ON QUANTUM FLUCTUATIONS AND BLACK HOLES

Thesis by

PILJIN YI

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ABSTRACT

We explore some exotic phenomena in charged black holes, arising from the second quantization of matter fields or from the first quantization of fundamental strings. Spherically symmetric magnetic black holes admit special modes of electrically charged fermions, known as Callan-Rubakov modes, which can be quantized efficiently. We find that the chargeless sector generates non-thermal quantum radiations from extremal Reissner-Nordström black holes, thereby reducing the black hole masses below the familiar classical bound in spite of the vanishing Hawking temperature. On the other hand, the charged sector induces a vacuum energy distribution and the gravitational backreaction thereof, which are particularly pronounced for extremal dilatonic black holes. Implications of these quantum effects are studied in great detail. Also considered is a string-inspired two-dimensional gravity with *nonsingular* charged black holes among its solutions. After a lengthy discussion on the stability of these novel space-times, we speculate on the possibility and implications of nonsingular black holes in full-blown string theories.

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1 Introduction and Summary

Since the discovery of General Relativity by Einstein, perhaps no other aspects of the theory captured the imagination of physicists more than the existence of black holes [1], which gained a firm astrophysical footing as the final evolutionary stage of very massive stars through the ground-breaking works of Chandrasekar [2]. While there exists no confirmed observation of these extraordinary astrophysical objects, many believe in their reality.

Yet, theoretical understanding of black holes is hardly complete after decades of research. Aside from such exotic and possibly unstable features as Cauchy horizons where initial value problems cease to exist, classical black holes possess two main characteristic structures: event horizons and curvature singularities, each of which is still a source of innumerable studies and debates.

The curvature singularities are, for instance, in direct contradiction with the equivalence principle, the very foundation of General Relativity that in effect assumes finite gravitational tidal forces everywhere. Efforts to deal with this inconsistency ultimately lead to celebrated singularity theorems, that singularities are inevitable products of General Relativity and cannot be avoided within the classical regime.

It is true that the classical description of gravity in terms of space-time geometry, governed by a simple local field theory such as General Relativity, is expected to be invalid under the extreme condition of infinite tidal forces, but we do not have a clue as to how a consistent quantum gravity would resolve the singularities.

On the other hand, for large black holes, event horizons are located at relatively small values of space-time curvature, where the low energy effective theory of gravity makes perfect sense. Instead, the difficulties associated with event horizons originate from quantization of matter fields around the black hole. Upon the quantization, a generic black hole loses its mass through seemingly thermal radiations of non-local origin, called Hawking radiation [3]. Essentially a one-loop effect, Hawking radiation exhibits the thermal behaviour not because of any coarse graining one opts for, but as an inevitable consequence of the non-trivial causal structure.

An apparent corollary proposed originally by Hawking [4] is that the process of the black hole formation and its subsequent evaporation may destroy quantum informations encoded in the initial state before the gravitational collapse. Going one step further, one can envision virtual processes where small black holes are spontaneously created and annihilated, just as any other particles, but destroying quantum coherence in the process. Whether and how such violations of the unitarity can be prevented are long-standing problems yet to be solved, often referred to as "the Information Puzzle" or "Problem of Information Loss."

Clearly we need to explore beyond the usual approximations in search of new and interesting structures which might shed some light on the problems of classical and semi-classical black holes. In this regard, arguably the two most important developments in recent years are the discovery of various stringy black hole solutions,[1] and the emergence of two-dimensional toy-models[2] where gravitational backreaction to the quantum radiation can be studied systematically.

While these new studies have not revealed a conclusive resolution of either the singularity problem or the information puzzle, it is also fair to say that we gained considerable insight to quantum aspects of black holes. For instance, the two-dimensional models dispelled the old belief that the information puzzle is a mere artifact of an adiabatic approximation that neglects the dynamics of gravitational backreaction.

This dissertation is a collection of our modest attempts to understand aspects of black holes better, beyond Einstein's general relativity and Hawking's thermal radiation, in the contexts of these new approaches. Mostly, we will be concerned with charged black holes of various origins.

[1]See chapter 2 and chapter 5.

[2]See chapter 2 and chapter 3.

In the second chapter, we review some well-known facts about black holes in three different contexts. First, static and spherically symmetric black hole solutions to Einstein's general relativity and the Einstein-Maxwell theory are presented with their Penrose diagrams. Secondly, we discuss how Hawking radiation arises as a one-loop effect in the framework of the effective action approach. Also discussed are the thermal description of late-time Hawking radiation, in canonical Hamiltonian approach, and its limitation. Finally, we discuss how string theories produce nontrivial background geometries, and in particular black holes.

In the following chapter [5], we explore semi-classical properties of external magnetic Reissner-Nordström black holes. By restricting the matter sector to that of chargeless Callan-Rubakov modes, we demonstrate that the thermal description of the black hole indeed breaks down in the extremal limit and that the extremal black holes with vanishing Hawking temperature do radiate away a finite amount of energy. Both analytic and numerical studies are presented as well as a discussion on the implication on black hole bifurcation processes. The numerical portion of the work was aided by Jaemo Park.

Chapter 4 [6] deals with a vacuum polarization of massive charged fermions in the background of an exotic magnetic black hole, known as the cornucopion. Isolating the charged sector of Callan-Rubakov modes, we study how the energetics of the vacuum polarization are drastically changed when the background configuration is modified from that of a monopole to the noncompact geometry of the cornucopion. The gravitational backreaction to the phenomenon is studied in great detail, and the generality of the conclusion is discussed at the end.

The final chapter [7] concerns even more exotic black hole solutions. In the context of low-energy effective string theories, we find a family of two-dimensional space-times with event horizons but without any curvature singularity. After a lengthy discussion on the stability of such space-times, we speculate on the possibility of nonsingular black holes in string theory. In particular, we show that the well-known exact string background supporting the $SL(2,R)_k/U(1)$ Wess-Zumino-Witten coset model possesses the same nonsingular geometry as one of our low-energy solutions.

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2 Quantum Fluctuations and Black Holes

One of the more challenging problems to modern theoretical physicists is understanding the quantum nature of gravity. While Einstein's General Relativity continues to be very successful in explaining a wide range of phenomena such as evolution of the universe [1] and decaying orbits of binary pulsars [2][3], we are yet to understand the microscopic behaviour of gravitational interactions.

Perhaps in no other system is this inability more frustrating than in black holes. Not only do the generic curvature singularities involve infinitesimal length scale, the realm of quantum gravity, but the apparent thermal behaviour of radiating black holes is seemingly difficult to reconcile with the unitarity of quantum mechanics. For sure, all the puzzles regarding black holes must be resolvable after a full understanding of quantum gravity is attained, but we are yet to have the luxury of a consistent and manageable quantum gravity in our hand.

While the quantization of gravity itself has eluded us so far, the quantization of matter degrees of freedom in the background of nontrivial geometry has been carried out in isolated cases. For instance, the quantization of (free) matter fields in black hole backgrounds leads to the celebrated Hawking radiation, while the consistent quantization of fundamental strings is known to restrict acceptable spacetime geometries. Therefore, for us who wish to study quantum aspects of black holes, it seems the logical starting point to explore quantum aspects of various matter degrees of freedom in black hole backgrounds.

It is the purpose of this chapter to introduce some of the well-established ideas and facts involving both quantum fluctuations and black holes. These somewhat fractured portraits are not meant to be a complete survey, but rather to set an undertone for the studies in the following chapters.

2.1 Black Holes and Penrose Diagrams

The simplest theory where black hole solutions are found is Einstein's general relativity [4]. The corresponding Einstein-Hilbert action can be written succinctly with the scalar curvature $R = g^{\mu\nu} R^{\alpha}_{\mu\alpha\nu}$ associated with the Levi-Civita connection of the metric g.[3]

$$S_{\textbf{Einstein-Hilbert}} = \frac{1}{16\pi} \int dx^4 \sqrt{-g}\, R \quad \rightarrow \quad G_{\mu\nu} \equiv R^{\alpha}_{\mu\alpha\nu} - g_{\mu\nu} R = 0. \tag{1}$$

The unique one-parameter family of static and spherically symmetric solutions can be written in the Schwarzschild gauge in the following way,

$$g = -(1 - \frac{2M}{r})\, dt^2 + (1 - \frac{2M}{r})^{-1}\, dr^2 + r^2\, d\Omega^2. \tag{2}$$

When the ADM mass [4] M is nonzero, the Riemann curvature tensor $R^{\alpha}_{\mu\beta\nu}$ diverges at $r = 0$ corresponding to a curvature singularity. Such curvature singularities involve arbitrarily small length scale and cannot be described by a low-energy effective theory such as General Relativity. It is a matter of great uncertainty whether such singular structure will appear in consistent quantum gravity as well.

For the observers far away, however, the more interesting feature of these black hole solutions is the event horizon at $r = 2M > 0$.[4] As is evident from the metric coefficients above, the "radial" coordinate r becomes timelike inside the horizon, and consequently the size of the transverse two-sphere $4\pi r^2$ changes monotonically along the worldline of any observer trapped inside the horizon $r < 2M$. For those falling into the black hole from outside, as a result, the singularity becomes unavoidable.[5]

Probably the most economical way of describing such nontrivial causal structure is to use Penrose diagrams [5]. In figure 2.1, we present two Penrose diagrams of the maximally extended space-times of the metric (2) for nonnegative M. These diagrams depict most clearly the trajectories of null lines, i.e., the worldline of

[3] In this chapter, we use the geometrized unit $c = G = 1$, while the signature of the metric is taken to be $(-1, 1, 1, 1)$ in this chapter.

[4] Here, we will not consider cases of naked singularities.

[5] See figure 2.1.

massless particles which always travel at a ± 45 degree angle. Hence, the bold line denoted by I^+ is future null infinity where all the outgoing massless particles are directed, while the intersection between the past and the future null infinities $I_\pm$ corresponds to $r = \infty$ with any finite t, namely space-like infinity..

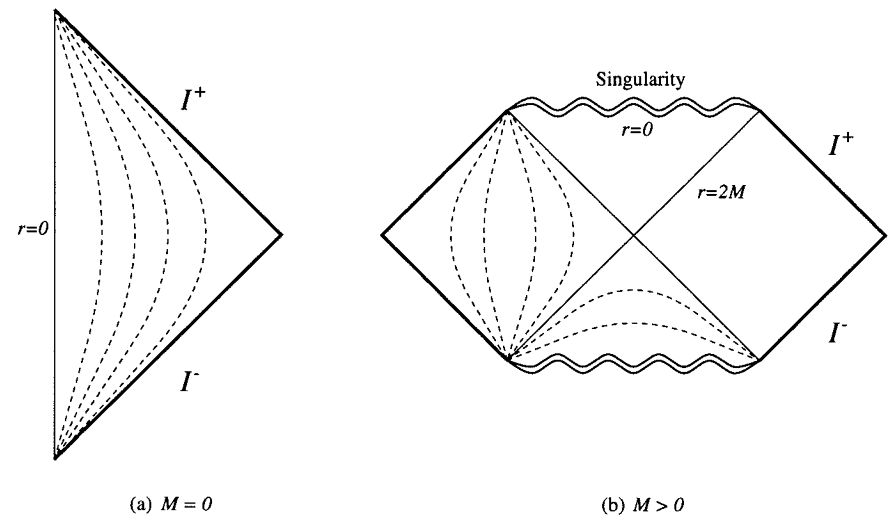


Figure 2.1: Penrose diagrams for (a) flat Minkowski space-time $M = 0$, and for (b) Schwarzschild black hole $M > 0$. The wavy lines indicate the curvature singularities at $r = 0$, while the bold lines correspond to null infinities. The dotted lines are those of constant r.

More complicated causal structures appear, if we consider Einstein's equations with a source term, i.e., energy-momentum tensor, on the right-hand-side,

$$G_{\mu\nu} = 8\pi T_{\mu\nu}. \tag{3}$$

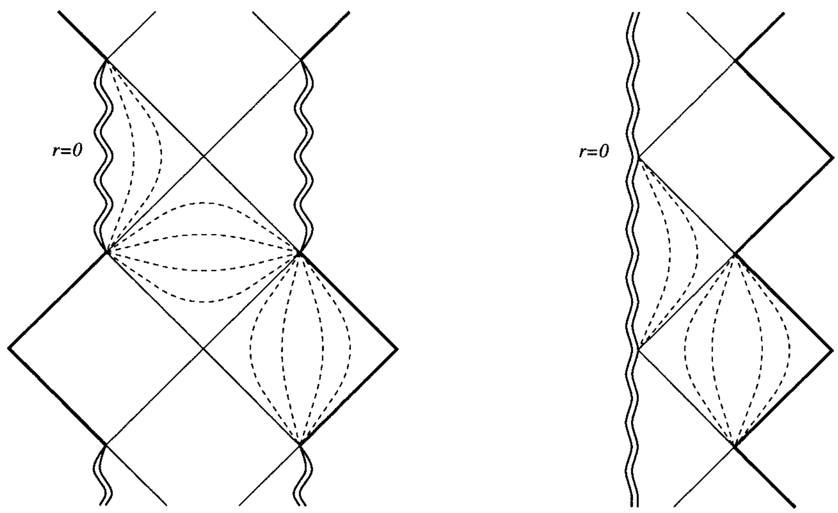
A particularly interesting example is when the black hole acquires an electromagnetic charges. Adding the Maxwell term to Einstein-Hilbert action,

$$S_{\text{Einstein-Maxwell}} = \frac{1}{16\pi} \int dx^4 \sqrt{-g}\, \{R - F^2\} \quad \rightarrow \quad G_{\mu\nu} = 8\pi T_{\mu\nu}^{\text{Maxwell}}, \tag{4}$$

the static and spherically symmetric solutions of which are similarly given by a unique two-parameter family of spacetimes, called Reissner-Nordström black holes,

$$g = -(1 - \frac{2M}{r} + \frac{Q^2}{r^2})\, dt^2 + (1 - \frac{2M}{r} + \frac{Q^2}{r^2})^{-1}\, dr^2 + r^2\, d\Omega^2. \quad (5)$$

Note that the value of the Maxwell energy-momentum T^{Maxwell} is invariant under the duality transformation exchanging the magnetic and the electric charge, and this solution describes black holes of either magnetic or electric charge Q provided that the singularity is not naked. Also note that this reduces to the Schwarzschild solution (2) as $Q \to 0$.



(a) Nonextremal

(b) Extremal

Figure 2.2: Penrose diagrams for Reissner-Nordström black holes with time-like singularities.

Again from the metric coefficients above, it is easy to see that the nature of r coordinate changes between space-like and time-like, except that, in this case, the transition can occur at two values of $r = r_{\pm} \equiv M \pm \sqrt{(M^2 - Q^2)} > 0$. In the extremal limit $r_+ = r_-$, in particular, r is spacelike everywhere and the extremal causal structure is different from that of nonextremal cases $r_+ > r_-$ as depicted in figure 2.2.

One interesting aspect of such causal structures with event horizons is that observers outside can see only part of the whole space-time. Whatever happens inside the future event horizon will not affect us simply because its future light-cones are confined to inside the event horizon at $r = 2M$ in the Schwarzschild case or r_+ in the Reissner-Nordström case.

An immediate consequence is that the worrisome curvature singularities and whatever else is hidden by the event horizon are irrelevant for physics outside. Not only can we continue to use the low-energy effective theory without worrying about the microscopic nature of gravity, but, for some applications such as the stability analysis [6], we can concentrate on fluctuations with support outside the horizon and simply ignore those inside since they are effectively decoupled. In a sense, the event horizon can be said to protect us from both the singularity and the resulting nontrivial causal structure.

But the protection comes at a price. The diagrams in the figures above are fictitious in that real black holes are expected to be created by gravitational collapse, which cannot be a static process. Instead, the region below the past event horizon is initially part of the smooth "outside" world as illustrated in figure 2.3 below. Through the formation of the black hole, then, part of the evolving Cauchy surface, contained in the regions denoted by III or IV, are engulfed by the future event horizon, and thus lost to us. While harmless classically, this inaccessibility is at the root of the so-called information puzzle associated with Hawking radiation of black holes [7][8].

2.2 Quantum Fields in Black Hole Backgrounds

Now that we have classical black hole solutions, let us consider quantization of free matter fields outside black holes. In addition to the classical part of the gravitating matter, there are always quantum fluctuations which necessarily carry energy-momentum of their own. One of the most famous examples would be the Casimir effect, where a vacuum bounded by two conducting plates develops a vacuum energy density through spontaneous creations and annihilations of quanta.

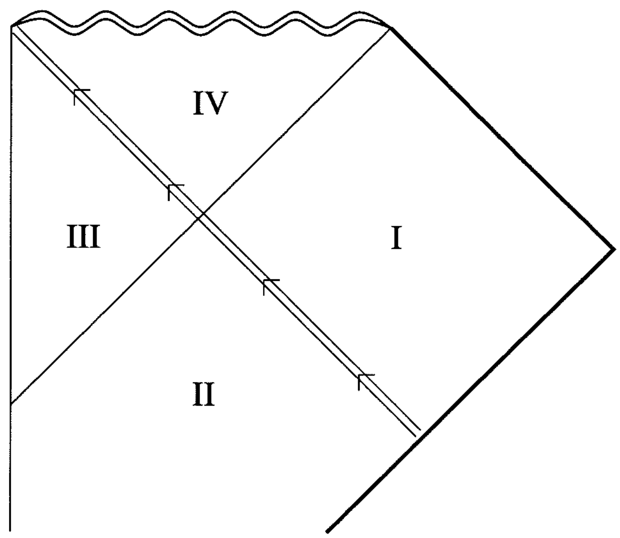


Figure 2.3: Penrose diagram for a collapsing massless shell. Region I is outside of the shell as well as outside of the future event horizon, while regions III and IV are trapped inside the horizon, thus inaccessible to observers in region I.

For the purpose of choosing a physically sensible vacuum state, it is convenient to consider an idealized gravitational collapse as depicted by figure 2.3, representing a Schwarzschild black hole formed by a gravitational collapse of imploding massless spherical shell of matter. Below the double line of the collapsing shell, or equivalently inside of the collapsing shell, the geometry is that of flat Minkowski space-time, while the outside geometry is that of a Schwarzshild black hole. To be consistent with this geometry, the classical energy-momentum must vanish completely except on the shell. This solution is known as the Vaidya metric [9].

For a free matter field ψ, the matter action $S_{\text{matter}}(\psi)$ is necessarily quadratic in ψ, and the only interaction is through gravity. Using the path-integral quantization, given an arbitrary background geometry, we can encode the gravitational effect of the quantized matter fields in the effective action $W(g)$,

$$e^{-iW(g)} = \int D\psi \; e^{iS_{\text{matter}}(\psi)/\hbar}, \tag{6}$$

which is just a sum of one-loop diagrams with arbitrary number of external graviton lines. The one-loop energy-momentum tensor from the quantized matter is then obtained by varying W with respect to the inverse metric, and the following equation summarizes the resulting effect on the background geometry.

$$\frac{1}{8\pi} G_{\mu\nu} = T_{\mu\nu}^{classical} + \langle T_{\mu\nu} \rangle, \qquad \langle T_{\mu\nu} \rangle = 2\hbar \frac{\delta W}{\delta g^{\mu\nu}} \tag{7}$$

By quantizing ψ in a given classical background such as shown in figure 2.3., we are effectively treating the second term, proportional to $\hbar$, as a small perturbation.

Within this approximation, the first step is to evaluate the one-loop contribution $\langle T_{\mu\nu} \rangle$ in the given classical geometry. In general, W is a complicated non-local functional of g [10] and this kind of straightfoward calculation is almost always impossible. However, our purpose being to provide a convincing illustration rather than a complete study of the phenomenon, we might as well introduce a toy-model where such an explicit calculation is possible. For instance, we can regard the Penrose diagram in figure 2.3 as depicting a solution to a two-dimensional dilatonic gravity, obtained by integrating Einstein-Hilbert action over the angular coordinates, and consider a conformal scalar ψ which propagates in such a two-dimensional world.

$$S_{\text{matter}}(\psi) = -\frac{1}{2\pi} \int dx^2 \sqrt{-g^{(2)}} \, (\nabla\psi)^2 \tag{8}$$

For the collapsing geometry of figure 2.3, the metric $g^{(2)}$ is expressed most conveniently by introducing a light-cone coordinate v which grows indefinitely toward future null infinity.

$$g^{(2)} = -(1 - \frac{2M\,\theta(v - v_0)}{r})\,dv^2 + 2\,dv\,dr \tag{9}$$

Obviously $v = v_0$ is the trajectory of the collapsing massless shell where $T^{classical}$ must be nonzero. Outside the shell $v > v_0$, the coordinate v is related to Schwarzschild time t by $v = t + \int dr/(1 - 2M/r)$.

Integrating out ψ, we obtain the famous Polyakov-Liouville action [11] as the effective action W and the one-loop energy-momentum thereof, in closed forms [12]. The detailed calculations are carried out in sections 1 and 2 of chapter 3, for a more general situation. Introducing another light-cone coordinate u in the region I of figure 2.3, $u = t - \int dr/(1 - 2M/r)$, growing indefinitely toward the future event horizon, we find the following outgoing energy flux in the region I.

$$\langle T_{uu} \rangle = \frac{\hbar}{12\pi}(\partial_u^2 \rho - (\partial_u \rho)^2) + t_{uu}(u), \qquad \rho \equiv \frac{1}{2}\,\log(1 - \frac{2M}{r}). \tag{10}$$

The first piece on the right-hand-size is necessary to satisfy the energy-momentum conservation in this curved space-time, while the second piece t_{uu} is to be determined by the initial condition on the initial Cauchy surface $v = v_0$.

On the other hand, one natural initial condition is to require vanishing energy-momentum on the initial Cauchy surface, except that of the collapsing shell, implying $\langle T_{uu} \rangle = 0$ on the surface $v = v_0$. In other words, there is no radiation emanating from the surface of collapsing shell. Accordingly, $t_{uu}(u)$ must be chosen such that it cancels the rest of the right-hand-side completely along $v = v_0$.

But note that the two contributions propagate in different fashion. The first piece, consisting of derivatives of ρ, propagates along constant r, while the second piece $t_{uu}(u)$ propagates outward at speed of light. Hence, an asymptotic observer at large fixed r will observe radiation t_{uu} only and nothing else. This emergence of outward radiation, apparently out of nothing, is clearly illustrated by figure 2.4. In particular, the resulting outward radiation flux eventually settles down to a finite limiting value.

$$\langle T_{uu}\rangle_{\text{asymptotic}} \quad \Rightarrow \quad t_{uu}(u \to +\infty) = \frac{\hbar}{12\pi}\left(\partial_u\rho\right)\Big|_{r=2M} = \frac{\hbar}{12\pi}\frac{1}{64M^2} \qquad (11)$$

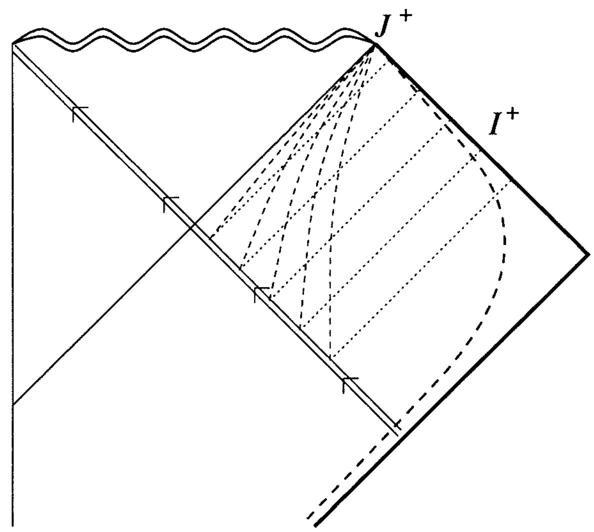


Figure 2.4: Propagation of one-loop energy-momentum. The broken lines are those of constant r while the dotted ones are those of constant u. An observer at large r, depicted by a thick broken line, detects outward radiation as encoded in t_{uu}, which becomes a constant as he approaches J^+, time-like future infinity.

From the derivation above, it is not difficult to notice that the crucial step is the choice of the initial state, but it is also easy to see that, as far as the late-time value of the flux $\sim \hbar/M^2$ is concerned, the initial condition of vanishing $\langle T_{uu}\rangle$ is unnecessarily strict. The same flux would have been found if we had required only that T_{uu} vanishes just near the horizon. But this condition is always satisfied, since T_{uu} near the future event horizon is roughly equal to $(r-2M)^2\sigma$, where σ is the energy density there as seen by freely falling observers and is necessarily finite.

A clear implication is the universality of this late-time radiation. The details of the gravitational collapse does not affect the late-time thermal radiation seen above. What if the collapsing body itself were radiating? An asymptotic observer would observe the energy content of this particular radiation diminishing rapidly, because the relative time-dilation between the surface of the collapsing body and the asymptotic observer, thus the red-shift factor, increases indefinitely as the collapsing body crosses the future event horizon.

This universality turns out to generalize to more realistic cases of four-dimensional black holes, and was fully exploited by Hawking in his original derivation. Before presenting Hawking's general idea, it is instructive to note that the late-time flux (11) is consistent with a steady thermal flux in two-dimensional space-time at temperature $T_{BH} = \hbar/8\pi M$, the Hawking temperature of Schwarzschild black holes [7][13].

$$\left\{\text{Thermal Flux}\right\}_{T_{BH}} = \int_0^\infty \frac{dp}{2\pi\hbar}\,\frac{p}{e^{p/T_{BH}}-1} = \frac{\pi}{12\hbar}\,T_{BH}^2 = \frac{\hbar}{12\pi}\,\frac{1}{64M^2}. \tag{12}$$

But is it possible for the final asymptotic state to be thermal when the initial state is chosen to be a pure quantum state? Recall the universality above allows all regular[6] initial states including coherent ones.

The key-word above is "asymptotic." Obviously, all the late-time asymptotic observers propagate in region I of figure 2.3, which is only part of the whole space-time, and are unable to see regions III and IV. Because some of the initial data, distributed along $v = v_0$ but inside the future event horizon, are inaccessible, the final "asymptotic" state is expressed in a reduced Hilbert space. Suppose that the initial state is a pure state of the following form with pairwise correlations between internal and external states.

$$|\text{initial}\rangle = \sum a_n\,|n;\text{in}\rangle \otimes |n;\text{out}\rangle \tag{13}$$

We denoted the basis of the internal and external reduced Hilbert spaces by $|n;\text{in}\rangle$ and $|n;\text{out}\rangle$. In other words, after the collapse, the reduced Hilbert space of region

[6]That is, regular at the future event horizon as seen by inertial observers.

I is spanned by $|n;\mathrm{out}\rangle$ only, while the complete Hilbert space is spanned by both $|n;\mathrm{in}\rangle$ and $|n;\mathrm{out}\rangle$. As seen by outside observers, then, the state is described by a density matrix obtained by tracing over the internal basis.

$$\mathrm{Tr}_{\mathrm{in}}\,|\mathrm{initial}\rangle\langle\mathrm{initial}| = \sum |a_n|^2\,|n;\mathrm{out}\rangle\langle n;\mathrm{out}| \qquad (14)$$

This is a completely mixed state without any correlation whatsoever, even though the complete state is a pure quantum state.

To explain the apparent thermal radiation from black holes, we need to establish two facts: First, almost complete pair-wise correlations between internal and external states. Second, $|a_n|^2 \simeq (e^{E_n/T_{BH}} - 1)^{-1}$ where E_n is the energy of the external eigenmodes $|n;\mathrm{out}\rangle$ as measured by an asymptotic observer. Of course, this cannot be true for arbitrary modes. Those modes with wavelength comparable to the length scales of the collapsing body will be sensitive to the details of the history, and cannot possibly produce such a universal result.

Rather, as we have seen above, the universal radiation is found only at late time, near J^+ in figure 2.4. The relevant modes are then easily shown to be those concentrated near future event horizon. Furthermore, as seen by freely infalling observers these modes are of ultrahigh frequencies, owing to the huge relative blueshift factor, encoded in the following relationship between two light-cone coordinates u and U, suitable for asymptotic observers and for infalling inertial observers respectively.

$$U \simeq -\frac{1}{4M}e^{-u/4M}, \quad \text{near the horizon } U = 0. \qquad (15)$$

Therefore, the late-time thermal radiation from a black hole arises through the following identity with $|a_{n'}|^2 \simeq (e^{E_{n'}/T_{BH}} - 1)^{-1}$:

$$|0\rangle \simeq \sum a_{n'}\,|n';\mathrm{in}\rangle \otimes |n';\mathrm{out}\rangle + \cdots, \qquad (16)$$

where $|0\rangle$ is any pure state smooth with respect to the local geodesic coordinate U, and the eigenmodes $|n' : \mathrm{out}\rangle$ are those concentrated near $U = 0$ and of finite positive frequency $E_{n'}$ with respect to the asymptotic coordinate u.

Now we have a Hilbert space description of Hawking radiation, it is time that we asked why. First of all, why are the coefficients a_n's nontrivial? For instance, when $|0\rangle$ is the vacuum of the complete Hilbert space, the naive expectation would be that $|0\rangle = |0;\mathrm{in}\rangle \otimes |0;\mathrm{out}\rangle$ with no radiation rather than the identity (16). The insightful observation of Hawking was that the concept of particles, or equivalently, the decomposition of a quantum field in terms of Fock space generators, depends on the fourier eigenmodes used. In the case at hand, the appropriate eigenmodes for the infalling inertial observers are $e^{\pm iw'U}$, while those appropriate for the asymptotic observers are $e^{\pm iwu}$. Using the transformation formula (15), it is not difficult to see that the corresponding change of basis of the Hilbert space, often called a Bogolubov Transformation, results in reshuffling of creation and annihilation operators. A state $|0\rangle$, devoid of excitations $e^{-iw'U}$ with ultrahigh w', does have excitations of the form e^{-iwu} with finite w, hence a_n's are nontrivial. We will not attempt to rederive this result, but simply state that the resulting distribution $|a_n|^2$ is indeed thermal for late-time observers [13][14].

Secondly, why are there the pair-wise correlations in (16)? A good way to illustrate the reason is to draw each mode $|n';\mathrm{out}\rangle$, $|n';\mathrm{in}\rangle$ and their tensor product. The Bogolubov transformation is induced in the process of expressing $e^{\pm iw'U}$ in terms of internal and external modes such as drawn in (a) and (b) above.

As a useful analogue, consider two possible choices of basis for a Hilbert space of functions on a unit interval $[-1,1]$: trigonometric functions $e^{in\pi x}$ and the Legendre polynomials $P_n(x)$. Suppose we split the unit interval into $[-1,0]$ and $[0,1]$, and create two reduced Hilbert spaces generated by "left" and "right" modes.

$$\begin{aligned} \langle x|P_n^l\rangle &\equiv P_n(x)\,\theta(-x) \\ \langle x|P_n^r\rangle &\equiv P_n(x)\,\theta(x) \end{aligned} \tag{17}$$

But this does not mean that $|P_n^l\rangle$ and $|P_n^r\rangle$ will enter the decomposition of $e^{in\pi x}$ separately. Rather, we know $P_n(x)$'s by themselves form a complete basis of the space, and only the tensor products $|P_n\rangle = |P_n^l\rangle \otimes |P_n^r\rangle$ will appear in the decomposition.

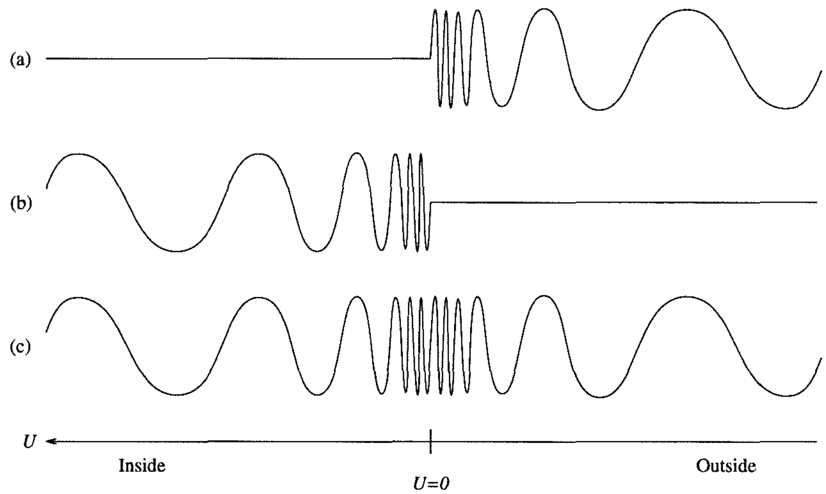


Figure 2.5: Schematic drawings for (a) an external and (b) an internal eigenmodes of same frequency, and (c) their tensor product. The oscillation at $U = 0$ is infinitely dense owing to the infinite blue-shift, though the drawings above show only a finite number of oscillations.

Similarly, as far as modes localized near horizon are concerned, the decomposition of $e^{\pm iw'U}$ in terms of the internal and the external modes should result in a decomposition in terms of their tensor products as drawn in (c) of figure 2.5., thus the pair-wise correlations of (16) [15].

Implications of Hawking radiation are profound. First of all, the one-loop Einstein equation (7) implies that black holes lose their mass continually through this late-time thermal radiation, since the total ADM mass must be conserved. For the case of Schwarzschild black holes, the result is particularly catastrophic since the corresponding Hawking temperature blows up like $\sim \hbar/M$, as the evaporation progresses. For astrophysical black holes, formed from giant stars and accreting matters continually, however, Hawking radiation is far too small to be of any consequence.

More fundamental issues arise from the fact that the asymptotic state of the radiation appears thermal. We have seen how the emergence of a mixed state is explained in terms of the reduced Hilbert space. Superficially, this is similar to how coarse-grained entropy arises from averaging over macroscopic scales, but there is a sharp distinction that the latter is just a convenient tool of trade while the other is forced upon asymptotic observers by the nontrivial causal structure. The resulting puzzle of information loss is genuine in this sense [8]

While we will not delve into the matters of information puzzle or its possible resolutions, it is worthwhile to examine the validity of Hawking's thermal radiation. The derivation above is based on a semiclassical approximation where the metric is treated classically, which is well-justified as long as the lengthscale of the geometry is much larger than Planck scale.

$$M^2 \gg L^2_{\text{Planck}} \equiv \hbar. \tag{18}$$

On the other hand, we are also using an adiabatic approximation where Hawking temperature change very slowly as the black hole evaporates. More precisely, in order to justify the thermal description, emission of a quantum, carrying a typical energy $\sim T_{BH}$, must not change the temperature drastically [16].

$$T_{BH} \gg |\delta T_{BH}| \sim \left|\frac{dT_{BH}}{dM}\right| T_{BH} \tag{19}$$

Of course, one realizes that this is, for Schwarzschild black holes, identical to the previous condition.

For charged black holes, we find somewhat different stories. Generalizing the formula (11) to arbitrary static black hole, we find the following simple prescription of calculating the Hawking temperature from a metric coefficient in the Schwarzschild gauge.

$$g = -A(r)\,dt^2 + \frac{dr^2}{A(r)} + \cdots \qquad \rightarrow \qquad T_{BH} = \left.\frac{A'(r)}{4\pi}\right|_{\text{horizon}} \tag{20}$$

In particular, Hawking temperature of Reissner-Nordström black holes with two horizons at $r = r_\pm$ is given by

$$T_{BH}\Big|_{\text{Reissner-Nordström}} = \frac{r_+ - r_-}{4\pi r_+^2}, \tag{21}$$

which vanishes in the extremal limit $r_+ \to r_-$. In the extremal limit, toward which the system is driven by the radiation, the necessary condition (19) is satisfied if and only if

$$M^2 \gg \frac{L^2_{\rm Planck}}{\sqrt{1 - Q^2/M^2}}. \tag{22}$$

Consequently, in the extremal limit, the thermal description of Hawking radiation need not be valid, even if the black hole remains macroscopic in its size.

In such a situation, we need to resort to the effective action approach we started with. While almost impossible to work with in four dimensions, the method has been successfully applied to a series of two-dimensional models of black holes, for the purpose of studying semiclassical properties, including gravitational backreaction, more systematically [17]. One of the test cases in these efforts involved stringy two-dimensional black holes associated with a Wess-Zumino-Witten coset model whose semiclassical properties are similar to those of four-dimensional charged black holes [18]. In the final section, we want to examine these classical space-times from the viewpoint of string theories, as a way of understanding the relationship between background geometries and fluctuating fundamental strings.

2.3 String Theories and Black Hole Geometries

Perhaps the most attractive feature of string theories [19] as opposed to conventional field theories is that it provides a way of quantizing gravity consistently without uncontrollable ultraviolet behaviour. Since some, if not all, of the puzzles associated with black holes are closely tied with our ignorance of microscopic nature of gravity, it is only natural to explore stringy features of black holes.

The quantization of fundamental strings is not only much more involved than that of point-like fields, but exhibits many unexpected features. For one thing, only after the first quantization, all the higher-loop Feynmann diagrams are naturally built-in, owing to the summation over world-sheet topologies, without further introduction of interaction vertices.

Another surprising aspect is that the consistent quantization of fundamental strings restricts the admissible ambient space-times in which strings propagate. More specifically, the conformal invariance of the classical string theories survives first quantization only for certain classes of space-time geometries.

In the limit where the inverse string tension α' is very small so that the typical size of fundamental string is also very small, this constraint translates to classical equations of motion for each and every background field [20]. An illuminating way of seeing this is to consider the string theory as a two-dimensional sigma-model with the space-time as the target manifold. Keeping the condensates of the bosonic graviton multiplet only, the sigma-model is written below with the worldsheet metric h_{ij},

$$S_\sigma = \frac{1}{4\pi\alpha'}\int d\sigma^2\sqrt{h}\left\{h^{ij}\partial_i x^\alpha\partial_j x^\beta g_{\alpha\beta} + \epsilon^{ij}\partial_i x^\alpha\partial_j x^\beta B_{\alpha\beta} + \alpha'\phi R^{(2)}_{\text{world-sheet}}\right\}. \quad (23)$$

Evidently, all the background fields play the role of couplings which happen to depend on the space-time position x^β with $\beta = 1,\ldots,D$. Since the conformal invariance implies vanishing renormalization group flow, beta-functions for each of these couplings should also vanish on any acceptable background field. A perturbative estimates of the beta-functions in α', then, provides field equations for each coupling.

Organizing these field equations into an action principle, we finally obtain a low-energy effective string field theory. With the Euclidean signature of the space-time, we find the following action for the gravity multiplet, containing the dilaton ϕ and the anti-symmetric 2-tensor B with its field strength 3-form H as well as the metric g.

$$S_{\text{effective}} \sim -\int dx^D\sqrt{g}\,e^{-2\phi}\left\{2\Lambda + R + 4(\nabla\phi)^2 + \frac{1}{12}H^2 + \cdots\right\} \quad (24)$$

The cosmological constant Λ is proportional to $1/\alpha'$ (the string tension) and the remaining terms of higher dimension are suppressed by positive powers of α'. In particular, the Maxwell term F^2 of dimension four appears with one factor of α'.

Now, as an illustration, let us consider the simplest possible case where a black hole geometry shows up as an admissible space-time geometry, namely when the space-time dimension is two. Since the 3-form H is trivial on two-dimensional manifolds, the action reduces to the following.

$$\int dx^2 \sqrt{g^{(2)}}\, e^{-2\phi} \Big\{ R^{(2)} + 4(\nabla\phi)^2 + 4\lambda^2 \Big\} \tag{25}$$

We dropped all the subleading terms and introduced $\Lambda = 2\lambda^2$ as our two-dimensional cosmological constant.

The dynamics are devoid of any local propagating degree of freedom and the solutions are given by a unique one-parameter family [21]. With a Minkowskian signature and a pair of Krustal coordinates (x^+, x^-),

$$g^{(2)} = -e^{2\phi} dx^+ dx^-, \qquad e^{-2\phi} = \frac{m}{\lambda} - \lambda^2 x^+ x^-. \tag{26}$$

The number m is a integration constant and can be interpreted as a "mass" of the solution. It is easy to see that the scalar curvature $R^{(2)}$ is divergent where $e^{-2\phi}$ vanishes. The corresponding singularities at $x^+x^- = m/\lambda^3$ are space-like, provided that the mass m is positive, and the solution represents a black hole with the event horizons at $x^+x^- = 0$.

To see that this is indeed a black hole solution, it is advantageous to introduce a Schwarzschild-like coordinate system (t, x), valid for nonzero m.

$$x^+ x^- = -\frac{m}{\lambda^3} \sinh^2 x, \qquad \frac{x^+}{x^-} = -e^{2\lambda t} \tag{27}$$

The resulting metric describes the geometry outside the horizon at $x = 0$, and is independent of the mass $m > 0$ [18].

$$g^{(2)} = g^{(2)}_{Witten} = \frac{1}{\lambda^2} dx^2 - \tanh^2 x\, dt^2, \qquad e^{-2\phi} = \frac{m}{\lambda} \cosh^2 x \tag{28}$$

It is only the value of the coupling e^{ϕ} at $x = 0$ that determines the energy content of the solution. In the extremal case of the massless solution, however, the geometry is flat with two asymptotic regions at $x^{\pm} = 0$ and at $x^{\pm} = \pm\infty$.

$$x^{\pm} = \pm e^{\pm\lambda\sigma^{\pm}} \quad \rightarrow \quad g^{(2)}\Big|_{m=0} = -d\sigma^+ d\sigma^- \tag{29}$$

Note that Hawking temperature T_{BH} of the black hole remains finite at $\lambda/2\pi$, while the flat extremal geometry of $m = 0$ corresponds to vanishing T_{BH}. The semiclassical behaviour of this black hole resembles roughly that of the Reissner-Nordström black holes, despite the superficial similarity to Schwarzschild black holes in terms of the causal structure.

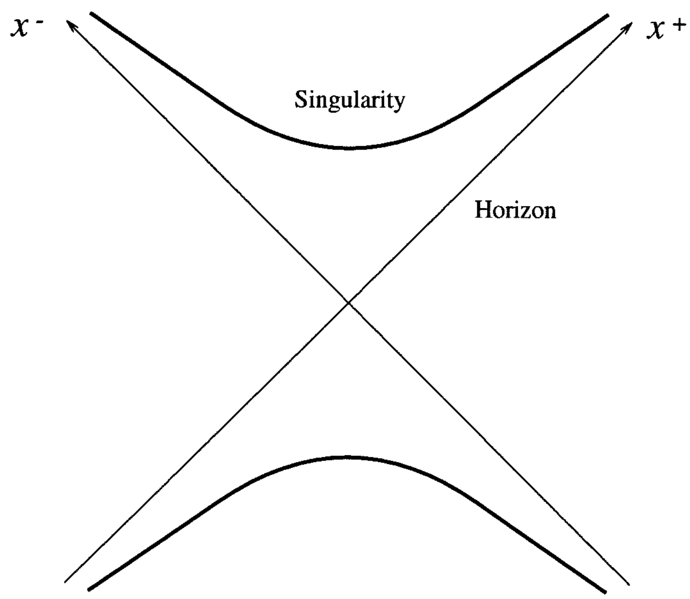


Figure 2.6: Causal structure of the two-dimensional black hole in (x^+, x^-) coordinates. The space-like singularities are hidden behind the event horizons along the axes $x^\pm = 0$.

Now, this geometry is supposed to represent a part of a space-time manifold which supports a consistent string theory, at least for certain values of λ^2. But, since we employed a hypothesis that the string tension $1/\alpha'$ is sufficiently high to justify ignoring large fluctuations of fundamental strings, the configuration is only approximately correct. As the lengthscale of the space-time approaches $\sqrt{\alpha'}$, the approximation eventually breaks down and the solutions such as (26) cannot be trusted near the singularities.

Finding configurations nonperturbative in α' is obviously impossible if we approach the problem from the field theory viewpoint as above, for we have no way of summing up all the higher corrections into a manageable action principle. But coming back to the string theories themselves, we realize that it is the conformal invariance that generated all the field equations in the small α' limit and that the first thing we need to do is to find a world-sheet conformal field theory whose background configuration reduces to the known approximate one in the low-energy limit.

In some isolated cases with a larger worldsheet symmetry algebra, conformal field theories on nontrivial background manifolds are known. One example would be Wess-Zumino-Witten models [22] with Kac-Moody algebras [23], which are essentially world-sheet sigma-models onto group manifolds. More interesting cases are obtained by gauging a subgroup and thereby effectively reducing the space-time dimension. Such models are known as Wess-Zumino-Witten coset models [24] whose lagrangian formulation has been known for some time [25][26][27].

As it turned out, the black hole configuration (28) corresponds to one of these coset models with the group manifold $SL(2,R)$ gauged by a $U(1)$ factor, as first discovered by Witten [18]. The nonperturbative version of the metric has been discovered by two independent groups [28][29] and the correction takes the following form,

$$g^{(2)}_{Witten} \quad \Rightarrow \quad g^{(2)}_{exact} = \frac{1}{\lambda^2}\, dx^2 - \frac{(k-2)\tanh^2 x}{k - 2\tanh^2 x}\, dt^2, \tag{30}$$

where $k = (2\lambda^2\alpha')^{-1} + 2 > 2$ is a central charge of $SL(2,R)$ affine Kac-Moody algebra used to construct the conformal field theory. In the small α' limit, note that this nonperturbative metric does reduce to $g^{(2)}_{Witten}$.

Again the geometry $g^{(2)}_{exact}$ possesses the event horizon at $x = 0$, for the metric coefficient of dt^2 vanishes there, but the behaviour inside the horizon is quite different. The proper analytic continuation of x inside the horizon is to follow the imaginary axis $x = iy$ causing $\tanh^2 x = -\tan^2 y$ to blow up at $y = \pi/2$, thus the singularity of $g^{(2)}_{Witten}$. With the nonperturbative correction, though, this divergence is canceled between the denominator and the numerator, hinting that the nonperturbative version is *nonsingular*[7] despite event horizons. The causal structure of this *exact* and *nonsingular* classical string background is presented in chapter 5.[8]

Here, let us remind ourselves that the discussion so far has been more or less concentrated on the condensates of massless string excitations as seen by the string center of mass. This means that the background fields are interpreted as seen by point-like observers and that the extended nature of strings is not fully taken into account. But near singularities, the sizes of the strings are never ignorable compared to the length scale of the space-time.

For instance, consider an orbifold model, the background manifold of which possesses conical singularities. While the background geometry is singular as seen by point-like observers, the conformal theory itself shows no sign of distress near the singularities. All that appear are extra winding modes associated with each singular point citeString. Though black hole singularities are admittedly a lot worse than conical ones, this simple example does illustrate the point that the conventional notion of geometry, such as seen by the string center of mass, may not be suitable in studying the singularity structure of black holes.

In conclusion, we observed how fluctuating strings shape the background geometries and are capable of producing nonsingular black holes. But, it is also imperative to understand better the interplay between fundamental strings and background geometries, in order to unravel the nature of black holes in the context of string theories.

[7]String theories are non-local in that the higher order corrections to the effective action, in terms of α', possesses arbitrary numbers of derivatives, and the singularity theorems [31] of Hawking and Penrose are not valid beyond the leading approximation. In this sense, there is no compelling reason to expect singularities inside stringy black holes.

[8]The nonsingular causal structure of g_{exact} is obtained by P.Y. and also independently by Perry and Teo[30].

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3 Quantum Radiation from Extremal Black Holes

In connection with the physics of black hole evaporation, extremal black holes with vanishing temperature provide interesting theoretical laboratories. Immune to Hawking's thermal radiation [1], they are the first clues as to what the final stage of the evaporation process might be. But this does not mean we can consider the classical extremal black holes as the final product of the process. For one thing, the thermal behavior is already expected to break down for near extremal cases [2]. Zero Hawking temperature simply means the leading quantum effect disappears. To obtain the next-to-leading quantum effect, one needs to find the celebrated Bogolubov transformation more accurately. On the other hand, it is possible to bypass this difficult problem by simplifying the matter sector to that of S-wave fermions, or Callan-Rubakov modes. By integrating out the chargeless combinations of them which behave as 2-D conformal scalars, we obtain a manageable one-loop effective action for the metric field. In this chapter, we will consider the effect of quantizing these S-wave fermion in the background of extremally charged black holes of Einstein-Maxwell theory.

3.1 Introduction to the Polyakov-Liouville Effective Action

While the physics behind the Hawking radiation is relatively well-understood, it is often the case that more systematic treatments of the phenomenon in the framework of perturbative quantum field theories are not available. For instance, the energy-momentum expectation values at one-loop level are reliably known only for late-time behaviour when the radiation from the black hole is thermal.

Keeping in mind that the Hawking radiation is at most a one-loop effect, one might be puzzled by this. After all, we are not talking about quantization of gravity at all, but rather quantization of matter fields in fixed background geometries. At least conceptually, the mathematics involved should not be very different from, say, those of quantizing charged matter field in background electromagnetic fields.

In fact, such one-loop effects can be studied, in principle, by carrying out the path integral of appropriate quadratic matter actions in arbitrary background fields, typically with zeta-function regularization procedure. If the matter field is a bosonic field ψ with the corresponding background-dependent kinetic operator Q, we find the induced effective action W of the background fields.

$$e^{-W} = \int [d\psi]\, e^{-\psi Q \psi} = \mathrm{Det}^{-1/2} Q. \tag{31}$$

For example, to extract the renormalization contribution of ψ to the electromagnetic coupling, all we need to do is to isolate an operator of the form F^2 in W, where F is the electromagnetic field strength.

What sets the Hawking radiation apart is that the phenomenon is essentially of non-local nature. Unless we keep track of the right vacuum state, we will end up with an ambiguous answer. The practical upshot from this is that we need to maintain explicitly non-local form of the effective action W. This is in sharp contrast with the previous example where we necessarily perform an expansion of W in terms of the naive scaling dimensions, which is also known as the Schwinger-DeWitt expansion.

Therefore, at least some of our technical problems in dealing with the Hawking radiation stems from the inability to calculate W maintaining its non-local form. While there is a known method of resumming the momentum part of the Schwinger-DeWitt expansion for certain elliptic Q with spacetime dimension larger than two [3], the resulting curvature expansion of W seems too complicated to be useful. In particular, in four dimensions, we need to analyze the logarithms of the Green's function of Q with the correct boundary condition in an arbitrary black hole background.

On the other hand, there is an exceptional case where W is not only exactly known, but also very much manageable. In two spacetime dimensions, any metric is conformally flat and any functional of the metric is determined by its dependence on the conformal factor only, up to possible topological terms. Therefore, W induced by integrating out a conformal scalar fields in two-dimension is completely determined by the conformal anomaly, a well-known local quantity. The resulting effective action $W = S_{PL}$ is called Polyakov-Liouville action and can be derived easily as follows [4].

Consider the case of a conformal scalar in a Euclidean two-dimensional space.

$$W = \log \mathrm{Det}^{1/2} Q = \frac{1}{2} \mathrm{Tr} \log Q, \qquad Q = -\frac{1}{\sqrt{g}}\, \partial_i \sqrt{g} g^{ij} \partial_j. \tag{32}$$

Using the zeta-function regularization scheme with an ultraviolet cut-off ϵ, and ignoring possible zero-modes,

$$W = -\frac{1}{2} \int_\epsilon^\infty \frac{dt}{t}\, \mathrm{Tr}\, e^{-tQ}. \tag{33}$$

Now writing the metric g in terms of the conformal mode ρ and a fixed flat metric $\hat{g}$, $g = e^{2\rho} \hat{g}$, we can take the variation of W with respect to ρ,

$$\delta W = -\int_\epsilon^\infty dt\, \mathrm{Tr}\Big\{ \delta\rho\, Q e^{-tQ} \Big\} = \int_\epsilon^\infty dt\, \frac{\partial}{\partial t} \mathrm{Tr}\Big\{ \delta\rho\, e^{-tQ} \Big\}. \tag{34}$$

By an integration by part, δW is expressed in terms of the heat kernel at the coincidence limit.

$$\delta W = -\int dx^2 \sqrt{g}\, \delta\rho \left\{ e^{-\epsilon Q} \right\}_{\text{coincidence}} = -\int dx^2 \sqrt{g}\, \delta\rho \left\{ \frac{1}{4\pi\epsilon} + \frac{1}{24\pi} R + \cdots \right\} \tag{35}$$

The first term merely renormalizes a cosmological constant, while the terms denoted by the ellipsis vanish as we remove the ultraviolet cut-off $\epsilon \to 0$. Integrating the only remaining term over $\delta\rho$, we finally arrive at the following effective action

$$W = S_{PL} \equiv \frac{1}{96\pi} \int dx^2 \sqrt{g^{(2)}}\, R^{(2)} \frac{1}{\nabla^2} R^{(2)}, \tag{36}$$

where we put the superscript (2) to emphasize that we are in two-dimensional world. In general, integrating out conformal matters with total conformal anomaly N produces the effective action $W = N S_{PL}$.

Once we know the explicit non-local form of W in terms of $g^{(2)}$, it is now a matter of algebra to derive the one-loop expectation value of the energy-momentum.

$$\langle T_{ij} \rangle = 2\hbar\, \frac{\delta W}{\delta g^{ij}} \tag{37}$$

If we were in two-dimensional world, we would be able to study the semi-classical properties of black holes complete with Hawking radiation and the *gravitational backreaction* thereof, by adding this expression to the right hand side of Einstein equations.

To find an application to real four-dimensional black holes, we need to find a case where the dimensional reduction makes sense physically. A good starting point would be spherically symmetric black holes. But the spherical symmetry of the background geometry by no means restricts the quantum fluctuations to be in S-wave sector, and, even if we managed to reduce the problem to S-wave sector, in general, the resulting two-dimensional modes are not conformal fields.

On the other hand, there are so-called Callan-Rubakov modes [5] around magnetically charged black hole, which appear to propagate freely along the radial direction. After a careful consideration of the electromagnetic backreaction [6], it turned out that the chargeless combinations of these lowest partial waves of massless fermions do behave as genuine conformal matter fields in two dimensions.

For most quantum fluctuations around a black hole geometry and even for most S-wave modes, there exists a potential barrier of some sort. In particular, some of the outgoing quanta from the event horizon is reflected by such barriers back to the black hole. The effect is especially very pronounced for low-energy quanta. This means that, when we are interested in physics dictated by the fluctuations of sufficient low energy fluctuation, we may be able to ignore most of these excitations.

Therefore, one hopes that there might be a situation where he can ignore everything but the effective two-dimensional conformal fields in the form of chargeless Callan-Rubakov modes. A now classic example of this is the case of an extremally charged dilatonic black hole, also known as the cornucopion. The generic potential barrier in this case induces a mass gap, and both the geometry near the horizon (which is asymptotically far away) and the low energy fluctuation there are dictated by an effective two-dimensional action [7][8].

A less dramatic case where this reduction to two dimensions might be useful is the extremal limit of magnetic Reissner-Nordström black hole, Hawking temperature of which is vanishingly small. The relevant energy scale decreases with decreasing Hawking temperature, and typical quanta of the corresponding low energy will find the potential barrier insurmountable. Then one may expect the leading radiation from the event horizon to consist of the chargeless Callan-Rubakov modes.

Of course, it is not crystal clear whether the extremal limit is actually dictated by low energy excitations only, for the thermal description based on the vanishing Hawking temperature is known to break down near the extremal limit [2]. However, we find the resulting two-dimensional system sufficiently interesting to study in-depth without further justification.

In the following sections, we want to study the dimensionally reduced system of Einstein-Maxwell theory coupled to N conformal scalars at one-loop level but beyond the usual late-time approximation by Hawking.

3.2 Quantum Radiation from a Zero-Temperature Black Hole

Here, we want to concentrate on the case of the extremal Reissner-Nordström black hole and to study how semiclassical effects modify one of the classical properties, namely ADM mass M. The model we consider is dimensionally reduced Einstein-Maxwell theory. By restricting to the spherically symmetric sector we obtain the following 2-D action,[7]

$$S_g = \frac{1}{4}\int d^2x\,\sqrt{-g^{(2)}}\,e^{-2\phi}\left\{R^{(2)} + 2\,(\nabla\phi)^2 + 2e^{2\phi} - F^2\right\}, \tag{38}$$

where the 4-D metric is split into 2-D metric $g^{(2)}$ and the dilaton part

$$g^{(4)} = g^{(2)} + e^{-2\phi}\,d\Omega^2. \tag{39}$$

The finite mass solutions with regular horizons are the well-known Reissner-Nordström solutions with mass M and charge Q satisfying the inequality $M \geq |Q|$.

[7] $G = c = 1$ in this chapter

$$g^{(4)} = -F(r)\,dt^2 + \frac{dr^2}{F(r)} + r^2\,d\Omega^2, \quad F(r) = 1 - \frac{2M}{r} + \frac{Q^2}{r^2}. \tag{40}$$

When the inequality is saturated, $F(r)$ has a double zero at the horizon $r = M = |Q|$ and the corresponding extremal black hole has zero Hawking temperature, hence no thermal radiation emanates from the horizon a long time after the black hole formation. This implies that the usual late time estimate of the Bogolubov transformation[1] is not the leading quantum correction. It vanishes identically and we need to study the next nonvanishing contribution, which may or may not depend on the history of the collapse.

For this purpose, we can follow CGHS and couple N conformal scalars to the above 2-D action. One can regard these 2-D scalars as chargeless combinations of Callan-Rubakov modes, which are discussed in complete detail in a later chapter. As shown in the previous section, one can integrate out these conformal matter completely to produce the non-local Polyakov-Liouville action[4] with a particular coefficient, which summarizes the effect of the quantized matter on gravity.[8]

$$-\hbar S_{PL} = -\frac{N\hbar}{96\pi}\int dx^2\sqrt{-g^{(2)}}\,R^{(2)}\frac{1}{\nabla^2}R^{(2)} \tag{41}$$

Furthermore, this semi-classical effective action can be conveniently handled with the introduction of a auxiliary scalar field z in the following manner[9],

$$S = S_g - \frac{N\hbar}{24\pi}\int d^2x\,\sqrt{-g^{(2)}}\,((\nabla z)^2 - zR^{(2)}). \tag{42}$$

Since the field equation for z reduces the second term to the original Polyakov-Liouville action of central charge N, solving this theory at tree level is equivalent to studying the semi-classical theory of 2-D gravity coupled to N scalars. These are the leading terms in the large N expansion of the full 2-D quantum theory, when $N \to \infty$ but $N\hbar$ is kept finite.

Notice that the scalar curvature $R^{(2)}$ acts as an external source coupled to the z field. This effectively induces the usual Hawking radiation in a classical black

[8]The extra minus sign, compared to the formula (35), is because the signature of the metric is now $(-1, 1)$.

hole geometry, i.e., a stationary point of S_g only. Because the classical black holes radiate, the theory does not have any static solution with finite mass and regular horizon of finite temperature. The only static solutions of finite mass are those of zero temperature, which were first studied by S. Trivedi[10].

But first let us consider the effect of quantizing the original N conformal scalars in a classical background. For example, given a classical geometry, the expectation value of the matter energy momentum tensor can be found simply by evaluating the classical energy momentum tensor of z-field on that classical background. A family of classical geometries known as the Vaidya metric[11] is particularly relevant to our discussion.

$$g^{(4)} = -(1 - \frac{2m(v)}{r} + \frac{e^2(v)}{r^2})\, dv^2 + 2dv\, dr + r^2\, d\Omega^2 \tag{43}$$

It represents a collapsing massless shell whose cumulative energy and charge at retarded time v are $m(v)$ and $e(v)$. For smooth m and e^2, the cosmic censorship is achieved by requiring the positive energy condition for the shell[12].[9]

For our purposes, however, it is appropriate to choose

$$m(v) = M\theta(v - v_0), \quad e^2(v) = Q^2\theta(v - v_0), \tag{44}$$

where θ is the usual step function. The geometry is then that of an initial Minkowski spacetime glued to a Reissner-Nordström black hole across an ingoing null shock wave located at $v = v_0$. In other words, this represents an idealized collapsing solution where a Reissner-Nordström black hole is created by a collapse of thin shell of massless charged matter. Introducing a new coordinate $u = v - 2\int F^{-1}(r)\, dr$ with $F(r)$ as in (39), (v, u) form a pair of light-cone coordinates above the shock,

$$g^{(2)} = -F(r)\, dv\, du, \quad v > v_0. \tag{45}$$

In these coordinates, $v \to \infty$ is the future null infinity, and $u \to \infty$ is the future event horizon.

[9] Of course, we have introduced an external charged matter source to create the shell itself.

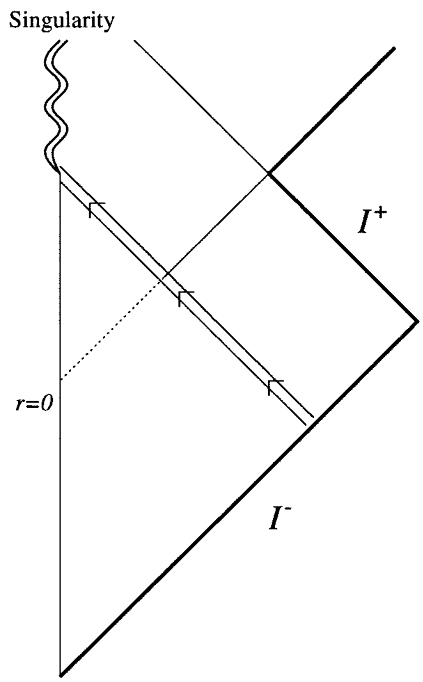

Figure 3.1: The Penrose diagram for the Vaidya solution with $M = |Q|$. The geometry outside the collapsing null shell is that of the extreme Reissner-Nordström black hole, while the inside is given by the Minkowski space-time.

This family of solutions represents idealized form of gravitational collapses leading to spherically symmetric Reissner-Nordström black holes. Once we have an explicit classical solution of gravitational collapse, the Unruh type boundary condition appears quite naturally. Namely, we expect the region inside the collapsing shell to behave regularly and the vacuum state before the collapse corresponds to having vacuum expectation value of the energy-momentum trivial. Therefore, we impose an initial condition on the N matter fields such that the expectation value of the energy-momentum tensor vanishes in the Minkowskian region. Using energy-momentum tensor conservation, this can be translated into

$$< T_{uu} >|_{v=v_0} \quad = \quad (\frac{N\hbar}{12\pi}(\partial_u^2\rho - (\partial_u\rho)^2) + t_{uu}(u))|_{v=v_0} = 0 \qquad (46)$$

$$\rho = (\frac{1}{2}\log F),$$

where t_{uu} comes from the homogeneous part of the solution to z field equation. This implies the following form of $< T_{uu} >$ as we approach the future null infinity.

$$< T_{uu} >|_{v\to\infty} = t_{uu}(u) = \frac{N\hbar}{12\pi}(\frac{1}{16}F'(r)^2 - \frac{1}{8}F(r)F''(r))|_{u=v_0-2\int F^{-1}dr}. \tag{47}$$

As $u \to \infty$, this clearly shows a steady flux proportional to the temperature squared ($F'(r \to \text{horizon}) \sim T_{Hawking}$). Also as expected, this asymptotically steady flux is absent, if M is equal to $|Q|$ so that $F'(r \to \text{horizon}) = 0$. However, there is a finite integrated flux; the total energy radiated is

$$-\Delta M = \int_{-\infty}^{\infty} < T_{uu} >|_{v\to\infty} du = \frac{N\hbar}{96\pi}\int_{|Q|}^{\infty}\frac{F'F'}{F}dr = \frac{k}{6}|Q|, \tag{48}$$

where we used $F'(r = |Q|) = 0$ for the $M = |Q|$ case. Since we ignored the gravitational backreaction, this estimate is valid only for small $k \equiv N\hbar/(12\pi Q^2)$, or equivalently for large black holes. Notice that $-\Delta M$ is positive for any $F \geq 0$ with a double zero at the event horizon. In short, we have found one-loop quantum radiation from the zero-temperature extremal Reissner-Nordström black holes in the form of S-wave fermions.

$-\Delta M$ represents the energy radiated away by the quantized matter, and after properly taking into account the gravitational backreaction, the Bondi Mass of the system should approach as $u \to \infty$

$$|Q|(1 - \frac{k}{6} + O(k^2)). \tag{49}$$

One might assert that it is not clear whether the estimated loss depends on the particular history of the collapse chosen. After all, the metric chosen can never be realized, since one cannot assimilate the collapsing process by a smooth version of the shock wave. As shown in [12], for smooth $m(v)$ and $e^2(v)$ satisfying the positive energy condition, the extremality can never be achieved in finite time. It might be that a realistic collapse scenario produces different $-\Delta M$. We will show that the above estimate of energy loss is robust by finding numerically the ADM mass of the semi-classical analogue of the extremal black hole which must be the end stage of the process described so far.

3.3 Masses of Semiclassical Extremal Black Holes

Semi-classical static solutions of (42) with extremal horizons have been studied near the horizon[10]. The requirement of zero-temperature horizon specifies a unique initial condition at the horizon for a given total charge, and the resulting static solution is known to be asymptotically flat. It is worth pointing out that, to have a static semi-classical solution with regular event horizon, it is necessary to immerse the black in a heat-bath, for the semiclassical effective action (42) simulates Hawking's thermal radiation automatically. As a result, the only case one can hope for a finite ADM mass is when the semiclassical horizon is of zero temperature.

There is no known analytical form of the extremal solution, but it is, in principle, possible to carry out numerical integration. However, before going into details of the simulations performed, it is helpful to discuss other static solutions of finite mass, all of which have naked singularities.

Those with smaller masses, to be called supercritical, are qualitatively similar to the classical ones with $M < |Q|$. The radius $e^{-\phi}$ monotonically decreases as we approach the naked singularity at near origin. On the other hand, solutions with larger masses, to be called subcritical, are quite different from classical analogues $M > |Q|$, which have curvature singularity at the center $e^{-\phi} = 0$ hidden by two layers of nonextremal horizons, since we assume no heat bath to support nonextremal horizons. (Heat bath makes ADM mass infinite.) More specifically, a semi-classical subcritical solution has a lower bound on the value of the radius $e^{-\phi}$ near would-be horizon. One can distinguish the two species by observing whether the simulation stops in the middle or continues all the way to the critical value of the radius $e^{-\phi_{cr}} \equiv \sqrt{kQ^2}$.

Coming back to the actual simulation, it turns out that static field equations can be decoupled to produce a single first-order differential equation with the following gauge choice.

$$g^{(2)} = -A^2\, dt^2 + B^2 Q^2\, dr^2, \quad e^{-2\phi} = Q^2 r^2. \tag{50}$$

In this gauge we can extract two independent first-order differential equations.

$$\begin{aligned} k(\frac{A'}{A})^2 + 2r(\frac{A'}{A}) + (1 - B^2 + \frac{B^2}{r^2}) &= 0 \\ (r^2 - k)(\frac{A'}{A} - \frac{B'}{B}) + (r^2 + k)\frac{B^2}{r^3} - r(B^2 - 1) &= 0. \end{aligned} \tag{51}$$

Solving for A'/A in terms of B produces a first-order differential equation for $1/B^2$.

We performed two independent simulations. First, we started from the asymptotic region with the initial condition determined by $M/|Q|$, and searched for the range of $M/|Q|$ producing an extremal horizon (a double zero of $1/B^2$). This involves a lot of trial and error. As mentioned above, the form of solution, in particular the metric coefficient B^2, exhibits qualitatively different behaviours for subcritical and supercritical cases. By narrowing the range of $M/|Q|$ where the transition from the subcritical to the supercritical occurs, one can easily find the approximate value of the mass of the semiclassical extremal black holes.

Secondly, we integrate outward from the known behavior near the extremal horizon and extract the ADM mass by fitting the curve in the asymptotic region. This second method turned out to be much more accurate and reliable.

Since the initial points are near, but not quite at the horizon or $r = \infty$, we needed to calculate accurate initial conditions. Symbolic expansions of $1/B^2$ in appropriate coordinates, solving the equation above approximately, are used for this purpose. Fortunately, the nonanalytic behavior of the metric near the horizon emphasized in [10] does not occur for $1/B^2$ as a function of r. We used MATHEMATICA for all numerical and symbolic calculations as well as preparation of the plot. As we improved the accuracy of the numerical calculation by supplying more accurate initial data, and also by increasing the intrinsic accuracy of the program used, the results from each simulation converge to each other. The data for $M/|Q|$ obtained by the two methods coincide to an accuracy of 10^{-6}.

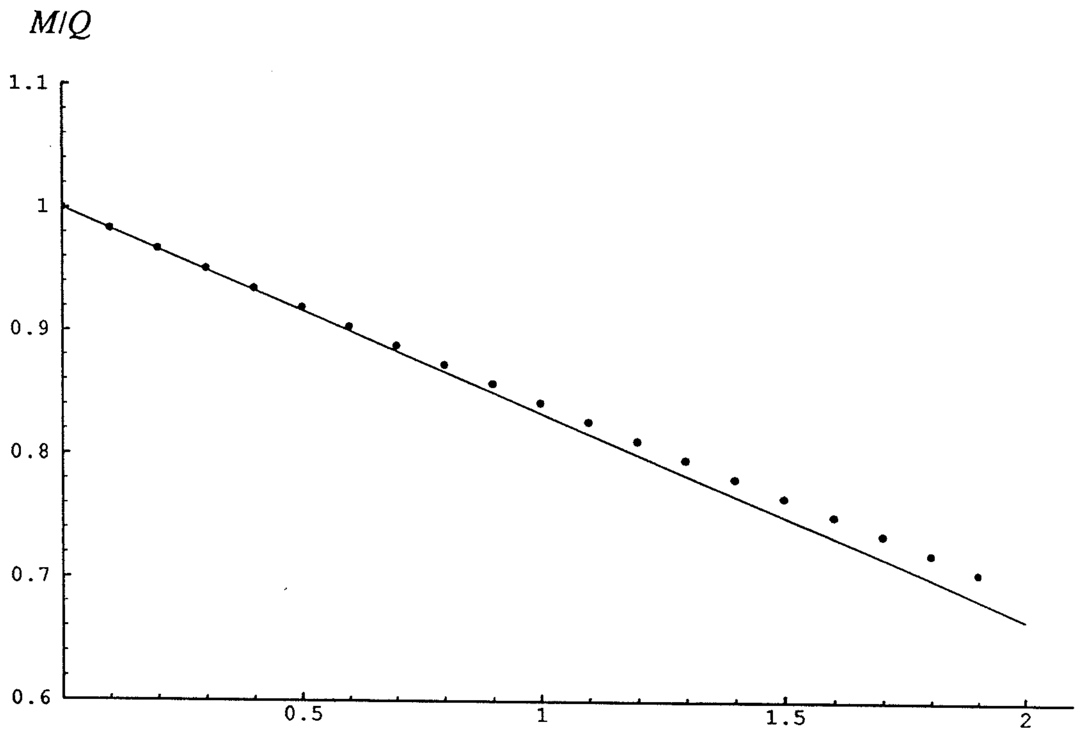

Figure 3.2: Plot of $M/|Q|$ versus $k \equiv N\hbar/(12\pi Q^2)$. The straight line shows the leading behaviour $M/|Q| = 1 - k/6$. The dots are the actual numerical results from the two independent simulations. Data points are at $k = n/10$ for $n = 1, ..., 19$ as well as $k = 0.001$.

The simulation is carried out only for $k < 2$ because the extremal horizon disappears beyond $k = 2$, when the horizon radius is equal to the critical value of the dilaton $e^{-\phi_{cr}} = \sqrt{kQ^2}$. The plot of $M/|Q|$ as a function of $k \equiv N\hbar/(12\pi Q^2)$ (Figure 3.1) clearly shows the initial slope of $-1/6$ as predicted in the previous section. Furthermore, up to $k = 2$, the ratio continues to drop as we increase $N\hbar$ or decrease the charge $|Q|$.

3.4 Discussion

What can we learn from this little demonstration? The first and foremost fact is that higher order corrections to Hawking's calculation must be taken account into even for such a crude operation as mass measurement. One should expect that a similar mechanism works for four-dimensional black holes and the classical bound $M \geq |Q|$ is modified, unless some unbroken extended supersymmetry protects it. But the model we used gives few clues as to what the modification might be. While 2-D conformal scalars can be interpreted as the S-wave modes of 4-D massless fermions, we cannot regard our model as a quantitative approximation to the full 4-D physics. There is no generic mass gap present to separate S-wave fermions out from the rest.

Nevertheless this doesn't prevent us from speculating on the effect of such a modified mass-charge relation in 4-D. In particular, suppose the same monotonic decreasing behavior is realized for the four-dimensionally extremal black holes. The possibility has been contemplated by J. Preskill with emphasis on charge renormalization[14]. The most immediate consequence would be to lift the well-known degeneracy for multi-extremal black hole configurations. Classically, a family of solutions known as the Papapetrou-Majumdar space-time[13], describes many extremal black holes at rest relative to one another. The total ADM mass of such a solution is the sum of the individual masses,

$$M = \sum_i |Q_i|. \tag{52}$$

This can easily be seen by imagining each hole separated from one another far away, so that whatever potential energy there might be becomes negligible. In fact, there is no potential between individual black holes, and the total mass is given by (51) for any finite separations. Therefore two different multi-black hole configurations in equilibrium have the same energy provided that the sum of absolute value of the charges are equal. But, with the modified $M/|Q|$ which decreases as Q^2 decreases, the same reasoning shows that it is energetically favorable to split one big black hole into many smaller ones. The classical degeneracy is lifted.

Classical physics forbids such a bifurcation process, since it violates the second law of black hole thermodynamics. However, there has been suggestions of possible finite action instantons mediating bifurcation of the extremal Reissner-Nordström black holes. In fact, D. Brill found an instanton of finite action interpolating between two Bertotti-Robinson metrics with different numbers of necks[15]. It is well known that a Bertotti-Robinson metric with a single neck approximates an extremal Reissner-Nordström black hole near the horizon. If the initial and the final states are of the same energy, the instanton will take infinite Euclidean time to make the transition, and the stationary state would be a linear combination of the two classical configurations. With the modified $M/|Q|$ relation however, a relevant Euclidean solution is a bounce solution and a big extremal Reissner-Nordström black would decay to many extremal black holes of smaller charges distantly separated.[10]

So far, we have completely ignored the possible presence of charged matter fields. Suppose there is an elementary charged particle of mass m and charge e and consider an extremal black hole of mass M and charge Q. For $m << |e|$, the Schwinger pair production near the horizon is always dominant over a possible bifurcation process and the black hole charge will eventually be wiped out. But for sufficiently large $m > m_{min}$, it will be kinematically impossible for an extremal black hole to lose its charge by emitting these charged particles[14]. For a large black hole $M \gg m$ in particular, we have $m_{min}/|e| \simeq M/|Q|$. Therefore the model we considered should be regarded as a possible scenario for magnetically charged extremal black holes in a world where the magnetic monopole comes with mass comparable to, or even larger than, its charge in Planck units.

In summary we have found that the classical inequality $M \geq |Q|$ can be modified through semi-classical effects. In the S-wave approximation, the corresponding

[10] A similar observation has been made in the context of classical dilatonic black holes with massive dilaton field[16].

modification is found to be such that bifurcations of a large extremally charged black hole to smaller ones, if possible, are energetically favorable.

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4 Fermionic θ Vacua and Long-Necked Remnants

In this chapter, we study another application of Callan-Rubakov modes to magnetic black holes. In the previous chapters, the chargeless combinations of Callan-Rubakov modes were used to study S-wave sector matter coupled to spherically symmetric black holes to one-loop level, for they can be integrated out completely, leaving a well-known effective action for the metric. On the other hand, there are other interesting physics associated with Callan-Rubakov modes on any magnetically charged object. One such example is the vacuum polarization effect of these S-wave fermions around magnetic monopoles when there exists a CP nonconserving vacuum angle θ. Studies of this vacuum polarization for various monopoles in flat spacetime has been carried out in detail, and one can expect that the qualitative features of these studies will carry over to some magnetically charged black holes which also have *compact* cores. However, in addition to the usual complications due to event horizons, there are cases where the geometry near the event horizon is literally noncompact, and naive expectations based on compact core structures breakdown. Here, we want to study *charged* Callan-Rubakov mode around such black holes and the resulting vacuum polarization effects on the black hole geometry itself.

4.1 Motivation

It is well-known that a magnetic monopole carries fractional electric charge [1][2] in the presence of a CP nonconserving angle θ [3]. For a spontaneously broken Yang-Mills theory (with monopoles as solitons), this angle parametrizes a continuum of vacua, known as θ vacua, the effect of which is naturally incorporated by including the Pontryagin density multiplied by θ in the Lagrangian. With such a setup, the origin of the fractional charge becomes quite clear. The Pontryagin density is essentially a product of electric and magnetic fields, through which a classical magnetic field acts as a source to the fluctuating electric field. As a result a monopole carries a long range electric field proportional to its magnetic field, and the corresponding electric charge must be proportional to θ. Allowing higher excitations, we arrive at the following Witten's quantization rule for unit magnetic monopoles.

$$q = N - \frac{\theta}{2\pi}, \qquad N \text{ is any integer.} \tag{53}$$

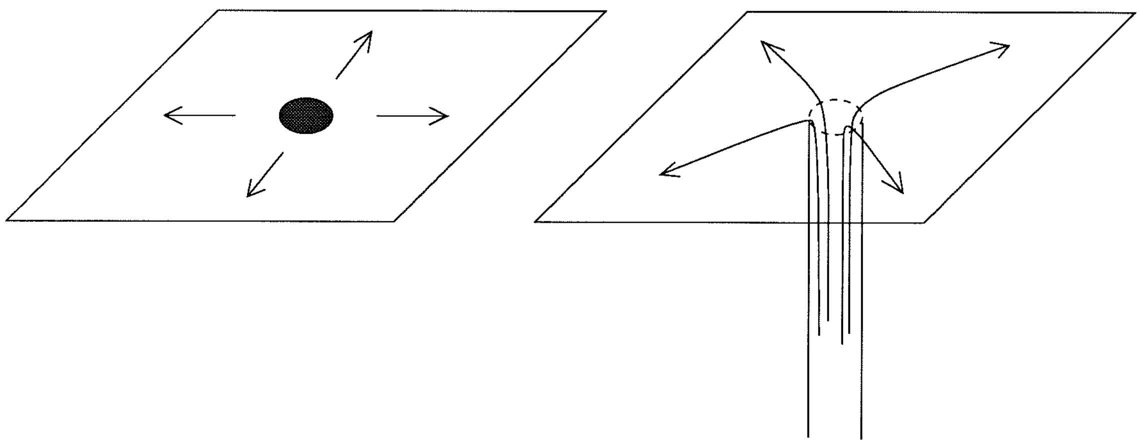
As usual, the quantization rule tells us nothing about how the electric charge should be realized in such dyons, which must depend on many details of the theory. One example where we can address this question of dyon core structure is a monopole coupled to charged fermions [4][5][6]. The lowest partial waves of such fermions, known to experience no potential barrier, can be used to study the static dyonic core structure.

As demonstrated by Callan [4], the vacuum fluctuation of the charged fermion field tries to shield the core from the radial electric field and, as a result, the fractional electric charge is effectively realized as a vacuum polarization cloud of size $\sim 1/m_\psi$ around the monopole, where m_ψ is the mass of the fermion. In particular, for small fermion mass, the electric charge distribution is concentrated on a thick and large shell of radius $\sim 1/m_\psi$ and the small magnetic core, being shielded from the extra radial electric field, is found to be essentially intact.

Witten's charge quantization is a topological statement, and the same fractional charge must appear also for magnetic black holes. On the other hand, when the magnetic black hole is much smaller than the fermion lengthscale $\sim 1/m_\psi$, we may also expect to find similar dilute clouds of vacuum polarization shielding the black hole from the extra radial electric field so that its geometry near the horizon is that of the pure magnetic black hole.

As far as light fermions are concerned, the leading effect in such a background is from the long-range magnetic field, and once the "core" region is shielded by the resulting dilute cloud, one may argue, there is little to which gravity reacts. While the horizon can affect the dynamics of the fermion fields nearby, this seems to matter only when the fermion fields are massive enough for the charged cloud to approach the event horizon.

However, there is a curious species of extremal magnetic black holes, known as cornucopia, whose "size" is both small and infinite simultaneously. Instead of a compact core, the cornucopion has an infinitely long neck with the transverse radius proportional to the total magnetic flux, as illustrated in Figure 4.1. It takes literally infinite proper time just to reach the event horizon at the bottom of the neck, let alone to cross it, while the transverse size of the neck can be arbitrarily small.



(a) (b)

Figure 4.1: Schematic diagrams for (a) a magnetic monopole in a flat space-time, and for (b) a cornucopion in an asymptotically flat space-time. Magnetic flux emanating from the central region is denoted by the arrows. A cornucopion, which is an extremally charged dilatonic black hole, has an infinitely long neck of fixed transverse radius threaded by the magnetic flux.

Then we may ask how the vacuum polarization behaves in such an exotic background. Should one expect to find the narrow and *infinite* neck surrounded by a large and dilute charged cloud of vacuum fluctuation, provided that the fermion mass is small enough? After all, far away from the black hole, there is little indication as to the existence of the infinite neck. In this chapter, we want to address this specific question, by studying a bosonized effective action for the S-wave charged fermions coupled to the dilaton gravity.

In section 2, we briefly discuss the dilatonic magnetic black holes as solutions to a dilatonic gravity in four dimensions, and also the cornucopion as their extremal limit. After the derivation of the effective action for a general spherically symmetric background in section 3, we shall return to the specific case of the cornucopion. The surprising result of section 4 is that the energy cost of the vacuum polarization, which should be balanced against the gain in the electrostatic energy, is actually divergent in such a noncompact background. The inevitable conclusion thereof is that the gravitational backreaction to this vacuum polarization process is never negligible and, for whatever $m_\psi \neq 0$ is, must have the characteristic long neck terminated, by creating an extremal horizon at finite physical distance.

In section 5, we study the effective action of S-wave fermions combined with the dilaton gravity in four dimension to investigate the self-consistent geometries with finite ADM masses. We find a useful and practical way of studying the solutions near the extremal horizon formed by the gravitational backreaction, and, using this method, we clarify the different roles the fractional electric charge plays in large and small fermion mass limit. In particular, we find the expected behaviour in the large fermion mass limit, where most of the fractional charge must be trapped by the black hole's gravitational pull. We conclude by discussing the generality of these results.

4.2 The Cornucopion

The magnetic black hole we are interested in is a solution to the following classical field theory of the metric, a $U(1)$ gauge field and a scalar degree of freedom ϕ called dilaton.[11]

$$-\frac{1}{16\pi\kappa^2}\int dx^4\sqrt{-g}\,e^{-2\phi}\Big\{R+4(\nabla\phi)^2+\kappa^2F^2\Big\} \tag{54}$$

From the form of the action, it is easy to see that the field e^{ϕ} plays the role of the *universal* coupling. This action can be regarded as a low-energy effective string theory in four dimension, but we will not make any connection to the string theory.

Similarly to the Einstein-Maxwell theory, this theory possesses a family of charged black hole solutions, of which those with the spherical symmetry is well-known. The solutions have been obtained with the following ansatz for the metric [7].

$$\tilde{g}\equiv e^{-2\phi}\,g=f^2(r)\,d\tau^2-\frac{dr^2}{f^2(r)}-\rho^2(r)\,d\Omega^2 \tag{55}$$

The new metric $\tilde{g}$ is sometimes called the Einstein metric because the action above reduces to the Einstein-Hilbert action in terms of $\tilde{g}$ plus matter fields with unconventional couplings.

For magnetically charged cases, we can easily extract the behaviour of the field strength 2-form F for objects with magnetic charge Q/κ.

$$F=\frac{Q}{\kappa}\sin\theta\,d\theta\,d\varphi \tag{56}$$

The static and radial field equations can be integrated exactly, and we find the spherically symmetric magnetic black holes with a regular event horizon. In terms of the ADM mass M/κ^2 and the asymptotic value of the coupling e^{ϕ_∞},

$$\tilde{g}=\left(1-\frac{2M}{r}\right)d\tau^2-\left(1-\frac{2M}{r}\right)^{-1}dr^2-r^2\left(1-\frac{Q^2e^{-2\phi_\infty}}{Mr}\right)d\Omega^2. \tag{57}$$

The corresponding coupling e^{ϕ} grows like r/ρ and is actually infinite at $\rho=0$.

$$e^{-2\phi}=e^{-2\phi_\infty}\left(1-\frac{Q^2e^{-2\phi_\infty}}{Mr}\right) \tag{58}$$

The curvature singularity at the vanishing ρ is spacelike and hidden behind a regular horizon at $r=2M$, provided that $\sqrt{2}\,M>Qe^{-\phi_\infty}$. But, in the extremal limit $\sqrt{2}\,M\rightarrow Qe^{-\phi_\infty}$, the singularity becomes null and naked.

[11] In this chapter, κ^{-2} is the gravitational constant, while $c=\hbar=1$. The signature of the metric is taken to be $(1,-1,-1,-1)$.

Converted back to the original variable $g = e^{2\phi}\tilde{g}$, however, the geometry looks quite different in the extremal limit. The vanishing behaviour of $\tilde{g}_{\tau\tau} = f^2$ and $\tilde{g}_{\theta\theta} = -\rho^2$, at the naked singularity $r = \sqrt{2}\,Qe^{-\phi_\infty}$, is completely canceled by the divergent coupling squared $e^{2\phi}$, and the resulting geometry is geodesically complete. Through a couple of coordinate redefinitions, we find the following metric for the extremal geometry.

$$
\begin{aligned}
g_{ex} &= dt^2 - dz^2 - R^2(z)\, d\Omega^2, \qquad R_{ex}(z) \equiv e^{\phi(r)}\rho(r)\,|_{extremal} \\
t &= e^{\phi_\infty}\tau \\
z &= e^{\phi_\infty}\int dr \left(1 - \frac{\sqrt{2}Qe^{-\phi_\infty}}{r}\right)^{-1}
\end{aligned}
\tag{59}
$$

Note that, as we approach $\rho = 0$, corresponding to a horizon infinitely far away, the transverse radius R_{ex} approaches a constant $\sqrt{2}\,Q$.

The geometry possesses two asymptotic regions: the first is the usual asymptotically Minkowskian region at $z, R_{ex} \to \infty$, while the second at $z \to -\infty$ looks like an infinitely long neck of the fixed transverse radius $\sqrt{2}\,Q$, threaded by the uniform magnetic flux $4\pi Q/\kappa$, as depicted in Figure 4.1. This extremal magnetic solution has been dubbed "the cornucopion" and is considered as a possible remnant candidate, because of its characteristic infinite-neck structure.

In the following sections, we want to explore the stability of this unusual geometry in the presence of a vacuum polarization effect associated with massive charged fermions.

4.3 Callan-Rubakov Modes in a Magnetic Black Hole Background

To be definite, let us consider a static dilatonic black hole solution with magnetic charge [7], written down in terms of the tortoise coordinate z.

$$
g = \lambda^2(z)\, dt^2 - \lambda^2(z)\, dz^2 - R^2(z)\, d\Omega^2, \qquad e^{-2\phi} = e^{-2\phi(z)}. \tag{60}
$$

Asymptotically $R \simeq z \to \infty$, while the event horizon is at $z = -\infty$ where the geometry is largely determined by the behaviour of λ. For a cornucopion which is a purely magnetic black hole, we can take $\lambda \equiv 1$. For black holes with an extremal horizon at finite physical distance, such as those with both electric and magnetic charges inside the event horizon [8], $\lambda^2 \sim 1/z^2$ as we approach the horizon. Finally ϕ is the dilaton field and e^ϕ plays the role of the coupling. Since we want to couple the matter system to gravity later on, we shall keep λ, R and ϕ unspecified for a while. As long as the solution is static and spherically symmetric, the detailed form of it does not enter the derivation in this section.

Now depending on the origin of the magnetic charge, we can introduce different kinds of charged fermions. The simplest case would be a Dirac fermion coupled to the $U(1)$ gauge field. However, we found it advantageous to work with a spontaneously broken $SU(2)$ theory so that the fermions are in the fundamental representation of $SU(2)$ and that the magnetic charge is realized as a topological quantum number of solitons. The background gauge field outside the horizon is still Abelian except that the Abelian $U(1)$ generator Q is expressed in terms of the $SU(2)$ generators T_a [9].

$$Q \equiv T_a n^a, \qquad \vec{n} \text{ is the unit radial vector field.}$$

With this choice, the derivation of the effective matter action is quite similar to that of Callan [4], up to the conventions regarding the spinor and the modifications due to the nontrivial geometry. We shall compare our results to those of Callan whenever appropriate.

Consider a $SU(2)$ doublet Dirac fermion ψ with nonzero mass m_ψ. As mentioned in the previous section, it is sufficient to focus on the lowest partial waves of the fermion, called Callan-Rubakov modes, which do not see any potential barrier of the geometry or of the spherically symmetric gauge field [4][10]. To isolate such modes, we use the following ansatz for $\psi_\pm$, positive and negative chiral eigenstates of ψ written in terms of Weyl 2-spinors.

$$\psi_+ = \frac{1}{\sqrt{4\pi\lambda R}}\chi_+(t,z), \qquad \psi_- = \frac{1}{\sqrt{4\pi\lambda R}}\gamma^t\chi_-(t,z). \tag{61}$$

The upper (lower) component of the two spinor χ_- (χ_+) has charge 1/2 while the other has $-1/2$, with respect to the unbroken $U(1)$ generator Q. We chose $\gamma^t = \sigma_x$ and $\gamma^z = i\sigma_y$ as our two-dimensional Dirac matrices. With this ansatz, the fermion action can be reduced to the following two-dimensional form.

$$\begin{aligned} S_\chi &= \int dtdz\,(i\overline{\chi}_+\gamma^i\partial_i\chi_+ + i\overline{\chi}_-\gamma^i\partial_i\chi_- + m_\psi\lambda\,(\overline{\chi}_+\chi_- + \overline{\chi}_-\chi_+)) \\ &+ \int dtdz\,4\pi\lambda^2R^2\,(a_tJ^t_Q + a_zJ^z_Q), \end{aligned} \tag{62}$$

where $\vec{a}$ is the fluctuating part of the radial $U(1)$ gauge field. We can set a_t equal to zero using the gauge degree of freedom and then the radial electric field in (t,z) coordinates is simply $E \equiv \partial_t a_z$. The relevant currents are,

$$J^t_Q = \frac{1}{8\pi\lambda^2R^2}(J^z_+ - J^z_-), \qquad J^z_Q = \frac{1}{8\pi\lambda^2R^2}(-J^t_+ + J^t_-), \tag{63}$$

where $J_\pm$ are the two-dimensional vector currents of $\chi_\pm$. The effective action S_χ above is incomplete since we neglected the action for the fluctuating electric field E so far. By isolating it from the full Yang-Mills action and integrating the angular part, we find

$$S_E = \int dtdz\,(\frac{\theta}{2\pi}E + \frac{R^2}{2\lambda^2e^{2\phi}}E^2). \tag{64}$$

The θ term is from the Pontryagin density and can be deduced from the fact that a unit magnetic monopole carries total magnetic flux 4π.

This effective action $S_\chi + S_E$ is different from that of Callan [4] in two respects. First, the effective couplings are changed due to the nontrivial geometry and the nonuniform coupling e^ϕ. In particular, the fermion mass term acquires a factor of λ. Second, the radial coordinate z extends from ∞ to all the way to $-\infty$. Because of this, we no longer need to impose a boundary condition at the origin. In fact, it is effectively a theory of fermions in flat 1+1 Minkowski space-time, coupled to a $U(1)$ field through an axial current, but with position-dependent mass and coupling.

Since the $U(1)$ field is coupled to a two-dimensional *axial* current,[12] we can bosonize $\chi_\pm$, through the following fundamental relationships between currents

[12] The two spinors $\chi_\pm$ are of opposite charges, so that the $U(1)$ gauge symmetry is not anomalous.

[4][11] which preserve the $U(1)$ current automatically,

$$\vec{J}_\pm = -\frac{1}{\sqrt{\pi}}\vec{\partial} f_\pm, \qquad \text{with respect to the flat metric} \quad dt^2 - dz^2. \tag{65}$$

Furthermore, to separate the charged and the uncharged sectors, it is convenient to perform a canonical transformation generated from

$$f \equiv (f_+ - f_-)/\sqrt{2}, \qquad \eta \equiv \eta(-\infty) + \int_{-\infty}^{z} (\dot{f}_+ + \dot{f}_-)/\sqrt{2}. \tag{66}$$

Once this is done, we can simply eliminate the electric field strength E through its equation of motion and express the effective action completely in terms of bosonic fields f and η. If we define μ to be the geometric mean of the normal ordering scales μ_f and μ_η for each field, the effective action $S_\chi + S_E$ is transformed into

$$\int dtdz \left\{ \frac{1}{2}(\partial f)^2 + \frac{1}{2}(\partial \eta)^2 - \frac{e^{2\phi}\lambda^2}{4\pi R^2}(f - \frac{\theta}{\sqrt{2\pi}})^2 + c m_\psi \mu \lambda \, \cos\sqrt{2\pi} f \, \cos\sqrt{2\pi}\eta \right\}. \tag{67}$$

The constant c is a number of order 1 and shall be kept unspecified. The electric charge density of fermions is now simply proportional to the spatial derivative of f and we find the total charge inside a radial coordinate z to be

$$q(z) = \int dz \, (4\pi\lambda R^2 \; J_Q^0) = \frac{1}{\sqrt{2\pi}}(f - \frac{\theta}{\sqrt{2\pi}})|_z, \tag{68}$$

where we fixed the integration constant by inspecting the electric energy term in the effective action above. Similarly, the fermion number inside z is, up to an additive constant, given by $\eta\sqrt{2/\pi}$ evaluated at z.

Coupling to gravity requires further considerations. First, the effective action above is not manifestly covariant. The actual two-dimensional metric $g^{(2)} \equiv \lambda^2\,(dt^2 - dz^2)$ has a conformal factor λ^2 and the only way to reconcile this with the present form of the action is to choose the normal ordering scales to be proportional to λ. On the other hand, the only two physical mass *parameters* of the effective theory are $e^\phi\lambda/R$ and $m_\psi\lambda$, both of which are proportional to λ. Hence it is appropriate to replace μ by $\lambda\overline{\mu}$, where the specific choice of $\overline{\mu}$ should not in principle affect the physics. But in practice, we will study the effective action at tree level only, which requires a judicious choice of the ordering scales. For example, we can take

$$\mu^2 = \lambda^2\overline{\mu}^2 \equiv m_f m_\eta, \quad m_f,\, m_\eta \text{ are } \textit{masses} \text{ of } f \text{ and } \eta, \text{ in the } (t,z) \text{ coordinate,}$$

automatically ensuring the covariance of the effective action. Secondly, there is the matter of zero-point energy, which is irrelevant before gravity is turned on. To ensure the existence of the cornucopion for trivial values of the vacuum angle θ, it is necessary to have vanishing vacuum energy density, whenever $\cos\theta = 1$, both at the asymptotic infinity and at the other asymptotic region deep inside the cornucopion. This can be achieved by adjusting the fermion mass term to have minimum at zero rather than at $-m_\psi\mu\lambda$. The resulting effective action $S_{\rm eff}$ in an arbitrary coordinate system is

$$\int dx^2\sqrt{-g^{(2)}}\Big\{\frac{1}{2}(\nabla f)^2+\frac{1}{2}(\nabla\eta)^2-\frac{e^{2\phi}}{4\pi R^2}(f-\frac{\theta}{\sqrt{2\pi}})^2-cm_\psi\overline{\mu}\,(1-\cos\sqrt{2\pi}f\cos\sqrt{2\pi}\eta)\Big\}. \tag{69}$$

4.4 A Vacuum Energy Distribution and the Gravitational Backreaction

To recover the charge quantization rule, it is sufficient to study the effective potential at spatial infinity. In the asymptotic region, the potential is dominated by the fermionic mass term, whose minima occur at $f = N\sqrt{\pi/2}$ with N even or odd depending on the asymptotic value of η. Therefore,

$$q_{total} = q(z=\infty) = \frac{N}{2} - \frac{\theta}{2\pi}, \qquad N \text{ is any integer.} \tag{70}$$

Obviously the dynamical fermion ψ is responsible for the new half-integral part, and odd N must correspond to an odd number of fermions.[13] It is not surprising to find

[13] The fermion number n_ψ is not conserved in the presence of a black hole. However, the odd and the even fermion numbers require different charge quantizations, and n_ψ modulo-two is a good quantum number. It is easy to see that the fermion number n_ψ modulo-two is given by $\eta\sqrt{2/\pi}$ evaluated at spatial infinity.

the same results as Callan did [4], since the asymptotic form of the effective theory is insensitive to whether the gravity is turned on or not.

However, the vacuum polarization effect as we approach the black hole can be very different from the case of a nonsingular monopole in a flat space-time. Specifically, we want to concentrate on the fermionic ground state in the background of the cornucopion, a purely magnetic extremal black hole. Suppose we want to find a parameter region where the noncompact core geometry of the cornucopion serves as the zero-th order approximation. Such a configuration would correspond to a narrow and infinite neck of cornucopion surrounded by harmless and dilute charged cloud of the vacuum polarization, just as the nonsingular monopole core is, according to Callan [4], surrounded by a harmless and dilute charged cloud of the vacuum polarization.

A quick look at the effective potential convinces us that this is not possible unless the fermion mass is actually zero. Deep inside the neck, the electric energy term is dominant so that $\sqrt{2\pi} f$ approaches θ, and as a result the ground state energy density behaves like

$$V_{min} \simeq c m_\psi \overline{\mu}(1 - |\cos\theta|) \; > 0.$$

Since the total vacuum energy inside the throat region is given by integrating V_{min} along the infinite neck, the ground state built on this background comes with infinite vacuum energy distributed along the infinite neck.[14]

Actually a similar phenomenon occurs for monopoles in a flat space-time, contributing a vacuum energy which scales like $m_\psi^2 L_c$, where L_c is the distance between the charged shell and the monopole center. On the other hand, the resulting electric charge distribution carries electrostatic energy which scales like $1/R_c$, where $4\pi R_c^2$ is the area of the charged shell. In a flat space-time, $L_c \simeq R_c$ and the balance

[14]While it is conceivable that some other θ-dependent effects may cancel V_{min}, such a cancellation, if possible, could occur only for very special values of parameters in the theory. The θ-independent part is fixed in the previous section by assuming that the purely magnetic configuration is given by the cornucopion.

between the two contributions fixes the order of magnitude of $L_c \simeq R_c$ at $1/m_\psi$, as mentioned earlier.

In a curved black hole geometry, however, L_c is now some measure of the physical distance between the charged shell and the event horizon, which needs not be proportional to R_c (the linear size of the charged shell) any more. With a cornucopion, in particular, $L_c \to \infty$ while R_c remains finite. It is not possible to achieve a balance between the two, and it is necessary to consider the gravitational backreaction to the vacuum polarization, to understand the true nature of the fermionic θ-vacua.

To understand the gravitational backreaction, let us digress a little bit and recall the energetics of the pure cornucopion solution. The infinite neck of a cornucopion is threaded by a constant flux of classical magnetic fields. While one would normally expect a uniform magnetostatic energy density associated with the flux, the energy density actually vanishes exponentially deep down the neck. The reason is simply that the electric coupling e^ϕ, inverse square of which appears in the megnetostatic energy density, is exponentially growing. Note that, in Einstein-Maxwell theory where the coupling is really a constant, an extremal horizon forms, hiding whatever divergent behaviour the energy-momentum may have.

In a sense, the purely magnetic cornucopion of finite ADM mass exists precisely because the divergent coupling prevents the magnetic energy-momentum from accumulating divergently. However, once we turn on the Callan-Rubakov modes with a generic θ, the energy-momentum given by V_{min} eventually dominates and does accumulate divergently deep inside the neck. On the other hand, since the energy-momentum far outside is completely determined by the total magnetic charge and the fractional electric charge, the ADM mass must be finite regardless of the vacuum polarization.

Now it is clear what must happen. The gravitational backreaction to the accumulated effect of V_{min} must eventually create a horizon somewhere down the would-be cornucopion, rendering L_c, thus the vacuum energy contribution to the ADM mass, finite. Hence, the infinite neck must be terminated by a zero-temperature horizon at finite physical distance. In such a self-consistent background, one should be able to find the true vacuum state of the fermion sea.

In fact, one can explicitly check this for small m_ψ. In this limit, the geometry near the throat region remains unchanged since the energy-momentum there is dominated by the magnetic flux, and the long-neck structure survives until the point where V_{min} is comparable to the magnetostatic energy density. Then as we travel down the would-be cornucopion, the dynamics effectively reduce to that of a 2-D dilatonic gravity [13]. The relevant 2-D action can be easily obtained by dimensionally reducing the complete action (70) to appear in the next section.

$$\int dx^2 \sqrt{-g^{(2)}} \left\{ e^{-2\phi} (-R^{(2)} - 4 (\nabla\phi)^2 + \frac{1}{2\kappa^2}) - 2V_{min} \right\}$$

For reasonable choices of $\overline{\mu}$, the static solutions of this action can be easily shown to possess two horizons generically, and the extremal limit thereof corresponds to a zero-temperature horizon at finite physical distance terminating the long neck.

4.5 Self-Consistent Geometries and the Fractional Charge

We concluded above that, for generic θ, the gravitational backreaction to the vacuum energy distribution is always important and that there exists an extremal horizon stopping indefinite growth of the would-be cornucopion.

In the discussion above however, the fractional electric charge itself does not seem to play a role as far as the core geometry is concerned. After all, not only is the charge cloud too large to approach even the throat region, but it is known that any electric charge faces an exponentially divergent potential barrier as it travels down a cornucopion, owing to the electromagnetic backreaction [12]. The electric flux tube attached to the charge costs more and more energy, proportional to the exponentially divergent coupling squared $e^{2\phi}$, and this tends to push away any electric charge.

On the other hand, the gravitational backreaction renders this potential barrier finite, since the coupling cannot be infinite at the regular extremal horizon, and at least part of the fractional charge should be expected to be trapped inside the horizon. This last observation raises a question whether one can explain the newly-formed extremal horizon entirely in terms of this trapped electric charge.

There are known *clean* dyonic black hole solutions of the dilaton gravity coupled to Maxwell fields [8], and their extremal limit comes with an extremal horizon.[15] Maybe, the vacuum energy density found above seemed so prominent only because we were using a wrong background. It is a logical possibility that the self-consistent geometry near the extremal horizon is dictated by the charges.

In fact, this is precisely what must happen in the large m_ψ limit. As m_ψ increase, the density of the charge cloud as well as V_{min} must increase accordingly. The gravitational backreaction to V_{min} creates the horizon more and more close to the fractional charge cloud which by now is itself dense enough to distort the geometry. With the increasingly weak potential barrier, the increasingly massive lump of the fractional charge will eventually fall into the black hole and the effect of V_{min} will disappear behind the horizon. Once this happens, an observer outside would be completely oblivious of the vacuum energy distribution and attribute the termination of the would-be cornucopion to the fact that the extremal horizon hides both electric and magnetic charges.

In this section we would like to investigate this possibility in both the small and large fermion mass limits, with the latter serving as a consistency check. The total action dictating the self-consistent geometry is given by the following.

$$S = S_{\text{eff}} - \frac{1}{16\pi\kappa^2}\int dx^4\sqrt{-g^{(4)}}\,e^{-2\phi}\,(R + 4\,(\nabla\phi)^2 + \kappa^2 F^2). \tag{71}$$

For the purpose of studying the properties of static and spherically symmetric solutions at the extremal horizon, which will turn out to be very informative, we can reduce the field equations to a set of algebraic ones involving various physical quantities at the horizon. The key to this simplification is the regularity of the

[15] In ref [8], electric and magnetic charges belong to different $U(1)$'s, but as far as the static and spherically symmetric geometry is concerned, it does not matter. One should be careful to distinguish these solutions from another known family of solutions with an axionic field [15].

horizon.[16]

If a function of the radial coordinate only is finite and differentiable at an extremal horizon with respect to a local geodesic coordinate, some of its covariant derivatives vanishes there just because λ vanish at the horizon. If we denote the evaluation at the extremal event horizon by the subscript h,

$$(\nabla^2 f)_h = (\nabla f)^2_h = 0, \qquad \text{the same for } \phi,\ R, \text{ and } \eta.$$

As a result, only terms without any derivative of f, η, ϕ, R survive the evaluation of the static field equations at the extremal horizon. For instance, combining the dilaton equation and an angular Einstein equation, we can easily deduce that $R_h^2 = 2\kappa^2$, showing that the transverse size of the neck remains unchanged. On the other hand, the equation for η tells us $\cos\sqrt{2\pi}\eta_h = \pm 1$. From some of the remaining equations, we find two algebraic equations for e^{ϕ_h} and f_h.

$$\frac{e^{2\phi_h}}{4\pi\kappa^2}(f_h - \frac{\theta}{\sqrt{2\pi}}) \pm \sqrt{2\pi} c m_\psi \overline{\mu}_h \sin\sqrt{2\pi} f_h \;=\; 0 \qquad (72)$$

$$\frac{e^{2\phi_h}}{8\pi\kappa^2}(f_h - \frac{\theta}{\sqrt{2\pi}})^2 + c m_\psi \overline{\mu}_h (1 \mp \cos\sqrt{2\pi} f_h) \;=\; \frac{e^{-2\phi_h}}{4\kappa^2} \qquad (73)$$

Now let us consider two limiting cases as promised, to unravel the role of the fractional electric charge in the formation of the extremal horizon.

When $\kappa m_\psi >> 1$, the first equation (71) tells us that the value $(\sin\sqrt{2\pi} f_h)$ is very small and that, for the lowest energy configuration, the left-hand-side of (72) is dominated by the electric energy term $\sim (f_h - \theta/\sqrt{2\pi})^2$. Then, we find the following relation between the coupling and the trapped charge q_h except for integral θ/π, when the argument above breaks down.

$$e^{-4\phi_h} = \frac{1}{2\pi}(f_h - \frac{\theta}{\sqrt{2\pi}})^2 + O(\frac{1}{c\kappa^2 m_\psi^2}) = q_h^2 + \cdots \qquad (74)$$

This is a nontrivial and significant piece of information, in that this is exactly what one would expect to be true if the horizon geometry is completely determined by

[16] In fact, possible mild singularities at the extremal horizon such as observed by Trivedi[14] in semiclassical extremal black holes do not interfere with this derivation.

the electromagnetic charges trapped inside. It is a matter of straightforward algebra to show that the *clean* dyonic black holes of [8] satisfies $(q/p)^2 = e^{-4\phi_h}$ with electric and magnetic charges given by q and p. Furthermore, small $(\sin\sqrt{2\pi} f_h)$ implies that most of the fractional charge is inside the black hole, confirming the assertions earlier in this section.

Finally, coming to the small fermion mass limit $\kappa m_\psi << 1$, we can easily see that the first equation (71) now predicts very small $q_h \sim (f_h - \theta/\sqrt{2\pi})$. Because of this, the left-hand-side of (72) is now dominated by the fermion mass term $\sim m_\psi \overline{\mu}$, and this in turn implies the following characteristic of the self-consistent geometry in the small fermion mass limit.

$$|q_h| \sim e^{-4\phi_h}, \quad \text{rather than} \quad q_h^2 \simeq e^{-4\phi_h}. \tag{75}$$

The implication is clear when compared to (73). Now the leading energy-momentum contribution closing off the infinite neck of the would-be cornucopion (making $e^{-2\phi}$ nonzero) is generated by V_{min} rather than by the trapped electric charge. Very small amounts of electric charge $q_h \sim e^{-4\phi_h}$ (not $\sim e^{-2\phi_h}$) are trapped by the newly-formed extremal horizon only as a secondary effect. The presence of the vacuum energy distribution V_{min} is very real unlike the previous case of large fermion limit.

A couple of remarks are in order. The key formulae (73) and (74) are derived without detailed knowledge of $\overline{\mu}$. All we needed was rough characteristics of it in each limit, such as $\overline{\mu}_h/m_\psi << e^{2\phi_h}$ for small m_ψ and $\overline{\mu} \sim m_\psi$ for large m_ψ. This is an important point because physics should not depend on the choice of the normal ordering scale, and indeed we managed to isolate such a μ-independent characterization of the self-consistent geometry in the form of these key formulae. Another fact we want to mention is the equations (71) and (72) above show the expected behaviour as $m_\psi \to 0$. Though the precise behaviour does depend on $\overline{\mu}$, the value $e^{-2\phi_h}$ can be shown to vanish rapidly as κm_ψ approaches zero, corresponding to a longer and longer neck. Eventually when m_ψ is identically zero, the limiting self-consistent geometry is that of a cornucopion, as it should be.

4.6 Conclusion

To summarize, we found that fermionic θ vacua tends to terminate the infinite neck of the cornucopion. This fact, by itself, should not be surprising, since a nontrivial θ implies the existence of the (fractional) electric charge which, if swallowed by the black hole, produces a *clean* dyonic black hole with an extremal horizon. In fact, this is exactly what happens when θ/π is non-integral and the fermion mass is sufficiently large. On the other hand, somewhat unexpected is the behaviour for a small fermion mass. In this case, the dilute charged clouds hovering far away from the throat region are shown to exist at the cost of a vacuum energy distribution inside. The energy-momentum associated with this energy density induces a strong gravitational backreaction, and the result is again the formation of an extremal horizon at finite distance. Unlike the case of large fermion mass, however, we found that the charge penetration to the black hole is at most of secondary effect.

One advantage of using the bosonic form of the matter is, among others, the exchange of the roles played by the coupling and the mass. The effective matter action (68) is such that the quantum fluctuation of the bosons f and η are increasingly costly deep down the would-be cornucopion, and our tree-level estimates of the energy-momentum are reliable in spite of, or we should say, because of the strong coupling which is inevitable for small m_ψ. Even though such a strong coupling may induce large gravitational fluctuations near the extremal horizon in small m_ψ limit, this should not disrupt our qualitative results. At most we expect a quantitative modification of the estimate (74).

While we worked with the case of a $SU(2)$ doublet fermion on a unit-charged would-be cornucopion, similar results should hold for some different models. The vacuum energy distribution trailing the fractionally charged cloud is a generic property of the screening and should exist regardless of the model. What makes this vacuum energy distribution dangerous enough to destabilize the core geometry is the noncompact nature of the unperturbed core structure, as emphasized in section 3. Therefore one should expect a similar destabilization to occur whenever he quantizes massive charged fermion field, in the background of θ and an infinite-neck geometry threaded by constant magnetic flux.

For instance, if we consider a single fermion coupled to $U(1)$ rather than $SU(2)$, the resulting bosonized matter action must be similar to our own but involve the charged sector only (corresponding to the f field above, with η frozen out). Since it is the effective potential associated with f which induces the screening and the vacuum energy distribution thereof, an analogue of V_{min} will appear in the background of a unit-charged cornucopion. Again the resulting gravitational backreaction will close off the would-be cornucopion, just as we observed above.

Another interesting case to consider is that of cornucopia of larger transverse sizes. Since the transverse size is proportional to the magnetic flux threading the infinite neck, these are highly charged magnetic black holes. In such a background, one finds analogues of the Callan-Rubakov modes in the form of zero-modes on the transverse two-spheres. Since the number of these zero-modes is proportional to the total magnetic flux threading the two-sphere, we need to deal with not just a pair of two-spinors $\chi_\pm$ but complicated multiplets of the generalized angular momentum [9]. Nevertheless, the separation of variables and the dimensional reduction must be possible, and upon a bosonization trick we expect to find only two kinds of bosonic fields: $\tilde{f}$, an analogue of f which keeps track of the electric charge and the vacuum energy distributions, and $\tilde{\eta}_a$'s, the rest of them. There are again two effective potentials: the electric energy term which is minimized for vanishing local electric fields and the fermionic mass term which is not. The upshot is again that the would-be cornucopion develops an extremal horizon at finite physical distance.

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5 Nonsingular Two-Dimensional Black Holes and Classical String Backgrounds

In this final chapter, we study a string-inspired classical 2-D effective field theory with *nonsingular* black holes as well as Witten's black hole among its static solutions. By a dimensional reduction, the static solutions are related to the $(SL(2,R)_k \otimes U(1))/U(1)$ coset model, or more precisely its $O((\alpha')^0)$ approximation known as the 3-D charged black string. The 2-D effective action possesses a propagating degree of freedom, and the dynamics are highly nontrivial. A collapsing shell is shown to bounce into another universe without creating a curvature singularity on its path, and the potential instability of the Cauchy horizon is found to be irrelevant in that some of the infalling observers never approach the Cauchy horizon. Finally a $SL(2,R)_k/U(1)$ nonperturbative coset metric, found and advocated by R. Dijkgraaf et al., is shown to be nonsingular and to coincide with one of the charged spacetimes found above. Implications of all these geometries are discussed in connection with black hole evaporation.

5.1 Introduction

The longstanding problem of black hole evaporation has been recently revitalized in the context of 2-D dilaton gravity [1], either on its own or as an effective S-wave sector theory of the 4-D black holes [2]. In 2-D, the coupling to conformal matter, hence the Hawking radiation, can be conveniently represented by the Polyakov-Liouville action of appropriate coefficient, enabling a systematic study of the gravitational backreaction [1][3]. But most of these studies are again plagued by curvature singularities, hidden or naked, and the inability to handle the singularities properly clouds any conclusion concerning the fate of the black holes [3][4].

The question arises, then, whether the singularity is an essential feature of black hole physics. For example the Hawking radiation is, strictly speaking, a property of event horizons, and singularities appear only as a result of the gravitational field equation. Is there a physically sensible theory, solutions of which are nonsingular spacetimes but with horizons? The answer is yes. In this chapter, we present such an 2-D effective field theory, static solutions of which are related to certain WZW coset models.

While WZW coset models provide us with string theories with interesting target spacetimes, it is often very difficult to study the dynamics since we have to solve the full string theory around nontrivial backgrounds. An easy way out is to abandon the exact model for an effective (approximate) local field theory by using the α' expansion of the sigma model [5]. Higher order corrections can be controlled by introducing extended supersymmetry in some cases, but not always. In particular, the α' expansion is also an expansion in the curvature of the target metric and a curvature singularity can be a signal of big higher order and nonperturbative corrections.

Apparently this is exactly what happens with Witten's black hole as an $O((\alpha')^0)$ approximation to the $SL(2,R)_k/U(1)$ coset model. So far, two independent attempts to find a nonperturbative *exact* geometry have yielded an identical static metric [6][7], and, quite surprisingly, this new metric has the causal structure of a *nonsingular* multi-universe spacetime with horizons, as will be shown in the final section of this chapter.

Furthermore we have found a string inspired local field theory in 2-D, whose charged static solutions are mostly of the same causal structure as this *exact* coset metric. The action is obtained by a dimensional reduction of the 3-D string effective action of the gravity multiplet and the static solutions are dimensionally reduced versions of the 3-D black strings of Horne et al. [8]. (These 3-D black strings are $O((\alpha')^0)$ approximations to the $(SL(2,R)_k \otimes U(1))/U(1)$ coset model.)

In short, we have not only an exact classical string background whose metric is a nonsingular black hole but also a local field theory, with essentially same type of static solutions, which can serve as a dynamical toy model. The form of the field theoretical solutions is identical to that of the $SL(2,R)_k/U(1)$ *exact* coset background up to an additive shift of one parameter. Not only the causal structures but the behaviour of the dilatons are the same. This presents us with a unique opportunity as well as a motivation to study these new kind of black holes. In this chapter we study the classical properties of the 2-D effective field theory for the most part and detailed discussions on the *exact* background is postponed until at the end.

In section 2, we introduce the 2-D effective field theory and the much advertised nonsingular 2-D spacetimes. We start with a compactified charged black string. The geodesic equation of motion is studied for radial motion, and the Penrose diagrams of the 2-D part of the metric are shown. After the calculation of the Hawking temperature, we elaborate on the relationship of these new spacetimes to Witten's black hole, with emphasis on the duality transformation used for the construction of the black string from Witten's black hole.

We devote the next two sections to the dynamics of the 2-D effective theory, in particular, the stability of the static solutions under the process of a gravitational collapse. In section 3, we investigate the response of the geometry to a collapsing shell of massless matter. Even though we are unable to solve the full partial differential equations governing the process, the massless nature of the gravitating source allows us to obtain useful information such as the difference of the curvature across the shell. Crucial to the calculation here is the asymptotic behaviour of the gravitational perturbation. We added an appendix at the end of the chapter to derive the necessary information. In section 4, we examine the instability of the Cauchy horizon and the consequences.

Section 5 is devoted to some of the unresolved issues of the gravitational collapse as well as implications of the results we obtained in the earlier sections. In particular, the *exact* metric of the $SL(2,R)_k/U(1)$, found in [6] and recently rediscovered in an entirely different approach by A.A. Tseytlin [7], is shown to have the same geometry as one of the nonextremal (hence nonsingular) solutions. Also discussed are nonperturbative black string solutions. Based on this new set of charged spacetimes, we speculate on the real nature of the 2-D black holes.

5.2 Compactified Black Strings as Geodesically Complete 2-D Spacetimes

We start with the charged black string solution by Horne et al.[8],

$$S^{(3)} = \int dx^3 \sqrt{-g^{(3)}}\, e^{-2\phi} \left\{ R^{(3)} + 4(\nabla\phi)^2 + 4\lambda^2 - \frac{1}{12}H^2 \right\}, \tag{76}$$

$$g^{(3)} = -(1-\frac{m}{r})\, dt^2 + \frac{1}{4\lambda^2 (r-m)(r-q)}\, dr^2 + (1-\frac{q}{r}) a^2\, d\theta^2, \tag{77}$$

$$e^{-2\phi} = \frac{r}{\lambda}, \tag{78}$$

and consider the case θ periodic with period 2π. Then a is the asymptotic radius of the internal circle. We define $q \equiv Q^2/m$ where Q is the axionic charge associated with the antisymmetric 2-tensor. This solution is actually $O((\alpha')^0)$ approximation to the coset model $(SL(2,R)_k \otimes U(1))/U(1)$ and $q = 0$ limit corresponds to $(SL(2,R)_k/U(1)) \otimes S^1$, the Witten's coset model multiplied by a circle of fixed radius. For $m > q > 0$, the metric describes singularities at $r = 0$ hidden inside two distinct sets of horizons at $r = m, q$. The leading divergence of the curvature near $r = 0$ is $\sim -(1/r^2)$. The causal structure is somewhat similar to that of the Reissner-Nordström black holes in that a countable number of asymptotically flat universes are connected through compact regions of inner and outer horizons as well as timelike singularities (for more detail, see reference [8]), but the analogy should not be taken too seriously since the nature of the inner horizons is very different. The Penrose diagram of (refeq:metth) cannot be accurately drawn in the r-t plane.

For small a, a big mass gap of order $1/a$ develops for the internal degrees of freedom and all the low energy states have circularly symmetric wave functions. For the most of this chapter we will consider the case of a comparable to the Planck length so that the only available internal state is the ground state. In other words, we will consider the circularly symmetric sector of the theory. In terms of the classical physics of the low energy observers, it means the only relevant worldlines are those without any angular momentum. In this section we want to explore the static geometry above as seen by such observers. To begin with, consider all geodesics of vanishing angular momenta. This is consistent with the geodesic equations of motion since θ is a Killing coordinate. With the help of two Killing vector fields, the geodesic equations of motion can be reduced to the following form for the r-coordinate as a function of an affine parameter τ [8].

$$(\frac{\dot{r}}{2\lambda r})^2 = (1 - \frac{q}{r})(\epsilon^2 + \alpha(1 - \frac{m}{r})). \tag{79}$$

ϵ is the covariant t-component of the 3-velocity and $\alpha = -1, 0, 1$ for timelike, null and spacelike geodesics. Immediately one finds a few revealing properties of $r(\tau)$. For generic values of the energy ϵ, r bounces at q quadratically, so that an initial inequality $r \geq q$ is maintained throughout the history of the worldline. The only exception occurs for spacelike geodesics with $1 + \epsilon^2 = m/q$, in which case it takes infinite amount of time τ for $r(\tau)$ to reach q. Another important fact is that, for $\alpha \neq 1$, $r = q$ is the unique extremal value inside $r = m$. In other words, the time coordinate r is monotonically increasing toward both future and past of $r = q$, while $r < m$. Obviously, no radial geodesics can penetrate the inner horizon and see the curvature singularity at $r = 0$ and the way it works out for timelike or null worldlines is that all radial observers bounce at the inner horizon into the next asymptotic region. In any case, one can conclude the restricted region defined by $r \geq q$ is by itself geodesically complete as far as radial geodesics are concerned. The singular structure is completely hidden inside a barrier at $r = q$. Therefore, the radial part of the metric (76) describes a nonsingular 2-D manifold with a countable number of universes as illustrated in figure 5.1.

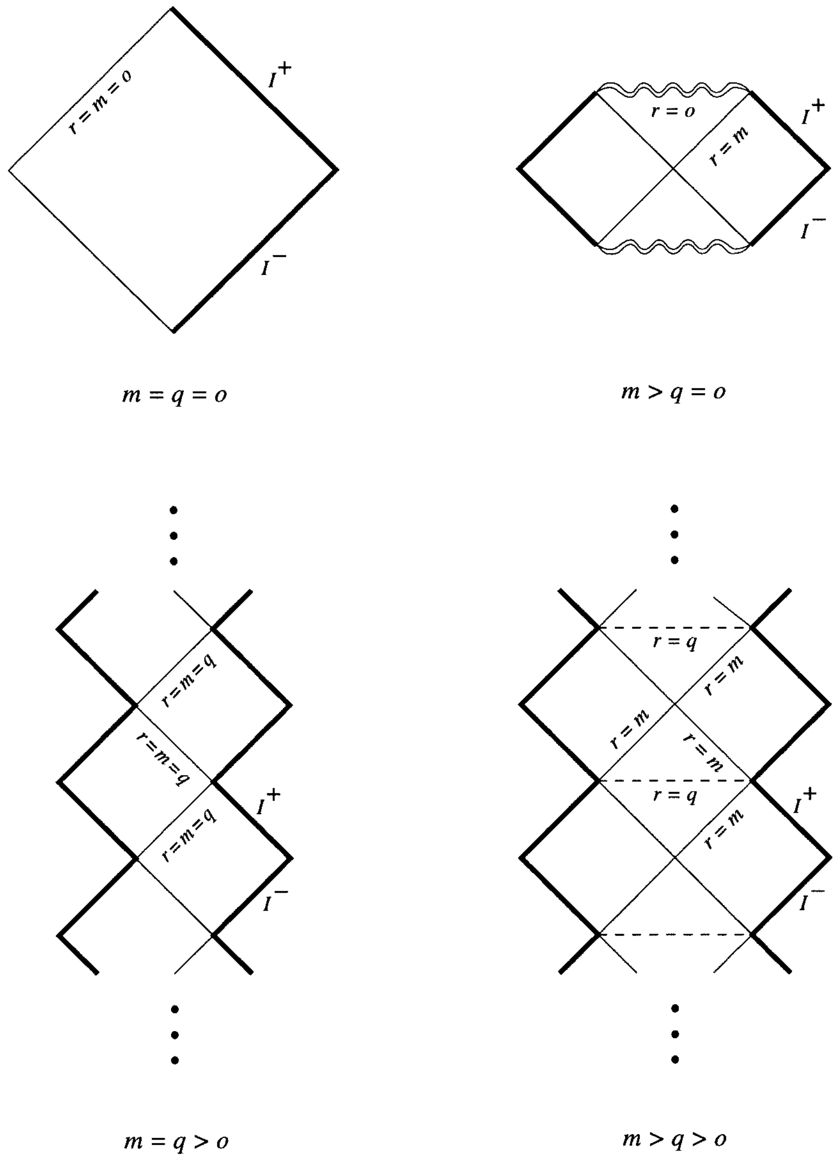

Figure 5.1: Penrose diagrams of $g^{(2)}$ for various parameter ranges. The bold lines indicate asymptotically far regions with vanishing effective coupling e^{χ}. $e^{\chi} = \infty$ at $r = q$. I^+ and I^- are the future and the past null infinities of a particular universe.

This unusual behaviour can be partially traced to the fact θ becomes a time coordinate inside the inner horizon, where a radially moving observer, freely falling or accelerated, can never satisfy the on-shell condition. Once we understand the peculiar nature of the inner horizons at $r = q$, it is not difficult to draw the Penrose diagram of the 2-D metric, or (76) without the $d\theta^2$ term, on the r-t plane. The nonextremal case is similar to that of the Reissner-Nordström geometry [9] with parts of it containing the inner horizons and the singularities cut out and the remaining pieces glued along spacelike hypersurfaces. For this case, the inner horizons at $r = q$ become spacelike lines, to be also called *the critical lines* from now on, contained in the compact region surrounded by outer horizons $r = m$. In figure 5.2, the dotted lines of the case $m > q > 0$ are these critical lines. For $m = q = 0$ they are null lines infinitely far away. In the extremal limit $q \longrightarrow m > 0$, half of the universes are decoupled and each remaining universe is successively attached to the next by the outer horizons at $r = m$. On the other hand the charge-zero limit can be easily shown to be Witten's black hole with the spacelike singularities forming along the critical lines (or the inner horizons) at $r = q$. The change of variable, $r/\lambda = e^{2\lambda x}$, is necessary to go back to a more familiar coordinate in which the dilaton ϕ is linear.

The action and the field equations governing the dynamics of these 2-D spacetimes are obtained by a dimensional reduction of 3-D effective string field theory. Choosing a convenient set of 2-D fields and splitting the metric to $g^{(2)}$ plus the internal part we obtain the following effective action.

$$S^{(2)} = \int \sqrt{-g^{(2)}} e^{-2\chi} (R^{(2)} + 4(\nabla\chi)^2 + 4\lambda^2 - (\nabla f)^2 - e^{2f} K^2), \tag{80}$$

$$g^{(3)} \equiv g^{(2)} + e^{-2f} a^2 \, d\theta^2,$$

$$e^{-2\chi} \equiv e^{-2\phi - f},$$

$$K_{ij} \equiv \frac{1}{2a} H_{ij\theta}.$$

Here K is an exterior derivative of a 1-form, hence a $U(1)$ gauge field strength. The field equations are, after appending charged source on the right-hand sides, the following: Q is the total charge inside the given value of r and is locally constant in the source-free region.

$$R^{(2)} + 4\nabla^2\chi - 4(\nabla\chi)^2 + 4\lambda^2 - (\nabla f)^2 + 2Q^2 e^{4\phi} = 0, \tag{81}$$

$$\nabla(e^{-2\chi}\nabla f) + 2Q^2 e^{4\phi - 2\chi} = 0, \tag{82}$$

$$-\nabla_i\nabla_j e^{-2\chi} + g_{ij}\nabla^2 e^{-2\chi} + \cdots = T_{ij}^{matter}. \tag{83}$$

The 3-D field $\phi = \chi - f/2$ is kept for later conveniences and $\cdots$ of (82) indicates terms of lower derivatives. The Bianchi identity of the left-hand side of (82) implies the energy momentum conservation of T^{matter}. It is not too difficult to see that the static solutions of finite mass are all given by the 2-parameter family above. For $q > 0$ we can define a new coordinate y by $r = q\cosh^2 \lambda y$, which shall be useful later on.

$$g^{(2)} = -(1 - \frac{m}{r})dt^2 + \frac{1}{4\lambda^2\,(r-m)(r-q)}dr^2 = -F\,dt^2 + \frac{1}{F}dy^2, \tag{84}$$

$$F \equiv (1 - \frac{m}{q\cosh^2 \lambda y}),$$

$$e^{-2\chi} = \frac{r}{\lambda}\sqrt{1 - \frac{q}{r}} = \frac{q}{\lambda}\cosh\lambda y \sinh\lambda y, \qquad e^{-f} = \sqrt{1 - \frac{q}{r}} = \tanh\lambda y,$$

$$R^{(2)} = 4\lambda m e^{2\phi} - 6mqe^{4\phi}.$$

Notice the inequality $e^{-2\phi} \equiv e^{-2\chi + f} \geq q/\lambda$. Even though the effective coupling e^{χ} is unbounded, the string coupling e^{ϕ} is bounded for charged cases. In the charge-zero limit, $f \equiv 0$, and $\chi \equiv \phi$ becomes the usual dilaton of [1]. The Hawking temperature [10] can be easily computed by requiring the Euclidean version of $g^{(2)}$ to be nonsingular at the horizon $r = m$. The periodicity of the Euclidean time coordinate is the inverse temperature.

$$T_{Hawking} = \frac{\lambda}{2\pi}\sqrt{\frac{m-q}{m}}. \tag{85}$$

This reduces to the expected value $\lambda/(2\pi)$ [1] as $q \to 0$.

To illustrate the relationship between these new spacetimes and Witten's, recall the construction of the charged black strings [8]. A neutral black string is a simple product of a Witten's black hole with a line. After a Lorentz boost along the line, one can make a duality transformation [11] to get a new classical solution to the effective string field theory with torsion, or charge as we call it in this chapter. We wrote the solution in (76) for a circle instead of a line. Without any boost ($q/m = 0$), the duality transformation is trivial and $g^{(2)}$ is the original Witten's black hole. The maximal boost corresponds to the extremal case $q/m = 1$. If the Witten's black hole we started with were an exact string background, the duality symmetry of string theory would imply all nonextremal solutions of (83) support one and the same string theory. But as we shall see in the final section this is not the case.

Simple counting reveals that this gravity system possesses one local degree of freedom. Classically, this means that there are numerous time dependent matter free solutions. This poses an essential difficulty in studying the dynamics and the question of stability becomes quite nontrivial. In particular, one needs to solve the full partial differential equations to study the process of gravitational collapses. For a system devoid of a local degree of freedom, such as the S-wave sector of Einstein-Maxwell theory or the dilaton gravity of [1], the solutions are locally static wherever the matter is absent and for thin shells of collapsing matter we can simply glue different static geometries across the history of the shell [9]. This statement is usually referred to as Birkhoff's theorem in general relativity. But given a local degree of freedom, the geometry will be fluctuating even after the matter part dies out and can be found only by explicitly integrating the nonlinear dynamical equations. For this reason we will investigate the gravitational collapse in a piecewise manner and try to come up with a physically reasonable scenario in sections 3 and 4.

5.3 Dynamics of a Collapsing Massless Shell

Discussions above raise an important question. Does the gravitational collapse of charged matter leave behind a nonsingular spacetime similar to the charged static solutions? A related question is whether the inflow of charged matter to a Witten's black hole will lift the curvature singularity just as the inflow of neutral matter in the linear dilaton vacuum creates a singularity. An even more immediate problem is whether the nonsingular nature of the charged spacetime is stable against extra (charged or neutral) matter inflow. A gravitational collapse can be divided into the following three stages.

(1) The immediate response of the geometry to the matter.

(2) The residual gravitational fluctuation for $(t + \int F^{-1}dy) < \infty$.

(3) The residual gravitational fluctuation as $(t + \int F^{-1}dy) \to \infty$.

While the general treatment of even small gravitational perturbation is a fairly complex problem, the physics of (3) is relatively well understood and is the subject of the next section. Such an asymptotic tail of the gravitational fluctuation is known to be responsible for the instability of the Cauchy horizon in the case of charged black holes of the Einstein gravity.

One needs to analyze the effect of (2) to investigate the stability of the event horizons, along the lines of Chandrasekar's analysis of the Kerr black holes[12]. But knowing that the 2-parameter family of solution (83) is the only finite mass static solution of the theory and that the ADM mass and the charge are conserved quantities, it is difficult to imagine a possible cause of instability that could change the structure outside the event horizons. Our static spacetime, however, has another source of instability under (2), namely the infinite effective coupling at the critical lines, which are the inner horizons of the 3-D metric. We will not pursue the matter here other than what is needed in this section concerning certain gravitational shock waves. The difficulty lies in reducing the linearized equations (3 dynamical and 2 constraint) to a single unconstrained dynamical equation in the region of interest. In other words we were not able to solve the constraint equations.

For part (1), consider a thin shell of conformal matter T^{matter} collapsing in an initially static spacetime specified by the two numbers (m_s, q_s). (It is important to assume that the spacetime is static initially in that we will need the explicit solution before the collapse.) In the limit of infinitesimal thickness, that is for a shock wave, the matter will induce discontinuous changes in normal[17] derivatives of various fields across the shock. Furthermore one can obtain a set of first order differential equations obeyed by these jumps from the set of the nonlinear field equations of section 2. All one has to do is to take the discontinuities of the homogeneous (source free) equations and to integrate the inhomogeneous equations across the shock.

Since we are dealing with massless matter, it is convenient to introduce a light-cone coordinate in the conformal gauge.

$$g^{(2)} = -e^{2\rho}\, dx^+ dx^- \tag{86}$$

Then $R^{(2)} = 8e^{-2\rho}\partial_+\partial_-\rho$ and $\nabla^2 = -4e^{-2\rho}\partial_+\partial_-$. Let $x^- = -\infty$ on past null infinity, to be concrete. Let the shock be centered at $x^+ = x_o^+$. The only nonvanishing component of the matter energy momentum tensor is T_{++}^{matter}, which has an infinitesimally thin support. Then we can assume that χ and f are continuous across the shock while the derivatives along x^+ may not be. Furthermore we assume the continuity of ρ as well, the consistency of which needs to be checked. First of all the effect of T_{++}^{matter} is felt by the dilaton χ through the δg^{++} equation of (82). If we define σ as the total flux, we have the equation

$$\sigma(x^-) \equiv \lim_{\delta\to 0}\int_{x_o^+-\delta}^{x_o^++\delta} T_{++}^{matter} dx^+ = \lim \int -\nabla_+\nabla_+ e^{-2\chi} + \cdots = -[\partial_+ e^{-2\chi}]. \tag{87}$$

Here, we introduced the notation [A] for the difference of A across the shock. Next, the evolution of σ is governed by δg^{+-} equation, whose difference across the shock is

$$\partial_-\sigma = -\partial_-[\partial_+ e^{-2\chi}] = -e^{4\phi_o+2\rho_o-2\chi_o}\frac{[Q^2]}{2}. \tag{88}$$

Here χ_o denotes χ restricted to $x^+ = x_o^+$ and similar for the other fields. Note that we already know values of these $\chi_o, \cdots$. They are simply the corresponding static

[17]Unlike the case of timelike shell, *normal* here does not mean orthogonal. Using a null coordinate pair (x^+, x^-), the normal derivative is ∂_+ for a shock propagating along a fixed value of x^+.

values owing to the continuity. Naturally all the quantities above are functions of x^- only. On the other hand the energy momentum conservation of T^{matter} implies

$$\partial_- \sigma = -[T^{matter}_{+-}] = 0. \tag{89}$$

Obviously the procedure is inconsistent for a charged shell ($[Q^2] \neq 0$). The assumption of continuous ρ, needed when we derived (86) ignoring the connection piece $\Gamma^+_{++}\partial_+ e^{-2\chi}$, is apparently too strong. A priori, there is no reason that ρ, a coordinate dependent function, should be continuous. However for the case of *massless and neutral* shell ($[Q^2] = 0$), we shall find no further inconsistency below. For this reason we will consider neutral conformal matter only, so that σ is constant.

We can also take the discontinuity of the equations (81) and (80) and the result can be written in the form (after substituting σ for $-[\partial_+ e^{-2\chi}]$)

$$\partial_-[\partial_+ f] - \partial_-\chi_o[\partial_+ f] - \frac{e^{2\chi_o}}{2}\sigma\partial_- f_o \;=\; 0, \tag{90}$$

$$\frac{e^{-2\chi_o+2\rho_o}}{8}[R^{(2)}] - \sigma\partial_-\chi_o + \frac{e^{-2\chi_o}}{2}\partial_- f_o[\partial_+ f] \;=\; 0. \tag{91}$$

Reduced to the usual dilaton gravity ($f \equiv 0, Q \equiv 0$), with the Kruskal coordinate in which $e^{-2\chi_o} = e^{-2\rho_o} = m/\lambda - \lambda^2 x_o^+ x^-$, these equations have the solution

$$[R^{(2)}] = 4\lambda\sigma x_o^+ e^{2\chi_o}. \tag{92}$$

$e^{2\chi_o} = e^{2\phi_o}$ is unbounded and shows a curvature singularity forming at infinite effective coupling $e^{\chi} = \infty$. It is easy to see the result reproduce what we would find by actually solving the full equations as done in [1]. In this particular case there is no gravitational fluctuation afterwards and the solution outside the shock is completely determined by this jump. All we have to do is to find the right static solution to match. It is worthwhile to notice that, in spite of the singular behaviour, σ is constant and therefore $e^{-2\chi}$ is continuous even at the singularity, a necessary behaviour for the self-consistency.

Now what happens if we start with a charged spacetime? Then we have to solve for $[\partial_+ f]$ first. Again for the sake of consistency we will consider a shock of neutral matter in an initially charged spacetime. Solving (89),

$$e^{-\chi_o}[\partial_+ f] = (C + \frac{\sigma}{2}\int_{-\infty}^{x^-} e^{\chi_o}\partial_- f_o \, dx^-). \tag{93}$$

Since the matter shock is neutral σ is independent of x^-. C is an integration constant. The lower bound of the integral is past null infinity where the shock wave originates.

Consider the effect of C on the jump of the curvature. Near the critical line $e^{-2\chi} = 0$ (or $y = 0$ in the coordinate of (83)), the corresponding jump scales like $\sim C/\sqrt{y^3}$ and is infinite at $y = 0$. This is rather strange in that not only the strength of the curvature singularity but also its sign are completely independent of the matter you are throwing in. (As to be shown later, the contribution from σ is finite at the critical line.) Even more disturbing is the jump of the internal part of the Ricci tensor $R_{\theta\theta} = -e^f \nabla^2 e^{-f}$. Asymptotically it is $\sim e^{-\lambda y}$, whereas the static value is $\sim e^{-2\lambda y}$. This seems to suggest that the gravitational fluctuation with nonzero C would cost an infinite amount of energy. It turns out that the problem is closely related to the asymptotic nature of the gravitational perturbation.

As demonstrated in the appendix, the gravitational perturbation around a static solution is asymptotically described by the following massive Klein-Gordon equation with $\Psi \equiv e^{-\chi_s}(f - f_s)$.

$$\nabla^2\Psi - \lambda^2\Psi = 0. \tag{94}$$

χ_s and f_s are the static values of χ and f. Across any null line $x^+ = x_o^+$, $[\partial_+\Psi] = C$ is allowed by (93). But it is quite deceptive since the behaviour of field as we move away from the null line is profoundly affected by the massive nature of the evolution equation. After all, we do not expect a massive field to propagate along a null line. As shown in the appendix, the bump of Ψ itself owing to the discontinuous derivative tends to diminish rapidly with time and becomes completely undetectible after an infinite amount of time. In other words, if we incorporate such a jump to the initial condition on past null infinity, the field configuration of Ψ in finite region does not show any sign of discontinuous derivatives. (In terms of the homogeneous solution presented in the appendix, this case corresponds to $u_o \to -\infty$ before taking the normal derivative.) On the other hand, if we insist that the field configuration show such a jump away from past null infinity, the required initial condition on past null infinity has divergent Ψ, and the energy content of the initial flux of the associated gravitational radiation is infinite. It is simply impossible to produce nonzero effective C in the finite region with a finite amount of energy.[18] C should be dropped.

While we discussed the problem in the context of a gravitational collapse of massless matter fields, the conclusion must hold in other contexts. After all, C represents a homogeneous solution to the gravitational field equations. We should not think of it as being generated by the matter shock. If $C \neq 0$ were allowed, it would imply the instability of the critical line under gravitational perturbation automatically, for example. In this sense we have eliminated one possible way for the infinite self-coupling to cause instability.

Rewriting (92),

$$[\partial_+ f] = \frac{\sigma}{2} e^{\chi_o} (I(x^-) - I(-\infty)) = \frac{\sigma}{2} e^{\chi_o} I(x^-). \tag{95}$$

I is the indefinite integral of $A(x^-) \equiv e^{\chi_o} \partial_- f_o$ over x^- and invariant under coordinate transformations $x^- \to z^-(x^-)$. One can ask whether the massive nature of the graviton affects this inhomogeneous part of the solution. The answer is no. It turns out that the asymptotic behaviour of the *source* term A is such that a separation of variables occurs. The form of Ψ near the shock is given by a product of two factors each of which is function of one of the null coordinates only. As a result, the estimate (94) is reliable even though we derived (89) without taking into account

[18] See appendix for detailed discussion on the asymptotics of the gravitational perturbation.

the massive nature of the gravitational perturbation. Inserting (94) to (90),

$$[R^{(2)}] = 2\sigma e^{-2\rho_o}(2\partial_- e^{2\chi_o} - e^{\chi_o}\partial_- f_o I). \tag{96}$$

No infinite jump is possible except maybe at the critical point $e^{\chi_o} = \infty$. But even at this point the leading terms of two parts cancel each other and the true leading term is a constant. For an initially nonextremal spacetime, an easy way to see this is to make a coordinate transformation $z^- \equiv y(x_o^+, x^-)$ where y is the coordinate used in (83) and recall that $e^{-2\chi_o} = qy + O(y^3)$, $e^{-f_o} = y + O(y^3)$.

$$-\infty < R^{(2)} < \infty \qquad \text{near} \quad x^+ = x_o^+. \tag{97}$$

The conclusion holds for all initially charged cases including the extremal case. Therefore the history of a collapsing shell of neutral conformal matter is qualitatively similar to that of a massless observer. It just bounces into the next universe. It is quite a surprising behaviour when compared to what happens in the usual dilaton gravity [1].

To conclude this section consider the case of an initially neutral spacetime. Will the singularity be lifted by the inflow of some charged matter? Can we open up a curvature singularity by a completely classical process? As we have seen earlier, we should be more careful in dealing with charged matter. In principle, one may proceed by considering two conformal factors inside and outside the shock, each of which is smooth but assumes values different from the other along the shock. However, we will postpone this problem of the gravitational collapse of charged matter in a neutral spacetime to a future project for the following reason. A neutral static spacetime has $e^{-2f} \equiv 1$, while the charged case has $e^{-2f} = \tanh^2 \lambda y$ ranging from 1 to 0. In the framework we are using, e^{-2f} is a scalar and should be continuous across the shock. Then the resulting immediate response of the initially neutral solution to the collapsing shell of charged matter must have constant e^{-2f} and hardly resembles a static charged spacetime. Even though it might be that the solution settles down to a nonsingular spacetime with varying e^{-2f} after a while, we can not detect that with this type of local analysis. At the singularity the scalar field can actually jump discontinuously to the desired value, but such a behaviour could imply an inconsistency. The same argument applies to the original 3-D dilaton field $e^{-2\phi}$. In this sense the shock wave analysis seems less promising for initially neutral cases. Even if we carried out similar calculations for the case of a charged shell, we would be unable to decide whether the spacelike singularity was lifted. Probably we need to go beyond the immediate neighborhood of the shock to probe the geometry outside the shock.

A different way of tackling this problem is to try to establish the stability of the critical lines, which would certainly imply *time dependent* charged solutions without a spacelike singularity. In this regard, the results above on the neutral matter shock in an initially charged spacetime are quite encouraging. We found that, as a consequence of the full nonlinear dynamics, no singularity met the shock. If the critical lines are unstable, it is difficult to imagine where the resulting spacelike singularity along the critical line might start. It has to start somewhere away from the shock.

5.4 Instability of the Cauchy Horizon

In section 3, we found that a collapsing shell in an initially charged static spacetime does not encounter a curvature singularity and bounces into the next universe. But, any gravitational collapse is followed by residual gravitational fluctuations and so far we have not addressed the effect of them on the subsequent evolution of the geometry. In principle these fluctuations could grow indefinitely to produce curvature singularities.

The most obvious place to look for the singularity is the Cauchy horizons, which are generically unstable because of the infinite blue shift of the asymptotic residual fluctuation entering the future event horizon at arbitrary late times. However, unlike the more familiar case of Reissner-Nordström black holes the Cauchy horizons of our geometry coincide with the outer horizons rather than the inner horizons. The inner horizons of the nonextremal geometry, also called the critical lines, are effectively spacelike for any radial observer and cannot be the Cauchy horizons. While the behaviour of the critical lines under the gravitational perturbation (as opposed to the matter inflow) still needs to be studied, we will first concentrate on the more obvious potential source of instability.

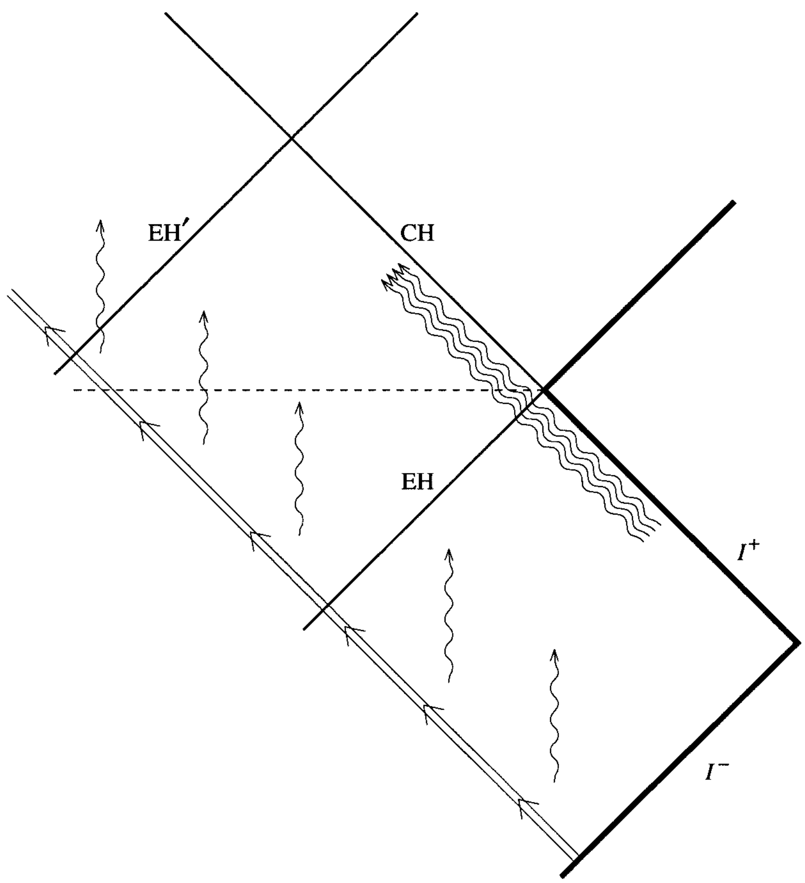

Figure 5.2: A collapsing shell of neutral conformal matter in a nonextremal spacetime. The shell simply bounces into the *next universe.* The wiggly arrows indicate the residual gravitational fluctuations, some of which enter the future event horizon (EH) at arbitrarily late times and propagate parallel to the Cauchy horizon (CH). Again, the dotted line is the critical line of infinite effective coupling.

A convenient way to analyze the instabilty of the Cauchy horizon is to consider ingoing null flux of energy stretched all the way to the infinite future [13]. Assuming an initially static geometry, choose a new set of coordinates (v, u).

$$g^{(2)} = -F(y)\, dt^2 + \frac{1}{F(y)} dy^2 = -F(y)\, dv du, \tag{98}$$

$$dv = dt + dy/F, \quad du = dt - dy/F.$$

With this coordinate system the Cauchy horizon of our universe is at $v = +\infty$ inside the future event horizon. In figure 5.2, we labeled the Cauchy horizon by CH and the future event horizon by EH. Outside EH, $v = +\infty$ corresponds to future null infinity I^+ and we are interested in asymptotic inflow of the radiation stretched all the way to I^+.

For simplicity, suppose that the energy inflow is due to matter rather than gravitational radiation.[19] This simplification is well justified by the universal nature of the instability. By the energy-momentum conservation, the energy-momentum tensor of a massless pure inflow is of the form

$$T^{matter} = \mu(v)\, dv dv. \tag{99}$$

As $v \to \infty$ (future null infinity if outside the event horizon) a typical $\mu(v)$ shows an inverse powerlike behaviour [14] with finite total energy and has little effect on the geometry outside the event horizon. But inside the event horizon, $v = \infty$ corresponds to CH at finite physical distance and the energy density measured by a freely falling observer could be infinite at CH. To be definite let us choose Kruskal type coordinates (V, U), to be defined later, good for both CH and EH. In addition, let $V = 0$ at CH,

$$\begin{aligned} g^{(2)} &= -e^{2\rho}\, dV dU \qquad (V = V(v), U = U(u)), \\ T^{matter} &= \mu(v)\, dv dv = \mu(v)\, (dv/dV)^2\, dV dV \equiv T_{VV}\, dV dV. \end{aligned} \tag{100}$$

[19]We are indebted to E. Poisson for suggesting this approach.

Notice that once the flux crosses future event horizon (EH), it looks more like a thin shell of massless matter, even though $\mu(v)$ has an infinite span outside. The energy density measured by a timelike observer crossing CH is roughly $\sim T_{VV}$ which diverges unless the fall-off of $\mu(v)$ is fast enough to overcome the infinite blueshift factor $(dv/dV)^2 \to \infty$.

A better estimate can be made using the machinery of the previous section. Since only the large v behaviour is important, let μ be nonzero only for $v > v_{min}$ with sufficiently big v_{min} , so that near CH the flux is that of an arbitrarily thin shell. Then certainly the formalism of section 3 is applicable inside EH. Using the definition (86), after identifying the corresponding null coordinates $x^+ \equiv V$ and $x^- \equiv U$,

$$\sigma = \int_{V(v_{min})}^{0} T_{VV}\, dV = \int_{v_{min}}^{\infty} \mu(v)\,(dv/dV)\, dv. \tag{101}$$

As long as σ is finite, we can estimate the jump of the curvature on CH induced by the neutral inflow using (90). For initially static spacetimes,

$$[R^{(2)}]_{V=0} = 0.$$

The reason is simply that the dilatons χ_s and f_s are uniform along CH and $\partial_U \chi_o = \partial_U f_o = 0$. This is a somewhat fictitious situation since we have assumed that before the on-set of the asymptotic inflow at $v = v_{min}$ the geometry is strictly static. For a realistic model, the asymptotic inflow originates from the collapsing matter and the geometry is non-static even for $v < v_{min}$. In particular CH does not necessarily coincide with an apparent horizon, the dilatons χ, f are no longer uniform along CH, and as a result the curvature jump is generically nonzero and finite, proportional to σ,

$$[R^{(2)}]_{V=0} \sim \sigma \quad \text{up to some finite factor as a function of } U.$$

A similar mechanism must work for infinite σ, even though the formalism of section 3 does not apply in this case. After taking into account other residual fluctuations, an infinite σ would most probably generate a null singularity along CH. The role of the extra radiation flowing into the Cauchy horizon rather than parallel to it has been first studied in [13] and found crucial for the instability to manifest itself.

For the particular case of the compactified charged black strings we are studying, we can choose the following Kruskal coordinates

$$\begin{aligned} V_{nonextremal} &= -\frac{1}{\beta}e^{-\beta v} \quad \beta = \lambda\sqrt{\frac{m-q}{m}}, \\ V_{extremal} &= -\frac{\lambda^2}{v}. \end{aligned} \tag{102}$$

(Obviously, this choice is not unique, but the same conclusion holds for any other coordinate regular on CH.) For any powerlike $\mu(v)$, $\sigma_{nonextremal}$ is infinite, and the Cauchy horizon is inherently unstable. Unless the asymptotic residual fluctuation turns out to be exponentially small ($\sim e^{-\lambda v}$ for example), a null curvature singularity will form along CH. But provided that $\mu(v) \sim v^{-n}$ with $n > 3$, $\sigma_{extremal}$ could be finite. Whether a finite $\sigma_{extremal}$ indicates a stable Cauchy horizon of an initially extremal solution is unclear. The extremal geometry can always be changed to a nonextremal one by an infinitesimal increase of the mass, and once that happens we must use the nonextremal estimate of σ.

(However, this classical picture might undergo a drastic modification once we include the semiclassical effect of Hawking radiation. Since the temperature is positive except for the extremal spacetime, the geometry outside the event horizon must approach the extremal geometry in the infinite future ($v \to \infty$). If the same is true for the geometry inside the event horizon, we must use the extremal estimate of σ and the Cauchy horizon might as well be stable for any initially charged geometry.)

What are the consequences of the null singularity along CH? Note that the Cauchy horizon of our universe (CH) meets the future event horizon of the next universe. But all the infalling observers escape to the next universe through its past horizon (EH'). That is, none of the observers from our universe, entering EH before the on-set of the asymptotic inflow , cross CH between EH and EH'. The singularity on CH hardly interferes with the observers entering the next universe. While some massive observers without sufficient momenta would be attracted to the singularity, the rest will probably continue to the asymptotic region away from it. This should be compared to what happens in case of 4-D Reissner-Nordström (nonextremal) black holes. For these 4-D black holes the phenomenon is well known as *mass inflation* [13]. The static geometries of nonextremal Reissner-Nordström black holes allow timelike observers (entering the future event horizon) to escape to the next universe, the static singularity being timelike. Once we include generic perturbations, however, the Cauchy horizon becomes singular all the way to the point where it meets the original timelike singularity and even the timelike observers eventually experience infinite tidal force.[20]

Of course, the past event horizon of the next universe (EH') overlaps the Cauchy horizon of another universe which is spacelikely separated from our universe. An asymptotic inflow from that universe can destabilize EH'. But if we suppose the charged spacetime is made by a gravitational collapse in our universe, the extended static structure before the collapse is fictitious just as the past event horizon of the Schwarzschild geometry is fictitious for a collapsing star. Then EH', the past event horizon of the next universe, would be as stable as EH.

5.5 Discussion: Nonsingular Exact String Backgrounds

The most curious feature of the charged black string is probably the way the inner horizons conceal the curvature singularities from the radial observers. The resulting 2-D geometry is geodesically complete and the scalar curvature is finite everywhere.

$$-\lambda^2\frac{2m}{q} \le R^{(2)} \le \lambda^2\frac{2m}{3q}. \tag{103}$$

In particular, as $q \to m$, the absolute value of the curvature is bounded by $2\lambda^2$. On the other hand, for Witten's black hole seen as the dimensional reduction of

[20]However, it has been shown the total impulse from the tidal force is finite and the would-be Cauchy horizon is traversable [15].

the 4-D dilatonic black hole, λ^{-1} corresponds to the total magnetic charge[2], or the radius of the long necked extremal solution called the cornucopion [16]. If we take this parameter λ much less than the Planck mass, then, as the black hole evaporation progresses $m \to q$, the geometry is of macroscopic scale everywhere including inside the event horizons. Furthermore the string coupling e^{ϕ} is bounded by $\sqrt{\lambda/q}$. It is possible that the quantum fluctuation of geometry does not change the causal structure qualitatively. Taken seriously, this last statement has a profound ramification in the black hole physics: Whatever the real answer is to the information puzzle[21][18], it has little to do with the details of the Planck scale physics. Either the information is completely recovered before the collapsing matter enters the future event horizon, or some will be lost forever to the next universe.

Any remnant scenario is likely to assume some nonsingular objects as the final products of the evaporation process. They are assumed to be able to carry macroscopic amounts of the information yet interact very little with the surroundings. All these assumptions may or may not be realized depending on the details of the Planck scale physics. Even in the case of the cornucopions we still need to explain how the information inside the horizon, far down at the tip of the long neck, propagates out to occupy the whole length of the cornucopions, which could happen only near the end of the process where the singular structure inside the horizon must be resolved somehow. The static extremal solution above could obviate all these details.

But we are hardly in a position to take these static solutions seriously. The Reissner-Nordström nonextremal solution, which is also known to allow some of the infalling observers (massive ones) to escape to the next universe, is dynamically unstable and the resulting generic geometry has a curvature singularity blocking the passage though perhaps not completely [13][15]. In the present case of the compactified black strings, while the instability of the Cauchy horizon due to the asymptotic inflow from our universe does not block the passage, there is a possibility that the critical lines of the infinite effective coupling become singular under the

[21]For a recent review see the reference [17].

generic gravitational perturbation. As mentioned at the end of section 3, the fact that a collapsing shell does not encounter a singularity even at the critical line seems to suggest no singularity whatsoever along the entire span of the critical line, but this might be an artifact of the initially static geometry.

Another difficulty concerns the creation of the charged spacetimes from vacua. If the linear dilaton background is the true vacuum, it is very difficult to imagine how the extended structures of the charged spacetime could be made from the gravitational collapse of thin shells without any singularity at the infinite effective coupling. More specifically, it is rather difficult to draw a sensible Penrose diagram describing the process. Even a qualitative understanding in this regard would be an important step.

At this point one can easily see that all the unanswered questions are in one way or another associated with the true nature of the critical lines (the inner horizons as seen by 3-D observers), which become curvature singularities under the duality transformation (see section 2). The ultimate question is then, *between the two geometrical pictures dual to each other, namely the Witten's causal structure and our nonsingular version, which one of them resembles the reality more closely* [11][8].

Of course the effective field theory (75) we started with is correct only up to $O(1)$ in the α' expansion of the sigma-model [5]. That is, the extended nature of the fundamental string is not properly taken into account. In particular, the static solutions we have found are related to the coset models $(SL(2,R)_k \otimes U(1))/U(1)$ for $m \geq q > 0$, or $(SL(2,R)_k/U(1)) \otimes S^1$ for $q = 0$, with $(k-2) = (2\alpha'\lambda^2)^{-1}$. k is larger than 2 provided that the total dimension of the spacetime, compact or not, is less than the critical dimension. A few months after Witten's derivation of the 2-D black hole geometry from $SL(2,R)_k/U(1)$ coset model, R. Dijkgraaf et al., attempted to find the exact geometry of the coset model by investigating the tachyon spectrum of the theory [6] with the assumption that the string on-shell condition $L_0 = 1$ should be interpreted as the tachyon field equation $\nabla^2 T = 0$. Recently, A.A. Tseytlin rediscovered the same metric in a functional integral approach where he replaced the classical effective action of the coset WZW model by a quantum effective action, which involved modifying the coefficients of the action from k to $k + c_G/2$ [7]. The change of metric can be easily written in the following form, in a gauge similar to those used in [19][6][7].

$$g^{(2)}_{Witten} = \frac{1}{\lambda^2}dx^2 - \tanh^2 x\, dt^2 \quad \rightarrow \quad g^{(2)}_{exact} = \frac{1}{\lambda^2}dx^2 - \frac{(1-2/k)\tanh^2 x}{1-(2/k)\tanh^2 x}dt^2 \quad (104)$$

In the gauge of (76), this can be rewritten

$$g^{(2)}_{exact} = -(1-\frac{m}{r})\, dt^2 + \frac{1}{4\lambda^2(r-m)(r-(2/k)m)}dr^2. \quad (105)$$

Amazingly, this looks exactly like a nonextremal metric, (83) with $q/m \rightarrow 1 > 2/k > 0$. The $SL(2,R)_k/U(1)$ coset metric has the causal structure of the Penrose diagram labeled by $m > q > 0$ instead of the singular one ($m > q = 0$) in figure 5.1. Furthermore the dilaton part is, according to Tseytlin's estimate,

$$e^{-2\chi_{exact}} = \frac{r}{\lambda}\sqrt{1 - \frac{(2/k)m}{r}}. \quad (106)$$

Again, it is identical to that of the nonextremal solution (83) with $q/m = 2/k$. The implication is very remarkable. The reality seems to be better described, at least classically, by the nonsingular causal structures we have found than by Witten's.

Another encouraging evidence for these nonsingular solutions comes from the duality symmetry of the string theory. The $(SL(2,R)_k \otimes U(1))/U(1)$ coset model has been considered by I. Bars and K. Sfetsos [20],[22] also nonperturbatively. The resulting metric is singular, unfortunately, and the causal structure is again that of timelike singularities hidden behind inner and outer horizons. However, if we consider the 2-D part of the metric as we did in section 2,

$$g^{(2)}_{BS} = -(1-\frac{m}{r})\, dt^2 + \frac{1}{4\lambda^2(r-m)(r-q-(2/k)(m-q))}dr^2, \quad (107)$$

[22]We thank G. Horowitz for drawing our attention to this work.

after an appropriate redefinition of the parameters in [20]. This metric is identical to the $O(1)$ approximation (83) except that $q \geq 0$ is replaced by $q + (2/k)(m-q) > 0$. The case of $q = 0$ coincides with $g^{(2)}_{exact}$ as it should, and the extremal limit is achieved by letting $q \rightarrow m$. (We need to perform a coordinate rescaling on the metric presented in [20] to achieve this limit.) Now since the $O(1)$ approximations to $g^{(2)}_{exact}$ and $g^{(2)}_{BS}$ are related to each other by a duality transformation, it is not unreasonable to expect these nonperturbative versions are dual to each other.

Unlike the $O(1)$ approximations, the 2-D part $g^{(2)}_{BS}$ of the nonperturbative black string metric is qualitatively self-dual. The causal structure of the 2-D part does not change upon $g_{\theta\theta} \rightarrow (1/g_{\theta\theta}), \cdots$, with the exception of the extremal case. We no longer have to choose between two entirely different causal structures dual to each other. It is true that there are many examples of different geometries (or even different topologies) supporting the same string theory, and a fundamental string must have a unique classical history independent of the particular geometry chosen. But here, instead of the horrendous task of figuring out the actual behaviour of test strings inside the event horizon, we are presented with a unique nonsingular causal structure invariant under the duality transformation. The naive expectations based on the behaviour of test particles are unambiguous and might as well indicate the actual behaviour of test strings. A radially moving classical fundamental string propagating into the future event horizon does not see a curvature singularity and simply propagates to another universe.

As for the exceptional case of the extremal solution, we can easily see that it is dual to the linear dilaton background. The relationship between masses of a pair of dual solutions is given by (according to the $O(1)$ estimate)

$$m_{\text{after}} = m_{\text{before}} \cosh^2 \alpha, \tag{108}$$

where α is the Lorentz boost parameter; α is infinite for the maximal boost [8]. Obviously a finite mass extremal solution is obtained by the duality transformation on a maximally boosted zero mass limit of $SL(2,R)_k/U(1) \otimes S^1$, the linear dilaton background up to a trivial internal part. The positive mass of the extremal solution must be an artifact of the effective field theory estimate. While this ambiguity prevents us from viewing either of them as the true ground state, it is quite interesting in that we have an alternative description for the end stage of the black holes, which has an event horizon at finite physical distance and another universe beyond it.

Of course we overlooked two important facts in the arguments above. First of all, the effective coupling is again unbounded.

$$e^{-2\phi_{BS}} = \frac{r}{\lambda}\sqrt{1 - \frac{(2/k)(m-q)}{r}},$$
$$e^{-2\chi_{BS}} = \frac{r}{\lambda}\sqrt{1 - \frac{q + (2/k)(m-q)}{r}},$$

which we deduced from [20]. The effective coupling e^{χ} is infinite at $r = q + (2/k)(m-q)$. In section 3, we have seen this infinity was not strong enough to create a curvature singularity out of a collapsing shell of matter. But we have not been able to address the problem of gravitational perturbation completely, let alone quantum fluctuations. Secondly, the 3-D metrics of the compactified charged black strings ($m > q > 0$), perturbative or nonperturbative, possess not only timelike singularities but also closed timelike worldlines inside inner horizons. While it is reassuring that the corresponding regions are completely inaccessible to 2-D observers and that they are causally disconnected from us by event horizons, it is quite unclear whether such a background is physically acceptable. It might be that we should be content with the effective field theory (79) rather than try to interpret the 2-D solution as a part of a classical string background.

Appendix: Asymptotics of the Effective Theory

To study the asymptotic behaviour of the gravitational perturbation, it is convenient to start with the ADM mass formula, obtained using the canonical formalism [23] on (79). Δ denotes the deviation from the linear dilaton vacuum.

$$M_{ADM} = 2\,N e^{-2\chi}\Delta(\frac{\chi'}{\sqrt{g}})\Big|_{x\to\infty},$$
$$g^{(2)} = N^2 dt^2 + g\,(dx + L\,dt)^2 \qquad N, g \to 1;\ L \to 0 \qquad \text{as} \quad x \to \infty,$$
$$\chi_{vacuum} = -\lambda x \qquad f_{vacuum} = 0. \tag{109}$$

Readers should be warned that (108) is derived with certain assumption on the allowed phase space [23]. More explicitly we need to restrict the phase space to $\Delta\chi_s \sim \Delta\sqrt{g_s} \sim \Delta f_s \sim e^{-2\lambda x}$ at most, which is sensible for the static solutions. Applied to (83), we obtain $M_{ADM} = m$.

Now consider a time-dependent solution. The requirement of finite mass restricts the possible asymptotic behaviours. Obviously the same asymptotic restrictions apply to $\delta\chi$, $\delta\rho$, the corresponding deviations from a static solution. To be definite, write a time dependent solution in the following form.

$$\begin{aligned} g^{(2)} &= e^{2\delta\rho}(-F(y)\,dt^2 + \frac{1}{F(y)}dy^2) \\ \chi &= \chi_s + \delta\chi, \qquad f = f_s + \delta f, \end{aligned} \tag{110}$$

$$(\delta\chi' + \lambda\delta\rho) \sim e^{-2\lambda y}, \qquad \text{at most.}$$

Notice that δf could be much larger than the static part $f_s \sim e^{-2\lambda y}$. Expanding the constraint equations $\delta S^{(2)}/\delta g^{01} = 0$ and $\delta S^{(2)}/\delta g^{00} = 0$ in $e^{-2\lambda y}$, and collecting the leading terms,

$$\partial_t(\delta\chi' + \lambda\delta\rho) = \frac{1}{2}\delta(f'\dot{f}), \tag{111}$$
$$(\delta\chi' + \lambda\delta\rho)' + 2\lambda(\delta\chi' + \lambda\delta\rho) = \frac{1}{4}\delta(f'f' + \dot{f}\dot{f}). \tag{112}$$

If $\delta f \sim e^{-\lambda y}$ then $(\delta\chi' + \lambda\delta\rho) \sim e^{-2\lambda y}$. If δf is exponentially smaller than $e^{-\lambda y}$, the leading part of $(\delta\chi' + \lambda\delta\rho) \sim e^{-2\lambda y}$ is static and the subleading time-dependent part of it is again exponentially smaller that $\delta\dot{f}$. Finally, the possibility of δf much larger than $\sim e^{-\lambda y}$ is excluded since it, combined with (110), will lead to infinite

ADM mass. After redefining the unperturbed fields to include the static parts of $\delta\chi, \cdots,$

$$(\delta\chi' + \lambda\delta\rho) \sim e^{-\lambda y}\delta f \qquad \text{at most.} \tag{113}$$

Therefore it is quite reasonable to expect $\delta\chi\,\delta f \ll \delta f$ as far as the time-dependent perturbations are concerned. The field equation for f can be expanded in the same fashion and up to the leading order δf is easily shown to be decoupled from the rest.

$$\nabla^2(\delta f) - 2\nabla\chi_s\nabla(\delta f) = 0 \tag{114}$$

After a field redefinition $\Psi \equiv e^{-\chi_s}\delta f$, this becomes,

$$\nabla^2\Psi - \lambda^2\Psi = 0. \tag{115}$$

The gravitational perturbation is asymptotically described by a massive Klein-Gordon equation.

Now consider solutions to this equation with the following characteristic data, in a flat[23] light-cone coordinate, $g^{(2)} = -dvdu$.

$$\begin{aligned} \Psi\Big|_{v=v_o} &= 0 \\ \Psi\Big|_{u=u_o} &= \psi \equiv (0 \quad \text{for} \quad v < v_o, \qquad C(v - v_o) \quad \text{for} \quad v_o < v < v_n). \end{aligned} \tag{116}$$

Since the geometry is flat, we can integrate the equation explicitly using the following recursive solution

$$\Psi(v, u) = -\frac{\lambda^2}{4}\int_{u_o}^{u}\int_{v_o}^{v}\Psi(v', u')\,dv'du' + \psi(v). \tag{117}$$

This uniquely determines Ψ up to $v < v_n$ in terms of a Bessel function.

$$\begin{aligned} \partial_v\Psi &= 0 \qquad v < v_o \\ \partial_v\Psi &= C\,J_0(\lambda\sqrt{(u - u_o)(v - v_o)}) \qquad v_o < v < v_n \end{aligned} \tag{118}$$

[23]The asymptotic geometry is flat within the same approximation

Notice $[\partial_v \Psi]_{v_o} = C$. However the massive nature of the field becomes important as we move away from v_o toward v_n. In the future direction $(u > u_o)$, Ψ is attenuated and spread in a powerlike manner and as $u \to \infty$ it would take an infinite resolution to see the ever so tiny bump of Ψ owing to the discontinuous derivative. Because of this tendency to spread the initial signal, if we had started with a smooth version of the jump C at $u = u_o$, it would have disappeared without trace for $\lambda\,(u - u_o) \gg 1$. The more striking feature is the behaviour in the past $(u < u_o)$. Then the square root is pure imaginary and

$$\Psi \sim e^{\lambda\sqrt{(u_o-u)(v-v_o)}} \tag{119}$$

up to some powerlike prefactor. On the past null infinity $u = -\infty$, Ψ is infinite in the interval (v_o, v_n).

In the asymptotically far region, the f field effectively decouples and the action (79) can be considered as the usual dilaton gravity coupled nonlinearly to matter fields f and K. Then the corresponding Einstein equation of the metric can be written with energy-momentum tensor of these matter fields on the right-hand side. δg^{vv} equation is given by the following. We did not write down the left-hand side explicitly.

$$\frac{\delta S^{grav}}{\delta g^{vv}} = e^{-2\chi}\partial_v f \partial_v f + \cdots, \tag{120}$$
$$S^{grav} \equiv \int \sqrt{-g^{(2)}} e^{-2\chi}(R^{(2)} + 4\,(\nabla\chi)^2 + 4\lambda^2).$$

But the leading contribution to the right-hand side is proportional to the square of (117) which is infinite on past null infinity in the finite interval (v_o, v_n). Physically this means an infinite amount of energy is sent in from null past infinity and subsequently the Bondi mass must be also infinite in the same interval.[24] (The reason the right-hand side of (119) is finite for $u > -\infty$ in spite of the initially infinite value can be understood again by considering the dispersive effect of the massive dynamical

[24]Unfortunately we were not able to derive the mass formula appropriate for the asymptotic behaviour of $\delta f \sim e^{-\lambda y}$. But if we naively extrapolate (108) to this case and evaluate on the past null infinity, the constraint equations (110,111) imply infinite Bondi mass.

equation. After a sufficient amount of time the initial flux of infinite density and infinite total energy becomes a flux of infinite energy yet of finite density spread all the way to infinite future.)

We can conclude that solutions of the type (117) should be excluded energetically since it involves an infinite amount of energy. Furthermore the same conclusion holds if we drop the assumption of initially static spacetime. Since the perturbation equation is linear asymptotically, we can subtract a smooth part from Ψ to reduce the problem to that of an initially static spacetime (i.e., $\Psi = 0$ before the shock).

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