ANALYTICAL STUDIES OF THE DYNAMIC RESPONSE OF CERTAIN STRUCTURES TO ASSUMED GROUND MOVEMENTS

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INTRODUCTION

This investigation is chiefly concerned with the effects of earthquakes upon various structures. Several of the chapters are quite far afield but are included with the thought that they deal either with earthquakes or with structures, and are, at least to some extent, related to the principal subject of this discussion.

The first chapter deals with the effect of hubs on the deflections of columns and girders. A hub is that part of an intersecting column and girder which is common to both. In computing deflections in frame buildings the effect of the hub may be important, expecially when it is relatively large.

The second chapter deals with the stresses in, and deflections of, a reinforced concrete building wall which is subjected to a horizontal load on its upper edge. This wall contains a large number of windows and rests on an elastic material. The problem is solved rather approximately by assuming it to act like a building bent on a series of elastic supports.

In California, buildings are now designed to be earthquake resistant by assuming them subjected to horizontal forces distributed as the weight of the building and of the order of five or ten per cent of this weight. The proportion constant is called the "Seismic Factor". In chapter three, Throop Hall (California Institute of Technology) is investigated in this fashion for its resistance to an earthquake in the East-West direction.

In chapter four, calculations are made showing the effect of harmonic ground movements of different amplitude, frequency and duration on symmetric structures with one flexible story. The effect of critical damping is briefly considered. There is included a copy of a discussion submitted to the "Proceedings of the American Society of Civil Engineers" written jointly by Paul L. Kartzke and the writer.

Chapter five deals with the rocking of blocks on hard, flat surfaces.

Chapter six contains a discussion of the natural frequencies of vibration of multi-story buildings which are either uniform or have no more than two sections of different stiffness and mass. Forced vibrations of uniform buildings are briefly considered. There is also a calculation of the amount of internal damping in an actual building, computed from test-data. There is included a calculation of natural frequencies of vibration of a uniform cantilever beam in which both bending and shear deflections are important.

The writer desires to express his indebtedness to Professor Martel, not only for advice and other material assistance but for the inspiration and interest without which what follows would not have been written.

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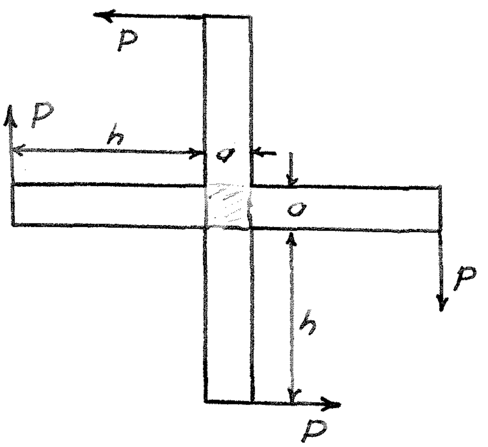
CHAPTER ONE

AN ATTEMPT TO DETERMINE THE EFFECT OF HUBS ON DEFLECTIONS OF COLUMNS AND GIRDERS

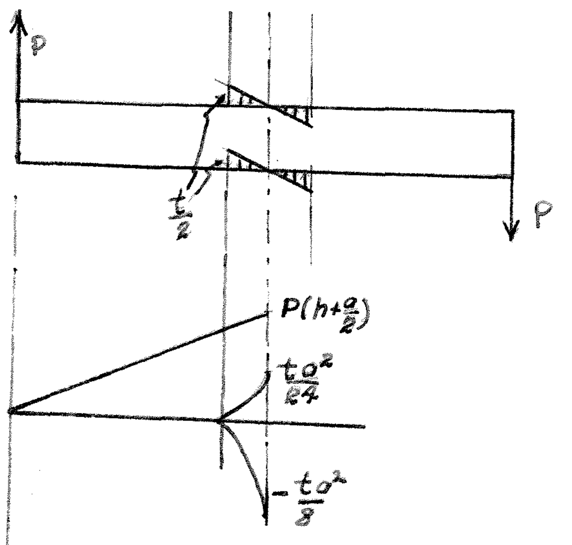


We consider the symmetric structure here shown and compute the deflections of the columns (or girders) under different assumptions.

First assumption: that no deflection occurs within the hub, then (if structure and loads are symmetric): $\delta = \frac{Ph^3}{3EI}$.

Second assumption: that hub is no stiffer than the members. In this case $\delta = \frac{P}{3EI}\left(h+\frac{a}{2}\right)^3$.

Third assumption: that hub is elastic and that vertical (or horizontal) members can be removed and replaced by the forces which they exert on the hub.



Moment Diagram

$$M_0^h = Px$$

$$M_h^{h+\frac{a}{2}} = Px - \frac{t}{2}(x-h)^2 + \frac{t}{3a}(x-h)^3$$

$$t = \frac{12}{a^2}\left(h+\frac{a}{2}\right)P$$

Deflection under P

$$EI\,\delta_p = \frac{P}{3}\left(h+\frac{a}{2}\right)^3 - \frac{ta^2}{8}\cdot\frac{a}{6}\left(h+\frac{3a}{8}\right) + \frac{ta^2}{24}\cdot\frac{a}{8}\left(h+\frac{4a}{16}\right)$$

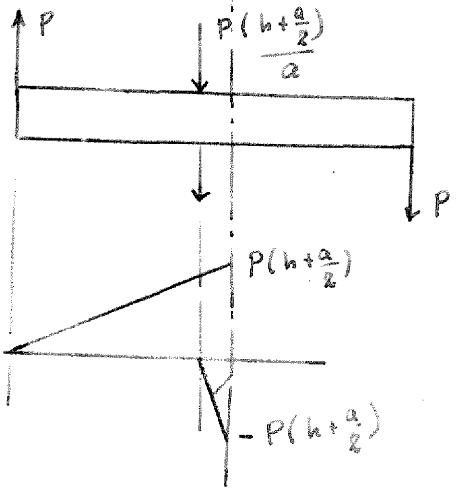
$$EI\,\delta_p = P\left(h+\frac{a}{2}\right)\left(\frac{h^2}{3}+\frac{7}{48}ah+\frac{7}{480}a^2\right)$$

(This has been checked by finding the equation of the elastic curve).

If $a = \kappa h$

$$\frac{EI\,\delta_p}{P} = \frac{h^3}{3}\left(1+\frac{15}{16}\kappa+\frac{21}{80}\kappa^2+\frac{21}{960}\kappa^3\right)$$

Fourth assumption: **the** same as preceding except that the vertical members are replaced by concentrated loads at their faces. This case would be approached by highly reinforced concrete beams or by steel I-beams with flange connections for moment.

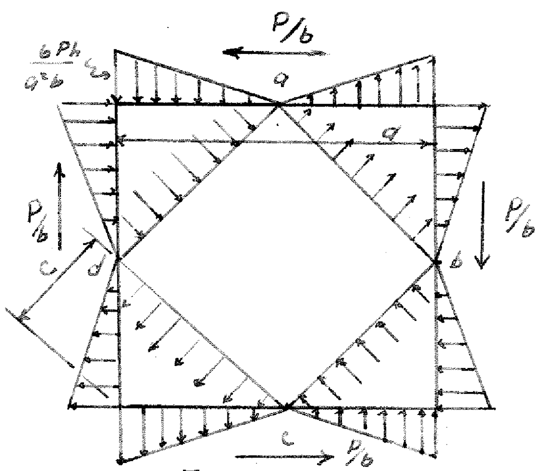


$$EI\,\delta_p = P\left(h+\frac{a}{2}\right)\left[\frac{1}{3}\left(h+\frac{a}{2}\right)^2 - \frac{ah}{4} - \frac{a^2}{12}\right]$$

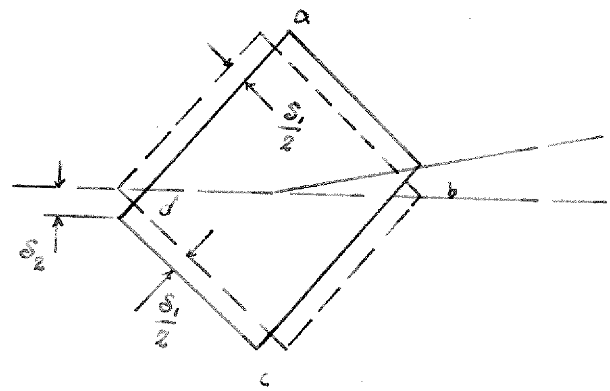
If $a = \kappa h$

$$\frac{EI\,\delta_p}{P} = \frac{h^3}{3}\left(1+\frac{3}{4}\kappa+\frac{1}{8}\kappa^2\right)$$

Fifth assumption: that the actual movement of P is made up of two parts; (1) bending deflection of the cantilever beam whose length equals h; (2) deformation of the hub, (shear deformation of the beam may be considered as well).



Forces on Hub



Uniform

On inner square we assume ~~stresses~~ forces to be ~~pure~~ tension and compression. The total force on one face of inner square

$$= \left(\frac{6Ph}{2a^2b} \cdot \frac{a}{2} - \frac{P}{2b} \right) \sqrt{2}\, b$$

P, h, a as before

b = dimension of hub ⊥ to paper

Unit force

$$= \frac{P}{ab} \left(\frac{3h}{a} - 1 \right)$$

$\delta_1 =$ elongation and compression of sides of inner square.

$$\delta_1 = \frac{1}{E} \frac{P}{ab} \left(3\frac{h}{a} - 1 \right) (1+\mu) \frac{a}{\sqrt{2}}$$

μ = Poissons Ratio

Before deformation each of the four outer triangles has an area of $\frac{a^2}{8}$.

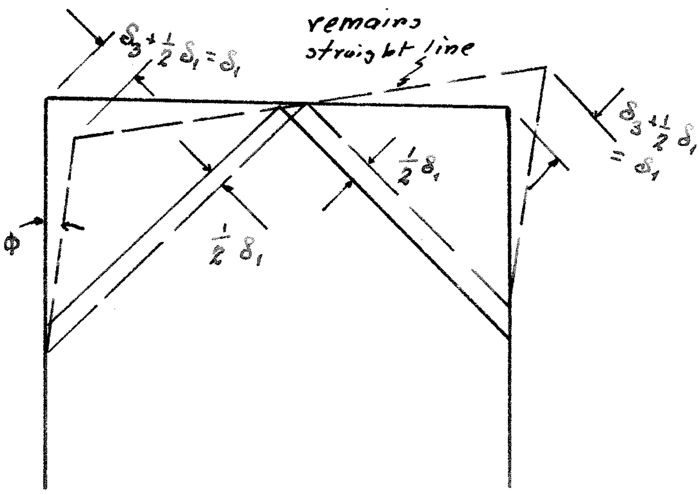
$$c = \frac{a^2}{8} \frac{\sqrt{2}}{a} \cdot 2$$

C = altitude of outer triangles before deformation

Each base is shortened or lengthened by an amount

$$\delta_1 = \frac{\sqrt{2}}{2E} (1+\mu) \frac{P}{b} \left(\frac{3h}{a} - 1 \right)$$

If $\delta_3 =$ lengthening or shortening of c, assuming that the triangles do not change area,



$$\delta_3 = \frac{\delta_1}{2}$$

$$\phi = \frac{\delta_1 \sqrt{2}}{a} = \frac{(1+\mu)}{E}\frac{P}{ab}\left(3\frac{h}{a}-1\right)$$

$\therefore$

$$\delta_P = \frac{Ph^3}{3EI} + h\cdot\phi - \delta_2$$

Since $I = \dfrac{ba^3}{12}$ and $a = kh$

$$\delta_P = \frac{Ph^3}{3EI}\left[1+(1+\mu)\left(\frac{3k}{4}-\frac{k^2}{2}+\frac{k^3}{8}\right)\right]$$

If we consider the shear deflection in length h

$$\delta_4 = \frac{nPh}{GA}$$

$$G = \frac{E}{2(1+\mu)} \qquad A = ab$$

For rectangular sections n = 1.2 (Str. of Mat.) (Timoshenko vol. I)

$$\delta_4 = \frac{nPh^3}{3EI}\,\frac{k^2}{2}(1+\mu)$$

adding δ_4 to δ_P already obtained

$$\delta_P' = \frac{Ph^3}{3EI}\left(1+(1+\mu)\left(\frac{3k}{4}+(n-1)k^2+\frac{k^3}{8}\right)\right)$$

We may now compare the results obtained from these five different assumptions. This is readily done by determining the length of an equivalent ideal cantilever which would have the same deflection.

First case: rigid hub

$$\delta = \frac{Ph^3}{3EI}$$

equivalent length $= h$

additional length $= 0$

Second case: hub no stiffer than members and receives no support from them.

$$\delta = \frac{P}{3EI} h^3 \left(1 + \frac{k}{2}\right)^3$$

equivalent length $= h + \frac{a}{2}$

additional length $= \frac{a}{2}$

Third case: that hub is as elastic as the rest of the structure but that it receives support in the form of triangular loads from all members.

$$\delta = \frac{Ph^3}{3EI}\left(1 + \frac{15}{16}k + \frac{21}{80}k^2 + \frac{21}{960}k^3\right)$$

$$\delta \cong \frac{Ph^3}{3EI}\left(1 + \frac{5}{16}k\right)^3$$

equivalent length $= h\left(1 + \frac{5}{16}k\right)$

Fourth case: same as preceding except that stiffening load instead of triangular is concentrated at outer edges of hub.

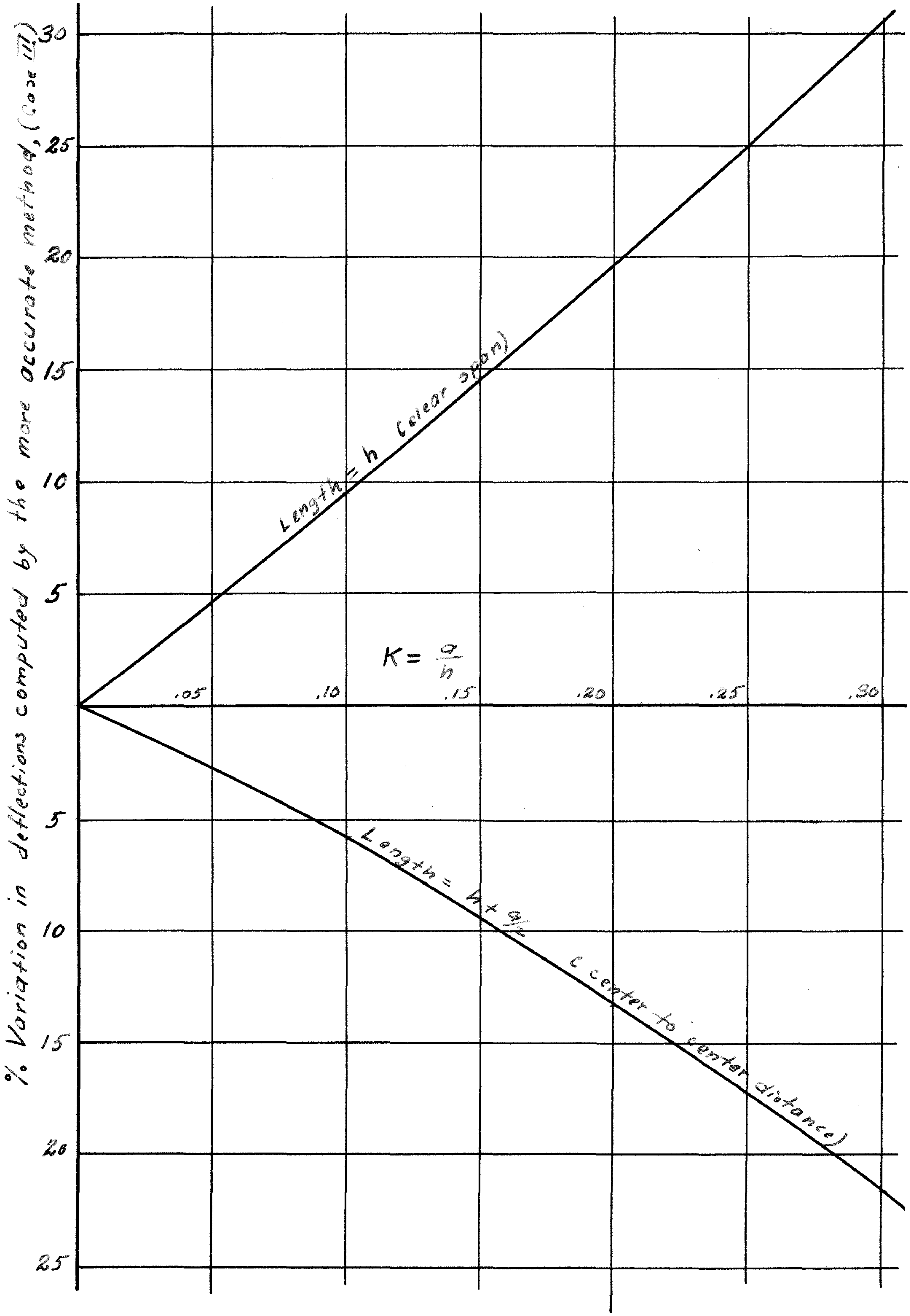
$$\delta = \frac{Ph^3}{3EI}\left(1 + \frac{3k}{4} + \frac{k^2}{8}\right)$$

$$\delta \cong \frac{Ph^3}{3EI}\left(1 + \frac{k}{4}\right)^3$$

equivalent length $= h\left(1 + \frac{k}{4}\right)$

Fifth case: considering deformations in hub resulting in rotations of the hub faces, together with the cantilever bending of the beam outside the hub.

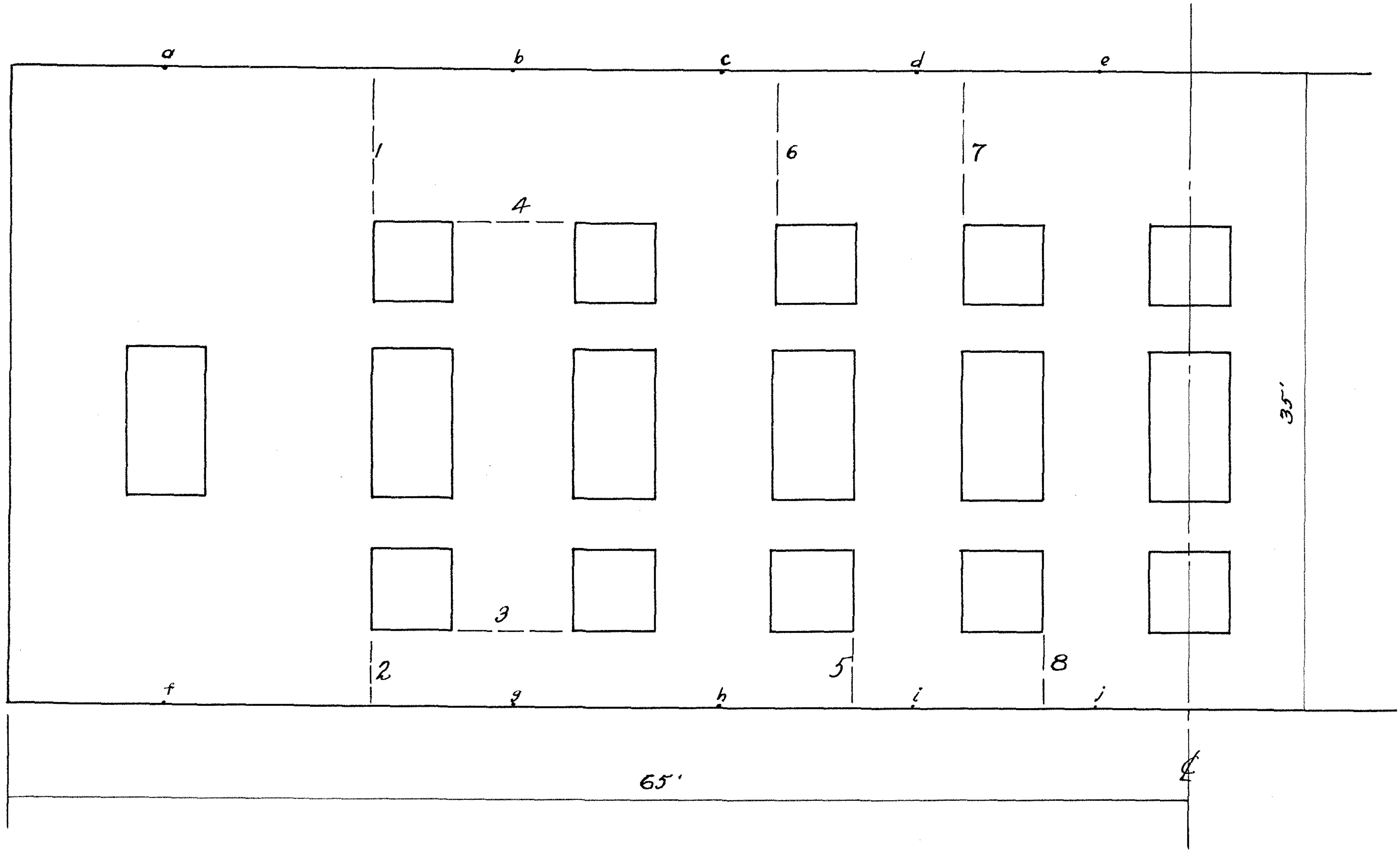
$$\delta = \frac{Ph^3}{3EI}\left(1 + (1+\mu)\left(\frac{3k}{4} + (\mu - 1)k^2 + \frac{k^2}{8}\right)\right) \cong \frac{Ph^3}{3EI}\left(1 + \frac{1+\mu}{4}k\right)^3$$



% Variation in deflections computed by the more accurate method, (Case III)
30
25
20
15
10
5
5
10
15
20
25
K = a/h
.05
.10
.15
.20
.25
.30
Length = h (clear span)
Length = h + a/2
(center to center distance)

equivalent length = $h\left(1+\frac{1+\mu}{4}K\right)$

This gives practically identical results with assumptions three and four.


a
b
c
d
e
1
6
7
4
3
2
5
8
f
g
h
i
j
35'
65'
℄

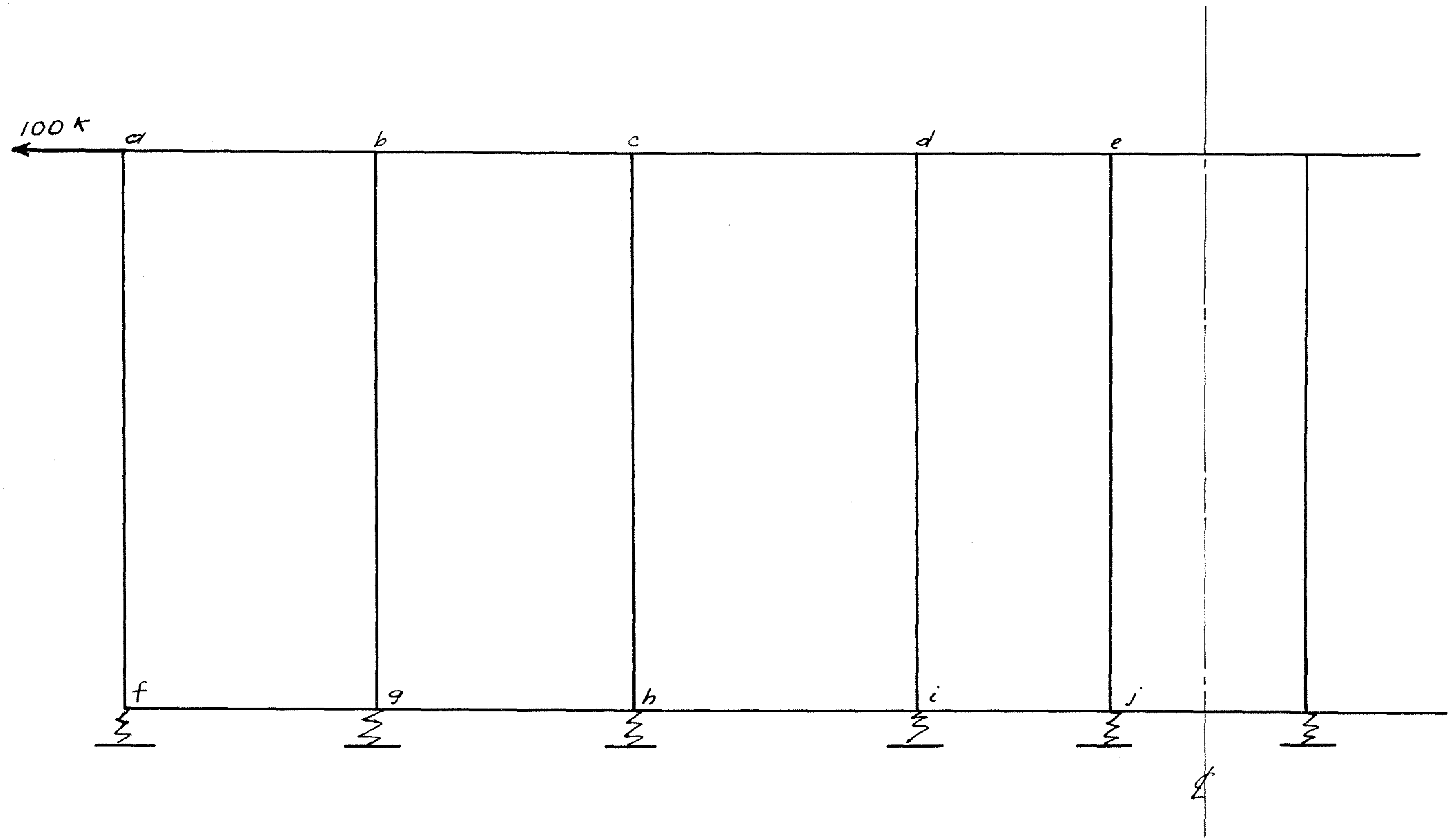
CHAPTER TWO

STRESSES AND DEFLECTIONS IN SUPPORTING WALL OF A REINFORCED CONCRETE BUILDING SUBJECTED TO LATERAL FORCES, AND RESTING ON AN ELASTIC FOUNDATION.

The total horizontal load on this wall was taken to be two hundred kips (200,000 pounds) acting along the upper edge of the wall, in the plane of the wall (100 kips on half wall). This value was given as part of the original data and supposedly is equivalent to the effect of a severe earthquake. The wall is one foot in thickness and contains several windows. The footings are three feet in width and are continuous along the wall. They were assumed to be in sections and to add no stiffness to the lower portion of the wall. The soil constant was given as 3000# / square foot/.05 inches, or 720,000#/square foot/foot. The other dimensions were scaled from the original of the diagram opposite this page.

Because of the presence of windows and the elasticity of the foundation this wall will not deform as a solid wall on a rigid foundation. An exact analytical analysis would be impossible but it was believed that an approximate analysis could be made which would at least give the order of magnitude of the stresses and deflections. This was done as follows.

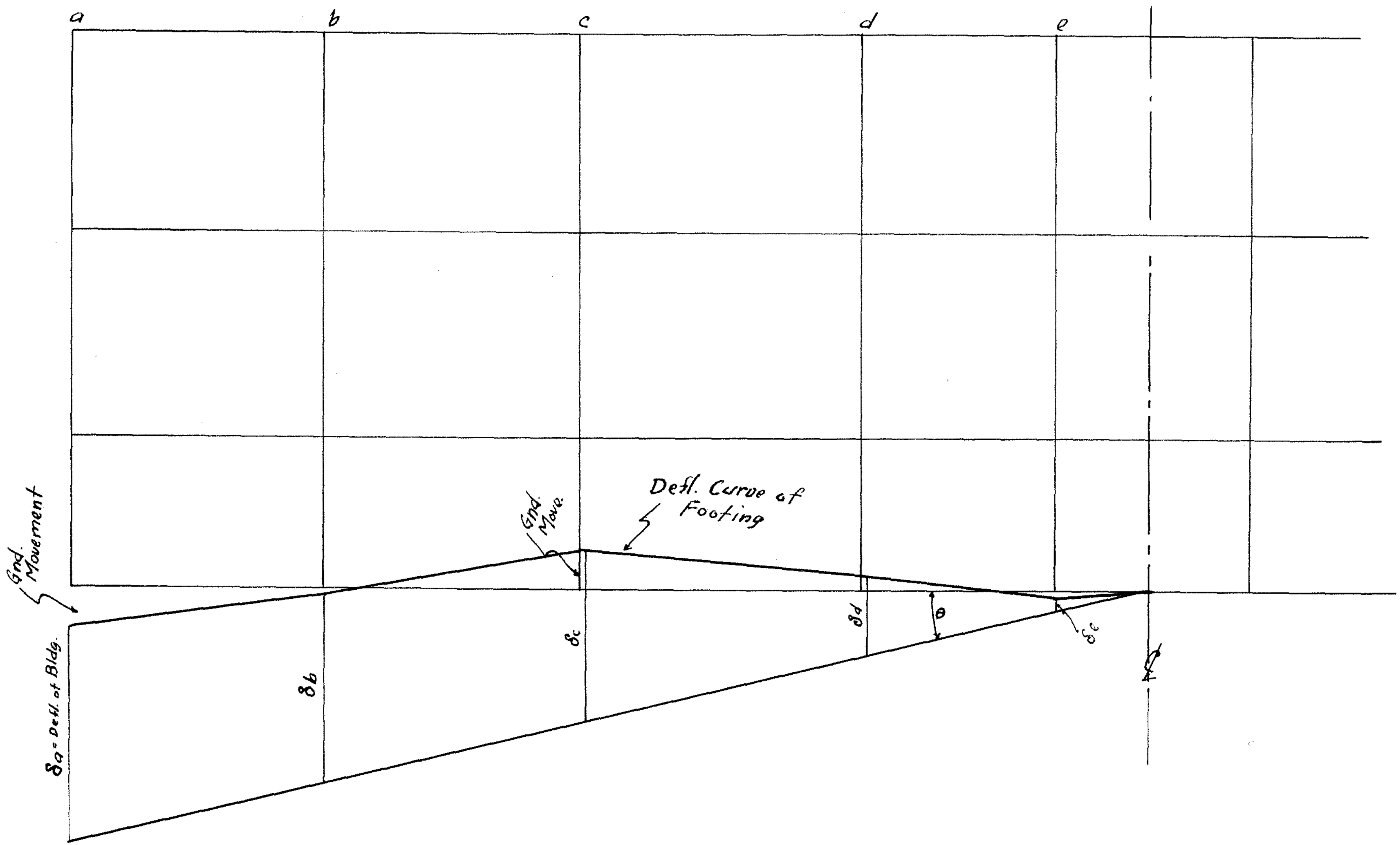
The wall was replaced by an equivalent bent (system of columns and girders). The large section at the end of the wall (a f) was taken as a single very stiff column. In computing its stiffness the


100 k
a
b
c
d
e
f
g
h
i
j
℄

effect of the hole was considered. The upper and lower edges of the wall became girders while the vertical strips between windows became columns. To simplify the calculations the short horizontal strips between windows were assumed to act as pin supports. That is, their effect in reducing side sway of the bent as a whole would be small while they would prevent any relative sideways movement of individual columns. Finally, the continuous elastic foundation was replaced by individual elastic supports under columns. The elastic constant of each support was made equal to the soil constant multiplied by the area of the corresponding section of footing. This equivalent bent has seven degrees of freedom (considering half bent only). Each support has its own deflection (5). Sidesway of the bent may occur. The bent as a whole may rotate about the mid-point of the base.

The seven conditions of equilibrium which correspond are: in each panel (between columns) the internal shear forces (due to bending of the girders) must balance the vertical external loads on one side of the section (5): the total horizontal shear in the columns must equal one hundred kips. The moment of the support reactions about the center line must equal one hundred kips multiplied by the height of the wall.

The Moment Distribution (Hardy Cross) Method was used to find the shears and moments at all points of the bent due to separate unit displacements of the five supports and to unit displacement of the upper girder horizontally with respect to the lower. Various methods (Least Work, Hardy Cross) were used to find the stiffnesses and carry-over factors which are used in the



a
b
c
d
e
Defl. Curve of Footing
Gnd. Move.
Gnd. Movement
δa = Defl. of Bldg.
δb
δc
δd
δe
θ
℄

Hardy Cross Method of calculation.

Having determined the moments and shears due to the individual unit displacements, it was possible to write seven simultaneous equations from the conditions outlined above. These equations contained the seven unknowns $\delta_A, \delta_B, \delta_C, \delta_D, \delta_E$ and δ_K and θ corresponding to the vertical deflections of the five supports, to the horizontal sidesway of the building and to the rotation of the whole building. These equations were solved and the deflections and resulting footing pressures are tabulated below and shown on the diagram opposite. A negative footing pressure means a reduction in the actual pressure due to dead loads.

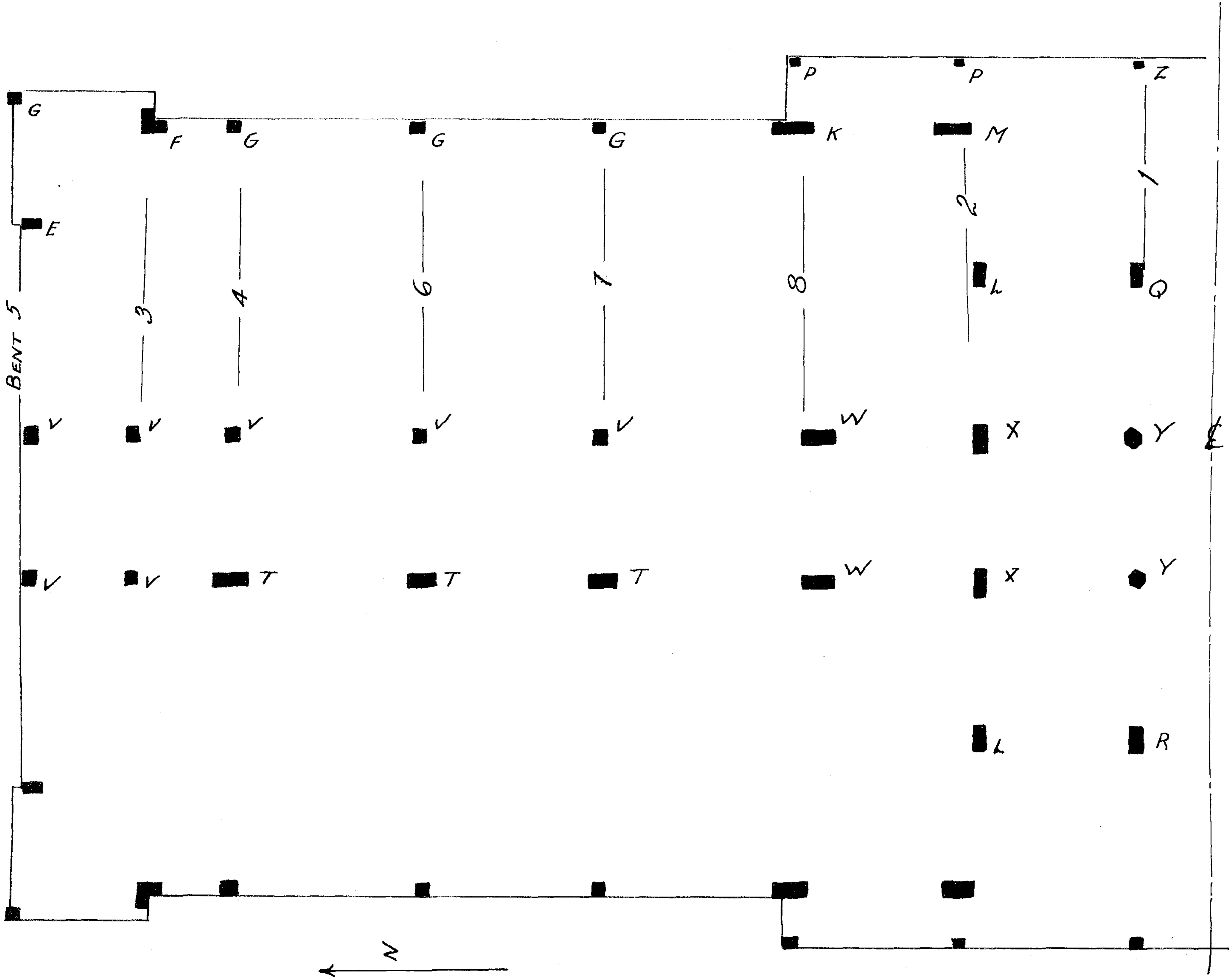
The next step was to calculate stresses at various points in the wall. Calculations were made for sections 1-8 (see diagram opposite page 1). These are tabulated below. Several of these stresses are quite high.

The results obtained, especially those which have to do with stresses and deflections of the building are not particularly exact. They give an indication of what occurs but they may be in considerable error. The effect of width of members was ignored, or in the case of certain members was treated quite casually by arbitrarily taking some reduced value of length instead of actual center to center or clear span length. The loads on the footings were actually distributed instead of concentrated as assumed. Shear deflections and shear stresses were important but were not considered except in one or two cases. Finally, the assumption of linear variation of stress across the section of a beam is quite inexact in such a case, because of the ratio of length to depth of most members and because of the large number of corners.

	Up, due to deflection $\delta \times 10^3$	Down, due to θ $\times 10^3$	Actual defl. ft.	Soil pressure kips/sq.ft.
A	14.23	-16.90	-2.67×10^{-3}	127.4
B	13.41	-14.30	-0.91×10^{-3}	21.12
C	12.58	-10.14	2.44×10^{-3}	- 58.55
D	6.54	- 5.96	0.58×10^{-3}	- 13.74
E	1.36	- 1.83	$-.47 \times 10^{-3}$	11.29
δ_K	.1024			
θ	.390 (radians)			

Stresses

Section	Stress #/in²
1	139
2	364
3	113
4	271
5	804
6	520
7	98
8	804



BENT 5
N
1
2
3
4
6
7
8
G
F
E
K
M
P
Z
Q
V
T
W
X
Y
R

CHAPTER THREE

EARTHQUAKE RESISTANCE OF THROOP HALL, CALIFORNIA INSTITUTE OF TECHNOLOGY

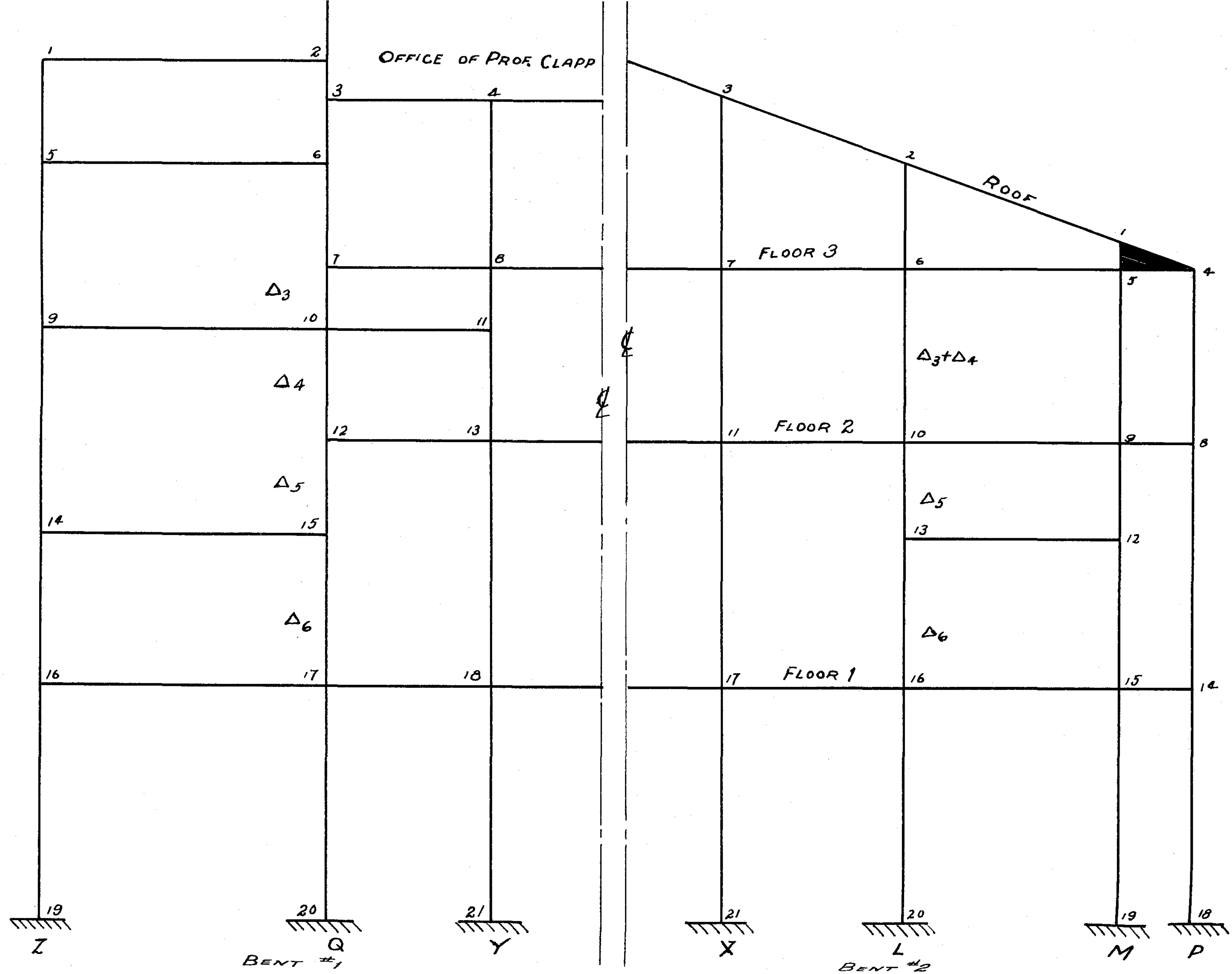
Throop Hall was built in, or about, 1908, long before present practice in regard to earthquake resistant design. It was suggested by Professor Martel that an investigation of the earthquake resistance of Throop Hall would be interesting and might be instructive. Such an investigation was carried out.

At present, a building is designed to be earthquake resistant by assuming the whole structure subjected to a constant horizontal acceleration. That is, horizontal forces are assumed acting on the building, proportional to, and distributed as, the weight of the building. The ratio assumed is usually of the order of five or ten per cent and a calculation is made for each principal axis of the building. These horizontal forces are carried to the footings by the frame or walls or both. It is usually assumed that each bent or bearing wall carries the horizontal force due to the weight of adjacent portions of the building. Ordinarily, little or no thought is given to the relative rigidities of different bents, or of walls and bents. Actually, these horizontal loads would be distributed to the vertical bents in accordance with their stiffnesses and in accordance with the stiffness of the horizontal systems (of girders). In case the horizontal systems are relatively very rigid concrete floors extending throughout the building all bents must deform with one another, and in case centers of mass and centers of rigidity coincide (or, more correctly, lie on the same vertical line) all bents will deflect equally. (Deflections of floors will be relatively small unless the building has very stiff bearing walls).

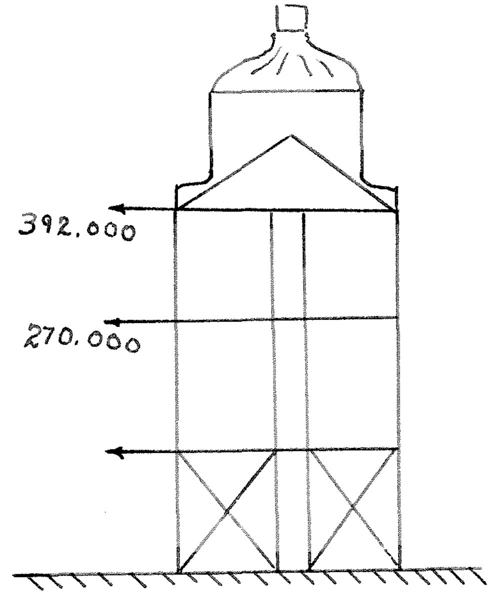
Throop Hall is nearly symmetric and has concrete floors. The walls (except in the basement) are of tile. Tile walls are very stiff but also quite weak. Their assistance, in case of an earthquake, would be unreliable. Except in the basement, all loads are assumed taken by the frame of the building. In the basement certain portions of each exterior wall appear to be of concrete (from the Architect's Drawings). These sections are large enough to withstand the horizontal forces assumed and stiff enough so that the deflection of the first floor will be negligible. However, this will not prevent bending in the basement columns.

Calculations were completed for forces in the East-West direction (parallel to the short sides of the building). Calculations were started for loads parallel to the long axis of the building but were not completed, since it was thought that the results would not be worth the effort. These unfinished calculations (65 pages) are in Professor Martel's files.

The first step in the calculations was the determination of loads. (Pages 1-4 and 41 of calculations) Weights were calculated and at each floor level horizontal forces were assumed equal to one-tenth the weight adjacent to that floor.



OFFICE OF PROF. CLAPP
ROOF
FLOOR 3
FLOOR 2
FLOOR 1
Δ_3
Δ_4
Δ_5
Δ_6
$\Delta_3+\Delta_4$
Z
Q
Y
X
L
M
P
BENT #1
BENT #2



The loads (for the whole building) are shown at the right.

Referring to diagram opposite page 14it can be seen that bents 1 and 2 are much the stiffest of the East-West bents. For this reason it was decided that the deflection of the building would depend chiefly on these two bents (and on their duplicates in the other half of the building). The deflections of the different floors were computed on this basis (i.e. that bents 1 and 2 carry the whole horizontal load). (Pages 11-20 and 22-24 of calculations) Then the shears which would result in all the other bents were determined (pp.25-34 of calculations) and finally the already computed deflections were reduced to make the internal shears equal the external shears (p.35 of calculations).

The deflections of bents 1 and 2 were found as follows. Knowing that the roof and floors are relatively rigid concrete slabs, that the first floor is held fast by stiff concrete basement walls, that the dome (Professor Clapp's office) is a relatively rigid reinforced concrete cylinder which is continuous with the roof on either side of it and that girder 14-15 in bent 1 must deflect with 12-13 in bent 2 since they are integral with a stair-landing slab, we may see, referring to the diagram opposite this page, that, with respect to horizontal deflections, these bents have four degrees of freedom. In bent 1 the relative deflection between points 14 and 16 is labeled Δ_6, that between 12 and 15 is Δ_5, Δ_4 between 10 and 12 and Δ_3 between 7 and 10 while in bent 1 there are three deflections: Δ_6, Δ_5, $\Delta_3 + \Delta_4$ as shown. The actual deflection of the roof would be $\Delta_3 + \Delta_4 + \Delta_5 + \Delta_6$.

Using the Hardy Cross method unit Δ's were applied succesively at the different elevations in each bent and for each Δ the shears at all elevations between floor 1 and floor 3 were calculated. (pp. 12-20 of calculations) Finally, four simultaneous equations were written with four unknowns ($\Delta_3 - \Delta_6$) from the condition that external shear must equal internal shear at each elevation between floors 1 and 3. (pp. 22-24 of calculations).

The Δ's obtained from these equations were actually too large because in obtaining them the stiffnesses of all the other bents were neglected. To obtain corrected values for the deflections these just-determined values of the deflections were applied to all other bents of the building and the resulting shears calculated. This process gave the following shears in the different bents. (p. 34)

Bent	V_6	V_5	V_4	V_3
1	141,000	83,400	80,000	109,200
2	192,600	256,800	128,200	128,200
3	49,800	49,800	23,700	23,700
4	27,800	27,800	33,000	33,000
5	42,300	42,300	53,700	53,700
6	27,800	27,800	33,000	33,000
7	27,800	27,800	33,000	33,000
8	30,000	30,000	30,000	30,000
Σ	539,000	546,000	435,000	444,000
2Σ	1,078,000	1,092,000	870,000	888,000

for whole building

But the shear between floors 2 and 3 must be 392,000 while the shear between 1 and 2 must be 660,000. Therefore, the deflections and moments already calculated must be decreased approximately as follows: between floors 1 and 2 the ratio must be $\frac{660}{1090} = .61$; between floors 2 and 3 the ratio must be $\frac{392}{870} = .45$; while for the girders in floor 2 the ratio will be $\frac{.61+.45}{2} = .53$. (P. 35)

While the values so obtained will not be exact, nevertheless, the error introduced will not be large and the accuracy of the results will be consistent with that of the assumptions and other calculations. (See sheet 35 of computations)

Calculations of dead load effects were made for various members which seemed critical. Because of the cumulation of dead load and live load, stresses in members in the neighborhood of the first floor are most likely to be dangerous. See computation sheets 36-41 for calculations of dead load moments in columns and girders and dead load direct stresses in columns. In this connection use was made of expressions for moments on the edges of uniformly loaded flat slabs, computed by H. M. Westergaard and published in "Standards of Design for Concreté" (U. S. Navy Department).

A maximum allowable combined stress of 1,000 lbs/sq.in. was used. In the case of girders, moments to produce this stress were calculated, (pp. 8-9b), from these were subtracted the dead load moments already found, the difference was available for earthquake resistance. In the case of columns tentative calculations were made by subtracting the dead load direct stress from 1,000 lbs/sq.in. and multiplying this value by the section modulus of the column, giving a very approximate value for allowable earthquake moment.

For those columns which this calculation showed to be most critical, further, more exact calculations were made neglecting the assistance of concrete under tension. (Computation sheet 44-49). In most cases dead load bending moments seem negligible but for several columns these were also included.

Finally (pp.40-41), knowing the moment which each member could carry in addition to its dead load, and the moment which would be induced in each member by horizontal forces equal to $\frac{1}{10}$ W we may calculate for each member the coefficient of W which will give limiting stress.

The results of these computations are tabulated on the following two pages. On the first page are the calculations for girders. The reader may refer to the diagrams opposite pages 14 and 16 for system of designating members. In column 3, L is the girder length in feet; in column 4, B is the panel width in feet; in column 5, q is the coefficient in the formula $M_1 = qwBL^2$ which is taken from Westergaard's paper. M_1 is the moment at the support, w the weight per square foot on the slab. M_1 (in inch kips) is given in the sixth column. In column 7, W is the additional weight on the girder (lbs/ft) due to partitions. In column 8, M_2 ($M_2 = {}^1/_{12}\,WL^2$) is the moment at the support due to W. Column 9 gives M_1 M_2. A calculation of dead load moments in bent 7 was made, using the Hardy Cross method. The moments obtained are shown in brackets. Column 10 gives the girder sections. Column 11 gives the ratio of slab thickness to girder depth (calculations are made assuming that 30" of slab acts as tee portion of tee-beam). Column 12 gives the coefficients which must be multiplied by bd^2 to give allowable bending moment. (b = width of tee section = 30"). These coefficients are taken from

"Concrete Engineer's Handbook" by Hool and Johnson. In column 13 are given allowable bending moments in inch kips. Column 14 is obtained by subtracting values in column 9 from those in column 13, giving the allowable additional moment. Column 15 gives the moments due to lateral forces equal to 1/10 the weight of the building. Column 16 gives ratios between values in column 14 and in column 16 for the worst cases, multiplied by 10. These values, multiplied by W (weight of building) give the lateral forces to produce 1000 lbs/sq.in. stresses in the different members.

The second of these two sheets is almost self-explanatory. Column 4 gives the direct load P; column 5 the average unit stress due to P; column 6 gives the difference between this stress and the maximum allowed; column 7 gives the section modulus ($Z = bd^2/6$) which, multiplied by the maximum allowable unit stress gives the allowable bending moment (column 8); column 9 gives the moments due to a lateral force of $W/10$; column 10 gives $\frac{M_1}{M} \times 10$ or the value of the lateral force to produce the allowable stress in the individual columns. The remaining five columns give the results of more accurate calculations. Column 11 gives the kind and amount of the steel reinforcement in each column; column 12 gives the size of core; column 13 gives the allowable total moment for the particular value of P; column 14 gives the dead load moment (when large); column 15 gives the values of $\frac{M_{DL(col\ 13)}}{M_{1\ (col\ 14)}} \times 10$ or the amount of lateral force to give 1000 lbs/sq.in. stress.

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
Bent	Girder	L	B	q	M_1	W/ft	M_2	M_1+M_2	Section	t/d	Coef of bd^2	M	$M-(M_1+M_2)$	M_{EQ}	
1	9-10	16.4	13′	1/15	224	600	162	386	$14\frac{1}{2}$x20	.25	150	1800	1400	460	
	10-11	12.0	13′	1/33	55	800	115	170	16	.31	162	1245	1100	390	
	12-13	12.0	13′	0	0	0	0	0	16	.31	162	1245	1250	413	
	14-15	16.4	13′	1/15	224	0	0	224	20	.25	150	1800	1575	1030	15.3%
	16-17	16.4	13′	1/33	112	800	215	317	20	.25	180	1860	1475	416	
	17-18	12.0	13′	1/16	102	0	0	112	16	.31	162	1245	1125	250	
2	8-9	4.5	m=.36 12.5	$\frac{wL^3}{33}$	3	800	16	20	42	.119	90				
	9-10	13.9	12.5	1/33	70	800	155	225	16	.31	162	1245	1025	410	
	10-11	12.0	12.5	1/36	48	800	115	165	16	.31	162	1245	1075	700	15.4%
	11-11′	12.0	12.5	1/16	108	0	0	108	16	.31	162	1245	1150	683	16.8%
	12-13	13.9	12.5	1/33	70	300	58	130	16	.31	162	1245	1125	831	13.5%
	14-15	4.5	m=.36 12.5	$\frac{wL^3}{33}$	3	800	16	20	16	.31	162	1245	1225	306	
3	2-3	26.0	m=.33 8.5	1/16	345	0	0	345	32	.16	114	3510	3150	428	
	4-5	26.0	m=.33 8.5	1/16	345	0	0	395	24	.21	136	2355	2000	572	
	5-5′	12.0	8.5	1/16	74	0	0	75	16	.31	162	1245	1175	111	
4	2-3	26.0	m=.46 12.0	1/16	487	400	270	757	32	.16	114	3510	2750	230	
	4-5	26.0	m=.46 12.0	1/16	487	800	540	1027	24	.21	136	2355	1325	419	
	5-5′	12.0	12.0	1/16	104	0	0	104	16	.31	162	1245	1100	381	
5	6-7	10.2	8.5	1/33	26	1500	156	182	24	.21	136	2355	2175	610	
	7-8	17.5	m=.5 8.5	1/33	76	1500	460	536	24	.21	136	2355	1800	376	
7	~~6-7~~ 2-3	26.0	m=.6 15.7	1/16	636	400	270	906	32	.16	114	3510	2600	225	
	4-5	26.0	m=.6 15.7	1/16	636	1100	745	1381	24	.21	136	2355	975	456	
	5-5′	12.0	15.7	1/16	136	0	0	136	16	.31	162	1245	1100	340	

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
									% G	Steel		Allowable	Dead	% G
Col.	Sect.	Area	P	P/A	$1000-\frac{P}{A}$	Z	M_1	M_{EQ}	$\frac{M_1}{M_{EQ}}\cdot 10$	(all round)	Core	M in Kips	Load M	
Z above 1	$14\frac{1}{2}$x24	348	85.400	245	755	1390	1050	1030	10.2	6-½	none			
below 1			90.100	259	740		1028	618	16.7	6-⅞	"			
Q	$14\frac{1}{2}$x$34\frac{1}{2}$	500	111,200	223	775	2870	2230	2500	8.9	10-⅝ 8-¾	11x31	2.075	small	8.3 %
			132.800	265	735		2110	1850	11.4	10-⅝ 12-¾	11"□ 11"□			
Y	16" oct.	201	54.900	273	725	400	290	263	11.0	4-⅞ 4-1	12" o	247	small	9.4 %
			75.100	374	625		250	24		8-1¼	13" o			
L	$14\frac{1}{2}$x$34\frac{1}{2}$	500	63.600	127	875	2870	2510	2710	9.3	8-½	11x31	2193	small	8.1 %
			88.300	177	825		2370	2250	10.5	8-⅞	none			
X	$14\frac{1}{2}$x$34\frac{1}{2}$	500	94.100	188	810	2870	2320	2500	9.3	10-⅝ 16-½	11x31 11□ 11□	2.212	350 estim	7.4 %
			124.200	248	750		2150	1990	10.8	10-⅝ 12-¾	11□, 11□			
K	$14\frac{1}{2}$x$37\frac{1}{2}$	544	60.700	112	890	1310	1065			8-½	none			
			67.200	124	875					6-⅝	none			
G_7	$14\frac{1}{2}$ □	210	92.000	437	565	508	287	318	9.0	8-½	$12\frac{1}{2}$ □	331	320	0.3 %
			109.800	522	480		244	55		4-⅞ 4-1	$12\frac{1}{2}$ □			
F	15·30 36·12	700	43.400	62	940	2250	2120	1480		10-⅜	none			
·			46.600	67	935		2100	1210		10-⅞	none			
T	$26\frac{1}{2}\cdot14\frac{1}{2}$	384	86.000	224	775	930	720	600±	12.0	12-½	11"			
			128.000	300	700		652	170±		16-¾	11"			
V_7	$14\frac{1}{2}$ □	210	86.000	410	590	508	300	590	5.1	8-⅞	12"	421	420	0 %
W	16 oct	201	128.000	635	365	400	146	80		8-1¼	13"			
E	$14\frac{1}{2}\cdot32\frac{1}{2}$	471	75.700	161	839	2555	955	642		6-⅜	none			
			79.500	169	831		946	110						

SUMMARY OF RESULTS:

All girders considered have very high margins of safety. For the weakest the lateral force factor was 13½%. Of the columns considered, all but two had factors ranging from 7% to 16%. However, two columns seem to be very highly stressed by dead load only. Their safety margin was zero. The dead loads on these columns (G and V) were obtained from an analysis of bent 7. The same designation is used on columns in other parts of the building but it is believed that those in bent 7 have the highest stresses. These columns lie between rooms 220 and 222 and between 207 and 207A. In case of a rather severe earthquake these columns would presumably be damaged. Complete failure would probably not occur. The effect of partial failure is to relieve the load on the column which fails and to increase the loads on neighboring columns. Both these critical columns are quite slight. Their help in resisting lateral forces is not great and in case this help were lost the additional loads to neighboring columns would not be important.

It appears, therefore, that a strong earthquake could be expected to damage columns G and V in bent 7 among the first. Columns G and V in other bents might also receive damage, especially if their dead loads are large. (In case their dead loads are zero, the seismic factor for column V is $\frac{420}{590} \times 10\%$ equals 7.1%, for column G, equals 10%).

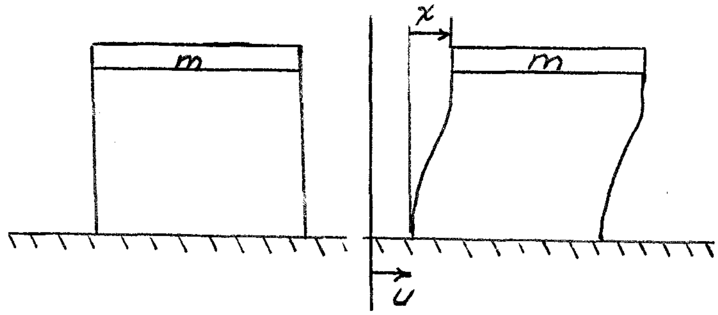
With these exceptions all the columns of Throop Hall seem capable of safely withstanding lateral forces of the order of at least 7% W, while the girders are safe for about twice as much.

CHAPTER FOUR

This chapter deals with the effects produced on simple one-story symmetric structures (such as elevated water tanks, one-story buildings or tall buildings with a flexible first story) by sinusoidal ground movements of limited duration. The effects of variations in amplitude, frequency and duration of movement (within limits) are shown in diagrams. A particular case of damping (critical damping) is considered as well.

The choice of simple harmonic ground movements is for the sake of simplicity and may be justified by the theory of Fourier Series and the fact that the differential equations governing the motions of the structures considered are linear and their solutions additive.

In "The Engineering News Record" (published by the American Society of Civil Engineers) for October 4, 1934 there appeared an article by A. L. Brown, Director of the Factory Mutual Laboratories, Boston, describing a series of experiments made at M. I. T. by Arthur C. Ruge on this problem (with particular reference to elevated tanks). Extracts from the results given in this article are shown on the diagram opposite page 32. These tests were made with a model on a shaking-table subjected to a cosine motion. The experimental results obtained at M. I. T. may be compared with the theoretical results shown on the following pages.



We may consider the structure shown. We let:

k = stiffness of columns (force to cause 1" deflection)

m = mass (W/g) carried by columns

$U = U_o \begin{cases} \sin \lambda t \\ \cos \lambda t \end{cases}$ ground displacement

x = deflection of columns, ∴ x + U = displacement of m

x_m = maximum deflection of columns in interval

$\frac{\lambda}{2\pi}$ = frequency of ground movement

$n = \frac{\lambda T}{\pi}$ where T = duration of ground motion (seconds) and n = number of half cycles of ground motion

$\frac{\nu}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{k}{m}}$ = natural frequency of structure

$c = \frac{\nu}{\lambda}$

From equilibrium:

1) $m\left(\frac{d^2x}{dt^2} + \frac{d^2u}{dt^2}\right) + kx = 0$ $\qquad \frac{d^2x}{dt^2} = \ddot{x} \qquad \frac{d^2u}{dt^2} = \ddot{u}$

$\ddot{x} + \nu^2 x = -\ddot{u}$

<u>If $u = u_o \sin \lambda t$</u>

1') $\ddot{x} + \nu^2 x = u_o \lambda^2 \sin \lambda t$

2) $x = A \sin(\nu t - \alpha) + \frac{u_o}{c^2-1} \sin \lambda t$

at time $t = 0$, $x = (\dot{x} + \dot{u}) = 0$

3) $\frac{x}{u_o} = \frac{1}{c^2-1}(\sin \lambda t - c \sin c\lambda t)$

when c = 0 structure has no stiffness

$\frac{X_m}{U_o} = 1$ m has no movement and the deflection in the columns is equal to the base movement.

when c = ∞ structure has infinite rigidity

$\frac{X_m}{U_o} = 0$ m moves with the base, no deflection can occur.

when c = 1 we have resonance. If we differentiate the numerator and denominator of (3) with respect to c and let c $\rightarrow$ 1, we get:

4) $\frac{X}{U_o} = -\frac{1}{2}(\sin \lambda t + \lambda t \cos \lambda t)$

For large values of λt $\frac{X_m}{U_o} \cong \frac{1}{2} \lambda t$ $(\lambda t = n\pi)$

For small values of λt we must equate $\frac{d}{dt}\left(\frac{X}{U_o}\right)$ to 0, which gives $\cot \lambda t = \frac{\lambda t}{2}$

Solutions:

$\lambda t =$	$61° - 40'$	$208° - 48'$	$377°$	$540° +$
$\frac{X_m}{U_o} =$	.645	1.841	3.246	4.11

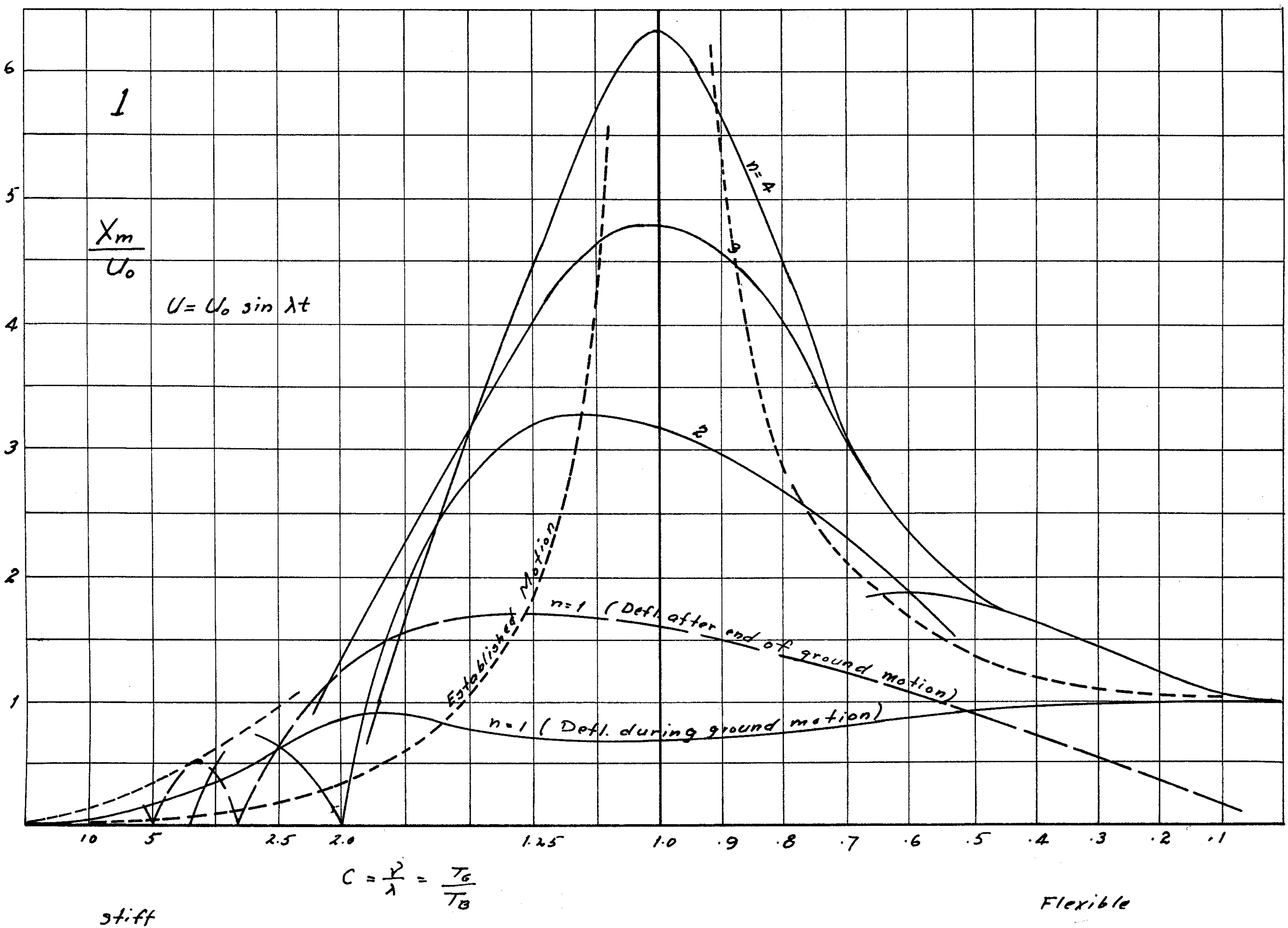
When c = $\frac{1}{2}$

5) $\frac{X}{U_o} = -\frac{4}{3}\left(\sin \lambda t - \frac{1}{2} \sin \frac{\lambda t}{2}\right)$

x_m occurs when cos $\lambda t = \frac{1}{4} \cos \frac{\lambda t}{2}$

Solutions:

$\lambda t =$	$78° - 50'$	$263° - 50'$	$2\pi + 77° - 10'$	$4\pi - 78° - 50'$
$\frac{X_m}{U_o} =$	.882	1.824	1.824	.882


1
Xm / Uo
U = Uo sin λt
n=4
3
2
Established Motion
n=1 (Defl. after end of ground motion)
n=1 (Defl. during ground motion)
6
5
4
3
2
1
10
5
2.5
2.0
1.25
1.0
.9
.8
.7
.6
.5
.4
.3
.2
.1
C = ν/λ = TG/TB
stiff
Flexible

This process is repeated for various values of c and λt. The values obtained will be the maximum deflections occuring during continued ground motion. If the ground motion stops after a certain number of cycles the building will continue to oscillate and may have greater deflection after the ground motion ceases than before. The deflections occuring after the end of the ground motion, assuming the ground motion to stop after a certain number (called n) of complete half cycles, have been calculated as well (see pp. 33-36 of calculations). The results of both sets of computations are shown in diagram 1 opposite this page. The two curves labeled n = 1. give maximum deflections during and after ground motions when the motion lasts for half a complete cycle. The other curves give whichever of the two values is the greater. For comparison the familiar resonance curve for established motion is also shown.

Since a Fourier Series resolutions always requireds completed cycles this is all right. However, it must be remembered that the ground motion is quite irregular. The effect of a sudden break in the motion would be given correctly enough by combining the effects of several completed sine or cosine cycles, but it would be much simpler to assume a single expression which need not complete its cycle. In other words we may find at what time the total energy (kinetic and potential) of the system is a maximum and assume that the ground motion ceases at that instant. Then, at some later time, all this energy will be potential, giving a maximum deflection. Since the ground motion may absorb energy these values will generally be greater than those previously found for continued motion.

At time t (at which ground motion ceases)

$$3) \quad \frac{x_t}{U_0} = \frac{c}{c^2-1}\left(\sin \lambda t - c \sin c\lambda t\right)$$

$$6) \quad \frac{\dot{x}_t}{U_0} = \frac{\lambda}{c^2-1}\left(\cos \lambda t - c^2 \cos c\lambda t\right)$$

If $x_n = x_{max}$ after end of ground motion

x_t, $\dot{x}_t$ are x and $\dot{x}$ at end of ground motion

$$x_n^2 = x_t^2 + \frac{m}{K}\left(\dot{x}_t + \dot{u}_t\right)^2$$

$$\dot{x}_t + \dot{u}_t = \frac{c^2\lambda}{c^2-1}\left(\cos \lambda t - \cos c\lambda t\right)$$

$\therefore$

$$7) \quad \frac{x_n^2}{U_0^2} = \frac{1}{(c^2-1)^2}\left[\sin^2\lambda t - 2c \sin \lambda t \sin c\lambda t + c^2\left(1 + \cos^2\lambda t - 2\cos \lambda t \cos c\lambda t\right)\right]$$

To find what value of λt will give a maximum value to x_n: call this x_{mn}

For x_{mn} $\quad \frac{d}{dt}\left(\frac{x_n}{U_0}\right)^2 = 0 \quad \therefore (1-c^2)\left(\sin \lambda t \cos \lambda t - c \cos \lambda t \sin c\lambda t\right) = 0$

$$\therefore \left.\begin{array}{l} 8) \quad \sin \lambda t = c \sin c\lambda t \\ 9) \quad \cos \lambda t = 0 \end{array}\right\} \text{ are conditions for } x_{mn}$$

From (8) and (7) we get:

$$\frac{x_{mn}}{U_0} = \frac{1}{c^2-1}\left[-\sin^2\lambda t\,(1+c^2) + 2c^2\left(1 - \cos \lambda t \cos c\lambda t\right)\right]^{\frac{1}{2}}$$

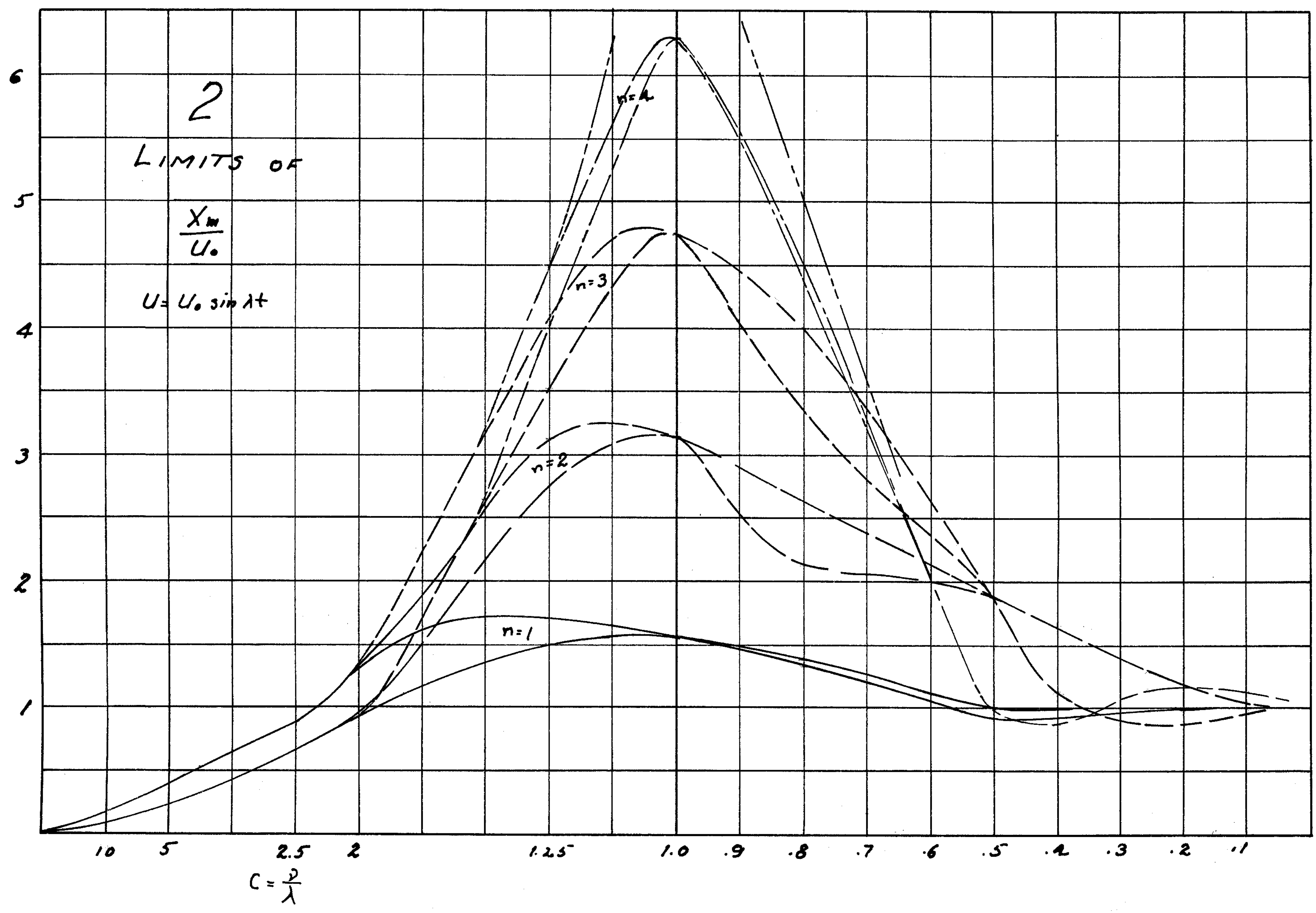
From (9) and (7) we get:

$$\frac{x_{mn}}{U_0} = \frac{1}{c^2-1}\left[1 + c^2 \mp 2c \sin c\,\frac{K\pi}{2}\right]^{\frac{1}{2}}$$

K = 1, 3, 5 . . . . . .

use − sign if K = 1, 5, 9 . . . .

\+ 3, 7, 11 . . . .


2
LIMITS OF
$\frac{X_m}{U_0}$
$U = U_0 \sin \lambda t$
n=4
n=3
n=2
n=1
6
5
4
3
2
1
10
5
2.5
2
1.25
1.0
.9
.8
.7
.6
.5
.4
.3
.2
.1
$C = \frac{\nu}{\lambda}$

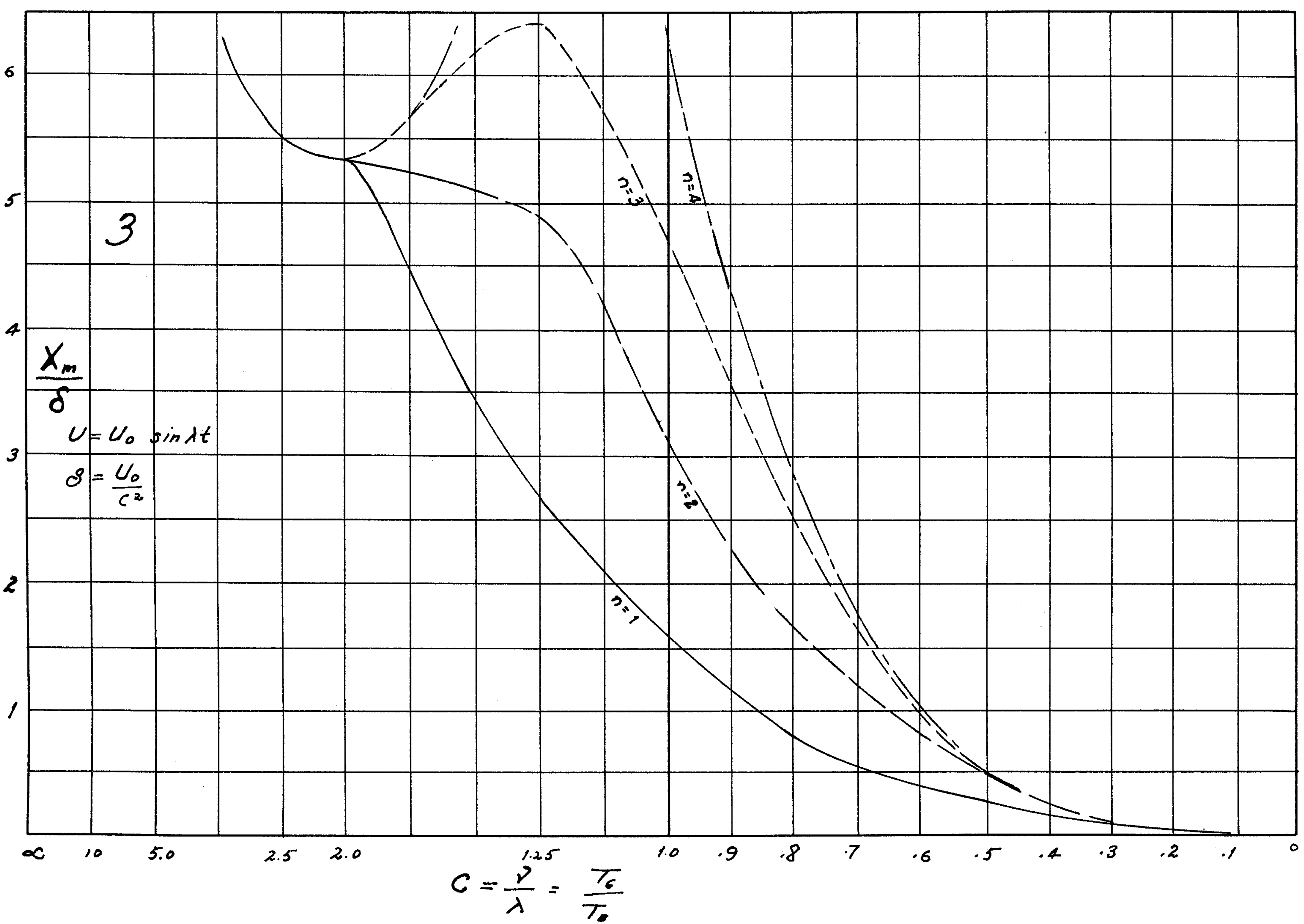
The expressions just found for x_{mn} have been evaluated for various values of c, and for different numbers of ground cycles. The results are shown in upper limit curves on the opposite page. These values give the deflections which correspond to the maximum total energy possessed by the system at some instant during each half cycle. The lower limit curves give the actual maximum deflections which occur during each half cycle (except the curve n = 1 which is taken from diagram 1). The difference between these limits is not great. Apparently, in general, there will be even less difference between results obtained from the maximum energy in each half cycle and those found by stopping the ground motion at the end of a given number of half cycles (as was done in finding the values shown opposite page 25).

We may give these results in a different form:
If δ= deflection produced by a constant acceleration equal to the maximum ground acceleration ($\lambda^2 U_0$):

$$\delta = U_0 \lambda^2 \cdot \frac{m}{K} = \frac{U_0}{c^2}$$

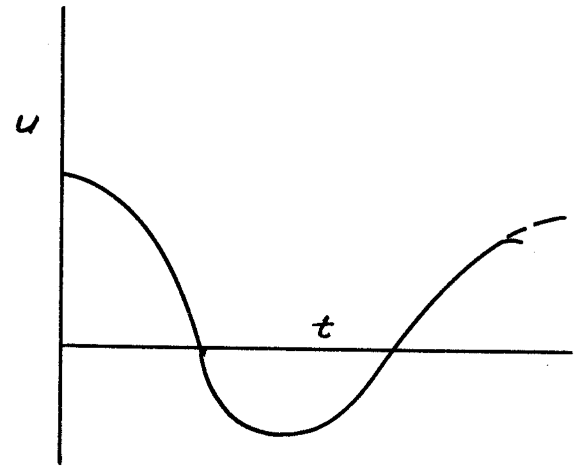
$$\therefore \quad \frac{X}{\delta} = c^2 \frac{X}{U_0}$$

The values of $\frac{X}{\delta}$ shown opposite page 28 were obtained by multiplying the upper limit values of $\frac{X}{U_0}$ opposite this page, by c^2. That these values become very large as c increases is unexpected but can be explained by the fact that our boundary conditions are impossible for a very stiff building.



3
$\frac{X_m}{\delta}$
$U = U_0 \sin \lambda t$
$\delta = \frac{U_0}{c^2}$
n=1
n=2
n=3
n=4
6
5
4
3
2
1
∞
10
5.0
2.5
2.0
1.25
1.0
.9
.8
.7
.6
.5
.4
.3
.2
.1
0
$C = \frac{\nu}{\lambda} = \frac{T_c}{T_0}$

We have required that at time t = 0 the building be at rest even though the ground is moving. Furthermore we have allowed the ground motion to stop suddenly at some instant. For flexible structures neither of these inconsistencies is important. Actually, the ground displacement curve must always have a horizontal tangent at the beginning and end of motion. For a structure of very great stiffness it is apparent that the particular manner in which the motion ceases will be important.



In case $u = u_0 \cos \lambda t$

'') $$\ddot{x} + \nu^2 x = u_0 \lambda^2 \cos \lambda t$$

$$x = B \cos (\nu t - \beta) + \frac{u_0}{c^2 - 1} \cos \lambda t$$

at time $t = 0$ $\quad x = \dot{x} = 0$

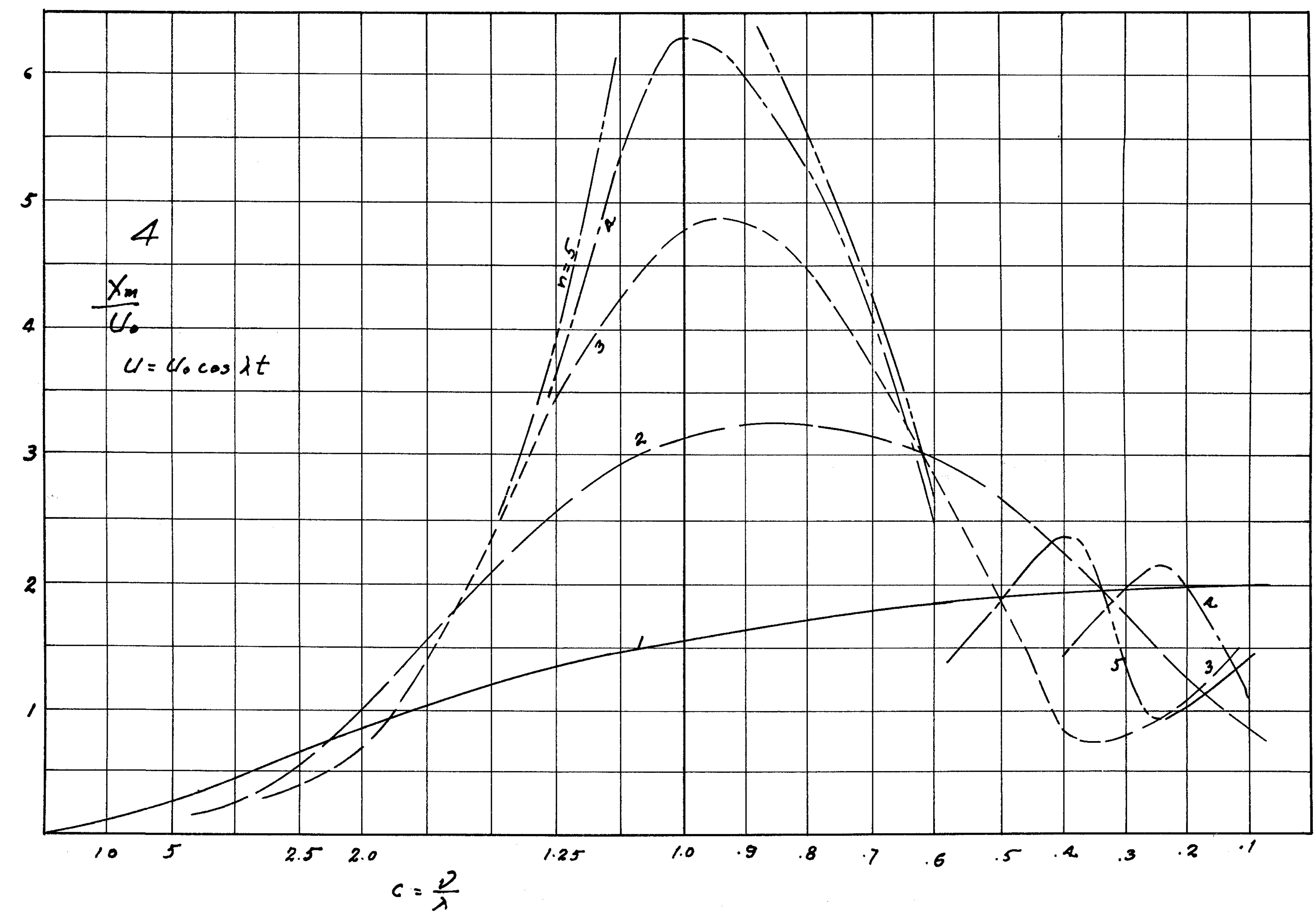
$$\therefore \quad \beta = 0; \quad B = -\frac{u_0}{c^2 - 1}$$

$$\therefore \quad \frac{x}{u_0} = \frac{1}{c^2 - 1} (\cos \lambda t - \cos c\lambda t)$$

$$\frac{\dot{x}}{u_0} = \frac{-\lambda}{c^2 - 1} (\sin \lambda t - c \sin c\lambda t)$$

If we consider the energy at any time t, and the deflection x_n which would correspond:

$$x_n^2 = x_t^2 + \frac{1}{\nu^2} (\dot{x}_t + \dot{u}_t)^2$$


4
$\frac{X_m}{U_0}$
$U = U_0 \cos \lambda t$
$n = 5$
4
3
2
1
5
3
6
5
4
3
2
1
10
5
2.5
2.0
1.25
1.0
.9
.8
.7
.6
.5
.4
.3
.2
.1
$c = \frac{\nu}{\lambda}$

$$\dot{X}_t + \dot{U}_t = \frac{-U_0 \lambda}{c^2-1} \left(c^2 \sin \lambda t - c \sin c\lambda t \right)$$

$\therefore$

$$12) \quad \left(\frac{X_n}{U_0} \right)^2 = \frac{1}{(c^2-1)^2} \left(1 + \cos^2 \lambda t - 2 \cos \lambda t \cos c\lambda t + c^2 \sin^2 \lambda t - 2c \sin \lambda t \sin c\lambda t \right)$$

For a maximum

$$\frac{d}{dt} \left(\frac{X_n}{U_0} \right)^2 = 0$$

$$\therefore \quad (1-c^2) \left(\sin \lambda t \cos c\lambda t - \sin \lambda t \cos \lambda t \right) = 0$$

or

$$13) \quad \sin \lambda t = 0 \quad \therefore \lambda t = n\pi \qquad n = 1.\ 2.\ 3 \ldots\ldots$$

$$14) \quad \cos \lambda t = \cos c\lambda t \quad \therefore \sin \lambda t = \pm \sin c\lambda t$$

Combining 12) and 13)

$$\frac{X_{mn}}{U_0} = \frac{1}{c^2-1} \left(2 - 2(-1)^n \cos c\lambda t \right)^{\frac{1}{2}}$$

or

$$15) \quad \frac{X_{mn}}{U_0} = \begin{cases} \dfrac{2}{c^2-1} \cos \dfrac{c\lambda t}{2} & \text{if } n = 1.\ 3.\ 5 \ldots\ldots \\[2ex] \dfrac{2}{c^2-1} \sin \dfrac{c\lambda t}{2} & \text{if } n = 2.\ 4.\ 6 \ldots\ldots \end{cases} \qquad \lambda t = n\pi$$

Combining 12) and 14)

if $\sin \lambda t = + \sin c\lambda t$

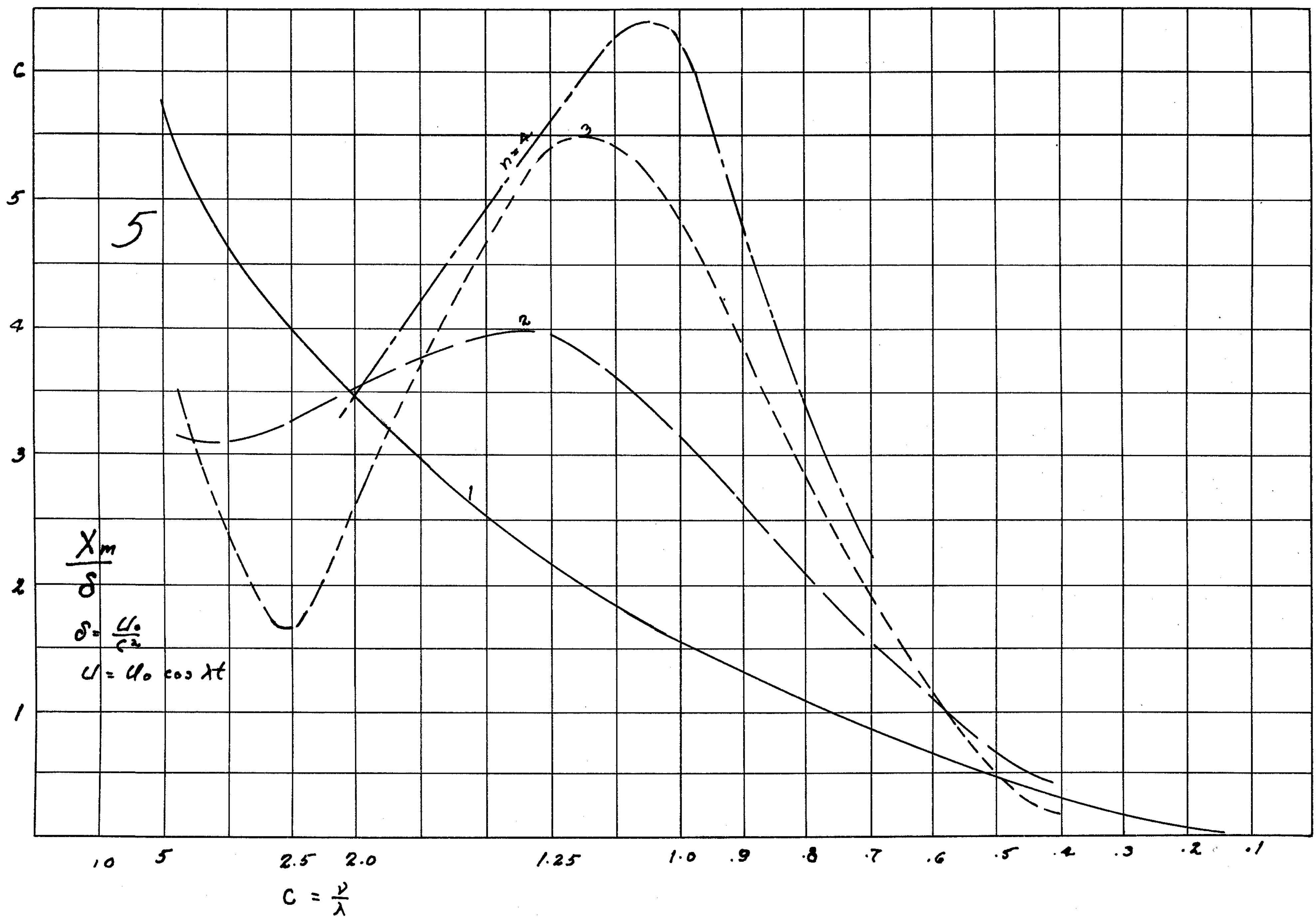
$$\lambda t = \frac{2K\pi}{\pm (c-1)} \qquad K = 1.\ 2.\ 3$$

$$16) \quad \frac{X_{mn}}{U_0} = \frac{\sin \lambda t}{c+1}$$

if $\sin \lambda t = - \sin c\lambda t$

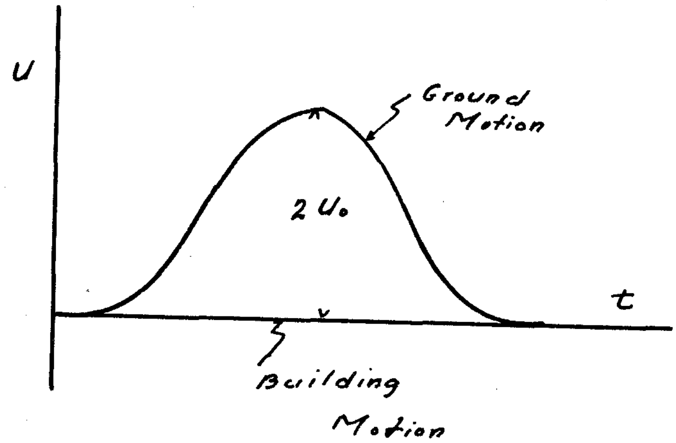
$$\lambda t = \frac{2(2K-1)\pi}{1+c} \qquad K = 1.\ 2.\ 3 \ldots\ldots$$

$$17) \quad \frac{X_{mn}}{U_0} = \frac{\sin \lambda t}{c-1}$$



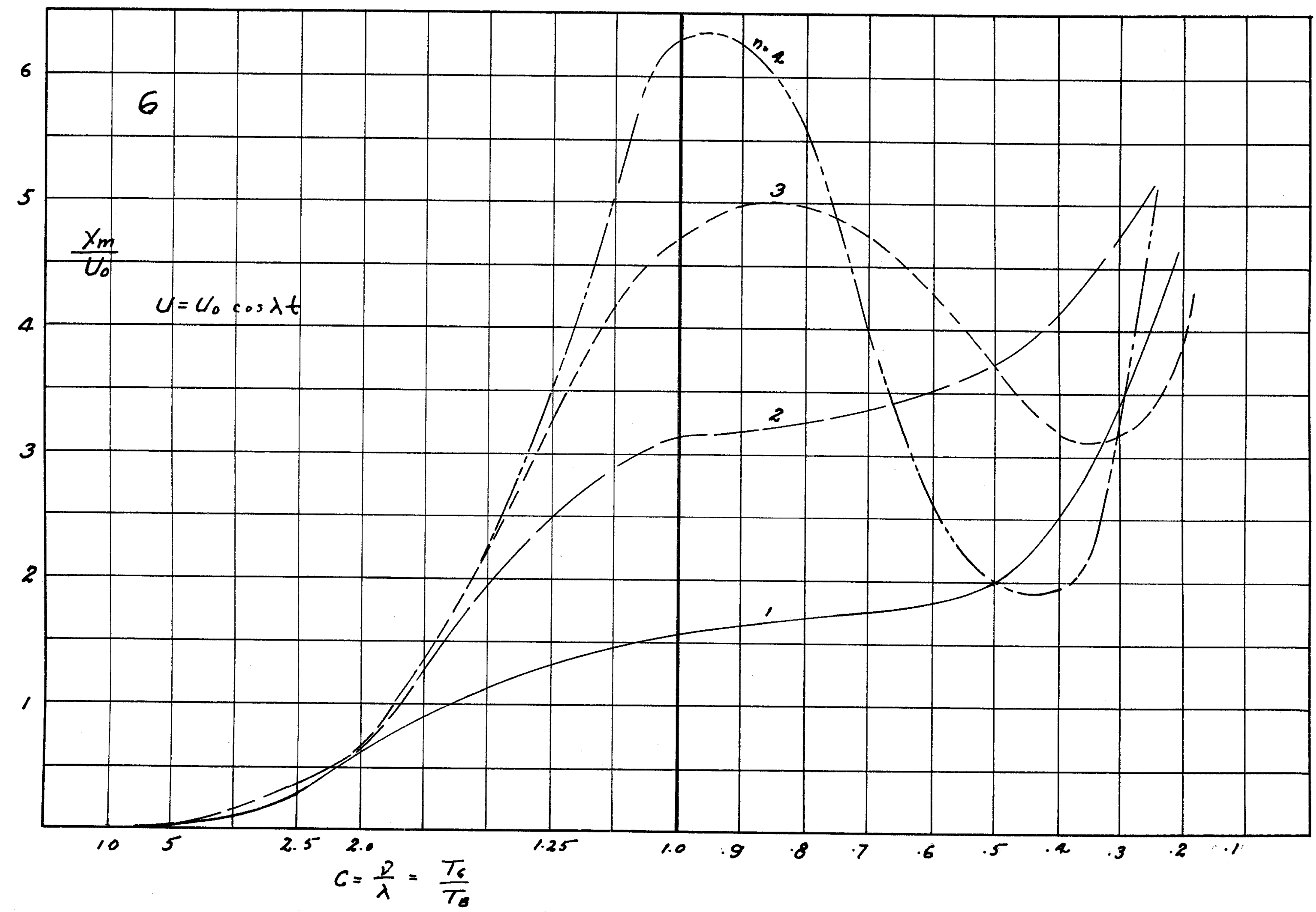
5
$\frac{X_m}{\delta}$
$\delta = \frac{U_0}{c^2}$
$U = U_0 \cos \lambda t$
n = 4
3
2
1
6
5
4
3
2
1
10
5
2.5
2.0
1.25
1.0
.9
.8
.7
.6
.5
.4
.3
.2
.1
$C = \frac{\nu}{\lambda}$

The graph opposite page 29 gives the values of $\frac{X_m}{U_0}$ when $U = U_0 \cos \lambda t$. The deflections with this kind of ground motion are very much the same as for $U = U_0 \sin \lambda t$, especially in the neighborhood of resonance. Only two differences are noticeable. The first is that for cosine motion maxima are in the region of flexibility, for sine motion in the region of stiffness. The second difference is that for very flexible structures the deflections due to a sine motion are equal to u_o, while for cosine motion they (or at least the first few) are equal to $2u_o$. That this is reasonable is obvious from the diagram on the right which shows the first cycle of ground motion (cosine) and building motion for a very flexible structure.



When we multiply the values on the graph opposite page 29 by c^2 we get the graph for $\frac{X_m}{\delta}$ shown opposite. That the ordinates would become very large for large values of c could have been expected from the fact that they did so for $\frac{X_m}{\delta}$ in the case of sine motion. That the values of $\frac{X_m}{\delta}$ (for c large) are much smaller in the case of cosine ground motion than for sine motion may be due to the fact that the initial (t = 0) conditions in this case are more reasonable.

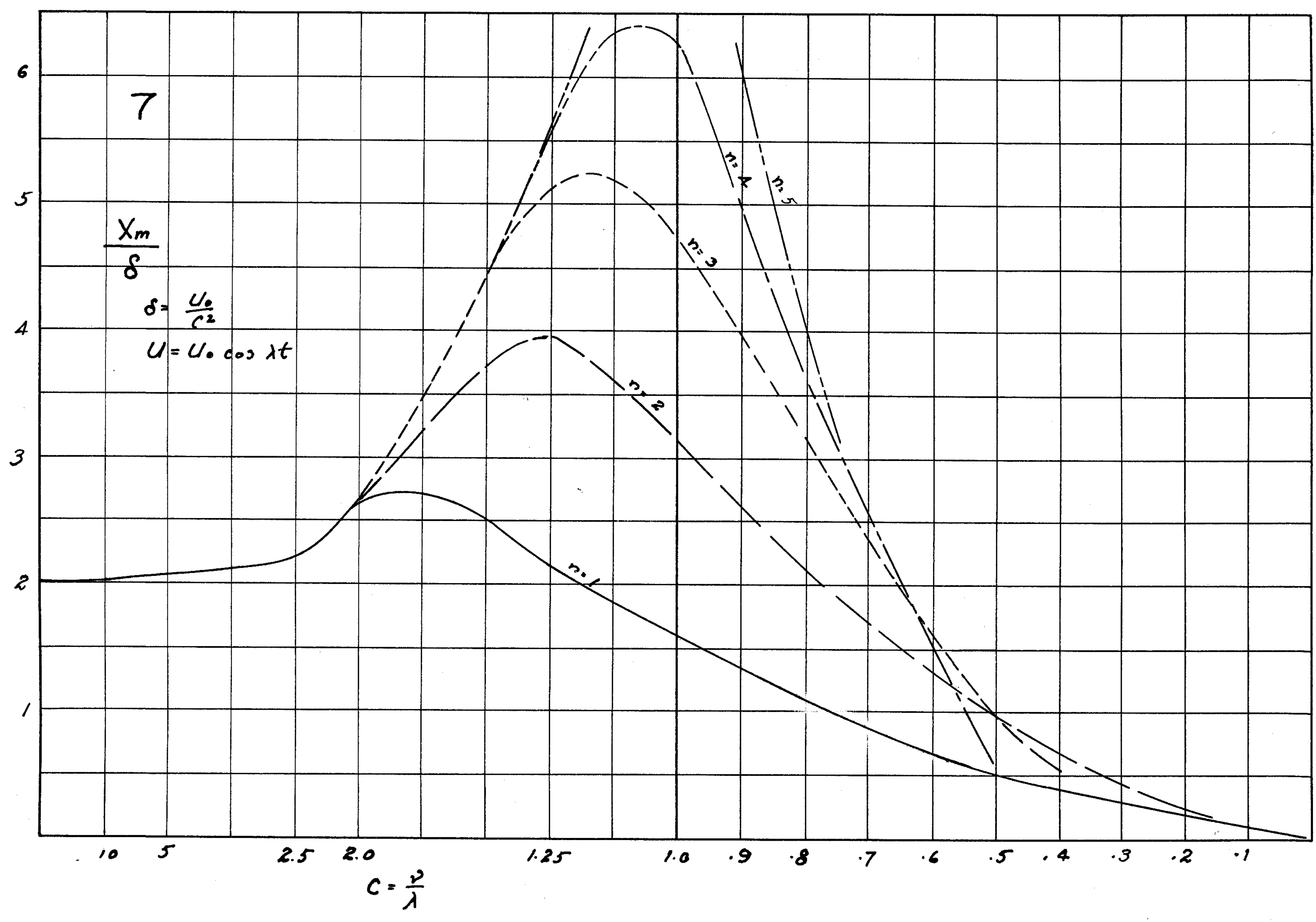
In order to make the work more complete the writer made calculations of $\frac{X_m}{U_0}$ and $\frac{X_m}{\delta}$ for $u = u_o \cos \lambda t$, neglecting the ground velocity in calculating the energy of the structure. In other respects the procedure was exactly as before. It appears


6
$\frac{X_m}{U_0}$
$U = U_0 \cos \lambda t$
n = 4
3
2
1
6
5
4
3
2
1
10
5
2.5
2.0
1.25
1.0
.9
.8
.7
.6
.5
.4
.3
.2
.1
$C = \frac{\nu}{\lambda} = \frac{T_c}{T_B}$

that for a very flexible structure the results so found will be erroneous, while for a stiff building this assumption will be more reasonable than the one previously made. That is, the infinite decelleration of the structure required by suddenly stopping the ground motion will not affect our results if we neglect the contribution of the ground motion to the total energy. These results are shown on the graphs opposite this and the following pages, together with the results obtained experimentally by Ruge at M.I.T. and referred to earlier. It is seen that Ruge's curves are very similar in shape to the writer's. There are rather large differences in values, especially in the neighborhood of resonance. The writer believes that Ruge's experiments were carried out for a continuous ground motion, and therefore only the actual maxima during the ground motion were recorded.

Considering the curves for cosine motion only (4, 5, 6, 7) it seems that for low values of c the first pair of graphs gives more reasonable results while for high values of c the second pair of graphs should be used.

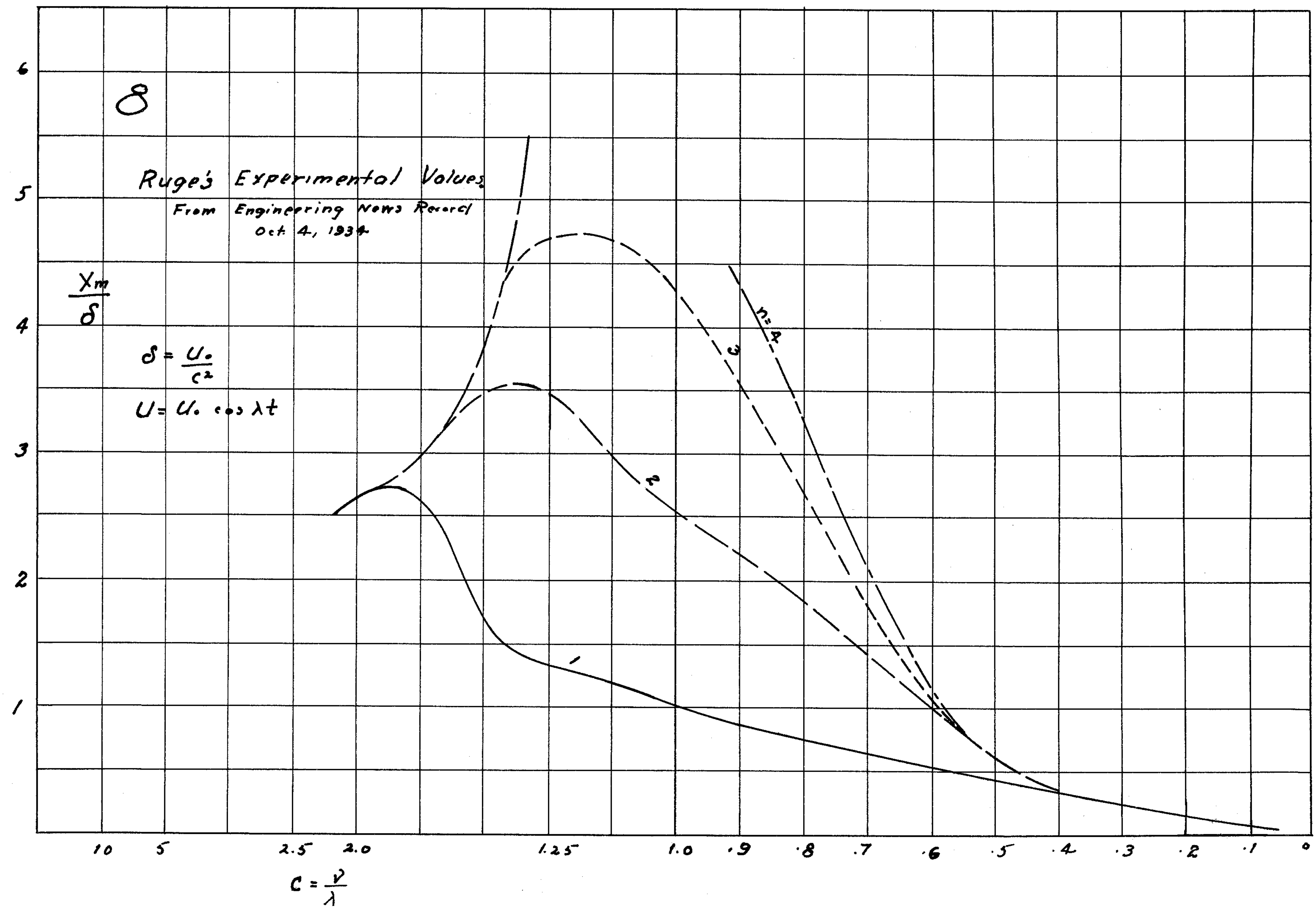
A consideration of these curves shows that the region of partial resonance (where the magnification factor is large) is quite wide and that in this region not many cycles are required to produce deflections several times as great as either the amplitude of the ground motion or the deflection produced by a constant acceleration equal to the maximum ground acceleration. An examination of diagram 1 will show that the critical region is much wider in this case of transient vibrations than it is for the case of a steady motion (shown by the inner broken lines). (Note that the left portions


7
$\frac{X_m}{\delta}$
$\delta = \frac{U_0}{C^2}$
$U = U_0 \cos \lambda t$
n=1
n=2
n=3
n=4
n=5
6
5
4
3
2
1
10
5
2.5
2.0
1.25
1.0
.9
.8
.7
.6
.5
.4
.3
.2
.1
$C = \frac{\nu}{\lambda}$

($c > 1$) of diagrams 1, 2, 3, are somewhat incorrect due to the exagerating effect of the boundary conditions assumed. This has been discussed in connection with the later diagrams (pp. 31 and 32).)

These facts, instead of explaining why earthquake damage is as great as it is, make us wonder why it is no greater. In the San Francisco Earthquake, Davison (A Manual of Seismology) states that the amplitude (half the range of motion) was about two inches with a period of one second. He does not say how long this motion continued or where these measurements were taken (or how obtained). In the Long Beach Earthquake (see Engineering News Record, April 6, 1933) records were obtained showing periods of 1 to 2 seconds associated with accelerations of .05 g ($g = 32.2$ ft/sec^2). If we assume that during this latter motion there were four consecutive ground cycles ($n = 8$) with the same period, buildings in the region of close resonance ($c = .9$ to 1.1) would be affected as by a constant acceleration of $(.05 \times 12.0) = .6$ g, while buildings for which $.7 < c < 1.5$ would be affected about half as much or as by a constant acceleration of .3 g. As was mentioned previously, the usual earthquake resistant design assumes a constant horizontal acceleration of the order of .1 g.

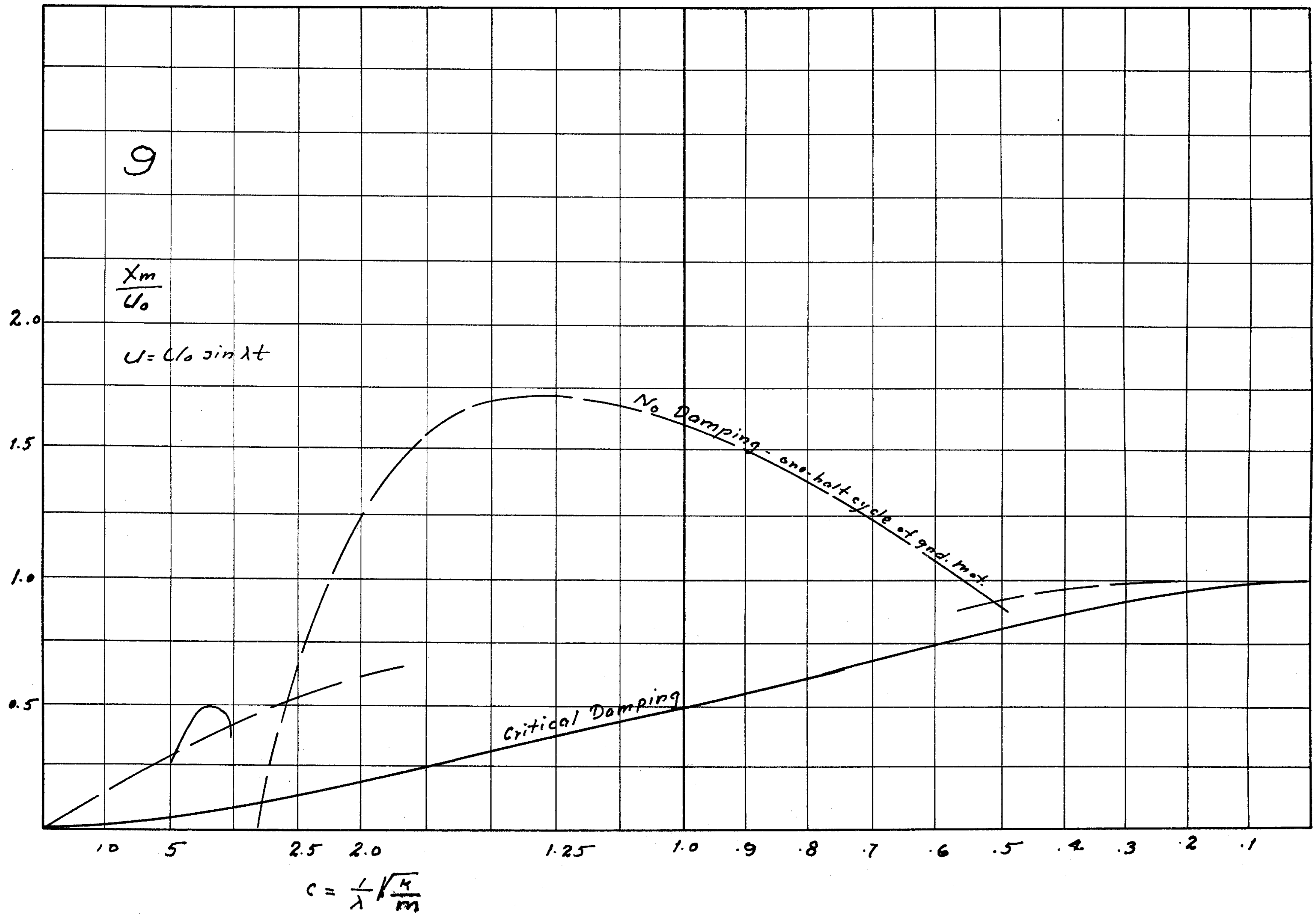
A reconciliation of earthquake records and earthquake damage with the results given above is difficult. Various explanations have occurred to the writer (or have been suggested to him by others). The structure considered in this discussion is quite unlike an actual building, even a one-story building. In a tall, more or less uniform building, it may be that ~~some~~ more time is required before large deflections can occur.



8
Ruge's Experimental Values
From Engineering News Record
Oct. 4, 1934
Xm / δ
δ = U₀ / c²
U = U₀ cos λt
n = 4
3
2
1
6
5
4
3
2
1
10
5
2.5
2.0
1.25
1.0
.9
.8
.7
.6
.5
.4
.3
.2
.1
0
c = ν / λ

Internal damping may possibly be important. However, it does not seem likely that the natural damping of an ordinary building would have much effect on its first oscillations. Failure of various inessential elements of a building (walls and partitions) would use up energy while the rubbing of fractured surfaces would give additional damping. If very high damping were present (as great as critical damping, for example) the effect would be important. It might be possible to artificially produce damping of this sort. A short discussion of this particular kind of damping follows after the next paragraph.

Another explanation of the discrepancy between earthquake records and earthquake damage may be that the records are incorrect. Earthquake accelerations are measured by fast motion seismographs which have fairly high damping. The response of one of these to the first motion of an earthquake might be quite different from the actual motion. Any change in ground motion might also give a distorted record. The writer suggests that a complete series of shaking table tests should be made to discover what effect change of motion would have on the record of an accelerograph.


9
$\frac{X_m}{U_o}$
$U = U_o \sin \lambda t$
No Damping - one-half cycle of gnd. mot.
Critical Damping
2.0
1.5
1.0
0.5
10
5
2.5
2.0
1.25
1.0
.9
.8
.7
.6
.5
.4
.3
.2
.1
$c = \frac{1}{\lambda}\sqrt{\frac{k}{m}}$

EFFECT OF CRITICAL DAMPING IN ONE-STORY BUILDING SUBJECTED TO SIMPLE HARMONIC GROUND MOTION

Let:

$U = U_o \sin \lambda t$

$x = (\dot{x} + \dot{u}) = 0$

at $t = 0$

Let notation be the same as before with the addition of a damping constant f such that

$$f^2 = \frac{K}{m} = (c\lambda)^2$$

then

$$\text{18)} \quad \ddot{x} + 2f\dot{x} + f^2 x = U_o \lambda^2 \sin \lambda t$$

is the equation of motion.

Solving: and using boundary conditions:

$$\frac{X}{U_o} = \frac{1}{1+c^2}\left[e^{-c\lambda t} c \left(\frac{2}{1+c^2} - c\lambda t \right) + \sin(\lambda t - \alpha) \right]$$

where u_o = amplitude of ground motion

$$c = \frac{1}{\lambda}\sqrt{\frac{K}{m}} = \frac{f}{\lambda}$$

$$\alpha = \sin^{-1}\frac{2c}{c^2+1} = \cos^{-1}\frac{c^2-1}{c^2+1}$$

The maximum values of $\frac{X_m}{U_o}$ were found for a single cycle of ground motion (n = 2). The steady state maxima ($t \to \infty$) were also found. These values are tabulated below.

c	0	.25	.50	.70	.75	1.0	2.0	5.0	10.0	∞
$\frac{X_m}{U_o}$ (one cycle)	1.0		.80			.49				0
$\frac{X_m}{U_o}$ (steady state)	1.0	.94	.80	.671	.64	.50	.20	.04	.01	0

Since the steady state values are the upper limit of the deflections they are plotted on the opposite page together with values of $\frac{X_m}{U_o}$ for n = 1 from diagram 1.

The following article by Paul L. Kartzke and the writer appeared in the "Proceedings of the American Society of Civil Engineers" for May 1934. It was written as a Discussion of a paper by Norman B. Green, Consulting Engineer of San Francisco, which appeared in the A. S. C. E. Proceedings for February 1934.

The original paper advocated the use of flexible first-story construction for earthquake resistance. That is: multi-story buildings would be stiffened in the upper stories and made flexible in the first story. The advantages of the design would be: absence of harmonic frequencies and less probability of resonance with the natural frequency.

This paper also suggested the use of acceleration diagrams (similar to the one on the following pages) which could be made up quite arbitrarily and which would represent hypothetical earthquakes. The accelerations from this diagram could then be applied to the base of the building and the resulting building deflections calculated by solving the equation of motion.

Although investigations have been carried on for several years at various laboratories in the United States and in Japan, there is still considerable to be learned in regard to the effect of earthquakes on structures. The author's assumptions that seismic acceleration is a linear function of time and that the initial shock is sudden seem as reasonable as any others that might be made for his analysis. Actually, the final results will be practically independent of either assumption. However, the evidence of an eye-witness of one of the Japanese earthquakes that copper coins in a can were thrown out at the first shock is not proof of a sudden initial acceleration unless the can happened to be so connected to the ground as to follow the ground motion.

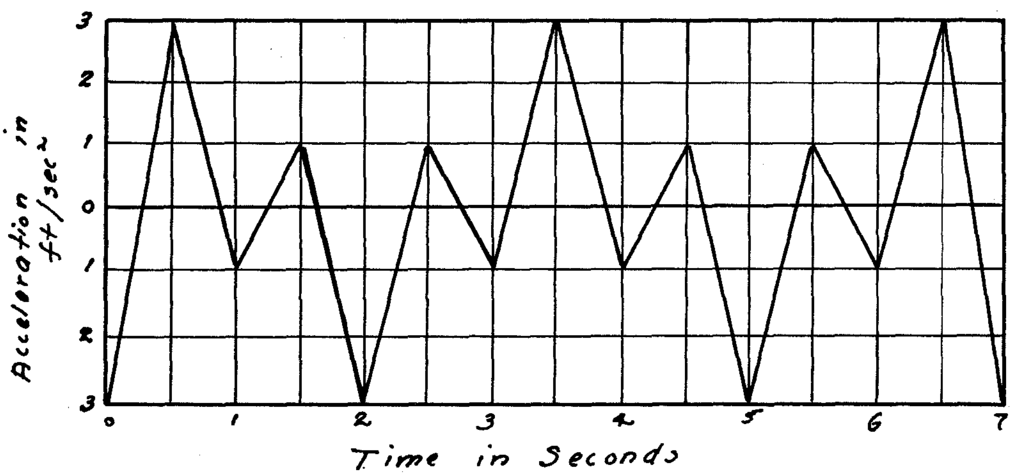
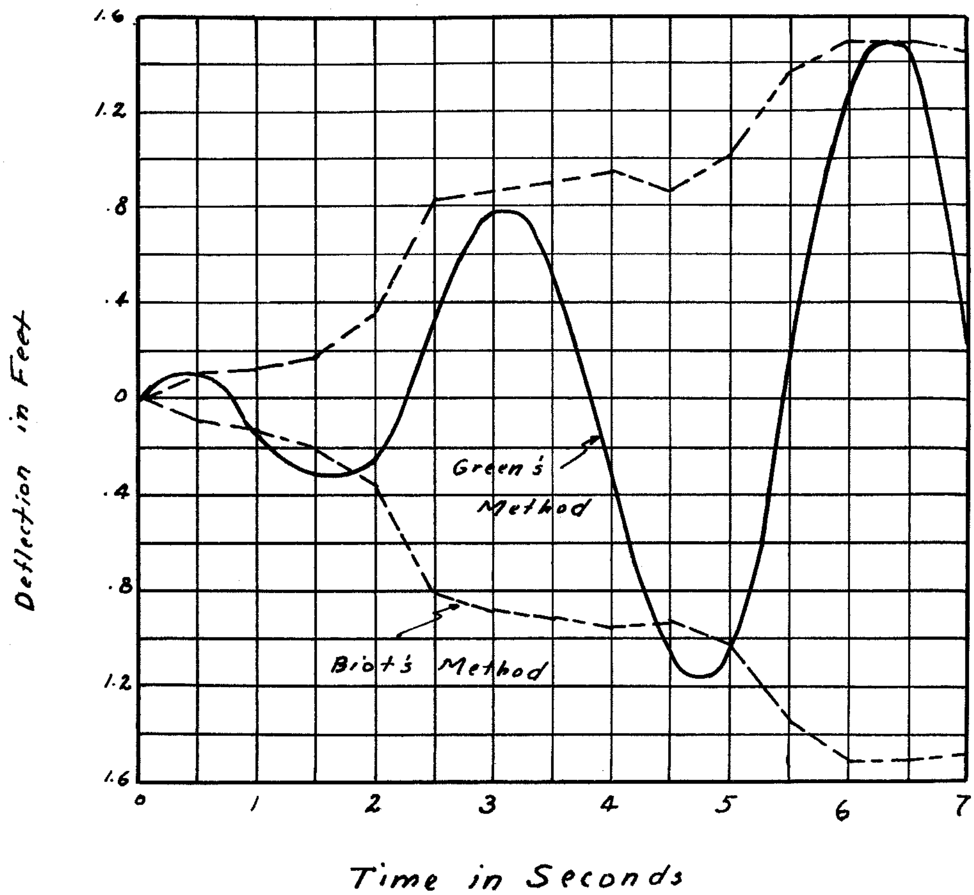


Fig. 9

The author's use of plus and minus signs is rather confusing. His values of Y_n and V_n at the end of each half-second period (Table 1) apparently are considered positive if in the direction of the acceleration at the beginning of that period, and are reversed in sign when they become initial conditions (Y_o, V_o) for the next period. It would seem more logical to give the plus sign to deflections and velocities to the right, and the minus sign to those to the left. Incidentally, if the second integral of the author's acceleration curve is taken it will be found that


1.6
1.2
.8
.4
0
.4
.8
1.2
1.6
Deflection in Feet
0
1
2
3
4
5
6
7
Time in Seconds
Green's Method
Biot's Method

the base of the building is displaced about 20 in. assuming it to start from rest.

Figure 9 shows an acceleration diagram somewhat similar to Figure 3, except for the presence of a rather long period acceleration ($3\frac{1}{2}$ sec), in addition to the shorter waves. This acceleration pattern was applied to Mr. Green's building. The computations, based on the following equations,

$$Y_n = .639\,Y_0 + .4385\,V_0 + .202\,\frac{a}{T} - .117\,a$$

and,

$$V_n = -1.350\,Y_0 + .6392\,V_0 + .117\,\frac{a}{T} - .4385\,a$$

yielded the deflections plotted in Figure 10. The maximum deflection is about 1.5 ft, which is more than six times the deflection obtained by Mr. Green. Probably the next swing would be still greater.

Objection may be taken to the writers' use of a long-period acceleration, particularly in view of Mr. Green's reference to Dr. Suyechiro's conclusion that in all earthquakes the period of acceleration in a particular locality is confined to a very narrow range. An examination of the Long Beach, Vernon, and Los Angeles, Calif., records of the Long Beach earthquake of March, 1933, shows the presence of waves of relatively long period. It is probable that these caused a great part of the damage to buildings with a long free period.

As an independent check on the foregoing results a calculation was made using a method of analysis developed by Dr. M. Biot. This analysis was developed for a building with any number of flexible stories, in which all the stories above the first have the same flexibility. Very briefly, this method consists of considering the actual ground displacement curve, applying appropriate constants which depend on the characteristics of the building, and, by using graphical integration, obtaining an envelope (limiting curve) which will give the limits of the actual displacement curve of the floor for which calculations are made. For the flexible first-story building the equation of this envelope becomes:

$$U_o^2 = \left[\int g(t) \sin 2\pi\nu_o t \, dt\right]^2 + \left[\int g(t) \cos 2\pi\nu_o t \, dt\right]^2$$

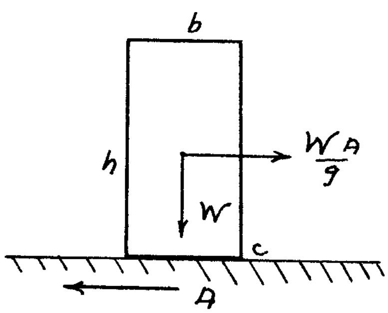
in which, g (t) is the ground displacement curve and ν_o, the fundamental frequency of the building. The results obtained are shown in Figure 10. Dr. Biot's method of analysis has been checked by comparison with actual deflections of a model upon a shaking-table. The agreement is quite satisfactory.

The deflections obtained in this example are obviously excessive. Furthermore, it can be seen that a careful choice of an acceleration pattern will give a similar result for any given structure. The results obtained from Mr. Green's method (or any similar method) will depend on the original choice of acceleration pattern. If one could be sure that all earthquakes in a given region would be alike, this method would be very valuable. Unfortunately, this is not the case and, hence, this method of analysis, as it stands, may be misleading.

CHAPTER FIVE

ROCKING OF A BLOCK ON A HARD FLAT SURFACE

The overthrow by earthquakes of pillars, gravestones and various kinds of blocks resting on flat surfaces is a very common occurrence. This fact is the basis of the "West Formula" for the maximum ground acceleration during an earthquake. This formula is derived by equating the moment of W about point c to the moment of the inertia force about the same point. When an earthquake overthrows a block of dimensions (b, h) it is concluded that the maximum acceleration of the ground during the earthquake motion was greater than A.



$$W \cdot \frac{b}{2} = W \frac{A}{g} \frac{h}{2}$$

$$\therefore \quad A = \frac{bg}{h}$$

A = acceleration of ground

g = acceleration of gravity

This is true, for obviously an acceleration smaller than A would cause no rocking. It is also obvious that a maximum acceleration just equal to A could not cause overturn, while for any greater acceleration the possibility of overturn will depend upon such factors as degree of resonance and duration of motion.

In making the very simple experiment of allowing a block to rock on a hard surface under the action of gravity only, two facts are immediately obvious: the first, that the amplitude of motion decreases very rapidly with time; the second, that the period depends very considerably on amplitude, decreasing with decrease in amplitude.

In an earthquake the initial motion has high frequency and high acceleration but small amplitude. This is followed by a series of slower, longer waves. Experiments with a block on a support which may be moved by hand will show that this kind of motion is ideal for overturning blocks. The high frequency, high acceleration initial motion can cause rocking but may be unable to cause overturn.

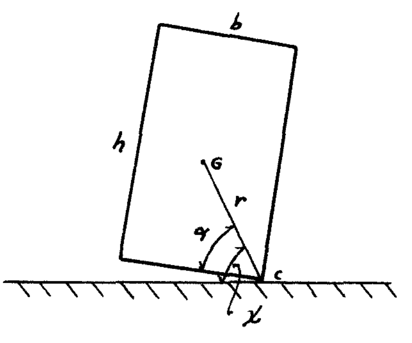
Once rocking starts however, the longer period waves have better chance of resonance while their greater amplitudes transfer more energy to the block during each oscillation. In other words, the initial high acceleration waves may be unable to do more than cause the block to rock while overthrow is due to later waves of greater amplitude and period but much less maximum acceleration. However, without the early waves to cause rocking the later waves could not cause overturning.

ROCKING OF A BLOCK UNDER THE ACTION OF GRAVITY ONLY

Two facts in connection with the free rocking of a block were mentioned on the previous page. The second of these, that period depends on amplitude, is readily explained mathematically.

W = weight of block

x = angle of elevation of center of gravity about point of rotation:



$T = K.E. = \frac{1}{2} I_c \dot{x}^2$

$V = P.E. = W r \sin x$

Lagrange's equation:

1) $I_c \ddot{x} + W r \cos x = 0$

Letting $\frac{W \cdot r}{I_c} = a^2$

$\ddot{x} + a^2 \cos x = 0$

Multiplying by $\dot{x}$ and integrating:

$\frac{\dot{x}^2}{2} + a^2 \sin x + C = 0$

If $x_0 =$ max value of x:

$\dot{x}^2 = 2 a^2 (\sin x_0 - \sin x)$

2) $\frac{dx}{a\sqrt{2}(\sin x_0 - \sin x)^{\frac{1}{2}}} = dt$

Let: $\chi = 2\theta - \frac{\pi}{2}$

$$d\chi = 2\, d\theta$$

$$\sin \chi = -\cos 2\theta = 2 \sin^2\theta - 1$$

Lower limit of motion $\quad \chi = \alpha \qquad \theta = \theta_1 = \frac{\alpha}{2} + \frac{\pi}{4}$

Upper limit of motion $\quad \chi = \chi_0 \qquad \theta = \theta_0 = \frac{\chi_0}{2} + \frac{\pi}{4}$

$$0 \leq \chi \leq \chi_0 \leq \frac{\pi}{2}$$

$$\therefore \quad \frac{\pi}{4} \leq \theta \leq \theta_0 \leq \frac{\pi}{2}$$

$$\therefore \quad \frac{\sin\theta}{\sin\theta_0} \leq 1$$

From (2)

$$3) \quad \frac{d\theta}{a \sin\theta_0 \left(1 - \frac{\sin^2\theta}{\sin^2\theta_0}\right)^{\frac{1}{2}}} = dt$$

since $\sin\theta \leq \sin\theta_0$ we may let:

$$\frac{\sin\theta}{\sin\theta_0} = \sin z$$

$$d\theta = \frac{\sin\theta_0 \cos z\, dz}{(1 - \sin^2\theta_0 \sin^2 z)^{\frac{1}{2}}}$$

From (3)

$$4) \quad \frac{dz}{a\left[1 - \sin^2\theta_0 \sin^2 z\right]^{\frac{1}{2}}} = dt$$

Lower limit of motion $\quad \chi = \alpha; \ \theta = \theta_1 = \frac{\alpha}{2} + \frac{\pi}{4}; \ z = z_1 = \sin^{-1}\left(\frac{\sin \frac{\alpha}{2} + \frac{\pi}{4}}{\sin \frac{\chi_0}{2} + \frac{\pi}{4}}\right)$

Upper limit of motion $\quad \chi = \chi_0; \ \theta = \theta_0 = \frac{\chi_0}{2} + \frac{\pi}{4}, \ z = z_0 = \frac{\pi}{2}$

Integrating (4):

$$T_2 - T_1 = \frac{1}{a}\int_1^2 \frac{dz}{\left[1-\sin^2\theta_0 \sin^2 z\right]^{\frac{1}{2}}}$$

The period of a semioscillation:

$$T_3 = \frac{2}{a}\int_{z_1}^{\pi/2} \frac{dz}{\left[1-\sin^2\theta_0 \sin^2 z\right]^{\frac{1}{2}}}$$

$$T_3 = \frac{2}{a}\left[K(k) - F(k, z_1)\right]$$

$$T_3 = \frac{2}{a} R$$

Where:

$$z_1 = \sin^{-1}\left(\frac{\sin\frac{\alpha}{2} + \frac{\pi}{4}}{\sin\frac{\gamma_0}{2} + \frac{\pi}{4}}\right)$$

$$\theta_0 = \frac{\gamma_0}{2} + \frac{\pi}{4}$$

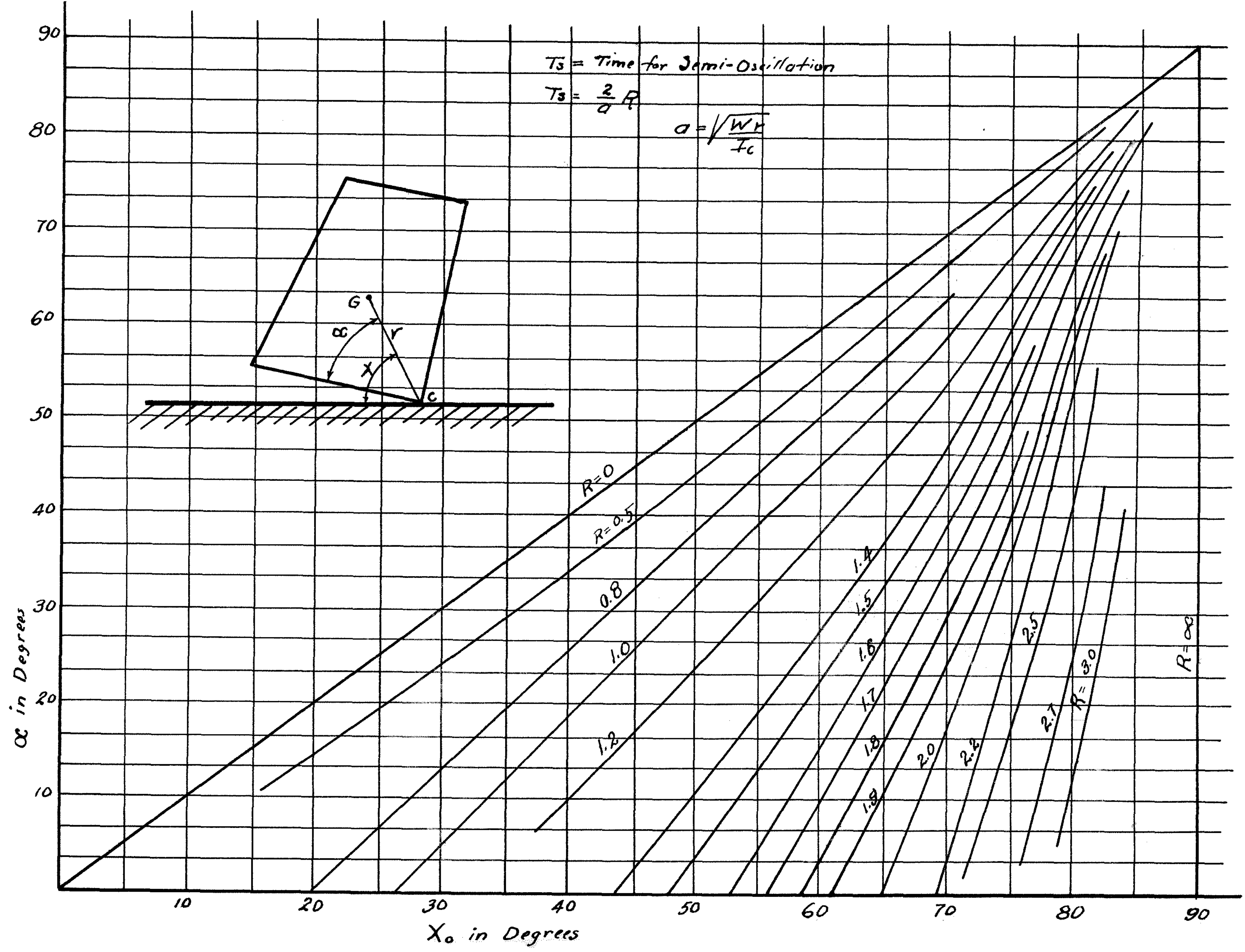
$$k = \sin^{-1}\theta_0$$

$$a^2 = \frac{W \cdot r}{I_c}$$

K = Complete Elliptic Integral of 1st kind

F = Elliptic Integral of 1st kind

$R = K(k) - F(k, z_1)$



T_s = Time for Semi-Oscillation
$T_s = \frac{2}{a} R$
$a = \sqrt{\frac{Wr}{I_c}}$
G
α
r
X
c
R=0
R=0.5
0.8
1.0
1.2
1.4
1.5
1.6
1.7
1.8
1.9
2.0
2.2
2.5
2.7
R=3.0
R=∞
X_o in Degrees
α in Degrees
10
20
30
40
50
60
70
80
90

The chart facing this page shows the dependence of period on amplitude. The values given are suitable for any shape of block which rotates about a line perpendicular to the paper through point c. On the figure: G is the center of gravity; r the distance from G to axis of rotation; α is the angle of elevation of G when block is in neutral position; x is the angle of elevation of G at any time, α is the lower limit of x, x_o (not shown) is the upper limit. T_s is the semi-period of oscillation of the block; that is, the time from one neutral position to the next. $a^2 = \frac{Wr}{I_c}$ where W = the weight of the block, and I_c equals the moment of inertia of the mass of the block about the axis of rotation. For a rectangular block of base b and height h, $a^2 = \frac{3g}{4r}$.

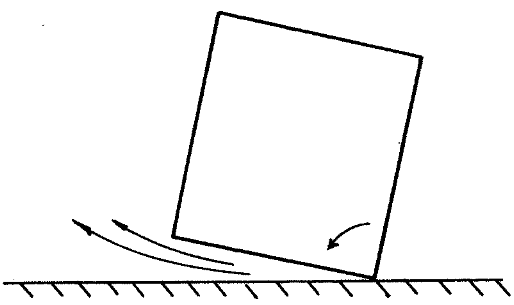
To use these curves, α and a must first be computed. Then, for any particular value of x_o (maximum value of x) R can be found. R, multiplied by $\frac{2}{a}$ will give T_s.

Example: consider a block 1' x 1'. Let x_o be 75°. Then a equals $\sqrt{\frac{3(32.2)}{4(.707)}} = 5.84$, $\alpha = 45°$, $R = 1.8$, $T_s = \frac{2(1.8)}{5.84} = .62$ sec.
The weight of the block is not needed.

On this chart the line R = 0 corresponds to $x_o = \alpha$ while the line $R = \infty$ corresponds to $x_o = 90°$.

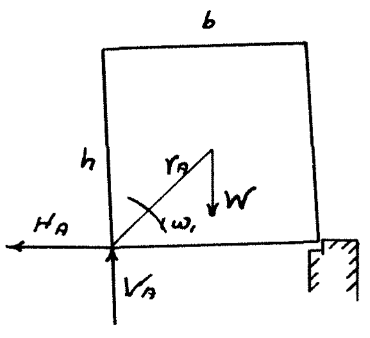
The other of the two phenomena mentioned in the first part of this discussion, namely that decrease in amplitude with time is very rapid, is more difficult to analyse mathematically. A solution is obtained, depending on a number of assumptions. The writer believes these to be reasonable but realizes that a much more exact method could be developed.

When a block oscillates to and fro on its base it loses energy rapidly. This loss is much too great to be caused by air resistance during the swing. Therefore it must occur during, or in the neighborhood of the passage through the neutral position. One of the first possibilities to suggest itself is that energy is used up in compressing and squeezing out the air under the block, especially just before impact, and in overcoming a slight vacuum after impact. This effect might be important in the case of light blocks with long bases. However,

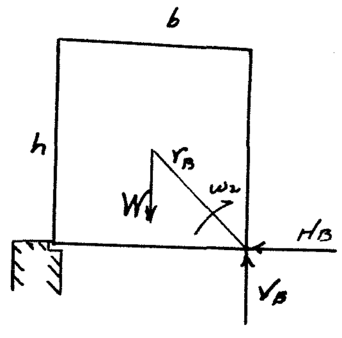


experiments with wooden blocks $3\frac{1}{2}$" x $3\frac{1}{2}$" x 16" (with the 16" dimension vertical) rocking on flat surfaces and on two parallel strips half an inch above a flat surface gave no noticeable difference between these cases. (But since no quantitative results were obtained, it is possible that this effect is measurable.)

A continuation of these simple experiments indicated that the relative energy loss during a transition depended primarily on the shape of the block, that the weight, material and size made little or no difference. The calculation which follows gives results which agree with these observations.



Before Impact



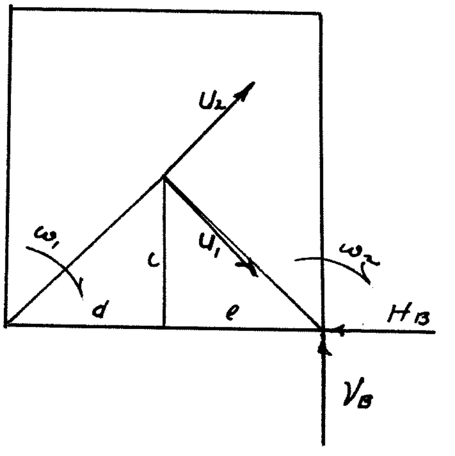
After Impact

First assumption: that reactions occur only at corners of block, and that no slipping occurs.

If $b \gg h$ we know that on impact the reaction at A will be increased by the block attempting to rotate in a counter clockwise fashion about B. If $b \ll h$ the block will rotate about B in a clockwise direction and the reaction at A will become zero. Let us first find the limiting case, in which the reaction at A becomes zero and the block stops completely.

Let the time required for completion of the transition from one motion to the other be Δt. During this time the reactions at A will change from maximum values of the order of magnitude of W (weight of block) to zero, while the reactions at B increase from zero to some maxima whose magnitudes are unknown. If the block and support are quite rigid then Δt will be small and the reactions at B will necessarily be large.

It is also assumed that in writing momenta equations the effect of the reactions at A and the weight W may be neglected in comparison with the effect of the reactions at B. This requires that the ratio $\frac{h}{b}$ is not great enough to cause any large increase in the forces at A.



Consider that during impact the angular velocity of the block changes from ω_1 about A to ω_2 about B while the linear velocity of G changes from u_1 to u_2.

$$\int H_B\,dt = \frac{W}{g}(c\omega_1 - c\omega_2)$$

$$\int V_B\,dt = \frac{W}{g}(d\omega_1 + e\omega_2)$$

$$\int (eV_B - cH_B)\,dt = I_G(\omega_1 - \omega_2)$$

Combining these to eliminate V and H

$$\frac{W}{g}(ed\omega_1 + e^2\omega_2 - c^2\omega_1 + c^2\omega_2) = I_G(\omega_1 - \omega_2)$$

If radius of gyration about G = k; $I_G = \frac{W}{g}k^2$

5) $$\frac{\omega_2}{\omega_1} = \frac{c^2 + k^2 - ed}{c^2 + k^2 + e^2}$$

If $\omega_2 = 0$ (limiting case)

$$c^2 + k^2 = ed$$

For a rectangular block

$$e = d = \frac{b}{2};\ c = \frac{h}{2};\ k^2 = \frac{1}{12}(b^2 + h^2) \qquad \text{Let } \frac{b}{h} = n$$

putting in these values

$$b = h\sqrt{2}$$

If $b > h\sqrt{2}$ block will not rotate about point B, and all its initial energy will be lost (used up in rebounds). If $e = d = 0$ obviously there is no loss. For intermediate shapes the loss will depend on the shape. For rectangular blocks:

n	0	.1	.2	.3	.4	.5	.7	1.0	$\sqrt{2}$	
$\frac{\omega_2}{\omega_1}$	1	.985	.943	.876	.793	.700	.507	.250	0	
$\left(\frac{\omega_2}{\omega_1}\right)^2$	1.00	.970	.888	.768	.629	.490	.257	.062	0	= Efficiency

For any other shape of block the efficiency may be easily calculated.

This section may be combined with the previous section to determine the motion of a block rocking under the action of gravity.

Example: Consider a block 8" x 20" with 20" dimension vertical. Its efficiency during the transition is .629 ; $\alpha = 68^\circ 12'$

Let $x_o = 80^\circ$.

Let us call E_1 the total energy (referred to neutral position of the block) during the first swing (until the first passage through the neutral position); E_2 the total energy during the second swing (until the second passage through the neutral position); E_3, E_4 . . . likewise.

Let us call T_1 the time to correspond to E_1, T_2 to E_2, etc.

Then: $E_1 : E_2 : E_3 : E_4 : \ldots = 1 : .629 : .396 : .249 \ldots$

Since $E = W\, r\,(\sin x_o - \sin\alpha)$ $r =$ semi-diagonal

$$X_{o1} = 80^\circ \quad \therefore \frac{E_1}{W\cdot r} = .9848 - .9285 = .0563$$

$$\frac{E_2}{W\cdot r} = .0563(.629) = .0354 \quad \therefore \sin X_{o2} = .0354 + .9285 = .9639$$

$$\therefore X_{o2} = 74^\circ 34'$$

$$\frac{E_3}{W\cdot r} = .0563(.629)^2 = .0223 \quad \therefore \sin X_{o3} = .0223 + .9285 = .9508$$

$$\therefore X_{o3} = 71^\circ 57'$$

$$\frac{E_4}{W\cdot r} = .0563(.629)^3 = .0141 \quad \therefore \sin X_{o4} = .0141 + .9285 = .9426$$

$$\therefore X_{o4} = 70^\circ - 31'$$

. . . .

$$a^2 = \frac{3\cdot 32.2\cdot 12}{4\cdot 10.77} \quad ; \quad a = 5.185$$

From diagram opposite page 43

$$R_1 = 1.75 \qquad T_1 = \frac{1.75}{5.185} = .34 \text{ sec.}$$

$$R_2 = .93 \qquad T_2 = \frac{2(.93)}{5.185} = .36 \text{ ''}$$

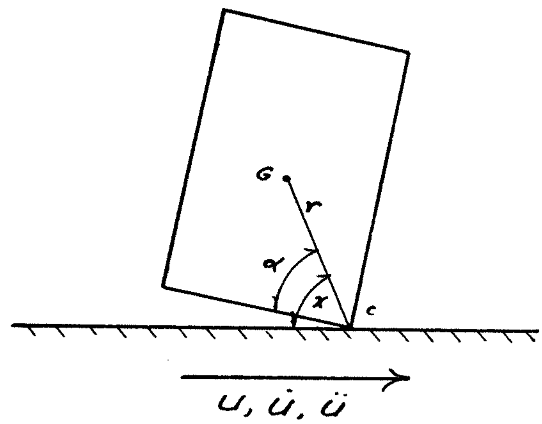
$$R_3 = .55 \qquad T_3 = \frac{2(.55)}{5.185} = .21 \text{ ''}$$

$$R_4 = .40 \qquad T_4 = \frac{2(.40)}{5.185} = .15 \text{ ''}$$

As yet, the analysis has no direct bearing on the problem mentioned at the beginning of this chapter, namely the determination of earthquake intensities (this word is used intentionally to include the effect of both acceleration and amplitude as well as duration of the ground motion) from data on the overthrow of blocks of various dimensions. Obviously no such determination is possible because of the fact, previously stated, that acceleration, amplitude and duration of the motion as well as the manner of variation of the kind of motion are all of fundamental importance, and, to a considerable extent, independent and unpredictable.

However, it will be possible to calculate the motion of a block whose base is subjected to a given displacement. For a given earthquake the effect on a particular block may be determined.

Consider a block on a moving base whose displacement is u. Other symbols may be carried over from previous work. The energy lost during each transition will be the same as before.



The velocity of G is given by:

$$U_G^2 = (r\dot{x})^2 + \dot{u}^2 + 2\dot{u} r \dot{x} \sin x$$

$$T_{KE} = \frac{1}{2} I_G \dot{x}^2 + \frac{W}{2g}\left[(r\dot{x})^2 + \dot{u}^2 + 2\dot{u} r \dot{x} \sin x\right]$$

$$T = \frac{1}{2} \dot{x}^2 I_c + \frac{W}{2g}\left(\dot{u}^2 + 2\dot{u} r \dot{x} \sin x\right)$$

V = W r sin x

From LaGrange's equations:

6) $$I_c \ddot{x} + W \cdot r \left(\cos x + \frac{\ddot{u}}{g} \sin x\right) = 0$$

If we assume that ü is constant and equal to n·g.

$$I_c \ddot{x} + W \cdot r (\cos x + n \sin x) = 0$$

Letting: $\frac{n}{\sqrt{1+n^2}} = \cos\beta$ $\quad \frac{1}{\sqrt{1+n^2}} = \sin\beta$

$$\ddot{x} + \frac{Wr}{I_c}\sqrt{1+n^2} \sin(x+\beta) = 0$$

Letting $\frac{Wr}{I_c}\sqrt{1+n^2} = a^2$ and $x+\beta = y$

7) $$\ddot{y} + a^2 \sin y = 0$$

Multiplying (7) by $\dot y$ and integrating:

$$8)\quad \frac{\dot y^2}{2} - a^2\cos y + C = 0$$

But $\dot y = \dot x = \omega_1$ when $y = y_1$

$$8')\quad \dot y = a\sqrt{2}\left(\cos y - \cos y_1 + \frac{\omega_1^2}{2a^2}\right)^{\frac{1}{2}}$$

Let $y = 2\theta$

$\cos y = 1 - 2\sin^2\theta$

$$9)\quad \frac{d\theta}{a\left[\sin^2\theta_1 + \frac{\omega_1^2}{4a^2} - \sin^2\theta\right]^{\frac{1}{2}}} = dt$$

Let $\sin^2\theta_1 + \frac{\omega_1^2}{4a^2} = B^2$

$$10)\quad \frac{d\theta}{aB\left(1 - \frac{\sin^2\theta}{B^2}\right)^{\frac{1}{2}}} = dt$$

Case I: $B^2 > 1$; Integrating (10) gives:

$$11)\quad T_2 - T_1 = \frac{1}{aB}\left[F\left(\tfrac{1}{B}, \theta_2\right) - F\left(\tfrac{1}{B}, \theta_1\right)\right]$$

F = Elliptic Integral of first kind.

Case II $B^2 < 1$

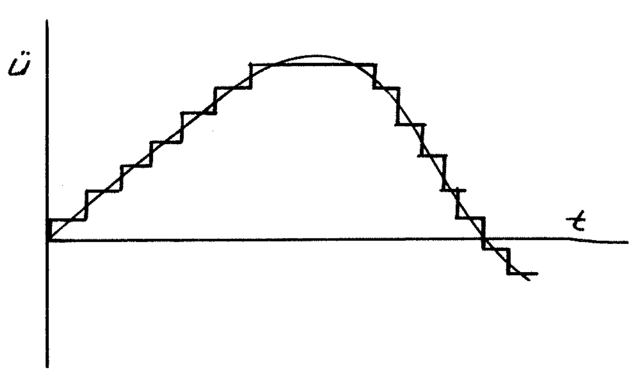
Let $\frac{\sin^2\theta}{B^2} = \sin^2 z$ giving:

$$dt = \frac{dz}{a(1 - B^2\sin^2 z)^{\frac{1}{2}}}$$ (see p. 420)

$$12)\quad T_2 - T_1 = \frac{1}{a}\left[F(B, \theta_1) - F(B, \theta_2)\right]$$

B can not be less than $\sin\theta$ because this would make the radical imaginary. $B = \sin\theta$ corresponds to maximum displacement of block. At this point $\frac{d\theta}{dt} = 0$ and the block prepares to reverse its direction.

These equations are derived for constant ground acceleration. In case the ground acceleration varies it will be necessary to consider the acceleration as a series of constant accelerations, each acting for a short time.

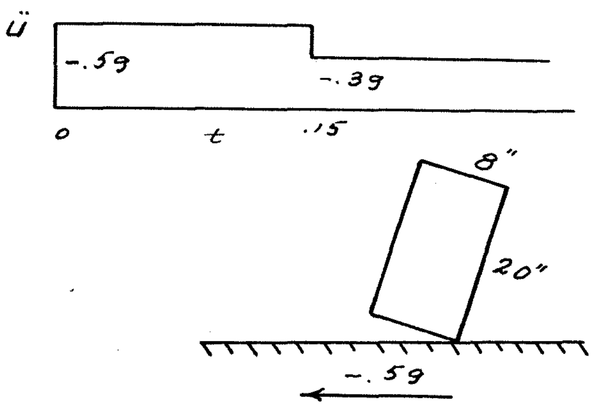


Example: consider same block previously used as an example starting from rest. (8" x 20"; α = 68°-12'; efficiency = .629) From the West Formula the smallest acceleration that will start rocking is:

$$A = \frac{bg}{h} = 0.4g$$

Let an acceleration of -.5g act for .15 seconds and be followed by an acceleration of -.3g.

1st Interval



$$\beta = \cos^{-1}\frac{-.5}{1.118} = 116°-35'$$

$$a^2 = \frac{3g}{4r}(1.118)$$

$$a = 5.49$$

$$\omega_1 = 0$$

$$\chi_1 = \alpha = 68°-12'$$

$$\gamma_1 = \chi_1 + \beta = 184°-47'$$

$$\theta_1 = \tfrac{1}{2}\gamma_1 = 92°-23\tfrac{1}{2}'$$

$$B^2 = \sin^2\theta_1 \ (\text{since } \omega_1 = 0)$$

$$B = .9991$$

There are three possibilities to consider:

1) Sin θ may become equal to B, in this case the motion will be reversed.

2) The acceleration will cease.

3) The value of x will exceed 90° and the equations must be rewritten for a block whose base is h and altitude b.

In this example case 1) is impossible because the lever arm of W about the point of rotation decreases as x increases while the ground [redacted] acceleration is constant.

Considering possibility 3)

$$X_2 = 90^\circ \quad y_2 = 90^\circ\text{-}116\text{-}35 \quad \theta_2 = 103^\circ\text{-}17$$

$$T = \frac{1}{5.49}\left[F(\beta, 103\text{-}17) - F(\beta, 92\text{-}23\tfrac{1}{2})\right]$$

$$\text{Since } F(\alpha, 90+\beta) = 2F(\alpha, 90) - F(\alpha, 90-\beta)$$

$$T = \frac{1}{5.49}\left[F(.9991, 87\text{-}36\tfrac{1}{2}) - F(.9991, 76\text{-}47)\right]$$

$$T = \frac{1}{5.49}(3.6346 - 2.1363) = 0.272 \text{ sec.}$$

Therefore acceleration ceases before $x = 90^\circ$ and possibility 2) must be considered.

$$\text{Let } 180^\circ - \theta_2 = \zeta$$

$$T_2 - T_1 = .15 = \frac{1}{5.49}\left[F(.9991, \zeta) - 2.1363\right] \quad \therefore F(.9991, \zeta) = 2.9598$$

$$\therefore \zeta = 84.7^\circ; \ \theta_2 = 95.3^\circ; \ y_2 = 190.6^\circ \quad x_2 = 74.0^\circ$$

2nd Interval:

$n = -.3 \quad x_1$ (x_2 from preceding) $= 74^\circ\text{-}00$

In preceding interval $\omega_2 = \frac{dy_2}{dt} = 5.49 r_2 (\cos y_2 - \cos y_1) = +.905$

which equals ω_1 in 2nd interval.

$$\beta = \cos^{-1}\frac{-.3}{\sqrt{1.09}} = 106^\circ - 42'$$

$$y_1 = x_1 + \beta = 180^\circ - 42 \quad \theta_1 = 90\text{-}21$$

$$a^2 = \frac{39}{4r}\sqrt{1.09} \qquad a = 5.30$$

$$B^2 = 1 + \frac{(.905)^2}{4(28.1)} = 1.0073 \qquad B = 1.0037$$

Since B > 1 we treat under case 1 (page 50).

$$T_2 - T_1 = \frac{1}{1.0037(5.30)}\left[F(.9963, \theta_2) - F(.9963, \theta_1)\right]$$

There are two possibilities:

1) The acceleration will cease.

2) The value of x will exceed 90°.

Investigating 2)

$$\begin{array}{ll} x_1 = 74\text{-}00 & x_2 = 90^\circ\text{-}00 \\ \beta = 106\text{-}42 & \beta = 106\text{-}42 \\ y_1 = 180\text{-}42 & y_2 = 196\text{-}42 \\ \theta_1 = 90\text{-}21 & \theta_2 = 98\text{-}21 \end{array}$$

$$T = \frac{1}{5.32}\left[F(.9963, 98\text{-}21) - F(.9963, 90\text{-}21)\right]$$

$$T = \frac{1}{5.32}\left[F(\sin 85\text{-}05, 89^\circ\text{-}39) - F(\sin 85\text{-}05, 81^\circ\text{-}39')\right]$$

*
$$T = \frac{1}{5.32}(4.2 - 2.49) = \underline{0.322} \text{ sec.}$$

Therefore x will equal 90° at the end of .322 seconds if the acceleration continues.

There is no reason for continuing this example. The procedure would be a repetition of the preceding. We must remember that the energy and therefore ω is reduced at each transition. Careful attention must be paid to algebraic signs, while a clear picture of the motion must be kept in mind.

Obviously this process is quite cumbersome. A calculation of the motion of a block during an earthquake of ordinary duration would probably require several days. Curves for the values of the elliptic function $F(k, \phi)$ would greatly facilitate this calculation.

The writer suggests that an experimental investigation of the formulae developed in this chapter would not be difficult. Such an investigation could be carried out by a graduate student as a portion of the research required of him for the Master of Science degree.

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* Note: Numerical values of elliptic integrals are from "Pierce's Tables", and because of interpolation are quite approximate.

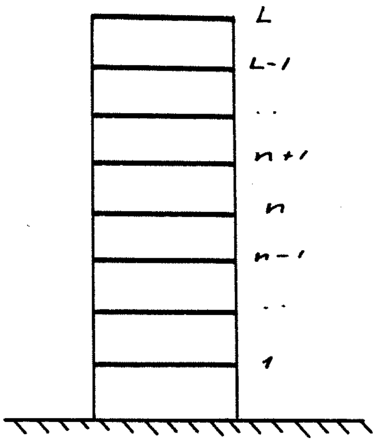
CONCLUSIONS:

Formulas, based on certain simplifying assumptions have been derived for the motion of a block rocking on its base on a hard surface which may be at rest or may move. For the block on a fixed base the calculations are quite simple. If the base is accelerated the calculations are somewhat complicated, chiefly because a variable acceleration must be approximated by a series of constant accelerations. For a particular base motion the motion of the block can be calculated. However, data on the overthrow of blocks during an earthquake are not sufficient to fix the intensity of the quake.

CHAPTER SIX

MULTI-STORY BUILDINGS

PART I: THE NATURAL FREQUENCIES OF VIBRATION OF A UNIFORM MULTI-STORY BUILDING



L = number of floors

K = stiffness of each story (force for unit deflection)

m = mass (W/g) of each floor

y_n = displacement of nth floor

T = kinetic energy

W = potential energy (due to deformations)

1) $T = \frac{m}{2}\dot{y}_1^2 + \cdots \frac{m}{2}\dot{y}_n^2 + \cdots\cdots \frac{m}{2}\dot{y}_L^2$

2) $W = \frac{K}{2}y_1^2 + \cdots \frac{K}{2}(y_n - y_{n-1})^2 + \frac{K}{2}(y_{n+1} - y_n)^2 + \cdots \frac{K}{2}(y_L - y_{L-1})^2$

or 3) $W = \cdots\cdots K\left(y_n^2 - y_n(y_{n-1} + y_{n+1})\right) + \cdots \frac{K}{2}(y_L - y_{L-1})^2$

Using La grange's equations:

$$\ddot{y}_1 + \frac{K}{m}(2y_1 - y_2) = 0$$

. . . .

4) $$\ddot{y}_n + \frac{K}{m}(2y_n - y_{n-1} - y_{n+1}) = 0$$

. . . .

$$\ddot{y}_L + \frac{K}{m}(y_L - y_{L-1}) = 0$$

Let (5) $y_n = A_n e^{i\omega t}$

and 6) $\left(2 - \omega^2 \frac{m}{K}\right) = C$

Substituting these in equation (4) we get the following homogeneous equations written as a determinant:

$$
7) \quad
\begin{array}{cccccccccccc}
A_1 & A_2 & A_3 & \ldots & & & & & A_L & & \\
c & -1 & 0 & 0 & 0 & . & 0 & 0 & 0 & = & 0 \\
-1 & c & -1 & 0 & 0 & . & 0 & 0 & 0 & = & 0 \\
0 & -1 & c & -1 & 0 & . & 0 & 0 & 0 & = & 0 \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & & \cdot \\
0 & 0 & 0 & 0 & 0 & . & -1 & c & -1 & = & 0 \\
0 & 0 & 0 & 0 & 0 & . & 0 & -1 & (c-1) & = & 0
\end{array}
$$

There are L homogeneous equations containing L unknown coefficients ($A_1 . . . . A_L$). In order to have any solutions (except $A_1 = A_2 = . . . = 0$) the determinant itself must equal zero, which means that c can have only certain definite values, determined from this fact. Since ω depends on c (equation (6)) we see that there is a certain definite frequency for each c. We expect L different values of c and therefore of ω^2.

Using a method and a notation to be found in many texts on Mechanics, we let:

D_n = value of a determinant consisting of the first n rows and columns of our Lth order determinant.

D_{L-1} = value of all but last row and last column of our determinant.

8) $D_L = (c-1)\, D_{L-1} - D_{L-2}$ = value of our determinant.

9) $D_{L-1} = c\, D_{L-2} - D_{L-3}$ (by expanding in terms of the minors of the last row and last column)

From the trigonometric relation:

$D \sin(\alpha+\beta) + D \sin(\alpha-\beta) = 2D \sin\alpha \cos\beta$

letting $D_{n-1} = D \sin n\theta$ and trying $c = 2\cos\theta$

we may rewrite (8) as: $D_L = (c-1)\, D \sin L\theta - D \sin (L-1)\theta$

When $L = 1$ $\quad D \sin 2\theta = C = 2\cos\theta$

$\therefore \quad D = \dfrac{1}{\sin\theta}$ and

$$10)\quad D_L = (2\cos\theta - 1)\frac{\sin L\theta}{\sin\theta} - \frac{\sin(L-1)\theta}{\sin\theta}$$

But $D_L = 0$

It can be shown that equation 10 becomes:

$$11)\quad \sin(L+1)\theta = \sin L\theta$$

$$\therefore\ (L+1)\theta + L\theta = \pi,\ 3\pi \cdots\cdots (2L-1)\pi$$

$$6)\quad \text{But } C = 2 - \omega^2\frac{m}{K} = 2\cos\frac{2n-1}{2L+1}\pi \qquad n = 1, 2 \cdots L$$

$$\omega_n^2\,\frac{m}{K} = 2\left(1 - \cos\frac{2n-1}{2L+1}\pi\right) = 4\sin^2\frac{2n-1}{2L+1}\,\frac{\pi}{2}$$

$\therefore$

$$12)\quad \omega_n = 2\sqrt{\frac{K}{m}}\,\sin\frac{2n-1}{2L+1}\,\frac{\pi}{2} \qquad n = 1, 2 \cdots\cdots L$$

If L = 1 (one story building)

$$\omega_0 = \sqrt{\frac{K}{m}}$$

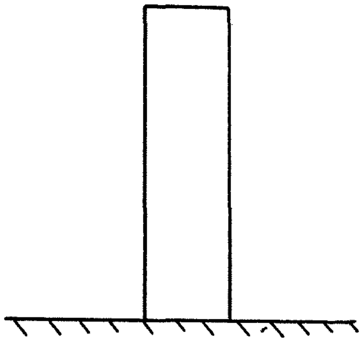
$\therefore$

$$13)\quad \frac{\omega_n}{\omega_0} = 2\sin\frac{2n-1}{2L+1}\,\frac{\pi}{2}$$

If L is large compared to n we get:

$$14)\quad \frac{\omega_n}{\omega_0} = (2n-1)\,\frac{\pi}{2L} \qquad n = 1 \cdots\cdots L$$

Vibration of uniform cantilever which deflects only in shear. A building of moderate height may be so represented.



Let m = mass per unit length

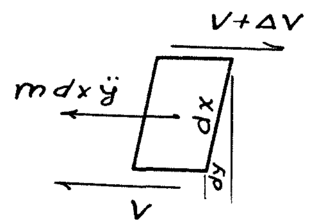
l = length

k = shear stiffness (force which will cause unit deflection in unit length)

y = deflection of any point

x = distance from base

$$y'' = \frac{d^2y}{dx^2} \qquad \ddot{y} = \frac{d^2y}{dt^2}$$



$$\Sigma F_y = 0$$

$$15)\quad \Delta V - m\,dx\,\ddot{y} = 0 \qquad \text{or} \qquad \frac{dV}{dx} = m\ddot{y}$$

From definition of k

$$V = K\frac{dy}{dx}$$

Combining these:

$$16)\quad K y'' = m\ddot{y} \qquad y'' = \frac{m}{K}\ddot{y}$$

Let y = X sin wt where X is a function of x only.

$$X'' = -\frac{m}{K}\omega^2 X$$

$$X = Y_0 \sin\left(\omega\sqrt{\frac{m}{K}}\,x - \alpha\right)$$

Boundary conditions:

1) X = 0 at x = 0

2) X′ = 0 at x = l

$$17\begin{cases} X = Y_0 \sin \omega\sqrt{\frac{m}{K}}\,x \\ X' = \omega\sqrt{\frac{m}{K}}\,Y_0 \cos \omega\sqrt{\frac{m}{K}}\,x \end{cases}$$

$$\therefore$$

$$y = Y_0 \sin \omega\sqrt{\frac{m}{K}}\,x \,\sin \omega t$$

From condition 2)

$$\omega \sqrt{\frac{m}{K}}\, l = (2n-1)\frac{\pi}{2} \qquad n = 1.\ 2.\ 3.\ \ldots\ldots$$

18) $\omega_n = \sqrt{\frac{K}{m}}\ \frac{2n-1}{l}\ \frac{\pi}{2}$ which agrees with (14)

19) $y = Y_0 \sin(2n-1)\frac{\pi x}{2l} \sin \omega_n t$

PART II: FORCED VIBRATION OF UNIFORM SHEAR PANEL

Use same nomeclature as before and let u (t) be displacement of base.

The acceleration of any point (x) will now be: $\ddot{y} + \ddot{u}$

By analogy with equation (16)

20) $y'' = \frac{m}{K}(\ddot{y} + \ddot{u})$

The solution of this equation will be the sum of a particular solution plus the solution of the homogeneous equation which we have already found (equation 19).

20a) Let $y = \sum q_i \sin\frac{i\pi x}{l} \qquad i = \frac{1}{2}, \frac{3}{2}, \frac{5}{2} \ldots\ldots$

then $\ddot{y} = \sum \ddot{q}_i \sin\frac{i\pi x}{l}$

and $y' = \sum \frac{i\pi}{l} q_i \cos\frac{i\pi x}{l}$

If base displacement is [redacted] u, the velocity of any point in building =

$$\dot{u} + \dot{y} = \dot{u} + \sum \dot{q}_i \sin\frac{i\pi x}{l}$$

Kinetic Energy

$$T = \frac{m}{2}\int_0^{l}\left(\dot{u} + \Sigma \dot{q}_i \sin\frac{i\pi x}{l}\right)^2 dx = \frac{ml}{2}\left[\dot{u}^2 + 2\dot{u}\Sigma\frac{\dot{q}_i}{i\pi} + \frac{1}{2}\Sigma \dot{q}_i^2\right]$$

Potential Energy

$$W = \frac{K}{2}\int_0^{l}\left(\Sigma \frac{i\pi}{l} q_i \cos\frac{i\pi x}{l}\right)^2 dx = \frac{K\pi^2}{4l}\Sigma i^2 q_i^2$$

Writing Lagrange's equation for any coordinate q_i;

$$23)\quad \ddot{q}_i + \frac{K}{m}\frac{\pi^2 i^2}{l^2} q_i = -\frac{2\ddot{u}}{i\pi}$$

Letting u = u_0 sinλt and solving:

$$q_i = A_i \sin\left(\frac{\pi i}{l}\sqrt{\frac{K}{m}}\, t - \alpha\right) + \frac{2u_0\lambda^2}{i\pi}\left(\frac{1}{\left(\frac{K}{m}\frac{\pi^2 i^2}{l^2} - \lambda^2\right)}\right)\sin\lambda t$$

at time t = o building has no deflection and no velocity.

$$q_i = 0 \qquad \therefore \qquad \alpha = 0$$

at $t = 0$

$$\dot{y} = \Sigma\left(A_i \frac{\pi i}{l}\sqrt{\frac{K}{m}} + \frac{2u_0\lambda^3}{i\pi}\frac{1}{\left(\frac{K}{m}\frac{\pi^2 i^2}{l^2} - \lambda^2\right)}\right)\sin\frac{i\pi x}{l}$$

Letting $\sqrt{\frac{K}{m}}\,\frac{\pi}{l} = c\lambda$

$$\text{at } t=0 \quad \dot{y} = \Sigma\left(A_i c i\lambda + \frac{2u_0\lambda}{i\pi(c^2i^2-1)}\right)\sin\frac{i\pi x}{l}$$

Velocity at any point x at $t = 0$

$$v_x = \dot{u} + \dot{y} = u_0\lambda + \Sigma\left(A_i c i\lambda + \frac{2u_0\lambda}{i\pi(c^2i^2-1)}\right)\sin\frac{i\pi x}{l} \equiv 0$$

multiplying by $\sin\frac{i\pi x}{l}dx$ and integrating from x = 0 to x = l.

$$A_i = \frac{-2u_0 ic}{i\pi(c^2i^2-1)}$$

$$24)\quad q_i = \frac{2u_0}{i\pi(c^2i^2-1)}\left(\sin\lambda t - ic\sin ic\lambda t\right)$$

Consider resonance with any q_i: that is $ic = 1$.

Since both numerator and denominator of (24) $\longrightarrow$ 0 as $ic \longrightarrow 1$, we must differentiate numerator and denominator separately. Doing this, and letting $ic = 1$ we get:

$$q_i = -\frac{u_0}{i\pi}\left(\sin\lambda t + \lambda t \cos\lambda t\right)$$

At end of a complete cycle $\lambda t = 2\pi$

$$\frac{q_i}{u_0} = -\frac{2}{i} \qquad i = \frac{1}{2} \cdot \frac{3}{2} \cdot \cdots$$

At end of n complete half-cycles: $\lambda t = n\pi$

$$\frac{q_i}{u_0} = \pm\frac{n}{i}$$

If resonance is with fundamental

$$i = \frac{1}{2} \quad \frac{q_i}{u_0} = 2n$$

For a one-story building we found (Chapter Four):

$$y_{max} = \frac{\pi}{2} n u_0$$

Actually, even though the ground motion has the same frequency as one of the natural modes of vibration of the building, any suddenly beginning ground motion will cause vibration in all the other modes. That is: from equation 20a and 24

$$26)\quad y = \sum q_i \sin\frac{i\pi x}{l} = 2u_0 \sum \frac{1}{i\pi(i^2c^2-1)}\left(\sin\lambda t - ic\sin ic\lambda t\right)\sin\frac{i\pi x}{l}$$

In case of resonance with fundamental mode c = 2 and we have, combining (25) with (26):

$$27)\quad y = 2u_0\Big\{\Big[-\frac{1}{\pi}\sin\lambda t - \frac{1}{\pi}\lambda t\cos\lambda t\Big]\sin\frac{\pi x}{2l} + \Big[\frac{1}{12\pi}\sin\lambda t - \frac{1}{4\pi}\sin 3\lambda t\Big]\sin\frac{3\pi x}{2l} + \Big[\frac{1}{60\pi}\sin\lambda t - \frac{1}{12\pi}\sin 5\lambda t\Big]\sin\frac{5\pi x}{2l} + \cdots\Big\}$$

$$28)\quad y = \frac{2u_0}{\pi}\Big\{\Big[-\sin\lambda t - \lambda t\cos\lambda t\Big]\sin\frac{\pi x}{2l} + \Big[.0833\sin\lambda t - .25\sin 3\lambda t\Big]\sin\frac{3\pi x}{2l} + \Big[.0167\sin\lambda t - .0833\sin 5\lambda t\Big]\sin\frac{5\pi x}{2l} + \cdots\Big\}$$

At the end of any number of complete half cycles of ground motion all terms will be zero except the one containing $\cos\lambda t$, so that our conclusions regarding the value of q_i at time $\lambda t = n\pi$ are equally good for y_{max} at that time, in the case of resonance with the fundamental frequency.

In the case of resonance with any higher mode (say the second, c = 2/3) it is clear that our equation corresponding to (28) will contain terms of the form $\beta \sin \frac{\lambda t}{3}$ which will not equal zero when $\lambda t = n\pi$ (unless n is divisible by 3).

We have seen that the deflection of a uniform building (in case of resonance with the fundamental frequency of the building) is $y_m = 2n\,U_o$ when $\lambda t = n\pi$. For a one-story building (in case of resonance) $y_m = \frac{\pi}{2} n U_o$. In each case the stress in any column will depend on the deflection of that column. For the one-story structure the deflection is obviously $\frac{\pi}{2} n U_o$. For a multi-story building the deflection will be approximately equal to the slope of the tangent at the base multiplied by the story height. If s = number of stories: $\delta_1 = \frac{\pi n U_o}{s}$

The preceding calculations are valid only for multi-story buildings of several stories (more than five perhaps). In the case of fewer stories we cannot consider the structure as continuous but must use the equations developed at the first of this chapter. If s = 6, $\delta_1 = \frac{\pi}{6} n U_o$ which, for the same u_o, is only 1/3 the deflection in a one-story structure at resonance.

PART III: THE EFFECT ON FREQUENCIES OF CHANGING THE STIFFNESS OF THE FIRST STORY

Using the same notation as before with the addition that the stiffness of the first story divided by the stiffness of any other story is d, we may write as before, the expressions for Energy.

$$T = \frac{m}{2}\left(\dot{y}_1^2 + \cdots\cdots \dot{y}_L^2\right)$$

$$29) \qquad W = \frac{K}{2}\left[d\, y_1^2 + (y_2 - y_1)^2 + \cdots\cdots (y_L - y_{L-1})^2\right]$$

Writing Lagrange's Equations:

$$\ddot{y}_1 + \frac{K}{m}\left((d+1)y_1 - y_2\right) = 0$$

$$\ddot{y}_2 + \frac{K}{m}\left(-y_1 + 2y_2 - y_3\right) = 0$$

$$\cdots\cdots\cdots\cdots$$

$$\ddot{y}_n + \frac{K}{m}\left(-y_{n-1} + 2y_n - y_{n+1}\right) = 0$$

$$\cdots\cdots\cdots\cdots$$

$$\ddot{y}_L + \frac{K}{m}\left(y_L - y_{L-1}\right) = 0$$

We may let $y_n = A_n \cos(\omega t - \alpha)$ and replace $(2 - \omega^2 \frac{m}{K})$ by c, giving us the following equations written as a determinant:

$$\begin{array}{ccccccccccc}
A_1 & A_2 & A_3 & \cdots & & & & & & & A_L \\
(c-1+d) & -1 & 0 & 0 & 0 & 0 & 0 & . & 0 & 0 & 0 = 0 \\
-1 & c & -1 & 0 & 0 & 0 & 0 & . & 0 & 0 & 0 = 0 \\
\cdots & & & & & & & & & & \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & . & -1 & c & -1 = 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & . & 0 & -1 & (c-1) = 0
\end{array}$$

Again we have L homogeneous equations with L unknowns (the coefficients $A_1 \ldots A_L$). In order to have a solution the value of the determinant itself must be zero, which means that only certain values of ω are possible.

If D_1 = value of the determinant without the last row and column.

D' = value of the determinant without the first row and column

D_1' = value of the determinant without first and last rows and columns

D_{11}, D_{11}', D'', D_{11}'', etc. are similarly defined;

we can write:

$$D \text{ (value of whole determinant)} = (c-1)\,D_1 - D_{11}$$

$$D_1 = (c-1+d)\,D_1' - D_1''$$

$$D_{11} = (c-1+d)\,D_{11}' - D_{11}''$$

Where D is a symmetric determinant with L - 4 rows and columns.

By algebraic manipulation we finally obtain, after replacing L by t+2:

$$D_{t+2} = [(c-2+d)c-d]\,D_t - [c-2+d]\,D_{t-1}$$

$$\text{But} \quad D_t = c\,D_{t-1} - D_{t-2} \qquad \text{(see eq. 9)}$$

Knowing that:

$$\sin(a+b) + \sin(a-b) = 2\sin a \cos b$$

we let

$$D_t = p \sin (t+1)\,\theta$$

$$b = \theta$$

$$a = t\,\theta$$

$$p \sin(t+1)\theta = 2\cos\theta \cdot p \sin t\theta - p \sin(t-1)\theta$$
$$D_t = c \cdot D_{t-1} - D_{t-2}$$

then $c = 2\cos\theta$

If $t = 1$ $\quad D_t = D_1 = c = 2\cos\theta = p\sin 2\theta$

$$\therefore \quad p = \frac{1}{\sin\theta}$$
$$\therefore \quad D_t = \frac{\sin(t+1)\theta}{\sin\theta}$$

and

$$32) \quad D_{t+2} = \left[(c-2+d)c - d\right]\frac{\sin(t+1)\theta}{\sin\theta} - (c-2+d)\frac{\sin t\theta}{\sin\theta}$$

and must equal zero.

By trigonometric manipulation [redacted] we obtain:

$$33) \quad \sin(L+1)\theta - \sin L\theta + (d-1)\left(\sin L\theta - \sin(L-1)\theta\right) = 0$$

In case d = 1 we obtain:

$$\sin(L+1)\theta - \sin L\theta = 0$$

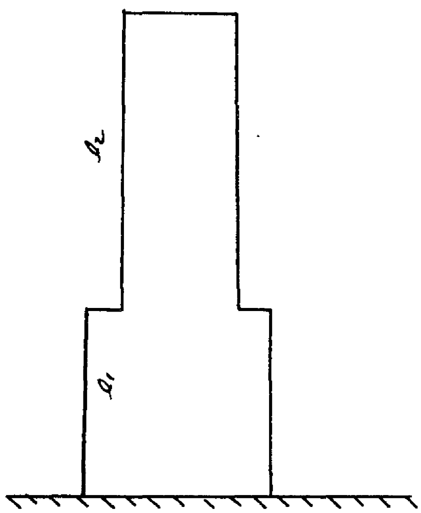
which is equation 11.

For any given values of L and d we may solve equation (33) for θ, from θ we determined c and finally ω. This has been done for a few values of d if L = 2. In case L = 1 we get $\omega^2 = \frac{Kd}{m}$ which was found in Chapter Four. For L = 2 there are two normal frequencies, call these ω_f (fundamental), and ω_h (harmonic).

d	very small	.1	.2	.5	.8	1.0	2.0	4.0	10.0	∞
$\omega_f^2 \frac{m}{K}$	d/2	.049	.095	.218	.324	.382	.586	.764	.900	1.00
$\omega_h^2 \frac{m}{K}$	2.00	2.051	2.105	2.228	2.476	2.618	3.414	5.236	9.100	∞

If L is larger than 2 the process of finding frequencies is quite tedious.

We may approach this problem in a different fashion by considering the building as continuous above and below the point where stiffness changes. We consider a cantilever shear panel with one discontinuity and uniform on each side of this point.



Let m_1 and m_2 be masses per unit length and k_1 and k_2 stiffnesses per unit length. Then:

$$34) \quad y_1'' = \frac{m_1}{k_1}\ddot{y}_1 \qquad y_2'' = \frac{m_2}{k_2}\ddot{y}_2$$

Boundary conditions:

at free end $\quad y_2' = 0$

at fixed end $\quad y_1 = 0$

at discontinuity $\quad y_1 = y_2 \; ; \; \frac{y_1'}{y_2'} = \frac{k_2}{k_1}$

Letting $\begin{matrix} y_1 = Y_1 \sin\omega t \\ y_2 = Y_2 \sin\omega t \end{matrix}$, where Y_1 and Y_2 are functions of x.

we get $Y_1 = A_1 \cos\sqrt{\frac{m_1}{k_1}}\,\omega x + B_1 \sin\sqrt{\frac{m_1}{k_1}}\,\omega x$

35) $\qquad Y_2 = A_2 \cos\sqrt{\frac{m_2}{k_2}}\,\omega x + B_2 \sin\sqrt{\frac{m_2}{k_2}}\,\omega x$

using our boundary conditions we get:

$$36) \quad \sqrt{\frac{k_1}{m_1}} \tan\sqrt{\frac{m_1}{k_1}}\,\omega l_1 = \sqrt{\frac{k_2}{m_2}} \tan\sqrt{\frac{m_2}{k_2}}\,\omega l_2$$

from which the ω's can be obtained

If only the first story is unlike the others we may say (following Biot*) that our boundary conditions are:

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* See: "Calculation of the Stresses Occurring in a Building during Earthquakes" by Dr. M. Biot, Publication no. 24 of the Guggenheim Aeronautics Laboratory, California Institute of Technology.

at free end $y' = 0$

at discontinuity $\frac{y_2'}{y_1'} = \frac{K_1}{K_2}$ where $y_1' = \frac{y_2}{l}$

$$\therefore \quad y_2' = \frac{K_1}{K_2} \frac{y_2}{l_1}$$

This is equivalent to saying that at the free end the shear equals zero, and that the shear in the first story is constant and equal to the shear just above the first story. Our structure now consists of a uniform shear-flexible cantilever supported on a one-story bent.

If we place our origin at free end we must write our condition equations at the discontinuity: $y_2' = -\frac{K_1}{K_2} \frac{y_2}{l_1}$

Letting $y_2 = Y_2(x) \sin \omega t$ and using our first boundary conditions we get:

$$Y_2 = A \cos \sqrt{\frac{m_2}{K_2}}\, \omega x$$

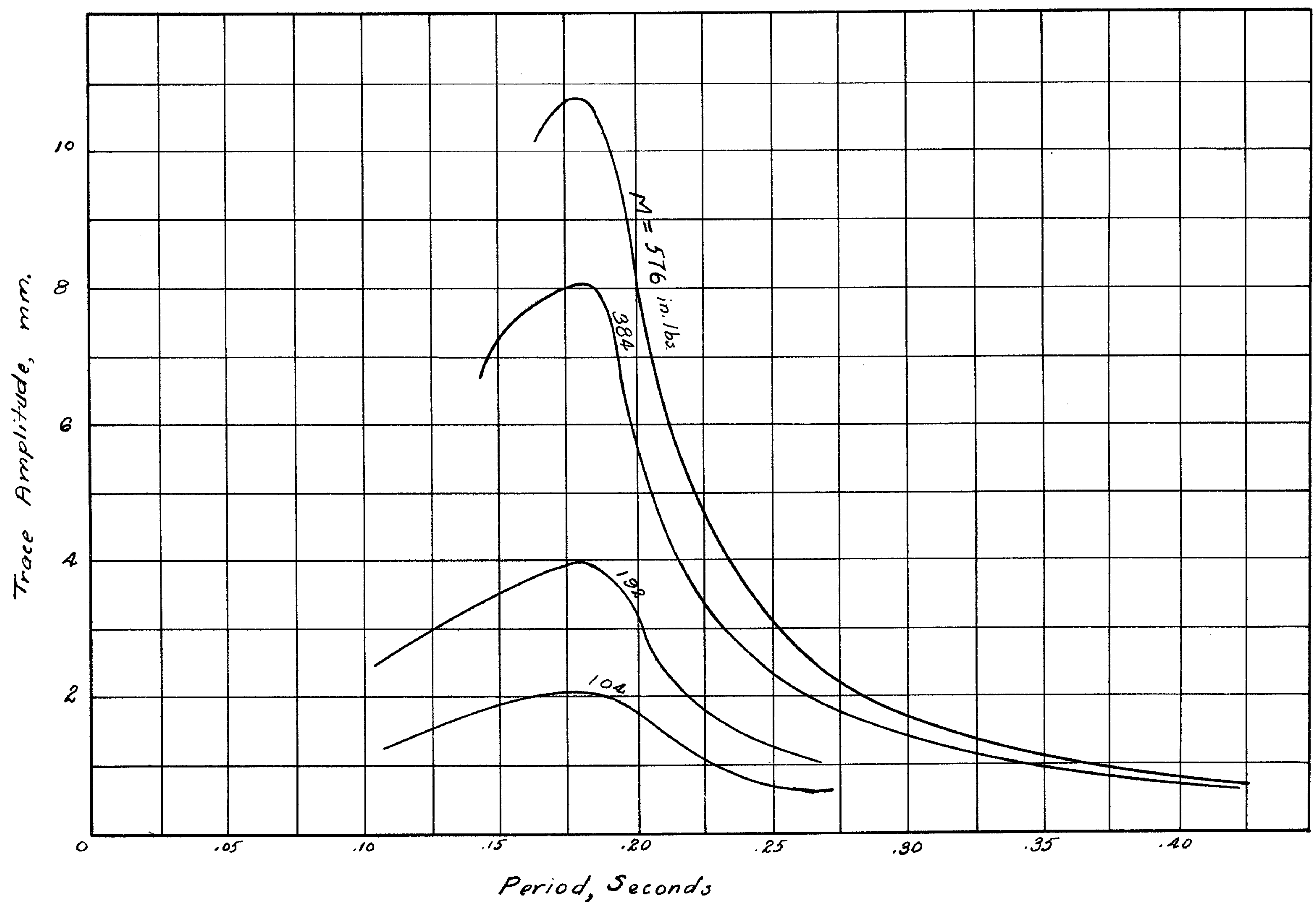
Using the second boundary equation and letting $\sqrt{\frac{m_2}{K_2}}\, \omega l_2 = \lambda$ we have the frequency equation:

$$37) \quad \lambda \tan \lambda = \frac{K_1}{K_2} \frac{l_2}{l_1}$$ (See Biot's paper)

In case of a building with a discontinuity at any point we may use equation (36):

$$36) \quad \sqrt{\frac{m_1 K_2}{m_2 K_1}} = \frac{\tan \sqrt{\frac{m_1}{K_1}}\, \omega l_1}{\tan \sqrt{\frac{m_2}{K_2}}\, \omega l_2}$$

to find the fundamental and harmonic frequencies of the structure.



M = 576 in. lbs.
384
192
104
Trace Amplitude, mm.
0
2
4
6
8
10
.05
.10
.15
.20
.25
.30
.35
.40
Period, Seconds

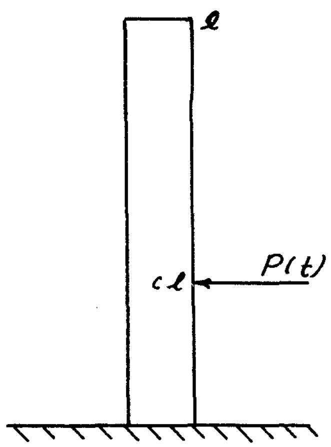
PART IV: INTERNAL DAMPING

The curves shown on the opposite page represent tests made on the Palo Alto Transfer and Storage Co. Building, in Palo Alto, to find its natural frequency. These tests were made with a device known as a "Shaking Machine" which consists essentially, of three unbalanced flywheels beside each other rotating at the same speed. The two outer flywheels (which are identical) turn in the same direction and their centers of gravity coincide. The inner wheel, whose unbalanced moment equals the sum of the other two, turns in the opposite direction. When all three centers of mass coincide the centrifugal forces add, when the phase difference is 180° all centrifugal forces cancel. In this way a sinusoidal force can be applied to a structure at any point where the machine can be set up. The direction of this force can be varied but would ordinarily be horizontal. The vibrations set up by this machine in the structure are easily recorded.

In obtaining the curves on the opposite page the machine was given a high initial angular velocity and was then allowed to come to rest gradually. The curves show the variation of the trace amplitude of a recording seismograph as the speed of rotation changed. The different curves are for different values of unbalanced moment (presumably the total unbalanced moment).

The building on which these tests were made is 40' x 75' in plan and about 50' high. The shaking machine acted in the direction of the long walls. Only one of these walls was shown in the photographs loaned to the writer. This wall was solid except for a large door and a large window near the ground, each about 10' x 20', and a small window. The walls are reinforced concrete and must furnish most of the stiffness of the building.

It occurred to the writer that data obtained from tests of this kind could be used to find the amount of internal damping in an actual building. Unfortunately, the information available was not sufficient, so the results obtained are necessarily incomplete. In addition to the information already given either in this discussion or on the graph opposite page 68, it would be necessary to know: the characteristics (magnification and degree of damping) of the recording seismograph; the location (height) of the shaking machine and of the recording seismograph in the building; and the dimensions of the building. It would also be very helpful to have a curve showing the relation between time and either amplitude of motion or frequency of the shaking machine.



We state our problem as follows: A shear panel, whose base is at rest is subjected to a periodic force P acting at a height $c\cdot l$ above the base. We assume a distributed friction force proportional to $\dot{y}$ (the rate of change of displacement) because this assumption gives by far the simplest solutions. We also assume that the rate of change of the speed of the shaking machine is small enough so that the building motion may be considered steady.

We write

$T = \Sigma \frac{V^2}{2} \Delta m$ Kinetic Energy

$W = \Sigma \frac{k}{2} (y')^2 \Delta x$ Potential Energy

$F = \Sigma \frac{f}{2} V^2 \Delta x$ Damping Function

Letting $y = \Sigma q_i \sin \frac{i\pi x}{l}$

where: m = mass per unit length

$T = \frac{ml}{4} \Sigma \dot{q}_i^2$ l = height

38) $W = \frac{k}{4} \frac{\pi^2}{l} \Sigma i^2 q_i^2$ k = stiffness (force to cause unit deflection in unit height)

$F = \frac{fl}{4} \Sigma \dot{q}_i^2$ f = a constant (damping coefficient)

R = $\frac{2 P_o \sin i\pi c}{ml}$

$\frac{k}{m} \left(\frac{\pi}{2l}\right)^2 = \omega^2 =$ $\overline{\text{natural frequency of undamped building}}^{\,2}$

If P = $P_o \sin \lambda t$

Due to a change δy_i of the ith coordinate the work done by P =

$$P \delta y_i = P \delta q_i \sin i\pi c = \delta work$$

39) $\therefore \quad Q_i = P \sin i\pi c = P_o \sin i\pi c \sin \lambda t$

where Q_i is a generalized force which corresponds to P.

Lagrange's Equation for any coordinate q_i:

$$\frac{d}{dt} \frac{\partial T}{\partial \dot{q}_i} + \frac{\partial F}{\partial \dot{q}_i} + \frac{\partial W}{\partial q_i} = Q_i$$

40 $\ddot{q}_i + \frac{f}{m} \dot{q}_i + \frac{\omega^2}{4} i^2 q_i = R \sin \lambda t$

Since the motion is steady we need find only the particular solution to this equation, which is:

$$q_i = \frac{R}{\left(\frac{f\lambda}{m}\right)^2 + (4\omega^2 i^2 - \lambda^2)^2}\left[(4\omega^2 i^2 - \lambda^2)\sin\lambda t - \frac{f\lambda}{m}\cos\lambda t\right]$$

If we let:

$$\frac{\frac{f\lambda}{m}}{\left[\left(\frac{f\lambda}{m}\right)^2 + (4\omega^2 i^2 - \lambda^2)^2\right]^{\frac{1}{2}}} = \sin\alpha$$

40a)

$$\frac{4\omega^2 i^2 - \lambda^2}{\left[\left(\frac{f\lambda}{m}\right)^2 + (4\omega^2 i^2 - \lambda^2)^2\right]^{\frac{1}{2}}} = \cos\alpha$$

we obtain

41) $$q_i = \frac{R}{\left[\left(\frac{f\lambda}{m}\right)^2 + (4\omega^2 i^2 - \lambda^2)^2\right]^{\frac{1}{2}}}\sin(\lambda t - \alpha)$$

$\therefore$

42) $$y_i = \frac{2 P_0 \sin i\pi c \sin\frac{i\pi x}{l}\sin(\lambda t - \alpha)}{m l\left[\left(\frac{f\lambda}{m}\right)^2 + (4\omega^2 i^2 - \lambda^2)^2\right]^{\frac{1}{2}}}$$

If our record is taken close to the shaking machine: x = c l and y = y_c.

43) $$\overline{y_c^i} = \frac{2 P_0 \sin^2 i\pi c}{m l\left(\lambda^4 + \lambda^2\left(\frac{f^2}{m^2} - 8\omega^2 i^2\right) + 16\omega^4 i^4\right)^{\frac{1}{2}}}$$

where $\overline{y_c^i}$ is the amplitude of the ith mode of vibration at the point x = c l.

As λ varies $\overline{y_c^i}$ will be a maximum when:

44) $$\lambda^2 = 4\omega^2 i^2 - \frac{f^2}{2m^2} \quad : \text{if } i = \tfrac{1}{2} \quad \lambda_1^2 = \omega^2 - \frac{f^2}{2m^2}$$

$\therefore$

45) $$\overline{\overline{y_c^i}} = \frac{2 P_0 \sin^2 i\pi c}{f l\left(4\omega^2 i^2 - \frac{f^2}{4m^2}\right)^{\frac{1}{2}}}$$

In the present case P_o is not constant but depends on the speed of the shaking machine. If M is the unbalanced moment of the machine:

$$P_o = \frac{M\lambda^2}{g}$$

In this case we must rewrite equation (43) to read:

$$46)\quad \bar{y}_c^i = \frac{2M\sin^2 i\pi c}{gml\left[1 + \frac{1}{\lambda^2}\left(\frac{f^2}{m^2} - 8\omega^2 i^2\right) + 16\frac{\omega^4 i^4}{\lambda^4}\right]^{\frac{1}{2}}}$$

For a maximum value of $\bar{y}_c^i$

$$47)\quad \lambda^2 = \frac{32\omega^4 i^4}{8\omega^2 i^2 - \frac{f^2}{m^2}} \quad \text{giving (48)} \quad \bar{\bar{y}}_c^i = \frac{8 i^2\omega^2 M\sin^2 i\pi c}{gfl\left(4\omega^2 i^2 - \frac{f^2}{4m^2}\right)^{\frac{1}{2}}}$$

If f is small

$$46a)\quad \lambda^2 = 4\omega^2 i^2 + \frac{f^2}{2m^2}$$

49) If i = ½ (fundamental mode) $\quad \lambda_1^2 = \dfrac{\omega^4}{\omega^2 - \frac{f^2}{2m^2}}$

Ordinarily damping would make $\lambda < \omega$ (ω= undamped frequency). However, in this case $\lambda > \omega$ because the magnitude of P (force applied) is proportional to λ^2. If P were proportional to λ, $\lambda = \omega$ would give maximum amplitude.

Then: $\quad$ if $\lambda = \lambda_1 = \dfrac{\omega^4}{\omega^2 - \frac{f^2}{2m^2}}$ $\quad$ (49) (Resonance with fundamental)

$$50)\quad \bar{\bar{y}}_{c\,1/2} = \frac{2\omega^2 M\sin^2\frac{\pi c}{2}}{gfl\left(\omega^2 - \frac{f^2}{4m^2}\right)^{\frac{1}{2}}} \quad \text{(fundamental maximum)}$$

for i = 3/2 but same λ as before:

(first harmonic maximum)

$$51)\quad \bar{y}_{c\,3/2} = \frac{2\omega^2 M\sin^2\frac{3\pi c}{2}}{gml\left(64\omega^4 - 71\omega^2\frac{f^2}{m^2} + \frac{79}{4}\frac{f^2}{m^2}\right)^{\frac{1}{2}}}$$

etc.

Letting $\frac{f}{m} = n\omega$; in case of resonance with fundamental

52)

	n = 0 (no damping)	n small	n = ½	n = 1	n = $\sqrt{2}$ (aperiodic)
$\dfrac{\bar{\bar{y}}_{c\,1/2}}{\bar{y}_{c\,3/2}} \cdot \dfrac{\sin^2\frac{3\pi c}{2}}{\sin^2\frac{\pi c}{2}}$	∞	$\dfrac{8}{n}$	14.2	4.1	1

From equation 42 it appears that the kind of motion which will be produced in a building will depend on the position of the shaking machine. The point where the force is applied can not be a node, therefore, all modes of vibration having nodes at this point must be absent. If $c = {}^{2}/_{3}$ the first harmonic will not be present ($\sin i\pi c = \sin \frac{3}{2}\pi \cdot \frac{2}{3} = 0$). Furthermore, it can be seen that the relative amplitudes of vibrations in different modes will depend on the position of the shaking machine as well as on its frequency.

In case $c = {}^{1}/_{3}$ and we have resonance with fundamental, from equation 52:

	$n = 0$ no damping	n small	$n = \frac{1}{2}$	$n = 1$	$n = \sqrt{2}$ critical damping
$\dfrac{\overline{\overline{y}}_c{}^{1/2}}{\overline{y}_c{}^{3/2}}$	∞	$\frac{2}{n}$	8.5	1.03	.25

It appears that if $c = {}^1/_3$ (the shaking machine is placed one-third of the way from the base to the top of the building) and if the damping coefficient is large, vibrations having the frequency of the fundamental will cause oscillations of the structure in the first harmonic mode (but with frequency λ_1) which are comparable in amplitude to the oscillations of the fundamental mode.

If c = 1 (machine at roof) the relative amplitudes of the fundamental and first harmonic modes when $\lambda = \lambda_1$ (for maximum amplitude of fundamental) are easily found from (52) by making $\sin^2 \frac{\pi c}{2} = \sin^2 \frac{3\pi c}{2} = 1$. Since, in an ordinary building, n will be small, the inaccuracy in calculation due to neglecting the first harmonic ($i = {}^3/_2$) mode will also be small if we place the machine at the roof. Higher harmonics will also be present but they will be small compared to the first (unless $c = {}^2/_3$ or f is very large indeed).

If desired the shaking machine may be placed at the height $c = {}^2/_3$ which will not decrease the amplitude of the fundamental mode greatly ($\sin^2 60 = .75$) but will completely eliminate the first harmonic.

We can make a very approximate calculation of the value of f (damping coefficient).

If we consider the second curve on the diagram opposite page 68.

M = 192 inch pounds

Maximum trace amplitude = 3.9 mm.

Period at maximum amplitude = .18 seconds.

From the period we calculate λ to be 35.0.

The weight of the building may be taken as about 12 pounds/cu.ft.

The magnification of the recorder (a seismograph) was about 200.

Assume that c = $^2/_3$.

Then:

$$\overline{\overline{y}}_{c^{\frac{1}{2}}} = \frac{3.9}{2000\,(2.54)} = .000768''$$

Equation (50) may be written:

$$53)\quad \overline{\overline{y}}_{c^{\frac{1}{2}}} = \frac{2\,\omega^2 M \sin^2\frac{\pi c}{2}}{g\,m\,e\left(\frac{f^2}{m^2}\omega^2 - \frac{f^4}{4m^4}\right)^{\frac{1}{2}}}$$

Equations (53) and (49) may be solved for the unknowns, f and ω.

Giving:

$$\frac{f}{m} = .207\,\lambda$$

$$\omega = .98\,\lambda$$

If $\frac{f}{m}$ is small we may write equation (53):

$$54)\quad \overline{\overline{y}}_{c^{\frac{1}{2}}} = \frac{2\,\omega M \sin^2\frac{\pi c}{2}}{g\,m\,e\left(\frac{f}{m}\right)} \qquad \text{which gives} \qquad \frac{f}{m} = .204\,\lambda$$

Since $\lambda = 35.0$

$$\frac{f}{m} = 7.07$$

$$f = \frac{7.07(12)\,40.75}{12(32.2)} = 660 \quad \frac{\text{lb sec}}{\text{in}^2}$$

We may calculate the energy which the structure will absorb from the shaking machine per cycle. H_i is the generalized damping force. In a displacement dq_i work is done by this force equal to $H_i\, dq_i$. In a cycle the work done is $\int_{\lambda t=0}^{\lambda t=2\pi} H_i\, dq_i$.

But $H_1 = \frac{\partial F}{\partial \dot{q}_i}$

$$W_i = \int \frac{\partial F}{\partial \dot{q}_i}\, dq_i = \int \frac{\partial F}{\partial \dot{q}_i}\cdot \dot{q}_i\, dt$$

Since $F = \frac{fl}{4} \sum \dot{q}_i^2$, F is a homogeneous function of the second order in the $\dot{q}_i$'s. By Euler's Theorem:

$$\sum \dot{q}_i \frac{\partial F}{\partial \dot{q}_i} = 2F$$

$$55) \quad \therefore \quad W_F = \int 2F\,dt$$

We may also write $(56) \quad W_F = \int P\,dy = \int \frac{M\lambda^2}{g}\,dy$

It can be shown that (55) and (56) are equal.

If D_1 = amplitude of motion at c, due to fundamental mode (but for any frequency λ):

$$q_{i=\frac{1}{2}} = \frac{D_1}{\sin\frac{\pi c}{2}} \sin(\lambda t - \alpha)$$

α is defined by equation (40a)

$$\dot{q}_{i=\frac{1}{2}} = \frac{\lambda D_1}{\sin\frac{\pi c}{2}} \cos(\lambda t - \alpha)$$

For a complete cycle $\lambda t = 2\pi$

$$W_{F_{i=\frac{1}{2}}} = \frac{fl}{2} \frac{\lambda D_1^2 \pi}{\sin^2\frac{\pi c}{2}} \text{ per cycle}$$

If D_3 = amplitude of motion at c in first harmonic mode (any frequency λ):

$$W_{F_{i=\frac{3}{2}}} = \frac{fl}{2} \frac{\lambda D_3^2 \pi}{\sin^2\frac{3\pi c}{2}}$$

From equation (52): If we have resonance with fundamental mode the ratio of rates of dissipation of energy due to fundamental and first harmonic modes is:

	$n \to 0$	n small	n = .5	n = 1	$n = \sqrt{2}$
$\dfrac{W_{F_{i=\frac{1}{2}}}}{W_{F_{i=\frac{3}{2}}}}$	∞	$\frac{8}{n}$	14.2	4.12	1.0

where $n = \frac{f}{m} \frac{1}{\omega}$

If a curve showing the relation between amplitude of motion and time (or, equivalent, a curve of energy absorption against amplitude and frequency) were given, a second method of obtaining f would be available.

The value of f obtained from the amplitude at resonance may be checked by choosing any other point on any of the curves in the amplitude-frequency diagram, or vice versa. In fact, it would be very illogical not to use the remainder of each curve. Equation (46) would be used for these calculations. Probably the portion of each curve for a frequency lower than the resonant frequency of the fundamental would be preferable, because the harmonic modes would have less effect.

If desired, ratios between ordinates on each curve could be used. From equation (46) it can be seen that this would eliminate all factors except λ, ω and $\frac{f}{m}$. If we can obtain λ_1 (for maximum amplitudes) from the curves, by using equation (49) we have an additional relation between $\frac{f}{m}$ and ω. Taking any three points on a curve we have two ratios of amplitudes from which we can obtain $\frac{f}{m}$ and ω.

NATURAL FREQUENCIES OF VIBRATION OF A UNIFORM CANTILEVER IN WHICH BOTH BENDING AND SHEAR DEFLECTIONS ARE IMPORTANT

Let y_b = deflection due to bending only.

Let y_s = deflection due to shear only.

Then: $y = y_b + y_s$

Let p = dynamic load due to vibration.

$p = -m\ddot{y}$, where m = mass per unit length.

$$p = EI\frac{d^4 y_b}{dx^4} = -K\frac{d^2 y_s}{dx^2}$$

$$\therefore \quad \frac{d^4 y}{dx^4} = \frac{p}{EI} - \frac{1}{K}\frac{d^2 p}{dx^2}$$

$$1) \quad \frac{d^4 y}{dx^4} = \frac{m}{K}\frac{d^4 y}{dx^2 dt^2} - \frac{m}{EI}\frac{d^2 y}{dt^2}$$

Let $y = Y(x)\sin(\omega t + \alpha)$ giving, $\qquad \frac{d^2 Y}{dx^2} = Y''$ etc.

$$2) \quad Y'''' + \frac{m}{K}\omega^2 Y'' - \frac{m}{EI}\omega^2 Y = 0$$

Let $Y = Y_0 e^{\beta x}$ giving,

$$3) \quad \beta^4 + \frac{m}{K}\omega^2\beta^2 - \frac{m}{EI}\omega^2 = 0$$

$$\therefore 4) \quad Y = Y_1 \cos\left\{\omega\sqrt{\frac{m}{2K}}\left[1 + \left(1 + \frac{4K^2}{EIm\omega^2}\right)^{\frac{1}{2}}\right]^{\frac{1}{2}} x\right\}$$
$$Y_2 \sin\left\{\omega\sqrt{\frac{m}{2K}}\left[1 + \left(1 + \frac{4K^2}{EIm\omega^2}\right)^{\frac{1}{2}}\right]^{\frac{1}{2}} x\right\}$$
$$Y_3 \cosh\left\{\omega\sqrt{\frac{m}{2K}}\left[-1 + \left(1 + \frac{4K^2}{EIm\omega^2}\right)^{\frac{1}{2}}\right]^{\frac{1}{2}} x\right\}$$
$$Y_4 \sinh\left\{\omega\sqrt{\frac{m}{2K}}\left[-1 + \left(1 + \frac{4K^2}{EIm\omega^2}\right)^{\frac{1}{2}}\right]^{\frac{1}{2}} x\right\}$$

Boundary Conditions:

1) At $x = 0$, $y = 0$

2) At $x = 0$, $y_b' = 0$ $\qquad y_s' = \lambda_0 = \frac{1}{K}\int_0^l p\,dx$

3) At $x = l$, $y_b'' = 0$ $\qquad y_s'' = -\frac{p}{K}$

4) At $x = l$, $y_b''' = 0$ $\qquad y_s''' = -\frac{1}{K}\frac{dp}{dx}$

Let: $$\frac{m\omega^2}{2K} = a^2$$

$$\frac{2K}{EI} = b$$

$$1 + \frac{b}{a^2} = c^2$$

Using 1st boundary condition: $Y_3 + Y_1 = 0$, giving:

5) $$Y = Y_1 \cos a\sqrt{c+1}\,x + Y_2 \sin a\sqrt{c+1}\,x + Y_4 \sinh a\sqrt{c-1}\,x - Y_1 \cosh a\sqrt{c-1}\,x$$

Using 2nd boundary condition:

$$\lambda_0 = \frac{1}{K}\int_0^l p\,dx$$ Since $p = m\omega^2 Y \sin(\omega t + \alpha)$

6) $$\lambda_0 = 2a\left[\frac{Y_1}{\sqrt{c+1}}\sin al\sqrt{c+1} + \frac{Y_2}{\sqrt{c+1}}(1-\cos al\sqrt{c+1}) - \frac{Y_1}{\sqrt{c-1}}\sinh al\sqrt{c-1} - \frac{Y_4}{\sqrt{c-1}}(1-\cosh al\sqrt{c-1})\right] = \frac{dY}{dx_{(0)}} = a\left(\sqrt{c+1}\,Y_2 + \sqrt{c-1}\,Y_4\right)$$

Using 3rd boundary condition:

7) $$-a^2\left[(c+1)\left(Y_1 \cos al\sqrt{c+1} + Y_2 \sin al\sqrt{c+1}\right) + (c-1)\left(Y_1 \cosh al\sqrt{c-1} - Y_4 \sinh al\sqrt{c-1}\right)\right] = -2a^2\left[Y_1 \cos al\sqrt{c+1} + Y_2 \sin al\sqrt{c+1} - Y_1 \cosh al\sqrt{c-1} + Y_4 \sinh al\sqrt{c-1}\right]$$

Using 4th boundary condition:

8) $$-a^3\left[(c+1)^{3/2}\left(-Y_1 \sin al\sqrt{c+1} + Y_2 \cos al\sqrt{c+1}\right) + (c-1)^{3/2}\left(Y_1 \sinh al\sqrt{c-1} - Y_4 \cosh al\sqrt{c-1}\right)\right]$$
$$= -2a^3\left[\sqrt{c+1}\left(-Y_1 \sin al\sqrt{c+1} + Y_2 \cos al\sqrt{c+1}\right) + \sqrt{c-1}\left(-Y_1 \sinh al\sqrt{c-1} + Y_4 \cosh al\sqrt{c-1}\right)\right]$$

Let:

$$\sin al\sqrt{c+1} = A$$

$$\cos al\sqrt{c+1} = B$$

$$\sinh al\sqrt{c-1} = C$$

$$\cosh al\sqrt{c-1} = D$$

$$a\sqrt{c+1} = E = a\sqrt{2}\,G$$

$$a\sqrt{c-1} = F = a\sqrt{2}\,H$$

$$\frac{K}{(EIm\omega^2)^{\frac{1}{2}}} = d$$

We may rewrite equations 6, 7, 8 as follows:

$$\begin{array}{cccc} Y_1 & Y_2 & Y_3 & \\ 2a^2\left(\frac{A}{E}-\frac{C}{F}\right) & \frac{2a^2}{E}(1-B)-E & -\frac{2a^2}{F}(1-D)-F & =0 \\ -BE^2-DF^2+2a^2(B-D) & -AE^2+2a^2A & CF^2+2a^2C & =0 \\ AE^3-CF^3-2a^2(AE+CF) & -BE^3+2BEa^2 & DF^3+2DFa^2 & =0 \end{array}$$

For solutions of Y_1, Y_2, $Y_3 \neq 0$ the determinant of their coefficients $= 0$. Making the determinant $= 0$ gives the following frequency equation. See calculation sheets for procedure.

$$9) \quad BD(1+2\alpha^2)-AC\alpha+2\alpha^2=0$$

One limiting case occurs when d = 0. This means that EI is very large and that only shear deflection occurs.

If d = 0, we have:

$B \cdot D = 0 \sim \cos al\sqrt{c+1} = 0$ or $\cos\sqrt{\frac{m}{K}}\omega l = 0$, which was found earlier in this chapter.

The other limit occurs when $d \to \infty$. In this case we have only bending deflection.

Then:

$B \cdot D + 1 = 0$

$$\cos al\sqrt{c+1}\,\cosh al\sqrt{c-1}+1=0$$

as $\alpha \to \infty$ $a \to 0$, $c \to \infty$ but $a\sqrt{c} \to \sqrt[4]{\frac{m\omega^2}{EI}}$

$$\therefore \quad \cos\sqrt[4]{\frac{m\omega^2}{EI}}\,l\,\cosh\sqrt[4]{\frac{m\omega^2}{EI}}\,l+1=0$$

Which may be found elsewhere for the natural frequencies of a uniform cantilever beam (bending only).

If beam is of rectangular section:

$k = \frac{AG}{1.2}$

A = Area of cross-section

G = Rigidity Modulus = .4 E (assumed)

1.2 is used because the shear stress across a section is not uniform (Vol. I, Timoshenko's Strength of Materials).

ρ = mass of building (or beam) per unit volume of wall (of beam).

h = depth of wall (beam)

z = a l

$I = \frac{1}{12} A h^2$

$$d = \frac{A \cdot .4E}{1.2\,(EA^2 \frac{h^2}{12} \rho)^{\frac{1}{2}} \omega} = \frac{1.155}{\omega h} \sqrt{\frac{E}{\rho}}$$

$$a = \frac{1}{2} \sqrt{\frac{k}{d^2}} = 1.225\, \omega \sqrt{\frac{\rho}{E}}$$

$d \cdot a \cdot l = 1.414\ l/h$

$c^2 = 1 + 4d^2$

From (9) we get:

$$10) \left[\cos z\sqrt{c+1}\, \cosh z\sqrt{c-1}\right](c^2-1) - \left[\sin z\sqrt{c+1}\, \sinh z\sqrt{c-1}\right]\sqrt{c^2-1} + c^2 - 1 = 0$$

c and z both depend on a, which is proportional to ω. By solving (10) for corresponding values of c and z we may obtain the ω's.

Solving for fundamental ω's,

If we let d = 0 (pure shear) we find:

$$z = \frac{\pi}{2\sqrt{2}} = 1.111 \qquad \omega = \frac{.907}{l} \sqrt{\frac{E}{\rho}}$$

$d = \frac{1}{2}$
$z = 1.057 \qquad \frac{l}{h} = .3744 \qquad \omega = \frac{.363}{l}\sqrt{\frac{E}{\rho}} = .323\frac{k}{l^2}\sqrt{\frac{E}{\rho}}$

$d = 1$
$z = .958 \qquad \frac{l}{h} = .678 \qquad \omega = \frac{.782}{l}\sqrt{\frac{E}{\rho}} = .530\frac{k}{l^2}\sqrt{\frac{E}{\rho}}$

$d = 2$
$z = .793 \qquad \frac{l}{h} = 1.123 \qquad \omega = \frac{.647}{l}\sqrt{\frac{E}{\rho}} = .726\frac{k}{l^2}\sqrt{\frac{E}{\rho}}$

$d = 5$
$z = .555 \qquad \frac{l}{h} = 1.96 \qquad \omega = \frac{.9515}{l}\sqrt{\frac{E}{\rho}} = .885\frac{k}{l^2}\sqrt{\frac{E}{\rho}}$

$d = \infty$ (Bending only)

$$\omega = 1.015\frac{k}{l^2}\sqrt{\frac{E}{\rho}}$$

An approximate method of calculating frequency in such a case follows from a consideration of the relative shear and bending deflections under uniform load.

Let: ω_1 = frequency due to bending only (for rectangular beam $= 1.015\frac{h}{l^2}\sqrt{\frac{E}{\rho}}$).

ω = frequency corrected for effect of shear deflections.

h = depth (rectangular beam).

l = length.

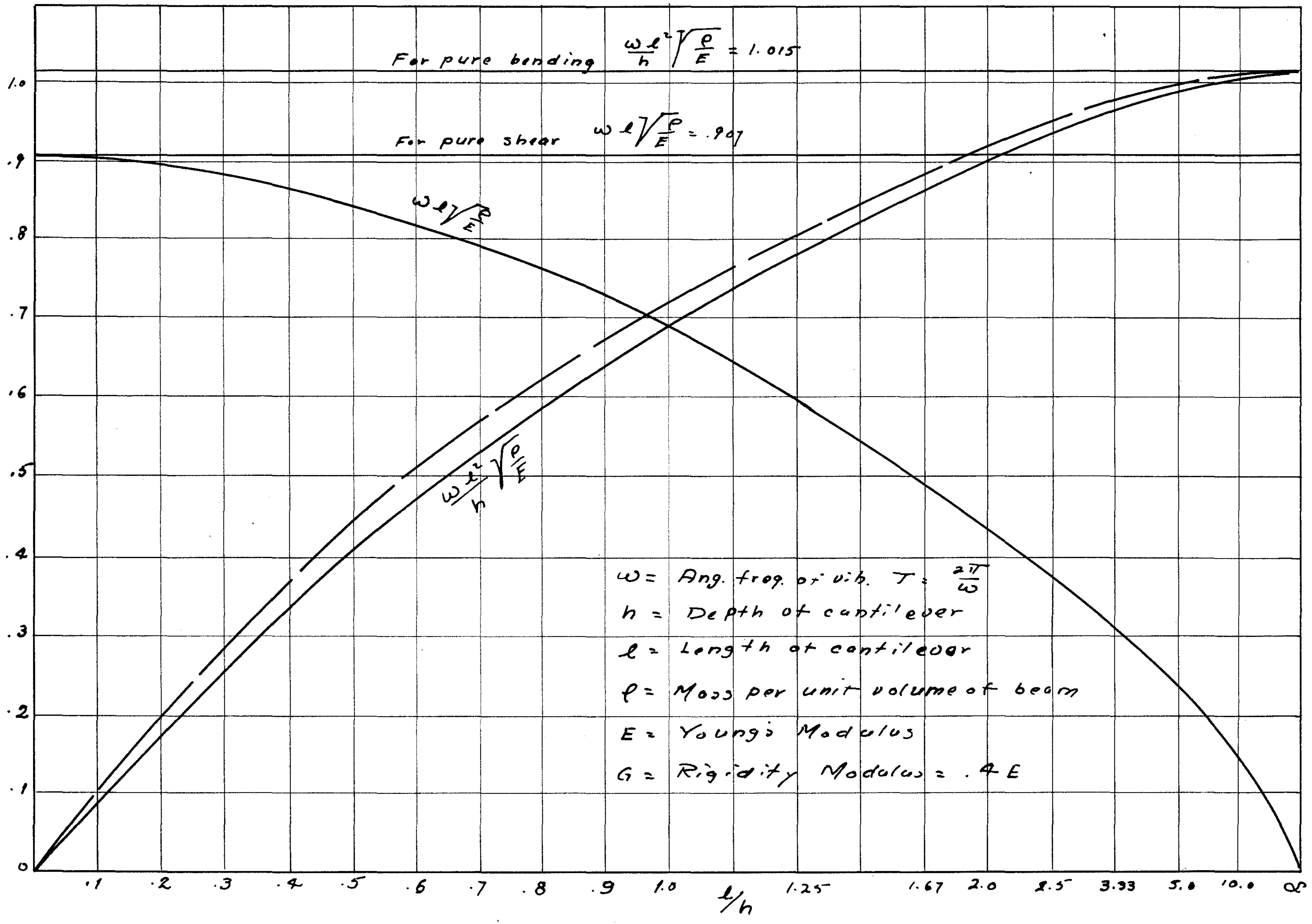
$c_1 = \frac{EI\,\delta_{max}}{w\,l^4}$

Then: for any rectangular, uniform beam:

$$\omega = \frac{\omega_1}{\left(1 + \frac{1}{8}\frac{h^2}{c_1 l^2}\right)^{\frac{1}{2}}}$$

For a cantilever $c_1 = \frac{1}{8}$

$$\therefore \quad \omega = \frac{\omega_1}{\left(1 + \frac{h^2}{l^2}\right)^{\frac{1}{2}}}$$


For pure bending $\frac{\omega l^2}{h}\sqrt{\frac{\rho}{E}} = 1.015$
For pure shear $\omega l\sqrt{\frac{\rho}{E}} = .907$
$\omega l\sqrt{\frac{\rho}{E}}$
$\frac{\omega l^2}{h}\sqrt{\frac{\rho}{E}}$
ω = Ang. freq. of vib. $T = \frac{2\pi}{\omega}$
h = Depth of cantilever
l = Length of cantilever
ρ = Mass per unit volume of beam
E = Youngs Modulus
G = Rigidity Modulus = .4 E
0
.1
.2
.3
.4
.5
.6
.7
.8
.9
1.0
1.25
1.67
2.0
2.5
3.33
5.0
10.0
∞
l/h

The full curves opposite show the variation of ω with the ratio $^{l}/_{h}$. Both curves are the same but give ω in terms of different factors. The broken line gives the results of the approximate solution just described.

SUMMARY

CHAPTER I: Effect of Hubs on Deflections of Columns and Girders. If hub and loads are symmetric, effective length of each member is approximately the average of the clear span and the center-to-center length. However, the effective length depends on the ratio of depth to length of members.

CHAPTER II: Stresses and Deflections in a Reinforced Concrete Building Wall containing openings and resting on an elastic material, due to a horizontal force along the upper edge. Rather large stresses were obtained. The method of calculation was very approximate, however, and therefore the numerical results are not particularly exact.

CHAPTER III: Earthquake Analysis of Throop Hall for forces in the East-West direction. All columns except four seem able to withstand horizontal loads of the order of 7% the weight of the building acting on the building in the East-West direction. The four exceptions are highly stressed by dead loads and although slender, might fail if large lateral loads came on the building. Partial failure of these columns would not cause great damage to the structure. All girders seem capable of withstanding lateral loads of the order of 14% of W.

CHAPTER IV: "Transient Vibrations of One-Story Structures." The region of large magnification is wider than for steady-state vibrations. Near resonance the magnification factor depends very much on the number of cycles of impressed force. At exact resonance the magnification factor is equal to $n\pi$ where n is the number of half-cycles of ground motion.

CHAPTER V: "Rocking of a Block on a Hard, Flat Surface." Calculations are made for the loss of energy suffered by a block rocking on a relatively unyielding surface. A method of calculating the motion of a such block is given in case the base on which the block rocks is at rest or moving with uniform acceleration. The effect of a variable acceleration of the base may be approximated by considering constant accelerations acting for short periods.

CHAPTER VI: Natural frenquencies of vibration of multi-story buildings with no more than two regions of different weight/stiffness ratio. Forced vibration of a uniform building. In this case resonance is somewhat less dangerous than for a one-story structure. A method of determining the amount of internal damping in a building from tests made with a shaking machine. Natural frequency of vibration of a tall building in which shear and bending deflections are comparable.