

ON THE THEORY OF NOISE-LIKE ELECTROMAGNETIC
FIELDS OF ARBITRARY SPECTRAL WIDTH

Thesis by
Alexander Donald Jacobson

In Partial Fulfillment of the Requirements
for the Degree of
Doctor of Philosophy

California Institute of Technology
Pasadena, California
1964

ACKNOWLEDGEMENT

The author wishes to acknowledge his indebtedness to his advisor, Professor C. H. Papas, for suggesting the subject matter of this research, and for his guidance and encouragement throughout the course of the research.

The author wishes to acknowledge several highly instructive discussions concerning the statistical aspect of this investigation with Professor D. N. Braverman and Professor H. Martel. The author also wishes to acknowledge a number of helpful discussions with his fellow graduate students, K. S. H. Lee, E. Nagelberg, and K. M. Mitzner.

Thanks are extended to Miss Marleen Mattern and to the Reports Group at the Hughes Research Laboratories who together typed the text, to Mrs. Benita Work, who supervised the art work, and to Mr. S. L. Greenberg, Manager of Technical Information at the Hughes Research Laboratories, for his generous co-operation throughout the preparation of the text. The author is grateful for the generous financial support and the considerate treatment he received under the Hughes Staff Doctoral Fellowship Program of the Hughes Aircraft Company. Last, but by no means least, the author wishes to acknowledge his great debt to his wife who gave so unstintingly her co-operation and support throughout the course of his doctoral studies.

ABSTRACT

A mathematical theory of noise-like electromagnetic fields of arbitrary spectral width is formulated. Attention is restricted to fields whose random fluctuations result exclusively from the chaotic nature of the source. The theory is expressed in terms of the second order moment of the field vector; hence, it is a tensor theory. Moreover, to make it applicable to fields of arbitrary spectral width, the theory is formulated in terms of a spectral representation, rather than directly in terms of the autocorrelation function of the vector field. The principal field quantity, the dyadic field spectral density (DFS), is interpreted from both a statistical and a physical standpoint. In particular, a statistical analysis of partial polarization is presented with the aim of providing a physical interpretation of the polarization of a quasi-monochromatic field. The differential equations that govern the behavior of the DFS are derived in the presence of a source, in a source free region, and in a generalized dielectric medium. Boundary conditions are derived for the DFS at a dielectric interface, at a perfectly conducting interface, and at infinity. The differential equations are integrated for various cases with the aid of the dyadic Green's function. The resulting integral representation for the DFS is employed to analyze an experiment that involves the measurement of a partially polarized, incoherent, discrete radio star by means of a two-element radio interferometer.

TABLE OF CONTENTS

ABSTRACT

INTRODUCTION	1
I THE DYADIC FIELD SPECTRAL DENSITY	7
1.1 Scope of the Problem	7
1.2 Statistical Considerations	10
1.3 The Definition and Properties of the Dyadic Field Spectral Density	16
1.4 The Physical Interpretation of the Dyadic Field Spectral Density	21
II THE STATISTICS OF THE INSTANTANEOUS ELLIPSE	65
2.1 Introductory Remarks	65
2.2 Statement of the Problem	68
2.3 The Statistics of the Amplitude and Phase Functions	71
2.4 Probability Distribution for the Instantaneous Ellipse	73
2.5 The Properties of $P(I, \theta, u)$	76
2.6 An Application	86
III THE EQUATION OF PROPAGATION AND THE BOUNDARY CONDITIONS FOR THE DFS	90
3.1 Introduction	90
3.2 The Dyadic Wave Equation	92
3.3 The Dyadic Wave Equation in a Source Free Medium	99
3.4 The Dyadic Wave Equation in a General Transparent Medium	101
3.5 The Boundary Conditions	107
IV THE INTEGRAL REPRESENTATION OF THE DFS	111
4.1 The Case of a Source Radiating in Free Space	111
4.2 The Vector Analog of the van Cittert-Zernike Formula	116
4.3 The DFS of a Source Radiating in the Presence of Perfect Conductors	126
4.4 The Case of an Incident Plane Wave	130
4.5 The Integral Representation of the DFS in a Source Free Region	136

V	AN APPLICATION -- ANALYSIS OF AN INTERFEROMETER EXPERIMENT	141
5.1	Introduction	141
5.2	Statement of the Problem	142
5.3	The Dyadic Green's Function for the Interferometer	145
5.4	The Formula for the Received Power Density	151
5.5	A Means for Studying the Detailed Character of the Source	166
APPENDICES		
A	TWO THEOREMS RELATING TO PROPERTIES OF THE DFS	169
B	SUBSIDIARY FORMULAS CONCERNING THE STATISTICS OF THE INSTANTANEOUS ELLIPSE	175
C	THE BEHAVIOR OF THE DFS AT INFINITY	186
D	THE SOURCE OF A PLANE WAVE	194
E	MEASUREMENT OF THE DFS	197
F	A THEOREM CONCERNING THE DEGREE OF POLARIZATION	200
	REFERENCES	206

INTRODUCTION

Ideally monochromatic electromagnetic fields do not exist in nature. All real sources have non-zero spectral widths. In many instances, the spectral spread occurs primarily because the source only radiates for a finite time. In other instances, it results from the inherent chaotic nature of the physical processes that give rise to the radiation. The former category includes the coherent sources commonly used in microwave technology such as the klystron oscillator or the traveling wave tube. Systems that employ these sources are analyzed either by means of the conventional monochromatic theory or else, if the transient effects at the beginning or end of the radiation period are of interest, by the combination of Fourier analysis and conventional monochromatic theory. The second category, the so-called noise-like sources, presents a considerably different problem. Examples of this type of source include the "line" sources of physical optics such as the sodium lamp, black body radiators, and the recently discovered cosmic radio sources. These sources are composed of a large number of independent radiators. Because the radiators emit independently, their fields fluctuate as a function both of time and position. Since experiments associated with sources of this type generally occupy time intervals that include many fluctuations of the field quantities, then from an observational standpoint, the fluctuating nature of these fields is one of their essential characteristics. Any analysis of noise-like fields must account for the effect which these fluctuations have on the basic electromagnetic phenomena associated with a radiation field. Broadly speaking, this is the particular problem posed by noise-like fields.

Interest in noise-like electromagnetic radiation fields has existed for well over 100 years. In the past, it was stimulated mainly by researches in the fields of optics and heat radiation. In these regions of the electromagnetic spectrum, all of the known sources (other than the very recently discovered laser) are noise-like sources. More recently, the advent of radio astronomy has extended interest in fluctuating fields down to the radio spectrum. Because of their wide range of occurrence, this class of fields has presented and still presents an important area for study.

Until very recently, the field theoretic considerations involved in the analysis of noise-like radiation have been based exclusively on the methods of geometrical optics. This was the logical approach since the problems originally arose in connection with optical and heat radiation fields. However, an effort has been made in recent years to develop a more complete theory on a more rigorous basis. To date, this research has resulted in a set of differential equations that describe the behavior of the primary field quantity -- the coherency tensor. The coherency tensor determines the intensity, the polarization, the degree of polarization, and the degree of spatial coherence of a quasi-monochromatic field as a function of position in space. The theory as it stands is rigorous; since it is derived in an orderly fashion starting from Maxwell's equations. However, it is both incomplete and of limited applicability. In the first place, the differential equations have not been integrated. Secondly, no consideration has been given to the solution of coherency tensor boundary value problems. But, most importantly, the present theory is restricted to quasi-monochromatic radiation fields. Thus, as it stands, it is not

applicable to problems involving white light or to the very important problem of radiation from celestial radio sources. It is the object of this dissertation to overcome some of these deficiencies by first extending the existing theory to fields of arbitrary spectral width and then integrating the resulting differential equations. Particular attention is given to the development of an adequate conceptual basis for the extended theory. The remaining paragraphs of this introduction survey the pertinent literature and introduce the topics treated in this paper.

The systematic study of irregularly fluctuating electromagnetic fields has been approached principally from the standpoint of optics. This work which covers a span of over 100 years has culminated in the theory of partially coherent fields developed by E. Wolf and his colleagues. The early work in this area was performed by Stokes (1852) (1), who originated the theory of partial polarization, by Verdet (1869) (2), Michelson (1892) (3), and von Lane (1907) (4), who performed early researches into the spatial coherence of extended, quasi-monochromatic sources, by van Cittert (1934) (5), who studied the statistics of light fluctuations at two different points of space, and finally, by Zernike (1938) (6), who formulated the first effective quantitative measure of the degree of spatial coherence of a scalar, quasi-monochromatic field. As a result of these researches, it was found that the effect of the irregular fluctuations of the source is to degrade the polarization and the spatial coherence of the field. To account for these effects, quantitative measures of the deterioration of the polarization and spatial coherence were formulated for the case of quasi-monochromatic fields. However, it was only recently that these effects were described by a systematic theory. Wolf (1955) (7) and,

independently, Blanc-Lapierre and Dumontet (1955) (8) generalized Zernike's theory and obtained the differential equations that govern the propagation of the spatial coherence of a scalar quasi-monochromatic field. At about the same time, Wolf (1955) (9), (10) introduced a new theory of partial polarization which paved the way for generalizing the theory of spatial coherence to include partial polarization. Finally, Wolf and Roman (11)-(14), in a series of four papers published in 1960 and 1961, formulated a unified tensor treatment of quasi-monochromatic vector noise-like fields.* The theory developed by Wolf and Roman consists of the differential equations that determine the behavior of the coherency tensor. The theory is expressed in terms of a tensor quantity because, as Wolf (9) points out, the observable quantities of a quasi-monochromatic optical field are quadratic time averages of the components of the field of which there are 9, in general. It is pointed out in Chapter I that a tensor quantity is also required on the basis of statistical considerations, since the critical quantity from a statistical standpoint is the auto-correlation function (or, equivalently, the spectrum) of the vector field, which is a tensor quantity.

The theory of Wolf and Roman applies only to quasi-monochromatic fields. However, in 1930, Wiener (16) published his celebrated theory of generalized harmonic analysis. Motivated by the earlier researches of Rayleigh, Gouy, and Schuster (17) into the harmonic structure of white

* Beran and Parrent (15), working independently, also reported results on a tensor theory of noise-like fields. However, their work is far less comprehensive than that of Wolf and Roman.

light, Wiener developed the spectral analysis of stationary noise-like signals. As a part of this work, he presented a spectral theory of polarization that is applicable to fields of arbitrary spectral width. This work provides the point-of-departure for the generalized theory of noise-like fields derived in this dissertation. It should be pointed out that Barakat (17), independently of this author, also recognized the fundamental importance of Wiener's work. In a paper published earlier this year, he discussed the significance of Wiener's coherency matrix and then employed it to analyze the interaction of a polarized light ray of arbitrary spectral width with an optical system. However, his study was essentially based on geometrical optics considerations. Hence, it is only remotely connected with the field theoretic study under consideration here.

The dissertation is divided into five chapters. In the first chapter, the problem is defined, statistical considerations are discussed, and the dyadic field spectral density, the primary field quantity, is introduced. The major portion of this chapter is devoted to an analysis of the properties of this quantity and to a discussion of its physical interpretation. Chapter II consists of a statistical analysis of partial polarization. The object of this chapter is to provide insight into the physical character of partial polarization. In Chapter III, the differential equations and the boundary conditions that the dyadic field spectral density must satisfy are derived. These differential equations are integrated in Chapter IV for the case of free space and for the case of perfectly conducting boundaries. The vector analog of the van Cittert-Zernike Theorem is obtained in this chapter, and a general approach is suggested for the solution of the tensor differential equations. The special case of a plane wave incident on perfect

conductors is also considered in Chapter IV. In the last chapter, Chapter V, the theory is applied to the measurement of the radiation field of an extended, incoherent, partially polarized source of arbitrary spectral width by means of an interferometer. This problem is approached from the standpoint of the radio astronomer. Thus, its solution is couched in the language of microwave technology.

I. THE DYADIC FIELD SPECTRAL DENSITY

1. Scope of the Problem

The electromagnetic fields incident on earth from cosmic and interplanetary sources induce voltages at the terminals of a receiving antenna which have the same appearance as the noise-like voltages commonly encountered in communication systems. Presumably, the field that induces such a voltage itself has noise-like properties, i.e., the intensity and direction of the field, fluctuate in a seemingly random manner with respect to both position and time. Fields such as these are referred to here as noise-like fields. They arise in a variety of circumstances other than the radio astronomical situation just mentioned. For example, a plane wave that propagates through a turbulent refractive medium acquires a noise-like character as it progresses through the medium. The fields radiated by the so-called monochromatic sources commonly used in physical optics (prior to the advent of the laser) are also examples of noise-like fields. The field scattered from a moving rough surface is still another example of a noise-like field. These fields share a common property: they all undergo a large number of apparently random fluctuations during the course of a typical measurement; they may also undergo random fluctuations over lengths characteristic of the experiment.

Noise-like fields occur whenever some element of the system possesses a random property. Rather than attempt to develop a general theory that encompasses all possible situations, interest here is directed to fields whose fluctuations are caused exclusively by the chaotic nature

of the source of the radiation field. All other elements of the system are assumed to be deterministic -- i.e., without any random characteristics.

Since the fields are the quantities of primary interest here, it is not intended that a specific detailed model of a physical source be formulated. The starting point of the analysis is the macroscopic current density $\underline{J}(\underline{r}, t)$. However, it is desirable to give some consideration to the nature of the radiating system in order to fix ideas.

The source is assumed to consist of a large number of "individual" radiators (e.g., the atoms of an ionized gas or the electrons of a plasma) which radiate independently of each other. Furthermore, the source is assumed to be in a *steady state*. That is, the *gross conditions that influence* the radiation processes of the source are assumed to be constant with respect to time. A source such as this is commonly referred to as a stationary, incoherent source. It is stationary because the radiators are assumed to be in an environment whose properties are constant (at least for periods long compared with the durations of an experiment). It is incoherent because the individual radiators emit independently of one another. Since the source is to be described by the macroscopic current density $\underline{J}(\underline{r}, t)$, the word "individual" requires some explanation. The "individual radiator" at point \underline{r} described by the current density $\underline{J}(\underline{r}, t)$ is, in actuality, the collection of atoms or electrons contained within a volume of macroscopic dimensions. Generally, this collection consists of a large number of elements. This characterization is essential to make effective use of Maxwell's equations in developing the theory of noise-like fields. However, it leads to questions of fundamental importance relating to the underlying physical processes responsible for the observed radiation

field. For example, in the case of the noise-radiation arising from a plasma in thermodynamic equilibrium, it is apparent that the number of electrons per unit time crossing an area of macroscopic dimension in one direction is equal to the number per unit time crossing in the opposite direction. Thus, in a strict physical sense, the macroscopic current in a plasma in equilibrium should be zero. The noise-fields must be attributed to the microscopic processes involved in the haphazard motion of the individual electrons. Similarly, many other radiation processes that give rise to noise-like fields are, in fact, microscopic processes. However, since interest here is directed toward the macroscopic fields, it is not desirable to treat the source on a microscopic basis. Consequently, the analysis that follows is developed on a phenomenological basis. The field is assumed to arise from a distribution of macroscopic current elements whose intensity and polarization fluctuate with respect to position and time. The theory is developed from Maxwell's macroscopic field equations.

Throughout this paper, no restrictions are placed on the spectral character of the radiation other than that the width of the spectrum must be non-zero.* Further, the polarization of the source is unrestricted. Hence, the theory to be developed here applies equally well to "white" light, to the so-called quasi-monochromatic fields of optics, and to the fields radiated by cosmic radio sources. On the other hand, it does not apply to incoherent sources that undergo significant changes in their gross

* The spectral width must be non-zero because the fields fluctuate with respect to time during the course of an observation.

behavior during the course of an experiment such as those radiated by solar flares or by lightning.

1.2 Statistical Considerations

Noise-like fields radiated by sources of the type just discussed are not adequately described by the spatial and temporal variation of the field vector alone (just as a noise-like voltage is not completely described by a single time record, $v(t)$). Because of the irregular fluctuations of the source, parallel experiments to determine the spatial and temporal variation of a noise-like field give rise to an ensemble of different results. Therefore, to specify a noise-like field completely, it is necessary to describe an entire ensemble of possible fields. This can be done most conveniently by employing a statistical description of the field.

By setting the analysis in a statistical context, the problem posed by a noise-like field is changed from the conventional one of determining the field vector as a function of position and time to one of determining the statistics of the field. This constitutes a radical change of viewpoint. To understand the distinction between the two types of analysis, consider the problem of characterizing a scalar random phenomenon, such as a noise-like voltage, by means of statistics. A single time record of the voltage measured from $t = -\infty$ to $t = +\infty$ is called a sample function of the voltage. Successive or parallel measurements of the voltage are capable of yielding any one of a variety of different voltage sample functions. The collection of all possible such sample functions is called a random (or stochastic) process, and is denoted simply as $v(t)$. Since the voltage measured at any instant t might be that associated with any

one of the possible sample functions, then $v(t_1)$ is a random variable. Therefore, to describe the entire ensemble of sample functions, it is necessary to specify the probability functions associated with the continuous sequence of random variables constituting $v(t)$.

A single random variable x is specified by $p(x)$, the probability distribution of x . Two random variables x, y are specified by their joint probability distribution $p(x, y)$. Similarly, the random process $v(t)$ is specified by the joint probability distribution of all the random variables of which it is composed. Thus, if $t_1 < \dots < t_N$ are any N instants of time and $v(t_1), \dots, v(t_N)$ are the associated random variables, then $p[v(t_1), \dots, v(t_N)]$ specifies these N random variables. To specify the random process, it is necessary to know all such joint probability distributions associated with the random process $v(t)$.

The statistical characterization of an electromagnetic field is carried out in the same manner as just described. However, instead of specifying a single random process, it is necessary to jointly specify three random processes, one for each component of the vector field. A process of this sort is called a vector random process. The situation is further complicated by the fact that in the case of an electromagnetic field, there are four independent variables x, y, z, t , instead of the single variable t that appeared in the voltage random process discussed above. Consequently, the statistics of a random electromagnetic field are specified only if $p[E_x(\underline{r}_1, t_1), E_y(\underline{r}_1, t_1), E_z(\underline{r}_1, t_1), \dots, E_x(\underline{r}_N, t_N), E_y(\underline{r}_N, t_N), E_z(\underline{r}_N, t_N)]$ is known for all possible sets of random variables $\{E(\underline{r}_i, t_i)\}$. Of course, the same remarks apply to the current density vector $\underline{J}(\underline{r}, t)$.

The usual problem in electromagnetic theory is to express the field in terms of a prescribed source. In the case of a noise-like field, the problem is to relate the noise-like field to its noise-like source. Since the noise-like source is described by its statistics, then to solve the problem, it is necessary to relate the probability distributions of the field to those of the source. Since the field is related to the source by means of a set of linear differential equations, it is possible to regard the problem as the passing of a noise-like input "signal", the current density vector, through a linear system, Maxwell's equations, to obtain the noise-like output signal, the electromagnetic field. From this point of view, the problem is similar to that of analyzing the interaction of a noise-like signal with a linear lumped parameter system. Although the mathematics are considerably different in the two cases, the over-all character of the problems is quite similar. Therefore, many of the ideas already developed to deal with communications systems can be carried over directly to the analysis of noise-like electromagnetic fields. For instance, it is commonly known in communication theory that no general, practicable method exists for relating the statistics of the signal obtained at the output of a linear filter to those of the input signal. The same statement applies to the electromagnetic field problem, it is not possible to obtain a general relationship between the probability functions of the field and those of the source. However, in the very important case of Gaussian statistics, the network problem can be solved exactly. Correspondingly, the noise-like electromagnetic field problem can also be solved exactly in this case (at least in principle). For this reason, it is tacitly assumed throughout the remainder of the dissertation that the field

quantities obey Gaussian statistics. As pointed out below, if this assumption is unfounded, then the results obtained here offer only a partial solution of the problem.

Since the theory that is developed here is based on the tacit assumption of Gaussian statistics, it is useful to consider briefly the definition and basic properties of a Gaussian random process. If for every finite collection of "points" $(\underline{r}_1, t_1), \dots, (\underline{r}_N, t_N)$ the vector random variables $\underline{E}(\underline{r}_i, t_i)$ (or $\underline{J}(\underline{r}_i, t_i)$) are specified by multivariate Gaussian distributions, then the vector field $\underline{E}(\underline{r}, t)$ (or $\underline{J}(\underline{r}, t)$) is said to be a Gaussian vector field. A Gaussian field has several important properties. It is convenient to discuss these properties in terms of a scalar process such as the noise voltage mentioned above. If $v(t)$ is a zero mean, stationary Gaussian process, then the complete description of the probability distributions which specify $v(t)$ is provided by the autocorrelation function $R_v(t) = \overline{v(t+\tau)v(t)}$, where the bar over the product on the right indicates an ensemble average.* Note that the autocorrelation function is assumed to be independent of t . This follows from the fact that $v(t)$ is a stationary process. Thus, it is only necessary to know the autocorrelation function of a Gaussian process to specify the process completely. A second important property of the Gaussian process is that it gives rise to a Gaussian output when it is passed through a linear system of any kind.

The extension of these properties to a vector stationary Gaussian random process is clear. The statistics of a Gaussian electric field are

* The mathematical characterization of a Gaussian process is given in Chapter 8 of Davenport and Root (19).

determined by the set of second moments $\overline{E_i(\underline{r}_1, t+\tau)E_j(\underline{r}_2, t)}$, $i, j = 1, 2, 3$. In this expression, \underline{r}_1 and \underline{r}_2 represent two different points in the field, and τ represents the time interval between the instants at which the field at \underline{r}_1 and at \underline{r}_2 is evaluated. Thus, the autocorrelation function of a vector electromagnetic field consists of a matrix of 9 elements. It is often convenient to write this matrix in vector notation, i.e., in the form

$$\underline{\underline{\xi}}(\underline{r}_1, \underline{r}_2, \tau) = \overline{\underline{E}(\underline{r}_1, t+\tau)\underline{E}(\underline{r}_2, t)} \quad (1.2.1)$$

where the field vectors $\underline{E}(\underline{r}_1, t+\tau)$ and $\underline{E}(\underline{r}_2, t)$ are juxtaposed. In other words, the autocorrelation function of a stochastic vector field is a dyadic quantity.* Expanded in a rectangular coordinate system, it has the form

$$\underline{\underline{\xi}}(\underline{r}_1, \underline{r}_2, \tau) = \sum_{i=1}^3 \sum_{j=1}^3 \overline{E_i(\underline{r}_1, t+\tau)E_j(\underline{r}_2, t)} \underline{e}_i \underline{e}_j \quad (1.2.2)$$

where $\underline{e}_1, \underline{e}_2, \underline{e}_3$ represent the unit vectors in the direction of the coordinates x_1, x_2, x_3 respectively. Observe that, because of the stationarity, the dyadic autocorrelation function depends on τ , but not on t . Thus, the autocorrelation function is independent of the time origin. The dyadic autocorrelation function of the source is written in the same manner, viz.,

* There is a formal identity between a dyadic and a second rank tensor. Thus, the autocorrelation function is also a second rank tensor. However, to maintain uniform terminology and to acknowledge the fact that the autocorrelation function is formed from the juxtaposition of vectors, the function $\underline{\underline{\xi}}(\underline{r}_1, \underline{r}_2, \tau)$ is referred to as a dyadic throughout this dissertation.

$$\underline{\underline{g}}(\underline{r}_1, \underline{r}_2, \tau) = \frac{\underline{J}(\underline{r}_1, t+\tau) \underline{J}(\underline{r}_2, t)}{\quad} \quad (1.2.3)$$

It also can be expanded in the form given in 1.2.2.

Since the source and the field are related by linear differential equations, then by virtue of the second property mentioned above, a Gaussian source radiates a Gaussian field and, conversely, the source of a Gaussian field is itself a Gaussian process. Consequently, if it is known "a priori" that either the field or the source is Gaussian, then the problem of determining the statistics of the source and the field reduces to one of determining the dyadic autocorrelation functions $\underline{\underline{\xi}}(\underline{r}_1, \underline{r}_2, \tau)$ and $\underline{\underline{g}}(\underline{r}_1, \underline{r}_2, \tau)$. It should be noted that if "a priori" knowledge of the statistics of the source or of the field is unavailable or if it is known "a priori" that the source or the field is not Gaussian, then determination of $\underline{\underline{\xi}}(\underline{r}_1, \underline{r}_2, \tau)$ and $\underline{\underline{g}}(\underline{r}_1, \underline{r}_2, \tau)$ provides only a partial solution of the problem. The complete solution of this more general problem is found only if all of the higher order moments of the field are known. Clearly, a problem of this magnitude is not tractable. Fortunately, however, it is found that in many situations of great interest, there is strong evidence to suggest that the fields are indeed described by Gaussian statistics.* On the basis of these considerations, this study is restricted to the analysis of the second moment of the field. Therefore, from the standpoint of statistical analysis, the results obtained here provide a complete solution of the Gaussian field problem, but only a partial

* See the early study by van Cittert (20). See also the studies by Janossy (21), (22) and by Hurwitz (23).

solution of the general problem of statistical electromagnetic fields.

1.3 The Definition and Properties of the Dyadic Field Spectral Density

Throughout the foregoing paragraphs, the discussion has been directed toward the statistical considerations that underlie the analysis of noise-like fields. It is now of interest to examine the problem from the standpoint of the theory of electromagnetic fields. Although it was found that the dyadic autocorrelation function is the quantity of primary interest from a statistical point of view, it is not clear that it is from either a mathematical or a physical standpoint. In fact, some thought on the matter indicates that it is not. In the first place, problems of interest often involve the interaction of noise-like fields with various types of physical systems (e.g., measuring instruments, non-vacuous propagating media, etc.). Since systems that interact with electromagnetic radiation are generally dispersive, i.e., they respond differently to radiation of different frequencies, then it is desirable to have available a spectral representation of the field. Recourse to a spectral field quantity becomes even more desirable if one considers the problem from a physical point of view. Physicists are not as interested in the statistics of the field and the relationship of these statistics to those of the source as they are in the physical nature of the source and the physical processes which give rise to the observed radiation field. Thus, they are interested in the spectrum of the source, the size of the source, the distribution of intensity over the source, the polarization of the source, etc. With information such as this, they can deduce the properties of the source as well as the characteristics of the sources' environment.

Therefore, to provide a theory that satisfies the requirements imposed by the physics of the problem as well as by the mathematical intricacies of electromagnetic theory without over-riding the statistical considerations discussed above, it is desirable to introduce a spectral representation of the field as the primary field quantity.

The spectral representation of an irregularly fluctuating optical field was originally introduced by Wiener (16) as an application of his theory of generalized harmonic analysis. Barakat (18) recently re-examined Wiener's representation, updated it into the context of modern statistical analysis, and offered this point of view as the natural extension of Wolf's theory of the coherency matrix to fields of arbitrary spectral width. Thus, the notion of a spectral representation of a fluctuating electromagnetic field has been considered before. However, the definition given here is considerably more general than that of either Wiener or of Barakat since it describes the spatial coherence as well as the polarization of the field. Because the spectral representation to be used here differs from that of Wiener, it is introduced below from first principles.

Consider first the definition of the spectrum of a scalar noise-like quantity such as the voltage signal mentioned earlier. The spectrum, or more precisely, the power spectral density of a stationary noise-like voltage is defined to be the Fourier transform of the autocorrelation function of the voltage.

$$\hat{R}_v(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} R_v(\tau) e^{i\omega\tau} d\tau, \quad (1.3.1)$$

where the caret indicates a spectral density function.

Since $\hat{R}_v(\omega)$ and $R_v(\tau)$ are a Fourier transform pair, then $R_v(\tau)$ can be expressed in terms of $\hat{R}_v(\omega)$ by means of the inversion theorem:

$$R_v(\tau) = \int_{-\infty}^{\infty} \hat{R}_v(\omega) e^{-i\omega\tau} d\omega. \quad (1.3.2)$$

The physical interpretation of the power spectral density $\hat{R}_v(\omega)$ can be easily demonstrated. Suppose the voltage $v(t)$ is passed through a band-pass filter of very narrow width $\Delta\omega$ that is centered at frequency ω , and terminated in a one-ohm resistor. Then it can be shown that the average power delivered to the resistor is $\hat{R}_v(\omega_0) \Delta\omega$ (provided $\Delta\omega$ is sufficiently small compared with the variation of $\hat{R}_v(\omega)$). But the filter only passes power in the frequency interval $\Delta\omega$ about the center frequency ω_0 . Therefore, $\hat{R}_v(\omega_0)$ represents the average power of $v(t)$ per unit frequency in this interval, hence, the name power spectral density.

The spectral representations of a noise-like field and its source follow immediately from the foregoing considerations. These spectral densities are defined to be the Fourier transform of the respective dyadic autocorrelation functions. In mathematical terms, these definitions are

$$\begin{aligned} \hat{\underline{\underline{\mathcal{E}}}}(\underline{r}_1, \underline{r}_2, \omega) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \underline{\underline{\mathcal{E}}}(\underline{r}_1, \underline{r}_2, \tau) e^{i\omega\tau} d\tau \\ \hat{\underline{\underline{\mathcal{G}}}}(\underline{r}_1, \underline{r}_2, \omega) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \underline{\underline{\mathcal{G}}}(\underline{r}_1, \underline{r}_2, \tau) e^{i\omega\tau} d\tau \end{aligned} \quad (1.3.3)$$

The new field quantity $\hat{\underline{\underline{\mathcal{E}}}}(\underline{r}_1, \underline{r}_2, \omega)$ is called the dyadic field spectral density. Hereafter, it is denoted by the abbreviation DFS. The

quantity $\hat{\underline{g}}(\underline{r}_1, \underline{r}_2, \omega)$ is referred to as the dyadic source spectral density and is denoted by the letters DSS.

Inversion of these equations provides the dyadic autocorrelation functions in terms of the respective spectra.

$$\begin{aligned}\underline{\underline{E}}(\underline{r}_1, \underline{r}_2, \tau) &= \int_{-\infty}^{\infty} \hat{\underline{\underline{E}}}(\underline{r}_1, \underline{r}_2, \omega) e^{-i\omega\tau} d\omega \\ \underline{\underline{g}}(\underline{r}_1, \underline{r}_2, \tau) &= \int_{-\infty}^{\infty} \hat{\underline{\underline{g}}}(\underline{r}_1, \underline{r}_2, \omega) e^{-i\omega\tau} d\omega\end{aligned}\tag{1.3.4}$$

Thus, to obtain the statistics of the field from $\hat{\underline{\underline{E}}}(\underline{r}_1, \underline{r}_2, \omega)$ (or the statistics of the source from $\hat{\underline{\underline{g}}}(\underline{r}_1, \underline{r}_2, \omega)$) it is only necessary to substitute $\hat{\underline{\underline{E}}}(\underline{r}_1, \underline{r}_2, \omega)$ ($\hat{\underline{\underline{g}}}(\underline{r}_1, \underline{r}_2, \omega)$) into 1.3.4, obtain the dyadic autocorrelation function, and then make use of the standard formula for the multivariate Gaussian probability distribution. The remaining paragraphs of this section are devoted to a presentation of the mathematical properties of the DFS.

The explicit relationship between the DFS and the field vectors themselves is obtained by substituting 1.2.1 into 1.3.3. Thus,

$$\hat{\underline{\underline{E}}}_{ij}(\underline{r}_1, \underline{r}_2, \omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \overline{E_i(\underline{r}_1, t+\tau) E_j(\underline{r}_2, t)} e^{i\omega\tau} d\tau\tag{1.3.5}$$

where $\hat{\underline{\underline{E}}}_{ij}(\underline{r}_1, \underline{r}_2, \omega)$ is the ij^{th} component of the DFS. Consider first the transpose property. From 1.3.5 it is seen that

$$\begin{aligned}\hat{\underline{\underline{E}}}_{ji}(\underline{r}_1, \underline{r}_2, \omega) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \overline{E_j(\underline{r}_1, t+\tau) E_i(\underline{r}_2, t)} e^{i\omega\tau} d\tau \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \overline{E_i(\underline{r}_2, t'-\tau) E_j(\underline{r}_1, t')} e^{i\omega\tau} d\tau\end{aligned}$$

where t has been replaced by $t' - \tau$. If τ is replaced by $-\tau'$ in the integrand, it is found that

$$\hat{\mathcal{E}}_{ij}(\underline{r}_1, \underline{r}_2, \omega) = \frac{1}{2\omega} \int_{-\infty}^{\infty} \frac{\underline{E}_i(\underline{r}_2, t' + \tau') \underline{E}_j(\underline{r}_1, t')}{\omega} e^{-i\omega\tau'} d\tau' \quad (1.3.6)$$

By comparing the right side of 1.3.6 with that of 1.3.5, it is seen that

$$\hat{\mathcal{E}}_{ji}(\underline{r}_1, \underline{r}_2, \omega) = \hat{\mathcal{E}}_{ij}(\underline{r}_2, \underline{r}_1, -\omega) \quad (1.3.7)$$

Since the field quantities are real functions of the real variables \underline{r} , t' , and τ' , then the right side of 1.3.6 is the complex conjugate of $\hat{\mathcal{E}}_{ij}(\underline{r}_2, \underline{r}_1, \omega)$, so that

$$\hat{\mathcal{E}}_{ji}(\underline{r}_1, \underline{r}_2, \omega) = \hat{\mathcal{E}}_{ij}^*(\underline{r}_2, \underline{r}_1, \omega). \quad (1.3.8)$$

Equations 1.3.7 and 1.3.8 can be written in vector form.

$$\hat{\underline{\mathcal{E}}}^T(\underline{r}_1, \underline{r}_2, \omega) = \hat{\underline{\mathcal{E}}}(\underline{r}_2, \underline{r}_1, -\omega) \quad (1.3.9)$$

or

$$\hat{\underline{\mathcal{E}}}^T(\underline{r}_1, \underline{r}_2, \omega) = \hat{\underline{\mathcal{E}}}^*(\underline{r}_2, \underline{r}_1, \omega) \quad (1.3.10)$$

where the superscript T indicates the transpose operation. That is,

$$[\sum_{ij} \hat{\mathcal{E}}_{ij}(\underline{r}_1, \underline{r}_2, \omega) \underline{e}_i \underline{e}_j]^T = [\sum_{ij} \hat{\mathcal{E}}_{ji}(\underline{r}_1, \underline{r}_2, \omega) \underline{e}_i \underline{e}_j].$$

A second important property of the DFS follow from 1.3.10. If both sides of that equation are evaluated at $\underline{r}_1 = \underline{r}_2 = \underline{r}$, it is seen that $\hat{\underline{\mathcal{E}}}(\underline{r}, \underline{r}, \omega) = \hat{\underline{\mathcal{E}}}^T(\underline{r}, \underline{r}, \omega)$. Therefore, $\hat{\underline{\mathcal{E}}}(\underline{r}, \underline{r}, \omega)$ is hermitian.* Observe that this is not true of the general DFS, $\hat{\underline{\mathcal{E}}}(\underline{r}_1, \underline{r}_2, \omega)$. Since the diagonal terms

* Although it is not the DFS, but its matrix, that is hermitian, it is convenient to overlook this nicety and, depending on the context, to discuss the DFS as though it were a dyadic or a matrix or even a tensor.

of an hermitian matrix are real, then $\text{Tr} \{ \hat{\underline{\underline{C}}}(\underline{r}, \underline{r}, \omega) \}$ is real. Furthermore, it can be shown that

$$\text{Tr} \{ \hat{\underline{\underline{C}}}(\underline{r}, \underline{r}, \omega) \} \geq 0. \quad (1.3.11)$$

To prove 1.3.11, it is only necessary to show that

$$\hat{\underline{\underline{C}}}_{ii}(\underline{r}, \underline{r}, \omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} [\underline{E}_i(\underline{r}, t+\tau) \underline{E}_i(\underline{r}, t)] e^{i\omega\tau} d\tau \geq 0.$$

But this follows from the well known fact that the power spectral density of a scalar stationary stochastic process such as $\underline{E}_i(\underline{r}, t)$ is a positive function of frequency (24).

A final property of the DFS relates to the determinant of the matrix formed from the coefficients of the dyadic $\hat{\underline{\underline{C}}}(\underline{r}, \underline{r}, \omega)$. It is always true that

$$\det \{ \hat{\underline{\underline{C}}}(\underline{r}, \underline{r}, \omega) \} \geq 0. \quad (1.3.12)$$

The proof of 1.3.12 is somewhat lengthy, and is relegated to Appendix A.

In summary, the basic mathematical properties of the DFS are:

- 1) $\hat{\underline{\underline{C}}}^T(\underline{r}_1, \underline{r}_2, \omega) = \hat{\underline{\underline{C}}}^*(\underline{r}_2, \underline{r}_1, \omega) = \hat{\underline{\underline{C}}}(\underline{r}_2, \underline{r}_1, -\omega).$
- 2) The coefficients of $\hat{\underline{\underline{C}}}(\underline{r}, \underline{r}, \omega)$ form a hermitian matrix so that in particular $\text{Tr} \{ \hat{\underline{\underline{C}}}(\underline{r}, \underline{r}, \omega) \}$ is real.
- 3) $\text{Tr} \{ \hat{\underline{\underline{C}}}(\underline{r}, \underline{r}, \omega) \} \geq 0.$
- 4) $\text{Det} \{ \hat{\underline{\underline{C}}}(\underline{r}, \underline{r}, \omega) \} \geq 0.$

1.4 The Physical Interpretation of the DFS

The subject of concern in this section is the physical interpretation of the DFS. To begin with, recall that the physical properties of the field that are of interest include the spectral density (hereafter simply called the spectrum), the polarization, and the pattern or spatial structure

of the field. It is the object of the following paragraphs to relate these properties to the elements of the DFS.

A. The Spectrum of the Field

The spectrum of an electromagnetic field having finite total energy (i.e., a field for which $\int_{-\infty}^{\infty} |\underline{E}(\underline{r}, t)|^2 dt < \infty$) is defined in terms of the Fourier transform of the field vector by means of Parseval's theorem. Consider a TEM wave. The instantaneous Poyntings vector for this wave at the point \underline{r} is given by

$$\underline{S}(\underline{r}, t) = \underline{E}(\underline{r}, t) \times \underline{H}(\underline{r}, t) = \sqrt{\frac{\epsilon_0}{\mu_0}} |\underline{E}(\underline{r}, t)|^2 \text{ watts/meter}^2, \quad (1.4.1)$$

where ϵ_0 and μ_0 are the permittivity and permeability of free space, respectively. Thus, the total energy crossing a surface of unit area oriented perpendicular to the direction of propagation is given by

$$W(\underline{r}) = \int_{-\infty}^{\infty} \underline{S}(\underline{r}, t) dt = \sqrt{\frac{\epsilon_0}{\mu_0}} \int_{-\infty}^{\infty} |\underline{E}(\underline{r}, t)|^2 dt. \quad (1.4.2)$$

If $\hat{\underline{E}}(\underline{r}, \omega)$ is the Fourier transform of $\underline{E}(\underline{r}, t)$ at the point \underline{r} , then by Parseval's theorem

$$W(\underline{r}) = \sqrt{\frac{\epsilon_0}{\mu_0}} \int_{-\infty}^{\infty} |\hat{\underline{E}}(\underline{r}, \omega)|^2 d\omega. \quad (1.4.3)$$

The quantity $\sqrt{\frac{\epsilon_0}{\mu_0}} |\hat{\underline{E}}(\underline{r}, \omega)|^2$ represents the spectral distribution of the energy crossing a unit area at the point \underline{r} . Its units are joules per meter squared per unit frequency. If it is denoted by $\hat{\mathcal{J}}(\underline{r}, \omega)$, then it can be written in the form

$$\hat{\mathcal{J}}(\underline{r}, \omega) = \sqrt{\frac{\epsilon_0}{\mu_0}} [\hat{E}_i(\underline{r}, \omega) \hat{E}_i^*(\underline{r}, \omega)] \quad i = 1, 2, \quad (1.4.4)$$

where the summation convention is implied.*

The extension of this definition to a noise-like field is straightforward. It is only necessary to observe that rather than having finite total energy, a stationary noise-like field has finite average power and, therefore, infinite total energy. Consequently, the energy spectrum is not defined. Instead, it is necessary to define the spectrum of a noise-like field in terms of the distribution of average power with respect to frequency. The average power per unit area carried by a given sample function of the wave is

$$P^{\alpha}(\underline{r}) = \sqrt{\frac{\epsilon_0}{\mu_0}} \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T E_i^{\alpha}(\underline{r}, t) E_i^{\alpha}(\underline{r}, t) dt, \quad (1.4.8)$$

where the superscript α indicates a particular sample function. Since the spectrum is defined to be a mean or average property of a noise-like variable, then the quantity of interest is the ensemble average of $P^{\alpha}(\underline{r})$. Thus, if $P(\underline{r})$ represents the ensemble average of $P^{\alpha}(\underline{r})$, then on interchanging the order of integration and averaging, it is found that

$$P(\underline{r}) = \sqrt{\frac{\epsilon_0}{\mu_0}} \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T E_i^{\alpha}(\underline{r}, t) E_i^{\alpha}(\underline{r}, t) dt = \sqrt{\frac{\epsilon_0}{\mu_0}} \text{Tr} \{ \underline{\underline{E}}(\underline{r}, \underline{r}, \tau) \Big|_{\tau=0} \} \quad (1.4.6)$$

* To eliminate the superabundance of summation signs that would ordinarily appear in a work of this sort, a somewhat liberalized version of the Einstein summation convention is employed throughout the dissertation. The principal rule to remember is that a subscript that appears once on each of two adjacent symbols is to be summed over its entire range. If this range is not obvious from the context, it will be stated explicitly. For example, each of the following terms is to be summed over "i": $E_i(\underline{r}, t) E_i(\underline{r}', t')$, $\hat{E}_{ij}(\underline{r}, \underline{r}, \omega) \hat{E}_{ik}(\underline{r}', \underline{r}', \omega)$, $\hat{E}_{ji}(\underline{r}, \underline{r}, \omega) \hat{E}_{ik}(\underline{r}', \underline{r}', \omega)$, etc. Repeated subscripts on the same symbol as in $\hat{E}_{ii}(\underline{r}, \underline{r}, \omega)$ do not imply summation.

where $\underline{\underline{\xi}}(\underline{r}, \underline{r}, \tau)$ is the dyadic autocorrelation function. Substitution of 1.3.4 into 1.4.6 yields

$$P(\underline{r}) = \sqrt{\frac{\epsilon_0}{\mu_0}} \int_{-\infty}^{\infty} \text{Tr} \hat{\underline{\underline{\xi}}}(\underline{r}, \underline{r}, \omega) d\omega. \quad (1.4.7)$$

Therefore, the spectrum of a noise-like field expressed in terms of the DFS is given by

$$\hat{P}(\underline{r}, \omega) = \sqrt{\frac{\epsilon_0}{\mu_0}} \text{Tr}[\hat{\underline{\underline{\xi}}}(\underline{r}, \underline{r}, \omega)]. \quad (1.4.8)$$

The units of $\hat{P}(\underline{r}, \omega)$ are watts per unit area per unit frequency. In physical terms $[\hat{P}(\underline{r}, \omega)\Delta\omega]$ represents the average power, contained in the narrow range of frequencies $\Delta\omega$, centered at frequency ω , that crosses a surface of unit area oriented perpendicular to the direction of propagation. Therefore, $\hat{P}(\underline{r}, \omega)$ is the spectral density per unit area.

Consider for a moment the measurement of the spectrum. Because the measurement of power involves the consideration of a single sample function of the random process, repeated power measurements will, in general, yield different results. This is true even if one considers ideal measurements that extend from $t = -\infty$ to $t = +\infty$. However, in the special case of an ergodic process, or in the less restrictive case of a process that satisfies the conditions of the ergodic theorem, the power contained in all sample functions is the same. Thus, in this case, it is unnecessary to distinguish between $P^*(\underline{r})$ and $P(\underline{r})$. These remarks pertain directly to the measurement of the spectrum of a stationary fluctuating electromagnetic field since such fields generally satisfy the conditions of the ergodic theorem. *

* As Yaglom (25) shows in his text on stationary processes, a Gaussian process that has a continuous spectrum satisfies the conditions of the ergodic theorem.

Hence, the spectrum of the fields considered here can be determined by a single experiment (i.e., without the necessity of statistical averaging).

Although no explicit reference was made to the fact that the wave is TEM in the foregoing derivation, it was tacitly assumed in writing 1.4.6. Therefore, in a strict sense the definition 1.4.8 applies only to TEM waves. However, for the sake of uniformity, this definition of the spectrum of the field at a point is assumed to apply regardless of the character of the wave.

B. Polarization of the Field

The definition of the polarization of a noise-like field at the point \underline{r} is a problem of considerably greater complexity than that of the spectrum of the field. In fact, it is sufficiently complicated that no attempt is made in this chapter to provide a physical interpretation. The physical interpretation is deferred to the following chapter. All that is done here is to obtain the mathematical expressions relating the polarization parameters to the elements of the DFS.

Briefly, the problem is to define the polarization of a polychromatic wave. It will be recalled that polarization is conventionally defined for time-harmonic field vectors, i.e., for fields of the type^{*}

$$\underline{E}(t) = A_1 \cos(\omega_0 t + \varphi_1) \underline{e}_1 + A_2 \cos(\omega_0 t + \varphi_2) \underline{e}_2, \quad (1.4.9)$$

where A_1 , A_2 , φ_1 , and φ_2 , the amplitudes and phases of the two transverse components of the electric vector, are independent of time. The

* Throughout the introductory material on polarization, the position variable \underline{r} is omitted from the argument of the field quantities for the sake of brevity.

vectors \underline{e}_1 and \underline{e}_2 are orthogonal unit vectors that describe coordinates in the transverse plane. The wave is assumed to propagate in the \underline{e}_3 direction. In this case, the polarization of the field is well defined. It is the locus of the electric vector in the transverse plane. If the coordinates of this curve are denoted by E_1 and E_2 , then the parametric representation of the polarization locus is, from 1.4.9

$$E_1(t) = A_1 \cos(\omega_0 t + \varphi_1)$$

$$E_2(t) = A_2 \cos(\omega_0 t + \varphi_2).$$

Upon elimination of t from these two equations, it is found that the equation of the locus is

$$\frac{E_1^2}{A_1^2} + \frac{E_2^2}{A_2^2} - 2 \left[\frac{\cos(\varphi_1 - \varphi_2)}{A_1 A_2} \right] E_1 E_2 = \sin^2(\varphi_1 - \varphi_2), \quad (1.4.11)$$

which is the well known polarization ellipse depicted in Figure 1.1. The ellipse can be uniquely specified by three parameters: 1) Its intensity, 2) Its ellipticity and sense, 3) Its orientation with respect to the chosen coordinate system. The first two of these parameters are defined in terms of A_M and a_m , the semi-major and semi-minor axes, respectively, of the ellipse:

$$I = A_M^2 + a_m^2 \quad 0 \leq I \leq \infty \quad (1.4.12)$$

$$\tan \beta = (-1)^p \frac{a_m}{A_M} \quad -\frac{\pi}{4} \leq \beta \leq \frac{\pi}{4}, \quad p = 0, 1 \quad (1.4.13)$$

where β is called the ellipticity angle. The sign of β determines the sense of the ellipse. If β is positive ($p = 0$), then the wave is said to be polarized in the positive or right-hand sense. If β is negative ($p = 1$), the wave is

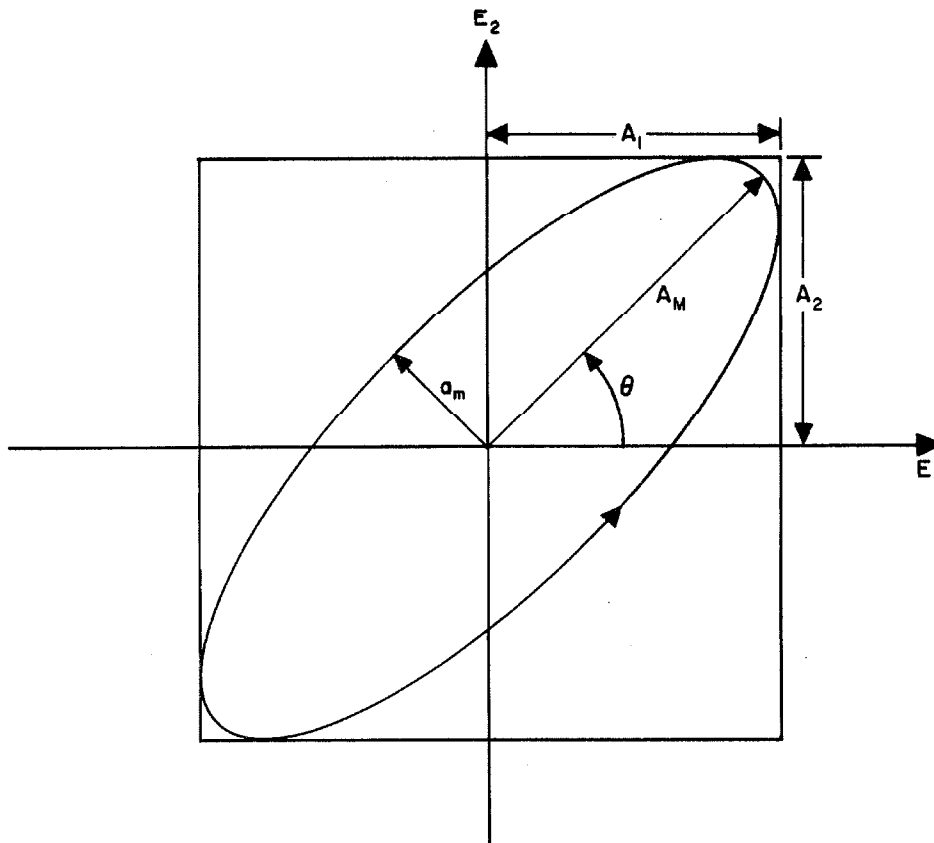


Figure 1.1. The geometry of the polarization ellipse.

polarized in the negative or left-hand sense. In practical terms, right-hand polarization means that the electric vector rotates in the clockwise sense as viewed by an observer looking toward the source of the radiation field. Correspondingly, left-hand polarization implies that the electric vector is rotating in the counter-clockwise sense as viewed by the same observer.

The intensity, ellipticity, sense, and orientation of the polarization ellipse can be expressed in terms of the amplitude and phase of the two orthogonal field components. A straightforward analysis yields the results

$$I = A_1^2 + A_2^2 \quad (1.4.14)$$

$$\tan 2\theta = \frac{2A_1A_2 \cos(\varphi_2 - \varphi_1)}{A_1^2 - A_2^2} \quad -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2} \quad (1.4.15)$$

$$\sin 2\beta = \frac{2A_1A_2 \sin(\varphi_2 - \varphi_1)}{A_1^2 + A_2^2} \quad -\frac{\pi}{4} \leq \beta \leq \frac{\pi}{4} \quad (1.4.16)$$

From 1.4.16, it is seen that if $0 \leq (\varphi_1 - \varphi_2) \leq \pi$, then the wave is polarized in the positive or right-hand sense. If the $\pi \leq (\varphi_1 - \varphi_2) \leq 2\pi$, then it is polarized in the left-hand sense. Equations 1.4.14, 1.4.15, and 1.4.16 show that the polarization of the wave is completely specified in terms of the four parameters

$$s_0 = A_1^2 + A_2^2 \quad (1.4.17)$$

$$s_1 = A_1^2 - A_2^2 \quad (1.4.18)$$

$$s_2 = 2A_1A_2 \cos(\varphi_2 - \varphi_1) \quad (1.4.19)$$

$$s_3 = 2A_1A_2 \sin(\varphi_2 - \varphi_1). \quad (1.4.20)$$

These are the well known Stokes' parameters originally introduced in 1852 by G. G. Stokes (1) to characterize the polarization of a partially polarized light ray. When applied to a completely polarized wave, the Stokes' parameters satisfy the equation

$$s_0^2 = s_1^2 + s_2^2 + s_3^2 \quad (1.4.21)$$

as is readily verified from 1.4.17 - 1.4.20. The parameters of the ellipse expressed in terms of the quantities s_0, \dots, s_3 take the simple form

$$I = s_0 \quad (1.4.22)$$

$$\tan 2\theta = \frac{s_2}{s_1} \quad (1.4.23)$$

$$\sin 2\beta = \frac{s_3}{\sqrt{s_1^2 + s_2^2 + s_3^2}} \quad (1.4.24)$$

Thus, given the amplitudes and phases of the wave, it is possible to obtain the Stokes' parameters from 1.4.17 - 1.4.20, and then the parameters that define the polarization ellipse from 1.4.22 - 1.4.24. When used in this manner, the Stokes' parameters act as the intermediary between the field quantities and the polarization ellipse. Although they are of questionable utility in connection with the analysis of ideal monochromatic wave, they are of significant value to the study of the polarization of noise-like fields as is shown below. Before continuing, it should be pointed out that the foregoing discussion of the polarization of time harmonic fields is not intended to exhaust the subject. The interested reader can find more

detailed presentations of this topic elsewhere.* The material presented above is included mainly to provide a point of departure for the discussion of the polarization of a noise-like field.

As mentioned previously, the Stokes' parameters were invented primarily to account for the polarization of waves that were not ideally harmonic, but that had amplitudes and phases which varied during the course of a measurement. The basic motivation for these studies was the need to analyze the polarization of the quasi-monochromatic sources used in physical optics experimentation. The electric field radiated by a source of this type can be expressed in the form

$$\underline{E}(t) = A_1(t)\cos[\omega_0 t + \varphi_1(t)]\underline{e}_1 + A_2(t)\cos[\omega_0 t + \varphi_2(t)]\underline{e}_2 \quad (1.4.25)$$

where a TEM wave has been assumed. Observe that this type of field has the same functional form as the time harmonic field, but now the amplitudes and phases vary with time. Because the spectral width of these sources is narrow, the quantities $A_1(t)$, $A_2(t)$, $\varphi_1(t)$, and $\varphi_2(t)$ vary slowly compared with $\cos \omega_0 t$. Thus, for times long compared with the period of $\cos \omega_0 t$, but short compared with the reciprocal of the spectral width (i.e., for intervals T satisfying the inequality $\frac{2\pi}{\omega_0} \ll T \ll \frac{2\pi}{\Delta\omega}$), the wave has the appearance of an ideal harmonic wave. However, continued observation of the wave for times greater than $\frac{2\pi}{\Delta\omega}$ shows that the character of

* One of the most complete textbook treatments of the polarization of time-harmonic fields is given by Born and Wolf (26) (pp. 24-31). An excellent treatment is also given by Chandrasekhar (27) as an introduction to his discussion of partial polarization.

the harmonic vibration changes at a rate fixed by $\Delta\omega$, the width of the spectrum. Since a single measurement of a noise-like field spans an interval $T_M \gg \frac{2\pi}{\Delta\omega}$, then the wave measured by the observer generally has occupied a number of different states of polarization. Stokes devised the quantitative means for determining the polarization observed by such a measurement.* He defined the four parameters s_0 , s_1 , s_2 , and s_3 in terms of $A_1(t), \dots, \varphi_2(t)$ by means of the equations

$$s_0 = \langle A_1^2(t) + A_2^2(t) \rangle \quad (1.4.26)$$

$$s_1 = \langle A_1^2(t) - A_2^2(t) \rangle \quad (1.4.27)$$

$$s_2 = 2 \langle A_1(t) A_2(t) \cos[\varphi_2(t) - \varphi_1(t)] \rangle \quad (1.4.28)$$

$$s_3 = 2 \langle A_1(t) A_2(t) \sin[\varphi_2(t) - \varphi_1(t)] \rangle \quad (1.4.29)$$

where the sharp brackets indicate the time average process defined as follows:

$$\langle f(t) \rangle = \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T f(t) dt \quad (1.4.30)$$

If $A_1(t), \dots, \varphi_2(t)$ are independent of time, then 1.4.26 - 1.4.29 reduce to

* Detailed discussions of the Stokes' parameters are given by Chandrasekhar (27) and by Ramachandran and Ramaseshan (28). Shurcliff (29) has prepared an elaborate bibliography on polarized light. Included in it are a number of interesting papers on the theory of polarization. See, in particular, Shurcliff's references B-29, B-43, C-18, F-3, F-5, J-14, M-2, M-3, M-28, W-3, and W-23.

1.4.17 - 1.4.20 in which case the polarization ellipse is defined by 1.4.22 - 1.4.24. If the amplitudes and phases are not independent of time, then the polarization ellipse is still defined by 1.4.22 - 1.4.24. However, in this case, 1.4.21 no longer applies. The four Stokes' parameters of a quasi-monochromatic wave are independent of one another. They are subject only to the restriction that

$$s_0^2 \geq s_1^2 + s_2^2 + s_3^2 \quad (1.4.31)$$

This inequality is readily established by means of a straightforward application of the Schwarz inequality. The proof proceeds by writing the quantity $s^2 = s_0^2 - s_1^2 - s_2^2 - s_3^2$ in the form

$$s^2 = \langle A_1^2(t) \rangle \langle A_2^2(t) \rangle - \langle A_1(t) A_2(t) \cos[\varphi_2(t) - \varphi_1(t)] \rangle^2 + \langle A_1(t) A_2(t) \sin[\varphi_2(t) - \varphi_1(t)] \rangle^2 \quad (1.4.32)$$

To prove $s^2 \geq 0$, it is convenient to introduce the notation

$$a_i(t) = A_i(t) e^{i\varphi_i(t)} \quad i = 1, 2.$$

Then in terms of the complex functions $a_i(t)$, s^2 becomes

$$s^2 = \langle |a_1(t)|^2 \rangle \langle |a_2(t)|^2 \rangle - \langle a_1(t) a_2^*(t) \rangle^2 \quad (1.4.33)$$

The Schwarz inequality states that if $f(x)$ and $g(x)$ are properly defined complex functions, then

$$\int_a^b |f(x)|^2 dx \int_a^b |g(x)|^2 dx \geq \left| \int_a^b f(x) g^*(x) dx \right|^2 \quad (1.4.34)$$

Thus, by comparing 1.4.34 and 1.4.33, it is seen that s^2 must be positive.

The equality sign in 1.4.31 applies if and only if

$$a_1(t) = K a_2(t) \quad (1.4.35)$$

where K is a constant that may be complex. Equation 1.4.35 can be expressed in the form

$$\frac{A_1(t)}{A_2(t)} = \alpha, \varphi_1(t) - \varphi_2(t) = \beta \quad (1.4.36)$$

where α and β are real constants. Of course, it follows immediately that the equality sign holds for ideal monochromatic signals, since in that case A_1 , A_2 , φ_1 , and φ_2 are individually constant with respect to time. However, it can also hold for a quasi-monochromatic field, in which case the orientation, the ellipticity, and the sense of the polarization ellipse are independent of time. This latter case is referred to as complete polarization since the only parameter of the instantaneous ellipse that varies with respect to time is the intensity.

If the equality sign in 1.4.31 does not apply, the wave is said to be partially polarized. The degree of polarization is defined to be*

$$p = \frac{\sqrt{s_1^2 + s_2^2 + s_3^2}}{s_0} \quad (1.4.37)$$

Thus, a partially polarized wave is characterized by the four parameters I , θ , β , and p , which are related to the four Stokes' parameters and, therefore, to the field quantities by 1.4.22 - 1.4.24 and 1.4.37.

* The physical meaning of the degree of polarization is discussed in Chapter II.

Using the notion of incoherent superposition, Stokes demonstrated that a partially polarized beam of light can be uniquely represented as the sum of an unpolarized and a completely polarized beam. The two beams are assumed to be mixed incoherently in the proportion determined by p . The polarized beam is characterized by the Stokes' parameters

$$s_0^{(c)} = p s_0, s_1^{(c)} = s_1, s_2^{(c)} = s_2, s_3^{(c)} = s_3, \quad (1.4.38)$$

while the unpolarized beam is characterized by the parameters

$$s_0^{(u)} = (1-p)s_0, s_1^{(u)} = s_2^{(u)} = s_3^{(u)} = 0. \quad (1.4.39)$$

Thus, the Stokes' parameters of the polarized beam satisfy 1.4.21, and by means of 1.4.22 - 1.4.24 determine the polarization ellipse that characterizes the wave. The unpolarized beam is characterized by only one non-zero parameter $s_0^{(u)}$. However, this does not imply that an unpolarized ray of light can be treated as a scalar quantity. For a discussion of this point, see page 82. It is important to observe that the decomposition of a partially polarized wave into a completely polarized part and an unpolarized part is not realizable experimentally.* It is purely a conceptual representation. However, it is a very useful representation from a theoretical standpoint, since in many situations it permits the analysis of a partially polarized field to be carried out in terms

* Although a partially polarized ray cannot be analyzed into its polarized and its unpolarized parts, the inverse or synthesis process can be achieved experimentally. It is done by combining a completely polarized ray from one quasi-monochromatic source with an unpolarized ray from a second quasi-monochromatic source. The resulting ray is partially polarized since, as is well known in optics, rays from physically independent quasi-monochromatic sources combine incoherently (i.e., their Stokes' parameters add).

of a completely polarized field and a completely unpolarized field, both of which are simpler to study than the more complex partially polarized field.

The Stokes' parameters are not the only means for analyzing the polarization of a quasi-monochromatic wave. A second method has been introduced by Wolf (26) which is more directly related to the field components. Wolf's theory is developed in terms of a two-by-two matrix obtained directly from the two transverse components of the electric field:

$$J = \begin{bmatrix} \langle E_1(t) E_1^*(t) \rangle & \langle E_1(t) E_2^*(t) \rangle \\ \langle E_2(t) E_1^*(t) \rangle & \langle E_2(t) E_2^*(t) \rangle \end{bmatrix} \quad (1.4.40)$$

where the sharp brackets indicate the time averaging process defined in 1.4.30. The asterisks denote complex conjugate. Wolf named the matrix J the coherency matrix of the field. It is essential to note that the coherency matrix is formed from complex field quantities rather than from the real fields introduced in 1.4.25. The generalization of the real field to the complex domain is carried out by means of the analytic signal representation originally introduced by Gabor (30), and later exploited by Wolf and others in the development of the theory of partially coherent fields. The analytic signal counterpart of the real field component $E_i^{(h)}(t)$, $i = 1, 2$ is defined in terms of the Hilbert transform of the real field component. That is,

$$E_i(t) = E_i^{(h)}(t) + i \left[\frac{1}{\pi} P \int_{-\infty}^{\infty} \frac{E_i^{(h)}(t')}{t' - t} dt' \right] \quad (1.4.41)$$

where P indicates the Cauchy principal value of the integral. The quantity in the braces is the Hilbert transform of $E^{(\wedge)}(t)^*$. In the following paragraphs, it is denoted by $E^{(l)}(t)$, so that the complex field can be written

$$E_i(t) = E_i^{(\wedge)}(t) + i E_i^{(l)}(t). \quad (1.4.42)$$

The function $E_i(t)$ is called an analytic signal because if t is replaced by the complex variable $t + i t'$, then it can be shown that $E_i(t + i t')$ is an analytic function throughout the half-plane $t' < 0$ (provided, of course, that $E_i^{(\wedge)}(t)$ satisfies mild regularity conditions).

It is evident from 1.4.40 that the coherency matrix analysis of partial polarization depends explicitly on the Hilbert transform of the real field components as well as on the real field itself. However, the basis for this dependence is not immediately obvious. Wolf (10) offered one explanation of this relationship in the process of resolving a question regarding the uniqueness of the Stokes' parameters. He analyzed a typical polarization measurement directly in terms of the real field components and deduced that the measured polarization must be expressed in terms both of the field and its Hilbert transform. Thus, he established the physical basis for the relationship between polarization and the analytic signal representation of the field. However, by directly considering the question of the uniqueness of the Stokes' parameters, one can obtain an alternative explanation that provides additional insight into this relationship.

* See Titchmarsh (31), Chapter 5, for a thorough discussion of the mathematical theory of the Hilbert transform.

The question of the uniqueness of the Stokes' parameters centers around the solution of the equation

$$E_i^{(h)}(t) = A_i(t) \cos [\omega_0 t + \varphi_i(t)], \quad i = 1, 2. \quad (1.4.43)$$

Since the Stokes' parameters are expressed in terms of $A_i(t)$ and $\varphi_i(t)$, then 1.4.43 must be solved for $A_i(t)$ and $\varphi_i(t)$ in terms of the known quantity, $E_i^{(h)}(t)$, in order to relate the Stokes' parameters to the field components. It is obvious, however, that 1.4.43 is not sufficient to determine $A_i(t)$ and $\varphi_i(t)$ uniquely (unless, of course, A_i and φ_i are independent of time). Thus, 1.4.43 by itself does not determine the Stokes' parameters uniquely. Interestingly, as shown below, this problem can be resolved by simply requiring that the functions $A_i(t)\cos\varphi_i(t)$ and $A_i(t)\sin\varphi_i(t)$ vary slowly compared with $\cos\omega_0 t$. Observe that this is a desirable restriction, since $A_i(t)$ and $\varphi_i(t)$ can be meaningfully identified as the amplitude and phase of a (slowly changing) sinusoid only if they remain constant for many periods of the carrier, $\cos\omega_0 t$. In addition to resolving the uniqueness question, this restriction leads to the important result that $A_i(t)$ and $\varphi_i(t)$, and, therefore, the Stokes' parameters are defined in terms of the analytic signal representation of the field. Thus, in a sense the dependence of polarization on the Hilbert transform of the field can be traced to the fact that polarization depends directly on the amplitude and phase of the field components, and that these quantities can only be meaningfully defined in terms of the analytic signal representation of the field.

The assertions made above will now be established in more detail. Assume that $E_i^{(h)}(t)$ represents a real quasi-monochromatic function of time. That is, assume that the spectrum of $E_i^{(h)}(t)$ has significant

amplitude only in the spectral range of width $\Delta\omega$ about the "center" frequency ω_0 .^{*} Assume also that $\Delta\omega < \omega_0$. Furthermore, assume that $A_i(t)\cos\varphi_i(t)$ and $A_i(t)\sin\varphi_i(t)$, hereafter denoted by $\underline{X}_i(t)$ and $\underline{Y}_i(t)$ respectively, have spectra restricted to the range $|\omega| < \omega_0$. Under these conditions it will be proven that 1.4.43 has the unique solution

$$A_i^2(t) = E_i(t) E_i^*(t) = E_i^{(h)}(t)^2 + E_i^{(l)}(t)^2 \quad (\text{summation not implied}) \quad (1.4.44)$$

$$\varphi_i(t) = \tan^{-1} \left[\frac{E_i^{(l)}(t)}{E_i^{(h)}(t)} \right] - \omega_0 t \quad (1.4.45)$$

where $E_i^{(l)}(t)$ is the Hilbert transform of $E_i^{(h)}(t)$. The proof follows. Let the field component $E_i^{(h)}(t)$ be represented in terms of its Fourier transform^{**}

$$E_i^{(h)}(t) = \underline{X}_i(t) \cos \omega_0 t - \underline{Y}_i(t) \sin \omega_0 t = \int_0^\infty \hat{a}_i(\omega) \cos[\hat{\varphi}_i(\omega) - \omega t] d\omega \quad (1.4.46)$$

By adding and subtracting $(\omega_0 t)$ from the argument of the cosine function under the integral, and performing a bit of manipulation, it is found that 1.4.46 can be written in the form

$$f_i(t) \cos \omega_0 t - g_i(t) \sin \omega_0 t = 0. \quad (1.4.47)$$

* The frequency ω_0 need not be precisely defined since, as proven elsewhere (33), the amplitude is independent of ω_0 and the phase is a linear function of ω_0 so that phase differences are also independent of ω_0 .

** As Wolf (10) points out, the following argument is not rigorous, since stationary fields do not possess Fourier integral representations. However, by employing a limiting process that in no way effects the logic of the argument, the proof can be placed on a sound mathematical basis.

where

$$\begin{aligned} f_i(t) &= \underline{X}_i(t) - \int_0^\infty \hat{a}_i(\omega) \cos[\hat{\psi}_i(\omega) - (\omega + \omega_0)t] d\omega \\ g_i(t) &= \underline{Y}_i(t) - \int_0^\infty \hat{a}_i(\omega) \cos[\hat{\psi}_i(\omega) - (\omega + \omega_0)t] d\omega. \end{aligned} \quad (1.4.48)$$

The general solution of 1.4.47 can be expressed in the form

$$f_i(t) = h_i(t) \sin \omega_0 t, \quad g_i(t) = h_i(t) \cos \omega_0 t. \quad (1.4.49)$$

where the function $h_i(t)$ is arbitrary (other than that it be sufficiently regular to represent a physical quantity). Thus, from 1.4.48 and 1.4.49

$$\begin{aligned} \underline{X}_i(t) &= h_i(t) \sin \omega_0 t + \int_0^\infty \hat{a}_i(\omega) \cos[\hat{\psi}_i(\omega) - (\omega + \omega_0)t] d\omega \\ \underline{Y}_i(t) &= h_i(t) \cos \omega_0 t + \int_0^\infty \hat{a}_i(\omega) \sin[\hat{\psi}_i(\omega) - (\omega + \omega_0)t] d\omega. \end{aligned} \quad (1.4.50)$$

The spectra of $\underline{X}_i(t)$ and $\underline{Y}_i(t)$ equal the sum of the spectra on the right side of the two equations 1.4.50. Since $\hat{a}_i(\omega)$ has appreciable magnitude only near ω_0 , the spectra of the two integrals can be shown to be restricted to the range $|\omega| < \omega_0$. However, the spectra of $h_i(t) \cos \omega_0 t$ and $h_i(t) \sin \omega_0 t$ have significant components at frequencies in the range $|\omega| > \omega_0$ due to the effect of the "shift" functions $\cos \omega_0 t$ and $\sin \omega_0 t$ unless, of course, $h_i(t) \equiv 0$. Thus, for $\underline{X}_i(t)$ and $\underline{Y}_i(t)$ to have spectra restricted to the range $|\omega| < \omega_0$, the function $h_i(t)$ must be identically zero. But in that case,

$$\begin{aligned} \underline{X}(t) &= \int_0^\infty \hat{a}_i(\omega) \cos[\hat{\psi}_i(\omega) - (\omega + \omega_0)t] d\omega \\ \underline{Y}_i(t) &= \int_0^\infty \hat{a}_i(\omega) \sin[\hat{\psi}_i(\omega) - (\omega + \omega_0)t] d\omega \end{aligned}$$

Therefore, one can write

$$\underline{X}_i(t) + \underline{Y}_i(t) = \left[\int_0^\infty \hat{a}_i(\omega) e^{i\hat{\phi}_i(\omega)} e^{-i\omega t} d\omega \right] e^{-i\omega_0 t} \quad (1.4.51)$$

This equation uniquely specifies $\underline{X}_i(t)$ and $\underline{Y}_i(t)$ in terms of the quantity on the right. The integral in the braces of 1.4.51 can be shown to be the analytic signal representation of $E_i^{(h)}(t)$ (see page 493 of Born and Wolf (26)). Thus, to obtain the results stated in 1.4.44 and 1.4.45, it is only necessary to replace $\underline{X}_i(t)$ and $\underline{Y}_i(t)$ in 1.4.51 with $A_i(t) \cos \phi_i(t)$ and $A_i(t) \sin \phi_i(t)$, respectively, and then to solve for $A_i(t)$ and $\phi_i(t)$. This completes the proof of the statements made above.

Equations 1.4.44 and 1.4.45 provide the basis for relating the Stokes' parameters and the coherency matrix. For example, from 1.4.26 and 1.4.44, it is seen that

$$s_0 = \langle A_1^2(t) + A_2^2(t) \rangle = \langle E_1(t) E_1^*(t) + E_2(t) E_2^*(t) \rangle$$

Hence, by subtracting from 1.4.40, one finds that

$$s_0 = J_{11} + J_{22}. \quad (1.4.52)$$

Similarly, it can be shown that

$$s_1 = J_{11} - J_{22} \quad (1.4.53)$$

$$s_2 = J_{12} + J_{21} = 2J_{12}^{(r)} \quad (1.4.54)$$

$$s_3 = i[J_{21} - J_{12}] = 2J_{12}^{(i)} \quad (1.4.55)$$

where $J_{12}^{(r)}$ and $J_{12}^{(i)}$ are the real and imaginary parts of J_{12} , respectively. Here, use has been made of the fact that $J_{21} = J_{12}^*$. Fano (32) has pointed out that the relationship between the Stokes' parameters and the coherency

matrix can be expressed in a particularly elegant form by expanding it in terms of the Pauli spin matrices used in quantum mechanics. He has shown that*

$$\underline{\underline{J}} = \begin{pmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{pmatrix} = \frac{1}{2} \left[s_0 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + s_1 \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + s_2 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + s_3 \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} \right] \quad (1.4.56)$$

Equation 1.4.56 connects the two theories of the polarization of quasi-monochromatic radiation. Thus, given the coherency matrix of the field, it is possible to obtain the Stokes' parameters from 1.4.56 and then the polarization from equations 1.4.14 - 1.4.16 and 1.4.37. Of course, it is not necessary to use the Stokes' parameters as an intermediary between the coherency matrix and the polarization ellipse. Both Wolf and Wiener describe the polarization of a wave directly in terms of the elements of their respective coherency matrices. However, because the Stokes' parameters are so simply related to the parameters of the polarization ellipse, they are used here as the link between the field quantities and the polarization ellipse.

The discussion of polarization presented so far has been limited to quasi-monochromatic radiation. Consider now the extension of these

* The Pauli spin matrices were also used by Wiener (16) in his analysis of partial polarization. However, because he was not familiar with Stokes' work, Wiener did not identify the coefficients of the spin matrices as the Stokes' parameters.

ideas to the case of polychromatic or broad-band radiation. The problem that one immediately encounters is that it is no longer useful to express the field in the form given in 1.4.25, since the amplitude and phase functions of a broad-band field vary just as rapidly as $\cos \omega_0 t$. As a result, the notions of amplitude and phase are not meaningful in this case. In other words, a broad-band field does not have the appearance of a time-harmonic wave over any time interval. Consequently, the locus of the electric vector of such a field is not an ellipse nor is it simply related to an ellipse. This reasoning leads one to conclude that it is not meaningful to characterize such a wave by a single state of polarization. That is, it is not satisfactory merely to extend the definition of polarization of a narrow-band, noise-like field given in 1.4.26 - 1.4.29 to include radiation of broad spectral width. A more suitable definition can be found by reconsidering the basic nature of polarization.

When examined from a somewhat abstract point of view, the polarization of a wave is seen to depend in a fundamental way on the phase difference between the two transverse components of the electric field. Equations 1.4.15 and 1.4.16 show this dependence explicitly. It also appears in connection with the polarization of quasi-monochromatic radiation in equations 1.4.28 and 1.4.29. In fact, it is because the phase difference between components of a quasi-monochromatic field is not constant with respect to time that the field suffers a loss of polarization (see Chapter II). Therefore, since the concept of polarization is intimately connected with that of phase, then the extension of the definition of polarization to a broad-band fields must be carried out in terms of the frequency domain representation of the field where phase is a well defined

quantity. In other words, the polarization of a broad-band field must be expressed in terms of the DFS. Since the elements of the DFS vary with frequency, then it is evident that the phase difference of the field components and, therefore, the polarization varies as a function of frequency. Consequently, defining the polarization of the wave in terms of the DFS leads to the fact that a broad-band field is characterized by an entire spectrum of polarizations, rather than by a single polarization as in the case of a quasi-monochromatic or an ideally monochromatic field. This implies that it is not meaningful to speak simply of the polarization of the broad-band field. It is necessary to qualify this remark and speak of the polarization of the wave at \underline{r} at the frequency ω . Thus, a complete description of the polarization of a broad-band field at some point \underline{r} is not determined unless the entire polarization spectrum is known at the point.

The foregoing remarks provide the background for the definition of the polarization of a broad-band field.* The DFS at \underline{r} , $\hat{\underline{\underline{C}}}(\underline{r}, \underline{r}, \omega)$, can be expanded in terms of the Pauli spin matrices.

$$\begin{aligned} \hat{\underline{\underline{C}}}(\underline{r}, \underline{r}, \omega) = \frac{1}{2} \left[\hat{\Delta}_0(\underline{r}, \omega) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \hat{\Delta}_1(\underline{r}, \omega) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + \hat{\Delta}_2(\underline{r}, \omega) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + \right. \\ \left. + \hat{\Delta}_3(\underline{r}, \omega) \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} \right] \end{aligned} \quad (1.4.57)$$

where $\hat{\Delta}_0(\underline{r}, \omega), \dots, \hat{\Delta}_3(\underline{r}, \omega)$ can be regarded as the spectral density of the Stokes' parameters. It is not useful to define the polarization spectrum

* The polarization of a broad-band field was originally defined by Wiener (16) and was later discussed by Barakat (18). However, these earlier presentations of the subject differ somewhat from the presentation given here.

directly in terms of $\hat{\underline{\underline{C}}}(\underline{r}, \underline{r}, \omega)$. Since the experimental determination of the polarization spectrum involves the use of equipment that has a finite bandwidth $\Delta\omega$, it is desirable to define the polarization at ω in terms of the total power contained in the band of frequencies of width $\Delta\omega$ about ω .

Thus, the polarization at frequency ω is defined by the Stokes' parameters.*

$$s_i(\underline{r}; \omega, \Delta\omega) = 2 \int_{\omega - \frac{\Delta\omega}{2}}^{\omega + \frac{\Delta\omega}{2}} \hat{\underline{\underline{C}}}_i(\underline{r}, \underline{r}, \omega') d\omega' \quad , \quad i = 0, 1, 2, 3 \quad (1.4.58)$$

The Stokes' parameters at frequency ω are related to the DFS by the equation

$$\begin{aligned} \underline{\underline{J}}(\underline{r}; \omega, \Delta\omega) = 2 \int_{\omega - \frac{\Delta\omega}{2}}^{\omega + \frac{\Delta\omega}{2}} \hat{\underline{\underline{C}}}(\underline{r}, \underline{r}, \omega') d\omega' = \frac{1}{4} \left[s_0(\underline{r}; \omega, \Delta\omega) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \right. \\ \left. + s_1(\underline{r}; \omega, \Delta\omega) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + s_2(\underline{r}; \omega, \Delta\omega) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + \right. \\ \left. + s_3(\underline{r}; \omega, \Delta\omega) \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} \right] \quad (1.4.59) \end{aligned}$$

where the symbol $\underline{\underline{J}}(\underline{r}; \omega, \Delta\omega)$ is introduced to extend Wolf's notation to fields of arbitrary spectral width. The factor $1/4$ (instead of $1/2$) is introduced so that the parameters $s_0(\underline{r}, \omega), \dots, s_3(\underline{r}, \omega)$ correspond

*

Observe that the image spectrum centered about frequency $-\omega$ is neglected in this definition. This is done in accord with the convention that the physical spectrum is meaningfully defined only for positive frequencies. The factor 2 is introduced to account for the power contained in the image spectrum.

exactly to the Stokes' parameters in the limit of a narrow band field (see below).

The properties of the instantaneous ellipse can be expressed in terms of $s_0(\underline{r};\omega,\Delta\omega)$, ..., $s_3(\underline{r};\omega,\Delta\omega)$:

$$I(\underline{r};\omega,\Delta\omega) = s_0(\underline{r};\omega,\Delta\omega) = 2[J_{11}(\underline{r};\omega,\Delta\omega) + J_{22}(\underline{r};\omega,\Delta\omega)] \quad (1.4.60)$$

$$\tan[2\theta(\underline{r};\omega,\Delta\omega)] = \frac{s_2(\underline{r};\omega,\Delta\omega)}{s_1(\underline{r};\omega,\Delta\omega)} = \frac{J_{12}(\underline{r};\omega,\Delta\omega) + J_{21}(\underline{r};\omega,\Delta\omega)}{J_{11}(\underline{r};\omega,\Delta\omega) - J_{22}(\underline{r};\omega,\Delta\omega)} \quad (1.4.61)$$

$$\begin{aligned} \sin[2\beta(\underline{r};\omega,\Delta\omega)] &= \frac{s_3(\underline{r};\omega,\Delta\omega)}{\sqrt{s_1^2(\underline{r};\omega,\Delta\omega) + s_2^2(\underline{r};\omega,\Delta\omega) + s_3^2(\underline{r};\omega,\Delta\omega)}} \\ &= \frac{i[J_{21}(\underline{r};\omega,\Delta\omega) - J_{12}(\underline{r};\omega,\Delta\omega)]}{\sqrt{[\text{Tr } \underline{\underline{J}}(\underline{r};\omega,\Delta\omega)]^2 - 4\text{Det } \underline{\underline{J}}(\underline{r};\omega,\Delta\omega)}} \end{aligned} \quad (1.4.62)$$

$$\begin{aligned} p(\underline{r};\omega,\Delta\omega) &= \frac{\sqrt{s_1^2(\underline{r};\omega,\Delta\omega) + s_2^2(\underline{r};\omega,\Delta\omega) + s_3^2(\underline{r};\omega,\Delta\omega)}}{s_0(\underline{r};\omega,\Delta\omega)} = \\ &= \sqrt{1 - 4 \frac{\text{Det } \underline{\underline{J}}(\underline{r};\omega,\Delta\omega)}{[\text{Tr } \underline{\underline{J}}(\underline{r};\omega,\Delta\omega)]^2}} \end{aligned} \quad (1.4.63)$$

where

$$J_{ij}(\underline{r};\omega,\Delta\omega) = 2 \int_{\omega - \frac{\Delta\omega}{2}}^{\omega + \frac{\Delta\omega}{2}} \hat{\mathcal{E}}_{ij}(\underline{r}, \underline{r}, \omega') d\omega' \quad (1.4.64)$$

It is evident from the above equations that the polarization of a broad-band wave at frequency ω depends on the bandwidth $\Delta\omega$ of the receiver. In general, this bandwidth is determined by factors other than those directly connected with the measurement of polarization.

Consequently, if the DFS varies appreciably over the range of frequencies $\Delta\omega$, then the measurement will determine the average polarization over this band of frequencies (see Appendix F). However, if $\Delta\omega$ is sufficiently small and if the DFS is continuous in the neighborhood of ω , then to a good approximation,

$$\int_{\omega - \frac{\Delta\omega}{2}}^{\omega + \frac{\Delta\omega}{2}} \hat{\underline{\underline{C}}}(\underline{r}, \underline{r}, \omega') d\omega' \approx \hat{\underline{\underline{C}}}(\underline{r}, \underline{r}, \omega) \Delta\omega \quad (1.4.65)$$

in which case the Stokes' parameters on the right of 1.4.59 are also linearly related to $\Delta\omega$. Since the polarization of the wave depends on the ratio of the Stokes' parameters (see 1.4.15, 1.4.16, and 1.4.37) then, in this case, the polarization is independent of the bandwidth of the receiver and a precise determination of the polarization density can be obtained.

Clearly, this is the optimum situation, since it provides a determination of polarization that is independent of the character of the measuring instrument. However, as mentioned above, it is sometimes impossible to achieve this precision, in which case one must be content to determine the average polarization over the bandwidth accepted by the receiver.

Besides smearing the polarization spectrum, wide receiver bandwidths also produce deleterious effects on the measured degree of polarization. As shown in Appendix F, the measured degree of polarization, $p(\underline{r}; \omega, \Delta\omega)$ is less than the mean value of the degree-of-polarization spectral density, $\hat{p}(\underline{r}, \omega)$ over the bandwidth $\Delta\omega$. Thus, $p(\underline{r}; \omega, \Delta\omega)$ can be significantly less than the maximum value of $\hat{p}(\underline{r}, \omega)$ over this bandwidth.

Moreover, this reduction of $p(\underline{r};\omega,\Delta\omega)$ results not only because it is bounded by the mean value of $\hat{p}(\underline{r},\omega)$ over the bandwidth $\Delta\omega$, but also because it is affected by the variations of the polarization of the wave over $\Delta\omega$. Although the inequality F.2 does not show this latter effect explicitly, the proof that follows F.2 shows that the variation of polarization over $\Delta\omega$ determines the disparity between the two sides of the inequality. For example, if two completely polarized beams of unlike polarization are added incoherently, the resulting beam is partially polarized. In fact, if the two completely polarized beams are orthogonally polarized, the resulting beam will be completely unpolarized.

In the event that the field of interest is narrow band, the integrated spectral coherency matrix, $\underline{\underline{J}}(\underline{r};\omega,\Delta\omega)$ defined above reduces to Wolf's coherency matrix $\underline{\underline{J}}$ except for a factor of 2. To see this, assume that in the case of a narrow-band field, the bandwidth of the receiver, $\Delta\omega$, exceeds the spectral width of the radiation. Thus $\underline{\underline{J}}_{N.B.}(\underline{r};\omega,\Delta\omega)$ can be written

$$\underline{\underline{J}}_{N.B.}(\underline{r};\omega,\Delta\omega) = 2 \int_0^{\infty} \underline{\underline{\hat{C}}}(\underline{r},\underline{r},\omega) d\omega \quad . \quad (1.4.66)$$

But it is readily shown that if $F(\omega)$ is the Fourier transform of the real function $f(t)$, then $\int_0^{\infty} F(\omega) e^{-i\omega t} dt$ is the analytic signal representation of $f(t)$. Therefore, the integral on the right in 1.4.66 is related to the analytic signal representation of the dyadic autocorrelation function $\underline{\underline{\hat{C}}}^{(r)}(\underline{r},\underline{r},\tau)$, where the superscript (r) indicates that the dyadic autocorrelation function is defined in terms of the real fields. In fact,

$$2 \int_0^{\infty} \underline{\underline{\hat{C}}}(\underline{r},\underline{r},\omega) d\omega = [\underline{\underline{\hat{C}}}^{(r)}(\underline{r},\underline{r},\tau) + i \underline{\underline{\hat{C}}}^{(i)}(\underline{r},\underline{r},\tau)] \Big|_{\tau=0} \quad (1.4.67)$$

To complete the argument, one needs only to observe that

$$\overline{\underline{E}^{(\kappa)}(\underline{r}, t+\tau) \underline{E}^{(\kappa)}(\underline{r}, t)} = \overline{\underline{E}^{(i)}(\underline{r}, t+\tau) \underline{E}^{(i)}(\underline{r}, t)} = \underline{\underline{\zeta}}^{(\kappa)}(\underline{r}, \underline{r}, \tau) \quad (1.4.68)$$

$$\overline{\underline{E}^{(i)}(\underline{r}, t+\tau) \underline{E}^{(\kappa)}(\underline{r}, t)} = - \overline{\underline{E}^{(\kappa)}(\underline{r}, t+\tau) \underline{E}^{(i)}(\underline{r}, t)} = \underline{\underline{\zeta}}^{(i)}(\underline{r}, \underline{r}, \tau) \quad (1.4.69)$$

where as before $\underline{E}^{(i)}(\underline{r}, t)$ represents the Hilbert transform of $\underline{E}^{(\kappa)}(\underline{r}, t)$.

Thus, substitution of 1.4.68 and 1.4.69 into the right side of 1.4.67 yields

$$2 \int_0^\infty \hat{\underline{\underline{\zeta}}}(\underline{r}, \underline{r}, \omega) d\omega = \frac{1}{2} [\overline{\underline{E}(\underline{r}, t+\tau) \underline{E}^*(\underline{r}, t)}]_{\tau=0} = 0$$

Finally, from this result and from Wolf's definition of the coherency matrix (equation 1.4.40), it is found that

$$2 \underline{\underline{J}}_{\text{N.B.}}(\underline{r}; \omega, \Delta\omega) = \underline{\underline{J}} \quad (1.4.70)$$

The factor 2 appears because $\underline{\underline{J}}_{\text{N.B.}}(\underline{r}; \omega, \Delta\omega)$ is derived from the real fields whereas $\underline{\underline{J}}$ is derived from the analytic fields; the analytic fields contain twice as much power as the real field.

Besides demonstrating the relationship between $\underline{\underline{J}}(\underline{r}; \omega, \Delta\omega)$ and Wolf's coherency matrix for the case of a narrow-band field, equation 1.4.70 also provides the basis for a simple interpretation of the polarization of a broad-band field. Although 1.4.70 shows that $\underline{\underline{J}}_{\text{N.B.}}(\underline{r}; \omega, \Delta\omega)$ equals 2 times Wolf's coherency matrix, it shows more generally that $\underline{\underline{J}}(\underline{r}; \omega, \Delta\omega)$ equals 2 times the coherency matrix associated with the quasi-monochromatic field obtained from the broad-band field by passing the latter through an ideal high-pass filter of width $\Delta\omega$ centered at frequency ω . Thus, $\underline{\underline{J}}(\underline{r}; \omega, \Delta\omega)$ represents the polarization of a quasi-monochromatic

field obtained by filtering a broad-band field. It seems natural to ask if there is any intrinsic difference between the polarization of a quasi-monochromatic field obtained by filtering and the polarization of one obtained as a result of inherent narrow-band nature of the source itself. The answer is that there is no difference. The only substantial difference between these two types of narrow-band fields is in the details of the form of the DFS over the bandwidth of interest. However, as indicated above, and as discussed in Chapter II, the polarization is dependent only on the integrated spectrum; it is insensitive to the variation of the spectrum over the bandwidth of the receiver. Thus, two quasi-monochromatic fields having the same integrated DFS, but different spectral variations, will still register the same polarization.

The fact just mentioned has been given a formal statement by Stokes in his famous principle of optical equivalence, which says in effect that two quasi-monochromatic light rays that have equal Stokes' parameters cannot be distinguished by means of any practicable experiment. The basis for this principle lies in the fact that optical instruments can only perform linear operations on the two rays. Consequently, the only properties of the light rays upon which the output of the instrument depends are the Stokes' parameters of the rays -- assuming that the spectral width of the instrument exceeds that of the light ray.

The principle of optical equivalence is a practical principle since it is based on the nature and limitations of available measuring equipment. If, on the other hand, one considers the question of optical equivalence from a theoretical standpoint -- in fact, from a theoretical statistics standpoint -- the situation is quite different.

Statistically speaking, two rays are equivalent provided the probability distributions that describe the fields are identical. If the fields are Gaussian, this requires that their respective DFS be equal frequency-by-frequency. But Stokes' principle of optical equivalence depends only on the integrated spectrum of the field. Thus, there is a basic difference between the two points of view. Of course, this dilemma is easily solved if the measuring instrument is incapable of resolving the details of the spectrum. In that case, one simply states the principle in terms of the integrated spectrum as Stokes did. However, an equivalence principle applicable to fields of arbitrary spectral width whose spectra are centered anywhere in the electromagnetic spectrum is more difficult to formulate. Perhaps the most straightforward procedure is to assume that $\Delta\omega$ in equation 1.4.59 represents the narrowest bandwidth that can be achieved for the measurement of the given radiation. Then, two noise-like fields, that satisfy Gaussian statistics, are equivalent provided that $\underline{\underline{J}}(\underline{r};\omega,\Delta\omega)$ of one is proportional to $\underline{\underline{J}}(\underline{r};\omega,\Delta\omega)$ of the other for all frequencies of interest. The justification for this statement rests on the fact that within the accuracy of measurement, the dyadic autocorrelation functions of the two rays and, therefore, the statistics of the two rays, cannot be distinguished from one another. Hence, there is no way for the rays themselves to be distinguished.

Before leaving the subject of polarization, it is of interest to point out several properties of the quantities defined in 1.4.59 - 1.4.64.

1) The parameter, $I(\underline{r};\omega,\Delta\omega)$, called the intensity of the field, is closely related to the spectrum defined in 1.4.8. In fact, it is proportional to the integral of the spectrum over the bandwidth of the receiver.

That is,

$$I(\underline{r}; \omega, \Delta\omega) = 2 \sqrt{\frac{\mu_0}{\epsilon_0}} \int_{\omega - \frac{\Delta\omega}{2}}^{\omega + \frac{\Delta\omega}{2}} \hat{P}(\underline{r}, \omega') d\omega' \quad (1.4.71)$$

2) The polarization ellipse at \underline{r} and ω is determined by the three parameters $s_1(\underline{r}; \omega, \Delta\omega)$, $s_2(\underline{r}; \omega, \Delta\omega)$, and $s_3(\underline{r}; \omega, \Delta\omega)$ as in the case of the quasi-monochromatic field. Thus, the polarized part of the field at \underline{r} is characterized by the four Stokes' parameters

$$s_0'(\underline{r}; \omega, \Delta\omega) \equiv \sqrt{s_1^2(\underline{r}; \omega, \Delta\omega) + s_2^2(\underline{r}; \omega, \Delta\omega) + s_3^2(\underline{r}; \omega, \Delta\omega)},$$

$$s_1(\underline{r}; \omega, \Delta\omega), s_2(\underline{r}; \omega, \Delta\omega), s_3(\underline{r}; \omega, \Delta\omega) \quad .$$

3) It can be shown that the degree of polarization $p(\underline{r}; \omega, \Delta\omega)$ of the field at (\underline{r}, ω) lies between 0 and 1, i.e.,

$$0 \leq p(\underline{r}; \omega, \Delta\omega) \leq 1 \quad . \quad (1.4.72)$$

At the lower limit, the field is unpolarized; at the upper limit, the field is completely polarized; and at intermediate values, the field is partially polarized. The inequality 1.4.72 is proven in Appendix A.

C. The Spatial Coherence of the Field

To complete the discussion of the properties and interpretation of the DFS, it is necessary to consider the general form $\hat{\underline{\underline{C}}}(\underline{r}_1, \underline{r}_2, \omega)$ where $\underline{r}_1 \neq \underline{r}_2$. The physical significance of this quantity is connected with the spatial structure of the field. Although $\hat{\underline{\underline{C}}}(\underline{r}_1, \underline{r}_2, \omega)$ appears to contain information concerning the polarization of the field (since it is a tensor quantity), this information is not of direct physical interest. The reason is that $\hat{\underline{\underline{C}}}(\underline{r}_1, \underline{r}_2, \omega)$ is formed from the field vector evaluated at two different

points of space, and, therefore, it cannot provide polarization information about either point. What it does describe is the spatial structure of the individual components of the field. Consequently, the individual elements of $\hat{\underline{\underline{C}}}(\underline{r}_1, \underline{r}_2, \omega)$ are generally of independent interest. Consider, then, the component $\hat{C}_{11}(\underline{r}_1, \underline{r}_2, \omega)$. This quantity provides a direct measure of the degree of coherence at frequency ω of the component of the field in the direction \underline{e}_1 at point \underline{r}_1 with respect to the same field component evaluated at point \underline{r}_2 . Similarly, $\hat{C}_{22}(\underline{r}_1, \underline{r}_2, \omega)$ and $\hat{C}_{33}(\underline{r}_1, \underline{r}_2, \omega)$ are the corresponding quantities associated with the components in the directions \underline{e}_2 and \underline{e}_3 , respectively. The off-diagonal terms provide equivalent information about mixed field components. Thus, $\hat{C}_{23}(\underline{r}_1, \underline{r}_2, \omega)$ relates to the degree of coherence at the frequency ω of E_2 evaluated at \underline{r}_1 with respect to E_3 evaluated at \underline{r}_2 .

The physical significance of the degree of coherence of the field is intimately connected with the phenomenon of interference.* As is well known, interference is a basic property of all wave phenomena. It is clearly demonstrated by means of the simple two-slit experiment of optics. Suppose, as in Figure 1.2a, that an opaque screen \mathcal{S} is interposed between a source σ and an image plane, \mathcal{L} . If two narrow slits are cut in \mathcal{S} a distance d apart, the radiation from the source passes through the screen and illuminates the image screen \mathcal{L} . If the source is an ideal monochromatic source, then the field oscillations in the slits are coherent with respect to each other, and a perfect interference pattern is

* The discussion of spatial coherence presented here is limited to the essential ideas and the basic formulas. A more complete discussion can be found in Born and Wolf (26), Chapter 10.

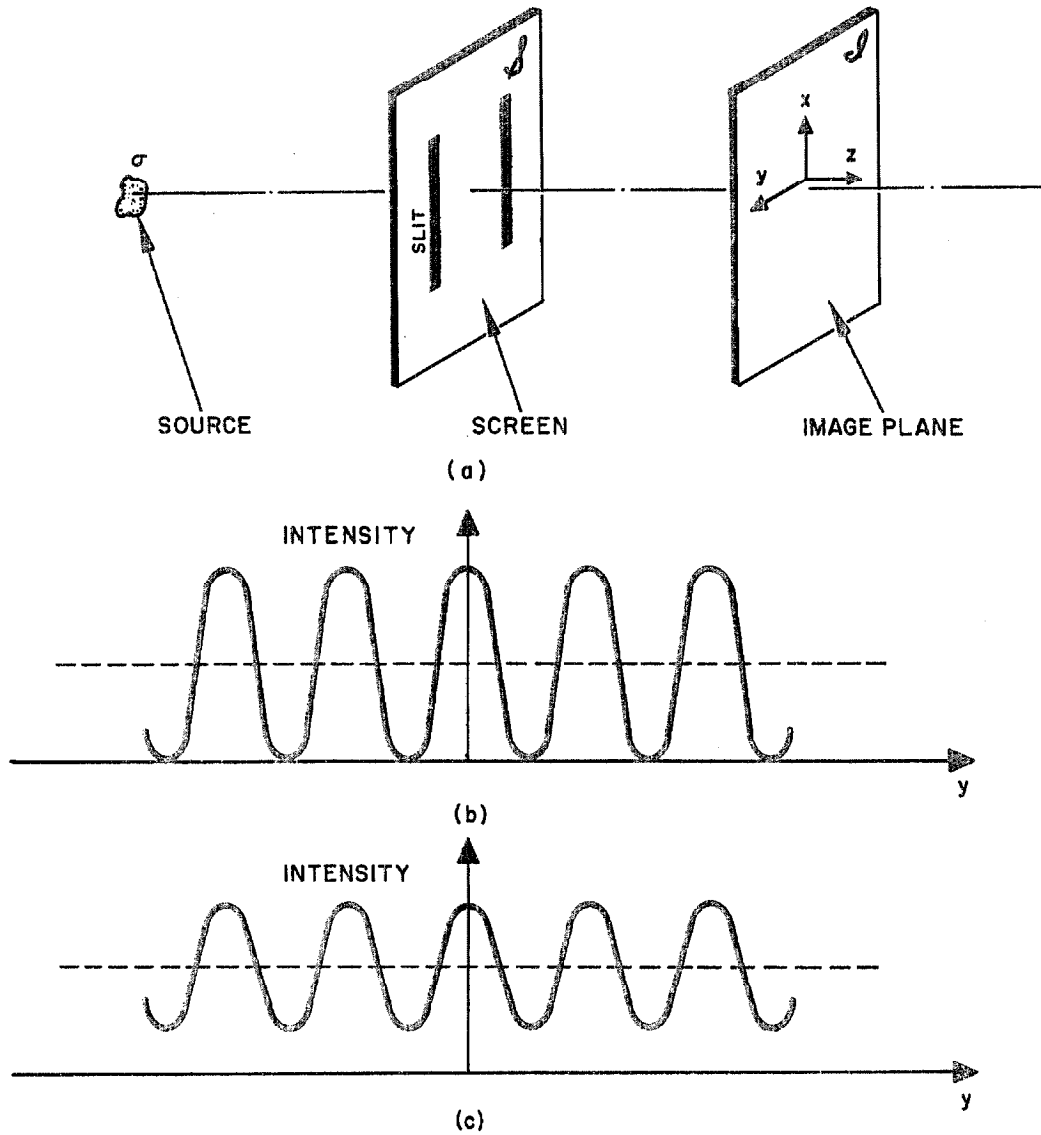


Figure 1.2. The interference experiment.

- (a) The general configuration
- (b) Interference pattern for ideal monochromatic radiation. Intensity versus position along the y axis.
- (c) Interference pattern for quasi monochromatic radiation.

observed. As shown in Figure 1.2b, this pattern is characterized by an alternating succession of intensity maxima and minima oriented transverse to the axis of the system. If the pattern is perfect, then the intensity maxima are twice as large as the sum of the intensities of the individual aperture fields at the screen, and the minima are perfect nulls. The position of the pattern relative to the axis of the spectrum is determined by the phase difference between the two aperture fields.

If the ideal monochromatic source is replaced by a quasi-monochromatic source, then the pattern that is projected through the slits onto the image screen changes noticeably. Basically, what is observed is that the minima fill in so that they are no longer perfect nulls, and the maxima decrease in magnitude. If the slits are moved farther apart, both effects become more pronounced until a critical separation is reached, at which point the pattern vanishes altogether leaving only a uniformly illuminated area on the image screen. The intensity of this uniform illumination equals the sum of the intensities of the individual aperture fields. Since at this critical separation the total intensity equals the sum of the individual intensities, the field fluctuations in the slits are said to be mutually incoherent. If the pattern is perfect (at least in the vicinity of the axis of the system), then the field fluctuations at the two apertures are said to be completely coherent. The intermediate cases are referred to as states of partial coherence.

Observe that the notion of coherence (or more precisely spatial coherence) refers to the relationship between the field fluctuations at two points of space (the apertures of the slits). If the location of these points is changed relative to the source without altering the slit

separation, a change may be observed in the state of coherence of the aperture fields. Thus, in general, the degree of spatial coherence is a function of position as well as of the separation of the slits. The calculation of the distribution of spatial coherence for a given quasi-monochromatic source was originally done by Zernike (6), and was subsequently elaborated by Wolf (7) and his co-workers. This work and its extension to broad-band fields is discussed in a later chapter.

It is convenient at this point to discuss briefly the mathematical analysis of partial coherence. In the literature, this analysis is carried out in terms of an unpolarized, or scalar, quasi-monochromatic field. This is a useful approach from the standpoint of exposition, since it eliminates the complications associated with polarization, thereby focusing attention directly on the interference phenomenon. Following this precedent, let us assume that the real scalar "field" fluctuations in the two slits are denoted by $V^{(h)}(\underline{r}_1, t)$ and $V^{(h)}(\underline{r}_2, t)$ respectively. Furthermore, let us assume that $V^{(h)}(\underline{r}, t)$ is a quasi-monochromatic function of time. It can be shown that the intensity of the radiation impinging on the image screen from the two slits is given by*

$$I(y) = |K_1|^2 I_1 + |K_2|^2 I_2 + 2 |K_1 K_2| \Gamma_{12}^{(h)}\left(\frac{s_2 - s_1}{c}\right) \quad (1.4.73)$$

where $I(y)$ represents the intensity as a function of position on the image screen, K_1 and K_2 are system factors that relate the field in the slits to the field on the image screen, s_1 and s_2 are optical path lengths from the slits to the point y on the image screen, and $\Gamma_{12}^{(h)}\left(\frac{s_1 - s_2}{c}\right)$ is the

* Born and Wolf (26), page 498.

real part of the mutual coherence function of the field. The quantities I_1 and I_2 are the average intensities of the fields in the slits at \underline{r}_1 and \underline{r}_2 , respectively. The mutual coherence function $\Gamma_{12}(\frac{s_1 - s_2}{c})$ is defined in terms of the analytic signal representation of the field:

$$\Gamma_{12}(\tau) = \langle V(\underline{r}_1, t + \tau) V^*(\underline{r}_2, t) \rangle \quad (1.4.74)$$

where $V(\underline{r}, t)$ is the analytic signal corresponding to $V^{(r)}(\underline{r}, t)$. The time delay τ appears because the difference of optical paths causes a delay in the signal arriving at the image screen from the more distant slit. The complex degree of spatial coherence or, more briefly, the degree of coherence is defined to be

$$\mu_{12} = \frac{\Gamma_{12}(0)}{[I_1 I_2]^{1/2}} = \frac{\langle V(\underline{r}_1, t) V^*(\underline{r}_2, t) \rangle}{[\langle |V(\underline{r}_1, t)|^2 \rangle \langle |V(\underline{r}_2, t)|^2 \rangle]^{1/2}} \quad (1.4.75)$$

It is shown by Born and Wolf (26) that $|\mu_{12}|$ is equal to the visibility of the interference fringes observed on the image screen, where the quantitative definition of visibility is given by

$$\mathcal{V} = \frac{I_{\max} - I_{\min}}{I_{\max} + I_{\min}} \quad (1.4.76)$$

In this expression, I_{\max} and I_{\min} are the maximum and minimum intensities, respectively, of the interference fringes in the vicinity of the axis of the system. Actually Born and Wolf present the more general relationship

$$\mathcal{V}_{(y)} = \frac{|\Gamma_{12}(\tau)|}{[I_1 I_2]^{1/2}} \quad (1.4.77)$$

which relates the visibility and mutual coherence at points off the optical axis of the system. However, for the purposes of this analysis, the specialized form given in 1.4.75 will suffice.

Equation 1.4.75 provides the mathematical relationship between spatial coherence and interference. The quantity μ_{12} is a direct measure of the similarity of the field fluctuations at \underline{r}_1 and at \underline{r}_2 . If the field fluctuations at the two points are independent of each other, then $|\mu_{12}| = 0$. On the other hand, if they are completely dependent, i.e., if $V(\underline{r}_1, t) \propto V(\underline{r}_2, t)$, then $|\mu_{12}| = 1$. Intermediate conditions of dependence yield values of $|\mu_{12}|$ between zero and one (as can be shown by means of the Schwarz inequality).

Equation 1.4.75 shows that the state of coherence of the two fluctuating fields directly affects the quality (visibility) of the interference pattern. Thus, by observing the interference pattern, it is possible to determine the distances over which the field fluctuations are at least partially coherent. Since there are many instances when it is of interest to know whether or not the field fluctuations at two points are independent, then this distance is an important property of the field. However, the phenomenon of partial coherence serves a more important function; it provides a means for studying the properties of the source of the radiation field. The physical basis for this relationship can be demonstrated by means of the interference pattern. One need only determine the properties of the source that cause the interference pattern to deteriorate to identify the cause of partial coherence. To this end, consider an incoherent source of finite extent that radiates an unpolarized, quasi-monochromatic field. The geometry of the source and the slits is shown

in Figure 1.3. Because the source is incoherent, the radiation field from each element of the source of area dA_s is independent of that arising from any other element of the source. Consequently, the total intensity of the field at some point is the sum of the intensities radiated from each element to the point in question. As can be seen from Figure 1.3, the interference pattern arising from a single element of the source will be perfect since the field fluctuations in the slits arise from a common elementary radiator and, therefore, must be completely dependent. The position of this elementary pattern depends on the phase difference of the radiation in the slits, which in turn depends on the position of the element relative to the slits. Clearly, the elementary interference patterns from different portions of the source lie in different positions on the image screen, since the slit-element geometry of the various elements composing the source varies from element to element. Consequently, the total interference pattern radiated by the entire extended source, being the superposition of the mutually displaced elementary patterns, is not a perfect interference pattern. Thus, partial coherence of the field radiated by an extended incoherent source is caused by the finite size of the source combined with the independence of the various radiators composing the source.*

The mathematical relationship between the degree of coherence and the source properties is embodied in the van Cittert-Zernike theorem.

* Observe that an extended monochromatic source will be completely coherent throughout space. It is necessary for the source to have a finite bandwidth to give any evidence of a loss of coherence.

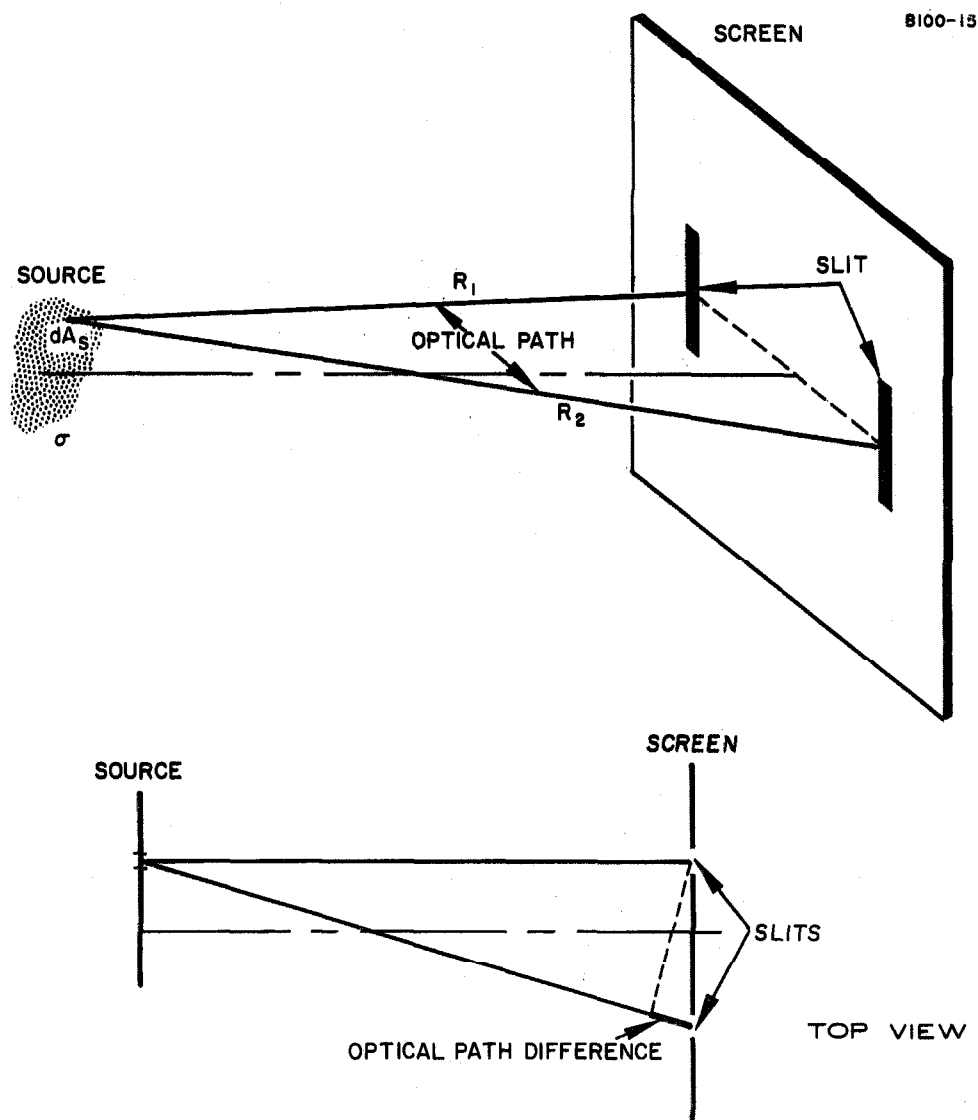


Figure 1.3. Configuration of the interference experiment with an extended source.

As stated by Born and Wolf (26), this theorem has the form

$$\mu_{12} = \frac{1}{\sqrt{I_1 I_2}} \int_{\sigma} I(s) \frac{e^{i k_o (R_1 - R_2)}}{R_1 R_2} dA_s \quad (1.4.78)$$

where $I(s)$ is the intensity per unit area of the source at point s , $k_o R_1$ and $k_o R_2$ are the optical path lengths from the point s in the source to the observation points \underline{r}_1 and \underline{r}_2 at the center frequency of the narrow-band spectrum, and I_1 and I_2 are the intensities of the field at slits 1 and 2, respectively. This formula describes the degree of coherence at any two points \underline{r}_1 and \underline{r}_2 in terms of the intensity distribution over the source. In theory, then, it is possible to solve this equation for $I(s)$ if μ_{12} is known. It is in this sense that the degree of coherence provides a means for studying the source. This idea forms the basis for the Michelson stellar interferometer. By observing the fringe visibility of a two element interferometer for a number of different element separations, Michelson (3) was able to estimate the diameter of stars. This same idea is widely used in radio astronomy to study the intensity distribution of radio stars, although in practice the technique of measurement is not always the same as it is at optical frequencies.* The important point here is that the mutual coherence function provides a direct means for determining the nature of the source. It is this aspect of partial coherence that is of greatest interest here.

The concept of partial coherence is meaningful only in connection with narrow-band radiation. The reason for this is apparent. The

* See, for example, the paper by Hanbury-Brown and Twiss (33).

concept was originally introduced to describe the effects of bandwidth (field fluctuations) on the interference phenomenon. But this phenomenon is inherently a narrow-band phenomenon, since no observable fringe pattern can be found if the bandwidth of the fields in the slits is too wide. Thus, any discussion of partial coherence tacitly assumes a narrow-band source. Clearly then, to extend the concept to a broad-band partially polarized source, it is necessary to account explicitly for the narrow-band character of the concept. In this respect (as in many others), the notion of spatial coherence is closely related to that of polarization. The extended definition of partial coherence is expressed most naturally in terms of the DFS. The degree of coherence at frequency ω of the field component in the direction \underline{e}_i at \underline{r}_1 with respect to the component in the direction \underline{e}_j at \underline{r}_2 is defined to be

$$\mu_{ij}(\underline{r}_1, \underline{r}_2, \omega) = \frac{\int_{\omega - \frac{\Delta\omega}{2}}^{\omega + \frac{\Delta\omega}{2}} \hat{E}_{ij}(\underline{r}_1, \underline{r}_2, \omega') d\omega'}{\left[\int_{\omega - \frac{\Delta\omega}{2}}^{\omega + \frac{\Delta\omega}{2}} \hat{E}_{ii}(\underline{r}_1, \underline{r}_1, \omega') d\omega' \int_{\omega - \frac{\Delta\omega}{2}}^{\omega + \frac{\Delta\omega}{2}} \hat{E}_{jj}(\underline{r}_2, \underline{r}_2, \omega') d\omega' \right]^{\frac{1}{2}}} \quad (1.4.79)$$

where $\Delta\omega \ll \omega$. The notation here differs from that in 1.4.78 in that the subscripts denote the field components whose mutual coherence is being determined. The position variables are now noted explicitly in the

argument of the degree of coherence. Recall that the repeated subscript on the terms in the demoninator of 1.4.79 does not indicate summation.

It can be seen from 1.4.79 that in the general case, it is necessary to identify the field components under consideration, the frequency, and the bandwidth when discussing spatial coherence. Thus, if a measurement is to be made of the size of a polarized broad-band source, it is possible to obtain different sizes depending on which field components are measured and the frequency to which the receiver is tuned. (The measurement of an extended, incoherent, polarized source with an interferometer is the subject of Chapter V.)

As in the case of quasi-monochromatic signals $\mu_{ij}(\underline{r}_1, \underline{r}_2, \omega)$ satisfies the inequality

$$0 \leq |\mu_{ij}(\underline{r}_1, \underline{r}_2, \omega)| \leq 1 \quad (1.4.80)$$

where the absolute value sign is required since $\mu_{ij}(\underline{r}_1, \underline{r}_2, \omega)$ is, in general, a complex quantity. A brief heuristic proof of 1.4.80 can be developed from the fact that the terms on the right of 1.4.79 represent the average intensity of the field after it has been passed through an ideal high frequency filter of bandwidth $\Delta\omega$ centered at frequency ω . Thus, if the analytic signal form of the filtered field is denoted by $\underline{E}^{(f)}(\underline{r}, t)$, then it follows that the power spectral density of the real part of $\underline{E}^{(f)}(\underline{r}, t)$ at positive frequencies is given by

$$\hat{\underline{E}}^{(f)}(\underline{r}_1, \underline{r}_2, \omega') = \hat{\underline{E}}(\underline{r}_1, \underline{r}_2, \omega') \left[\mathcal{U}(\omega' - \omega + \frac{\Delta\omega}{2}) - \mathcal{U}(\omega' - \omega - \frac{\Delta\omega}{2}) \right] \quad (1.4.81)$$

where \mathcal{U} denotes the unit step function:

$$\begin{aligned} \mathcal{U}(x) &= 1 \quad x > 0 \\ &= 0 \quad x < 0 \end{aligned} \quad (1.4.82)$$

But from 1.3.4, it is seen that*

$$\langle E_i^{(f)}(\underline{r}_1, t) E_j^{(f)*}(\underline{r}_2, t) \rangle = 4 \int_0^\infty \hat{\xi}_{ij}^{(f)}(\underline{r}_1, \underline{r}_2, \omega') d\omega',$$

or

$$\langle E_i^{(f)}(\underline{r}_1, t) E_j^{(f)*}(\underline{r}_2, t) \rangle = 4 \int_{\omega - \frac{\Delta\omega}{2}}^{\omega + \frac{\Delta\omega}{2}} \hat{\xi}_{ij}^{(f)}(\underline{r}_1, \underline{r}_2, \omega') d\omega' \quad (1.4.83)$$

To establish the inequality on the right side of 1.4.80, it is sufficient to show that

$$\left| \int_{\omega - \frac{\Delta\omega}{2}}^{\omega + \frac{\Delta\omega}{2}} \hat{\xi}_{ij}^{(f)}(\underline{r}_1, \underline{r}_2, \omega') d\omega' \right| \leq \left[\int_{\omega - \frac{\Delta\omega}{2}}^{\omega + \frac{\Delta\omega}{2}} \hat{\xi}_{ii}^{(f)}(\underline{r}_1, \underline{r}_1, \omega') d\omega' \int_{\omega - \frac{\Delta\omega}{2}}^{\omega + \frac{\Delta\omega}{2}} \hat{\xi}_{jj}^{(f)}(\underline{r}_2, \underline{r}_2, \omega') d\omega' \right]^{\frac{1}{2}} \quad (1.4.84)$$

This inequality can be expressed in terms of the field components of $E^{(f)}(\underline{r}, t)$ with the aid of 1.4.83

$$\left| \langle E_i^{(f)}(\underline{r}_1, t) E_j^{(f)*}(\underline{r}_2, t) \rangle \right| \leq \left[\langle E_i^{(f)}(\underline{r}_1, t) E_i^{(f)*}(\underline{r}_1, t) \rangle \langle E_j^{(f)}(\underline{r}_2, t) E_j^{(f)*}(\underline{r}_2, t) \rangle \right]^{\frac{1}{2}} \quad (1.4.85)$$

But the validity of this inequality follows at once from the Schwarz inequality.

This completes the presentation of the DFS and the discussion of

*

Dugundji (34) shows that the power spectral density of an analytic signal equals 4 times the power spectral density of the real part for $\omega > 0$ and zero for $\omega < 0$.

its definition, its mathematical properties, and its physical interpretation. In the following chapters, the mathematical theory of the DFS is developed. The differential equations governing its spatial behavior are derived and integrated, and these results are applied to the analysis of an interferometer experiment. However, before this general theory is developed, a statistical analysis of partial polarization is carried out with the purpose of providing insight into the nature of partial polarization. This analysis forms the subject matter of Chapter II.

II. THE STATISTICS OF THE INSTANTANEOUS ELLIPSE

2.1 Introductory Remarks

The polarization of a monochromatic field is completely specified by the three parameters that characterize the polarization ellipse. However, to describe the polarization of a quasi-monochromatic field it is necessary to introduce an additional parameter - the degree of polarization. The need for this additional parameter arises from the fact that quasi-monochromatic fields are not necessarily completely polarized, but may to some degree be unpolarized. That is, the field at a point may evidence the characteristics of both a polarized field and an unpolarized field. But if this is the case the question may be asked: In what way is the behavior of the field at the point modified when it exhibits incomplete polarization? That is, in what sense does a partially polarized field differ from a monochromatic field?

The conventional answer to this question is based on the mathematical analysis of partial polarization. It can be proven that a partially polarized field can be uniquely decomposed into the incoherent sum of a completely polarized wave and a completely unpolarized wave (27). Thus it is quite natural to think of a partially polarized wave as the incoherent superposition of an unpolarized wave and a completely polarized wave. However, this decomposition can not be achieved experimentally. There is no way of dividing a partially polarized ray into two component rays, one of which is completely polarized and the other of which is completely unpolarized. Thus from a physical standpoint this is an inadequate model of a partially polarized wave, since it is not posed in terms of observable properties of the field.

To obtain more insight into the physical character of partial polarization, it is fruitful to consider the temporal behavior of the field vector. In the case of a monochromatic field, the field vector traces out a polarization ellipse once every period. Furthermore, it traces the same ellipse from period to period since the ellipse is determined by the amplitude and phase of the field components and these quantities are independent of time for a monochromatic field. However, in the case of a quasi-monochromatic field, the amplitude and phase are not constant with respect to time. Thus, in this case the polarization ellipse traced out by the field vector need not be invariant with respect to time. In fact, as will be shown below, it is the tendency for the ellipse to assume different shapes and orientations during the course of a measurement that causes the observed deterioration of the polarization.

The notion of a time-varying polarization locus is not new. In one of the earliest models of unpolarized or natural light, it was assumed that the electric vector is plane (linearly) polarized over short intervals of time, but that the plane of polarization varies rapidly over all possible orientations as a function of time. This model, which was originally proposed by Fresnel, is still widely accepted. However, it is limited in the sense that it restricts the instantaneous polarization locus to the degenerate case of linear polarization. A more general model can be formulated by making use of the properties of quasi-monochromatic time signals.

Recall from Chapter I that by writing the quasi-monochromatic field component $E_i^{(\nu)}(t)$ at some point \underline{r} of space in the form

$$E_i^{(\nu)}(t) = A_i(t) \cos[\omega_0 t + \phi_i(t)]$$

and by defining $A_i(t)$ and $\varphi_i(t)$ in terms of the analytic signal representation of the field, it is found that the amplitude $A_i(t)$ and the phase $\varphi_i(t)$ are slowly varying functions of time (compared with $\cos \omega_0 t$). In fact, it can be shown that if the bandwidth of $E_i^{(r)}(t)$ is $\Delta\omega$, then for periods small compared with $1/\Delta\omega$ but large compared with $1/\omega_0$, $A_i(t)$ and $\varphi_i(t)$ are essentially constant. Thus for times $\frac{1}{\omega_0} \ll t \ll \frac{1}{\Delta\omega}$, the field has the appearance of an ideal monochromatic wave. The polarization of this field during this interval is determined by $A_1(t)$, $A_2(t)$, and $[\varphi_1(t) - \varphi_2(t)]$. This polarization may or may not be linear. In fact, Hurwitz (23) in 1945 calculated the probability distribution of the instantaneous ellipse (as he called the transitory polarization figure) of an unpolarized wave. He found that there was a substantial probability that the instantaneous ellipse is elliptically rather than linearly polarized. In fact, he found that the median ellipticity is 0.268*. Thus, ellipses with ellipticity greater than 0.268 ($a_m/A_m > 0.268$) occur about half the time. Although Hurwitz didn't bother to calculate the probability distribution of θ , the orientation angle, it is shown below to be a uniform distribution for the case of unpolarized light.

The translation of Hurwitz's results into a description of the temporal behavior of the polarization ellipse of unpolarized light is straightforward. His results show that the semi-major axis of the instantaneous ellipse lies in each possible orientation for an equal portion of the duration of the experiment. Furthermore, as shown below, the shape of the instantaneous ellipse varies over all possible values, also

* Ramachandrau and Ramaseshan (28) obtain the same value for the median eccentricity angle on the basis of a simple geometrical argument.

with equal likelihood. Thus, an unpolarized field is one that is elliptically polarized for intervals of time that are small compared with $1/\Delta\omega$, but whose polarization ellipse changes its shape and orientation so thoroughly during the course of a typical measurement that the field evidences no preferred polarization locus.

This point of view proves to be remarkably fruitful when it is extended to a partially polarized field. By calculating the statistics of the instantaneous ellipse for a partially polarized field, it is possible to relate the idea of polarized part, unpolarized part, complete polarization, partial polarization, loss of polarization, etc., to the temporal behavior of the field. In this way, insight is gained into the physical character of partial polarization.

2.2 Statement of the Problem

Let $\underline{E}^{(\lambda)}(\underline{r}, t)$ represent the real, quasi-monochromatic TEM field that is radiated by a noise-like source*. Suppose further that the statistics of the field are Gaussian (at least to second order distributions). Employing the quasi-monochromatic representation used above, and suppressing the position variable as before, one can write $\underline{E}^{(\lambda)}(t)$ in the form

* As pointed out in the previous chapter, it is immaterial whether the spectral width of $\underline{E}^{(\lambda)}(\underline{r}, t)$ is narrow because of the intrinsic nature of the source or because it is obtained by filtering a field of broad spectral width. The polarization will be the same in both cases provided the integrated spectra are equal. Therefore, the statistical analysis of polarization to be presented in this chapter applies equally well to the spectrum of polarization of a broad-band field and to the polarization of an intrinsically quasi-monochromatic field.

$$\underline{E}^{(\kappa)}(t) = A_1(t)\cos[\omega_0 t + \varphi_1(t)]\underline{e}_1 + A_2(t)\cos[\omega_0 t + \varphi_2(t)]\underline{e}_2 \quad (2.2.1)$$

where \underline{e}_1 and \underline{e}_2 are orthogonal unit vectors in the plane perpendicular to the direction of propagation. The amplitudes and phases $A_1(t)$, $A_2(t)$, $\varphi_1(t)$, and $\varphi_2(t)$ are assumed to vary very slowly compared with $\cos\omega_0 t$. The instantaneous locus of the field vector is obtained by eliminating $\omega_0 t$ between the two field components $E_1^{(\kappa)}(t)$ and $E_2^{(\kappa)}(t)$. Proceeding in this manner one obtains

$$\frac{E_1^{(\kappa)2}}{A_1^2(t)} + \frac{E_2^{(\kappa)2}}{A_2^2(t)} - \frac{2E_1^{(\kappa)}E_2^{(\kappa)}\cos[\varphi_1(t) - \varphi_2(t)]}{A_1(t)A_2(t)} = \sin^2[\varphi_1(t) - \varphi_2(t)] \quad (2.2.2)$$

Thus, the instantaneous ellipse, which is fixed for periods of time less than $1/\Delta\omega$, is defined in terms of the four quantities $A_1(t)$, $A_2(t)$, $\varphi_1(t)$, and $\varphi_2(t)$.

The instantaneous ellipse is characterized by the three parameters: 1) The angle $\theta(t)$ between the major axes of the instantaneous ellipse and the E_1 axis (see Figure 1.1). 2) The ellipticity angle $\beta(t)$ defined by $\tan\beta(t) = (-1)^P [a_m(t)/A_M(t)]$. 3) The intensity or "size" of the ellipse $I(t) = a_m^2(t) + A_M^2(t)$. These parameters are related to the instantaneous amplitude and phase of the field by equations of the same form as 1.4.14 - 1.4.16. That is,

$$I(t) = A_1^2(t) + A_2^2(t) \quad 0 \leq I(t) < \infty \quad (2.2.3)$$

$$\tan[2\theta(t)] = \frac{2A_1(t)A_2(t)\cos[\varphi_1(t) - \varphi_2(t)]}{A_1^2(t) - A_2^2(t)}, \quad -\frac{\pi}{2} \leq \theta(t) \leq \frac{\pi}{2} \quad (2.2.4)$$

$$u(t) \equiv \sin[2\beta(t)] = \frac{2A_1(t)A_2(t)\sin[\varphi_2(t) - \varphi_1(t)]}{A_1^2(t) + A_2^2(t)}, \begin{cases} -1 \leq u(t) \leq 1 \\ -\frac{\pi}{4} \leq \beta(t) \leq \frac{\pi}{4} \end{cases} \quad (2.2.5)$$

The parameter $u(t)$ is introduced for reasons that become evident as the analysis progresses.

The field quantities $E_1^{(\lambda)}(\underline{r}, t)$ and $E_2^{(\lambda)}(\underline{r}, t)$ are statistical variables. Therefore, $A_1(t), \dots, \varphi_2(t)$ are also statistical quantities, since they are functionally related to $E_1^{(\lambda)}(\underline{r}, t)$ and $E_2^{(\lambda)}(\underline{r}, t)$ (see 1.4.44, 1.4.45). This in turn implies that $I(t)$, $\theta(t)$, and $u(t)$ are statistical parameters. The problem then is to calculate the statistics of I , θ , and u given the joint statistics of $E_1^{(\lambda)}(\underline{r}, t)$ and $E_2^{(\lambda)}(\underline{r}, t)$.

The statistics of the field are assumed to be Gaussian. Thus at any instant t

$$P[E_1^{(\lambda)}(t), E_2^{(\lambda)}(t)] = \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}} \exp\left\{-\frac{1}{2(1-\rho^2)}\left[\frac{E_1^{(\lambda)}(t)^2}{\sigma_1^2} + \frac{E_2^{(\lambda)}(t)^2}{\sigma_2^2} - \frac{2\rho}{\sigma_1\sigma_2}E_1^{(\lambda)}(t)E_2^{(\lambda)}(t)\right]\right\} \quad (2.2.6)$$

where σ_1 is the variance of $E_1^{(\lambda)}(t)$, σ_2 is the variance of $E_2^{(\lambda)}(t)$, and ρ is the covariance of $E_1^{(\lambda)}(t)$ and $E_2^{(\lambda)}(t)$. Note that the mean values $\overline{E_1^{(\lambda)}(t)}$ and $\overline{E_2^{(\lambda)}(t)}$ are assumed to be zero, and that σ_1, σ_2 , and ρ are independent of time since the process is assumed to be stationary. By starting from 2.2.6 and employing the relationships 1.4.44, 1.4.45, and 2.2.3-2.2.5, it is possible to calculate the joint statistics of I , θ , and u .

2.3 The Statistics of the Amplitude and Phase Functions

Since $A_1(t)$, $A_2(t)$, $\varphi_1(t)$, and $\varphi_2(t)$ are defined in terms of the analytic signal representation of the field, it is necessary to know the joint statistics of the field components and their Hilbert transforms. According to Arens (35), this probability function can be expressed in the concise form

$$P[E_1(t), E_2(t)] = \frac{\text{Det } C}{(2\pi)^2} \exp \left\{ -\frac{1}{2} \left[\sum_{i,k=1,2} C_{ik} E_i^*(t) E_k(t) \right] \right\} \quad (2.3.1)$$

where $E_i(t)$ is the analytic signal associated with the real field component $E_i^{(r)}(t)$. The matrix C is the inverse of the coherency matrix

$$J_{ik}(\underline{r}) = \frac{1}{2} \overline{E_i(t) E_k^*(t)}$$

Thus,

$$C = \frac{2}{\text{Det } \underline{\underline{J}}(\underline{r})} \begin{pmatrix} J_{22}(\underline{r}) & -J_{12}(\underline{r}) \\ -J_{12}^*(\underline{r}) & J_{11}(\underline{r}) \end{pmatrix}, \quad (2.3.2)$$

and

$$\text{Det } C = \frac{4}{\text{Det } \underline{\underline{J}}(\underline{r})} \quad (2.3.3)$$

The substitution of 2.3.2 and 2.3.3 into 2.3.1 yields

$$\begin{aligned} P[E_1^{(r)}(t), E_1^{(i)}(t), E_2^{(r)}(t), E_2^{(i)}(t)] &= \frac{1}{\pi^2 \text{Det } \underline{\underline{J}}} \exp \left\{ -\frac{1}{\text{Det } \underline{\underline{J}}} [J_{22} |E_1|^2 + \right. \\ &\quad \left. + J_{11} |E_2|^2 - 2 \text{Re}(J_{12} E_1^* E_2)] \right\} \end{aligned} \quad (2.3.4)$$

Observe that for convenience variable t has been suppressed on the right side of 2.3.4.

The probability function for $A_1(t), A_2(t), \varphi_1(t)$, and $\varphi_2(t)$ is obtained from 2.3.4 by means of the transformation of variables

$$\begin{aligned} E_i^{(\wedge)}(t) &= A_i(t) \cos[\omega_o t + \varphi_i(t)] \\ E_i^{(\vee)}(t) &= A_i(t) \sin[\omega_o t + \varphi_i(t)] \end{aligned} \quad (2.3.5)$$

Following the standard procedure for transforming the probability function to a new set of variables, one obtains*

$$\begin{aligned} P[A_1, A_2, \varphi_1, \varphi_2] &= \frac{A_1 A_2}{\pi^2 \text{Det} J} \exp\left\{-\frac{1}{\text{Det} J} [J_{22} A_1^2 + J_{11} A_2^2 - 2A_1 A_2 |J_{12}| \cdot \right. \\ &\quad \left. \cdot \cos(\varphi_1 - \varphi_2 - \arg J_{12})]\right\} \end{aligned} \quad (2.3.6)$$

Note that $P[A_1, A_2, \varphi_1, \varphi_2]$ depends only on the phase difference $(\varphi_1 - \varphi_2)$. By introducing the transformation $\psi = \varphi_1 - \varphi_2$, $\xi = \varphi_2$ and then integrating over ξ , 2.3.6 simplifies slightly.

* Briefly, the formula for transforming probability functions is given by

$$P(y_1, \dots, y_n) = \{P(x_1, \dots, x_n) |\mathcal{J}|\} \Big|_{x_i = f_i(y_1, \dots, y_n)} \quad (2.3.8)$$

where \mathcal{J} is the Jacobian of the one-to-one transformation $x_i = f_i(y_1, \dots, y_n)$. A detailed discussion of this formula is given in Chapter 3 of Davenport and Root(19).

$$P(A_1, A_2, \psi) = \frac{2A_1 A_2}{\pi \text{Det} \underline{\underline{J}}} \exp \left\{ -\frac{1}{\text{Det} \underline{\underline{J}}} [J_{22} A_1^2 + J_{11} A_2^2 - 2A_1 A_2 |J_{12}| \cos(\psi - \arg J_{12})] \right\} \quad (2.3.7)$$

Equation 2.3.7 describes the joint statistics of the amplitudes and phases of a Gaussian quasi-monochromatic field at the point \underline{r} in space.

2.4 Probability Distribution for the Instantaneous Ellipse

If 2.2.3 - 2.2.5 are regarded as a transformation of variables from the coordinates A_1, A_2, ψ to new coordinates I, θ, u , then $P(I, \theta, u)$ can be derived from 2.3.7 by the transformation formula used above. However, before considering this transformation, it is desirable to rewrite 2.3.7 in terms of "polar" coordinates. Thus, if A_1 and A_2 are replaced by I, ϕ , where

$$\begin{aligned} I &= A_1^2 + A_2^2, & 0 \leq I &\leq \infty \\ \tan \phi &= \frac{A_2}{A_1}, & 0 \leq \phi &\leq \frac{\pi}{2} \end{aligned} \quad (2.4.1)$$

Then 2.13 can be written

$$P(I, \phi, \psi) = \frac{I \sin 2\phi}{2\pi \text{Det} \underline{\underline{J}}} \exp \left\{ -\frac{I}{2} [(J_{11} + J_{22}) - (J_{11} - J_{22}) \cos 2\phi - 2|J_{12}| \sin 2\phi \cdot \cos(\psi - \arg J_{12})] \right\} \quad (2.4.2)$$

where once again 2.3.8 has been used. Now consider the transformation of coordinates from (I, Φ, Ψ) to (I, θ, u) . Equation 2.4.1, combined with 2.2.3 - 2.2.5, leads to the result

$$\begin{cases} I = I \\ \tan 2\theta = \tan 2\Phi \cos \Psi \\ u = \sin 2\Phi \sin \Psi \end{cases} \begin{cases} 0 \leq \Phi \leq \frac{\pi}{2} \\ 0 \leq \Psi \leq 2\pi \\ 0 \leq I \leq \infty \end{cases} \quad (2.4.3)$$

To obtain $P(I, \theta, u)$, it is necessary to determine the inverse of the transformation 2.4.3. The inverse transformation must be defined carefully to insure that it is single valued, i.e., to insure that a particular set of amplitudes and phases correspond to a unique ellipse and vice versa.

The desired inverse transformation is

$$\begin{aligned} \Phi &= \frac{1}{2} \cos^{-1}[(\cos 2\theta) \sqrt{1-u^2}] & -1 \leq u \leq 1, -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2} \\ \Psi &= \begin{cases} \sin^{-1} \frac{u}{\sqrt{u^2 \cos^2 2\theta + \sin^2 2\theta}} & -1 \leq u \leq 1, 0 \leq \theta \leq \frac{\pi}{2} \\ \pi - \sin^{-1} \frac{u}{\sqrt{u^2 \cos^2 2\theta + \sin^2 2\theta}} & -1 \leq u \leq 1, -\frac{\pi}{2} \leq \theta \leq 0 \end{cases} & (2.4.4) \\ I &= I & 0 \leq I < \infty \end{aligned}$$

The ranges for I, θ , and u are chosen so that each possible ellipse is described by only one set of parameters. (The complicated relationship between Ψ and (u, θ) results from the fact that if θ is negative, then Ψ must be in either the second or the third quadrant.)

The Jacobian of the transformation 2.4.4 is

$$|\mathcal{J}| = \frac{1}{[u^2 \cos^2 2\theta + \sin^2 2\theta]^{1/2}} \quad (2.4.5)$$

Repeating the transformation procedure, one obtains, after some manipulation, the result

$$P(I, \theta, u) = \frac{1}{2\pi \text{Det } \underline{\underline{J}}} \exp \left\{ -\frac{I}{2 \text{Det } \underline{\underline{J}}} [(J_{11} + J_{22}) - 2J_{12}^{(i)} u - \sqrt{1-u^2}] \right. \\ \left. \cdot \sqrt{(J_{11} - J_{22})^2 + 4J_{12}^{(i)2}} \cos(2\theta - \tan^{-1}(\frac{2J_{12}^{(i)}}{J_{11} - J_{22}})) \right\} \quad (2.4.6)$$

This equation can be put in a more concise form by introducing the Stokes' parameters. The final result, then, is obtained from 1.4.37, 1.4.52 - 1.4.55, and the fact that $\text{Det } \underline{\underline{J}} = \frac{1}{4} (s_0^2 - s_1^2 - s_2^2 - s_3^2) = \frac{s_0^2}{4} (1-p^2)$

$$P(I, \theta, u) = \frac{2I}{\pi s_0^2 (1-p^2)} \exp \left\{ -\frac{2I}{s_0^2 (1-p^2)} [s_0 - s_3 u - \sqrt{s_1^2 + s_2^2} \sqrt{1-u^2}] \right. \\ \left. \cdot \cos(2\theta - \tan^{-1} \frac{s_2}{s_1}) \right\} \quad (2.4.7)$$

Equation 2.4.7 represents the joint probability distribution of the three parameters (I, θ, u) that characterize the instantaneous ellipse. Observe that $P(I, \theta, u)$ is a positive function over the entire range of the random variables I, θ, u , as it must be if it is to represent a probability distribution.

The result given in 2.4.7 is verified in Appendix B by integration over the entire range of the variables I, θ, u . As expected, the integral has the value unity. Appendix B also contains the calculation of the various marginal distributions associated with $P(I, \theta, u)$. Some of the more interesting properties of 2.4.7 are discussed in the following section.

2.5 The Properties of $P(I, \theta, u)$

Equation 2.4.7 provides the statistics of the instantaneous ellipse of a partially polarized field. Observe that the probability distribution is completely specified by the Stokes' parameters s_0, s_1, s_2 , and s_3 . Thus, an experiment designed to measure the instantaneous ellipses of two rays having identical Stokes' parameters will be unable to distinguish between the rays, since the instantaneous ellipse of each ray is completely characterized by its probability distribution and, in this case, the probability distributions of the two rays would be identical. This result is in agreement with Stokes' principle of optical equivalence. But, more importantly, it shows that two quasi-monochromatic light rays, that have equal Stokes' parameters, cannot even be distinguished from the details of their temporal variations.

Consider next the two limiting cases of complete polarization and complete absence of polarization. The latter case is characterized by $p = 0, s_1 = s_2 = s_3 = 0$. Thus, for unpolarized radiation

$$P(I, \theta, u) = \frac{2I}{\pi s_0} \cdot \exp\left[-\frac{2I}{s_0}\right] \quad \begin{array}{l} 0 \leq I \leq \infty \\ -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2} \\ -1 \leq u \leq 1 \end{array} \quad (2.5.1)$$

First observe that in this limit the variables I, θ , and u are statistically independent. Furthermore, u and θ are uniformly distributed. That is,

$$p(I, \theta, u) = p(I) p(\theta) p(u) \quad (2.5.2)$$

where

$$p(I) = \frac{4I}{s_o^2} \exp\left[-\frac{2I}{s_o}\right] \quad 0 \leq I \leq \infty \quad (2.5.3)$$

$$p(\theta) = \frac{1}{\pi} \quad -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2} \quad (2.5.4)$$

$$p(u) = \frac{1}{2} \quad -1 \leq u \leq 1 \quad (2.5.5)$$

These results are consistent with those found by Hurwitz (23). Their implication is clear. The instantaneous ellipse of an unpolarized wave assumes all possible elliptical configurations with equal likelihood. Thus, an unpolarized wave is one whose instantaneous ellipse shows absolutely no preference for a specific geometry.

Although these results are intuitively satisfying, they require some qualification. The difficulty relates to the ellipticity or shape of the instantaneous ellipse. In 2.5.2 and 2.5.5, the shape is represented by u , which is uniformly distributed in the case of unpolarized light. However, the ellipticity a_m/A_M also provides a measure of the shape of the ellipse. These two parameters are related by the equation

$$u = \frac{2 \left(\frac{a_m}{A_M} \right)}{1 + \left(\frac{a_m}{A_M} \right)^2} \quad 0 \leq \frac{a_m}{A_M} \leq 1 \quad (2.5.6)$$

Thus, from 2.5.5 and 2.5.6, it is seen that

$$P\left(\frac{a_m}{A_M}\right) = \frac{1 - \left(\frac{a_m}{A_M}\right)^2}{\left[1 + \left(\frac{a_m}{A_M}\right)^2\right]^2} \quad 0 \leq \frac{a_m}{A_M} \leq 1 \quad (2.5.7)$$

Equation 2.5.7 shows that, although u is uniformly distributed, (a_m/A_m) is not. In fact, $P\left(\frac{a_m}{A_M}\right)$ is a monotonically decreasing function on the interval $0 \leq \frac{a_m}{A_M} \leq 1$. The average value of (a_m/A_M) is 0.307, its most probable value is zero, and its median value is 0.268. The median value indicates that a_m is less than one-fourth A_M about one-half the time.

The two parameters u and a_m/A_M both describe the shape of the instantaneous ellipse, but they are not distributed in the same way in probability. Consequently, it is desirable to select one or the other as the shape parameter. Since u is so simply related to the Stokes' parameters, and since it is uniformly distributed in the case of unpolarized light, it appears to be the more desirable choice. Hence, hereafter u , along with I , θ , and p , will be regarded as the principal polarization parameters.

Before proceeding to the other limiting case, it is worthwhile to consider $p(\psi)$, the probability distribution of the phase difference $\varphi_1(t) - \varphi_2(t)$, in the limit of unpolarized light. To this end, it is necessary to express 2.4.2 in terms of the Stokes' parameters. Once again, using 1.4.37, 1.4.52 - 1.4.55, and $\text{Det} \underline{J} = \frac{1}{4} (s_0^2 - s_1^2 - s_2^2 - s_3^2)$, one finds that

$$P(I, \Phi, \psi) = \frac{2I \sin 2\Phi}{\pi s_0^2 (1-p^2)} \exp \left\{ -\frac{2I}{s_0^2 (1-p^2)} \left[s_0 - s_1 \cos 2\Phi - \frac{1}{2} \sqrt{s_2^2 + s_3^2} \sin 2\Phi \right. \right. \\ \left. \left. \cdot \cos \left(\psi - \tan^{-1} \frac{s_3}{s_2} \right) \right] \right\} \quad (2.5.8)$$

Thus in the limit $p = s_1 = s_2 = s_3 = 0$,

$$P(I, \Phi, \psi) = \frac{2I \sin 2\Phi}{\pi s_0^2} \exp \left[-\frac{2I}{s_0} \right] \quad (2.5.9)$$

In other words, ψ is statistically independent of I and Φ , and is uniformly distributed between 0 and 2π . Thus, in unpolarized light, the phase difference between the components of the field takes on every value between 0 and 2π with equal probability. In other words, an unpolarized wave is one for which there is no preferred phase difference between components.

Consider now the case of complete polarization. This limit is characterized by $p = 1$. If 2.4.7 is written in the concise form

$$P(I, \theta, u) = \text{const.} \frac{I}{(1-p^2)} \exp \left[-\frac{f(I, \theta, u)}{(1-p^2)} \right] \quad (2.5.10)$$

it is seen that

$$\lim_{p \rightarrow 1} P(I, \theta, u) = 0 \quad f(I, \theta, u) \neq 0 \\ = \infty \quad f(I, \theta, u) = 0. \quad (2.5.11)$$

Since the integral of $P(I, \theta, u)$ is unity, then in this limit the probability function has the characteristics of a delta function. The singular "point" is defined by

$$\frac{I}{s_o} [s_o - s_3 u - \sqrt{s_1^2 + s_2^2} \sqrt{1-u^2} \cos(2\theta - \tan^{-1} \frac{s_2}{s_1})] = 0 \quad (2.5.12)$$

Neglecting the trivial case $I = 0$, this equation requires that

$$\frac{s_o - s_3 u}{\sqrt{s_1^2 + s_2^2} \sqrt{1-u^2}} = \cos(2\theta - \tan^{-1} \frac{s_2}{s_1}) \quad (2.5.13)$$

It is readily found that over the range $-1 \leq u \leq 1$ the left side of 2.5.13 has an absolute minimum at $u_o = s_3/s_o$. If $p < 1$, this minimum is greater than 1. However, as $p \rightarrow 1$, the minimum does also. Hence, in the limit $p \rightarrow 1$, 2.5.12 has a single solution: $u_o = s_3/s_o, \tan 2\theta_o = s_2/s_1$. The solution of 2.5.12 is independent of I . To determine the dependence of $P(I, \theta, u)$ on I in the limit $p \rightarrow 1$, it is necessary to calculate the marginal distribution $P_M(I)$ and then approach the limit. This distribution is calculated in Appendix C. It is given by

$$P_M(I) = \frac{2}{\sqrt{s_1^2 + s_2^2 + s_3^2}} \sinh \left[\frac{2I}{s_o^2(1-p^2)} \sqrt{s_1^2 + s_2^2 + s_3^2} \right] \exp \left[-\frac{2I}{s_o(1-p^2)} \right] \quad (2.5.14)$$

Thus, *

$$\lim_{p \rightarrow 1} P_M(I) = \frac{1}{s_o} \exp \left[\frac{-I}{s_o} \right] \quad (2.5.15)$$

* The value of $P_M(I)$ at the origin in the limit $p = 1$ depends on which order the limits are taken. $\lim_{p \rightarrow 1} \lim_{I \rightarrow 0} P_M(I) = 0$
 $\lim_{I \rightarrow 0} \lim_{p \rightarrow 1} P_M(I) = 2/s_o$. Since interest here is restricted to positive values of I , then the latter value is defined to be the probability density at $I = 0$ in the limit $p \rightarrow 1$.

The above results can be expressed mathematically in the form

$$\lim_{p \rightarrow 1} P(I, \theta, u) = \frac{1}{s_0} \delta(u - \frac{s_3}{\sqrt{s_1^2 + s_2^2 + s_3^2}}) \delta(\theta - \frac{1}{2} \tan^{-1} \frac{s_2}{s_1}) \exp[-\frac{I}{s_0}] \quad (2.5.16)$$

where $\sqrt{s_1^2 + s_2^2 + s_3^2}$ has been substituted for s_0 in the denominator of the singular point u_0 . The significance of this result is clear. The ellipse with parameters u_0, θ_0 occurs with probability 1 while all other ellipses occur with probability zero. Thus, the orientation and ellipticity of the instantaneous ellipse of a completely polarized wave are independent of time (with probability 1)--a result which is in complete agreement with the conventional analysis of complete polarization. On the other hand, the "size" of the ellipse is variable, its distribution being exponential in character. Observe that the most likely value of I (as determined from the maximum of the distribution of I) is zero. This is consistent with the fact that the vector field is assumed to have zero mean value. Interestingly, the average value of I equals s_0 . Thus, the Stokes' parameter s_0 represents the average intensity of the instantaneous ellipse of the field.

It is interesting to note from 2.5.15 that the amplitude of the field has the Rayleigh distribution, i.e., if ρ denotes the amplitude of the field, then $\rho = \sqrt{I}$, and under this transformation $P_M(I)$ becomes

$$P_M(\rho) = \frac{2\rho}{s_0} \exp[-\frac{\rho^2}{s_0}] \quad (2.5.17)$$

But the amplitude of scalar, narrow-band, Gaussian noise is Rayleigh distributed. Thus, 2.5.17 suggests that from a statistical standpoint a completely polarized noise-like field has the character of scalar narrow-

band noise. This conclusion is consistent with the result stated in 1.4.35 --namely that the two orthogonal components of a completely polarized field are proportional, so that such a field is represented by a single random process. Thus, for instance, the statistics of a completely polarized ray can be treated separately from the vector considerations in taking such a ray through a deterministic optical system. It is this fact that forms the basis for the Jones calculus of optical systems (36).

The foregoing considerations indicate that an unpolarized ray cannot be treated as a scalar quantity when considering the interaction of such a ray with an optical system. Equation 2.5.3 shows that the amplitude, \sqrt{I} , of an unpolarized ray is not Rayleigh distributed. In fact, it is distributed as the amplitude of the sum of two independent Gaussian random processes that have equal statistics. These two processes, of course, are the two instantaneous components of the field vector. But an optical system can affect the two instantaneous components of a field vector differently. Therefore, since the temporal variations of the two components are independent, the two components must both be considered when carrying such a ray through an optical system. In other words, the vector character of an unpolarized ray can only be ignored if the ray propagates undisturbed through free space.

By considering the two limiting cases $p = 0$ and $p = 1$, it has been shown that the degree of polarization determines the tendency for the geometry of the instantaneous ellipse to vary with respect to time. In the case of unpolarized radiation ($p = 0$), the statistics show that the instantaneous ellipse varies uniformly over all possible configurations.

On the other hand, in the case of complete polarization ($p = 1$), the geometry of the instantaneous ellipse is constant with respect to time, so that in this case there is a unique polarization locus associated with the field vector. Clearly then, partial polarization must be characterized by some intermediate behavior. More specifically, it will now be shown that although the instantaneous ellipse of a partially polarized field fluctuates with respect to time, it has a preferred geometry. It will also be shown that this preferred geometry corresponds to the geometry of the ellipse that characterizes the completely polarized part of the field.

Consider 2.4.7, the probability distribution of the instantaneous ellipse associated with a partially polarized wave. The maximum of this function (which defines the most likely ellipse) is determined from the simultaneous solution of

$$\frac{\partial P}{\partial I} = 0, \quad \frac{\partial P}{\partial \theta} = 0, \quad \frac{\partial P}{\partial u} = 0 \quad .$$

These three conditions lead to the following equations:

$$s_o^2(1-p^2) - I[s_o - s_3 u - \sqrt{1-u^2} \sqrt{s_1^2 + s_2^2} \cos(2\theta - \tan^{-1} \frac{s_2}{s_1})] = 0 \quad (2.5.18)$$

$$\sqrt{1-u^2} s_3 - u \sqrt{s_1^2 + s_2^2} \cos(2\theta - \tan^{-1} \frac{s_2}{s_1}) = 0 \quad (2.5.19)$$

$$\sqrt{1-u^2} \sin(2\theta - \tan^{-1} \frac{s_2}{s_1}) = 0 \quad (2.5.20)$$

Starting with 2.5.20, one can readily show that 2.5.18 - 2.5.20 have three solutions:

$$\begin{aligned}
 I_1 &= s_o(1+p) \quad , \quad u_1 = \frac{s_3}{\sqrt{s_1^2 + s_2^2 + s_3^2}} \quad , \quad \theta_1 = \frac{1}{2} \tan^{-1} \frac{s_2}{s_1} \\
 I_2 &= \frac{s_o^2(1-p^2)}{s_o - s_3} \quad , \quad u_2 = 1 \quad , \quad \theta_2 = \frac{\pi}{4} + \frac{1}{2} \tan^{-1} \frac{s_2}{s_1} \\
 I_3 &= \frac{s_o^2(1-p^2)}{s_o + s_3} \quad , \quad u_3 = -1 \quad , \quad \theta_3 = \frac{\pi}{4} + \frac{1}{2} \tan^{-1} \frac{s_2}{s_1}
 \end{aligned}
 \tag{2.5.21}$$

By substituting 2.5.21 into 2.4.7, it is found that $P(I_1, \theta_1, u_1)$ is greater in magnitude than either $P(I_2, \theta_2, u_2)$ or $P(I_3, \theta_3, u_3)$. Thus, I_1, θ_1, u_1 characterizes the most probable ellipse. But observe that this is exactly the polarization that is defined by the Stokes' parameters of the completely polarized part of the wave (see 1.4.38). Therefore, it is seen that the polarization defined by the Stokes' parameters corresponds to the polarization locus most frequently traced out by the field vector in the course of its time variation. This result clearly demonstrates that the Stokes' parameters determine the preferred state of polarization of the field rather than some average polarization. Furthermore, although $p(I, \theta, u)$ has a single well defined maximum if $p < 1$, it is not zero for points (I, θ, u) that are different from (I_1, θ_1, u_1) . Therefore, a partially polarized wave is characterized not only by a preferred polarization, but also by the fact that the instantaneous ellipse assumes other orientations and shapes as well. It is due to the fact that the instantaneous ellipse varies that an unpolarized component exists. That is, incomplete

polarization is caused by the excursions of the instantaneous ellipse from its preferred state. From a statistical standpoint, the extent to which the instantaneous ellipse deviates from its preferred state is determined by the width of the probability distribution. But, as seen from 2.4.7, the width of $P(I, \theta, u)$ is determined largely by the degree of polarization (i.e., p behaves like the variance of ordinary statistical analysis). Thus, the degree of polarization can be interpreted as a measure of the extent to which the instantaneous ellipse deviates from the preferred, or completely polarized, state.

To summarize the results obtained in this section, it was shown that a quasi-monochromatic field vector traces out an instantaneous ellipse which, in general, changes its character over time intervals in excess of $2\pi/\Delta\omega$. In the event that the instantaneous ellipse shows a preference for one particular polarization, the wave is partially polarized. The polarized part of the wave, as determined by the Stokes' parameters, is characterized by the preferred polarization state of the instantaneous ellipse. The degree of polarization is determined by the extent to which the instantaneous ellipse seeks other states of polarization during the course of a measurement. In the limiting case of complete polarization, the instantaneous ellipse remains fixed in geometry--only its intensity fluctuates. The geometry of the preferred ellipse corresponds to that of the ellipse determined from the Stokes' parameters. In the other limiting case, that of completely unpolarized radiation, the instantaneous ellipse fluctuates in a uniform fashion over all possible polarization states. Thus, on the basis of this analysis, polarization is seen to be a measure of the tendency for the field locus to prefer a particular configuration. The unpolarized

part of the wave results from the tendency for the instantaneous ellipse to deviate from this preferred geometry.

2.6 An Application

To complete this chapter it is useful to consider a simple application of the results obtained above. An interesting problem, which requires explicit use of the model and the statistics for an unpolarized quasi-monochromatic plane wave, concerns the calculation of the differential scattering cross section of a free electron excited by an unpolarized wave. This calculation has been carried out by Landau and Lifshitz (37). However, they use the Fresnel model of an unpolarized wave rather than the more credible instantaneous ellipse model that was discussed above. The statistics derived in this chapter provide a means for verifying the applicability of the Fresnel model to the calculation of the scattering cross section. The procedure followed here is to assume that the instantaneous power scattered into some direction by the electrons results from an elliptically polarized incident plane wave. The average power scattered in this direction is then computed using the statistics derived above.

Suppose an unpolarized quasi-monochromatic plane wave is incident upon a free electron situated at the origin of coordinates. The problem posed here is to calculate the effective differential scattering cross section of the electron. By definition, the effective scattering cross section is (37)

$$d\sigma = \frac{\langle dI \rangle}{\langle S \rangle} \quad (2.6.1)$$

i.e., $d\sigma$ is the ratio of the time average of the power scattered into the solid angle $d\Omega$ to the average power density of the incident wave. For the elliptically polarized monochromatic wave

$$\underline{E}(t) = \underline{A}_M \cos(\omega t + \alpha) + \underline{a}_m \sin(\omega t + \alpha) \quad (2.6.2)$$

(where \underline{A}_M and \underline{a}_m represent the vector semi-major and semi-minor axes of the ellipse), the scattering cross section is*

$$d\sigma_{\text{mono}} = \frac{e^4}{m^2 c^4} \frac{(\underline{A}_M \times \underline{e}_n)^2 + (\underline{a}_m \times \underline{e}_n)^2}{A_M^2 + a_m^2} d\Omega \quad (2.6.3)$$

where \underline{e}_n is a unit vector in the direction of observation. Thus the scattering cross section of a quasi-monochromatic wave can be written

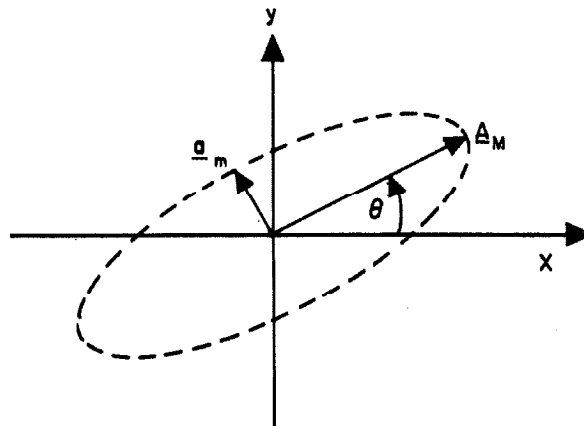
$$d\sigma = \frac{e^4}{m^2 c^4} \frac{\langle [\underline{A}_M(t) \times \underline{e}_n]^2 + [\underline{a}_m(t) \times \underline{e}_n]^2 \rangle}{\langle A_M^2(t) + a_m^2(t) \rangle} d\Omega \quad (2.6.4)$$

where, as before, the sharp brackets indicate a time average. By appealing to the ergodicity hypothesis, it is possible to rewrite 2.6.4 as an ensemble average.

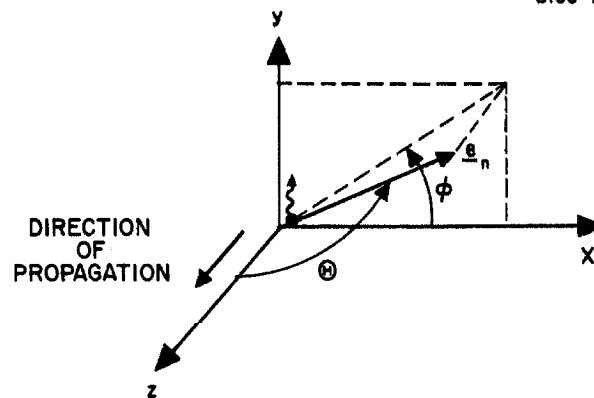
$$d\sigma = \frac{e^4}{m^2 c^4} \frac{\overline{[\underline{A}_M \times \underline{e}_n]^2 + [\underline{a}_m \times \underline{e}_n]^2}}{I_{\text{inc}}} d\Omega \quad (2.6.5)$$

where I_{inc} , the intensity of the incident wave, is defined by 1.4.22 and 1.4.26. To introduce the parameters of the instantaneous ellipse, consider the geometry depicted in Figure 2.1. Equation 2.6.5 can be

* See Problem 1, page 234 of Landau and Lifshitz (37).



(a) Geometry of the instantaneous ellipse.



(b) Geometry of the scattering process.

Figure 2.1. Scattering by a free electron.

written in the form

$$\begin{aligned}
 d\sigma &= \frac{e^4}{m^2 c^4 I_{inc}} \left[A_M^2 \sin^2(\underline{e}_n, \underline{A}_M) + a_m^2 \sin^2(\underline{e}_n, \underline{a}_m) \right] d\Omega \\
 &= \frac{e^4}{m^2 c^4 I_{inc}} \left[I_{inc} - A_M^2 \cos^2(\underline{e}_n, \underline{A}_M) + a_m^2 \cos^2(\underline{e}_n, \underline{a}_m) \right] d\Omega
 \end{aligned} \tag{2.6.6}$$

But from Figure 2.1 it is seen that

$$\begin{aligned}
 \cos(\underline{e}_n, \underline{A}_M) &= \cos\theta \sin\Theta \cos\varphi + \sin\theta \sin\Theta \sin\varphi = \sin\Theta \cos(\varphi - \theta) \\
 \cos(\underline{e}_n, \underline{a}_m) &= -\sin\theta \sin\Theta \cos\varphi + \cos\theta \sin\Theta \sin\varphi = \sin\Theta \sin(\varphi - \theta)
 \end{aligned} \tag{2.6.7}$$

Therefore, 2.6.6 can be written

$$d\sigma = \frac{e^4}{m^2 c^4 I_{inc}} \left\{ I_{inc} - \frac{\sin^2\Theta}{2} \left[(A_M^2 + a_m^2) + (A_M^2 - a_m^2) \cos 2(\theta - \varphi) \right] \right\} d\Omega \tag{2.6.8}$$

From the statistics derived on page 77, it is seen that the angle θ , the orientation angle of the instantaneous ellipse, is uniformly distributed between 0 and 2π . Furthermore, the statistics of A_M and a_m are independent of those of θ in the case of unpolarized field. Hence, the second term under the expectation bar is zero, and

$$\begin{aligned}
 d\sigma &= \frac{e^4}{m^2 c^4} \left[1 - \frac{\sin^2\Theta}{2} \right] d\Omega \\
 &= \frac{1}{2} \left(\frac{e^2}{mc^2} \right)^2 [1 + \cos^2\Theta] d\Omega
 \end{aligned}$$

where Θ is the angle between the direction of propagation and the direction of observation. This is the same result obtained by Landau and Lifshitz.

III. THE EQUATION OF PROPAGATION AND THE BOUNDARY CONDITIONS FOR THE DFS

3.1 Introduction

One of the most noteworthy achievements in the development of the theory of partial coherence was Zernike's calculation of the functional relationship between the mutual coherence function and the intensity distribution of the source. By means of this calculation, he was able to show that the spatial coherence of the field has the form of the diffraction pattern radiated by an aperture of the same size and shape as the source that is illuminated by a spherical wave originating at a particular point behind the aperture.* This observation is quite suggestive: since a diffraction pattern satisfies the wave equation (or the reduced wave equation in the case of sinusoidal time variations), perhaps the mutual coherence function satisfies a similar type of equation. On the basis of this sort of reasoning, Wolf (7) was led to derive the differential equations that the mutual coherence function must satisfy. As a result of this analysis, he found that the mutual coherence function,

$\Gamma(P_1, P_2, t_1, t_2)$, must simultaneously satisfy

$$\nabla_1^2 \Gamma(P_1, P_2, \tau) = \frac{1}{C^2} \frac{\partial^2}{\partial \tau^2} \Gamma(P_1, P_2, \tau) \quad (3.1.1)$$

$$\nabla_2^2 \Gamma(P_1, P_2, \tau) = \frac{1}{C^2} \frac{\partial^2}{\partial \tau^2} \Gamma(P_1, P_2, \tau)$$

where

$$\Gamma(P_1, P_2, \tau) = \langle V(P_1, t_1) V^*(P_2, t_2) \rangle,$$

* See Born and Wolf (26), page 507.

$V(P, t)$ being the complex scalar disturbance at point P at time t . As usual, the symbol ∇_i^2 represents the Laplacean operator $\frac{\partial^2}{\partial x_i^2} + \frac{\partial^2}{\partial y_i^2} + \frac{\partial^2}{\partial z_i^2}$, where (x_i, y_i, z_i) are the rectangular coordinates of the point $P_i (i = 1, 2)$. The symbol, c , represents the velocity of light. Thus, it is seen that the spatial coherence of the field propagates like a scalar wave. Of course, since Γ depends on two independent field points, it is defined by two wave equations. Nevertheless, the analogy with wave propagation is clear. As will be found below, this analogy is very strongly reflected in the mathematics of the theory of partial coherence.

Shortly after this work, Wolf and his colleagues generalized 3.1.1 to vector quasi-monochromatic fields (9). This analysis originated from Maxwell's equations. Hence, it was the first derivation of the equations of propagation of the coherency matrix that started from first principles. This work was restricted to quasi-monochromatic fields, because the coherency matrix that Wolf introduced had meaning only for narrow-band radiation. However, in Chapter I the concept of coherence (and the coherency matrix) was extended to vector fields of broad spectral width. Consequently, it is now possible to extend Wolf's derivation of the wave equation for the coherency matrix to fields of arbitrary spectral width. The procedure employed in the derivation of the differential equation is basically that developed by Wolf. The form of the results given here differs from Wolf's in that dyadic notation is introduced for the purpose of simplifying the subsequent integration of the differential equation.

3.2 The Dyadic Wave Equation

The wave equation for the DFS is most readily derived by starting with Maxwell's equations expressed in tensor notation. It is assumed that the domain of interest is free space, and that the field is radiated by a distribution of real macroscopic sources. Since it is the object of this analysis to derive the wave equation directly, without first obtaining the tensor analog of Maxwell's equations, then only the current density $\underline{J}(\underline{r}, t)$ is required to specify the source. In fact, the only equations of interest here are

$$\nabla \times \underline{E}^{(n)}(\underline{r}, t) = -\mu_0 \frac{\partial \underline{H}^{(n)}(\underline{r}, t)}{\partial t} \quad (3.2.1)$$

$$\nabla \times \underline{H}^{(n)}(\underline{r}, t) = \underline{J}^{(n)}(\underline{r}, t) + \epsilon_0 \frac{\partial \underline{E}^{(n)}(\underline{r}, t)}{\partial t} \quad (3.2.2)$$

In tensor notation 3.2.1 and 3.2.2 become

$$\eta_{ijk} \partial_j E_k^{(n)}(\underline{r}, t) = -\mu_0 \frac{\partial}{\partial t} H_i^{(n)}(\underline{r}, t) \quad i, j, k, = 1, 2, 3$$

$$\eta_{ijk} \partial_j H_k^{(n)}(\underline{r}, t) = J_i^{(n)}(\underline{r}, t) + \epsilon_0 \frac{\partial}{\partial t} E_i^{(n)}(\underline{r}, t) \quad (3.2.4)$$

where η_{ijk} is the Levi-Civita symbol

$$\eta_{ijk} = \begin{cases} 1 & \text{for an even permutation of } i, j, \text{ and } k. \\ -1 & \text{for an odd permutation of } i, j, \text{ and } k. \\ 0 & \text{if two or more indices are the same.} \end{cases} \quad (3.2.5)$$

The symbol ∂_i represents the differentiation operation $\frac{\partial}{\partial x_i}$, where x_i is one of the three rectangular coordinates. The quantities ϵ_0 and μ_0 are the permittivity and permeability of free space, respectively.

The field quantity $E_i^{(n)}(\underline{r}, t)$, appearing in 3.2.3 and 3.2.4, represents the component of the field in the direction of the coordinate x_i . It should be observed that real field components are used here. This represents a departure from Wolf's derivation which is based on the analytic signal representation of the field. Because interest here is concerned with the DFS rather than with the dyadic autocorrelation function, it is not necessary to employ complex field quantities.

The first step in the derivation of the dyadic wave equation is to multiply 3.2.3, evaluated at point \underline{r} , and time t_1 , by 3.2.3, evaluated at \underline{r}_2, t_2 . This procedure yields the equation*

$$\eta_{ijk} \eta_{lmn} \partial_j^{(1)} \partial_m^{(2)} E_k(\underline{r}_1, t_1) E_n(\underline{r}_2, t_2) = \mu_o^2 \frac{\partial}{\partial t_1} H_i(\underline{r}_1, t_1) \frac{\partial}{\partial t_2} H_l(\underline{r}_2, t_2) \quad (3.2.6)$$

where the superscripts (1) and (2) indicate the point with respect to which the differentiation applies. If the ensemble average of 3.2.6 is formed and if use is made of the fact that the field is stationary, then 3.2.6 reduces to

$$\eta_{ijk} \eta_{lmn} \partial_j^{(1)} \partial_m^{(2)} \mathcal{E}_{kn}(\underline{r}_1, \underline{r}_2, \tau) = \mu_o^2 \overline{\left[\frac{\partial}{\partial t_1} H_i(\underline{r}_1, t_1) \frac{\partial}{\partial t_2} H_l(\underline{r}_2, t_2) \right]} \quad (3.2.7)$$

where $\tau = t_1 - t_2$. The right side of 3.2.7 can be simplified by interchanging the orders of averaging and differentiation and then making use

*

Since only real functions of time appear in this analysis, the superscript (n) on the field quantities is suppressed throughout this chapter.

of stationarity. That is,

$$\begin{aligned}
 \frac{\partial H_i(\underline{r}_1, t_1)}{\partial t_1} \frac{\partial H_l(\underline{r}_2, t_2)}{\partial t_2} &= \frac{\partial^2}{\partial t_1 \partial t_2} \overline{H_i(\underline{r}_1, t_1) H_l(\underline{r}_2, t_2)} \\
 &= \frac{\partial^2}{\partial t_1 \partial t_2} \mathcal{H}_{il}(\underline{r}_1, \underline{r}_2, t_1 - t_2) \\
 &= \frac{-\partial^2}{\partial \tau^2} \mathcal{H}_{il}(\underline{r}_1, \underline{r}_2, \tau)
 \end{aligned} \tag{3.2.8}$$

where $\mathcal{H}_{ij}(\underline{r}_1, \underline{r}_2, \tau)$ is an element of the magnetic field dyadic autocorrelation function. It should be noted that $\underline{\mathcal{H}}(\underline{r}_1, \underline{r}_2, \tau)$ plays a secondary role in this analysis and does not appear in the final results. The interested reader can find a detailed treatment of this quantity in the papers by Wolf and Roman (11)-(14).

From 3.2.7 and 3.2.8, it is seen that

$$\eta_{ijk} \eta_{lmn} \frac{\partial^{(1)}}{\partial t_j} \frac{\partial^{(2)}}{\partial t_m} \mathcal{E}_{kn}(\underline{r}_1, \underline{r}_2, \tau) = -\mu_0 \frac{\partial^2}{\partial \tau^2} \mathcal{H}_{il}(\underline{r}_1, \underline{r}_2, \tau) \tag{3.2.9}$$

The second dyadic differential equation is obtained by repeating the above procedure, starting this time with 3.2.4. The equation analogous to 3.2.6 is

$$\begin{aligned}
 \eta_{ijk} \eta_{lmn} \frac{\partial^{(1)}}{\partial t_j} \frac{\partial^{(2)}}{\partial t_m} H_k(\underline{r}_1, t_1) H_n(\underline{r}_2, t_2) &- \epsilon_0 \eta_{ijk} \frac{\partial^{(1)}}{\partial t_j} \frac{\partial}{\partial t_2} H_k(\underline{r}_1, t_1) E_l(\underline{r}_2, t_2) - \\
 - \epsilon_0 \eta_{lmn} \frac{\partial^{(2)}}{\partial t_m} \frac{\partial}{\partial t_1} E_i(\underline{r}_1, t_1) H_n(\underline{r}_2, t_2) &+ \epsilon_0 \frac{\partial^2}{\partial t_1 \partial t_2} E_i(\underline{r}_1, t_1) E_l(\underline{r}_2, t_2) = \\
 &= J_i(\underline{r}_1, t_1) J_l(\underline{r}_2, t_2)
 \end{aligned} \tag{3.2.10}$$

Taking the ensemble average of 3.2.10, one finds that

$$\begin{aligned} & \eta_{ijk} \eta_{lmn} \partial_j^{(1)} \partial_m^{(2)} \mathcal{H}_{kn}(\underline{r}_1, \underline{r}_2, \tau) + \epsilon_o \eta_{ijk} \partial_j^{(1)} \frac{\partial}{\partial \tau} \mathcal{G}_{kl}(\underline{r}_1, \underline{r}_2, \tau) - \\ & - \epsilon_o \eta_{lmn} \partial_m^{(2)} \frac{\partial}{\partial \tau} \mathcal{G}_{ni}(\underline{r}_2, \underline{r}_1, -\tau) - \epsilon_o^2 \frac{\partial^2}{\partial \tau^2} \mathcal{E}_{ij}(\underline{r}_1, \underline{r}_2, \tau) = \mathcal{G}_{ij}(\underline{r}_1, \underline{r}_2, \tau) \end{aligned} \quad (3.2.11)$$

where $\mathcal{G}_{ij}(\underline{r}_1, \underline{r}_2, \tau)$ is an auxiliary dyadic defined by

$$\mathcal{G}_{ij}(\underline{r}_1, \underline{r}_2, \tau) = \overline{H_i(\underline{r}_1, t_1) E_j(\underline{r}_2, t_2)} \quad (3.2.12)$$

and where $\mathcal{G}_{ij}(\underline{r}_1, \underline{r}_2, \tau)$ represents the dyadic autocorrelation function of the source. To remove the term involving $\mathcal{H}_{kn}(\underline{r}_1, \underline{r}_2, \tau)$ from 3.2.11, it is necessary to multiply it by $-\mu_o^2 \frac{\partial^2}{\partial \tau^2}$ and then make use of 3.2.9. Proceeding in this manner, one obtains

$$\begin{aligned} & \eta_{ijk} \eta_{lmn} \eta_{kab} \eta_{ncd} \partial_j^{(1)} \partial_a^{(1)} \partial_m^{(2)} \partial_c^{(2)} \mathcal{E}_{bd}(\underline{r}_1, \underline{r}_2, \tau) - \epsilon_o \mu_o^2 \eta_{ijk} \partial_j^{(1)} \frac{\partial^3}{\partial \tau^3} \mathcal{G}_{kl}(\underline{r}_1, \underline{r}_2, \tau) \\ & + \epsilon_o \mu_o^2 \eta_{lmn} \partial_m^{(2)} \frac{\partial^3}{\partial \tau^3} \mathcal{G}_{ni}(\underline{r}_2, \underline{r}_1, -\tau) + \mu_o^2 \epsilon_o^2 \frac{\partial^4}{\partial \tau^4} \mathcal{E}_{ij}(\underline{r}_1, \underline{r}_2, \tau) = \\ & = \mu_o^2 \frac{\partial^2}{\partial \tau^2} \mathcal{G}_{ij}(\underline{r}_1, \underline{r}_2, \tau) \end{aligned} \quad (3.2.13)$$

The terms involving $\{\mathcal{G}_{ij}(\underline{r}_1, \underline{r}_2, \tau)\}$ can be eliminated by returning to

3.2.3. Expressed in terms of the coordinates \underline{r}_1, t_1 , this equation can be written

$$\eta_{kab} \partial_a^{(1)} E_b(\underline{r}_1, t_1) = -\mu_o \frac{\partial}{\partial t_1} H_k(\underline{r}_1, t_1) \quad (3.2.14)$$

If 3.2.14 is multiplied by $E_l(\underline{r}_2, t_2)$ and averaged, the result is

$$\eta_{kab} \partial_a^{(1)} \mathcal{E}_{bl}(\underline{r}_1, \underline{r}_2, \tau) = -\mu_0 \frac{\partial}{\partial \tau} \mathcal{H}_{kl}(\underline{r}_1, \underline{r}_2, \tau) \quad (3.2.15)$$

The term involving $\mathcal{H}_{ni}(\underline{r}_1, \underline{r}_2, \tau)$ is treated by writing 3.2.3 in the form

$$\eta_{ncd} \partial_c^{(2)} E_d(\underline{r}_2, t_2) = -\mu_0 \frac{\partial}{\partial t_2} H_n(\underline{r}_2, t_2) \quad , \quad (3.2.16)$$

multiplying through by $E_i(\underline{r}_1, t_1)$, and averaging. As a result, it is found that

$$\eta_{ncd} \partial_c^{(2)} \mathcal{E}_{id}(\underline{r}_1, \underline{r}_2, \tau) = \mu_0 \frac{\partial}{\partial \tau} \mathcal{H}_{ni}(\underline{r}_2, \underline{r}_1, -\tau) \quad . \quad (3.2.17)$$

The "wave equation" is obtained by substituting 3.2.15 and 3.2.17 into 3.2.13:

$$\begin{aligned} & \eta_{ijk} \eta_{lmn} \eta_{kab} \eta_{ncd} \partial_j^{(1)} \partial_a^{(1)} \partial_m^{(2)} \partial_c^{(2)} \mathcal{E}_{bd}(\underline{r}_1, \underline{r}_2, \tau) + \\ & + \frac{1}{c^2} \eta_{ijk} \eta_{kab} \partial_j^{(1)} \partial_a^{(1)} \frac{\partial^2}{\partial \tau^2} \mathcal{E}_{bl}(\underline{r}_1, \underline{r}_2, \tau) + \frac{1}{c^2} \eta_{lmn} \eta_{ncd} \partial_m^{(2)} \partial_c^{(2)} \frac{\partial^2}{\partial \tau^2} \mathcal{E}_{id}(\underline{r}_1, \underline{r}_2, \tau) + \\ & + \frac{1}{c^4} \frac{\partial^4}{\partial \tau^4} \mathcal{E}_{il}(\underline{r}_1, \underline{r}_2, \tau) = -\mu_0^2 \frac{\partial^2}{\partial \tau^2} \mathcal{G}_{il}(\underline{r}_1, \underline{r}_2, \tau) \quad . \quad (3.2.18) \end{aligned}$$

The DFS satisfies the reduced "wave equation" which is obtained from 3.2.18 by means of the Fourier transform. If 3.2.18 is multiplied by $e^{i\omega\tau}$ and integrated with respect to τ from $\tau = -\infty$ to $\tau = +\infty$, then by following the usual procedure of interchanging the order of differentiation and integration, it is found that

$$\begin{aligned} & \eta_{ijk} \eta_{kab} \eta_{lmn} \eta_{ncd} \partial_j^{(1)} \partial_a^{(1)} \partial_m^{(2)} \partial_c^{(2)} \hat{C}_{bd}(\underline{r}_1, \underline{r}_2, \omega) - k^2 \eta_{ijk} \eta_{kab} \partial_j^{(1)} \partial_a^{(1)} \hat{C}_{bf}(\underline{r}_1, \underline{r}_2, \omega) - \\ & - k^2 \eta_{lmn} \eta_{ncd} \partial_m^{(2)} \partial_c^{(2)} \hat{C}_{id}(\underline{r}_1, \underline{r}_2, \omega) + k^4 \hat{C}_{il}(\underline{r}_1, \underline{r}_2, \omega) = \mu_o^2 \omega^2 \hat{Q}_{il}(\underline{r}_1, \underline{r}_2, \omega) \end{aligned} \quad (3.2.19)$$

where $k^2 = \frac{\omega^2}{c^2}$, and where the DFS and the DSS have been introduced by means of the definitions given in 1.3.3.

Equation 3.2.19 represents the dyadic wave equation expressed in tensor notation. Although this notation provides a convenient means for deriving the dyadic wave equation, it is not particularly useful for the solution of the equation. Vector notation provides a more effective form. To make the transition from one notation to the other, it is merely necessary to observe that

$$\eta_{ijk} \eta_{klm} \partial_j \partial_l A_m(\underline{r}) = \{\nabla \times \nabla \times \underline{A}(\underline{r})\}_i \quad (3.2.20)$$

where $\underline{A}(\underline{r})$ is a vector field, and the subscript i denotes the i^{th} component of the vector $\nabla \times \nabla \times \underline{A}(\underline{r})$. The corresponding relationship for a dyadic field is

$$\eta_{ijk} \eta_{klm} \partial_j \partial_l Q_{mn}(\underline{r}) \equiv \{\nabla \times \nabla \times \underline{Q}(\underline{r})\}_{in} \quad (3.2.21)$$

where now the quantity on the left represents one term of the dyadic $\nabla \times \nabla \times \underline{Q}(\underline{r})$. In addition to terms like that given on the left of 3.2.21, 3.2.19 includes terms of the form

$$\eta_{hjk} \eta_{klm} \partial_j \partial_l Q_{im}(\underline{r}).$$

Since

$$\eta_{ijk} \eta_{klm} \partial_j \partial_l Q_{nm}(\underline{r}) \equiv \{\nabla_x \nabla_x Q^T(\underline{r})\}_{in}$$

then it follows that

$$\eta_{njk} \eta_{klm} \partial_j \partial_l Q_{im}(\underline{r}) \equiv \{[\nabla_x \nabla_x Q^T(\underline{r})]^T\}_{in} \quad (3.2.22)$$

Interestingly, the right side of 3.2.22 can be written more simply in the form $\{Q(\underline{r}) \times \nabla_x \nabla\}_{in}$. However, convention demands that differential operators be written such that they operate from the left rather than from the right. Therefore, the more complicated form given in 3.2.22 is adopted here.

The vector form of the wave equation is obtained from 3.2.19 by means of 3.2.21 and 3.2.22. Rewriting 3.2.19 directly in vector notation yields

$$\begin{aligned} & \nabla_1 \times \nabla_1 \times \{[\nabla_2 \times \nabla_2 \times \hat{\underline{C}}^T(\underline{r}_1, \underline{r}_2, \omega)]^T\} - k^2 \nabla_1 \times \nabla_1 \times \hat{\underline{C}}(\underline{r}_1, \underline{r}_2, \omega) - \\ & - k^2 [\nabla_2 \times \nabla_2 \times \hat{\underline{C}}^T(\underline{r}_1, \underline{r}_2, \omega)]^T + k^4 \hat{\underline{C}}(\underline{r}_1, \underline{r}_2, \omega) = \mu_0^2 \omega^2 \hat{\underline{Q}}(\underline{r}_1, \underline{r}_2, \omega). \end{aligned} \quad (3.2.23)$$

where ∇_1 and ∇_2 operate on the position variables \underline{r}_1 and \underline{r}_2 , respectively. The final result is obtained by factoring the left side of 3.2.23.

It is found to be

$$(\nabla_1 \times \nabla_1 \times - k^2 \underline{\underline{I}}) \{[\nabla_2 \times \nabla_2 \times - k^2 \underline{\underline{I}}] \hat{\underline{C}}^T(\underline{r}_1, \underline{r}_2, \omega)\} = \mu_0^2 \omega^2 \hat{\underline{Q}}(\underline{r}_1, \underline{r}_2, \omega) \quad (3.2.24)$$

The symbol $\underline{\underline{I}}$ represents the unit dyadic.

The symmetry of this equation is obscured by the cumbersome notation introduced for the sake of convention. To illustrate this symmetry (which shows itself quite clearly in the integrated form of the differential equations), it is worthwhile to consider the unconventional notation mentioned briefly above. In this notation 3.25 becomes

$$(\nabla_1 \times \nabla_1 \times - k^2 \underline{\underline{I}}) [\hat{\underline{\underline{E}}}(\underline{r}_1, \underline{r}_2, \omega)] (\underline{\underline{I}} k^2 - \nabla_2 \times \nabla_2) = -\mu_0^2 \omega^2 \hat{\underline{\underline{Q}}}(\underline{r}_1, \underline{r}_2, \omega) \quad (3.2.25)$$

where the operator in the parenthesis on the right operates on $\hat{\underline{\underline{E}}}(\underline{r}_1, \underline{r}_2, \omega)$ from the right. The minus sign on the right side of 3.2.25 appears because the right-hand operator on the left of 3.2.25 has been multiplied by (-1) for notational convenience.

In this form, it is seen that $\hat{\underline{\underline{E}}}(\underline{r}_1, \underline{r}_2, \omega)$ must simultaneously satisfy two reduced vector wave equation operators. For this reason, $\hat{\underline{\underline{E}}}(\underline{r}_1, \underline{r}_2, \omega)$ is said to propagate like a wave, and equation 3.2.24 is referred to as a dyadic wave equation. The wave-like character of $\hat{\underline{\underline{E}}}(\underline{r}_1, \underline{r}_2, \omega)$ is more clearly emphasized by the integral representation of $\hat{\underline{\underline{E}}}(\underline{r}_1, \underline{r}_2, \omega)$ which is derived in the next chapter. A second interesting aspect of 3.2.25 is the symmetry it exhibits with respect to the two vector operators. This symmetry is not unexpected considering that the two observation points \underline{r}_1 and \underline{r}_2 are in every respect equivalent. Nevertheless, it is satisfying to obtain such a simply stated result.

3.3 The Dyadic Wave Equation in a Source Free Region

Equation 3.2.24 represents the dyadic wave equation in a region containing a source. Because of the presence of the source, the differential operators that relate to \underline{r}_1 and \underline{r}_2 are coupled into a single equation.

If the region of interest is free of sources, this equation separates into two parts as will now be shown.

Although the separated equations for a source free region can be derived directly from

$$(\nabla_1 \times \nabla_1 \times - k_1^2) \cdot [(\nabla_2 \times \nabla_2 \times - k_2^2) \cdot \hat{\underline{\underline{\epsilon}}}^T(\underline{r}_1, \underline{r}_2, \omega)]^T = 0, \quad (3.3.1)$$

it is more instructive to begin directly from Maxwell's equations. In a source free domain, 3.2.3 and 3.2.4 reduce to

$$\eta_{ijk} \partial_j E_k(\underline{r}, t) = -\mu_0 \frac{\partial}{\partial t} H_i(\underline{r}, t) \quad (3.3.2)$$

$$\eta_{lmi} \partial_m H_i(\underline{r}, t) = \epsilon_0 \frac{\partial}{\partial t} E_l(\underline{r}, t) \quad (3.3.3)$$

Multiply both 3.3.2 and 3.3.3, evaluated at (\underline{r}_1, t_1) , by $E_n(\underline{r}_2, t_2)$, average both, and take the curl with respect to \underline{r}_1 of the first of the two equations. This results in the equations

$$\eta_{lmi} \eta_{ijk} \partial_j^{(1)} \partial_m^{(1)} \mathcal{E}_{kn}(\underline{r}_1, \underline{r}_2, \tau) = -\mu_0 \frac{\partial}{\partial \tau} \eta_{lmi} \partial_m^{(1)} \mathcal{H}_{in}(\underline{r}_1, \underline{r}_2, \tau) \quad (3.3.4)$$

$$\eta_{lmi} \partial_m^{(1)} \mathcal{H}_{in}(\underline{r}_1, \underline{r}_2, \tau) = \epsilon_0 \frac{\partial}{\partial \tau} \mathcal{E}_{ln}(\underline{r}_1, \underline{r}_2, \tau) \quad (3.3.5)$$

Substitute 3.3.5 into 3.3.4 to obtain

$$\eta_{lmi} \eta_{ijk} \partial_j^{(1)} \partial_m^{(1)} \mathcal{E}_{kn}(\underline{r}_1, \underline{r}_2, \tau) + \frac{1}{c^2} \frac{\partial^2}{\partial \tau^2} \mathcal{E}_{ln}(\underline{r}_1, \underline{r}_2, \tau) = 0. \quad (3.3.6)$$

Take the Fourier transform of 3.3.6 and write the result in vector notation:

$$(\nabla_1 \times \nabla_1 \times - k_1^2) \cdot \hat{\underline{\underline{\epsilon}}}(\underline{r}_1, \underline{r}_2, \omega) = 0$$

This is the first of the two separate wave equations. If this procedure

is repeated starting with 3.3.2 and 3.3.3 evaluated at (\underline{r}_2, t_2) , and by multiplying through by $E_n(\underline{r}_1, t_1)$, the second wave equation is found

$$(\nabla_2 \times \nabla_2 \times - k^2) \hat{\underline{\underline{C}}}^T(\underline{r}_1, \underline{r}_2, \omega) = 0 \quad (3.3.8)$$

Since the divergence of the electric field is zero in a source free region, then it can be shown that

$$\nabla_1 \cdot \hat{\underline{\underline{C}}}(\underline{r}_1, \underline{r}_2, \omega) = \nabla_2 \cdot \hat{\underline{\underline{C}}}^T(\underline{r}_1, \underline{r}_2, \omega) = 0$$

Consequently, 3.3.7 and 3.3.8 reduce to

$$(\nabla_1^2 + k^2) \hat{\underline{\underline{C}}}(\underline{r}_1, \underline{r}_2, \omega) = 0 \quad (3.3.9)$$

$$(\nabla_2^2 + k^2) \hat{\underline{\underline{C}}}(\underline{r}_1, \underline{r}_2, \omega) = 0 \quad (3.3.10)$$

The significance of 3.3.9 and 3.3.10 is that the differential operator has been reduced to a scalar operator. Observe also that these equations are the vector analog of 3.1.1, which defines the propagation of the mutual coherence function. The foregoing analysis shows that this analogy holds only in the absence of sources. If sources are present, the two differential equations couple together and must be dealt with as a single dyadic differential equation.

3.4 The Dyadic Wave Equation in a General Transparent Medium

Up to this point, consideration has been given only to propagation in free space. To generalize this result, it is desirable to consider the propagation law of the DFS in a material medium. Suppose the medium is characterized by a dyadic electric susceptibility of the type that is usually employed for a cold plasma biased by a constant magnetic

field. In this case, the presence of the medium is introduced into Maxwell's equations by means of the constitutive equation

$$\hat{\underline{D}}(\underline{r}, \omega) = \epsilon_0 \hat{\underline{\chi}}(\omega) \cdot \hat{\underline{E}}(\underline{r}, \omega) \quad (3.4.1)$$

where the $\hat{\underline{E}}(\underline{r}, \omega)$ represents the amplitude and phase of a time harmonic field. The quantity $\hat{\underline{\chi}}(\omega)$ represents the dyadic dielectric susceptibility at frequency ω . The more general form of this equation is written

$$\underline{D}(\underline{r}, t) = \epsilon_0 \int_0^\infty \underline{\chi}(t') \cdot \underline{E}(\underline{r}, t-t') dt' \quad (3.4.2)$$

where the field quantities $\underline{E}(\underline{r}, t)$ and $\underline{D}(\underline{r}, t)$ are no longer restricted to be harmonic functions of time. The quantity $\underline{\chi}(t)$ is the Fourier transform of $\hat{\underline{\chi}}(\omega)$. It represents the response of the medium to an impulsive electric field. The mathematical relationship between $\hat{\underline{\chi}}(\omega)$ and $\underline{\chi}(t)$ is given by

$$\underline{\chi}(t) = \int_{-\infty}^{\infty} \hat{\underline{\chi}}(\omega) e^{-i\omega t} d\omega \quad (3.4.3)$$

Correspondingly,

$$\hat{\underline{\chi}}(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \underline{\chi}(t) e^{i\omega t} dt \quad (3.4.4)$$

Since 3.4.2 is valid for stationary noise-like fields as well as for fields that have well defined Fourier transforms, it is the natural starting point for the analysis of the propagation of the DFS in a dispersive anisotropic medium.

The field equations in such a medium may be written in the form

$$\eta_{ijk} \partial_j E_k(\underline{r}, t) = -\mu_0 \frac{\partial}{\partial t} H_i(\underline{r}, t) \quad (3.4.5)$$

$$\eta_{ijk} \partial_j H_k(\underline{r}, t) - \frac{\partial D_i(\underline{r}, t)}{\partial t} = 0 \quad (3.4.6)$$

$$D_i(\underline{r}, t) = \epsilon_0 \int_0^\infty \chi_{ij}(t') E_j(\underline{r}, t - t') dt' \quad (3.4.7)$$

Employing the procedure used in Section 3.2, one obtains from 3.4.5 as before the equation

$$\eta_{ijk} \eta_{lmn} \partial_j^{(1)} \partial_m^{(2)} \mathcal{E}_{kn}(\underline{r}_1, \underline{r}_2, \tau) = -\mu_0^2 \frac{\partial^2}{\partial \tau^2} \mathcal{H}_{il}(\underline{r}_1, \underline{r}_2, \tau) \quad (3.2.9)$$

However, when this procedure is applied to 3.4.6, a new equation results, which involves $D_i(\underline{r}, t)$ in place of $\epsilon_0 E_i(\underline{r}, t)$. Thus, by properly modifying 3.2.11, it is found that

$$\begin{aligned} & \eta_{ijk} \eta_{lmn} \partial_j^{(1)} \partial_m^{(2)} \mathcal{H}_{kn}(\underline{r}_1, \underline{r}_2, \tau) + \eta_{ijk} \partial_j^{(1)} \frac{\partial}{\partial \tau} \mathcal{F}_{kl}(\underline{r}_1, \underline{r}_2, \tau) - \\ & - \eta_{lmn} \partial_m^{(2)} \frac{\partial}{\partial \tau} \mathcal{F}_{ni}(\underline{r}_2, \underline{r}_1, -\tau) - \frac{\partial^2}{\partial \tau^2} \mathcal{D}_{il}(\underline{r}_1, \underline{r}_2, \tau) = 0 \end{aligned} \quad (3.4.8)$$

where

$$\mathcal{F}_{ij}(\underline{r}_1, \underline{r}_2, \tau) = \overline{H_i(\underline{r}_1, t_1) D_j(\underline{r}_2, t_2)} \quad (3.4.9)$$

$$\mathcal{D}_{ij}(\underline{r}_1, \underline{r}_2, \tau) = \overline{D_i(\underline{r}_1, t_1) D_j(\underline{r}_2, t_2)} \quad (3.4.10)$$

The dyadics $\underline{\mathcal{F}}$ and $\underline{\mathcal{D}}$ can be related to $\underline{\mathcal{H}}$ and $\underline{\mathcal{E}}$, respectively, by means of 3.4.7. Consider $\underline{\mathcal{F}}$ first.

$$\mathcal{F}_{ij}(\underline{r}_1, \underline{r}_2, \tau) = \epsilon_0 \left[\overline{H_i(\underline{r}_1, t_1) \int_0^\infty \chi_{jk}(t') E_k(\underline{r}_2, t_2 - t') dt'} \right] \quad (3.4.11)$$

It is assumed that the properties of the medium are fixed with respect to time. Therefore, the medium is deterministic, and the orders of averaging and integration can be interchanged. This yields the result

$$\tilde{F}_{ij}(\underline{r}_1, \underline{r}_2, \tau) = \epsilon_0 \int_0^\infty \chi_{jk}(t') \mathcal{D}_{ik}(\underline{r}_1, \underline{r}_2, t' + \tau) dt' \quad (3.4.12)$$

Correspondingly,

$$\tilde{F}_{ij}(\underline{r}_2, \underline{r}_1, -\tau) = \epsilon_0 \int_0^\infty \chi_{jk}(t') \mathcal{D}_{ik}(\underline{r}_2, \underline{r}_1, t' - \tau) dt' \quad (3.4.13)$$

The equivalent result for $\mathcal{D}_{ij}(\underline{r}_1, \underline{r}_2, \tau)$, found in the same manner, is

$$\mathcal{D}_{ij}(\underline{r}_1, \underline{r}_2, \tau) = \epsilon_0^2 \int_0^\infty \int_0^\infty \chi_{ik}(t') \chi_{jm}(t'') \mathcal{E}_{km}(\underline{r}_1, \underline{r}_2, \tau - t' + t'') dt' dt'' \quad (3.4.14)$$

These results, when substituted into 3.4.4, yield the equation

$$\begin{aligned} & \eta_{ijk} \eta_{lmn} \frac{\partial^{(1)}}{\partial j} \frac{\partial^{(2)}}{\partial m} \mathcal{H}_{kn}(\underline{r}_1, \underline{r}_2, \tau) + \epsilon_0 \eta_{ijk} \frac{\partial^{(1)}}{\partial j} \frac{\partial}{\partial \tau} \int_0^\infty \chi_{lp}(t') \mathcal{D}_{kp}(\underline{r}_1, \underline{r}_2, t' + \tau) dt' - \\ & - \epsilon_0 \eta_{lmn} \frac{\partial^{(1)}}{\partial m} \frac{\partial}{\partial \tau} \int_0^\infty \chi_{ip}(t') \mathcal{D}_{np}(\underline{r}_2, \underline{r}_1, t' - \tau) dt' - \\ & - \epsilon_0^2 \frac{\partial^2}{\partial \tau^2} \int_0^\infty \int_0^\infty \chi_{ip}(t') \chi_{lq}(t'') \mathcal{E}_{pq}(\underline{r}_1, \underline{r}_2, \tau - t' + t'') dt' dt'' = 0 \quad (3.4.15) \end{aligned}$$

The quantities $\mathcal{H}_{kn}(\underline{r}_1, \underline{r}_2, \tau)$, $\mathcal{D}_{km}(\underline{r}_1, \underline{r}_2, t' + \tau)$, and $\mathcal{D}_{nk}(\underline{r}_2, \underline{r}_1, t' - \tau)$ are eliminated by means of 3.2.9, 3.2.15, and 3.2.17. The resulting expression is

$$\begin{aligned}
 & \eta_{ijk} \eta_{lmn} \eta_{kab} \eta_{ncd} \partial_j^{(1)} \partial_a^{(1)} \partial_m^{(2)} \partial_c^{(2)} \mathcal{E}_{bd}(\underline{r}_1, \underline{r}_2, \tau) + \\
 & + \frac{1}{c^2} \eta_{ijk} \eta_{kab} \partial_j^{(1)} \partial_a^{(1)} \frac{\partial^2}{\partial \tau^2} \int_0^\infty \chi_{lp}(t') \mathcal{E}_{bp}(\underline{r}_1, \underline{r}_2, t' + \tau) dt' + \\
 & + \frac{1}{c^2} \eta_{lmn} \eta_{ncd} \partial_m^{(2)} \partial_c^{(2)} \frac{\partial^2}{\partial \tau^2} \int_0^\infty \chi_{ip}(t') \mathcal{E}_{pd}(\underline{r}_1, \underline{r}_2, \tau - t') dt' + \\
 & + \frac{1}{c^4} \frac{\partial^4}{\partial \tau^4} \int_0^\infty \int_0^\infty \chi_{ip}(t') \chi_{lq}(t'') \mathcal{E}_{pq}(\underline{r}_1, \underline{r}_2, \tau - t' + t'') dt' dt'' = 0
 \end{aligned}
 \tag{3.4.16}$$

The Fourier transform of this equation provides the wave equation for the DFS in the medium

$$\begin{aligned}
 & \eta_{ijk} \eta_{lmn} \eta_{kab} \eta_{ncd} \partial_j^{(1)} \partial_a^{(1)} \partial_m^{(2)} \partial_c^{(2)} \hat{\mathcal{E}}_{bd}(\underline{r}_1, \underline{r}_2, \omega) - \\
 & - k^2 \hat{\chi}_{lp}^*(\omega) \eta_{ijk} \eta_{kab} \partial_j^{(1)} \partial_a^{(1)} \hat{\mathcal{E}}_{bp}(\underline{r}_1, \underline{r}_2, \omega) - \\
 & - k^2 \hat{\chi}_{ip}(\omega) \eta_{lmn} \eta_{ncd} \partial_m^{(2)} \partial_c^{(2)} \hat{\mathcal{E}}_{pd}(\underline{r}_1, \underline{r}_2, \omega) + \\
 & + k^4 \hat{\chi}_{ip}(\omega) \hat{\chi}_{lq}^*(\omega) \hat{\mathcal{E}}_{pq}(\underline{r}_1, \underline{r}_2, \omega) = 0
 \end{aligned}
 \tag{3.4.17}$$

This equation can be expressed in vector notation by making use of 3.2.21 and 3.2.22.

$$[\nabla_1 \times \nabla_1 \times - k^2 \hat{\underline{\chi}}(\omega)] \cdot [[\nabla_2 \times \nabla_2 \times - k^2 \hat{\underline{\chi}}^*(\omega)] \cdot \hat{\underline{\mathcal{E}}}^T(\underline{r}_1, \underline{r}_2, \omega)]^T = 0 \tag{3.4.18}$$

It is possible to show, starting from 3.4.5 and 3.4.6 that

$$\nabla_1 \times \nabla_1 \times \hat{\underline{\underline{C}}}(\underline{r}_1, \underline{r}_2, \omega) - k^2 \hat{\underline{\underline{\chi}}}(\omega) \cdot \hat{\underline{\underline{C}}}(\underline{r}_1, \underline{r}_2, \omega) = 0 \quad (3.4.19)$$

$$\nabla_2 \times \nabla_2 \times \hat{\underline{\underline{C}}}^T(\underline{r}_1, \underline{r}_2, \omega) - k^2 \hat{\underline{\underline{\chi}}}^*(\omega) \cdot \hat{\underline{\underline{C}}}^T(\underline{r}_1, \underline{r}_2, \omega) = 0 \quad (3.4.20)$$

Thus, as in free space, the dyadic wave equation in a material medium factors into two conventional wave equations if there are no sources present in the medium.

It can be shown by straightforward calculation that if in addition to being anisotropic and dispersive the medium is also inhomogeneous, then 3.4.18 takes the form

$$[\nabla_1 \times \nabla_1 \times - k^2 \hat{\underline{\underline{\chi}}}(\underline{r}_1, \omega)] \{ [\nabla_2 \times \nabla_2 \times - k^2 \hat{\underline{\underline{\chi}}}^*(\underline{r}_2, \omega)] \hat{\underline{\underline{C}}}^T(\underline{r}_1, \underline{r}_2, \omega) \}^T = 0, \quad (3.4.21)$$

where $\hat{\underline{\underline{\chi}}}(\underline{r}, \omega)$ is the dyadic electric susceptibility at point \underline{r} and frequency ω . As before, this equation factors into the two ordinary wave equations

$$\nabla_1 \times \nabla_1 \times \hat{\underline{\underline{C}}}(\underline{r}_1, \underline{r}_2, \omega) - k^2 \hat{\underline{\underline{\chi}}}(\underline{r}_1, \omega) \cdot \hat{\underline{\underline{C}}}(\underline{r}_1, \underline{r}_2, \omega) = 0 \quad (3.4.22)$$

$$\nabla_2 \times \nabla_2 \times \hat{\underline{\underline{C}}}^T(\underline{r}_1, \underline{r}_2, \omega) - k^2 \hat{\underline{\underline{\chi}}}^*(\underline{r}_2, \omega) \cdot \hat{\underline{\underline{C}}}^T(\underline{r}_1, \underline{r}_2, \omega) = 0 \quad (3.4.23)$$

This completes the derivation of the differential equations satisfied by the DFS. It should be noted that the above results by no means exhaust the subject of the differential equations for noise-like fields. Wolf and Roman (11) - (14) derive a number of other equations not discussed here. However, the point of view adopted in this dissertation regards the DFS $\hat{\underline{\underline{C}}}(\underline{r}_1, \underline{r}_2, \omega)$ as the primary field variable for a noise-like field.

Consequently, the objective here is to determine the mathematical theory of the DFS alone. As a result, the extension of the general set of differential equations that describe the second order moments of $\underline{H}(\underline{r}, t)$ and the mixed moments of $\underline{E}(\underline{r}, t)$ and $\underline{H}(\underline{r}, t)$ to broad-band fields is not undertaken here. The basis for this viewpoint lies in the fact that most measurements of radiation fields take place in the far zone of the source where power is expressible directly in terms of one of the field vectors alone. Therefore, it is sufficient to develop a theory that determines the power spectrum of one or the other of the field vectors in terms of the characteristics of the source and the various parameters of the surrounding system. However, the more elaborate theory--one that involves all of the second moments--can be derived by a simple application of the methods and concepts introduced here.

3.5 The Boundary Conditions

The usual boundary conditions imposed on an electromagnetic field at a dielectric interface are

$$\underline{m} \times [\underline{E}^{\text{out}}(\underline{r}, t) - \underline{E}^{\text{in}}(\underline{r}, t)] \big|_S = 0 \quad (3.5.1)$$

$$\underline{m} \times [\underline{H}^{\text{out}}(\underline{r}, t) - \underline{H}^{\text{in}}(\underline{r}, t)] \big|_S = 0 \quad (3.5.2)$$

where the superscript "out" refers to one side of the boundary S and "in" refers to the other side of the boundary. The vector \underline{m} is the unit vector normal to the surface S at the point \underline{r} , pointing from the inside to the outside of the surface S (see Figure 3.1). In tensor notation, 3.5.1 and 3.5.2 become

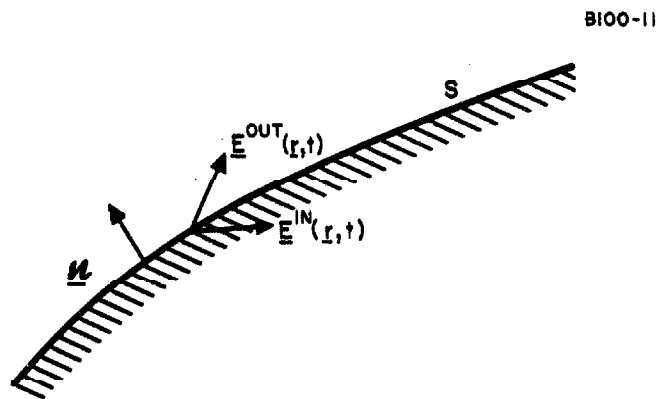


Figure 3.1. Configuration at a dielectric interface.

$$\eta_{ijk} m_j [E_k^{\text{out}}(\underline{r}, t) - E_k^{\text{in}}(\underline{r}, t)]|_S = 0 \quad (3.5.3)$$

$$\eta_{ijk} m_j [H_k^{\text{out}}(\underline{r}, t) - H_k^{\text{in}}(\underline{r}, t)]|_S = 0 \quad (3.5.4)$$

If the term that involves $E_k^{\text{in}}(\underline{r}, t)$ in 3.5.3 is transposed to the right side, and if the resulting equation, evaluated at (\underline{r}_1, t_1) is multiplied by itself, evaluated at (\underline{r}_2, t_2) , then it is found that

$$\eta_{ijk} \eta_{lmn} m_j^{(1)} m_m^{(2)} [E_k^{\text{out}}(\underline{r}_1, t_1) E_n^{\text{out}}(\underline{r}_2, t_2) - E_k^{\text{in}}(\underline{r}_1, t_1) E_n^{\text{in}}(\underline{r}_2, t_2)]|_S = 0 \quad (3.5.5)$$

where the superscripts (1) and (2) on the unit vector \underline{m} refer to the points \underline{r}_1 , and \underline{r}_2 on S, respectively. If the ensemble average of 3.5.5 is expressed in vector notation, the following boundary condition is obtained

$$\underline{m}^{(1)} \times \{ \underline{m}^{(2)} \times [\underline{\underline{\mathcal{E}}}^{\text{out}}(\underline{r}_1, \underline{r}_2, \tau) - \underline{\underline{\mathcal{E}}}^{\text{in}}(\underline{r}_1, \underline{r}_2, \tau)]^T \}^T|_{\underline{r}_1, \underline{r}_2 \text{ on } S} = 0 \quad (3.5.6)$$

The corresponding equation expressed in terms of the DFS is found by taking the Fourier transform of 3.5.6

$$\underline{m}^{(1)} \times \{ \underline{m}^{(2)} \times [\underline{\underline{\hat{\mathcal{E}}}}^{\text{out}}(\underline{r}_1, \underline{r}_2, \omega) - \underline{\underline{\hat{\mathcal{E}}}}^{\text{in}}(\underline{r}_1, \underline{r}_2, \omega)]^T \}^T|_{\underline{r}_1, \underline{r}_2 \text{ on } S} = 0 \quad (3.5.7)$$

By repeating this procedure with 3.5.4 and making use of 3.2.9, it is found that

$$\underline{m}^{(1)} \times \nabla_1 \times \{ \underline{m}^{(2)} \times \nabla_2 \times [\underline{\underline{\hat{\mathcal{E}}}}^{\text{out}}(\underline{r}_1, \underline{r}_2, \omega) - \underline{\underline{\hat{\mathcal{E}}}}^{\text{in}}(\underline{r}_1, \underline{r}_2, \omega)]^T \}^T|_{\underline{r}_1, \underline{r}_2 \text{ on } S} = 0 \quad (3.5.8)$$

The remaining two boundary conditions are obtained by multiplying 3.5.3 and 3.5.4 together--first with 3.5.3 evaluated at (\underline{r}_1, t_1) and 3.5.4 at (\underline{r}_2, t_2) , then with 3.5.3 evaluated at (\underline{r}_2, t_2) and 3.5.4 at (\underline{r}_1, t_1) . By using 3.2.15 and 3.2.17, these hybrid boundary conditions are found to be

$$\underline{m}^{(1)} \times \underline{\nabla}_1 \times \{ \underline{m}^{(2)} \times [\hat{\underline{C}}^{\text{out}}(\underline{r}_1, \underline{r}_2, \omega) - \hat{\underline{C}}^{\text{in}}(\underline{r}_1, \underline{r}_2, \omega)]^T \}^T \big|_{\underline{r}_1, \underline{r}_2 \text{ on } S} = 0 \quad (3.5.9)$$

$$\underline{m}^{(1)} \times \{ \underline{m}^{(2)} \times \underline{\nabla}_2 \times [\hat{\underline{C}}^{\text{out}}(\underline{r}_1, \underline{r}_2, \omega) - \hat{\underline{C}}^{\text{in}}(\underline{r}_1, \underline{r}_2, \omega)]^T \}^T \big|_{\underline{r}_1, \underline{r}_2 \text{ on } S} = 0 \quad (3.5.10)$$

Equations 3.5.7 - 3.5.10 constitute the complete set of boundary conditions which $\hat{\underline{C}}(\underline{r}_1, \underline{r}_2, \omega)$ must satisfy at a dielectric interface (it has been assumed that $\mu_{\text{in}} = \mu_{\text{out}} = \mu_0$). Observe that the set includes four independent conditions, rather than just the two equations 3.5.1 and 3.5.2 that apply to vector problems.

The boundary condition at a conducting interface is readily obtained from the usual vector condition:

$$\underline{m} \times \underline{E}(\underline{r}, t) \big|_{\underline{r} \text{ on } S} = 0 \quad (3.5.11)$$

Proceeding as above, one can show that at a conducting interface

$$[\underline{m}^{(1)} \times \hat{\underline{C}}(\underline{r}_1, \underline{r}_2, \omega) \times \underline{m}^{(2)}] \big|_{\underline{r}_1, \underline{r}_2 \text{ on } S} = 0 \quad (3.5.12)$$

IV. INTEGRAL REPRESENTATION OF THE DFS

In the previous chapter, the differential equations and the boundary conditions, which define the propagation of the DFS, were derived. These equations constitute a dyadic boundary value problem. Unfortunately, the author has found that because of the complexity of this set of equations the widely used separation-of-variables technique for solving electromagnetic boundary value problems is not practicable for this problem. It leads to a prohibitive amount of work when applied to even the simplest problem (e.g., the problem of a plane wave incident from an arbitrary angle on a plane dielectric interface). However, this difficulty can be circumvented if the dyadic problem can be reduced to an equivalent vector problem. It is the object of this chapter to develop this method of attack on the dyadic boundary value problem. The procedure is based on the integral representation of the DFS which will now be derived.

4.1 The Case of a Source Radiating in Free Space

In this section, the differential equation

$$(\nabla_1 \times \nabla_1 \times - k^2 \underline{\underline{I}} \cdot) \{ (\nabla_2 \times \nabla_2 \times - k^2 \underline{\underline{I}} \cdot) \hat{\underline{\underline{C}}}^T(\underline{\underline{r}}_1, \underline{\underline{r}}_2, \omega) \}^T = \mu_0^2 \omega^2 \hat{\underline{\underline{Q}}}(\underline{\underline{r}}_1, \underline{\underline{r}}_2, \omega) \quad , \quad (4.1.1)$$

which holds throughout space, will be integrated in order to obtain an expression for $\hat{\underline{\underline{C}}}(\underline{\underline{r}}_1, \underline{\underline{r}}_2, \omega)$ in terms of $\hat{\underline{\underline{Q}}}(\underline{\underline{r}}_1, \underline{\underline{r}}_2, \omega)$. It is assumed that the source is confined to a finite volume of space, V_s . Therefore, $\hat{\underline{\underline{Q}}}(\underline{\underline{r}}_1, \underline{\underline{r}}_2, \omega)$ is zero if either $\underline{\underline{r}}_1$ or $\underline{\underline{r}}_2$ lies outside this volume. Equation 4.1.1 is integrated by means of the vector analog of Green's second

identity. If $\underline{A}(\underline{r})$ and $\underline{B}(\underline{r})$ and their first and second derivatives are continuous throughout V and on the bounding surface S , it can be shown that*

$$\begin{aligned} \int_V \{ \underline{A}(\underline{r}') \cdot \nabla' \times \nabla' \times \underline{B}(\underline{r}') - \underline{B}(\underline{r}') \cdot \nabla' \times \nabla' \times \underline{A}(\underline{r}') \} dv' = \\ = \oint_S \{ \underline{A}(\underline{r}') \times \nabla' \times \underline{B}(\underline{r}') - \underline{B}(\underline{r}') \times \nabla' \times \underline{A}(\underline{r}') \} \cdot \underline{ds}' \end{aligned} \quad (4.1.2)$$

where \underline{ds}' has the direction of the outward directed normal to S . To apply 4.1.2 to the solution of 4.1.1, it is necessary to introduce the free space dyadic Green's function $\hat{\underline{\underline{\Gamma}}}(\underline{r}, \underline{r}', \omega)$, defined by the differential equation (39)

$$(\nabla' \times \nabla' \times - k^2 \underline{\underline{I}} \cdot) \hat{\underline{\underline{\Gamma}}}(\underline{r}', \underline{r}, \omega) = \underline{\underline{I}} \delta(\underline{r}' - \underline{r}) \quad , \quad (4.1.3)$$

where $\delta(\underline{r}' - \underline{r})$ represents the Dirac delta function. In order to simplify notation in the subsequent derivation, it is also useful to introduce an auxiliary dyadic field function $\hat{\underline{\underline{Q}}}(\underline{r}_1, \underline{r}_2, \omega)$, defined by the relation

$$\hat{\underline{\underline{Q}}}^T(\underline{r}_1, \underline{r}_2, \omega) = (\nabla_2 \times \nabla_2 \times - k^2 \underline{\underline{I}} \cdot) \hat{\underline{\underline{C}}}^T(\underline{r}_1, \underline{r}_2, \omega) \quad (4.1.4)$$

Therefore, it is seen from 4.1.1 that $\hat{\underline{\underline{Q}}}(\underline{r}_1, \underline{r}_2, \omega)$ satisfies the differential equation:

$$(\nabla_1 \times \nabla_1 \times - k^2 \underline{\underline{I}} \cdot) \hat{\underline{\underline{Q}}}(\underline{r}_1, \underline{r}_2, \omega) = \mu_0^2 \omega^2 \hat{\underline{\underline{Q}}}(\underline{r}_1, \underline{r}_2, \omega) \quad (4.1.5)$$

Now let

$$\underline{A}(\underline{r}_1') \equiv \hat{\underline{\underline{Q}}}(\underline{r}_1', \underline{r}_2^0, \omega) \cdot \underline{a} \quad (4.1.6)$$

$$\underline{B}(\underline{r}_1') \equiv \hat{\underline{\underline{\Gamma}}}(\underline{r}_1', \underline{r}_1, \omega) \cdot \underline{b} \quad (4.1.7)$$

* See Stratton (38), page 464.

where \underline{a} and \underline{b} are arbitrary constant vector fields, and where the superscript "o" on \underline{r}_2 is introduced to indicate that \underline{r}_2 is a fixed point for this integration. If 4.1.6 and 4.1.7 are substituted into 4.1.2, it is found that

$$\begin{aligned} \int_V \{ [\hat{\underline{Q}}(\underline{r}'_1, \underline{r}_2^o, \omega) \cdot \underline{a}] \cdot [\nabla_1 \times \nabla_1 \times \hat{\underline{P}}(\underline{r}'_1, \underline{r}_1, \omega)] \cdot \underline{b} - [\hat{\underline{P}}(\underline{r}'_1, \underline{r}_1, \omega) \cdot \underline{b}] \cdot \\ \cdot [\nabla_1 \times \nabla_1 \times \hat{\underline{Q}}(\underline{r}'_1, \underline{r}_2^o, \omega)] \cdot \underline{a} \} d\underline{v}'_1 = \\ = \oint_S [\hat{\underline{Q}}(\underline{r}'_1, \underline{r}_2^o, \omega) \cdot \underline{a}] \times [\nabla_1 \times \hat{\underline{P}}(\underline{r}'_1, \underline{r}_1, \omega)] \cdot \underline{b} - [\hat{\underline{P}}(\underline{r}'_1, \underline{r}_1, \omega) \cdot \underline{b}] \times \\ \times [\nabla_1 \times \hat{\underline{Q}}(\underline{r}'_1, \underline{r}_2^o, \omega)] \cdot \underline{a} \} \cdot d\underline{s}'_1 \end{aligned} \quad (4.1.8)$$

Since the differential equation holds throughout space, it is desirable to extend S to infinity. Therefore, it is necessary to consider the behavior of the surface integral in the limit as S approaches infinity. It is shown in Appendix C that if the electromagnetic field $\underline{E}(\underline{r}, t)$, $\underline{H}(\underline{r}, t)$ satisfies the radiation condition, then this surface integral approaches zero as S approaches infinity. Thus, 4.1.8 can be written

$$\begin{aligned} \int_{a.s.} \{ [\hat{\underline{Q}}(\underline{r}'_1, \underline{r}_2^o, \omega) \cdot \underline{a}] \cdot [\nabla_1 \times \nabla_1 \times \hat{\underline{P}}(\underline{r}'_1, \underline{r}_1, \omega)] \cdot \underline{b} - [\hat{\underline{P}}(\underline{r}'_1, \underline{r}_1, \omega) \cdot \underline{b}] \cdot \\ \cdot [\nabla_1 \times \nabla_1 \times \hat{\underline{Q}}(\underline{r}'_1, \underline{r}_2^o, \omega)] \cdot \underline{a} \} d\underline{v}'_1 = 0 \end{aligned} \quad (4.1.9)$$

where a.s. means all space. This result can be reduced further by substituting 4.1.3 and 4.1.5 into the integrand. It is easily shown that

$$[\hat{\underline{Q}}(\underline{r}'_1, \underline{r}_2^o, \omega) \cdot \underline{a}] \cdot \underline{b} = \mu_0^2 \omega^2 \int_V [\hat{\underline{J}}(\underline{r}'_1, \underline{r}_2^o, \omega) \cdot \underline{a}] \cdot [\hat{\underline{P}}(\underline{r}'_1, \underline{r}_1, \omega) \cdot \underline{b}] d\underline{v}'_1 \quad (4.1.10)$$

Since dyadics have the property that $\underline{A} \cdot \underline{\hat{A}} = \underline{\hat{A}}^T \cdot \underline{A}$, then

$$\underline{a} \cdot \underline{\hat{A}}^T(\underline{r}_1, \underline{r}_2^o, \omega) \cdot \underline{b} = \mu_o^2 \omega^2 \int_{V_s} \underline{a} \cdot \underline{\hat{G}}^T(\underline{r}'_1, \underline{r}_2^o, \omega) \cdot \underline{\hat{\Gamma}}(\underline{r}'_1, \underline{r}_1, \omega) \cdot \underline{b} \, dv'_1 \quad (4.1.11)$$

Now it is possible to remove the constant vectors \underline{a} and \underline{b} from the equation. This leaves the result

$$\underline{\hat{A}}^T(\underline{r}_1, \underline{r}_2^o, \omega) = \mu_o^2 \omega^2 \int_{V_s} \underline{\hat{G}}^T(\underline{r}'_1, \underline{r}_2^o, \omega) \cdot \underline{\hat{\Gamma}}(\underline{r}'_1, \underline{r}_1, \omega) \, dv'_1 \quad (4.1.12)$$

Observe that the limits of integration have been reduced to the volume V_s , since $\underline{\hat{G}}(\underline{r}'_1, \underline{r}_2^o, \omega)$ is zero if \underline{r}'_1 is outside V_s .

The final result is obtained by substituting 4.1.12 into 4.1.4. If \underline{r}_1 is now regarded to be the fixed point and \underline{r}_2 the variable point, this substitution yields the equation

$$(\nabla_2 \times \nabla_2' \times -k^2 \underline{I}) \cdot \underline{\hat{E}}^T(\underline{r}_1^o, \underline{r}_2', \omega) = \mu_o^2 \omega^2 \int_{V_s} \underline{\hat{G}}^T(\underline{r}'_1, \underline{r}_2', \omega) \cdot \underline{\hat{\Gamma}}(\underline{r}'_1, \underline{r}_1^o) \, dv'_1 \quad (4.1.13)$$

This equation has the same form as 4.1.5. Therefore, it can be integrated by the procedure described above. However, it is important to note that there is one important difference between these two equations; 4.1.13 involves the transpose of the DFS, whereas 4.1.5 involves the DFS itself. The consequence of this distinction can be made more apparent by recalling from 1.3.9 that

$$\underline{\hat{E}}^T(\underline{r}_1^o, \underline{r}_2', \omega) = \underline{\hat{E}}^*(\underline{r}_2', \underline{r}_1^o, \omega) \quad ,$$

i.e., 4.1.13 involves the complex conjugate of the DFS. Because of this, it is necessary to use the complex conjugate of the dyadic Green's function, $\underline{\hat{\Gamma}}^*(\underline{r}'_2, \underline{r}_2, \omega)$, to insure that the surface integral approaches zero as S approaches infinity (see Appendix C). In other words, it is necessary to

use $\underline{\underline{\hat{\Gamma}}}^*(\underline{r}', \underline{r}, \omega)$ for the integration of 4.1.13 to insure that $\underline{\underline{\hat{\Gamma}}}(\underline{r}_1, \underline{r}_2, \omega)$ represents the DFS of an electromagnetic field that behaves like an outgoing wave at infinity. With these considerations in mind, let

$$\underline{A}(\underline{r}'_2) = \underline{\underline{\hat{\Gamma}}}^T(\underline{r}_1^o, \underline{r}'_2, \omega) \cdot \underline{a} \quad (4.1.14)$$

$$\underline{B}(\underline{r}'_2) = \underline{\underline{\hat{\Gamma}}}^*(\underline{r}'_2, \underline{r}_2, \omega) \cdot \underline{b} \quad (4.1.15)$$

If these expressions are substituted into 4.1.2 and if V is extended to include all space, then it is found that

$$\int_{a.s.} \{ [\underline{\underline{\hat{\Gamma}}}^T(\underline{r}_1^o, \underline{r}'_2, \omega) \cdot \underline{a}] \cdot [\nabla'_2 \times \nabla'_2 \times \underline{\underline{\hat{\Gamma}}}^*(\underline{r}'_2, \underline{r}_2, \omega)] \cdot \underline{b} - [\underline{\underline{\hat{\Gamma}}}^*(\underline{r}'_2, \underline{r}_2, \omega) \cdot \underline{b}] \cdot [\nabla'_2 \times \nabla'_2 \times \underline{\underline{\hat{\Gamma}}}^T(\underline{r}_1^o, \underline{r}'_2, \omega)] \cdot \underline{a} \} dv'_2 = 0 \quad (4.1.16)$$

where the surface integral has been neglected, since it approaches zero as S approaches infinity. Substituting from 4.1.13 and from the complex conjugate of 4.1.3, evaluated at $(\underline{r}'_2, \underline{r}_2)$, one obtains the result

$$[\underline{\underline{\hat{\Gamma}}}^T(\underline{r}_1^o, \underline{r}_2, \omega) \cdot \underline{a}] \cdot \underline{b} = \mu_o^2 \omega^2 \int_V \left\{ \left[\int_V \underline{\underline{\hat{J}}}^T(\underline{r}'_1, \underline{r}'_2, \omega) \cdot \underline{\underline{\hat{\Gamma}}}(\underline{r}'_1, \underline{r}_1^o) dv'_1 \right] \cdot \underline{a} \right\} \cdot [\underline{\underline{\hat{\Gamma}}}^*(\underline{r}'_2, \underline{r}_2, \omega)] \cdot \underline{b} dv'_2 \quad (4.1.17)$$

The final result is obtained by eliminating \underline{a} and \underline{b} as before:

$$\underline{\underline{\hat{\Gamma}}}(\underline{r}_1, \underline{r}_2, \omega) = \mu_o^2 \omega^2 \int_V \int_V \underline{\underline{\hat{\Gamma}}}^T(\underline{r}'_1, \underline{r}_1, \omega) \cdot \underline{\underline{\hat{J}}}(\underline{r}'_1, \underline{r}'_2, \omega) \cdot \underline{\underline{\hat{\Gamma}}}^*(\underline{r}'_2, \underline{r}_2, \omega) dv'_1 dv'_2 \quad (4.1.18)$$

Equation 4.1.18 provides the formal representation for the DFS in terms of the source distribution at the frequency ω for a source radiating in

free space. In the next section, this formula is developed further and a far-zone expression is derived. Before proceeding, it is worthwhile to note the symmetry inherent in the representation 4.1.18. The two Green's functions $\hat{\underline{\underline{\Gamma}}}(\underline{r}', \underline{r}_1, \omega)$ and $\hat{\underline{\underline{\Gamma}}}(\underline{r}', \underline{r}_2, \omega)$ enter the integrand in completely equivalent manners. As mentioned above, this reflects the equivalence of the field points \underline{r}_1 and \underline{r}_2 . However, 4.1.18 presents a more direct illustration of this equivalence than was available from the differential equation.

4.2 The Vector Analog of the van Cittert-Zernike Theorem

Equation 4.1.18 can be expanded by using the explicit formula for the free space dyadic Green's function (39):

$$\hat{\underline{\underline{\Gamma}}}(\underline{r}', \underline{r}, \omega) = \frac{1}{4\pi} \left(\underline{\underline{I}} + \frac{1}{k^2} \nabla \nabla \right) \frac{e^{ikR}}{R}, \quad R = |\underline{r}' - \underline{r}|. \quad (4.2.1)$$

where, in this case, \underline{r} represents the source point and \underline{r}' the observation point. By carrying out the differentiation, it can be shown that

$$\hat{\Gamma}_{ij}(\underline{r}', \underline{r}, \omega) = \frac{1}{4\pi} \left\{ \left[1 - \frac{1}{k^2 R^2} + \frac{i}{kR} \right] \delta_{ij} - \left[1 - \frac{3}{k^2 R^2} + \frac{3i}{kR} \right] \frac{(x'_i - x_i)(x'_j - x_j)}{R^2} \right\} \frac{e^{ikR}}{R} \quad (4.2.2)$$

In vector notation, this has the form

$$\hat{\underline{\underline{\Gamma}}}(\underline{r}', \underline{r}, \omega) = \frac{1}{4\pi} \left\{ \left[1 - \frac{1}{k^2 R^2} + \frac{i}{kR} \right] \underline{\underline{I}} - \left[1 - \frac{3}{k^2 R^2} + \frac{3i}{kR} \right] \underline{e}_R \underline{e}_R \right\} \frac{e^{ikR}}{R} \quad (4.2.3)$$

where

$$\underline{e}_R = \frac{\underline{r}' - \underline{r}}{R} \quad (4.2.4)$$

Thus formally

$$\begin{aligned} \hat{\underline{\underline{C}}}(\underline{r}_1, \underline{r}_2, \omega) = & \left(\sqrt{\frac{\mu_0}{\epsilon_0}} \frac{k}{4\pi} \right)^2 \int_{V_s} \int_{V_s} \{ [\alpha(R_{11}) \underline{I} - \beta(R_{11}) \underline{e}_{R_{11}} \underline{e}_{R_{11}}] \cdot \hat{\underline{\underline{G}}}(\underline{r}_1, \underline{r}_2, \omega) \cdot \\ & [\alpha^*(R_{22}) \underline{I} - \beta^*(R_{22}) \underline{e}_{R_{22}} \underline{e}_{R_{22}}] \} \frac{e^{ik(R_{11} - R_{22})}}{R_{11} R_{22}} \cdot \\ & dv'_1 dv'_2 \end{aligned} \quad (4.2.5)$$

where

$$R_{ii} = |\underline{r}'_i - \underline{r}_i| \quad (\text{see Figure 4.1}),$$

$$\alpha(R_{ii}) = \left[1 - \frac{1}{k^2 R_{ii}^2} + \frac{i}{k R_{ii}} \right] \quad (4.2.6)$$

$$\beta(R_{ii}) = \left[1 - \frac{3}{k^2 R_{ii}^2} + \frac{3i}{k R_{ii}} \right] \quad i = 1, 2 \quad (4.2.7)$$

If the source is incoherent, the DSS can be written

$$\hat{\underline{\underline{G}}}(\underline{r}'_1, \underline{r}'_2, \omega) = \hat{\underline{\underline{G}}}(\underline{r}'_1, \omega) \delta(\underline{r}'_2 - \underline{r}'_1) \quad (4.2.8)$$

Thus, in this case

$$\begin{aligned} \hat{\underline{\underline{C}}}(\underline{r}_1, \underline{r}_2, \omega) = & \left(\sqrt{\frac{\mu_0}{\epsilon_0}} \frac{k}{4\pi} \right)^2 \int_{V_s} [\alpha(R_1) \underline{I} - \beta(R_1) \underline{e}_{R_1} \underline{e}_{R_1}] \cdot \hat{\underline{\underline{G}}}(\underline{r}', \omega) \cdot \\ & [\alpha^*(R_1) \underline{I} - \beta^*(R_1) \underline{e}_{R_2} \underline{e}_{R_2}] \frac{e^{ik(R_1 - R_2)}}{R_1 R_2} dv' \end{aligned} \quad (4.2.9)$$

where

$$R_i = |\underline{r}_i - \underline{r}'|$$

B100-10

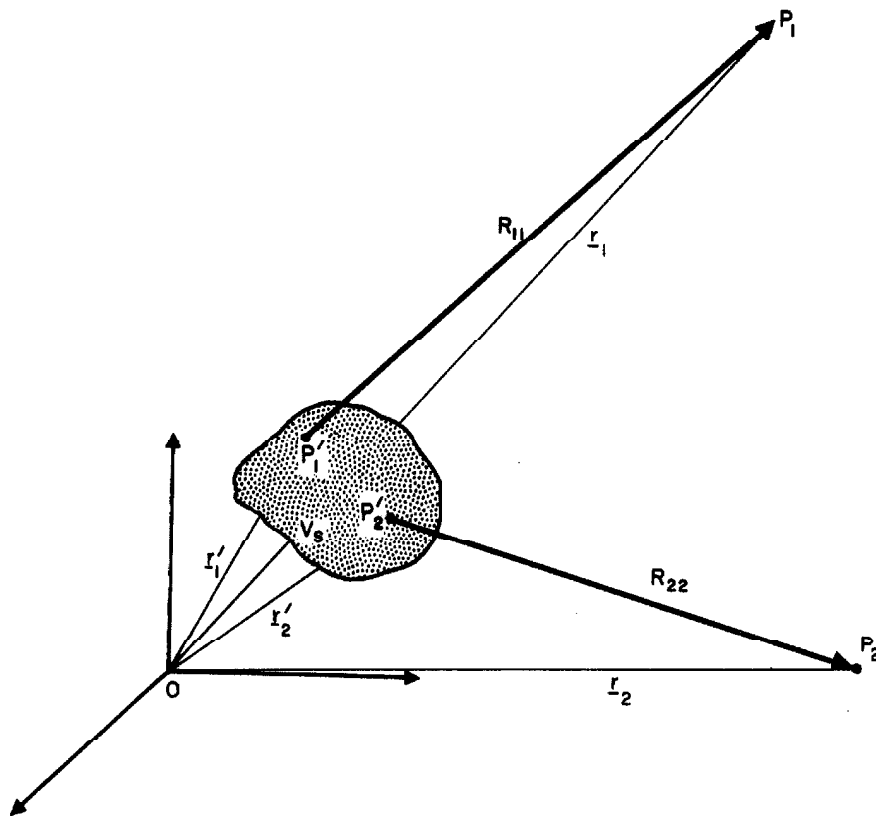


Figure 4.1. Geometry for a source radiating in free space.

Equation 4.2.9 represents the vector analog of the van Cittert-Zernike theorem for a field of arbitrary spectral width. It determines the mutual coherence between the various field components at point \underline{r}_1 and those at \underline{r}_2 at the frequency ω in terms of the source of the radiation field.

Equation 4.2.9 is of significant academic interest since it represents the exact propagation law for the DFS of a stationary noise-like field. However, in its present complicated form, it is of little practical interest. Since the physically meaningful properties of the radiation field are usually measured in the far zone of the source, it is desirable to simplify 4.2.9 by calculating the far-zone representation of the DFS. In performing this calculation, the assumption is made that

$$\begin{aligned} k R_i &\gg 1 \\ R_i &\gg a \qquad i = 1, 2 \\ R_i &\gg |\underline{r}_1 - \underline{r}_2| \end{aligned}$$

where "a" represents the maximum dimensions of the source. The geometry is depicted in Figure 4.2. The origin of coordinates is assumed to be centered in the volume V_s . The auxiliary vectors \underline{r} and $\underline{L} = \frac{1}{2}(\underline{r}_2 - \underline{r}_1)$ are used in order to introduce a local coordinate system in the neighborhood of P_1 and P_2 . If terms of order $1/kR_i$ or higher are ignored compared with 1, then

$$\frac{1}{R_i} \approx \frac{1}{r} \qquad i = 1, 2 \qquad (4.2.11)$$

where r represents the magnitude of \underline{r} . Furthermore,

$$\alpha(R_i) \approx 1 \qquad , \qquad \beta(R_i) \approx 1. \qquad (4.2.12)$$

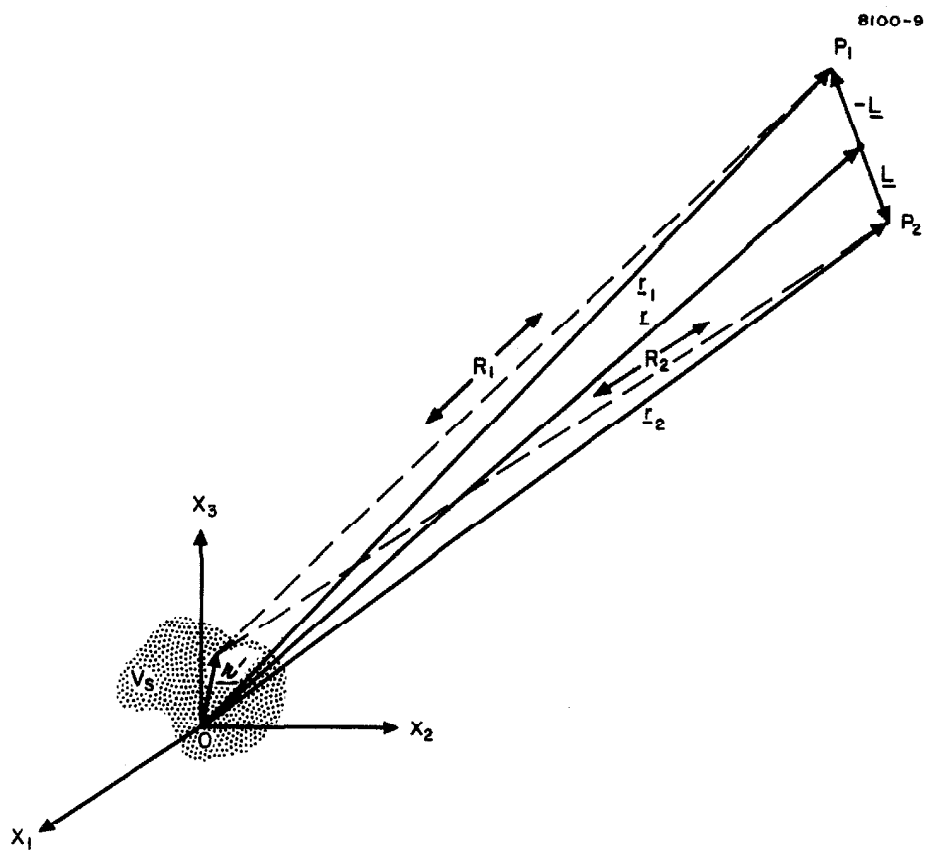


Figure 4.2. Geometry for the calculation of the far-zone DFS.

Furthermore, if terms of order $\frac{1}{kr}$ are neglected compared with 1,

$$\frac{\underline{e}_{R_i}}{R_i} = \frac{\underline{r}_i - \underline{r}'}{R_i} = \frac{\underline{r} + (-1)^i \underline{L} - \underline{r}'}{R_i} \approx \underline{e}_r + (-1)^i \frac{\underline{L}}{r} - \frac{\underline{r}'}{r} \approx \underline{e}_r \quad (4.2.13)$$

The phase function $k(R_1 - R_2)$ must be approximated more carefully, since it involves the difference of two large quantities.

$$\begin{aligned} R_i &= \sqrt{r_i^2 + r'^2 - 2 \underline{r}' \cdot \underline{r}_i} = r_i \sqrt{1 + \left(\frac{r'}{r_i}\right)^2 - 2 \frac{r'}{r_i} \cdot \underline{e}_{r_i}} \\ &\approx r_i \left[1 - \frac{\underline{r}' \cdot \underline{e}_{r_i}}{r_i} - \frac{1}{2} \frac{r'}{r_i} \cdot \underline{e}_{r_i} \times \underline{e}_{r_i} \times \left(\frac{r'}{r_i}\right) \right] \end{aligned} \quad (4.2.14)$$

But

$$\begin{aligned} r_i &= r \left[1 + \left(\frac{L}{r}\right)^2 + (-1)^i 2 \left(\frac{L}{r}\right) \cdot \underline{e}_r \right]^{1/2} \\ &\approx r \left[1 + (-1)^i \frac{L}{r} \cdot \underline{e}_r - \frac{1}{2} \frac{L}{r} \cdot \underline{e}_r \times \underline{e}_r \times \frac{L}{r} \right], \quad i = 1, 2 \end{aligned} \quad (4.2.15)$$

and

$$\begin{aligned} (r_i)^{-1} &= \frac{1}{r} \left[1 + \left(\frac{L}{r}\right)^2 + (-1)^i 2 \frac{L}{r} \cdot \underline{e}_r \right]^{-1/2} \\ &\approx \frac{1}{r} \left[1 - (-1)^i \frac{L}{r} \cdot \underline{e}_r \right], \end{aligned} \quad (4.2.16)$$

and

$$\begin{aligned} \underline{e}_{r_i} &= \frac{\underline{r} + (-1)^i \underline{L}}{r_i} \\ &\approx \underline{e}_r + (-1)^i \frac{\underline{L}}{r} - (-1)^i \frac{(\underline{L} \cdot \underline{e}_r)}{r} \underline{e}_r = \underline{e}_r - (-1)^i \underline{e}_r \times (\underline{e}_r \times \left(\frac{\underline{L}}{r}\right)) \end{aligned} \quad (4.2.17)$$

Hence, to second order in $\frac{1}{r}$, R_i becomes

$$R_i \approx r \left[1 - \frac{\underline{r}' \cdot \underline{e}_r}{r} + (-1)^i \frac{\underline{L} \cdot \underline{e}_r}{r} + (-1)^i \frac{\underline{L}}{r} \cdot \underline{e}_r \times \underline{e}_r \times \frac{\underline{r}'}{r} - \frac{1}{2} \frac{\underline{r}'}{r} \cdot \underline{e}_r \times \underline{e}_r \times \frac{\underline{r}'}{r} - \frac{1}{2} \frac{\underline{L}}{r} \cdot \underline{e}_r \times \underline{e}_r \times \frac{\underline{L}}{r} \right] \quad (4.2.18)$$

Clearly, then

$$R_1 - R_2 \approx -2 \left[\underline{L} \cdot \underline{e}_r + \underline{L} \cdot \underline{e}_r \times \underline{e}_r \times \frac{\underline{r}'}{r} \right] \quad (4.2.19)$$

The far-zone expression for $\hat{\underline{\xi}}(\underline{r}_1, \underline{r}_2, \omega)$ is obtained by substituting 4.2.11-4.2.13, and 4.2.19 into 4.2.9. The resulting formula is

$$\hat{\underline{\xi}}(\underline{r}_1, \underline{r}_2, \omega) = \left[k \sqrt{\frac{\mu_0}{\epsilon_0}} \frac{e^{-i(k\underline{L} \cdot \underline{e}_r)}}{4\pi r} \right]^2 \times (\underline{I} - \underline{e}_r \underline{e}_r) \cdot \left[\int_V \hat{\underline{\xi}}(\underline{r}', \omega) e^{-i2k\underline{L} \cdot \underline{e}_r \times \underline{e}_r \times \frac{\underline{r}'}{r}} dv' \right] \cdot (\underline{I} - \underline{e}_r \underline{e}_r) \quad (4.2.20)$$

This formula has several interesting aspects. However, before discussing them, note that \underline{e}_r is the unit vector at the origin in the direction of the midpoint of the line connecting P_1 and P_2 , and \underline{L} is the directed line segment from this midpoint to P_2 . Thus, as mentioned above, $\underline{r}_1 = \underline{r} - \underline{L}$ and $\underline{r}_2 = \underline{r} + \underline{L}$.

Since 4.2.20 represents the DFS of the far-zone field, it is not surprising that when $\underline{r}_1 = \underline{r}_2$, only those terms of $\hat{\underline{\xi}}(\underline{r}, \underline{r}, \omega)$ that are transverse to \underline{e}_r are non-zero. However, because 4.2.20 was derived under the assumption $|\underline{R}_1 - \underline{R}_2| \ll r$, the DFS is "transverse" to the direction \underline{e}_r even if $\underline{r}_1 \neq \underline{r}_2$. Thus, if 4.2.20 is expanded in a Cartesian coordinate system

oriented so that the x_3 axis coincides with \underline{e}_r , then only the terms $\hat{\epsilon}_{11}$, $\hat{\epsilon}_{12}$, $\hat{\epsilon}_{21}$, and $\hat{\epsilon}_{22}$ are non-zero.

A second property of 4.2.20 relates to the exponential factor in the integrand. If the vector product $\underline{L} \cdot \underline{e}_r \times \underline{e}_r \times \underline{r}'$ is expanded in the coordinate system just described, it is found that

$$\underline{L} \cdot \underline{e}_r \times \underline{e}_r \times \underline{r}' = (L_i \underline{e}_i) \cdot [x'_3 \underline{e}_r - x_i \underline{e}_i] = -(L_1 x'_1 + L_2 x'_2) \quad .$$

Thus, in terms of these coordinates, the integral on the right side of 4.2.20 becomes

$$\int_{V_s} \hat{\mathcal{L}}(\underline{x}'_1, \underline{x}'_2, \underline{x}'_3; \omega) e^{i \frac{2k}{r} [L_1 x'_1 + L_2 x'_2]} dx'_1 dx'_2 dx'_3$$

This indicates that the far-zone DFS is insensitive to changes of P_1 and P_2 along the direction \underline{e}_r —i.e., along the line of sight from the observation points to the source. This result is closely connected with the well known fact that an interferometer can only determine the intensity distribution of a source in the plane perpendicular to its line of sight. In fact, the above integral implies that an interferometer is only capable of determining the quantity

$$\int \hat{\mathcal{L}}(\underline{x}'_1, \underline{x}'_2, \underline{x}'_3) dx'_3 \quad ,$$

which represents the distribution of the source over the transverse plane weighted with respect to the distribution in the direction along the line of sight. This result is considered further in the next chapter in connection with the analysis of an interferometric measurement.

Next, observe that if the source is unpolarized, then so is the field. That is, if

$$\underline{\underline{\hat{\mathcal{L}}}}(\underline{r}', \omega) = \underline{\underline{I}} \hat{\mathcal{L}}(\underline{r}', \omega) \quad (4.2.21)$$

Then,

$$\underline{\underline{\hat{\mathcal{C}}}}(\underline{r}, \underline{r}, \omega) \approx \left(\frac{k}{4\pi r} \sqrt{\frac{\mu_0}{\epsilon_0}} \right)^2 \left[\int_{V_s} \hat{\mathcal{L}}(\underline{r}', \omega) dv' \right] (\underline{\underline{I}} - \underline{e}_r \underline{e}_r) \quad (4.2.22)$$

Hence, 4.2.22, expanded in the coordinate system described above, has the form

$$\underline{\underline{\hat{\mathcal{C}}}}(\underline{r}, \underline{r}, \omega) \approx \left(\frac{k}{4\pi r} \sqrt{\frac{\mu_0}{\epsilon_0}} \right)^2 \left[\int_{V_s} \hat{\mathcal{L}}(\underline{r}', \omega) dv' \right] \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad (4.2.23)$$

which is the DFS of an unpolarized TEM wave. Furthermore, since only one scalar function appears on the right side of 4.2.23, then it is possible to represent the far-zone field of an unpolarized source that radiates in free space, as a scalar field.*

Another interesting property of the far-zone DFS can be seen from 4.2.20. If $\underline{r}_1 = \underline{r}_2 = \underline{r}$, then

$$\underline{\underline{\hat{\mathcal{C}}}}(\underline{r}, \underline{r}, \omega) \approx \left(k \sqrt{\frac{\mu_0}{\epsilon_0}} \frac{1}{4\pi r} \right)^2 (\underline{\underline{I}} - \underline{e}_r \underline{e}_r) \cdot \left[\int_{V_s} \underline{\underline{\hat{\mathcal{L}}}}(\underline{r}', \omega) dv' \right] \cdot (\underline{\underline{I}} - \underline{e}_r \underline{e}_r) \quad (4.2.24)$$

* It is not clear whether the near-zone field of an unpolarized source is unpolarized. Examination of 4.2.9 for the case $\underline{r}_1 = \underline{r}_2$, indicates that under the condition 4.2.21 the dyadic $\underline{\underline{\hat{\mathcal{C}}}}(\underline{r}, \underline{r}, \omega)$ is not necessarily diagonal. However, whether or not this implies that the near-zone field is polarized cannot be decided here, since the analysis of polarization considered in this dissertation applies only to TEM fields.

This expression can be expanded in spherical coordinates centered at the origin 0 (see Figure 4.2), in which case

$$\hat{\underline{\underline{C}}}_{ij}(\underline{r}, \underline{r}, \omega) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \hat{C}_{\theta\theta}(\underline{r}, \underline{r}, \omega) & \hat{C}_{\theta\varphi}(\underline{r}, \underline{r}, \omega) \\ 0 & \hat{C}_{\varphi\theta}(\underline{r}, \underline{r}, \omega) & \hat{C}_{\varphi\varphi}(\underline{r}, \underline{r}, \omega) \end{pmatrix} \quad (4.2.25)$$

where

$$\hat{\underline{\underline{C}}}_{ij}(\underline{r}, \underline{r}, \omega) \approx \left(k \sqrt{\frac{\mu_0}{\epsilon_0}} \frac{1}{4\pi r} \right)^2 \int_{V_s} \hat{\underline{\underline{C}}}_{ij}(\underline{r}', \omega) dV' \quad (4.2.26)$$

Thus, the far-zone DFS of an incoherent source is isotropic even if the source is partially polarized. The polarization of the field is independent of direction (i.e., of θ, φ), and is equal to the sum of the polarizations of the various elements that constitute the source. This result is not true if the source is partially coherent.

Finally, observe from 4.2.26 that even if the source is completely polarized (i.e., if $\text{Det} \hat{\underline{\underline{C}}}(\underline{r}', \omega) = 0$ for all \underline{r}' in V_s), its field is not necessarily completely polarized. In fact, the field of a completely polarized incoherent source is itself completely polarized only if the source is uniformly polarized (see Appendix F). It may seem somewhat surprising to the reader who is unfamiliar with coherence theory that a completely polarized source can radiate a partially polarized field. However, if the reader recalls that $\hat{\underline{\underline{C}}}(\underline{r}, \underline{r}, \omega)$ determines the power in the small band of frequencies about ω , then it becomes more reasonable; for, then, the field can be considered as the sum of a number of harmonic waves that differ in frequency as well as in polarization. Since the polarization locus of the

field formed from the sum of two harmonic fields of different polarization and frequency varies with respect to time, then such a field is partially polarized. Clearly, then, the field formed from the sum of many such harmonic waves will be partially polarized.

The foregoing remarks complete the discussion of the DFS in free space. The remainder of this chapter is devoted to the analysis of the DFS that is radiated in the presence of perfectly conducting bodies. These results will then be applied to the analysis of an interferometer experiment in the next chapter.

4.3 The DFS of a Source Radiating in the Presence of Perfect Conductors

In the previous sections, the radiation problem for the DFS was solved formally. That is, the DFS of a source radiating in free space was expressed in terms of an integral of the source distribution. Because it is often necessary to account for the effect on the DFS of various bodies that surround the source, it is desirable to extend this analysis to the case of a source that radiates in the presence of perfectly conducting bodies. This problem is a dyadic boundary value problem. If it is recalled that the complexity of vector boundary value problems makes their solution considerably more difficult than the corresponding scalar problem, then it becomes clear that the direct solution of the dyadic boundary value problem should be circumvented if possible. Fortunately, it is possible. By representing the solution in terms of the dyadic Green's function for the given system, the problem is reduced to one of determining this Green's function. This remains a dyadic boundary value problem; however, it is a dyadic problem of a very special sort. Its special property results from the fact that the

source of the dyadic Green's function forms a diagonal matrix. As a result, the three column vectors which form the dyadic Green's function can be determined independently. Thus, the DFS can be expressed in terms of the solution of three vector problems. The effectiveness of this viewpoint is demonstrated in the next chapter where it is applied to the solution of a particular problem.

Suppose a source, characterized by the DSS $\hat{\underline{\underline{G}}}(\underline{r}_1, \underline{r}_2, \omega)$, occupies the finite volume V_s . Suppose also that this source radiates in the presence of a perfectly conducting body, which occupies a volume V .^{*} The problem is to find the DFS in the region outside V .

Although this problem can be solved by application of 4.1.2, a simpler approach is to assume, by analogy with 4.1.17, that $\hat{\underline{\underline{C}}}(\underline{r}_1, \underline{r}_2, \omega)$ can be represented in the form

$$\hat{\underline{\underline{C}}}(\underline{r}_1, \underline{r}_2, \omega) = \mu_0^2 \omega^2 \int_{V_s} \int_{V_s} \hat{\underline{\underline{G}}}^T(\underline{r}'_1, \underline{r}_1, \omega) \cdot \hat{\underline{\underline{G}}}(\underline{r}'_1, \underline{r}'_2, \omega) \cdot \hat{\underline{\underline{G}}}^*(\underline{r}'_2, \underline{r}_2, \omega) dv'_1 dv'_2 \quad (4.3.1)$$

where $\hat{\underline{\underline{G}}}$ has been substituted for $\hat{\underline{\underline{C}}}$ to distinguish the Green's function of this problem from the free space Green's function. When formulated from this point of view, the problem becomes one of defining $\hat{\underline{\underline{G}}}(\underline{r}', \underline{r}, \omega)$ such that $\hat{\underline{\underline{C}}}(\underline{r}_1, \underline{r}_2, \omega)$ satisfies the dyadic wave equation

$$(\nabla_1 \times \nabla_1 \times - k^2 \underline{\underline{I}} \cdot) \left\{ (\nabla_2 \times \nabla_2 \times - k^2 \underline{\underline{I}} \cdot) \hat{\underline{\underline{C}}}^T(\underline{r}_1, \underline{r}_2, \omega) \right\}^T = \mu_0^2 \omega^2 \hat{\underline{\underline{G}}}(\underline{r}_1, \underline{r}_2, \omega) \quad (4.3.2)$$

* The extension of this derivation to more than one conductor is straightforward.

in the region outside V, and the boundary condition

$$\underline{m}^{(1)} \times \underline{\hat{G}}(\underline{r}_1, \underline{r}_2, \omega) \times \underline{m}^{(2)} \Big|_{\underline{r}_1, \underline{r}_2 \text{ on } S} = 0 \quad (4.3.3)$$

on the surface that bounds V.

The equations that define $\underline{\hat{G}}(\underline{r}'_1, \underline{r}, \omega)$ are readily found. Substitute 4.3.1 into 4.3.2 and interchange the orders of differentiation and integration. If the operator $(\nabla \times \nabla \times - k^2 \underline{I} \cdot)$ is abbreviated by the symbol \underline{L} , then

$$\underline{L}_1 \cdot [\underline{L}_2 \int_{V_s} \int_{V_s} \underline{\hat{G}}^{*T}(\underline{r}'_2, \underline{r}_2, \omega) \cdot \underline{\hat{G}}^T(\underline{r}'_1, \underline{r}'_2, \omega) \cdot \underline{\hat{G}}(\underline{r}'_1, \underline{r}_1, \omega) dv'_1 dv'_2]^T = \underline{\hat{G}}(\underline{r}_1, \underline{r}_2, \omega), \quad (4.3.4)$$

or

$$\int_{V_s} \int_{V_s} [\underline{L}_1 \cdot \underline{\hat{G}}^T(\underline{r}'_1, \underline{r}_1, \omega)] \cdot \underline{\hat{G}}(\underline{r}'_1, \underline{r}'_2, \omega) \cdot [\underline{L}_2 \cdot \underline{\hat{G}}^{*T}(\underline{r}'_2, \underline{r}_2, \omega)]^T dv'_1 dv'_2 = \underline{\hat{G}}(\underline{r}_1, \underline{r}_2, \omega). \quad (4.3.5)$$

If the right side of 4.3.5 is represented in the form

$$\underline{\hat{G}}(\underline{r}_1, \underline{r}_2, \omega) = \int_{V_s} \int_{V_s} \underline{\hat{G}}(\underline{r}'_1, \underline{r}'_2, \omega) \delta(\underline{r}'_1 - \underline{r}_1) \delta(\underline{r}'_2 - \underline{r}_2) dv'_1 dv'_2, \quad ,$$

Then 4.3.5 can be written

$$\int_{V_s} \int_{V_s} \{ [\underline{L}_1 \cdot \underline{\hat{G}}^T(\underline{r}'_1, \underline{r}_1, \omega)] \cdot \underline{\hat{G}}(\underline{r}'_1, \underline{r}'_2, \omega) \cdot [\underline{L}_2 \cdot \underline{\hat{G}}^{*T}(\underline{r}'_2, \underline{r}_2, \omega)] - \delta(\underline{r}'_1 - \underline{r}_1) \underline{\hat{G}}(\underline{r}'_1, \underline{r}'_2, \omega) \delta(\underline{r}'_2 - \underline{r}_2) \} dv'_1 dv'_2 = 0 \quad . \quad (4.3.6)$$

This equation is satisfied if

$$(\nabla_1' \times \nabla_1' - k^2 \underline{\underline{I}} \cdot) \underline{\underline{\hat{D}}}^T(\underline{\underline{r}}_1', \underline{\underline{r}}_1, \omega) = \underline{\underline{I}} \delta(\underline{\underline{r}}_1' - \underline{\underline{r}}_1) \quad (4.3.7)$$

$$(\nabla_2' \times \nabla_2' - k^2 \underline{\underline{I}} \cdot) \underline{\underline{\hat{D}}}^{*T}(\underline{\underline{r}}_2', \underline{\underline{r}}_2, \omega) = \underline{\underline{I}} \delta(\underline{\underline{r}}_2' - \underline{\underline{r}}_2) \quad (4.3.8)$$

in the region outside V.

To insure that $\underline{\underline{\hat{G}}}(\underline{\underline{r}}_1, \underline{\underline{r}}_2, \omega)$ satisfies 4.3.3, it is necessary that

$$\int_{V_s} \int_{V_s} [\underline{\underline{m}}^{(1)} \times \underline{\underline{\hat{D}}}^T(\underline{\underline{r}}_1', \underline{\underline{r}}_1, \omega)] \cdot \underline{\underline{\hat{G}}}(\underline{\underline{r}}_1', \underline{\underline{r}}_2, \omega) \cdot [\underline{\underline{m}}^{(2)} \times \underline{\underline{\hat{D}}}^{*T}(\underline{\underline{r}}_2', \underline{\underline{r}}_2, \omega)] dv_1' dv_2' = 0 \quad (4.3.9)$$

where $\underline{\underline{r}}_1$ and $\underline{\underline{r}}_2$ lie on the surface of the conductor. But 4.3.9 is satisfied if

$$\underline{\underline{m}}^{(1)} \times \underline{\underline{\hat{D}}}^T(\underline{\underline{r}}_1', \underline{\underline{r}}_1, \omega) = 0 \quad \text{for } \underline{\underline{r}}_1 \text{ on } S \quad (4.3.10)$$

$$\underline{\underline{m}}^{(2)} \times \underline{\underline{\hat{D}}}^{*T}(\underline{\underline{r}}_2', \underline{\underline{r}}_2, \omega) = 0 \quad \text{for } \underline{\underline{r}}_2 \text{ on } S \quad (4.3.11)$$

Levine and Schwinger (39) show that a dyadic function satisfying a differential equation of the form 4.3.7 and the boundary condition 4.3.10 has the property

$$\underline{\underline{\hat{D}}}^T(\underline{\underline{r}}', \underline{\underline{r}}, \omega) = \underline{\underline{\hat{D}}}(\underline{\underline{r}}, \underline{\underline{r}}', \omega) \quad (4.3.12)$$

Thus, it is clear that if $\underline{\underline{\hat{D}}}(\underline{\underline{r}}, \underline{\underline{r}}', \omega)$ satisfies the differential equation

$$(\nabla \times \nabla - k^2 \underline{\underline{I}} \cdot) \underline{\underline{\hat{D}}}(\underline{\underline{r}}, \underline{\underline{r}}', \omega) = \underline{\underline{I}} \delta(\underline{\underline{r}} - \underline{\underline{r}}') \quad (4.3.13)$$

in the region excluding the conductor, and the boundary condition

$$\underline{n} \times \underline{\hat{D}}(\underline{r}, \underline{r}', \omega) = 0 \quad (4.3.14)$$

for points \underline{r} on the surface of the conductor, then it leads to a DFS that satisfies 4.3.2 and 4.3.3. The quantity, $\underline{\hat{D}}(\underline{r}, \underline{r}', \omega)$ represents the dyadic Green's function of the electric type.

It is necessary to point out that in addition to satisfying 4.3.12 and 4.3.13, $\underline{\hat{D}}(\underline{r}, \underline{r}', \omega)$ must also satisfy a radiation condition at infinity. This condition is easily derived from the radiation condition on the electromagnetic field. The result, stated here without proof, is

$$\lim_{r \rightarrow \infty} r [\underline{e}_r \times \nabla \times \underline{\hat{D}}(\underline{r}, \underline{r}', \omega) + ik \underline{\hat{D}}(\underline{r}, \underline{r}', \omega)] = 0 \quad (4.3.15)$$

Equation 4.3.15 can be interpreted as requiring that the three electromagnetic fields, which compose $\underline{\hat{D}}(\underline{r}, \underline{r}', \omega)$, behave like outgoing waves at infinity.

Corresponding representations of the DFS can be derived for the case of a dielectric scatterer. Although their derivation is somewhat more complicated than that presented here, they do not present any startling new information. Therefore, these additional results are neglected here.

4.4 The Case of an Incident Plane Wave

The formal representation for the DFS developed in the previous section is one step removed from an explicit formula. The remaining step involves an integration over the source distribution which, in general, can be just as difficult as the determination of the Green's function

itself. However, if the incident field happens to be a plane wave, then this formula reduces to a relatively simple explicit representation. Since the incident plane wave problem is important, it is worthwhile to develop this simplified representation here.

Equation 4.3.1 represents the response of a system of scatterers to the field of a general dyadic source $\hat{\underline{\underline{J}}}(\underline{r}_1, \underline{r}_2, \omega)$; it is now desired to obtain the corresponding formula for the special case of an incident noise-like plane wave. Because this problem is mainly one of interpretation (as will become evident below), it is necessary to lay some groundwork before proceeding to the main problem. To this end, consider the familiar problem of an ordinary time-harmonic source radiating in the presence of a system of perfectly conducting scatterers. A general representation for the total electric field in this case is

$$\underline{E}(\underline{r}, \omega) = ik \sqrt{\frac{\mu_0}{\epsilon_0}} \int_{V_S} \underline{\underline{G}}(\underline{r}, \underline{r}', \omega) \cdot \hat{\underline{J}}(\underline{r}', \omega) dv' \quad (4.4.1)$$

where $\hat{\underline{E}}(\underline{r}, \omega)$ represents the total electric field at the point \underline{r} , $\underline{\underline{G}}(\underline{r}, \underline{r}', \omega)$ is the dyadic Green's function of the electric type that is appropriate to the given problem, and $\hat{\underline{J}}(\underline{r}', \omega)$ is the source of the incident field. For example, if $\hat{\underline{J}}(\underline{r}', \omega)$ represents an elementary dipole of unit strength that is located at \underline{r}_0 and oriented in the direction \underline{e}_1 then the field becomes

$$\hat{\underline{E}}_1(\underline{r}, \omega) = ik \sqrt{\frac{\mu_0}{\epsilon_0}} \underline{\underline{G}}(\underline{r}, \underline{r}_0, \omega) \cdot \underline{e}_1$$

The dyadic Green's function in this equation can be expressed in terms of its column vectors:

$$\underline{\underline{G}}(\underline{r}, \underline{r}_0, \omega) = \hat{\underline{g}}(\underline{r}, \underline{r}_0, \omega) \underline{e}_i \quad (\text{summation implied}) \quad (4.4.2)$$

Hence, the field of the dipole can be written

$$\hat{\underline{E}}_d(\underline{r}, \omega) = ik \sqrt{\frac{\mu_0}{\epsilon_0}} \hat{\underline{g}}_1(\underline{r}, \underline{r}_0, \omega) \quad (4.4.3)$$

But this equation suggests the interpretation that the column vector $\hat{\underline{g}}_1(\underline{r}, \underline{r}_0, \omega)$ of the dyadic Green's function is the total electric field at \underline{r} due to an elementary dipole of strength $(ik \sqrt{\frac{\mu_0}{\epsilon_0}})^{-1}$ that is oriented in the \underline{e}_1 direction at \underline{r}_0 . Corresponding interpretations can be given to $\hat{\underline{g}}_2(\underline{r}, \underline{r}_0, \omega)$ and $\hat{\underline{g}}_3(\underline{r}, \underline{r}_0, \omega)$. Therefore, these results verify the statement made above that the dyadic Green's function is formed from the solution of three vector problems.

Equation 4.4.1 represents the total electric field associated with the incident field radiated by the source $\hat{\underline{J}}(\underline{r}', \omega)$. Hence, it should be possible to obtain the response to an incident plane wave simply by substituting the source distribution that gives rise to a plane wave for $\hat{\underline{J}}(\underline{r}', \omega)$. It is shown in Appendix D that the source

$$\hat{\underline{J}}(\underline{r}, \omega) = \lim_{z_0 \rightarrow -\infty} \left[2 \sqrt{\frac{\epsilon_0}{\mu_0}} \hat{\underline{E}}_0(\omega) e^{ikz_0} \delta(z - z_0) \right] \quad (4.4.4)$$

radiates the plane wave field

$$\hat{\underline{E}}(\underline{r}_1, \omega) = \hat{\underline{E}}_0(\omega) e^{ikz} \quad -\infty \leq z \leq \infty \quad (4.4.5)$$

in free space. The response of a system of scatterers to this plane wave can be expressed in terms of the Green's function for the problem by substituting 4.4.4 into 4.4.1. This yields the result

$$\hat{\underline{E}}(\underline{r}, \omega) = 2ik \left\{ \lim_{\zeta_0 \rightarrow -\infty} \left[e^{ik\zeta_0} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \hat{\underline{D}}(\underline{r}; \xi, \eta, \zeta_0; \omega) d\xi d\eta \right] \right\} \cdot \hat{\underline{E}}_0(\omega) \quad (4.4.6)$$

where the dummy variables (ξ, η, ζ) are chosen such that ζ lies in the direction of propagation of the incident wave and ξ, η are rectangular coordinates in the plane perpendicular to this direction. It is assumed above that $\underline{E}_0(\omega)$ lies in the (ξ, η) plane.

The significance of 4.4.6 is that it represents the response of the system of scatterers to an incident plane wave in terms of the dyadic Green's function for the system. Thus, it provides the connection between the two most fundamental scattering problems-- scattering due to a dipole source and scattering due to an incident plane wave. Equation 4.4.6 also displays an interesting subsidiary property of the dyadic Green's function. It was mentioned above that the constant vector $\hat{\underline{E}}_0$ lies in the ξ, η plane. Therefore, 4.6.5 is independent of the third column vector of $\hat{\underline{G}}(\underline{r}, \underline{r}', \omega)$, i.e., the column vector associated with a dipole oriented along the direction of propagation. However, it is certainly possible to equate $\hat{\underline{E}}_0$ to \underline{e}_ζ , the unit vector in the direction of propagation, formally in 4.4.4, and to substitute that result into 4.4.1. In this case, 4.4.6 reduces to

$$\hat{\underline{E}}(\underline{r}, \omega) = 2ik \left\{ \lim_{\zeta_0 \rightarrow \infty} \left[e^{ik\zeta_0} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \hat{\underline{G}}(\underline{r}; \xi, \eta, \zeta_0; \omega) d\xi d\eta \right] \right\} \cdot \underline{e}_\zeta \quad (4.4.7)$$

But according to the interpretation given above, 4.4.7 represents the total field associated with an incident plane wave polarized in the direction of propagation. Since such an incident wave cannot exist in free space, then there cannot be an associated scattered field. Hence, it must be that

$$\left[\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \hat{\underline{G}}(\underline{r}; \xi, \eta, \zeta_0; \omega) d\xi d\eta \right] \cdot \underline{e}_\zeta \equiv 0 \quad (4.4.8)$$

This argument by no means proves the validity of 4.48, but it does provide an intuitive basis upon which to construct a mathematical proof.

To apply the foregoing ideas to the DFS, it is necessary to have an expression for the source of a noise-like plane wave. Since the source of a monochromatic plane wave is written in the form 4.4.4, then the source of a noise-like plane wave can be expressed in the form

$$\underline{J}(\underline{r}, t) = \lim_{\zeta_o \rightarrow -\infty} \left[2i \sqrt{\frac{\epsilon_o}{\mu_o}} \underline{E}_o(t) \delta(\zeta - \zeta_o) \right] \quad (4.4.9)$$

where the constant vector $\hat{\underline{E}}_o(\omega)$ that appears in 4.4.4 has been replaced by the time varying, but spatially constant, vector $\underline{E}_o(t)$. The phase factor e^{ikz_o} appearing in 4.4.4 is unnecessary here, since the phase of the DFS is not referred to a fixed reference point. The DSS associated with this current is

$$\hat{\underline{\underline{G}}}(\underline{r}'_1, \underline{r}'_2, \omega) = \lim_{\zeta_o \rightarrow -\infty} \left\{ 4 \left(\frac{\epsilon_o}{\mu_o} \right) \hat{\underline{\underline{C}}}_o(\omega) \delta(\zeta'_1 - \zeta_o) \delta(\zeta'_2 - \zeta_o) \right\} \quad (4.4.10)$$

where

$$\hat{\underline{\underline{C}}}_o(\omega) = \int_{-\infty}^{\infty} \underline{E}_o(t+\tau) \underline{E}_o^*(t) e^{i\omega\tau} d\tau \quad (4.4.11)$$

Since $\underline{E}_o(t)$ is parallel to the surface current $\underline{J}(\underline{r}, t)$, then the dyadic $\hat{\underline{\underline{C}}}_o(\omega)$ has only four independent elements. Thus, when expanded in the preferred coordinate system, it takes the form

$$\begin{pmatrix} \hat{\underline{\underline{\zeta}}}_{\underline{\underline{o}}}(\omega) \end{pmatrix} = \begin{pmatrix} \hat{\underline{\underline{\zeta}}}_{\underline{\underline{o}}_{11}}(\omega) & \hat{\underline{\underline{\zeta}}}_{\underline{\underline{o}}_{12}}(\omega) & 0 \\ \hat{\underline{\underline{\zeta}}}_{\underline{\underline{o}}_{12}}^*(\omega) & \hat{\underline{\underline{\zeta}}}_{\underline{\underline{o}}_{22}}(\omega) & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad (4.4.12)$$

If 4.4.10 is substituted into 4.3.1, the following representation is obtained;*

$$\begin{aligned} \hat{\underline{\underline{\zeta}}}_{\underline{\underline{p}},\omega}(\underline{\underline{r}}_1, \underline{\underline{r}}_2, \omega) = \lim_{\zeta_o \rightarrow -\infty} & \left[2ik e^{ik\zeta_o} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \hat{\underline{\underline{D}}}(\underline{\underline{r}}_1; \xi, \eta, \zeta_o; \omega) d\xi d\eta \right] \cdot \hat{\underline{\underline{\zeta}}}_{\underline{\underline{o}}}(\omega) \cdot \\ & \left[(-2ik) e^{-ik\zeta_o} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \hat{\underline{\underline{D}}}^*(\underline{\underline{r}}_2; \xi, \eta, \zeta_o; \omega) d\xi d\eta \right] \end{aligned} \quad (4.4.13)$$

If 4.4.13 is expressed in terms of the column vectors of the dyadic Green's function, it can be written in the form

$$\hat{\underline{\underline{\zeta}}}_{\underline{\underline{p}},\omega}(\underline{\underline{r}}_1, \underline{\underline{r}}_2, \omega) = \hat{\underline{\underline{G}}}_i(\underline{\underline{r}}_1, \omega) \hat{\underline{\underline{G}}}_j^*(\underline{\underline{r}}_2, \omega) \hat{\underline{\underline{\zeta}}}_{ij_o}(\omega) \quad \begin{matrix} \text{(summation over } i \text{ and } j \\ \text{implied)} \end{matrix} \quad (4.4.14)$$

*

It is worth noting that if 4.4.10 is substituted into 4.1.17 and the resulting integration is carried out, the formula for the plane wave DFS is obtained. This formula is

$$\hat{\underline{\underline{\zeta}}}_{\underline{\underline{p}},\omega}(\underline{\underline{r}}_1, \underline{\underline{r}}_2, \omega) = \hat{\underline{\underline{\zeta}}}_{\underline{\underline{o}}}(\omega) e^{ik(\zeta_1 - \zeta_2)}$$

where

$$\hat{\underline{G}}_i(\underline{r}, \omega) = \lim_{\zeta_0 \rightarrow -\infty} \left[2ik e^{ik\zeta_0} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \hat{\underline{G}}(\underline{r}; \xi, \eta, \zeta_0; \omega) d\xi d\eta \right] \cdot \underline{e}_i, \quad i = \xi, \eta \quad (4.4.15)$$

Clearly, $\hat{\underline{G}}_i(\underline{r}, \omega)$ represents the total field at \underline{r} due to a plane wave traveling in the $+\zeta$ direction and polarized in the direction \underline{e}_i impinging on the scatterer. Thus, 4.4.14 provides an algebraic relationship between the DFS and the solutions of the vector, monochromatic plane wave problem associated with the given system of scatterers. The coefficient $\hat{\underline{C}}_{ij_0}(\omega)$ determines the strength of the incoming plane wave. The usefulness of 4.4.15 is clear. By solving the vector problem of a monochromatic plane wave incident on the given scatterers for both linear polarizations, it is possible to write down the explicit solution of the corresponding dyadic noise-field boundary value problem.

4.5 The Integral Representation of the DFS in a Source Free Region

To complete this chapter, the representation of the DFS in a source free region will be represented in terms of the DFS and its derivatives evaluated on the bounding surface of the region. This representation has application in any situation where the source of the radiation field is not specified, but instead the field is given over some surface that excludes the source from the region of interest. An example of this sort of problem is provided by a calculation and experiment carried out by N. George (40). He considered the radiation field emitted from a slot in a ground plane which is excited by a gaseous discharge located behind the ground plane. A problem of this sort can be analyzed by means of the representation to

be developed here once the DFS in the slot is determined.

The problem here is to integrate the dyadic wave equations in a source free region. The differential equations were derived in section 3.3. They are:

$$(\nabla_1 \times \nabla_1 \times - k^2 \underline{\underline{I}} \cdot) \hat{\underline{\underline{C}}}(\underline{r}_1, \underline{r}_2, \omega) = 0 \quad (4.5.1)$$

$$(\nabla_2 \times \nabla_2 \times - k^2 \underline{\underline{I}} \cdot) \hat{\underline{\underline{C}}}^T(\underline{r}_1, \underline{r}_2, \omega) = 0 \quad (4.5.2)$$

Let $\underline{A}(\underline{r}'_1) = \hat{\underline{\underline{C}}}(\underline{r}'_1, \underline{r}_2^o, \omega) \cdot \underline{a}$ and $\underline{B}(\underline{r}'_1) = \hat{\underline{\underline{C}}}(\underline{r}'_1, \underline{r}_1, \omega) \cdot \underline{b}$, and then substitute \underline{A} and \underline{B} into 4.1.2. With the aid of the differential equations 3.3.7 and 4.1.3, it can be shown that

$$\begin{aligned} \underline{a} \cdot \hat{\underline{\underline{C}}}^T(\underline{r}_1, \underline{r}_2^o, \omega) \cdot \underline{b} &= \oint_S \{ [\hat{\underline{\underline{C}}}(\underline{r}'_1, \underline{r}_2^o, \omega) \cdot \underline{a}] \times [\nabla_1' \times \hat{\underline{\underline{C}}}(\underline{r}'_1, \underline{r}_1, \omega) \cdot \underline{b}] - \\ &\quad - [\hat{\underline{\underline{C}}}(\underline{r}'_1, \underline{r}_1, \omega) \cdot \underline{b}] \times [\nabla_1' \times \hat{\underline{\underline{C}}}(\underline{r}'_1, \underline{r}_2^o, \omega) \cdot \underline{a}] \} \cdot \underline{m}^{(l)} ds'_1 \end{aligned} \quad (4.5.3)$$

The vectors \underline{a} and \underline{b} can be eliminated from 4.5.3 by using the vector identity $\underline{A} \cdot \underline{B} \times \underline{C} = \underline{B} \cdot \underline{C} \times \underline{A} = \underline{C} \cdot \underline{A} \times \underline{B}$ and the associative property of dyadics. The resulting equation, after a transpose operation, is

$$\begin{aligned} \hat{\underline{\underline{C}}}^T(\underline{r}_1, \underline{r}_2, \omega) &= - \oint_S \{ [\underline{m}^{(l)} \times \nabla_1' \times \hat{\underline{\underline{C}}}(\underline{r}'_1, \underline{r}_1, \omega)]^T \cdot \hat{\underline{\underline{C}}}(\underline{r}'_1, \underline{r}_2, \omega) + [\underline{m}^{(l)} \times \hat{\underline{\underline{C}}}(\underline{r}'_1, \underline{r}_1, \omega)]^T \cdot \\ &\quad \cdot [\nabla_1' \times \hat{\underline{\underline{C}}}(\underline{r}'_1, \underline{r}_2, \omega)] \} ds'_1 \end{aligned} \quad (4.5.4)$$

This equation provides a representation of the DFS in terms of an integration with respect to \underline{r}'_1 alone over the bounding surface S . By integrating 4.5.2, it is possible to obtain a representation in terms of an integration over \underline{r}'_2 . Here again, the Green's function $\hat{\underline{\underline{\Gamma}}}^*(\underline{r}'_2, \underline{r}_2, \omega)$ is used due to the presence of the transpose operation in 3.3.8. Thus,

$$\hat{\underline{\underline{\Gamma}}}(\underline{r}_1, \underline{r}_2, \omega) = \oint_S \{ \hat{\underline{\underline{\Gamma}}}(\underline{r}_1, \underline{r}'_2, \omega) \cdot [\underline{m}^{(2)} \underline{x} \nabla'_2 \hat{\underline{\underline{\Gamma}}}^*(\underline{r}'_2, \underline{r}_2, \omega)] + [\nabla'_2 \hat{\underline{\underline{\Gamma}}}^T(\underline{r}_1, \underline{r}'_2, \omega)]^T \cdot [\underline{m}^{(2)} \underline{x} \hat{\underline{\underline{\Gamma}}}^*(\underline{r}'_2, \underline{r}_2, \omega)] \} ds'_2 \quad (4.5.5)$$

Although 4.5.4 and 4.5.5 are two valid representations of $\hat{\underline{\underline{\Gamma}}}(\underline{r}_1, \underline{r}_2, \omega)$ they are not particularly useful since they don't express $\hat{\underline{\underline{\Gamma}}}(\underline{r}_1, \underline{r}_2, \omega)$ in terms of its value on the boundary. To obtain a more satisfactory representation, it is necessary to substitute 4.5.5 into 4.5.4. This yields the result

$$\begin{aligned} \hat{\underline{\underline{\Gamma}}}(\underline{r}_1, \underline{r}_2, \omega) = & \oint_S \oint_S \{ [\underline{m}^{(1)} \underline{x} \nabla'_1 \hat{\underline{\underline{\Gamma}}}(\underline{r}'_1, \underline{r}_1, \omega)]^T \cdot \hat{\underline{\underline{\Gamma}}}(\underline{r}'_1, \underline{r}'_2, \omega) \cdot [\underline{m}^{(2)} \underline{x} \nabla'_2 \hat{\underline{\underline{\Gamma}}}^*(\underline{r}'_2, \underline{r}_2, \omega)] + \\ & + [\underline{m}^{(1)} \underline{x} \nabla'_1 \hat{\underline{\underline{\Gamma}}}(\underline{r}'_1, \underline{r}_1, \omega)]^T \cdot [\nabla'_2 \hat{\underline{\underline{\Gamma}}}^T(\underline{r}'_1, \underline{r}'_2, \omega)]^T \cdot \\ & \cdot [\underline{m}^{(2)} \underline{x} \hat{\underline{\underline{\Gamma}}}^*(\underline{r}'_2, \underline{r}_2, \omega)] + [\underline{m}^{(1)} \underline{x} \hat{\underline{\underline{\Gamma}}}(\underline{r}'_1, \underline{r}_1, \omega)]^T \cdot [\nabla'_1 \hat{\underline{\underline{\Gamma}}}(\underline{r}'_1, \underline{r}'_2, \omega)] \cdot \\ & \cdot [\underline{m}^{(2)} \underline{x} \nabla'_2 \hat{\underline{\underline{\Gamma}}}^*(\underline{r}'_2, \underline{r}_2, \omega)] + [\underline{m}^{(1)} \underline{x} \hat{\underline{\underline{\Gamma}}}(\underline{r}'_1, \underline{r}_1, \omega)]^T \cdot \\ & \cdot [\nabla'_1 \hat{\underline{\underline{\Gamma}}}(\underline{r}'_1, \underline{r}'_2, \omega)]^T \cdot [\nabla'_2 \hat{\underline{\underline{\Gamma}}}^T(\underline{r}'_1, \underline{r}'_2, \omega)]^T \cdot [\underline{m}^{(2)} \underline{x} \hat{\underline{\underline{\Gamma}}}^*(\underline{r}'_2, \underline{r}_2, \omega)] \} ds'_1 ds'_2 \end{aligned} \quad (4.5.6)$$

Thus, given $\hat{\underline{\underline{E}}}(\underline{r}_1, \underline{r}_2, \omega)$ on the surface S , it is possible to obtain $\hat{\underline{\underline{E}}}(\underline{r}_1, \underline{r}_2, \omega)$ at all points within S by means of 4.5.6. Observe that 4.5.6 involves the free space Green's function. If instead the Green's function for the particular surface S is used, this formula can be greatly simplified. For example, suppose $\hat{\underline{\underline{G}}}(\underline{r}, \underline{r}, \omega)$ satisfies 4.1.3 and also the boundary condition

$$\underline{m} \times \hat{\underline{\underline{G}}}(\underline{r}', \underline{r}, \omega) = 0 \quad \text{on } S,$$

then 4.5.6 reduces to the relatively simple form

$$\begin{aligned} \hat{\underline{\underline{E}}}(\underline{r}_1, \underline{r}_2, \omega) = & \oint_S \oint_S [\underline{m}^{(1)} \times \nabla'_1 \times \hat{\underline{\underline{G}}}_e(\underline{r}'_1, \underline{r}_1, \omega)]^T \cdot \hat{\underline{\underline{E}}}(\underline{r}'_1, \underline{r}'_2, \omega) \cdot \\ & \cdot [\underline{m}^{(2)} \times \nabla'_2 \times \hat{\underline{\underline{G}}}_e^*(\underline{r}'_2, \underline{r}_2, \omega)] ds'_1 ds'_2 \end{aligned} \quad (4.5.7)$$

where the subscript "e" on the dyadic Green's function indicates it is of the electric type. A similar expression can be obtained by requiring that $\underline{m} \times \nabla' \times \hat{\underline{\underline{G}}}(\underline{r}', \underline{r}, \omega) = 0$ on S .

In that case

$$\begin{aligned} \hat{\underline{\underline{E}}}(\underline{r}_1, \underline{r}_2, \omega) = & \oint_S \oint_S [\underline{m}^{(1)} \times \hat{\underline{\underline{G}}}_m(\underline{r}'_1, \underline{r}_1, \omega)]^T \cdot [\nabla'_1 \times (\nabla'_2 \times \hat{\underline{\underline{E}}}^T(\underline{r}'_1, \underline{r}'_2, \omega))^T] \cdot \\ & \cdot [\underline{m}^{(2)} \times \hat{\underline{\underline{G}}}_m^*(\underline{r}'_2, \underline{r}_2, \omega)] ds'_1 ds'_2 \end{aligned} \quad (4.5.8)$$

where the subscript "m" indicates the magnetic type dyadic Green's function. Equations 4.5.7 and 8 demonstrate that only the DFS or its derivatives are required on a surface to specify the DFS within the enclosed

volume. Furthermore, these equations once again show that the dyadic boundary value problem can be solved in terms of the corresponding vector problem.

To close this chapter, observe that formulas 4.5.6, 4.5.7, and 4.5.8 apply regardless of whether the volume bounded by S lies within or outside of S --provided, of course, that in the latter case the fields are required to satisfy the radiation condition at infinity and all sources are confined to the excluded portion of space. The proof of this statement can be performed in a straightforward manner by writing $S = S_1 + S_2$ where S_1 is the surface over which $\hat{\underline{\underline{\xi}}}$ (or its curl) is specified, and S_2 is a large sphere that encloses S_1 . By letting the radius of S_2 approach infinity, and by appealing to the results proved in Appendix C, the integrals over S_2 approach zero, thereby leaving $\hat{\underline{\underline{\xi}}}(\underline{r}_1, \underline{r}_2, \omega)$ expressed in terms of an integral over S_1 alone.

V. AN APPLICATION — ANALYSIS OF AN INTERFEROMETER EXPERIMENT

5.1 Introduction

The object of the present chapter is to illustrate the considerations involved in the application of the foregoing theory by applying it to a particular problem. The problem chosen for this purpose is the analysis of the measurement of the DFS by means of a two element interferometer. This particular experiment was chosen for two reasons:

1) It is the basic means for measuring the DFS, and therefore is fundamental to the theory of the DFS. 2) It provides a sensitive means for studying the source of a partially polarized noise-like field. Furthermore, the experiment is sufficiently simple that it clearly illustrates the general considerations involved in the application of the theory.

Recently, Ko(41) analyzed the measurement of narrow-band, partially polarized, noise-like fields by means of a single receiving antenna. With the concepts and theory developed here it is possible to extend this work to broad-band fields measured by an interferometer (two-element receiving antenna). The value of an interferometric measurement lies in the fact that it provides (in principle) detailed information concerning the distribution of polarization over the plane of the source. Moreover, by determining the polarization of small regions of the source it provides a more sensitive means for identifying polarized sources, than a single receiver does. This is because the measured degree of polarization decreases as the resolution of the receiver decreases unless the source is uniformly polarized (see Appendix F). Before proceeding with the analysis, it should be noted that a great deal of attention has been given to the theory

of interferometers (see e.g., Born and Wolf (26)). However, this material has been largely restricted to scalar, narrow-band fields or else to ideal monochromatic vector fields. The analysis that follows extends this work to partially polarized broad-band fields such as those studied in radio astronomy.

5.2 Statement of the Problem

Suppose a partially polarized, incoherent, noise-like source of arbitrary spectral width radiates in the presence of a two element interferometer as depicted in Figure 5.1. Suppose further that the two elements of the interferometer are identical, and that the baseline of the interferometer is oriented in an arbitrary direction with respect to the line-of-sight to the source. For purposes of measurement it is desirable to introduce a variable phase shift, $\psi(\omega)$, into one of the elements of the interferometer. Finally, assume that the interferometer lies in the far-zone of the source and vice-versa. The problem then is to calculate the power carried to the receiver via the interferometer at the frequency ω . Since an experiment of this type involves a non-zero band of frequencies, it is necessary in general to calculate the integrated DFS, $\underline{J}(\underline{r}; \omega, \Delta\omega)$ (see 1.4.59). However, if the DFS is continuous at frequency ω and if the bandwidth of the receiver is sufficiently small, then the DFS and the interferometer pattern can be regarded as constant over the bandwidth of interest. In that case one can assume that

$$P_{\text{rec}} = 2 \int_{\omega - \Delta\omega/2}^{\omega + \Delta\omega/2} \hat{W}_{\text{rec}}(\omega') d\omega' \approx 2 \hat{W}_{\text{rec}}(\omega) \Delta\omega, \quad \omega > 0.$$

where $\hat{W}_{\text{rec}}(\omega)$ represents the power spectral density of the input signal to

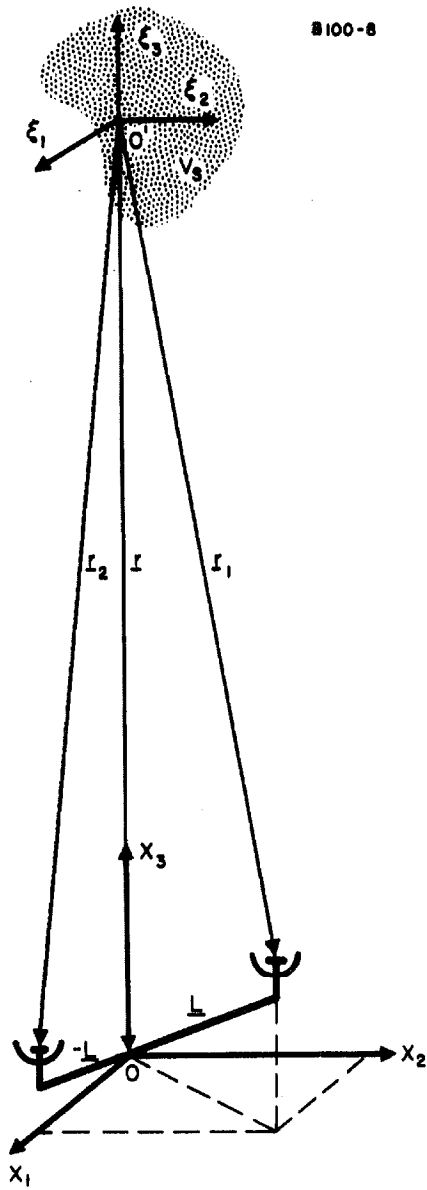


Figure 5.1. Configuration of the problem.

the receiver. The following calculation is restricted to this assumption. Hence, the experiment is assumed to be an optimum spectral measurement in the sense defined on page 46.

The problem just described is solved by means of the integral representation derived in Chapter IV:

$$\hat{\underline{\underline{C}}}(\underline{r}_1, \underline{r}_2, \omega) = \mu_0^2 \omega^2 \int_{V_s} \int_{V_s} \underline{\underline{G}}^T(\underline{r}', \underline{r}_1, \omega) \cdot \underline{\underline{G}}(\underline{r}', \underline{r}_2, \omega) \cdot \underline{\underline{G}}^*(\underline{r}', \underline{r}_2, \omega) dv'_1 dv'_2 \quad (5.2.1)$$

where $\underline{\underline{G}}(\underline{r}, \underline{r}', \omega)$ is the dyadic Green's function of the electric type. Since the source is assumed to be incoherent, equation 4.2.8 applies and 5.2.1 can be written

$$\hat{\underline{\underline{C}}}(\underline{r}_1, \underline{r}_2, \omega) = \mu_0^2 \omega^2 \int_{V_s} \underline{\underline{G}}^T(\underline{r}', \underline{r}_1, \omega) \cdot \underline{\underline{J}}(\underline{r}', \omega) \cdot \underline{\underline{G}}^*(\underline{r}', \underline{r}_2, \omega) dv' \quad (5.2.2)$$

The quantity $\hat{\underline{\underline{C}}}(\underline{r}_1, \underline{r}_2, \omega)$ represents the total DFS of the source-interferometer system as a function of any two points $\underline{r}_1, \underline{r}_2$ in space. However, since interest here is directed toward the power delivered to the receiver, it is only necessary to find $\hat{\underline{\underline{C}}}(\underline{r}_1, \underline{r}_2, \omega)$ for points \underline{r}_1 and \underline{r}_2 inside the wave guide that carries the power from the interferometer to the receiver. Therefore, to solve this problem it is necessary to determine the dyadic Green's function for observation points situated in the guide with the source in the far zone of the interferometer. This result, when substituted into 5.2.2 yields the DFS at points within the guide from which the power spectrum delivered to the receiver can be obtained.

5.3 The Dyadic Green's Function for the Interferometer

The dyadic Green's function is found by treating three separate vector boundary value problems — each associated with the dipole source oriented along one of the three coordinate directions, and located at some point in the far-zone of the interferometer (see Figure 5.2). Therefore, consider the following problem: an elementary dipole, with moment \underline{p} , radiates in the far zone of an antenna system. What is the amplitude of the dominant mode excited in the wave-guide that connects the antenna to the receiver? This problem is most readily solved by a method that is analogous to the method used to derive the Lorentz reciprocity condition. This technique was introduced by Brown(41) to obtain the response of an antenna to an incident plane wave. Since the source is not a plane wave in this case, the analysis is presented here in detail.

Let $\hat{\underline{E}}_d(\underline{r}, \omega)$, $\hat{\underline{H}}_d(\underline{r}, \omega)$ represent the total electromagnetic field that results from the dipole radiating in the presence of the antenna with the antenna in its receiving state. Let $\hat{\underline{E}}_A(\underline{r}, \omega)$, $\hat{\underline{H}}_A(\underline{r}, \omega)$ represent the field radiated by the antenna in free space. Assume that the waveguide is perfectly matched to both the receiver and the antenna, and that the antenna, when acting as a transmitter, is excited by a dominant mode carrying 1 watt of power. In the region enclosed by the surfaces S_W , S_A , and S_∞ , the field $\hat{\underline{E}}_d$, $\hat{\underline{H}}_d$ satisfies

$$\nabla \times \hat{\underline{E}}_d(\underline{r}, \omega) = i\omega\mu_o \hat{\underline{H}}_d(\underline{r}, \omega) \quad (5.3.1)$$

$$\nabla \times \hat{\underline{H}}_d(\underline{r}, \omega) = \underline{p}\delta(\underline{r} - \underline{r}') - i\omega\epsilon_o \hat{\underline{E}}_d(\underline{r}, \omega) \quad , \quad (5.3.2)$$

8100-7

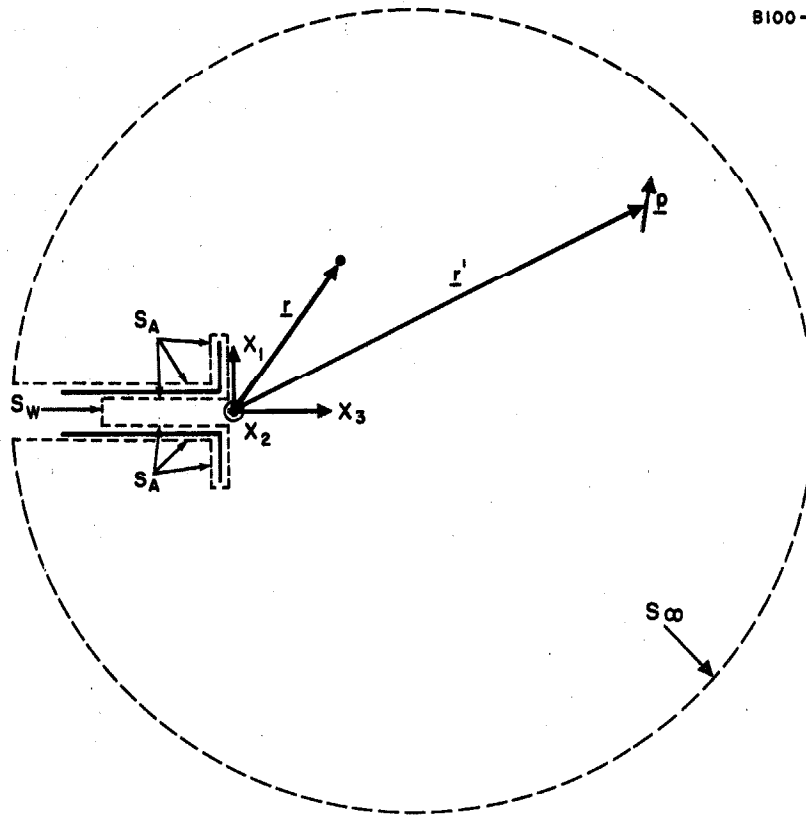


Figure 5.2. Configuration of the Green's Function Vector Boundary Value Problem.

the condition

$$\underline{m} \times \underline{\hat{E}}_d(\underline{r}, \omega) = 0 \quad \text{on } S_A, \quad (5.3.3)$$

and the radiation condition at infinity. Furthermore, if S_w is sufficiently far from the junction of the waveguide and the antenna, then at this cross section

$$\underline{\hat{E}}_d(\underline{r}, \omega) = B \underline{\hat{E}}_D(x_1, x_2, \omega) e^{-i\beta_D x_3} \quad (5.3.4)$$

$$\underline{\hat{H}}_d(\underline{r}, \omega) = B \underline{\hat{H}}_D(x_1, x_2, \omega) e^{-i\beta_D x_3}, \quad (5.3.5)$$

where $\underline{\hat{E}}_D, \underline{\hat{H}}_D$ is the normalized dominant mode of the guide, β_D is the propagation constant of this mode, and B represents the amplitude of the mode.* The factor B is the quantity to be determined in this analysis.

The field radiated by the antenna in free space satisfies

$$\nabla \times \underline{\hat{E}}_A(\underline{r}, \omega) = i\omega\mu_0 \underline{\hat{H}}_A(\underline{r}, \omega), \quad (5.3.6)$$

$$\nabla \times \underline{\hat{H}}_A(\underline{r}, \omega) = -i\omega\epsilon_0 \underline{\hat{E}}_A(\underline{r}, \omega), \quad (5.3.7)$$

$$\underline{m} \times \underline{\hat{E}}_A(\underline{r}, \omega) = 0 \quad \text{on } S_A, \quad (5.3.8)$$

and the radiation condition. At the surface S_w this field is given by

$$\underline{\hat{E}}_A(\underline{r}, \omega) = \underline{\hat{E}}_D(x_1, x_2, \omega) e^{i\beta_D x_3} \quad (5.3.9)$$

*

The dominant mode is normalized such that it carries unit power.

$$\underline{\hat{H}}_A(\underline{r}, \omega) = \underline{\hat{H}}_D(x_1, x_2, \omega) e^{i\beta_D x_3} \quad (5.3.10)$$

Multiply 5.3.1 by $\underline{\hat{H}}_A(\underline{r}, \omega)$, 5.3.2 by $\underline{\hat{E}}_A(\underline{r}, \omega)$, 5.3.4 by $\underline{\hat{H}}_d(\underline{r}, \omega)$, and 5.3.5 by $\underline{\hat{E}}_d(\underline{r}, \omega)$ and subtract the fourth equation from the first and the second from the third. This results in the two equations

$$\nabla \cdot \left[\underline{\hat{E}}_d \times \underline{\hat{H}}_A \right] = i\omega\mu_0 \underline{\hat{H}}_d \cdot \underline{\hat{H}}_A + i\omega\epsilon_0 \underline{\hat{E}}_d \cdot \underline{\hat{E}}_A \quad (5.3.11)$$

$$\nabla \cdot \left[\underline{\hat{E}}_A \times \underline{\hat{H}}_d \right] = i\omega\mu_0 \underline{\hat{H}}_d \cdot \underline{\hat{H}}_A - \delta(\underline{r} - \underline{r}') (\underline{p} \cdot \underline{\hat{E}}_A) + i\omega\epsilon_0 \underline{\hat{E}}_d \cdot \underline{\hat{E}}_A. \quad (5.3.12)$$

The difference of these two equations is

$$\nabla \cdot \left[\underline{\hat{E}}_d(\underline{r}, \omega) \times \underline{\hat{H}}_A(\underline{r}, \omega) - \underline{\hat{E}}_A(\underline{r}, \omega) \times \underline{\hat{H}}_d(\underline{r}, \omega) \right] = \left[\underline{p} \cdot \underline{\hat{E}}_A(\underline{r}, \omega) \right] \delta(\underline{r} - \underline{r}'). \quad (5.3.13)$$

If this equation is integrated over the volume enclosed by S_w , S_A , and S_∞ , and the divergence theorem is used, then

$$\oint_{S_w + S_\infty + S_A} \left[\underline{\hat{E}}_d(\underline{r}, \omega) \times \underline{\hat{H}}_A(\underline{r}, \omega) - \underline{\hat{E}}_A(\underline{r}, \omega) \times \underline{\hat{H}}_d(\underline{r}, \omega) \right] \cdot \underline{n} \, da = \underline{p} \cdot \underline{\hat{E}}_A(\underline{r}, \omega). \quad (5.3.14)$$

In view of 5.3.3 and 5.3.8, the integral over S_A is zero. Moreover, the radiation condition which is satisfied by both fields, implies that the integral over S_∞ is zero. This latter point can be established as follows: The integral over S_∞ can be written

$$I_{\infty} = \int_0^{2\pi} \int_0^{\pi} \lim_{r \rightarrow \infty} \left\{ \left[\underline{\hat{E}}_A(\underline{r}, \omega) \cdot \underline{e}_r \times \underline{\hat{H}}_d(\underline{r}, \omega) - \underline{\hat{E}}_d(\underline{r}, \omega) \cdot \underline{e}_r \times \underline{\hat{H}}_A(\underline{r}, \omega) \right] r^2 \right\} d\Omega \quad (5.3.15)$$

where the origin of coordinates is at the center of the sphere of radius r .

But the radiation condition requires that

$$\lim_{r \rightarrow \infty} r \left[\underline{e}_r \times \underline{\hat{H}}_A(\underline{r}, \omega) \right] = \sqrt{\frac{\epsilon_0}{\mu_0}} \lim_{r \rightarrow \infty} \left[r \underline{\hat{E}}_A(\underline{r}, \omega) \right] \quad (5.3.16)$$

$$\lim_{r \rightarrow \infty} r \left[\underline{e}_r \times \underline{\hat{H}}_d(\underline{r}, \omega) \right] = \sqrt{\frac{\epsilon_0}{\mu_0}} \lim_{r \rightarrow \infty} \left[r \underline{\hat{E}}_d(\underline{r}, \omega) \right] \quad (5.3.17)$$

Therefore

$$\begin{aligned} \lim_{r \rightarrow \infty} \left\{ r^2 \left[\underline{\hat{E}}_A(\underline{r}, \omega) \cdot \underline{e}_r \times \underline{\hat{H}}_d(\underline{r}, \omega) - \underline{\hat{E}}_d(\underline{r}, \omega) \cdot \underline{e}_r \times \underline{\hat{H}}_A(\underline{r}, \omega) \right] \right\} \\ = \lim_{r \rightarrow \infty} \left\{ r^2 \sqrt{\frac{\epsilon_0}{\mu_0}} \left[\underline{\hat{E}}_A(\underline{r}, \omega) \cdot \underline{\hat{E}}_d(\underline{r}, \omega) - \underline{\hat{E}}_d(\underline{r}, \omega) \cdot \underline{\hat{E}}_A(\underline{r}, \omega) \right] \right\} = 0, \end{aligned} \quad (5.3.18)$$

which proves that $I_{\infty} = 0$. Thus,

$$\int_{S_A} \left[\underline{\hat{E}}_d \times \underline{\hat{H}}_A - \underline{\hat{E}}_A \times \underline{\hat{H}}_d \right] \cdot \underline{n} da = \underline{p} \cdot \underline{\hat{E}}_A(\underline{r}', \omega) \quad (5.3.19)$$

Substituting 5.3.4, 5.3.5, 5.3.9, and 5.3.10 into 5.3.19, one obtains

$$- 2B \iint \underline{\hat{E}}_D(x_1, x_2, \omega) \times \underline{\hat{H}}_D(x_1, x_2, \omega) \cdot \underline{e}_3 dx_1 dx_2 = \underline{p} \cdot \underline{\hat{E}}_A(\underline{r}', \omega).$$

But for a propagating mode $\underline{e}_3 \cdot \underline{\hat{E}}_D \times \underline{\hat{H}}_D = - \underline{e}_3 \cdot \underline{\hat{E}}_D \times \underline{\hat{H}}_D^*$. Thus, since

$\hat{\underline{E}}_D, \hat{\underline{H}}_D$ carries unit power, then

$$B = \frac{1}{4} \underline{p} \cdot \hat{\underline{E}}_A(\underline{r}', \omega) \quad (5.3.20)$$

If \underline{r}' is in the far-zone of the antenna, then 5.3.20 can be written

$$B = \left(\frac{i\omega\mu_0}{16\pi R'} \right) \left[e^{ikR'} \hat{\underline{F}}_A(\theta', \phi', \omega) \cdot \underline{p} \right] \quad (5.3.21)$$

where $\hat{\underline{F}}_A(\theta', \phi', \omega)$, the interferometer pattern at frequency ω , is given by

$$\hat{\underline{F}}_A(\theta', \phi', \omega) = \underline{e}_{r'} \times \underline{e}_{r'} \times \int \hat{\underline{J}}_A(\underline{\xi}, \omega) e^{-ik\underline{\xi} \cdot \underline{e}_{r'}} d\underline{v} \quad (5.3.22)$$

In 5.3.22, $\underline{e}_{r'}$ is the unit vector at 0 directed toward the field point \underline{r}' . Equation 5.3.21, combined with equation 5.3.4 provides the solution of the vector boundary value problem in terms of the transmitting pattern of the interferometer. The complete formula is

$$\hat{\underline{E}}_d(\underline{r}, \underline{r}'; \underline{p}, \omega) = \frac{i\omega\mu_0}{16\pi R'} e^{ikR'} \left[\hat{\underline{F}}_A(\theta', \phi', \omega) \cdot \underline{p} \right] \hat{\underline{E}}_D(x_1, x_2, \omega) e^{-i\beta_D x_3} \quad (5.3.23)$$

where (R', θ', ϕ') represents the source point and (x_1, x_2, x_3) the observation point in the waveguide. Since $\hat{\underline{F}}_A$ represents the transmitting pattern of an interferometer, it can be expressed in terms of an element pattern and an array factor. That is,

$$\hat{\underline{E}}_d(\underline{r}, \underline{r}'; \underline{p}, \omega) = \frac{i\omega\mu_0}{16\pi R'} e^{ikR'} \left[e^{-ik\underline{L} \cdot \underline{e}_{r'}} + e^{ik\underline{L} \cdot \underline{e}_{r'} + i\psi(\omega)} \right] \left[\hat{\underline{f}}_e(\theta', \phi', \omega) \cdot \underline{p} \right]$$

$$\hat{\underline{E}}_D(x_1, x_2, \omega) e^{-i\beta_D x_3}$$

(5.3.24)

where $\hat{\underline{f}}_e(\theta', \phi', \omega)$ is the element pattern of the interferometer. It should be noted that $\hat{\underline{f}}_e(\theta', \phi', \omega)$ is not the usual normalized antenna pattern. It is the angular pattern of each element of the interferometer when the interferometer as a whole radiates one watt of power.

The dyadic Green's function consists of the juxtaposition of the three fields associated with the source oriented in each of the three coordinate directions \underline{e}_1 , \underline{e}_2 , and \underline{e}_3 . In addition, there is a factor of $(i\omega\mu_0)^{-1}$ that relates the column vectors of $\hat{\underline{\mathcal{G}}}(\underline{r}, \underline{r}', \omega)$ and the field $\hat{\underline{E}}_d(\underline{r}, \underline{r}'; \underline{p}, \omega)$ since the source of the Green's function is dimensionless, whereas that of the electric field is not. Thus, for points \underline{r} on S_A

$$\hat{\underline{\mathcal{G}}}(\underline{r}, \underline{r}', \omega) = \frac{1}{i\omega\mu_0} \left[\hat{\underline{E}}_d(\underline{r}, \underline{r}'; \underline{e}_i, \omega) \underline{e}_i \right] \quad (\text{summation implied}).$$

5.4. The Formula for the Received Power Density

The total power per unit frequency that crosses S_A is given by*

$$\hat{W}_{\text{rec}} = \frac{1}{Z_D} \int_{S_A} \text{Tr} \left[\hat{\underline{\mathcal{E}}}(\underline{r}, \underline{r}, \omega) \right] da, \quad (5.4.1)$$

*

This formula assumes that the dominant mode is an H-mode.

where \underline{r} lies on S_A and Z_D is the impedance of the dominant mode of the guide. But from 5.2.2 and 5.3.25 it is seen that

$$\hat{\underline{\underline{C}}}(\underline{r}, \underline{r}, \omega) = \int_{V_s} \left\{ \hat{\underline{E}}_d(\underline{r}, \underline{r}'; \underline{e}_m, \omega) \left[\underline{e}_m \cdot \hat{\underline{\mathcal{L}}}(\underline{r}', \omega) \cdot \underline{e}_n \right] \hat{\underline{E}}_d^*(\underline{r}, \underline{r}'; \underline{e}_n, \omega) \right\} dv' \quad (5.4.2)$$

which can be written

$$\hat{\underline{\underline{C}}}(\underline{r}, \underline{r}, \omega) = \left\{ \left(\frac{\omega \mu_0}{16\pi} \right)^2 \int_{V_s} \frac{|e^{-ik\underline{L} \cdot \underline{e}_{r'}} + e^{ik\underline{L} \cdot \underline{e}_{r'}}|^2}{R'^2} \left[\hat{\underline{f}}_e^*(\theta', \phi', \omega) \underline{e}_m \right] \cdot \left[\hat{\underline{f}}_e(\theta', \phi', \omega) \cdot \underline{e}_n \right] \hat{\underline{\mathcal{L}}}_{mn}(\underline{r}', \omega) dv' \right\} \hat{\underline{E}}_D(x_1, x_2, \omega) \hat{\underline{E}}_D^*(x_1, x_2, \omega) \quad (5.4.3)$$

where the quantity in the braces is a scalar function that is independent of \underline{r} . If this scalar is denoted by A , then 5.4.3 can be written

$$\text{Tr } \hat{\underline{\underline{C}}}(\underline{r}, \underline{r}, \omega) = A |\hat{\underline{E}}_D(x_1, x_2, \omega)|^2 \quad (5.4.4)$$

Therefore,

$$\hat{W}_{\text{rec}} = 2A \left\{ \frac{1}{2Z_D} \iint |\hat{\underline{E}}_D(x_1, x_2, \omega)|^2 dx_1 dx_2 \right\} \quad (5.4.5)$$

If $\hat{\underline{E}}_D$ is the electric field associated with the lowest order H-mode of the guide, then

$$\frac{1}{Z_D} |\hat{\underline{E}}_D(x_1, x_2, \omega)|^2 = \hat{\underline{E}}_D(x_1, x_2, \omega) \times \hat{\underline{H}}_D^*(x_1, x_2, \omega) \cdot \underline{e}_3 \quad .$$

But $\hat{\underline{E}}_D, \hat{\underline{H}}_D$ was defined such that it carries unit power. Thus, the quantity in the braces in 5.4.5 is unity

$$\hat{W}_{\text{rec}} = \frac{\omega^2 \mu_o^2}{128\pi^2} \int_{V_s} \frac{|e^{-ik\underline{L} \cdot \underline{e}_{r'}} + e^{ik\underline{L} \cdot \underline{e}_{r'} + i\psi}|^2}{R'^2} \left[\hat{\underline{f}}_e(\theta', \phi', \omega) \cdot \underline{e}_m \right] \left[\hat{\underline{f}}_e^*(\theta', \phi', \omega) \cdot \underline{e}_n \right] \hat{\mathcal{J}}_{mn}(\underline{r}', \omega) dv' \quad (5.4.6)$$

This expression relates the power incident on a matched load at the terminus of the wave guide to the distribution $\hat{\mathcal{J}}_{mn}(\underline{r}', \omega)$ of the source at frequency ω , as well as to the element pattern of the interferometer at ω .

If attention is restricted to sources that subtend small angles (compared with the beam width of the element pattern) at the interferometer, then 5.4.6 can be expressed directly in terms of the free-space DFS of the source. Since the discrete celestial radio sources satisfy this restriction, this case is not without interest. In the case of a small source, 5.4.6 can be written

$$\hat{W}_{\text{rec}} \approx \frac{1}{8} \left(\frac{k}{4\pi} \sqrt{\frac{\mu_o}{\epsilon_o}} \right)^2 \left[\hat{\underline{f}}_e^*(\theta_o, \phi_o, \omega) \hat{\underline{f}}_e(\theta_o, \phi_o, \omega) \right] : \int_{V_s} \left[2 + e^{i2k\underline{L} \cdot \underline{e}_{r'} + i\psi} + e^{-i2k\underline{L} \cdot \underline{e}_{r'} - i\psi} \right] \frac{\hat{\mathcal{J}}(\underline{r}', \omega) dv'}{R'^2}, \quad (5.4.7)$$

since by assumption $\hat{\underline{f}}_e(\theta, \phi, \omega)$ does not vary appreciably over the disc of the source.* The position (θ_o, ϕ_o) defines the angular orientation of a

* It is also assumed that the source is not situated near a null of the pattern.

reference point in the source with respect to the electrical axis of element pattern of the interferometer (see the insert in Figure 5.3). To introduce the DFS into 5.4.7 it is necessary to expand the integrand of 5.4.7 about $0'$, the reference point in the source. The coordinate geometry is illustrated in Figure 5.3. Note that the coordinate system in Figure 5.3 is aligned with respect to the line-of-sight to the source rather than with respect to the base-line of the interferometer. This is done because the interferometer is only sensitive to variations of the source in the plane perpendicular to the line-of-sight, which implies that the line-of-sight is a preferred direction.

Clearly, if the diameter of the source is small compared with $|\underline{r}|$, then

$$\frac{1}{r'} \approx \frac{1}{r} \quad . \quad (5.4.8)$$

Furthermore, it can be shown by the method used to derive 4.2.17, that

$$\underline{e}_{r'} \approx \underline{e}_r - \underline{e}_r \times \underline{e}_r \times \frac{\underline{\xi}'}{r} \quad . \quad (5.4.9)$$

Consequently, 5.4.7 can be written

$$\begin{aligned} \hat{W}_{\text{rec}} \approx \frac{1}{8} \left[\hat{\underline{f}}_e^* (\theta_o, \phi_o, \omega) \hat{\underline{f}}_e (\theta_o, \phi_o, \omega) \right] : \left\{ 2 \left(\frac{k}{4\pi r} \sqrt{\frac{\mu_o}{\epsilon_o}} \right)^2 \int_{V_s} \hat{\underline{g}}(\underline{r}', \omega) d\underline{v}' \right. \\ \left. + e^{i\psi} \left[e^{i2k\underline{L} \cdot \underline{e}_r} \left(\frac{k}{4\pi r} \sqrt{\frac{\mu_o}{\epsilon_o}} \right)^2 \int_{V_s} \hat{\underline{g}}(\underline{r}', \omega) e^{-i2k\underline{L} \cdot \underline{e}_r \times \underline{e}_r \times \underline{\xi}'/r} d\underline{v}' \right] \right. \\ \left. + e^{-i\psi} \left[e^{-i2k\underline{e}_r \cdot \underline{L}} \left(\frac{k}{4\pi r} \sqrt{\frac{\mu_o}{\epsilon_o}} \right)^2 \int_{V_s} \hat{\underline{g}}(\underline{r}', \omega) e^{i2k\underline{L} \cdot \underline{e}_r \times \underline{e}_r \times \underline{\xi}'/r} d\underline{v}' \right] \right\} \quad (5.4.10) \end{aligned}$$

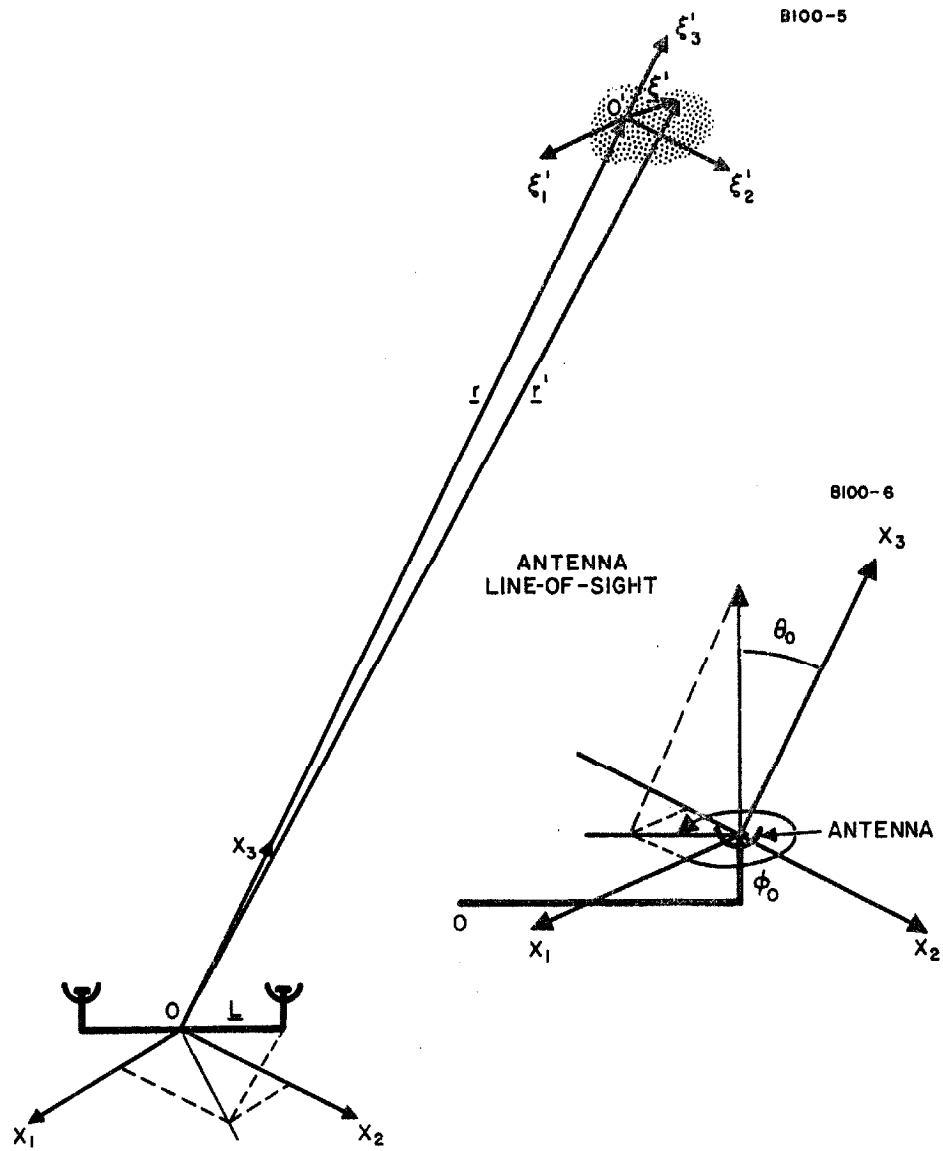


Figure 5.3. Geometry of the interferometer problem.

Since, to the accuracy of this calculation,

$$\underline{e}_r \cdot \hat{\underline{f}}_e(\theta_o, \phi_o, \omega) = 0, \quad (5.4.11)$$

then the components of $\hat{\underline{J}}(\underline{r}', \omega)$ that involve the direction \underline{e}_r do not influence the value of \hat{W}_{rec} . In mathematical terms this means that in 5.4.10 $\hat{\underline{J}}(\underline{r}', \omega)$ can be replaced by

$$(\underline{I} - \underline{e}_r \underline{e}_r) \cdot \hat{\underline{J}}(\underline{r}', \omega) \cdot (\underline{I} - \underline{e}_r \underline{e}_r)$$

without altering the value of \hat{W}_{rec} . Therefore,

$$\begin{aligned} \hat{W}_{rec} \approx \frac{1}{8} \left[\hat{\underline{f}}_e^*(\theta_o, \phi_o, \omega) \hat{\underline{f}}_e(\theta_o, \phi_o, \omega) \right] &: \left\{ 2 \left(\frac{k}{4\pi r} \sqrt{\frac{\mu_o'}{\epsilon_o}} \right)^2 (\underline{I} - \underline{e}_r \underline{e}_r) \cdot \left[\int_{V_s} \hat{\underline{J}}(\underline{r}', \omega) d\underline{v}' \right] \right. \\ &\cdot (\underline{I} - \underline{e}_r \underline{e}_r) + e^{i\psi} e^{i2kL \cdot \underline{e}_r} \left(\frac{k}{4\pi r} \sqrt{\frac{\mu_o'}{\epsilon_o}} \right)^2 (\underline{I} - \underline{e}_r \underline{e}_r) \cdot \left[\int_{V_s} \hat{\underline{J}}(\underline{r}', \omega) \times \right. \\ &\times e^{-i2kL \cdot \underline{e}_r \times \underline{e}_r \times \underline{r}'/r} d\underline{v}' \cdot (\underline{I} - \underline{e}_r \underline{e}_r) + e^{-i\psi} e^{-i2kL \cdot \underline{e}_r} \left(\frac{k}{4\pi r} \sqrt{\frac{\mu_o'}{\epsilon_o}} \right)^2 \\ &\left. \left. (\underline{I} - \underline{e}_r \underline{e}_r) \cdot \left[\int_{V_s} \hat{\underline{J}}(\underline{r}', \omega) e^{i2kL \cdot \underline{e}_r \times \underline{e}_r \times \underline{r}'/r} d\underline{v}' \right] \cdot (\underline{I} - \underline{e}_r \underline{e}_r) \right\} \right. \\ &\left. \right\}. \end{aligned} \quad (5.4.12)$$

If \underline{e}_r is replaced by $-\underline{e}_{r_o}$, where \underline{e}_{r_o} is the unit vector directed from $0'$ to 0 , and the result is compared with 4.2.20, it is seen that

$$\hat{W}_{\text{rec}} \approx \frac{1}{8} \left[\hat{\underline{f}}_{\underline{e}}^*(\theta_o, \phi_o, \omega) \hat{\underline{f}}_{\underline{e}}(\theta_o, \phi_o, \omega) \right] : \left[2 \hat{\underline{\underline{C}}}_{\underline{s}}(\underline{r}, \underline{r}, \omega) + \hat{\underline{\underline{C}}}_{\underline{s}}(\underline{r}_1, \underline{r}_2, \omega) e^{i\psi} + \hat{\underline{\underline{C}}}_{\underline{s}}^T(\underline{r}_1, \underline{r}_2, \omega) e^{-i\psi} \right] , \quad (5.4.13)$$

where \underline{r}_1 and \underline{r}_2 are defined in Figure 5.1. To complete this formula it is desirable to express it in terms of the parameters conventionally used to describe a receiving antenna. The directional gain of an antenna system is defined to be the power per unit solid angle radiated in the direction (θ, ϕ) divided by the average power radiated per unit solid angle. Since the field $\hat{\underline{E}}_A(\underline{r}, \omega)$ results from the radiation of 1 watt of power, then, neglecting losses in the antenna system,

$$\hat{G}_A(\theta_o, \phi_o, \omega) = \frac{2\pi R^2}{Z_o} \left| \hat{\underline{E}}_A(\underline{r}, \omega) \right|^2 = \frac{2\pi Z_o}{\lambda^2} \left| \hat{\underline{f}}_{\underline{e}}(\theta_o, \phi_o, \omega) \right|^2 \cos^2 k\underline{L} \cdot \underline{e}_r \quad (5.4.14)$$

where $Z_o = \epsilon_o / \mu_o$. Since the effective cross section of an antenna in the direction (θ, ϕ) is related to the directional gain in that direction by the formula $\hat{A}_A(\theta_o, \phi_o, \omega) = (\lambda^2 / 4\pi) \hat{G}_A(\theta_o, \phi_o, \omega)$, then

$$\left| \hat{\underline{f}}_{\underline{e}}(\theta_o, \phi_o, \omega) \right|^2 = \frac{2 \hat{A}_A(\theta_o, \phi_o, \omega)}{Z_o \cos^2 k\underline{L} \cdot \underline{e}_r} . \quad (5.4.15)$$

If the dyad $\hat{\underline{f}}_{\underline{e}} \hat{\underline{f}}_{\underline{e}}^* / |\hat{\underline{f}}_{\underline{e}}|^2$ is denoted by $\hat{\underline{\underline{P}}}_{\underline{e}}$, then

$$W_{\text{rec}} \approx \frac{\hat{A}_A(\theta_o, \phi_o, \omega)}{4Z_o \cos^2(k\underline{L} \cdot \underline{e}_r)} \text{Tr} \left\{ \hat{\underline{\underline{P}}}_{\underline{e}}^T(\theta_o, \phi_o, \omega) \cdot \left[2 \hat{\underline{\underline{C}}}_{\underline{s}}(\underline{r}, \underline{r}, \omega) + \hat{\underline{\underline{C}}}_{\underline{s}}(\underline{r}_1, \underline{r}_2, \omega) e^{i\psi} + \hat{\underline{\underline{C}}}_{\underline{s}}^T(\underline{r}_1, \underline{r}_2, \omega) e^{-i\psi} \right] \right\} \quad (5.4.16)$$

In 5.4.16, $\hat{\underline{\underline{p}}}_e(\theta_o, \phi_o, \omega)$ represents the polarization of the elements of the interferometer in the direction (θ_o, ϕ_o) at the frequency ω . The transpose of $\hat{\underline{\underline{p}}}_e$ appears because polarization is defined for a wave traveling toward the observer (in this case the origin 0), whereas the transmitted field of the interferometer travels away from the observer. The factor $\cos^{-2}(k\underline{L} \cdot \underline{e}_r)$ serves to cancel the zeros of $\hat{A}_A(\theta_o, \phi_o, \omega)$. These zeros are already present in the field quantity within the bracket of 5.4.16.

Equation 5.4.16 can be written in component form.

$$\hat{W}_{\text{rec}} \approx \frac{\hat{A}_A(\theta_o, \phi_o, \omega)}{4Z_o \cos^2(k\underline{L} \cdot \underline{e}_r)} \left\{ \hat{\underline{\underline{p}}}_{ij_e}(\theta_o, \phi_o, \omega) \left[2 \hat{\underline{\underline{E}}}_{ij_s}(\underline{r}, \underline{r}, \omega) + \hat{\underline{\underline{E}}}_{ij_s}(\underline{r}_1, \underline{r}_2, \omega) e^{i\psi} + \hat{\underline{\underline{E}}}_{ji_s}^*(\underline{r}_1, \underline{r}_2, \omega) e^{-i\psi} \right] \right\} \quad (5.4.17)$$

where the summation convention is implied. This formula can be expressed more concisely. In general, one can write the DFS evaluated at a single point in terms of the Stoke's parameters:

$$\hat{\underline{\underline{E}}}_s(\underline{r}, \underline{r}, \omega) = \hat{\underline{\underline{E}}}_s^{(u)}(\underline{r}, \underline{r}, \omega) + \hat{\underline{\underline{E}}}_s^{(c)}(\underline{r}, \underline{r}, \omega) \quad (5.4.18)$$

where from 1.4.38, 1.4.39, and 1.4.57

$$\hat{\underline{\underline{E}}}_s^{(u)}(\underline{r}, \underline{r}, \omega) = \frac{1}{2} \left[1 - \hat{p}(\underline{r}, \omega) \right] \hat{\underline{\underline{A}}}_{o_s}(\underline{r}, \omega) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad (5.4.19)$$

$$\hat{\underline{\underline{E}}}_s^{(c)}(\underline{r}, \underline{r}, \omega) = \frac{1}{2} \left[\hat{p}(\underline{r}, \omega) \hat{\underline{\underline{A}}}_{o_s}(\underline{r}, \omega) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \hat{\underline{\underline{A}}}_{1_s}(\underline{r}, \omega) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + \hat{\underline{\underline{A}}}_{2_s}(\underline{r}, \omega) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + \hat{\underline{\underline{A}}}_{3_s}(\underline{r}, \omega) \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} \right] \quad (5.4.20)$$

Similarly, one can write

$$\begin{aligned} \underline{\hat{P}}_e(\theta_o, \phi_o, \omega) = \frac{1}{2} \left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \hat{\Delta}_{1e}(\theta_o, \phi_o, \omega) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + \hat{\Delta}_{2e}(\theta_o, \phi_o, \omega) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right. \\ \left. + \hat{\Delta}_{3e}(\theta_o, \phi_o, \omega) \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} \right] \end{aligned} \quad (5.4.21)$$

where

$$\hat{\Delta}_{1e}^2 + \hat{\Delta}_{2e}^2 + \hat{\Delta}_{3e}^2 = 1 \quad (5.4.22)$$

(In the following material the argument of the Stokes' parameters will be suppressed.) A straightforward calculation shows that

$$\begin{aligned} \text{Tr} \left[\underline{\hat{P}}_e^T(\theta_o, \phi_o, \omega) \cdot \underline{\hat{E}}_s(\underline{r}, \underline{r}, \omega) \right] = \frac{1}{2} \left[(1-p) \hat{\Delta}_{os} + p \hat{\Delta}_{os} \hat{\Delta}_{oe} + \hat{\Delta}_{1s} \hat{\Delta}_{1e} \right. \\ \left. + \hat{\Delta}_{2s} \hat{\Delta}_{2e} - \hat{\Delta}_{3s} \hat{\Delta}_{3e} \right] \end{aligned} \quad (5.4.23)$$

where the quantity on the left is proportional to the power received by one antenna. By regarding Δ_1, Δ_2 , and Δ_3 as rectangular coordinates one can identify each point of a "polarization space" with the polarization of a completely polarized wave of a specific intensity. Since all possible polarizations of a wave of a given intensity define the surface of a sphere, it is convenient to express the polarization "point" in terms of spherical coordinates. This leads directly to the "Poincaré"

sphere representation of polarization.* Thus, if one writes (see Figure 5.4a)

$$\Delta_1 = p \Delta_0 \sin \alpha \cos \beta \quad (5.4.24)$$

$$\Delta_2 = p \Delta_0 \sin \alpha \sin \beta \quad (5.4.25)$$

$$\Delta_3 = p \Delta_0 \cos \alpha \quad (5.4.26)$$

Then, the coordinates $(p \Delta_0, \alpha, \beta)$ uniquely specify the polarization of the wave. In fact, the equations

$$I = p \Delta_0 \quad (5.4.27)$$

$$\tan 2\theta = \tan \beta \quad (5.4.28)$$

$$\sin 2\beta = \cos \alpha \quad (5.4.29)$$

provide the relationship between the parameters of the polarization ellipse and what may be called the Poincaré coordinates of the polarization. If 5.4.24-5.4.26 are substituted into 5.4.23, one finds

$$\begin{aligned} \text{Tr} \left[\hat{\underline{\underline{P}}}_e^T(\theta_o, \phi_o, \omega) \cdot \hat{\underline{\underline{E}}}_s(\underline{r}, \underline{r}, \omega) \right] = \frac{1}{2} \left\{ \hat{\Delta}_o + p \hat{\Delta}_o \left[\sin \alpha_s \sin(\pi - \alpha_e) \cos \beta_e \cos \beta_s \right. \right. \\ \left. \left. + \sin \alpha_s \sin(\pi - \alpha_e) \sin \beta_e \sin \beta_s + \cos \alpha_s \cos(\pi - \alpha_e) \right] \right\} . \end{aligned} \quad (5.4.30)$$

* Shurcliff (29) presents a detailed discussion of the Poincaré representation of polarization.

Denoting the polar angle between the two points (α_s, β_s) and $(\pi - \alpha_s, \beta_e)$ on the Poincaré sphere (see Figure 5.4b) by δ , one can write

$$\text{Tr} \left[\hat{\underline{\underline{P}}}_e^T(\theta_o, \phi_o, \omega) \cdot \hat{\underline{\underline{E}}}_s(\underline{r}, \underline{r}, \omega) \right] = \left[\text{Tr} \hat{\underline{\underline{E}}}_s(\underline{r}, \underline{r}, \omega) \right] \left[\frac{1 + p \cos \delta}{2} \right].$$

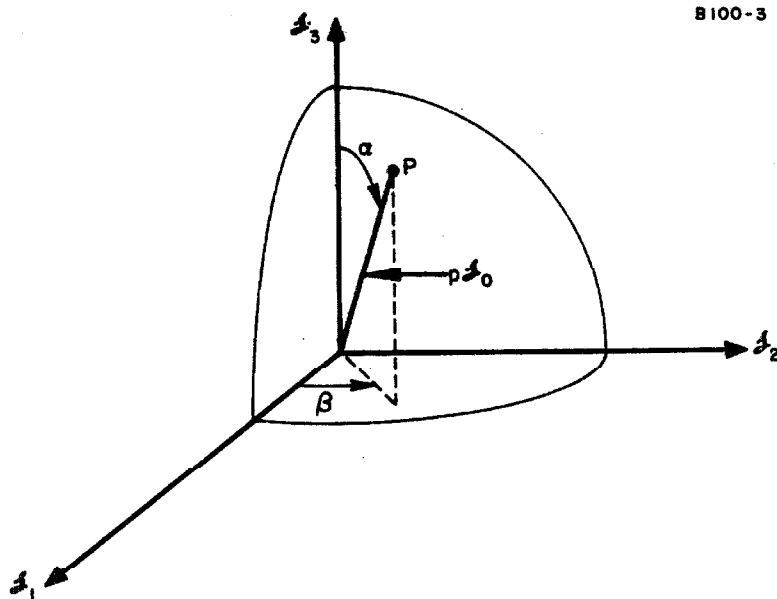
This result indicates that $\text{Tr} \left[\hat{\underline{\underline{P}}}_e^T(\theta_o, \phi_o, \omega) \cdot \hat{\underline{\underline{E}}}_s(\underline{r}, \underline{r}, \omega) \right]$ is a maximum when $\delta = 0$, i.e., when the polarization of the antenna is the same as that of the polarized part of the incoming wave, except for sense. As shown in Figure 5.5, the senses must be opposite for maximum received power since under this condition the polarization of the antenna lies at the image point of the polarization of the incoming wave.

If 5.4.31 is substituted into 5.4.17, and if 1.4.79 is introduced, then

$$\hat{W}_{\text{rec}} \approx \frac{\hat{A}_A(\theta_o, \phi_o, \omega)}{4Z_o \cos^2(kL \cdot \underline{e}_r)} \left\{ \text{Tr} \hat{\underline{\underline{E}}}_s(\underline{r}, \underline{r}, \omega) \left[1 + \hat{p}_s(\underline{r}, \omega) \cos \hat{\delta}(\underline{r}, \omega) \right] + \hat{P}_{ij}(\theta_o, \phi_o, \omega) \left[\hat{E}_{ii_s}(\underline{r}, \underline{r}, \omega) \hat{E}_{jj_s}(\underline{r}, \underline{r}, \omega) \right]^{1/2} \cdot \left[\hat{\mu}_{ij_s}(\underline{r}_1, \underline{r}_2, \omega) e^{i\psi} + \hat{\mu}_{ji_s}^*(\underline{r}_1, \underline{r}_2, \omega) e^{-i\psi} \right] \right\} \quad (5.4.32)$$

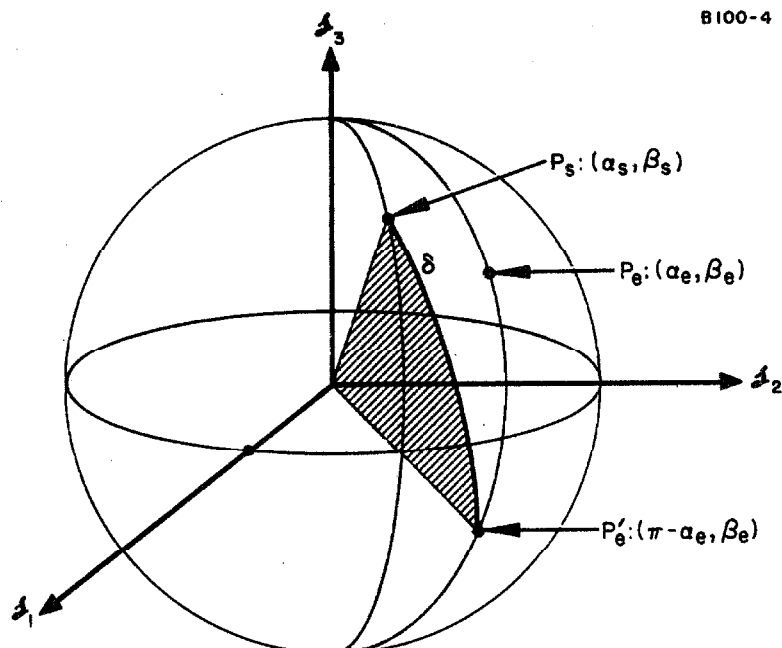
where the last term is summed over the indices i and j . It is tempting to attempt to separate the polarization properties from the spatial coherence properties in the last term on the right of 5.4.32. However, this can not be done in general since the spatial coherence need not be the same for the various components of the field as can be seen by

B100-3



(a) Polarization coordinates on the Poincare sphere.

B100-4



(b) The geometry of a polarization measurement.

Figure 5.4. The Poincare Sphere.

carefully examining 4.2.32. Nevertheless, there is one special case where the spatial coherence and the polarization may be factored — the case of a uniformly, polarized source. If the source is uniformly polarized, then

$$\underline{\underline{\hat{\mathcal{E}}}}(\underline{r}', \omega) = \underline{\underline{\hat{\mathcal{P}}}}_s(\omega) \hat{I}(\underline{r}', \omega) \quad (5.4.33)$$

where $\underline{\underline{\hat{\mathcal{P}}}}_s(\omega)$ represents the polarization of the source, and $\hat{I}(\underline{r}', \omega)$ represents the distribution of intensity of the source. In this case

$$\underline{\underline{\hat{\mathcal{E}}}}(\underline{r}_1, \underline{r}_2, \omega) = \left\{ e^{-2ik(\underline{L} \cdot \underline{e}_r)} \int_{V_s} \hat{I}(\underline{r}') e^{-i2k\underline{L} \cdot \underline{e}_r \underline{x}_r \underline{x}_r' / r} dv' \right\} \cdot \frac{\underline{\underline{\hat{\mathcal{E}}}}(\underline{r}, \underline{r}, \omega)}{\int_{V_s} \hat{I}(\underline{r}', \omega) dv'} \quad (5.4.34)$$

as can be seen by substituting 5.4.33 into 4.2.20. Combining 5.4.34 with 5.4.32, or more directly with 5.4.16, one finds that

$$\hat{W}_{\text{rec}} \approx \frac{\hat{A}_A(\theta_o, \phi_o, \omega)}{4Z_o \cos^2(k\underline{L} \cdot \underline{e}_r)} \text{Tr} \underline{\underline{\hat{\mathcal{E}}}}_s(\underline{r}, \underline{r}, \omega) \left[1 + \hat{p}_s(\underline{r}, \omega) \cos \hat{\delta}(\underline{r}, \omega) \right] \cdot \left\{ 1 + \frac{\text{Re} \left[e^{-i[k\underline{L} \cdot \underline{e}_r - \psi]} \int_{V_s} \hat{I}(\underline{r}', \omega) e^{-i2k\underline{L} \cdot \underline{e}_r \underline{x}_r \underline{x}_r' / r} dv' \right]}{\int_{V_s} \hat{I}(\underline{r}') dv'} \right\} \quad (5.4.35)$$

If the second term in the braces is expanded in a rectangular coordinate system oriented so that the x_3 axis is parallel to \underline{e}_r , then this term takes the form

$$\frac{e^{-i(2kL_3 - \psi)} \iint \hat{I}_0(x'_1, x'_2, \omega) e^{-ik \left[2(L_1/r)x'_1 + 2(L_2/r)x'_2 \right]} dx'_1 dx'_2}{\iint \hat{I}_0(x'_1, x'_2, \omega) dx'_1 dx'_2}$$

where

$$\hat{I}_0(x'_1, x'_2, \omega) = \int \hat{I}(x'_1, x'_2, x'_3, \omega) dx'_3$$

This expression is closely related to the far-zone representation of the complex degree of coherence of a scalar partially coherent field derived in Born and Wolf(26), page 508. In fact, apart from unimportant differences in geometry, * the only difference between the two expressions is that the formula of Born and Wolf represents degree of spatial coherence in a narrow range of frequencies about ω of a field of arbitrary spectral width. In terms of the notation of Born and Wolf, \hat{W}_{rec} for the case of a uniformly polarized source becomes

$$\hat{W}_{\text{rec}} \approx \frac{\hat{A}_A(\theta_o, \phi_o, \omega)}{4Z_o \cos^2(k\underline{L} \cdot \underline{e}_r)} \text{Tr} \underline{\hat{C}}_s(\underline{r}, \underline{r}, \omega) \left[1 + \hat{p}(\underline{r}, \omega) \cos \hat{\delta}(\underline{r}, \omega) \right] \left\{ 1 + \text{Re} \left[\hat{\mu}(\underline{r}_1, \underline{r}_2, \omega) e^{i\psi} \right] \right\} \quad (5.4.36)$$

* Born and Wolf locate their observation points P_1 and P_2 such that $\overline{P_1 P_2} \cdot \underline{e}_R = 0$ and $\overline{O' P_1} \neq \overline{O' P_2}$ (see Figure 10.3 of B and W). As a result their phase factor ψ differs from the phase factor $2kL_3$ obtained here.

Thus in the case of a uniformly polarized source the polarization and the spatial coherence properties of the field factor into two distinct terms.

If the source is completely, as well as uniformly polarized, and if the field is spatially coherent, then

$$\hat{p}(\underline{r}, \omega) = 1, \quad |\hat{\mu}(\underline{r}_1, \underline{r}_2, \omega)| = 1. \quad (5.4.37)$$

If, in addition, the polarization of the receiver is conjugate matched to that of the field, then

$$\hat{W}_{\text{rec}}(\omega) \approx \frac{\hat{A}_{A(\theta_o, \phi_o, \omega)}}{2Z_o \cos^2(k\underline{L} \cdot \underline{e}_r)} \text{Tr} \hat{\underline{\underline{E}}}_s(\underline{r}, \underline{r}, \omega) \left\{ 1 + \cos[\arg \hat{\mu}(\underline{r}_1, \underline{r}_2, \omega) + \psi] \right\}. \quad (5.4.38)$$

By choosing ψ such that

$$\psi = -\arg \left\{ \iiint \hat{I}_o(\underline{x}'_1, \underline{x}'_2, \omega) e^{-i2k \frac{(L_1 \underline{x}'_1 + L_2 \underline{x}'_2) \cdot \underline{r}}{r}} dv' \right\} \quad (5.4.39)$$

one obtains the formula

$$\hat{W}_{\text{rec}} \approx \hat{A}_{A(\theta_o, \phi_o, \omega)} \left[\frac{1}{Z_o} \text{Tr} \hat{\underline{\underline{E}}}_s(\underline{r}, \underline{r}, \omega) \right]. \quad (5.4.40)$$

That is, in the limit of complete coherence (spatial as well as polarization), the received power equals the incident power density times the effective receiving cross section of the interferometer. If, on the other hand, the completely incoherent limit is considered, then

$$W_{\text{rec}} \approx \frac{1}{4} \hat{A}_{A(\theta_o, \phi_o, \omega)} \left[\frac{1}{Z_o} \text{Tr} \hat{\underline{\underline{E}}}_s(\underline{r}, \underline{r}, \omega) \right] \quad (5.4.41)$$

In the two intermediate cases: 1) ($\hat{p} = 1, |\hat{\mu}|=0$). 2) ($\hat{p} = 0, |\hat{\mu}| = 1$),

$$\hat{W}_{\text{rec}} \approx \frac{1}{2} \hat{A}_{A(\theta_o, \phi_o, \omega)} \left[\frac{1}{Z_o} \text{Tr} \hat{\underline{\underline{E}}}_s(\underline{r}, \underline{r}, \omega) \right] . \quad (5.4.42)$$

In other words, the interferometer receives four times the power in the completely coherent case than it does in the completely incoherent case. If there is either complete polarization or complete spatial coherence, but not both, then the interferometer receives twice the power as in the completely incoherent case. Observe that these latter remarks apply equally well to a non-uniformly, polarized source. However, in that case it is not possible to obtain a simple relationship of the form 5.4.36 between the received power, the receiving cross section, and the various properties of the incident field.

5.5. A Means for Studying the Detailed Character of the Source

The foregoing discussion describes the measurement of the DFS of a particular source. However, it is often of interest to obtain direct information regarding the source of the field. This is particularly true in the case of astronomy where it is desired to obtain information regarding the distribution of polarization over the source. It is evident from the preceding calculation that if a single antenna were used rather than an interferometer, then the measured quantity would be $\hat{\underline{\underline{E}}}(\underline{r}, \underline{r}, \omega)$ so that in this case

$$\hat{W}_{\text{rec}} \propto \int_{V_s} \hat{\underline{\underline{I}}}(\underline{r}', \omega) d\omega . \quad (5.5.1)$$

Thus, a single antenna determines the average polarization of the source. In this manner information concerning the distribution of polarization is lost. Furthermore, the measured degree of polarization is considerably less than the maximum degree of polarization of the source (see Appendix F). As a result the sensitivity of detection of polarized radiation sources is reduced. As will now be shown interferometric measurement overcomes this limitation — at least in principle — by providing information directly about the distribution of polarization of the source, thereby improving the resolution of the experiment.

By proceeding in the manner outlined in Appendix E, the real and imaginary part of $\hat{\mathcal{C}}_{ij_s}(\underline{r}_1, \underline{r}_2, \omega)$ can be measured. But this quantity is related to the source by means of 4.2.32. Thus, in the preferred coordinate system

$$\hat{\mathcal{C}}_{ij_s}(\underline{r}_1, \underline{r}_2, \omega) = \left(\frac{k}{4\pi r} \sqrt{\frac{\mu_o}{\epsilon_o}} e^{-ik\underline{L} \cdot \underline{e}_r} \right)^2 \int_{V_s} \hat{\mathcal{C}}_{ij}(\underline{r}', \omega) e^{-i2k\underline{L} \cdot \underline{e}_r \underline{x} \underline{e}_r \underline{x} \underline{r}' / r} dv' \quad (5.5.2)$$

If the integrand is expanded in this coordinate system, then

$$\hat{\mathcal{C}}_{ij}(\underline{r}_1, \underline{r}_2, \omega) = \left(\frac{k}{4\pi r} \sqrt{\frac{\mu_o}{\epsilon_o}} e^{-ik\underline{L} \cdot \underline{e}_r} \right)^2 \int_{A_s} \mathcal{C}_{ij_o}(\underline{x}'_1, \underline{x}'_2, \omega) e^{-i2k\underline{r}(L_1 \underline{x}'_1 + L_2 \underline{x}'_2)} dx'_1 dx'_2, \quad (5.5.3)$$

where A_s is the disc of the source seen along the line of sight, and

$$\hat{\mathcal{L}}_{ij_0}(x'_1, x'_2, \omega) = \int \hat{\mathcal{L}}_{ij}(\underline{r}', \omega) dx'_3 \quad . \quad (5.5.4)$$

That is, $\hat{\mathcal{L}}_{ij_0}(x'_1, x'_2, \omega)$ is the distribution over the source of the ij component of the DSS weighted with respect to the direction along the line of sight. The essential feature of the above formula is that the integral in 5.5.3 is the Fourier transform of the source distribution $\hat{\mathcal{L}}_{ij_0}(x'_1, x'_2, \omega)$. Thus by measuring this integral for a large number of interferometer spacings $(\underline{r}_1, \underline{r}_2)$, it is possible, in principle, to invert the transform to obtain $\hat{\mathcal{L}}_{ij_0}(x'_1, x'_2, \omega)$. Note that because $\hat{\mathcal{L}}_{ij_0}(x'_1, x'_2, \omega)$ is limited to a finite region of space use can be made of the sampling theorem of Fourier transform theory to reduce the labor involved in inverting the transform. Assuming the DSS $\hat{\mathcal{L}}_{\underline{\underline{0}}}(x'_1, x'_2, \omega)$ is known, then the distribution of polarization over the disc of the source can be studied by means of the Stokes' parameters to complete the analysis of the source.

APPENDIX A: TWO THEOREMS RELATING TO PROPERTIES OF THE DFS

Theorem 1: The matrix formed from the DFS evaluated at $\underline{r}_1 = \underline{r}_2 = \underline{r}$ has a non-negative determinant for all \underline{r}, ω . That is,

$$\det \left\{ \hat{\mathcal{E}}_{ij}(\underline{r}, \underline{r}_1, \omega) \right\} \geq 0 \quad . \quad (\text{A. 1})$$

Proof: It is seen from 1.3.8 on setting $\underline{r}_1 = \underline{r}_2 = \underline{r}$ that the matrix $\left\{ \hat{\mathcal{E}}_{ij}(\underline{r}, \underline{r}, \omega) \right\}$ is hermitian. Therefore $\left\{ \hat{\mathcal{E}}_{ij}(\underline{r}, \underline{r}, \omega) \right\}$ can be diagonalized by a unitary transformation. More specifically, if $\left\{ \hat{U}_{ij}(\underline{r}, \omega) \right\}$ is the transformation that diagonalizes the DFS evaluated at $\underline{r}_1 = \underline{r}_2 = \underline{r}$, then

$$\hat{U}_{im}^{-1}(\underline{r}, \omega) \hat{\mathcal{E}}_{mn}(\underline{r}, \underline{r}, \omega) \hat{U}_{nj}(\underline{r}, \omega) = \hat{\Lambda}_{ij}(\underline{r}, \omega) \quad i, j, m, n = 1, 2, 3. \quad (\text{A. 2})$$

where

$$\begin{aligned} \hat{\Lambda}_{ij}(\underline{r}, \omega) &= \hat{\lambda}^i(\underline{r}, \omega) \quad i = j \\ &= 0 \quad i \neq j \end{aligned} \quad (\text{A. 3})$$

The quantity $\hat{\lambda}^i(\underline{r}, \omega)$ is the i^{th} eigen value of $\left\{ \hat{\mathcal{E}}_{ij}(\underline{r}, \underline{r}, \omega) \right\}$. Since the two matrices in A.2 are equal then so are their determinants. Thus,

$$\det \left\{ \hat{\mathcal{E}}_{ij}(\underline{r}, \underline{r}, \omega) \right\} = \prod_{i=1}^3 \hat{\lambda}^i(\underline{r}, \omega) \quad (\text{A. 4})$$

Hence, to prove A.1 it is only necessary to show that the eigen values of $\hat{C}_{ij}(\underline{r}, \underline{r}, \omega)$ are positive.

The eigen values $\hat{\lambda}_i(\underline{r}, \omega)$ are defined by the equation

$$\hat{C}_{ij}(\underline{r}, \underline{r}, \omega) \hat{u}_j^a(\underline{r}, \omega) = \hat{\lambda}^a(\underline{r}, \omega) \hat{u}_i^a(\underline{r}, \omega) \quad (A.5)$$

where $\hat{u}^a(\underline{r}, \omega)$ is the eigen vector of $\hat{C}_{ij}(\underline{r}, \underline{r}, \omega)$ that corresponds to the eigen value $\hat{\lambda}^a(\underline{r}, \omega)$. Multiply A.5 by $\hat{u}_i^a(\underline{r}, \omega)$ and sum over the subscript i (summation over j on the left is implied by the summation convention). Since the eigen vectors of an hermitian matrix form an orthonormal set, then $\hat{u}_i^a(\underline{r}, \omega) \hat{u}_i^{a*}(\underline{r}, \omega) = 1$. Therefore, A.5 becomes

$$\hat{u}_i^{a*}(\underline{r}, \omega) \hat{C}_{ij}(\underline{r}, \underline{r}, \omega) \hat{u}_j^a(\underline{r}, \omega) = \hat{\lambda}^a(\underline{r}, \omega) . \quad (A.6)$$

But from 1.3.5 it is seen that A.6 can be written

$$\hat{u}_i^{a*}(\underline{r}, \omega) \left[\int_{-\infty}^{\infty} \overline{E_i(\underline{r}, t + \tau) E_j(\underline{r}, t)} e^{i\omega\tau} d\tau \right] \hat{u}_j^a(\underline{r}, \omega) = \hat{\lambda}^a(\underline{r}, \omega) \quad (A.7)$$

Since the eigen vectors are deterministic quantities which are independent of τ , it is possible to take them inside both the integration and the averaging process to obtain

$$\int_{-\infty}^{\infty} \hat{u}_i^{a*}(\underline{r}, \omega) E_i(\underline{r}, t + \tau) \hat{u}_j^a(\underline{r}, \omega) E_j(\underline{r}, t) e^{i\omega\tau} d\tau = \hat{\lambda}^a(\underline{r}, \omega) . \quad (A.8)$$

This equation can be written in vector form:

$$\int_{-\infty}^{\infty} \overline{\left[\hat{\underline{u}}^{a*}(\underline{r}, \omega) \cdot \underline{E}(\underline{r}, t + \tau) \right] \left[\hat{\underline{u}}^a(\underline{r}, \omega) \cdot \underline{E}(\underline{r}, t) \right]} e^{i\omega\tau} d\tau = \hat{\lambda}^a(\underline{r}, \omega) \quad (\text{A. 9})$$

But $\overline{\left[\hat{\underline{u}}^{a*} \cdot \underline{E}(\underline{r}, t + \tau) \right] \left[\hat{\underline{u}}^a \cdot \underline{E}(\underline{r}, t) \right]}$ is the autocorrelation function of the stationary scalar random process $\hat{\underline{u}}^a(\underline{r}, \omega) \cdot \underline{E}(\underline{r}, t)$, where \underline{r} is a fixed point. Thus, $\hat{\lambda}^a(\underline{r}, \omega)$ is the power spectral density of the projection of $\underline{E}(\underline{r}, t)$ on the eigen vector $\hat{\underline{u}}^a(\underline{r}, \omega)$. Since the power spectral density is a positive function of frequency, then the eigen values at \underline{r} are positive functions of ω^* .

Q. E. D.

Theorem 2: The degree of polarization $p(\underline{r}; \omega, \Delta\omega)$ at any point \underline{r} satisfies the inequality

$$0 \leq p(\underline{r}; \omega, \Delta\omega) \leq 1 \quad (\text{A. 10})$$

for all ω .

Proof: From 1.4.63 $p(\underline{r}; \omega, \Delta\omega)$ can be written in the form

$$p(\underline{r}; \omega, \Delta\omega) = \frac{\sqrt{\text{Tr } \underline{\underline{J}}(\underline{r}; \omega, \Delta\omega)^2 - 4 \text{Det } \underline{\underline{J}}(\underline{r}; \omega, \Delta\omega)}}{\text{Tr } \underline{\underline{J}}(\underline{r}; \omega, \Delta\omega)} \quad (\text{A. 11})$$

* Davenport and Root (19) prove that the spectrum of a stationary random process is a positive function of frequency.

Clearly, $p(\underline{r}; \omega, \Delta\omega)$ is positive since $\text{Tr } \underline{\underline{J}}(\underline{r}; \omega, \Delta\omega)$ is the integral of $\text{Tr } \hat{\underline{\underline{C}}}(\underline{r}, \underline{r}, \omega)$ which is positive for all ω (see 1.3.11), and the numerator is by definition the positive square root of the real, positive quantity $s_1^2(\underline{r}; \omega, \Delta\omega) + s_2^2(\underline{r}; \omega, \Delta\omega) + s_3^2(\underline{r}; \omega, \Delta\omega)$. Thus, $p(\underline{r}; \omega, \Delta\omega) \geq 0$.

The right hand inequality, $p(\underline{r}; \omega, \Delta\omega) \leq 1$ can be proved by the approach used in Theorem 1. If $p(\underline{r}; \omega, \Delta\omega) \leq 1$, then

$$\text{Tr } \underline{\underline{J}}(\underline{r}; \omega, \Delta\omega) \geq \sqrt{[\text{Tr } \underline{\underline{J}}(\underline{r}; \omega, \Delta\omega)]^2 - 4 \text{Det } \underline{\underline{J}}(\underline{r}; \omega, \Delta\omega)}$$

But this latter inequality is satisfied provided that

$$\text{Det } \underline{\underline{J}}(\underline{r}; \omega, \Delta\omega) = \text{Det} \left\{ \int_{\omega - \frac{\Delta\omega}{2}}^{\omega + \frac{\Delta\omega}{2}} \hat{\underline{\underline{C}}}_{ij}(\underline{r}, \underline{r}, \omega') d\omega' \right\} \geq 0 \quad (\text{A.12})$$

Thus, it is sufficient to prove A.12 to complete the proof of theorem 2.

It is seen that since $\underline{\underline{J}}(\underline{r}; \omega, \Delta\omega)$ is the integral of an hermitian matrix, it also is hermitian. If the eigen values of $\underline{\underline{J}}(\underline{r}; \omega, \Delta\omega)$ are denoted by $\gamma^a(\underline{r}; \omega, \Delta\omega)$, and the eigen vectors by $\underline{v}^a(\underline{r}; \omega, \Delta\omega)$, then according to the argument used in theorem 1 it is seen that

$$\begin{aligned} \underline{v}_i^{a*}(\underline{r}; \omega, \Delta\omega) \left[\int_{\omega - \frac{\Delta\omega}{2}}^{\omega + \frac{\Delta\omega}{2}} \int_{-\infty}^{\infty} \overline{E_i(\underline{r}, t + \tau)} E_j(\underline{r}, t) e^{i\omega' \tau} d\tau d\omega' \right] \underline{v}_j^a(\underline{r}; \omega, \Delta\omega) \\ = \gamma^a(\underline{r}; \omega, \Delta\omega) \end{aligned} \quad (\text{A.13})$$

Equation A.13 corresponds to A.7 of Theorem 1. They differ only in that the quantity within the braces in A.7 has been replaced by its integral with respect to ω in A.13. Of course, the eigen values and eigen vectors are not the same in the two cases since the two matrices are not necessarily equal. If the eigen vectors are taken within the integrals and inside the averaging process, A.13, expressed in vector notation, becomes

$$\int_{\omega - \frac{\Delta\omega}{2}}^{\omega + \frac{\Delta\omega}{2}} \int_{-\infty}^{\infty} \frac{\overline{[\underline{v}^a \cdot \underline{E}(\underline{r}, t + \tau)]^* [\underline{v}^a \cdot \underline{E}(\underline{r}, t)]}}{e^{i\omega'\tau} d\tau d\omega'} = \gamma^a(\underline{r}; \omega, \Delta\omega) \quad (\text{A.14})$$

Since $\underline{v}^a(\underline{r}; \omega, \Delta\omega) \cdot \underline{E}(\underline{r}, t)$ is a scalar random process derived from the projection of the electric vector at \underline{r} onto the fixed vector $\underline{v}^a(\underline{r}; \omega, \Delta\omega)$ defined at \underline{r} , then the quantity

$$\hat{F}(\underline{r}, \omega') = \int_{-\infty}^{\infty} \frac{\overline{[\underline{v}^a \cdot \underline{E}(\underline{r}, t + \tau)]^* [\underline{v}^a \cdot \underline{E}(\underline{r}, t)]}}{e^{i\omega'\tau} d\tau} \quad (\text{A.15})$$

is a power spectral density and therefore is a real, positive function of ω' (See Theorem 1). Consequently,

$$\gamma^a(\underline{r}; \omega, \Delta\omega) = \int_{\omega - \frac{\Delta\omega}{2}}^{\omega + \frac{\Delta\omega}{2}} \hat{F}(\underline{r}, \omega') d\omega' \quad (\text{A.16})$$

is positive

To complete the proof it is only necessary to recall that the determinant of an hermitian matrix equals the product of the matrix's eigenvalues. Thus,

$$\text{Det} \left\{ \int_{\omega - \frac{\Delta\omega}{2}}^{\omega + \frac{\Delta\omega}{2}} \hat{\xi}_{ij}(\underline{r}, \underline{r}, \omega') d\omega' \right\} = \prod_{i=1}^3 \gamma^i(\underline{r}; \omega, \Delta\omega) \geq 0 \quad (\text{A. 17})$$

Q. E. D.

APPENDIX B: FURTHER RESULTS CONCERNING THE STATISTICS OF THE INSTANTANEOUS ELLIPSE

In this Appendix it is shown that the integral of $P(I, \theta, u)$ over its domain of definition is unity. The various marginal distribution are derived also. The results are summarized at the end of this appendix.

Denote the integral of $P(I, \theta, u)$ by α . Then from 2.4.7

$$\alpha = \frac{2}{\pi s_o^2 (1-p^2)} \int_{-1}^1 \int_{-\pi/2}^{\pi/2} \int_0^\infty I \exp \left\{ -\frac{2I}{s_o^2 (1-p^2)} \left[s_o - s_3 u - \sqrt{s_1^2 + s_2^2} \sqrt{1-u^2} \cos \left(2\theta - \tan^{-1} s_2/s_1 \right) \right] \right\} dI d\theta du \quad (B.1)$$

Integrating first with respect to I , one obtains

$$\alpha = \frac{s_o^2 (1-p^2)}{2\pi} \int_{-1}^1 \int_{-\pi/2}^{\pi/2} \left[s_o - s_3 u - \sqrt{s_1^2 + s_2^2} \sqrt{1-u^2} \cos \left(2\theta - \tan^{-1} \frac{s_2}{s_1} \right) \right]^{-2} d\theta du \quad (B.2)$$

To integrate with respect to θ , introduce the substitution $\psi = (\pi + 2\theta - \tan^{-1} s_2/s_1)$ which reduces α to the form

$$\alpha = \frac{s_o^2 (1-p^2)}{4\pi} \int_{-1}^1 \int_{-\theta}^{2\pi-\theta} \left[s_o - s_3 u + \sqrt{s_1^2 + s_2^2} \sqrt{1-u^2} \cos \psi \right]^{-2} d\psi du$$

$$= \frac{s_o^2 (1-p^2)}{4\pi} \int_{-1}^1 \left\{ \left(\int_{-\phi}^0 + \int_0^{2\pi} - \int_{2\pi-\phi}^{2\pi} \right) \cdot \left[s_o - s_3 u + \sqrt{s_1^2 + s_2^2} \sqrt{1-u^2} \cos \psi \right]^{-2} d\psi \right\} du \quad (B.3)$$

where $\phi = \tan^{-1} s_2/s_1$. The integral from $-\phi$ to 0 cancels the integral from $2\pi - \phi$ to 2π . Thus,

$$a = \frac{s_o^2 (1-p^2)}{4\pi} \int_{-1}^1 \int_0^{2\pi} \frac{1}{\left[s_o - s_3 u + \sqrt{s_1^2 + s_2^2} \sqrt{1-u^2} \cos \psi \right]^2} d\psi du \quad (B.4)$$

From 444.01 of Dwight (42), it is found that

$$a = \frac{s_o^2 (1-p^2)}{2} \int_{-1}^1 \frac{(s_o - s_3 u)}{\left[(s_o - s_3 u)^2 - (1-u^2)(s_1^2 + s_2^2) \right]^{3/2}} du \quad (B.5)$$

Equation B.5 can be expressed in a more convenient form.

$$a = p^2 (1-p^2) \int_{-1}^1 \frac{(\frac{1}{p} - u_1 u)}{\left[(u - \frac{u_1}{p})^2 + (\frac{1}{p^2} - 1)(1-u_1^2) \right]^{3/2}} du \quad (B.6)$$

where p is the degree of polarization and $u_1 = s_3 / \sqrt{s_1^2 + s_2^2 + s_3^2}$.

It is tedious but straightforward to show (see Dwight 200.03 and 201.9)

that B.6 reduces to $a = 1$.

The marginal distributions include: $p(I, \theta)$, $p(I, u)$, $p(u, \theta)$, $p(I)$, $p(\theta)$, $p(u)$. Of these, $p(\theta, u)$ has been evaluated in B.2, and $p(u)$ is given in B.6. That is,

$$p(u, \theta) = \frac{s_o^2 (1-p^2)}{2\pi} \frac{1}{\left[s_o - s_3 u - \sqrt{s_1^2 + s_2^2} \sqrt{1-u^2} \cos \left(2\theta - \tan^{-1} \frac{s_2}{s_1} \right) \right]^2} \quad (B.7)$$

$$p(u) = p^4 (1-p^2) \left\{ \frac{(1 - p u_1 u)}{\left[(p u - u_1)^2 + (1 - p^2) (1 - u_1^2) \right]^{3/2}} \right\} \quad (B.8)$$

The distribution $p(I, u)$ is found by integrating 2.4, 7 with respect to θ . This requires that the integral

$$\beta = \int_{-\pi/2}^{\pi/2} \exp \left\{ \frac{2I}{s_o^2 (1-p^2)} \sqrt{s_1^2 + s_2^2} \sqrt{1-u^2} \cos \left(2\theta - \tan^{-1} \frac{s_2}{s_1} \right) \right\} d\theta \quad (B.9)$$

be evaluated. The substitution $\psi = \pi + 2\theta - \phi$, where $\phi = \tan^{-1} \frac{s_2}{s_1}$, leads to the integral

$$\beta = \frac{1}{2} \int_{-\phi}^{2\pi-\phi} e^{z \cos \psi} d\psi \quad (B.10)$$

where

$$z = \frac{2I}{s_o^2 (1-p^2)} \sqrt{s_1^2 + s_2^2} \sqrt{1-u^2}, \quad (B.11)$$

As before the integral from $-\phi$ to 0 cancels with the integral from $(2\pi - \phi)$ to 2π . Thus

$$\beta = \frac{1}{2} \int_0^{2\pi} e^{z \cos \psi} d\psi, \quad (\text{B. 12})$$

which can be written

$$\beta = \int_0^{\pi} \cosh(z \cos \psi) d\psi$$

This integral represents the modified Bessel function of order zero, i.e., *

$$\beta = \pi I_0(z) \quad (\text{B. 13})$$

Thus,

$$p(I, u) = \frac{2I}{s_o^2 (1 - p^2)} \exp \left[- \frac{2I}{s_o^2 (1 - p^2)} (s_o - s_3 u) \right] \cdot I_0 \left(\frac{2I}{s_o^2 (1 - p^2)} \sqrt{s_1^2 + s_2^2} \sqrt{1 - u^2} \right) \quad (\text{B. 14})$$

Unfortunately the marginal distribution $p(I, \theta)$ cannot be obtained in closed form. This distribution involves the integral

* See, e.g., Watson (43) page 79.

$$\gamma = \int_{-1}^1 \exp \left[au + b\sqrt{1 - u^2} \right] du \quad (\text{B. 15})$$

where

$$a = \frac{2Is_3}{s_o^2 (1 - p^2)}, \quad b = \frac{2I\sqrt{s_1^2 + s_2^2}}{s_o^2 (1 - p^2)} \cos \left(2\theta - \tan^{-1} \frac{s_2}{s_1} \right) \quad (\text{B. 16})$$

Since the integrand is well behaved on the interval $(-1, 1)$, there is little reason to consider this integral further. Although it can be expressed in various forms, the most desirable form depends on the information required from the integral. Therefore, $p(I, \theta)$ will simply be written as

$$p(I, \theta) = \frac{2I}{\pi s_o^2 (1 - p^2)} \gamma(I, \theta) \exp \left[- \frac{2I}{s_o (1 - p^2)} \right] \quad (\text{B. 17})$$

where $\gamma(I, \theta)$ is given by B.15.

The marginal distribution of I is most easily determined from $p(I, u)$. If B.14 is integrated with respect to u , it is found that

$$p(I) = \frac{2I}{\pi s_o^2 (1 - p^2)} \exp \left[- \frac{2I}{s_o (1 - p^2)} \right] \int_{-1}^1 I_o \left(u\sqrt{1 - u^2} \right) e^{\beta u} du \quad (\text{B. 18})$$

where

$$\alpha = \frac{2\sqrt{s_1^2 + s_2^2}}{s_o(1-p^2)} I \quad , \quad \beta = \frac{2s_3}{s_o(1-p^2)} I \quad (B.19)$$

Let $u = \cos \theta$; then the integral in B.18 becomes

$$\eta = \int_0^\pi I_o(\alpha \sin \theta) e^{\beta \cos \theta} \sin \theta d\theta \quad (B.20)$$

This is a specific example of a class of integral evaluated in Watson (43) (equation (1), page 379). For this case Watson's formula reduces to

$$\eta = \sqrt{\frac{2\pi}{i\sqrt{\alpha^2 + \beta^2}}} J_{1/2}\left(i\sqrt{\alpha^2 + \beta^2}\right) \quad (B.21)$$

which can be written more simply as

$$\eta = \frac{2 \sinh \sqrt{\alpha^2 + \beta^2}}{\sqrt{\alpha^2 + \beta^2}} \quad (B.22)$$

If B.19 is substituted into B.22, and that formula is substituted into B.18 it is found that

$$p(I) = \frac{1}{\pi s_o p} \sinh \left[\frac{2Ip}{s_o(1-p^2)} \right] \exp \left[- \frac{2I}{s_o(1-p^2)} \right] \quad (B.23)$$

which can be written in the alternative form

$$p(I) = \frac{1}{2\pi\sqrt{s_1^2 + s_2^2 + s_3^2}} \left\{ \exp \left[- \frac{2I}{s_o(1+p)} \right] - \exp \left[- \frac{2I}{s_o(1-p)} \right] \right\} \quad (B.24)$$

To obtain $p(\theta)$ it is most convenient to begin with $p(u, \theta)$. It is seen from B.7 that

$$p(\theta) = \frac{s_o^2 (1 - p^2)}{2\pi} \int_{-1}^1 \frac{du}{\left[s_o - s_3 u - \sqrt{s_1^2 + s_2^2} \sqrt{1-u^2} \cos\left(2\theta - \tan^{-1} \frac{s_2}{s_1}\right) \right]^2} \quad (\text{B.25})$$

After the transformation of variables $u = \cos\psi$ it is found that

$$p(\theta) = \frac{s_o^2 (1 - p^2)}{2\pi} \int_0^\pi \frac{\sin\psi \, d\psi}{\left[s_o - s_3 \cos\psi - \sin\psi \sqrt{s_1^2 + s_2^2} \cos\left(2\theta - \tan^{-1} \frac{s_2}{s_1}\right) \right]^2} \quad (\text{B.26})$$

Rather than deal directly with this integral, it is easier to consider

$$\xi = \int_0^\pi \frac{dx}{[a + b \cos x + c \sin x]} \quad (\text{B.27})$$

Then

$$p(\theta) = - \frac{s_o^2 (1 - p^2)}{2\pi} \left. \frac{\partial \xi}{\partial c} \right| \begin{array}{l} a = s_o \\ b = -s_3 \\ c = \sqrt{s_1^2 + s_2^2} \cos\left(2\theta - \tan^{-1} \frac{s_2}{s_1}\right) \end{array} \quad (\text{B.28})$$

The integral ξ can be written

$$\xi = \int_{\phi}^{\pi+\phi} \frac{dy}{a + r \sin y} \quad (\text{B. 29})$$

where

$$r = \sqrt{b^2 + c^2}, \quad \phi = \tan^{-1} \frac{b}{c} \quad (\text{B. 30})$$

Dwight (42) provides this integral. Since $a^2 > r^2$

$$\xi = \frac{2}{\sqrt{a^2 - b^2 - c^2}} \left\{ \tan^{-1} \left[\frac{(-a \cot \frac{\phi}{2}) + r}{\sqrt{a^2 - b^2 - c^2}} \right] - \tan^{-1} \left[\frac{(a \tan \frac{\phi}{2}) + r}{\sqrt{a^2 - b^2 - c^2}} \right] \right\} \quad (\text{B. 31})$$

By considering the tangent of the quantity in the braces it is found that

B. 31 reduces to

$$\xi = \frac{2}{\sqrt{a^2 - r^2}} \tan^{-1} \left[\frac{\sqrt{a^2 - r^2}}{c} \right] \quad (\text{B. 32})$$

where r is defined in B. 30 . If B. 32 is substituted into B. 28 , the following formula results:

$$\begin{aligned}
 p(\theta) = \frac{q^2 (1 - p^2)}{2\pi} & \left\{ \frac{2}{\left[q^2 (1 - p^2) + \sin^2 \left(2\theta - \tan^{-1} \frac{s_2}{s_1} \right) \right]} \right. \\
 & - \frac{\cos \left(2\theta - \tan^{-1} \frac{s_2}{s_1} \right)}{\left[q^2 (1 - p^2) + \sin^2 \left(2\theta - \tan^{-1} \frac{s_2}{s_1} \right) \right]^{3/2}} \\
 & \left. + \tan^{-1} \left[\frac{\sqrt{q^2 (1 - p^2) + \sin^2 \left(2\theta - \tan^{-1} \frac{s_2}{s_1} \right)}}{\cos \left(2\theta - \tan^{-1} \frac{s_2}{s_1} \right)} \right] \right\} \quad (B. 33)
 \end{aligned}$$

where

$$q^2 = \frac{s_o^2}{s_1^2 + s_2^2}$$

This completes the analysis of the probability distributions of the instantaneous ellipse. The pertinent formulas are summarized below.

$$\begin{aligned}
 p(I, \theta, u) = \frac{2I}{\pi s_o^2 (1 - p^2)} \exp & \left\{ - \frac{2I}{s_o^2 (1 - p^2)} \left[s_o - s_3 u - \sqrt{s_1^2 + s_2^2} \sqrt{1 - u^2} \right. \right. \\
 & \left. \left. \cdot \cos \left(2\theta - \tan^{-1} \frac{s_2}{s_1} \right) \right] \right\}
 \end{aligned}$$

$$p(\theta, u) = \frac{s_o^2(1-p^2)}{2\pi} \left\{ \frac{1}{\left[s_o - s_3 u - \sqrt{s_1^2 + s_2^2} \sqrt{1-u^2} \cos \left(2\theta - \tan^{-1} \frac{s_2}{s_1} \right) \right]^2} \right\}$$

$$p(I, u) = \frac{2I}{s_o^2(1-p^2)} \left\{ \exp \left[- \frac{2I}{s_o^2(1-p^2)} (s_o - s_3 u) \right] \cdot I_o \left(\frac{2\sqrt{s_1^2 + s_2^2}}{s_o^2(1-p^2)} I \sqrt{1-u^2} \right) \right\}.$$

$$p(I, \theta) = \frac{2I}{\pi s_o^2(1-p^2)} \left\{ \exp \left[- \frac{2I}{s_o^2(1-p^2)} \right] \int_{-1}^1 \exp \left[au + b \sqrt{1-u^2} \right] du \right\}$$

where

$$a = \frac{2s_3}{s_o^2(1-p^2)} I, \quad b = \frac{2\sqrt{s_1^2 + s_2^2}}{s_o^2(1-p^2)} I \cos \left(2\theta - \tan^{-1} \frac{s_2}{s_1} \right)$$

$$p(I) = \frac{1}{\pi s_o p} \left\{ \sinh \left[\frac{2Ip}{s_o^2(1-p^2)} \right] \exp \left[- \frac{2I}{s_o^2(1-p^2)} \right] \right\}$$

$$p(\theta) = \frac{q^2 (1 - p^2)}{2\pi} \left\{ \frac{2}{\left[q^2 (1 - p^2) + \sin^2 \left(2\theta - \tan^{-1} \frac{s_2}{s_1} \right) \right]} - \frac{\cos \left(2\theta - \tan^{-1} \frac{s_2}{s_1} \right)}{\left[q^2 (1 - p^2) + \sin^2 \left(2\theta - \tan^{-1} \frac{s_2}{s_1} \right) \right]^{3/2}} \tan^{-1} \left[\frac{\sqrt{q^2 (1 - p^2) + \sin^2 \left(2\theta - \tan^{-1} \frac{s_2}{s_1} \right)}}{\cos \left(2\theta - \tan^{-1} \frac{s_2}{s_1} \right)} \right] \right\}$$

where

$$q^2 = \left(\frac{s_o^2}{s_1^2 + s_2^2} \right) .$$

$$p(u) = p^4 (1 - p^2) \left\{ \frac{(1 - pu_1 u)}{\left[(pu - u_1)^2 + (1 - p^2) (1 - u_1^2) \right]^{3/2}} \right\}$$

where

$$u_1 = \frac{s_3}{\sqrt{s_1^2 + s_2^2 + s_3^2}} .$$

APPENDIX C: THE BEHAVIOR OF THE DFS AT INFINITY

To determine the DFS uniquely in the case of exterior boundary value problems, it is necessary to specify the behavior of the DFS at infinity. In other words it is necessary to have a radiation condition for the DFS in order to specify the DFS uniquely. This condition can be derived directly from the radiation condition on the vector field. However, the vector radiation condition is generally stated for monochromatic fields, whereas in this problem the field quantities are stationary, irregularly varying functions of time which do not possess Fourier transforms in the ordinary sense. Therefore, it is necessary to start from a radiation condition for fields having non-sinusoidal time variations. In the following paragraphs this condition is obtained and from it the radiation condition for the DFS is derived.

The radiation condition for electro magnetic fields is intimately connected with the vanishing of the surface integral

$$\hat{I}_S(\underline{r}, \omega) = \oint \left\{ i\omega\mu_0 \left[\underline{m} \times \underline{\hat{H}}(\underline{r}', \omega) \right] \psi + \left[\underline{m} \times \underline{\hat{E}}(\underline{r}', \omega) \right] \times \nabla \psi + \left[\underline{m} \cdot \underline{\hat{E}}(\underline{r}', \omega) \right] \nabla \psi \right\} dS', \quad (C. 1)$$

in the limit as S approaches the sphere at infinity. In this expression ψ represents the free space scalar Green's function,

$$\frac{\pi}{4} \frac{\exp[ik|\underline{r} - \underline{r}'|]}{|\underline{r} - \underline{r}'|}.$$

The integral $\hat{I}_S(\underline{r}, \omega)$ can be generalized to non-sinusoidal fields by taking its Fourier transform. To do this it is convenient to rewrite C.1 in the form

$$\begin{aligned} \hat{\underline{I}}_S(\underline{r}, \omega) = & \oint_S \left[i \omega \mu_o \left\{ \underline{m} \times \hat{\underline{H}}(\underline{r}', \omega) + \sqrt{\frac{\epsilon_o}{\mu_o}} \left[\underline{m} \times \hat{\underline{E}}(\underline{r}', \omega) \right] \times \underline{e}_R + \right. \right. \\ & \left. \sqrt{\frac{\epsilon_o}{\mu_o}} \left[\underline{m} \cdot \hat{\underline{E}}(\underline{r}', \omega) \right] \right\} \frac{e^{ikR}}{R} - \left\{ \left[\underline{m} \times \hat{\underline{E}}(\underline{r}', \omega) \right] \times \underline{e}_R + \right. \\ & \left. \left[\underline{m} \cdot \hat{\underline{E}}(\underline{r}', \omega) \right] \underline{e}_R \right\} \frac{e^{ikR}}{R^2} \right] da' \end{aligned} \quad (C.2)$$

where $R = |\underline{r} - \underline{r}'|$ and \underline{e}_R is the unit vector directed from \underline{r}' toward \underline{r} .

The vector \underline{m} is the outward normal to S . The Fourier transform of

C.2 is

$$\begin{aligned} \underline{I}(\underline{r}, t) = & \oint_S \left[\frac{\mu_o}{R} \frac{\partial}{\partial t'} \left\{ \underline{m} \times \underline{H}(\underline{r}', t') + \sqrt{\frac{\epsilon_o}{\mu_o}} \left[\underline{m} \times \underline{E}(\underline{r}', t') \right] \times \underline{e}_R + \right. \right. \\ & \left. \sqrt{\frac{\epsilon_o}{\mu_o}} \left[\underline{m} \cdot \underline{E}(\underline{r}', t') \right] \right\} - \frac{1}{R^2} \left\{ \left[\underline{m} \times \underline{E}(\underline{r}', t') \right] \times \underline{e}_R + \right. \\ & \left. \left[\underline{m} \cdot \underline{E}(\underline{r}', t') \right] \underline{e}_R \right\} \right] da' \end{aligned} \quad (C.3)$$

where t' is the retarded time $(t - \frac{R}{C})$. Let S be a sphere centered at \underline{r} of very large radius. Thus, $\underline{m} = -\underline{e}_R$ so that C.3 becomes

$$\begin{aligned} \underline{I}_S(\underline{r}, t) = & \oint_S \left\{ \frac{\mu_o}{r'} \frac{\partial}{\partial t'} \left[\underline{e}_{r'} \times \underline{H}(\underline{r}', t') + \sqrt{\frac{\epsilon_o}{\mu_o}} \underline{E}(\underline{r}', t') \right] - \right. \\ & \left. \frac{1}{r'^2} \underline{E}(\underline{r}', t') \right\} r'^2 d\Omega \end{aligned} \quad (C.4)$$

where $d\Omega$ is an element of solid angle on the sphere of radius R . This integral vanishes if

$$\lim_{r' \rightarrow \infty} \underline{E}(\underline{r}', t') = 0 \quad (C. 5)$$

$$\lim_{r' \rightarrow \infty} r' \left[\underline{e}_{r'} \times \underline{H}(\underline{r}', t') + \sqrt{\frac{\epsilon_0}{\mu_0}} \underline{E}(\underline{r}', t') \right] = 0 \quad (C. 6)$$

where t' and r' are related such that $t = t' + r'/C$ is constant. Thus,

C. 6 requires that the non-sinusoidal field behave like an outgoing wave at infinity where the amplitude of the wave is evaluated at the retarded time t' .

Since C. 6 must hold for $-\infty \leq t \leq \infty$, then it can be replaced by the simpler condition

$$\lim_{r \rightarrow \infty} r \left[\underline{e}_r \times \underline{H}(\underline{r}, t) + \sqrt{\frac{\epsilon_0}{\mu_0}} \underline{E}(\underline{r}, t) \right] = 0, \quad -\infty \leq t \leq \infty. \quad (C. 7)$$

Equation C. 7 is the starting point for the derivation of the radiation condition for the DFS. In view of C. 7 it is certainly true that

$$\lim_{r_1 \rightarrow \infty} r_1 \left[\underline{e}_{r_1} \times \underline{H}(\underline{r}_1, t_1) \underline{E}(\underline{r}_2, t_2) + \sqrt{\frac{\epsilon_0}{\mu_0}} \underline{E}(\underline{r}_1, t_1) \underline{E}(\underline{r}_2, t_2) \right] = 0 \quad (C. 8)$$

where \underline{r}_2 is an arbitrary fixed point located a finite distance from the origin. In the case of a noise-like field this result must hold for each sample function; thus it must hold on the average. After interchanging the order of averaging and limit taking, it is found that

$$\lim_{r_1 \rightarrow \infty} r_1 \left[\underline{e}_{r_1} \times \underline{\hat{H}}(\underline{r}_1, \underline{r}_2, \tau) + \sqrt{\frac{\epsilon_0}{\mu_0}} \underline{\hat{E}}(\underline{r}_1, \underline{r}_2, \tau) \right] = 0. \quad (C. 9)$$

The corresponding condition in the frequency domain is

$$\lim_{r_1 \rightarrow \infty} r_1 \left[\underline{e}_{r_1} \times \underline{\hat{H}}(\underline{r}_1, \underline{r}_2, \omega) + \sqrt{\frac{\epsilon_0}{\mu_0}} \underline{\hat{E}}(\underline{r}_1, \underline{r}_2, \omega) \right] = 0. \quad (C. 10)$$

This is the desired condition. A corresponding result can be obtained for the limit as $r_2 \rightarrow \infty$. It is

$$\lim_{r_2 \rightarrow \infty} r_2 \left[\underline{e}_{r_2} \times \underline{y}^T(\underline{r}_1, \underline{r}_2, \omega) + \sqrt{\frac{\epsilon_0}{\mu_0}} \underline{\xi}^T(\underline{r}_1, \underline{r}_2, \omega) \right] = 0. \quad (C. 11)$$

Since the DFS involves two points, a third radiation condition can be derived. This condition involves the limit as both \underline{r}_1 and \underline{r}_2 approach infinity. It is of direct use primarily in connection with the uniqueness theorem for the DFS in an unbounded medium. However, since that theorem is not considered here the third radiation condition will not be derived.

Equations C.10 and C.11 can be used to show that the surface integrals that arise in connection with the integration of 4.1.5 and 4.1.13 approach zero as the surface approaches infinity. The surface integral associated with 4.1.5 is

$$I_S^{(1)} = \oint_S \left[\hat{\underline{Q}}(\underline{r}'_1, \underline{r}_2^0, \omega) \cdot \underline{a} \right] \times \left[\underline{\nabla}'_1 \times \hat{\underline{\Gamma}}(\underline{r}'_1, \underline{r}_1, \omega) \right] \cdot \underline{b} - \left[\hat{\underline{\Gamma}}(\underline{r}'_1, \underline{r}_1, \omega) \cdot \underline{b} \right] \times \left[\underline{\nabla}'_1 \times \hat{\underline{Q}}(\underline{r}'_1, \underline{r}_2^0, \omega) \right] \cdot \underline{a} \} \cdot \underline{ds}'_1 \quad (C. 12)$$

By using the vector identity $\underline{A} \cdot \underline{B} \times \underline{C} = \underline{B} \cdot \underline{C} \times \underline{A} = \underline{C} \cdot \underline{A} \times \underline{B}$ and the associative property of dyadics it is found that C.12 can be written

$$I_S^{(1)} = - \oint_S \underline{a} \cdot \left\{ \hat{\underline{Q}}^T(\underline{r}'_1, \underline{r}_2^0, \omega) \cdot \left[\underline{m}^{(1)} \times \underline{\nabla}'_1 \times \hat{\underline{\Gamma}}(\underline{r}'_1, \underline{r}_1, \omega) \right] + \left[\underline{\nabla}'_1 \times \hat{\underline{Q}}(\underline{r}'_1, \underline{r}_2^0, \omega) \right]^T \cdot \left[\underline{m}^{(1)} \times \hat{\underline{\Gamma}}(\underline{r}'_1, \underline{r}_1, \omega) \right] \right\} \cdot \underline{b} \, ds'_1 \quad (C. 13)$$

where $\underline{m}^{(1)}$ is the outward normal to S at \underline{r}'_1 .

If a and b are removed from the integral, then the dyadic integral that remains is

$$\begin{aligned} I_S^{(1)} = & - \oint_S \left[(\nabla_2^0 \times \nabla_2^0 \times - k^2 \underline{\underline{I}} \cdot) \hat{\underline{\underline{E}}}^T(\underline{r}_1', \underline{r}_2^0, \omega) \right] \cdot \left[\underline{m}^{(1)} \times \nabla_1' \times \hat{\underline{\underline{A}}}(\underline{r}_1', \underline{r}_1, \omega) \right] + \\ & \left[\nabla_1' \times \left[(\nabla_2^0 \times \nabla_2^0 \times - k^2 \underline{\underline{I}} \cdot) \hat{\underline{\underline{E}}}^T(\underline{r}_1', \underline{r}_2^0, \omega) \right]^T \right]^T \cdot \left[\underline{m}^{(1)} \times \hat{\underline{\underline{A}}}(\underline{r}_1', \underline{r}_1, \omega) \right] \Big\} ds_1' \end{aligned} \quad (C. 14)$$

where 4.1.4 has been used to eliminate $\hat{\underline{\underline{Q}}}(\underline{r}_1', \underline{r}_2^0, \omega)$.

Since

$$\begin{aligned} \left\{ \nabla_1' \times \left[(\nabla_2^0 \times \nabla_2^0 \times - k^2 \underline{\underline{I}} \cdot) \hat{\underline{\underline{E}}}^T(\underline{r}_1', \underline{r}_2^0, \omega) \right]^T \right\}^T = & (\nabla_2^0 \times \nabla_2^0 \times - k^2 \underline{\underline{I}} \cdot) \\ & \left[\nabla_1' \times \hat{\underline{\underline{E}}}(\underline{r}_1', \underline{r}_2^0, \omega) \right]^T, \end{aligned}$$

then, after interchanging orders of integration and differentiation, it is seen that

$$\begin{aligned} I_S^{(1)} = & - (\nabla_2^0 \times \nabla_2^0 \times - k^2 \underline{\underline{I}} \cdot) \oint_S \left\{ \hat{\underline{\underline{E}}}^T(\underline{r}_1', \underline{r}_2^0, \omega) \cdot \left[\underline{m}^{(1)} \times \nabla_1' \times \hat{\underline{\underline{A}}}(\underline{r}_1', \underline{r}_1, \omega) \right] + \right. \\ & \left. \left[\nabla_1' \times \hat{\underline{\underline{E}}}(\underline{r}_1', \underline{r}_2^0, \omega) \right]^T \cdot \left[\underline{m}^{(1)} \times \hat{\underline{\underline{A}}}(\underline{r}_1', \underline{r}_1, \omega) \right] \right\} ds_1' \end{aligned} \quad (C. 15)$$

Suppose now that S represents a large sphere centered at \underline{r}_1 . In that event

$$\underline{m}^{(1)} \times \hat{\underline{\underline{A}}}(\underline{r}_1', \underline{r}_1, \omega) = \underline{e}_{r_1'} \times (\underline{e}_{\theta_1'} \underline{e}_{\theta_1'} + \underline{e}_{\phi_1'} \underline{e}_{\phi_1'}) \frac{e^{ikr_1'}}{r_1'} + O\left(\frac{1}{r_1'^2}\right). \quad (C. 16)$$

$$\begin{aligned} \underline{I}_S^{(2)} = - \oint_S \left\{ \hat{\underline{\xi}}(\underline{r}_1^0, \underline{r}_2', \omega) \cdot \left[\underline{m}^{(2)} \times \nabla_2' \times \underline{\Gamma}^*(\underline{r}_2', \underline{r}_2, \omega) \right] + \right. \\ \left. \left[\nabla_2' \times \hat{\underline{\xi}}^T(\underline{r}_1^0, \underline{r}_2', \omega) \right]^T \cdot \left[\underline{m}^{(2)} \times \underline{\Gamma}^*(\underline{r}_2', \underline{r}_2, \omega) \right] \right\} ds_2' . \end{aligned} \quad (C. 20)$$

Choosing S to be a sphere of large radius centered at \underline{r}_2 , one obtains for the complex conjugate dyadic Green's function

$$\underline{m}^{(2)} \times \underline{\Gamma}^*(\underline{r}_2', \underline{r}_2, \omega) = \underline{e}_{r_2'} \times (\underline{e}_{\theta_1} \underline{e}_{\theta_1} + \underline{e}_{\phi_1} \underline{e}_{\phi_1}) \frac{e^{-ikr_2'}}{r_2'} + o\left(\frac{1}{r_2'}\right) \quad (C. 21)$$

$$\underline{m}^{(2)} \times \nabla_2' \times \underline{\Gamma}^*(\underline{r}_2', \underline{r}_2, \omega) = ik (\underline{e}_{\theta_1} \underline{e}_{\theta_1} + \underline{e}_{\phi_1} \underline{e}_{\phi_1}) \frac{e^{-ikr_2'}}{r_2'} + o\left(\frac{1}{r_2'}\right) \quad (C. 22)$$

Thus, the far zone approximation of C. 20 is

$$\begin{aligned} \underline{I}_S^{(2)} \approx - \oint_S \left\{ ik \hat{\underline{\xi}}(\underline{r}_1^0, \underline{r}_2', \omega) + \left[\nabla_2' \times \hat{\underline{\xi}}^T(\underline{r}_1^0, \underline{r}_2', \omega) \right]^T \right\} \cdot \\ (\underline{e}_{\theta_1} \underline{e}_{\theta_1} + \underline{e}_{\phi_1} \underline{e}_{\phi_1}) e^{-ikr_2'} r_2' d\Omega' . \end{aligned} \quad (C. 23)$$

But from 3.2.17 it is seen that

$$\left[\nabla_2' \times \hat{\underline{\xi}}^T(\underline{r}_1^0, \underline{r}_2', \omega) \right]^T = ik \sqrt{\frac{\mu_0}{\epsilon_0}} \hat{\underline{g}}^T(\underline{r}_1^0, \underline{r}_2', \omega) . \quad (C. 24)$$

Thus

$$\begin{aligned} \underline{I}_S^{(2)} \approx - ik \oint_S r_1' \left\{ \hat{\underline{\xi}}^T(\underline{r}_1^0, \underline{r}_2', \omega) + \sqrt{\frac{\mu_0}{\epsilon_0}} \hat{\underline{g}}^T(\underline{r}_1^0, \underline{r}_2', \omega) \right\}^T \cdot \\ (\underline{e}_{\theta_1} \underline{e}_{\theta_1} + \underline{e}_{\phi_1} \underline{e}_{\phi_1}) e^{-ikr_2'} d\Omega' . \end{aligned} \quad (C. 25)$$

The radiation condition C.11 implies that this integral approaches zero as S approaches infinity. Note that if the ordinary dyadic Green's function $\hat{\underline{\underline{G}}}(\underline{r}_2, \underline{r}_2', \omega)$ were used instead of its complex conjugate, the sign of C.22 would have been different so that C.11 would not have implied the vanishing of $\underline{I}_S^{(2)}$.

APPENDIX D: THE SOURCE FOR A PLANE WAVE

In most problems that involve an incident plane wave, the analysis is formulated directly in terms of the plane wave field with no attention given to the source of the wave. However, because of the approach taken in this dissertation it is desirable to characterize an incident plane wave in terms of its source. Rather than attempt to deduce the nature of the source of a plane wave from the character of the field, it is easier to propose a source and then verify that this source does indeed lead to a plane wave.

The proposed source is the surface current

$$\hat{\underline{J}}(\underline{r}, \omega) = \lim_{z_0 \rightarrow -\infty} \left[2 \sqrt{\frac{\epsilon_0}{\mu_0}} \hat{\underline{E}}_0 e^{ikz_0} \delta(z - z_0) \right] \quad (D. 1)$$

where the limit operation is included to place the source of the plane wave at infinity; the factor 2 accounts for the fact that the current sheet radiates its power to both sides of the plane $z = z_0$; the factor e^{ikz_0} is introduced to refer the phase of the plane wave to the origin; and the vector $\hat{\underline{E}}_0$, a constant (possibly complex) vector, is assumed to lie parallel to the plane $z = z_0$.

To verify that the field associated with this source is a plane wave observe that in general the radiation field associated with a prescribed harmonic current is given by

$$\hat{\underline{E}}(\underline{r}, \omega) = ik \sqrt{\frac{\epsilon_0}{\mu_0}} \int_{V_s} \hat{\underline{J}}(\underline{r}', \omega) \cdot \hat{\underline{J}}(\underline{r}, \omega) d\mathbf{x}' d\mathbf{y}' d\mathbf{z}' \quad (D. 2)$$

where $\hat{\underline{\underline{G}}}$ represents the free space dyadic Green's function. In this case after integration with respect to \underline{z}' D.2 becomes

$$\hat{\underline{\underline{E}}}(\underline{r}, \omega) = \frac{ik}{2\pi} \lim_{z_o \rightarrow -\infty} \left\{ \left(\underline{\underline{I}} + \frac{1}{k^2} \nabla \nabla \right) e^{ikz_o} \cdot \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{e^{ik\sqrt{(x-x')^2 + (y-y')^2 + (z-z_o)^2}}}{\sqrt{(x-x')^2 + (y-y')^2 + (z-z_o)^2}} dx' dy' \right\} \cdot \hat{\underline{\underline{E}}}_o \quad (D.3)$$

If $(x-x')$ and $(y-y')$ are replaced by u and v respectively, and if the transformation $u = \rho \cos \phi$, $v = \rho \sin \phi$ is then made, D.3 becomes

$$\hat{\underline{\underline{E}}}(\underline{r}, \omega) = ik \lim_{z_o \rightarrow -\infty} \left\{ \left(\underline{\underline{I}} + \frac{1}{k^2} \nabla \nabla \right) e^{ikz_o} \cdot \int_0^{\infty} \frac{e^{ik\sqrt{\rho^2 + (z-z_o)^2}}}{\sqrt{\rho^2 + (z-z_o)^2}} \rho d\rho \right\} \cdot \hat{\underline{\underline{E}}}_o \quad (D.4)$$

The further substitution $t^2 = \rho^2 + (z-z_o)^2$ yields

$$\hat{\underline{\underline{E}}}(\underline{r}, \omega) = ik \lim_{z_o \rightarrow -\infty} \left\{ \left(\underline{\underline{I}} + \frac{1}{k^2} \nabla \nabla \right) e^{ikz_o} \int_{|z-z_o|}^{\infty} e^{ikt} dt \right\} \cdot \hat{\underline{\underline{E}}}_o \quad (D.5)$$

Since in any real medium there is always some dissipation, the propagation constant k will have a small, but finite (and positive) imaginary part.

Thus the integral is zero at the upper limit and

$$\underline{\hat{E}}(\underline{r}, \omega) = \lim_{z_0 \rightarrow -\infty} \left\{ \left(\underline{I} + \frac{1}{k^2} \nabla \nabla \right) e^{ikz_0} e^{ik|z-z_0|} \right\} \cdot \underline{\hat{E}}_0 \quad (D. 6)$$

Only points $z > z_0$ are of interest in this formula; therefore,

$$\begin{aligned} \underline{\hat{E}}(\underline{r}, \omega) &= \left[\left(\underline{I} + \frac{1}{k^2} \nabla \nabla \right) e^{ikz} \right] \cdot \underline{\hat{E}}_0, \\ &= \left[\underline{I} - \frac{\underline{e} \underline{e}}{z-z} \right] \cdot \underline{\hat{E}}_0 e^{ikz} \end{aligned} \quad (D. 7)$$

Since $\underline{\hat{E}}_0$ lies in the plane parallel to $z = z_0$, then

$$\underline{\hat{E}}(\underline{r}, \omega) = \underline{\hat{E}}_0 e^{ikz} \quad (D. 8)$$

which is a plane wave field that propagates in the $+z$ direction, and is polarized according to the complex vector $\underline{\hat{E}}_0$. Observe that the plane wave propagates in the direction perpendicular to the plane of the source. Thus, to obtain a wave that propagates in some direction other than the $+z$ direction it is only necessary to orient the surface current perpendicular to this direction.

APPENDIX E: MEASUREMENT OF THE DFS

Equation 5.4.17 provides a relationship between the power received by an interferometer and the DFS. The problem is to develop a procedure for solving this relationship for the individual terms of the DFS. Consider first the determination of the two diagonal terms $\hat{\mathcal{E}}_{11s}(\underline{r}_1, \underline{r}_2, \omega)$ and $\hat{\mathcal{E}}_{22s}(\underline{r}_1, \underline{r}_2, \omega)$.^{*} If the two elements of the interferometer are linearly polarized along \underline{e}_1 , then

$$\hat{W}_{\text{rec}} \approx K \left\{ \hat{\mathcal{E}}_{11s}(\underline{r}, \underline{r}, \omega) + \text{Re} \left[\hat{\mathcal{E}}_{11s}(\underline{r}_1, \underline{r}_2, \omega) e^{i\psi} \right] \right\} \quad (\text{E. 1})$$

where $K = \frac{\hat{Q}_s(\theta_o, \phi_o, \omega)}{2 Z_o \cos^2(k\underline{L} \cdot \underline{e}_r)}$. By measuring the output from either of the elements of the interferometer $\hat{\mathcal{E}}_{11s}(\underline{r}, \underline{r}, \omega)$ can be found. This value substituted into (E. 1) then provides $\text{Re} \left[\hat{\mathcal{E}}_{11s}(\underline{r}_1, \underline{r}_2, \omega) e^{i\psi} \right]$. If this measurement is carried out for two values of ψ , say $\psi = 0$, $\psi = -\frac{\pi}{2}$, then both the real and imaginary parts of $\hat{\mathcal{E}}_{11s}(\underline{r}_1, \underline{r}_2, \omega)$ can be found. An equivalent procedure provides $\hat{\mathcal{E}}_{22s}(\underline{r}, \underline{r}, \omega)$ and the real and imaginary parts of $\hat{\mathcal{E}}_{22s}(\underline{r}_1, \underline{r}_2, \omega)$.

The diagonal terms are more difficult to determine. If 5.4.17 is expanded, it takes the form

* The formula 5.4.17 holds only in the far zone of the source in which case the third diagonal term, corresponding to the field component along the line of sight, is negligible.

$$\hat{W}_{\text{rec}} \approx \frac{\hat{a}_A(\theta_o, \phi_o, \omega)}{2 Z_o \cos^2(k \underline{L} \cdot \underline{e}_r)} \left\{ \hat{P}_{11} \left[\hat{\mathcal{E}}_{11s}(\underline{r}, \underline{r}, \omega) + \text{Re} \left(\hat{\mathcal{E}}_{11s}(\underline{r}_1, \underline{r}_2, \omega) e^{i\psi} \right) \right] + \right. \\ \left. \hat{P}_{22} \left[\hat{\mathcal{E}}_{22s}(\underline{r}, \underline{r}, \omega) + \text{Re} \hat{\mathcal{E}}_{22s}(\underline{r}_1, \underline{r}_2, \omega) \right] + \text{Re} \left[\hat{P}_{12} \left[\hat{\mathcal{E}}_{12s}(\underline{r}, \underline{r}, \omega) + \right. \right. \right. \\ \left. \left. \left. \hat{\mathcal{E}}_{12s}(\underline{r}_1, \underline{r}_2, \omega) e^{i\psi} + \hat{\mathcal{E}}_{21s}^*(\underline{r}_1, \underline{r}_2, \omega) e^{-i\psi} \right] \right] \right\}. \quad (\text{E. 2})$$

Since the first two terms on the right of [E. 2] can be measured independently for a given value of ψ these quantities can be placed on the left of [E. 2], thereby leaving all of the unknown quantities on the right. Then

$$K_{\text{rec}}(\psi) = P_{12}^{(r)} \left\{ \hat{\mathcal{E}}_{12s}^{(r)}(\underline{r}, \underline{r}, \omega) + \left[\hat{\mathcal{E}}_{12s}^{(r)}(\underline{r}_1, \underline{r}_2, \omega) + \hat{\mathcal{E}}_{21s}^{(r)}(\underline{r}_1, \underline{r}_2, \omega) \right] \cos \psi - \right. \\ \left. \left[\hat{\mathcal{E}}_{12s}^{(i)}(\underline{r}_1, \underline{r}_2, \omega) + \hat{\mathcal{E}}_{21s}^{(i)}(\underline{r}_1, \underline{r}_2, \omega) \right] \sin \psi \right\} - P_{12}^{(i)} \left\{ \hat{\mathcal{E}}_{12s}^{(i)}(\underline{r}, \underline{r}, \omega) + \right. \\ \left. \left[\hat{\mathcal{E}}_{12s}^{(i)}(\underline{r}_1, \underline{r}_2, \omega) - \hat{\mathcal{E}}_{21s}^{(i)}(\underline{r}_1, \underline{r}_2, \omega) \right] \cos \psi + \left[\hat{\mathcal{E}}_{12s}^{(r)}(\underline{r}_1, \underline{r}_2, \omega) - \hat{\mathcal{E}}_{21s}^{(r)}(\underline{r}_1, \underline{r}_2, \omega) \right] \sin \psi \right\}. \quad (\text{E. 3})$$

where

$$K_{\text{rec}}(\psi) = \frac{2 Z_o \cos^2(k \underline{L} \cdot \underline{e}_r) \hat{W}_{\text{rec}}}{\hat{a}_s(\theta_o, \phi_o, \omega)} - \hat{P}_{11} \left[\hat{\mathcal{E}}_{11s}(\underline{r}, \underline{r}, \omega) + \right. \\ \left. \text{Re} \left(\hat{\mathcal{E}}_{11s}(\underline{r}_1, \underline{r}_2, \omega) e^{i\psi} \right) \right] - \hat{P}_{22} \left[\hat{\mathcal{E}}_{22s}(\underline{r}, \underline{r}, \omega) + \text{Re} \left(\hat{\mathcal{E}}_{22s}(\underline{r}_1, \underline{r}_2, \omega) e^{-i\psi} \right) \right] \quad (\text{E. 4})$$

which is directly measurable. By determining K_{rec} for $\hat{P}_{12}^{(i)} = 0$,

$\psi = 0$, $\psi = \pi/2$ and for $\hat{P}_{12}^{(r)} = 0$; $\psi = 0$, $\psi = \pi/2$, one obtains four equations

in the four unknowns $\hat{\mathcal{E}}_{12}^{(n)}(\underline{r}_1, \underline{r}_2, \omega)$, $\hat{\mathcal{E}}_{12}^{(i)}(\underline{r}_1, \underline{r}_2, \omega)$, $\hat{\mathcal{E}}_{21}^{(n)}(\underline{r}_1, \underline{r}_2, \omega)$, and $\hat{\mathcal{E}}_{21}^{(i)}(\underline{r}_1, \underline{r}_2, \omega)$. The two terms $\hat{\mathcal{E}}_{12}^{(n)}(\underline{r}, \underline{r}, \omega)$, $\hat{\mathcal{E}}_{12}^{(i)}(\underline{r}, \underline{r}, \omega)$ can again be measured by observing the output of one of the elements of the interferometer.

This is not the only procedure for determining $\hat{\mathcal{E}}_{ij}(\underline{r}_1, \underline{r}_2, \omega)$. One could have the two antennas of the interferometer polarized independently. This would somewhat simplify the procedure for determining $\hat{\mathcal{E}}_{12}(\underline{r}_1, \underline{r}_2, \omega)$ and $\hat{\mathcal{E}}_{21}(\underline{r}_1, \underline{r}_2, \omega)$. However, there is little point in describing this technique in detail, since in the final analysis the particular method chosen will depend entirely on the instrumentation available and on the nature of the measurement. The main thing is that the individual terms can be obtained explicitly.

APPENDIX F: A THEOREM CONCERNING THE DEGREE OF POLARIZATION

In Chapter I it is pointed out that the measurement of polarization determines the integral of the DFS over a finite band of frequencies of width $\Delta\omega$. Similarly, the expression obtained in Chapter IV for the far zone DFS that is radiated by an incoherent partially polarized source of finite extent involves an integral of the polarization of the source with respect to position. In both cases the quantity of interest is expressed in terms of the incoherent superposition of partially polarized elements. It will now be shown that the degree of polarization of the sum of incoherent partially polarized elements is less than or equal to the mean degree of polarization of the elements themselves. The proof presented here is based on an approach set forth by Ramachandran and Ramaseshan (28).

Theorem: If

$$\underline{\underline{J}}(\underline{r};\omega, \Delta\omega) = \int_{\omega-\frac{\Delta\omega}{2}}^{\omega+\frac{\Delta\omega}{2}} \underline{\underline{\hat{C}}}(\underline{r}, \underline{r}, \omega') d\omega' \quad (\text{F. 1})$$

Then the degree of polarization, $p(\underline{r};\omega, \Delta\omega)$, of $\underline{\underline{J}}(\underline{r};\omega, \Delta\omega)$ and the degree of polarization $\hat{p}(\underline{r}, \omega)$ of $\underline{\underline{\hat{C}}}(\underline{r}, \underline{r}, \omega)$ satisfy the following inequality:

$$p(\underline{r};\omega, \Delta\omega) \leq \int_{\omega-\frac{\Delta\omega}{2}}^{\omega+\frac{\Delta\omega}{2}} \hat{f}(\underline{r}, \omega') \hat{p}(\underline{r}, \omega') d\omega' \quad (\text{F. 2})$$

where

$$f(\underline{r}, \omega') = \frac{\text{Tr } \hat{\underline{\underline{\xi}}}(\underline{r}, \underline{r}, \omega')}{\int_{\omega - \frac{\Delta\omega}{2}}^{\omega + \frac{\Delta\omega}{2}} \text{Tr } \hat{\underline{\underline{\xi}}}(\underline{r}, \underline{r}, \omega') d\omega'} , \quad (\text{F. 3})$$

i. e. $\hat{f}(\underline{r}, \omega')$ is the ratio of the intensity of the wave at frequency ω' to the total intensity of the wave over the bandwidth $\Delta\omega$. The equality sign holds if the polarization of $\hat{\underline{\underline{\xi}}}(\underline{r}, \underline{r}, \omega)$ is constant over the frequency interval $\Delta\omega$ - i. e., if

$$\hat{\underline{\underline{\xi}}}(\underline{r}, \underline{r}, \omega') = \hat{I}(\omega') \hat{\underline{\underline{\xi}}}_0(\underline{r}, \underline{r}, \omega) \quad (\text{F. 4})$$

Proof: The polarization spectrum can be represented in terms of the Stokes parameter spectrum introduced in 1.4.57. Thus the four functions $\hat{\Delta}_0(\underline{r}, \omega'), \hat{\Delta}_1(\underline{r}, \omega'), \hat{\Delta}_2(\underline{r}, \omega'), \hat{\Delta}_3(\underline{r}, \omega')$ uniquely specify the polarization of the field at frequency ω' . The polarization of $\underline{\underline{J}}(\underline{r}; \omega, \Delta\omega)$ can also be represented in terms of Stokes parameters. The relationship between the two sets of Stokes parameters is given in 1.4.58.

If a polarization vector, $\underline{p}(\underline{r}; \omega, \Delta\omega)$, is defined for $\underline{\underline{J}}(\underline{r}; \omega, \Delta\omega)$ by the equation

$$\underline{p}(\underline{r}; \omega, \Delta\omega) = \frac{s_1(\underline{r}; \omega, \Delta\omega) \underline{e}_1 + s_2(\underline{r}; \omega, \Delta\omega) \underline{e}_2 + s_3(\underline{r}; \omega, \Delta\omega) \underline{e}_3}{s_0(\underline{r}; \omega, \Delta\omega)} \quad (\text{F. 5})$$

then the magnitude of $\underline{p}(\underline{r};\omega, \Delta\omega)$ equals the degree of polarization of $\underline{\hat{C}}(\underline{r}, \underline{r}, \omega)$. But from 1.4.58, it is seen that

$$\underline{p}(\underline{r};\omega, \Delta\omega) = \frac{\int_{\omega-\frac{\Delta\omega}{2}}^{\omega+\frac{\Delta\omega}{2}} \left[\hat{\Delta}_1(\underline{r}, \omega') \underline{e}_1 + \hat{\Delta}_2(\underline{r}, \omega') \underline{e}_2 + \hat{\Delta}_3(\underline{r}, \omega') \underline{e}_3 \right] d\omega'}{\int_{\omega-\frac{\Delta\omega}{2}}^{\omega+\frac{\Delta\omega}{2}} \hat{\Delta}_0(\underline{r}, \omega') d\omega'} \quad (\text{F. 6})$$

Let $\hat{\underline{p}}(\underline{r}, \omega')$ represent the polarization vector that is associated with $\hat{\underline{C}}(\underline{r}, \underline{r}, \omega)$. Then

$$\hat{\underline{p}}(\underline{r}, \omega') = \frac{\hat{\Delta}_1(\underline{r}, \omega') \underline{e}_1 + \hat{\Delta}_2(\underline{r}, \omega') \underline{e}_2 + \hat{\Delta}_3(\underline{r}, \omega') \underline{e}_3}{\hat{\Delta}_0(\underline{r}, \omega')} \quad (\text{F. 7})$$

and the magnitude of $\hat{\underline{p}}(\underline{r}, \omega')$ equals the degree of polarization of $\hat{\underline{C}}(\underline{r}, \underline{r}, \omega)$. Equation F. 7 can be substituted into F. 6 to obtain

$$\underline{p}(\underline{r};\omega, \Delta\omega) = \int_{\omega-\frac{\Delta\omega}{2}}^{\omega+\frac{\Delta\omega}{2}} \left[\frac{\hat{\Delta}_0(\underline{r}, \omega')}{\int_{\omega-\frac{\Delta\omega}{2}}^{\omega+\frac{\Delta\omega}{2}} \hat{\Delta}_0(\underline{r}, \omega') d\omega'} \right] \hat{\underline{p}}(\underline{r}, \omega') d\omega' \quad (\text{F. 8})$$

Let

$$\hat{f}(\underline{r}, \omega') = \frac{\hat{\Delta}_o(\underline{r}, \omega')}{\int_{\omega - \frac{\Delta\omega}{2}}^{\omega + \frac{\Delta\omega}{2}} \hat{\Delta}_o(\underline{r}, \omega') d\omega'} = \frac{\text{Tr } \hat{\underline{\underline{\Delta}}}(\underline{r}, \underline{r}, \omega')}{\int_{\omega - \frac{\Delta\omega}{2}}^{\omega + \frac{\Delta\omega}{2}} \text{Tr } \hat{\underline{\underline{\Delta}}}(\underline{r}, \underline{r}, \omega') d\omega'} \quad (\text{F. 9})$$

Then $0 < \hat{f}(\underline{r}, \omega') < 1$ and

$$\int_{\omega - \frac{\Delta\omega}{2}}^{\omega + \frac{\Delta\omega}{2}} \hat{f}(\underline{r}, \omega') d\omega' = 1 \quad . \quad (\text{F. 10})$$

If F. 8 is expressed in terms of $\hat{f}(\underline{r}, \omega')$, then

$$\underline{p}(\underline{r}; \omega, \Delta\omega) = \int_{\omega - \frac{\Delta\omega}{2}}^{\omega + \frac{\Delta\omega}{2}} \hat{f}(\underline{r}, \omega') \underline{\hat{p}}(\underline{r}, \omega') d\omega' \quad (\text{F. 11})$$

Thus, the polarization of the integrated DFS equals the weighted sum of the polarization of the DFS over the interval of integration. The weighting factor is the ratio of the intensity of the field per unit frequency at frequency ω' to the total intensity of the field over this frequency interval. Since the direction of the polarization vector determines the polarization of the completely polarized part of the wave (on the Poincare sphere), then the polarization of the integrated DFS equals the average or mean polarization of the DFS over $\Delta\omega$.

The proof of the Theorem follows immediately from F.11, since the magnitude of the sum of the vectors $\hat{f}(\underline{r}, \omega') \hat{p}(\underline{r}, \omega')$ is less than or equal to the sum of the magnitudes of the vectors. That is;

$$p(\underline{r}; \omega, \Delta\omega) \leq \int_{\omega - \frac{\Delta\omega}{2}}^{\omega + \frac{\Delta\omega}{2}} \hat{f}(\underline{r}, \omega') \hat{p}(\underline{r}, \omega') d\omega' . \quad (F.12)$$

where by definition $p(\underline{r}; \omega, \Delta\omega)$ represents the degree of polarization of $\underline{J}(\underline{r}, \omega, \Delta\omega)$ and $\hat{p}(\underline{r}, \omega')$ represents the degree of polarization of $\hat{\underline{\underline{C}}}(\underline{r}, \underline{r}, \omega')$. The equality holds only if the vectors are colinear — i.e., only if the polarization of $\hat{\underline{\underline{C}}}(\underline{r}, \underline{r}, \omega')$ is independent of frequency over the interval $\omega - \Delta\omega/2 \leq \omega' \leq \omega + \Delta\omega/2$. The greater the diversity of polarization, the greater the disparity will be between the two sides of the inequality. Furthermore, the upper bound on the degree of polarization of $\underline{J}(\underline{r}, \omega, \Delta\omega)$ is determined not by the maximum degree of polarization of $\hat{\underline{\underline{C}}}(\underline{r}, \underline{r}, \omega)$ over the range of integration but by the average polarization over this range.

Q. E. D.

Observe that the foregoing analysis applies equally well to integration with respect to r . Thus if one replaces $\hat{\underline{\underline{C}}}(\underline{r}, \underline{r}, \omega')$ with $\hat{\underline{\underline{I}}}(\underline{r}, \omega')$, the DSS of an incoherent source, and recalls that

$$\hat{\underline{\underline{C}}}(\underline{r}, \underline{r}, \omega) \propto \int_{V_s} \hat{\underline{\underline{I}}}(\underline{r}, \omega') d\omega'$$

he is lead to the conclusion that the upper bound of the degree of polarization of the DFS at the point \underline{r} in the far-zone of the source equals the average degree of polarization over the source; and that if the polarization of the source varies with respect to position, then the degree of polarization of the field will not achieve this upper bound.

REFERENCES

- (1) G. G. Stokes, "On the composition and resolution of streams of polarized light from different sources," Trans. Cam. Phil. Soc., 9, 399-416 (1852). Mathematical and Physical Papers, vol. 3, Cambridge University Press, Cambridge, England, 1901, page 233.
- (2) M. E. Verdet, "Étude sur la constitution de la lumière non polarisée," Ann. Scientif. l'École Normale Supérieure, 2, 291-307 (1865).
- (3) A. A. Michelson, Phil. Mag. Series 5, 30, 1-22 (1890); 31, 256-273 (1891).
- (4) M. von Laue, "Die Entropie von partiell kohärenten Strahlenbunden," Ann. d. Physik, 23, 1-43 (1907).
- (5) P. H. von Cittert, "Die wahrscheinliche Schwingungsverteilung...", Physica, 1 201-210 (1934).
- (6) F. Zernike, "Concepts of degree of coherence and its application to optical problems," Physica, 5, 785-795 (1938).
- (7) E. Wolf, "A macroscopic theory of interference and diffraction of light from finite sources. II — Fields with spectral range of arbitrary width," Proc. Roy. Soc., A, 230, 246-265 (1955).
- (8) A. Blanc-Lapierre, P. Dumontet, "The concept of coherence in optics," Rev. Opt., 34, 1-21 (1955).
- (9) E. Wolf, "Optics in terms of observable quantities," Nuovo Cimento, 12, 884-888 (1954).
- (10) E. Wolf, "Coherence properties of partially polarized electromagnetic radiation," Nuovo Cimento, 13, 1165-1181 (1959).

- (11) P. Roman, E. Wolf, "Correlation theory of stationary electromagnetic fields. Part I — The basic field equations," *Nuovo Cimento*, 17, 462-476 (1960).
- (12) P. Roman, E. Wolf, "Correlation theory of stationary electromagnetic fields. Part II — Conservation laws," *Nuovo Cimento*, 17, 477-489 (1960).
- (13) P. Roman, "Correlation theory of stationary electromagnetic fields. Part III — The presence of random sources," *Nuovo Cimento*, 20, 759-772 (1961).
- (14) P. Roman, "Correlation theory of stationary electromagnetic fields. Part IV — Second order conservation laws," *Nuovo Cimento*, 22, 1005-1011 (1961).
- (15) M. Beran, G. Parrent, Jr., "On the equations governing the coherence of the radiation emitted by random charge and current distributions," *J. of Opt. Soc. of Amer.*, 52, 98-99 (1962).
- (16) N. Wiener, "Generalized Harmonic Analysis," *Acta Mathematica*, 55, 118-258 (1930).
- (17) See reference (16) for a discussion and bibliography of this work.
- (18) R. Barakat, "Theory of the coherency matrix for light of arbitrary spectral bandwidth," *J. Opt. Soc. of Amer.*, 53, 317-323 (1963).
- (19) Davenport and Root, Random Signals and Noise, McGraw-Hill Book Company, Inc., New York (1958).

- (20) P. H. van Cittert, "Kohaerenz - Probleme," *Physica*, 6, 1129-1138 (1939).
- (21) L. Janossy, "On the classical fluctuation of a beam of light," *Nuovo Cimento*, 6, 111-124 (1957).
- (22) L. Janossy, "The fluctuations of intensity of an extended source," *Nuovo Cimento*, 12, 369-384 (1959).
- (23) H. Hurwitz, Jr., "The statistical properties of unpolarized light," *J. Opt. Soc. of Amer.*, 35, 525-531 (1945).
- (24) Reference (19), page 106, contains a detailed proof of this statement.
- (25) A. M. Yaglom, An Introduction to the Theory of Stationary Random Functions, Prentice-Hall, Inc., New Jersey (1962).
- (26) M. Born, E. Wolf, Principles of Optics, Pergamon Press, New York (1959).
- (27) S. Chandrasekhar, Radiative Transfer, Dover Publications, New York (1960).
- (28) G. N. Ramachandran, S. Ramasechan, "Crystal Optics," Encyclopedia of Physics, vol. 25/1, S. Flugge, ed., Springer, Berlin (1961).
- (29) W. A. Shurcliff, Polarized Light, Harvard University Press, Cambridge, Massachusetts (1962).
- (30) D. Gabor, "Theory of information," *J. Inst. Elec. Engrs.*, Part III, 93, 429-441 (1946).
- (31) E. C. Titchmarsh, Introduction to the Theory of Fourier Integrals, Oxford University Press, London (1948), 2nd ed.

- (32) U. Fano, "A Stokes parameter technique for the treatment of polarization in quantum mechanics," *Phys. Rev.*, 93, 121-3 (1954).
- (33) R. Hanbury-Brown, R. O. Twiss, "A new kind of Radio Interferometer," *Phil. Mag.*, 45, 663-682 (1954).
- (34) J. Dugundji, "Envelopes and pre-envelopes of real waveforms," *I.R.E. Transactions on Information Theory*, IT-4, 53-57 (1958).
- (35) R. Arens, "Complex processes for envelopes of normal noise," *I.R.E. Trans. on Information Theory*, IT-3, 204-207 (1957).
- (36) See references (28) and (29) for discussions of the Jones Calculus.
- (37) L. D. Landau, E. M. Lifshitz, The Classical Theory of Fields, Pergamon Press, New York (1960).
- (38) J. A. Stratton, Electromagnetic Theory, McGraw-Hill, New York (1941).
- (39) H. Levine, J. Schwinger, "On the theory of electromagnetic wave diffraction by an aperture in an infinite plane conducting screen," *Comm. Pure and Appl. Math.*, 3, 355-391 (1950).
- (40) N. George, "Spatial distribution of thermal radiation at micro-wave frequencies," Ph.D. Thesis, Department of Electrical Engineering, California Institute of Technology (1959), 138 pp.
- (41) J. Brown, "A generalized form of the aerial reciprocity theorem," *Proc. of Inst. Elec. Engr.*, Part C, 105, 471-475 (1958).
- (42) Dwight, Tables of Integrals and Other Mathematical Data, 3rd ed., The Macmillan Company, New York (1957).
- (43) G. N. Watson, Theory of Bessel Functions, 2nd ed., Cambridge University Press, London (1958).