \frac{1}{R_e}\Big\{-\frac{1}{\sigma_0}\Big[(1+\bar K)\left(\frac{\partial^2\theta}{\partial\xi^2}+\frac{\partial^2\theta}{\partial\eta^2}\right) + \frac{\partial\bar K}{\partial\eta}\frac{\partial\theta}{\partial\eta}$$

$$+\frac{\partial\bar\theta}{\partial\eta}\frac{\partial K}{\partial\eta}+\frac{\partial^2\bar\theta}{\partial\eta^2}K\Big] + f\Big[(1+\bar q)\left(\bar U\frac{\partial\bar\theta}{\partial x_1}+\bar V\frac{\partial\bar\theta}{\partial\eta}\right)r + (1+\bar r)\left(\bar U\frac{\partial\bar\theta}{\partial x_1}+\bar V\frac{\partial\bar\theta}{\partial\eta}\right)q$$

$$+(1+\bar r)(1+\bar q)\left(\frac{\partial\bar\theta}{\partial x_1}u+\bar V\frac{\partial\theta}{\partial\eta}\right)\Big]\Big\} + \frac{f}{\sigma_0 R_e^2}\Big\{-\left[\frac{\partial\bar K}{\partial x_1}\frac{\partial\theta}{\partial\xi}+\frac{\partial\bar\theta}{\partial x_1}\frac{\partial K}{\partial\xi}\right]\Big\} + \frac{f^2}{\sigma_0 R_e^3}\Big\{-\frac{\partial^2\bar\theta}{\partial x_1^2}K\Big\}$$

$$+\frac{U_m^2}{C_{p0}\Delta T}\Big\{-\left[\frac{\partial\pi}{\partial\tau}+\bar U\frac{\partial\pi}{\partial\xi}+\frac{\partial\bar\pi}{\partial\eta}v\right]\Big\} + \frac{U_m^2}{C_{p0}\Delta T R_e}\Big\{-\Big[2(1+\bar\omega)\frac{\partial\bar U}{\partial\eta}\left(\frac{\partial u}{\partial\eta}+\frac{\partial v}{\partial\xi}\right)+\left(\frac{\partial\bar U}{\partial\eta}\right)^2\omega\Big]$$

$$-f\left[\frac{\partial\bar\pi}{\partial x_1}u+\bar V\frac{\partial\pi}{\partial\eta}\right]\Big\} + \frac{U_m^2 f}{C_{p0}\Delta T R_e^2}\Big\{[1+\bar\omega]\Big[\frac{4}{3}\left(\frac{\partial\bar V}{\partial\eta}\frac{\partial u}{\partial\xi}+\frac{\partial\bar U}{\partial x_1}\frac{\partial v}{\partial\eta}\right) - \frac{8}{3}\Big(\frac{\partial\bar U}{\partial x_1}\frac{\partial u}{\partial\xi}$$

$$+\frac{\partial\bar V}{\partial\eta}\frac{\partial v}{\partial\eta}\Big)\Big]\Big\} + \frac{U_m^2 f^2}{C_{p0}\Delta T R_e^3}\Big\{-2[1+\bar\omega]\left[\frac{\partial\bar V}{\partial x_1}\left(\frac{\partial u}{\partial\eta}+\frac{\partial v}{\partial\xi}\right)\right] - \Big[-\frac{4}{3}\frac{\partial\bar U}{\partial x_1}\frac{\partial\bar V}{\partial\eta}+\frac{4}{3}\left(\frac{\partial\bar U}{\partial x_1}\right)^2$$

$$+\frac{4}{3}\left(\frac{\partial\bar V}{\partial\eta}\right)^2+2\frac{\partial\bar V}{\partial x_1}\frac{\partial\bar U}{\partial\eta}\Big]\omega\Big\} + \frac{U_m^2 f^4}{C_{p0}\Delta T R_e^5}\Big\{-\left(\frac{\partial\bar V}{\partial x_1}\right)^2\omega\Big\} = 0 \qquad (2\text{-}4b)$$

$$r - \Big\{\frac{1}{\rho_0}\left(\frac{\partial\rho}{\partial T}\right)_p\Big|_{\rho=\bar\rho}\Delta T\Big\}\theta - \Big\{U_m^2\left(\frac{\partial\rho}{\partial p}\right)_T\Big|_{\rho=\bar\rho}\Big\}\pi = 0 \qquad (2\text{-}5b)$$

In these equations u and $\bar{u}$ are nondimensional forms of U^* and $\bar{U}$, v and $\bar{v}$ are nondimensional forms of V^* and $\bar{V}$, ξ and η correspond to x and y, τ to t, π and $\bar{\pi}$ to p^* and $\bar{p}$, r and $1+\bar{r}$ to ρ^* and $\bar{\rho}$, ω and $1+\bar{\omega}$ to μ^* and $\bar{\mu}$, q and $1+\bar{q}$ to c_p^* and $\bar{c}_p$, θ and $\bar{\theta}$ to T^* and $\bar{T}$, and κ and $1+\bar{\kappa}$ to k^* and $\bar{k}$. x_1 is a second nondimensional form of x in addition to ξ. f is a dimensionless function of σ_0 related to the rate of growth of the mean-flow boundary layer. R_e is the Reynolds number based on the maximum velocity U_m in the boundary layer and the boundary layer thickness δ at the particular value of x at which the stability is to be studied. ΔT is the absolute value of the difference between the temperatures of the plate and the ambient fluid or fluid which is unaffected by the plate. σ_0 is the Prandtl number of the ambient fluid. The exact definitions of these terms are given in Appendix 1.

One can see by reference to the notation table of Appendix 1 that the boundary layer character of the mean flow has been utilized in defining the dimensionless variables and that $\bar{u}$, $\bar{v}$, $\bar{\theta}$, and their derivatives are of the same order of magnitude. $\bar{r}$, $\bar{\omega}$, $\bar{\kappa}$, $\bar{q}$, and their derivatives are much smaller unless the temperature difference between the plate and the ambient fluid is extremely large.

r, $\left(\frac{\partial \rho}{\partial T}\right)_p\Big|_{\rho=\bar{\rho}}$, and $\left(\frac{\partial \rho}{\partial p}\right)_T\Big|_{\rho=\bar{\rho}}$ are each expanded in Taylor series of two terms as functions of the temperature to obtain

$$\bar{r} = -\varepsilon\bar{\theta} + \gamma\varepsilon^2\bar{\theta}^2 \tag{2-6}$$

and

$$r = (-\varepsilon + 2\gamma\varepsilon^2\bar{\theta})\theta + U_m^2(\lambda_1 + \lambda_2\bar{\theta})\pi. \tag{2-7}$$

ε. is equal to the product of the ambient fluid coefficient of thermal expansion and the absolute value of the temperature difference between the plate and the ambient fluid. ε and the symbols δ, λ_1, and λ_2 are defined in Appendix 1.

With the use of these relations and a utilization of the fact that $\omega = \frac{d\bar{\omega}}{d\bar{\theta}}\theta$ because viscosity is considered to be a function of temperature only, the momentum equations 1-2b and 1-3b can be cross-differentiated with respect to η and ξ and combined to eliminate the terms in the dimensionless pressure π. The resulting "combined momentum equation" is

$$
\begin{aligned}
&\Big\{[1-\varepsilon\bar{\theta}]\Big[\frac{\partial^2 u}{\partial\eta\partial\tau} + \frac{\partial\bar{u}}{\partial\eta}\Big(\frac{\partial u}{\partial\xi}+\frac{\partial v}{\partial\eta}\Big) + \bar{u}\frac{\partial^2 u}{\partial\eta\partial\xi} + \frac{\partial^2\bar{u}}{\partial\eta^2}v - \Big(\frac{\partial^2 v}{\partial\xi\partial\tau} + \bar{u}\frac{\partial^2 v}{\partial\xi^2}\Big)\Big] - \varepsilon\frac{\partial\bar{\theta}}{\partial\eta}\Big(\frac{\partial u}{\partial\tau} \\
&+ \bar{u}\frac{\partial u}{\partial\xi} + \frac{\partial\bar{u}}{\partial\eta}v\Big)\Big\} + \frac{\varepsilon}{F}\Big\{-a_1\frac{\partial\theta}{\partial\eta} - a_2\frac{\partial\theta}{\partial\xi}\Big\} + \frac{1}{Re}\Big\{-\frac{\partial^3 u}{\partial\eta\partial\xi^2} + \frac{\partial^3 v}{\partial\eta^2\partial\xi} - \frac{\partial^3 u}{\partial\eta^3} \\
&+ \frac{\partial^3 v}{\partial\xi^3} + f\Big[-\frac{\partial^2\bar{u}}{\partial x_1\partial\eta}u + \bar{v}\Big(\frac{\partial^2 u}{\partial\eta^2} - \frac{\partial^2 v}{\partial\xi\partial\eta}\Big)\Big] + \bar{\omega}\Big[-\frac{\partial^3 u}{\partial\eta\partial\xi^2} + \frac{\partial^3 v}{\partial\eta^2\partial\xi} - \frac{\partial^3 u}{\partial\eta^3} + \frac{\partial^3 v}{\partial\xi^3}\Big] \\
&+ \frac{d\bar{\omega}}{d\bar{\theta}}\Big[-2\frac{\partial\bar{\theta}}{\partial\eta}\Big(\frac{\partial^2 u}{\partial\xi^2} + \frac{\partial^2 u}{\partial\eta^2}\Big) - \frac{\partial^2\bar{\theta}}{\partial\eta^2}\Big(\frac{\partial u}{\partial\eta} + \frac{\partial v}{\partial\xi}\Big) - \frac{\partial^3\bar{u}}{\partial\eta^3}\theta - 2\frac{\partial^2\bar{u}}{\partial\eta^2}\frac{\partial\theta}{\partial\eta} - \frac{\partial\bar{u}}{\partial\eta}\Big(\frac{\partial^2\theta}{\partial\xi^2} + \frac{\partial^2\theta}{\partial\eta^2}\Big)\Big] \\
&+ \frac{d^2\bar{\omega}}{d\bar{\theta}^2}\Big[-\Big(\frac{\partial\bar{\theta}}{\partial\eta}\Big)^2\Big(\frac{\partial u}{\partial\eta} + \frac{\partial v}{\partial\xi}\Big) - \Big(2\frac{\partial\bar{\theta}}{\partial\eta}\frac{\partial^2\bar{u}}{\partial\eta^2} + \frac{\partial^2\bar{\theta}}{\partial\eta^2}\frac{\partial\bar{u}}{\partial\eta}\Big)\theta - 2\frac{\partial\bar{\theta}}{\partial\eta}\frac{\partial\bar{u}}{\partial\eta}\frac{\partial\theta}{\partial\eta}\Big] \\
&+ \frac{d^3\bar{\omega}}{d\bar{\theta}^3}\Big[-\Big(\frac{\partial\bar{\theta}}{\partial\eta}\Big)^2\frac{\partial\bar{u}}{\partial\eta}\theta\Big]\Big\} = O\Big\{\varepsilon^2, \frac{U_m^2\lambda_{1,2}}{F}, \frac{\varepsilon f}{Re}, \frac{f^2}{Re^2}\Big\}. \qquad (2\text{-}8)
\end{aligned}
$$

Here it is assumed that $\varepsilon \ll 1$ and $Re \gg 1$. The latter inequality should be true at least when air is the convecting fluid, since Reference 17 indicates that instability waves first appear at an Re of approximately 2.4×10^2 for free convection along a vertical flat plate in air. Unless the Prandtl number is very small, f is on the order of unity, as Figure 1 shows. The values of f indicated there were computed as described in Appendix 3 from information given in Reference 14. Also, it is assumed that the terms of order $\frac{U_m^2 \lambda_{1,2}}{F}$ are very small. This is equivalent to considering that mean velocities in the flow are sufficiently low so that dynamic pressure changes have little effect on the fluid density. Thus, the terms inside the order-of-magnitude brackets are small.

By employment of the relations 2-6 and 2-7, the energy equation 2-4b can be simplified to become

$$\left\{(1+\bar{q})\left(\frac{\partial \theta}{\partial \tau} + \bar{u}\frac{\partial \theta}{\partial \xi} + \frac{\partial \bar{\theta}}{\partial \eta} v\right)\right\} + \frac{1}{Re}\left\{\frac{-1}{\sigma_0}\left[(1+\bar{\kappa})\left(\frac{\partial^2 \theta}{\partial \xi^2} + \frac{\partial^2 \theta}{\partial \eta^2}\right) + \frac{d\bar{\kappa}}{d\bar{\theta}}\left(2\frac{\partial \bar{\theta}}{\partial \eta}\frac{\partial \theta}{\partial \eta} + \frac{\partial^2 \bar{\theta}}{\partial \eta^2}\theta\right)\right]\right.$$

$$\left. + f\left[(1+\bar{q})\left(\frac{\partial \bar{\theta}}{\partial x_1} u + \bar{v}\frac{\partial \theta}{\partial \eta}\right) + \frac{d\bar{q}}{d\bar{\theta}}\left(\bar{u}\frac{\partial \bar{\theta}}{\partial x_1} + \bar{v}\frac{\partial \bar{\theta}}{\partial \eta}\right)\theta\right]\right\}$$

$$= O\left\{\varepsilon^2, \frac{\varepsilon f}{Re}, \frac{U_m^2}{C_{p_0}\Delta T}, \frac{f}{\sigma_0 Re^2}, \frac{U_m^2 \lambda_{1,2} f}{Re}\right\}. \qquad (2\text{-}4c)$$

In the simplification process use has been made of the relations $q = \frac{d\bar{q}}{d\bar{\theta}}\theta$ and $\kappa = \frac{d\bar{\kappa}}{d\bar{\theta}}\theta$, which hold by virtue of the dependence upon temperature alone of the specific heat and the thermal conductivity. $\frac{d\bar{q}}{d\bar{\theta}}$ and $\frac{d\bar{\kappa}}{d\bar{\theta}}$ are taken to be constant.

Of interest is the similarity between equations 2-8 and 2-4c and the corresponding ones that would have appeared if Schlichting's method of treating the stability of a boundary layer with stratified density had been used. Schlichting neglected the energy equation, considered all fluid properties except the density ρ to be constant, and took the flow to be parallel. His assumptions regarding parallelism of the flow and constancy of viscosity would, of course, have given a much simpler combined momentum equation than equation 2-8. Instead of using the energy equation, he took $\frac{D\rho}{Dt} = 0$, which states that a fluid particle retains its steady-state density during the disturbance motion. In terms of the present non-dimensional variables this relation is, with the assumption of linear dependence of density upon temperature,

$$\frac{\partial \theta}{\partial \tau} + \bar{u}\frac{\partial \theta}{\partial \xi} + \frac{\partial \bar{\theta}}{\partial \eta} v = 0.$$

This equation holds only for parallel flow and is linearized in the disturbance quantities. One can easily see that the energy equation 2-4c reduces to this relation for parallel flow if only the convective terms are retained. If the Prandtl number is large, then, Schlichting's assumption concerning fluid density is approximately equivalent to the energy equation with pressure work and dissipation terms neglected.

A stream function for the disturbance motion will now be defined, and the magnitude of the error introduced by using it in the present case of variable fluid density will be estimated. The continuity equation 2-1b and the state equation 2-7 can be combined to give

$$\frac{\partial u}{\partial \xi} + \frac{\partial v}{\partial \eta} = \varepsilon\left\{\frac{\partial \theta}{\partial \tau} + \bar{u}\frac{\partial \theta}{\partial \xi} + \frac{\partial \bar{\theta}}{\partial \eta} v\right\} + O\left\{\varepsilon^2, \frac{\varepsilon f}{Re}, U_m^2 \lambda_{1,2}\right\}. \tag{2-9}$$

Similarly, the energy equation 1-4c can be written as

$$\frac{\partial \theta}{\partial \tau} + \bar{u}\frac{\partial \theta}{\partial \xi} + \frac{\partial \bar{\theta}}{\partial \eta} v = O\left\{\varepsilon^2, \frac{f}{Re}, \frac{1}{\sigma_o Re}, \frac{f}{\sigma_o Re^2}, \frac{U_m^2}{C_{po}\Delta T}\right\}. \tag{2-10}$$

From a combination of equation 2-9 and 2-10,

$$\frac{\partial u}{\partial \xi} + \frac{\partial v}{\partial \eta} = O\left\{\varepsilon^2, U_m^2 \lambda_{1,2}, \frac{\varepsilon}{\sigma_o Re}, \frac{\varepsilon f}{Re}, \frac{\varepsilon U_m^2}{C_{po}\Delta T}\right\}. \tag{2-11}$$

In order for the use of a disturbance stream function to be feasible, the terms within the brackets on the right-hand side of this last equation must be small compared with unity. Empirical studies [16] of heat-transfer rates for the free convection about a vertical flat plate of fluids with different Prandtl numbers may indicate that the Reynolds number for transition is inversely proportional to the fourth root of the

Prandtl number. Since transition when air is the fluid occurs at a Reynolds number of approximately 2.4×10^2, the terms on the right-hand side of this equation which have the Reynolds number in their denominators should be small, except possibly when the Prandtl number is very low. The expressions $U_m^2 \lambda_{1,2}$ and $\frac{\varepsilon U_m^2}{C_{p_0} \Delta T}$ represent, respectively, the orders of magnitude of the change in density because of pressure fluctuations and the pressure work term in the energy equation. The order of magnitude of the viscous dissipation term of the energy equation is equal to $\frac{\varepsilon U_m^2}{C_{p_0} \Delta T Re}$. A numerical check indicated that these terms are completely negligible with air as the fluid for velocities on the order of those at transition.

The disturbance is next taken to be in the form of a wave traveling parallel to the plate surface. It can be considered to be resolved into Fourier components which are examined individually for stability. The customary complex notation is employed, and the disturbance stream function for a given component is taken to be of the form

$$\Psi(\xi, \eta, \tau) = \varphi(\eta)\, e^{i\alpha(\xi - c\tau)} \tag{2-12}$$

α is the wave number of this component of the disturbance wave and is equal to $\frac{2\pi\delta}{\Lambda}$, Λ being the wave length. c is the phase velocity of this component. The disturbance velocities u and v parallel and perpendicular to the plate can be written in terms of the stream function and correction terms, the orders of magnitude of which are found from equation 2-11:

$$u = \frac{d\varphi(\eta)}{d\eta}\, e^{i\alpha(\xi - c\tau)} + O\left\{ \varepsilon^2, U_m^2 \lambda_{1,2}, \frac{\varepsilon}{\sigma_0 Re}, \frac{\varepsilon f}{Re}, \frac{\varepsilon U_m^2}{C_{p_0} \Delta T} \right\} \tag{2-13}$$

$$v = -i\alpha\varphi(\eta)\, e^{i\alpha(\xi - c\tau)} + O\left\{ \varepsilon^2, U_m^2 \lambda_{1,2}, \frac{\varepsilon}{\sigma_0 Re}, \frac{\varepsilon f}{Re}, \frac{\varepsilon U_m^2}{C_{p_0} \Delta T} \right\} \tag{2-14}$$

As is usual in this type of stability analysis, the sign of the imaginary part of the complex constant c determines whether the disturbance will be amplified or damped. When $\mathcal{I}m(c) > 0$, the disturbance is amplified and when $\mathcal{I}m(c) < 0$, it is damped. A real value of c corresponds to a neutral disturbance, i. e., a disturbance that is neither amplified nor damped. The temperature disturbance ϑ is also expressed as a wave-type variable by the equation

$$\vartheta(\xi,\eta,\tau) = S(\eta)\, e^{i\alpha(\xi - c\tau)} \tag{2-15}$$

After substitution of the relations for u, v, and ϑ given by equations 2-13, 2-14, and 2-15 into equations 2-8 and 2-4c, the combined momentum and energy equations are obtained as

$$\begin{aligned}
&\Big\{[1-\varepsilon\bar{\vartheta}][(\bar{u}-c)(\varphi''-\alpha^2\varphi)-\bar{u}''\varphi]-\varepsilon\bar{\vartheta}'[(\bar{u}-c)\varphi'-\bar{u}'\varphi]\Big\}+\frac{\varepsilon}{\alpha F}\Big\{i a_1 S'-\alpha a_2 S\Big\}\\
&+\frac{i}{\alpha Re}\Big\{(1+\bar{\omega})(\varphi^{IV}-2\alpha^2\varphi''+\alpha^4\varphi)+f[-\bar{u}'_{x_1}\varphi'-\bar{v}(\varphi'''-\alpha^2\varphi')]+\frac{d\bar{\omega}}{d\bar{\vartheta}}\Big[2\bar{\vartheta}'(\varphi'''-\alpha^2\varphi')\\
&+\bar{\vartheta}''(\varphi''+\alpha^2\varphi)+\bar{u}'''S+2\bar{u}''S'+\bar{u}'(S''-\alpha^2 S)\Big]+\frac{d^2\bar{\omega}}{d\bar{\vartheta}^2}\Big[(\bar{\vartheta}')^2(\varphi''+\alpha^2\varphi)+(2\bar{\vartheta}'\bar{u}''+\bar{\vartheta}''\bar{u}')S\\
&+2\bar{\vartheta}'\bar{u}'S'\Big]+\frac{d^3\bar{\omega}}{d\bar{\vartheta}^3}\Big[(\bar{\vartheta}')^2\bar{u}'S\Big]\Big\} = O\Big\{\varepsilon^2; U_m^2\lambda_{1,2}, \frac{\varepsilon f}{Re}, \frac{\varepsilon}{\sigma_0 Re}, \frac{f^2}{Re^2}, \frac{\varepsilon U_m^2}{C_{p_0}\Delta T}\Big\}
\end{aligned} \tag{2-16}$$

and

$$\begin{aligned}
&\Big\{[1+\bar{q}][(\bar{u}-c)S-\bar{\vartheta}'\varphi]\Big\}+\frac{i}{\alpha Re}\Big\{\frac{1}{\sigma_0}\Big[(1+\bar{k})(S''-\alpha^2 S)+\frac{d\bar{k}}{d\bar{\vartheta}}(\bar{\vartheta}''S+2\bar{\vartheta}'S')\Big]\\
&+f\Big[-(1+\bar{q})(\bar{\vartheta}_{x_1}\varphi'+\bar{v}S')-\frac{d\bar{q}}{d\bar{\vartheta}}(\bar{u}\,\bar{\vartheta}_{x_1}+\bar{v}\,\bar{\vartheta}')S\Big]\Big\}\\
&= O\Big\{\varepsilon^2; U_m^2\lambda_{1,2}, \frac{U_m^2}{C_{p_0}\Delta T}, \frac{\varepsilon f}{Re}, \frac{\varepsilon}{\sigma_0 Re}, \frac{f}{\sigma_0 Re^2}\Big\}
\end{aligned} \tag{2-17}$$

Primes on the fluctuation quantities φ and S in the preceding equations denote total differentiation with respect to η; on the steady-state dimensionless velocity $\bar{u}$ and temperature $\bar{\vartheta}$ they signify partial differentiation with respect to η. Also, the subscript x_1 appended to a steady-state term indicates partial differentiation with respect to x_1.

For fluids having dynamic viscosities which are sensitive functions of temperature, $\bar{\omega}$ and its derivatives with respect to $\bar{\theta}$ can be quite appreciable at even moderate values of ΔT, while $\bar{q}$, $\bar{\kappa}$, and their derivatives will almost always be very small unless ΔT is extremely large. With neglect of $\bar{q}$, $\frac{\bar{\kappa}}{Re}$, and their derivatives, as well as the terms within the order-of-magnitude brackets, equations 2-16 and 2-17 become

$$\begin{aligned}&\left\{[1-\varepsilon\bar{\theta}]\left[(\bar{u}-c)(\varphi''-\alpha^2\varphi)-\bar{u}''\varphi\right]-\varepsilon\bar{\theta}'\left[(\bar{u}-c)\varphi'-\bar{u}'\varphi\right]\right\}+\frac{\varepsilon}{\alpha F}\left\{i a_1 s'-\alpha a_2 s\right\}\\&+\frac{i}{\alpha Re}\Big\{(1+\bar{\omega})(\varphi^{IV}-2\alpha^2\varphi''+\alpha^4\varphi)+f\left[-\bar{u}'_{x_1}\varphi'-\bar{v}(\varphi'''-\alpha^2\varphi')\right]+\frac{d\bar{\omega}}{d\bar{\theta}}\Big[2\bar{\theta}'(\varphi'''-\alpha^2\varphi')\\&+\bar{\theta}''(\varphi''+\alpha^2\varphi)+\bar{u}'''s+2\bar{u}''s'+\bar{u}'(s''-\alpha^2 s)\Big]+\frac{d^2\bar{\omega}}{d\bar{\theta}^2}\Big[(\bar{\theta}')^2(\varphi''+\alpha^2\varphi)\\&+(2\bar{\theta}'\bar{u}''+\bar{\theta}''\bar{u}')s+2\bar{\theta}'\bar{u}'s'\Big]+\frac{d^3\bar{\omega}}{d\bar{\theta}^3}\left[(\bar{\theta}')^2\bar{u}'s\right]\Big\}=0\end{aligned} \tag{2-16a}$$

and

$$\left\{(\bar{u}-c)s-\bar{\theta}'\varphi\right\}+\frac{i}{\alpha Re}\left\{\frac{1}{\sigma_0}(s''-\alpha^2 s)+f\left(-\bar{\theta}_{x_1}\varphi'-\bar{v}s'\right)\right\}=0. \tag{2-17a}$$

Further simplifications which should not seriously diminish the validity of the two preceding equations can be made. Disregarding $\bar{v}$ and derivatives of steady-state quantities with respect to x_1 should not change the most essential characteristics of the equations, since very successful stability analyses have been made of the Blasius boundary layer with the assumption of parallel flow. In addition, if the viscosity is a reasonably insensitive function of temperature, as is the case for gases, $\bar{\omega}$ and its derivatives will be sufficiently small to be dropped from the combined momentum equation 2-16a unless ΔT is extremely large. A final simplification is concerned with the first bracketed expression of the combined momentum equation, which can be written as

$$\left\{[1-\varepsilon\bar{\theta}][(\bar{u}-c)\varphi'-\bar{u}'\varphi]\right\}'-\alpha^2(1-\varepsilon\bar{\theta})(\bar{u}-c)\varphi.$$

It can be seen that neglecting ε in the above expression will cause only minor changes if $\varepsilon \ll 1$. ε apparently cannot be set equal to zero with such impunity in the second bracketed expression for all values of the Prandtl number σ_0. This is because $\frac{\varepsilon}{F} = \frac{f(\sigma_0)}{Re}$ as mentioned in Section C1 and explained in Appendix 3, and f is very large if σ_0 is either very small or very large.

Simplifying equations 2-16a and 2-17a as discussed gives

$$\left\{(\bar{u}-c)(\varphi''-\alpha^2\varphi)-\bar{u}''\varphi\right\}+\frac{\varepsilon}{\alpha F}\left\{ia_1 s'-\alpha a_2 s\right\}+\frac{i}{\alpha Re}\left\{\varphi^{IV}-2\alpha^2\varphi''+\alpha^4\varphi\right\}=0 \qquad (2\text{-}16b)$$

and

$$\left\{(\bar{u}-c)s-\bar{\theta}'\varphi\right\}+\frac{i}{\sigma_0\alpha Re}\left\{s''-\alpha^2 s\right\}=0. \qquad (2\text{-}17b)$$

which are the equations upon which the material in Section C and Appendix 4 is based.

APPENDIX 3

DERIVATIONS OF EXPRESSIONS FOR $f(\sigma_0)$ AND $\mathfrak{f}(\sigma_0)$

In Appendix 1, the notation table, the following definitions are stated:

$$\delta = \frac{1}{U_m}\int_0^\infty \bar{U}\,dy$$

$$Re = \frac{U_m \delta}{\nu_0}$$

$$F = \frac{U_m^2}{g\delta}$$

$$f = \frac{\delta}{x} Re$$

$$\mathfrak{f} = \frac{\varepsilon}{F} Re$$

$$Gr_x = \frac{|a_1| g \varepsilon x^3}{\nu_0^2}$$

Through appropriate combinations of the first five of these relations one can show that

$$f = \frac{U_m \delta^2}{\nu_0 x}$$

and

$$\mathfrak{f} = \frac{g \varepsilon \delta^3}{\nu_0 U_m}.$$

Reference 14 contains the nondimensional velocity $\dfrac{\bar{U} x}{2\nu_0 (Gr_x)^{\frac{1}{2}}}$ expressed as a function of the similarity variable $\left(\dfrac{Gr_x}{4}\right)^{\frac{1}{4}} \dfrac{y}{x}$ for seven values of the Prandtl number ranging from 0.01 to 1000. Also presented there is the integral $\displaystyle\int_0^\infty \left\{\frac{\bar{U} x}{2\nu_0 (Gr_x)^{\frac{1}{2}}}\right\} d\left\{\left(\frac{Gr_x}{4}\right)^{\frac{1}{4}} \frac{y}{x}\right\}$. Because the maximum value of $\dfrac{\bar{U} x}{2\nu_0 (Gr_x)^{\frac{1}{2}}}$ at a given value of the Prandtl number is $\dfrac{U_m x}{2\nu_0 (Gr_x)^{\frac{1}{2}}}$, one can write

$$\delta = \frac{x\displaystyle\int_0^\infty \left\{\frac{\bar{U} x}{2\nu_0 (Gr_x)^{\frac{1}{2}}}\right\} d\left\{\left(\frac{Gr_x}{4}\right)^{\frac{1}{4}} \frac{y}{x}\right\}}{\left\{\dfrac{Gr_x}{4}\right\}^{\frac{1}{4}} \left\{\dfrac{U_m x}{2\nu_0 (Gr_x)^{\frac{1}{2}}}\right\}}.$$

With the use of the preceding equations for U_m, δ, and Gr_x, one can obtain the two final relations

$$f = \frac{4\left[\int_0^\infty \left\{ \frac{\overline{U}x}{2\nu_0 (Gr_x)^{\frac{1}{2}}} \right\} d\left\{ \left(\frac{Gr_x}{4}\right)^{\frac{1}{4}} \frac{y}{x} \right\} \right]^2}{\frac{U_m x}{2\nu_0 (Gr_x)^{\frac{1}{2}}}}$$

and

$$f = \frac{\left[\int_0^\infty \left\{ \frac{\overline{U}x}{2\nu_0 (Gr_x)^{\frac{1}{2}}} \right\} d\left\{ \left(\frac{Gr_x}{4}\right)^{\frac{1}{4}} \frac{y}{x} \right\} \right]^2}{|a_1| \left\{ \frac{U_m x}{2\nu_0 (Gr_x)^{\frac{1}{2}}} \right\}^3} .$$

APPENDIX 4

DEVELOPMENT OF SIMPLIFIED METHODS FOR SOLVING APPROXIMATELY THE FREE CONVECTION STABILITY PROBLEM

In Section C a general procedure was outlined for classifying combinations of the wave number α of a disturbance and the Reynolds number Re of the mean flow as stable or unstable. Six variations of this general method will be developed in this appendix. All of these methods will be derived for the case of neutral oscillations only, that is, in each case except the simplest one the object will be to find the relation between α and Re which defines in the α-Re plane a neutral stability or indifference curve that divides this plane into stable and unstable pairs of values of α and Re. These variations of the general method differ from each other according to the simplifying assumptions made regarding the effects of different flow and fluid properties on the problem. In all of these six methods it is assumed that the Reynolds number is large, and in all of the approximate solutions which are developed for the differential equations this assumption is basic.

Because of their extreme complexity, no application of four of these six methods to any given flow was attempted. They are, however, available for the use of a future investigator having extensive high-speed computing facilities at hand. The remaining two methods differ from each other in that for one, viscous forces are taken into account, while for the other they are neglected. The method considering viscosity was applied to the case of the free convection of air, and this application is described in Section D. In a sense this method includes the nonviscous one also because the two become equivalent when the Reynolds number becomes infinite.

Before the individual methods of solution are presented, the temperature variable s and its derivatives will be eliminated between the combined momentum equation

$$(\bar{u}-c)(\varphi''-\alpha^2\varphi)-\bar{u}''\varphi+\frac{f}{\alpha Re}\left\{ia_1 s'-\alpha a_2 s\right\}+\frac{i}{\alpha Re}\left\{\varphi^{IV}-2\alpha^2\varphi''+\alpha^4\varphi\right\}=0 \tag{1a}$$

and the energy equation

$$(\bar{u}-c)s-\bar{\theta}'\varphi+\frac{i}{\sigma_0\alpha Re}\left\{s''-\alpha^2 s\right\}=0. \tag{2}$$

This is done by appropriate differentiation and substitution as described in Part 4. 3. 1 of this appendix to give

$$\begin{aligned}
&\left[\frac{ia_1^2}{\sigma_0\alpha Re}(\bar{u}-c)+\frac{\alpha^2}{(\sigma_0\alpha Re)^2}\right]\left[(\bar{u}-c)(\varphi''-\alpha^2\varphi)-\bar{u}''\varphi+\frac{i}{\alpha Re}\left\{\varphi^{IV}-2\alpha^2\varphi''+\alpha^4\varphi\right\}\right]''\\
&+\left[\frac{-ia_1^2\bar{u}'}{\sigma_0\alpha Re}\right]\left[(\bar{u}-c)(\varphi''-\alpha^2\varphi)-\bar{u}''\varphi+\frac{i}{\alpha Re}\left\{\varphi^{IV}-2\alpha^2\varphi''+\alpha^4\varphi\right\}\right]'+\Big[a_1^2(\bar{u}-c)^2\\
&-\frac{1}{\sigma_0\alpha Re}\left\{\alpha a_1 a_2\bar{u}'+i\alpha^2(1+a_1^2)(\bar{u}-c)\right\}-\frac{\alpha^4}{(\sigma_0\alpha Re)^2}\Big]\Big[(\bar{u}-c)(\varphi''-\alpha^2\varphi)-\bar{u}''\varphi\\
&+\frac{i}{\alpha Re}\left\{\varphi^{IV}-2\alpha^2\varphi''+\alpha^4\varphi\right\}\Big]+\left[\frac{f}{\alpha Re}\right]\Big[\Big\{-ia_1^3\bar{u}'-\alpha a_2\Big[a_1^2(\bar{u}-c)-\frac{i\alpha^2}{\sigma_0\alpha Re}\Big]\Big\}\left\{\bar{\theta}'\varphi\right\}\\
&+ia_1\Big\{a_1^2(\bar{u}-c)-\frac{i\alpha^2}{\sigma_0\alpha Re}\Big\}\left\{\bar{\theta}''\varphi+\bar{\theta}'\varphi'\right\}\Big]=0.
\end{aligned} \tag{4-1}$$

If the plate is vertical, that is, if $a_1=\pm 1$ and $a_2=0$ in equation 1a, the elimination of s and its derivatives between equations 1a and 2 gives the simpler relation

$$\begin{aligned}
&\left[\frac{i}{\sigma_0\alpha Re}(\bar{u}-c)+\frac{\alpha^2}{(\sigma_0\alpha Re)^2}\right]\left[(\bar{u}-c)(\varphi''-\alpha^2\varphi)-\bar{u}''\varphi+\frac{i}{\alpha Re}\left\{\varphi^{IV}-2\alpha^2\varphi''+\alpha^4\varphi\right\}\right]''\\
&+\left[\frac{-i\bar{u}'}{\sigma_0\alpha Re}\right]\left[(\bar{u}-c)(\varphi''-\alpha^2\varphi)-\bar{u}''\varphi+\frac{i}{\alpha Re}\left\{\varphi^{IV}-2\alpha^2\varphi''+\alpha^4\varphi\right\}\right]'
\end{aligned}$$

$$+\left[(\bar{u}-c)\ -\frac{i\alpha^2}{\sigma_0 \alpha Re}\right]^2\left[(\bar{u}-c)(\varphi''-\alpha^2\varphi)-\bar{u}''\varphi\ +\frac{i}{\alpha Re}\left\{\varphi^{IV}-2\alpha^2\varphi''+\alpha^4\varphi\right\}\right]$$

$$+\left[\frac{\pm f}{\alpha Re}\right]\left[-i\bar{u}'\bar{\theta}'\varphi+\left\{i(\bar{u}-c)+\frac{\alpha^2}{\sigma_0\alpha Re}\right\}\left\{\bar{\theta}''\varphi+\bar{\theta}'\varphi'\right\}\right] = 0. \qquad (4-2)$$

The derivation of this equation as well is outlined in Part 4. 3. 1.

Eliminating s and its derivatives rather than φ and its derivatives between equations 1a and 2 was done for two reasons. The first is that the viewpoint was taken that instability in free convection is basically a dynamic process modified by thermal effects. Because of this it was thought best to leave the combined momentum equation affected as little as possible by the elimination of one of the two variables. Eliminating φ rather than s would have made the combined momentum equation completely unrecognizable in the resulting equations corresponding to 4-1 and 4-2; but by eliminating s, the inertial and viscous terms of the combined momentum equation can be clearly identified. The second reason is that methods of solution previously developed for the Orr-Sommerfeld stability equation, which involves only φ and describes the stability of a viscous, incompressible flow, can be adapted to the present case when φ instead of s is the dependent variable.

4. 1 Methods of Approximate Solution of the Free Convection Stability Problem with Consideration of Only the Combined Momentum Equation

One can see in Figure 2 that $f(\sigma_0)$ has its minimum value when σ_0 lies between 0. 5 and 1. 0. Also $|a_1| f(\sigma_0)$ is less than 20 for $0.2<\sigma_0<2.0$, a_1 being the cosine of the angle between the body force vector and the plate surface. One might consider simplifying the boundary-value problem involving equations 1a and 2 by neglecting the term $\frac{f}{\alpha Re}\{i a_1 s' - \alpha a_2 s\}$

in equation 1a if αRe is much larger than f. This simplification is equivalent to assuming that only inertial, pressure, and viscous forces are important in the combined momentum equation because neglecting $\frac{f}{\alpha^3 Re}\{ia_1 s' - \alpha a_2 s\}$ means that the forces resulting from interaction of the body-force field with the density perturbations are ignored. The combined momentum equation is reduced to the Orr-Sommerfeld equation, and the energy equation is not considered. One may well ask why the term $\frac{i}{\alpha Re}\{\varphi^{IV} - 2\alpha^2\varphi'' + \alpha^4\varphi\}$ of the combined momentum equation is retained when $\frac{f}{\alpha^3 Re}\{ia_1 s' - \alpha a_2 s\}$ is neglected, especially in view of the fact that $\frac{f}{\alpha^3 Re} > \frac{1}{\alpha Re}$. The reason for doing so is that retaining the highest derivative of φ, namely φ^{IV}, is taken to be more important than retaining the terms representing forces due to temperature perturbations. With the coupling term neglected, solution of the stability problem requires solving the Orr-Sommerfeld equation,

$$(\bar{u}-c)(\varphi''-\alpha^2\varphi) - \bar{u}''\varphi + \frac{i}{\alpha Re}\{\varphi^{IV} - 2\alpha^2\varphi'' + \alpha^4\varphi\} = 0, \tag{4-3}$$

with appropriate boundary conditions.

4.1.1 The Uncoupled, Nonviscous Case

4.1.1.1 The Differential Equation

If the Reynolds number in the Orr-Sommerfeld equation 4-3 is very large, the equation obtained by letting $\alpha Re = \infty$,

$$(\bar{u}-c)(\varphi''-\alpha^2\varphi) - \bar{u}''\varphi = 0, \tag{4-4}$$

should approximately describe the flow stability. This equation specifically describes the stability of a two-dimensional, nonviscous, incompressible, parallel flow. The only forces considered are inertial and pressure forces; viscous and body forces have been neglected.

4.1.1.2 Boundary Conditions

Because the order of equation 4-4 is two, only two boundary conditions on its solutions can be specified. The appropriate boundary condition at the plate surface for a nonviscous flow is that there can be no flow across the surface; in terms of φ this is

$$\varphi(0) = 0, \tag{4-5a}$$

since the component of the disturbance velocity normal to the wall, v, is equal to $-i\alpha\varphi(\eta)e^{i\alpha(\xi - c\tau)}$. Far from the plate the disturbance must disappear; this requirement is expressed by

$$\varphi(\infty) = 0. \tag{4-5b}$$

4.1.1.3 Solutions of the Differential Equation

Two linearly independent solutions of equation 4-4 can be written as

$$\left.\begin{aligned} \varphi_1 &= (\bar{u}-c)\sum_{m=0}^{\infty}\alpha^{2m}\mathcal{L}_m(\eta) \\ \text{and}\quad \varphi_2 &= (\bar{u}-c)\sum_{m=0}^{\infty}\alpha^{2m}\mathcal{J}_m(\eta), \end{aligned}\right\} \tag{4-6}$$

in which

$$\left.\begin{aligned} \mathcal{L}_0 &= 1, \\ \mathcal{J}_0 &= \int_b^{\eta} d\Omega(\bar{u}-c)^{-2}, \\ \mathcal{L}_{m \geq 1} &= \int_b^{\eta} d\Omega_1(\bar{u}-c)^{-2}\int_b^{\Omega_1} d\Omega_2(\bar{u}-c)^2\mathcal{L}_{m-1}, \end{aligned}\right\} \tag{4-7}$$

and

$$\mathcal{J}_{m \geq 1} = \int_b^{\eta} d\Omega_1(\bar{u}-c)^{-2}\int_b^{\eta} d\Omega_2(\bar{u}-c)^2\mathcal{J}_{m-1}.$$

according to Heisenberg[6]. The correct path of integration in the neighborhood of a critical point η_{c_j}, which is a value of η where $\bar{u} = c$, as indicated in Figure 3, is determined by the requirements

4-26 and 4-29 of Part 4.1.2.3. These requirements 4-26 and 4-29 are developed in the treatment of the uncoupled, viscous case. The definitions 4-7 of the λ_m's and ϑ_m's are obtained by substituting the solutions 4-6 into the original equation 4-4 and equating the coefficients of successive powers of α^2 separately to 0. The lower limit of integration is chosen as b, the edge of the boundary layer beyond which $\bar{u}$ and $\bar{\theta}$ are 0, for the purpose of simplifying the application of the infinite boundary conditions.

4.1.1.4 The Boundary-Condition Equation

In order to apply the boundary condition at $\eta = \infty$, the form of equation 4-4 outside the boundary layer, or for $\eta > b$, will be examined. Because $\bar{u} = 0$ outside the boundary layer, the equation becomes

$$\varphi'' - \alpha^2 \varphi = 0, \quad \eta > b. \tag{4-8}$$

Of the linearly independent solutions $e^{\alpha(\eta - b)}$ and $e^{-\alpha(\eta - b)}$ of this equation, only the second can be retained because of the boundary condition at $\eta = \infty$. Therefore, a solution $\varphi = C_1\varphi_1 + C_2\varphi_2$ valid both inside and outside the boundary layer must reduce to a constant times $e^{-\alpha(\eta - b)}$ when $\eta > b$. By reference to equations 4-6 and 4-7 one can determine that

$$\varphi_1 = -c \sum_{m=0}^{\infty} \frac{\alpha^{2m}(\eta - b)^{2m}}{(2m)!}, \quad \eta \geq b;$$

and

$$\varphi_2 = \frac{-1}{c} \sum_{m=0}^{\infty} \frac{\alpha^{2m}(\eta - b)^{2m+1}}{(2m+1)!}, \quad \eta \geq b;$$

so that

$$\{\varphi_1 - \alpha c^2 \varphi_2\}\Big|_{\eta \geq b} = -c \sum_{m=0}^{\infty} \frac{(-)^m \alpha^m (\eta - b)^m}{m!}$$

$$= -c\, e^{-\alpha(\eta - b)}.$$

One must thus choose $C_2 = -\alpha c^2$ if C_1 is taken to be unity for simplicity. The problem now reduces to the task of solving the equation

$$\phi_1(0) - \alpha c^2 \phi_2(0) = 0, \tag{4-9}$$

which represents the application of the boundary condition at the wall.

Rayleigh(23) deduced a classical theorem relating to neutral oscillations for an inviscid, constant-density flow. It states that the disturbance phase velocity c cannot be greater than the maximum value of $\bar{u}$ or less than its minimum value. In addition, it is possible to show that for a free convection profile, $0 < c < \bar{u}_f$, $\bar{u}_f$ being the value of $\bar{u}$ for which $\bar{u}'' = 0$, if $c \neq 0$ *. The derivation of this fact is given in Part 4. 3. 2 of this appendix.

In order to find what, if any, combinations of values of c and α correspond to neutral oscillations for this case in which $\alpha Re = \infty$, one should in general attempt to solve equation 4-9 by first choosing a value of c and then computing enough of the $\mathcal{Z}_m$'s and $\mathcal{Y}_m$'s evaluated at $\eta = 0$ to insure that $\phi_1(0)$ and $\phi_2(0)$ will be approximated well. Then values of α should be chosen to determine $\phi_1(0)$ and $\phi_2(0)$ in attempting to solve the equation. One can see almost by inspection that equation 4-9 is solved when both c and α disappear. When both these parameters are 0, the original disturbance equation 4-4 simplifies to

$$\bar{u}\phi'' - \alpha^2 \phi = 0. \tag{4-10}$$

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* Subsequent to deducing this, the author found that Stuart(24) had recently derived a very similar requirement on the value of c for a class of velocity profiles of which the typical free convection profile is a member.

In this case

$$\varphi_1 = \bar{u}$$

and

$$\varphi_2 = \bar{u}\int_{b}^{\eta} d\Omega\,(\bar{u})^{-2}$$

so that equation 4-9 becomes

$$\bar{u}(0) = 0,$$

a requirement satisfied by the velocity profile.

Of interest is the fact that solving equation 4-9 is a means of investigating stability in cases of a finite Reynolds number Re when $Re \rightarrow \infty$. Since the reciprocal of the Reynolds number is a factor in the combined momentum equation 1a of both the viscous term and the term which couples the equation to the energy equation, any of the cases described later must reduce essentially to equation 4-4 for $Re \rightarrow \infty$. That is, no matter what initial assumptions are made regarding the sizes of the Reynolds and Prandtl numbers and whether or not the combined momentum and energy equations are coupled, equation 4-4 must describe the stability when $Re \rightarrow \infty$.

4.1.2 The Uncoupled, Viscous Case

4.1.2.1 The Differential Equation

The equation which describes the stability of a viscous, incompressible flow is the complete Orr-Sommerfeld equation

$$(\bar{u}-c)(\varphi''-\alpha^2\varphi) - \bar{u}''\varphi + \frac{i}{\alpha Re}\left\{\varphi^{IV} - 2\alpha^2\varphi'' + \alpha^4\varphi\right\} = 0. \qquad (4-3)$$

In this equation viscous, pressure, and inertial forces are considered, while not taking into account the coupling with the energy equation is the same as neglecting the effects of body forces.

4.1.2.2 Boundary Conditions

The boundary conditions at the wall for this viscous case are that there can be flow neither across nor parallel to the surface; in terms of φ and φ' these are

$$\left.\begin{aligned} &\varphi(0)=0 \\ \text{and}\qquad &\varphi'(0)=0. \end{aligned}\right\} \quad (4\text{-}11a)$$

If both the y and x-components of the velocity disturbance disappear far from the plate, the other two boundary conditions must be

$$\left.\begin{aligned} &\varphi(\infty)=0 \\ \text{and}\qquad &\varphi'(\infty)=0. \end{aligned}\right\} \quad (4\text{-}11b)$$

The no-slip condition of viscous flow is the reason for the presence of the boundary condition $\varphi'(0)=0$, which is not present in the non-viscous problem. That the additional boundary condition $\varphi'(\infty)=0$ is really superfluous in this viscous case will be apparent when it is shown how the solutions of equation 4-3 behave as $\eta\to\infty$. That is, it will be shown that the condition $\varphi(\infty)=0$ and the boundary conditions 4-11a are sufficient to determine the problem. One might suspect that since the nonviscous problem does not have the boundary condition $\varphi'(\infty)=0$ the component of velocity parallel to the wall does not die away as $\eta\to\infty$ in the nonviscous case. Such is not the situation; the behavior of the solutions for both the viscous and nonviscous equations is such that the condition $\varphi'(\infty)=0$ is implied by the condition $\varphi(\infty)=0$ *.

* Such an implication is not necessarily true for all functions of η. An example of a function $\mathcal{H}(\eta)$ satisfying the requirement $\mathcal{H}(\infty)=0$ but not $\mathcal{H}'(\infty)=0$ is $\mathcal{H}(\eta) = \frac{1}{\eta}\sin(\eta^3)$.

4.1.2.3 Solutions of the Differential Equation

In accordance with the customary method of treating the Orr-Sommerfeld equation, two solutions of the equation are taken to be the nonviscous solutions φ_1 and φ_2 defined by equations 4-6 and 4-7. These are useful only when the Reynolds number is sufficiently large so that the nonviscous equation is a good approximation to the complete equation. In addition, they fail to be good approximations when η is close to the critical points η_{c_j} at which $\bar{u} = c$. Near the critical points the viscous part of equation 4-3, $\frac{i}{\alpha Re}\{\varphi^{IV} - 2\alpha^2\varphi'' + \alpha^4\varphi\}$, is of the same order as the nonviscous part, $(\bar{u} - c)(\varphi'' - \alpha^2\varphi) - \bar{u}''\varphi$; and the viscous part cannot be neglected with impunity.

In order that four homogeneous boundary conditions imposed on the solution of a fourth-order linear differential equation be satisfied, it is in general necessary that the complete solution be written as a linear, homogeneous combination of four linearly independent solutions. With two linearly independent approximate solutions φ_1 and φ_2 available, it is necessary to find two additional solutions. The procedure will be to determine approximations to these additional solutions by two different methods. One of these methods gives the solutions in a form in which their behavior as $\eta \to \infty$ can be studied, but the solutions fail to be valid in the neighborhoods of the critical points. The other method is useful near the critical points but is of no use far from the critical points. In actual calculations the second method is used, but the first must be employed in determining how to satisfy the boundary conditions at $\eta = \infty$. How the solutions obtained by the two methods are related will be shown after both pairs of solutions are developed.

The method which gives approximate solutions valid far from the critical points is due to Heisenberg[6]. One assumes an asymptotic form for φ good when $\alpha Re \gg 1$ as

$$\varphi = \exp\left[\int^{\eta} d\Omega \left\{(\alpha Re)^{\frac{1}{2}} w_0(\Omega) + w_1(\Omega) + (\alpha Re)^{-\frac{1}{2}} w_2(\Omega) + O\left[(\alpha Re)^{-1}\right]\right\}\right] \quad (4\text{-}12)$$

and substitutes this into the original equation 4-3. The procedure then is to equate coefficients of successive powers of $(\alpha Re)^{\frac{1}{2}}$ to zero. Doing this for αRe and $(\alpha Re)^{\frac{1}{2}}$ gives

$$\alpha Re: \qquad (\bar{u}-c)\, w_0^2 + i\, w_0^4 = 0 \quad (4\text{-}13a)$$

and

$$(\alpha Re)^{\frac{1}{2}}: \quad (\bar{u}-c)(w_0' + 2 w_0 w_1) + i(4 w_0^3 w_1 + 6 w_0^2 w_0') = 0. \quad (4\text{-}13b)$$

The solutions of equation 4-13a other than $w_0 = 0$ are

$$w_{0_{3,4}} = \mp\left\{i(\bar{u}-c)\right\}^{\frac{1}{2}}. \quad (4\text{-}14a)$$

Substituting either of these expressions for w_0 into equation 4-13b gives

$$w_{1_{3,4}} = -\frac{5}{4}\frac{\bar{u}'}{(\bar{u}-c)}, \quad (4\text{-}14b)$$

and one can write

$$\varphi_3 = (\bar{u}-c)^{-\frac{5}{4}} \exp\left[-\int_{b}^{\eta} d\Omega \left\{i\alpha Re(\bar{u}-c)\right\}^{\frac{1}{2}} + O\left\{(\alpha Re)^{-\frac{1}{2}}\right\}\right]$$

and

$$\varphi_4 = (\bar{u}-c)^{-\frac{5}{4}} \exp\left[\int_{b}^{\eta} d\Omega \left\{i\alpha Re(\bar{u}-c)\right\}^{\frac{1}{2}} + O\left\{(\alpha Re)^{-\frac{1}{2}}\right\}\right] \quad (4\text{-}15)$$

One will immediately notice that φ_3 and φ_4 have singularities at the critical points where $\bar{u} = c$. Also, the critical points are algebraic branch points of the function $\left\{i\alpha Re(\bar{u}-c)\right\}^{\frac{1}{2}}$.

After the solutions valid near the critical points are developed, they will be used to determine the sectors with centers at the critical points in which the asymptotic approximations φ_3 and φ_4 are valid.

Practically, this determination of the sectors of validity specifies the sign of the argument of $(\bar{u}-c)$ when $(\bar{u}-c)<0$. Since φ_3 and φ_4 are functions of $(\bar{u}-c)$ raised to fractional powers, whether $\arg(\bar{u}-c)$ is taken to be $+\pi$ or $-\pi$ when $(\bar{u}-c)<0$ will affect the values of the two solutions. Because $(\bar{u}-c)$ can be expanded in a Taylor series as $\bar{u}'_{cj}(\eta-\eta_{cj})+\frac{1}{2}\bar{u}''_{cj}(\eta-\eta_{cj})^2+\ldots$ in the neighborhood of a critical point η_{cj}, one can investigate the allowable limits on $\arg(\eta-\eta_{cj})$ in order to determine the sign of $\arg(\bar{u}-c)$ when $(\bar{u}-c)<0$. One essentially confines his attention concerning the change of sign of $(\bar{u}-c)$ to a neighborhood of η_{cj} sufficiently small for $(\bar{u}-c)$ to be approximated well by $\bar{u}'_{cj}(\eta-\eta_{cj})$. For the nonviscous solution φ_2 also, the sign of $\arg(\eta-\eta_{cj})$ when $(\bar{u}-c)<0$ is of importance. This is because logarithmic terms such as $\frac{\bar{u}''_{cj}}{(\bar{u}'_{cj})^3}\log(\eta-\eta_{cj})$ (which occurs in $\mathcal{J}_0(\eta)$) appear when the integrations for the $\mathcal{J}_m(\eta)$'s are performed around the singular points.

For the purpose of developing the viscous solutions valid in the neighborhoods of the critical points, one transforms the independent variable η in a manner designed to simplify the original equation by retaining the term having the highest derivative, $\frac{i}{\alpha Re}\varphi^{IV}$, as well as at least one term which is of order zero in αRe. The equation defining the transformation to the new independent variable, ζ, is

$$(\eta-\eta_{cj}) = \nu\zeta, \tag{4-16}$$

ν being considered to be proportional to a negative power of αRe and hence to be small. $(\bar{u}-c)$ and $\bar{u}''$ are expanded in power series in $(\eta-\eta_{cj})$ or $\nu\zeta$, and $\varphi(\eta)$ is replaced by $\Phi(\zeta)$ to produce

$$\frac{i\nu^{-4}}{\alpha Re}\Phi^{IV} + \nu^{-1}\bar{u}'_{cj}\,\zeta\Phi'' = O\left\{\frac{\nu^{-2}}{\alpha Re}, 1\right\}. \tag{4-17a}$$

In this equation, as in similar equations developed elsewhere, it has been effectively assumed that the velocity profile is a straight line passing through the point $\bar{u} = c$, $\eta = \eta_{cj}$, and having the slope $\bar{u}'_{cj}$. Its solutions therefore best approximate solutions of the original Orr-Sommerfeld equation 4-3 in an interval near η_{cj} in which this linear approximation to the velocity profile is least in error.

In equation 4-17a, $\Phi^{(n)} = \dfrac{d^n\Phi}{d\zeta^n}$ and $\bar{u}'_{cj} = \left.\dfrac{d\bar{u}}{d\eta}\right|_{\eta=\eta_{cj}}$. Now it is specified that $j = 1$ so that $\bar{u}'_{cj} = \bar{u}'_{c_1} > 0$. ν is chosen equal to $(\bar{u}'_{c_1}\alpha Re)^{-\frac{1}{3}}$, and the equation is multiplied by $-\dfrac{i\nu}{\bar{u}'_{c_1}}$ to obtain

$$\Phi^{IV} - i\zeta\Phi'' = O\left\{(\alpha Re)^{-\frac{1}{3}}\right\}. \tag{4-17b}$$

If one assumes that αRe is so large that terms of order $(\alpha Re)^{-\frac{1}{3}}$ can be neglected, the equation becomes

$$\Phi^{IV} - i\zeta\Phi'' = 0, \tag{4-17c}$$

which has the solutions

$$\left.\begin{aligned} \Phi_1 &= 1, \\ \Phi_2 &= \zeta, \\ \Phi_3 &= \int_{\infty}^{\zeta} d\Omega_1 \int_{\infty}^{\Omega_1} d\Omega_2\, \Omega_2^{\frac{1}{2}} H_{\frac{1}{3}}^{(1)}\left\{\tfrac{2}{3}(i\Omega_2)^{\frac{3}{2}}\right\}, \\ \text{and} \quad \Phi_4 &= \int_{-\infty}^{\zeta} d\Omega_1 \int_{-\infty}^{\Omega_1} d\Omega_2\, \Omega_2^{\frac{1}{2}} H_{\frac{1}{3}}^{(2)}\left\{\tfrac{2}{3}(i\Omega_2)^{\frac{3}{2}}\right\}. \end{aligned}\right\} \tag{4-18}$$

These solutions are really valid only in the immediate neighborhood of the critical point, and one must investigate their asymptotic equivalence with other solutions valid for $\eta \rightarrow \infty$. This must be done in order that the proper solution or solutions of the group Φ_1 , Φ_2 , Φ_3, and Φ_4

be retained along with φ_1 and φ_2 in consideration of the infinite boundary conditions.

One can show after Lin[10] that Φ_3 is asymptotic to φ_3 and Φ_4 to φ_4 for αRe large and $(\bar{u}-c)$ approximated by $\bar{u}'_{c_1}(\eta-\eta_{c_1})$. This is done by utilizing the asymptotic expansions given in Reference 25 for the Hankel functions:

$$H_j^{(1)}(z) \sim \left(\frac{2}{\pi z}\right)^{\frac{1}{2}}\left\{\exp\left[i\left(z-\frac{j\pi}{2}-\frac{\pi}{4}\right)\right]\right\}\left\{1+\sum_{m=1}^{\infty}\frac{(-)^m(j,m)}{(i2z)^m}\right\} \tag{4-19}$$

with

$$-\pi < \arg z < 2\pi$$

$$H_j^{(2)}(z) \sim \left(\frac{2}{\pi z}\right)^{\frac{1}{2}}\left\{\exp\left[-i\left(z-\frac{j\pi}{2}-\frac{\pi}{4}\right)\right]\right\}\left\{1+\sum_{m=1}^{\infty}\frac{(j,m)}{(i2z)^m}\right\} \tag{4-20}$$

with

$$-2\pi < \arg z < \pi$$

In these expressions

$$(j,m) = \frac{(4j^2-1^2)(4j^2-3^2)(\ldots)(4j^2-[2m-1]^2)}{2^{2m}\,m!}$$

If the substitution $z=\frac{2}{3}(i\zeta)^{\frac{3}{2}}$ is made in equations 4-19 and 4-20 and the results are multiplied by $\zeta^{\frac{1}{2}}$, Φ_3'' and Φ_4'' can be expressed in asymptotic forms as

$$\Phi_3'' = \zeta^{\frac{1}{2}}H_{\frac{1}{3}}^{(1)}\left\{\frac{2}{3}(i\zeta)^{\frac{3}{2}}\right\} \sim \text{const.}\times\zeta^{-\frac{1}{4}}\left\{\exp\left[\frac{2}{3}e^{i\frac{5\pi}{4}}\zeta^{\frac{3}{2}}\right]\right\}\left\{1+O(\zeta^{-\frac{3}{2}})\right\} \tag{4-21}$$

with

$$\frac{-7\pi}{6} < \arg z < \frac{5\pi}{6}$$

and

$$\Phi_4'' = \zeta^{\frac{1}{2}}H_{\frac{1}{3}}^{(2)}\left\{\frac{2}{3}(i\zeta)^{\frac{3}{2}}\right\} \sim \text{const.}\times\zeta^{-\frac{1}{4}}\left\{\exp\left[\frac{2}{3}e^{i\frac{\pi}{4}}\zeta^{\frac{3}{2}}\right]\right\}\left\{1+O(\zeta^{-\frac{3}{2}})\right\} \tag{4-22}$$

with

$$\frac{-11\pi}{6} < \arg\zeta < \frac{\pi}{6}.$$

After integrating the asymptotic series for Φ_3'' and Φ_4'' appropriately by parts one can write

$$\Phi_3 \sim \text{const.} \times \zeta^{-\frac{5}{4}} \left\{ \exp\left[\tfrac{2}{3} e^{i\frac{5\pi}{4}} \zeta^{\frac{3}{2}}\right]\right\}\left\{1 + O(\zeta^{-\frac{3}{2}})\right\} \tag{4-23a}$$

with

$$-\frac{7\pi}{6} < \arg \zeta < \frac{5\pi}{6}$$

and

$$\Phi_4 \sim \text{const.} \times \zeta^{-\frac{5}{4}} \left\{ \exp\left[\tfrac{2}{3} e^{i\frac{\pi}{4}} \zeta^{\frac{3}{2}}\right]\right\}\left\{1 + O(\zeta^{-\frac{3}{2}})\right\} \tag{4-24a}$$

with

$$-\frac{11\pi}{6} < \arg \zeta < \frac{\pi}{6}.$$

With the substitution $\zeta = (\eta - \eta_{c_1})(\bar{u}'_{c_1} \alpha Re)^{\frac{1}{3}}$, these relations become

$$\Phi_3 \sim \text{const.} \times (\eta - \eta_{c_1})^{-\frac{5}{4}} \left\{ \exp\left[-\tfrac{2}{3}(i\bar{u}'_{c_1}\alpha Re)^{\frac{1}{2}}(\eta-\eta_{c_1})^{\frac{3}{2}}\right]\right\}\left\{1 + O\left[(\alpha Re)^{-\frac{1}{2}}\right]\right\} \tag{4-23b}$$

with

$$-\frac{7\pi}{6} < \arg(\eta - \eta_{c_1}) < \frac{5\pi}{6}$$

and

$$\Phi_4 \sim \text{const.} \times (\eta - \eta_{c_1})^{-\frac{5}{4}} \left\{ \exp\left[\tfrac{2}{3}(i\bar{u}'_{c_1}\alpha Re)^{\frac{1}{2}}(\eta-\eta_{c_1})^{\frac{3}{2}}\right]\right\}\left\{1 + O\left[(\alpha Re)^{-\frac{1}{2}}\right]\right\} \tag{4-24b}$$

with

$$-\frac{11\pi}{6} < \arg(\eta - \eta_{c_1}) < \frac{\pi}{6}.$$

If in the definitions 4-15 of the exponential asymptotic viscous solutions the substitution $(\bar{u} - c) = \bar{u}'_{c_1}(\eta - \eta_{c_1})$ is made and the indicated integrations are performed, the results are

$$\varphi_3 = \text{const.} \times (\eta - \eta_{c_1})^{-\frac{5}{4}} \left\{ \exp\left[-\tfrac{2}{3}(i\bar{u}'_{c_1}\alpha Re)^{\frac{1}{2}}(\eta-\eta_{c_1})^{\frac{3}{2}} + O\left\{(\alpha Re)^{-\frac{1}{2}}\right\}\right]\right\} \tag{4-25a}$$

and

$$\varphi_4 = \text{const.} \times (\eta - \eta_{c_1})^{-\frac{5}{4}} \left\{ \exp\left[\tfrac{2}{3}(i\bar{u}'_{c_1}\alpha Re)^{\frac{1}{2}}(\eta-\eta_{c_1})^{\frac{3}{2}} + O\left\{(\alpha Re)^{-\frac{1}{2}}\right\}\right]\right\}. \tag{4-25b}$$

Comparing equations 4-23b and 4-24b with equations 4-25a, b shows that Φ_3 is asymptotic to $\text{const.} \times \varphi_3$ and Φ_4 to $\text{const.} \times \varphi_4$, within the interval about η_{c_1} in which $(\bar{u}-c)$ is approximated well by $\bar{u}'_{c_1}(\eta-\eta_{c_1})$.

It is assumed that the same limits on $\arg(\eta-\eta_{c_1})$ must hold in the neighborhood of η_{c_1} for the asymptotic solutions φ_1, φ_2, φ_3, and φ_4 as for the asymptotic representations 4-23b and 4-24b of Φ_3 and Φ_4. These limits are

$$-\frac{7\pi}{6} < \arg(\eta-\eta_{c_1}) < \frac{\pi}{6}, \tag{4-26}$$

which are obtained by applying simultaneously the limits which must be observed for $\arg(\eta-\eta_{c_1})$ in the asymptotic expansions 4-23b and 4-24b for Φ_3 and Φ_4. This method of obtaining the limits on $\arg(\eta-\eta_{c_1})$ for the asymptotic solutions was first used by Heisenberg[6]. Wasow[26] much later obtained the same result by means of a considerably more abstract approach.

The situation in the vicinity of the outer critical point η_{c_2} where $\bar{u}'_{c_2} < 0$ is slightly more complicated. If one simply uses the definition 4-16 to derive equation 4-17c with solutions 4-18, the sector of the complex η-plane in which the asymptotic solutions are invalid rotates through an angle of $\frac{\pi}{3}$ to include a part of the real axis. This occurs because $\nu = e^{-i\frac{\pi}{3}}|\nu|$ when $\bar{u}'_c < 0$. Consequently, a new independent variable ζ_* is prescribed to replace ζ. The defining equation for it is

$$(\eta-\eta_{c_2}) = -(|\bar{u}'_{c_2}|\alpha Re)^{-\frac{1}{3}}\zeta_*. \tag{4-27}$$

In this case the equation corresponding to equation 4-17c is

$$\Phi^{IV} - i\zeta_*\Phi'' = 0, \tag{4-28}$$

in which $\Phi^{(n)} = \dfrac{d^n\Phi(\zeta_*)}{d\zeta_*^n}$. The solutions of this equation are equivalent to the solutions 4-18 of equation 4-17c with ζ replaced by ζ_*. The limits on $\arg \zeta_*$ for the asymptotic forms of the solutions corresponding to Φ_3 and Φ_4 are the same as those on $\arg \zeta$ for the earlier case. Because $(\eta-\eta_{c_2})$ and ζ_* are of opposite sign while $(\eta-\eta_{c_1})$ and ζ have the same sign, the restriction on $\arg(\eta-\eta_{c_2})$ must be

$$-\frac{\pi}{6} < \arg(\eta-\eta_{c_2}) < \frac{7\pi}{6}. \tag{4-29}$$

The sectors in the neighborhoods of η_{c_1} and η_{c_2} in which the asymptotic solutions are invalid are shown in Figure 4.

If the convention is made that $\arg(\bar{u}-c)=0$ when $\bar{u}>c$, the stipulation 4-29 requires that $\arg(\bar{u}-c)=-i\pi$ when $\bar{u}<c$ and $\eta>\eta_{c_2}$. This is found by letting $(\bar{u}-c)=\bar{u}'_{c_2}(\eta-\eta_{c_2})+O\{(\eta-\eta_{c_2})^2\}$ and considering how $\arg(\bar{u}-c)$ changes when η moves in the complex η-plane from a point on the real axis on the left of η_{c_2} to a point on the real axis on the right of η_{c_2}, indentation being performed in the upper half of the plane around η_{c_2} according to the requirement 4-29. With the requirement met that $\arg(\bar{u}-c)=-i\pi$ when $\eta>\eta_{c_2}$, it follows from the definitions of φ_3 and φ_4 in equation 4-15 that $\lim_{\eta\to\infty} |\varphi_3|, |\varphi_3'| = 0$ and $\lim_{\eta\to\infty} |\varphi_4|, |\varphi_4'| = \infty$.

4.1.2.4 The Boundary-Condition Equation

One might suppose that the boundary conditions at the plate could be applied to the sum $C_1(\varphi_1-\alpha c^2\varphi_2)+C_3\varphi_3$ in order to determine an indifference curve in the $\alpha-Re$ plane, since φ_3 and the combination $(\varphi_1-\alpha c^2\varphi_2)$ satisfy the boundary conditions at $\eta=\infty$. However, the singularity of φ_3 at η_{c_1} prevents such a process from giving valid results, so Φ_3 is used in place of φ_3, to which Φ_3 is asymptotically equivalent. The form of φ is now chosen to be

$$\varphi = C_1(\varphi_1 - \alpha c^2 \varphi_2) + C_3 \Phi_3. \tag{4-30}$$

Applying the boundary conditions $\varphi(0)=0$ and $\varphi'(0)=0$ gives

and

$$\left.\begin{aligned} &C_1\{\varphi_1(0)-\alpha c^2\varphi_2(0)\} + C_3\,\Phi_3(\zeta_0) = 0 \\ &C_1\{\varphi_1'(0)-\alpha c^2\varphi_2'(0)\} + C_3\,(\bar{u}'_{c_1}\alpha Re)^{\frac{1}{3}}\,\Phi_3'(\zeta_0) = 0\,. \end{aligned}\right\} \tag{4-31}$$

Here $\zeta_0 = -\eta_{c_1}(\bar{u}'_{c_1}\alpha Re)^{\frac{1}{3}}$. The condition for the existence of values of C_1 and C_3 not both zero is that the determinant of the coefficients of C_1 and C_3 be zero. One way in which this requirement can be written is

$$\frac{-\{\varphi_1(0)-\alpha c^2\varphi_2(0)\}}{\eta_{c_1}\{\varphi_1'(0)-\alpha c^2\varphi_2'(0)\}} = \frac{\Phi_3(\zeta_0)}{\zeta_0\,\Phi_3'(\zeta_0)}\,. \tag{4-32}$$

This is the form normally employed in analyzing the stability of boundary layer profiles by means of the Orr-Sommerfeld equation. With the use of this form of the boundary-condition equation, first the right-hand side is calculated as a function of ζ_0 and is plotted in the complex plane. Then a value of c is chosen, and the left-hand side is plotted as a function of α with this value of c. The process is repeated with different values chosen for c. Intersections of the curves for the left-hand and right-hand sides specify combinations of ζ_0, α, and c from which pairs of values of α and Re determining an indifference curve can be found.

The application of this method of attack upon the free convection stability problem to the case in which air is the convecting fluid is described in Section D.

4.2 Methods of Approximate Solution of the Free Convection Stability Problem with Consideration of Both the Combined Momentum Equation and the Energy Equation

4.2.1 The Coupled, Nonviscous, Non-Heat-Conducting Case

4.2.1.1 The Differential Equations

The problem of the stability of laminar free convection at high Reynolds numbers of a fluid having a very high Prandtl number is somewhat simpler than the cases involving smaller values of these parameters. If the Reynolds and Prandtl numbers are set equal to ∞ in all terms of the combined momentum and energy equations 1a and 2 in which they appear, the results are simply

$$(\bar{u}-c)(\varphi''-\alpha^2\varphi)-\bar{u}''\varphi \tag{4-4}$$

and

$$(\bar{u}-c)s-\bar{\theta}'\varphi=0. \tag{4-33}$$

The uncoupled, nonviscous combined momentum equation 4-4 has been discussed in Part 4.1.1. According to the energy equation 4-33, the temperature disturbance function s is specified if φ is known. φ may be considered to be determined by equation 4-4 only; thus s has no effect of its own on the stability. That is, the problem is really the same as the case in which the energy equation is completely neglected and only the nonviscous combined momentum equation is considered to determine the stability of the flow.

In order to include the effect of the body force term in coupling the combined momentum and energy equations, αRe in the coupling term $\frac{f}{\alpha Re}\{ia_1 s'-\alpha a_2 s\}$ of the combined momentum equation must be considered to remain finite. Equation 4-1 is the result of eliminating s and its derivatives between the combined momentum equation 1a and the energy equation 2. If, then, $\sigma_0\alpha Re$ and αRe are set equal to ∞ in equation 4-1, except where αRe appears in the term $\frac{f}{\alpha Re}$, the resulting equation should describe the stability of a nonviscous, non-heat-conducting fluid in free convection about an inclined plate. This resulting equation when divided by a_1^2, which is never zero, is

$$(\bar{u}-c)^2\{(\bar{u}-c)(\varphi''-\alpha^2\varphi)-\bar{u}''\varphi\} + \frac{f}{\alpha^2 Re}\{[-ia_1\bar{u}'-\alpha a_2(\bar{u}-c)][\bar{\theta}'\varphi] + ia_1(\bar{u}-c)(\bar{\theta}''\varphi+\bar{\theta}'\varphi')\} = 0. \tag{4-34}$$

This equation could have been derived by eliminating s and s' between the coupled, nonviscous combined momentum equation

$$(\bar{u}-c)(\varphi''-\alpha^2\varphi)-\bar{u}''\varphi + \frac{f}{\alpha^2 Re}\{ia_1 s'-\alpha a_2 s\} = 0 \tag{4-35}$$

and the non-heat-conducting energy equation

$$(\bar{u}-c)s - \bar{\theta}'\varphi = 0. \tag{4-33}$$

If equation 4-34 is divided by $(\bar{u}-c)^2$ and is rearranged, it can be written as

$$\{(\bar{u}-c)\varphi'-\bar{u}'\varphi\}' + ia_1\frac{f}{\alpha^2 Re}\left\{\frac{\bar{\theta}'\varphi}{(\bar{u}-c)}\right\}' = \alpha a_2\frac{f}{\alpha^2 Re}\left\{\frac{\bar{\theta}'\varphi}{(\bar{u}-c)}\right\} + \alpha^2(\bar{u}-c)\varphi. \tag{4-34a}$$

4.2.1.2 Boundary Conditions

The boundary conditions that must be applied on φ to correspond to no disturbance far from the plate and no flow across the surface of the plate are

$$\left.\begin{aligned} &\varphi(\infty) = 0 \\ \text{and}\quad &\varphi(0) = 0. \end{aligned}\right\} \tag{4-36}$$

Because equation 4-34a is of second order, no boundary conditions on s can be specified in addition to these on φ. However, by reference to the non-heat-conducting energy equation 4-33, one can note that if $c \neq 0$ the boundary conditions 4-36 on φ imply that

$$\left.\begin{aligned} &s(\infty)=0 \\ \text{and} \qquad &s(0)=0, \end{aligned}\right\} \qquad (4\text{-}37)$$

which is to say that the temperature disturbances die away far from the plate and the plate remains absolutely isothermal.

4.2.1.3 Solutions of the Differential Equation in φ

Two linearly independent solutions of equation 4-34a can be developed in a manner similar to that used for finding the solutions of the nonviscous Orr-Sommerfeld equation. One sets

$$\varphi = (\bar{u}-c)\sum_{m=1}^{\infty}\alpha^m \mathcal{F}_m(\eta)$$

and substitutes into equation 4-34a. Equating coefficients of equal powers of α to 0 gives

$$\left\{(\bar{u}-c)^2\mathcal{F}_0' + i a_1 \frac{f}{\alpha^0 R_e}\bar{\theta}'\mathcal{F}_0\right\}' = 0 \qquad (4\text{-}38a)$$

and

$$\left\{(\bar{u}-c)^2\mathcal{F}_m' + i a_1 \frac{f}{\alpha^0 R_e}\bar{\theta}'\mathcal{F}_m\right\}' = a_2\frac{f}{\alpha^0 R_e}\bar{\theta}'\mathcal{F}_{m-1} + (\bar{u}-c)^2\mathcal{F}_{m-2},\ m\geq 1,\ \mathcal{F}_{-1}=0. \qquad (4\text{-}38b)$$

Equation 4-38a has two linearly independent solutions which will be designated by $\mathcal{J}_0(\eta)$ and $\mathcal{W}_0(\eta)$, and which are defined by

$$\mathcal{J}_0(\eta) = \exp\left[-i a_1 \frac{f}{\alpha^0 R_e}\int_b^\eta d\Omega\, \bar{\theta}'(\bar{u}-c)^{-2}\right] \qquad (4\text{-}39a)$$

and

$$\mathcal{W}_0(\eta) = \left\{\exp\left[-i a_1\frac{f}{\alpha^0 R_e}\int_b^\eta d\Omega\,\bar{\theta}'(\bar{u}-c)^{-2}\right]\right\}\int_b^\eta d\Omega_1(\bar{u}-c)^{-2}\exp\left[i a_1\frac{f}{\alpha^0 R_e}\int_b^{\Omega_1} d\Omega_2\,\bar{\theta}'(\bar{u}-c)^{-2}\right]. \qquad (4\text{-}39b)$$

If the symbols $\mathcal{J}_m(\eta)$ and $\mathcal{W}_m(\eta)$ are substituted separately for $\mathcal{F}_m(\eta)$ in equation 4-38b, solving the resulting equations yields

$$\mathscr{S}_m(\eta)=\left\{\exp\left[-ia_1\frac{f}{\alpha^{\mathrm{!}}Re}\int_b^\eta d\Omega\,\bar{\theta}'(\bar{u}-c)^{-2}\right]\right\}\int_b^\eta d\Omega_1(\bar{u}-c)^{-2}\left\{\exp\left[ia_1\frac{f}{\alpha Re}\int_b^{\Omega_1} d\Omega_2\,\bar{\theta}'(\bar{u}-c)^{-2}\right]\right\}$$
$$\times\left\{\int_b^{\Omega_1} d\Omega_2\left[a_2\frac{f}{\alpha Re}\bar{\theta}'\mathscr{S}_{m-1}(\Omega_2)+(\bar{u}-c)^2\mathscr{S}_{m-2}(\Omega_2)\right]\right\},\ m\geq 1,\ \mathscr{S}_{-1}=0, \qquad (4\text{-}40a)$$

and

$$\mathscr{W}_m(\eta)=\left\{\exp\left[-ia_1\frac{f}{\alpha Re}\int_b^\eta d\Omega\,\bar{\theta}'(\bar{u}-c)^{-2}\right]\right\}\int_b^\eta d\Omega_1(\bar{u}-c)^{-2}\left\{\exp\left[ia_1\frac{f}{\alpha Re}\int_b^{\Omega_1} d\Omega_2\,\bar{\theta}'(\bar{u}-c)^{-2}\right]\right\}$$
$$\times\left\{\int_b^{\Omega_1} d\Omega_2\left[a_2\frac{f}{\alpha Re}\bar{\theta}'\mathscr{W}_{m-1}(\Omega_2)+(\bar{u}-c)^2\mathscr{W}_{m-2}(\Omega_2)\right]\right\},\ m\geq 1,\ \mathscr{W}_{-1}=0. \qquad (4\text{-}40b)$$

The restrictions 4-26 on $\arg(\eta-\eta_{c1})$ and 4-29 on $\arg(\eta-\eta_{c2})$ are to be observed in performing the integrations to determine the $\mathscr{S}_m(\eta)$'s and the $\mathscr{W}_m(\eta)$'s . These restrictions are taken to apply in the non-viscous, non-heat-conducting case because, as indicated in the following Parts 4.2.2, 4.2.3, and 4.2.4, they must be observed when solutions of this form are used in the cases in which viscosity and heat conduction are considered.

One can now express the two linearly independent solutions of the disturbance equation 4-34a as

$$\varphi_1=(\bar{u}-c)\sum_{m=0}^{\infty}\alpha^m\mathscr{S}_m(\eta) \qquad (4\text{-}41)$$

and

$$\varphi_2=(\bar{u}-c)\sum_{m=0}^{\infty}\alpha^m\mathscr{W}_m(\eta). \qquad (4\text{-}42)$$

For $\eta>b$, $\bar{\theta}'$ and $\bar{u}=0$; and, according to the definitions 4-39a, b and 4-40a, b of the $\mathscr{S}_m$'s and $\mathscr{W}_m$'s, φ_1 and φ_2 become

$$\varphi_1=-c\sum_{m=0}^{\infty}\frac{\alpha^{2m}(\eta-b)^{2m}}{(2m)!}$$

and

$$\varphi_2=-\frac{1}{c}\sum_{m=0}^{\infty}\frac{\alpha^{2m}(\eta-b)^{2m+1}}{(2m+1)!}.$$

As in the case of the solutions of the nonviscous Orr-Sommerfeld equation of Part 4.1.1, one can combine φ_1 and φ_2 to write

$$\varphi_1 - \alpha c^2 \varphi_2 = -c\, e^{-\alpha(\eta-b)}, \quad \eta > b,$$

$e^{-\alpha(\eta-b)}$ being one of the two solutions $e^{\pm\alpha(\eta-b)}$ of equation 4-34a holding when $\eta > b$. Thus, if φ is set equal to the combination $\varphi_1 - \alpha c^2 \varphi_2$, the boundary condition $\varphi(\infty) = 0$ is satisfied.

4.2.1.4 The Boundary-Condition Equation

The relation defining an indifference curve is

$$\varphi_1(0) - \alpha c^2 \varphi_2(0) = 0, \tag{4-43}$$

which expresses the boundary condition at the wall. For the case of a given Prandtl number, which specifies f, both $\varphi_1(0)$ and $\varphi_2(0)$ depend on c, α, and αRe in quite complicated ways. A possible method of solving equation 4-43 would be to choose values of c and αRe, compute the corresponding values at $\eta = 0$ of an appropriate number of the $\mathcal{S}_m$'s and $\mathcal{W}_m$'s, and then attempt to find a value of α for which the equation is satisfied. At the same value of c one would repeat this process with different choices of values of αRe, since one would expect to find at most only isolated pairs of values of α and αRe for which equation 4-43 is satisfied at a given value of c. There is no reason to expect that a finite value of α can be found for which equation 4-43 is satisfied at a given value of c for an arbitrary choice of αRe. Therefore, in order to define an indifference curve, an excessive amount of calculation that would be prohibitive without the use of an electronic computer is anticipated. There seems to be no method of separating the terms of equation 4-43 into one part dependent on α and c and another part dependent on a single variable such as can be done when the disturbance equation is the complete Orr-Sommerfeld equation.

4.2.1a The Case of the Vertical Plate

Equation 4-43, the boundary-condition equation for this coupled, nonviscous, non-heat-conducting case, was derived on the assumption that the plate is inclined. If it is desired to treat the situation in which the plate is vertical, the simplifications appropriate to considering the fluid to be nonviscous and non-heat-conducting should be applied to equation 4-2 rather than to equation 4-1. Since equation 4-2 is equivalent to equation 4-1 with a_1 set equal to ± 1 and a_2 set equal to 0, the final boundary-value problem for the vertical plate will be the same as equation 4-43 if one sets $a_1 = \pm 1$ and $a_2 = 0$ in the $\mathscr{J}_m(0)$'s and $\mathscr{W}_m(0)$'s upon which $\varphi_1(0)$ and $\varphi_2(0)$ in equation 4-43 depend.

4.2.2 The Coupled, Viscous, Non-Heat-Conducting Case

4.2.2.1 The Differential Equations

If the Prandtl number of a fluid in free convection about an inclined plate is very large and the Reynolds number of the flow is not so large as to make viscous terms negligible, the following forms of the combined momentum and energy equations should be descriptive of the flow stability:

$$(\bar{U}-c)(\varphi''-\alpha^2\varphi)-\bar{U}''\varphi+\frac{f}{\alpha Re}\{ia_1 s'-\alpha a_2 s\}+\frac{i}{\alpha Re}\{\varphi^{IV}-2\alpha^2\varphi''+\alpha^4\varphi\}=0 \qquad (1a)$$

$$(\bar{U}-c)s-\bar{\Theta}'\varphi=0 \qquad (4\text{-}33)$$

The only difference between this pair of equations and the original pair 1a and 2 is that in equation 2 $\sigma_0\alpha Re$ has been set equal to ∞ to produce equation 4-33. If $\sigma_0\alpha Re$ is set equal to ∞ in equation 4-1, which was obtained by eliminating s and its derivatives between equations 1a and 2, the result can be written after division by

$a_1^2(\bar{u}-c)^2$ as

$$(\bar{u}-c)(\varphi''-\alpha^2\varphi)-\bar{u}''\varphi+\frac{1}{\alpha Re}\left\{ia_1\left[\frac{\bar{\theta}'\varphi}{(\bar{u}-c)}\right]'-\alpha a_2\frac{\bar{\theta}'\varphi}{(\bar{u}-c)}\right\}$$
$$+\frac{i}{\alpha Re}\left\{\varphi^{IV}-2\alpha^2\varphi''+\alpha^4\varphi\right\}=0. \qquad (4\text{-}44)$$

Another way of deriving this equation would have been to solve equation 4-33 for s and differentiate in order to obtain expressions for s and s' in terms of known functions, φ and φ'. Substitution would then have been made for s and s' in equation 1a.

4.2.2.2 Boundary Conditions

With viscosity considered, the appropriate boundary conditions to be met by φ and φ' are

$$\left.\begin{aligned}\varphi(0)&=0,\\ \varphi'(0)&=0,\\ \varphi(\infty)&=0,\\ \text{and}\quad \varphi'(\infty)&=0.\end{aligned}\right\} \qquad (4\text{-}45)$$

As discussed in Part 4.1.2.2, these boundary conditions state that at the plate surface there is flow neither across nor parallel to the surface and that the components of disturbance velocity both normal and parallel to the plate vanish far from the plate.

Because equation 4-44 is of fourth order, no boundary conditions on s or s' can be specified if those of equations 4-45 are applied on φ and φ'. What is implied regarding s and s' by the conditions 4-45 on φ and φ' can be found by solving equation 4-33 for s to obtain

$$s=\frac{\bar{\theta}'\varphi}{(\bar{u}-c)} \qquad (4\text{-}46)$$

and differentiating to secure

$$s'=\frac{-\bar{u}'\bar{\theta}'\varphi}{(\bar{u}-c)^2}+\frac{\bar{\theta}''\varphi+\bar{\theta}'\varphi'}{(\bar{u}-c)}. \qquad (4\text{-}47)$$

Applying the boundary conditions 4-45 on φ and φ' in equations 4-46 and 4-47 gives

$$\left.\begin{aligned} &S(0) = 0, \\ &S'(0) = 0, \\ &S(\infty) = 0, \\ \text{and}\quad &S'(\infty) = 0. \end{aligned}\right\} \qquad (4\text{-}48)$$

if $c \neq 0$. Physically, these equations mean that the temperature disturbances die away far from the plate, that the isothermal condition of the plate is not changed by the presence of temperature disturbances in the flow, and that the temperature disturbances do not cause periodic changes in the rate of energy transport from the plate surface.

4.2.2.3 Solutions of the Differential Equation in φ

Two approximate solutions of equation 4-44 are φ_1 and φ_2 which are defined by equations 4-39a through 4-42 of Part 4.2.1.3. φ_1 and φ_2 are the exact solutions of the equation obtained from equation 4-44 by neglecting the term $\frac{i}{\alpha Re}\left\{\varphi^{IV} - 2\alpha^2\varphi'' + \alpha^4\varphi\right\}$, and they best represent solutions of the complete equation when $\alpha Re \gg 1$ and $\bar{u} \neq c$.

Two exponential asymptotic solutions of the form indicated by equation 4-12,

$$\varphi = \exp\left[\int^{\eta} d\Omega \left\{ (\alpha Re)^{\frac{1}{2}} w_0(\Omega) + w_1(\Omega) + (\alpha Re)^{-\frac{1}{2}} w_2(\Omega) + O\left[(\alpha Re)^{-1}\right] \right\} \right], \qquad (4\text{-}12)$$

can be developed for the present equation 4-44. As in the uncoupled, viscous case, these asymptotic solutions are developed for use in considering how the boundary conditions far from the plate are to be satisfied. If one substitutes the solution indicated by equation 4-12 into equation 4-44 and equates the coefficients of αRe and $(\alpha Re)^{\frac{1}{2}}$ separately to 0 with the assumption that f is of order lower than αRe , the resulting expressions for w_0 and w_1 will be those given by equations 4-14a and 4-14b. Although f is very large when $\sigma_0 >> 1$, for simplicity it will be assumed that αRe is so large that f can be considered to be of order lower than unity in αRe. With this simplification, the exponential asymptotic solutions of equation 4-44 are identical with the exponential asymptotic solutions 4-15 of the simpler Orr-Sommerfeld equation 4-3 when only the terms $(\alpha Re)^{\frac{1}{2}} w_0$ and w_1 are retained. As in the previous case, these solutions are unsatisfactory near the critical points, and another method must be used in order to have "viscous" solutions for use in the boundary-condition equation.

Solutions developed by changing the independent variable according to the relation $(\eta - \eta_{c_1}) = (\bar{U}'_{c_1} \alpha Re)^{-\frac{1}{3}} \zeta$ and expanding known functions in Taylor series about $\eta = \eta_{c_1}$ are affected somewhat more seriously than are the asymptotic solutions when $f >> 1$. When one sets $\varphi(\eta) = \Phi(\zeta)$, $(\eta - \eta_{c_1}) = (\bar{U}'_{c_1} \alpha Re)^{-\frac{1}{3}} \zeta$, and expands $(\bar{U} - c)$, $\bar{U}''$, $\bar{\theta}'$, and $\bar{\theta}''$ in Taylor series about $\eta = \eta_{c_1}$ in equation 4-44, the resulting relation is

$$\Phi^{IV} - i\zeta\Phi'' = -a_1 f \bar{\theta}'_{c_1} (\bar{U}'_{c_1})^{-\frac{5}{3}} (\alpha Re)^{-\frac{2}{3}} \left\{\frac{\Phi}{\zeta}\right\}' + O\left\{(\alpha Re)^{-\frac{1}{3}}, \frac{f}{\alpha Re}\right\} . \tag{4-49}$$

If the first term on the right-hand side of the equation is not neglected, the order of the equation must be taken to be four rather than effectively two, as in the corresponding equation 4-17b of the uncoupled, viscous case. The appearance of the regular singular point in equation 4-49 at $\zeta = 0$ is a result of taking $\sigma_0 \alpha Re = \infty$. This can be verified by comparing equation 4-44 with equation 4-1; if $\sigma_0 \alpha Re$ were taken to be finite, the denominators $(\bar{U} - c)$ of the fractions multiplied by $\frac{f}{\alpha Re}$ in equation 4-44 would be replaced by functions which do not disappear when $\bar{u} = c$ on account of the presence of additional terms of orders $\frac{1}{\sigma_0 \alpha Re}$ and $\frac{1}{(\sigma_0 \alpha Re)^2}$.

Although a power-series type of solution could be developed for equation 4-49 with the terms of order $(\alpha Re)^{-\frac{1}{3}}$ and $\frac{f}{\alpha' Re}$ neglected but with the first term on the right-hand side retained, it appears to be more practical to consider the solution of the equation with the entire right-hand side neglected because the solutions of this simpler equation are already known. Neglecting the entire right-hand side of equation 4-49 is allowable if αRe is taken to be so large that $\frac{f}{(\alpha' Re)^{\frac{2}{3}}} << 1$. If the results of calculating an indifference curve were to show that such an assumption were very poor, the next logical step would be to initiate a perturbation procedure in which Φ were assumed to be of the form $\Phi_{(0)} + \frac{f}{(\alpha' Re)^{\frac{2}{3}}} \Phi_{(1)}$, $\Phi_{(0)}$ being a solution of the homogeneous equation resulting from setting the entire right-hand side of equation 4-49 equal to 0, and $\Phi_{(1)}$ being the solution of the inhomogeneous equation obtained by substituting $\Phi_{(0)}$ into the first term of the right-hand side of equation 4-49 and neglecting the other terms on that side of the equation.

With neglect of its entire right-hand side, equation 4-49 reduces to equation 4-17c, which has solutions 4-18. As shown in Part 4.1.2.3, the exponential asymptotic solution φ_3 as well as its first derivative φ_3' dies away exponentially as $\eta \rightarrow \infty$ on account of the limits 4-29 on $\arg(\eta - \eta_{c_2})$. Also shown in Part 4.1.2.3 is that the solution Φ_3 of equation 4-17c is asymptotically equivalent to φ_3 .

4.2.2.4 <u>The Boundary-Condition Equation</u>

Because Φ_3 is asymptotically equivalent to a solution satisfying the infinite boundary conditions and the sum $(\varphi_1 - \alpha c^2 \varphi_2)$ dies away exponentially as $\eta \to \infty$, the combination

$$\varphi = C_1(\varphi_1 - \alpha c^2 \varphi_2) + C_3 \Phi_3 \tag{4-50}$$

is considered to approximate near $\eta = 0$ a solution of equation 4-44 which satisfies the boundary conditions at $\eta = \infty$. In this form of an approximate solution of equation 4-44, the effect of coupling has been included only in the nonviscous solutions φ_1 and φ_2 because $\frac{f}{\alpha^{\prime} Re}$ has effectively been set equal to 0 in arriving at the viscous solution Φ_3. The boundary-condition equation resulting from applying the boundary conditions $\varphi(0) = 0$ and $\varphi'(0) = 0$ at the plate is, in the form corresponding to equation 4-32,

$$\frac{-\{\varphi_1(0) - \alpha c^2 \varphi_2(0)\}}{\eta_{c_1}\{\varphi_1'(0) - \alpha c^2 \varphi_2'(0)\}} = \frac{\Phi_3(\zeta_0)}{\zeta_0 \Phi_3'(\zeta_0)} . \tag{4-51}$$

Here $\zeta_0 = -\eta_{c_1}(\bar{u}'_{c_1} \alpha Re)^{\frac{1}{3}}$ as in equation 4-32. The right-hand side of the present equation is the same function of ζ_0 as is the right-hand side of equation 4-32. Unfortunately, the left-hand side of the present equation is a function of α, c, and αRe rather than of only α and c, as is the left-hand side of equation 4-32. In order to solve the present equation, its right-hand side would first be plotted as a function of ζ_0 in the complex plane. Then a value of c would be chosen and the integrals upon which $\varphi_1(0)$, $\varphi_2(0)$, $\varphi_1'(0)$, and $\varphi_2'(0)$ depend would be calculated for an assumed value of αRe. The left-hand side would be plotted in the complex plane as a function of α, and the intersections, if any, of the curves for the right and left-hand sides would specify values of α and ζ_0. The value of αRe corresponding to the value of ζ_0 at an intersection would be determined for the chosen value of c, and a comparison would be made between it and the value of αRe assumed in calculating the left-hand side of the equation. In the probable case of serious disagreement between the values of αRe for the two sides of the equation, another value of αRe would be chosen for computing the left-hand side of the equation, and the process would be iterated until the values of αRe for both sides of the equation agreed sufficiently well. Thus points defining an indifference curve in the α-Re plane would be determined. Such a process would, however, be extremely laborious and would probably necessitate the use of electronic computing equipment.

4.2.2a <u>The Case of the Vertical Plate</u>

If the plate is vertical rather than inclined, a_1 becomes equal to ± 1 and a_2 goes to 0. The only effect of this is on the solutions φ_1 and φ_2 ; that is, a_1 would be set equal to ± 1 and a_2 to 0 in performing the integrations for the functions $\mathcal{J}_m(\eta)$ and $\mathcal{W}_m(\eta)$ and their derivatives which determine φ_1 and φ_2 according to equations 4-41 and 4-42.

4.2.3 <u>The Coupled, Nonviscous, Heat-Conducting Case</u>

4.2.3.1 <u>The Differential Equations</u>

In the section immediately preceding, a method was outlined for the study of a free convection flow in which the effect of viscosity is appreciable but for which the thermal conductivity is negligible. It is of some interest to consider the contrasting case for which the viscosity is negligible but the thermal conductivity is appreciable. An example of such a flow would be the case of a liquid metal in laminar free convection at a very high Reynolds number. The basic equations 1a and 2 in this case are simplified by neglecting the viscous term $\frac{i}{\alpha Re}\left\{\varphi^{IV}-2\alpha^2\varphi''+\alpha^4\varphi\right\}$ of the combined momentum equation 1a. The coupling term $\frac{f}{\alpha \upsilon Re}\left\{i a_1 s'-\alpha a_2 s\right\}$ in this equation is retained, as is the conduction term $\frac{i}{\sigma_0 \alpha Re}\left\{s''-\alpha^2 s\right\}$ in the energy equation 2. One essentially considers the Reynolds number to be large enough to cause the viscous term of the combined momentum equation to be negligible but not so large that the equations are uncoupled. One takes the Prandtl number to be so small that the product $\sigma_0 \alpha Re$, although large, is much smaller than αRe. Even if the minimum Reynolds number for instability is very high for the free convection of a fluid with a very low Prandtl number, as is assumed, the coupling between the two equations is apt to have an appreciable effect because the parameter f, which is a factor of the coupling term, is very large when the Prandtl number is very small, as shown in Figure 2.

With the viscous term of the combined momentum equation neglected, the pair of equations to be solved simultaneously becomes

$$(\bar{u}-c)(\varphi''-\alpha^2\varphi)-\bar{u}''\varphi+\frac{f}{\alpha \upsilon Re}\left\{i a_1 s'-\alpha a_2 s\right\}=0 \tag{4-35}$$

and

$$(\bar{u}-c)s'-\bar{\theta}'\varphi+\frac{i}{\sigma_0 \alpha Re}\left\{s''-\alpha^2 s\right\}=0. \tag{2}$$

The equation resulting from eliminating s and its derivatives between these two equations is the same as that obtained by setting $\frac{1}{\alpha Re}=0$ in the complete equation 4-1, while retaining $\frac{1}{\sigma_0 \alpha Re}$ and $\frac{f}{\alpha \upsilon Re}$ as finite. This equation is

$$\left\{\frac{i a_1^2}{\sigma_0 \alpha Re}(\bar{u}-c)+\frac{\alpha^2}{(\sigma_0 \alpha Re)^2}\right\}\left\{(\bar{u}-c)(\varphi''-\alpha^2\varphi)-\bar{u}''\varphi\right\}''$$

$$+\left\{\frac{-i a_1^2 \bar{u}'}{\sigma_0 \alpha Re}\right\}\left\{(\bar{u}-c)(\varphi''-\alpha^2\varphi)-\bar{u}''\varphi\right\}'+\left\{a_1^2(\bar{u}-c)^2-\frac{1}{\sigma_0 \alpha Re}\left[\alpha a_1 a_2 \bar{u}'+i\alpha^2(1+a_1^2)(\bar{u}-c)\right]\right.$$

$$-\frac{\alpha^4}{(\sigma_0 \alpha Re)^2}\Big\}\Big\{(\bar{U}-c)(\varphi''-\alpha^2\varphi)-\bar{U}''\varphi\Big\}+\frac{f}{\alpha Re}\Big\{\Big[-i a_1^3 \bar{U}'-\alpha a_2\Big\{a_1^2(\bar{U}-c)$$

$$-\frac{i\alpha^2}{\sigma_0 \alpha Re}\Big\}\Big]\Big[\bar{\theta}'\varphi\Big]+i a_1\Big[a_1^2(\bar{U}-c)-\frac{i\alpha^2}{\sigma_0\alpha Re}\Big]\Big[\bar{\theta}''\varphi+\bar{\theta}'\varphi'\Big]\Big\} = 0. \qquad (4\text{-}52)$$

4.2.3.2 Boundary Conditions

The boundary conditions which are to be applied to the solutions of this equation should be derived from stipulations on both the disturbance motion and the temperature disturbance. Since the flow is considered to be nonviscous, two of the appropriate boundary conditions are that there is no flow across the surface of the plate and that the disturbance motion dies away far from the plate. In terms of φ these are

$$\varphi(0) = 0 \qquad (4\text{-}5a)$$

and

$$\varphi(\infty) = 0. \qquad (4\text{-}5b)$$

That the temperature disturbance should vanish far from the plate is one of the boundary conditions to be imposed upon S. Formally, it is

$$S(\infty) = 0. \qquad (4\text{-}53)$$

At the surface of the plate a boundary condition on the temperature disturbance is more difficult to formulate. An exact boundary condition involving S and its derivatives could supposedly be derived by taking into account the heat flow process within the plate, but the analysis as well as the results would probably be quite complex. In order to avoid this complication, two alternative boundary conditions are proposed. The first of these is that the plate remains absolutely isothermal;

mathematically this is

$$S(0) = 0. \tag{4-54}$$

The other proposed boundary condition is that the plate is adiabatic with respect to the temperature disturbances. Since the process of heat transfer at the plate surface is by conduction alone, this boundary condition implies that the temperature gradient in the fluid is not changed by a disturbance in the flow. Because the dimensionless temperature gradient is equal to $\bar{\theta}'(\eta) + s'(\eta)\, e^{i\alpha(\xi - c\tau)}$, in which $\bar{\theta}'(\eta)$ is the steady-state part, this boundary condition is expressed as

$$S'(0) = 0. \tag{4-55}$$

In Part 4. 3. 3 expressions for S and S' in terms of φ, its derivatives, and known functions of η are developed. Inasmuch as equation 4-52 is a differential equation in φ rather than in S, it is necessary to have the boundary conditions 4-53 and 4-54 or 4-55 expressed in terms of φ and its derivatives. If in equation 4-97 of Part 4. 3. 3 one lets $\eta = \infty$, sets $\frac{1}{\alpha Re} = 0$ while keeping $\frac{f}{\alpha^{?} Re}$ and $\frac{1}{\sigma_0 \alpha Re}$ finite, and applies the boundary condition 4-5b, the boundary condition 4-53 on S becomes, for $c \neq 0$,

$$-a_1\{\varphi''' - \alpha^2\varphi'\}\Big|_{\eta=\infty} + i\alpha a_2\{\varphi''\}\Big|_{\eta=\infty} = 0. \tag{4-53a}$$

Here use has been made of the fact that $\bar{u}$, its derivatives, and $\bar{\theta}'$ all disappear outside the boundary layer. The boundary conditions 4-54 and 4-55 on S and S' at $\eta = 0$ are also expressible in terms of the $\varphi^{(n)}$'s and known functions of η according to equations 4-97 and 4-98 of Part 4. 3. 3. The isothermal boundary condition 4-54 becomes

$$a_1\{\bar{u}'\varphi'' - c(\varphi''' - \alpha^2\varphi') - \bar{u}''\varphi'\}\Big|_{\eta=0} - i\alpha a_2\{-c\varphi''\}\Big|_{\eta=0} = 0, \tag{4-54a}$$

and the adiabatic boundary condition 4-55 becomes

$$\left\{-a_i^2\bar{u}'\left[a_i\{\bar{u}'\varphi''-c(\varphi'''-\alpha^2\varphi')-\bar{u}''\varphi'\}-i\alpha a_2\{-c\varphi''\}\right]\right\}\Big|_{\eta=0}$$
$$+\left\{-a_i^2c-\frac{i\alpha^2}{\sigma_0\alpha Re}\right\}\Big\{\sigma_0 a_i^2 f\,\bar{\theta}'\varphi'+a_1\left[2\bar{u}'(\varphi'''-\alpha^2\varphi')\right.$$
$$\left.-c(\varphi^{IV}-\alpha^2\varphi'')-2\bar{u}'''\varphi'\right]-i\alpha a_2\left[\bar{u}'\varphi''-c(\varphi'''-\alpha^2\varphi')-\bar{u}''\varphi'\right]\Big\}\Big|_{\eta=0}=0. \tag{4-55a}$$

4.2.3.3 Solutions of the Differential Equation in φ

Approximate solutions of equation 4-52 are developed by methods similar to those used in previous cases. Two approximate solutions can be obtained by setting $\frac{1}{\sigma_0\alpha Re}=0$. One might suppose that the terms multiplied by $\frac{f}{\alpha Re}$ should be retained, but Figure 2 indicates that $|a_1|f$ and $\frac{1}{\sigma_0}$ are of the same order of magnitude for $\sigma_0 \ll 1$. Therefore, terms having $\frac{f}{\alpha Re}$ as a factor should be dropped also if terms of order $\frac{1}{\sigma_0\alpha Re}$ are to be neglected. Equation 4-52 divided by $a_1^2(\bar{u}-c)^2$ and with both $\frac{f}{\alpha Re}$ and $\frac{1}{\sigma_0\alpha Re}$ set equal to 0 is

$$(\bar{u}-c)(\varphi''-\alpha^2\varphi)-\bar{u}''\varphi=0, \tag{4-4}$$

which has the solutions φ_1 and φ_2 defined by equations 4-6 and 4-7 of Part 4.1.1.3. As shown in Part 4.1.1.4, the combination $(\varphi_1-\alpha c^2\varphi_2)$ is proportional to $e^{-\alpha(\eta-b)}$ when $\eta > b$.

Obtaining two additional solutions is done in essentially the same manner as in the coupled and uncoupled viscous cases previously discussed. One again investigates two pairs of solutions, one pair of which are good approximations far from the critical points but which have singularities at the critical points, while the other pair are valid only close to the inner critical point.

In the present problem, the appropriate large parameter to be used in the expansion for the exponential asymptotic solutions is $(\sigma_0 \alpha Re)^{\frac{1}{2}}$ rather than $(\alpha Re)^{\frac{1}{2}}$, which was used in the previous cases. Assuming an exponential asymptotic solution of the form 4-12 with αRe replaced by $\sigma_0 \alpha Re$, substituting into the original equation 4-52, and equating the coefficients of $\sigma_0 \alpha Re$ and $(\sigma_0 \alpha Re)^{\frac{1}{2}}$ to 0, considering $\frac{f}{\alpha^\nu Re}$ to be of order $\frac{1}{\sigma_0 \alpha Re}$, give

$$w_{0_{3,4}} = \mp\{i(\bar{u}-c)\}^{\frac{1}{2}} \tag{4-56a}$$

and

$$w_{1_{3,4}} = \frac{-7}{4}\frac{\bar{u}'}{(\bar{u}-c)}. \tag{4-56b}$$

The exponential asymptotic solutions are thus defined by

$$\left.\begin{aligned} \varphi_3 &= (\bar{u}-c)^{\frac{-7}{4}} \exp\left[-\int_b^\eta d\Omega\{i\sigma_0\alpha Re(\bar{u}-c)\}^{\frac{1}{2}} + O\{(\sigma_0\alpha Re)^{-\frac{1}{2}}\}\right] \\ \text{and} \quad \varphi_4 &= (\bar{u}-c)^{\frac{-7}{4}} \exp\left[\int_b^\eta d\Omega\{i\,\sigma_0\alpha Re(\bar{u}-c)\}^{\frac{1}{2}} + O\{(\sigma_0\alpha Re)^{-\frac{1}{2}}\}\right]. \end{aligned}\right\} \tag{4-57}$$

Solutions valid near the inner critical point are developed as previously by using the transformations $(\eta-\eta_{c_1}) = \nu\zeta$ and $\varphi(\eta) = \Phi(\zeta)$, ν being small. For the present case ν is taken to be $(\bar{u}'_{c_1}\sigma_0\alpha Re)^{-\frac{1}{3}}$ rather than $(\bar{u}'_{c_1}\alpha Re)^{-\frac{1}{3}}$ as before. Using these substitutions and expanding the known functions of η in Taylor series about η_{c_1} give

$$\zeta^2\Phi^{IV} + \zeta\Phi''' - (1+i\zeta^3)\Phi'' = O\left\{\frac{\sigma_0^{\frac{1}{3}} f}{(\alpha^\nu Re)^{\frac{2}{3}}}, \frac{1}{(\sigma_0\alpha Re)^{\frac{1}{3}}}, \frac{f}{\alpha^\nu Re}\right\}. \tag{4-58a}$$

Since $f = O\{\frac{1}{\sigma_0}\}$, $\frac{\sigma_0^{\frac{1}{3}} f}{(\alpha^\nu Re)^{\frac{2}{3}}} = O\left\{\frac{1}{(\sigma_0\alpha Re)^{\frac{2}{3}}}\right\}$ and $\frac{f}{\alpha^\nu Re} = O\left\{\frac{1}{\sigma_0\alpha Re}\right\}$, so that the equation may be rewritten as

$$\zeta^2\Phi^{IV} + \zeta\Phi''' - (1+i\zeta^3)\Phi'' = O\left\{\frac{1}{(\sigma_0\alpha Re)^{\frac{1}{3}}}\right\}. \tag{4-58b}$$

Neglecting the right-hand side gives

$$\zeta^2 \Phi^{IV} + \zeta\Phi''' - (1+i\zeta^3)\Phi'' = 0. \tag{4-58c}$$

Four linearly independent solutions of this equation are

$$
\begin{aligned}
{}_1\Phi_1 &= 1, \\
{}_1\Phi_2 &= \zeta, \\
{}_1\Phi_3 &= \int_{\infty}^{\zeta} d\Omega_1 \int_{\infty}^{\Omega_1} d\Omega_2\, H^{(1)}_{\frac{2}{3}}\left\{\tfrac{2}{3}(i\Omega_2)^{\frac{3}{2}}\right\}, \\
\text{and} \quad {}_1\Phi_4 &= \int_{-\infty}^{\zeta} d\Omega_1 \int_{-\infty}^{\Omega_1} d\Omega_2\, H^{(2)}_{\frac{2}{3}}\left\{\tfrac{2}{3}(i\Omega_2)^{\frac{3}{2}}\right\}.
\end{aligned} \tag{4-59}
$$

Asymptotic forms of ${}_1\Phi_3$ and ${}_1\Phi_4$ are used as in previous cases for determining which of these solutions approximates in the neighborhood of the critical point η_{c_1} a solution satisfying the boundary conditions at $\eta = \infty$. If the Hankel functions $H^{(1)}_{\frac{2}{3}}\left\{\frac{2}{3}(i\zeta)^{\frac{3}{2}}\right\}$ and $H^{(2)}_{\frac{2}{3}}\left\{\frac{2}{3}(i\zeta)^{\frac{3}{2}}\right\}$ are expressed as asymptotic series according to equations 4-19 and 4-20 and if these series are each integrated appropriately twice, term-by-term, the results after substituting $\zeta = (\eta-\eta_{c_1})(\bar{u}'_{c_1}\sigma_0\alpha Re)^{\frac{1}{3}}$ are

$${}_1\Phi_3 \sim \text{const.} \times (\eta-\eta_{c_1})^{-\frac{7}{4}}\left\{\exp\left[-\tfrac{2}{3}(i\sigma_0\alpha Re\,\bar{u}'_{c_1})^{\frac{1}{2}}(\eta-\eta_{c_1})^{\frac{3}{2}}\right]\right\}\left\{1 + O\left[(\sigma_0\alpha Re)^{-\frac{1}{2}}\right]\right\} \tag{4-60}$$

with

$$\frac{-7\pi}{6} < \arg(\eta-\eta_{c_1}) < \frac{5\pi}{6}$$

and

$${}_1\Phi_4 \sim \text{const.} \times (\eta-\eta_{c_1})^{-\frac{7}{4}}\left\{\exp\left[\tfrac{2}{3}(i\sigma_0\alpha Re\,\bar{u}'_{c_1})^{\frac{1}{2}}(\eta-\eta_{c_1})^{\frac{3}{2}}\right]\right\}\left\{1 + O\left[(\sigma_0\alpha Re)^{-\frac{1}{2}}\right]\right\} \tag{4-61}$$

with

$$-\frac{11\pi}{6} < \arg(\eta-\eta_{c_1}) < \frac{\pi}{6}.$$

If in the exponential asymptotic solutions $(\bar{u}-c)$ is approximated by $\bar{u}'_{c_1}(\eta-\eta_{c_1})$, the relations 4-57 become

$$_1\varphi_3 = const.\times(\eta-\eta_{c_1})^{-\frac{7}{4}}\exp\left[-\tfrac{2}{3}(i\sigma_0\alpha Re\bar{U}'_{c_1})^{\frac{1}{2}}(\eta-\eta_{c_1})^{\frac{3}{2}}+O\{(\sigma_0\alpha Re)^{-\frac{1}{2}}\}\right] \tag{4-62a}$$

and

$$_1\varphi_4 = const.\times(\eta-\eta_{c_1})^{-\frac{7}{4}}\exp\left[\tfrac{2}{3}(i\sigma_0\alpha Re\bar{U}'_{c_1})^{\frac{1}{2}}(\eta-\eta_{c_1})^{\frac{3}{2}}+O\{(\sigma_0\alpha Re)^{-\frac{1}{2}}\}\right]. \tag{4-62b}$$

Thus, one concludes that $_1\Phi_3 \sim const.\times\, _1\varphi_3$ and $_1\Phi_4 \sim const\times\, _1\varphi_4$. Assuming that all asymptotic solutions must be valid in the same sector in the neighborhood of η_{c_1}, one finds upon the simultaneous application of the limits on $\arg(\eta-\eta_{c_1})$ of the relations 4-60 and 4-61 that

$$\frac{-7\pi}{6} < \arg(\eta-\eta_{c_1}) < \frac{\pi}{6} \tag{4-63}$$

must be true for the asymptotic solutions φ_1, φ_2, $_1\varphi_3$, and $_1\varphi_4$.

To find the sector about the outer critical point η_{c_2} in which the asymptotic solutions are valid, one proceeds in a manner very similar to that of the uncoupled, viscous case of Part 4. 1. 2. In the present case the appropriate transformation of the independent variable is

$$(\eta-\eta_{c_2}) = -(|\bar{U}'_{c_2}|\sigma_0\alpha Re)^{-\frac{1}{3}}\zeta_*. \tag{4-64}$$

Applying this transformation, using $\Phi = \Phi(\zeta_*)$, and expanding the known functions of η in Taylor series about η_{c_2} in equation 4-52 give an equivalent to equation 4-58c if one replaces ζ with ζ_*. The solutions of this equation are the same as those of equation 4-58c, which are denoted by equations 4-59, if ζ is replaced by ζ_*. The limits on $\arg\zeta_*$ in the asymptotic series for $_1\Phi_3$ and $_1\Phi_4$ are the same as those that apply on $\arg\zeta$; but since ζ_* and $(\eta-\eta_{c_2})$ are of opposite sign while ζ and $(\eta-\eta_{c_1})$ have the same sign, the limits on $\arg(\eta-\eta_{c_2})$ differ from those on $\arg(\eta-\eta_{c_1})$ by π. A comparison with the limits 4-63 on $\arg(\eta-\eta_{c_1})$ indicates that

$$-\frac{\pi}{6} < \arg(\eta-\eta_{c_2}) < \frac{7\pi}{6} \tag{4-65}$$

must hold for the asymptotic solutions φ_1, φ_2, ${}_1\varphi_3$, and ${}_1\varphi_4$. It is worth noting that these limits 4-63 and 4-65 on $\arg(\eta-\eta_{c_1})$ and $\arg(\eta-\eta_{c_2})$ for the present coupled, nonviscous, heat-conducting case are the same as the limits 4-26 and 4-29 to be met in the previous cases involving different assumptions regarding the characteristics of the flow and the convecting fluid. That these limits should be the same is not necessarily to be expected because they are determined in the present case by considering the effect of heat conduction and in the previous cases by taking viscosity into account.

As in Part 4.1.2.3, the convention that $\arg(\bar{u}-c)=0$ when $\bar{u}>c$ and the restriction 4-65 on $\arg(\eta-\eta_{c_2})$ requires that $\arg(\bar{u}-c) = -i\pi$ when $\bar{u}<c$ and $\eta>\eta_{c_2}$. This implies that $|{}_1\varphi_3| \to 0$ exponentially and $|{}_1\varphi_4| \to \infty$ exponentially as $\eta \to \infty$.

4.2.3.4 The Boundary-Condition Equations

Because the solution ${}_1\varphi_3$ and the sum $(\varphi_1-\alpha c^2\varphi_2)$ both die away exponentially as $\eta\to\infty$, any linear combination of these two functions satisfies the boundary conditions 4-5b and 4-53a at $\eta=\infty$. On account of the singularities of ${}_1\varphi_3$ at the critical points, it is better to use ${}_1\Phi_3$, which is asymptotic to ${}_1\varphi_3$, in place of ${}_1\varphi_3$ when the boundary conditions 4-5a and 4-54a or 4-55a are applied. One takes

$$\varphi = C_1(\varphi_1-\alpha c^2\varphi_2) + C_3\,{}_1\Phi_3 \tag{4-66}$$

and first applies the boundary condition 4-5a to obtain

$$\varphi(0) = C_1\{\varphi_1(0)-\alpha c^2\varphi_2(0)\} + C_3\,{}_1\Phi_3(\zeta_0) = 0, \tag{4-67}$$

in which $\zeta_o = -\eta_{c_1}(\bar{U}'_{c_1}\sigma_o \alpha Re)^{\frac{1}{3}}$. For convenience, the boundary conditions 4-54a and 4-55a will be expressed in operational notation as $D_{I_o}(\varphi) = 0$ and $D_{II_o}(\varphi) = 0$, respectively, the operators D_{I_o} and D_{II_o} representing the linear operations of differentiation with respect to η, multiplication by various terms, and evaluation at $\eta = 0$ according to equations 4-54a and 4-55a. With this notation the isothermal disturbance condition 4-54a becomes

$$C_1\{D_{I_o}(\varphi_1) - \alpha c^2 D_{I_o}(\varphi_2)\} + C_3 D_{I_o}({}_1\Phi_3) = 0 \tag{4-68}$$

so that the boundary-condition equation for the existence of non-trivial values of C_1 and C_2 satisfying equations 4-67 and 4-68 can be written as

$$\frac{D_{I_o}(\varphi_1) - \alpha c^2 D_{I_o}(\varphi_2)}{\varphi_1(0) - \alpha c^2 \varphi_2(0)} = \frac{D_{I_o}({}_1\Phi_3)}{{}_1\Phi_3(\zeta_o)}. \tag{4-69}$$

In operational notation the adiabatic disturbance condition 4-55a is

$$C_1\{D_{II_o}(\varphi_1) - \alpha c^2 D_{II_o}(\varphi_2)\} + C_3 D_{II_o}({}_1\Phi_3) = 0. \tag{4-70}$$

If C_1 and C_2 are to be determined by this equation and equation 4-67, the equation which states that the determinant of the coefficients of C_1 and C_2 vanishes is

$$\frac{D_{II_o}(\varphi_1) - \alpha c^2 D_{II_o}(\varphi_2)}{\varphi_1(0) - \alpha c^2 \varphi_2(0)} = \frac{D_{II_o}({}_1\Phi_3)}{{}_1\Phi_3(\zeta_o)}. \tag{4-71}$$

The solution of either equation 4-69 or equation 4-71 for the purpose of determining an indifference curve would require a very great amount of work. Besides the work of calculating the integrals upon which φ_1, φ_2, and their derivatives depend, there is the necessity for computing the function ${}_1\Phi_3$ and its derivatives, which to the knowledge of the author have not been tabulated.

4.2.3a The Case of the Vertical Plate

As in previous cases, the analysis for the inclined plate applies to the vertical plate if one sets $a_1 = \pm 1$ and $a_2 = 0$.

4.2.4 The Coupled, Viscous, Heat-Conducting Case

4.2.4.1 The Differential Equations

In this case one considers that in the disturbance equations 1a and 2, αRe and $\sigma_0 \alpha Re$ are not of extremely different size, that neither is so large that terms multiplied by its reciprocal are always negligible, and that the effect of coupling between the combined momentum and energy equations must be taken into account. With these assumptions none of the terms in equations 1a and 2 may be neglected until one begins to simplify in order to construct approximate solutions. These basic equations are rewritten below for convenience:

$$(\bar{u}-c)(\varphi''-\alpha^2\varphi) - \bar{u}''\varphi + \frac{f}{\alpha^2 Re}\left\{ i a_1 s' - \alpha a_2 s \right\} + \frac{i}{\alpha Re}\left\{ \varphi^{IV} - 2\alpha^2\varphi'' + \alpha^4 \varphi \right\} = 0 \qquad (1a)$$

$$(\bar{u}-c)s - \bar{\theta}'\varphi + \frac{i}{\sigma_0 \alpha Re}\left\{ s'' - \alpha^2 s \right\} = 0 \qquad (2)$$

As described in Part 4.3.1 of this appendix, s and its derivatives can be eliminated between the above equations in order to obtain an equation involving only φ and its derivatives as unknowns. This equation is

$$\left[\frac{i a_1^2}{\sigma_0 \alpha Re}(\bar{u}-c) + \frac{\alpha^2}{(\sigma_0 \alpha Re)^2} \right] \left[(\bar{u}-c)(\varphi''-\alpha^2\varphi) - \bar{u}''\varphi + \frac{i}{\alpha Re}\left\{ \varphi^{IV} - 2\alpha^2\varphi'' + \alpha^4\varphi \right\} \right]''$$

$$+ \left[\frac{-i a_1^2 \bar{u}'}{\sigma_0 \alpha Re} \right] \left[(\bar{u}-c)(\varphi''-\alpha^2\varphi) - \bar{u}''\varphi + \frac{i}{\alpha Re}\left\{ \varphi^{IV} - 2\alpha^2\varphi'' + \alpha^4\varphi \right\} \right]' + \Big[a_1^2 (\bar{u}-c)^2$$

$$- \frac{1}{\sigma_0 \alpha Re}\left\{ \alpha a_1 a_2 \bar{u}' + i\alpha^2(1+a_1^2)(\bar{u}-c) \right\} - \frac{\alpha^4}{(\sigma_0 \alpha Re)^2} \Big] \Big[(\bar{u}-c)(\varphi''-\alpha^2\varphi) - \bar{u}''\varphi$$

$$+\frac{i}{\alpha Re}\left\{\varphi^{IV}-2\alpha^2\varphi''+\alpha^4\varphi\right\}\Big] + \left[\frac{1}{\alpha Re}\right]\Big[-i a_1^3\bar{u}' - \alpha a_2\left[a_1^2(\bar{u}-c)-\frac{i\alpha^2}{\sigma_0\alpha Re}\right]\Big\}\left\{\bar{\theta}'\varphi\right\}$$

$$+ i a_1\left\{a_1^2(\bar{u}-c) - \frac{i\alpha^2}{\sigma_0\alpha Re}\right\}\left\{\bar{\theta}''\varphi+\bar{\theta}'\varphi'\right\}\Big] = 0.$$

4.2.4.2 Boundary Conditions

Equation 4-1 being a sixth-order differential equation, six boundary conditions must be specified on φ and its derivatives in order to derive a boundary-condition equation for the determination of an indifference curve in the $\alpha - Re$ plane. Four of these boundary conditions are the same as those specified for the uncoupled, viscous case of Part 4.1.2. In order that the disturbance velocity components parallel and perpendicular to the plate disappear both far from the plate and at its surface, the requirements are

$$\left.\begin{aligned}\varphi(0) &= 0,\\ \varphi'(0) &= 0,\end{aligned}\right\} \qquad (4\text{-}11a)$$

and

$$\left.\begin{aligned}\varphi(\infty) &= 0,\\ \varphi'(\infty) &= 0.\end{aligned}\right\} \qquad (4\text{-}11b)$$

The remaining two boundary conditions are based on requirements that the temperature disturbances must meet. For the temperature disturbances to die away far from the plate, one specifies as in Part 4.2.3.2 that

$$s(\infty) = 0. \qquad (4\text{-}53)$$

If s is expressed in terms of φ, its derivatives, and known functions according to Part 4.3.3, this stipulation on $s(\infty)$ can be written as

$$a_1\left\{-c\varphi''' + \frac{i}{\alpha Re}\left[\varphi^{V}-2\alpha^2\varphi'''\right]\right\}\Big|_{\eta=\infty}$$

$$-i\alpha a_2\left\{-c\varphi'' + \frac{i}{\alpha Re}\left[\varphi^{IV}-2\alpha^2\varphi''\right]\right\}\Big|_{\eta=\infty} = 0. \qquad (4\text{-}72)$$

At the surface of the plate either an isothermal or an adiabatic boundary condition regarding the temperature disturbances will be assumed. The isothermal boundary condition, which says that the flow temperature disturbances do not affect the plate temperature, is

$$s(0) = 0, \tag{4-54}$$

which, in terms of φ and its derivatives according to Part 4.3.3, is

$$a_1\left\{\bar{u}'\varphi'' - c\varphi''' + \frac{i}{\alpha Re}\left[\varphi^{V} - 2\alpha^2\varphi'''\right]\right\}\Big|_{\eta=0} - i\alpha a_2\left\{-c\varphi'' + \frac{i}{\alpha Re}\left[\varphi^{IV} - 2\alpha^2\varphi''\right]\right\}\Big|_{\eta=0} = 0. \tag{4-73}$$

On the other hand, if the plate is assumed to be adiabatic with respect to the temperature disturbances, that is, if the temperature disturbances are not considered to affect the rate of heat transfer from the plate, the appropriate boundary condition is

$$s'(0) = 0. \tag{4-55}$$

According to Part 4.3.3, this condition in terms of φ and its derivatives is

$$\begin{aligned} &\left\{-a_i^2\bar{u}'\left[a_1\left\{\bar{u}'\varphi'' - c\varphi''' + \frac{i}{\alpha Re}(\varphi^{V} - 2\alpha^2\varphi''')\right\} - i\alpha a_2\left\{-c\varphi'' + \frac{i}{\alpha Re}(\varphi^{IV} - 2\alpha^2\varphi'')\right\}\right]\right\}\Big|_{\eta=0} \\ &+\left\{-a_i^2 c - \frac{i\alpha^2}{\sigma_0\alpha Re}\right\}\left\{a_1\left[2\bar{u}'\varphi''' - c(\varphi^{IV} - \alpha^2\varphi'') + \frac{i}{\alpha Re}\left\{\varphi^{VI} - 2\alpha^2\varphi^{IV} + \alpha^4\varphi''\right\}\right]\right. \\ &\left.- i\alpha a_2\left[\bar{u}'\varphi'' - c\varphi''' + \frac{i}{\alpha Re}\left\{\varphi^{V} - 2\alpha^2\varphi'''\right\}\right]\right\}\Big|_{\eta=0} = 0. \end{aligned} \tag{4-74}$$

The six boundary conditions, then, to be met by φ and its derivatives are those indicated by equations 4-11a, 4-11b, 4-72, and either 4-73 or 4-74.

4.2.4.3 Solutions of the Differential Equation in φ

Approximate solutions of equation 4-1 that should be useful when αRe and $\sigma_0 \alpha Re$ are large are obtained by methods very similar to those employed in the preceding cases. Two of these solutions can be secured by letting $\frac{1}{\sigma_0 \alpha Re}$ and $\frac{1}{\alpha Re}$ go to zero with retention of $\frac{f}{\alpha Re}$ as remaining finite, since f is quite large even when σ_0 is on the order of unity. Doing this in equation 4-1 and dividing by a_1^2 gives

$$(\bar{u}-c)^2\left\{(\bar{u}-c)(\varphi''-\alpha^2\varphi)-\bar{u}''\varphi\right\} + \frac{f}{\alpha Re}\Big\{[-ia_1\bar{u}'-\alpha a_2(\bar{u}-c)][\bar{\theta}'\varphi] + ia_1(\bar{u}-c)(\bar{\theta}''\varphi+\bar{\theta}'\varphi')\Big\} = 0. \qquad (4\text{-}34)$$

This equation is treated in Part 4.2.1.3 in the study of the coupled, non-viscous, non-heat-conducting case. In that part two solutions φ_1 and φ_2 defined by equations 4-39a through 4-42 were developed, and it was shown that the combination $(\varphi_1 - \alpha c^2 \varphi_2)$ behaves as $const. \times e^{-\alpha(\eta-b)}$ when $\eta \to \infty$. Thus the two approximate solutions φ_1 and φ_2 of equation 4-34 are already available; and, in addition, the combination $(\varphi_1 - \alpha c^2 \varphi_2)$ satisfies the infinite boundary conditions 4-11b and 4-72.

Four additional approximate solutions of equation 4-1 are secured by basically the same method as that used in the other cases in which either the fluid thermal conductivity or viscosity was taken to be finite. That is, four solutions are developed which are valid approximations far from the critical points but not at them, and four solutions valid near the inner critical point but not far from it are also constructed. Then a comparison is made between the two groups of four solutions in order to find which of the solutions valid near the inner critical point approximate solutions which satisfy the boundary conditions far from the plate. Also, as previously, the solutions valid near the critical point are used to determine the sector with center at the inner critical point in which asymptotic solutions such as φ_1 and φ_2 are valid. This determination of the sector of validity is extended to the region near the outer critical point.

The solutions valid far from the critical points but not at them are developed in the familiar exponential asymptotic form,

$$\varphi = \exp\left[\int^{\eta} d\Omega \left\{ (\alpha Re)^{\frac{1}{2}} w_0(\Omega) + w_1(\Omega) + (\alpha Re)^{-\frac{1}{2}} w_2(\Omega) + O\left[(\alpha Re)^{-1}\right] \right\}\right]. \tag{4-12}$$

When $\sigma_0 \neq 1$, substituting this expression for φ into equation 4-1 and equating the coefficients of the two highest powers of $(\alpha Re)^{\frac{1}{2}}$, namely αRe and $(\alpha Re)^{\frac{1}{2}}$, to zero give

$$\left.\begin{aligned} w_{0_{3,4}} &= \mp\{i(\bar{u}-c)\}^{\frac{1}{2}}, \\ w_{0_{5,6}} &= \mp\{i\sigma_0(\bar{u}-c)\}^{\frac{1}{2}}, \\ \text{and} \qquad w_{1_{3,4}} &= -\frac{5}{4}\frac{\bar{u}'}{(\bar{u}-c)}, \\ w_{1_{5,6}} &= -\frac{7}{4}\frac{\bar{u}'}{(\bar{u}-c)}, \quad \sigma_0 \neq 1. \end{aligned}\right\} \tag{4-75}$$

Here is has been assumed that f is of lower order than $(\alpha Re)^{\frac{1}{2}}$. When $\sigma_0 = 1$, one must consider the coefficient of $(\alpha Re)^0$ in order to determine $w_{1_{3,4}}$ and $w_{1_{5,6}}$ because the coefficient of $(\alpha Re)^{\frac{1}{2}}$ is identically zero. One finds that the values of $w_{0_{3,4}}$, $w_{0_{5,6}}$, and $w_{1_{3,4}}$ indicated in equations 4-75 apply in this case but that

$$w_{1_{5,6}} = -\frac{5}{4}\frac{\bar{u}'}{(\bar{u}-c)} + \frac{1}{\eta+K}, \quad \sigma_0 = 1, \tag{4-75a}$$

K being an arbitrary constant.

Using these values for the w's, one may write

$$\left.\begin{aligned}
{}_2\varphi_3 &= (\bar{u}-c)^{-\frac{5}{4}} \exp\left[-\int_b^\eta d\Omega \left\{ i\alpha Re(\bar{u}-c)\right\}^{\frac{1}{2}} + O\left\{(\alpha Re)^{-\frac{1}{2}}\right\}\right], \\
{}_2\varphi_4 &= (\bar{u}-c)^{-\frac{5}{4}} \exp\left[\int_b^\eta d\Omega \left\{ i\alpha Re(\bar{u}-c)\right\}^{\frac{1}{2}} + O\left\{(\alpha Re)^{-\frac{1}{2}}\right\}\right], \\
&\left.\begin{aligned}
{}_2\varphi_5 &= (\bar{u}-c)^{-\frac{7}{4}} \exp\left[-\int_b^\eta d\Omega \left\{ i\sigma_0\alpha Re(\bar{u}-c)\right\}^{\frac{1}{2}} + O\left\{(\alpha Re)^{-\frac{1}{2}}\right\}\right], \\
{}_2\varphi_6 &= (\bar{u}-c)^{-\frac{7}{4}} \exp\left[\int_b^\eta d\Omega \left\{ i\sigma_0\alpha Re(\bar{u}-c)\right\}^{\frac{1}{2}} + O\left\{(\alpha Re)^{-\frac{1}{2}}\right\}\right],
\end{aligned}\right\} \sigma_0 \neq 1 \\
&\left.\begin{aligned}
{}_2\varphi_5 &= (\eta+K)(\bar{u}-c)^{-\frac{5}{4}} \exp\left[-\int_b^\eta d\Omega \left\{ i\alpha Re(\bar{u}-c)\right\}^{\frac{1}{2}} + O\left\{(\alpha Re)^{-\frac{1}{2}}\right\}\right], \\
&\text{and} \\
{}_2\varphi_6 &= (\eta+K)(\bar{u}-c)^{-\frac{5}{4}} \exp\left[\int_b^\eta d\Omega \left\{ i\alpha Re(\bar{u}-c)\right\}^{\frac{1}{2}} + O\left\{(\alpha Re)^{-\frac{1}{2}}\right\}\right].
\end{aligned}\right\} \sigma_0 = 1
\end{aligned}\right\} \quad (4\text{-}76)$$

These solutions, like all others of the exponential asymptotic form, have the disadvantage of being invalid near the critical points. Solutions valid near the critical points are constructed in the same manner as that used in other cases. If one desires to construct solutions valid near the inner critical point η_{c_1}, the proper transformation of the independent variable is

$$(\eta-\eta_{c_1}) = (\bar{u}'_{c_1}\alpha Re)^{-\frac{1}{3}}\zeta. \qquad (4\text{-}16a)$$

Letting $\varphi(\eta) = \Phi(\zeta)$ and expanding known functions of η in Taylor series about η_{c_1} as before allow one to write equation 4-1 as

$$\left\{\frac{d^2}{d\zeta^2} - \frac{1}{\zeta}\frac{d}{d\zeta} - i\sigma_0\zeta\right\}\left\{\frac{d^4\Phi}{d\zeta^4} - i\zeta\frac{d^2\Phi}{d\zeta^2}\right\} = O\left\{\frac{f}{(\alpha Re)^{\frac{2}{3}}}, (\alpha Re)^{-\frac{1}{3}}\right\} \qquad (4\text{-}77)$$

Here it has been assumed that αRe and $\sigma_0\alpha Re$ do not differ greatly in size. If αRe is considered to be sufficiently large for the right-hand side of equation 4-77 to be neglected, the equation becomes

$$\left\{\frac{d^2}{d\zeta^2} - \frac{1}{\zeta}\frac{d}{d\zeta} - i\sigma_0\zeta\right\}\left\{\frac{d^4\Phi}{d\zeta^4} - i\zeta\frac{d^2\Phi}{d\zeta^2}\right\} = 0. \tag{4-77a}$$

Obviously, four of the solutions of this sixth-order equation are solutions of the fourth-order equation

$$\frac{d^4\Phi}{d\zeta^4} - i\zeta\frac{d^2\Phi}{d\zeta^2} = 0, \tag{4-17c}$$

which first appeared in Part 4. 1. 2. 3. As indicated by equations 4-18 of that section, four linearly independent solutions of this equation are

$$\left.\begin{aligned} {}_2\Phi_1 &= 1, \\ {}_2\Phi_2 &= \zeta, \end{aligned}\right\} \tag{4-78a}$$

and

$$\left.\begin{aligned} {}_2\Phi_3 &= \int_{\infty}^{\zeta} d\Omega_1 \int_{\infty}^{\Omega_1} d\Omega_2\, \Omega_2^{\frac{1}{2}} H^{(1)}_{\frac{1}{3}}\left\{\tfrac{2}{3}(i\Omega_2)^{\frac{3}{2}}\right\}, \\ {}_2\Phi_4 &= \int_{-\infty}^{\zeta} d\Omega_1 \int_{-\infty}^{\Omega_1} d\Omega_2\, \Omega_2^{\frac{1}{2}} H^{(2)}_{\frac{1}{3}}\left\{\tfrac{2}{3}(i\Omega_2)^{\frac{3}{2}}\right\}. \end{aligned}\right\} \tag{4-78b}$$

The remaining two solutions of equation 4-77a can be secured by setting

$$\frac{d^4\Phi}{d\zeta^4} - i\zeta\frac{d^2\Phi}{d\zeta^2} = Y \tag{4-79}$$

so that equation 4-77a becomes

$$\frac{d^2 Y}{d\zeta^2} - \frac{1}{\zeta}\frac{dY}{d\zeta} - i\sigma_0\zeta Y = 0. \tag{4-80}$$

Two linearly independent solutions of this equation are

$$Y_5 = \zeta H^{(1)}_{\frac{2}{3}}\left\{\tfrac{2}{3}(ia_0\zeta)^{\frac{3}{2}}\right\} \tag{4-81a}$$

and

$$Y_6 = \zeta H^{(2)}_{\frac{2}{3}}\left\{\tfrac{2}{3}(ia_0\zeta)^{\frac{3}{2}}\right\}, \tag{4-81b}$$

with $a_0 = \sigma_0^{\frac{1}{3}}$. The inhomogeneous equation 4-79 can now be solved by the method of variation of parameters in order to obtain

$$_2\Phi_5 = \frac{i\pi}{6}\int_{\infty}^{3} d\Omega_1 \int_{\infty}^{\Omega_1} d\Omega_2 \left\{ {}_2\Phi_4'' \int_{\infty}^{\Omega_2} d\Omega_3 \, Y_5 \, {}_2\Phi_3'' - {}_2\Phi_3'' \int_{\pm\infty,0}^{\Omega_2} d\Omega_3 \, Y_5 \, {}_2\Phi_4'' \right\}$$

and (4-78c)

$$_2\Phi_6 = \frac{i\pi}{6}\int_{-\infty}^{3} d\Omega_1 \int_{-\infty}^{\Omega_1} d\Omega_2 \left\{ {}_2\Phi_4'' \int_{\mp\infty,0}^{\Omega_2} d\Omega_3 \, Y_6 \, {}_2\Phi_3'' - {}_2\Phi_3'' \int_{-\infty}^{\Omega_2} d\Omega_3 \, Y_6 \, {}_2\Phi_4'' \right\}.$$

In the integral $\int_{\pm\infty,0}^{\Omega_2} d\Omega_3 Y_5 \, {}_2\Phi_4''$ of the expression for ${}_2\Phi_5$ one must take $+\infty$ to be the lower limit of integration when $\sigma_0 > 1$, $-\infty$ when $\sigma_0 < 1$, and a finite value such as 0 when $\sigma_0 = 1$. Similarly, the lower limit of the integral $\int_{\mp\infty,0}^{\Omega_2} d\Omega_3 Y_6 \, {}_2\Phi_3''$ of the expression for ${}_2\Phi_6$ is to be taken as $-\infty$ when $\sigma_0 > 1$, $+\infty$ when $\sigma_0 < 1$, and 0 as a definite finite value when $\sigma_0 = 1$.

In part 4. 3. 4 it is shown that the asymptotic relations below hold between the exponential asymptotic solutions and the solutions valid near η_{c_1}:

$$\begin{aligned} {}_2\Phi_3 &\sim \text{const.} \times {}_2\varphi_3 \\ {}_2\Phi_4 &\sim \text{const.} \times {}_2\varphi_4 \\ {}_2\Phi_5 &\sim \text{const.} \times {}_2\varphi_3 + \text{const.} \times {}_2\varphi_5 \\ {}_2\Phi_6 &\sim \text{const.} \times {}_2\varphi_4 + \text{const.} \times {}_2\varphi_6 \end{aligned} \tag{4-82}$$

Also in that part it is shown that in the neighborhoods of the critical points η_{c_1} and η_{c_2} one must observe the requirements

$$-\frac{7\pi}{6} < \arg(\eta - \eta_{c_1}) < \frac{\pi}{6} \tag{4-26}$$

and

$$-\frac{\pi}{6} < \arg(\eta - \eta_{c_2}) < \frac{7\pi}{6}. \tag{4-29}$$

for the asymptotic solutions. Finally, it follows from these limits and the definitions 4-76 that if one takes $\arg(\bar{u}-c)=0$ when $\bar{u}>c$, then

$$\left.\begin{aligned}
&\lim_{\eta\to\infty} |{}_2\varphi_3| = 0, \\
&\lim_{\eta\to\infty} |{}_2\varphi_4| = \infty, \\
\text{and}\quad &\lim_{\eta\to\infty} |{}_2\varphi_5| = 0, \\
&\lim_{\eta\to\infty} |{}_2\varphi_6| = \infty.
\end{aligned}\right\} \tag{4-83}$$

4.2.4.4 The Boundary-Condition Equations

From the forms of ${}_2\varphi_3$ and ${}_2\varphi_5$ indicated by equations 4-76, one can see that these functions die away exponentially as $\eta\to\infty$ under the requirement 4-29 so that their derivatives to any order also disappear at $\eta=\infty$. The same is true for the combination $({}_1\varphi_1 - \alpha c^2\, {}_1\varphi_2)$ because it behaves as $const.\times e^{-\alpha(\eta-b)}$ outside the boundary layer. Therefore, any linear combination of $({}_1\varphi_1 - \alpha c^2\, {}_1\varphi_2)$, ${}_2\varphi_3$, and ${}_2\varphi_5$ satisfies the boundary conditions 4-11b and 4-72 at $\eta=\infty$. In applying the boundary conditions at the plate surface it should be better to replace ${}_2\varphi_3$ and ${}_2\varphi_5$ with ${}_2\Phi_3$ and ${}_2\Phi_5$, respectively, because of the singularities of ${}_2\varphi_3$ and ${}_2\varphi_5$ at η_{c_1}. One takes

$$\varphi = C_1({}_1\varphi_1 - \alpha c^2\, {}_1\varphi_2) + C_3\, {}_2\Phi_3 + C_5\, {}_2\Phi_5 \tag{4-84}$$

and applies the boundary conditions 4-11a on the disturbance motion at the plate surface with the results that

$$C_1\{{}_1\varphi_1(0) - \alpha c^2\, {}_1\varphi_2(0)\} + C_3\, {}_2\Phi_3(\zeta_0) + C_5\, {}_2\Phi_5(\zeta_0) = 0 \tag{4-85}$$

and

$$C_1\{{}_1\varphi_1'(0) - \alpha c^2\, {}_1\varphi_2'(0)\} + C_3(\bar{u}_{c_1}'\alpha Re)^{\frac{1}{3}}\, {}_2\Phi_3'(\zeta_0) + C_5(\bar{u}_{c_1}'\alpha Re)^{\frac{1}{3}}\, {}_2\Phi_5'(\zeta_0) = 0. \tag{4-86}$$

Here $\zeta_0 = -\eta_{c_1}(\bar{u}_{c_1}'\alpha Re)^{\frac{1}{3}}$. The boundary conditions 4-73 and 4-74 resulting from taking the plate to be isothermal and adiabatic, respectively, concerning the temperature disturbances are linear in φ and its derivatives. For simplification, the boundary condition 4-73 will now be denoted by $D_{III0}(\varphi) = 0$, and the boundary condition 4-74 will be indicated by $D_{IV0}(\varphi) = 0$. Applying these boundary conditions to φ expressed as indicated by equation 4-84 gives

$$C_1\{D_{III0}(\varphi_1) - \alpha c^2 D_{III0}(\varphi_2)\} + C_3 D_{III0}({}_2\Phi_3) + C_5 D_{III0}({}_2\Phi_5) = 0 \qquad (4\text{-}87a)$$

and

$$C_1\{D_{IV0}(\varphi_1) - \alpha c^2 D_{IV0}(\varphi_2)\} + C_3 D_{IV0}({}_2\Phi_3) + C_5 D_{IV0}({}_2\Phi_5) = 0. \qquad (4\text{-}87b)$$

If one considers that the plate remains isothermal, equations 4-85, 4-86, and 4-87a must be satisfied. Setting the determinant of the coefficients of C_1, C_3, and C_5 in these three equations equal to zero gives

$$\begin{aligned}
&\{\varphi_1(0) - \alpha c^2 \varphi_2(0)\}\{[(\bar{U}'_{c1}\alpha Re)^{\frac{1}{3}}{}_2\Phi_3'(\zeta_0)][D_{III0}({}_2\Phi_5)] - [(\bar{U}'_{c1}\alpha Re)^{\frac{1}{3}}{}_2\Phi_5'(\zeta_0)][D_{III0}({}_2\Phi_3)]\} \\
&+\{{}_2\Phi_3(\zeta_0)\}\{[(\bar{U}'_{c1}\alpha Re)^{\frac{1}{3}}{}_2\Phi_5'(\zeta_0)][D_{III0}(\varphi_1) - \alpha c^2 D_{III0}(\varphi_2)] - [\varphi_1'(0) - \alpha c^2 \varphi_2'(0)][D_{III0}({}_2\Phi_5)]\} \\
&+\{{}_2\Phi_5(\zeta_0)\}\{[\varphi_1'(0) - \alpha c^2 \varphi_2'(0)][D_{III0}({}_2\Phi_3)] - [(\bar{U}'_{c1}\alpha Re)^{\frac{1}{3}}{}_2\Phi_3'(\zeta_0)][D_{III0}(\varphi_1) - \alpha c^2 D_{III0}(\varphi_2)]\} \\
&= 0. \qquad (4\text{-}88a)
\end{aligned}$$

For the case in which the plate remains adiabatic with respect to temperature disturbances in the flow, the appropriate boundary-condition relations are equations 4-85, 4-86, and 4-87b. In this case, setting the determinant of the coefficients of the C_j's equal to zero gives

$$\begin{aligned}
&\{\varphi_1(0) - \alpha c^2 \varphi_2(0)\}\{[(\bar{U}'_{c1}\alpha Re)^{\frac{1}{3}}{}_2\Phi_3'(\zeta_0)][D_{IV0}({}_2\Phi_5)] - [(\bar{U}'_{c1}\alpha Re)^{\frac{1}{3}}{}_2\Phi_5'(\zeta_0)][D_{IV0}({}_2\Phi_3)]\} \\
&+\{{}_2\Phi_3(\zeta_0)\}\{[(\bar{U}'_{c1}\alpha Re)^{\frac{1}{3}}{}_2\Phi_5'(\zeta_0)][D_{IV0}(\varphi_1) - \alpha c^2 D_{IV0}(\varphi_2)] - [\varphi_1'(0) - \alpha c^2 \varphi_2'(0)][D_{IV0}({}_2\Phi_5)]\} \\
&+\{{}_2\Phi_5(\zeta_0)\}\{[\varphi_1'(0) - \alpha c^2 \varphi_2'(0)][D_{IV0}({}_2\Phi_3)] - [(\bar{U}'_{c1}\alpha Re)^{\frac{1}{3}}{}_2\Phi_3'(\zeta_0)][D_{IV0}(\varphi_1) - \alpha c^2 D_{IV0}(\varphi_2)]\} \\
&= 0. \qquad (4\text{-}88b)
\end{aligned}$$

Solving either equation 4-88a or equation 4-88b in order to obtain an indifference curve in the α-Re plane would be an extremely complicated process. To find values of c, α, and αRe satisfying one of the equations, one would have to construct the four solutions ${}_1\varphi_1$, ${}_1\varphi_2$, ${}_2\Phi_3$, and ${}_2\Phi_5$ along with their derivatives through the fifth order for equation 4-88a and through the sixth order for equation 4-88b. Such an amount of calculation would probably be prohibitive without the use of electronic computing equipment.

4.2.4a The Case of the Vertical Plate

As in each of the preceding cases, treating the case of a vertical rather than inclined plate involves only setting $a_1 = \pm 1$ and $a_2 = 0$ in the analysis leading to equations 4-88a and 4-88b.

4.3 Mathematical Details Concerning Methods Developed for Approximately Solving the Free Convection Stability Problem

4.3.1 Elimination of s and its Derivatives Between the Combined Momentum and Energy Equations

4.3.1.1 The Case of the Inclined Plate

For this case the combined momentum and energy equations are, respectively,

$$(\bar{u}-c)(\varphi''-\alpha^2\varphi)-\bar{u}''\varphi+\frac{f}{\alpha Re}\{ia_1 s'-\alpha a_2 s\}+\frac{i}{\alpha Re}\{\varphi^{IV}-2\alpha^2\varphi''+\alpha^4\varphi\}=0 \qquad (1a)$$

and

$$(\bar{u}-c)s-\bar{\theta}'\varphi+\frac{i}{\sigma_0\alpha Re}\{s''-\alpha^2 s\}=0. \qquad (2)$$

One method of eliminating s and its derivatives between this pair of equations requires that first s be eliminated between the pair in order to obtain an equation involving s' and s'' as well as $\varphi^{(n)}$'s * and known functions of η. Secondly, equation 1a is differentiated to produce an equation involving $\frac{\partial}{\partial \alpha \bar{U} Re} s'$, $\frac{\partial}{\partial \alpha \bar{U} Re} s''$, $\varphi^{(n)}$'s, and known functions. Then s'' is eliminated between these two equations involving s' and s'', and the resulting equation is solved for $\frac{\partial}{\partial \alpha \bar{U} Re} s'$ in terms of $\varphi^{(n)}$'s and known functions. This relation is differentiated to obtain an expression in $\varphi^{(n)}$'s and known functions for $\frac{\partial}{\partial \alpha \bar{U} Re} s''$. Substitution is next made for $\frac{\partial}{\partial \alpha \bar{U} Re} s'$ and $\frac{\partial}{\partial \alpha \bar{U} Re} s''$ in the equation obtained by differentiating equation 1a to secure a sixth-order differential equation in φ, one form of which is denoted as equation 4-1 in the introduction to this appendix.

In the algebraic manipulation necessary to derive this equation 4-1, division by the terms α, a_1, a_2, and $\left\{a_1^2(\bar{U}-c) - \frac{i\alpha^2}{\sigma_0 \alpha Re}\right\}$ is involved. Therefore, equation 4-1 is invalid when any one of these four terms is zero.

4.3.1.2 The Case of the Vertical Plate

One might suppose that setting a_1, the cosine of the angle of inclination of the plate with the vertical, equal to ± 1 and a_2, the sine of that angle, equal to 0 in equation 4-1 would be a valid way to get the corresponding equation for a vertical plate. Although the correct equation for a vertical plate would be obtained, such a procedure is not strictly legitimate because in deriving equation 4-1 division by a_2 was performed. No method of obtaining equation 4-1 was found that

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* $\varphi^{(n)}$ symbolizes a derivative of φ of the n^{th} order, n having integral values from 0 to whatever is the order of the highest derivative of φ that appears in the equation.

did not either involve division by a_2 or require that a_2 be left as a factor in each term of the final equation. Therefore, in order to preserve mathematical validity for the final equation, one must eliminate s and its derivatives between the original combined momentum and energy equations which describe the case of the vertical plate. These equations for a vertical plate are

$$(\bar{u}-c)(\varphi''-\alpha^2\varphi) - \bar{u}''\varphi + \frac{f}{\alpha\sqrt{Re}}\{\pm i s'\} + \frac{i}{\alpha Re}\{\varphi^{IV} - 2\alpha^2\varphi'' + \alpha^4\varphi\} = 0 \qquad (4\text{-}89)$$

and

$$(\bar{u}-c)s - \bar{\theta}'\varphi + \frac{i}{\sigma_0 \alpha Re}\{s'' - \alpha^2 s\} = 0. \qquad (2)$$

s and its derivatives can be eliminated between equations 4-89 and 2 by first solving equation 2 for s, multiplying by $\frac{if}{\alpha\sqrt{Re}}$, and differentiating to obtain an expression for $\frac{if}{\alpha\sqrt{Re}}s'$ which involves s'', s''', $\varphi^{(n)}$'s, and known functions. Then equation 4-89 is rearranged to express $\frac{if}{\alpha\sqrt{Re}}s'$ in terms of $\varphi^{(n)}$'s and known functions. This relation is differentiated once and twice in order to obtain $\frac{if}{\alpha\sqrt{Re}}s''$ and $\frac{if}{\alpha\sqrt{Re}}s'''$ in terms of $\varphi^{(n)}$'s and known functions. Lastly, substitution for $\frac{if}{\alpha\sqrt{Re}}s'$, $\frac{if}{\alpha\sqrt{Re}}s''$, and $\frac{if}{\alpha\sqrt{Re}}s'''$ in the equation derived from equation 2 is made. One of the forms in which this final equation can be written is given as equation 4-2 of the introduction to the appendix. In securing this relation, division by the term $\left\{(\bar{u}-c) - \frac{i\alpha^2}{\sigma_0 \alpha Re}\right\}$ was performed, so the equation ceases to be valid when $(\bar{u}-c) = \frac{i\alpha^2}{\sigma_0 \alpha Re}$.

4.3.2 <u>Determination of Bounds on the Phase Velocity c for a Neutral Oscillation in the Uncoupled, Nonviscous Case</u>

This analysis is an adaptation of the method used by Foote and

Lin[27] in studying the stability of the free jet profile.

In Part 4.1.1.3 solutions denoted as φ_1 and φ_2 of the non-viscous Orr-Sommerfeld equation

$$(\bar{u}-c)(\varphi''-\alpha^2\varphi)-\bar{u}''\varphi=0 \tag{4-4}$$

are presented; and it is shown that a linear combination of these two solutions, $(\varphi_1-\alpha C^2\varphi_2)$, satisfies the boundary condition $\varphi(\infty)=0$. The first step here will be to write other pairs of linearly independent solutions of equation 4-4 which are more useful in the neighborhoods of the critical points η_{c_1} and η_{c_2} at which $\bar{u}=c$. These solutions are obtained by development in power series about the critical points, which is the method used by Tollmien[7] in his stability study of the Blasius boundary layer. They can be written as

$$\left.\begin{aligned}
\varphi_{jI} &= (\eta-\eta_{cj})+O\{(\eta-\eta_{cj})^2\} = P_1\{(\eta-\eta_{cj})\} \\
\text{and} \qquad & \\
\varphi_{jII} &= P_2\{(\eta-\eta_{cj})\}+[P_1\{(\eta-\eta_{cj})\}]\times\frac{\bar{u}''_{cj}}{\bar{u}'_{cj}}\log(\eta-\eta_{cj}),
\end{aligned}\right\} \tag{4-90}$$

with $P_2\{(\eta-\eta_{cj})\}=1+O\{(\eta-\eta_{cj})\}$ and $j=1,2$. These two pairs of representations of the solutions are valid only when $0\leq\eta<b$ and η lies within the appropriate circle of convergence of the power series. In the neighborhood of each critical point one can write

$$\varphi=(\varphi_1-\alpha C^2\varphi_2)=\left\{\begin{aligned} &K_{1I}\,\varphi_{1I}+K_{1II}\,\varphi_{1II}, \ |\eta-\eta_{c_1}|<R_1 \\ &K_{2I}\,\varphi_{2I}+K_{2II}\,\varphi_{2II}, \ |\eta-\eta_{c_2}|<R_2 \end{aligned}\right\}\ 0\leq\eta<b, \tag{4-91}$$

since any solution of equation 4-4 can be written as a linear combination of any two linearly independent solutions. Here R_1 and R_2 are the radii of convergence of the power series. Of importance is the fact that $K_{jII}\neq 0$, $j=1,2$, if $\bar{u}''_{cj}$, α, and $c\neq 0$. This follows from the necessity for having the logarithmic singularity present both when φ is written in terms of φ_1 and φ_2 and when it is expressed as a combination of φ_{jI} and φ_{jII}.

Because the coefficients of φ'' and φ in equation 4-4 are real, φ_r and φ_i, the real and imaginary parts of φ, must separately satisfy the equation; and one can use equation 4-4 to write

$$\left.\begin{aligned} \varphi_i \varphi_r'' &= \frac{\bar{u}''\varphi_i\varphi_r}{(\bar{u}-c)} + \alpha^2\varphi_i\varphi_r \\ \text{and} \qquad \varphi_r\varphi_i'' &= \frac{\bar{u}\,\varphi_r\varphi_i}{(\bar{u}-c)} + \alpha^2\varphi_r\varphi_i \end{aligned}\right\} \qquad (4\text{-}92)$$

Subtracting the second of these equations from the first and integrating over an interval not including a critical point yields

$$\varphi_r'\varphi_i - \varphi_r\varphi_i' = \text{const.} \qquad (4\text{-}93)$$

The value of this constant can be different in the three intervals $0<\eta<\eta_{c_1}$, $\eta_{c_1}<\eta<\eta_{c_2}$, and $\eta_{c_2}<\eta$. Boundary conditions at $\eta=0$ and $\eta=\infty$ require that it disappear for $0<\eta<\eta_{c_1}$ and for $\eta_{c_2}<\eta$. Consequently,

$$\left\{\varphi_r'\varphi_i-\varphi_r\varphi_i'\right\}\Big|_{\eta_{c_1}-\ell}^{\eta_{c_1}+\ell} + \left\{\varphi_r'\varphi_i-\varphi_r\varphi_i'\right\}\Big|_{\eta_{c_2}-\ell}^{\eta_{c_2}+\ell} = 0, \qquad (4\text{-}94)$$

in which

$$\left\{\varphi_r'\varphi_i-\varphi_r\varphi_i'\right\}\Big|_{\eta_{c_1}-\ell}^{\eta_{c_1}+\ell} = \left\{\varphi_r'\varphi_i-\varphi_r\varphi_i'\right\}\Big|_{\eta_{c_1}<\eta<\eta_{c_2}} - \left\{\varphi_r'\varphi_i-\varphi_r\varphi_i'\right\}\Big|_{0<\eta<\eta_{c_1}}$$

and

$$\left\{\varphi_r'\varphi_i-\varphi_r\varphi_i'\right\}\Big|_{\eta_{c_2}-\ell}^{\eta_{c_2}+\ell} = \left\{\varphi_r'\varphi_i-\varphi_r\varphi_i'\right\}\Big|_{\eta_{c_2}<\eta} - \left\{\varphi_r'\varphi_i-\varphi_r\varphi_i'\right\}\Big|_{\eta_{c_1}<\eta<\eta_{c_2}}.$$

By the use of equations 4-90 and 4-91 with fulfillment of the requirements 4-26 and 4-29 which $\arg(\eta-\eta_{c_1})$ and $\arg(\eta-\eta_{c_2})$ must meet, one can show that

$$\left\{\varphi_r'\varphi_i - \varphi_r\varphi_i'\right\}\Big|_{\eta_{c_1}-\ell}^{\eta_{c_1}+\ell} = |K_{1\mathrm{II}}|^2\, \pi \, \frac{\bar{u}''_{c_1}}{\bar{u}'_{c_1}}$$

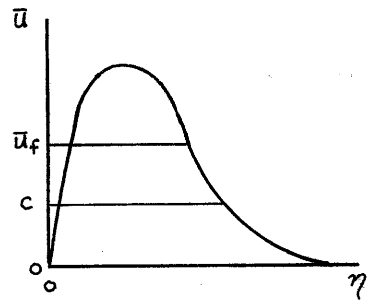
and

$$\left\{\varphi_r'\varphi_i - \varphi_r\varphi_i'\right\}\Big|_{\eta_{c_2}-\ell}^{\eta_{c_2}+\ell} = -|K_{2\mathrm{II}}|^2\, \pi \, \frac{\bar{u}''_{c_2}}{\bar{u}'_{c_2}}$$

so that equation 4-6 becomes

$$\pi\left\{|K_{1\mathrm{II}}|^2 \frac{\bar{u}''_{c_1}}{\bar{u}'_{c_1}} - |K_{2\mathrm{II}}|^2 \frac{\bar{u}''_{c_2}}{\bar{u}'_{c_2}}\right\} = 0. \tag{4-95}$$

Because $\bar{u}'_{c_1} > 0$ and $\bar{u}'_{c_2} < 0$, $\bar{u}''_{c_1}$ and $\bar{u}''_{c_2}$ must be of different signs. Hence, $0 < c < \bar{u}_f$ if $\alpha, c \neq 0$, $\bar{u}_f$ being the value of $\bar{u}$ for which $\bar{u}'' = 0$, as indicated:



4.3.3 Expression of s and s' in Terms of Known Functions, φ, and Derivatives of φ

With the abbreviation

$$W = (\bar{u}-c)(\varphi'' - \alpha^2\varphi) - \bar{u}''\varphi + \frac{i}{\alpha Re}\left\{\varphi^{IV} - 2\alpha^2\varphi'' + \alpha^4\varphi\right\} = 0, \tag{4-96}$$

the combined momentum equation 1a and the energy equation 2 can be written as

$$W + \frac{f}{\alpha^{\prime} Re}\left\{i a_1 s' - \alpha a_2 s\right\} = 0 \tag{1b}$$

and

$$(\bar{u}-c)s - \bar{\vartheta}'\varphi + \frac{i}{\sigma_0 \alpha Re}\left\{s'' - \alpha^2 s\right\} = 0. \tag{2}$$

One method for obtaining s in terms of known functions, φ, and derivatives of φ begins with differentiating equation 1b to secure an equation in s'', s', various $\varphi^{(n)}$'s, and known functions of η. Then s' is eliminated between this equation and equation 1b in order to obtain an equation involving s'', s, $\varphi^{(n)}$'s, and known functions. Finally, s'' is eliminated between this equation and equation 2 so that s can be expressed in terms of $\varphi^{(n)}$'s and known functions. This final relation can be written as

$$s = \left\{ \frac{f}{\alpha\sqrt{Re}} \left[a_1^2(\bar{u}-c) - \frac{i\alpha^2}{\sigma_0 \alpha Re} \right] \right\}^{-1} \left\{ \frac{f}{\alpha\sqrt{Re}} (a_1^2 \bar{\theta}' \varphi) + \frac{1}{\sigma_0 \alpha Re} (a_1 W' - i\alpha a_2 W) \right\} . \tag{4-97}$$

Differentiating this equation gives

$$\begin{aligned} s' = & -\frac{f}{\alpha\sqrt{Re}} \{a_1^2 \bar{u}'\} \left\{ \frac{f}{\alpha\sqrt{Re}} \left[a_1^2(\bar{u}-c) - \frac{i\alpha^2}{\sigma_0 \alpha Re} \right] \right\}^{-2} \left\{ \frac{f}{\alpha\sqrt{Re}} (a_1^2 \bar{\theta}' \varphi) + \frac{1}{\sigma_0 \alpha Re} (a_1 W' - i\alpha a_2 W) \right\} \\ & + \left\{ \frac{f}{\alpha\sqrt{Re}} \left[a_1^2(\bar{u}-c) - \frac{i\alpha^2}{\sigma_0 \alpha Re} \right] \right\}^{-1} \left\{ \frac{f}{\alpha\sqrt{Re}} \left[a_1^2(\bar{\theta}'' \varphi + \bar{\theta}' \varphi') \right] + \frac{1}{\sigma_0 \alpha Re} (a_1 W'' - i\alpha a_2 W') \right\} . \end{aligned} \tag{4-98}$$

The corresponding expressions for s and s' in the case of the vertical plate are identical with equations 4-97 and 4-98 except that is replaced by ± 1 and a_2 by 0.

4.3.4 Determination of Regions of Validity of Asymptotic Solutions in Neighborhoods of Critical Points and Equivalence of Exponential Asymptotic Solutions with Solutions Valid in Neighborhood of Inner Critical Point

4.3.4.1 Determinations of Regions of Validity of Asymptotic Solutions in Neighborhoods of Critical Points

In Part 4.2.4.3, six approximate solutions of the disturbance equation are developed which are valid in the neighborhood of the inner critical point η_{c_1}. Four of these six solutions, ${}_2\Phi_3$, ${}_2\Phi_4$, ${}_2\Phi_5$, and ${}_2\Phi_6$, which are exact solutions of the approximate disturbance equation 4-77a, are expanded in asymptotic series. These asymptotic series are obtained by utilizing the asymptotic expansions 4-19 and 4-20 for the Hankel functions of the first and second kinds in conjunction with the definitions 4-78b and 4-78c of ${}_2\Phi_3$, ${}_2\Phi_4$, ${}_2\Phi_5$, and ${}_2\Phi_6$. The results are

$${}_2\Phi_3 \sim \text{const.} \times \zeta^{-\frac{5}{4}} \left\{ \exp\left[\tfrac{2}{3} e^{i\frac{5\pi}{4}} \zeta^{\frac{3}{2}}\right]\right\} \left\{1 + O(\zeta^{-\frac{3}{2}})\right\} \tag{4-99}$$

with

$$-\frac{7\pi}{6} < \arg \zeta < \frac{5\pi}{6},$$

$${}_2\Phi_4 \sim \text{const.} \times \zeta^{-\frac{5}{4}} \left\{ \exp\left[\tfrac{2}{3} e^{i\frac{\pi}{4}} \zeta^{\frac{3}{2}}\right]\right\} \left\{1 + O(\zeta^{-\frac{3}{2}})\right\} \tag{4-100}$$

with

$$-\frac{11\pi}{6} < \arg \zeta < \frac{\pi}{6},$$

$${}_2\Phi_5 \sim \left\{ \begin{array}{l} \text{const.} \times \zeta^{-\frac{7}{4}} \left\{ \exp\left[\tfrac{2}{3} e^{i\frac{5\pi}{4}} \sigma_0^{\frac{1}{2}} \zeta^{\frac{3}{2}}\right]\right\} \left\{1 + O(\zeta^{-\frac{3}{2}})\right\}, \quad \sigma_0 \neq 1 \\ \left\{\text{const.} \times \zeta^{-\frac{1}{4}} + \text{const.} \times \zeta^{-\frac{5}{4}} + \text{const.} \times \zeta^{-\frac{7}{4}}\right\} \left\{ \exp\left[\tfrac{2}{3} e^{i\frac{5\pi}{4}} \zeta^{\frac{3}{2}}\right]\right\} \left\{1 + O(\zeta^{-\frac{3}{2}})\right\}, \sigma_0 = 1 \end{array} \right\} \tag{4-101}$$

with

$$-\frac{7\pi}{6} < \arg \zeta < \frac{\pi}{6},$$

and

$${}_2\Phi_6 \sim \left\{ \begin{array}{l} \text{const.} \times \zeta^{-\frac{7}{4}} \left\{ \exp\left[\tfrac{2}{3} e^{i\frac{\pi}{4}} \sigma_0^{\frac{1}{2}} \zeta^{\frac{3}{2}}\right]\right\} \left\{1 + O(\zeta^{-\frac{3}{2}})\right\}, \quad \sigma_0 \neq 1 \\ \left\{\text{const.} \times \zeta^{-\frac{1}{4}} + \text{const.} \times \zeta^{-\frac{5}{4}} + \text{const.} \times \zeta^{-\frac{7}{4}}\right\} \left\{ \exp\left[\tfrac{2}{3} e^{i\frac{\pi}{4}} \zeta^{\frac{3}{2}}\right]\right\} \left\{1 + O(\zeta^{-\frac{3}{2}})\right\}, \sigma_0 = 1 \end{array} \right\} \tag{4-102}$$

with

$$-\frac{7\pi}{6} < \arg \zeta < \frac{\pi}{6}.$$

In these expressions $\zeta = (\bar{U}'_{c_1} \alpha Re)^{\frac{1}{3}} (\eta - \eta_{c_1})$.

Applying the limits on $\arg \zeta$ simultaneously gives the familiar requirement

$$-\frac{7\pi}{6} < \arg(\eta - \eta_{c_1}) < \frac{\pi}{6}, \tag{4-26}$$

which must be met in the neighborhood of η_{c_1} for the asymptotic solutions. In the neighborhood of the outer critical point η_{c_2} , the limits on $\arg(\eta-\eta_{c_2})$ are

$$-\frac{\pi}{6} < \arg(\eta-\eta_{c_2}) < \frac{7\pi}{6}. \tag{4-29}$$

These latter limits are established as in other cases by making the transformations $(\eta-\eta_{c_2}) = -(|\bar{U}'_{c_2}|\alpha Re)^{-\frac{1}{3}}\zeta_*$ and $\varphi(\eta) = \Phi(\zeta_*)$ in the original disturbance equation 4-1. The procedure is essentially the same as that first used in Part 4.1.2.3. That is, one finds approximate solutions ${}_2\Phi_3(\zeta_*)$, ${}_2\Phi_4(\zeta_*)$, ${}_2\Phi_5(\zeta_*)$ and ${}_2\Phi_6(\zeta_*)$ by solving an equation identical to equation 4-77a, except that ζ is replaced with ζ_*. Then limits on $\arg\zeta_*$ in asymptotic forms of these solutions are established, from which the limits 4-29 on $\arg(\eta-\eta_{c_2})$ follow.

4.3.4.2 Equivalence of Exponential Asymptotic Solutions with Solutions Valid in Neighborhood of Inner Critical Point

Substituting $\zeta = (\bar{U}'_{c_1}\alpha Re)^{\frac{1}{3}}(\eta-\eta_{c_1})$ in equations 4-99 through 4-102 gives

$$ {}_2\Phi_3 \sim \text{const.} \times (\eta-\eta_{c_1})^{-\frac{5}{4}}\left\{\exp\left[-\frac{2}{3}(i\bar{U}'_{c_1}\alpha Re)^{\frac{1}{2}}(\eta-\eta_{c_1})^{\frac{3}{2}}\right]\right\}\left\{1+O\left[(\alpha Re)^{-\frac{1}{2}}\right]\right\} \tag{4-99a}$$

with

$$-\frac{7\pi}{6} < \arg(\eta-\eta_{c_1}) \frac{5\pi}{6},$$

$$ {}_2\Phi_4 \sim \text{const.} \times (\eta-\eta_{c_1})^{-\frac{5}{4}}\left\{\exp\left[\frac{2}{3}(i\bar{U}'_{c_1}\alpha Re)^{\frac{1}{2}}(\eta-\eta_{c_1})^{\frac{3}{2}}\right]\right\}\left\{1+O\left[(\alpha Re)^{-\frac{1}{2}}\right]\right\} \tag{4-100a}$$

with $$-\frac{11\pi}{6} < \arg(\eta-\eta_{c_1}) < \frac{\pi}{6},$$

$$ {}_2\Phi_5 \sim \left\{ \begin{array}{l} \text{const.} \times (\eta-\eta_{c_1})^{-\frac{7}{4}}\left\{\exp\left[-\frac{2}{3}(i\bar{U}'_{c_1}\sigma_0\alpha Re)^{\frac{1}{2}}(\eta-\eta_{c_1})^{\frac{3}{2}}\right]\right\}\left\{1+O\left[(\alpha Re)^{-\frac{1}{2}}\right]\right\}, \ \sigma_0 \neq 1 \\ \left\{\text{const.} \times (\eta-\eta_{c_1})^{-\frac{1}{4}} + \text{const.} \times (\eta-\eta_{c_1})^{-\frac{5}{4}}\right\} \\ \quad \times \left\{\exp\left[-\frac{2}{3}(i\bar{U}'_{c_1}\alpha Re)^{\frac{1}{2}}(\eta-\eta_{c_1})^{\frac{3}{2}}\right]\right\}\left\{1+O\left[(\alpha Re)^{-\frac{1}{2}}\right]\right\}, \ \sigma_0 = 1 \end{array} \right\} \tag{4-101a}$$

with $$-\frac{7\pi}{6} < \arg(\eta-\eta_{c_1}) < \frac{\pi}{6},$$

and

$$
{}_2\Phi_6 \sim \left\{ \begin{array}{ll} \text{const.}\times(\eta-\eta_{c_1})^{-\frac{7}{4}}\left\{\exp\left[\frac{2}{3}(i\bar{u}'_{c_1}\sigma_0\alpha Re)^{\frac{1}{2}}(\eta-\eta_{c_1})^{\frac{3}{2}}\right]\right\}\left\{1+O\left[(\alpha Re)^{-\frac{1}{2}}\right]\right\}, & \sigma_0 \neq 1 \\ \left\{\text{const.}\times(\eta-\eta_{c_1})^{-\frac{1}{4}}+\text{const.}\times(\eta-\eta_{c_1})^{-\frac{5}{4}}\right\} & \\ \quad \times \exp\left[\frac{2}{3}(i\bar{u}'_{c_1}\alpha Re)^{\frac{1}{2}}(\eta-\eta_{c_1})^{\frac{3}{2}}\right]\Big\}\left\{1+O\left[(\alpha Re)^{-\frac{1}{2}}\right]\right\}, & \sigma_0 = 1 \end{array} \right\} \quad (4\text{-}102\text{a})
$$

with $$\frac{-7\pi}{6} < \arg(\eta-\eta_{c_1}) < \frac{\pi}{6}.$$

If one sets $(\bar{u}-c) = \bar{u}'_{c_1}(\eta-\eta_{c_1})$ in equations 4-76 of Part 4.2.4.3, one will obtain the relations

$$
\left.\begin{array}{l}
{}_2\varphi_3 = \text{const.}\times(\eta-\eta_{c_1})^{-\frac{5}{4}}\exp\left[-\frac{2}{3}(i\bar{u}'_{c_1}\alpha Re)^{\frac{1}{2}}(\eta-\eta_{c_1})^{\frac{3}{2}}+O\left\{(\alpha Re)^{-\frac{1}{2}}\right\}\right], \\
{}_2\varphi_4 = \text{const.}\times(\eta-\eta_{c_1})^{-\frac{5}{4}}\exp\left[\frac{2}{3}(i\bar{u}'_{c_1}\alpha Re)^{\frac{1}{2}}(\eta-\eta_{c_1})^{\frac{3}{2}}+O\left\{(\alpha Re)^{-\frac{1}{2}}\right\}\right], \\
{}_2\varphi_5 = \left\{ \begin{array}{ll} \text{const.}\times(\eta-\eta_{c_1})^{-\frac{7}{4}}\exp\left[-\frac{2}{3}(i\bar{u}'_{c_1}\sigma_0\alpha Re)^{\frac{1}{2}}(\eta-\eta_{c_1})^{\frac{3}{2}}+O\left\{(\alpha Re)^{-\frac{1}{2}}\right\}\right], & \sigma_0\neq 1 \\ \left\{\text{const.}\times(\eta-\eta_{c_1})^{-\frac{1}{4}}+\text{const.}\times(\eta-\eta_{c_1})^{-\frac{5}{4}}\right\} & \\ \quad \times\exp\left[-\frac{2}{3}(i\bar{u}'_{c_1}\alpha Re)^{\frac{1}{2}}(\eta-\eta_{c_1})^{\frac{3}{2}}+O\left\{(\alpha Re)^{-\frac{1}{2}}\right\}\right], & \sigma_0 = 1, \end{array}\right. \\
\text{and} \\
{}_2\varphi_6 = \left\{ \begin{array}{ll} \text{const.}\times(\eta-\eta_{c_1})^{-\frac{7}{4}}\exp\left[\frac{2}{3}(i\bar{u}'_{c_1}\sigma_0\alpha Re)^{\frac{1}{2}}(\eta-\eta_{c_1})^{\frac{3}{2}}+O\left\{(\alpha Re)^{-\frac{1}{2}}\right\}\right], & \sigma_0\neq 1 \\ \left\{\text{const.}\times(\eta-\eta_{c_1})^{-\frac{1}{4}}+\text{const.}\times(\eta-\eta_{c_1})^{-\frac{5}{4}}\right\} & \\ \quad \times\exp\left[\frac{2}{3}(i\bar{u}'_{c_1}\alpha Re)^{\frac{1}{2}}(\eta-\eta_{c_1})^{\frac{3}{2}}+O\left\{(\alpha Re)^{-\frac{1}{2}}\right\}\right], & \sigma_0 = 1, \end{array}\right.
\end{array}\right\} \quad (4\text{-}76\text{a})
$$

which hold in a neighborhood about η_{c_1} in which $(\bar{u}-c)$ is approximated well by $\bar{u}'_{c_1}(\eta-\eta_{c_1})$.

Comparing equations 4-76a with equations 4-99a through 4-102a shows that the following asymptotic relations are true:

$$
\begin{aligned}
{}_2\Phi_3 &\sim \text{const.} \times {}_2\varphi_3 \\
{}_2\Phi_4 &\sim \text{const.} \times {}_2\varphi_4 \\
{}_2\Phi_5 &\sim \text{const.} \times {}_2\varphi_3 + \text{const.} \times {}_2\varphi_5 \\
{}_2\Phi_6 &\sim \text{const.} \times {}_2\varphi_4 + \text{const.} \times {}_2\varphi_6
\end{aligned}
\tag{4-82}
$$

APPENDIX 5

ESTIMATE OF THE EFFECT OF THE COUPLING TERM IN THE COMBINED MOMENTUM EQUATION WITH THE ASSUMPTION THAT $\frac{f}{\alpha Re}$ IS SMALL

5.1 The Equations Considered

The object of this investigation is to estimate quantitatively the effect of coupling between the combined momentum and energy equations in the case of free convection of air. The complete combined momentum equation is

$$(\bar{u}-c)(\varphi''-\alpha^2\varphi)-\bar{u}''\varphi+\frac{f}{\alpha Re}\left\{ia_1 S'-\alpha a_2 S\right\}+\frac{i}{\alpha Re}\left\{\varphi^{IV}-2\alpha^2\varphi''+\alpha^4\varphi\right\}=0, \tag{1a}$$

while the uncoupled form of it, the Orr-Sommerfeld equation, is

$$(\bar{u}-c)(\varphi''-\alpha^2\varphi)-\bar{u}''\varphi+\frac{i}{\alpha Re}\left\{\varphi^{IV}-2\alpha^2\varphi''+\alpha^4\varphi\right\}=0. \tag{4-3}$$

This simpler form was employed in the study of the stability of the laminar free convection of air which is described in Section D. The energy equation is rewritten for reference:

$$(\bar{u}-c)S-\bar{\theta}'\varphi+\frac{i}{\sigma_0\alpha Re}\left\{S''-\alpha^2 S\right\}=0 \tag{2}$$

5.2 Modification of Nonviscous Solutions of the Orr-Sommerfeld Equation by Coupling

Nonviscous solutions of the Orr-Sommerfeld equation 4-3 are developed in Part 4.1.1 of Appendix 4. Solutions corresponding to these but with coupling taken into account are obtained by considering the combined momentum equation 1a with the viscous term

$\frac{i}{\alpha Re}\left\{\varphi^{IV}-2\alpha^2\varphi''+\alpha^4\varphi\right\}$ neglected and the energy equation 2 with the conduction term $\frac{i}{\sigma_0 \alpha Re}\{s'' - \alpha^2 s\}$ dropped. Neglecting this conduction term in the energy equation is consistent with dropping the viscous term of the combined momentum equation since σ_0 should be on the order of unity when f is of moderate size, which is implied by the assumption that $\frac{f}{\alpha^{\nu} Re}$ is small. Solutions in terms of φ of the nonviscous combined momentum equation and the nonconducting energy equation are developed in Part 4. 2. 1 of Appendix 4. An examination of them shows that they reduce to the nonviscous solutions of the Orr-Sommerfeld equation as $\frac{f}{\alpha^{\nu} Re} \longrightarrow 0$. Just how much one of the coupled nonviscous solutions differs from its corresponding uncoupled nonviscous solution cannot be determined without computing both solutions for comparison. However, one might estimate that the relative difference would be small when $\frac{f}{\alpha^{\nu} Re}$ is, for instance, less than 0. 1.

5. 3 Effect of Coupling on Viscous Solutions of the Orr-Sommerfeld Equation

For simplicity, the situation in the neighborhood of the inner critical point η_{c_1} will be considered. Viscous solutions of the Orr-Sommerfeld equation valid close to η_{c_1} are developed in Part 4. 1. 2 of Appendix 4 through the application of the transformations $(\eta - \eta_{c_1}) = (\bar{u}'_{c_1} \alpha Re)^{-\frac{1}{3}} \zeta$ and $\varphi(\eta) = \Phi(\zeta)$. These are the solutions utilized in the investigation of free convection in air which is described in Section D. A similar procedure can be used for obtaining an approximation to the temperature disturbance function s . It is assumed in the energy equation 2 that a viscous solution $\varphi_0(\eta)$ or $\Phi_0(\zeta)$ has been obtained from the Orr-Sommerfeld equation. One sets $s(\eta) = S(\zeta_+)$, in which $\zeta_+ = \ell^{-1}(\eta - \eta_{c_1})$, ℓ being proportional to a negative power of $\sigma_0 \alpha Re$, and expands known functions in Taylor series about $\eta = \eta_{c_1}$. The result is

$$\frac{i}{\sigma_0 \alpha Re}\left\{ \ell^{-2} S''(\zeta_+) - \alpha^2 S(\zeta_+) \right\} + \bar{U}'_{c_1} \ell \zeta_+ S(\zeta_+) - \left\{ \bar{\Theta}'_{c_1} + \bar{\Theta}''_{c_1} \ell \zeta_+ \right\} \Phi_0(\zeta) + O(\ell^2) = 0 . \tag{5-1}$$

To formalize the retention of $S''(\zeta_+)$ and the largest other term when $\sigma_0 \alpha Re \gg 1$, one can take $\ell = (\bar{\Theta}'_{c_1} \sigma_0 \alpha Re)^{-\frac{1}{2}}$. Then with the neglect of higher-order terms in ℓ the equation becomes

$$S''(\zeta_+) + i \Phi_0(\zeta) = 0, \tag{5-2}$$

from which one can determine that

$$s'(\eta) = -i \bar{\Theta}'_{c_1} \sigma_0 (\bar{U}'_{c_1})^{-\frac{1}{3}} (\alpha Re)^{\frac{2}{3}} \int^{\zeta} d\Omega \, \Phi_0(\Omega) \tag{5-3}$$

and

$$s(\eta) = -i \bar{\Theta}'_{c_1} \sigma_0 (\bar{U}'_{c_1})^{-\frac{2}{3}} (\alpha Re)^{\frac{1}{3}} \int^{\zeta} d\Omega_1 \int^{\Omega_1} d\Omega_2 \, \Phi_0(\Omega_2) \, . \tag{5-4}$$

Substituting these expressions for s' and s into equation 1a and applying the transformations $(\eta - \eta_{c_1}) = (\bar{U}'_{c_1} \alpha Re)^{-\frac{1}{3}} \zeta$ and $\varphi(\eta) = \Phi_1(\zeta)$ give the equation

$$\Phi_1^{IV} - i \zeta \Phi_1'' = O\left\{ (\alpha Re)^{-\frac{1}{3}}, \ f \cdot (\alpha Re)^{-\frac{2}{3}} \right\} . \tag{5-5}$$

Unless f is much larger than $(\alpha Re)^{\frac{1}{3}}$, the coupling terms can be neglected within the same degree of approximation as that used in obtaining the uncoupled, viscous solutions which satisfy the equation

$$\Phi^{IV} - i \zeta \Phi'' = 0. \tag{4-17c}$$

5.4 Effect of Coupling at the Plate Surface

Because boundary conditions are applied at the plate surface, it is desirable to investigate the effect of coupling there. Very close to the plate surface the known functions of η that appear in the combined momentum and energy equations can be considered to be constant and to have their values at the plate surface, which are denoted by the subscript p. The combined momentum and energy equations 1a and 2 become

$$-c(\varphi''-\alpha^2\varphi)-\bar{u}''_p\varphi+\frac{f}{\alpha\sqrt{Re}}\left\{ia_1 s'-\alpha a_2 s\right\}+\frac{i}{\alpha Re}\left\{\varphi^{IV}-2\alpha^2\varphi''+\alpha^4\varphi\right\}=0 \tag{5-6}$$

and

$$-cs-\bar{\theta}'_p\varphi+\frac{i}{\sigma_0\alpha Re}\left\{s''-\alpha^2 s\right\}=0. \tag{5-7}$$

This pair of differential equations with constant coefficients has in general six linearly independent solutions of the form $e^{A\eta}$, in which A is a constant determined by the algebraic equation

$$\left\{A^4\left(\frac{i}{\alpha Re}\right)+A^2\left(-c-\frac{i2\alpha^2}{\alpha Re}\right)+\left(\alpha^2c-\bar{u}''_p+\frac{i\alpha^4}{\alpha Re}\right)\right\}\left\{A^2\left(\frac{i}{\sigma_0\alpha Re}\right)+\left(-c-\frac{i\alpha^2}{\sigma_0\alpha Re}\right)\right\}$$
$$-\left\{A\left(ia_1\frac{f}{\alpha\sqrt{Re}}\right)+\left(-\alpha a_2\frac{f}{\alpha\sqrt{Re}}\right)\right\}\left\{-\bar{\theta}'_p\right\}=0. \tag{5-8}$$

With coupling neglected, the combined momentum equation in the region very close to the wall simplifies to

$$-c(\varphi''-\alpha^2\varphi)-\bar{u}''_p\varphi+\frac{i}{\alpha Re}\left\{\varphi^{IV}-2\alpha^2\varphi''+\alpha^4\varphi\right\}=0, \tag{5-9}$$

which has four linearly independent solutions $e^{B\eta}$, in which four values of B are determined by the algebraic equation

$$B^4\left(\frac{i}{\alpha Re}\right)+B^2\left(-c-\frac{i2\alpha^2}{\alpha Re}\right)+\left(\alpha^2c-\bar{u}''_p+\frac{i\alpha^4}{\sigma_0\alpha Re}\right)=0. \tag{5-10}$$

This equation becomes the same as one of the two factors of the first line of equation 5-8 if B is replaced with A. Since the first line of equation 5-8 is of order 1 and the second line is of order $\frac{f}{\alpha Re}$, one can expect when $\frac{f}{\alpha Re}$ is small that four of the solutions A of equation 5-8 will closely correspond to the four solutions B of equation 5-10. Thus, if $\frac{f}{\alpha Re}$ is small the solutions of the uncoupled combined momentum equation are good approximations at the plate surface to four of the solutions of the coupled combined momentum and energy equations.

By considering only the combined momentum equation and boundary conditions on the velocity disturbance, one cannot in general expect to satisfy boundary conditions on the temperature disturbance exactly. However, a consideration of the general energy equation 2 indicates that the isothermal and adiabatic boundary conditions on the temperature disturbance are satisfied within an order of magnitude of $\frac{1}{\alpha Re}$ if $c \neq 0$ and the usual viscous boundary conditions on the velocity disturbance are applied.

APPENDIX 6

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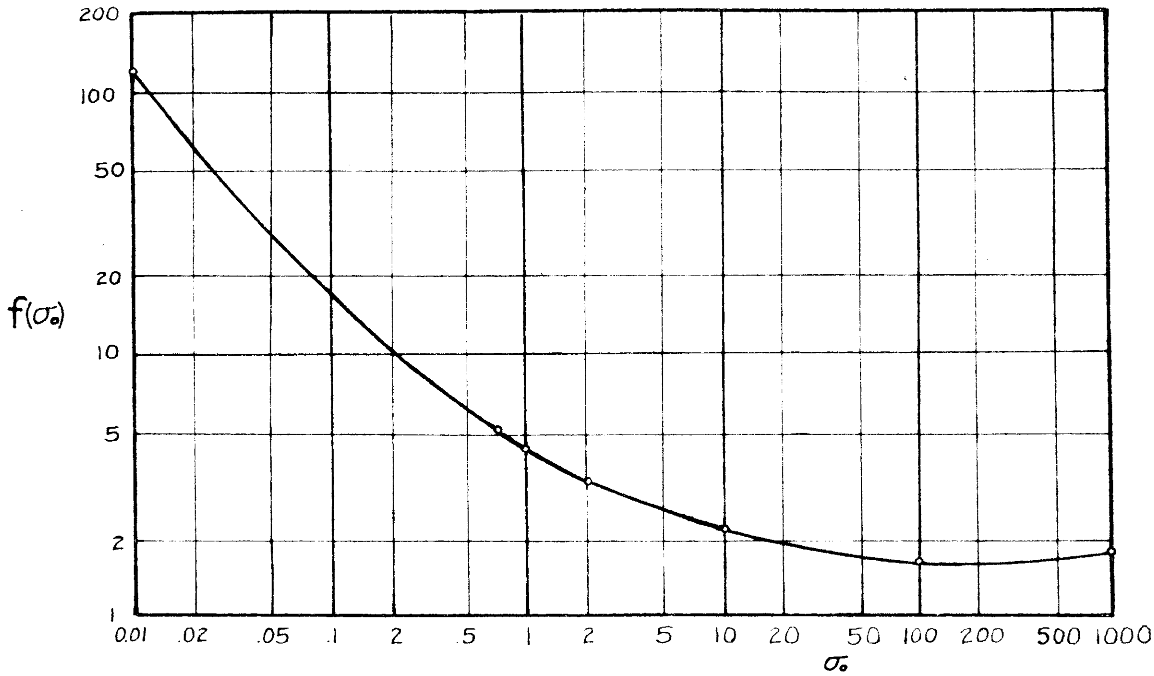

Figure 1

$f(\sigma_0)$ Computed from Reference 14 According to Appendix 3

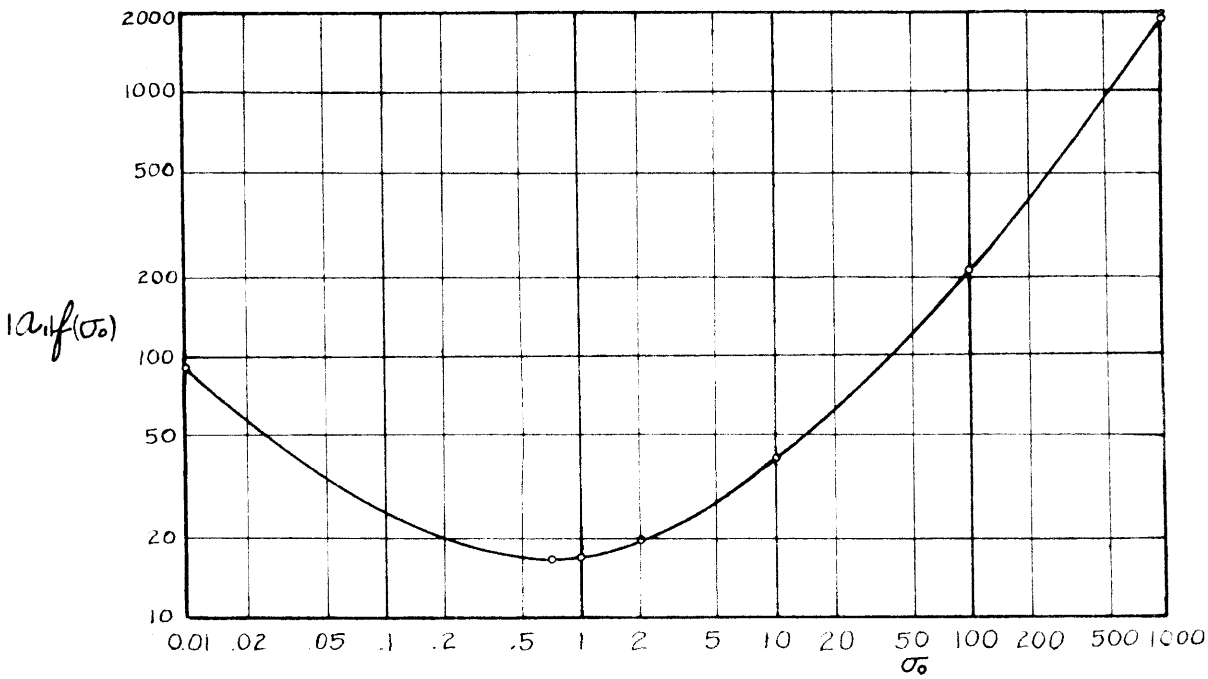

Figure 2

$|a_{11}|f(\sigma_0)$ Computed From Reference 14 According to Appendix 3

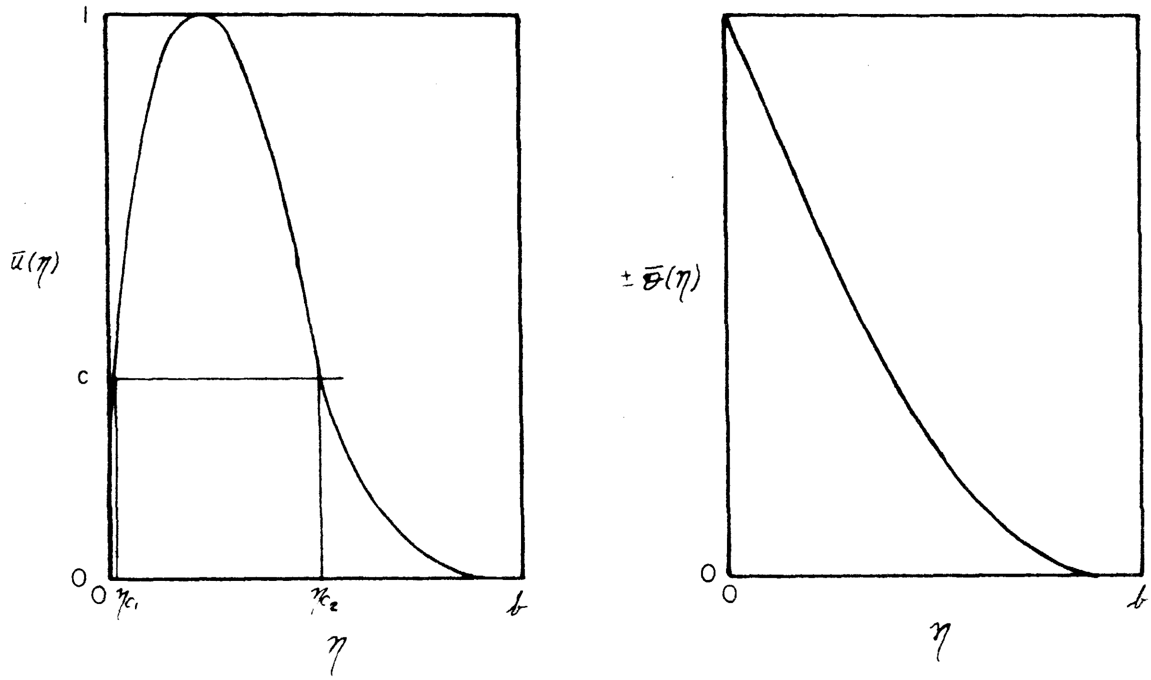

Figure 3

General Shapes of Dimensionless Velocity and Temperature Profiles

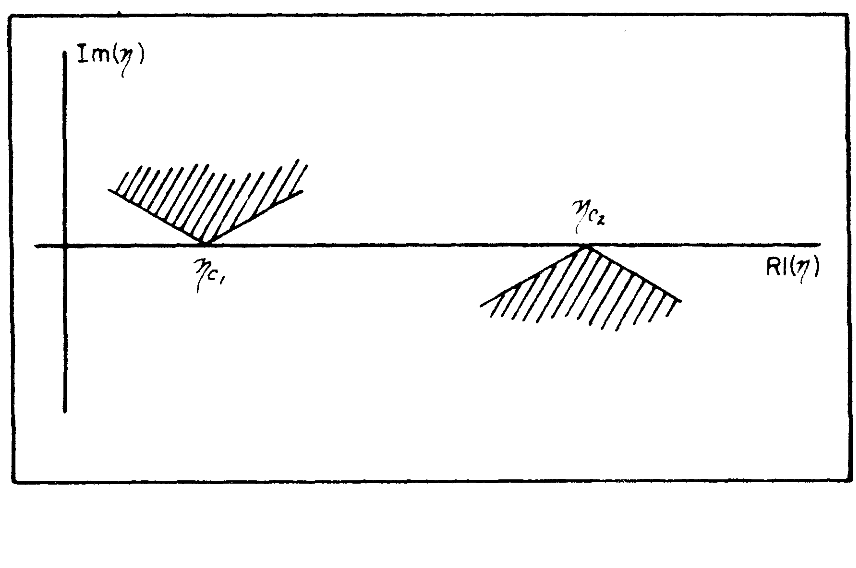

Figure 4

Regions of Invalidity of Asymptotic Solutions in Complex η Plane

(Solutions Invalid in Shaded Areas)

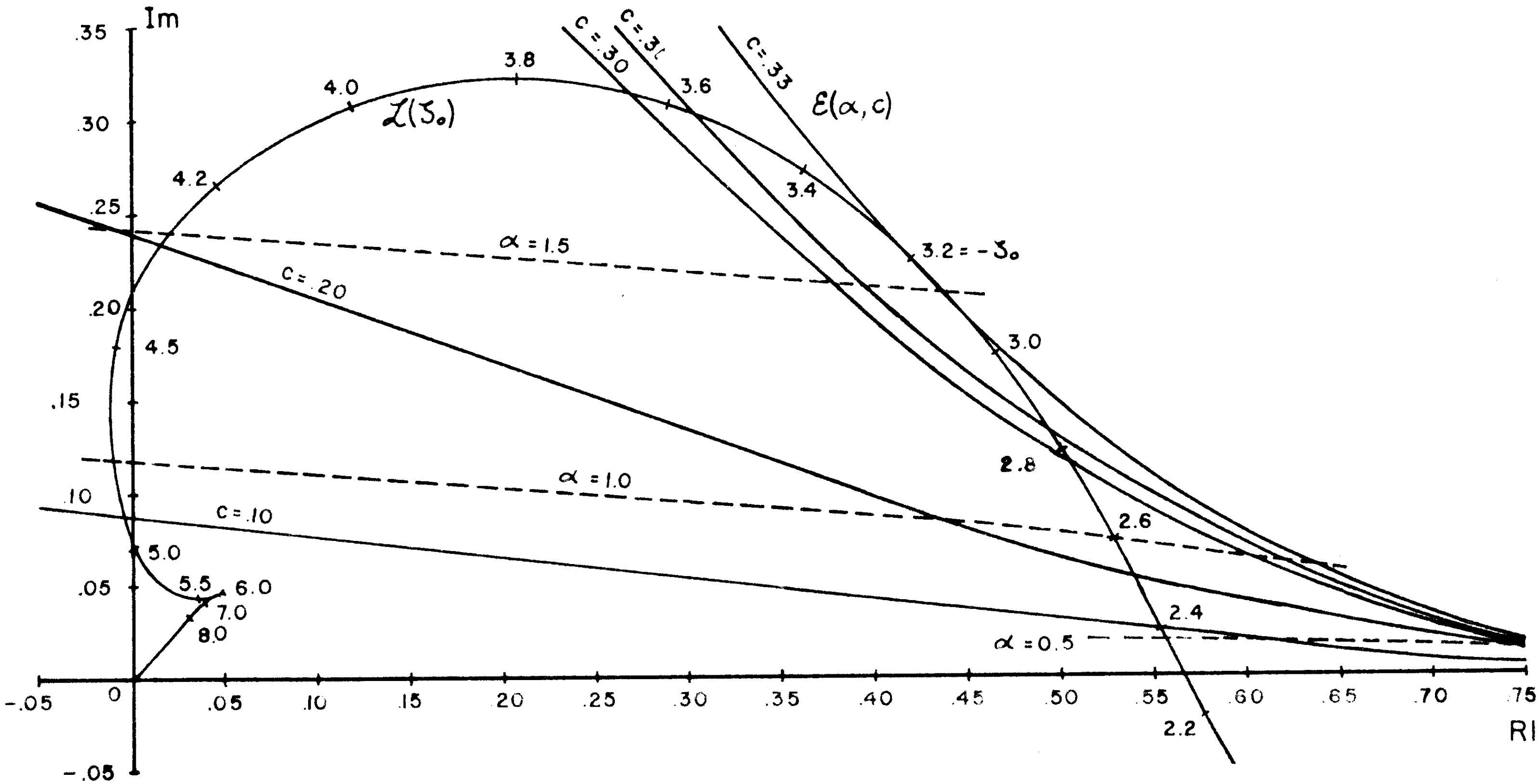

Figure 5

Solution of the Equation $\mathcal{E}(\alpha, c) = \mathcal{L}(\zeta_0)$ of Section **D**

For the Cubic Polynomial Velocity Profile with $\sigma_0 = 072$

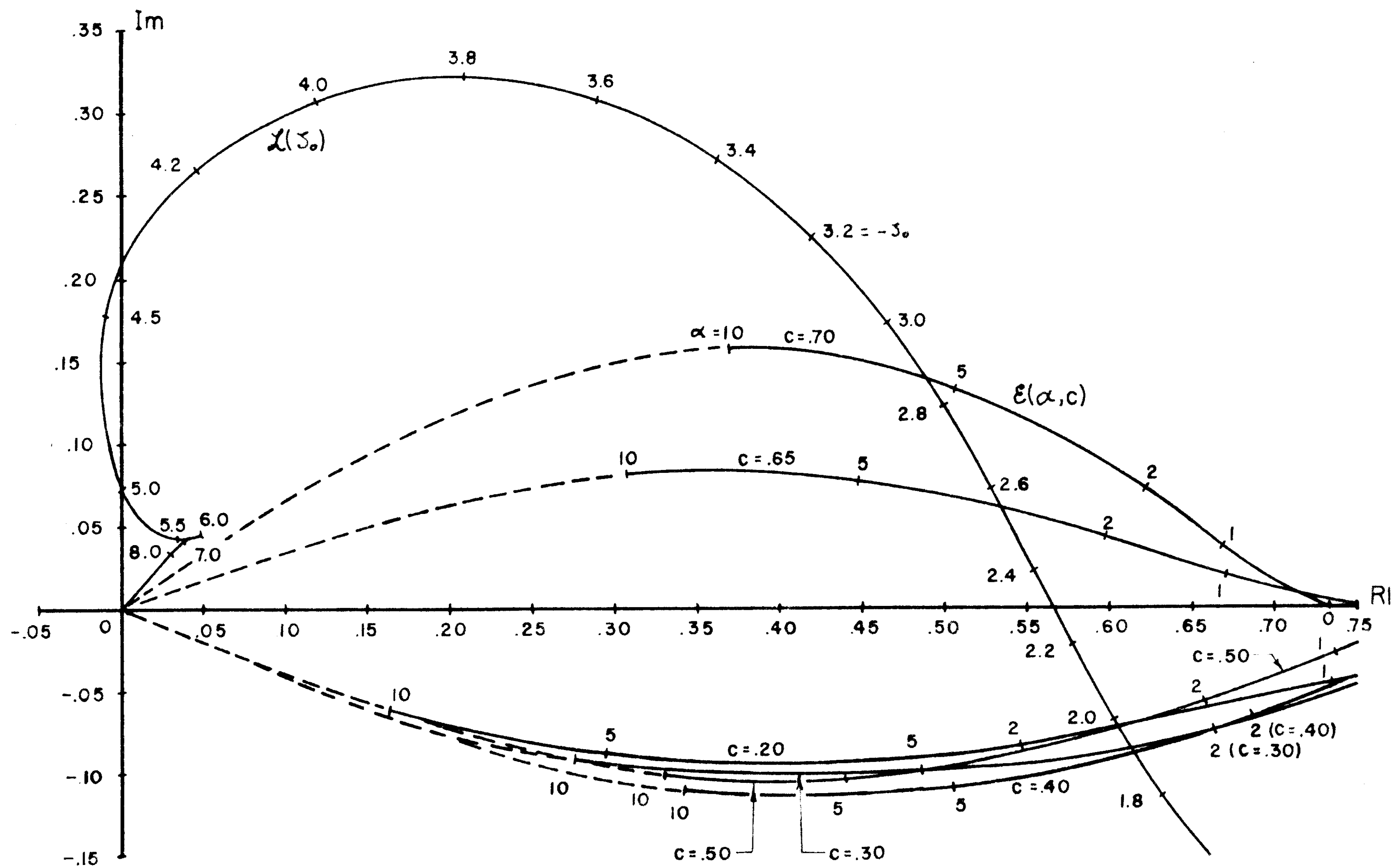

Figure 6

Solution of the Equation $\mathcal{E}(\alpha,c) = \mathcal{L}(\zeta_o)$ of Section D
For the Exact Velocity Profile with $\sigma_o = 0.72$

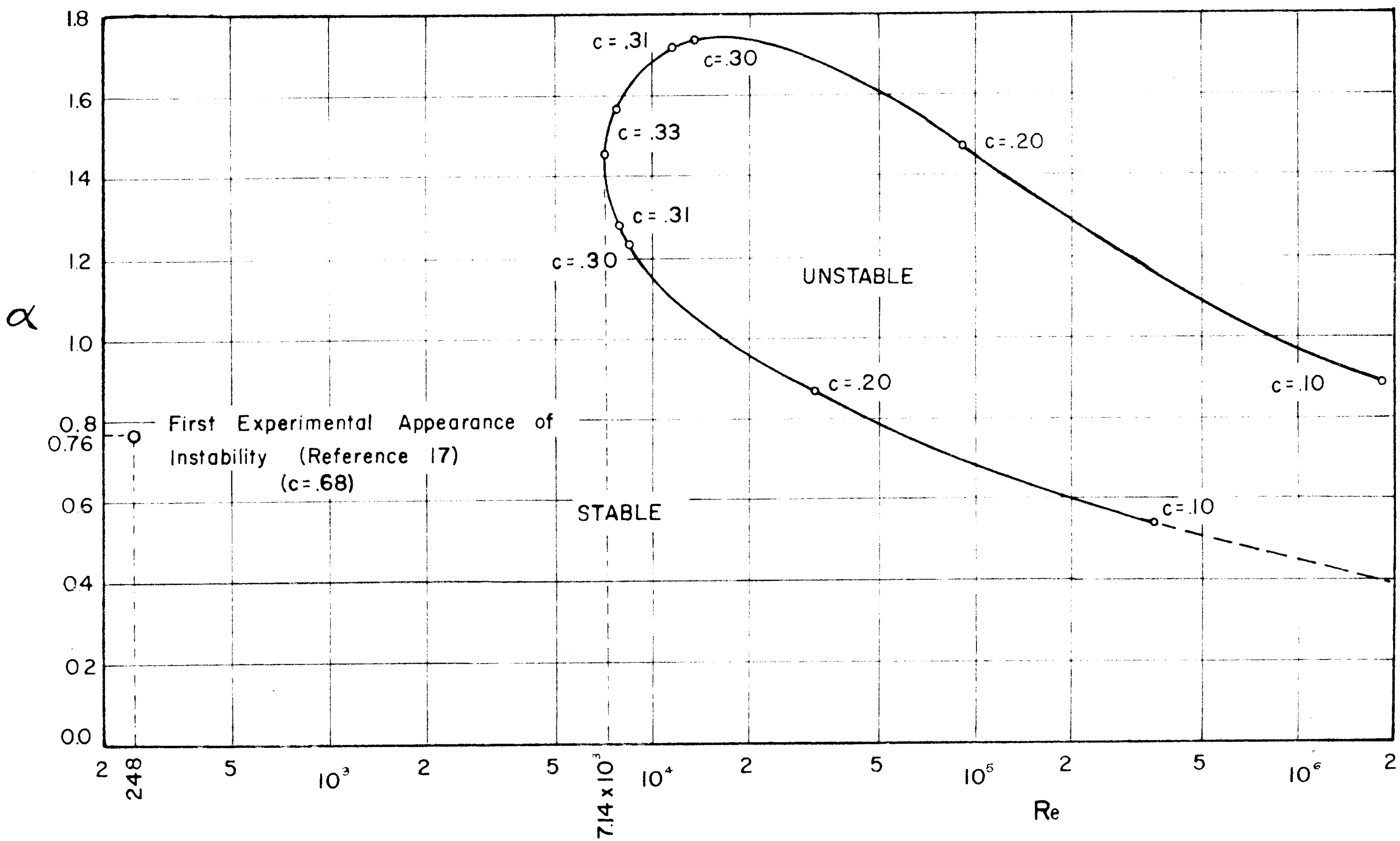

Figure 7

Indifference Curve for Cubic Polynomial Velocity Profile for

$\sigma_0 = 0.72$

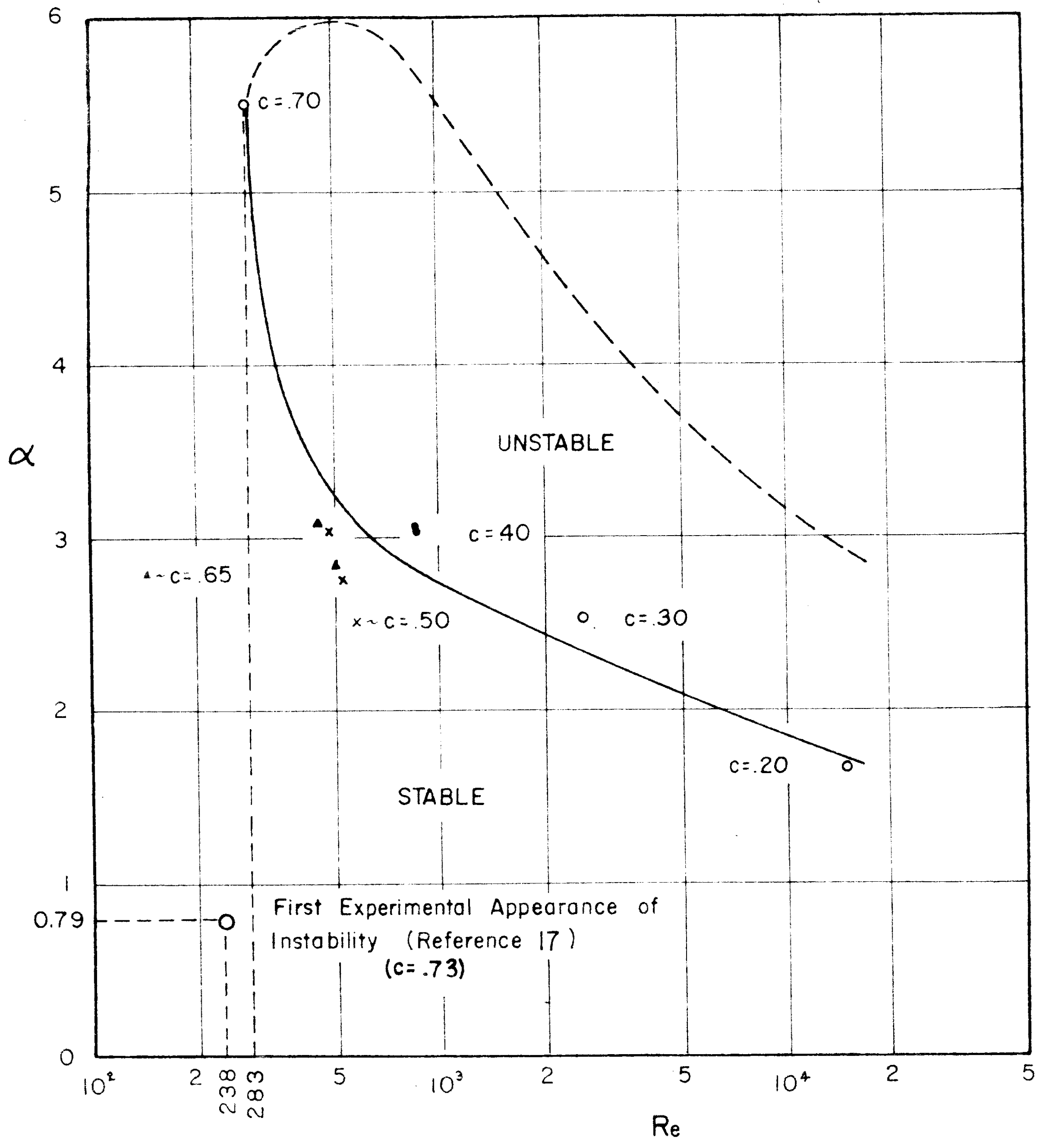

Figure 8
Indifference Curve for "Exact" or Tabulated Velocity
Profile for $\sigma_o = 0.72$

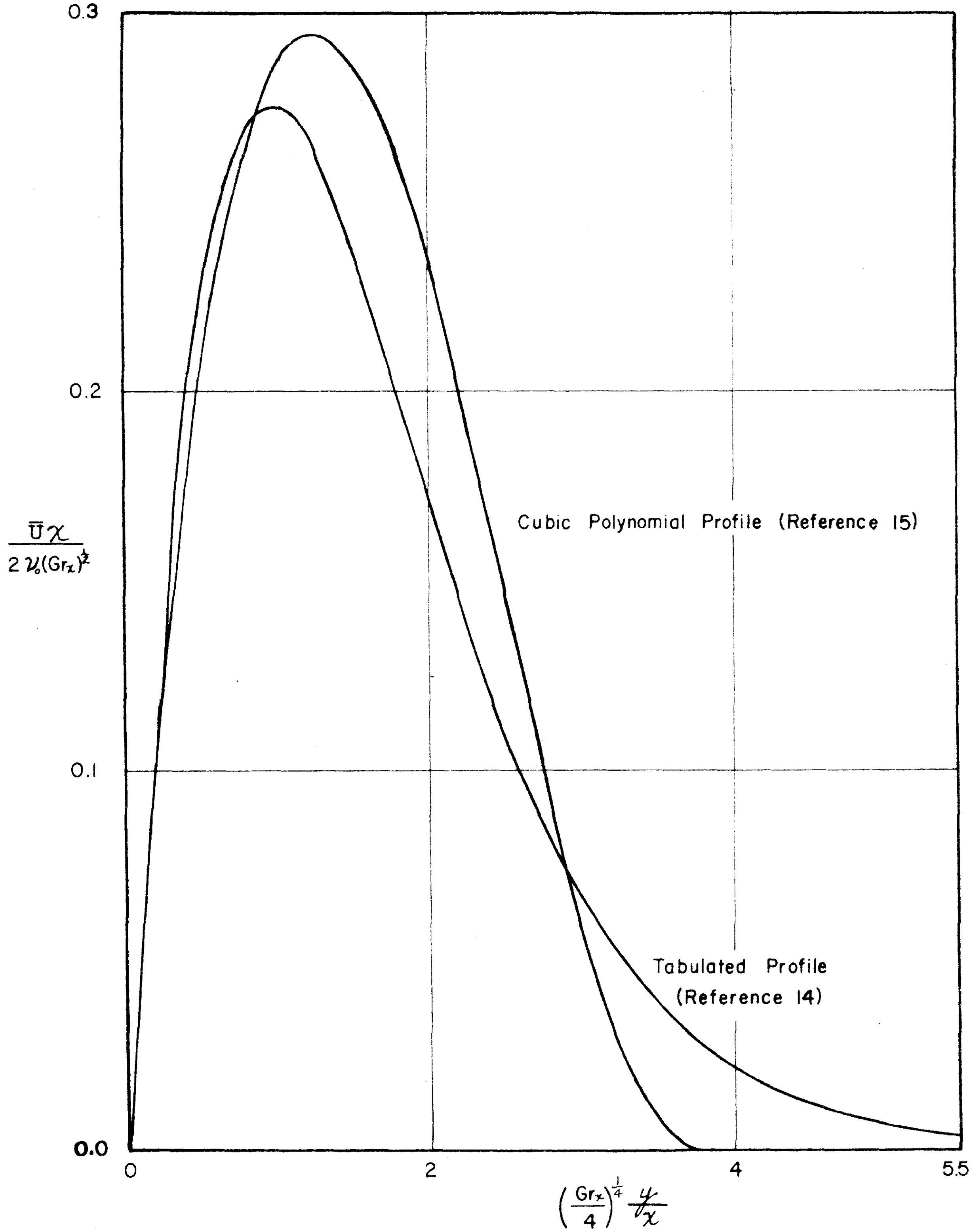

Figure 9

Comparison of Cubic Polynomial and "Exact" or Tabulated Velocity Profiles

II

LAMINAR FREE CONVECTION WITH VARIABLE FLUID PROPERTIES

A

SUMMARY

This work is an analytic study of laminar free convection about an inclined* or vertical*, isothermal, semi-infinite, flat plate in a fluid which far from the plate is at rest and at a temperature different from that of the plate. In contrast to previous analyses of essentially the same configuration in which the only consideration given to variations in fluid properties was to take the density to be linearly dependent on the temperature, all fluid properties in the present treatment were considered to vary with temperature.

On the basis of experimental observations that the velocity and temperature fields in the fluid surrounding such a plate are of the boundary layer type, the general equations of fluid mechanics with the assumption of variable fluid properties were simplified to the forms describing free convection flow. It was found that the process of laminar free convection about an inclined plate is fundamentally the same as that about a vertical plate if the driving body force is taken to be the component of the total body force that is parallel to the plate.

Through the introduction of similarity variables into these boundary layer equations in the same manner as had been done earlier by others in studying the essentially constant-property case, a reduction of the relations to two simultaneous nonlinear differential equations in one independent variable and two dependent variables was accomplished.

* "Inclined" or "vertical" indicates that the body force, such as that of gravity, which produces the flow is either inclined or parallel to the surface of the plate.

An examination of these equations and the definitions of the similarity variables used to derive them indicated that the fluid property variations with temperature in addition to the Prandtl number of the fluid far from the plate and an appropriately defined Grashof number determine the local Nusselt number at a given distance from the edge of the plate. These equations with suitable boundary conditions were not solved for two reasons, the first being that any solution of the variable-property problem applies only to the case of a particular fluid with the plate and the fluid far from it at specific temperatures, and the second being that very extensive numerical computation would be necessary to effect such a solution.

In place of this similarity method of securing an exact numerical solution, an integral method was developed for obtaining approximately the dependence of the Nusselt number on the Grashof number for a given fluid with stated temperatures of the plate and of the fluid far from it. This method is an adaption of a process developed by another investigator for the constant-property problem, and its application requires the assumption of approximate velocity and temperature profiles for the boundary layer. With its use heat transfer between a plate and an oil with a Prandtl number varying from 10 to 100 was investigated for the cases of both heating and cooling the fluid. Also, an experimental study of heat transfer from a vertical heated plate to a transformer oil which was made by an earlier researcher was treated by this integral method in order to compare the findings of analysis with those of observation. The analytic treatment gave a heat-transfer rate 13 per cent higher than the experimentally-determined value, a discrepancy that was explained by the presence of errors inherent in the approximate method and the lack of exact correspondence between the assumed and experimental situations.

By comparing the heat-transfer rates determined by the variable-property method with those found by another investigator on constant-property assumptions it was concluded that even when the variations in fluid properties are extreme the constant-property analyses give good results if they are based on the properties of the fluid at the plate. In addition, it was suggested that applying the results of constant-property analyses or of experiments with small property variations to cases of free convection about isothermal surfaces other than flat plates in an infinite fluid should give accurate values of heat-transfer rates when the property variation is large if the constant-property results are based on the fluid properties at the surface.

B

INTRODUCTION

B1 Heat Transfer by Laminar Free Convection

As stated in the preceding Part I of this thesis, "free" or "natural" convection is herein taken to be the process of energy transfer between a convecting fluid and a bounding surface in the case that the motion of the fluid is caused by the interaction of a body force field, usually that of gravity, with a variable density field in the fluid. The process is inherently difficult to describe mathematically because to do so requires the simultaneous solution of the energy, continuity, and momentum equations describing the flow as well as the equation of state of the convecting fluid.

In general, when free convection occurs in the transfer of energy between a solid object of finite size and a surrounding fluid of infinite extent, the flow and the temperature changes in the fluid can be considered to be confined to a boundary layer adjacent to the surface of the object insofar as the process of heat transfer from or to the object is concerned. The flow is laminar near the stagnation point (or points) at which it originates, and unless the object is of an extremely irregular shape the flow will remain laminar for some distance as it proceeds along the surface of the object. If the Grashof number* based on, for instance, the maximum linear dimension of the object and the maximum temperature difference between its surface and the fluid far away is sufficiently

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* The Grashof number is a free convection parameter analogous to the Reynolds number of forced convection. For the case discussed here the Grashof number would be taken to be $\frac{g\beta_0(\Delta T)_M \ell_M^3}{\nu_0^2}$,

g being the magnitude of the driving body force, β_0 being the large, the flow will become turbulent while yet remaining as a boundary layer adjacent to the surface. If the object is of a regular shape and has a smooth surface one can expect that there will be a minimum Grashof number below which the boundary layer will be laminar over the entire surface.

B2 Historical Survey of the Principal Previous Analytic Treatments of Laminar Free Convection

In a paper by Schmidt and Beckmann(1), Polhausen's solution of the problem of laminar, two-dimensional free convection about an isothermal, semi-infinite, vertical flat plate was presented. The Prandtl number of the fluid was taken to be 0.733, which corresponds to that of air. A linear variation of fluid density with temperature was adopted for treating the convective driving force term of the momentum equation; otherwise, constant fluid properties were assumed. It was assumed that the temperature and velocity fields were of the boundary layer type, and similarity relations were found which reduced the problem to that of solving numerically two simultaneous nonlinear ordinary differential equations in one independent variable and two dependent variables. Good agreement with Schmidt and Beckmann's measurements of temperature and velocity profiles in air was obtained.

Hermann(2) applied boundary layer theory to the problem of laminar, two-dimensional free convection about a horizontal circular cylinder and obtained fairly good agreement with experimental observa-

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coefficient of thermal expansion of the fluid far from the plate, $(\Delta T)_M$ being the greatest temperature difference between the surface of the object and the fluid far from it, l_M being the greatest linear dimension of the object, and ν_o being the kinematic viscosity of the fluid far from the object.

tions. His assumptions regarding fluid properties were the same as those of Polhausen.

Squire[3] solved the laminar problem for an isothermal vertical plate approximately by integrating the momentum and energy equations with respect to distance normal to the plate with the use of simple polynomial representations of the velocity and temperature profiles in the boundary layer. He also treated the fluid properties as did Polhausen. For the velocity and temperature profiles, similarity relations were found which gave the correct dependence of the maximum velocity in the boundary layer and the boundary layer thickness on the distance from the leading edge of the plate. In addition, the ratio at a given Grashof number of Squire's local or average Nusselt number to the corresponding one obtained by solving Polhausen's equations numerically does not vary by more than some 10 per cent from unity over the range of Prandtl number variation 0.01 to 1000, according to Ostrach[4]. This good agreement is somewhat surprising because of the wide discrepancy for Prandtl numbers greater than 5 or 10 between Squire's assumed velocity and temperature profiles and those obtained by the numerical solution of Polhausen's equations.

Saunders[5] secured approximate solutions of Polhausen's differential equations by a different method. He first reduced them to a single equation in terms of a temperature variable* and derivatives of that temperature variable with respect to the similarity independent variable. He then assumed that the first derivative of the temperature

* This temperature variable was the temperature of the fluid suitably nondimensionalized.

variable could be written as a simple polynomial in the temperature variable which satisfied the boundary conditions both at the plate surface and far from it. Substituting at appropriate values of the independent variable into the differential equation was employed to determine unknown constants appearing in the assumed polynomial. With these constants determined, Saunders was able to obtain the Nusselt number as a function of the product of the Grashof and Prandtl numbers for fluids of different Prandtl numbers.

Compared with the results of Ostrach's solutions[4] of Polhausen's equations, the heat-transfer coefficients found by the most accurate treatment of Saunders range from approximately 4 per cent high at a Prandtl number of 0.01 to 16 per cent high at a Prandtl number of 40. (Forty is the maximum value of the Prandtl number for which Saunders presented the results of his calculations.)

The solution of Polhausen's equations for Prandtl numbers of 0.73, 10, 100, and 1000 was accomplished by Schuh[6], who used a numerical method analogous to the Stodola-Vianello method of obtaining characteristic values and characteristic solutions in linear eigenvalue problems. Included in his work was a brief discussion of the effect of inclination of the plate. In addition, he derived similarity equations describing the situation in the neighborhood of the stagnation point on a body in free convection as well as for the cases of free convection resulting from a heat source distribution in a vertical plane and on a vertical axis of symmetry. These latter three cases were solved for a Prandtl number of 0.7 with Polhausen's assumptions concerning fluid properties.

Ostrach and Albers[4] solved Polhausen's equations with

Prandtl numbers of 0.01, 0.72, 0.733, 1, 2, 10, 100, and 1000. These solutions were obtained with the use of an electronic digital computer and should be quite accurate.

Very recently Sparrow and Gregg[7] solved the problem of laminar free convection from a semi-infinite vertical flat plate with a uniform heat flux rather than with a constant surface temperature. They made the same assumptions regarding the fluid properties as did the other investigators mentioned here. It is of interest that they utilized similarity relations different from those of Polhausen in order to apply their boundary condition of constant heat flux at the plate surface. By means of these similarity relations they arrived at a pair of nonlinear ordinary differential equations somewhat similar to, but not identical with, those of Polhausen. For values of the Prandtl number of 0.1, 1, 10, and 100 these equations were solved by utilizing an electronic computer as did Ostrach and Albers.

B3 Free Convection About an Isothermal Flat Plate with Large Variations in Properties of the Convecting Fluid

To the knowledge of the author, assuming that the fluid density varies linearly with temperature in the body force term of the momentum equation is the only consideration given to variation of fluid properties in previous analytic treatments of free convection. The density in this body force term of the momentum equation must be considered to vary in order to take into account the hydrostatic driving force which is characteristic of free convection. In all other terms of the equations of fluid motion the density is considered to remain constant, as are the viscosity, the specific heat, and the thermal conductivity.

These assumptions regarding fluid properties best describe an actual situation when the relative variation in each of the fluid properties between the fluid at the solid boundary and far from it is small. In the case of laminar free convection between an isothermal solid boundary and a fluid which is isothermal far from the boundary*, one can assume that the fluid properties are functions of temperature only rather than of both temperature and pressure, since experiments have indicated that transition to turbulent flow occurs at velocities sufficiently low so that attendant pressure variations in the flow field have only negligible effects on fluid properties. Thus, these assumptions concerning constancy of fluid properties are restricted to cases in which the difference in temperature between the bounding surface and the surrounding fluid is small. How large this temperature difference can be is dependent on the rate of variation of the fluid properties with temperature, the allowable temperature difference being large when the rate of property variation is small.

With the mentioned simple assumptions respecting fluid properties, analysis such as that done by Polhausen(1) indicates that for given boundary conditions and a given configuration the rate of heat transfer expressed as a Nusselt number is a function of only the Prandtl number of the fluid and a Grashof number related to the situation under consideration. (The values of fluid properties which enter into these parameters can be considered taken at the temperature of the fluid far from the surface.) In a sense the only variation in fluid properties which is considered, the change of density, is contained in the Grashof number in that the coefficient of thermal expansion is one of the terms which

* The fluid at this isothermal condition far from the boundary will be designated as "ambient" fluid, or it will be said to be at "ambient conditions".

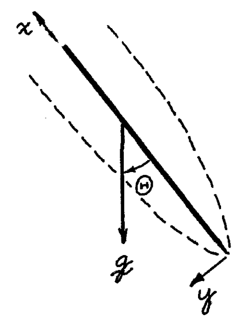
appear in this dimensionless group.

In attempting to correlate the results of analysis and experiment or to apply analytic results to actual situations in free convection, one is led to ask what effects large variations in fluid properties have on the process. For the free convection of an oil the question is particularly significant because of the rapid change of the viscosity and hence the Prandtl number of the fluid with temperature. This part of the thesis is an attempt to answer to some extent this question.

C

THE EQUATIONS DESCRIBING LAMINAR FREE CONVECTION ABOUT AN ISOTHERMAL INCLINED PLATE AND THEIR SOLUTION

C1 The Physical Configuration Treated



Above is sketched the two-dimensional configuration to be treated analytically. This configuration was chosen for the relative simplicity of its analysis and the availability of both experimental observations and earlier analyses made with the assumption of constant fluid properties for comparison with the present work. The semi-infinite plate, shown in cross section, is oriented with respect to the body force field represented by the vector g so that flow proceeds away from the leading edge in the region indicated by the dotted lines. In the sketch the orientation of the body force is such that the plate temperature would have to be higher than the ambient fluid temperature in order that the flow proceed away from the leading edge. If the temperature of the plate were below that of the ambient fluid, the component of g parallel to the plate would have to be reversed in order that flow proceed away from the leading edge.

The co-ordinate system shown on the sketch is established for a consideration of the situation on the lower surface of the plate. x is measured from the leading edge, and y is measured from the surface normal to the surface.

Some factors of static instability enter into a treatment of this type of flow. Consider that for the situation shown in the sketch the plate temperature is above the ambient fluid temperature and that the angle Θ between the body force vector and the x- axis has a value between 0 and 90 degrees. Then the density of the fluid above the upper surface of the plate will increase with distance away from the plate. In the limiting case of Θ = 90 degrees an adequately large value of the difference between the plate and ambient fluid temperatures might produce an irregular free convection flow of somewhat the same nature as the cellular pattern which appears in a horizontal layer of fluid heated sufficiently strongly from below. One is led to suspect that the statically unstable condition in which the density increases in a direction opposite to the direction of the body force field could be responsible for eventual separation of the flow on one side of an inclined, rather than horizontal, plate, although extrapolating from the static to the dynamic case must be done cautiously.

Only two reports of experimental studies for inclined plates in which this factor of hydrostatic instability enters were found by the author. Schlieren photographs by Schmidt[9] show possibly only incipient separation on the upper surface of a heated plate in air with a value of Θ of 79 degrees at Grashof numbers based on length from the plate leading edge and the component of g parallel to the plate of roughly 4 to 5 x 10^6. Interferometric studies by Rich[10] on an inclined plate in air were apparently done with the boundary layer still attached.

There are reasons to suspect that transition to turbulent flow will occur on the upper and lower surfaces of an inclined plate at values of the Grashof number Gr_{x_0}* different from the value for which it occurs on a vertical plate. These reasons are the results of Prandtl's simplified energy considerations(11) based on Richardson's work(12), (13) concerning turbulence in a flow having stratification of density with a body force field present and Schlichting's findings(14) regarding the stability with respect to small oscillations of similarly stratified forced convection flows. One should expect that compared with the case of a vertical plate, for which transition is considered to occur at a value of Gr_{x_0} of about 10^9, the process will occur at lower values of the parameter on the side of an inclined plate for which hydrostatic forces act to push the fluid away from the plate and at higher values on the other side.

The present analysis is restricted to the case of an attached, laminar flow.

C2 Boundary Layer Equations Describing Laminar Free Convection with Variable Fluid Properties Along an Inclined Isothermal Flat Plate

The continuity equation, components of the vector momentum equation parallel to and normal to the plate, and energy equation for steady flow with variable fluid properties are, respectively,

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* Gr_{x_0} is the Grashof number based on the ambient fluid properties, the distance from the leading edge of the plate, and the component of the body force vector along the plate. It is equal to $\frac{|\alpha_1| g \varepsilon_0 x^3}{\nu_0^2}$, α_1 being the cosine of Θ, g being the magnitude of the body force vector, ε_0 being the product $\beta_0 \Delta T$ in which β_0 is the coefficient of thermal expansion of the ambient fluid and ΔT is the absolute value of the difference between the plate and ambient temperatures, x being the distance from the leading edge of the plate, and ν_0 being the kinematic viscosity of the ambient fluid.

$$\frac{\partial(\rho U)}{\partial x} + \frac{\partial(\rho V)}{\partial y} = 0, \tag{1}$$

$$\rho\left\{U\frac{\partial U}{\partial x} + V\frac{\partial U}{\partial y}\right\} = -\frac{\partial p}{\partial x} + \alpha_1 g\rho - \frac{2}{3}\frac{\partial}{\partial x}\left\{\mu\left(\frac{\partial U}{\partial x} + \frac{\partial V}{\partial y}\right)\right\}$$
$$+2\frac{\partial}{\partial x}\left\{\mu\frac{\partial U}{\partial x}\right\} + \frac{\partial}{\partial y}\left\{\mu\left(\frac{\partial U}{\partial y} + \frac{\partial V}{\partial x}\right)\right\}, \tag{2}$$

$$\rho\left\{U\frac{\partial V}{\partial x} + V\frac{\partial V}{\partial y}\right\} = -\frac{\partial p}{\partial y} + \alpha_2 g\rho - \frac{2}{3}\frac{\partial}{\partial y}\left\{\mu\left(\frac{\partial U}{\partial x} + \frac{\partial V}{\partial y}\right)\right\}$$
$$+2\frac{\partial}{\partial y}\left\{\mu\frac{\partial V}{\partial y}\right\} + \frac{\partial}{\partial x}\left\{\mu\left(\frac{\partial U}{\partial y} + \frac{\partial V}{\partial x}\right)\right\}, \tag{3}$$

and

$$\rho\left\{U\frac{\partial h}{\partial x} + V\frac{\partial h}{\partial y}\right\} = \frac{\partial}{\partial x}\left\{k\frac{\partial T}{\partial x}\right\} + \frac{\partial}{\partial y}\left\{k\frac{\partial T}{\partial y}\right\} + \text{negligible pressure}$$

work and viscous dissipation terms.* (4)

In these equations x and y are Cartesian co-ordinates measured from the leading edge of the plate and the plate surface, respectively. U and V are the respective components of fluid velocity in the x and y - directions, p is the pressure, and T is the temperature. ρ is the mass density of the fluid, μ is its dynamic viscosity, h is its enthalpy, and k is its thermal conductivity. ρ, μ, h, and k are taken to be functions of only the temperature. g is the magnitude of the body force vector, α_1 is its direction cosine relative to the x - axis, and α_2 is its direction cosine relative to the y - axis. These and other symbols are defined in Appendix 1.

The assumption is now made that the pressure gradient along

* Unless laminar flow is maintained for Grashof numbers much higher than those experimentally observed for transition, velocities and velocity gradients in the flow will be too small for consideration of these terms to be necessary.

the plate is determined by the hydrostatic relation far from it; mathematically this is expressed by

$$\frac{\partial p}{\partial x} = \alpha_1 g \rho_o{}^{*}. \tag{5}$$

The hydrostatic principle is also used to obtain the equation

$$\frac{\partial p}{\partial y} = \alpha_2 g \rho \tag{6}$$

for the pressure gradient normal to the plate. It should be noted that in the hydrostatic relation 5 for the pressure gradient along the plate the density of the fluid far from the plate is employed, while in the relation 6 for the pressure gradient normal to the plate the local fluid density is used. The former relation was assumed by Polhausen[1] in his analysis of free convection in air along a vertical plate, the results of which were confirmed well by the experiments of Schmidt and Beckmann[1]. The assumption of the latter relation is consistent with the fact that the velocity component normal to the plate should be very small in comparison with the velocity component along the plate.

In Appendix 2 it is shown that appropriately simplifying equations 1, 2, and 4 in accordance with assuming that the velocity and temperature fields are of the boundary layer type gives

$$\frac{\partial(\rho U)}{\partial x} + \frac{\partial(\rho V)}{\partial y} = 0, \tag{7}$$

$$\rho\left\{U\frac{\partial U}{\partial x} + V\frac{\partial U}{\partial y}\right\} = \alpha_1 g \rho_o r + \frac{\partial}{\partial y}\left\{\mu\frac{\partial U}{\partial y}\right\}, \tag{8}$$

and

$$\rho\left\{U\frac{\partial h}{\partial x} + V\frac{\partial h}{\partial y}\right\} = \frac{\partial}{\partial y}\left\{k\frac{\partial T}{\partial y}\right\}. \tag{9}$$

* ρ_o is the density of the fluid far from the plate. The subscript $_o$ refers to the ambient condition, while the subscript $_p$ refers to the condition at the plate surface.

Here the term π is equal to $\frac{\rho-\rho_0}{\rho_0}$. Also in Appendix 2 it is shown that the relation 6 between the pressure gradient normal to the plate and the component of the body force term normal to the plate in addition to the boundary layer assumptions allow equation 3 to be neglected in comparison with equations 1, 2, and 4.

The boundary conditions which solutions of equations 7, 8, and 9 have to meet are

$$\left.\begin{aligned} U, V = 0,\ T = T_P \quad &\text{at } y = 0 \\ \text{and} \qquad U = 0,\ T = T_0 \quad &\text{at } y = \infty. \end{aligned}\right\} \quad (10)$$

These boundary conditions are obtained from the requirements that the velocity components both normal and parallel to the plate must disappear at the plate surface, the fluid temperature at the plate surface must be equal to the plate temperature, the velocity component parallel to the plate must disappear far from the plate, and the fluid temperature far from the plate must be that of the ambient fluid.

C3 Solution of the Boundary Layer Equations

Equations 7, 8, and 9 differ from those which Polhausen[1], Squire[3], Schuh[6], and Ostrach[4] solved in that the present relations are valid when the plate is inclined as well as vertical and when all the fluid properties are functions of the temperature, while the equations of the earlier investigators were derived for a vertical plate and with only the nondimensional density change π considered to vary with the temperature.

In Appendix 3 a method is presented for the solution of these equations which is based on similarity transformations. The method is essentially an adaption to the case involving variable fluid properties of the method developed by Polhausen, and it would require an extreme amount of numerical computation for its application.

An alternative means of treating the equation is an integral procedure which is an extension of Squire's method[3] for solving the constant-property case. An approximate expression for the Nusselt number, which is the parameter of most engineering interest, is obtained, although the velocity and temperature profiles in the boundary layer cannot be determined but have to be assumed.

Before this method is presented, the local and average Nusselt numbers based on the distance from the leading edge of the plate and the fluid properties at both ambient and plate temperatures will be defined. These definitions are

$$Nu_{Lo} = \frac{H_L x}{k_o}, \tag{11}$$

$$Nu_{Ao} = \frac{H_A x}{k_o}, \tag{12}$$

$$Nu_{LP} = \frac{H_L x}{k_P}, \tag{13}$$

and

$$Nu_{AP} = \frac{H_A x}{k_P}. \tag{14}$$

Nu_{Lo} is the local Nusselt number based on the ambient fluid thermal conductivity, Nu_{Ao} is the average Nusselt number based on the thermal conductivity of the ambient fluid, Nu_{LP} is the local parameter based on the thermal conductivity of the fluid at the plate, and Nu_{AP} is the average parameter based on the conductivity of the fluid at the plate. H_L is the local heat-transfer coefficient, which is equal to $\frac{-k_P}{(T_P-T_o)}\left(\frac{\partial T}{\partial y}\right)\Big|_{y=0}$, and H_A is the average heat-transfer coefficient defined to be $\frac{1}{x}\int_0^x H_L d\tau$. k_o and k_P are the values of the thermal conductivity of the fluid at the ambient and plate temperatures, T_0 and T_P, respectively.

The first step in applying the integral method is to rewrite the boundary layer momentum and energy equations 8 and 9 with the use of the continuity equation 7 as

$$\frac{\partial(\rho U^2)}{\partial x} + \frac{\partial(\rho UV)}{\partial y} = \alpha_1 g \rho_0 \pi + \frac{\partial}{\partial y}\left(\mu \frac{\partial U}{\partial y}\right) \tag{15}$$

and

$$\frac{\partial(\rho h U)}{\partial x} + \frac{\partial(\rho h V)}{\partial y} = \frac{\partial}{\partial y}\left(k \frac{\partial T}{\partial y}\right). \tag{16}$$

Integrating these equations with respect to y between the limits 0 and ∞ gives

$$\frac{d}{dx}\int_0^\infty \rho U^2 dy + (\rho UV)\Big|_{y=0}^{y=\infty} = \alpha_1 g \rho_0 \int_0^\infty \pi\, dy + \left(\mu \frac{\partial U}{\partial y}\right)\Big|_{y=0}^{y=\infty} \tag{17}$$

and

$$\frac{d}{dx}\int_0^\infty \rho h U dy + (\rho h V)\Big|_{y=0}^{y=\infty} = \left(k \frac{\partial T}{\partial y}\right)\Big|_{y=0}^{y=\infty}. \tag{18}$$

If the boundary conditions 10 are applied and the enthalpy h is taken to be zero at $T = T_0$, the ambient fluid temperature, the equations simplify to

$$\frac{d}{dx}\int_0^\infty \rho U^2 dy = \alpha_1 g \rho_0 \int_0^\infty \pi\, dy - \left(\mu \frac{\partial U}{\partial y}\right)\Big|_{y=0} \tag{19}$$

and

$$\frac{d}{dx}\int_0^\infty \rho h U dy = -\left(k \frac{\partial T}{\partial y}\right)\Big|_{y=0}. \tag{20}$$

Use is now made of the similarity property of the velocity and temperature profiles, which is demonstrated in Appendix 3, to write

$$U(x,y) = \{U_m(x)\}\cdot\{u(\eta)\}$$

and

$$T(x,y) = T_0 + \{T_P - T_0\}\cdot\{\theta(\eta)\} \tag{21}$$

with

and
$$\left.\begin{aligned}\eta &= \tfrac{y}{\delta}\\ \delta &= \delta(x).\end{aligned}\right\} \quad (22)$$

In these relations U_m is the maximum value of U in the boundary layer at a given value of x, and δ is a boundary layer thickness defined to be $\frac{1}{U_m}\int_0^\infty U\,dy$.

Substituting for U and T as indicated by the relations 21 into equations 19 and 20 yields

$$\rho_o G_1 \frac{d(U_m^2\delta)}{dx} = \rho_o \alpha_1 g \delta G_2 - \mu_P \frac{U_m}{\delta} F_1 \quad (23)$$

and

$$\rho_o C_{p_o} G_3 \frac{d(U_m\delta)}{dx} = \frac{k_P}{\delta} F_2, \quad (24)$$

in which μ_P and k_P are the values of μ and k at the plate. The abbreviations F_1, F_2, G_1, G_2, and G_3 are defined by

$$\left.\begin{aligned} F_1 &= \frac{du}{d\eta}\Big|_{\eta=0},\\ F_2 &= -\frac{d\theta}{d\eta}\Big|_{\eta=0},\end{aligned}\right\} \quad (25)$$

$$G_1 = \int_0^\infty (1+r)\,u^2\,d\eta,$$

$$G_2 = \int_0^\infty r\,d\eta, \quad (26)$$

and

$$G_3 = \int_0^\infty (1+r)\,u\left\{\int_0^\theta q\,d\tau\right\}d\eta,$$

the enthalpy h being taken equal to $\int_{T_0}^{T} C_p\,d\tau$, in which C_p is the specific heat at constant pressure. The dimensionless term q is defined to be $\frac{C_p}{C_{p_o}}$.

The solutions

and
$$\left.\begin{aligned} U_m &= C_1 x^{\frac{1}{2}}\\ \delta &= C_2 x^{\frac{1}{4}}\end{aligned}\right\} \quad (27)$$

satisfy equations 23 and 24 similarly to the case of constant fluid properties. Substituting these expressions for U_m and δ into equations 23 and 24 yields two simultaneous equations for C_1 and C_2, the solutions of which are

$$C_1 = \left\{ \frac{4 K_P F_2 G_2 \alpha_1 g}{3 \sigma_0 m_P F_1 G_3 + 5 K_P F_2 G_1} \right\}^{\frac{1}{2}} \tag{28}$$

and

$$C_2 = \left\{ \frac{4 \nu_0^2 K_P F_2 (3 \sigma_0 m_P F_1 G_3 + 5 K_P F_2 G_1)}{9 \sigma_0^2 G_2 G_3^2 \alpha_1 g} \right\}^{\frac{1}{4}}. \tag{29}$$

In these relations K_P is equal to $\frac{k_P}{k_0}$, m_P is equal to $\frac{\mu_P}{\mu_0}$, ν_0 is the kinematic viscosity, $\frac{\mu_0}{\rho_0}$, of the ambient fluid, and σ_0 is the Prandtl number $\frac{c_{p_0} \mu_0}{k_0}$ of the ambient fluid.

According to the definitions 11 through 14 of the different Nusselt numbers and the relations 21, 22, 25, 27, 28, and 29,

$$Nu_{L0} = \frac{K_P F_2 x^{\frac{3}{4}}}{C_2} \tag{30a}$$

or

$$Nu_{L0} = A_{L0} (Gr_{x0})^{\frac{1}{4}} \tag{30b}$$

with

$$A_{L0} = K_P F_2 \left\{ \frac{9 \sigma_0^2 |G_2| G_3^2}{4 \varepsilon_0 K_P F_2 (3 \sigma_0 m_P F_1 G_3 + 5 K_P F_2 G_1)} \right\}^{\frac{1}{4}} \tag{30c}$$

and

$$Nu_{A0} = A_{A0} (Gr_{x0})^{\frac{1}{4}} \tag{31a}$$

with

$$A_{A0} = \frac{4}{3} A_{L0} \tag{31b}$$

for this variable-property analysis. An alternative way of expressing the heat-transfer rate for a given fluid with given ambient and plate temperatures is to write the Nusselt number based on the fluid properties at the plate in terms of the Grashof number also based on the fluid properties at the plate. The relations are

$$Nu_{LP} = A_{LP}(Gr_{xP})^{\frac{1}{4}} \tag{32a}$$

with
$$A_{LP} = \frac{1}{K_P}\left\{\frac{\varepsilon_o}{\varepsilon_P}\left(\frac{m_P}{1+\eta_P}\right)^2\right\}^{\frac{1}{4}} A_{Lo} \tag{32b}$$

and
$$Nu_{AP} = A_{AP}(Gr_{xP})^{\frac{1}{4}} \tag{33a}$$

with
$$A_{AP} = \frac{4}{3} A_{LP} \tag{33b}$$

or
$$A_{AP} = \frac{1}{K_P}\left\{\frac{\varepsilon_o}{\varepsilon_P}\left(\frac{m_P}{1+\eta_P}\right)^2\right\}^{\frac{1}{4}} A_{Ao}. \tag{33c}$$

The terms ε_o and ε_P are equal to $\beta_o \Delta T$ and $\beta_P \Delta T$, β_o and β_P being the coefficients of thermal expansion of the fluid at its ambient and plate temperatures, and ΔT being $|T_P - T_o|$. The Grashof number Gr_{xo} is equal to $\frac{|\alpha_1| g \varepsilon_o x^3}{\nu_o^2}$, and Gr_{xP} is equal to $\frac{|\alpha_1| g \varepsilon_P x^3}{\nu_P^2}$.

Equations 32b and 33c expressing A_{LP} and A_{AP} in terms of A_{Lo} and A_{Ao}, respectively, result simply from the definitions of the Nusselt and Grashof numbers based on the fluid properties at the ambient and plate temperatures. That is, these two equations are not the results of a heat-transfer analysis: they follow from definitions of dimensionless parameters. They may therefore be used in relating the Nusselt and Grashof numbers based on ambient and plate fluid properties when the heat-transfer rate is determined by any analytic or experimental method.

A final definition is

$$Nu_{AP} = B_{AP}(\sigma_P Gr_{xP})^{\frac{1}{4}} \tag{34a}$$

with
$$B_{AP} = \frac{A_{AP}}{\sigma_P^{\frac{1}{4}}}, \tag{34b}$$

which is made for use in comparing the results of the present analysis with the experimental studies of free convection about a vertical plate in an oil made by Lorenz[8], who presented his findings in the form of equation 34a.

The terms F_1, F_2, G_1, G_2, and G_3 upon which A_{Lo}, A_{Ao}, A_{LP}, and A_{AP} depend are functions of the nondimensional velocity and temperature profiles, which must be assumed. Since in general the assumed profiles will not be the exact profiles which would be obtained by an accurate numerical solution of the boundary layer equations such as that outlined in Appendix 3, one should expect some error to be present in the dependence of the Nusselt numbers on the Grashof numbers as determined by this integral method. However, this error should not be very large, even when the assumed velocity and temperature profiles are only moderately good approximations to the exact profiles. This can be expected because heat-transfer rates resulting from Squire's application of the integral method with rather crude velocity and temperature profiles to the constant-property problem are quite close to those found from the exact solutions over a very wide range of Prandtl numbers. (Squire's impressive success with the integral method is discussed at greater length in Section B2.)

While the value of the ratio of the Nusselt number to the one-quarter power of the Grashof number according to the approximate analysis should not differ greatly percentagewise from the determination by the exact method, the error can be quite serious in making a comparison between the variable and constant-property cases. This is because the effects of variable fluid properties can be expected to be small unless the fluid property variation between the plate and ambient temperatures is very large. Thus, errors due to assuming other than exact profiles could easily obscure the effects of variable properties unless the property changes are extreme.

D

LAMINAR FREE CONVECTION WITH LARGE TEMPERATURE DIFFERENCES IN OILS

D1 Comparison of Constant and Variable-Property Analyses for the Case of Laminar Free Convection in an Oil with a Prandtl Number Varying by a Factor of Ten

The most interesting application of variable-property analyses of free convection was considered to be in comparing the results of variable and constant-property analyses in situations for which the fluid properties are widely variable. This comparison must be made by considering cases of specific fluids with specific plate and ambient temperatures, since the variation in fluid properties can usually be prescribed only by fixing the fluid and the two temperatures.

Ideally, the dependence of the Nusselt number on the Grashof number for the variable-property case should be determined by an exact numerical solution of the boundary layer equations as discussed in Appendix 3. This was not done because the author lacked the necessary computing facilities. Instead, the integral method described in Section C was applied to examples of heating and cooling an oil with a large temperature difference between the plate and the ambient fluid. The fluid and the size of the temperature difference were chosen so that the property variation would be sufficiently large to be detectable among the errors caused by assuming inexact velocity and temperature profiles.

The fluid selected for study was a crude oil fraction having moderate linear changes with temperature of density, specific heat, and thermal conductivity and a relatively large nonlinear variation of dynamic viscosity with temperature over the range considered. At a temperature of 102 degrees Fahrenheit this oil has a Prandtl number of 100, while the value of the parameter drops to 10 at a temperature of 408 degrees, principally on account of the reduction of viscosity with temperature. Appendix 4 contains more information concerning the oil properties.

The results of the present variable-property analyses for heating and cooling this oil are presented in Table 1, which follows Appendix 4, along with corresponding results based on constant-property analyses. The terms A_{AO}, A_{AP}, and B_{AP}, values of which are given in this table, are defined by the equations

$$Nu_{AO} = A_{AO}(Gr_{xo})^{\frac{1}{4}}, \tag{31a}$$

$$Nu_{AP} = A_{AP}(Gr_{xP})^{\frac{1}{4}}, \tag{33a}$$

and

$$Nu_{AP} = B_{AP}(\sigma_P\, Gr_{xP})^{\frac{1}{4}} \tag{34a}$$

of Section C2.

The variable-property analyses were performed for three different assumed pairs of velocity and temperature profiles in the cases of both heating and cooling the oil. Three pairs of profiles rather than one were used in order to obtain some estimate of the magnitude of the effect of the profile shapes on the dependence of the Nusselt number on the Grashof number. In the column of the table entitled "Deviation from Mean of Variable-Property Analyses" the deviations of the values of A_{AO}, A_{AP}, and for the variable-property analyses using each of the three pairs of profiles and for the constant-property analyses are presented as percentages of the means of the values for the variable-property analyses. These percentage deviations are the same for all three terms A_{Ao}, A_{AP}, and B_{AP} for a given analysis. In the cases of both heating and cooling, the three pairs of velocity and temperature profiles for the variable-property analyses were the exact profiles found for the constant-property case for a fluid with a Prandtl number equal to that of the ambient oil, the exact constant-property ones for a Prandtl number equal to that of the oil at the plate, and the simple polynomial profiles used by Squire[3] in his integral treatment of the constant-property problem. The exact constant-property profiles were obtained from Reference 4. All three pairs of profiles are plotted for comparison in Figures 3, 4, and 5; and their use in the analyses is described in Appendix 4.

A_{Ao}, A_{AP}, and B_{AP} were determined according to constant-property treatments for two values of the Prandtl number. One of these values of the Prandtl number was that of the ambient fluid, while the other corresponded to the fluid at the plate surface. For the cases based on the ambient Prandtl number, A_{Ao} was determined according to the constant-property relations between the Nusselt, Prandtl, and Grashof numbers given in Reference 4. Then equation 33c relating A_{Ao} and A_{AP} in terms of the ambient and plate fluid properties was used to secure A_{AP}; and this number was divided by $\sigma_P^{\frac{1}{4}}$ to get B_{AP} as indicated by equation 34b. Similarly, for the cases based on the Prandtl number of the fluid at the plate, the value of A_{AP} was found from information in Reference 4, B_{AP} was obtained by dividing A_{AP} by $\sigma_P^{\frac{1}{4}}$, and A_{Ao} was gotten from A_{AP} through the use of equation 33c.

As can be seen by reference to Table 1, the values of A_{Ao}, A_{AP}, and B_{AP} as determined by the variable-property analyses are within $\pm$ 8 per cent of their means for the case of heating the oil and are within $\pm$ 3 per cent for cooling. The corresponding constant-property terms are approximately 41 per cent of the variable-property means lower than those means for heating and 69 per cent higher for cooling when based on the Prandtl number of the ambient fluid, but they are only 4.50 per cent higher for heating and 4.61 per cent higher for cooling when based on the Prandtl number of the fluid at the plate.

D2 Comparison of Results of Integral Method of Analysis with Experimental Findings

Lorenz[8] obtained the empirical relation

$$Nu_{AP} = 0.555\,(\sigma_P\, Gr_{xP})^{\frac{1}{4}} \tag{35}$$

describing heat transfer by laminar free convection from a heated vertical plate to a transformer oil by plotting experimental points over a range of Prandtl numbers of the oil at the plate ranging from 75.5 to 442. The constant 0.555 corresponds to the terms labeled B_{AP} in Tables 1 and 2. It is interesting to note that although the properties of Lorenz's oil were considerably different from those assumed in the variable-property analyses of Table 1, the resulting values of B_{AP} do not differ greatly from Lorenz's value. This fact is not presented as an argument for the validity of the analysis; it is rather an indication that the term B_{AP} in the relation

$$Nu_{AP} = B_{AP}(\sigma_P\, Gr_{xP})^{\frac{1}{4}} \tag{34a}$$

may be fairly constant for different oils even when there are large differences between the properties of the oils at the plate and ambient temperatures. B_{AP} is, as a matter of fact, often taken in empirical formulas for heat transfer by laminar free convection to be a constant for a wide variety of fluids of vastly different properties.

So that Lorenz's experimental results could be compared with analytic work an attempt was made to duplicate analytically the test run of those he reported that had the largest difference between the plate and the ambient fluid temperatures. For this case σ_o, the Prandtl number of the ambient oil, was 309; and σ_P, the Prandtl number of the oil at the plate surface, was 75.5. The experimental and analytic results can be checked against one another by comparing the corresponding values of B_{AP} presented in Table 2.

Making this comparison indicates that the analysis with variable fluid properties gives a value of B_{AP} that is 13 per cent higher than the experimental value and that the constant-property analysis gives a value 21 per cent lower when based on the ambient fluid properties and 15 per cent higher when based on the properties of the fluid at the plate. It is doubtful that all of the discrepancy between the variable-property analysis and the experimental observation should be ascribed to the inexactness of the assumed velocity and temperature profiles, which were those obtained by Ostrach(4) in his constant-property analysis for a Prandtl number of 100.

At least part of the discrepancy should be due to the difference between Lorenz's experimental situation and that assumed in the analysis. Lorenz's flat plate was only 12 centimeters high by 25 centimeters wide, and a Schlieren photograph included with his paper shows that the temperature boundary layer at the lower or leading edge of the plate has a thickness roughly half of that of the boundary layer at the top of the plate. In the analysis, which is based on boundary layer theory, the assumption is made that the boundary layer begins with zero thickness at the leading edge; and the local heat-transfer coefficient is therefore infinite there. For the actual physical situation the local heat-transfer coefficient must be infinite nowhere, and one should expect that even if the theory describes the situation well far from the leading edge, the calculated average heat-transfer coefficient will be larger than one which is measured. This leading edge effect should be particularly noticeable with short plates such as Lorenz used for which the ratio of the plate length to the boundary layer thickness is not very large. Lateral edge effects in the experiment could also be partly responsible for the discrepancy; but it is difficult to say just what these effects should be, although one might expect that one of them would be an increase rather than a decrease in the rate of heat transfer for the experimental case compared with the strictly two-dimensional case.

The values of the term B_{AP} obtained according to the constant-property analyses are related to the values found by the variable-property method similarly to the previous case of heating the California crude fraction. That is, the value obtained according to the constant-property analysis based on ambient fluid properties is considerably less than the variable-property value; and the value found according to the constant-property analysis based on the fluid at the plate is approximately equal to the variable-property value.

Details of treating this comparison between analysis and experiment are given in Part 4.2 of Appendix 4.

D3 Conclusions and Discussion of Conclusions

D3.1 Conclusions

I. Heat-transfer rates for laminar free convection about an inclined or vertical flat plate in cases for which the difference between the Prandtl number of the fluid at the plate surface and far from the plate is large can be expected to be predicted accurately by using the results of analytic studies based on the assumption that the fluid properties are essentially constant. In the application of the results of a constant-property analysis to a variable-property situation, the properties of the fluid of the analysis should be taken equal to those of the variable-property fluid at the temperature of the plate.

II. It should be expected that the results of both analytic and experimental studies of heat transfer in laminar free convection with small temperature differences between isothermal bounding surfaces in general and a surrounding fluid of infinite extent can be successfully applied to identical configurations when the variations in the Prandtl number of the fluid are large. When the results of a study made with a small temperature difference and an essentially constant Prandtl number are applied to a situation in which the temperature difference is large, the Prandtl number of the fluid of the study should be equal to that of the fluid at the bounding surface of the situation in question. Also, the flow must be of the boundary layer type in both the study and the situation in question.

D3.2 Discussion of Conclusions

The first conclusion was made after a comparison of the determinations of the dependence of the Nusselt numbers on the Grashof and Prandtl numbers by the variable and constant-property methods for the cases of heating and cooling the California crude fraction and heating Lorenz's transformer oil. By reference to Tables 1 and 2 one can see that the ratios of the Nusselt numbers to the one-quarter power of the Grashof numbers or to the one-quarter power of the products of the Grashof and Prandtl numbers when determined by the constant-property analyses based on the fluid properties at the plate are within a few per cent of the values determined by the variable-property analyses. One might be able to suggest using some temperature other than simply the plate temperature for specifying the fluid properties in order to obtain a more accurate determination of the relations between the dimensionless heat-transfer parameters if a considerably more precise means of performing the variable-property analysis were used for comparison instead of the approximate integral method.

One could suppose that for liquid metals, which have very low Prandtl numbers, the present conclusions regarding constant and variable-property analyses might not be true, since they are based on cases for which the Prandtl numbers are comparatively much greater. However, in a sense they should remain valid because the effects of variable fluid properties in free convection situations involving liquid metals should not be important unless temperature differences are extremely large. A check made on the properties of mercury is the basis of this statement.

The second conclusion is an extrapolation of the results obtained for laminar free convection about a flat plate to the general case of laminar free convection about any finite isothermal body surrounded by a fluid of infinite extent. This extrapolation is reasonable because the free convection velocity and temperature fields should be of the boundary layer type for a finite body which is not too small, and one should expect that the process would behave similarly to that of the flat plate case in regard to the relation between heat transfer with small and large fluid property variations. Also, Lorenz[8] obtained a good correlation of his experimental data for heating an oil over a wide range of temperatures by basing his Nusselt, Prandtl, and Grashof numbers on the properties of the fluid at the bounding surface.

APPENDIX 1

NOTATION*

1.1 Latin Letters

$A_{A0,P}$ Dimensionless terms defined by the equations
$$Nu_{A0,P} = A_{A0,P}\,(Gr_{x_{0,P}})^{\frac{1}{4}}$$

B_{AP} Dimensionless term defined by the equation
$$Nu_{AP} = B_{AP}\,(\sigma_P\, Gr_{xP})^{\frac{1}{4}}$$

c_p Specific heat of fluid at constant pressure, assumed to be a function of temperature only

F_1 Dimensionless term equal to $\left.\frac{du}{d\eta}\right|_{\eta=0}$

F_2 Dimensionless term equal to $-\left.\frac{d\theta}{d\eta}\right|_{\eta=0}$

G_1 Dimensionless term equal to $\int_0^\infty (1+\pi)u^2 d\eta$

G_2 Dimensionless term equal to $\int_0^\infty \pi\, d\eta$

G_3 Dimensionless term equal to $\int_0^\infty (1+\pi)u\left\{\int_0^\theta q\, d\tau\right\}d\eta$

Gr_x Grashof number based on component of body force parallel to plate and distance from leading edge of plate. It is equal to $\frac{|\alpha| g\, \varepsilon\, x^3}{\nu^2}$

Gr_{L_0} Grashof number equal to $\frac{|\alpha_1|\, g\, \varepsilon_0 L^3}{\nu_0^2}$

$\mathbf{g}$ Vector representing body force per unit mass

g Magnitude of $\mathbf{g}$

H_L Local heat-transfer coefficient equal to $\frac{-k_P}{(T_P - T_0)}\left.\left(\frac{\partial T}{\partial y}\right)\right|_{y=0}$

H_A Average heat-transfer coefficient equal to $\frac{1}{x}\int_0^x H_L\, d\tau$

h Enthalpy of fluid, assumed to be a function of temperature only and to be equal to $\int_{T_0}^{T} c_p\, d\tau$

k Thermal conductivity of fluid, assumed to be a function of temperature only

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* The subscript $_0$ appended to symbols for fluid properties or parameters defined in terms of fluid properties indicates that the properties are those of "ambient" fluid or fluid far from the plate. Similarly, the subscript $_P$ refers to properties of fluid at the plate surface.

L	Characteristic fixed length equal to distance from leading edge of plate to point of application of flow equations
m	Dimensionless dynamic viscosity equal to $\frac{\mu}{\mu_o}$
Nu_L	Nusselt number based on local heat-transfer coefficient. It is defined to be $\frac{H_L x}{k}$.
Nu_A	Nusselt number based on average heat-transfer coefficient. It is defined to be $\frac{H_A x}{k}$.
p	Pressure in fluid
q	Dimensionless specific heat at constant pressure equal to $\frac{c_p}{c_{p_o}}$
Re	Reynolds number equal to $\frac{U_m \delta^+}{\nu_o}$
$1+r$	Dimensionless density equal to $\frac{\rho}{\rho_o}$
T	Fluid temperature
T_p	Plate temperature
T_o	Temperature of ambient fluid (fluid far from plate)
ΔT	Temperature difference defined to be $\lvert T_p - T_o \rvert$
U	x - component of fluid velocity
U_m	Maximum value of U in the boundary layer at a given distance from the edge of the plate
u	Dimensionless velocity component equal to $\frac{U}{U_m}$
V	y - component of fluid velocity
v	Dimensionless velocity component equal to $\frac{L}{\delta^+}\frac{V}{U_m}$
x	Cartesian co-ordinate representing distance from leading edge of plate measured along an axis parallel to the plate
y	Cartesian co-ordinate representing distance from plate surface measured along an axis perpendicular to the plate surface

1.2 Greek Letters

α_1 Direction cosine of body force vector relative to x- axis

α_2 Direction cosine of body force vector relative to y- axis

β Coefficient of thermal expansion of fluid equal to $-\frac{1}{\rho}\left(\frac{\partial \rho}{\partial T}\right)_P$

δ Boundary layer thickness defined to be $\frac{1}{U_m}\int_0^\infty U dy$

δ^+ Value of δ at $x = L$

ε Dimensionless parameter equal to $\beta \Delta T$

ζ Similarity independent variable defined by equations 3-5 and 3-6 of Appendix 3

η Dimensionless Cartesian co-ordinate equal to $\frac{y}{\delta}$

η^+ Dimensionless Cartesian co-ordinate equal to $\frac{y}{\delta^+}$

Θ Angle between body force vector and x- axis

ϑ Dimensionless temperature equal to $\frac{T-T_0}{T_P-T_0}$

κ Dimensionless thermal conductivity equal to $\frac{k}{k_0}$

μ Dynamic viscosity of fluid, assumed to be a function of temperature only

ν Kinematic viscosity of fluid, a function of temperature only

ξ Dimensionless Cartesian co-ordinate equal to $\frac{x}{L}$

ρ Mass density of fluid, assumed to be a function of temperature only

σ Prandtl number of fluid equal to $\frac{c_P \mu}{k}$

τ Dummy variable of integration

φ Similarity dependent variable defined by equation 3-7 of Appendix 3

Ψ Mass flow function defined by equations 3-1 and 3-2 of Appendix 3

ω Similarity dependent variable defined by equation 3-8 of Appendix 3

APPENDIX 2

SIMPLIFICATION OF FLOW EQUATIONS ACCORDING TO BOUNDARY LAYER THEORY

In Section C2 the flow equations for free convection with variable fluid properties along an inclined semi-infinite flat plate are given as

$$\frac{\partial(\rho U)}{\partial x} + \frac{\partial(\rho V)}{\partial y} = 0, \tag{1}$$

$$\begin{aligned}\rho\left\{U\frac{\partial U}{\partial x} + V\frac{\partial U}{\partial y}\right\} = & -\frac{\partial p}{\partial x} + \alpha_1 g\rho - \frac{2}{3}\frac{\partial}{\partial x}\left\{\mu\left(\frac{\partial U}{\partial x} + \frac{\partial V}{\partial y}\right)\right\} \\ & + 2\frac{\partial}{\partial x}\left\{\mu\frac{\partial U}{\partial x}\right\} + \frac{\partial}{\partial y}\left\{\mu\left(\frac{\partial U}{\partial y} + \frac{\partial V}{\partial x}\right)\right\},\end{aligned} \tag{2}$$

$$\begin{aligned}\rho\left\{U\frac{\partial V}{\partial x} + V\frac{\partial V}{\partial y}\right\} = & -\frac{\partial p}{\partial y} + \alpha_2 g\rho - \frac{2}{3}\frac{\partial}{\partial y}\left\{\mu\left(\frac{\partial U}{\partial x} + \frac{\partial V}{\partial y}\right)\right\} \\ & + 2\frac{\partial}{\partial y}\left\{\mu\frac{\partial V}{\partial y}\right\} + \frac{\partial}{\partial x}\left\{\mu\left(\frac{\partial U}{\partial y} + \frac{\partial V}{\partial x}\right)\right\},\end{aligned} \tag{3}$$

and $$\rho\left\{U\frac{\partial h}{\partial x} + V\frac{\partial h}{\partial y}\right\} = \frac{\partial}{\partial x}\left\{k\frac{\partial T}{\partial x}\right\} + \frac{\partial}{\partial y}\left\{k\frac{\partial T}{\partial y}\right\} \tag{4}$$

+ negligible terms.

For use in the determination of appropriate simplifications of these equations to conform with the assumptions of boundary layer flow, the following nondimensional variables are defined:

$$\xi = \frac{x}{L} \tag{2-1}$$

$$\eta^+ = \frac{y}{\delta^+} \tag{2-2}$$

$$u = \frac{U}{U_m} \tag{2-3}$$

$$v = \frac{L}{\delta^+}\frac{V}{U_m} \tag{2-4}$$

$$1+r = \frac{\rho}{\rho_0} \tag{2-5}$$

$$m = \frac{\mu}{\mu_0} \tag{2-6}$$

$$q = \frac{C_p}{C_{p_0}} = \frac{\frac{dh}{dT}}{\left.\frac{dh}{dT}\right|_0} \tag{2-7}$$

$$\theta = \frac{T-T_0}{T_p-T_0} \tag{2-8}$$

$$\kappa = \frac{k}{k_0} \tag{2-9}$$

In these definitions L is a fixed characteristic length equal to x, the distance from the leading edge of the plate to the point in the flow at which the equations are applied; and δ^+ is a fixed characteristic length defined to be $\frac{1}{U_m}\int_0^\infty U\,dy$, U_m being the maximum value of U at $x=L$, and $\int_0^\infty U\,dy$ being evaluated at $x=L$. C_p is the specific heat of the fluid at constant pressure.

Writing equations 1, 2, 3, and 4 in terms of these dimensionless variables and using the relations 5 and 6 for the pressure gradients in the x and y- directions give

$$\frac{\partial}{\partial\xi}\left\{(1+r)u\right\} + \frac{\partial}{\partial\eta^+}\left\{(1+r)v\right\} = 0, \tag{2-10}$$

$$\left\{1+r\right\}\left\{u\frac{\partial u}{\partial\xi} + v\frac{\partial u}{\partial\eta^+}\right\} = \frac{\alpha_1 g L}{U_m^2}\, r + \frac{1}{Re}\frac{L}{\delta^+}\frac{\partial}{\partial\eta^+}\left\{m\frac{\partial u}{\partial\eta^+}\right\}$$
$$+\frac{1}{Re}\frac{\delta^+}{L}\left\{-\frac{2}{3}\frac{\partial}{\partial\xi}\left[m\left(\frac{\partial u}{\partial\xi}+\frac{\partial v}{\partial\eta^+}\right)\right] + 2\frac{\partial}{\partial\xi}\left(m\frac{\partial u}{\partial\xi}\right) + \frac{\partial}{\partial\eta^+}\left(m\frac{\partial v}{\partial\xi}\right)\right\}, \tag{2-11}$$

$$\frac{\delta^+}{L}\left\{1+r\right\}\left\{u\frac{\partial v}{\partial\xi} + v\frac{\partial v}{\partial\eta^+}\right\} = \frac{1}{Re}\left\{-\frac{2}{3}\frac{\partial}{\partial\eta^+}\left[m\left(\frac{\partial u}{\partial\xi}+\frac{\partial v}{\partial\eta^+}\right)\right]\right.$$
$$\left. + 2\frac{\partial}{\partial\eta^+}\left(m\frac{\partial v}{\partial\eta^+}\right) + \frac{\partial}{\partial\xi}\left(m\frac{\partial u}{\partial\eta^+}\right)\right\} + \frac{1}{Re}\left(\frac{\delta^+}{L}\right)^2\left\{\frac{\partial}{\partial\xi}\left(m\frac{\partial v}{\partial\xi}\right)\right\}, \tag{2-12}$$

and
$$\left\{g(1+\pi)\right\}\left\{u\frac{\partial\theta}{\partial\xi}+v\frac{\partial\theta}{\partial\eta^+}\right\}=\frac{1}{\sigma_o Re}\frac{L}{\delta^+}\left\{\frac{\partial}{\partial\eta^+}\left(K\frac{\partial\theta}{\partial\eta^+}\right)\right\}+\frac{1}{\sigma_o Re}\frac{\delta^+}{L}\left\{\frac{\partial}{\partial\xi}\left(K\frac{\partial\theta}{\partial\xi}\right)\right\}, \qquad (2\text{-}13)$$

in which Re, the Reynolds number based on the maximum velocity in the boundary layer and the boundary layer thickness, is equal to $\frac{U_m\delta^+}{\nu_o}$, and σ_o, the Prandtl number of the ambient fluid, is equal to $\frac{C_{po}\mu_o}{k_o}$.

These equations are now simplified by considering that $\frac{\delta^+}{L} \ll 1$, $Re \gg 1$, and σ_o is sufficiently large so that $\sigma_o Re \gg 1$. The assumption that $\frac{\delta^+}{L} \ll 1$ is equivalent to considering that the flow is of the boundary layer type, while taking Re to be very large means that Gr_{Lo}*, the Grashof number based on L, must be very large. Gr_{Lo} must be large when Re is large because for a given fluid, with given temperatures of the plate and of the ambient fluid, Gr_{Lo} is proportional to $(Re)^4$, when $\frac{\delta^+}{L} \ll 1$ and $Re \gg 1$, as indicated in Appendix 3.

The dimensionless continuity equation 2-10 is independent of the sizes of $\frac{\delta^+}{L}$ and $\frac{1}{Re}$. However, the dimensionless component of the momentum equation in the x- direction, equation 2-11, has terms of orders unity, $\frac{1}{Re}\frac{L}{\delta^+}$, and $\frac{1}{Re}\frac{\delta^+}{L}$, as well as the driving force term $\frac{\alpha_1 g L \pi}{U_m^2}$. Of these, the terms of order $\frac{1}{Re}\frac{\delta^+}{L}$ are dropped for simplicity, while the others are retained.

In the derivation of equations 2-11 and 2-12, the dimensionless components of the vector momentum equation, both equations were multiplied by the same factor after the dimensionless variables were introduced. By comparing the two equations one can see that dropping

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* Gr_{Lo} is equal to $\frac{|\alpha_1| g \varepsilon_o L^3}{\nu_o^2}$.

equation 2-12 in comparison with equation 2-11 is equivalent to neglecting vectors with lengths of orders $\frac{\delta^+}{L}$ and $\frac{1}{Re}$ compared with the left-hand side of equation 2-11, which is of order unity. Therefore, the y-component of the momentum equation is neglected because doing so is consistent with retaining only the most important terms of the x-component of the equation.

The term representing the conduction of heat parallel to the plate surface in the dimensionless energy equation 2-13 is to be neglected, since it is of order $\frac{1}{\sigma_0 Re}\,\frac{\delta^+}{L}$, while the other terms are of order unity and $\frac{1}{\sigma_0 Re}\,\frac{L}{\delta^+}$.

With the indicated simplifications the dimensional equations reduce to

$$\frac{\partial(\rho U)}{\partial x} + \frac{\partial(\rho V)}{\partial y} = 0, \tag{2-14}$$

$$\rho\left\{U\frac{\partial U}{\partial x} + V\frac{\partial U}{\partial y}\right\} = \alpha_1 g \rho_0 \pi + \frac{\partial}{\partial y}\left\{\mu\frac{\partial U}{\partial y}\right\}, \tag{2-15}$$

and

$$\rho\left\{U\frac{\partial h}{\partial x} + V\frac{\partial h}{\partial y}\right\} = \frac{\partial}{\partial y}\left\{k\frac{\partial T}{\partial y}\right\} \tag{2-16}$$

with neglect of the y-momentum equation. These equations fail to be a good description of the free convection process in the region close to the leading edge of the plate where the boundary layer assumptions are invalid.

APPENDIX 3

SIMILARITY METHOD FOR SOLVING BOUNDARY LAYER EQUATIONS

Section C2 contains the continuity, momentum, and energy equations for the free convection boundary layer in the forms

$$\frac{\partial(\rho U)}{\partial x} + \frac{\partial(\rho V)}{\partial y} = 0, \tag{7}$$

$$\rho\left\{U\frac{\partial U}{\partial x} + V\frac{\partial U}{\partial y}\right\} = \alpha_1 g \rho_0 r + \frac{\partial}{\partial y}\left\{\mu\frac{\partial U}{\partial y}\right\}, \tag{8}$$

and

$$\rho\left\{U\frac{\partial h}{\partial x} + V\frac{\partial h}{\partial y}\right\} = \frac{\partial}{\partial y}\left\{k\frac{\partial T}{\partial y}\right\} \tag{9}$$

with the boundary conditions

and

$$\left.\begin{array}{l} U, V = 0,\ T = T_P \text{ at } y = 0 \\ U = 0,\ T = T_0 \text{ at } y = \infty. \end{array}\right\} \tag{10}$$

As suggested by W. D. Rannie, the continuity equation 7 is used to define a mass flow function Ψ such that

$$\frac{\partial\Psi}{\partial y} = \rho U \tag{3-1}$$

and

$$\frac{\partial\Psi}{\partial x} = -\rho V. \tag{3-2}$$

Substituting into the momentum equation 8 for U and V in terms of this function and for ρ and μ in terms of the dimensionless density $1+r$ and viscosity m gives

$$\frac{\partial\Psi}{\partial y}\cdot\frac{\partial}{\partial x}\left\{(1+r)^{-1}\frac{\partial\Psi}{\partial y}\right\} - \frac{\partial\Psi}{\partial x}\cdot\frac{\partial}{\partial y}\left\{(1+r)^{-1}\frac{\partial\Psi}{\partial y}\right\}$$
$$= \alpha_1 g \rho_0^2 r + \mu_0\frac{\partial}{\partial y}\left\{m\frac{\partial}{\partial y}\left[(1+r)^{-1}\frac{\partial\Psi}{\partial y}\right]\right\}. \tag{3-3}$$

Substituting for U and V similarly as well as for T, k, and C_P in terms of their dimensionless forms ϑ, κ, and q into the energy equation 9, after using the relation $h=\int_{T_0}^{T} c_p d\tau$, yields

$$q\left\{\frac{\partial\Psi}{\partial y}\cdot\frac{\partial\vartheta}{\partial x}-\frac{\partial\Psi}{\partial x}\cdot\frac{\partial\vartheta}{\partial y}\right\}=\frac{k_0}{c_{p0}}\frac{\partial}{\partial y}\left\{\kappa\frac{\partial\vartheta}{\partial y}\right\}. \qquad (3\text{-}4)$$

With a given fluid and with fixed values of the plate temperature T_P and the ambient temperature T_0, the nondimensional fluid properties n, m, q, and κ can be considered to be functions of ϑ only.

Polhausen's similarity independent variable ζ is defined by

$$\zeta = c\,y\,x^{-\frac{1}{4}} \qquad (3\text{-}5)$$

with

$$c = \left(\frac{|\alpha_1|\, g\, \varepsilon_0}{4\,\nu_0^2}\right)^{\frac{1}{4}}. \qquad (3\text{-}6)$$

His transformations of the dependent variables,

$$\Psi(x,y) = 4\mu_0 c\, x^{\frac{3}{4}}\varphi(\zeta) \qquad (3\text{-}7)$$

and

$$\vartheta(x,y) = \omega(\zeta), \qquad (3\text{-}8)$$

are also used in order to write equations 3-3 and 3-4 in terms of a single independent variable.

Making these substitutions and noting that n, m, q, and κ are functions of ζ only, by virtue of equation 3-8 and their dependence on ϑ alone, give

$$\{m[(1+n)^{-1}\varphi']'\}' + 3\{(1+n)^{-1}\varphi\varphi'\}' - 5\{(1+n)^{-1}(\varphi')^2\} \mp \frac{n}{\varepsilon_0} = 0 \qquad (3\text{-}9)$$

and

$$\{\kappa\omega'\}' + 3\sigma_0\{q\varphi\omega'\} = 0. \qquad (3\text{-}10)$$

In these equations primes indicate differentiation with respect to ζ. Also, in the term $\frac{\mp n}{\varepsilon_0}$ the minus sign is to be used when the plate is heated compared with the ambient fluid and the plus sign when the plate is cooled.

The appropriate boundary conditions to be met by the solutions of equations 3-9 and 3-10 are

$$\left.\begin{aligned} &\varphi(0)=0,\\ &\varphi'(0)=0,\\ &\omega(0)=1,\\ &\varphi'(\infty)=0,\\ \text{and}\qquad &\omega(\infty)=0. \end{aligned}\right\} \tag{3-11}$$

These are simply restatements of the boundary conditions 10.

It should be noted that the definition 3-5 of the similarity variable ζ implies that the boundary conditions $\varphi'(\infty)=0$ and $\omega(\infty)=0$, which result from requirements at $y=\infty$, are also to be satisfied at $x=0$. These requirements are consistent with the physical situation everywhere except at the leading edge of the plate, where the boundary layer approximations themselves are inherently invalid.

Through a consideration of the definitions 3-5, 3-6, and 3-8 of ζ, C, and $\omega(\zeta)$ along with the fact that $\theta(x,y)=\dfrac{T-T_0}{T_P-T_0}$, one can determine that the local Nusselt number Nu_{Lo} based on the thermal conductivity of the ambient fluid and the distance from the leading edge of the plate is given by

$$Nu_{Lo} = \frac{-k_P x}{k_0(T_P-T_0)}\left(\frac{\partial T}{\partial y}\right)\bigg|_{y=0} \tag{3-12a}$$

or

$$Nu_{Lo} = -\frac{K_P}{\sqrt{2}}\left\{\omega'(\zeta)\big|_{\zeta=0}\right\}(Gr_{x_0})^{\frac{1}{4}}, \tag{3-12b}$$

K_P being the dimensionless thermal conductivity of the fluid at the plate surface, Gr_{x_0} being the Grashof number based on the ambient fluid properties and the distance from the leading edge, and $\omega(\zeta)$ being defined by the differential equations 3-9 and 3-10 with the boundary conditions 3-11. By reference to these equations 3-9 and 3-10 one can see that the variations of the dimensionless fluid properties with temperature as well as the value of the Prandtl number σ_0 of the ambient fluid determine $\omega(\zeta)$ and hence $\omega'(\zeta)|_{\zeta=0}$. Hence, it is not surprising that the Nusselt number depends on the variation of fluid properties with temperature in the variable-property case in addition to being a function of the ambient fluid Prandtl number and being proportional to the one-quarter power of the Grashof number as in the constant-property case.

Of importance also is the relation between the Reynolds number Re and the Grashof number Gr_{L_0} that appear in Appendix 2. Noting that Re is defined to be $\frac{U_m \delta^+}{\nu_0}$ and using equations 3-1, 3-5, 3-6, 3-7, and 3-8 along with the definitions of the nondimensional temperature, one can apply the definition $\delta^+ = \left\{ \frac{1}{U_m} \int_0^\infty U dy \right\} \Big|_{x=L}$ to obtain

$$Re = 2\sqrt{2} \left\{ \int_0^\infty \frac{\Phi'(\zeta) d\zeta}{[1 + h\{\omega(\zeta)\}]} \right\} (Gr_{L_0})^{\frac{1}{4}} \tag{3-13a}$$

or

$$Gr_{L_0} = \text{const.} \times (Re)^4, \tag{3-13b}$$

the value of the constant being dependent on the fluid and on the plate and ambient temperatures.

Equations 3-9 and 3-10 are generalizations to the variable-property case of simpler equations derived by Polhausen(1) on the assumption that the only effect of variable fluid properties is to produce a driving term for the convection process. The simpler equations have in the past been solved principally by finite difference techniques(1), (4) and by an iteration method developed by Schuh[6], which is somewhat similar to the Stodola-Vianello method for solving linear boundary-value problems.

An attempt was made to solve the present equations by Schuh's method for the case of heating the California crude oil fraction described in Section D and Appendix 4, but the process was found to be divergent. It is possible, however, that the method could be used for cases in which the viscosity of the fluid does not change greatly across the boundary layer, since for the unsuccessful attempt the viscosity changed by a factor of almost 15 between the plate and ambient conditions, and the method has been used successfully for the constant-property problem in the same range of Prandtl numbers. In view of this failure of Schuh's method, it is suggested that any future investigator attempting to solve the present equations should try an adaption of the finite-difference method outlined by Albers[4] for solving the simpler Polhausen equations with the use of electronic computing facilities.

APPENDIX 4

DETAILS CONCERNING THE APPLICATION OF THE INTEGRAL METHOD TO THE LAMINAR FREE CONVECTION OF OILS

4.1 Heating and Cooling the California Crude Oil Fraction with a Prandtl Number Varying Between 10 and 100

4.1.1 Properties of the Oil Treated

An oil having the desired variation of its Prandtl number over a realistic temperature range is the California crude fraction the viscosity of which is plotted as a function of temperature on Page 165 of Reference 15 and which is denoted there as "Number 4". This oil has an API gravity of 25.0 and a boiling point of 392-437 degrees Fahrenheit in a Hempel vacuum column. Its Prandtl number is 100 at 102 degrees and drops to 10 with an increase of temperature to 408 degrees.

For the case of heating the oil with the ambient temperature 102 degrees and the plate temperature 408 degrees, the dependencies of the dimensionless density $1+\pi$, specific heat q, and thermal conductivity κ on the dimensionless temperature ϑ are expressible as

$$1+\pi = 1-(0.1357)\vartheta, \tag{4-1}$$

$$q = 1+(0.3169)\vartheta, \tag{4-2}$$

and

$$\kappa = 1-(0.1006)\vartheta. \tag{4-3}$$

The dimensionless dynamic viscosity m is a more complicated function of ϑ and is plotted in Figure 1.

For cooling the oil with 408 degrees as the ambient temperature and 102 degrees as the plate temperature, the dimensionless properties are expressible as

$$1 + \pi = 1 + (0.1570)\theta, \qquad (4\text{-}4)$$

$$q = 1 - (0.2406)\theta, \qquad (4\text{-}5)$$

and

$$\kappa = 1 + (0.1119)\theta, \qquad (4\text{-}6)$$

the dimensionless dynamic viscosity m being plotted in Figure 2.

Information for determining q, κ, and m was obtained from Reference 15, while $1+\pi$ was found from formulas in Reference 16.

4.1.2 Determination of the Dependence of the Nusselt Numbers on the Grashof and Prandtl Numbers

Equations 30b, 30c, 31a, and 31b of Section C2 indicate that one can write

$$Nu_{AO} = A_{AO}\left(Gr_{x_0}\right)^{\frac{1}{4}} \qquad (31a)$$

with

$$A_{AO} = \frac{4}{3}\kappa_p F_2 \left\{ \frac{9\sigma_0^2 \, |G_2| \, G_3^2}{4\varepsilon_0 \kappa_p F_2 \left(3\sigma_0 m_p F_1 G_3 + 5\kappa_p F_2 G_1\right)} \right\}. \qquad (4\text{-}7)$$

F_1, F_2, G_1, G_2, and G_3, which are defined by equations 25 and 26, depend on the nondimensional velocity and temperature profiles as well as on the nondimensional density and specific heat. Figures 3, 4, and 5 are plots of the three pairs of profiles of the nondimensional velocity u and temperature θ which were assumed. The profiles shown in Figures 3 and 4 were taken from the exact solutions of the constant-property problem for Prandtl numbers of 10 and 100 as obtained from information in Reference 4, and the polynomial profiles of Figure 5 are those used by Squire[3] in his integral treatment of the constant-property problem. The integrations performed to obtain G_1, G_2, and G_3 were done numerically by the use of Simpson's Rule in the cases of the exact solutions of the constant-property problem; and they were done analytically for the polynomial profiles.

Once A_{Ao} was found, A_{Ap} and B_{Ap} were secured by using the relations

$$A_{AP} = \frac{1}{K_P}\left\{\frac{\mathcal{E}_o}{\mathcal{E}_P}\left(\frac{m_P}{1+\mathcal{R}_P}\right)^2\right\}^{\frac{1}{4}} A_{Ao} \tag{33c}$$

and

$$B_{AP} = \frac{A_{AP}}{\sigma_P^{\frac{1}{4}}} . \tag{34b}$$

4.2 Lorenz's Experiment on Heating a Transformer Oil with a Prandtl Number Varying Between 75.5 and 309

4.2.1 Properties of the Oil Treated

In his paper describing an experimental study of free convection in a transformer oil, Lorenz[8] tabulated a number of the properties of the oil at different temperatures. For the experimental run which he designated as "H," the average plate and oil temperatures were 70.2 and 25.7 degrees centigrade, respectively. Knowing these temperatures and the variation of the oil properties with temperature, one can determine that for the analytic treatment of the experimental run H, the variations of the dimensionless density $1+\mathcal{R}$ and specific heat q are described by the equations

$$1+\mathcal{R} = 1-(0.0278)\,\theta \tag{4-8}$$

and

$$q = 1+(0.11)\,\theta, \tag{4-9}$$

while the values of the dimensionless dynamic viscosity m and thermal conductivity K at the plate surface are given by

$$m_p = 0.217 \tag{4-10}$$

and

$$\kappa_p = 0.974 . \tag{4-11}$$

The Prandtl numbers of the oil at the ambient and plate temperatures were 309 and 75.5, respectively.

4.2.2 Determination of the Dependence of the Nusselt Number on the Product of the Prandtl and Grashof Numbers

The term B_{AP} in the equation

$$Nu_{AP} = B_{AP} (\sigma_P Gr_{xP})^{\frac{1}{4}} \tag{34a}$$

was found for the variable-property treatment of Lorenz's oil by essentially the same method as that used for the previous cases of heating and cooling the California crude oil fraction. In the present case the fluid properties are those given in the preceding Part 4.2.1 of this appendix, and the only velocity and temperature profiles used were those corresponding to the solution of the constant-property problem for a Prandtl number of 100. These profiles are plotted in Figure 4.

APPENDIX 5

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TABLE 1

HEATING AND COOLING AN OIL WITH A PRANDTL NUMBER VARYING BETWEEN 10 AND 100 IN LAMINAR FREE CONVECTION ABOUT A FLAT PLATE

I. Heating (σ_o = 100, σ_p = 10)	A_{Ao}^*	A_{Ap}^*	B_{Ap}^*	Deviation from Mean of Variable-Property Analyses (In Per Cent of Mean)
A. Variable-Property Analyses				
1. Velocity and Temperature Profiles Taken from Constant-Property Solutions for Ambient Fluid (σ = 100)	3.246	0.9760	0.5488	-7.49
2. Velocity and Temperature Profiles Taken from Constant-Property Solutions for Fluid at Plate (σ = 10)	3.507	1.054	0.5930	-0.05
3. Velocity and Temperature Profiles Taken To Be Squire's Polynomial Approximations	3.773	1.134	0.6380	+7.53
4. Means for the Three Sets of Profiles	3.508	1.055	0.5933	-
B. Constant-Property Analyses				
1. Based on Ambient Fluid (σ = 100)	2.066	0.6212	0.3493	-41.12
2. Based on Fluid at Plate (σ = 10)	3.666	1.102	0.6200	+4.50

* A_{Ao}, A_{Ap}, and B_{Ap} are defined by the equations

$$Nu_{Ao} = A_{Ao}(Gr_{xo})^{\frac{1}{4}}, \quad (31a)$$

$$Nu_{Ap} = A_{Ap}(Gr_{xp})^{\frac{1}{4}}, \quad (33a)$$

and

$$Nu_{Ap} = B_{Ap}(\sigma_p Gr_{xp})^{\frac{1}{4}}. \quad (34a)$$

Nu_{Ao} is the Nusselt number based on the distance from the

II. Cooling (σ_o = 10, σ_P = 100)	A_{Ao}	A_{AP}	B_{AP}	Deviation from Mean of Variable-Property Analyses (In Per Cent of Mean)
A. Variable-Property Analyses				
1. Velocity and Temperature Profiles Taken from Constant-Property Solutions for Ambient Fluid (σ = 10)	0.6394	2.126	0.6724	-1.80
2. Velocity and Temperature Profiles Taken from Constant-Property Solutions for Fluid at Plate (σ = 100)	0.6470	2.151	0.6803	-0.65
3. Velocity and Temperature Profiles Taken To Be Squire's Polynomial Approximations	0.6672	2.218	0.7016	+2.45
4. Means for the Three Sets of Profiles	0.6512	2.165	0.6848	-
B. Constant-Property Analyses				
1. Based on Ambient Fluid (σ = 10)	1.102	3.666	1.159	+69.31
2. Based on Fluid at Plate (σ = 100)	0.6212	2.066	0.6532	+4.61

leading edge of the plate, the average heat-transfer coefficient over that distance, and the thermal conductivity of the ambient fluid; and Nu_{AP} is the same except that it is defined in terms of the thermal conductivity of the fluid at the plate surface. Gr_{xo} and Gr_{xP} are the Grashof numbers based on distance from the edge of the plate and the component of the body force parallel to the plate, Gr_{xo} being defined in terms of the ambient fluid properties and Gr_{xP} in terms of the properties of the fluid at the plate.

TABLE 2

COMPARISON OF ANALYTIC AND EXPERIMENTAL STUDIES OF HEATING A TRANSFORMER OIL WITH A PRANDTL NUMBER VARYING FROM 75.5 TO 309

	B_{AP}^{*}	Deviation from Experimental Value (In Per Cent of Experimental Value)
A. Variable-Property Analysis with Velocity and Temperature Profiles Taken from Constant-Property Solutions with $\sigma = 100$	0.641	+13
B. Constant-Property Analyses		
1. Based on Ambient Fluid ($\sigma = 309$)	0.45	-21
2. Based on Fluid at Plate ($\sigma = 75.5$)	0.65	+15
C. Experimental (Run "H" of Reference 8)	0.567	-

* B_{AP} is defined as in the footnote to Table 1.

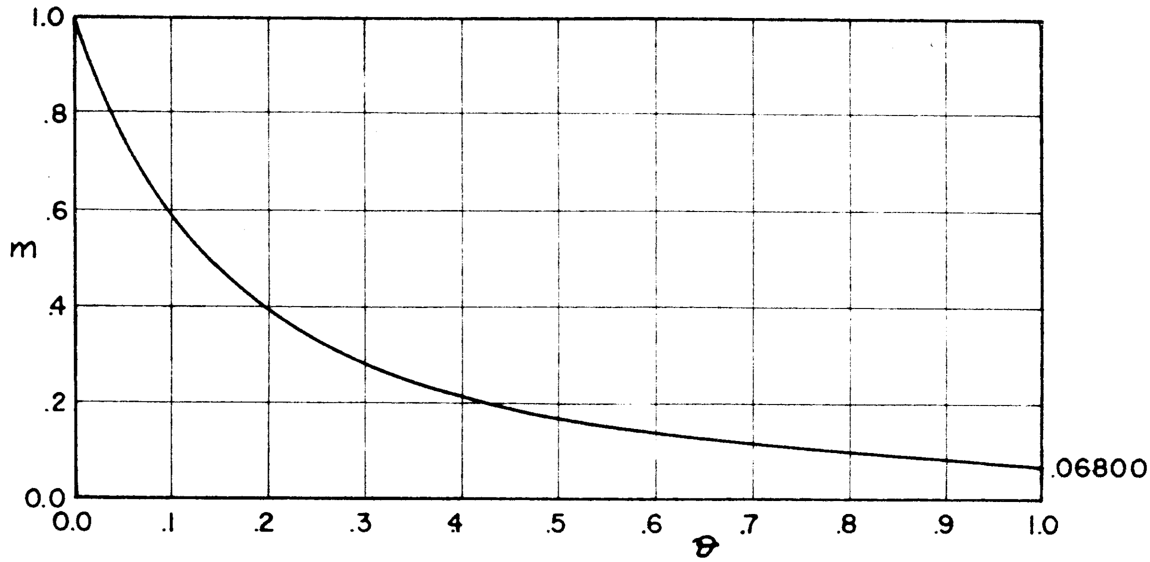

Figure 1

m vs. ϑ for Case of Heating California Crude Fraction

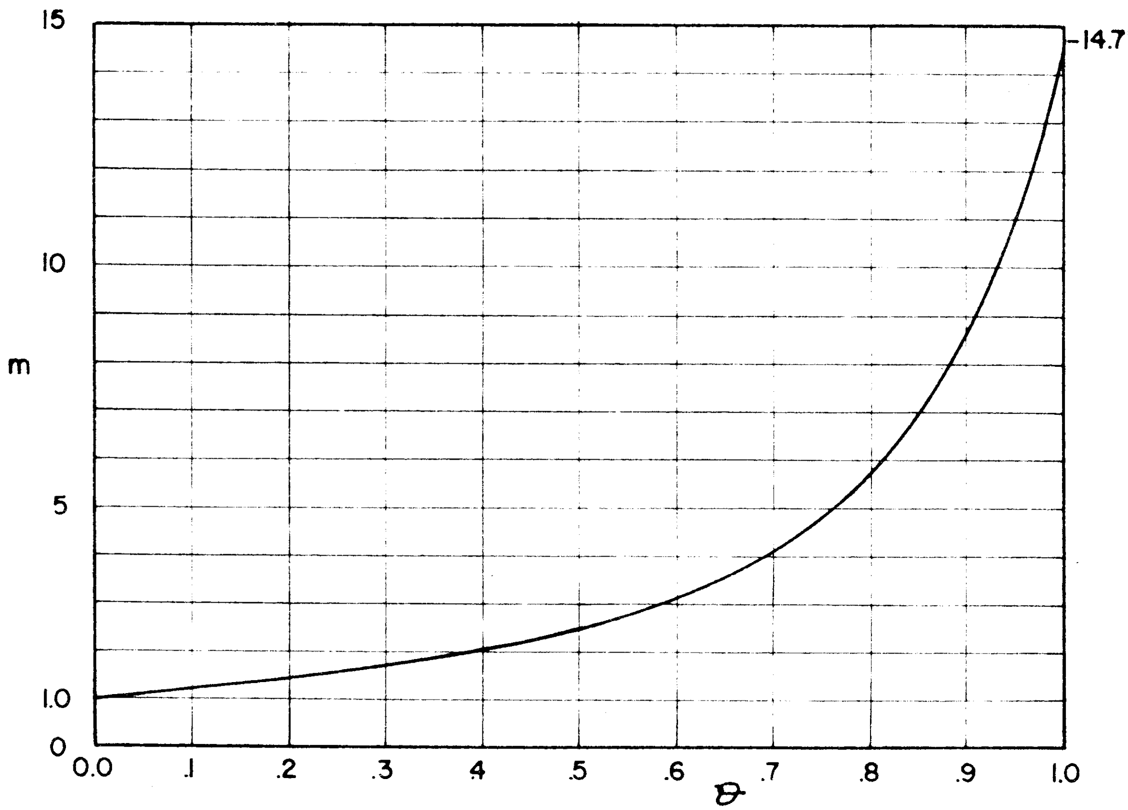

Figure 2

m vs. ϑ for Case of Cooling California Crude Fraction

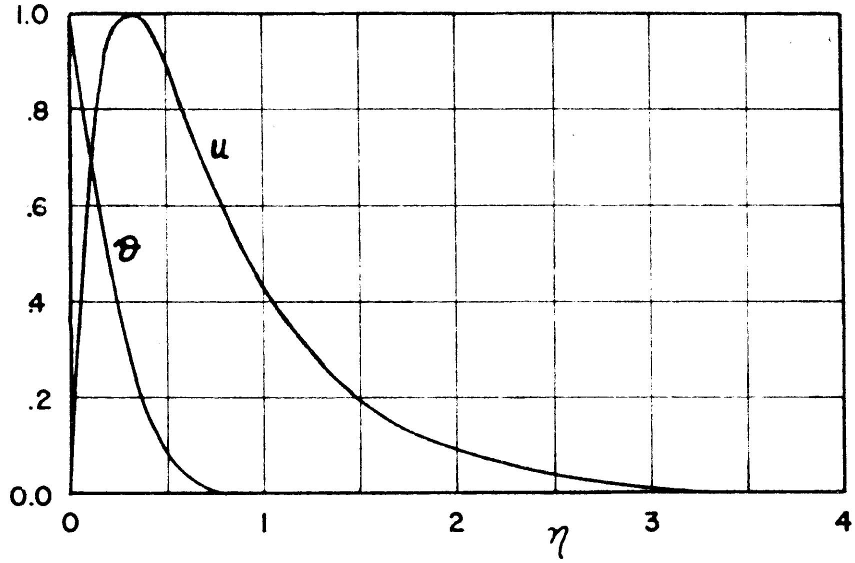

Figure 3

u and ϑ from Solution of Constant-Property Problem for $\sigma = 10$

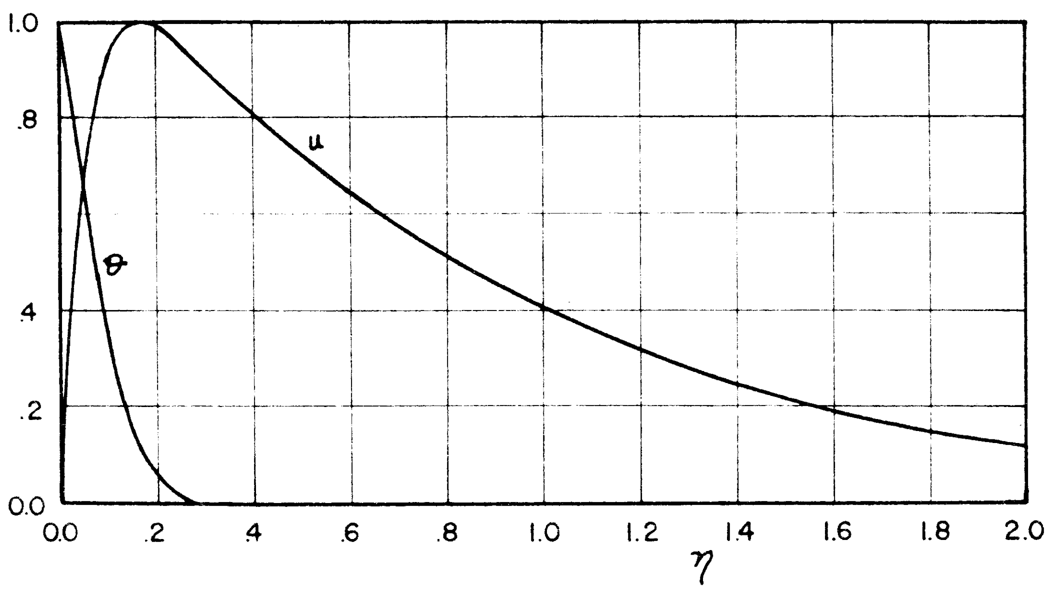

Figure 4

u and ϑ from Solution of Constant-Property Problem for $\sigma = 100$

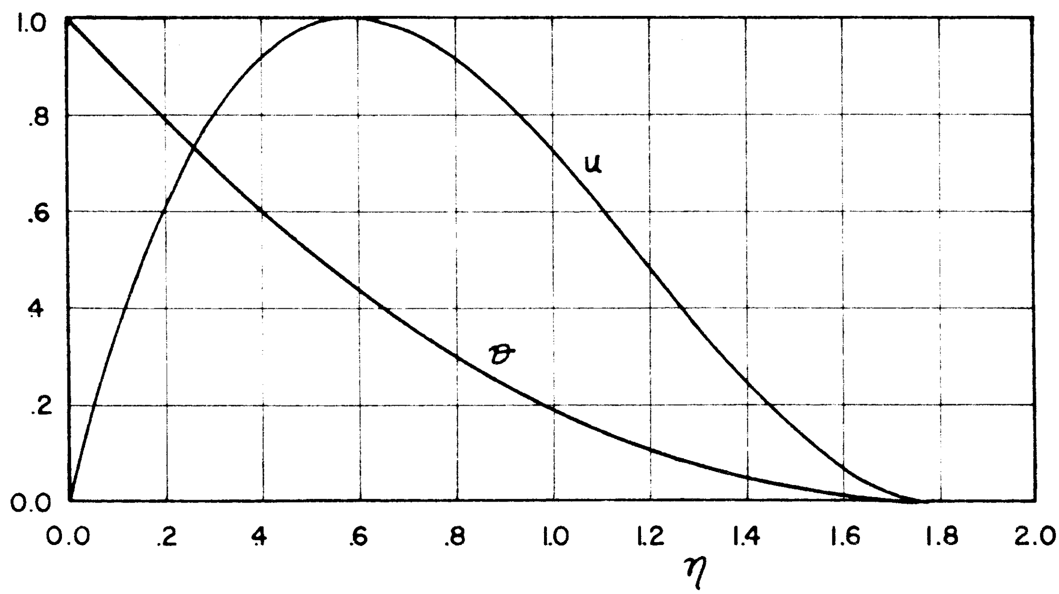

Figure 5

u and ϑ as Assumed by Squire (Reference 3)

$$u = \begin{cases} \frac{243}{64}\eta\left(1-\frac{9}{16}\eta\right)^2, & 0 \leq \eta \leq \frac{16}{9} \\ 0, & \frac{16}{9} \leq \eta \end{cases}$$

$$\vartheta = \begin{cases} \left(1-\frac{9}{16}\eta\right)^2, & 0 \leq \eta \leq \frac{16}{9} \\ 0, & \frac{16}{9} \leq \eta \end{cases}$$