FINITE PLANE AND ANTI-PLANE ELASTOSTATIC FIELDS WITH DISCONTINUOUS DEFORMATION GRADIENTS NEAR THE TIP OF A CRACK

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SUMMARY

In this paper the fully nonlinear theory of finite deformations of an elastic solid is used to study the elastostatic field near the tip of a crack. The special elastic materials considered are such that the differential equations governing the equilibrium fields may lose ellipticity in the presence of sufficiently severe strains.

The first problem considered involves finite anti-plane shear (Mode III) deformations of a cracked incompressible solid. The analysis is based on a direct asymptotic method, in contrast to earlier approaches which have depended on hodograph procedures.

The second problem treated is that of plane strain of a compressible solid containing a crack under tensile (Mode I) loading conditions. The material is characterized by the so-called Blatz-Ko elastic potential. Again, the analysis involves only direct local considerations.

For both the Mode III and Mode I problems, the loss of equilibrium ellipticity results in the appearance of curves ("elastostatic shocks") issuing from the crack-tip across which displacement gradients and stresses are discontinuous.

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INTRODUCTION

Among a number of recent papers devoted to the study of the structure of finite elastostatic fields near the tip of a crack,[1] the investigations summarized in [4], [5] and [6] are concerned in particular with elastic materials whose corresponding equilibrium equations are capable of losing ellipticity at sufficiently severe strains. All three of these papers treat the so-called "small-scale nonlinear crack problem" assocaited with the finite anti-plane shear (Mode III) of an infinite slab containing a crack of finite length and deformed to a state of simple shear at infinity.

The principal feature of the results reported in [4], [5] and [6] is the appearance of two curves, issuing from each crack-tip and terminating in the interior of the body, across which the displacement gradient and the stresses are discontinuous. Such "elastostatic shocks" have been discussed in general terms elsewhere [7].

The analysis in [4], [5] and [6] depends critically on the fact that the finite anti-plane shear problem is governed by a second order quasi-linear partial differential equation and can therefore be successfully treated with the help of the hodograph transformation. For the more important problem of the plane deformation of a cracked slab by tensile loading at infinity (the Mode I problem) the hodograph transformation is not available because the associated quasi-linear system of differential equations is of the fourth order.

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[1] See [1], [2], [3] for reviews of recent work in this area.

In the present paper, we first show that the qualitative features of the results in [4] and [5] for the Mode III crack problem can be obtained by a direct local asymptotic analysis which makes no use of the hodograph transformation. The advantage of the procedure used here lies in its applicability to the plane strain Mode I problem which constitutes our main objective. We determine the qualitative structure of the crack-tip field in the latter problem for the so-called Blatz-Ko strain energy density introduced in [8] in connection with experiments on a highly compressible rubber-like material. It is known (see [9]) that this material is capable of a loss of equilibrium ellipticity at severe deformations.

In Section 1 we quote pertinent results from the nonlinear equilibrium theory of homogeneous and isotropic elastic solids and we introduce the special deformations and materials appropriate to this study. Section 2 begins with the formulation of the Mode III crack problem dealt with in [4]. We obtain the asymptotic representation near the crack-tip of a number of solutions of the displacement equation of equilibrium valid on overlapping domains. The final solution is then generated by a consistent matching across two symmetrically located elastostatic shocks. In Section 3 the tension crack problem is treated in an analogous manner and we find the corresponding asymptotic solutions to leading order. Section 4 is devoted to higher order considerations and a discussion of the results.

1. PRELIMINARIES FROM NONLINEAR ELASTOSTATICS

In this section we present a brief summary of the equilibrium theory of finitely deformed, homogeneous and isotropic elastic solids. The two special deformations and materials relevant to this study are then discussed.

Let $\mathcal{R}$ be an open region occupied by the interior of a body in its undeformed configuration and denote by $\underset{\sim}{x}$ the position vector of a material point in $\mathcal{R}$. A deformation of the body, indicated by

$$\underset{\sim}{y} = \underset{\sim}{y}(\underset{\sim}{x}) = \underset{\sim}{x} + \underset{\sim}{u}(\underset{\sim}{x}) \quad \text{for all} \quad \underset{\sim}{x} \in \mathcal{R} \quad , \tag{1.1}$$ [1]

is a mapping of $\mathcal{R}$ onto a domain $\mathcal{R}^*$ in which $\underset{\sim}{u}(\underset{\sim}{x})$ is the displacement field. We assume the transformation (1.1) to be invertible.

Let $\underset{\sim}{F}$ be the deformation-gradient tensor field associated with the mapping (1.1) and J its Jacobian determinant, so that

$$\underset{\sim}{F} = \nabla \underset{\sim}{y} \ , \ J = \det \underset{\sim}{F} > 0 \quad \text{on} \quad \mathcal{R} \ . \tag{1.2}$$

For an <u>incompressible</u> material the deformation must be locally volume preserving, whence $J = 1$ on $\mathcal{R}$. Define $\underset{\sim}{C}$ and $\underset{\sim}{G}$ by

$$\underset{\sim}{C} = \underset{\sim}{F}^T \underset{\sim}{F} \ , \ \underset{\sim}{G} = \underset{\sim}{F}\,\underset{\sim}{F}^T \quad . \tag{1.3}$$

These deformation tensors have common fundamental scalar invariants given by

[1] Letters underlined by a tilde represent three-dimensional vectors and tensors.

$$I_1 = \mathrm{tr}(\underset{\sim}{C}) \ , \ I_2 = \frac{1}{2}[(\mathrm{tr}\,\underset{\sim}{C})^2 - \mathrm{tr}(\underset{\sim}{C}^2)] \ , \ I_3 = \det \underset{\sim}{C} \quad . \qquad (1.4)$$

To continue, we let $\underset{\sim}{\tau}$ be the actual (Cauchy) stress tensor field on $\mathcal{R}^*$ and $\underset{\sim}{\sigma}$ the corresponding nominal (Piola) stress field on $\mathcal{R}$; $\underset{\sim}{\tau}$ and $\underset{\sim}{\sigma}$ are related by

$$\underset{\sim}{\tau} = J^{-1}\underset{\sim}{\sigma}\underset{\sim}{F}^T \ , \ \underset{\sim}{\sigma} = J\underset{\sim}{\tau}(\underset{\sim}{F}^T)^{-1} \quad . \qquad (1.5)$$

The balance of linear momentum, in the absence of body forces, leads to the following equivalent alternative forms of the local equilibrium equations:

$$\mathrm{div}\,\underset{\sim}{\tau} = \underset{\sim}{0} \quad \text{on} \quad \mathcal{R}^* \quad \text{or} \quad \mathrm{div}\,\underset{\sim}{\sigma} = \underset{\sim}{0} \quad \text{on} \quad \mathcal{R} \quad . \qquad (1.6)$$

Suppose $\mathcal{S}$ is a surface in $\mathcal{R}$ and $\mathcal{S}^*$ its deformation image in $\mathcal{R}^*$. Let $\underset{\sim}{n}$ and $\underset{\sim}{n}^*$ be the respective unit normals to $\mathcal{S}$ and $\mathcal{S}^*$ so that the associated nominal and true surface tractions are given by

$$\underset{\sim}{s} = \underset{\sim}{\sigma}\underset{\sim}{n} \quad \text{on} \quad \mathcal{S} \ , \ \underset{\sim}{t} = \underset{\sim}{\tau}\underset{\sim}{n}^* \quad \text{on} \quad \mathcal{S}^* \quad . \qquad (1.7)$$

It can be shown that

$$\underset{\sim}{t} = \underset{\sim}{0} \quad \text{on} \quad \mathcal{S}^* \quad \text{if and only if} \quad \underset{\sim}{s} = \underset{\sim}{0} \quad \text{on} \quad \mathcal{S} \quad ; \qquad (1.8)$$

this proposition is important because it allows the boundary condition on a traction free surface $\mathcal{S}^*$ to be specified on the undeformed surface $\mathcal{S}$.

Let W be the stored energy per unit undeformed volume characteristic of a given elastic material. For <u>compressible materials</u> $W = \mathring{W}(I_1, I_2, I_3)$ and the corresponding constitutive relation can be given in either of the following equivalent forms

$$\underset{\sim}{\tau} = \Sigma_1 \underset{\sim}{G} + \Sigma_2 \underset{\sim}{G}^2 + \Sigma_3 \underset{\sim}{1} \quad , \qquad (1.9)^1$$

$$\underset{\sim}{\sigma} = J\, \underset{\sim}{F}(\Sigma_1 + \Sigma_2 \underset{\sim}{C} + \Sigma_3 \underset{\sim}{C}^{-1}) \quad , \qquad (1.10)$$

where

$$\Sigma_1 = \frac{2}{\sqrt{I_3}}\left(\frac{\partial \overset{\circ}{W}}{\partial I_1} + I_1 \frac{\partial \overset{\circ}{W}}{\partial I_2}\right) , \quad \Sigma_2 = \frac{-2}{\sqrt{I_3}} \frac{\partial \overset{\circ}{W}}{\partial I_2} \quad ,$$

$$\Sigma_3 = 2\sqrt{I_3}\, \frac{\partial \overset{\circ}{W}}{\partial I_3} \quad . \qquad (1.11)$$

For incompressible materials, $W = \overset{*}{W}(I_1, I_2)$ and the corresponding constitutive law takes the form

$$\underset{\sim}{\tau} = 2\left[\frac{\partial \overset{*}{W}}{\partial I_1} \underset{\sim}{G} + \frac{\partial \overset{*}{W}}{\partial I_2}(I_1 \underset{\sim}{1} - \underset{\sim}{G})\underset{\sim}{G}\right] - P\, \underset{\sim}{1} \quad , \qquad (1.12)$$

or, equivalently,

$$\underset{\sim}{\sigma} = 2\left[\frac{\partial \overset{*}{W}}{\partial I_1} \underset{\sim}{F} + \frac{\partial \overset{*}{W}}{\partial I_2}(I_1 \underset{\sim}{1} - \underset{\sim}{G})\underset{\sim}{F}\right] - P(\underset{\sim}{F}^T)^{-1} \quad . \qquad (1.13)$$

Here P is the arbitrary hydrostatic pressure required to accommodate the constraint $J = \sqrt{I_3} = 1$.

We now turn to two special classes of finite deformations: anti-plane shear and plane strain. For these the region $\mathcal{R}$ occupied by the undeformed body is taken to be cylindrical and a fixed cartesian

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[1] $\underset{\sim}{1}$ stands for the idem tensor.

coordinate frame is chosen so that the x_3-axis is parallel to the generators of $\mathcal{R}$. Let $\mathcal{D}$ be the cross-section of $\mathcal{R}$ in the plane $x_3 = 0$.

To begin, we assume that the material occupying $\mathcal{R}$ is incompressible. A deformation (1.1) on $\mathcal{R}$ is an <u>anti-plane</u> <u>shear</u> if it is of the form

$$y_\alpha = x_\alpha \ , \ y_3 = x_3 + u(x_1, x_2) \quad . \tag{1.14}$$ [1]

As shown in detail in [10], the deformation (1.14) can in general only be sustained by materials whose strain energy density $W = \overset{*}{W}(I_1, I_2)$ is of suitably restricted form. Here we consider a class of materials shown in [10] to be compatible with (1.14) for which $\overset{*}{W}$ depends only on I_1:

$$W = \overset{*}{W}(I_1) \ , \ \overset{*}{W}'(I_1) > 0 \quad \text{for} \quad I_1 \geq 3 \quad , \tag{1.15}$$ [2]

$$\overset{*}{W}(3) = 0 \quad ; \tag{1.16}$$

here $\overset{*}{W}'$ is the derivative of $\overset{*}{W}$ with respect to I_1. As indicated in [4], substituting (1.14) into the constitutive law for incompressible materials (1.12), (1.13) permits the reduction of the field equations to

$$[\overset{*}{W}'(I_1) u_{,\alpha}]_{,\alpha} = 0 \quad \text{on} \quad \mathcal{D} \quad , \tag{1.17}$$ [3]

[1] Greek subscripts take the range 1,2 while Latin subscripts assume the values 1,2,3. Repeated subscripts are summed.

[2] Note that $I_1 = 3$ in the undeformed state.

[3] Subscripts preceded by a comma indicate partial differentiation with respect to the corresponding material cartesian coordinate.

with

$$I_1 = 3 + |\nabla u|^2 \ , \ |\nabla u|^2 = u_{,\alpha} u_{,\alpha} \ . \tag{1.18}$$

The components of Cauchy stress are given by

$$\tau_{3\alpha} = \tau_{\alpha 3} = 2\overset{*}{W}'(I_1) u_{,\alpha} \ , \ \tau_{33} = 2\overset{*}{W}'(I_1)|\nabla u|^2 \ , \tag{1.19}$$

$$\tau_{\alpha\beta} = 0 \ .$$

An elementary solution of (1.17), corresponding to a state of simple shear with amount of shear γ, is given by

$$u(x_1,x_2) = \gamma x_2 \quad \text{for all} \quad (x_1,x_2) \in \mathcal{D} \ . \tag{1.20}$$

For this deformation

$$\tau_{23} = \tau_{32} = \tau(\gamma) = 2\overset{*}{W}'(3+\gamma^2)\gamma \ , \ \tau_{33} = \tau(\gamma)\gamma \qquad (0 \le \gamma < \infty) \ . \tag{1.21}$$

We refer to the graph of $\tau(\gamma)$ vs. γ as the response curve in simple shear for the material at hand. Because of the inequality (1.15), the displacement equation of equilibrium (1.17) can be shown to be elliptic at a solution u and at a point (x_1,x_2) if and only if

$$\tau'(\gamma) > 0 \ , \ \gamma = |\nabla u(x_1,x_2)| \ . \qquad (1.22)^{1}$$

Thus, the condition of ellipticity is satisfied if the slope of the response curve in simple shear is positive at an amount of shear equal to

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[1] See [11].

the magnitude of the local displacement gradient $\nabla u(x_1,x_2)$.

The incompressible material to be considered here was introduced in [4]; its response function $\tau(\gamma)$ in simple shear is given by

$$\tau(\gamma) = \begin{cases} \mu\gamma & \text{for } 0 \le \gamma \le 1 \ , \\ \mu\gamma^{-1/2} & \text{for } 1 \le \gamma < \infty \ , \end{cases} \tag{1.23}$$

where μ is the shear modulus at infinitesimal deformations. The response curve (1.23) is described in Fig.1. The elastic potential $\overset{*}{W}(I_1)$ associated with this material reduces to

$$\overset{*}{W}(I_1) = \begin{cases} \frac{\mu}{2}(I_1 - 3) & (3 \le I_1 \le 4) \ , \\ -\frac{3\mu}{2} + 2\mu(I_1 - 3)^{1/4} & (4 \le I_1 < \infty) \ . \end{cases} \tag{1.24}$$

We observe for this material, the differential equation (1.17) is elliptic at a solution u and at a point (x_1,x_2) if $|\nabla u| < 1$; it is hyperbolic if $|\nabla u| > 1$. In this paper we shall study weak solutions u of (1.17) which are continuous and have piecewise continuous first and second partial derivatives on $\mathcal{D}$. Clearly, these continuity requirements allow for the possibility of finite jump discontinuities in ∇u as well as in the stresses across curves in $\mathcal{D}$. Equilibrium requires that, across such a curve $\mathbb{C}$, the axial component of Piola traction given by

$$s_3 = \sigma_{3\alpha} n_\alpha = 2\overset{*}{W}'(I_1)\frac{\partial u}{\partial n} \ , \tag{1.25}$$

must be continuous. In (1.25) $\partial u/\partial n$ is the normal derivative of u

on $\mathcal{C}$. We call such a curve an "equilibrium shock".

Let us now suppose the material occupying the cylindrical region $\mathcal{R}$ to be compressible. A plane deformation of $\mathcal{R}$ parallel to the (x_1, x_2)-plane is described by

$$y_\alpha = x_\alpha + u_\alpha(x_1, x_2) \quad , \qquad y_3 = x_3 \quad . \tag{1.26}$$
[1]

For a plane deformation, the fundamental scalar invariants are related by $I_2 = I_1 + I_3 - 1$, so that the strain energy density $W(I_1, I_2, I_3)$ for compressible materials can be written as

$$W(I_1, I_2, I_3) = \mathring{W}(I, J) \quad , \tag{1.27}$$

where

$$I = I_1 - 1 \ , \ J = \sqrt{I_3} \quad . \tag{1.28}$$

Equations (1.26)-(1.28), in conjunction with (1.9)-(1.11) and (1.6), provide the coordinate equations of equilibrium:

$$2\frac{\partial \mathring{W}}{\partial I}\nabla^2 y_\alpha + 2\left(\frac{\partial^2 \mathring{W}}{\partial I^2} I_{,\beta} + \frac{\partial^2 \mathring{W}}{\partial I \partial J} J_{,\beta}\right) y_{\alpha,\beta} + \left(\frac{\partial^2 \mathring{W}}{\partial J^2} J_{,\beta} + \frac{\partial^2 \mathring{W}}{\partial I \partial J} I_{,\beta}\right)\varepsilon_{\beta\gamma}\varepsilon_{\alpha\rho}\, y_{\rho,\gamma} = 0 \quad , \tag{1.29}$$
[2]

[1] One may consult [12] for a detailed treatment of plane-strain deformations.

[2] $\varepsilon_{\alpha\beta}$ are the components of the two-dimensional altenator.

together with the in plane stress components

$$\tau_{\alpha\beta} = \tau_{\beta\alpha} = \frac{2}{J}\frac{\partial \overset{\circ}{W}}{\partial I} y_{\alpha,\rho} y_{\beta,\rho} + \frac{\partial \overset{\circ}{W}}{\partial J}\delta_{\alpha\beta} \quad \text{on} \quad \mathcal{D} \ , \qquad (1.30)^{1}$$

$$\sigma_{\alpha\beta} = 2\frac{\partial \overset{\circ}{W}}{\partial I} y_{\alpha,\beta} + \frac{\partial \overset{\circ}{W}}{\partial J}\varepsilon_{\beta\gamma}\varepsilon_{\alpha\rho} y_{\rho,\gamma} \quad \text{on} \quad \mathcal{D} \ . \qquad (1.31)$$

Furthermore,

$$J = y_{1,1}y_{2,2} - y_{1,2}y_{2,1} \ , \ I = y_{1,1}^2 + y_{1,2}^2 + y_{2,2}^2 + y_{2,1}^2 \quad \text{on} \quad \mathcal{D} \ . \ (1.32)$$

The differential equations (1.29) may suffer a loss of ellipticity at solutions that have sufficiently severe local deformations. The ellipticity conditions for the system (1.29) are discussed in [7] and due to their complicated nature will not be reproduced here.

The special compressible material that concerns us has the elastic potential

$$\overset{\circ}{W}(I,J) = \frac{\mu}{2}(IJ^{-2} + 2J - 4) \qquad (\mu > 0) \ , \qquad (1.33)$$

where μ is a constant. This strain energy density was proposed by Blatz and Ko [8] to model a highly compressible rubber-like material. The basic properties of the Blatz-Ko material are investigated in [9]. It is shown in [9] that, for the material characterized by (1.33), the coordinate equations of equilibrium (1.29), are elliptic at a solution y_α and at a point (x_1,x_2) if and only if the invariants I and J, found through (1.32), satisfy

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[1] $\delta_{\alpha\beta}$ is the two-dimensional Kronecker delta.

$$2J \le I < 4J \quad . \tag{1.34}$$

In view of the potential loss of ellipticity, it is natural to require of the coordinates $y_\alpha(x_1,x_2)$ only the relaxed smoothness specified for u on $\mathcal{D}$. Thus, there may be curves $\mathcal{C}$ across which discontinuities in the deformation gradients $y_{\alpha,\beta}$ occur. Across such equilibrium shocks, the Piola tractions

$$s_\alpha = \sigma_{\alpha\beta} n_\beta \quad , \tag{1.35}$$

must be continuous to maintain equilibrium.

Finally, the homogeneous deformation corresponding to a state of plane strain uniaxial stress parallel to the x_2-axis is of interest. For the Blatz-Ko material (1.33), such a deformation is described by

$$y_1 = \lambda^{-1/3} x_1 \ , \ y_2 = \lambda x_2 \ , \ \lambda > 0 \ \text{ for all } \ (x_1,x_2) \in \mathcal{D} \quad , \qquad (1.36)^1$$

where the stretch λ is a constant. The associated true and nominal stresses are

$$\begin{aligned} \tau_{22} &= \mu(1 - \lambda^{-8/3}) \ , \ \tau_{11} = \tau_{12} = \tau_{21} = 0 \ \text{ on } \ \mathcal{D} \ , \\ \sigma_{22} &= \mu(\lambda^{-1/3} - \lambda^{-3}) \ , \ \sigma_{11} = \sigma_{12} = \sigma_{21} = 0 \ \text{ on } \ \mathcal{D} \ . \end{aligned} \tag{1.37}$$

Suppose a solution to the governing differential equations (1.29) corresponds locally to a state of plane strain uniaxial stress with principal stretch ratio λ. If λ lies on the interval $(\lambda_0^{-1},\lambda_0)$,

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[1] See [9].

where

$$\lambda_0 = (2 + \sqrt{3})^{3/4} \quad , \tag{1.38}$$

the field equations will be locally elliptic (see Fig.2).

2. ANTI-PLANE SHEAR DEFORMATIONS NEAR THE TIP OF A CRACK

In the present section we give the formulation and asymptotic solution of a nonlinear crack problem involving finite anti-plane shear deformations. This analysis is included to provide valuable insight into the more challenging tension crack problem considered in the next section.

Let the open cross-section $\mathcal{D}$ be the plane domain exterior to the line-segment

$$\mathcal{L} = \{(x_1, x_2) \mid -a \le x_1 \le a \ , \ x_2 = 0\} \ , \tag{2.1}$$

as depicted in Fig.3. Thus, the region $\mathcal{R}$ corresponds to the undeformed configuration of an all-around infinite body with a plane, infinitely long crack of width $2a$.

The body is composed of the special incompressible material introduced in Section 1 and characterized by the response curve in simple shear (1.23). Suppose the body is subjected at infinity to a simple shear parallel to the edges of the crack. An assumption, consistent with the loading and the particular material, is that the deformation is entirely one of anti-plane shear as given in (1.14). Referring to Eqs. (1.17), (1.18) and (1.24) we obtain the governing differential equation for the unknown out-of-plane displacement u:

$$\nabla^2 u = 0 \quad \text{for} \quad |\nabla u| < 1 \ , \tag{2.2a}$$

$$[|\nabla u|^{-3/2} u_{,\alpha}]_{,\alpha} = 0 \quad \text{for} \quad |\nabla u| > 1 \ . \tag{2.2b}$$

At infinity we stipulate

$$u = \gamma x_2 + o(1) \quad \text{on} \quad \text{as} \quad x_\alpha x_\alpha \to \infty \ , \tag{2.3}$$

where the positive constant γ is the amount of shear. Further, we assume that the crack boundary in the deformed body remains traction free. According to (1.8), (1.25) and the inequality in (1.15), this requirement is equivalent to

$$u_{,2}(x_1, 0\pm) = 0 \qquad (-a < x_1 < a) \ . \tag{2.4}$$

Note that the solution u to the boundary value problem (2.2)-(2.4) is to obey the continuity requirements set down in Section 1. Also, for the problem at hand we require that u be bounded within any circle of finite radius centered at the crack-tips and the limits $\nabla u(x_1, 0\pm)$ are to exist and be continuous for $-a < x_1 < a$.

The nonlinear crack problem formulated above is one of considerable difficulty. Knowles and Sternberg [4] considered the so-called "small-scale nonlinear crack problem" associated with the global problem described above. In the small scale problem, γ is assumed to be small compared to unity and the finite crack is replaced by a semi-infinite one. One seeks a solution of (2.2) which satisfies (2.4) on the semi-infinite crack and which, at infinity, "matches" the near-tip field predicted by the linearized theory for the original global problem associated with the finite crack. An <u>exact</u> solution to this small-scale problem was constructed in [4] by means of a hodograph transformation. Here, we consider the global problem and use a direct asymptotic approach to study the field near the crack-tip.

Let (r,θ) be local polar coordinates in the undeformed configuration as shown in Fig.3. Then,

$$x_1 - a = r\cos\theta \ , \ x_2 = r\sin\theta \quad . \tag{2.5}$$

The tip of the crack is denoted by $r=0$, while $\theta=\pi$ and $\theta=-\pi$ represent the upper and lower crack faces, respectively.

We first investigate the possibility of "smooth" solutions near the crack-tip. Suppose the global solution to the crack problem admits the asymptotic representation

$$u \sim r^m v(\theta) \quad \text{as} \quad r \to 0 \quad , \qquad (2.6)^1$$

uniformly on $-\pi \le \theta \le \pi$. In (2.6) m is a constant in the range

$$0 \le m < 1 \quad , \tag{2.7}$$

and v is a "smooth" function on $[-\pi,\pi]$. Here, we say a function is smooth if it is at least twice continuously differentiable on its domain of definition. In the present problem, symmetry implies

$$v(\theta) = -v(-\theta) \qquad (-\pi \le \theta \le \pi) \quad . \tag{2.8}$$

The boundary condition (2.4) yields

$$\dot{v} = 0 \quad \text{on} \quad \theta = -\pi, \pi \quad , \tag{2.9}$$

where the dot denotes the differentiation with respect to θ. The restrictions (2.7) on m guarantee that the displacement remains bounded

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[1]The asymptotic equality symbol "$\sim$" is used in the following sense; $u \sim r^m v(\theta)$ is equivalent to $u = r^m v(\theta) + o(r^m)$ as $r \to 0$.

near $r=0$ for $-\pi \le \theta \le \pi$, while permitting unbounded displacement gradients.

Equations (2.6) and (1.18) imply

$$|\nabla u| \sim p(\theta) r^{m-1} \quad \text{as} \quad r \to 0 \quad , \tag{2.10}$$

where

$$p(\theta) = \sqrt{m^2 v^2(\theta) + \dot{v}^2(\theta)} \quad , \tag{2.11}$$

for $-\pi \le \theta \le \pi$. Clearly, if $p(\theta) > 0$, then $|\nabla u| > 1$ as $r \to 0$, $-\pi \le \theta \le \pi$, so that (2.2b) applies. Assume this to be the case and substitute (2.6) into (2.2b). Retaining only the dominant power of r provides the governing differential equation for $v(\theta)$:

$$2p\ddot{v} - 3\dot{p}\dot{v} + m(3-m)pv = 0 \quad , \tag{2.12}$$

on $(-\pi,\pi)$.

If $m=0$ in (2.12), we obtain the solution $v(\theta) = c\theta + b$ (b and c are constants). Consideration of (2.8) and (2.9) requires $v(\theta) \equiv 0$ for $-\pi \le \theta \le \pi$. We may thus restrict m to the interval $(0,1)$.

The differential equation (2.12) can be analyzed in the phase plane. Details of this analysis are given in [12]. One finds in this way that the general solution of (2.12) may be written in the form

$$mv(\theta) = V \cos\psi(\theta) \, |\omega_0 + \cos 2\psi(\theta)|^{-1/2(1-m)} \quad , \tag{2.13}$$

where $\psi(\theta)$ is the general solution of

$$(1/3 + \cos 2\psi)\dot{\psi} + (\omega_0 + \cos 2\psi) = 0 \quad , \tag{2.14}$$

for $-\pi < \theta < \pi$. In (2.13), V is constant which we assume to be positive and

$$\omega_0 = 1/3(3 - 2m) \quad . \tag{2.15}$$

Observe that for $0 < m < 1$, one has $1/3 < \omega_0 < 1$. In Eq.(2.13) we must have $\omega_0 + \cos 2\psi \neq 0$ to assure bounded displacements. Further, we note that if p is calculated from (2.13), (2.11), it vanishes nowhere on $[-\pi,\pi]$, as was assumed.

In view of Eqs. (2.8), (2.13) and (2.14) we choose

$$\psi(0) = \pi/2 \quad , \tag{2.16}$$

so that (2.9) holds if and only if

$$\psi(\pi) = 2n\pi \ , \ \psi(-\pi) = (2n+1)\pi \ , \ n = 0,\pm 1,\pm 2,\ldots \tag{2.17}$$

In Fig.4 we sketch the curve $\psi = \psi(\theta)$ governed by (2.14), (2.16) and (2.17) for the case $n = 0$. It is apparent from the figure that there are no continuous solutions of the boundary value problem for ψ — this result is found to be independent of the choice of n. Consequently, there does not exist a smooth function v on $[-\pi,\pi]$ that satisfies (2.12), (2.8) and (2.9).

We now wish to generalize the Ansatz (2.6) concerning the form of u near $r = 0$ in such a way as to permit discontinuities in ∇u across certain curves issuing from the crack-tips. Accordingly, we suppose that there are two curves, $\mathcal{S}^+$ and $\mathcal{S}^-$, defined in a neighborhood of the origin by

$$\mathcal{S}^{\pm}: \quad \theta = \pm\Theta(r) \ , \qquad 0 < r \le r_0 \ , \tag{2.18}$$

where $\Theta(r)$ is a smooth positive function such that

$$\Theta(r) \sim Ar^{s} \quad \text{as} \quad r \to 0 \ , \tag{2.19}$$

and $A > 0$, $s \ge 0$ are constants to be determined. If $s = 0$ in (2.19), then A is required to satisfy $0 < A < \pi$ and the curves $\mathcal{S}^{\pm}$ are tangent to the rays $\theta = \pm A$ at the origin. When $s > 0$, $\mathcal{S}^{+}$ and $\mathcal{S}^{-}$ are both tangent to the x_1-axis at the origin. Since our interest lies in reproducing the results in [4], in which the shocks $\mathcal{S}^{\pm}$ are found to be tangent to the x_1-axis at $r = 0$, we treat here only the case $s > 0$ in (2.19). The case $s = 0$ is discussed in Appendix A.

It is convenient to introduce the regions $\mathcal{H}$, ξ^{+} and ξ^{-} (Fig.5) as follows

$$\left.\begin{aligned} \mathcal{H} &= \{(r,\theta) \mid -\Theta(r) < \theta < \Theta(r) \ , \quad 0 < r \le r_0\} \ , \\ \xi^{+} &= \{(r,\theta) \mid \Theta(r) < \theta < \pi \ , \qquad 0 < r \le r_0\} \ , \\ \xi^{-} &= \{(r,\theta) \mid -\pi < \theta < -\Theta(r) \ , \quad 0 < r \le r_0\} \ . \end{aligned}\right\} \tag{2.20}$$

We first investigate solutions of the differential equation (2.2b) which, in the region $\mathcal{H}$, have the asymptotic form (2.6), where $v(\theta)$ is a smooth odd function defined for $-\theta_0 \le \theta \le \theta_0$ for some θ_0, $0 < \theta_0 < \pi$. The exponent m is now permitted to be negative, but $r^m v(\theta)$ must remain bounded as (r,θ) approaches the origin from within $\mathcal{H}$.

If $m < 0$, we find that (2.10), (2.11) still apply as $r \to 0$ in $\mathcal{H}$ and $v(\theta)$ must again satisfy (2.12), which leads to the same implicit representation given in (2.13)-(2.15) except that now $m < 0$. Note that, in this case, $\omega_0 > 1$ and hence $\omega_0 + \cos 2\psi \neq 0$. As before, (2.16) accounts for the parity condition (2.8). Integrating (2.14), subject to (2.16), yields $\psi(\theta)$ implicitly through

$$\theta = -(\psi - \pi/2) + \frac{(1-m)}{(3-m)}\omega_1 \tan^{-1}[\omega_1 \tan(\psi - \pi/2)] \quad (-\theta_0 < \theta < \theta_0) \ , \quad (2.21)^1$$

where

$$\omega_1 = \left[\frac{3-m}{-m}\right]^{1/2} \qquad (m < 0) \ , \qquad (2.22)$$

and

$$\theta_0 = -1/2 \cos^{-1}\left(\tfrac{1}{3}\right) + \frac{(1-m)}{(3-m)}\omega_1 \tan^{-1}\left(\frac{\omega_1}{\sqrt{2}}\right), \quad 0 < \theta_0 < \pi \ . \quad (2.23)$$

At this point Eq.(2.21) cannot be inverted explicitly to furnish $\psi = \psi(\theta)$. It does, however, indicate that $\psi(\theta)$ is a continuously differentiable function which increases monotonically on $(-\theta_0, \theta_0)$ from a value $\psi_0 = 1/2 \cos^{-1}(-1/3)$ to $\pi - \psi_0$ and is antisymmetric about $\theta = 0$, $\psi = \pi/2$. In view of these facts, we confirm through (2.13), (2.21) the existence of a smooth function v on $(-\theta_0, \theta_0)$ which satisfies (2.12) and has the appropriate symmetry. Thus,

$$u \sim r^m v(\theta) \quad (m < 0) \quad \text{as} \quad r \to 0 \ , \qquad (2.24)$$

represents an asymptotic solution to the <u>hyperbolic</u> differential equation (2.2b) on $\mathcal{H}$.

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[1]Here, the inverse trigonometric functions take their principal values.

For $|\theta| \ll 1$, Eqs. (2.13) and (2.21) provide

$$v = p_0[\theta + \frac{m}{6}(3-4m)\theta^3 + O(\theta^5)] \quad \text{as} \quad \theta \to 0 \quad , \tag{2.25}$$

where

$$p_0 = p(0) \quad . \tag{2.26}$$

From (2.18), (2.19), (2.24) and (2.25) we observe that the displacement remains bounded as the crack-tip is approached from within $\mathcal{N}$ if $s + m \geq 0$.

We now seek a solution to (2.2) valid on the region ξ^+. Assume the displacement on ξ^+ admits the general form (2.6) where the exponent $m \geq 0$ takes on the smallest possible value — boundedness of the displacement as the origin is approached in ξ^+ necessitates m be nonnegative. We find that

$$u \sim b + rv(\theta) \quad \text{as} \quad r \to 0 \quad \text{on} \quad \xi^+ \quad , \tag{2.27}$$

where b is a positive constant and now v is a smooth function on $(0,\pi]$ that obeys the boundary condition $\dot{v}(\pi) = 0$. Note that the constant b is the solution corresponding to $m = 0$ and recall, for $0 < m < 1$ in (2.6), there are no smooth solutions of (2.2) on $(0,\pi]$ satisfying the boundary condition on $\theta = \pi$.

Substituting (2.27) into either (2.2a) or (2.2b) leads to the same differential equation:

$$\ddot{v} + v = 0 \quad (0 < \theta < \pi) \quad . \tag{2.28}$$

The most general solution of (2.28) for which the boundary condition $\dot{v}(\theta) = 0$ is satisfied is

$$v(\theta) = c\cos\theta\ , \quad 0 < \theta \leq \pi\ , \tag{2.29}$$

where c is a constant. In ξ^-, the dominant term analogous to (2.29) is obtained by symmetry.

It is convenient at this point to present the results of a higher order analysis. If $|c| < 1$ we find

$$u \sim b + cr\cos\theta + dr^k \cos k(\theta - \pi) \quad (k > 1) \quad \text{as} \quad r \to 0 \quad \text{on} \quad \xi^+\ . \tag{2.30}$$

Note that $|\nabla u| \sim |c| < 1$ as $r \to 0$ which assures the ellipticity of the displacement equations of equilibirum on ξ^+.

What remains now is to construct an asymptotic solution to the crack problem near the crack-tip — continuous, piecewise smooth and with continuous nominal tractions across $\mathcal{S}^{\pm}$. To this end, we confine our attention to $0 \leq \theta \leq \pi$ and note that (2.8) generates the solution in the lower half-plane.

The requirement that u be continuous across $\mathcal{S}^+$, together with (2.18), (2.19), (2.24), (2.25) and (2.30), clearly implies

$$b \sim Ap_0 r^{s+m}\ , \tag{2.31}$$

so that necessarily

$$A = p_0^{-1} b\ , \ s = -m\ . \tag{2.32}$$

We now address the continuity of the non-zero nominal traction s_3

(1.25) across the shock $\mathcal{S}^+$. The asymptotic forms of the traction (1.25) on the two sides of $\mathcal{S}^+$ are evaluated through (1.18), (1.24) where the displacement on the hyperbolic side of the shock is given by (2.24), (2.25) while, on the elliptic side, we use (2.30). The tractions are found to be continuous across $\mathcal{S}^+$ if and only if

$$p_o^{1/2} r^{1/2(m-1)} \sim (m-1)bcr^{-1} + dp_o k \sin k\pi r^{k+m-2} \quad . \qquad (2.33)$$

The possible relationships between the exponents k and m that permit an asymptotic balance in Eq.(2.33) are described in Fig.6. Asymptotic solutions to the next order of the displacement equation of equilibrium (2.2a), (2.2b) were found on ξ^+ and $\mathcal{H}$ in an attempt to eliminate some of the cases I-V in the figure. However, the solutions had sufficient flexibility so as to allow the necessary higher order matching across an appropriately chosen shock $\mathcal{S}^+$ for all the five cases. Our purpose, in this paper, is to reproduce the solution in [4] in the vicinity of the crack-tip and hence we assume case II to hold, so that

$$m = -1 \;,\; k = 2 \quad . \qquad (2.34)$$

Equations (2.33), (2.34) reveal that

$$-1 < \left(c = -\frac{p_o^{1/2}}{2b} \right) < 1 \quad . \qquad (2.35)$$

We note that for $m = -1$, (2.21) can be inverted analytically and on substitution into (2.13) furnishes, after some algebra

$$v(\theta) = 2^{-5/2} p_0 \frac{(3\cos\theta + P(\theta))^{3/2}}{(\cos\theta + P(\theta))^{1/2}} \sin\theta \qquad (-\theta_0 < \theta < \theta_0) \quad , \qquad (2.36)$$

where

$$P(\theta) = (9\cos^2\theta - 8)^{1/2} \qquad (-\theta_0 < \theta < \theta_0) \quad , \qquad (3.37)$$

and $\theta_0 = \cos^{-1}(2\sqrt{2}/3)$. This completes the matching to leading order.

To conclude, we summarize the asymptotic representation for the solution to the crack problem (2.2)-(2.4) in the vicinity of the right crack-tip. Equations (2.24), (2.30) together with (2.34), (2.35) give

$$\left.\begin{aligned} u &\sim b - \frac{p_0^{1/2}}{2b} r\cos\theta + dr^2\cos 2\theta \quad \text{as} \quad r \to 0 \quad \text{on} \quad \xi^+ \ , \\ u &\sim r^{-1} v(\theta) \quad \text{as} \quad r \to 0 \quad \text{on} \quad \mathcal{H} \ , \end{aligned}\right\} \qquad (2.38)$$

where $b>0$, $0<p_0<\sqrt{2b}$ and d are unknown constants and $v(\theta)$ is given in (2.36). In (2.38), ξ^+ is the elliptic region and $\mathcal{H}$ the hyperbolic region described in (2.20). The displacements on ξ^- are obtained through symmetry. Equations (2.18), (2.19) and (2.32) indicate the elastostatic shocks $\mathcal{S}^+$ and $\mathcal{S}^-$ have the following asymptotic form

$$\mathcal{S}^{\pm}: \ \theta \sim \pm \frac{b}{p_0} r \quad \text{as} \quad r \to 0 \quad . \qquad (2.39)$$

The above results are consistent with those found in [4] on taking the limit $r \to 0$. However, in the local analysis presented here, the constants b, d and p_0 in (2.38) remain undetermined. In contrast, the specific small-scale nonlinear crack problem treated in [4] leads to fully determined values of b, d and p_o. Furthermore, for each successively higher order calculation we obtain one additional unknown constant. The reader may refer to [4] to obtain representations for the stresses near the crack-tip.

3. PLANE STRAIN DEFORMATIONS NEAR THE TIP OF A CRACK

In this section we present the formulation and first order asymptotic analysis of the nonlinear plane strain crack problem which constitutes the main purpose of this study.

Let $\mathcal{R}$ be the undeformed configuration of a cylindrical body whose cross-section $\mathcal{D}$ in the (x_1,x_2)-plane is described in Fig.3. The line segment $\mathcal{L}$ (2.1) is the boundary of $\mathcal{D}$ and represents a crack of width 2a. The material making up the body is <u>compressible</u> and has the Blatz-Ko plane strain elastic potential (1.33).

Suppose the body is subjected to uniform uniaxial tension at infinity perpendicular to the crack faces, so that the actual stresses satisfy

$$\tau_{\alpha\beta}(x_1,x_2) = \delta_{2\alpha}\delta_{2\beta}\tau + o(1) \quad \text{as} \quad x_\alpha x_\alpha \to \infty \quad , \tag{3.1}$$

where $\tau > 0$ is the magnitude of the applied load. The crack faces are to remain traction free. Applying (1.7) and (1.8) results in the boundary conditions

$$\sigma_{\alpha 2}(x_1,0\pm) = 0 \qquad (-a < x_1 < a) \ . \tag{3.2}$$

As a consequence of (3.1) and (3.2) the deformation conforms to (1.26), corresponding to plane strain parallel to the (x_1,x_2)-plane. Accordingly, the relevant coordinates y_α must satisfy the equations of equilibrium (1.29), together with (1.32) and (1.33). The conditions at infinity (3.1) and boundary condition (3.2) are represented in terms of the coordinates through (1.30)-(1.32). The solution y_α to the boundary value problem must be continuous and piecewise smooth on $\mathcal{D}$ as indicated in Section 1. We insist also, that the solution be bounded in the vicinity of the crack-tip and the limits $y_{\alpha,\beta}(x_1, 0\pm)$ exist and are continuous for $-a < x_1 < a$.

An analytical solution to the global problem is not attempted here. We consider instead, the asymptotic character of the solution in the neighborhood of the crack-tip.[1]

Let (r,θ) be the polar coordinates introduced in (2.5) and $y_\alpha = y_\alpha(r,\theta)$ be the <u>local</u> deformation near the right crack-tip. From (1.29), with $\overset{\circ}{W}$ given by (1.33), we obtain the appropriate form for the governing differential equations, namely

$$J\nabla^2 y_\alpha - 2(J_{,r} y_{\alpha,r} + r^{-2} J_{,\theta} y_{\alpha,\theta}) + (-)^\alpha r^{-1} J^3 (R_{,\theta} y_{\beta,r} - R_{,r} y_{\beta,\theta}) = 0 \qquad (\alpha \neq \beta) \quad , \tag{3.3}$$

for $r > 0$, $-\pi < \theta < \pi$. In (3.3),

$$R(r,\theta) = 1 - IJ^{-3} \quad , \tag{3.4}$$

where (1.32) gives

$$J(r,\theta) = r^{-1}(y_{1,r} y_{2,\theta} - y_{1,\theta} y_{2,r}) > 0 \quad , \tag{3.5}$$

$$I(r,\theta) = y_{1,r}^2 + y_{2,r}^2 + r^{-2}(y_{1,\theta}^2 + y_{2,\theta}^2) \quad , \tag{3.6}$$

[1]The existence of a solution is assumed.

for $r>0$, $-\pi \le \theta \le \pi$. The loading at infinity (3.1) indicates the following coordinate symmetries:

$$y_1(r,\theta) = y_1(r,-\theta)\ ,\ y_2(r,\theta) = -y_2(r,-\theta) \qquad (r>0,\ -\pi \le \theta \le \pi)\ . \quad (3.7)$$

Substituting (1.31), (1.33) into (3.2) provides the boundary conditions on the crack faces

$$\left.\begin{array}{l} J^3 - y_{1,r}^2 - y_{2,r}^2 = 0 \\ y_{1,r}y_{1,\theta} + y_{2,r}y_{2,\theta} = 0 \end{array}\right\} \text{ on } \theta = -\pi,\pi \text{ for } r>0\ . \qquad (3.8)$$

To complete the formulation we stipulate that the resulting deformations have the same smoothness as that specified for the global problem.

Now turn to the asymptotic solution of the crack problem. We follow the anti-plane shear example in the previous section and assume that the local deformation field is represented by

$$y_\alpha \sim r^{m_\alpha} v_\alpha(\theta) \quad \text{as} \quad r \to 0 \quad (\text{no sum on } \alpha)\ , \qquad (3.9)$$

where m_α are constants. In Appendix B we consider $0 \le \min[m_1, m_2] < 1$ in an attempt to find a solution, consistent with (3.3)-(3.8), in which v_α are smooth functions on $[-\pi,\pi]$. We conclude from the analysis that no such deformations exist. Motivated by the anti-plane shear problem, we now let $\mathcal{S}^+$ and $\mathcal{S}^-$ be the curves originating at $r=0$ described in (2.18), (2.19). Also, define the regions $\mathcal{H}$, ξ^+ and ξ^-, in the neighborhood of the crack-tip through (2.20) (see Fig.5).

Suppose (3.9) holds on $\mathcal{H}$ with

$$m_2 = m < 0 \ , \ m < m_1 \quad , \tag{3.10}$$ [1]

and v_α are smooth functions on $(-\theta_0, \theta_0)$ $(0 < \theta_0 < \pi)$. We assume here, that the deformations are "most severe" in the x_2-direction. We let

$$J \sim r^\nu q(\theta) \quad \text{as} \quad r \to 0 \quad \text{on} \quad \mathcal{N} \ , \tag{3.11}$$

where q is a continuously differentiable function for $-\theta_0 < \theta < \theta_0$ and (3.5), (3.9) and (3.10) infer the constant $\nu > 2(m-1)$. Further, (3.6), (3.9) and (3.10) provide the asymptotic form of the scalar invariant I, namely

$$I \sim r^{2(m-1)} p^2(\theta) \quad \text{as} \quad r \to 0 \quad \text{on} \quad \mathcal{N} \ , \tag{3.12}$$

where

$$p(\theta) = \sqrt{m^2 v_2^2(\theta) + \dot{v}_2^2(\theta)} \quad \text{on} \quad (-\theta_0, \theta_0) \quad . \tag{3.13}$$

Proceeding as in the case (B.23) in Appendix B we find that (B.25)-(B.27) hold on $(-\theta_0, \theta_0)$ for $m < 0$. Equations (B.29)-(B.32) provide expressions for v_2 and q on $(-\theta_0, \theta_0)$ while the symmetry (3.7) and smoothness across the ray $\theta = 0$ is guaranteed by (B.34). In (B.34) we choose $\varepsilon_1 = 0$ and hence (B.29)-(B.32) reduce to[2]

$$mv_2(\theta) = V_2 \cos\psi(\theta) \left|\varepsilon_0 + \cos 2\psi(\theta)\right|^{1/2(m-1)} \quad , \tag{3.14}$$

---

[1] The case $m_1 = m_2 < 0$ is treated in a manner similar to that of the case of positive equal exponents discussed in Appendix B. We find that the coordinates (3.9) and the parity condition (3.7) are incompatible.

[2] The case $\varepsilon_1 \geq 1$ is not treated here. We find this leads to inconsistent result $J(r,0) = 0$ for <u>all</u> sufficiently small $r > 0$.

$$q(\theta) = Q|\varepsilon_0 + \cos 2\psi(\theta)|^{1/3(m-1)} \quad , \tag{3.15}$$

for $-\theta_0 < \theta < \theta_0$, where V_2 and Q are positive constants and $\psi(\theta)$ satisfies

$$(1/2 + \cos 2\psi)\dot{\psi} + (\varepsilon_0 + \cos 2\psi) = 0 \ , \ \psi(0) = \pi/2 \quad . \tag{3.16}$$

Furthermore,

$$\varepsilon_0 = 1 - m/2 \ , \ \nu = 2/3(m-1) \quad . \tag{3.17}$$

Integrating (3.16) yields

$$\theta = -(\psi - \pi/2) + \frac{(1-m)}{(4-m)}\varepsilon_4 \tan^{-1}[\varepsilon_4 \tan(\psi - \pi/2)] \quad (-\theta_0 < \theta < \theta_0) \ , \tag{3.18}$$

where

$$\varepsilon_4 = \left[\frac{m-4}{m}\right]^{1/2} \quad (m < 0) \quad , \tag{3.19}$$

and

$$\theta_0 = -\pi/6 + \frac{(1-m)}{(4-m)}\varepsilon_4 \tan^{-1}(\varepsilon_4 \tan \pi/6) \ , \quad 0 < \theta_0 < \pi \quad . \tag{3.20}$$

The expressions (3.18)-(3.20) indicate that $\psi(\theta)$ is a smooth function on $(-\theta_0, \theta_0)$ and hence $v_2(\theta)$ (3.14), $q(\theta)$ (3.15) are also smooth with the correct parity for $-\theta_0 < \theta < \theta_0$.

We now evaluate the y_1-coordinate to leading order. Note that (3.13), (3.14) and (3.15) imply

$$q(\theta) = Q_0 p^{2/3}(\theta) > 0 \quad \text{on} \quad (-\theta_0, \theta_0) \quad , \tag{3.21}$$

with $Q_0 = QV_2^{-2/3}$. The Jacobian is evaluated using (3.5), (3.9), (3.10) and again through (3.11), (3.21). Equating the two results provides the asymptotic equality

$$r^{m+m_1-2}(m_1 v_1 \dot{v}_2 - m\dot{v}_1 v_2) \sim Q_0 r^{2/3(m-1)} p^{2/3} \quad \text{as} \quad r \to 0 \quad , \tag{3.22}$$

on $\mathcal{N}$. Assume first, $m_1 < 1/3(4-m)$ and find $v_1(\theta) = V_1 | v_2(\theta)|^{m_1/m}$ on $(-\theta_0, \theta_0)$ where $V_1 (>0)$ is a constant. The restrictions (3.10) imply that $m_1 = 0$ and $v_1(\theta) = V_1$ is the only smooth solution across $\theta = 0$. Taking $y_1(0,\theta) = 0$ $(-\pi \leq \theta \leq \pi)$ requires $V_1 = 0.$[1] Now let

$$m_1 = 1/3(4-m) \quad , \tag{3.23}$$

and obtain a nonhomogeneous first order differential equation for v_1 from (3.22). We establish that the homogeneous solution is unbounded on $\theta = 0$ and is neglected, while the smooth particular solution gives

$$v_1(\theta) = -\frac{Q_0}{m} |v_2(\theta)|^{\varepsilon_5 - 1} \int^{|\theta|} p^{2/3}(\varphi) v_2^{-\varepsilon_5}(\varphi) d\varphi \quad \text{on} \quad (-\theta_0, \theta_0) \ , \tag{3.24}$$

where

$$\varepsilon_5 = \frac{2(2+m)}{3m} < 1 \quad \text{for} \quad m < 0 \quad . \tag{3.25}$$

Equations (3.9), (3.10), (3.14), (3.17), (3.23)-(3.25) provide an asymptotic solution to the differential equation (3.3)-(3.6) such that

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[1] The deformation is normalized so that the crack-tip does not move with respect to the x_1-axis.

$$\left.\begin{aligned} y_1 &= h_1 \sim r^{1/3(4-m)} v_1(\theta) \\ y_2 &= h_2 \sim r^m v_2(\theta) \end{aligned}\right\} (m<0) \quad \text{as} \quad r \to 0 \quad \text{on} \quad \mathscr{H} \ . \qquad (3.26)$$

Further, (3.11), (3.17) and (3.21) give

$$J \sim Q_o r^{2/3(m-1)} p^{2/3}(\theta) \quad \text{as} \quad r \to 0 \quad \text{on} \quad \mathscr{H} \ , \qquad (3.27)$$

while substituting (3.26) and (3.27) into (3.3) $(\alpha = 1)$ reveals

$$R = 1 - Q_o^{-3} + O(r^{8/3(1-m)}) \quad \text{as} \quad r \to 0 \quad \text{on} \quad \mathscr{H} \ . \qquad (3.28)$$

It is apparent from (3.12) and (3.27) that $I \gg J$ on $\mathscr{H}$ and thus the coordinate equations of equilibrium are <u>hyperbolic</u> at the solution (3.26) on $\mathscr{H}$.

For $|\theta| \ll 1$, (3.14)-(3.16) and (3.24) provide

$$\left.\begin{aligned} v_1 &= \frac{3Q_o p_o^{-1/3}}{(4-m)}[1 - \tfrac{1}{6}m(4-m)\theta^2 + O(\theta^4)] \\ v_2 &= p_o[\theta - \tfrac{1}{6}m(5m-4)\theta^3 + O(\theta^5)] \\ q &= Q_o p_o^{2/3}[1 - \tfrac{4}{3}m(m-1)\theta^2 + O(\theta^4)] \end{aligned}\right\} \text{as} \quad \theta \to 0 \ , \qquad (3.29)$$

where

$$p_o = p(0) \ . \qquad (3.30)$$

Equations (2.18), (2.19), (3.26) and (3.29) infer that the y_2-coordinate remains bounded on $\mathscr{H}$ in the limit $r \to 0$ if $s + m \geq 0$.

We now seek solutions to the governing equations (3.3) on ξ^+. Suppose

$$y_\alpha = \overset{+}{e}_\alpha \sim b_\alpha + r v_\alpha(\theta) \quad \text{as} \quad r \to 0 \quad \text{on} \quad \xi^+ \quad , \tag{3.31}$$

where b_α are constants and v_α are now smooth functions on $(0,\pi]$. As in the anti-plane shear problem we obtain the representation (3.31) after first assuming (3.9) holds on ξ^+ with $0 \le \min[m_1,m_2] < 1$ and draw on results from Appendix B. The constants b_α are the solutions (B.5) for $m_1 = m_2 = 0$. The case $m_1 = m_2 = m$ $(0 < m < 1)$ yields $v_1(\theta) = cv_2(\theta)$ on $(0,\pi]$, where c is a constant. If $c = 0$ we recover the unequal exponent case. The boundary conditions on $\theta = \pi$ are given in (B.28). After a tedious analysis we prove that no smooth solution v_2 exists on $(0,\pi]$ that is compatible with (B.28) and can be matched to the hyperbolic solution on $\mathscr{H}$ across $\mathscr{S}^+$.

Substituting (3.31) into (3.3)-(3.6) and retaining the leading order terms gives

$$\left.\begin{aligned}
&[(v_1^2 + 2v_2^2)\dot{v}_2 + v_1 v_2 \dot{v}_1](\ddot{v}_1 + v_1) \\
&\quad -[(v_2^2 + 2v_1^2)\dot{v}_1 + v_1 v_2 \dot{v}_2](\ddot{v}_2 + v_2) = 0 \quad , \\
&[3(v_1^2 + v_2^2)v_2 + (v_1\dot{v}_1 + v_2\dot{v}_2)\dot{v}_2](\ddot{v}_1 + v_1) \\
&\quad -[3(v_1^2 + v_2^2)v_1 + (v_1\dot{v}_1 + v_2\dot{v}_2)\dot{v}_1](\ddot{v}_2 + v_2) = 0 \quad ,
\end{aligned}\right\} \tag{3.32}$$

for $0 < \theta < \pi$. The differential equations (3.32) are equivalent to

$$\ddot{v}_\alpha + v_\alpha = 0 \quad \text{on} \quad (0,\pi) \quad , \tag{3.33}$$

unless the determinant of the coefficients of $(\ddot{v}_\alpha + v_\alpha)$ vanishes on $(0,\pi)$.[1] Assume (3.33) holds and thus

$$\left.\begin{aligned} v_1(\theta) &= \lambda_1 \cos\theta - \lambda_2(\lambda_1^2 + \lambda_2^2)^{-2/3}\sin\theta \quad , \\ v_2(\theta) &= \lambda_2 \cos\theta + \lambda_1(\lambda_1^2 + \lambda_2^2)^{-2/3}\sin\theta \quad , \end{aligned}\right\} \tag{3.34}$$

for $0 < \theta \leq \pi$, where λ_α are constants. The solution (3.34) obeys the boundary conditions (3.8).

Observe from (3.26), (3.29), (3.31), (3.34) and (2.18) that $h_1 \sim 3Q_0 p_0^{-1/3}(4-m)^{-1} r^{1/3(4-m)}$, $\overset{+}{e}_1 \sim b_1 + \lambda_1 r$ as $r \to 0$ on $\mathcal{S}^+$. As $1/3(4-m) > 1$ for $m < 0$ we arrive at

$$b_1 = 0 \ , \ \lambda_1 = 0 \quad , \tag{3.35}$$

as a prerequisite for continuous coordinates across $\mathcal{S}^+$. Letting the scalar invariants take on their leading order values calculated through (3.5), (3.6), (3.31), (3.34) and (3.35), the ellipticity condition (1.32) reduces to $(2-\sqrt{3})^{3/4} < |\lambda_2| < (2+\sqrt{3})^{3/4}$. Physically, in the vicinity of the crack face, we expect the leading order homogeneous deformation (3.31), (3.34) and (3.35) to represent a state of <u>uniaxial tension</u> parallel to the y_2-direction. This limits the range of the stretch ratio further, so that <u>elliptic</u> solutions of interest obey

$$1 < \lambda < (2+\sqrt{3})^{3/4} \ , \ \lambda = -\lambda_2 > 0 \quad . \tag{3.36}$$

---

[1] It can be shown that functions v_α that produce a vanishing determinant are not consistent with either the boundary conditions on the crack face or the matching across $\mathcal{S}^+$.

In what follows we require the solution on ξ^+ to higher order. Noting (3.31) and (3.34)-(3.36), we now assume

$$\left.\begin{aligned} y_1 = \overset{+}{e}_1 &\sim \lambda^{-1/3} r\sin\theta + r^k f_1(\theta) \\ y_2 = \overset{+}{e}_2 &\sim b_2 - \lambda r\cos\theta + r^k f_2(\theta) \end{aligned}\right\} \text{ as } r \to 0 \text{ on } \xi^+ \quad , \qquad (3.37)$$

where $k > 1$ is constant and f_α are smooth functions on $(0,\pi]$. Equation (3.37) satisfies the differential equations (3.3) to the appropriate order if

$$L_\alpha(f_1, f_2; k, \lambda) = 0 \quad \text{on} \quad (0,\pi) \quad , \qquad (3.38)$$

where L_α is the differential operator,

$$\begin{aligned} L_1(f_1,f_2;k,\lambda) = {} & (2+K+K\cos 2\theta)\ddot{f}_1 + 2K(k-1)\sin 2\theta \dot{f}_1 \\ & + k[k(2+K) - K(k-2)\cos 2\theta] f_1 \\ & + N[\sin 2\theta \ddot{f}_2 - 2(k-1)\cos 2\theta \dot{f}_2 - k(k-2)\sin 2\theta f_2] \quad , \end{aligned} \qquad (3.39)$$

$$\begin{aligned} L_2(f_1,f_2;k,\lambda) = {} & (2+M-M\cos 2\theta)\ddot{f}_2 - 2M(k-1)\sin 2\theta \dot{f}_2 \\ & + k[k(2+M) + M(k-2)\cos 2\theta] f_2 \\ & + N[\sin 2\theta \ddot{f}_1 - 2(k-1)\cos 2\theta \dot{f}_1 - k(k-2)\sin 2\theta f_1] \quad . \end{aligned} \qquad (3.40)$$

In (3.39) and (3.40)

$$K = 3\lambda^{8/3} - 1 \ , \ M = 3\lambda^{-8/3} - 1 \ , \ N = \lambda^{4/3} + \lambda^{-4/3} \quad . \qquad (3.41)$$

The boundary conditions (3.8) infer

$$3\dot{f}_1(\pi) - \lambda^{-4/3}kf_2(\pi) = 0 \ , \ \dot{f}_2(\pi) - \lambda^{-4/3}kf_1(\pi) = 0 \quad . \tag{3.42}$$

Furthermore, (3.7) implies

$$y_1 = \bar{e}_1 = \overset{+}{e}_1(r,-\theta) \ , \ y_2 = \bar{e}_2 = -\overset{+}{e}_2(r,-\theta) \quad \text{on} \quad \xi^- \quad . \tag{3.43}$$

We now generate the first order asymptotic solution with the appropriate smoothness to the small scale nonlinear crack problem (3.3)-(3.8). Consider $0 \le \theta \le \pi$ and refer to (3.43) to provide the deformation in the lower half-plane. Accordingly, the matching conditions are

$$\overset{+}{e}_\alpha = h_\alpha \quad \text{on} \quad \mathcal{S}^+ \quad , \tag{3.44}$$

together with continuous Piola tractions across $\mathcal{S}^+$. We compute the general expression for the nominal tractions through (1.7), (1.31) and (1.33):

$$s_\alpha/\mu = J^{-2}(y_{\alpha,r}n_r + r^{-1}y_{\alpha,\theta}n_\theta) + (-)^\alpha R(y_{\beta,r}n_\theta - r^{-1}y_{\beta,\theta}n_r) \ . \tag{3.45}$$

(n_r, n_θ) are the components of the normal vector $\underset{\sim}{n}$ in polar coordinates.

Let $\alpha = 2$ in (3.44) and use (2.18), (2.19), (3.26), (3.29) and (3.37) to find

$$b_2 \sim Ap_o r^{\mathcal{s}+m} \quad \text{as} \quad r \to 0 \quad . \tag{3.46}$$

Take $b_2 > 0$, then

$$b_2 = Ap_o \ , \ \mathcal{s} = -m \quad . \tag{3.47}$$

Similarly, let $\alpha = 1$ in (3.44) and use (3.47) to obtain

$$\lambda^{-1/3}Ar^{-m} + f_1(0)r^{k-1} \sim \frac{3Q_o p_o}{(4-m)}^{-1/3} r^{-1/3(m-1)} \quad \text{as} \quad r \to 0 \ . \qquad (3.48)$$

Here and in what follows we assume that the functions f_α are analytic on $[0,\pi]$ and hence can be represented by a Taylor series about $\theta = 0$.

Before matching tractions on $\mathcal{S}^+$ we modify the result (3.28) so that

$$Q_o = 1 \quad \text{and} \quad R = O(r^{8/3(1-m)}) \quad \text{as} \quad r \to 0 \quad \text{on} \quad \mathcal{X} \ . \qquad (3.49)$$

We confirm (3.49) by first assuming the existence of an equilibrium solution in the neighborhood of the crack-tip that has the general form (3.26) and (3.37) with the curve $\mathcal{S}^+$ given by (2.18) and (3.47). Force equilibrium in the x_1-direction on a circular region in $\mathcal{D}$ centered at $r = 0$ necessitates that (3.49) holds on taking the limit $r \to 0$.[1]

Equating the tractions (3.45) across $\mathcal{S}^+$ and drawing on (3.47) and (3.49) yields

$$\left.\begin{aligned} &3\dot{f}_1(0) - \lambda^{-4/3}kf_2(0) = 0 \quad , \\ &\lambda^{-1/3}(m-1)(\lambda^{-8/3}-1)Ar^{-m} \\ &\quad + \lambda^{-4/3}(\dot{f}_2(0) - \lambda^{-4/3}kf_1(0))r^{k-1} \sim p_0^{-1/3}r^{-1/3(m-1)} \quad \text{as} \quad r \to 0 \ . \end{aligned}\right\} \qquad (3.50)$$

The relationships between the exponents k and m that could provide

[1] The result (3.49) assures no concentrated forces act at the crack-tip.

a solution to (3.48) and (3.50) are presented in Fig.7. We note the similarity between this figure and the diagram obtained for the anti-plane shear problem (see Fig.6). Again, we choose k and m through a distinguished limit and thus

$$m = -1/2 \ , \ k = 3/2 \quad . \tag{3.51}$$

In view of (3.51), the equations (3.48) and (3.50) give

$$A = \lambda^{1/3}\left(\frac{2}{3}p_0^{-1/3} - f_1(0)\right) \quad , \tag{3.52}$$

$$\dot{f}_1(0) - \frac{1}{2}\lambda^{-4/3}f_2(0) = 0 \quad , \quad \dot{f}_2(0) - \frac{3}{2}\lambda^{4/3}f_1(0) = \lambda^{-4/3}p_0^{-1/3} \ . \tag{3.53}$$

Recall in (2.18), (2.19) that A is a positive constant and hence

$$f_1(0) < \frac{2}{3}p_0^{-1/3} \quad . \tag{3.54}$$

Equations (3.38)-(3.42), (3.51) and (3.53) constitute a boundary value problem for f_α on $[0,\pi]$. For convenience set

$$\bar{f}_\alpha(\theta) = p_0^{1/3}f_\alpha(\theta) \qquad (0 \le \theta \le \pi) \quad , \tag{3.55}$$

where $\bar{f}_\alpha$ satisfies

$$L_\alpha(\bar{f}_1,\bar{f}_2;3/2,\lambda) = 0 \quad \text{on} \quad (0,\pi) \quad , \tag{3.56}$$

with

$$\left.\begin{array}{l}\dot{\bar{f}}_1(0) - \frac{1}{2}\lambda^{-4/3}\bar{f}_2(0) = 0 \ , \ \ \dot{\bar{f}}_2(0) - \frac{3}{2}\lambda^{4/3}\bar{f}_1(0) = \lambda^{-4/3} \ , \\ \\ \dot{\bar{f}}_1(\pi) - \frac{1}{2}\lambda^{-4/3}\bar{f}_2(\pi) = 0 \ , \ \ \dot{\bar{f}}_2(\pi) - \frac{3}{2}\lambda^{-4/3}\bar{f}_1(\pi) = 0 \quad . \end{array}\right\} \tag{3.57}$$

A closed form solution to the differential equation (3.56) appears to be difficult to construct. However, a numerical analysis indicates that for values of λ obeying the inequality (3.36) the nondimensional boundary value problem (3.56), (3.57) does possess analytic solutions. For $\lambda \neq \lambda^*$, where

$$\lambda^* \doteq 2.519 \quad , \tag{3.58}$$

the solutions depend uniquely on λ. We find that $\lambda = \lambda^*$ is an eigenvalue associated with the homogeneous version of (3.56), (3.57). Only one eigenfunction exists together with the particular solution. We note that in all the solutions, $\bar{f}_1(0) < 2/3$, complying with (3.54).[1]

We now present a summary of the asymptotic solution to the crack problem considered here. Applying the results (3.47), (3.51) and (3.53) to (3.26), (3.37) and (3.43) gives

$$\left.\begin{array}{l}\overset{+}{e}_1 \sim \lambda^{-1/3} r \sin\theta + p_o^{-1/3} r^{3/2}\bar{f}_1(\theta) \\ \\ \overset{+}{e}_2 \sim \lambda^{1/3} p_o^{2/3}(\frac{2}{3} - \bar{f}_1(0)) \\ \\ \qquad - \lambda r\cos\theta + p_o^{-1/3} r^{3/2}\bar{f}_2(\theta) \end{array}\right\} \text{as} \quad r \to 0 \quad \text{on} \quad \xi^+ \quad , \tag{3.59}$$

[1]Analytic solutions to (3.38)-(3.42) and (3.53) were found for integer values of k with λ arbitrary and for arbitrary values of k with $\lambda = 1$. These solutions were used to check the numerical procedure.

$$\left.\begin{aligned} h_1 &\sim r^{3/2} v_1(\theta) \\ h_2 &\sim r^{-1/2} v_2(\theta) \end{aligned}\right\} \text{as} \quad r \to 0 \quad \text{on} \quad \mathscr{H} \;. \tag{3.60}$$

$$\bar{e}_1 = \overset{+}{e}_1(r,-\theta) \;\;,\; \bar{e}_2 = -\overset{+}{e}_2(r,-\theta) \quad \text{on} \quad \xi^- \;, \tag{3.61}$$

where p_o is a positive constant and λ a constant satisfying (3.36). In (3.59), $\bar{f}_\alpha(\theta)$ is a smooth function governed by (3.56), (3.57). For $m = -1/2$ the implicit equation for $\psi(\theta)$ (3.18) can be inverted analytically as in the anti-plane shear problem. This result and (3.14) eventually yields

$$v_2(\theta) = \frac{2^{1/2}}{3} p_o \frac{(2\cos\theta + P(\theta))}{(\cos\theta + P(\theta))^{\frac{1}{2}}} \sin\theta \qquad (-\frac{\pi}{6} < \theta < \frac{\pi}{6}) \quad , \tag{3.62}$$

where

$$P(\theta) = (4\cos^2\theta - 3)^{1/2} \qquad (-\frac{\pi}{6} < \theta < \frac{\pi}{6}) \quad . \tag{3.63}$$

Further, (3.13), (3.24), (3.25) and (3.62) provide

$$v_1(\theta) = 2^{1/2} p_o^{-1/3} \frac{(\cos\theta + P(\theta))^{\frac{1}{2}}}{(2\cos\theta + P(\theta))} \qquad (-\frac{\pi}{6} < \theta < \frac{\pi}{6}) \quad . \tag{3.64}$$

Alternative representations for v_α are obtained from (3.29), (3.49) and (3.51) in the neighborhood of $\theta = 0$:

$$\left.\begin{aligned} v_1 &= \frac{2}{3}p_0^{-1/3}[1+\frac{3}{8}\theta^2+O(\theta^4)] \\ v_2 &= p_0[\theta-\frac{13}{24}\theta^3+O(\theta^5)] \end{aligned}\right\} \text{as } \theta \to 0 \quad . \tag{3.65}$$

Finally, in (3.59)-(3.61), $\mathcal{E}^+$ and $\mathcal{E}^-$ are elliptic regions and $\mathcal{H}$ the hyperbolic region given in (2.20). The domains are separated by the elastostatic shocks $\mathcal{S}^+$ and $\mathcal{S}^-$. Referring to (2.18), (2.19), (3.47), (3.51) and (3.52) we find

$$\mathcal{S}^{\pm}: \ \theta = \pm\Theta(r) \ , \ \Theta(r) \sim \left(\frac{\lambda}{p_0}\right)^{1/3}\left(\frac{2}{3}-\bar{f}_1(0)\right)r^{1/2} \quad \text{as} \quad r \to 0 \quad . \tag{3.66}$$

4. HIGHER ORDER CONSIDERATIONS AND RESULTS

What remains to complete the first order analysis is the evaluation of the constant λ and hence the functions $\overline{f}_\alpha(\theta)$ on $[0,\pi]$. In this section we calculate these unknowns numerically through higher order considerations and the subsequent results are presented.

To begin, we replace (3.66) by the two term asymptotic representation

$$\mathcal{S}^{\pm}:\ \theta=\pm\Theta(r)\ ,\ \Theta(r)\sim\left(\frac{\lambda}{p_0}\right)^{1/3}\left(\frac{2}{3}-\overline{f}_1(0)\right)r^{1/2}+Br^{t}\quad \text{as}\quad r\to 0\ , \tag{4.1}$$

where B and $t>1/2$ are undetermined constants. The domains $\mathcal{K}$, ξ^+ and ξ^- remain as defined by (2.20). Similarly, (3.60) is modified, such that

$$\left.\begin{aligned} h_1 &\sim r^{3/2}v_1(\theta)+r^{n_1}w_1(\theta)\\ h_2 &\sim r^{-1/2}v_2(\theta)+r^{n_2}w_2(\theta)\end{aligned}\right\}\text{as}\quad r\to 0\quad \text{on}\quad \mathcal{K}\ . \tag{4.2}$$

In (4.2) $n_1>3/2$ and $n_2>-1/2$ are constants while $w_1(\theta)$ and $w_2(\theta)$ are smooth functions on $(-\pi/6,\pi/6)$. Equations (3.27), (3.49) and (3.51) provide the Jacobian to leading order. Now assume

$$J\sim r^{-1}p^{2/3}(\theta)+r^{\nu_1}q_1(\theta)\quad \text{as}\quad r\to 0\quad \text{on}\quad \mathcal{K}\ . \tag{4.3}$$

The constant ν_1 is greater than -1 and $q_1(\theta)$ has the same continuity as $p(\theta)$ (3.13) on $(-\pi/6, \pi/6)$.

Observe that (3.49) and (3.51) imply

$$R \sim r^4 u(\theta) \quad \text{as} \quad r \to 0 \quad \text{on} \quad \mathscr{K} \quad , \tag{4.4}$$

with $u(\theta)$ a continuously differentiable function for $-\pi/6 < \theta < \pi/6$. A further asymptotic representation for R is generated from the definition (3.4)-(3.6), together with (4.2) and (4.3). On equating this expression to (4.4) we arrive at

$$3r^{\nu_1+1} p^{-2/3} q_1 - 2r^{n_2+1/2} p^{-2}(\dot{v}_2 \dot{w}_2 - \tfrac{1}{2} n_2 v_2 w_2) \sim r^4 u \quad \text{as} \quad r \to 0 \quad , \tag{4.5}$$

on $\mathscr{K}$. Suppose

$$\nu_1 = n_2 - 1/2 < 3 \quad , \tag{4.6}$$

and thus (4.5) and (4.6) gives

$$3p^{4/3} q_1 - 2(\dot{v}_2 \dot{w}_2 - \tfrac{1}{2} n_2 v_2 w_2) = 0 \quad \text{on} \quad (-\tfrac{\pi}{6}, \tfrac{\pi}{6}) \quad . \tag{4.7}$$

The governing differential equation (3.3) with $\alpha = 2$ infers, when the coordinates are replaced by (4.2) and the Jacobian by (4.3), (4.6), that w_2, q_1 further satisfy

$$p^{2/3}\ddot{w}_2 - \tfrac{4}{3} p^{-1/3} \dot{p} \dot{w}_2 + n_2(n_2 + 2) p^{2/3} w_2 - 2\dot{v}_2 \dot{q}_1$$
$$+ [\ddot{v}_2 + (n_2 - \tfrac{1}{4}) v_2] q_1 = 0 \quad , \tag{4.8}$$

for $-\pi/6 < \theta < \pi/6$.

It is convenient at this point to assume suitably smooth solutions exist to (4.7) and (4.8) on $(-\pi/6,\pi/6)$. For small $|\theta|$, we find by making use of (3.29), (3.51) and (3.65) in (4.7) and (4.8), that

$$\left.\begin{aligned} w_2 &= p_1\theta + O(\theta^3) \\ q_1 &= \frac{2}{3}p_0^{-1/3}p_1 + O(\theta^2) \end{aligned}\right\} \text{ as } \theta\to 0 \ , \tag{4.9}$$

where p_1 is a positive constant of integration.

Next, we consider the Jacobian to higher order. Through (3.5), (4.2), (4.3) and (4.6) we find

$$r^{n_1-2}(\tfrac{1}{2}v_2\dot{w}_1 + n_1\dot{v}_2w_1) \sim r^{n_2}(q_1 - \tfrac{3}{2}v_1\dot{w}_2 + n_2\dot{v}_1w_2) \quad \text{as} \quad r\to 0 \ , \tag{4.10}$$

on $\mathscr{K}$. Take n_1 to have its smallest possible value

$$n_1 = n_2 + 2 \ , \tag{4.11}$$[1]

whence

$$v_2\dot{w}_1 + 2(n_2+2)\dot{v}_2w_1 = 2q_1 - 3v_1\dot{w}_2 + 2n_2\dot{v}_1w_2 \quad \text{on} \quad (-\tfrac{\pi}{6},\tfrac{\pi}{6}) \ . \tag{4.12}$$

Again we assume the existence of a smooth solution,while for small $|\theta|$ equations (4.9) and (4.12) give

$$w_1 = -\frac{p_0^{-4/3}p_1}{3(n_2+2)} + O(\theta^2) \quad \text{as} \quad \theta\to 0 \ . \tag{4.13}$$

Now turn to the solutions of the differential equations (3.3) on

[1]Note, if $n_1 < n_2 + 2$, w_1 is unbounded on $\theta = 0$.

the elliptic domain ξ^+. We replace (3.59) by

$$\left.\begin{aligned} \overset{+}{e}_1 &\sim \lambda^{-\frac{1}{3}} r \sin\theta + p_0^{-\frac{1}{3}} r^{\frac{3}{2}} \bar{f}_1(\theta) + p_0^{-\frac{2}{3}} r^2 \bar{g}_1(\theta) \\ \overset{+}{e}_2 &\sim \lambda^{\frac{1}{3}} p_0^{\frac{2}{3}} (\tfrac{2}{3} - \bar{f}_1(0)) - \lambda r \cos\theta + p_0^{-\frac{1}{3}} r^{\frac{3}{2}} \bar{f}_2(\theta) \\ &\qquad + p_0^{-\frac{2}{3}} r^2 \bar{g}_2(\theta) \end{aligned}\right\} \text{ as } r \to 0 \text{ on } \xi^+ , \tag{4.14}$$

where $\bar{g}_\alpha(\theta)$ are analytic functions on $[0,\pi]$. In (3.3) we assume the coordinates conform to the representation (4.14) and obtain the following differential equations for $\bar{g}_\alpha$:

$$L_\alpha(\bar{g}_1, \bar{g}_2; 2, \lambda) = H_\alpha(\bar{f}_1, \bar{f}_2; \lambda) \quad \text{for} \quad 0 < \theta < \pi \quad , \tag{4.15}$$

where the differential operators L_α are defined in (3.39) and (3.40). Furthermore,

$$\begin{aligned} H_1(\bar{f}_1, \bar{f}_2; \lambda) &= 4\lambda^{-\frac{2}{3}}[(J_1 \dot{\bar{f}}_1)^{\boldsymbol{\cdot}} + 3J_1\bar{f}_1] - 6\lambda^{-\frac{5}{3}}(\sin\theta\, J_1 + 2\cos\theta\, \dot{J}_1)J_1 \\ &+ \lambda^{\frac{4}{3}}[3\dot{R}_1\bar{f}_2 - R_1\dot{\bar{f}}_2 + 2\sin\theta(2\lambda^{-\frac{7}{3}}J_2 - \lambda R_2) + 2\cos\theta(2\lambda^{-\frac{7}{3}}J_2 - \lambda R_2)^{\boldsymbol{\cdot}}] \quad , \end{aligned} \tag{4.16}$$

$$\begin{aligned} H_2(\bar{f}_1, \bar{f}_2; \lambda) &= 4\lambda^{-\frac{2}{3}}[(J_1 \dot{\bar{f}}_2)^{\boldsymbol{\cdot}} + 3J_1\bar{f}_2] - 6\lambda^{-\frac{1}{3}}(2\sin\theta\, \dot{J}_1 - \cos\theta\, J_1)J_1 \\ &- \lambda^{\frac{4}{3}}[3\dot{R}_1\bar{f}_1 - R_1\dot{\bar{f}}_1 + 2\cos\theta(2\lambda^{-1}J_2 - \lambda^{-\frac{1}{3}}R_2) - 2\sin\theta(2\lambda^{-1}J_2 - \lambda^{-\frac{1}{3}}R_2)^{\boldsymbol{\cdot}}], \end{aligned} \tag{4.17}$$

on $(0,\pi)$ with

$$\left.\begin{aligned} J_1 &= \lambda[\sin\theta(\tfrac{3}{2}\bar{f}_1 + \lambda^{-\frac{4}{3}}\dot{\bar{f}}_2) + \cos\theta(\dot{\bar{f}}_1 - \tfrac{3}{2}\lambda^{-\frac{4}{3}}\bar{f}_2)] \ , \\ J_2 &= \tfrac{3}{2}(\bar{f}_1\dot{\bar{f}}_2 - \dot{\bar{f}}_1\bar{f}_2) \ , \end{aligned}\right\} \quad (4.18)$$

$$R_1 = \lambda^{-1}\{\sin\theta[\tfrac{3}{2}\lambda^{-\frac{4}{3}}(2+K)\bar{f}_1 + (2+M)\dot{\bar{f}}_2] + \cos\theta[\lambda^{-\frac{4}{3}}(2+K)\dot{\bar{f}}_1 - \tfrac{3}{2}(2+M)\bar{f}_2]\} \ , \quad (4.19)$$

$$\begin{aligned} R_2 = -\lambda^{-2}\{&\tfrac{9}{4}(2+K-3\lambda^{\frac{8}{3}}\cos 2\theta)\bar{f}_1^2 + \tfrac{9}{4}(2+M+3\lambda^{-\frac{8}{3}}\cos 2\theta)\bar{f}_2^2 \\ &+ (2+K+3\lambda^{\frac{8}{3}}\cos 2\theta)\dot{\bar{f}}_1^2 + (2+M-3\lambda^{-\frac{8}{3}}\cos 2\theta)\dot{\bar{f}}_2^2 \\ &+ 9(\lambda^{\frac{8}{3}}\bar{f}_1\dot{\bar{f}}_1 - \lambda^{-\frac{8}{3}}\bar{f}_2\dot{\bar{f}}_2)\sin 2\theta - 3N[\tfrac{3}{2}(\bar{f}_1\dot{\bar{f}}_2 + \dot{\bar{f}}_1\bar{f}_2)\cos 2\theta \\ &- (\dot{\bar{f}}_1\dot{\bar{f}}_2 - \tfrac{9}{4}\bar{f}_1\bar{f}_2)\sin 2\theta]\} \ . \end{aligned} \quad (4.20)$$

In (4.18)-(4.20) the constants K, M and N are supplied by (3.41). The boundary conditions on the crack face $\theta = \pi$ (3.8) suggest

$$\begin{aligned} 3\dot{\bar{g}}_1(\pi) - 2\lambda^{-\frac{4}{3}}\bar{g}_2(\pi) &= \tfrac{3}{2}\lambda^{-\frac{7}{3}}(3\bar{f}_1^2(\pi) - \bar{f}_2^2(\pi)) \ , \\ \dot{\bar{g}}_2(\pi) - 2\lambda^{-\frac{4}{3}}\bar{g}_1(\pi) &= -3\lambda^{-\frac{7}{3}}\bar{f}_1(\pi)\bar{f}_2(\pi) \ . \end{aligned} \quad (4.21)$$

Associated with (4.15) is the homogeneous solution

$$\left.\begin{aligned} \bar{g}_1^* &= G_1[K + (2+K)\cos 2\theta] + G_2 M\sin 2\theta - G_3 N\cos 2\theta \quad , \\ \bar{g}_2^* &= G_4[M - (2+M)\cos 2\theta] + G_2 N\cos 2\theta + G_3 K\sin 2\theta \quad , \end{aligned}\right\} \quad (4.22)$$

on $[0,\pi]$, where $G_1 - G_4$ are arbitrary constants. The formulation of the boundary value problem for $\bar{g}_\alpha$ on $[0,\pi]$ is completed once boundary conditions on $\theta = 0$ have been established.

We proceed now with matching the higher order solutions on $\mathcal{N}$ and ξ^+ across the elastostatic shock $\mathcal{S}^+$. Recall that the smoothness of the complete solution to the problem (3.3)-(3.8) requires continuous coordinates and Piola tractions across $\mathcal{S}^+$.

Equating the coordinates on $\mathcal{S}^+$ using (3.44), (4.1), (4.2), (4.9), (4.11), (4.13) and (4.14) leads to

$$\left.\begin{aligned} &-\lambda r \sim p_0 B r^{t-\frac{1}{2}} + \left(\frac{\lambda}{p_0}\right)^{\frac{1}{3}}\left(\frac{2}{3} - \bar{f}_1(0)\right) p_1 r^{n_2+\frac{1}{2}} \\ &\lambda^{-\frac{1}{3}} B r^{t+1} \\ &\quad + p_0^{-\frac{2}{3}}\left[\lambda^{\frac{1}{3}}\left(\frac{2}{3} - \bar{f}_1(0)\right)\dot{\bar{f}}_1(0) + \bar{g}_1(0)\right] r^2 \sim \frac{-p_0^{-\frac{4}{3}} p_1}{3(n_2+2)} r^{n_2+2} \end{aligned}\right\} \text{as} \quad r \to 0 \quad . \quad (4.23)$$

In (4.23) we let

$$n_2 = 0 \;, \; t = 1 \quad , \qquad (4.24)$$

and referring to (3.57) suggests

$$\left.\begin{aligned} B &= -\left(\frac{\lambda}{p_o^4}\right)^{\frac{1}{3}}\left(\frac{2}{3}-\bar{f}_1(0)\right)p_1 \quad , \\ p_1 &= \frac{p_o^{\frac{2}{3}}}{\left(\frac{1}{2}-\bar{f}_1(0)\right)}\left[\frac{1}{2}\lambda^{-1}\left(\frac{2}{3}-\bar{f}_1(0)\right)\bar{f}_2+\bar{g}_1(0)\right] \quad . \end{aligned}\right\} \qquad (4.25)^{1}$$

(4.24), (4.25) and continuity of the tractions (3.45) across $\mathcal{S}^+$ reveals, after considerable algebra

$$\left.\begin{aligned} &3\dot{\bar{g}}_1(0)-2\lambda^{-\frac{4}{3}}\bar{g}_2(0) = -\frac{1}{3}\lambda^{-5}+\frac{1}{2}\lambda^{-\frac{7}{3}}\left(\lambda^{-\frac{8}{3}}-3\right)\bar{f}_1(0)-\frac{9}{2}\lambda^{\frac{1}{3}}\bar{f}_1^2(0) \\ &\quad +\frac{3}{2}\lambda^{-\frac{7}{3}}\bar{f}_2^2(0), \quad \left(\frac{1}{2}-\bar{f}_1(0)\right)\dot{\bar{g}}_2(0)+\left[\frac{1}{3}\lambda^{-\frac{4}{3}}-2\lambda^{\frac{4}{3}}\left(\frac{1}{2}-\bar{f}_1(0)\right)\right]\bar{g}_1(0) \\ &\quad =\lambda^{\frac{1}{3}}\left[\frac{14}{9}\lambda^{-\frac{8}{3}}-\frac{1}{6}\left(9+25\lambda^{-\frac{8}{3}}\right)\bar{f}_1(0)+\left(3+2\lambda^{-\frac{8}{3}}\right)\bar{f}_1^2(0)\right]\bar{f}_2(0) \quad , \end{aligned}\right\} \quad (4.26)$$

What remains is to solve the boundary value problem (4.15)-(4.21) and (4.26). Recalling (4.21), (4.22) and (4.26) we note that the <u>homogeneous</u> system has the nontrivial solution

$$\left.\begin{aligned} \bar{g}_1^e(\theta) &= 2\lambda^{-\frac{4}{3}}M\lambda_1\sin 2\theta \quad , \\ \bar{g}_2^e(\theta) &= \left\{M\left(\lambda^{-\frac{4}{3}}N-3M\right)+\left[2\lambda^{-\frac{4}{3}}N+(2+M)\left(3M-\lambda^{-\frac{4}{3}}N\right)\right]\cos 2\theta\right\}\lambda_1 \quad , \end{aligned}\right\} \quad (4.27)$$

[1]For λ obeying (3.36), $1/2-\bar{f}_1(0)\neq 0$.

where λ_1 is an arbitrary constant. Consequently, in order for a solution to the nonhomogeneous problem to exist the eigenfunction (4.27) must be orthogonal to the right hand sides in (4.15).

The particular solution to (4.15) is found in terms of λ and f_α through the method of variation of parameters.[1] This expression, with (4.22), facilitates a numerical investigation into the existence of a solution to the boundary value problem. We find functions $\bar{g}^p_\alpha(\theta)$, satisfying the nonhomogeneous differential equations (4.15)-(4.20) and the boundary conditions (4.21), (4.26), if

$$\lambda = \Lambda \doteq 2.657 < (2+\sqrt{3})^{\frac{3}{4}} \quad (\doteq 2.685) \quad . \qquad (4.28)^2$$

This result, together with the eigenfunction (4.27), provides the complete solution:

$$\bar{g}_\alpha(\theta) = \bar{g}^e_\alpha(\theta) + \bar{g}^p_\alpha(\theta) \quad \text{on} \quad [0,\pi] \quad . \qquad (4.29)$$

The numerical solutions obtained for $\bar{f}_\alpha$ and $\bar{g}^p_\alpha$ with $\lambda = \Lambda$ are shown Fig.8. Some pertinent results from the figure are

$$\left.\begin{array}{l} \bar{f}_1(0) \doteq -.065, \ \bar{f}_2(0) = 0, \ \bar{f}_1(\pi) = 0, \ \bar{f}_2(\pi) \doteq -.090 \quad , \\ \bar{g}^p_1(0) \doteq -.020, \ \bar{g}^p_2(0) = 0, \ \bar{g}^p_1(\pi) \doteq .105, \ \bar{g}^p_2(\pi) \doteq -.152 \quad . \end{array}\right\} \qquad (4.30)$$

---

[1] Recall that the functions $\bar{f}_\alpha$ on $[0,\pi]$ depend uniquely on λ for $\lambda \neq \lambda^*$ given in (3.58). If $\lambda = \lambda^*$, $\bar{f}_\alpha$ comprises of an eigenfunction and a particular solution.

[2] The result (4.28) is confirmed by an independent numerical computation in which (4.15)-(4.20) subject to (4.21), (4.26) were treated directly using a standard differential equation solving routine.

The differential equations (4.7) and (4.8), in which the exponents n_2 and t are given by (4.24), yield

$$w_2(\theta) = p_1 p_o^{\frac{2}{3}} \int^{\theta} e^{-\int^{\chi} \vartheta(\varphi)d\varphi} d\chi \qquad \left(-\frac{\pi}{6} < \theta < \frac{\pi}{6}\right) \quad . \tag{4.31}$$

where

$$\vartheta(\theta) = \frac{2\dot{p}(\theta)\left(\frac{1}{3}\dot{v}_2^2(\theta) + \frac{3}{4}v_2^2(\theta)\right)}{3p(\theta)\left(\frac{1}{3}\dot{v}_2^2(\theta) - \frac{1}{4}v_2^2(\theta)\right)} \quad \text{on} \quad \left(-\frac{\pi}{6}, \frac{\pi}{6}\right) \quad . \tag{4.32}$$

The function $v_2(\theta)$ (3.62) satisfies

$$\frac{1}{3}\dot{v}_2^2(\theta) - \frac{1}{4}v_2^2(\theta) \begin{cases} > 0 \quad \text{for} \quad -\frac{\pi}{6} < \theta < \frac{\pi}{6} \quad , \\ = 0 \quad \text{for} \quad \theta = \pm\frac{\pi}{6} \quad , \end{cases} \tag{4.33}$$

and thus w_2 is a smooth function on $(-\pi/6, \pi/6)$. Furthermore, (4.12) and (4.24) provides

$$w_1(\theta) = v_2^{-4}(\theta) \int^{|\theta|} v_2^3(\varphi)\left(\frac{4}{3}p^{-\frac{4}{3}}(\varphi)\dot{v}_2(\varphi) - 3v_1(\varphi)\right)\dot{w}_2(\varphi)d\varphi \quad , \tag{4.34}$$

which also has the appropriate continuity on $(-\pi/6, \pi/6)$. The constant p_1 is evaluated through (4.25), (4.28) and (4.30) to reveal

$$p_1 \doteq -.035 \quad , \tag{4.35}$$

which, in conjunction with (4.9), (4.13) and (4.24), gives

$$\left.\begin{aligned} w_1 &= W_1 p_o^{-\frac{2}{3}} + O(\theta^2), \quad W_1 \doteq .006 \\ w_2 &= W_2 p_o^{\frac{2}{3}} \theta + O(\theta^3), \quad W_2 \doteq -.035 \end{aligned}\right\} \text{as} \quad \theta \to 0 \ . \tag{4.36}$$

Summary of the final results

From Eqs. (4.2), (4.11), (4.14), (4.24) and (4.28) we observe that the second order asymptotic solution to the local crack problem (3.3)-(3.8) is as follows:

$$\left.\begin{aligned} \bar{y}_1 &\sim d_1 \bar{r} \sin\theta + \bar{r}^{\frac{3}{2}} \bar{f}_1(\theta) + \bar{r}^2 \bar{g}_1(\theta) \\ \bar{y}_2 &\sim c_2 + d_2 \bar{r} \cos\theta + \bar{r}^{\frac{3}{2}} \bar{f}_2(\theta) + \bar{r}^2 \bar{g}_2(\theta) \end{aligned}\right\} \text{as} \quad \bar{r} \to 0 \quad \text{on} \quad \xi^+ \ , \tag{4.37}$$

$$\left.\begin{aligned} \bar{y}_1 &\sim \bar{r}^{\frac{3}{2}} \bar{v}_1(\theta) + \bar{r}^2 \bar{w}_1(\theta) \\ \bar{y}_2 &\sim \bar{r}^{-\frac{1}{2}} \bar{v}_2(\theta) + \bar{w}_2(\theta) \end{aligned}\right\} \text{as} \quad \bar{r} \to 0 \quad \text{on} \quad \mathcal{N} \ , \tag{4.38}$$

where

$$\bar{y}_\alpha = p_o^{-\frac{2}{3}} y_\alpha \ , \ \bar{r} = p_o^{-\frac{2}{3}} r \ , \tag{4.39}$$

and the coordinates on ξ^- are obtained through symmetry. The constants c_2, d_α are found to be

$$c_2 \doteq 1.013, \; d_1 \doteq .722, \; d_2 \doteq -2.657 \quad . \tag{4.40}$$

Recall that $\bar{g}_\alpha(\theta) = \bar{g}^e_\alpha(\theta) + \bar{g}^p_\alpha(\theta)$ on $[0,\pi]$ in (4.37). An analytical expression for $\bar{g}^e_\alpha(\theta)$ is given in (4.27) while $\bar{g}^p_\alpha(\theta)$, together with $\bar{f}_\alpha(\theta)$, are described in Fig.8. In (4.38)

$$\left.\begin{aligned} \bar{v}_1(\theta) &= p_0^{\frac{1}{3}} v_1(\theta), \; \bar{v}_2(\theta) = p_0^{-1} v_2(\theta) \quad , \\ \bar{w}_1(\theta) &= p_0^{\frac{2}{3}} w_1(\theta), \; \bar{w}_2(\theta) = p_0^{-\frac{2}{3}} w_2(\theta) \quad , \end{aligned}\right\} \tag{4.41}$$

where $v_1(\theta)$ and $v_2(\theta)$ are found in (3.62) and (3.64), respectively. We refer to Eqs. (4.31)-(4.36) to obtain expressions for $w_\alpha(\theta)$. In addition, we note that the domains ξ^+ and $\mathscr{H}$ (2.20) have the elastostatic shock $\mathscr{S}^+$ (2.18), (2.19) as a common boundary. In view of (4.1), (4.24), (4.25), (4.28), (4.35) and (4.39) the asymptotic representation of the shock reduces to

$$\mathscr{S}^+ : \; \theta \sim c_2 \bar{r}^{\frac{1}{2}} + c_3 \bar{r} \quad \text{as} \quad \bar{r} \to 0 \quad . \tag{4.42}$$

The constant c_2 is given in (4.40) and

$$c_3 \doteq .036 \quad . \tag{4.43}$$

In the analysis summarized above the remaining unknown constants are p_0 (>0) and λ_1 in $\bar{g}_\alpha$ (see (4.27), (4.29)).

The results (4.37)-(4.43) represent a locally one-to-one map of the neighborhood of the crack-tip. Calculating the deformation image of the crack face $\theta = \pi$ in terms of the nondimensionalized spatial coordinates leads to

$$\bar{y}_2 \sim c_2 + c_4 \bar{y}_1^{\frac{1}{2}} \quad (c_2 \doteq 1.013,\ c_4 \doteq 8.24) \quad \text{as} \quad \bar{y}_1 \to 0^+ \quad . \tag{4.44}$$

The crack-tip $(r = 0)$ deforms onto the line $-c_2 \le \bar{y}_2 \le c_2$, $\bar{y}_1 = 0$ and the shock $\mathcal{S}^+$ (4.42) maps onto $\overset{*}{\mathcal{S}}{}^+$ which is found to satisfy

$$\bar{y}_2 \sim c_2 - c_5 \bar{y}_1^{\frac{2}{3}} \quad (c_5 \doteq 3.48) \quad \text{as} \quad \bar{y}_1 \to 0^+ \quad . \tag{4.45}$$

The curves (4.44), (4.45) are sketched in Figure 9 where $\overset{*}{\mathcal{N}}$ and $\overset{*}{\xi}{}^+$ correspond to the respective images of $\mathcal{N}$ and ξ^+. It is apparent from the figure that, in the vicinity of $r = 0$, the traction free faces $(\theta = \pm\pi)$ open to almost flat surfaces oriented in the direction of the applied load at infinity. The fact that the curves extend slightly into the half-plane $\bar{y}_1 > 0$ cannot be explained by this local analysis.

We now list the Cauchy stresses as functions of the material coordinates (r, θ) in the upper half-plane computed from (1.30), (1.33) and (4.37), (4.38):

$$\left.\begin{aligned}
\frac{\tau_{11}}{\mu} &\sim \begin{cases} p_0^{-1/3} r^{1/2} E_{11}(\theta) & \text{on } \xi^+ \ , \\ o(r^4) & \text{on } \mathcal{N} \ , \end{cases} \\
\frac{\tau_{22}}{\mu} &\sim \begin{cases} (1 - \Lambda^{-8/3}) = .926 & \text{on } \xi^+ \ , \\ 1 & \text{on } \mathcal{N} \ , \end{cases} \\
\frac{\tau_{12}}{\mu} &\sim \begin{cases} p_0^{-1/3} r^{1/2} E_{12}(\theta) & \text{on } \xi^+ \ , \\ o(r^2) & \text{on } \mathcal{N} \ , \end{cases}
\end{aligned}\right\} \tag{4.46}$$

where

$$\left.\begin{aligned} E_{11}(\theta) &= 3\Lambda^{\frac{1}{3}}(\cos\theta\dot{\overline{f}}_1 + \tfrac{3}{2}\sin\theta\overline{f}_1) + \Lambda^{-1}(\sin\theta\dot{\overline{f}}_2 - \tfrac{3}{2}\cos\theta\overline{f}_2) \quad , \\ E_{12}(\theta) &= \Lambda^{-1}(\sin\theta\dot{\overline{f}}_1 - \tfrac{3}{2}\cos\theta\overline{f}_1) + \Lambda^{-\frac{7}{3}}(\cos\theta\dot{\overline{f}}_2 + \tfrac{3}{2}\sin\theta\overline{f}_2) \quad , \end{aligned}\right\} \qquad (4.47)$$

and recall $\Lambda \doteq 2.657$. In the hyperbolic region $\mathcal{H}$ the leading terms in the coordinates do not contribute directly to the stresses τ_{11} and τ_{12}. The functions $E_{11}(\theta)$ and $E_{12}(\theta)$ on $[0,\pi]$ are plotted in Fig.10. The stresses (4.46) indicate that the material in the neighborhood of the crack-tip is subjected to deformation that approximates uniaxial tension parallel to the y_2-axis.

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APPENDIX A

In this Appendix solutions to the anti-plane shear crack problem are found which include two elastostatic shocks that are asymptotically tangent to rays issuing from the crack-tips.

To this end, let $\mathcal{S}^+$ and $\mathcal{S}^-$ be curves defined in (2.18), (2.19) with $s = 0$ and $A = \theta^*$, so that

$$\mathcal{S}^{\pm}: \ \theta = \pm\Theta(r) \ , \quad 0 < r \leq r_0 \ , \tag{A.1}$$

where

$$\Theta(r) \sim \theta^* \ \text{ as } \ r \to 0 \ . \tag{A.2}$$

The regions $\mathcal{N}_1$, $\mathcal{N}_2^+$ and $\mathcal{N}_2^-$ described in Figure 11 are defined locally as follows

$$\left.\begin{aligned} \mathcal{N}_1 &= \{(r,\theta) \mid -\Theta(r) < \theta < \Theta(r) \ , \quad 0 < r \leq r_0\} \ , \\ \mathcal{N}_2^+ &= \{(r,\theta) \mid \Theta(r) < \theta < \pi \ , \quad 0 < r \leq r_0\} \ , \\ \mathcal{N}_2^- &= \{(r,\theta) \mid -\pi < \theta < -\Theta(r) \ , \quad 0 < r \leq r_0\} \ . \end{aligned}\right\} \tag{A.3}$$

Keeping in mind the symmetry imposed by the deformation at infinity (2.3) we assume the displacements admit the representation

$$\left.\begin{aligned} &u \sim r^{m_1} v_1(\theta)\ ,\ v_1(\theta) = -v_1(-\theta) \quad \text{as} \quad r \to 0 \quad \text{on} \quad \mathcal{K}_1\ , \\ &u \sim r^{m_2} v_2(\theta) \quad \text{as} \quad r \to 0 \quad \text{on} \quad \mathcal{K}_2^+\ , \\ &u \sim -r^{m_2} v_2(-\theta) \quad \text{as} \quad r \to 0 \quad \text{on} \quad \mathcal{K}_2^-\ , \end{aligned}\right\} \tag{A.4}$$

where the exponents m_α satisfy

$$0 < m_\alpha < 1 \quad . \tag{A.5}$$

Furthermore, $v_1(\theta)$ and $v_2(\theta)$ are smooth functions on the overlapping intervals $(-\theta_1, \theta_1)$ and $[\pi, \theta_2)$, respectively — the angles θ_1 and θ_2 (yet to be determined) obey $\theta^* \le \theta_1 \le \pi$ and $0 < \theta_2 \le \theta^*$. The boundary conditions (2.4) imply

$$\dot{v}_2(\pi) = 0 \quad . \tag{A.6}$$

We repeat the analysis resulting in Eqs. (2.10)-(2.15) for the present case and find the general solutions for $v_\alpha(\theta)$:

$$m_\alpha v_\alpha(\theta) = V_\alpha \cos\psi_\alpha(\theta)\,|\nu_\alpha + \cos 2\psi_\alpha(\theta)|^{-1/2(1-m_\alpha)} \quad , \tag{A.7}$$

where $\psi_\alpha(\theta)$ is the solution of

$$(\tfrac{1}{3} + \cos 2\psi_\alpha)\dot{\psi}_\alpha + (\nu_\alpha + \cos 2\psi_\alpha) = 0 \quad . \tag{A.8}$$

The constant V_α is assumed positive while

$$\nu_\alpha = \frac{1}{3}(3 - 2m_\alpha) \qquad (\frac{1}{3} < \nu_\alpha < 1) \quad . \tag{A.9}$$

We note also that $\nu_\alpha + \cos 2\psi_\alpha \neq 0$ in order that the displacements be bounded.

In what follows, we consider $0 \leq \theta \leq \pi$ and refer to (A.4) for the displacement in the lower half-plane. As in (2.16), let

$$\psi_1(0) = \pi/2 \quad , \tag{A.10}$$

so that $v_1(0) = 0$. The boundary condition (A.6) is satisfied, without loss of generality, by stipulating

$$\psi_2(\pi) = 0 \quad . \tag{A.11}$$

The solutions $\psi_1(\theta)$ and $\psi_2(\theta)$ governed by (A.8)-(A.11) are sketched in Figure 12. The function $\psi_1(\theta)$ decreases monotonically from a value of $\pi/2$ at $\theta = 0$ (as specified by (A.10)) and approaches the asymptote $\bar{\psi} = 1/2 \cos^{-1}(-\nu_1)$ in the limit $\theta \to \infty$. $\psi_2(\theta)$ is a decreasing single valued expression on the interval $[\theta_0, \pi]$ where

$$\theta_0 = \pi - \psi_0 + \frac{(1 - m_2)}{m_2 \nu^*} \tanh^{-1}(\frac{1}{\nu^*}\tan\psi_0) \quad ,$$

$$\nu^* = \left[\frac{3 - m_2}{m_2}\right]^{\frac{1}{2}} , \quad \psi_0 = \frac{1}{2}\cos^{-1}(-\frac{1}{3}) \quad . \tag{A.12}$$

Further, we note that $\psi_2(\theta_0) = \psi_0$ while $\psi_2(\pi) = 0$.

On substituting $\psi_\alpha(\theta)$ into (A.7) we find $v_1(\theta)$ and $v_2(\theta)$ are smooth functions defined on $[0,\pi)$ and $(\theta_0,\pi]$ respectively — the interval (θ_0,π) is common to both domains in accordance with the original assumption (where $\theta_1=\pi$ and $\theta_2=\theta_0$). Also $V_\alpha>0$ implies v_α have the correct sign, namely

$$v_1(0)>0 \quad \text{for} \quad 0<\theta<\pi, \; v_2(\theta)>0 \quad \text{for} \quad \pi\leq\theta<\theta_0 \quad . \tag{A.13}$$

Accordingly, the shock angle θ^* must satisfy

$$\theta_0<\theta^*<\pi \quad . \tag{A.14}$$

An asymptotic equilibrium solution displaying adequate smoothness is now generated to this anti-plane shear crack problem by matching the displacements and associated tractions across the shock $\mathcal{S}^+$.

Continuity of displacements (A.4) requires

$$r^{m_1}v_1(\theta)\sim r^{m_2}v_2(\theta) \quad \text{as} \quad r\to 0 \quad \text{on} \quad \mathcal{S}^+ \quad . \tag{A.15}$$

On referring to (A.1), (A.2) and (A.13) we conclude that asymptotic equality (A.15) holds if and only if

$$m_1=m_2=m, \; v_1(\theta^*)=v_2(\theta^*) \quad . \tag{A.16}$$

This result, together with (A.7) provides one matching condition across the shock:

$$\frac{V_1}{V_2}\left[\frac{\nu+\cos 2\psi_1(\theta^*)}{\nu+\cos 2\psi_2(\theta^*)}\right]^{-1/2(1-m)}=\frac{\cos\psi_2(\theta^*)}{\cos\psi_1(\theta^*)} \quad , \tag{A.17}$$

where

$$\nu = \nu_\alpha = \frac{1}{3}(3 - 2m) \qquad (\frac{1}{3} < \nu < 1) \quad . \tag{A.18}$$

Further, (A.14) indicates that

$$\bar{\psi} < \psi_1(\theta^*) < \frac{\pi}{2} \ , \ 0 < \psi_2(\theta^*) < \psi_0 \qquad (\psi_0 < \bar{\psi}) \quad . \tag{A.19}$$

The remaining condition to hold on $\mathbb{S}^+$ pertains to the matching of tractions. Equations (1.18), (1.24), (1.25), (A.4) and (A.7) imply

$$\frac{V_1}{V_2}\left[\frac{\nu + \cos 2\psi_1(\theta^*)}{\nu + \cos 2\psi_2(\theta^*)}\right]^{-1/2(1-m)} = \frac{\sin^2\psi_1(\theta^*)}{\sin^2\psi_2(\theta^*)} \quad . \tag{A.20}$$

In view of (A.17), (A.20) is replaced by

$$\sin^2\psi_1(\theta^*)\cos\psi_1(\theta^*) = \sin^2\psi_2(\theta^*)\cos\psi_2(\theta^*) \quad , \tag{A.21}$$

which, in turn, can be written in the form

$$\psi_1(\theta^*) = \mathfrak{F}(\psi_2(\theta^*)) \quad \text{for} \quad 0 < \psi_2(\theta^*) < \psi_0 \quad . \tag{A.22}$$

Here, $\mathfrak{F}(\cdot)$ is a monotonically decreasing function on $[0,\psi_0]$ with the end points $\mathfrak{F}(0) = \pi/2$ and $\mathfrak{F}(\psi_0) = \psi_0$.

What remains is to prove the existence of a solution to (A.22) for an admissible value of θ. The shock condition (A.17) can be made to hold merely through an appropriate choice of the ratio V_1/V_2. From the sketch of $\mathfrak{F}(\psi_2(\theta))$ in Fig.12 it is apparent that we can find an angle θ^* (depending uniquely on m) for which the curves $\psi_1(\theta)$ and $\mathfrak{F}(\psi_2(\theta))$

intersect and thus satisfy (A.22).

Consequently, there exists a system of solutions to the crack problem (2.2)-(2.4) in which the displacement field near the crack-tip is of the form

$$u \sim r^m v(\theta) \quad \text{as} \quad r \to 0 \quad (-\pi \le \theta \le \pi) \quad , \tag{A.23}$$

where

$$v(\theta) = \begin{cases} v_1(\theta) & \text{for} \quad -\theta^* < \theta < \theta^* \quad , \\ v_2(\theta) & \text{for} \quad \theta^* < \theta \le \pi \quad . \end{cases} \tag{A.24}$$

In (A.23) the exponent $0 < m < 1$ remains undetermined from the leading order analysis. The function $v(\theta)$ on $[-\pi,\pi]$ (A.24) possesses a discontinuous first derivative across the elastostatic shock $\mathcal{S}^+$ (A.1), (A.2). The shock angle θ^* and $v(\theta)$ depend on the unknown exponent m while $v(\theta)$ is also arbitrary with respect to a multiplicative constant. An appeal to symmetry reveals the displacement on $\mathcal{X}_2^-$. Furthermore, the displacement equation of equilibrium is everywhere <u>hyperbolic</u> as $r \to 0$ for $-\pi \le \theta \le \pi$.

We now consider the stresses associated with the deformation (A.23). The nontrivial components of the actual stress field are obtained, in terms of the polar coordinates $(r,\theta,z) \equiv (r,\theta,x_3)$, from (1.18) and (1.19):

$$\tau_{rz} = 2\overset{*}{W}'(I_1)\frac{\partial u}{\partial r} \; , \quad \tau_{\theta z} = 2\overset{*}{W}'(I_1)\frac{1}{r}\frac{\partial u}{\partial \theta}$$
$$\tau_{zz} = 2\overset{*}{W}'(I_1)|\nabla u|^2 \quad , \tag{A.25}$$

where $I_1 = 3 + |\nabla u|^2$. Referring to (1.24), (A.7), (A.8) and (A.23) yields

$$\tau_{rz} \sim \mu r^{\frac{1}{2}(1-m)} \cos\psi(\theta) p^{-\frac{1}{2}}(\theta), \quad \tau_{\theta z} \sim \mu r^{\frac{1}{2}(1-m)} \sin\psi(\theta) p^{-\frac{1}{2}}(\theta) \quad ,$$

$$\tau_{zz} \sim \mu r^{m-1} p^{\frac{1}{2}}(\theta) \quad \text{as} \quad r \to 0 \quad (-\pi \le \theta \le \pi) \quad , \tag{A.26}$$

with

$$p(\theta) = V|\nu + \cos 2\psi(\theta)|^{-\frac{1}{2}(1-m)} \quad . \tag{A.27}$$

In (A.26) and (A.27)

$$\psi(\theta) = \begin{cases} \psi_1(\theta) & \text{for} \quad -\theta^* < \theta < \theta^* \quad , \\ \psi_2(\theta) & \text{for} \quad \theta^* < \theta \le \pi \quad , \\ \pi - \psi_2(-\theta) & \text{for} \quad -\pi \le \theta < -\theta^* \quad . \end{cases} \tag{A.28}$$

Finally, $V = V_1$ on $(-\theta^*, \theta^*)$ and $V = V_2$ on $(\theta^*, \pi]$, $[-\pi, -\theta^*)$ such that the ratio V_1/V_2 is given in (A.17)

The asymptotic solution presented in this Appendix was not found in [4]. In that study the applied shear at infinity γ was assumed small compared with one and the solutions to the appropriate displacement equation of equilibrium in the vicinity of the crack-tip where chosen so as to facilitate matching onto the linear elasticity solution valid elsewhere. We note that the hodograph transformation used in [4] to solve the small-scale problem may not yield all possible solutions. Furthermore, for more severe deformations at infinity in which γ is not necessarily small, the results in [4] are no longer appropriate. In such cases, displacements of the form (A.23) may occur near the crack-tips.

APPENDIX B

Here, we establish that no <u>smooth</u> solutions of the form (3.9) exist for the problem (3.3)-(3.8). Accordingly, we let

$$y_\alpha \sim r^{m_\alpha} v_\alpha(\theta) \quad \text{as} \quad r \to 0 \quad \text{(no sum)} \quad , \tag{B.1}$$

uniformly for $-\pi \le \theta \le \pi$, where m_α are constants such that

$$0 \le \min[m_1, m_2] < 1 \quad . \tag{B.2}$$

Further, we stipulate

$$v_\alpha \not\equiv 0 \quad , \tag{B.3}$$

are smooth functions on $[-\pi,\pi]$. The symmetry (3.7) associated with this problem implies

$$v_1(\theta) = v_1(-\theta) \ , \ v_2(\theta) = -v_2(-\theta) \quad . \tag{B.4}$$

The inequality (B.2) assures singular deformation gradients and bounded displacements at $r = 0$.

Suppose $m_1 = m_2 = 0$ in (B.1), whereupon the coordinate equations of equilibrium (3.3) are satisfied, to leading order, if

$$\tilde{p}^2(\theta) = \dot{v}_1^2(\theta) + \dot{v}_2^2(\theta) \equiv 0 \quad \text{on} \quad [-\pi,\pi] \quad , \tag{B.5a}$$

or

$$J \sim Qr^{-2/3}\tilde{p}(\theta) \ , \ \tilde{p}(\theta) \neq 0 \quad \text{on} \quad [-\pi,\pi] \quad \text{as} \quad r \to 0 \ , \tag{B.5b}$$

where Q is a positive constant. Equations (B.3), (B.4) are not consistent with (B.5a) on $[-\pi,\pi]$. Now assume (B.5b) holds and note that equilibrium of an arbitrary region $\hat{\mathcal{D}}=\{(r,\theta)|0<r<r_0,\theta_1<\theta<\theta_2,\theta_1<\theta_2 \in (-\pi,\pi)\}$ requires, in the limit as $r_0 \to 0$, that

$$\tilde{p}^2(\theta) = Q^{-6} \quad \text{on} \quad (-\pi,\pi) \ . \tag{B.6}$$

According to the boundary conditions (3.8),

$$\tilde{p}^2(\theta) = 0 \quad \text{on} \quad \theta = -\pi,\pi \ . \tag{B.7}$$

The results (B.6) and (B.7) are incompatible with the assumed smoothness of v_α and thus eliminating the case $m_1 = m_2 = 0$.

Now let

$$m_1 = m_2 = m \ , \quad 0 < m < 1 \ , \tag{B.8}$$

in (B.1). Replacing the coordinates in (3.3) by (B.1) and retaining dominant powers of r, provides two coupled nonlinear differential equations. Assume the equations hold on $(-\pi,\pi)$ and let

$$v_1(\theta) = \rho(\theta)\cos\varphi(\theta), \ \ v_2(\theta) = \rho(\theta)\sin\varphi(\theta) \quad \text{on} \quad [-\pi,\pi] \ . \tag{B.9}$$

If

$$\eta(\theta) = \dot{\rho}(\theta)/\rho(\theta), \ \ \chi(\theta) = \dot{\varphi}(\theta) \quad \text{on} \quad [-\pi,\pi] \ , \tag{B.10}$$

the differential equations reduce to

$$(3m^2 + \eta^2)\dot{\chi} - [\eta\dot{\eta} - m(m+4)\eta - \eta^3]\chi + \eta\chi^3 = 0 \quad , \tag{B.11}$$

$$2m\eta\dot{\chi} - [m\dot{\eta} + m^3 - (4-m)\eta^2]\chi + (4-3m)\chi^3 = 0 \quad , \tag{B.12}$$

on $(-\pi,\pi)$. The boundary conditions on the crack faces (3.8) imply

$$v_1^2(\theta) + v_2^2(\theta) = 0 \ \ \underline{\text{or}} \ \ \dot{v}_1^2(\theta) + \dot{v}_2^2(\theta) = 0 \quad \text{on} \quad \theta = -\pi,\pi \quad . \tag{B.13}$$

In view of (B.9) and (B.10), (B.13) is equivalent to

$$\rho(\theta) = 0 \quad \text{or} \quad \chi(\theta) = 0,\ \dot{\rho}(\theta) = 0 \quad \text{on} \quad \theta = -\pi,\pi \quad . \tag{B.14}$$

Solutions of (B.11) and (B.12) subject to (B.14) can be examined in the phase plane. For convenience, let

$$\Omega(\theta) = \eta^2(\theta),\ X(\theta) = \chi^2(\theta) \quad \text{on} \quad [-\pi,\pi] \quad , \tag{B.15}$$

and derive, from (B.11) and (B.12), the trajectory equation

$$\frac{dX}{d\Omega} = \frac{2XF(\Omega,X)}{G(\Omega,X)} \qquad (X > 0, \Omega > 0) \quad , \tag{B.16}$$

with

$$F(\Omega,X) = 2(1-m)X - [m^2(2+m) - (2-m)\Omega] \quad , \tag{B.17}$$

$$G(\Omega,X) = [3m^2(4-3m) + (4-5m)\Omega]X + [(4-3m)\Omega - 3m^3](m^2+\Omega) \quad . \tag{B.18}$$

The trajectories $X = X(\Omega)$ are sketched in Figs. 13 and 14. From the figures we note that if $X(\theta^*) = \chi^2(\theta^*) = 0$ $(\Omega(\theta^*) \neq \Omega_0)$ for some $-\pi \leq \theta^* \leq \pi$ then, $X(\theta) \equiv 0$ on $[-\pi,\pi]$. Further, a solution to (B.11) and (B.12) such that Ω and X lie on the separatrix can only attain the values at the equilibrium point, $\Omega = \Omega_0$ and $X = 0$, in the limit $\theta \to -\infty$ or $+\infty$. Consequently, the boundary condition (B.14), in which $\chi(\theta) = 0$ on $\theta = -\pi, \pi$, can only be satisfied if $\chi(\theta) = \dot{\varphi}(\theta) \equiv 0$ on $[-\pi, \pi]$. From (B.9) we conclude $v_\alpha(\theta) = c_\alpha \rho(\theta)$ on $[-\pi, \pi]$ while (B.4) implies the constants $c_\alpha = 0$.

Assume now the first condition in (B.14) holds and consider, in particular, the crack face $\theta = \pi$. From (B.10)-(B.12) and (B.16)-(B.18) we obtain an implicit representation for $\rho(\theta)$:

$$\rho(\theta) = e^{\mathcal{J}(\theta)} \tag{B.19}$$

on $[-\pi, \pi]$, with

$$\mathcal{J}(\theta) = \int_{X(\pi)}^{X(\theta)} \frac{m(3m^2 - \Omega(X))}{4XF(\Omega(X), X)}\, dX \quad . \tag{B.20}$$

In (B.20), the appropriate trajectory $\Omega = \Omega(X)$ $(X > 0)$ is chosen with regard to the boundary condition $\rho(\pi) = 0$, which dictates

$$\mathcal{J}(\theta) \to -\infty \quad \text{as} \quad \theta \to \pi \quad . \tag{B.21}$$

We investigate the possibility of satisfying (B.21). Suppose T is a curve $\Omega = \Omega(X)$ in the phase plane that intersects the straight line $F(\Omega, X) = 0$ when $\theta = \pi$ (see Fig.13). Thus, the integral (B.20) is singular at the boundary. Further analysis indicates the integrand is $O[(X - X(\pi))^{-1/2}]$ as $X \to X(\pi)$ on T which implies the condition (B.21) is not attained. Now let $X(\pi) = \infty$. It can be shown from

(B.16)-(B.18) that

$$\left.\begin{array}{lll}\Omega = O\left(X^{\frac{(4-5m)}{4(1-m)}}\right) & & (0<m<4/5) \\ \Omega = O(\log X) & & (m=4/5) \\ \Omega = O(1) & & (4/5<m<1)\end{array}\right\} \text{as} \quad X \to \infty \quad . \qquad \text{(B.22)}^{1}$$

Using (B.22) in (B.20) we again find the integral remains bounded for $0<m<1$. It is apparent then, that suitably smooth solutions of (B.11) and (B.12) cannot be made to satisfy (B.14).

What remains is to account for the case $m_1 \neq m_2$. Accordingly, let

$$m_2 = m < m_1 , \quad 0<m<1 \quad . \qquad \text{(B.23)}$$

Define the function $q(\theta)$ on $[-\pi,\pi]$ through

$$J \sim r^{\nu} q(\theta) \quad \text{as} \quad r \to 0 \quad , \qquad \text{(B.24)}$$

where (B.1), (B.23) and (3.5) indicate that the constant $\nu > 2(m-1)$. Further, we assume q is continuously differentiable on $(-\pi,\pi)$ and is continuous up to the boundaries $\theta = -\pi,\pi$.

Substituting (B.1), (B.23) and (B.24) into the coordinate equilibrium equations (3.3)-(3.6) and considering only the leading order terms we obtain two nonlinear differential equations for q and v_2. Eliminating q from one equation yields

---

[1] Note that $0<(4-5m)/4(1-m)<1$ for $0<m<4/5$.

$$mv_2(4m^2v_2^2 - p^2)(\ddot{v}_2 + m^2v_2) - 2(m-1)p^2(2m^2v_2^2 + p^2) + 3\nu p^4 = 0 \quad , \tag{B.25}$$

$$(2m^2v_2^2 + p^2)\dot{q} - \frac{1}{2}(2\nu m v_2^2 + p^2)q = 0 \quad , \tag{B.26}$$

for $-\pi < \theta < \pi$, where

$$p^2(\theta) = m^2v_2^2(\theta) + \dot{v}^2(\theta) \quad , \tag{B.27}$$

on $[-\pi,\pi]$. Equations (B.1), (B.23), (B.24) infer the boundary conditions

$$\left.\begin{array}{l} r^{3\nu}q^3(\theta) \sim r^{2(m-1)}m^2v_2^2(\theta) \quad \text{as} \quad r \to 0 \\ \\ v_2(\theta)\dot{v}_2(\theta) = 0 \end{array}\right\} \text{on} \quad \theta = -\pi, \pi \quad . \tag{B.28}$$

The technique adopted to solve the differential equation (2.12) is applied to (B.25). We obtain

$$mv_2(\theta) = V_2|\cos\psi(\theta)|^{\varepsilon_1}|\varepsilon_0 + \cos 2\psi(\theta)|^{\varepsilon_2}\cos\psi(\theta) \quad , \tag{B.29}$$

$$q(\theta) = Q|\cos\psi(\theta)|^{\varepsilon_1}|\varepsilon_0 + \cos 2\psi(\theta)|^{\varepsilon_3} \quad , \tag{B.30}$$

on $[-\pi,\pi]$, where V_2 and Q are positive constants and $\psi(\theta)$ satisfies

$$(\tfrac{1}{2} + \cos 2\psi)\dot{\psi} + (\varepsilon_0 + \cos 2\psi) = 0 \quad , \tag{B.31}$$

on $(-\pi,\pi)$. The constants $\varepsilon_0 - \varepsilon_3$ are given by

$$\left.\begin{aligned} \varepsilon_0 &= 2 - \tfrac{3}{2}(m - \nu) \ , \quad \varepsilon_1 = \frac{m}{2(1-\varepsilon_0)} - 1 \\ \varepsilon_2 &= \frac{m(1-2\varepsilon_0)}{4(1-\varepsilon_0)} \ , \quad \varepsilon_3 = \frac{2(\nu+1)(1-\varepsilon_0) - m}{4(1-\varepsilon_0)} \end{aligned}\right\} (\varepsilon_0 \neq 1) \ , \qquad \text{(B.32)}$$

for $0<m<1$. We note that if $0<\varepsilon_0<1$ any solution $\psi(\theta)$ of (B.31) on the finite interval $[-\pi,\pi]$ must necessarily satisfy

$$\psi \neq \psi_1 \ , \ \psi_1 = \tfrac{1}{2}\cos^{-1}(-\varepsilon_0) \qquad (0<\psi_1<\tfrac{\pi}{2}) \ . \qquad \text{(B.33)}$$

This condition is violated only in the limit $\theta \to -\infty$ or $+\infty$.

Equation (B.29) accommodates the smoothness requirement and the parity condition (B.4) if we choose

$$\psi(0) = \pi/2 \quad \text{and} \quad \varepsilon_1 = 0 \quad \text{or} \quad \varepsilon_1 \geq 1 \ . \qquad \text{(B.34)}$$

Taking $\varepsilon_1 = 0$ in (B.32) gives $\varepsilon_0 = 1 - m/2$ and $\nu = 2/3(m-1)$. The boundary conditions (B.28), together with (B.29), (B.30) and (B.32), then imply without loss of generality, that

$$\psi(\pi) = 0 \ , \ \psi(-\pi) = \pi \quad \text{and} \quad Q^3 = V_2^2 \ . \qquad \text{(B.35)}$$

Note that for $0<m<1$, $1/2<\varepsilon_0<1$ and assume the existence of a smooth solution to the boundary value problem (B.31), (B.34) and (B.35). Accordingly, for some value of θ on $(-\pi,\pi)$ ψ must attain the value $\psi_1 = 1/2\cos^{-1}(-\varepsilon_0)$ and thus contradicting the result (B.33). We conclude then, that no smooth solutions exist for ψ and hence V_2 on $[-\pi,\pi]$. A similar conclusion is reached for the case $\varepsilon_1 \geq 1$.

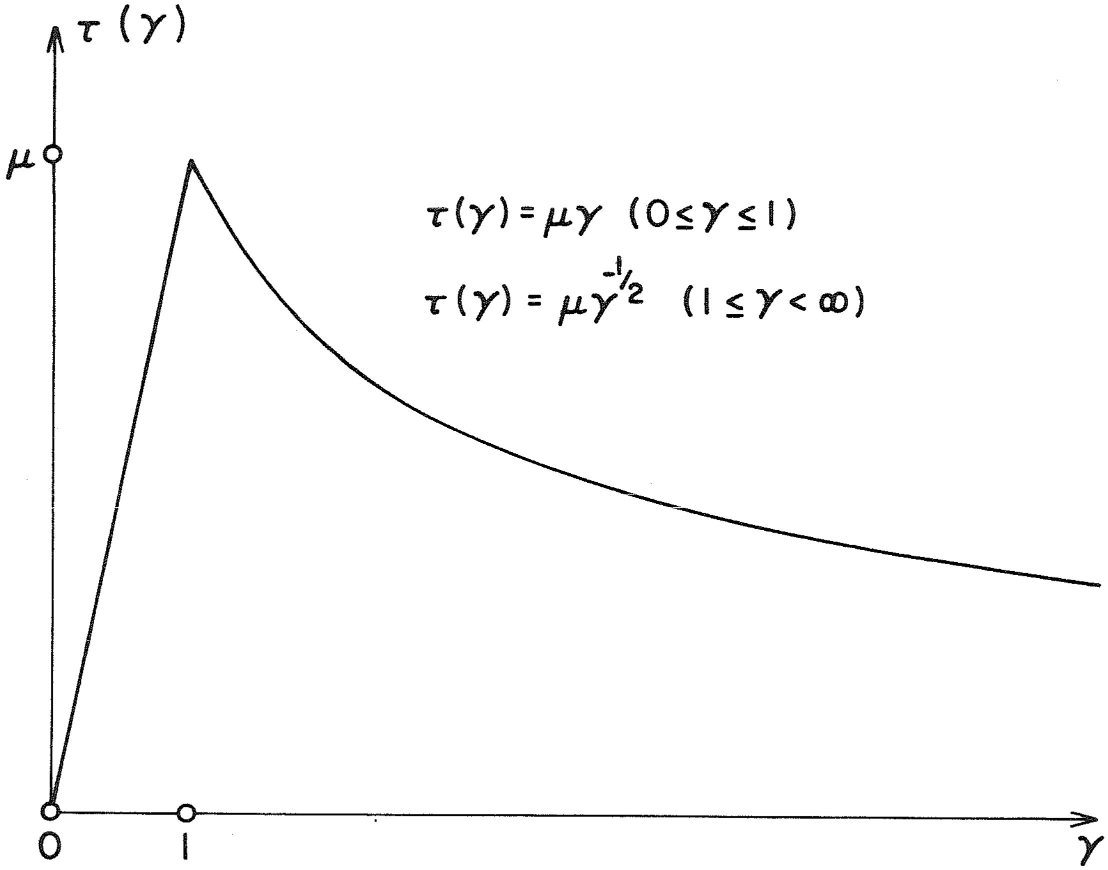

FIGURE 1. RESPONSE CURVE IN SIMPLE SHEAR FOR THE SPECIAL INCOMPRESSIBLE MATERIAL CHARACTERIZED BY (1:23)

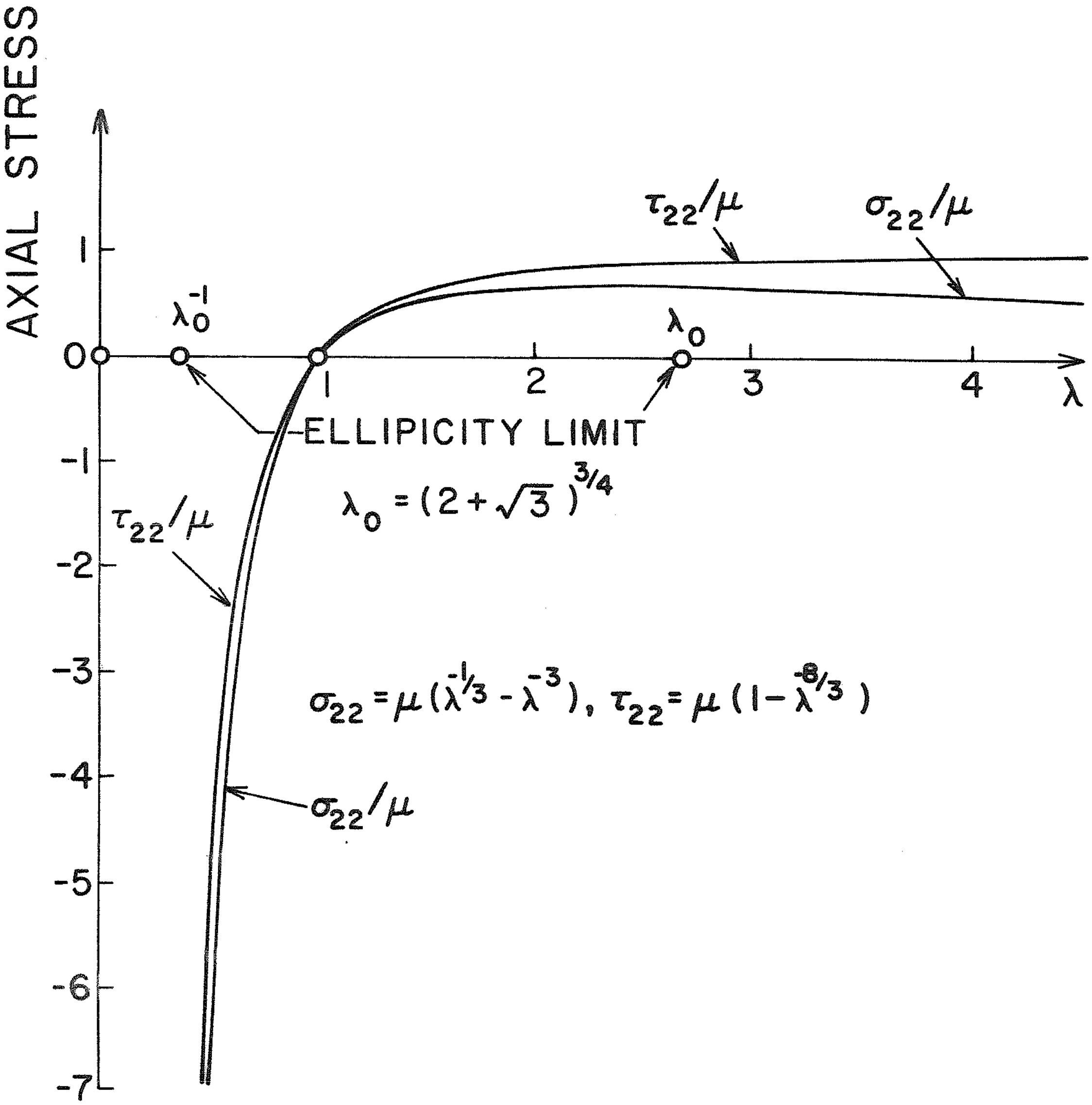

FIGURE 2. BLATZ-KO MATERIAL. PLANE STRAIN UNIAXIAL STRESS: $\tau_{11}=0$. NOMINAL AND ACTUAL STRESS VS. AXIAL STRETCH

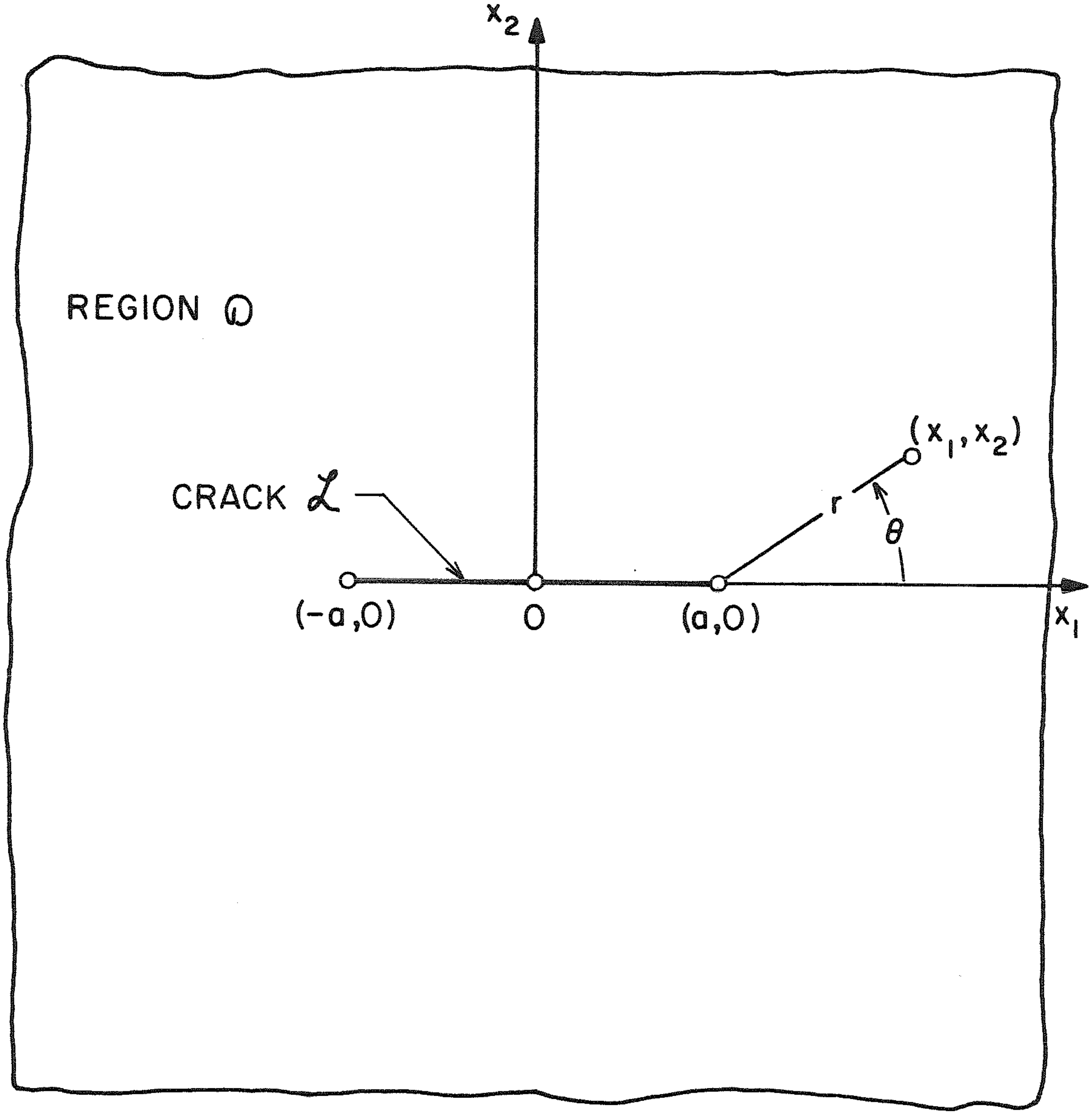

FIGURE 3. GEOMETRY OF THE GLOBAL CRACK PROBLEM

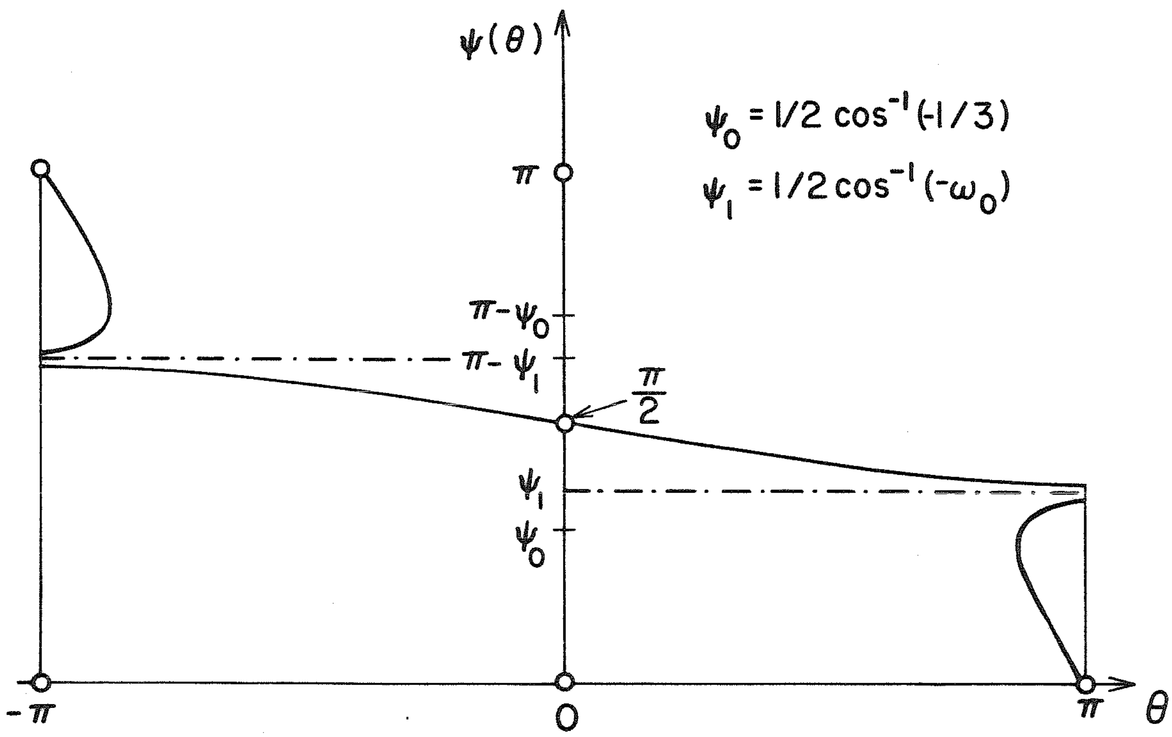

FIGURE 4. SKETCH OF THE PHASE ANGLE $\psi(\theta)$ VS. θ FOR $0 < m < 1$

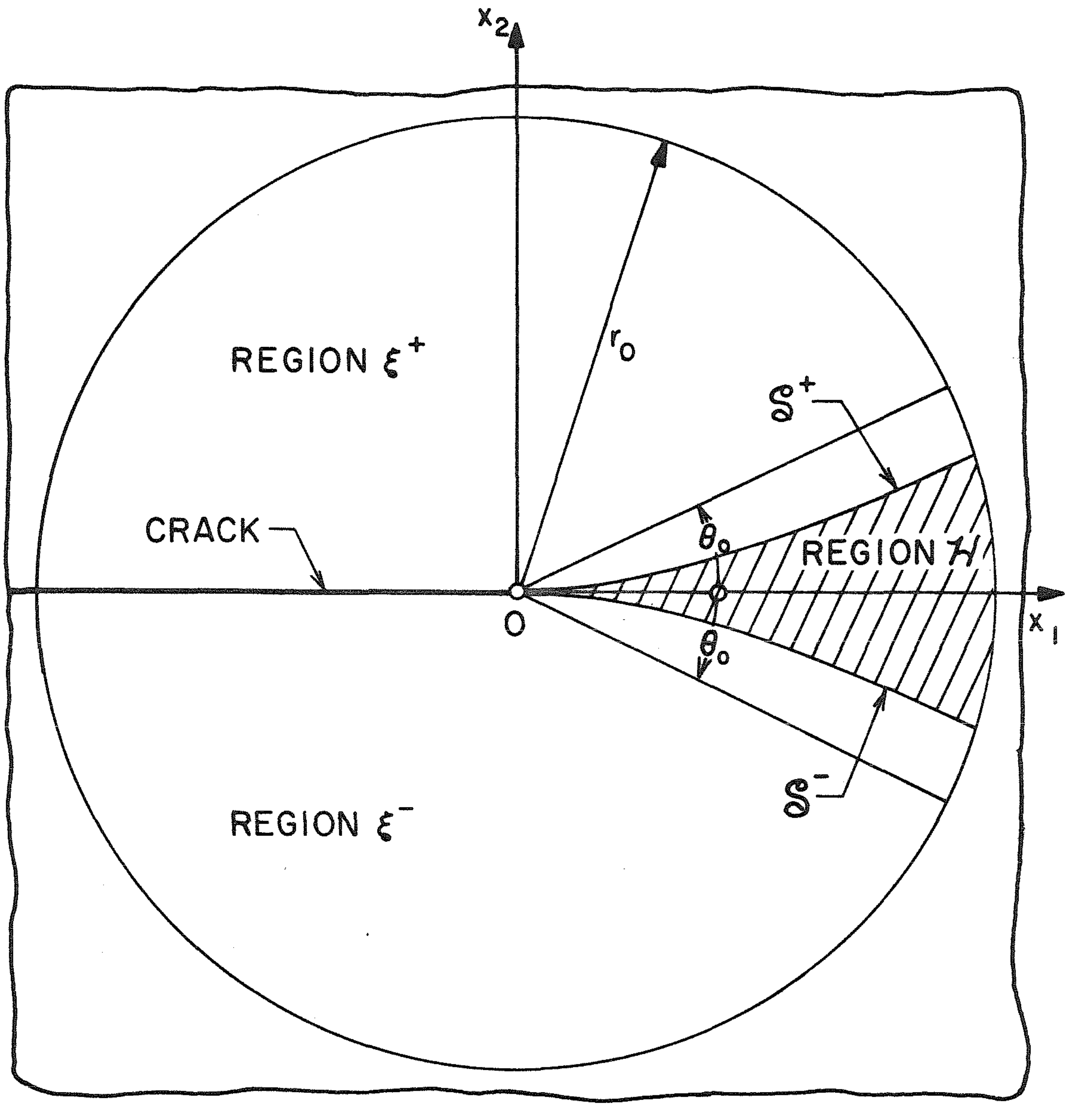

FIGURE 5. UNDEFORMED CONFIGURATION NEAR THE RIGHT CRACK-TIP

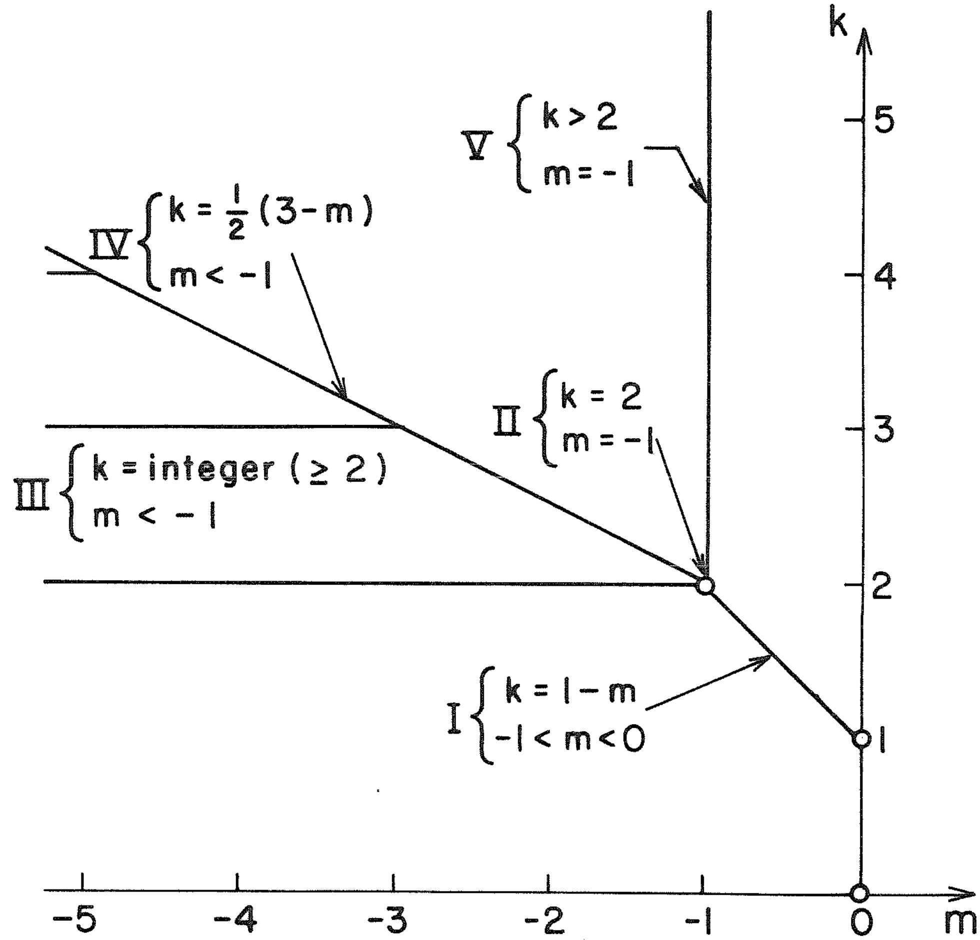

FIGURE 6. (m, k)-DIAGRAM FOR THE ANTI-PLANE SHEAR CRACK PROBLEM

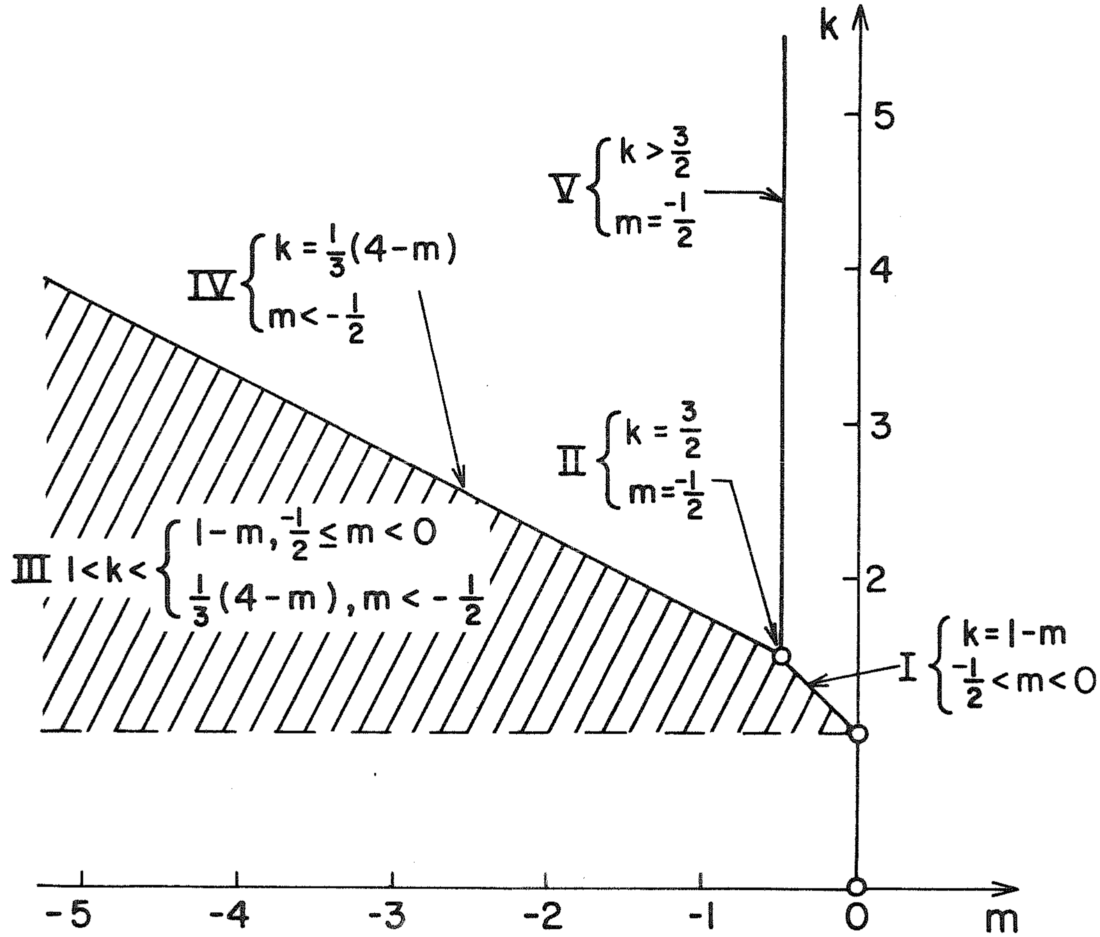

FIGURE 7. (m,k)-DIAGRAM FOR THE PLANE STRAIN CRACK PROBLEM

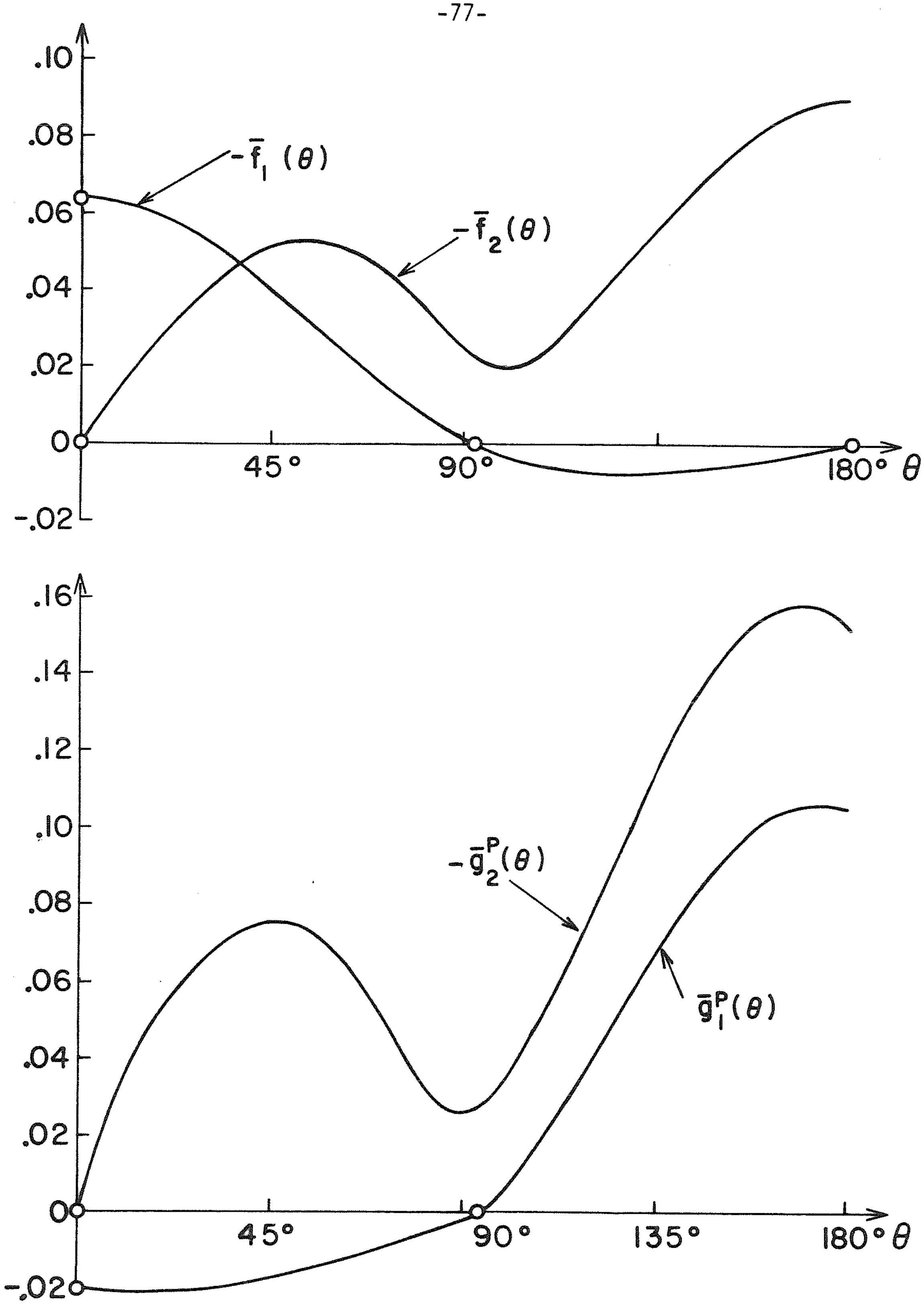

FIGURE 8. NUMERICAL SOLUTIONS FOR $\bar{f}_\alpha(\theta)$ AND $\bar{g}_\alpha^P(\theta)$: $\lambda = \Lambda$

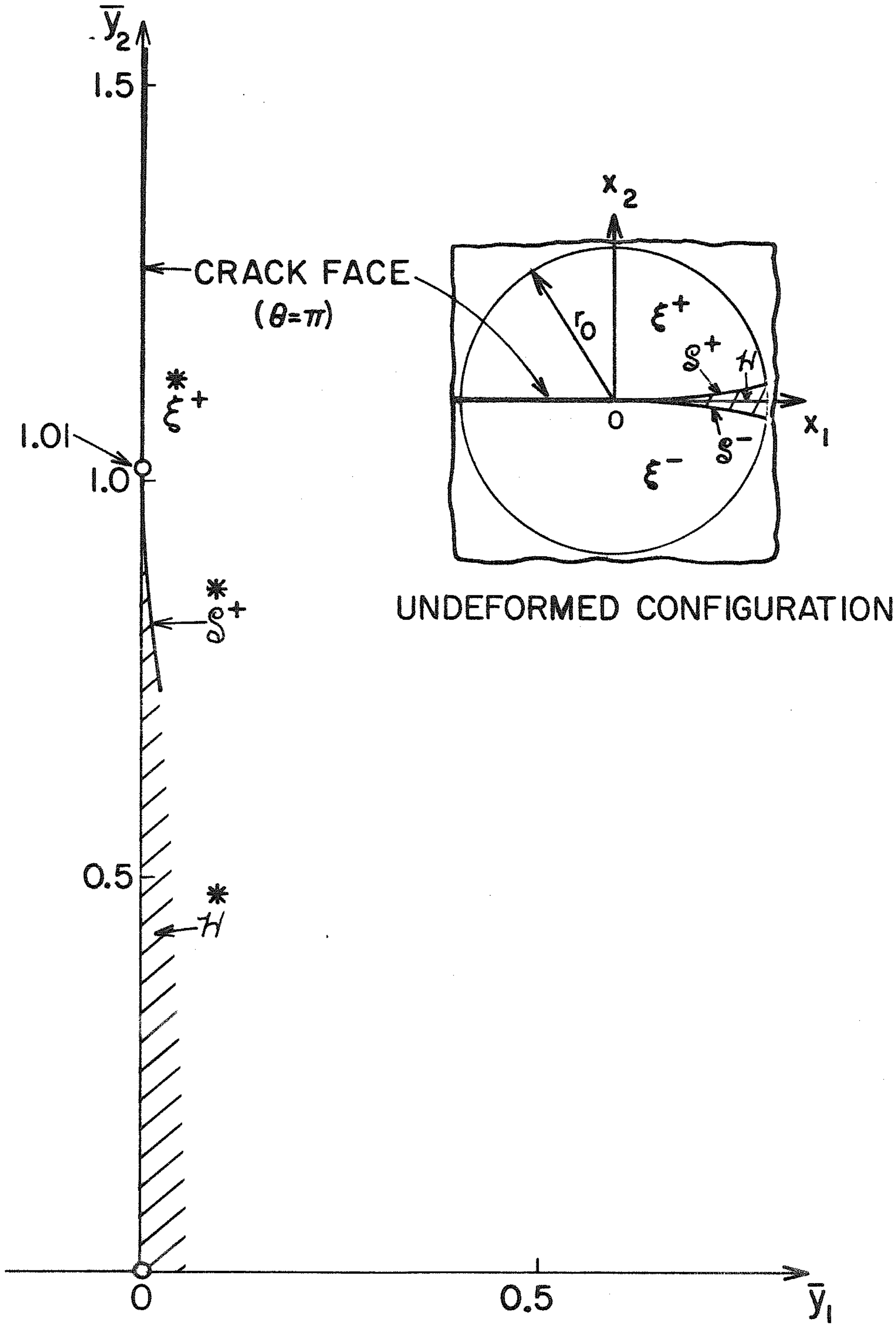

FIGURE 9. DEFORMATION IMAGE OF THE PLANE STRAIN CRACK PROBLEM

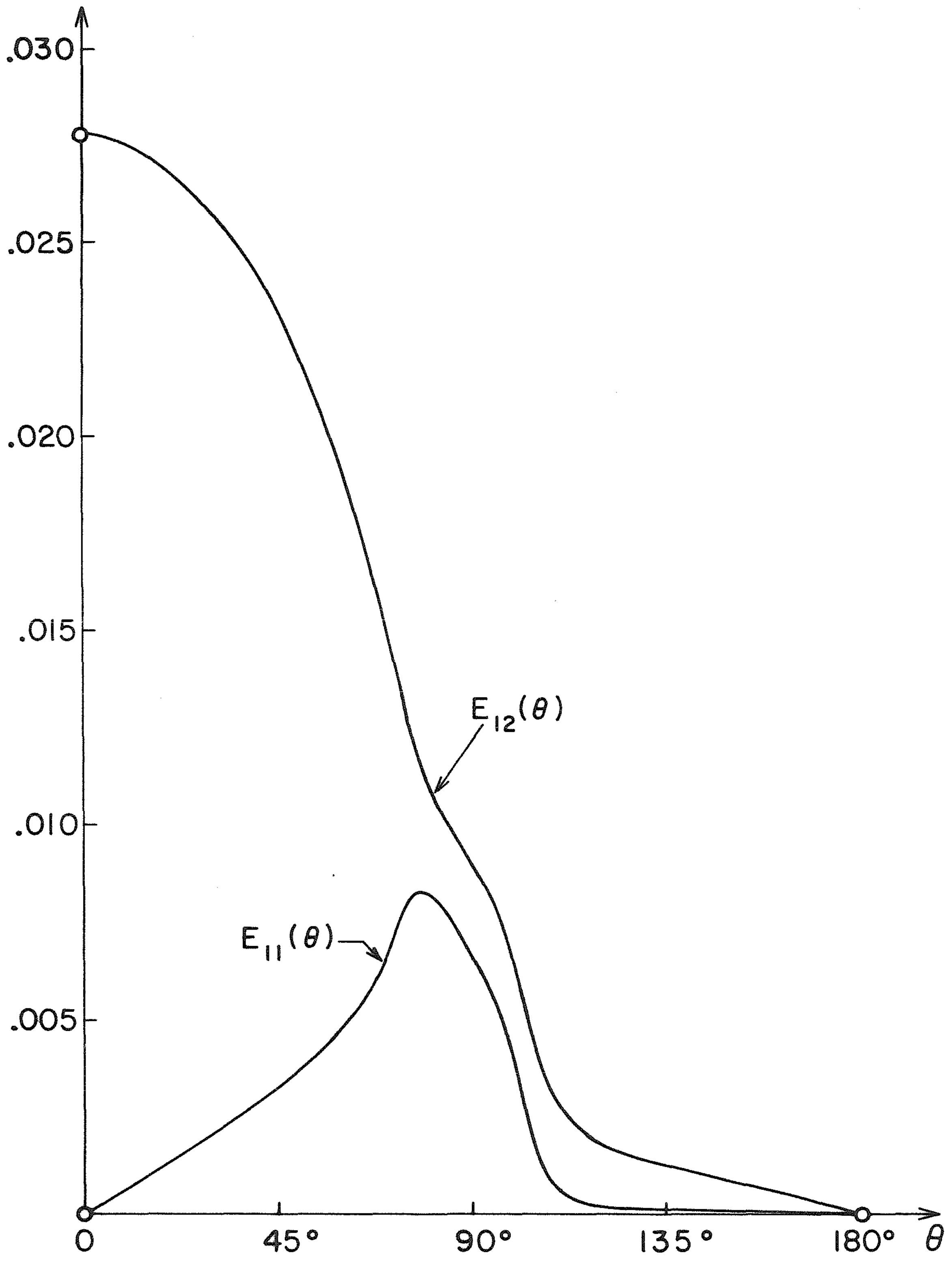

FIGURE 10. NUMERICAL SOLUTIONS FOR $E_{11}(\theta)$ AND $E_{12}(\theta)$.

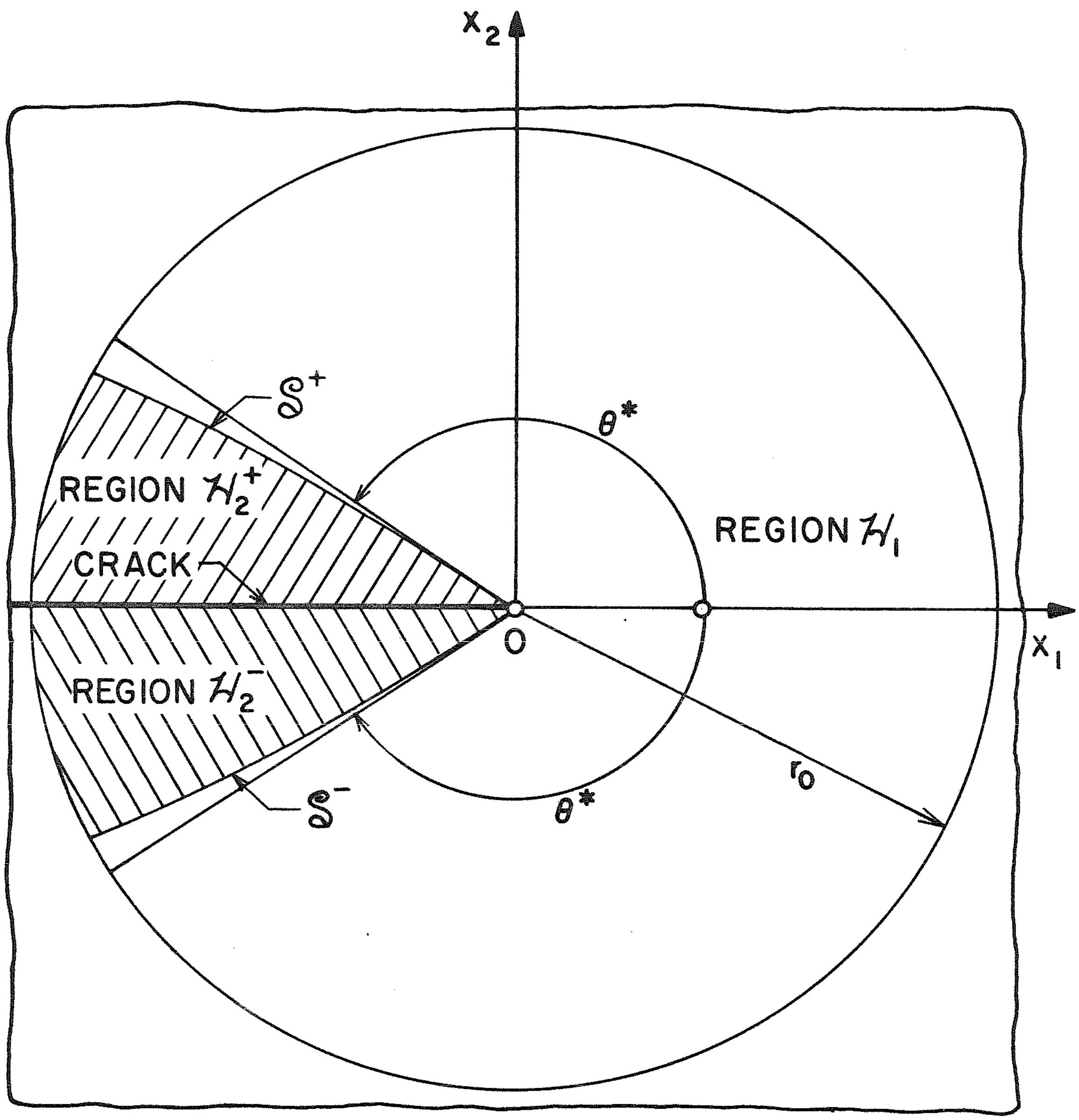

FIGURE 11. LOCAL GEOMETRY NEAR THE CRACK-TIP FOR THE ANTI-PLANE SHEAR PROBLEM TREATED IN APPENDIX A

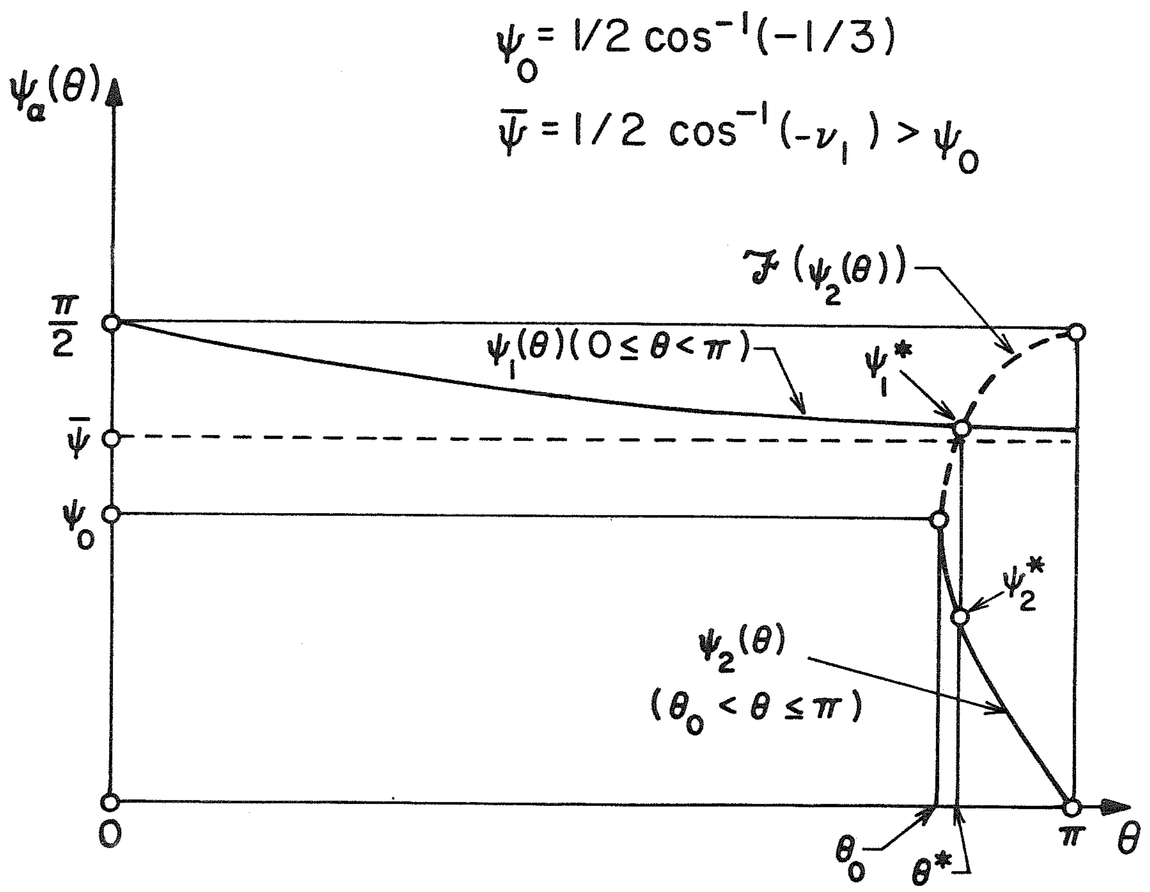

FIGURE 12. SKETCH OF THE PHASE ANGLE $\psi_a(\theta)$ VS. θ REFERRED TO IN APPENDIX A

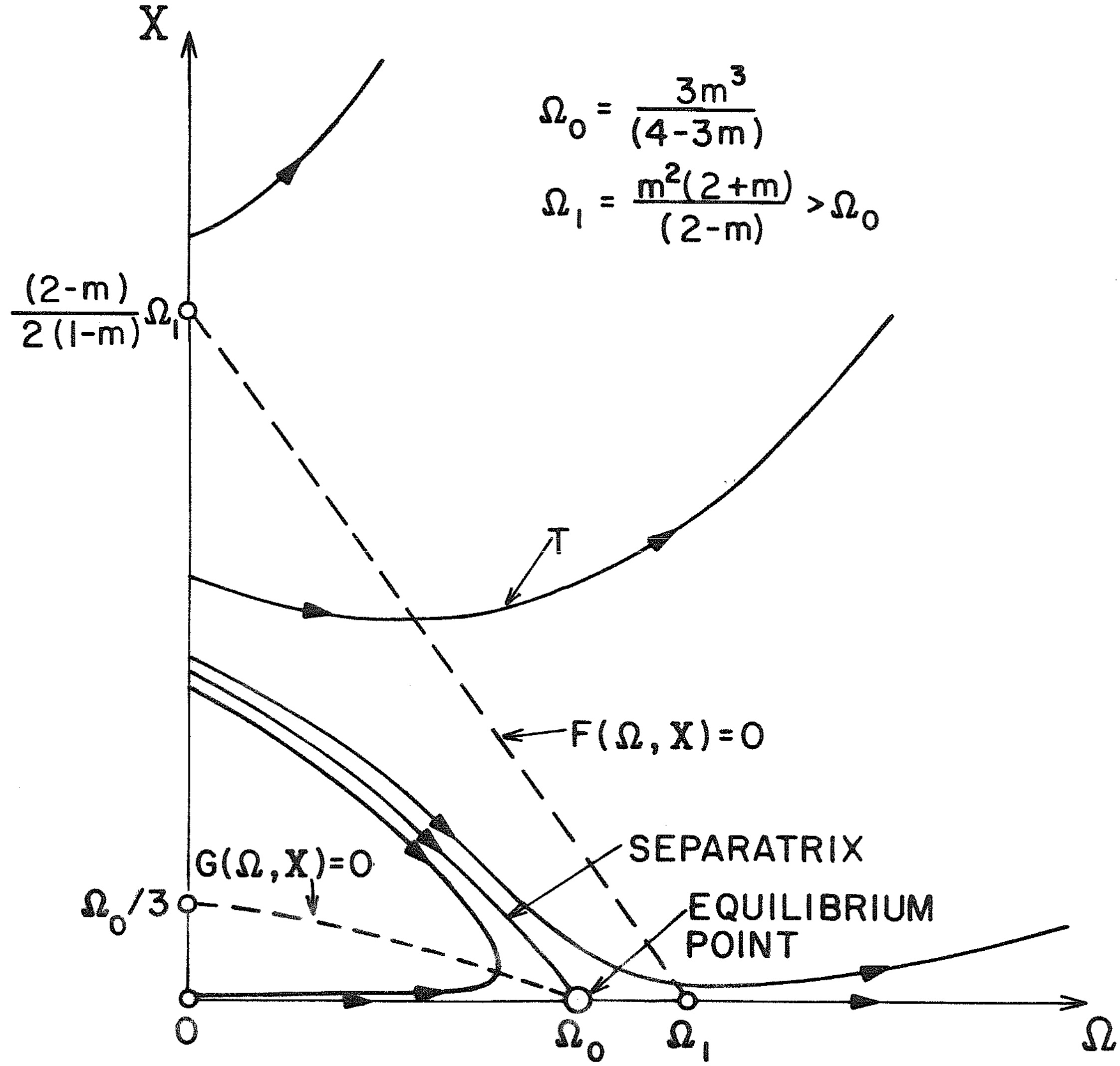

FIGURE 13. SKETCH OF THE TRAJECTORIES IN THE PHASE PLANE FOR EQUATION (B.16): $0 < m < 4/5$

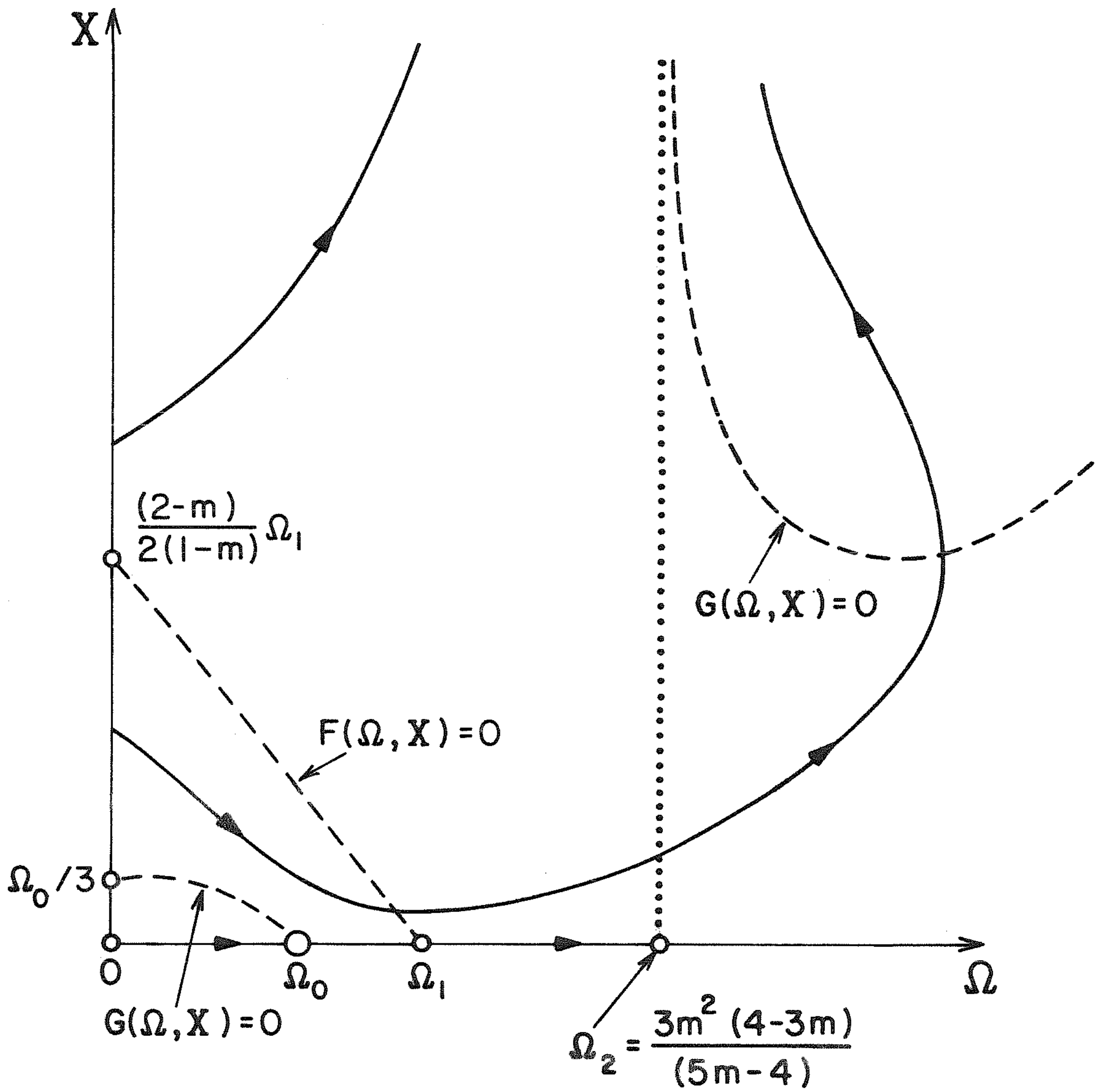

FIGURE 14. SKETCH OF THE TRAJECTORIES IN THE THE PHASE PLANE FOR EQUATION (B.16): $4/5 < m < 1$