REGGE POLES IN FIELD THEORY
AND CRITERIA FOR ELEMENTARITY

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ABSTRACT

It is known that in the field theory description of the Compton scattering of nucleons by vector mesons the nucleon lies on a Regge trajectory. In this work it is found that the vector meson channel of the same theory exhibits no such Regge-like behavior. However, in electron-graviton scattering the spinor particle is again Reggeized.

The various problems that arise when zero mass mesons are used in the scattering are discussed and it is shown that the Reggeization generally proceeds as in the massive case. A field theory of massive gravitons is discussed.

The dynamical criterion of vanishing renormalization constants is applied to Reggeized particles and it is shown that this criterion successfully distinguishes an elementary Reggeized particle from a true dynamical state. This provides a dynamical test which can replace the postulate that a "bootstrapped" theory is characterized by the absence of Kronecker delta terms in all channels. We may thus recover the one-to-one correspondence between dynamical particles and certain moving poles in the partial wave amplitudes.

Appendices B and C review the generation of Regge trajectories by iteration through unitarity and Mandelstam's treatment of Reggeization using the N/D equations.

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I. INTRODUCTION

One of the main distinctions between ordinary field theories and S-matrix theory has been the treatment of "elementary" particles. In a field theory elementary particles are introduced a priori into the theory; their masses, coupling constants, and spins are arbitrary except for renormalization corrections which may arise. In S-matrix theory, on the other hand, every particle is regarded as a bound state of all the possible states of particles which have its quantum numbers; the masses and couplings, and even the existence of particles, are regarded as determined in a self-consistent (and hopefully unique) way by their mutual interactions.

One of the ways in which the theories might be contrasted is in the analytic structure of scattering amplitudes. For example, we may consider scattering in a channel having the quantum numbers of an elementary particle A in a conventional field theory. In the lowest order of perturbation theory (the "Born approximation") there is a fixed singularity of the form $\delta_{J\,J_A}$ in the partial wave amplitude (analytically continued in the angular momentum plane). However, in an amplitude obtained using S-matrix theory one would expect to find only moving poles in the angular momentum plane.(1) One of these Regge poles would give rise to particle A at energy $m_A c^2$ by passing through $J = J_A$ at that energy; but the analogy from non-relativistic theory would suggest that the poles would all move with energy. In principle, one could study experimental scattering amplitudes to see whether the singularities moved or not, and thus hope to distinguish between the two approaches.

However, Gell-Mann and Goldberger[2] suggested that when higher order diagrams were included the field theory amplitude might not exhibit fixed singularities. The effect of radiative corrections might be to form a Regge trajectory instead of a Kronecker delta in the partial wave amplitudes.* If this occurs, the effect of the elementary particle on the entire scattering amplitude resembles the effect of a composite particle, and the particle is said to be "Reggeized."

Compton scattering of vector mesons (heavy "photons") by spin 1/2 or spin 0 "nucleons" was treated in a series of papers by Gell-Mann, Goldberger, Low, Zachariasen, Singh, and Marx,[2, 3, 4, 5] using two different approaches. One method was to calculate directly the scattering amplitude by means of perturbation theory, adding together selected (hopefully dominant) terms in each order of the coupling constant for large momentum transfer t, at fixed $(\text{energy})^2$ s. If the amplitude is dominated by a Regge pole it will have the asymptotic form $t^{\alpha(s)}$ at large t and the trajectory $\alpha(s)$ of the pole can be determined.

The other approach was to iterate the Born approximation through elastic unitarity in the s channel, examining the partial wave amplitude near $J = J_{\text{Nucleon}}$ for terms of the form $\beta(s) / \left[J_N - \alpha(s) \right]$.

*In fact, in a theory with only scalar particles, the higher order terms do add up to a Regge trajectory; however, it starts at $J = -1$ at small couplings and thus has no relation to the particles already in the theory. See Reference 7, J. C. Polkinghorne, J. Math. Phys. 4, 503 (1963).

They found two necessary conditions for Reggeizing the nucleon in this manner: the existence of a "nonsense" (unphysical) channel at $J = J_N$, and factorization of the amplitudes near the pole. The latter condition assures that only one Regge trajectory is generated and that its residues factor. The former requirement makes the sense-sense amplitude reduce to the $\delta_{J\,J_N}$ given by the Born approximation when the coupling constant is small, so that all of the Kronecker delta term arises from the Regge pole. The two conditions are not sufficient for the Reggeization, however, unless the subtraction constants in the dispersion relations for the amplitudes are known.

It did turn out, however, that the same form for $\alpha(s)$ could be obtained in both treatments; the perturbation theory verifies the correctness of the unitarity treatment. The spinor nucleon is Reggeized but a scalar nucleon does not lie on the trajectory that is generated.

Mandelstam(6) has used analyticity to study the exact partial wave amplitudes in this problem, independent of the perturbation expansion. He has shown that the exact amplitude contains a Regge trajectory on which the spin 1/2 nucleon lies. He found that the scalar nucleon does not have to be Reggeized and that it would lie on a trajectory only by sheer accident. The previous work of Gell-Mann et al. showed that indeed it did not lie on a trajectory.

This paper continues where the previous work leaves off. We will find a conspicuous failure of Reggeization for the vector meson (in nucleon-antinucleon scattering) and an amazing success for the spinor nucleon once again (in nucleon-graviton scattering). Mandelstam's arguments will not serve to remove the miraculousness of this success as they were able to do in the vector-spinor problem. Finally, we will also make the important generalization to zero mass photons and gravitons.

Our interpretation of these results will be that the absence of fixed poles in a scattering amplitude gives no assurance that the particles involved are dynamical. Particles which have been "Reggeized" are clearly still "elementary" in the intuitive sense; i.e., the Reggeization occurs for all values of the particle's mass and coupling constant, whereas these quantities should be calculable for a true dynamical particle. We show that our notions of true "elementary" are supported by some evidence regarding vanishing renormalization constants.

Extended calculations and background material have been placed in the appendices to allow the main text to read as smoothly as possible.

II. VECTOR MESONS

The common feature of the scattering processes considered by Gell-Mann et al. was the existence of a nonsense channel; this leads to a trajectory being generated when the Born approximation is iterated through unitarity in the s channel. However, it is also possible that a particle could be Reggeized by the sum of more complicated perturbation diagrams, not just the lowest order diagrams iterated in the s channel. To study this possibility we will consider nucleon-antinucleon scattering, coupling the nucleon to vector mesons.

The intrinsic parity of the vector meson is -, so the amplitude must be asymptotically antisymmetric under the interchange of t and u if it is dominated by a vector meson Regge trajectory. This requires that at large t the amplitude have the form $t^{\alpha(s)} - u^{\alpha(s)}$ if this is the dominant trajectory, where $\alpha(s) = 1$ when s equals the rest energy squared of the vector particle. If we assume that $\alpha(s) = 1 + \gamma(s)$, with $\gamma(s) \to 0$ as $g^2 \to 0$, the perturbation expansion of the scattering amplitude should have the form

$$t^{\alpha(s)} - u^{\alpha(s)} = t \left\{ 1 + \gamma(s) \ln t + \tfrac{1}{2} \left[\gamma(s) \ln t \right]^2 + \ldots \right\} - u \left\{ 1 + \gamma(s) \ln u + \tfrac{1}{2} \left[\gamma(s) \ln u \right]^2 + \ldots \right\} . \quad \text{(II-1)}$$

Thus we see that if the scattering amplitude is to be dominated by a Regge trajectory asymptotically,. the higher orders of perturbation theory must provide amplitudes proportional to $(t \ln t - u \ln u)$, $(t \ln^2 t - u \ln^2 u)$, and so on. We begin by looking for the asymptotic form $t \ln t - u \ln u$ in the lowest possible orders of perturbation theory.

The fourth order diagrams cannot provide such a form. Some individual diagrams in this order have the form $t \ln t$, but they combine in the form $t \ln t + u \ln u$ and the total fourth order contribution is asymptotically only of order t. Therefore, the fourth order diagrams change the residue at the vector meson singularity in higher orders of the coupling constant, but do not contribute to the formation of a Regge trajectory. The lowest nontrivial order in which the form $t \ln t - u \ln u$ could occur is therefore sixth order, which we now examine.

As we have seen, the sum of the sixth order scattering diagrams must have the asymptotic form $t \ln t - u \ln u$ at large t if they are to contribute to Reggeizing the vector meson. One can easily eliminate all sixth order diagrams except those of Figure 1 (and the corresponding diagrams with t and u interchanged) as giving contributions of lower asymptotic order. If one approximates these by using two-particle unitarity in the t channel, however, it appears that the leading terms at large t from these diagrams cancel each other. This suspicious behavior makes it imperative to calculate the exact amplitudes using field theory, which we will first carry out using spinless nucleons.

The calculation is much simpler if the denominators are combined in a particular order when parameterizing the integrals. The nucleon propagators from the "top half" of each diagram are first combined into a single denominator, then the other nucleon propagators are combined similarly to form another denominator; finally, these two terms are combined with the three vector propagators to form a single denominator. (This procedure is useful for similar ladder-type diagrams in higher orders.)

The form of the Feynman amplitude found in each case for the diagram of Fig. 1 is

$$M = \frac{i\ g^6}{16\pi^4} \int_0^1 dx_1 dx_2 dy_1 dy_2 dz_1 dz_2 dz_3 dz_4 dz_5\ \delta(x_1 + x_2 - 1)$$

$$\delta(y_1+y_2-1)\ \delta(z_1+z_2+z_3+z_4+z_5-1)\ z_4\ z_5\ t^3\ C\ D^{-3}\ . \qquad \text{(II-2)}$$

Here we have suppressed all factors in the numerator which we do not contribute to the leading term at large t. The denominator function D has the convenient form (because of our parameterization)

$$D = A + B\ s + z_4 z_5\ F\ t\ ; \qquad \text{(II-3)}$$

A, B, C, and F are certain polynomials in the x_i, y_i, z_i.

In the case of diagram 1a, the function F is semidefinite and we can directly apply Polkinghorne's method[7] to find the form of M_a at large t. The result is

$$M_a \xrightarrow[t \to \infty]{} \frac{i\ g^6}{32\pi^4} \int_0^1 dx_1 dx_2 dy_1 dy_2 dz_1 dz_2 dz_3\ \delta(x_1+x_2-1)\ \delta(y_1+y_2-1)$$

$$\delta(z_1+z_2+z_3-1)\ A_1\ (z_1 z_2 z_3 s - A_1)^{-1}\ f_a^{-2}\ t \ln t \equiv g(s)\ t \ln t, \qquad \text{(II-4)}$$

where $A_1 = z_1 z_2 + z_2 z_3 + z_3 z_1$ (II-5)

and $f_a = z_2 + z_1 x_2 y_2 + z_3 x_1 y_1$. (II-6)

This diagram was correctly calculated by Freund and Oehme[8].

The function g(s) may also be written

$$g(s) = \frac{i\ g^6}{32\pi^4} \int_0^1 dz_1 dz_2 dz_3\ \delta(z_1+z_2+z_3-1) \qquad \text{(II-7)}$$

$$(z_1 z_2 z_3 s - A_1)^{-1} \ln \frac{(z_1+z_2)(z_2+z_3)}{z_2^2}$$

Diagrams lb and lc give identical contributions. Finding the asymptotic form of these amplitudes at large t is a difficult task, which has been approached at several levels of sophistication. The trouble comes from the fact that in this case F can vanish within the region of integration; thus at large t there may be contributions to the integral other than those near the boundaries, which are calculated in Polkinghorne's method. The evaluation of the asymptotic form of the amplitude for these diagrams is discussed in Appendix A.

The result is that the leading contribution from diagrams lb and lc is again proportional to t ln t, and it is of the form

$$M_b + M_c \xrightarrow[t\to\infty]{} \left[-g(s) + 2i\pi\, h(s)\right] \; t \ln t \; . \qquad \text{(II-8)}$$

The second term is the additional contribution caused by the contour of integration being pinched by two poles in the asymptotic limit, as treated by Polkinghorne(9), and is explicitly

$$h(s) = -\frac{i\; g^6}{32\,\pi^4} \int_0^1 dz_1 dz_2 dz_3 \; \delta(z_1+z_2+z_3-1)(z_1 z_2 z_3 s - A_1)^{-1}. \qquad \text{(II-9)}$$

The crossed diagram obtained from diagram la by the exchange of t and u, which we denote a', contributes of course

$$M_{a'} \xrightarrow[t\to\infty]{} - g(s)\, u \ln u \; . \qquad \text{(II-10)}$$

However, in finding the contribution to the diagrams b' and c' obtained from lb and lc by crossing, we cannot simply replace t by u and affix an overall negative sign. The problem is that the asymptotic limit $t \to +\infty + i\epsilon$ implies that $u \to -\infty - i\epsilon$; this has the effect of changing the relative sign of the pinch contribution, as discussed in Appendix A. The result is that

$$M_{b'} + M_{c'} \underset{t \to \infty}{\longrightarrow} - u \ln u \left[- g(s) - 2i\pi h(s) \quad . \right] \qquad \text{(II-11)}$$

When all the sixth order graphs are summed, there are two types of cancellation occurring. The "real" parts (from g(s)) of diagrams 1b and 1c are canceled by diagram 1a; the "real" parts of diagrams b' and c' are canceled by diagram a'. Secondly, the "imaginary" parts (from h(s)) of diagrams b and c are canceled by the "imaginary" parts of diagrams b' and c'.

Freund and Oehme(8) failed to notice the first cancellation; Ahmed(10) considered the pinch effect but failed to notice the second cancellation.

The result is that asymptotically the sixth order amplitude is of lower order than the t ln t required for a Regge-like form. In fact, an estimate of the total contribution of these diagrams using two-particle unitarity in the t channel gives an asymptotic form of $(\ln t)^2$ or less. We are forced to conclude that the vector meson singularity is not affected by sixth order corrections and that the vector meson remains unReggeized in this order. Freund(11, 12) and Ahmed(13) agree with this conclusion.

The contributions from the eighth order perturbation theory diagrams were calculated approximately using two-particle unitarity in the t channel and found to give no asymptotic contributions of order t ln t.

The next order of perturbation theory which might contribute terms like t ln t is thus the tenth order. Two particle unitarity in both s and t channels was used to calculate the amplitudes approximately. The only possibilities are diagrams of the general type depicted in Figure 2.

The leading terms of diagrams 2a and 2b cancel in the approximation of two-particle unitarity in the t channel, and cannot give a net contribution as large as t ln t. Mandelstam studied these diagrams with the better approximation of three particle unitarity in the s channel and obtained the same result.

Although diagram 2c has the asymptotic form t ln t, the net contribution from the sum of diagrams 2c, 2d, 2e, and all similar ones is again of lower order than t, in the approximation of two-particle unitarity. It thus appears that exactly the same cancellation occurs in tenth order as occurred in sixth order. However, although diagram 2c has been computed exactly, diagrams 2d, 2e, etc. are known only in the elastic approximation.

The attempt to Reggeize the vector meson in a theory with scalar nucleons thus seems a total failure. Of course, it is true that any one of these diagrams (for example, diagram 1a) could be iterated in the s channel to form a Regge trajectory; this is objectionable for two reasons.

For one thing, we are no longer taking the sum of the dominant contributions to the amplitude in each order of perturbation theory; we are merely selecting out specific terms from each order (which are in fact canceled by other diagrams) and forming a Regge trajectory from their sum.

Secondly, we then find additional trajectories being started in each order of perturbation theory in rapidly increasing numbers, all passing through the point $J = 1$. For example, any of the tenth order diagrams of Figure 2 could be iterated in the s channel to form a Regge trajectory. The convergence of such a procedure is obviously suspect.

Giving the nucleons spin 1/2 does not appear to help either. Freund[12] finds that the same integrals occur in the sixth order calculation of nucleon-antinucleon scattering and they cancel again to leave the vector meson singularity unaffected. Mandelstam[6] has used more general arguments but reaches the same conclusion.

The result is that when the vector meson is coupled to either spin 0 or spin 1/2 "nucleons," its effect on the scattering amplitude still has the form that we would ascribe to an elementary particle. Thus, not only does the vector meson appear "elementary," but its pole actually gives the dominant asymptotic contribution to nucleon-antinucleon scattering. Apparently no trajectory is even generated in the vicinity of its pole, as occurred for the scalar nucleon in the Compton scattering treated by Gell-Mann et al.

III. ZERO MASS PARTICLES

The spinless and spin 1/2 nucleon-antinucleon scattering amplitudes just considered are examples of theories in which a two-particle nonsense state does not exist at the spin of the elementary particle we are hoping to Reggeize, i.e., at $J = 1$. Therefore, we will now return to considering theories with two-particle nonsense states available -- specifically, we will consider nucleon Reggeization in nucleon-vector meson Compton scattering. The reader who is not familiar with the generation of Regge trajectories by iteration through unitarity and the general arguments of Mandelstam should at this point consult Appendices B and C, where the important points are reviewed.

One important generalization that we wish to make at this point is to the case of zero mass vector mesons. The work of Gell-Mann et al.[2-5] and Mandelstam[6] assumed massive photons to avoid certain difficulties, but clearly the zero mass case is significant. In their works, a non-zero vector meson mass λ was retained while the limit $t \to \infty$ was taken to extract the Regge asymptotic form. If one wishes to discusss real photons then one must take $\lambda = 0$ before taking the large t limit.

There are three difficulties that arise when the vector meson mass is zero. One is the "infrared problem:" some diagrams now diverge, and also in any experiment with finite energy resolution infinitely many additional soft photons can be radiated undetected. It is necessary not only to eliminate the infinities, but to take account of any residual terms which may be important in the asymptotic limit.

A second problem is that the vector mesons of zero mass are restricted to only two helicities, so that some of the helicity amplitudes for the scattering vanish identically. Finally, the two- and many-particle thresholds are no longer distinct and Mandelstam's use of elastic unitarity must be re-examined.

<u>The Infrared Problem</u>: In the case of photon-nucleon scattering, the infrared problem is best approached using the work of Eriksson[(15)] and Yennie, Frautschi, and Suura[(16)].*

Eriksson shows that the observed differential cross sections can be obtained from a nondivergent amplitude $\overline{M}$. He points out that as long as one remains in one Lorentz frame one may define two different kinds of photons whose field operators commute and hence which may be treated independently. Soft photons, which are undetected experimentally, are defined as photons with wave vector k_μ satisfying

$$\begin{cases} k_o \leq \epsilon \,, \\ |\vec{k}| \leq \epsilon \,, \end{cases} \qquad \text{(III-1)}$$

where ϵ is an arbitrarily chosen positive energy. All others are called hard photons.

He then shows that for ϵ small compared with the masses of the heavy particles in the theory (in our case, $\epsilon << m_{\text{Nucleon}}$), one may write

$$\overline{M} = \widehat{M} \exp\left(\tfrac{1}{2}A - \tfrac{1}{2}\widehat{A}\right) . \qquad \text{(III-2)}$$

In this expression, $\widehat{M}$ is the scattering amplitude calculated from all the diagrams involving only hard photons and is therefore not infrared divergent. The factor $\frac{1}{2}A - \frac{1}{2}\widehat{A}$ is a geometrical factor depending only upon the external non-soft particle in the process. It is a common factor for all perturbation theory diagrams and is

*Eriksson glosses over the proof of which diagrams are infrared divergent, but this question is discussed in detail by Yennie, Frautschi, and Suura.

explicitly

$$\tfrac{1}{2}A - \tfrac{1}{2}\hat{A} = \frac{1}{4\pi i}\int_R \frac{d^4k}{k^2}\; s_\mu(k)\; s^\mu(-k) \quad . \tag{III-3}$$

The region of integration R is given by

$$k \in R \text{ if and only if } \begin{cases} |\vec{k}| \leq \epsilon, \\ k_o \leq \epsilon \end{cases} \tag{III-4}$$

If the initial and final nucleons have four-momenta p_1 and p_2, the function $s_\mu(k)$ for our problem is

$$s_\mu(k) = \frac{2ie}{(2\pi)^{3/2}}\left(\frac{p_{1\mu}}{k^2 + 2k\cdot p_1} + \frac{p_{2\mu}}{k^2 - 2k\cdot p_2}\right) \tag{III-5}$$

We may thus explicitly calculate the contribution of infrared effects.

The $\bar{M}$ defined above is of course finite; moreover, for $\epsilon << m_{\text{Nucleon}}$ it is independent of the quantity ϵ.

If we now specify that the experimenter is to determine $\bar{M}$ from cross sections having an energy uncertainty Δ E, and require that he take the limit $\Delta E \to 0$ before $t \to \infty$, then the remaining amplitude is exactly given by the formula III-2. This is clearly an experimentally accessible limit, and just requires making successively more accurate measurements of the cross section at each value of t until it is clear that the limit $\Delta E \to 0$ has been approached closely enough to determine the limiting value for $\bar{M}$. Of course, the physical cross section vanishes as a power of ΔE(15), but the vanishing factor can be factored out to determine $\bar{M}$, which is nonvanishing.

We will digress a moment on the implications of the experimental limit which is here implied. It is clear that we expect a

Reggeistic behavior to manifest itself in the amplitude only if we define an amplitude which depends only upon s and t, and not upon the detailed experimental technique. If Regge trajectories are an intrinsic property of the amplitudes, we expect them to be independent of the experimental setup.

This leaves us two choices: either to take the limit $\Delta E \to 0$ first, then $t \to \infty$; or we could take large t at various fixed values of ΔE and then last of all take $\Delta E \to 0$ for the asymptotic forms thus obtained.

We have chosen the former limit to define how we wish the experiment to be performed, because then all the infrared terms factor out in a simple way. However, it would be possible to also study the other limiting procedure.

Unfortunately, the alternate limiting procedure has severe drawbacks. Since ΔE is kept fixed until the end, to obtain finite amplitudes we must make the separation into hard and soft photons carried out by Eriksson. However, this separation is not invariant under a Lorentz transformation or under crossing symmetry; thus, the amplitude obtained will not satisfy unitarity in the crossed channel, at least before the limit $\Delta E \to 0$. The undesirability of working with noninvariant amplitudes as well as the calculational difficulty of doing the necessary four-dimensional integrals over the region R of equation III-4 for the hard photon diagrams together prompt us to reject this alternate limiting procedure as unrealistic.

With the understanding that the limit $\Delta E \to 0$ is to be taken first experimentally, we may then proceed to find the effects of the infrared factors upon a possibly Reggeistic amplitude.

The infrared factor may be explicitly calculated from equations III-3 and III-5. We must choose a specific Lorentz frame, and we choose the CM system (for definiteness, one may consider the polar axis to be along p_1, but of course the separation into hard and soft photons is invariant under pure spatial rotations so that it is not necessary to specify this).

We have

$$\tfrac{1}{2}A - \tfrac{1}{2}\widehat{A} = \frac{1}{4\pi i}\int_R \frac{d^4k}{k^2}\left(\frac{-4e^2}{(2\pi)^3}\right)$$

$$\left(\frac{m^2}{(k^2+2k\cdot p_1)(k^2-2k\cdot p_1)} + \frac{m^2}{(k^2-2k\cdot p_2)(k^2+2k\cdot p_2)}\right. \tag{III-6}$$

$$\left. + \frac{p_1\cdot p_2}{(k^2+2k\cdot p_1)(k^2+2k\cdot p_2)} + \frac{p_1\cdot p_2}{(k^2-2k\cdot p_1)(k^2-2k\cdot p_2)}\right)$$

Let $$p_1 = \begin{pmatrix} E \\ 0 \\ 0 \\ p \end{pmatrix}, \quad p_2 = \begin{pmatrix} E \\ p\sin\Theta \\ 0 \\ p\cos\Theta \end{pmatrix}. \tag{III-7}$$

Then we see by inspection that the integral of the first term depends only upon E and ϵ. The integral of the second term may be reduced to the same form by a suitable rotation of coordinates and therefore it depends only upon E and ϵ also. Finally, the change of variables $k \leftrightarrow -k$ converts the third term to the form of the fourth term, so that they are also equal. At this point we thus have

$$\tfrac{1}{2}A - \tfrac{1}{2}\widehat{A} = g(\epsilon, E) + h(\epsilon, E, z), \quad \text{with } z = \cos\Theta. \tag{III-8}$$

The function $h(\epsilon, E, z)$ arises from the third and fourth terms and is explicitly

$$h(\epsilon, E, z) = -\frac{1}{2\pi i}\frac{4e^2}{(2\pi)^3}$$

$$\int_R \frac{d^4k}{k^2} \frac{p_1\cdot p_2}{(k^2+2k\cdot p_1)(k^2+2k\cdot p_2)} \tag{III-9}$$

$$= -\frac{1}{2\pi i}\frac{4e^2}{(2\pi)^3}\left(E^2 - p^2\cos\Theta\right)$$

$$\int_R \frac{d^4k}{k^2(k^2+2k_oE-2k_zp)(k^2+2k_oE-2k_xp\sin\Theta-2k_zp\cos\Theta)}$$

We now analytically continue this expression to (unphysically) large z, i.e., we take $t \to \infty$. This is justified by the convergence of the resulting integral. The continuation is chosen so that $\sin\Theta \to +iz$. We obtain

$$h(\epsilon,E,z) \xrightarrow[z\to\infty]{} i\,\frac{2e^2p}{(2\pi)^4}\int_R \frac{d^4k}{k^2(k^2+2k_oE-2k_zp)(k_z+ik_x)}$$

$$\equiv \bar{h}(\epsilon,E)\,z^0 . \tag{III-10}$$

However, we may show that $\bar{h}(\epsilon,E)$ is purely imaginary:

$$\mathrm{Re}\,\bar{h}(\epsilon,E) = \frac{2e^2p}{(2\pi)^4}\int_R \frac{d^4k\;\;k_x}{k^2(k^2+2k_oE-2k_zp)(k_z^2+k_x^2)}$$

$$= 0 \tag{III-11}$$

since the integrand changes sign under the change of variables $k_x \leftrightarrow -k_x$ but the region of integration does not change.

We thus find that $h(\epsilon,E,z)$, which contains all the z dependence of $\frac{1}{2}A - \frac{1}{2}\hat{A}$, is purely imaginary at large z and contributes only an overall phase factor to the expression III-2.

We may express this feature of $\frac{1}{2}A - \frac{1}{2}\hat{A}$ by writing

$$\tfrac{1}{2}A - \tfrac{1}{2}\hat{A} \xrightarrow[t \to \infty]{} i\, f(\epsilon, s, t) + w(\epsilon, s) \qquad \text{(III-12)}$$

where $f(\epsilon, s, t)$ is a real function, and both f and w are proportional to e^2. In this particular problem, concerning photons, the t dependence of $f(\epsilon, s, t)$ is in fact the trivial dependence t^0, as we see from eq. (III-10).

This amazingly simple form allows us to dispose of the entire infrared problem. At large t, all of the t dependence of the infrared contribution appears merely as a phase factor in the expression III-2 for $\overline{M}$ and does not affect the cross section. Moreover, the residual term $\exp\left[w(\epsilon, s)\right]$ contributes only in higher orders of g^2; thus, although it changes the residue at the Regge pole in higher orders it does not at all affect its position.

This convenient form for the infrared factor leads to a great calculational simplification, as we shall now demonstrate. We note that if $\overline{M}$ is dominated by a Regge pole at large t, then increasing powers of ln t appear in each order of the perturbation expansion. Since what we are doing is taking the dominant term in each order of perturbation theory and adding them together, the dominant term in each order of a Reggeistic amplitude is unaffected by the infrared factors. It is true that there are terms in each order of perturbation theory (proportional to fewer powers of ln t) containing various powers of the infrared correction w, but as long as the amplitude is dominated by a Regge pole these will be of lower order. Since all the infrared infinities were removed in the infrared factor, the leading terms in t will all be finite.

Thus we have a very simple prescription; if increasing powers of ln t occur in each order of perturbation theory, as is necessary to form a Regge trajectory, the correct amplitude, complete with infrared corrections, is found by extracting the leading terms at large t. Some of the terms discarded will be infrared divergent, but these are canceled by the infrared factors anyway; the coefficients of the leading powers of ln t will always be finite. We therefore can avoid doing four-dimensional integrals over the noninvariant region R of eq. III-4. This procedure gives the correct position of the Regge pole but its residue is correct only to lowest order in g^2; finding the exact residue requires explicitly evaluating the infrared corrections (and doing integrals over R). However, we may note that in the previous work(2, 3, 4,5,8, 11) the residues were usually only treated in lowest order anyway.

We have thus managed to reduce the massless photon case to that of massive photons as far as Reggeization goes.

This entire argument may be carried out for gravitons instead of photons. The only changes are that now the geometrical infrared factor is

$$\tfrac{1}{2}A - \tfrac{1}{2}\hat{A} = \frac{1}{4\pi i} \int_R \frac{d^4k}{k^2} \left[s_{\mu\nu}(k)\, s^{\mu\nu}(-k) - \tfrac{1}{2}\, s_{\mu\mu}(k)\, s^{\nu\nu}(-k) \right], \qquad \text{(III-13)}$$

where

$$s_{\mu\nu}(k) = \frac{\kappa}{2} \frac{i}{(2\pi)^{3/2}} \left(\frac{p_{1\mu}p_{1\nu}}{k^2+2k\cdot p_1} + \frac{p_{2\mu}p_{2\nu}}{k^2-2k\cdot p_2} \right) \qquad \text{(III-14)}$$

In these expressions, the coupling constant κ is given by

$$\kappa^2 = 32\pi G \qquad \text{(III-15)}$$

in terms of the gravitational constant G.

These minor changes are irrelevant, however; when we do the integrals we again find the form III-12 for the infrared factor, except that now $f(\epsilon, s, t)$ is asymptotically proportional to t. Thus, the same simple prescription for treating the massless spin 1 meson works for the massless spin 2 meson and the infrared problem is essentially disposed of.

Other Zero Mass Problems: Another difficulty with treating zero mass mesons is that their helicities are restricted to only two values and therefore a number of the scattering amplitudes vanish identically in the zero mass limit. For example, in the vector-spinor problem there are only three amplitudes instead of six when the zero helicity vector state is excluded.

The approach suggested by Gell-Mann et al.[(4)] is to take the limit of zero mass in the nucleon-vector meson problem before taking the asymptotic limit, but retaining the lowest order terms in the photon mass instead of letting them vanish, by factoring out the vanishing mass. The factoring of these amplitudes would Reggeize the nucleon. Our present view is that this procedure would be an interesting way to study the relation between the massive and massless theories; however, the question of Reggeization in the zero mass limit cannot depend upon having taken some limiting form from a massive theory.

What we are interested in is just a massless theory, regardless of whether it can be obtained from a massive theory in a proper limit. The essential point is that if whatever amplitudes remain at zero mass actually do factor, a trajectory is automatically generated passing through the nucleon, in the sense that the asymptotic form of the scattering amplitude near the pole looks just like it would look if dominated by such a trajectory. The unitarity equation, of course, involves only these surviving amplitudes.

If the factorization does not hold at finite mass, but only in the zero mass limit, the Reggeization may still occur. Our interpretation would be as follows: for nonzero mass several Regge trajectories are generated near $\ell = 0$ (or $J = \frac{1}{2}$). At zero mass those trajectories whose residues don't vanish must merge into a single one, on which the nucleon lies, or else the factoring at zero mass could not occur. Again our arguments must assume that possible subtraction constants do not affect the generation of the trajectory by iteration through unitarity. This same reasoning even applies to nucleon-graviton scattering, where all of the amplitudes for zero mass gravitons are nonsense-nonsense.

Thus, our viewpoint is that it is just by the "accident" of zero mass that some of the helicity amplitudes are not experimentally available. The amplitudes that can be examined still reveal the presence or absence of a Regge trajectory, upon which the nucleon may or may not lie.

We find therefore that the conditions for Reggeization using zero mass mesons are not different because of the absence of some of the helicity amplitudes. Apart from possible subtractions, we may conclude that the nucleon Reggeizes when the amplitudes have the correct asymptotic form and factor properly, as long as a reasonable form is obtained for the Regge trajectory (i.e., it must pass through $J = \frac{1}{2}$ at $s = m^2_{\text{Nucleon}}$).

However, the vanishing of some of the helicity amplitudes causes more difficulty in Mandelstam's arguments. In nucleon-photon scattering there is only one sense-sense amplitude instead of three, and the number of threshold conditions that must be satisfied is correspondingly reduced. The situation in nucleon-graviton scattering is even worse; since there are no sense-sense amplitudes, one might say that Mandelstam's entire argument "vanishes."

The only recourse seems to be to assume the existence of a consistent field theory with massive mesons of which the zero mass theory is the limit, in the sense that the amplitudes of the massive theory smoothly approach the corresponding amplitudes of the massless theory as the limit of zero mass is taken. We then must know that the nonsense-nonsense amplitudes factor among themselves for nonzero masses as well as for zero mass in order for Mandelstam's arguments to show that the nucleon is Reggeized. However, in the vector-spinor problem, of course, this condition is trivially satisfied.

The final zero mass difficulty also affects Mandelstam's arguments. Now that the two- and many-particle thresholds are not separated in energy there is less reason to believe that two-particle unitarity can be used. The best we can do to justify two-particle unitarity is to speculate as follows. Eriksson's treatment of the infrared problem shows that "soft" photons contribute only to the infrared factor in front of the amplitude; the physically important amplitude $\bar{M}$ depends only upon "hard" photons, except for a known geometrical factor. Thus, the scattering amplitude "just above" threshold might still be accurately represented by the contribution from only two-particle intermediate states, even though there is no longer a discrete separation in energy between the regions where the two-particle approximation can be used and the region where it is invalid.

Our conclusions are that Mandelstam's arguments in their present form are valuable in lending greater validity to the Reggeization arguments (in that they are independent of perturbation theory); however, we can apply them to the zero mass case only by the unsatisfying assumption that we can obtain this case by taking a limit in a massive theory.

It might be possible to repair this difficulty in the spinor-vector case by finding additional conditions to determine the amplitudes in the zero mass limit; these would take the place of the threshold conditions on the vanishing sense-sense amplitudes. However, it does not seem at all possible to directly apply such arguments to cases of higher spin massless mesons (such as the spinor-graviton problem) because of the fact that all of the sense-sense amplitudes vanish. The additional confidence provided by Mandelstam's arguments in nucleon-vector meson scattering does not seem to be available to us in massless problems involving higher spin mesons.

We will summarize the changes that occur when treating the case of massless mesons. Infrared divergences can be removed by properly defining the experimental technique, and they have no effect upon the Reggeization or non-Reggeization. However, Mandelstam's arguments can be applied only as a limit from a massive theory, which we do not consider a realistic way to treat a true massless problem. In massless meson problems we will therefore have to continue to assume the absence of subtraction constants as was done in references 2-5.

The general treatment of zero mass mesons that we have just given does not keep us from encountering difficulties in particular problems. Specifically, let us consider what happens when we let the vector meson mass approach zero in vector-spinor scattering. For nonzero mass the trajectory in this problem is given by[4]

$$\mathrm{Im}\,\alpha = \gamma^2 (E + m)(W - m) / 8\pi k W .$$

Doing the dispersion integral to determine α, we obtain

$$\alpha = \gamma^2 \left[(W-m) / 8\pi^2 \right] \left[(W + m) I_o - W I_1 \right] .$$

The quantity I_1 is well behaved as the vector meson mass approaches zero, becoming

$$I_1 = - (1/s) \ln (1 - s/m^2) .$$

However, I_0 is given by

$$I_0 = \int_{(m+\lambda)^2}^{\infty} \frac{ds'}{k'W'(s'-s)}$$

$$= - (1/kW) \ln \left[(k W + EW - m^2)/(-m\lambda) \right] ,$$

which diverges logarithmically as the vector mass approaches zero. Thus the trajectory α is stretched off to infinity in the ℓ plane, becoming singular in the limit.

It should be emphasized that this behavior is not an infrared divergence, or even a general feature of zero mass problems; the trajectories found in photon-scalar nucleon scattering and in graviton-spinor scattering are finite. This singular behavior is apparently an accidental feature of the vector-spinor problem, and it means that the assumed Regge-like form is inconsistent with the form of the amplitude in the limit of zero mass. We do not understand why this peculiar behavior occurs in this theory.

Otokozawa and Suura[23], noticing this divergence, have treated the analogous spinless problem, and they find not only trajectories but also fixed single and double poles. Perhaps in the problem with spin as well this singular behavior marks the formation of higher order poles in the vicinity of $\ell = 0$. This is a difficult problem and to my knowledge no one has yet attempted to solve it. However, it is still important for our purposes that in graviton scattering the zero mass does not lead to any divergences, and that this difficulty is as far as we know confined to the case of vector meson-spinor scattering.

IV. GRAVITON-NUCLEON COMPTON SCATTERING

The failure of Reggeization to occur in theories without two-particle nonsense states prompts us to concentrate on only those theories possessing such states. Then we could expect Reggeization to occur in the amplitude formed by iterating the Born approximation in the s-Channel, as discussed in Appendix B.

A very interesting theory to study is the scattering of gravitons by spin $\frac{1}{2}$ or spin 0 nucleons. Besides the obvious universal presence of gravity, this field theory has several interesting features. Since massless gravitons have spin projections of ± 2 there are no sense channels available; however, in scattering by spin $\frac{1}{2}$ nucleons (or electrons, if you wish) there are nonsense channels available at both $J = \frac{1}{2}$ and $J = 3/2$. In general, therefore, we would expect a dynamical trajectory to be generated near $J = 3/2$, not corresponding to any particle in the theory. Near $J = \frac{1}{2}$ the work of Appendix B would lead us to expect the generation of two Regge trajectories, one of which might "accidentally" pass through the nucleon.

Thus, not only do we not expect the nucleon to be Reggeized in this theory, but we anticipate the generation of a trajectory near $J = 3/2$ which would dominate the scattering amplitudes at large t; the effects of the trajectories near $J = \frac{1}{2}$ would have to be extracted from the lower order terms in the large t limit.

In light of these expectations the results that are obtained are quite surprising.

The partial wave decompositions of reference 4 may be used straightforwardly to obtain the effect on the scattering amplitudes of a Regge pole in the partial wave amplitudes. We first treat the case of spin $\frac{1}{2}$ nucleons; it is sufficient to take helicities of + $\frac{1}{2}$ for both initial and final nucleons. The MacDowell symmetry relates + and - parity amplitudes so that amplitudes of only one parity need be considered, but we will explicitly state the asymptotic forms for both parities. We suppress the nucleon helicity indices on the amplitudes, and set $\ell = J - \frac{1}{2}$.

We assume a Regge pole contribution to the partial wave amplitudes of the form

$$F^{\ell}_{\lambda\mu} = g_{\lambda}(W)\, g_{\mu}(W) \left[\ell - \alpha(W)\right]^{-1} \qquad \text{(IV-1)}$$

where W is the total energy in the CM system; we have explicitly shown the factorization of the residues. The indices range over the helicities -2, -1, 0, + 1, and + 2, so that there are 15 independent amplitudes in the symmetric matrix F^{ℓ}. We remove fixed branch points in ℓ from the residues by defining quantities η_{μ} as follows:

$$g_{0,1}(W) = \eta_{0,1}(W) \qquad \text{(IV-2)}$$

$$g_{2,-1}(W) = \eta_{2,-1}(W) \left[\ell\,(\ell + 2)\right]^{\frac{1}{2}} \qquad \text{(IV-3)}$$

$$g_{-2}(W) = \eta_{-2}(W) \left[(\ell - 1)\,\ell\,(\ell + 2)\,(\ell + 3)\right]^{\frac{1}{2}}. \qquad \text{(IV-4)}$$

The parity-conserving scattering amplitudes are given in terms of the partial wave amplitude by

$$f^{\pm}_{\lambda\mu} = \sum_{\ell} 2(\ell + 1) \left[e^{J+}_{\lambda'\mu'}(z)\, F^{J\pm}_{\lambda\mu} + e^{J-}_{\lambda'\mu'}(z)\, F^{J\pm}_{\lambda\mu}\right], \qquad \text{(IV-5)}$$

where z is the cosine of the scattering angle between the initial and final nucleons in the CM system and $\lambda' = \lambda - \frac{1}{2}$, $\mu' = \mu - \frac{1}{2}$. The e's are given in Appendix D and just involve derivatives of Legendre functions.

We now Reggeize in the usual manner as described in reference 4. We find that if the partial wave amplitudes are dominated by a Regge pole near $J - \frac{1}{2} = \alpha$, the scattering amplitudes have the asymptotic form

$$\begin{aligned}
f_{-2,-2} &\approx -N_\alpha \, \eta_{-2} \, \eta_{-2} \, (\pi/\sin\pi\alpha)\, \alpha^2 (\alpha-1)^2 (\alpha+1)\, (-z)^{\alpha-2} \\
f_{-2,a} &\approx -N_\alpha \, \eta_{-2} \, \eta_a \, (\pi/\sin\pi\alpha)\, \alpha^2 (\alpha-1)\, (\alpha-1)(\alpha+1)\, (-z)^{\alpha-2} \\
f_{-2,\mu} &\approx -N_\alpha \, \eta_{-2} \, \eta_\mu (\pi/\sin\pi\alpha)\, \alpha(\alpha-1)(\alpha+1)\, (-z)^{\alpha-2} \\
f_{a,b} &\approx \epsilon_{ba} N_\alpha \, \eta_a \, \eta_b \, (\pi/\sin\pi\alpha)\, \alpha^2 (\alpha+1)\, (-z)^{\alpha-1} \\
f_{a,\mu} &\approx \epsilon_{a\mu} N_\alpha \, \eta_a \, \eta_\mu \, (\pi/\sin\pi\alpha)\, \alpha(\alpha+1)\, (-z)^{\alpha-1} \\
f_{\mu\nu} &\approx -\epsilon_{\mu\nu} N_\alpha \, \eta_\mu \, \eta_\nu \, (\pi/\sin\pi\alpha)\, (\alpha+1)\, (-z)^{\alpha}
\end{aligned} \tag{IV-6}$$

at large t. In these equations,

$$N_\alpha = \frac{\sqrt{2}\, \Gamma(\alpha+3/2)\, 2^{\alpha+1}}{\sqrt{\pi}\, \Gamma(\alpha+2)} \tag{IV-7}$$

The subscripts a and b take the values -1 and 2; the subscripts μ and ν take the values 0 and 1. All of the ϵ's are +1 except

$$\epsilon_{-1,2} = \epsilon_{2,1} = \epsilon_{0,1} = -1 . \tag{IV-8}$$

<u>Zero Mass Gravitons</u>: We first treat the physically interesting case of massless gravitons. The graviton helicities are now restricted to ±2 and we give the nucleon and graviton + intrinsic parities to be definite. We then explicitly calculate the Born approximation from the lowest order diagrams in field theory, which appear in Figure 3.

The calculation is performed using Feynman's field theory for gravitons(17, 18); the details appear in Appendix E.

The results we obtain for the + parity amplitudes are

$$f^{+}_{2,2} \longrightarrow A\,(m^2/W^2)\,z^{-1}\,, \qquad \text{(IV-9)}$$

$$f^{+}_{-2,2} \longrightarrow A\,(-m/W)\,z^{-2}\,,$$

$$f^{+}_{-2,-2} \longrightarrow A\quad z^{-2}\,,$$

at large z. The quantity A is given by

$$A = \frac{G}{2\sqrt{2}\,W\,p}\,(W^3 - m^3)\;. \qquad \text{(IV-10)}$$

A comparison with equations IV-6 shows that this is precisely the form that would arise from a Regge trajectory near $J = \frac{1}{2}$. This result is doubly surprising; not only do the amplitudes properly factor to Reggeize the spinor nucleon, but no dynamical trajectory is generated near $J = 3/2$ and the nucleon trajectory actually dominates the amplitude at large t.

From Appendix B and equations IV-2, IV-3, and IV-4 we find that the trajectory $\alpha(W)$ is given by

$$\text{Im}\,\alpha(s) = \frac{\kappa^2}{64\,\pi\,W^3}\,(W^3 - m^3)\,(3W^2 - m^2) \qquad \text{(IV-11)}$$

This bears a striking similarity to the trajectory obtained for the nucleon in the spinor nucleon-vector meson problem:

$$\text{Im}\;\alpha(s) = \gamma^2\;\frac{(E+m)\;(W-m)}{8\,\pi\,p\,W}\;. \qquad \text{(IV-12)}$$

Both expressions IV-11 and IV-12 vanish at W = m, as is necessary if the nucleon is to lie on the trajectory obtained, and both are positive above that energy. Graviton-nucleon scattering thus Reggeizes the spinor nucleon in much the same way that vector meson-nucleon scattering does.

A completely analogous calculation may be carried out for the scattering of scalar nucleons by gravitons. In this problem there is only one independent scattering amplitude and thus factorization is automatically satisfied. The asymptotic forms of the parity-conserving amplitudes are obtained in Appendix F, and are

$$f^{+}_{2,2} \to \frac{\kappa^2 E}{16\pi} \; z^{-2}$$
$$f^{-}_{2,2} \to \frac{\kappa^2}{32\pi p} \; (E^2 + p^2) \; z^{-2} \quad . \tag{IV-13}$$

Iteration through unitarity clearly will produce trajectories in this case in the amplitudes of both parities. The trajectories are given by

$$\text{Im } \alpha^{-} = (\kappa^2/8\pi)\;(E^2 + p^2) \tag{IV-14}$$

and

$$\text{Im } \alpha^{+} = (\kappa^2/4\pi)\; E\, p \; . \tag{IV-15}$$

However, the trajectory in the + parity amplitudes, α^+, does not pass through the scalar nucleon at W = m and thus, although dynamical trajectories are generated in this case, the nucleon is not Reggeized. A similar result occurred in the scalar nucleon-photon problem (zero mass photons), where(5)

$$\text{Im } \alpha^{-} = (\gamma^2/4\pi W)\; p \tag{IV-16}$$

and

$$\text{Im } \alpha^{+} = (\gamma^2/4\pi W)\; E \quad . \tag{IV-17}$$

Note that no trajectory is generated near J = 3/2 in the spinor nucleon scattering.

We thus find that the Compton scattering of nucleons by gravitons of zero mass behaves quite similarly to the Compton scattering of nucleons by photons. In these theories a nucleon with spin 1/2 lies on a Regge trajectory, which moreover dominates the asymptotic amplitude. If the nucleon is spinless, trajectories are generated but the nucleon remains unReggeized.

Massive Gravitons: We have seen that the Reggeization of the nucleon occurs almost identically in scattering by gravitons and photons. Let us therefore try to calculate the scattering of nucleons by massive spin 2 particles to see whether the results resemble those obtained in vector meson scattering. For example, it would be interesting to see whether the scalar-massive graviton amplitudes fail to factor in lowest order as the scalar-vector meson amplitudes did.

If gravitational quanta really have mass, of course, the mass must be extremely small since we know gravitational forces extend over astronomical distances. If gravity is responsible for attractions between members of clusters of galaxies[19], for example, the mass of the graviton must be less than about 10^{-35} electron masses.

We will formulate a massive graviton theory by modifying Feynman's theory. We will not go so far as to calculate any scattering amplitudes, because it will turn out that the simplest and most natural theory we can write down has unique properties that make it difficult to calculate even the simplest processes.

In the case of vector mesons interacting with scalar nucleons, it turns out that one obtains a consistent field theory by using the same vertex couplings for vector mesons as one would for photons, merely changing the photon propagator to allow for the nonzero mass. In graviton theory this simple prescription doesn't work -- one obtains amplitudes in which the gravitational current is not conserved.

It is therefore necessary to go back to the field equations of Feynman's theory to see how to introduce a mass term. The tensor notation is fairly standard and is given in Appendix G.

We make use of the fact that experimentally the metric tensor of the known universe is very nearly pseudo-Euclidean, and therefore write

$$g_{\mu\nu} = \delta_{\mu\nu} + \kappa h_{\mu\nu} \quad . \qquad \text{(IV-18)}$$

The gravitational coupling constant is κ, presumed to be small, and the gravitational field makes its effects felt as the correction $h_{\mu\nu}$.

Feynman's Lagrangian for the free gravitational field is

$$L = -\frac{2}{\kappa^2} \quad R\sqrt{-g} \quad , \qquad \text{(IV-19)}$$

which may be expanded to lowest order in κ to obtain

$$L = L_2 + \kappa L_3 + \ldots , \qquad \text{(IV-20)}$$

with

$$L_2 = \tfrac{1}{2} h_{\mu\nu,\sigma} h_{\mu\nu,\sigma} - \tfrac{1}{2} h_{\mu\mu,\sigma} h_{\nu\nu,\sigma} - h_{\mu\sigma,\mu} h_{\nu\sigma,\nu} + h_{\mu\nu,\mu} h_{\sigma\sigma,\nu} . \qquad \text{(IV-21)}$$

The equation for the free gravitational field is

$$\frac{\delta L}{\delta h_{\mu\nu}} + \frac{\delta L}{\delta h_{\nu\mu}} = 0 \quad , \qquad \text{(IV-22)}$$

which yields in lowest order

$$\bar{h}_{\mu\nu,\sigma\sigma} - \bar{h}_{\mu\sigma,\nu\sigma} - \bar{h}_{\nu\sigma,\mu\sigma} + \bar{h}_{\rho\sigma,\rho\sigma} \delta_{\mu\nu} = 0 , \qquad \text{(IV-23)}$$

where

$$\bar{h}_{\mu\nu} \equiv h_{\mu\nu} - \tfrac{1}{2} \delta_{\mu\nu} h_{\sigma\sigma} \quad . \qquad \text{(IV-24)}$$

The solutions of equation IV-23 in the gauge $\bar{h}_{\mu\nu,\nu} = 0$ have the form of plane waves:

$$\bar{h}_{\mu\nu} = \bar{e}_{\mu\nu} \quad e^{ik\cdot x} \quad , \tag{IV-25}$$

$$k\cdot k = 0 \ . \tag{IV-26}$$

The analogous equations for the electromagnetic field in the gauge $A_{\mu,\mu} = 0$ are $\quad A_{\mu,\sigma\sigma} = 0$, (IV-27)

with solutions

$$A_{\mu} = e_{\mu} \, e^{ik\cdot x} \ , \tag{IV-28}$$

$$k\cdot k = 0. \tag{IV-29}$$

When the photons are made massive, equation IV-27 is modified by the addition of a term $m^2 A_{\mu}$ to the left hand side. The obvious change for gravity is therefore

$$\bar{h}_{\mu\nu,\sigma\sigma} + \lambda^2 \bar{h}_{\mu\nu} = 0. \tag{IV-30}$$

This has solutions

$$\bar{h}_{\mu\nu} = \bar{e}_{\mu\nu} \quad e^{ik\cdot x} \quad , \tag{IV-31}$$

$$k\cdot k = \lambda^2 \quad , \tag{IV-32}$$

with λ the mass of the graviton.

We therefore wish to modify the Lagrangian by adding to L_2 a term of the form

$$- \tfrac{1}{2} \ \lambda^2 \ h_{\mu\nu} \, h_{\mu\nu} \ + \tfrac{1}{4} \ \lambda^2 \ h_{\mu\mu} \ h_{\nu\nu} \ . \tag{IV-33}$$

The addition of this term changes the stress-energy tensor and it will therefore be necessary to change L_3 and higher order terms in the Lagrangian for this tensor to remain divergenceless.

That is, the addition of a mass term to the equation of the free gravitational field requires that the three-graviton vertex appearing in diagram d of figure 3 must be changed if the gravitational current is to be conserved.

The lowest order correction, that appearing in L_3, should be obtainable by adding all possible terms involving the product of three h's with arbitrary coefficients. The coefficients are then adjusted to make the stress-energy tensor divergenceless. A similar procedure was used by Feynman[17, 18] to obtain L_3 originally.

If we confine ourselves to terms involving reasonable numbers of derivatives (i.e., two, since L_3 contains only terms with two derivatives) this procedure fails to work. It is impossible to correct L_3 by such terms to obtain a conserved stress-energy tensor.

The reason is clear from Feynman's discussion of what terms can be present in the Lagrangian if the stress-energy tensor is to be exactly conserved, to all orders. He shows that the only terms which can appear are scalar densities formed from the curvature tensor R_{abcd} and the metric tensor $g_{\mu\nu}$. The simplest such term involving derivatives of the metric tensor is the free gravitational field Lagrangian in equation IV-19.

To produce the mass term in equation IV-30, we want a term in the Lagrangian involving no derivatives of the metric tensor. The only such term leading to a conserved stress-energy tensor is $\sqrt{-g}$. We thus find that the appropriate form of the Lagrangian yielding the second-order terms of equations IV-21 and IV-33 is

$$L' = -\frac{2}{\kappa^2} R\sqrt{-g} + \frac{2}{\kappa^2}\lambda^2\sqrt{-g} \quad . \tag{IV-34}$$

Unfortunately, we cannot make a valid expansion like that of equation IV-18 now. If we try to so expand the Lagrangian, we get not only terms of second and higher order in $h_{\mu\nu}$ but also the lower order terms

$$\frac{2\lambda^2}{\kappa^2} + \frac{\lambda^2}{\kappa} h_{\mu\mu} \quad . \tag{IV-35}$$

The first of these yields no term in the field equation IV-22, but the second gives a term which make s the field equation no longer homogeneous in $h_{\mu\nu}$. In the Lagrangian, this term appears to create gravitons out of nothing and is very troublesome.

Even this problem can be circumvented, however, by simply making a more intelligent expansion of the metric tensor. We write

$$g_{\mu\nu} = g^o_{\mu\nu} + \kappa \quad h_{\mu\nu}, \tag{IV-36}$$

where $g^o_{\mu\nu}$ is the solution of the exact gravitational field equation in the limit of no interactions with other particles. For small λ, $g^o_{\mu\nu}$ should somehow resemble $\delta_{\mu\nu}$.

Let us consider the interaction of gravitons with scalar particles; the additional term needed in the Lagrangian is the covariant form of the usual expression for scalar particles,

$$L_{scalar} = \tfrac{1}{2}(\phi_{,\mu}\phi_{,\mu} - m^2\phi^2). \tag{IV-37}$$

Our total Lagrangian is then

$$L_{total} = L' + L_s \quad , \tag{IV-38}$$

where L' appears in equation IV-34 and

$$L_s = \tfrac{1}{2} \sqrt{-g} \quad (g^{\mu\nu} \phi_{,\mu} \phi_{,\nu} - m^2 \phi) \quad . \qquad \text{(IV-39)}$$

The equations IV-22 may be computed exactly and are

$$g_{\mu\nu} R - 2 R_{\mu\nu} - \lambda^2 g_{\mu\nu} = \tfrac{1}{4} \kappa^2 g_{\mu\nu} (g^{\alpha\beta} \phi_{,\alpha} \phi_{,\beta} - m^2 \phi^2) - \tfrac{1}{2} \kappa^2 \phi_{,\mu} \phi_{,\nu} . \qquad \text{(IV-40)}$$

The scalar field satisfies the equation

$$\frac{\delta L}{\delta \phi} = -(\sqrt{-g} \; g^{\alpha\beta} \phi,\alpha),\beta - \sqrt{-g} \; m^2 \phi = 0 , \qquad \text{(IV-41)}$$

which may be written

$$g^{\alpha\beta} \phi_{,\alpha\beta} + m^2 \phi + g^{\alpha\beta}{}_{,\beta} \phi_{,\alpha} + \tfrac{1}{2} g^{\mu\nu} \; g_{\mu\nu \;,\beta} \, g^{\alpha\beta} \phi_{,\alpha} = 0. \qquad \text{(IV-42)}$$

These equations appear to be the correct extension of the massless theory, as we notice that in the limit of no interactions ($\kappa = 0$), and zero mass ($\lambda = 0$), $g^o_{\mu\nu} = \delta_{\mu\nu}$ and $\phi = c \, e^{ik\cdot x}$ are solutions of equations IV-40 and IV-42 provided that $k \cdot k = 0$.

We may now find the zero order metric tensor $g^o_{\mu\nu}$ which our world possesses in the absence of gravity-matter interactions ($\kappa = 0$). Let R^o_{abcd} denote R_{abcd}, but computed using $g^o_{\mu\nu}$; then equation IV-40 becomes

$$(R^o - \lambda^2) \, g^o_{\mu\nu} = 2 \, R^o_{\mu\nu} \, . \qquad \text{(IV-43)}$$

Multiplying by $g^{\mu\nu o}$ we get

$$R^o = 2\lambda^2 , \qquad \text{(IV-44)}$$

i.e., a space of constant curvature.

It is well known[20] that the coordinates of such a space can always be chosen so that the metric tensor assumes the Riemannian form:

$$g^{o}_{ij} = e_i \ \bar{\delta}_{ij} \left[1 + \frac{\lambda^2}{24} \sum_{k=1}^{4} e_k (x^k)^2 \right]^{-2} . \qquad \text{(IV-45)}$$

In this expression, the e_i are all either + 1 or - 1, and $\bar{\delta}_{ij}$ denotes the ordinary Euclidean Kronecker delta. There is no summation on the index i.

This metric tensor is completely specified by requiring that at small distances it reduces to the pseudo-Euclidean metric

$$\delta_{ij} = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix} . \qquad \text{(IV-46)}$$

Thus, we see that when the interaction of gravity with the scalar field is "turned off," the metric tensor of a space containing massive gravitons assumes the form

$$g^{o}_{\mu\nu} = \delta_{\mu\nu} / \psi \qquad \text{(IV-47)}$$

$$\psi = (1 + \frac{\lambda^2}{24} \delta_{\alpha\beta} x^{\alpha} x^{\beta})^2 \qquad \text{(IV-48)}$$

It reduces to the usual pseudo-Euclidean metric at distances $x << 1/\lambda$.

By making the coordinate transformation

$$\begin{aligned} x_0 &= f \cosh \alpha \\ x_1 &= f \sinh \alpha \cos \beta \\ x_2 &= f \sinh \alpha \sin \beta \cos \gamma \\ x_3 &= f \sinh \alpha \sin \beta \sin \gamma, \end{aligned} \qquad \text{(IV-49)}$$

where

$$f = (2\sqrt{6}/\lambda) \ \tan \ (\lambda t/2\sqrt{6}), \qquad \text{(IV-50)}$$

we can write the line element in the form

$$ds^2 = g^o_{\mu\nu} \, x^\mu \, x^\nu \qquad \text{(IV-51)}$$
$$= dt^2 - R(t)^2 \left[d\alpha^2 + \sinh^2 \alpha \, (d\beta^2 + \sin^2\beta d\gamma^2) \right] ,$$

with

$$R(t) = (\sqrt{6}/\lambda) \sin (\lambda t/\sqrt{6}). \qquad \text{(IV-52)}$$

The 3-space spanned by the coordinates α, β and γ is the 3-dimensional analog of a two-dimensional (unbounded) space of constant negative gaussian curvature. This universe is periodic in time.

Thus our space is curved even in the absence of anything except the massive gravitons themselves. This is an interesting feature of the massive gravity problem and might deserve further study. Unfortunately, plane waves are no longer solutions of the free-field equations for the scalar field; the solutions only look like plane waves for distances small compared to $1/\lambda$.

We can exhibit how complicated these free-particle solutions are. In the limit $\kappa = 0$ equation IV-42 may be written

$$\psi \, \phi_{,\mu\mu} - \psi_{,\mu} \phi_{,\mu} + m^2 \phi = 0 , \qquad \text{(IV-53)}$$

where our notation is the usual one:

$$\phi_{,\mu\mu} = \delta^{\alpha\beta} \, \phi_{,\alpha\beta} , \qquad \text{(IV-54)}$$

$$\psi_{,\mu} \phi_{,\mu} = \delta^{\alpha\beta} \, \psi_{,\alpha} \phi_{,\beta} . \qquad \text{(IV-55)}$$

The natural solutions one would try, exp (i k·x) or exp (i k·x/ψ), do not satisfy equation IV-53. Its solutions are fairly complicated;* the following expression satisfies the equation to order

*Plane waves in a spherical 4-space are treated by Schrodinger, Reference 21. Such a space has a metric like that of equation (IV-51), but with sinh α replaced by sin α.

λ^2 in the (small) graviton mass:

$$\phi = c \exp \left[i\, k\cdot x + (1/72)\,(\lambda^2/m^2)(-3\, m^2 x\cdot x + 12\, k\cdot x\, k\cdot x - 3i\, k\cdot x\, x\cdot x\, m^2 + 2i\, [k\cdot x]^3) \right] . \qquad \text{(IV-56)}$$

The field equations for gravity, whose solutions are gravitational waves which look like equation IV-25 at small distances, are obtained by substituting equation IV-36 into equation IV-40, using the explicit form for g^o in equations IV-47 and IV-48, and extracting the terms proportional to κ. The result is just equation IV-23 with the additional term $+ \lambda^2 \bar{h}_{\mu\nu}$ on the left-hand side, plus terms of order λ^4 and higher.

Thus we have circumvented the problem caused by the additional terms of equation IV-35 in the Lagrangian, but only at the expense of having more complicated free-particle states now.

The entire question of developing rules for calculating diagrams is more difficult now, since we can't simply Fourier transform everything. It would be interesting to proceed from this point to calculate scattering amplitudes, but I have not been able to devise a sufficiently elegant approach to make the calculation practical.

This discussion at least explains why the extension to massive spin 2 particles is a problem of entirely a different sort than going from massless to massive spin 1 particles. Since real gravitons appear to be very nearly massless, further study of the massive gravity theory does not seem necessary at this time.

V. REGGEIZED PARTICLES AND ELEMENTARITY

We have seen that the generation of Regge trajectories from elementary particle poles is a phenomenon occurring in a number of field theories, including zero mass theories of physical interest. However, some particles conspicuously fail to Reggeize (the vector meson and the scalar nucleon) and presumably contribute fixed singularities to the exact, all-orders partial wave amplitudes.

Does this mean that even in field theories, certain "elementary" particles are to be regarded as being dynamical in some sense? No. It is true that we can no longer view Regge trajectories as being associated exclusively with dynamical states, but it appears that Reggeized particles should still be regarded as "elementary."

There are two ways we can look at the problem of elementarity. One is the intuitive one -- an elementary particle is one upon whose mass and coupling constant there are no restraints.* Since Reggeization occurs (when it works) for arbitrary values of the mass and coupling constant of the particle, no such restraints are placed on them. We feel that we should have some way of calculating these quantities for a true dynamical particle, by some self-consistent argument perhaps.

Another way we could approach the problem of elementarity may be more satisfying to the reader who enjoys seeing calculations. It has been shown (see reference (22) and work there cited) that a useful definition of a dynamical particle in field theories is that $Z_1/Z_3 = 0$, where Z_1 is the vertex renormalization associated with the coupling

*My thanks to Professor F. Zachariasen for this helpful comment.

of the particle and Z_3 is its wave function renormalization. This combination may be found from the high energy limit of the form factor, which we will now proceed to compute.

We specifically consider the interesting case of spin $\frac{1}{2}$ nucleons interacting with spin 1 massless photons. The requirement of massless photons simplifies our work by making only one intermediate state contribute to the unitarity equation. Cases with lower spin are rather easier to treat and we will mention the results obtained in those cases.

Our approach is depicted in figure 4. An initial nucleon of four-momentum p and helicity h and a photon of four-momentum k and helicity λ interact to produce a nucleon of four-momentum σ. We take $\sigma^2 = s$, not on the mass shell, of course. Because the photon is free and massless the vertex function cannot contain terms of the form k_μ, and the most general form for the vertex function is just

$$T_\mu = F(\sigma) \gamma_\mu = \left[A(s) + \not\sigma B(s) \right] \gamma_\mu . \qquad \text{(V-1)}$$

Our principal approximation is the use of two-particle unitarity. The Cutkosky prescription allows us to write down an expression for the discontinuity in s of the amplitude depicted in figure 4. What we want to do is to convert the expression to one involving helicities rather than four-vectors and spinors, as much as possible. We can't entirely avoid spinors, because one of the spinor particles is off the mass shell, but we do the best we can and define

$$T_{h,\lambda}(p) = T_\mu \left| u_{p,h} \right) e_\mu^{(\lambda)} . \qquad \text{(V-2)}$$

$T_{h,\lambda}$ is a Dirac spinor and both T's are functions of s and $\not\sigma$.

To avoid introducing spurious discontinuities in s from the fermion spinors we are inserting, we need the special notation

$$"\mathrm{Im}\ T_{h,\lambda}" \equiv \left[\mathrm{Im}\ T_{\mu}\right] |u_{p,h}) \quad e_{\mu}^{(\lambda)} \ , \tag{V-3}$$

where the second "Im" does not apply to the spinor $|u_{p,h})$. Finally, we let $T_{h',\lambda',h,\lambda}$ $(s, \cos\theta)$ be the scattering amplitude for nucleon-photon scattering. Then we obtain for the unitarity equation

$$"\mathrm{Im}\ T_{h,\lambda}(p)" = \frac{1}{32\pi^2} \frac{p}{W} \int d\Omega q \sum_{h'=\pm\frac{1}{2}} \sum_{\lambda'=\pm 1} T_{h',\lambda'}(\sigma - q) \, T^{*}_{h',\lambda',h,\lambda}(s, \cos\theta_q) \ . \tag{V-4}$$

We can express $T_{h',\lambda',h,\lambda}$ as $8\pi W\, f_{h',\lambda',h,\lambda}$ and use the formulas of reference 4 to further simplify. We choose the polar axis along the direction of p in the CM system and explicitly insert the necessary Dirac spinors. Eventually we get

$$\left[\mathrm{Im}\ F(\sigma)\right] \ 2^{-\frac{1}{2}}(E+m)^{-\frac{1}{2}} \begin{pmatrix} 0 \\ -2p \\ 0 \\ 2(E+m) \end{pmatrix} = (1/32\pi^2)(p/W)\ 8\pi W \ F(\sigma)\ 2^{-1}\ (E+m)^{-\frac{1}{2}} \int d\Omega \begin{pmatrix} -2p \sin\theta\ f^{+}_{1\frac{1}{2}1\frac{1}{2}} \\ -2p\ f^{+}_{1\frac{1}{2}1\frac{1}{2}} - 2p\cos\theta\ f^{-}_{1\frac{1}{2}1\frac{1}{2}} \\ 2(E+m)\sin\theta\ f^{+}_{1\frac{1}{2}1\frac{1}{2}} \\ 2(E+m)\ f^{+}_{1\frac{1}{2}1\frac{1}{2}} + 2(E+m)\cos\theta\ f^{-}_{1\frac{1}{2}1\frac{1}{2}} \end{pmatrix} . \tag{V-5}$$

We have avoided doing some obvious simplifying to make it clear where all the factors come from.

The angular integrals can be explicitly done with some help from the partial wave expansions and their inversion formulas. The result of the angular integration is to replace the spinor on the right-hand side of the equation by

$$4\pi\sqrt{2}\begin{pmatrix}(2/3)\ p\ (F_{11}^{0+} - F_{11}^{1-}) \\ -\ 2\ p\ F_{11}^{0+} \\ -\ (2/3)\ (E+m)\ (F_{11}^{0+} - F_{11}^{1-}) \\ 2\ (E+m)\ F_{11}^{0+}\end{pmatrix}. \qquad (V\text{-}6)$$

Apparently we can find a consistent solution only if we assume the symmetry $F_{11}^{1-} = F_{11}^{0+}$ in whatever expression we decide to put in for the Compton scattering, but since we are interested only in the + parity amplitudes anyway that seems to be acceptable. This difficulty does not arise for spinless nucleons interacting with spin 0 or spin 1 photons.

We can easily see that the equations for A(s) and B(s) are uncoupled (for example, by choosing coordinates so that

$\sigma_\mu = (\sqrt{s}\ ,\ 0,\ 0,\ 0)$). They have the simple form

$$\text{Im } A(s) = k\ F_{11}^{0+}\ A(s)\ , \qquad (V\text{-}7)$$

$$\text{Im } B(s) = k\ F_{11}^{0+}\ B(s)\ . \qquad (V\text{-}8)$$

These equations can be explicitly solved by making the substitution $A(s) = c \exp\left[g(s)\right]$. The result is

$$A(s) = c \exp\left\{-\frac{1}{2\pi i}\int\frac{ds'}{s'-s}\ \ln\left[1 - 2\,i\,k(s')\ F_{11}^{0+}(s')\right]\right\} \qquad (V\text{-}9)$$

and similarly for B(s). The arbitrary multiplicative constants are determined by the condition that when $s = m^2$, $F(\sigma)$ must simply reduce to the coupling constant e. Thus $A(m^2) = e$, $B(m^2) = 0$ and as a result B(s) vanishes identically.

We may now find

$$A(s) \xrightarrow[s \to \infty]{} e\, Z_1/Z_3 \qquad \text{(V-10)}$$

for any assumed form of the Compton scattering contribution $F_{11}^{0+}(s)$.

To see what happens, suppose that the Compton scattering is dominated by a single Regge pole contribution from a dynamical state. From Appendix B we would expect the partial wave amplitude to have the form

$$F_{11}^{\ell+} \approx \beta(s)/\left[\ell - \alpha(s)\right] , \qquad \text{(V-11)}$$

where

$$\text{Im}\, \alpha(s) = k\, \beta(s) \qquad \text{(V-12)}$$

and

$$\text{Im}\, \beta(s) = 0. \qquad \text{(V-13)}$$

It is then clear that the integral in equation V-9 converges for $s = m^2$ so that c is some finite nonzero number which we won't need to find.

Now to be definite we will have to assume some form for $\beta(s)$ at large s. We take

$$k\ \beta(s) \xrightarrow[s \to \infty]{} a\, s^n , \qquad \text{(V-14)}$$

where n is an integer greater than or equal to -1. This form is applicable to trajectories found in relativistic problems. Then $\alpha(s)$ has the asymptotic form

$$\alpha(s) \xrightarrow[s \to \infty]{} -(1/\pi)\, a\, s^n \ln s , \qquad \text{(V-15)}$$

so that

$$k\, F_{11}^{0+} \approx -k\, \beta(s)/\alpha(s) \xrightarrow[s \to \infty]{} +\pi/\ln s . \qquad \text{(V-16)}$$

The algebraic sign of this asymptotic form is crucial. The + sign is due specifically to the photon's having spin 1; it appears in the analogous equation for scalar nucleon-photon scattering, but the corresponding expression in the scalar nucleon-scalar "photon" case is $-\pi/\ln s$.

We can now proceed to find the form of A(s) at large s from equation V-9:

$$\begin{aligned} A(s) &\approx c \exp\left\{ - \frac{1}{2\pi i} \int \frac{ds}{s'-s} \ln\left[1 - 2i(\pi/\ln s')\right]\right\} \\ &\approx c \exp\left\{ + \int \frac{ds}{s'-s} \frac{1}{\ln s'}\right\} \\ &\approx c \exp\left[- \ln \ln s\right] \\ &\approx c (\ln s)^{-1} \\ &\xrightarrow[s \to \infty]{} 0 \quad . \end{aligned} \qquad \text{(V-17)}$$

Thus $Z_1/Z_3 = 0$ and the "Z test" verifies that the Regge trajectory represents a dynamical particle. The same result is obtained for spin 0 nucleons interacting with photons. However, if all the particles are spinless, the photon included, the form factor F(s) (corresponding to A(s)) is asymptotically c ln s; then $Z_1/Z_3 = \infty$ in this approximation and the "Z test" doesn't work.

Now we proceed to apply this test to a true elementary particle; for the Compton scattering we will just use that calculated from the Born approximation. The appropriate diagrams look like a and b of figure 3 with gravitons replaced by photons. The result is quoted in reference 4, and its asymptotic form is

$$k\ F_{11}^{0+} \xrightarrow[s \to \infty]{} - \frac{e^2 m^2}{16\pi s} \quad . \qquad \text{(V-18)}$$

In this case the numerical factor in front is irrelevant; the 1/s dependence itself is enough to obtain

$$A(s) \xrightarrow[s \to \infty]{} c . \qquad (V\text{-}19)$$

Again the integral converges at $s = m^2$ so c is finite and we have Z_1/Z_3 = some constant. Hence the "Z test" says that the nucleon is elementary, as it should.

However, consider what happens when the form we assume for $F_{11}^{\ell +}$ is that appropriate for a Reggeized particle as obtained in Appendix B:

$$F_{11}^{\ell +} \approx \beta(s)\, \alpha(s) \,/ \left[\ell - \alpha(s)\right] , \qquad (V\text{-}20)$$

with

$$\text{Im}\, \beta(s) = 0. \qquad (V\text{-}21)$$

The contribution to the vertex function comes only from the point $\ell = 0$, and $\alpha(s)$ disappears from sight. Specifically, if we take the $F_{11}^{\ell +}$ obtained in reference 4 for the Reggeized spin $\frac{1}{2}$ nucleon, the asymptotic form of $k\, F_{11}^{0+}$ is precisely that given by equation V-18 and the Z test shows the Reggeized nucleon to be elementary.

It is even plausible that a Reggeized particle always will appear to be elementary by this test. The reason is that the form of equation V-20 was originally conceived so that at small $\alpha(s)$ a $\delta_{\ell 0}$ term would be produced, retrieving the $\delta_{\ell 0}$ term which must be present in the Born approximation amplitude. In other words, the residues of the Regge pole were postulated to be such that at precisely $J = \frac{1}{2}$ the contribution to the scattering amplitude would be just that due to the original elementary particle.

However, the total spin of the intermediate state must always be just the spin of the particle we are trying to Reggeize, which occurs off its mass shell in the final state; e.g. the spin of the intermediate state is confined to $J = \frac{1}{2}$ in the nucleon problem. Thus, the vertex function is only able to "observe" the Regge pole exactly at that spin at which its contribution looks like that due to an elementary particle; the vertex function cannot feel out the Reggeness of the particle in the J-plane, and thus the Z test will always say that it is elementary.

It turns out that for scalar nucleons interacting with either spin 0 or spin 1 photons these expectations are borne out; the Z test verifies that the nucleon is elementary.

We conclude that Reggeized particles must still be regarded as elementary in any reasonable sense of the word and that therefore moving poles in scattering amplitudes apparently are not necessarily associated with dynamical states.

VI. CONCLUDING REMARKS

We will now summarize our results and discuss the conclusions that may be drawn from them. We have encountered three different types of asymptotic behavior in the scattering amplitudes that we have studied. When spin $\frac{1}{2}$ particle scatter vector mesons or gravitons the scattering amplitudes are dominated asymptotically by a single Regge trajectory, on which the spin $\frac{1}{2}$ particle lies.

A second type of behavior occurs when scalar particles scatter vector mesons or gravitons. In such processes the amplitude is dominated asymptotically not only by a dynamically formed Regge trajectory, but also by a fixed singularity in the angular momentum plane corresponding to the elementary scalar particles.

The third type of asymptotic dependence appears in the vector meson channel in a theory of vector mesons and "nucleons." In the case of both spin $\frac{1}{2}$ and spin 0 "nucleons," the scattering amplitude is dominated asymptotically just by the singularity corresponding to the elementary vector meson; no Regge trajectory appears.

In all cases, however, the Z test gives results that are intuitively sensible: the nucleon or vector meson in these field theories is still elementary, as defined by the Z test, whether or not it lies on a Regge trajectory.

The most obvious conclusion that may be drawn is that moving poles in the partial wave amplitudes do not necessarily correspond to dynamical particles, but may instead be associated with elementary particles. This result was certainly suggested by the previous work on Compton scattering with vector mesons (in references 2-7) but could have been due to the special features of the nucleon-vector meson problem. (For example, the existence of only one nonsense state in this problem makes the required nonsense-nonsense factorization trivial.) However, we have seen that elementary particles are Reggeized in other theories, and it seems likely that this may be a generally valid feature of some field theories.

We should point out here that the stronger form of Chew and Frautschi's dynamical criterion[1] may still be valid; that is, that a truly dynamical theory should yield amplitudes containing only moving poles, in <u>all</u> possible channels. Thus even though Regge poles in a given channel need not correspond to dynamical particles, the occurrence of only Regge poles in all channels might be a feature of a bootstrapped, self-consistent scattering amplitude. This stronger postulate would disqualify the nucleon-vector meson theory as an example of a field theory in which Regge poles correspond to elementary rather than dynamical particles, since the vector meson fails to Reggeize in this theory. This postulate was discussed by Mandelstam[7] and we will refer to it as the CFM postulate.

Our present view is that the CFM postulate is not completely satisfactory. The field theories we have studied discredit any one-to-one correspondence between moving poles and dynamical particles; when this esthetically pleasing one-to-oneness is destroyed it seems artificial to try to use moving poles as dynamical criteria at all. We would instead prefer a modified statement of the connection between moving poles and elementary or dynamical particles, which we will discuss in connection with the Z test.

However, if one does wish to retain the CFM postulate, that the absence of Kronecker delta terms in all channels implies a dynamical theory, it would be desirable to study a field theory in which all the particles in the theory could be formed using two-particle nonsense states. After all, the only Reggeization we have yet found occurs by iterations of the Born approximation and requires the presence of nonsense-nonsense amplitudes; so possibly it will turn out that this is the only way that Reggeization can occur in field theories.

If we consider a theory with conserved particles A and mesons B, with spins S and s, we can find the required spins of these particles if two-particle nonsense states are to be available for each particle. The process $A + B \rightarrow A \rightarrow A + B$ requires $S + s - 1 \geq S$ for a two-particle nonsense state, and the process $A + \bar{A} \to B \to \bar{A} + A$ requires $2S - 1 \geq s$. Thus the meson B must have spin 1, 2, 3, ... and the particle A must have spin 1, 3/2, 2, ... Although there are difficulties in dealing with particles of such high spins in field theories, one might at least study factorization of the Born approximation. Such theories, with nonsense channels for both particles, seem most likely to provide a counter-example to the CFM postulate, if counter-examples indeed exist in conventional field theories.

Probably a better test of elementarity or compositeness than the CFM criterion, however, is the Z test. In our two-particle approximation this was capable of correctly distinguishing between a Reggeized elementary particle and a true dynamical state. This test allows one to retain the one-to-one correspondence between certain moving poles and dynamical particles; only those moving poles which pass the Z test will be called "Regge poles."

More generally, the point we are making here is that the trajectory corresponding to a Reggeized elementary particle has definite features which distinguish it from a dynamically produced trajectory. The specific feature that the Z test examines is the high energy behavior of the trajectory. However, we could just as well have chosen the condition $\mathrm{Im}\,\beta = 0$ to distinguish a dynamical trajectory, since the discussion of Appendix B makes it clear that this criterion would serve to differentiate between a dynamical state and a Reggeized particle.

An alternative to the CFM postulate is then the postulate that all amplitudes in a dynamical theory must satisfy the Z test. In fact, as far as we know, it would be sufficient to postulate that all moving poles in a dynamical theory must have residues without discontinuities. One of these postulates may prove to be a satisfactory way to characterize a dynamical scattering theory.

Even if we have been able to resolve some of the confusion about moving poles as related to elementarity, however, we would like a better understanding of why so many different kinds of asymptotic behavior are found in field theories. Why the dynamics should choose to "Reggeize" some particles and not others, and why dynamical trajectories which dominate the amplitudes are only sometimes formed are most interesting questions.

One possibility is that the asymptotic behaviors we have seen are characteristic of the elementary particle that can occur in the intermediate state. The asymptotic amplitudes for a scalar in the intermediate state contained both a dynamical trajectory and a fixed sigularity. For a spin $\frac{1}{2}$ particle the amplitudes contained only the Reggeized trajectory, and for the spin 1 particle the amplitudes contained only the fixed singularity. Although such a universal form for the scattering amplitudes seems unlikely, it would be simple enough to try it out on a few examples -- such as the reaction $\pi + N \rightarrow N \rightarrow \pi + N$.

Another, highly speculative, viewpoint to take with regard to these varying asymptotic behaviors is that they provide a dynamical criterion in certain processes. We would suppose that dominance by a single fixed or moving sigularity is a general feature of theories describing the real world, and constitutes a "normal" behavior for the asymptotic scattering amplitudes. Our results thus far might suggest that in any process in which non-strongly interacting particles interact by electromagnetic or gravitational forces, this normal behavior is always obtained. For example, our previous work shows that this "normal" behavior is exhibited in both the fermion and the photon channel in a theory of photons interacting with electrons or muons, and in the fermion channel when gravitons interact with electrons or muons.

The fact that all amplitudes are postulated to have this behavior in the real world, however, provides a dynamical condition on some processes, such as the pion channels in graviton-scalar or vector-scalar scattering. Presumably these processes will always involve at least one hadron (strongly interacting particle).

In principle at least the dynamical condition here implied can easily be determined. The exact field theory amplitude is solved for in the N/D equations (as they are used in Appendix C) and the dynamical pole generated by the nonsense channel is analytically continued to the position of the elementary particle (at $\ell = 0$ for pions). The couplings and mass of this state are adjusted relative to those of the elementary particle until they coincide; thus, instead of saying as Mandelstam does that these poles could only coincide by accident, we adjust the parameters of the theory until they do coincide. This provides the necessary dynamical condition.

The results obtained thus far suggest that dynamical conditions may be obtained only in processes involving hadrons. Such a result would be satisfying because it is only in such processes that we have yet encountered practical difficulties in distinguishing between elementary and composite particles (i.e., γ, e, μ, and ν give no sign at all of being composite). An obvious place to try out such a dynamical condition is in pion-nucleon scattering, where it should yield the mass ratio and the coupling constant.

Lest there be some confusion about obtaining this dynamical condition, let us point out that it cannot be obtained directly from the Born approximation of field theory. We are not proposing trying to adjust the parameters available until the Born approximation factors in the proper way. Not only would such attempts be fruitless for determining the coupling constant, but they would be manifestly wrong.

We cannot use perturbation theory at all when we expect to determine the coupling constant dynamically. One way to visualize this is to recall that we took an asymptotic form in the field theory amplitude, equated it to the asymptotic form expected from a Regge pole, and then equated the coefficients of corresponding powers of the coupling constant on each side of the equation. This equating of coefficients tacitly assumed that the equation was valid for a continuous range of values of the coupling constant.

When we set a dynamical condition as we do now, however, we recognize that the equation will be true only for some particular value or values of the coupling constant, which we must then find. This illustrates why we should not rely upon such principles as factorization of the Born approximation as a simple means of obtaining the dynamical conditions imposed by our postulate of "normal" behavior of the scattering amplitudes. We are still working in the framework of field theory in that we use the N/D equations to study field theory amplitudes, but we can no longer simply apply a perturbation expansion.

The remarks we have just made concerning dynamical conditions should be regarded as purely speculative, however.

Where do we go from here? Several possible problems to consider have been referred to in the text: it would be desirable to apply analyticity arguments such as Mandelstam's directly to the case of zero mass without taking any limits from a massive theory. The infrared problem could be treated at least approximately by taking large t before letting the energy resolution go to zero, to see whether the amplitudes exhibited the same asymptotic behavior. The massive graviton field theory would be interesting to pursue in connection with cosmology. It would also be appropriate to study Reggeization in a field theory with two-particle nonsense states available to each of the particles in the theory, in an attempt to learn more about how the CFS hypothesis applies to field theories.

- END -

APPENDIX A

Calculation of Sixth-Order Non-Planar Diagrams

This appendix is specifically concerned with determining the asymptotic form of the scattering amplitude calculated from diagram b of figure 1 and the corresponding crossed diagram b' . The theory is vector mesons coupled to spin 0 nucleons.

The amplitude which is obtained when the denominators are properly combined has the form given in equations II-2 and II-3. For this diagram, the function F can vanish within the region of integration; this appears to be a general feature of nonplanar diagrams. Thus at large t the dominant contributions to the integral come not only from the region where z_4 and z_5 are small, but also from the region where F is small.

It was suggested by Ahmed[10] that at large t the vanishing of F may cause the contour of integration to be pinched by two poles in the manner that has been discussed by Polkinghorne[9] and others. The fact that a double pole is encountered on the path of integration if one blindly applies Polkinghorne's method for planar graphs[7] to calculate the asymptotic form suggests that this may indeed occur.

Ahmed has attempted to calculate these amplitudes including the pinching effect[10], but it appears that some errors appear in his work, some of which he has acknowledged[13]. It is hoped that the following discussion is completely accurate.

It is not easy to calculate the pinch contribution using the conventional parameterization of the four-dimensional integrals, in which all the propagators are combined into a single denominator using a single delta function. The reason is this: When calculating the pinch contribution, a few integrals must first be evaluated to obtain logarithmic terms; these terms then yield the purely imaginary pinch contribution when the integration contour is pulled across one of the pinching poles[9]. However, the ranges of integration depend upon many of the parameters of integration and the pinch does not occur except over certain parts of the range of the other integrations.

We may avoid this by parameterizing the integrals as discussed in the text to obtain expressions of the form in equation II-2. The additional contribution from the pinch can then be calculated by exactly doing the x_1, x_2, y_1, and y_2 integrals (it is sufficient to take z_4 and z_5 to be small in the functions A, B, C, and F if only the leading asymptotic term in t is ultimately desired). One then obtains a number of terms in the integrand; the only one that is affected by pulling the contour of integration across a pole is the logarithmic term. It has the form

$$\ln \frac{\left[d + z_4 z_5 t z_2\right]\left[d + z_4 z_5 t z_1\right]}{\left[d - z_4 z_5 t z_3\right]\left[d + z_4 z_5 t(z_1+z_2)\right]} , \tag{A-1}$$

where

$$d = z_1 z_2 z_3 s - A_1 . \tag{A-2}$$

When the limit $t \to +\infty + i\epsilon$ is taken, this yields a contribution $-i\pi$ (rather than $-2i\pi$ as occurred in the examples treated by Polkinghorne[9]). The total contribution of the pinch to the amplitude for diagram 1b is then asymptotically

$$+ i\pi\, h(s)\ t \ln t , \tag{A-3}$$

where h(s) is given by equation II-9.

We remark here that if the integral for this amplitude is parameterized in the conventional way and the pinch contribution is calculated without proper attention to the limits of integration, the result is (-2) times that quoted above, as obtained by Ahmed[10].

The treatment of diagram b' is quite similar, with t replaced by u, up to the point of calculating the pinch contribution arising from the logarithmic term. This term is given by equation (A-1) with t replaced by u. However, in the limit $t \rightarrow +\infty + i\epsilon$, the limit for u is $u \rightarrow -\infty - i\epsilon$. The result of this difference is that the pinch contribution from the logarithmic term is $+i\pi$ for diagram b'. The net asymptotic contribution of the pinch to diagram b' is thus

$$-(-i\pi \; h(s))\, u \ln u \qquad \text{(A-4)}$$

as remarked in equation II-11.

What has happened is the following; although the scattering amplitude from diagram b' is indeed equal to minus the amplitude from diagram 1b, but with t replaced by u, the asymptotic limit taken in the variables t and u leads one to different sheets of the function in the large t limit. Thus the real parts are indeed related by a simple $t \longleftrightarrow u$ prescription but the imaginary contributions enter with opposite relative signs.

APPENDIX B

Generation of Trajectories by Iteration Through Unitarity

In this appendix we will summarize how the generation of Regge trajectories can occur when the Born approximation amplitude is iterated through the unitarity relation in the s channel. The discussion follows the treatment in section 5 of reference 4 and contains essentially no new work.

If we consider scattering of nucleons (either scalar or spinor) by mesons of spin s, the work of reference 4 shows us how to analyze the amplitudes in partial waves. The partial wave amplitudes for any spin J will be denoted by $\underline{\underline{F}}$, a symmetric matrix of order 2s+1. The unitarity equation is then given by

$$\mathrm{Im}\,\underline{\underline{F}} = k\,\underline{\underline{F}}^{+}\,\underline{\underline{F}}\ , \qquad \text{(B-1)}$$

where k is the momentum of either particle in the C.M. system. We wish to examine the conditions under which the unitarity equation is consistent with a Regge-like form for the amplitudes $\underline{\underline{F}}$.

Consider first the generation of a dynamical trajectory. Since the residues of Regge poles must factor, near the pole the amplitude is expected to have the form

$$\underline{\underline{F}} = \underline{\eta}\,(s)\,\underline{\eta}^{+}(s)\,\left[\ell - \alpha(s)\right]^{-1} \qquad \text{(B-2)}$$

In this expression, $\underline{\eta}\,(s)$ is a 2s + 1 dimensional vector.

Then near the pole we have

$$\mathrm{Im}\,\underline{\underline{F}}\ - k\underline{\underline{F}}^{+}\underline{\underline{F}} = \left[\underline{\eta}^{*}\,\underline{\eta}^{+*}\ \ \mathrm{Im}\,\alpha - k\,\underline{\eta}\,\underline{\eta}^{+}\,(\underline{\eta}^{+}\cdot\underline{\eta})\right]\left|\ell - \alpha\right|^{-2}. \qquad \text{(B-3)}$$

We see that this vanishes and so is consistent with unitarity provided that

$$\text{Im}\ \underline{\eta}(s) = 0 \tag{B-4}$$

and

$$\text{Im}\ \alpha(s) = k\underline{\eta}^{+}(s)\cdot\underline{\eta}(s). \tag{B-5}$$

(The dot product is the ordinary Euclidean product of the two vectors.)

Thus we see that if the Born approximation yields a form like the lowest order of equation (B-2) expanded in powers of the coupling constant, a Regge trajectory is generated by the unitarity relation; only a lack of knowledge about the possible subtraction constants in the higher-order terms in $\underline{\underline{F}}$ obtained by iterating through unitarity keeps this condition from being rigorously sufficent.

However, a form for the amplitude like that given in equations (B-2) and (B-4) cannot represent one of the elementary particles in the theory, because for small coupling constant this sort of expression does not reduce to the Kronecker delta term found in the Born approximation. That is, the Born approximation yields a Kronecker delta term of the form $\delta_{\ell 0}$ in the analytically continued partial wave amplitude, and this can only generate more terms of the same form when iterated through unitarity, rather than generating an expression like equation (B-2).

If a theory has nonsense channels available at the spin of the nucleon, however, there is a way out of this difficulty. Gell-Mann et al. conjectured(4) that the correct Reggeistic expression for $\underline{\underline{F}}$ that is consistent with (B-2) but still reduces to the Kronecker delta at small coupling constant is

$$\underline{\underline{F}}_{NN} = \underline{A}_N(s)\ \underline{A}_N^+(s)\left[\ell - \alpha(s)\right]^{-1} \qquad (B\text{-}6)$$

$$\underline{\underline{F}}_{NS} = \underline{A}_N(s)\ \underline{A}_S^+(s)\ \ell^{\frac{1}{2}}\left[\ell - \alpha(s)\right]^{-1} \qquad (B\text{-}7)$$

$$\underline{\underline{F}}_{SS} = \underline{A}_S(s)\ \underline{A}_S^+(s)\ \alpha(s)\left[\ell - \alpha(s)\right]^{-1} \qquad (B\text{-}8)$$

Here the subscripts S and N denote sense and nonsense states at $J = J_{Nucleon}$. If $\alpha(s)$ is proportional to g^2, this clearly gives the Kronecker delta term in the lowest order expansion of the sense-sense amplitudes in powers of the coupling constant.

To see how this is consistent with the unitarity equation, let us look at the dominant terms near the pole, near $\ell = 0$. The unitarity equation for the nonsense-nonsense amplitudes is then approximately

$$\text{Im}\ \underline{\underline{F}}_{NN} = k\ \underline{\underline{F}}_{NN}^+\ \underline{\underline{F}}_{NN}\ , \qquad (B\text{-}9)$$

because $\underline{\underline{F}}_{NN}$ gives a more singular term near $\ell = 0$ than $\underline{\underline{F}}_{SN}$ does. The form given in equation (B-6) is consistent with this equation provided that

$$\text{Im}\ \underline{A}_N(s) = 0 \qquad (B\text{-}10)$$

and

$$\text{Im}\ \alpha(s) = k\ \underline{A}_N^+(s)\cdot\underline{A}_N(s)\ . \qquad (B\text{-}11)$$

It is in this sense that we may say that a Regge trajectory is generated by the nonsense channels as long as they factor and have the proper form at small g^2.

We digress to remark at this point that if the nonsense-nonsense amplitudes do not factor, several Regge trajectories are generated, in the sense that the amplitude has a form that can be analyzed in terms of several trajectories rather than one.

In fact, in Compton scattering mesons of spin s by nucleons of spin 0 or $\frac{1}{2}$, there are 2s+1 independent states, of which only two are "sense" at $J = J_{\text{Nucleon}}$. Thus the symmetry of the $\underline{\underline{F}}$-matrix implies that there are

$$\sum_{i=1}^{2s-1} i = s\,(2s - 1) \qquad \text{(B-12)}$$

nonsense-nonsense amplitues at $J = J_{\text{Nucleon}}$. Factorization means that only 2s-1 of the residues of a Regge pole will be independent; thus in general s Regge trajectories are generated in such a problem. That is, if the amplitude is assumed to be dominated by Regge poles, in the absence of any particular factoring properties in the Born approximation amplitudes, it is necessary to use s Regge poles to properly represent the amplitude.

In a similar manner we may show that the forms given for $\underline{\underline{F}}_{NS}$ and $\underline{\underline{F}}_{SS}$ by equations (B-7) and (B-8) are consistent with unitarity. The most singular term in the unitarity equation for $\underline{\underline{F}}_{NS}$ is the term $k\,\underline{\underline{F}}^{+}_{NN}\ \underline{\underline{F}}_{NS}$, and we get the condition

$$\text{Im}\,\underline{A}_S\,(s) = 0\ . \qquad \text{(B-13)}$$

Likewise, the most singular term in the unitarity equation for $\underline{\underline{F}}_{SS}$ is the term $k\,\underline{\underline{F}}^{+}_{SN}\,\underline{\underline{F}}_{NS}$ and the resulting equation is identically satisfied.

Thus we see that when nonsense channels are available the amplitude can contain a Regge trajectory on which the elementary particle lies, in the sense that <u>near</u> the spin of the particle the partial wave amplitudes yield the correct Kronecker delta terms found in the

Born approximation amplitude. Since the approximation made does not depend on whether g^2 or ℓ is taken to be small first, this result should also be rigorous at the pole except for possible subtractions.

APPENDIX C

Mandelstam's Treatment of Reggeization

In this appendix we summarize the arguments Mandelstam applies to the problem of Reggeizing the nucleon in Compton scattering with vector mesons. His discussion appears in reference 6, but this slightly different treatment may be helpful to the reader. We will also indicate the slight changes which occur when the Compton scattering involves mesons of higher spins.

Mandelstam's approach is to assume the existence of a field theory whose amplitudes have certain properties and then to study these properties using unitarity and the N/D equations; discussions of the N/D equations appear in many places, for example, reference 14. We follow Mandelstam by first specifically treating the interaction of spin $\frac{1}{2}$ nucleons with (massive) vector mesons.

First, we consider the N/D equations for the partial wave amplitudes F at large enough J that all the states are "sense" states. We consider the N/D equations as merely being a convenient exact way to describe the analytic properties of the exact field theory amplitudes to all orders; for the moment, we make the approximation of elastic unitarity so that F is only a 3 by 3 matrix.

What analytic information do we need about the exact amplitudes to correctly write the N/D equations describing their properties? The limits on the dispersion integrals are known from the field theory amplitudes, at least in perturbation theory. We know that no explicit poles of the CDD type are to be inserted in the expression for D, since in general these add bound states to the solution and we could imagine studying the structure of the N/D equations in a region of the J plane devoid of poles. (For example, for forces of finite range we need only go to large enough real J and no bound states will occur, at least in the analogous non-relativistic problem.) Possible subtractions in D are eliminated by choosing $D(s) \to 1$ at large s in the usual manner.

The only other quantities that must be specified to determine the partial wave amplitudes are the possible subtraction constants in N. The condition that the elements of F approach constants as $s \to \infty$ means that there are six such constants at large J (this property of the $\underline{F}$'s can be seen from the lowest order diagrams in field theory, for example).

We fix these quantities (constants in s but functions of J) by requiring that the partial wave amplitudes have the proper threshold behavior. There are 6 threshold conditions for the positive-parity amplitudes which may be satisfied by fixing the 6 subtraction constants. Since these conditions serve to completely determine that the $\underline{F}$ obtained agrees with field theory in all orders, and since we assumed that a field theory with the proper threshold behaviors existed, the 6 threshold conditions on the negative-parity amplitudes are then automatically satisfied. It is crucial that we have enough freedom to satisfy all these conditions.

Our procedure will now be to analytically continue these partial wave amplitudes, which agree with field theory for large enough J that all the states are sensible, to the point $J = \frac{1}{2}$; we will compare these analytically continued amplitudes with those obtained at $J = \frac{1}{2}$ directly from the field theory. If we can identify the analytic continuation of one of the dynamical poles produced at higher values of J with the elementary particle pole required by the field theory at $J = \frac{1}{2}$, we will have demonstrated that the elementary particle in question lies on a Regge trajectory.

The treatment of Appendix B tells us that near $J = \frac{1}{2}$ the single nonsense state will generate a dynamical Regge trajectory in the partial wave ampltudes; this trajectory is present in the amplitudes that are becoming sensible, for all J in the vicinity of $J = \frac{1}{2}$. Therefore, even though the N/D equations and the unitarity equation for the sense-sense amplitudes are uncoupled from the nonsense states precisely at $J = \frac{1}{2}$, this dynamically produced pole is still present in the analytically continued sense-sense amplitudes.

Thus the analytically continued sense-sense amplitudes are now known; we know that they have the correct threshold behaviors at $J = \frac{1}{2}$ and in addition contain a pole with some particular positions and residues. We may now write down N/D equations describing these analytically continued amplitudes, involving only the sensible amplitudes. However, since the amplitudes we want to describe have this known pole, we must insert a single CDD pole in the denominator.

In these N/D equations, there are three sense-sense amplitudes and so there are six constants to determine: the three subtraction constants, the position of the CDD pole, and its couplings to the two sense states. There are nine conditions the amplitudes must satisfy: they must have the proper threshold conditions (three conditions for the positive parity amplitudes and three conditions for the negative parity amplitudes) and the pole in the amplitudes must have the proper position and proper couplings to the two sense states.

However, we can clearly fix the six constants to satisfy the nine conditions. The reason is that as soon as we have fixed the six constants to satisfy six of the conditions, we have just reconstructed the amplitudes we obtained a minute ago by analytical continuation from large J; hence the last three conditions will then be automatically satisfied. This is again a crucial point.

Now that we have obtained N/D-type equations describing the analytically continued sense-sense amplitudes, let us instead consider the amplitudes obtained directly from the perturbation theory at $J = \frac{1}{2}$.

Since the sense-sense amplitudes couple only to themselves at $J = \frac{1}{2}$, we may describe <u>these</u> amplitudes also by N/D equations involving only sense states. Now we have 6 threshold conditions to satisfy (three for the amplitudes of each parity) but only 3 subtraction constants available to fix them. It is not obvious that we can fix the three constants to satisfy the six conditions, and the reason is clear: the presence of an elementary particle in the theory tells us that there must be a pole in the amplitude, and to ensure this analytic form for the amplitudes we must introduce a CDD pole which now represents the nucleon. Its position and couplings to the two sense states gives just three more parameters available, and we can now fix all the subtraction constants and CDD pole parameters to satisfy all the threshold conditions.

We have thus found that we may write similar N/D-type equations for both the analytically continued amplitudes obtained from large J and the field theory amplitudes calculated exactly at $J = \frac{1}{2}$ with an elementary nucleon present. Likewise, they satisfy the same unitarity equations and each have one CDD pole. However, all the subtraction constants and CDD parameters are uniquely determined by the six threshold conditions which both sets of amplitudes satisfy. Thus the equations are in fact identical in every respect and the pole due to the elementary nucleon is identical to the analytical continuation of the dynamical Regge pole found in the amplitude at other values of J.

This ingenious argument due to Mandelstam demonstrates that the nucleon has to be Reggeized in a theory in which it is coupled to vector mesons. Since his arguments are independent of perturbation theory, the main assumption at this point has been elastic unitarity.

We can even remove the restriction of elastic unitarity by agreeing to consider the amplitudes at energies below the lowest three-particle threshold. We then see that the scattering amplitude will still contain a Regge trajectory on which the nucleon lies (even though it might not dominate the amplitude) and it could in principle be picked out by studying the amplitude with sufficient accuracy in the energy range below the three-particle thresholds.

It is now easy to see why the spinless nucleon can fail to Reggeize. The positive-parity amplitudes are no longer related to the negative-parity amplitudes by the MacDowell symmetry as they were in the spinor nucleon case.

At J = 0 there is only one sense-sense amplitude, which needs to satisfy only one threshold condition. Thus when calculating the amplitudes directly from the field theory at J = 0 there is no need to introduce a CDD pole to satisfy all the threshold conditions; or, if one is introduced, the equations will in general have solutions for all values of its position and residue and we have no assurance that it will be identical to some trajectory appearing in the analytically continued amplitude. In fact, the work of Gell-Mann et al. in reference 5. showed that the elementary particle did not lie on the dynamically produced trajectory.

We may also apply similar arguments to problems of higher spin mesons. For example, suppose that we consider Compton scattering of spin $\frac{1}{2}$ nucleons by mesons of integer spin s. There are now nonsense states at $J = s - \frac{1}{2}$, $s-3/2, \ldots,$. In fact, if n is an integer, $1 \leq n \leq s$, at $J = s + \frac{1}{2} - n$ there are 2n-1 nonsense states. From the work of Appendix B we see that in general n trajectories are generated near $J = s + \frac{1}{2} - n$ by the nonsense channels.

In particular, at $J = \frac{1}{2}$ we thus have s trajectories generated except in the special case that the nonsense-nonsense amplitudes factor at $J = \frac{1}{2}$; if that occurs, only one trajectory is generated. Since s trajectories would require introducing s CDD poles to obtain the analytically continued sense-sense amplitudes from an N/D equation, in general we would not be able to conclude that one of these had to be identical to the nucleon pole in the field theory amplitudes.

Our conclusion is that only if the nonsense-nonsense amplitudes factor among themselves does the nucleon have to be Reggeized in a higher spin theory. (Of course, in the case of vector mesons there is only one nonsense-nonsense amplitude at $J = \frac{1}{2}$ and the factorization is valid trivially.) If these amplitudes factor among themselves, then <u>all</u> the amplitudes must factor and the nucleon lies on a Regge trajectory.

If the factorization is nonexistent or incomplete, it seems unlikely that the dynamics of the problem would be just right to make one of the dynamical poles coincide with the nucleon pole, especially since we are comparing amplitudes obtained from N/D equations with different numbers of explicit CDD poles. However, it could occur, and only further inspection of the field theory would be able to settle that question.

When higher spin mesons interact with spin 0 nucleons the results are very similar to the case of vector mesons; there are not enough conditions to fix the position of the nucleon pole and again we conclude that it could coincide with the position of one of the dynamically produced poles only by sheer chance.

The conclusion is that Mandelstam's arguments make it clear that the nucleon with spin $\frac{1}{2}$ has to be Reggeized in Compton scattering with vector mesons; that a spin $\frac{1}{2}$ nucleon might be Reggeized in Compton scattering with higher integral spin mesons (this occurs if the nonsense-nonsense amplitudes factor); and that the spin 0 nucleon appears unlikely to be Reggeized in any Compton scattering.

APPENDIX D

Coefficients in the Graviton-Nucleon Partial Wave Expansions

$$e^{J+}_{\frac{1}{2}\frac{1}{2}} = \frac{P'_{\ell+1}}{\sqrt{2}(\ell+1)} \qquad e^{J-}_{\frac{1}{2}\frac{1}{2}} = \frac{-P'_{\ell}}{\sqrt{2}\,(\ell+1)}$$

$$e^{J+}_{\frac{1}{2}\frac{3}{2}} = \frac{P''_{\ell+1}}{\sqrt{2}\,(\ell+1)\,\sqrt{\ell(\ell+2)}} \qquad e^{J-}_{\frac{1}{2}\frac{3}{2}} = \frac{-P''_{\ell}}{\sqrt{2}\,(\ell+1)\,\sqrt{\ell(\ell+2)}}$$

$$e^{J+}_{\frac{3}{2}\frac{3}{2}} = \frac{P''_{\ell+1} + z\,P'''_{\ell+1} + P'''_{\ell}}{\sqrt{2}\,\ell\,(\ell+1)(\ell+2)} \qquad e^{J-}_{\frac{3}{2}\frac{3}{2}} = \frac{-P'''_{\ell+1} - z\,P'''_{\ell} - P''_{\ell}}{\sqrt{2}\,\ell\,(\ell+1)(\ell+2)}$$

$$e^{J+}_{\frac{1}{2}\frac{5}{2}} = \frac{P'''_{\ell+1}}{\sqrt{2}\,(\ell+1)\sqrt{(\ell-1)\,\ell(\ell+2)(\ell+3)}} \qquad e^{J-}_{\frac{1}{2}\frac{5}{2}} = \frac{-P'''_{\ell}}{\sqrt{2}\,(\ell+1)\sqrt{(\ell-1)\,\ell\,(\ell+2)(\ell+3)}}$$

$$e^{J+}_{\frac{3}{2}\frac{5}{2}} = \frac{-P''''_{\ell+1} - zP''''_{\ell} - P'''_{\ell}}{\sqrt{2}\,\ell\,(\ell+1)(\ell+2)\sqrt{(\ell-1)(\ell+3)}} \qquad e^{J-}_{\frac{3}{2}\frac{5}{2}} = \frac{P'''_{\ell+1} + zP''''_{\ell+1} + P''''_{\ell}}{\sqrt{2}\,\ell\,(\ell+1)(\ell+2)\sqrt{(\ell-1)(\ell+3)}}$$

$$e^{J+}_{\frac{5}{2}\frac{5}{2}} = \frac{P'''_{\ell+1} + 2z\,P''''_{\ell+1} + z^2 P'''''_{\ell+1} + 2P''''_{\ell} + P'''''_{\ell+1} + 2z\,P'''''_{\ell}}{\sqrt{2}\quad(\ell-1)\,\ell\,(\ell+1)(\ell+2)(\ell+3)}$$

$$e^{J-}_{\frac{5}{2}\frac{5}{2}} = \frac{-2P''''_{\ell+1} - 2z\,P'''''_{\ell+1} - P'''_{\ell} - 2z\,P''''_{\ell} - z^2 P'''''_{\ell} - P'''''_{\ell}}{\sqrt{2}\;(\ell-1)\,\ell(\ell+1)\,(\ell+2)(\ell+3)}$$

The remaining functions required may be found from the symmetries satisfied by these quantities:

$$e^{J\pm}_{-\mu,\,-\lambda}\ (z) = e^{J\pm}_{\lambda\mu}\ (z)$$

$$e^{J\pm}_{\mu,\,\lambda}\ (z) = (-1)^{\lambda-\mu}\, e^{J\pm}_{\lambda\mu}\ (z)$$

$$e^{J\pm}_{\lambda,\,-\mu}\ (z) = \pm(-1)^{\lambda+\lambda_m}\ e^{J\pm}_{\lambda\mu}\ (z)\ ,$$

where

$$\lambda_m = \max\left(|\lambda|,\ |\mu|\right)\ .$$

APPENDIX E

Calculation of Graviton-Nucleon Compton Scattering

This appendix presents the details of the calculation of the lowest order scattering amplitudes when (massless) gravitons are scattered by spin $\frac{1}{2}$ nucleons (or electrons). We use Feynman's gravitational field theory[17, 18] and must compute the amplitudes due to the four perturbation theory diagrams of figure 3, which we will represent by M_a, M_b, M_c, and M_d.

Since all the amplitudes are nonsense-nonsense for massless gravitons,

$$M_a = 0.$$

The other amplitudes can be simplified to the forms given below after much labor:

$$M_b = \frac{\kappa^2}{u-m^2} \frac{1}{4} \left[2p_{2\alpha}\, p_{2\beta}\, \gamma_\delta \; p_{1\gamma} - p_{2\alpha}\, \gamma_\beta \not{k}_1\, \gamma_\delta\, p_{1\gamma} \right]$$

$$M_c = \frac{\kappa^2}{16}\, \delta_{\beta\gamma} \left[-3p_{2\alpha}\, \gamma_\delta - 3p_{1\delta}\, \gamma_\alpha \right.$$
$$\left. + \frac{1}{2} \left(\gamma_\alpha \left[\not{k}_1 + \not{k}_2\right] \gamma_\delta - \gamma_\delta \left[\not{k}_1 + \not{k}_2\right] \gamma_\alpha \right) \right]$$

$$M_d = \frac{\kappa^2}{8\,t} \left[4 \not{k}_1\, \delta_{\alpha\gamma}\, p_{2\beta}\, p_{1\delta} - 2\,(u-m^2)(\gamma_\beta\, p_{1\delta} + \gamma_\delta\, p_{2\beta})\,\delta_{\alpha\gamma} \right.$$
$$\left. + 4\, p_{2\alpha}\, p_{2\beta}\, p_{1\gamma}\, \gamma_\delta + 4\, \gamma_\alpha\, p_{2\beta}\, p_{1\gamma}\, p_{1\delta} + (s-u)\, \delta_{\alpha\gamma}\, \delta_{\beta\delta} \not{k}_1 \right] .$$

In these expressions, the initial four-momenta of the nucleon and graviton are p_1 and k_1, and the final momenta are p_2 and k_2. The invariants s, t, and u are defined in the usual way:

$$s = (p_1 + k_1)^2 ,$$

$$t = (p_2 - p_1)^2 ,$$

$$u = (k_2 - p_1)^2 .$$

The coupling constant is $\kappa^2 = 32\pi G$ and m is the nucleon mass. Current conservation has been verified.

These expressions are to be multiplied by the (traceless) polarization tensors $e^1_{\alpha\beta}$ and $e^2_{\gamma\delta}$ for the initial and final gravitons, and then sandwiched between final and initial spinors. The resulting M's are related to the ordinary (non-parity-conserving) amplitudes defined by Gell-Mann et al.[4] by $M = 8\pi W f$.

We will explicitly give the results for large t in terms of the quantities $z = \cos\Theta$ (the angle between p_1 and p_2 in the CM system), E (the nucleon energy in the CM system), and p (the momentum of either particle in the CM system). We will let $M(2a, h, 2b, \frac{1}{2})$ denote the amplitude obtained when a graviton of helicity 2b scatters from a nucleon of helicity $+\frac{1}{2}$ to produce a graviton of helicity 2a and a nucleon of helicity h.

The asymptotic form of the M's follows.

$$M_b\left(2a, -\tfrac{1}{2}, 2b, \tfrac{1}{2}\right) \rightarrow \kappa^2\, mp\left(i\sqrt{\tfrac{z}{2}}\right)\Big(-\frac{1}{8} z^2 + \frac{1}{8}\frac{E}{p} z$$

$$-\frac{1}{8}\frac{E^2}{p^2} - b\frac{1}{16} z + b\frac{1}{16}\frac{E}{p} + a\frac{1}{16} z - a\frac{1}{16}\frac{E}{p}$$

$$- ab\frac{1}{16} z + ab\frac{1}{16}\frac{E}{p} + \frac{11}{64} - a\frac{1}{32} + b\frac{1}{32} + ab\frac{1}{32}\Big) + O(z^{-\frac{1}{2}})$$

$$M_c\,(2a,\ -\tfrac{1}{2},\ 2b,\ \tfrac{1}{2}) \rightarrow \kappa^2\, mp\,(i\sqrt{\tfrac{z}{2}})\,(\tfrac{1}{16}\, z^2 + \tfrac{1}{32}\, z$$

$$+ b\,\tfrac{1}{16}\, z - a\,\tfrac{1}{16}\, z + ab\,\tfrac{1}{16}\, z - \tfrac{5}{128} - b\,\tfrac{3}{32} + a\,\tfrac{3}{32}$$

$$+ ab\,\tfrac{1}{32}) + O(z^{-\frac{1}{2}})$$

$$M_d\,(2a,\ -\tfrac{1}{2},\ 2b,\ \tfrac{1}{2}) \rightarrow \kappa^2\, mp\,(i\sqrt{\tfrac{z}{2}})\,(\tfrac{1}{16}\, z^2 - \tfrac{1}{32}\, z$$

$$-\tfrac{1}{8}\,\tfrac{E}{p}\, z - \tfrac{25}{128} - \tfrac{1}{16}\,\tfrac{E}{p} - ab\,\tfrac{1}{8} - ab\,\tfrac{1}{4}\,\tfrac{E}{p}) + O\,(z^{-\frac{1}{2}})$$

$$M_b\,(2a,\ +\tfrac{1}{2},\ 2b,\ \tfrac{1}{2}) \rightarrow \kappa^2\, p^2\,(\sqrt{\tfrac{z}{2}})\,(-\tfrac{1}{8}\, z^2 - b\,\tfrac{1}{8}\, z^2$$

$$-a\,\tfrac{1}{8}\, z^2 + \tfrac{1}{8}\,\tfrac{E}{p}\, z^2 + \tfrac{1}{8}\, z + ab\,\tfrac{1}{16}\, z + \tfrac{1}{8}\,\tfrac{E}{p}\, z$$

$$+ b\,\tfrac{1}{16}\,\tfrac{E}{p}\, z + a\,\tfrac{1}{16}\,\tfrac{E}{p}\, z + ab\,\tfrac{1}{16}\,\tfrac{E}{p}\, z - \tfrac{1}{8}\,\tfrac{E^2}{p^2}\, z$$

$$+ \tfrac{15}{64} + b\,\tfrac{11}{64} + a\,\tfrac{11}{64} + ab\,\tfrac{1}{32} - \tfrac{19}{64}\,\tfrac{E}{p} - b\,\tfrac{1}{32}\,\tfrac{E}{p}$$

$$-a\,\tfrac{1}{32}\,\tfrac{E}{p} - ab\,\tfrac{1}{32}\,\tfrac{E}{p} - \tfrac{1}{8}\,\tfrac{E^2}{p^2} - b\,\tfrac{1}{16}\,\tfrac{E^2}{p^2}$$

$$-a\,\tfrac{1}{16}\,\tfrac{E^2}{p^2} - ab\,\tfrac{1}{16}\,\tfrac{E^2}{p^2} + \tfrac{1}{8}\,\tfrac{E^3}{p^3}) + O\,(z^{-\frac{1}{2}})$$

$$M_c\,(2a,\ +\tfrac{1}{2},\ 2b,\ \tfrac{1}{2}) \rightarrow \kappa^2\, p^2\,(\sqrt{\tfrac{z}{2}})\,(\tfrac{3}{16}\, z^2 + a\,\tfrac{1}{8}\, z^2$$

$$+ b\,\tfrac{1}{8}\, z^2 - \tfrac{1}{16}\,\tfrac{E}{p}\, z^2 - \tfrac{3}{32}\, z + b\,\tfrac{1}{8}\, z + a\,\tfrac{1}{8}\, z$$

$$+ ab\,\tfrac{3}{16}\, z + \tfrac{1}{32}\,\tfrac{E}{p}\, z + b\,\tfrac{1}{16}\,\tfrac{E}{p}\, z + a\,\tfrac{1}{16}\,\tfrac{E}{p}\, z$$

$$- ab\,\tfrac{1}{16}\,\tfrac{E}{p}\, z - \tfrac{15}{128} - b\,\tfrac{3}{64} - a\,\tfrac{3}{64} - ab\,\tfrac{3}{32}$$

$$+ \tfrac{5}{128}\,\tfrac{E}{p} + b\,\tfrac{3}{32}\,\tfrac{E}{p} + a\,\tfrac{3}{32}\,\tfrac{E}{p} + ab\,\tfrac{1}{32}\,\tfrac{E}{p}) + O\,(z^{-\frac{1}{2}})$$

$$M_d\,(2a,\ +\tfrac{1}{2},\ 2b,\ \tfrac{1}{2}) \rightarrow \kappa^2 p^2 \left(\sqrt{\tfrac{z}{2}}\right) \Big(-\frac{1}{16} z^2$$

$$- \frac{1}{16}\frac{E}{p} z^2 - \frac{1}{32} z - b\,\frac{1}{8} z - a\,\frac{1}{8} z - ab\,\frac{1}{4} z$$

$$- \frac{5}{32}\frac{E}{p} z - b\,\frac{1}{8}\frac{E}{p} z - a\,\frac{1}{8}\frac{E}{p} z + \frac{1}{8}\frac{E^2}{p^2}$$

$$- \frac{7}{128} - b\,\frac{3}{16} + \frac{33}{128}\frac{E}{p} - b\,\frac{3}{16}\frac{E}{p} - a\,\frac{3}{16}\frac{E}{p}$$

$$+ ab\,\frac{1}{8}\frac{E}{p} + \frac{3}{16}\frac{E^2}{p^2} + ab\,\frac{1}{4}\frac{E^2}{p^2} \Big) + O(z^{-\frac{1}{2}})$$

It is necessary to keep terms to very low orders in z because the leading two powers of z all cancel when the contributions from the diagrams are added. The total contribution at large z is easily verified to be as follows:

$$M\,(+2,\ -\tfrac{1}{2},\ +2,\ +\tfrac{1}{2}) \rightarrow \frac{\kappa^2 m}{8p}\left(-i\sqrt{\tfrac{z}{2}}\right)(E+p)^2$$

$$M\,(-2,\ -\tfrac{1}{2},\ +2,\ +\tfrac{1}{2}) \rightarrow \frac{\kappa^2 m}{8p}\left(-i\sqrt{\tfrac{z}{2}}\right)(E-p)^2$$

$$M\,(+2,\ -\tfrac{1}{2},\ -2,\ +\tfrac{1}{2}) \rightarrow \frac{\kappa^2 m}{8p}\left(-i\sqrt{\tfrac{z}{2}}\right)(E-p)(E+p)$$

$$M\,(-2,\ -\tfrac{1}{2},\ -2,\ +\tfrac{1}{2}) \rightarrow \frac{\kappa^2 m}{8p}\left(-i\sqrt{\tfrac{z}{2}}\right)(E+p)^2$$

$$M\,(+2,\ +\tfrac{1}{2},\ +2,\ +\tfrac{1}{2}) \rightarrow \frac{\kappa^2}{8p}\left(\sqrt{\tfrac{z}{2}}\right)(E+p)^2(E-p)$$

$$M\,(-2,\ +\tfrac{1}{2},\ +2,\ +\tfrac{1}{2}) \rightarrow \frac{\kappa^2}{8p}\left(\sqrt{\tfrac{z}{2}}\right)(E+p)(E-p)^2$$

$$M\,(+2,\ +\tfrac{1}{2},\ -2,\ +\tfrac{1}{2}) \rightarrow \frac{\kappa^2}{8p}\left(\sqrt{\tfrac{z}{2}}\right)(E+p)(E-p)^2$$

$$M\,(-2,\ +\tfrac{1}{2},\ -2,\ +\tfrac{1}{2}) \rightarrow \frac{\kappa^2}{8p}\left(\sqrt{\tfrac{z}{2}}\right)(E+p)^3$$

When we combine these to find the parity-conserving helicity amplitudes we make use of the simplifying relations

$$W = E + p$$
$$E - p = m^2/W \, .$$

We then find at large t (= large z):

$$f^{\pm}_{22}(z) \to \frac{1}{8\pi W} \frac{\kappa^2}{8p\sqrt{2}} (W^3 \mp m^3) \frac{m^2}{W^2} \frac{1}{z}$$

$$f^{\pm}_{2-2}(z) = f^{\pm}_{-22}(z) \to \frac{1}{8\pi W} \frac{\kappa^2}{8p\sqrt{2}} (W^3 \mp m^3)(\mp \frac{m}{W}) \frac{1}{z^2}$$

$$f^{\pm}_{-2,-2}(z) \to \frac{1}{8\pi W} \frac{\kappa^2}{8p\sqrt{2}} (W^3 \mp m^3) \frac{1}{z^2}$$

APPENDIX F

Calculation of Graviton-Scalar Compton Scattering

In this appendix we present the essential steps in computing the scattering amplitudes for scattering massless gravitons by spin 0 "nucleons." In lowest order the four diagrams of figure 3 contribute terms which we denote M_a, M_b, M_c, and M_d. The notation follows that of Appendix E.

Again, all the amplitudes are nonsense-nonsense and $M_a = 0$. The other amplitudes can be simplified to the form

$$M_b = \frac{\kappa^2}{u-m^2} \; p_{2\alpha} \, p_{2\beta} \, p_{1\gamma} \, p_{1\delta}$$

$$M_c = \kappa^2 \left[\tfrac{1}{2}(p_1 \cdot p_2 - m^2) \; \delta_{\alpha\gamma} \; \delta_{\beta\delta} - \delta_{\alpha\gamma} \; p_{1\delta} \; p_{2\beta} \right]$$

$$M_d = \frac{\kappa^2}{2t} \left\{ \delta_{\alpha\gamma} \; \delta_{\beta\delta} \left[\tfrac{1}{2} t^2 + \tfrac{1}{2} t(s-m^2) + \tfrac{1}{2} (s-m^2)^2 \right] \right.$$

$$\left. + \; 2 \; \delta_{\alpha\gamma} \, p_{2\beta} \, p_{1\delta} \; (s + t - m^2) + 2 \, p_{2\alpha} \, p_{2\beta} \, p_{1\gamma} \; p_{1\delta} \right\} .$$

Current conservation has been verified for these expressions.

If we let M (2a, 2b) denote the amplitude obtained for graviton helicities of 2b initially and 2a finally, the form of the amplitudes at large t is as follows:

$$M_b \,(2a,\; 2b) \;\rightarrow\; \frac{1}{8} \; \kappa^2 \, p^2 \; \left(-z^3 + \frac{E}{p} \, z^2 - \frac{E^2}{p^2} \, z + 2z \right.$$

$$\left. + \frac{E^3}{p^3} - 2 \, \frac{E}{p} \right) \; + \; O\,(z^{-1})$$

$$M_c\,(2a,\;2b) \rightarrow \frac{1}{8}\;\kappa^2\,p^2\,(z^3 + z^2 - 3z + ab\;2z - ab\;2 + 1) + O\,(z^{-1})$$

$$M_d\,(2a,\;2b) \rightarrow \frac{1}{8}\;\kappa^2\,p^2\,(-z^2 - \frac{E}{p}\;z^2 + z + \frac{E^2}{p^2}\;z - ab\;2z$$

$$+ ab\;2\;\frac{E}{p} + ab\;2\;\frac{E^2}{p^2} + 3\;\frac{E}{p} + \frac{E^2}{p^2} + ab\;2) + O\,(z^{-1})\;.$$

Again the leading two powers of z cancel when the amplitudes are added together. One obtains asymptotically

$$M\,(+2,\;+2) = M\,(-2,\;-2) \rightarrow \frac{\kappa^2}{8p}\;(E + p)^3$$

$$M\,(+2,\;-2) = M\,(-2,\;+2) \rightarrow \frac{\kappa^2}{8p}\;(E + p)\,(E - p)^2\;.$$

In this problem, there is only one independent parity-conserving amplitude of each parity because of the relations

$$f^{\pm}_{2,\,2} = f^{\pm}_{-2,\,-2} = \mp\;f^{\pm}_{2,\,-2} = \mp\;f^{\pm}_{-2,\,2}$$

derivable from the equations analogous to equation IV-5 and the symmetries listed in Appendix D. The leading terms at large t in these amplitudes are

$$f^{+}_{2,\,2} \;\rightarrow\; \frac{\kappa^2\,E}{16\,\pi}\quad z^{-2}\;,$$

$$f^{-}_{2,\,2} \;\rightarrow\; \frac{\kappa^2}{32\pi p}\;(E^2 + p^2)\;z^{-2}\;.$$

APPENDIX G

Definitions of Certain Tensors

The metric tensor is denoted $g_{\mu\nu}$ and its determinant is denoted g. For the metric of special relativity this reduces to the Kronecker delta

$$\delta_{\mu\nu} = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}$$

which we use to define all direct products.

The curvature tensor is defined by

$$R^{a}_{\cdot\,bcd} = \left\{ {a \atop bc} \right\}_{,d} - \left\{ {a \atop bd} \right\}_{,c} + \left\{ {e \atop bc} \right\} \left\{ {a \atop de} \right\} - \left\{ {e \atop bd} \right\} \left\{ {a \atop ce} \right\} ,$$

where

$$\left\{ {a \atop bc} \right\} = \tfrac{1}{2} g^{ad} \left(-g_{bc,d} + g_{bd,c} + g_{cd,b} \right) .$$

The commas denote partial derivatives. Finally,

$$R_{\mu\nu} = R^{\tau}_{\cdot\mu\nu\tau}$$

$$R_{\mu\nu\sigma\tau} = g_{\mu\lambda} \; R^{\lambda}_{\cdot\,\nu\sigma\tau}$$

$$R = g^{\mu\nu} \; R_{\mu\nu}$$

The quantity R is called the scalar curvature. These definitions have been chosen to agree with references 17 and 18.

If $g_{\mu\nu} = \delta_{\mu\nu} + \epsilon_{\mu\nu}$, to lowest order in $\epsilon_{\mu\nu}$ we have

$$g^{\mu\nu} = \delta_{\mu\nu} - \epsilon_{\mu\nu} + O(\epsilon^2)$$

$$\sqrt{-g} = 1 + \tfrac{1}{2}\epsilon_{\sigma\sigma} + O(\epsilon^2)$$

$$R^{a}_{\cdot\,bcd} = \tfrac{1}{2}\left[-\epsilon_{bc,ad} + \epsilon_{ca,bd} + \epsilon_{bd,ac} - \epsilon_{da,bc}\right] + O(\epsilon^2)$$

$$R = \epsilon_{ab,ab} - \epsilon_{aa,bb} + O(\epsilon^2) \ .$$

Finally, the following relations are useful for reference:

$$\frac{\delta\sqrt{-g}}{\delta g_{\mu\nu}} = \tfrac{1}{2}\, g^{\mu\nu} \sqrt{-g}$$

$$\frac{\delta R}{\delta g_{\mu\nu}} = -R^{\mu\nu} \ .$$

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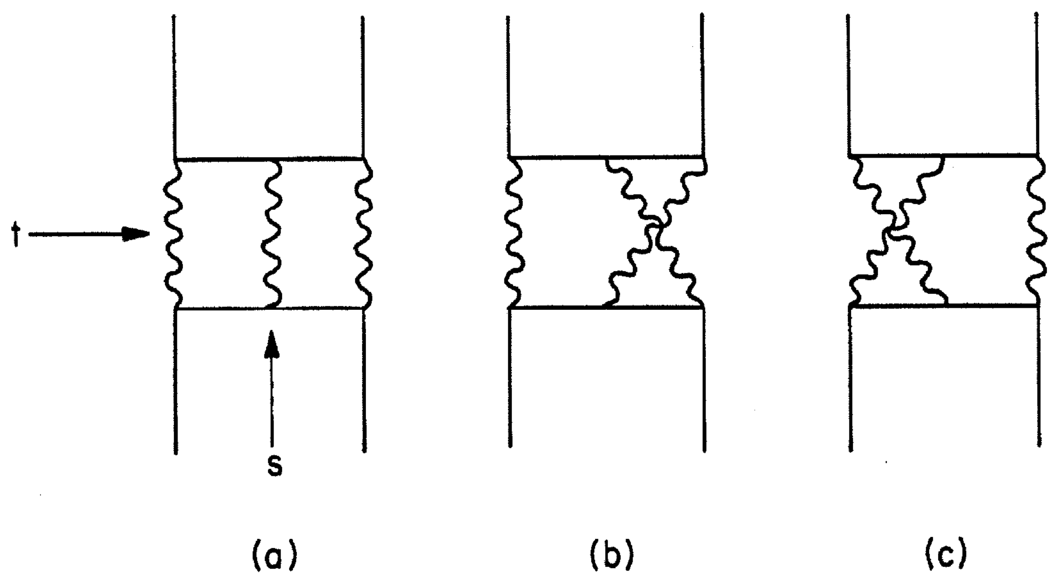

FIGURE 1

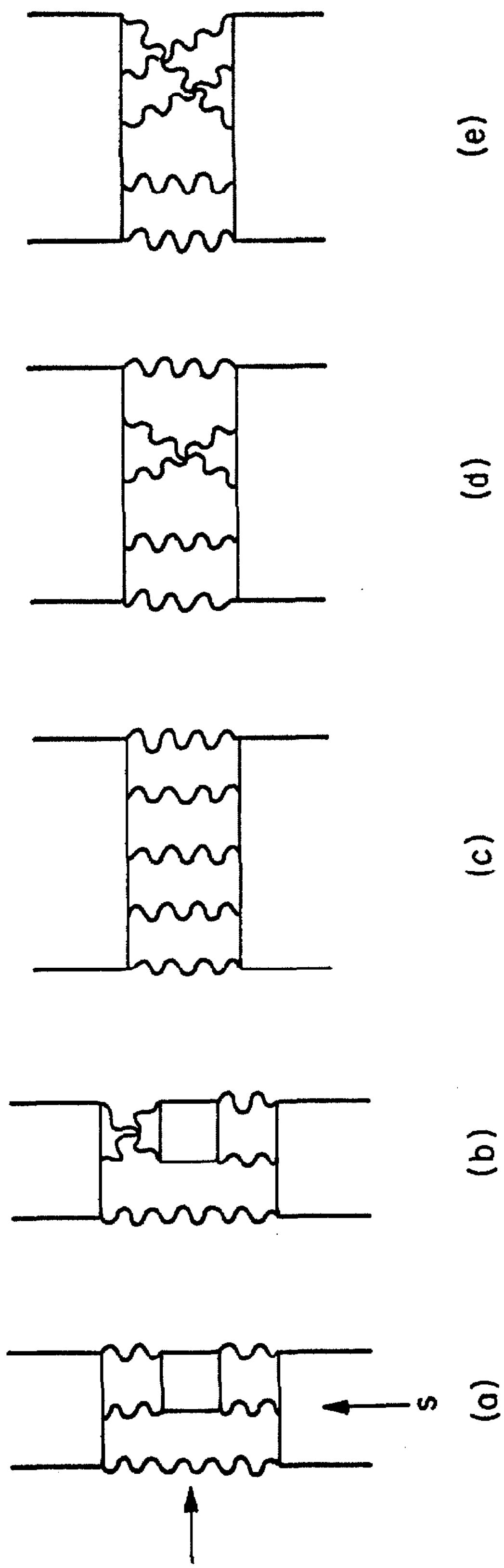

FIGURE 2

FIGURE 2

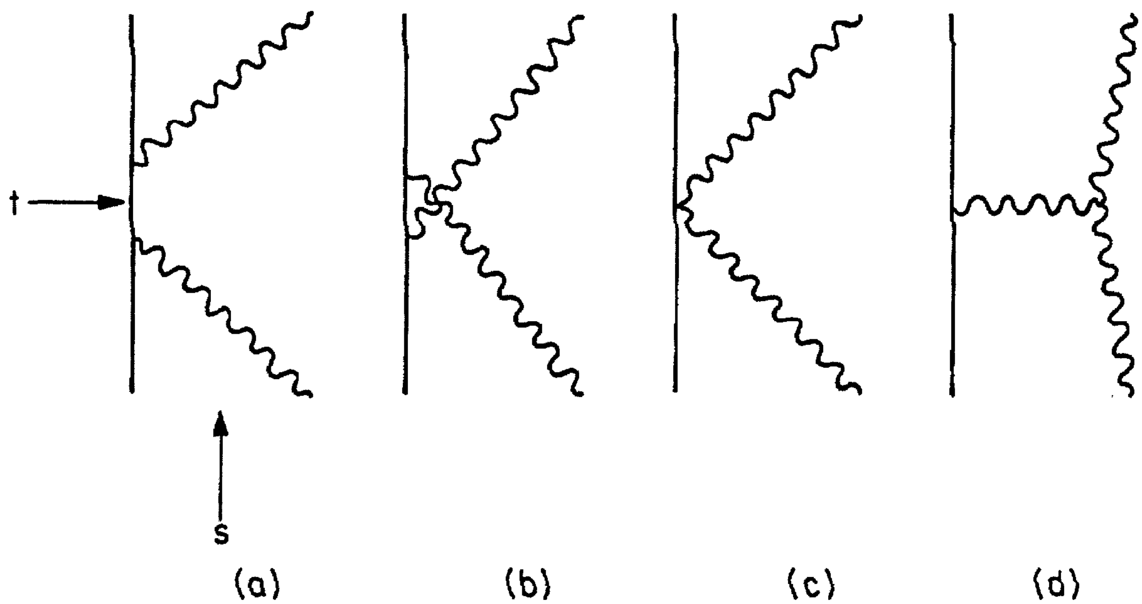

FIGURE 3

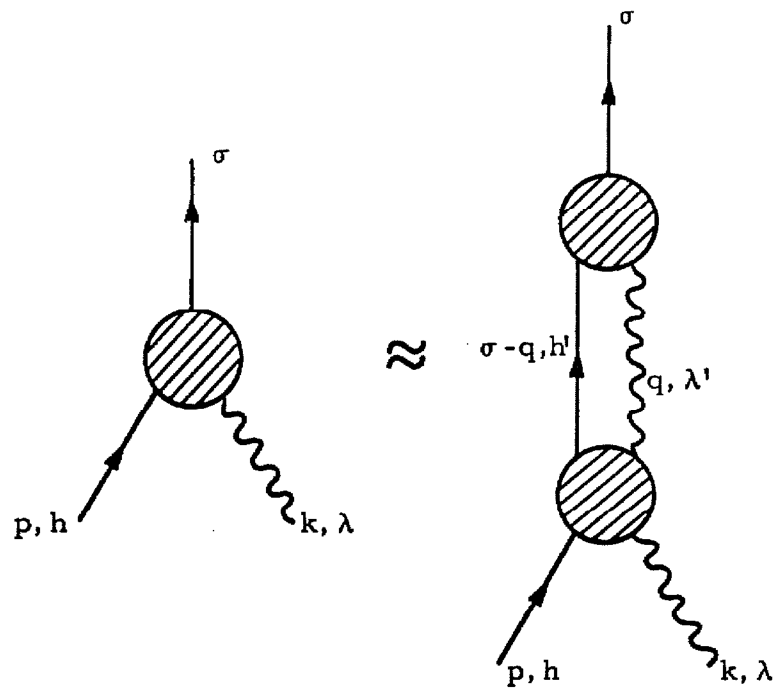

FIGURE 4