ON SUBLATTICES

OF PARTITION LATTICES

Thesis by

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Abstract

We characterize strongly independent sets in an arbitrary geometric lattice in terms of properties of minimal dependent sets of points. The minimal dependent sets of points in partition lattices are identified. It turns out that strongly independent sets in the partition lattice on S correspond in a one - one fashion with systems of subsets of S characterized by certain properties. Lattice properties of partitions are obtained through application of a careful study of these subset systems. We show, for example, that the complete sublattice of a partition lattice generated by the ideals corresponding to a strongly independent set is isomorphic to the direct union of those ideals. Necessary and sufficient conditions are given for two ideals $\alpha/0$ and $\beta/0$ in a partition lattice to generate the entire ideal $\alpha \cup \beta/0$. The problem dual to this one is also solved. We characterize a large class of complete sublattices of a partition lattice, namely, those in which the union of all of the points of the partition lattice contained in the sublattice is the unit partition. The characterization takes the form of a system of subsets of S, of the type mentioned above, together with a suitable equivalence relation between the subsets comprising that system.

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INTRODUCTION

Whitman [6] and Jonsson [3] have shown that every lattice is isomorphic to a sublattice of a partition lattice. They used transfinite methods which leave the following question unsettled: Is every finite lattice isomorphic to a sublattice of a finite partition lattice? This question consitutes one of the most important unsolved problems in lattice theory at the present time. Possibly a study of sublattices of partition lattices would contribute to a solution. It is for this reason that the investigations leading to this thesis were begun. However, sublattices of partition lattices are of considerable interest in their own right.

Several of the properties of partition lattices which we shall need may be proved in the more general case of geometric lattices. These constitute the core of chapter I. Chapter II translates the principal result of Chapter I, the characterization of strongly independent sets, into the context of partition lattices. This leads to the derivation of a number of results of a combinatorial nature which provide the foundation for the latter three chapters. In chapter III we obtain our basic results on sublattices of partition lattices. A characterization of a large class of sublattices is given in chapter IV. Finally, in chapter V we obtain several lattice theoretical properties of the sublattices studied in chapter IV. Each chapter contains an individual introduction. The thesis as a whole is self-contained.

We now establish the notation that will be used and develop certain fundamental background material. Lattices, and subsets of lattices, are denoted by capital Greek letters. The partition lattice on the arbitrary set S is denoted by Π. Small Greek letters represent elements of lattices. When they exist, the unit element of a lattice is denoted by i and the zero element, by O. Lattice operations and relations are indicated by the rounded symbols $\cup$, $\cap$, $\subseteq$, and $\supseteq$. Capital Roman letters denote subsets of S; small Roman letters denote elements of S. Set operations and relations are indicated by the angular symbols $\vee$, $\wedge$, $\leq$, and $\geq$.

A <u>partition</u>, that is an element of Π, is a decomposition of S into disjoint subsets called <u>blocks</u>. A block of a partition is called <u>non-trivial</u> if it contains more than one element of S. A partition α with exactly one non-trivial block A is said to be <u>singular</u>. In this case we write

$$\alpha = (A).$$

If the partition β has only a finite number of non-trivial blocks $B_1, B_2, \ldots, B_k$, we write

$$\beta = (B_1)(B_2)\cdots(B_k).$$

Sometimes we shall even include trivial blocks in parentheses, as in

$$\sigma = (x)(S-x).$$

If the non-trivial block of a singular partition γ contains just two elements, say a and b, of S, then we call γ a <u>point</u> and write

$$\gamma = (a,b).$$

Let α and β be two partitions. We write

$$\alpha \subseteq \beta$$

if every block of α is a subset of some block of β. For the partition 0 whose blocks are all singletons, we have $0 \subseteq \sigma$ for every $\sigma \in \Pi$. For the partition $i = (S)$, we have $\sigma \subseteq i$ for all $\sigma \in \Pi$. Suppose now that we are given an arbitrary subset

$$\Lambda \leq \Pi.$$

Let $x \in S$. Then each $\sigma \in \Lambda$ has a block $B_\sigma(x)$ which contains x. Set

$$C(x) = \bigwedge\{B_\sigma(x) : \sigma \in \Lambda\}$$

and let γ be the partition whose blocks are all sets of the form $C(x)$. Then clearly $\gamma \subseteq \sigma$ for all $\sigma \in \Lambda$. Now let α be an arbitrary partition such that $\alpha \subseteq \sigma$ for all $\sigma \in \Lambda$. Let $A(x)$ be the block of α containing x. Then $A(x) \leq B_\sigma(x)$ for all $\sigma \in \Lambda$. Hence $A(x) \leq C(x)$. It follows that $\alpha \subseteq \gamma$. Hence the arbitrary subset Λ contains a unique greatest lower bound $\bigcap\Lambda$ and $\bigcap\Lambda = \gamma$. Since Π has a unit element, we conclude that Π is a complete lattice.

Clearly Π may also be regarded as the set of all <u>equivalence relations</u> on S. We write

$$x \equiv y (\text{mod } \alpha)$$

if there is a block of α containing both x and y. $\alpha \subseteq \beta$ if and only if

$$x \equiv y(\text{mod } \alpha) \quad \text{implies} \quad x \equiv y(\text{mod } \beta)$$

for all x and y in S. Suppose that for an arbitrary subset Λ of Π we have

$$
(1) \qquad
\begin{aligned}
a &\equiv g_1 (\mathrm{mod}\ \sigma_1),\\
g_1 &\equiv g_2 (\mathrm{mod}\ \sigma_2),\\
&\ \vdots\\
g_n &\equiv b\ (\mathrm{mod}\ \sigma_n),
\end{aligned}
$$

where $\{\sigma_1, \sigma_2, \ldots, \sigma_n\} \leq \Lambda$. Since $\sigma_i \subseteq \bigcup\Lambda$ for each i, each equivalence mod σ_i may be replaced by an equivalence mod $\bigcup\Lambda$. Since an equivalence relation must be transitive, we have

$$
(2) \qquad a \equiv b (\mathrm{mod}\ \bigcup\Lambda).
$$

On the other hand, if we put $a \sim b$ whenever there is a finite chain as in (1), then $a \sim b$ is already an equivalence relation containing every equivalence relation in Λ. We conclude that (2) implies the existence of a finite chain of type (1). It follows that (1) and (2) are equivalent. Now (1) implies that

$(a,b) \subseteq \bigcup\Lambda$ if and only if
$(a,b) \subseteq \bigcup\Lambda'$ for some finite subset
Λ' of Λ.

We also deduce from (1) that

$(a,b) \subseteq \bigcup\Lambda$ if and only if
there is a finite sequence of overlapping
blocks of partitions in Λ
$B_1 \between B_2 \between \cdots \between B_n$ such that
$a \in B_1$ and $b \in B_n$.

(We write $B_i \between B_{i+1}$ to indicate that $B_i \wedge B_{i+1} \neq \emptyset$).

In an arbitrary lattice Γ, we say that α <u>covers</u> β if $\alpha \supset \beta$ and if $\alpha \supseteq \sigma \supseteq \beta$ implies that $\sigma = \alpha$ or $\sigma = \beta$. Elements covering the zero element, if Γ has a zero element, are called <u>points</u>. (This is clearly consistent with our definition of points in Π.) If every element of Γ is a union of points, then Γ is called a <u>point lattice</u>. If $\sigma \in \Pi$, then

$$\sigma = \bigcup\{(x,y) : (x,y) \subseteq \sigma\}$$

and so Π is a point lattice.

An element α of the arbitrary complete lattice Γ is called <u>compact</u> if $\alpha \subseteq \bigcup\Lambda$ always implies that $\alpha \subseteq \bigcup\Lambda'$ for some finite subset Λ' of Λ. If every element of Γ is a union of a suitable collection of compact elements, then Γ is called <u>compactly generated</u>. It is showed above that in Π, the points (a,b) are compact. Since Π is a point lattice, it is compactly generated.

A compactly generated lattice Γ has the property that for any two elements α and β in Γ, there is a maximal element $\mu \supseteq \alpha$ such that $\mu \cap \beta = \alpha \cap \beta$. To see this, let Σ be a chain (totally ordered set) of elements σ such that $\sigma \supseteq \alpha$ and $\sigma \cap \beta = \alpha \cap \beta$. Let γ be a compact element such that $\gamma \subseteq (\bigcup\Sigma) \cap \beta$. Then $\gamma \subseteq \bigcup\Sigma'$ where Σ' is a finite subset of Σ. It follows that γ is contained in the maximum element τ of Σ'. Hence $\gamma \subseteq \tau \cap \beta = \alpha \cap \beta$. Then by compact generation

$$(\bigcup\Sigma) \cap \beta = \bigcup\{\gamma : \gamma \text{ compact}, \gamma \subseteq (\bigcup\Sigma) \cap \beta\}$$
$$\subseteq \alpha \cap \beta.$$

But $(\bigcup\Sigma) \cap \beta \supseteq \alpha \cap \beta$ is trivial. Hence $(\bigcup\Sigma) \cap \beta = \alpha \cap \beta$. From Zorn's lemma, it follows that there is a maximal element $\mu \supseteq \alpha$ such that $\mu \cap \beta = \alpha \cap \beta$. This proves our assertion.

A subset Δ of the arbitrary lattice Γ with a zero element is called <u>independent</u> if $(\bigcup\Delta_1) \cap (\bigcup\Delta_2) = 0$ for all disjoint subsets Δ_1 and Δ_2 of Δ. If Δ is not independent, it is called <u>dependent</u>.

We return to the case of the partition lattice Π. Suppose that $(a,b) \not\subseteq \alpha$. Then a and b occur in different blocks, say A and B, of α. Among all of the blocks of (a,b) and α, only three overlap,

$$A \between \{a,b\} \between B.$$

Hence $\alpha \cup (a,b)$ is obtained by joining the blocks A and B of α. It follows that $\alpha \cup (a,b)$ covers α.

Suppose now that α covers $\alpha \cap \beta$. The dual ideal $i/\alpha \cap \beta$ is clearly a partition lattice containing α and β. α is a point in $i/\alpha \cap \beta$. By what we proved above, $\alpha \cup \beta$ covers β in $i/\alpha \cap \beta$ and hence also in Π. We therefore have showed that in Π

α covers $\alpha \cap \beta$ always implies that
$\alpha \cup \beta$ covers β.

Such a lattice is called <u>semi-modular</u>.

CHAPTER I: GEOMETRIC LATTICES

Partition lattices are examples of a more general class of lattices called geometric (or matroid) lattices. In this chapter we introduce the concept of a cycle in a geometric lattice (definition 2). It turns out that cycles are very effective tools in the analysis of geometric lattices. Much of the chapter will be devoted to characterizing lattice properties of geometric lattices in terms of properties of cycles. The result most crucial to the remainder of this thesis is the characterization of strongly independent sets (definition 5 and theorem 3). This theorem, together with the especially simple nature of cycles in partition lattices, provides the foundation for nearly all of the remainder of the thesis.

We give two further applications of the theory of cycles. In the first we obtain the direct decomposition of geometric lattices which was originally derived, by a different method, by Sasaki and Fujiwara [5]. In the second we characterize geometric lattices which are modular. Our methods give short proofs and considerable insight into the nature of these theorems. The apparently new results on cycles which we use are lemma 5.2 and theorem 6.

We first define geometric lattices.

Definition 1: A lattice Γ is called geometric if it is a compactly generated semi-modular point lattice.

For completeness we include a proof of the following well-known theorem:

<u>Theorem 1</u>: A geometric lattice Γ is relatively complemented.

<u>Proof</u>: Since any quotient σ/τ in a geometric lattice is clearly a geometric lattice, we need only show that an arbitrary geometric lattice Γ is complemented.

Let $\alpha \in \Gamma$. Then by compact generation there is a maximal element $\beta \in \Gamma$ such that $\alpha \cap \beta = 0$. We claim that $\alpha \cup \beta = i$ (the unit element of Γ). Suppose otherwise, that $\alpha \cup \beta \subset i$. Then there is a point σ such that $\sigma \nsubseteq \alpha \cup \beta$. Then $\beta \cup \sigma \succ \beta$. Then by the maximality of β we have $\alpha \cap (\beta \cup \sigma) \supset 0$. Hence there is a point τ contained in $\alpha \cap (\beta \cup \sigma)$. Then $\tau \subseteq \alpha$ which implies $\tau \nsubseteq \beta$ since $\alpha \cap \beta = 0$. But $\tau \subseteq \beta \cup \sigma$. By semi-modularity $\beta \cup \tau = \beta \cup \sigma$. But then $\alpha \cup \beta = (\alpha \cup \tau) \cup \beta = \alpha \cup (\tau \cup \beta) = \alpha \cup (\sigma \cup \beta) \supseteq \sigma$, a contradiction. Thus we must have $\alpha \cup \beta = i$ whence Γ is complemented completing the proof.

Suppose that we are given a dependent set Δ of points. Then there are disjoint subsets Δ_1 and Δ_2 of Δ such that $(\bigcup\Delta_1) \cap (\bigcup\Delta_2) \supset 0$. Let α be a point contained in $(\bigcup\Delta_1) \cap (\bigcup\Delta_2)$. Since α is compact, there are finite subsets Δ_1' and Δ_2' of Δ_1 and Δ_2 respectively such that $\alpha \subseteq (\bigcup\Delta_1') \cap (\bigcup\Delta_2')$. Hence the set $\Delta_1' \vee \Delta_2'$ is a finite dependent subset of Δ. Thus every dependent set of points contains a finite dependent subset. It follows that every dependent set of points contains a minimal dependent subset, and that every minimal dependent set is finite. We state this as a lemma.

<u>Lemma 2.1</u>: Every dependent set of points contains a minimal dependent subset, and every minimal dependent set is finite.

We are now prepared to define cycles.

Definition 2: A minimal dependent set of points in a geometric lattice will be called a cycle.

We did not use semi-modularity in proving lemma 2.1. Semi-modularity implies that cycles have a large amount of structure.

Lemma 2.2: Let Σ be a cycle. Then for each $\alpha \in \Sigma$ we have that $\alpha \subseteq \bigcup(\Sigma-\alpha)$.

Proof: Let Σ_1 and Σ_2 be disjoint subsets of Σ such that $(\bigcup\Sigma_1) \cap (\bigcup\Sigma_2) \supset 0$. Since Σ is a cycle it is clear that $\Sigma_1 \vee \Sigma_2 = \Sigma$. Hence α is an element of, say, Σ_1. We distinquish two cases.

Case I: $\alpha \subseteq \bigcup\Sigma_2$.

Then $\{\alpha\} \vee \Sigma_2$ is a dependent subset of $\Sigma = \Sigma_1 \vee \Sigma_2$. Since Σ is a cycle, we have $\{\alpha\} \vee \Sigma_2 = \Sigma_1 \vee \Sigma_2$. It follows that $\{\alpha\} = \Sigma_1$ and $(\Sigma-\alpha) = \Sigma_2$. Then $\alpha \subseteq \bigcup(\Sigma-\alpha)$.

Case II: $\alpha \not\subseteq \bigcup\Sigma_2$.

Then there is a point σ contained in $(\bigcup\Sigma_1) \cap (\bigcup\Sigma_2)$ and $\sigma \neq \alpha$. Since Σ is a cycle we have that $\bigcup(\Sigma_1-\alpha) \cap \bigcup\Sigma_2 = 0$. Hence $\sigma \subseteq \bigcup\Sigma_1$ but $\sigma \not\subseteq \bigcup(\Sigma_1-\alpha)$. By semi-modularity $\sigma \cup \bigcup(\Sigma_1-\alpha) = \alpha \cup \bigcup(\Sigma_1-\alpha) = \bigcup\Sigma_1$. Hence $\alpha \subseteq \sigma \cup \bigcup(\Sigma_1-\alpha)$. But $\sigma \subseteq \bigcup\Sigma_2$. Hence $\alpha \subseteq \bigcup(\Sigma_1-\alpha) \cup \bigcup\Sigma_2 = \bigcup(\Sigma-\alpha)$ and this completes the proof.

A simple consequence is the following.

Lemma 2.3: Let α and β be distinct points in the cycle Σ. Then $\bigcup\Sigma = \bigcup(\Sigma-\alpha) = \bigcup(\Sigma-\beta)$. On the other hand $\alpha \not\subseteq \bigcup(\Sigma-\alpha-\beta)$.

Proof: By 2.2 $\alpha \subseteq \bigcup(\Sigma-\alpha)$ whence $\bigcup(\Sigma-\alpha) \subseteq \bigcup\Sigma = \alpha \cup \bigcup(\Sigma-\alpha) = \bigcup(\Sigma-\alpha)$. Similarly $\bigcup(\Sigma-\beta) = \bigcup\Sigma$.

$\Sigma - \beta$ is independent since Σ is a minimal dependent set. Hence $\alpha \cap \bigcup(\Sigma-\alpha-\beta) = 0$ and $\alpha \not\subseteq \bigcup(\Sigma-\alpha-\beta)$.

The next lemma shows the natural way in which cycles arise in geometric lattices.

Lemma 2.4: Let α be a point such that $\alpha \subseteq \bigcup\Delta$ where Δ is a set of points not containing α. Then there exists a minimal set of points $\Delta' \leq \Delta$ such that $\alpha \subseteq \bigcup\Delta'$. Furthermore $\{\alpha\} \vee \Delta'$ is a cycle.

Proof: Since α is compact, there is a finite subset Δ' of Δ such that $\alpha \subseteq \bigcup\Delta'$. It is then clear that Δ' may be chosen minimally.

In any case, the set $\Sigma = \{\alpha\} \vee \Delta'$ is dependent. Hence it contains a cycle Σ'. Suppose $\Sigma' \leq \Delta'$. Then by 2.2 for any $\beta \in \Sigma'$ we have $\beta \subseteq \bigcup(\Sigma'-\beta) \subseteq \bigcup(\Delta'-\beta)$. Hence $\alpha \subseteq \bigcup\Delta' = \beta \cup \bigcup(\Delta'-\beta) = \bigcup(\Delta'-\beta)$ contrary to the selection of Δ'. It follows that $\Sigma' \not\leq \Delta'$ and that $\alpha \in \Sigma'$. But by 2.2 we have $\alpha \subseteq \bigcup(\Sigma'-\alpha)$ where $\Sigma' - \alpha \leq \Delta'$. Hence $\Sigma' - \alpha = \Delta'$ and $\Sigma' = \{\alpha\} \vee \Delta'$ is a cycle, completing the proof.

The first lattice property we characterize is modularity of pairs.

Definition 3: Let α and β belong to the lattice Γ. We call α and β a modular pair if for all $\sigma \subseteq \alpha$ we have that $\sigma \cup (\alpha\cap\beta) = \alpha \cap (\sigma\cup\beta)$. We also indicate this by writing $(\alpha,\beta)M$.

Theorem 2: Suppose $\alpha \cap \beta = 0$. Then $(\alpha,\beta)M$ holds if and only if the following condition holds: whenever Σ is a cycle each point of which is contained in α or in β, then either $\bigcup\Sigma \subseteq \alpha$ or $\bigcup\Sigma \subseteq \beta$.

Proof: (Necessity) Let Σ be a cycle violating the above condition. Then $\Sigma = \Sigma_1 \vee \Sigma_2$ where $\alpha \supseteq \bigcup\Sigma_1 \supset 0$ and $\beta \supseteq \bigcup\Sigma_2 \supset 0$. Let $\sigma \in \Sigma_1$. Then $\bigcup\Sigma_1 \supset \bigcup(\Sigma_1-\sigma)$ since Σ_1 is independent. But $\bigcup(\Sigma-\sigma) = \bigcup\Sigma$ since Σ is a cycle. Hence $\bigcup(\Sigma_1-\sigma) \cup \beta = (\bigcup\Sigma_1) \cup \beta$. But then $\alpha \cap [\bigcup(\Sigma_1-\sigma)\cup\beta] = \alpha \cap [(\bigcup\Sigma_1)\cup\beta]$ and, using $(\alpha,\beta)M$, $\bigcup(\Sigma_1-\sigma) = \bigcup\Sigma_1$, a contradiction. Thus the stated condition must hold.

(Sufficiency) Suppose the condition holds. We must show that $\sigma = (\sigma\cup\beta) \cap \alpha$ for each $\sigma \subseteq \alpha$. Let π be a point such that $\pi \subseteq (\sigma\cup\beta) \cap \alpha$. Suppose that $\pi \not\subseteq \sigma$. Then $\pi \subseteq \sigma \cup \beta$ and $\pi \subseteq \alpha$. $\pi \not\subseteq \beta$ since $\alpha \cap \beta = 0$. Then there is a minimal set of points $\Sigma_1 \vee \Sigma_2$ such that $\pi \subseteq (\bigcup\Sigma_1) \cup (\bigcup\Sigma_2)$ where $\bigcup\Sigma_1 \subseteq \sigma \subseteq \alpha$ and $\bigcup\Sigma_2 \subseteq \beta$. $\pi \not\subseteq \bigcup\Sigma_1$ and $\pi \not\subseteq \bigcup\Sigma_2$ since $\pi \not\subseteq \sigma$ and $\pi \not\subseteq \beta$. Hence $\Sigma_1 \neq \emptyset$, $\Sigma_2 \neq \emptyset$, and $\pi \notin \Sigma_1 \vee \Sigma_2$. Then $\{\pi\} \vee \Sigma_1 \vee \Sigma_2$ is a cycle by lemma 2.4. Also $\alpha \supseteq \pi \cup \bigcup\Sigma_1 \supset 0$ and $\beta \supseteq \bigcup\Sigma_2 \supset 0$. Since $\alpha \cap \beta = 0$, neither α nor β contains $\pi \cup \bigcup\Sigma_1 \cup \bigcup\Sigma_2$. Thus the cycle $\{\pi\} \vee \Sigma_1 \vee \Sigma_2$ violates the hypothesis. Thus every point contained in $(\sigma\cup\beta) \cap \alpha$ is also contained in σ. Hence $(\sigma\cup\beta) \cap \alpha \subseteq \sigma$. The opposite containment is trivial. Hence $(\sigma\cup\beta) \cap \alpha = \sigma$ and $(\alpha,\beta)M$ holds.

Corollary 2.5: In a geometric lattice $(\alpha,\beta)M$ and $(\beta,\alpha)M$ are equivalent.

Proof: The condition of theorem 2 is symmetric in α and β. Hence 2.5 holds whenever $\alpha \cap \beta = 0$. But it is easily shown that $i/\alpha\cap\beta$ is a geometric lattice and that $(\alpha,\beta)M$ holds in Γ if and only if $(\alpha,\beta)M$ holds in $i/\alpha\cap\beta$. 2.5 is now apparent.

In a finite dimensional semi-modular lattice a set $\Delta = \{\alpha_1, \alpha_2, \cdots, \alpha_n\}$ is often defined to be independent if

$$(1) \qquad d(\alpha_1 \cup \alpha_2 \cup \cdots \cup \alpha_n) = d(\alpha_1) + d(\alpha_2) + \cdots + d(\alpha_n)$$

where d is the rank function. This condition implies that Δ is an independent set (by our definition) in which $\bigcup \Delta_1$ and $\bigcup \Delta_2$ are a modular pair for all disjoint subsets Δ_1 and Δ_2 of Δ (see [2]). We shall need an analog of condition (1) which is meaningful in an infinite dimensional geometric lattice. We give such a condition in theorem 3, but first we make some appropriate definitions.

<u>Definition 4</u>: We say that the elements α and β are <u>perpendicular</u> if $\alpha \cap \beta = 0$ and α and β are a modular pair. We shall indicate perpendicularity by writing $(\alpha, \beta)\perp$.

<u>Definition 5</u>: A set Δ of non-zero elements in a lattice will be called <u>strongly independent</u> if $\bigcup \Delta_1$ and $\bigcup \Delta_2$ are perpendicular for each pair of disjoint subsets Δ_1 and Δ_2 of Δ.

Thus a strongly independent set Δ is an independent set in which $\bigcup \Delta_1$ and $\bigcup \Delta_2$ are a modular pair for all disjoint subsets Δ_1 and Δ_2 of Δ.

<u>Theorem 3</u>: Let Δ be a subset of a geometric lattice. Then Δ is strongly independent if and only if

(i) $0 \notin \Delta$.

(ii) If α and β are distinct elements of Δ, then $\alpha \cap \beta = 0$.

(iii) If Σ is a cycle such that $\sigma \in \Sigma$ implies $\sigma \subseteq \delta$ for some $\delta \in \Delta$, then there is a $\tau \in \Delta$ such that $\bigcup \Sigma \subseteq \tau$.

<u>Proof</u>: (Necessity) We need verify only property (iii). Let Σ be a cycle violating property (iii); that is, suppose each element of Σ is contained in some element of Δ but that $\bigcup\Sigma \not\subseteq \delta$ for all $\delta \in \Delta$. Let $\alpha \in \Sigma$ and let $\alpha \subseteq \beta \in \Delta$. Set $\Delta_1 = \{\beta\}$ and set $\Delta_2 = \Delta - \beta$. Then each element of Σ is contained in either $\bigcup\Delta_1$ or $\bigcup\Delta_2$. By assumption $\bigcup\Sigma \not\subseteq \bigcup\Delta_1 = \beta$. By strong independence $(\bigcup\Delta_1) \cap (\bigcup\Delta_2) = 0$. But then by theorem 2 $\bigcup\Delta_1$ and $\bigcup\Delta_2$ are not a modular pair contrary to the strong independence of Δ. It follows that (iii) must hold, completing the proof.

(Sufficiency) Suppose that conditions (i), (ii), and (iii) hold. Let Δ_1 and Δ_2 be disjoint sets of Δ. We need only show that $\bigcup\Delta_1$ and $\bigcup\Delta_2$ are perpendicular.

Set $\Omega = \Omega_1 \vee \Omega_2$ where

$\Omega_1 = \{\sigma : \sigma$ is a point, $\sigma \subseteq$ some $\delta \in \Delta_1\}$ and

$\Omega_2 = \{\sigma : \sigma$ is a point, $\sigma \subseteq$ some $\delta \in \Delta_2\}$.

In view of property (ii), $\Delta_1 \wedge \Delta_2 = \emptyset$ implies $\Omega_1 \wedge \Omega_2 = \emptyset$. It is also clear that $\bigcup\Omega_1 = \bigcup\Delta_1$ and $\bigcup\Omega_2 = \bigcup\Delta_2$.

We prove the perpendicularity of $\bigcup\Delta_1$ and $\bigcup\Delta_2$ in two in two parts.

<u>Part I</u>: $(\bigcup\Delta_1) \cap (\bigcup\Delta_2) = 0$.

Suppose that $(\bigcup\Delta_1) \cap (\bigcup\Delta_2) \supset 0$. Let α be a point such that $\alpha \subseteq (\bigcup\Delta_1) \cap (\bigcup\Delta_2)$. Then there exist minimal sets $\Sigma_1 \leq \Omega_1$ and $\Sigma_2 \leq \Omega_2$ such that $\alpha \subseteq \bigcup\Sigma_1$ and $\alpha \subseteq \bigcup\Sigma_2$. If α is an element of one of the sets Σ_1 and Σ_2, then α is not an element of the other since $\Omega_1 \wedge \Omega_2 = \emptyset$. We therefore consider just two cases:

<u>Case A</u>: $\alpha \in \Sigma_1$ but $\alpha \notin \Sigma_2$.

Let $\beta \in \Sigma_2$. Then $\alpha \subseteq \alpha'$ for some $\alpha' \in \Delta_1$ and $\beta \subseteq \beta'$ for some $\beta' \in \Delta_2$. Then $\alpha' \neq \beta'$ since $\Delta_1 \wedge \Delta_2 = \emptyset$. $\{\alpha\} \vee \Sigma_2$ is a cycle contained in Ω. Hence there is a $\tau \in \Delta$ such that $\alpha \cup \bigcup\Sigma_2 \subseteq \tau$ by property (iii). By property (ii) $\tau = \alpha'$. Hence $\alpha \cup \bigcup\Sigma_1 \subseteq \alpha'$. But then $\alpha' \cap \beta' \supseteq \beta \supset 0$ whence $\alpha' = \beta'$ by property (ii). This is a contradiction. (The case $\alpha \notin \Sigma_1$ but $\alpha \in \Sigma_2$ is similar.)

<u>Case B</u>: $\alpha \notin \Sigma_1$ and $\alpha \notin \Sigma_2$.

Σ_1 and Σ_2 are disjoint since Ω_1 and Ω_2 are disjoint. Let $\beta \in \Sigma_1$. Then, since $\{\alpha\} \vee \Sigma_1$ and $\{\alpha\} \vee \Sigma_2$ are cycles, we have $\bigcup(\Sigma_1 \vee \Sigma_2 - \beta) \supseteq \bigcup\Sigma_2 \supseteq \alpha$ and $\bigcup(\Sigma_1 \vee \Sigma_2 - \beta) \supseteq \alpha \cup \bigcup(\Sigma_1 - \beta) = \alpha \cup \bigcup\Sigma_1 \supseteq \beta$. Let Σ be a minimal subset of $\Sigma_1 \vee \Sigma_2 - \beta$ such that $\bigcup\Sigma \supseteq \beta$. We know that $\{\alpha\} \vee \Sigma_1$ is a cycle and that $\beta \in \Sigma_1$. Hence $\bigcup(\Sigma_1 - \beta) = \bigcup(\{\alpha\} \vee \Sigma_1 - \alpha - \beta) \not\supseteq \beta$ by lemma 2.3. Hence Σ contains at least one point in Σ_2. We may then show that the cycle $\{\beta\} \vee \Sigma$ violates condition (iii) by the argument used in case A.

We have therefore proved that $(\bigcup\Delta_1) \cap (\bigcup\Delta_2) = 0$.

<u>Part II</u>: $\bigcup\Delta_1$ and $\bigcup\Delta_2$ are a modular pair.

Suppose that $\bigcup\Delta_1$ and $\bigcup\Delta_2$ do not form a modular pair. Let $\Sigma = \Sigma_1 \vee \Sigma_2$ be a cycle, containing the smallest possible number of points not in Ω, such that $0 \subset \bigcup\Sigma_1 \subseteq \bigcup\Delta_1$ and $0 \subset \bigcup\Sigma_2 \subseteq \bigcup\Delta_2$. (Such a cycle exists by theorem 2.)

We claim that $\Sigma \leq \Omega$. For suppose the contrary, that some $\alpha \in \Sigma$, say $\alpha \in \Sigma_1$, but $\alpha \notin \Omega$. Then $\bigcup\Omega_1 = \bigcup\Delta_1 \supseteq \alpha$. Let $\beta \in \Sigma_2$. Set

$$\theta = (\Sigma - \alpha - \beta) \vee \Omega_1.$$

Then $\bigcup\theta \supseteq \bigcup\Omega_1 \supseteq \alpha$ and $\bigcup\theta \supseteq \alpha \cup \bigcup(\Sigma - \alpha - \beta) = \bigcup(\Sigma - \beta) = \bigcup\Sigma \supseteq \beta$. Let θ' be a minimal subset of θ such that $\bigcup\theta' \supseteq \beta$. Since Σ is a cycle we have $\bigcup(\Sigma-\alpha-\beta) \not\supseteq \beta$. Hence θ' contains at least one point γ in Ω_1. Then $\gamma \subseteq \bigcup\Omega_1 = \bigcup\Delta_1$. Also $\beta \subseteq \bigcup\Delta_2$. Hence $\{\beta\} \vee \theta'$ is a cycle of the form $\{\beta\} \vee \theta' = \theta_1 \vee \theta_2$ where $0 \subset \bigcup\theta_1 \subseteq \bigcup\Delta_1$ and $0 \subset \bigcup\theta_2 \subseteq \bigcup\Delta_2$. Also $\{\beta\} \vee \theta' \leq (\Sigma-\alpha) \vee \Omega$. Hence $\{\beta\} \vee \theta'$ does not contain α. Hence $\{\beta\} \vee \theta'$ contains fewer points not in Ω than does Σ, and this contradicts the selection of Σ. Thus $\Sigma \leq \Omega$ as claimed.

The cycle Σ then violates condition (iii) by the same argument as that used above in case A. Hence $\bigcup\Delta_1$ and $\bigcup\Delta_2$ form a modular pair. This completes the proof.

It will be convenient to have the following result on strongly independent sets.

<u>Theorem 4</u>: Let $\Delta = \{\alpha_p : p \in P\}$ be a strongly independent set. Let $\Sigma = \{\sigma_p : p \in P\}$ be such that $\sigma_p \subseteq \alpha_p$ for each $p \in P$. Then $\alpha_p \cap \bigcup\Sigma = \sigma_p$ for each $p \in P$.

<u>Proof</u>: Using the fact that α_p and $\bigcup(\Delta-\alpha_p)$ are perpendicular, we have

$$\begin{aligned} \alpha_p \cap \bigcup\Sigma &= \alpha_p \cap [\sigma_p \cup \bigcup(\Sigma-\sigma_p)] \\ &\subseteq \alpha_p \cap [\sigma_p \cup \bigcup(\Delta-\alpha_p)] \\ &= [\alpha_p \cap \bigcup(\Delta-\alpha_p)] \cup \sigma_p \\ &= \sigma_p. \end{aligned}$$

But it is obvious that $\alpha_p \cap \bigcup\Sigma \supseteq \sigma_p$. Hence $\alpha_p \cap \bigcup\Sigma = \sigma_p$ for each $p \in P$.

Direct Decomposition of Geometric Lattices

Suppose we are given an (unrestricted) direct decomposition of the geometric lattice Γ. The component lattices may be taken to be ideals in Γ so that we may write

$$\Gamma \cong \bigotimes \{\delta/0 : \delta \in \Delta\}.$$

It is clear that the set Δ determines an equivalence relation on the set Ω of all points in Γ. We simply call the points α and β equivalent if they are contained in the same element of Δ. We make the following definition.

Definition 6: An equivalence relation on the set Ω of all points of Γ will be called direct if it corresponds to a direct decomposition of Γ.

It is natural to ask when two points are equivalent by every direct equivalence. We have the following lemma.

Lemma 5.1: Let α and β belong to the cycle Σ. Then α and β are equivalent by every direct equivalence.

Proof: Let the direct equivalence $\equiv$ correspond to the decomposition

$$\Gamma \cong \bigotimes \{\delta/0 : \delta \in \Delta\}.$$

Suppose that $\alpha \not\equiv \beta$. Then there is a $\delta \in \Delta$ such that $\alpha \subseteq \delta$ but $\beta \not\subseteq \delta$. Then for the congruence relation on Γ defined by putting σ congruent to τ if and only if $\sigma \cap \delta = \tau \cap \delta$, we have that β is congruent to 0 but α is not congruent to 0. This is impossible since $\alpha/0$ and $\beta/0$ have the common transpose $\bigcup\Sigma/\bigcup(\Sigma-\alpha-\beta)$. Thus we must have $\alpha \equiv \beta$ for every direct equivalence.

The somewhat surprising truth is that the converse of lemma 5.1 holds. This is more troublesome because it is not obvious that the property of belonging to the same cycle defines an equivalence relation. This result is contained in the following lemma, which is of considerable interest in itself.

<u>Lemma 5.2</u>: Let α and β be points in Γ. Suppose there exists a finite sequence of cycles $\Sigma_1,\Sigma_2,\ldots,\Sigma_n$ such that $\alpha \in \Sigma_1$, $\beta \in \Sigma_n$, and $\Sigma_i \wedge \Sigma_{i+1} \neq \emptyset$ for each i. Then there is a single cycle Σ containing both α and β. Furthermore we may take $\Sigma \leq \Sigma_1 \vee \Sigma_2 \vee \cdots \vee \Sigma_n$.

<u>Proof</u>: Clearly we need consider only the case in which n = 2.

Hence suppose $\alpha \in \Sigma_1$ and $\beta \in \Sigma_2$ where $\Sigma_1 \wedge \Sigma_2 \neq \emptyset$. We may also suppose that $\Sigma_1 \vee \Sigma_2$ is minimal in the sense that if Σ_1' and Σ_2' are cycles such that $\Sigma_1' \vee \Sigma_2' \leq \Sigma_1 \vee \Sigma_2$ where $\alpha \in \Sigma_1'$, $\beta \in \Sigma_2'$, and $\Sigma_1' \wedge \Sigma_2' \neq \emptyset$; then $\Sigma_1' \vee \Sigma_2' = \Sigma_1 \vee \Sigma_2$.

It will suffice to show that $\Sigma_1 \vee \Sigma_2$ contains a cycle containing both α and β. Suppose it does not. We will obtain a contradiction. Since α and β are not contained in a cycle contained in $\Sigma_1 \vee \Sigma_2$, we have $\alpha \in \Sigma_1 - \Sigma_2$ and $\beta \in \Sigma_2 - \Sigma_1$. Let $\sigma \in \Sigma_1 \wedge \Sigma_2$ and set

$$\Theta = \Sigma_1 \vee \Sigma_2 - \{\alpha,\beta,\sigma\}.$$

We distinguish two cases:

<u>Case I</u>: $\bigcup\Theta \supseteq \alpha$.

Then $\bigcup\Theta \supseteq \bigcup(\Sigma_1-\sigma) = \bigcup\Sigma_1 \supseteq \sigma$ and $\bigcup\Theta \supseteq \bigcup(\Sigma_2-\beta) = \bigcup\Sigma_2 \supseteq \beta$. Now let

θ_1 and θ_2 be minimal subsets of θ such that $\bigcup\theta_1 \supseteq \alpha$ and $\bigcup\theta_2 \supseteq \beta$. We shall show that the cycles $\{\alpha\} \vee \theta_1$ and $\{\beta\} \vee \theta_2$ violate the minimality of $\Sigma_1 \vee \Sigma_2$. Clearly $(\{\alpha\} \vee \theta_1) \vee (\{\beta\} \vee \theta_2) < \Sigma_1 \vee \Sigma_2$ since σ is not contained in the left member. Hence we need only show that $(\{\alpha\} \vee \theta_1) \wedge (\{\beta\} \vee \theta_2) \neq \emptyset$. $\theta_1 \not\leq \Sigma_1$ since otherwise $\alpha \subseteq \bigcup(\Sigma_1-\alpha-\sigma)$ contrary to fact that Σ_1 is a cycle. Similarly $\theta_2 \not\leq \Sigma_2$. Hence

$$\theta_1 \wedge (\theta-\Sigma_1) > \emptyset \text{ and } \theta_2 \wedge (\theta-\Sigma_2) > \emptyset.$$

Then $\{\alpha\} \vee \theta_1$ and Σ_2 are overlapping cycles contained in $\Sigma_1 \vee \Sigma_1$. By the minimality of $\Sigma_1 \vee \Sigma_2$ we have $(\{\alpha\} \vee \theta_1) \vee \Sigma_2 = \Sigma_1 \vee \Sigma_2$ whence $\{\alpha\} \vee \theta_1 \vee \Sigma_2 \geq (\theta-\Sigma_2)$. It follows that

$$\theta_1 \geq (\theta-\Sigma_2).$$

Hence $(\{\alpha\} \vee \theta_1) \wedge (\{\beta\} \vee \theta_2) \geq \theta_1 \wedge \theta_2 \geq (\theta-\Sigma_2) \wedge \theta_2 > \emptyset$. Thus the cycles $\{\alpha\} \vee \theta_1$ and $\{\beta\} \vee \theta_2$ overlap and contradict the minimality of $\Sigma_1 \vee \Sigma_2$.

<u>Case II</u>: $\bigcup\theta \not\supseteq \alpha$.

Then $\beta \cup \bigcup\theta \supseteq \bigcup(\Sigma_2-\sigma) = \bigcup\Sigma_2 \supseteq \sigma$ and $\beta \cup \bigcup\theta = \beta \cup \sigma \cup \bigcup\theta \supseteq \bigcup(\Sigma_1-\alpha) = \bigcup\Sigma_1 \supseteq \alpha$. Now let θ' be a minimal subset of $\{\beta\} \vee \theta$ such that $\bigcup\theta' \supseteq \alpha$. Then $\beta \in \theta'$ since $\bigcup\theta \not\supseteq \alpha$. But then $\{\alpha\} \vee \theta'$ is a cycle containing both α and β. Furthermore $\{\alpha\} \vee \theta' \leq \Sigma_1 \vee \Sigma_2$.

Thus our assumption that there is no cycle contained in $\Sigma_1 \vee \Sigma_2$ which contains both α and β is untenable. This completes the proof of 5.2.

<u>Definition 7</u>: Let α and β be points. We write $\alpha \sim \beta$ or if there is a cycle Σ containing both α and β.

Lemma 5.3: The relation $\alpha \sim \beta$ is a direct equivalence on the set Ω of all points in Γ.

Proof: The relation $\alpha \sim \beta$ is an equivalence relation by lemma 5.2. Denote the collection of equivalence classes by $\{\Lambda_p : p \in P\}$. Let $\alpha_p = \bigcup \Lambda_p$ for each $p \in P$. We must show that

$$\Gamma \cong \bigotimes \{\alpha_p/0 : p \in P\}.$$

Since Γ is a point lattice and since each point is contained in some α_p, we see that for each σ in Γ, $\sigma = \bigcup \{\sigma \cap \alpha_p : p \in P\}$. By theorem 3, $\{\alpha_p : p \in P\}$ is strongly independent. Hence, by theorem 4, the mapping taking each element σ of Γ into the element of $\bigotimes \{\alpha_p/0 : p \in P\}$ whose $p^{\underline{th}}$ component is $\sigma \cap \alpha_p$ for each $p \in P$ is one - one and onto. Since this mapping is also order-preserving, it is an isomorphism. This completes the proof.

We may summarize our results as follows.

Theorem 5: A geometric lattice has a unique representation as a direct union of irreducible geometric lattices. The direct equivalence corresponding to this representation identifies precisely those pairs of points which occur in a common cycle.

The usual characterization of the direct equivalence follows.

Corollary 5.4: $\alpha \sim \beta$ if and only if α and β have a common complement.

Proof: Suppose $\alpha \sim \beta$ and $\alpha \neq \beta$. Then there is a cycle Σ containing α and β. By relative complementation there is an element γ such that

$\gamma \cup \bigcup\Sigma = 1$ and $\gamma \cap \bigcup\Sigma = \bigcup(\Sigma-\alpha-\beta)$. Then $\alpha \cup \gamma \supseteq \alpha \cup \bigcup(\Sigma-\alpha-\beta) = \bigcup(\Sigma-\beta)$ $= \bigcup\Sigma$ and $\alpha \cup \gamma \supseteq \gamma \cup \bigcup\Sigma = 1$ whence $\alpha \cup \gamma = 1$. Also $\alpha \cap \gamma \subseteq (\bigcup\Sigma) \cap \gamma$ $= \bigcup(\Sigma-\alpha-\beta)$ and $\alpha \cap \gamma \subseteq \alpha \cap \bigcup(\Sigma-\alpha-\beta) = 0$ whence $\alpha \cap \gamma = 0$. Thus γ is a complement of α and, similarly, of β.

Conversely, suppose α and β have a common complement. Then, by the argument of lemma 5.1, α and β are equivalent by every direct equivalence. In particular, $\alpha \sim \beta$.

The transitivity of the relation given in 5.4 is usually rather troublesome to obtain. Using similar methods we shall easily derive the transitivity of another relation in corollary 6.3.

Modularity in Geometric Lattices

We shall obtain a characterization of those geometric lattices Γ which are modular in terms of a property of the cycles in Γ. We require the following lemmas.

Lemma 6.1: In any lattice $(\alpha,\beta)M$ and $\gamma \subseteq \alpha$ imply $(\alpha,\beta\cup\gamma)M$.

Proof: Let $\sigma \subseteq \alpha$. Applying $(\alpha,\beta)M$ twice we obtain $\sigma \cup [\alpha\cap(\beta\cup\gamma)]$ $= \sigma \cup [\gamma\cup(\beta\cap\alpha)] = (\sigma\cup\gamma) \cup (\beta\cap\alpha) = \alpha \cap [(\beta\cup\gamma)\cup\sigma]$. Thus $(\alpha,\beta\cup\gamma)M$ holds.

Lemma 6.2: Let Γ be a relatively complemented lattice with a zero element. Then if $(\alpha,\beta)M$ holds for all β such that $\alpha \cap \beta = 0$, then $(\alpha,\sigma)M$ holds for all σ.

Proof: Let $\sigma \in \Gamma$. Then there is an element τ in Γ such that $(\alpha \cap \sigma) \cup \tau = \sigma$ and $(\alpha \cap \sigma) \cap \tau = 0$. Then $\tau \subseteq \sigma$ whence $\alpha \cap \tau \subseteq \alpha \cap (\sigma \cap \tau) = (\alpha \cap \sigma) \cap \tau = 0$ and $\alpha \cap \tau = 0$. Then $(\alpha,\tau)M$ holds by hypothesis. By lemma 6.2, $(\alpha,\tau \cup (\alpha \cap \sigma))M$ holds. But $\tau \cup (\alpha \cap \sigma) = \sigma$ and so $(\alpha,\sigma)M$ holds as was to be proved.

Theorem 6: A geometric lattice Γ is modular if and only if for every cycle $\Sigma = \Sigma_1 \vee \Sigma_2$ where $\Sigma_1 \neq \emptyset$ and $\Sigma_2 \neq \emptyset$, we have that $(\bigcup \Sigma_1) \cap (\bigcup \Sigma_2) \supset 0$.

Proof: (Necessity) Suppose there is a cycle $\Sigma = \Sigma_1 \vee \Sigma_2$ such that $\Sigma_1 \neq \emptyset$, $\Sigma_2 \neq \emptyset$, and $(\bigcup \Sigma_1) \cap (\bigcup \Sigma_2) = 0$. Then, by theorem 2, $\bigcup \Sigma_1$ and $\bigcup \Sigma_2$ are not a modular pair. Hence Γ is not a modular lattice.

(Sufficiency) The stated condition and theorem 2 imply that $(\alpha,\beta)M$ holds for all α and β such that $\alpha \cap \beta = 0$. By lemma 6.2, $(\sigma,\tau)M$ holds for all σ and τ in Γ. But then the lattice Γ is modular. This completes the proof.

The direct equivalence of theorem 5 takes an interesting form when the geometric lattice Γ is modular.

Corollary 6.3: Let Γ be a modular geometric lattice. Then the distinct points α and β are equivalent by the direct equivalence of theorem 5 if and only if there is a third point γ such that $\gamma \subseteq \alpha \cup \beta$.

Proof: Suppose $\alpha \sim \beta$. By theorem 5 there is a cycle Σ_1 containing α and β. Set $\Sigma_1 = \{\alpha,\beta\}$ and set $\Sigma_2 = \Sigma - \{\alpha,\beta\}$. Then by theorem 6

$(\bigcup\Sigma_1) \cap (\bigcup\Sigma_2) \supset 0$. Let γ be a point such that $\gamma \subseteq (\bigcup\Sigma_1) \cap (\bigcup\Sigma_2)$. Then $\gamma \subseteq \bigcup\Sigma_1 = \alpha \cup \beta$, and γ is different from α and β since $\gamma \subseteq \bigcup\Sigma_2 = \bigcup(\Sigma-\alpha-\beta)$.

Conversely, if there is a third point γ such that $\gamma \subseteq \alpha \cup \beta$; then clearly the set $\{\alpha,\beta,\gamma\}$ is a cycle. By theorem 5 we have $\alpha \sim \beta$. This completes the proof.

CHAPTER II: CYCLES IN PARTITION LATTICES, STRUCTURES

Π will always be understood to be the partition lattice on the (possibly infinite) set S. We have already shown that Π is a geometric lattice. In this chapter we identify the cycles in Π. We then find that the conditions of theorem 3 define a system of subsets of S characterized by certain properties. Such systems we call <u>structures</u>. The remainder of the thesis will depend heavily on a careful study of structures.

<u>Lemma 7.1</u>: Let Σ be a set of points in the partition lattice Π. Then Σ is a cycle if and only if Σ is of the form $\Sigma = \{(a_1,a_2), (a_2,a_3),\ldots,(a_{r-1},a_r),(a_r,a_1)\}$ where the a_i are distinct elements of S.

<u>Proof</u>: It is clear that a set of the given form must be a cycle. Conversely, let Σ be a cycle and let $(a,b) \in \Sigma$. Then $\Sigma - (a,b)$ is a minimal set such that $(a,b) \subseteq \bigcup(\Sigma-(a,b))$. Hence there is a sequence of points $(a,a_1),(a_1,a_2),\ldots,(a_n,b)$ in $\Sigma - (a,b)$. From these we may select a subset Σ' such that the a_i are all distinct. Then $\{(a,b)\} \vee \Sigma'$ is of the required form. But Σ is a minimal dependent set, and so $\{(a,b)\} \vee \Sigma' = \Sigma$ completing the proof.

Suppose now that we are given a strongly independent set Δ in Π. In view of lemma 7.1, condition (iii) of theorem 3 takes the following form:

Let $a_1, a_2, \ldots, a_r$ be distinct elements of S. Set $a_1 = a_{r+1}$.
If for each i there exists a $\tau_i \in \Delta$ such that $(a_i, a_{i+1}) \subseteq \tau_i$;
then there exists an $\alpha \in \Delta$ such that $(a_1, a_2, \ldots, a_r) \subseteq \alpha$.

It follows that the <u>set</u> $\{a_1, a_2, \ldots, a_r\}$ is contained in some <u>block</u> U of α. Letting $\mathfrak{H}$ denote the set of all non-trivial blocks of partitions in Δ, we may then write the conditions of theorem 3 in the following form:

(i) Each set in $\mathfrak{H}$ contains at least two elements.

(ii) If $T \in \mathfrak{H}$ and $U \in \mathfrak{H}$ where $T \neq U$, then $T \wedge U$ is at most a singleton.

(iii) Let $a_1, a_2, \ldots, a_r$ be distinct elements in S. Set $a_1 = a_{r+1}$. If for each i there exists some $T_i \in \mathfrak{H}$ such that $\{a_i, a_{i+1}\} \leq T_i$; then there exists a $U \in \mathfrak{H}$ such that $\{a_1, a_2, \ldots, a_r\} \leq U$.

<u>Definition 8</u>: A <u>structure</u> on the set S is a collection $\mathfrak{H}$ of subsets of S having the preceding properties (i), (ii), and (iii).

An immediate consequence of theorem 3 is:

<u>Theorem 7</u>: Let Δ be a set of non-trivial partitions in Π. Suppose no two partitions in Δ have a common non-trivial block. Let $\mathfrak{H}$ be the set of all non-trivial blocks of partitions in Δ. Then Δ is strongly independent if and only if $\mathfrak{H}$ is a structure on S.

Structures therefore have a great deal of lattice theoretical significance in partition lattices. We shall devote the rest of this chapter to a study of structures.

We consider the empty set to be a structure. Since, however, all of our results become trivialities in this case, we shall always assume that $\mathfrak{S}$ is non-empty.

If a and b are distinct elements of S contained in some set T in $\mathfrak{S}$, then a and b characterize T by property (ii). We therefore may write $T = F(a,b)$. Also as a consequence of (ii), the conclusion of (iii) may be strengthened to $U = T_1 = T_2 = \cdots = T_r$.

We also mention that structures might well be called geometries, but in our context this is an unnecessary distraction.

We next introduce a property analogous to property (iii):

(iii)*: Let $T_1, T_2, \ldots, T_r$ be distinct sets in $\mathfrak{S}$. Set $T_1 = T_{r+1}$. If $T_i \between T_{i+1}$ for each i, then all of the intersections $T_i \wedge T_{i+1}$ are equal.

The following theorem exhibits the interdependence between (iii) and (iii)*:

<u>Theorem 8</u>: If (i) and (ii) hold, then (iii) and (iii)* are equivalent.

<u>Proof</u>: We first show that (iii) implies (iii)*. Suppose (iii) holds but (iii)* does not. Let $T_1, T_2, \ldots, T_r$ be a minimal sequence of distinct sets in $\mathfrak{S}$ such that, after setting $T_1 = T_{r+1}$, we have that no $T_i \wedge T_{i+1}$ is empty and that not all $T_i \wedge T_{i+1}$ are equal. Clearly $r \geq 2$. By (ii) all intersections are of the form $T_i \wedge T_{i+1} = \{a_i\}$

for each i. Set $a_1 = a_{r+1}$. We shall show that $a_i \neq a_{i+1}$ for each i. Suppose $a_i = a_{i+1}$. Then $T_i \wedge T_{i+1} = T_{i+1} \wedge T_{i+2} = T_i \wedge T_{i+2}$ using (ii). Hence after dropping T_{i+1} from the sequence $T_1, T_2, \ldots, T_{r+1}$, adjacent sets still overlap and still not all adjacent intersections are equal. This violates the minimality of the original sequence. Thus $a_i \neq a_{i+1}$ for each i. Now let a_k be the first element in $a_1, a_2, \ldots, a_{r+1}$ which equals a predecessor. Then a_k = some a_j where $j < k$. Then $a_j, a_{j+1}, \ldots, a_{k-1}$ are all distinct. Also $a_{j+1} \neq a_k$ since $a_j \neq a_{j+1}$. This sequence satisfies the hypothesis of (iii). Hence by (iii) and (ii) $T_{j+1} = T_{j+2}$ contrary to the assumed distinctness of $T_1, T_2, \ldots, T_r$. Thus (iii) implies (iii)*.

Next suppose that (iii)* holds but (iii) does not. Let $a_1, a_2, \ldots, a_r$ be a minimal sequence of distinct elements in S such that, after setting $a_1 = a_{r+1}$, we have that each $\{a_i, a_{i+1}\} \leq$ some $T_i \in \mathfrak{S}$, but that $\{a_1, a_2, \ldots, a_r\} \not\leq U$ for all $U \in \mathfrak{S}$. Let $T_1 = T_{r+1}$. We shall show that $T_i \neq T_{i+1}$ for each i. Suppose $T_i = T_{i+1}$. Then $\{a_i, a_{i+2}\} \leq T_i \vee T_{i+1} = T_i$. Hence after dropping a_{i+1} for the sequence $a_1, a_2, \ldots, a_r$, we still have a violation of (iii) contrary to the assumed minimality. Thus $T_i \neq T_{i+1}$ for each i. Now let T_k be the first set in $T_1, T_2, \ldots, T_{r+1}$ which equals a predecessor. Then T_k = some T_j where $j < k$. Then $T_j, T_{j+1}, \ldots, T_{k-1}$ are all distinct. Also $T_j \between T_{j+1} \between \cdots \between T_{k-1} \between T_1$. Hence by property (iii)* we have $\{a_{j+1}\} = T_j \wedge T_{j+1} = T_{j+1} \wedge T_{j+2} = \{a_{j+2}\}$ and $a_{j+1} = a_{j+2}$ contrary to the assumed distinctness of $a_1, a_2, \ldots, a_r$. This completes the proof.

<u>Definition 9</u>: Let a and b be elements of S. We write $a\mathfrak{S}b$ if $a = b$ or if there is a set T in $\mathfrak{S}$ which contains both a and b.

Given the structure $\mathfrak{S}$ on the set S, a structure on the set $\mathfrak{S}$ arises in a very natural way:

<u>Theorem 9</u>: Let A be the set of those elements of S which belong to more than one set in the structure $\mathfrak{S}$. For each $\alpha \in A$ let $\mathfrak{S}(a) = \{T \in \mathfrak{S} : a \in T\}$. Then the set $\mathfrak{J} = \{\mathfrak{S}(a) : a \in A\}$ is a structure on the set $\mathfrak{S}$. Furthermore $T\mathfrak{J}U$ if and only if T and U overlap as subsets of S.

<u>Proof</u>: Property (i) follows from the selection of A. To verify (ii) let $\mathfrak{S}(a)$ and $\mathfrak{S}(b)$ be sets in $\mathfrak{J}$ which share the distinct sets T and U of $\mathfrak{S}$. Then by property (ii) for $\mathfrak{S}$ we have $\{a\} = T \wedge U = \{\beta\}$ and $a = b$. Thus $\mathfrak{S}(a) = \mathfrak{S}(b)$. Thus $\mathfrak{J}$ satisfies property (ii). Property (iii) for $\mathfrak{J}$ is simply property (iii)* for $\mathfrak{S}$. Therefore $\mathfrak{J}$ is a structure on $\mathfrak{S}$.

The final statement is obvious.

<u>Definition 10</u>: We call the elements a and b of S <u>related</u> (with respect to $\mathfrak{S}$) whenever there exists a finite sequence of elements $a_1, a_2, \ldots, a_r$ in S such that $a\mathfrak{S}a_1\mathfrak{S}a_2\mathfrak{S}\cdots\mathfrak{S}a_r\mathfrak{S}b$.

<u>Definition 11</u>: We call the sets T and U in $\mathfrak{S}$ <u>related</u> whenever they are related with respect to the structure $\mathfrak{J}$ of theorem 9.

The sets T and U in $\mathfrak{S}$ are clearly related precisely when there is a finite sequence of sets in $\mathfrak{D}$ such that $T \between T_1 \between T_2 \between \cdots \between T_r \between U$.

<u>Theorem 10</u>: If a and b are related elements of S, then there exists a unique minimal chain $a\mathfrak{D}g_1\mathfrak{D}g_2\mathfrak{D}\cdots\mathfrak{D}g_n\mathfrak{D}b$. The elements $a, g_1, g_2, \ldots, g_n, b$ are all distinct. If also $a\mathfrak{D}d_1\mathfrak{D}d_2\mathfrak{D}\cdots\mathfrak{D}d_m\mathfrak{D}b$, then there exists a correspondence $j \to k(j)$ such that $g_1 = d_{i(1)}, g_2 = d_{i(2)}, \ldots, g_n = d_{i(n)}$ and $i(1) < i(2) < \cdots < i(n)$.

<u>Proof</u>: Clearly the elements in a minimal chain are distinct. Also note that the last statement is contained in the first since any chain may be shortened to a minimal chain in which the order of the elements is preserved. Thus we need only prove the first statement. Suppose that it is false. Let

$$a\mathfrak{D}g_1\mathfrak{D}g_2\mathfrak{D}\cdots\mathfrak{D}g_n\mathfrak{D}b \quad \text{and}$$
$$a\mathfrak{D}d_1\mathfrak{D}d_2\mathfrak{D}\cdots\mathfrak{D}d_m\mathfrak{D}b$$

be distinct minimal chains. Set $a = g_0 = d_0$ and set $b = g_{n+1} = d_{m+1}$. Since the chains differ, we may choose k minimally so that $g_k \neq d_k$. Let l be minimal so that $k \leq l$ and g_l = some d_j where $k \leq j$. Let l' be minimal so that $k \leq l'$ and $g_l = d_{l'}$. Either l or l' is greater than k since $g_k \neq d_k$. Without loss of generality suppose that $k < l$. Then

$$g_{k-1}\mathfrak{D}g_k\mathfrak{D}\cdots\mathfrak{D}g_l = d_{l'}\mathfrak{D}d_{l'-1}\mathfrak{D}\cdots\mathfrak{D}d_{k-1} = g_{k-1}$$

and the elements $g_{k-1}, g_k, \ldots, g_l, d_{l'-1}, \ldots, d_k$ are distinct. Hence (iii) is applicable and

$$g_{k-1}\mathfrak{D}g_l.$$

But $k < 1$ implies that $k + 1 \leq 1$, and this contradicts our assumption that the first chain was minimal. Thus every two minimal chains must be identical and this proves the theorem.

<u>Theorem 11</u>: If A and B are related sets in $\mathfrak{H}$, then there exists a unique minimal sequence of sets G_i in $\mathfrak{H}$ such that

$$A \between G_1 \between G_2 \between \cdots \between G_n \between B.$$

The sets $A, G_1, G_2, \ldots, G_n, B$ are distinct. If also $A \between D_1 \between D_2 \between \cdots \between D_m \between B$ where the D_i are in $\mathfrak{H}$, then there exists a correspondence $j \to i(j)$ such that $G_1 = D_{i(1)}, G_2 = D_{i(2)}, \ldots, G_n = D_{i(n)}$ and $i(1) < i(2) < \cdots < i(n)$.

<u>Proof</u>: We apply theorem 10 to the structure of theorem 9.

<u>Theorem 12</u>: If a and b are distinct related elements of S, then there exists a unique minimal sequence of sets $G_1, G_2, \ldots, G_n$ in $\mathfrak{H}$ such that $a \in G_1$ and $b \in G_n$ and $G_1 \between G_2 \between \cdots \between G_n$. The sets G_i are distinct. If also $D_1 \between D_2 \between \cdots \between D_m$ where the D_i are in $\mathfrak{H}$ and $a \in D_1$ and $b \in D_m$, then there exists a correspondence $j \to i(j)$ such that $G_1 = D_{i(1)}, G_2 = D_{i(2)}, \ldots, G_n = D_{i(n)}$ and $i(1) < i(2) < \cdots < i(n)$.

<u>Proof</u>: Again it is clear that we need prove only the first statement. Let

$$G_1 \between G_2 \between \cdots \between G_n,\ a \in G_1,\ b \in G_n \quad \text{and}$$
$$D_1 \between D_2 \between \cdots \between D_m,\ a \in D_1,\ b \in D_m$$

be minimal chains of sets in $\mathfrak{H}$. The first chain is clearly minimal between G_1 and G_n in the sense of the previous theorem. If $1 < j < n$, then $G_j \neq D_1$ otherwise $a \in D_1 = G_j$ would violate the minimality of the first chain. Similarly $G_j \neq D_m$. Since $a \in G_1 \wedge D_1$ and $b \in G_n \wedge D_m$, we have

$$G_1 \between D_1 \between D_2 \between \cdots \between D_m \between G_n.$$

By theorem 11 and our observation that $G_j \neq D_1$ and $G_j \neq D_m$ for $1 < j < n$, we have $\{G_2, G_3, \ldots, G_{n-1}\} \leq \{D_2, D_3, \ldots, D_{m-1}\}$. The opposite containment follows similarly. Thus $n = m$. By the order preserving correspondence of theorem 11 we see that

$$G_2 = D_2, G_3 = D_3, \ldots, G_{n-1} = D_{n-1}.$$

Now set $G_1 \wedge G_2 = \{x\}$ and $D_1 \wedge D_2 = y$. Then $x \mathfrak{S} y$ since $\{x,y\} \leq G_2 \vee D_2 = G_2$. Hence $a \mathfrak{S} x \mathfrak{S} g \mathfrak{S} a$. If $x \neq y$, then by (iii) $\{a,x,y\}$ is contained in some set in $\mathfrak{S}$ and this set must be G_2 since G_2 contains x and y. But then $a \in G_2$ and this contradicts the minimality of the first chain. Thus $x = y$ and therefore G_1 and D_1 share the distinct elements a and x. Therefore $G_1 = D_1$. $G_n = D_n$ similarly, and this completes the proof.

<u>Definition 12</u>: Let $a \mathfrak{S} g_1 \mathfrak{S} g_2 \mathfrak{S} \cdots \mathfrak{S} g_n \mathfrak{S} b$ be the minimal chain from a to b in the sense of theorem 10. We set

$$C(a,b) = \{a, g_1, g_2, \ldots, g_n, b\}.$$

<u>Definition 13</u>: Let $A \between G_1 \between G_2 \between \cdots \between G_n \between B$ be the minimal chain from A to B in the sense of theorem 11. We set

$$\mathfrak{C}(A,B) = \{A, G_1, G_2, \ldots, G_n, B\}.$$

<u>Definition 14</u>: Let $G_1 \between G_2 \between \cdots \between G_n$ be the minimal chain from a to b in the sense of theorem 12. We set

$$\mathfrak{C}(a,b) = \{G_1,G_2,\ldots,G_n\}.$$

Whenever we list the elements in a minimal chain, we shall list them in the meaningful order. For example, if we write $C(a,b) = \{a,z,p,q,b\}$, we mean that $a\mathfrak{R}z\mathfrak{R}p\mathfrak{R}q\mathfrak{R}b$.

The preceding three theorems have the following formal consequences:

(1) For any related triple a, b, and c in S we have

$$C(a,c) \leq C(a,b) \vee C(b,c).$$

(2) For any related triple A, B, and C in $\mathfrak{D}$ we have

$$\mathfrak{C}(A,C) \leq \mathfrak{C}(A,B) \vee \mathfrak{C}(B,C).$$

(3) For any related triple a, b, and c in S we have

$$\mathfrak{C}(a,c) \leq \mathfrak{C}(a,b) \vee \mathfrak{C}(b,c).$$

The sets $\mathfrak{C}(a,b)$ and $C(a,b)$ are related in the obvious manner. We shall prove this.

<u>Theorem 13</u>: Let a and b be related elements of S.

(1) Let $\mathfrak{C}(a,b) = \{D_1,D_2,\ldots,D_n\}$. Set $D_i \wedge D_{i+1} = \{g_i\}$ for each $i < n$. Set $a = g_0$ and $b = g_n$. Then $C(a,b) = \{g_0,g_1,\ldots,g_n\}$.

(2) Let $C(a,b) = \{g_0,g_1,\ldots,g_n\}$. Let D_i be the set in $\mathfrak{D}$ containing g_{i-1} and g_i for each $i > 0$. Then $\mathfrak{C}(a,b) = \{D_1,D_2,\ldots,D_n\}$.

Proof of (1): $\{g_i, g_{i+1}\} \leq D_{i+1}$ for each $i < n + 1$. Hence $g_0 \mathfrak{S} g_1 \mathfrak{S} \cdots \mathfrak{S} g_n$. We claim this set is minimal, for suppose it is not. Then some $g_j \mathfrak{S} g_k$ where $j + 1 < k$. If $g_j = g_k$, then $D_j \between D_{k+1}$ since $g_j \in D_j$ and $g_j = g_k \in D_{k+1}$. Hence D_{j+1} is superfluous in $\mathfrak{C}(a,b)$, a contradiction. Suppose now that $g_j \mathfrak{S} g_k$ where $g_j \neq g_k$. Let A be the set in $\mathfrak{S}$ containing g_j and g_k. Then

$$D_1 \between \cdots \between D_j \between A \between D_{k+1} \between \cdots \between D_n$$

is a chain of $n + j + 1 - k$ sets where a is in the first and b is in the last (either of which may be A). But $n + j + 1 - k < n$ since $j + 1 < k$ and this contradicts the minimality of $\mathfrak{C}(a,b)$. Thus it is not possible that any $g_j \mathfrak{S} g_k$ where $j + 1 < k$. Hence the chain $\{g_0, g_1, \ldots, g_n\}$ is minimal and the theorem is proved.

(2): We are given that $g_0 \mathfrak{S} g_1 \mathfrak{S} \cdots \mathfrak{S} g_n$ is the minimal chain from a to be. Let A_i be the set in $\mathfrak{S}$ containing the elements g_{i-1} and g_i for $i = a, 2, \ldots, n$. Then $A_1 \between A_2 \between \cdots \between A_n$ where $a \in A_1$ and $b \in A_n$. This chain may be shortened to the unique minimal chain $\mathfrak{C}(a,b)$ by dropping out all superfluous sets (see theorem 12). Suppose $\mathfrak{C}(a,b)$ has m sets. Then by part (1), C(a,b) has m + 1 elements. But by hypothesis *C(a,b) has n + 1 elements. Hence $\mathfrak{C}(a,b)$ must have m = n* sets. Hence none of the A_i are superfluous, i.e. the chain $A_1 \between A_2 \between \cdots \between A_n$ is minimal. This completes the proof.

We shall show that the elements in C(a,b) and the sets in $\mathfrak{C}(a,b)$ are together linearly ordered in a natural way. Let C(a,b) $= \{a = g_1, g_2, \ldots, g_n = b\}$. Let D_i be the set in $\mathfrak{S}$ containing g_i and

g_{i+1} for $i = 1,2,\ldots,n-1$. Then by theorem 13, $\mathfrak{C}(a,b) = \{D_1,D_2,\ldots,D_{n-1}\}$. We may then order the elements and sets according to position in the sequence

$$g_1, D_1, g_2, D_2, g_3, \ldots, g_n.$$

We write

$$g_i \trianglelefteq g_j \quad \text{if} \quad i \leq j,$$
$$D_i \trianglelefteq D_j \quad \text{if} \quad i \leq j,$$
$$g_i \trianglelefteq D_j \quad \text{if} \quad i \leq j, \quad \text{and}$$
$$D_i \trianglelefteq g_j \quad \text{if} \quad i + 1 \leq j.$$

The ordered pair with respect to which we are ordering must always be clearly understood. In the above we ordered relative to (a,b).

The next theorem will be needed in chapter IV.

<u>Theorem 14</u>: Let A and B be an arbitrary pair of related sets in $\mathfrak{D}$. Then we may choose $a \in A$ and $b \in B$ so that A and B are in $\mathfrak{C}(a,b)$.

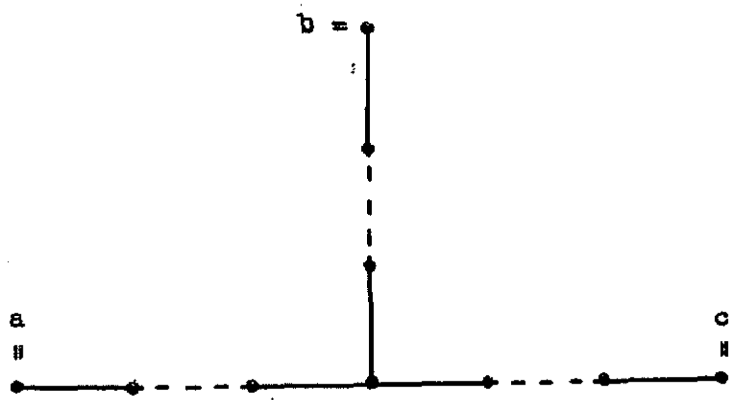
<u>Proof</u>: Let $\mathfrak{C}(A,B) = \{T_1,T_2,\ldots,T_n\}$ where, of course, $A = T_1$ and $B = T_n$. Choose $a \in T_1 - T_1 \wedge T_2$ and choose $b \in T_n - T_{n-1} \wedge T_n$ which is feasible since any set in a structure contains more than one element. Since $a \in A$ and $b \in B$ we have $\mathfrak{C}(a,b) \leq \mathfrak{T}(A,B)$ by theorem 12. Let U be the set in $\mathfrak{C}(a,b)$ which contains a. Then U equals some T_j. Since a belongs to both T_1 and T_j we have $T_1 \between T_j$. If $j > 2$ then T_2 is not needed in $\mathfrak{C}(A,B)$, a contradiction. Thus $j = 2$ or $j = 1$. If $j = 2$ then $a \in T_1 \wedge T_2$ contrary to the choice of a. Thus $j = 1$ and $A = T_1 = U \in \mathfrak{C}(a,b)$. Similarly $B \in \mathfrak{C}(a,b)$.

The following concept will simplify the statement of later theorems.

Definition 15: The graph associated with the structure $\mathfrak{S}$ is the graph having the elements of S as vertices and the pairs $\{x,y\}$ where $x \neq y$ and $x\mathfrak{S}y$ as edges.

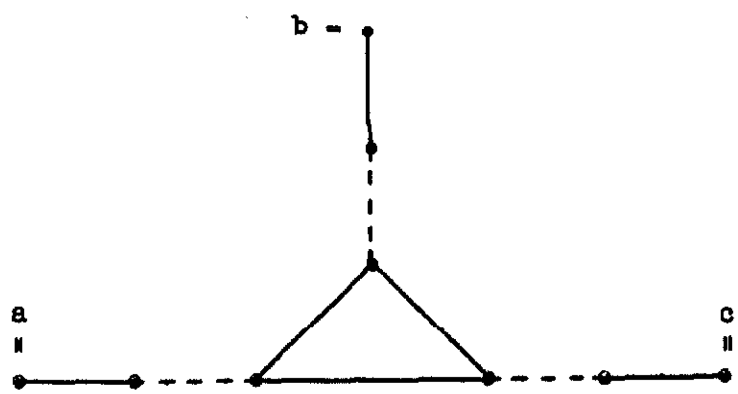
Theorem 15: The graph of the set $C(a,b) \vee C(a,c) \vee C(b,c)$ takes one of the following two forms:

(I)



We include in (I) the cases where $a \in C(b,c)$ or $b \in C(a,c)$ or $c \in C(a,b)$.

(II)



__Proof__: Suppose $b \notin C(a,c)$ since otherwise we have case (I). Let $C(a,c) = \{a = g_0, g_1, \dots, g_n = b\}$. By theorem 10, $C(a,c) \leq C(a,b) \vee C(b,c)$. Let l be maximal so that $g_l \in C(a,b)$ and let m be minimal so that $g_m \in C(b,c)$. Using theorem 10, $C(a,b)$ and $C(b,c)$ have the following form:

$$C(a,b) = \{a = g_0, g_1, \dots, g_l, d_1, d_2, \dots, d_r = b\} \text{ and}$$
$$C(b,c) = \{b = e_s, e_{s-1}, \dots, e_1, g_m, g_{m+1}, \dots, g_n = c\}$$

where no g_i occurs among the d_i or the e_i. Choose p minimally so that d_p equals some e_i. Choose q minimally so that $d_p = e_q$. Then

$$C(g_l, g_m) \vee \{d_1, d_2, \dots, d_p\} \vee \{e_1, e_2, \dots, e_{q-1}\}$$

is a set of distinct elements to which structure property (iii) is applicable. Hence there is a set in $\mathfrak{S}$ containing all of these elements and in particular containing $\{g_l, g_m, d_p\}$. Then $g_l \mathfrak{S} d_p$ from which it follows that $p = 1$. Also $g_m \mathfrak{S} d_p = e_q$ whence $q = 1$. Hence $d_1 = e_1$ and consequently $d_i = e_i$ for all i. Also $g_l \mathfrak{S} g_m$ whence $g_m = g_l$ or $g_m = g_{l+1}$. This proves the theorem. $g_m = g_l$ corresponds to type (I) and $g_m = g_{l+1}$ corresponds to type (II).

We have now proved all of the results on structures that we shall need. However, we mention some additional properties. The set Λ of all structures on the set S forms a lattice under the ordering defined by setting $\mathfrak{S}_1 \subseteq \mathfrak{S}_2$ whenever each set in $\mathfrak{S}_1$ is a subset of some set in $\mathfrak{S}_2$. The structure lattice Λ is in many respects similar to the partition lattice Π. In particular, it is a compactly generated point lattice.

It has no proper congruence relations provided S has at least four elements. Λ however differs from Π in that neither Λ nor its dual is semi-modular. We omit the proofs of these results.

CHAPTER III: SUBLATTICES OF PARTITION LATTICES

As mentioned in the introduction, every lattice is isomorphic to a sublattice of a partition lattice. It is not known, however, whether a finite lattice is necessarily isomorphic to a sublattice of a finite partition lattice. A solution to this embedding problem would probably involve a profound understanding of the nature of sublattices of a partition lattice. In this connection it is natural to ask for information about the sublattice Γ generated by a given set Σ of partitions. In this chapter we obtain satisfactory answers to this question for sets Σ of certain special types. We determine the sublattice generated by the ideals corresponding to a strongly independent set (theorem 16). We give necessary and sufficient conditions for two ideals $\alpha/0$ and $\beta/0$ to generate the ideal $\alpha \cup \beta/0$ (theorem 17). Finally we solve the same problem for dual ideals $1/\alpha$ and $1/\beta$ (theorem 18). Every sublattice of the partition lattice Π contains a pair of fundamental subsets Δ and Δ_0 whose existence is deduced from theorems 17 and 18 respectively. This result is the principal content of theorems 19 and 21. It turns out that the set Δ is strongly independent, which enables us to prove that every sublattice of Π which is generated by points is isomorphic to a direct union of partition lattices.

Our results hold regardless of the order of the set S. However, in the case in which S is infinite, our methods seem to require that we consider sublattices generated under the <u>complete</u> operations.

We give now our first application of the structure theory developed in chapter II.

Theorem 16: Let $\{\alpha_p : p \in P\}$ be a strongly independent subset of the partition lattice Π. Let Γ be the sublattice generated by the set $\bigvee\{\alpha_p/0 : p \in P\}$ under the complete lattice operations. Then Γ is isomorphic to $\bigotimes\{\alpha_p/0 : p \in P\}$, the direct union of the ideals $\alpha_p/0$.

Proof: Let Σ be the set of all partitions of the form $\bigcup_p \sigma_p$ where each $\sigma_p \subseteq \alpha_p$. We claim that $\Sigma = \Gamma$. Clearly we have $\bigvee\{\alpha_p/0 : p \in P\} \leq \Sigma \leq \Gamma$. Hence it suffices to show that Σ is a complete sublattice of Π. Let $\{\bigcup_p \sigma_{pq} : q \in Q\}$ be an arbitrary subset of Σ. It is obvious that $\bigcup_q(\bigcup_p \sigma_{pq}) \in \Sigma$. Next we show that $\bigcap_q\bigcup_p\sigma_{pq} = \bigcup_p\bigcap_q\sigma_{pq}$. $\bigcap_q\bigcup_p\sigma_{pq} \supseteq \bigcup_p\bigcap_q\sigma_{pq}$ always holds. Now let (a,b) be an arbitrary point contained in $\bigcap_q\bigcup_p\sigma_{pq}$. Then for each particular q, $(a,b) \subseteq \bigcup_p \sigma_{pq}$. Thus there exists a chain of blocks of the σ_{pq}, $B_1 \between B_2 \between \cdots \between B_k$, such that $a \in B_1$ and $b \in B_k$. For each p, σ_{pq} is contained in α_p, and the blocks of all of the α_p form a structure $\mathfrak{S}$. Hence each B_i is contained in some set in $\mathfrak{S}$. Let $d_i \in B_i \wedge B_{i+1}$ for each i. It follows that $a\mathfrak{S}d_1\mathfrak{S}d_2\mathfrak{S}\cdots\mathfrak{S}d_{k-1}\mathfrak{S}b$. By theorem 10, $C(a,b) \leq \{a,d_1,d_2,\ldots,d_{k-1},b\}$. Hence the partition $(C(a,b)) \subseteq (a,d_1,d_2,\ldots,d_{k-1},b) \subseteq \bigcup_p \sigma_{pq}$. Let $C(a,b) = \{a = g_0, g_1, \ldots, g_n = b\}$. Then each minimal partition (g_j,g_{j+1}) is contained in some α_{p_j}. By theorem 4 we have

$$\begin{aligned}(g_j,g_{j+1}) &= \alpha_{p_j} \cap \bigcup_i (g_i,g_{i+1}) \\ &= \alpha_{p_j} \cap (C(a,b)) \\ &\subseteq \alpha_{p_j} \cap \bigcup_p \sigma_{pq} \\ &= \sigma_{p_j q}.\end{aligned}$$

But the partitions (g_i, g_{i+1}) are <u>independent</u> of q. Hence $(g_i, g_{i+1}) \subseteq \bigcap_q \sigma_{p_i q}$ and $(a,b) \subseteq (C(a,b)) = \bigcup_i (g_i, g_{i+1}) \subseteq \bigcup_p \bigcap_q \sigma_{pq}$. It follows that $\bigcap_q \bigcup_p \sigma_{pq} = \bigcup_p \bigcap_q \sigma_{pq}$. Hence Σ is a complete sublattice of Π and, consequently, $\Sigma = \Gamma$.

It is now clear that the natural mapping of Γ into $\bigotimes \{\alpha_p/0 : p \in P\}$ is one - one, onto, and order preserving. Hence it is an isomorphism and the proof is complete.

The preceding theorem is of an entirely lattice-theoretical nature. It is therefore interesting that its conclusion does not hold in a general geometric lattice. Since there seems to be some confusion in the literature over this point, we include an example of a geometric lattice in which theorem 16 does not hold. Consider the set Λ of all partitions on $\{1,2,3,4,5\}$ which are unions of the minimal partitions (1,2), (1,3), (2,4), (3,4), (4,5), and (1,5). Λ is clearly a geometric lattice (although not a <u>sublattice</u> of the partition lattice). The elements

$$\sigma_1 = (1,2,3,4)(5) \quad \text{and} \quad \sigma_2 = (1)(2)(3)(4,5)$$

are strongly independent in Λ. Set

$$\alpha = (1,2,4)(3)(5) \quad \text{and} \quad \beta = (1,3,4)(2)(5).$$

Then clearly α and β belong to the ideal $\sigma_1/0$ in Λ. We easily verify that

$$(\alpha \cup \sigma_2) \cap (\beta \cup \sigma_2) = (1,4,5)(2)(3) \quad \text{but}$$
$$(\alpha \cap \beta) \cup \sigma_2 = (1)(2)(3)(4,5),$$

where we have used the fact that $\alpha \cap \beta = 0$ in Λ. Therefore

$$(\alpha \cup \sigma_2) \cap (\beta \cup \sigma_2) \supset (\alpha \cap \beta) \cup \sigma_2.$$

This shows that the ideals $\sigma_1/0$ and $\sigma_2/0$ do not generate a sublattice isomorphic with their direct union. Hence theorem 16 does not hold in the geometric lattice Λ.

Next we consider complete sublattices generated by ideals from a somewhat different viewpoint.

<u>Theorem 17</u>: Let α and β be singular partitions with non-trivial blocks A and B. Let Γ be the complete sublattice of Π generated by the ideals $\alpha/0$ and $\beta/0$. Then $\Gamma = \alpha \cup \beta/0$ if and only if $\alpha \cap \beta \supset 0$ or $A \wedge B = \emptyset$.

<u>Proof</u>: (Necessity) Suppose that $\alpha \cap \beta = 0$ and $A \wedge B \neq \emptyset$. $\alpha \cup \beta$ is a singular partition since $A \wedge B \neq \emptyset$. Then $(\alpha \cup \beta)/0$ is a partition lattice and hence directly undecomposable. On the other hand $\alpha \cap \beta = 0$ implies that $A \wedge B$ is at most a singleton. But then $\{A,B\}$ is a structure and $\Gamma \cong a/0 \otimes b/0$ by theorem 16. It follows that $\Gamma \neq (a \cup b)/0$.

(Sufficiency) This is obvious if $A \wedge B = \emptyset$. Suppose that $\alpha \cap \beta \supset 0$. It will suffice to show that for every x and y in $A \vee B$, the minimal partition (x,y) is in Γ. This is because every partition in $(\alpha \cup \beta)/0$ is a union of such minimal partitions. If both x and y are in the same set, A or B, then (x,y) is in $\alpha/0$ or $\beta/0$ and hence in Γ. Suppose now that $x \in A - B$ and $y \in B - A$. Since $\alpha \cap \beta \supset 0$, there exist distinct elements c and d in $A \wedge B$. Then (x,c) and (x,d) are in $\alpha/0$, and (y,c) and (y,d) are in $\beta/0$. Hence $(x,y) = [(x,c) \cup (c,y)] \cap [(x,d) \cup (d,y)] \in \Gamma$ completing the proof.

With considerably more difficulty we solve the analogous problem for dual ideals.

<u>Theorem 18</u>: Suppose that α and β are incomparable partitions in Π. Let Γ be the complete sublattice of Π generated by the dual ideals i/α and i/β. Then $\Gamma = i/(\alpha \cap \beta)$ if and only if no block of α overlaps every block of β and no block of β overlaps every block of α.

<u>Proof</u>: We clearly lose no generality in taking $\alpha \cap \beta = 0$. Then $i/(\alpha \cap \beta) = \Pi$.

(Sufficiency) Let (a,b) be an arbitrary point. (a,b) is the meet of all partitions of the form $(x)(S-x)$ where $x \in S$ and $\{a,b\} \leq (S-x)$. Hence in order to show that $\Gamma = \Pi$ it will suffice to show that every partition of the form $(x)(S-x)$ is in Γ.

Let x be an arbitrary element of S. x is in some block A_1 of α and in some block B_1 of β. The partition $(A_1)(S-A_1)$ is in τ/α and $(B_1)(S-B_1)$ is in τ/β. Therefore the partition $(A_1)(S-A_1) \cap (B_1)(S-B_1)$, which we set equal to σ, is in Γ. $A_1 \wedge B_1 = \{x\}$ since $\alpha \cap \beta = 0$. Thus we have

$$(1) \qquad \sigma = (x)(A_1-x)(B_1-x)(S-A_1 \vee B_1) \in \Gamma.$$

If A_1 or $B_1 = \{x\}$, then $(x)(S-x)$ is in τ/α or τ/β and hence in Γ as desired. Suppose therefore that $A_1 > \{x\}$ and $B_1 > \{x\}$. Then A_1 contains an element $y \neq x$. y belongs to some block B_2 of β different from B_1. By hypothesis there is a block A_2 of α disjoint from B_1. A_2 is non-empty, so there exists a block B_k of β which overlaps A_2. Let $A_2 \wedge B_k = \{z\}$. Clearly $y \in (A_1-x)$ and $z \in (S-A_1 \vee B_1)$. We now set $\mu = [A_1 \vee A_2] \cap [B_2 \vee B_k]$ where $[A_1 \vee A_2]$ denotes the partition in τ/α obtained by joining the blocks A_1 and A_2 of α and where $[B_2 \vee B_k]$ denotes the partition in τ/β obtained by joining the blocks B_2 and B_k of β (possibly $B_2 = B_k$). Clearly $\mu \in \Gamma$. *x is contained the block* $(A_1 \vee A_2) \wedge B_1$ of μ. But $(A_1 \vee A_2) \wedge B_1 = (A_1 \wedge B_1) \vee (A_2 \wedge B_1) = \{x\}$ since $A_2 \wedge B_1 = \emptyset$. Thus $\{x\}$ is a block of μ. Hence $(x)(S-x) \supseteq \sigma \cup \mu$. $\{y,z\} \leq A_1 \vee A_2$ and $\{y,z\} \leq B_2 \vee B_k$. Therefore $\mu \supseteq (y,z)$ and $\sigma \cup \mu \supseteq \sigma \cup (y,z) = (x)(B_1-x)(S-B_1)$. Thus

(2) $$(x)(S-x) \supseteq \sigma \cup \mu \supseteq (x)(B_1-x)(S-B_1).$$

By a similar argument, there exists a ν in Γ such that

(3) $$(x)(S-x) \supseteq \sigma \cup \nu \supseteq (x)(A_1-x)(S-A_1).$$

$(S-B_1)$ and $(S-A_1)$ overlap. Therefore $\sigma \cup \mu \cup \nu = (x)(S-x) \in \Gamma$. This implies, as observed, that $\Gamma = \Pi$ completing the first part of the proof.

As part of the converse, we prove the following lemmas.

Lemma 18.1: Suppose the block A of α overlaps every block of β. Then each σ in Γ satisfies the following for every $\{a,b\} \leq A$:

(P): $$(a,b) \subseteq \sigma \cup \beta \quad \text{implies} \quad (a,b) \subseteq \sigma.$$

Proof of the lemma: Partitions in i/α or i/β clearly satisfy (P). Hence it suffices to show that (P) is preserved under complete lattice operations.* Suppose that the partitions σ_i, $i \in I$, satisfy property (P). We show that $\bigcap_i \sigma_i$ and $\bigcup_i \sigma_i$ satisfy (P).

Suppose $(a,b) \subseteq (\bigcap_i \sigma_i) \cup \beta$ where $\{a,b\} \leq A$. Then for each i, $(a,b) \subseteq \sigma_i \cup \beta$. But for each i, σ_i satisfies (P) and therefore $(a,b) \subseteq \sigma_i$. Hence $(a,b) \subseteq \bigcap_i \sigma_i$ which shows that $\bigcap_i \sigma_i$ satisfies property (P).

Now suppose that $(a,b) \subseteq (\bigcup_i \sigma_i) \cup \beta$ where $\{a,b\} \leq A$. Then $(a,b) \subseteq \bigcup_i (\sigma_i \cup \beta)$. Hence there exists a chain of sets $T_1 \between T_2 \between \cdots \between T_n$ where each T_j is a block of some $\sigma_i \cup \beta$ and where $a \in T_1$ and $b \in T_n$. Each set $T_k \wedge T_{k+1}$ is a block of a partition of the form $(\sigma_i \cup \beta) \cap (\sigma_j \cup \beta)$. Since $(\sigma_i \cup \beta) \cap (\sigma_j \cup \beta) \supseteq \beta$, each set $T_k \wedge T_{k+1}$ must contain a (non-empty) block of β. But each block of β contains an element of A. Hence each set $T_k \wedge T_{k+1}$ contains at least one element a_k of A. Each partition (a,a_1), $(a_1,a_2), \ldots, (a_{n-1},b)$ is contained in a partition of the form $\sigma_i \cup \beta$ where σ_i satisfies (P). Therefore each partition $(a,a_1),(a_1,a_2),\ldots,(a_{n-1},b)$ is contained in some σ_i. But then $(a,b) \subseteq (a,a_1) \cup (a_1,a_2) \cup \cdots \cup (a_{n-1},b) \subseteq \bigcup_i \sigma_i$. Hence $(a,b) \subseteq \bigcup_i \sigma_i$. Thus $\bigcup_i \sigma_i$ satisfies property (P) and the lemma is proved.

<u>Lemma 18.2</u>: Suppose the block A of α overlaps every block of β. If $\sigma \subseteq \beta$ and $i/\sigma \leq \Gamma$, then $\sigma = \beta$.

<u>Proof</u>: Suppose the assertion is false. Then $i/\sigma \leq \Gamma$ for some $\sigma \subset \beta$. Since each block of β that $\tau/\sigma \leq \Gamma$ for some $\sigma \subset \beta$. Since each block of β contains exactly one element of A (because $\alpha \cap \beta = 0$), σ must have

*This may be deduced from theorem 23 in chapter IV.

a block C_1 such that $C_1 \wedge A = \emptyset$. But C_1 is contained in some block B_1 of β. Let B_2 be a second block of β. Let C_2 be a block of σ which is contained in B_2. Let $B_1 \wedge A = \{a_1\}$ and let $B_2 \wedge A = \{a_2\}$. Let δ denote the minimal partition in i/σ obtained by joining the blocks C_1 and C_2 of σ. Then $\delta \in \Gamma$. But $(a_1,a_2) \subseteq \delta \cup \beta$ where $(a_1,a_2) \nsubseteq \delta$ contrary to lemma 18.1. This proves lemma 18.2.

<u>Proof of theorem 18</u>: (Necessity) Suppose α has a block which overlaps every block of β. $\alpha \cap \beta \subset \beta$ since α and β are incomparable. By lemma 18.2 $i/(\alpha \cap \beta) \nsubseteq \Gamma$ and hence $i/(\alpha \cap \beta) \neq \Gamma$. The case in which β has a block overlapping every block of α is, of course, similar. This completes the proof of theorem 18.

<u>Corollary 18.3</u>: Suppose $\alpha \cup \beta \neq i$. Then the complete sublattice generated by the ideals i/α and i/β is $i/\alpha \cap \beta$.

<u>Proof</u>: If, say, α had a block overlapping every block of β, then $\alpha \cup \beta$ would equal i. With this observation the result follows from theorem 18.

One should note, however, that $\alpha \cup \beta = i$ does not imply that one of the partitions α and β has a block overlapping every block of the other. It is therefore entirely possible that i/α and i/β generate $i/(\alpha \cap \beta)$ even though $\alpha \cup \beta = i$.

The remaining results in this chapter are just applications of the preceding theorems.

<u>Theorem 19</u>: Any complete sublattice Γ of Π contains a unique set Δ of singular partitions with the following properties:

(1) $\alpha/0 \leq \Gamma$ for each $\alpha \in \Delta$.

(2) If β is singular and $\beta \supset \alpha$ where $\alpha \in \Delta$, then $\beta/0 \not\leq \Gamma$.

(3) Let Ω be the set of all points in Π. If $\sigma \in \Gamma \wedge \Omega$, then there is an $\alpha \in \Delta$ such that $\sigma \subseteq \alpha$.

Proof: For each $\omega \in \Gamma \wedge \Omega$ let $\Delta(\omega) = \{\sigma \in \Gamma : \sigma$ is singular, $\sigma \supseteq \omega$, and $\sigma/0 \leq \Gamma\}$. Let Σ be a chain in $\Delta(\omega)$. Then $\bigcup\Sigma$ is singular and $\bigcup\Sigma \supseteq \omega$. If (a,b) is a point contained in $\bigcup\Sigma$, then by compactness $(a,b) \subseteq$ some $\sigma \in \Sigma$. Then $(a,b) \in \Gamma$ since $\sigma/0 \leq \Gamma$. It follows that $\bigcup\Sigma/0 \leq \Gamma$ and that $\bigcup\Sigma \in \Delta(\omega)$. By Zorn's lemma $\Delta(\omega)$ contains a maximal element α.

Suppose now that both α and β are maximal elements of $\Delta(\omega)$. Then $\alpha \cup \beta$ is singular since $\alpha \supseteq \omega$ and $\beta \supseteq \omega$. Also $\alpha \cap \beta \neq 0$, $\alpha/0 \leq \Gamma$, and $\beta/0 \leq \Gamma$. Hence by theorem 17 $\alpha \cup \beta/0 \leq \Gamma$. But then $\alpha \cup \beta \in \Delta(\omega)$. By maximality $\alpha = \alpha \cup \beta = \beta$ and $\alpha = \beta$. Thus each set $\Delta(\omega)$ contains a unique maximal element $\alpha(\omega)$.

Set $\Delta = \{\alpha(\omega) : \omega \in \Gamma \wedge \Omega\}$. Δ clearly satisfies the three stated conditions. Suppose that Δ' also satisfies the three conditions. Let $\alpha \in \Delta$. Then $\alpha \supseteq$ some ω where $\omega \in \Gamma \wedge \Omega$. By (3) there is an $\alpha' \in \Delta'$ such that $\alpha' \supseteq \omega$. But then $\alpha \cup \alpha'$ is singular and $(\alpha \cup \alpha')/0 \leq \Gamma$ by (1) and theorem 17. By (2) we have $\alpha = \alpha \cup \alpha' = \alpha'$. Hence $\alpha \in \Delta'$ and $\Delta \leq \Delta'$. Similarly $\Delta' \leq \Delta$. Thus $\Delta = \Delta'$, which proves the asserted uniqueness.

Additional information about the set Δ is easily obtained.

Theorem 20: The set Δ of theorem 19 is strongly independent.

Proof: By theorem 7 we need only show that the set $\{A : (A) \in \Delta\}$, which we denote by $\mathfrak{H}$, is a structure. $\mathfrak{H}$ clearly satisfies structure property (i). To verify (ii) suppose that $T \in \mathfrak{H}$ and $U \in \mathfrak{H}$ and that $T \wedge U$ contains at least two elements. Then $(T) \cap (U) \supset 0$ and $(T)/0 \leq \Gamma$ and $(U)/0 \leq \Gamma$. By theorem 17, $(T) \cup (U)/0 \leq \Gamma$. But $(T) \cup (U)$ is singular and so, by property (2) of Δ, $(T) = (T) \cup (U) = (U)$. Hence $T = U$, from which it follows that (ii) holds.

To prove (iii) let

$$a_1, a_2, \ldots, a_r$$

be a sequence of distinct elements of S such that, after setting $a_1 = a_{r+1}$, each (a_i, a_{i+1}) is contained in some element of Δ. Let α be the element of Δ containing (a_1, a_2) and let $\sigma = (a_1, a_2, \ldots, a_r)$. Each point (a_i, a_j), $i < j$, is contained in Γ since (a_i, a_j) $= [(a_i, a_{i+1}) \cup \cdots \cup (a_{j-1}, a_j)] \cap [(a_j, a_{j+1}) \cup \cdots \cup (a_r, a_1) \cup \cdots \cup (a_{i-1}, a_i)]$. It follows that $\sigma/0 \leq \Gamma$. Also $\alpha/0 \leq \Gamma$ because $\alpha \in \Delta$. $\alpha \cap \sigma \supset 0$ and $\alpha \cup \sigma$ is singular since $\alpha \supseteq (a_1, a_2)$ and $\sigma \supseteq (a_1, a_2)$. By theorem 17, $\alpha \cup \sigma/0 \leq 0$. By property (2) of Δ, $\alpha \cup \sigma = \alpha$. Hence $\sigma = (a_1, a_2, \ldots, a_r) \subseteq \alpha \in \Delta$, which verifies (iii) and proves the theorem.

As a simple corollary, we characterize all complete sublattices of Π which are generated by points.

Corollary 20.1: Let $\Omega' \leq \Omega$ where Ω is the set of all points in Π. Let Γ be the complete sublattice of Π which is generated by Ω'. Then Γ is isomorphic to a direct union of partition lattices. In particular, Γ is a point lattice (every element of Γ is a union of points).

<u>Proof</u>: The set Δ of theorem 19 corresponding to Γ is strongly independent by theorem 20. Let $\bar{\Delta} = \bigvee \{\alpha/0 : \alpha \in \Delta\}$. Then $\Omega' \leq \bar{\Delta}$ by theorem 19 (3). $\bar{\Delta} \leq \Gamma$ by theorem 19 (1). $\bar{\Delta}$ therefore generates Γ and, by theorem 16, $\Gamma \cong \bigotimes \{\alpha/0 : \alpha \in \Delta\}$. This completes the proof.

Next we give a theorem which is, in a sense, dual to theorem 19.

<u>Theorem 21</u>: Any complete sublattice Γ of Π contains a unique subset Δ_0 with the following properties:

(1) $i/\alpha \leq \Gamma$ for each $\alpha \in \Delta_0$.

(2) If $\beta \subset \alpha$ where $\alpha \in \Delta_0$, then $i/\beta \not\leq \Gamma$.

(3) Let Ω_0 be the set of all maximal partitions (dual points) in Π. If $\sigma \in \Gamma \wedge \Omega_0$, then there is an $\alpha \in \Delta_0$ such that $\alpha \subseteq \sigma$.

<u>Proof</u>: For each $\gamma \in \Gamma \wedge \Omega_0$ let

$$\Delta(\gamma) = \{\sigma \in \Gamma : \sigma \subseteq \gamma \text{ and } i/\sigma \leq \Gamma\}.$$

Let Σ be a chain in $\Delta(\gamma)$. Then $\bigcap \Sigma \subseteq \gamma$. For notational convenience suppose $\bigcap \Sigma = 0$. Then for each a and b in S,

$$(a,b) = \bigcap \{\sigma \cup (a,b) : \sigma \in \Sigma\} \in \Gamma,$$

which implies that $i/\bigcap\Sigma \leq \Gamma$ and hence that $\bigcap\Sigma \in \Delta(\gamma)$. By Zorn's lemma $\Delta(\gamma)$ has a minimal element α.

Suppose now that both α and β are minimal elements of $\Delta(\gamma)$. Then $i/\alpha \leq \Gamma$ and $i/\beta \leq \Gamma$ and $\alpha \cup \beta \subseteq \gamma \subset i$. Hence by corollary 18.3, $i/(\alpha \cap \beta) \leq \Gamma$. By the minimality of α and β we have $\alpha = \alpha \cap \beta = \beta$.

Thus each set $\Delta(\gamma)$ has a unique minimal element $\delta(\gamma)$.

Set $\Delta_0 = \{\delta(\gamma) : \gamma \in \Gamma \wedge \Omega_0\}$. Δ_0 clearly has the three stated properties. The uniqueness of the set Δ_0 is easy to verify.

The set Δ_0 seems to be much less regular than the set Δ of theorem 19. From theorem 18 it is clear that if α and β are distinct elements of Δ_0, then $\alpha \cup \beta = i$. In fact either α or β has a block overlapping every block of the other. However we cannot deduce a result like corollary 20.1 for lattices generated by dual points. In fact, one may give examples of sublattices Γ of Π generated by dual points which are not dual point lattices and which are not dense in Π.

CHAPTER IV: SUBLATTICES Γ WHERE $\bigcup(\Gamma \wedge \Omega) = i$.

Again, we let Ω denote the collection of all points in Π, and we let Ω_0 denote the collection of all dual points (maximal partitions) in Π.

In this chapter we characterize all complete sublattices Γ of Π such that $\bigcup(\Gamma \wedge \Omega) = i$. The principal result is theorem 23. The rather involved proof depends upon an extensive application of the theory developed in chapters II and III.

We gain access to the problem through the observation that if $\bigcup(\Gamma \wedge \Omega) = i$, then the set Δ_0 which corresponds to Γ by theorem 21 in fact characterizes the lattice Γ. We see this as follows.

<u>Lemma 22.1</u>: If Γ is a complete sublattice of Π in which $\bigcup(\Gamma \wedge \Omega) = i$, then $\bigcap(\Gamma \wedge \Omega_0) = 0$.

<u>Proof</u>: Suppose that i covers (in Γ) α. Since $\bigcup(\Gamma \wedge \Omega) = i$, there is a $\beta \in \Gamma \wedge \Omega$ such that $\beta \not\leq \alpha$. It follows that $\alpha \cup \beta = i$ and, by semi-modularity, that i covers (in Π) α. Hence every maximal partition in Γ is also maximal in Π. By corollary 20.1 the set $\Gamma \wedge \Omega$ generates a geometric lattice Γ_1. In any geometric lattice the intersection of all the maximal elements is 0. But the maximal elements in Γ_1 are clearly maximal in Γ, and hence also maximal in Π. It follows that

$0 = \bigcap(\Gamma_1 \wedge \Omega_0) \supseteq \bigcap(\Gamma \wedge \Omega_0)$ whence $0 = \bigcap(\Gamma \wedge \Omega_0)$ completing the proof.

<u>Lemma 22.2</u>: Let $\bigcup(\Gamma \wedge \Omega) = i$. Then for each $\alpha \in \Gamma$ we have that

$$\alpha = \bigcap[\Gamma \wedge (i/\alpha) \wedge \Omega_0].$$

<u>Proof</u>: i/α is a partition lattice and

$$\bigcup\{\sigma \cup \alpha : \sigma \in \Gamma \wedge \Omega\} = i.$$

The partitions $\sigma \cup \alpha$ are points in i/α. Also every maximal partition in i/α is also maximal in Π. We therefore replace Γ in lemma 22.1 by $\Gamma \wedge (i/\alpha)$ and obtain the desired result.

Hence every element of Γ (except i) is the intersection of a suitable subset of $\Gamma \wedge \Omega_0$. Now consider the set Δ_0 corresponding to Γ by theorem 21. We have

$$\Gamma \wedge \Omega_0 \leq \bigvee\{i/\sigma : \sigma \in \Delta_0\} \leq \Gamma,$$

which proves the following lemma.

<u>Lemma 22.3</u>: If $\bigcup(\Gamma \wedge \Omega) = i$, then Γ is completely determined by its subset Δ_0.

We therefore introduce the following definition.

<u>Definition 16</u>: Any subset Σ of Π will be called <u>characteristic</u> if for the complete sublattice Γ generated by $\overline{\Sigma} = \bigvee\{i/\sigma : \sigma \in \Sigma\}$ we have

(1) $\bigcup(\Gamma \wedge \Omega) = i$.

(2) $\beta \subset \alpha \in \Sigma$ implies $i/\beta \not\leq \Gamma$.

(3) $\Gamma \wedge \Omega_0 \leq \bar{\Sigma}$.

<u>Theorem 22</u>: Complete sublattices Γ of Π where $\bigcup(\Gamma \wedge \Omega) = i$ and characteristic subsets of Π are in one - one correspondence.

<u>Proof</u>: Let Γ be a sublattice such that $\bigcup(\Gamma \wedge \Omega) = i$. The subset Δ_0 of theorem 21 then has the required three properties. In addition, by lemma 22.3, the set

$$\bar{\Delta}_0 = \{i/\sigma : \sigma \in \Delta_0\}$$

generates Γ. Hence Δ_0 is a characteristic set.

Conversely, a characteristic set corresponds to a complete sublattice Γ where $\bigcup(\Gamma \wedge \Omega) = i$ by definition. The one - one nature of the correspondence follows from the uniqueness part of theorem 21 and from the observation that a given subset of Π generates only one complete sublattice.

In the statement of theorem 23 we shall need the following definitions.

<u>Definition 17</u>: A structure $\mathfrak{H}$ relative to which every two elements of S are related (chapter II) will be called a <u>connected structure</u>.

<u>Definition 18</u>: Let $A \sim B$ define an equivalence relation on the set $\mathfrak{H}$. Let $\mathfrak{H}'$ and $\mathfrak{H}''$ be equivalence classes. We say that $\mathfrak{H}'$ <u>splits</u> $\mathfrak{H}''$ (relative to $A \sim B$) if for some $T \in \mathfrak{H}'$ and $U_1, U_2 \in \mathfrak{H}''$ we have that $T \in \mathfrak{C}(U_1, U_2)$.

Definition 19: A structure $\mathfrak{H}$ together with an equivalence relation $A \sim B$ on $\mathfrak{H}$ will be called an equivalence structure if

(1) $\mathfrak{H}$ is connected.

(2) If $A \sim B$ and $A \wedge B \neq \emptyset$, then $A = B$.

(3) If $\mathfrak{H}'$ and $\mathfrak{H}''$ are equivalence classes such that $\mathfrak{H}'$ splits $\mathfrak{H}''$ and $\mathfrak{H}''$ splits $\mathfrak{H}'$, then $\mathfrak{H}' = \mathfrak{H}''$.

We now state the main theorem of this chapter.

Theorem 23: Let Ω be the set of all points in the partition lattice Π on the set S. Then the set of all complete sublattices Γ of Π such that $\bigcup(\Gamma \wedge \Omega) = i$ is in one - one correspondence with the set of all equivalence structures on S.

Part I: Suppose we are given the complete sublattice Γ where $\bigcup(\Gamma \wedge \Omega) = i$. Let Δ_0 be its characteristic subset. Then there is an equivalence structure $(\mathfrak{H},\sim)$ such that Δ_0 is derived from $(\mathfrak{H},\sim)$ in the following way. The equivalence relation on $\mathfrak{H}$ may clearly be considered an equivalence relation on the set $\Delta = \{(A) \in \Pi : A \in \mathfrak{H}\}$. For each $\alpha \in \Delta$ set $\mu(\alpha) = \bigcup\{\sigma \in \Delta : \sigma \not\sim \alpha\}$. (Note that by theorem 7, Δ is strongly independent. Hence after setting $\sigma^* = \bigcup\{\tau \in \Delta : \tau \neq \sigma\}$ for each $\sigma \in \Delta$, we have

$$\mu(\alpha) = \bigcup\{\sigma \in \Delta : \sigma \not\sim \alpha\} = \bigcap\{\sigma^* : \sigma \in \Delta, \sigma \sim \alpha\}.)$$

Then $\Delta_0 = \{\mu(\alpha) : \alpha \in \Delta\}$; and this expresses the characteristic set Δ_0, and hence also Γ, as a function of the equivalence structure $(\mathfrak{H},\sim)$.

Part II: Suppose we are given the equivalence structure $(\mathfrak{S},\sim)$. Defining $\mu(\alpha)$, as above, we show that the set $\Sigma = \{\mu(\alpha) : \alpha \in \Delta\}$ is characteristic, and hence corresponds to a complete sublattice Γ where $\bigcup(\Gamma \wedge \Omega) = i$. Furthermore, we show that $(\mathfrak{S},\sim)$ is the only equivalence structure which gives the sublattice Γ.

Proof: Both parts of the proof require considerable effort.

Part I: Let Γ be a complete sublattice of Π where $\bigcup(\Gamma \wedge \Omega) = i$. Let Δ_0 be its characteristic subset (see definition 16 and theorem 22).

Lemma 23.1: The complete sublattice Γ_1 generated by $\Gamma \wedge \Omega$ corresponds to a unique connected structure $\mathfrak{S}$.

Proof: By 20.1, Γ_1 is a point lattice and corresponds to a unique structure $\mathfrak{S}$. We will show that $\mathfrak{S}$ is connected. Let $\Delta = \{(A) \in \Pi : A \in \mathfrak{S}\}$. For each point α of Γ_1 there is an $(A) \in \Delta$ such that $\alpha \subseteq (A)$ by theorem 19. Since the union of the points of Γ_1 is i, we have also that $\bigcup\Delta = i$. Let x and y be arbitrary elements of S. Then $(x,y) \subseteq \bigcup\Delta$ whence (x,y) is contained in a finite union of partitions in Δ. Hence there are distinct $(A_i) \in \Delta$ such that $(x,y) \subseteq (A_1) \cup (A_2) \cup \cdots \cup (A_n)$ where $A_1 \between A_2 \between \cdots \between A_n$ and $x \in A_1$ and $y \in A_n$. Let $A_i \wedge A_{i+1} = \{a_i\}$ for each i. Then $x\mathfrak{S}a_1\mathfrak{S}a_2\mathfrak{S}\cdots\mathfrak{S}a_{n-1}\mathfrak{S}y$. The arbitrary elements x and y are related, and $\mathfrak{S}$ is a connected structure.

Definition 20: *Let $x \in S$ and $y \in S$.*

We define $\lambda(x,y)$ to be the singular partition $(C(x,y))$.

Any two elements of S are related since $\mathfrak{L}$ is connected. Hence $\lambda(x,y)$ is defined for any pair of elements x and y in S. It is easy to see that $(x,y) \subseteq \lambda(x,y) = \bigcap\{\sigma \in \Gamma_1 : \sigma \supseteq (x,y)\}$. In particular, $\lambda(x,y)$ is always in the lattice Γ.

Lemma 23.2: The characteristic set Δ_0 is contained in the point sublattice Γ_1.

Proof: Suppose that for some $\alpha \in \Delta_0$ we have $\alpha \notin \Gamma_1$. Then $\lambda(x,y) \nsubseteq \alpha$ for some $(x,y) \subseteq \alpha$ since otherwise

$$\alpha = \bigcup\{(x,y) \subseteq \alpha\} \subseteq \bigcup\{\lambda(x,y) : (x,y) \subseteq \alpha\} \subseteq \alpha$$

and $\alpha = \bigcup\{\lambda(x,y) : (x,y) \subseteq \alpha\} \in \Gamma_1$. Choose (a,b) so that $C(a,b)$ has the fewest possible elements such that $(a,b) \subseteq \alpha$ but $\lambda(a,b) \nsubseteq \alpha$. Let

$$C(a,b) = \{a, g_1, g_2, \ldots, g_n, b\} \text{ and}$$

let T be the block of α which contains $\{a,b\}$. Then

$$T \wedge C(a,b) = \{a,b\}$$

by the minimality of $C(a,b)$. Also $n \geq 1$ since $\lambda(a,b) \nsubseteq \alpha$.

If there were a partition β such that $\alpha \cap \beta \subset \alpha$, $\alpha \cup \beta \neq i$, and $i/\beta \leq \Gamma$; then it would follow from 18.3 that $i/\alpha \cap \beta \leq \Gamma$. This would contradict the fact that α is in Δ_0, the characteristic set of Γ. Hence our proof will be complete when we have constructed such a partition β. Set

$$\mu = (T)(S-T),$$

$$\nu = (A)(S-A) \text{ where } A = \{x \in S : g_1 \notin C(x,a)\},$$

$$\beta = (T \wedge A)(T-A)(S-T).$$

$\mu \in \Gamma$ since $\mu \supseteq \alpha$ where $\alpha \in \Delta_0$. We next prove that $\nu \in \Gamma$. Let y be an arbitrary element of $C(x,a)$ where $x \in A$. By theorem 10, $C(y,a) \leq C(x,a)$. Then $g_1 \not\leq C(x,a)$ implies $g_1 \not\leq C(y,a)$. Hence $y \in A$. Thus for every $x \in A$, we have $C(x,a) \leq A$. Hence $(A) = \bigcup\{\lambda(x,a) : x \in A\} \in \Gamma$. By a similar argument $(S-A) \in \Gamma$. Hence

$$\nu = (A) \cup (S-A) \in \Gamma.$$

Now we show that $\beta \in \Gamma$. We have

$$\mu \cap \nu = (T \wedge A)(T-A)(A-T)(S-T \vee A) \in \Gamma.$$

If $A - T = \emptyset$, then $\beta = \mu \cap \nu \in \Gamma$. Hence suppose $A - T \neq \emptyset$. Let $d \in A - T$. Let

$$\delta = (\mu \cap \nu) \cup (\mu \cap \lambda(d,g_1)).$$

Clearly $\delta \in \Gamma$. We claim $\delta = \beta$. Both d and g_1 belong to the block $S - T$ of μ. Hence $(d,g_1) \subseteq \mu \cap \lambda(d,g_1)$. Hence

$$\delta \geq (\mu \cap \nu) \cup (d,g_1) = \beta.$$

Next we claim that $C(d,g_1) \wedge (T-A) = \emptyset$. By theorem 10, $C(d,g_1) \leq C(d,a) \vee C(a,g_1) = C(d,a) \vee \{g_1\}$. Since $d \in A$ we have $C(d,a) \leq A$ (shown above). Hence $C(d,a) \wedge (T-A) \leq A \wedge (T-A) = \emptyset$. Now $T \wedge C(a,b) = \{a,b\}$ and so $g_1 \not\leq T$. Hence $g_1 \not\leq T - A$. Therefore $C(d,g_1) \wedge (T-A) \leq [C(d,a) \vee \{g_1\}] \wedge (T-A) = \emptyset$ and $C(d,g_1) \wedge (T-A) = \emptyset$ as claimed. But then $\delta = (\mu \cap \nu) \cup (\mu \cap \lambda(d,g_1)) \subseteq (\mu \cap \nu) \cup \lambda(d,g_1)$ $= (T-A)(T \wedge A)(A-T)(S-T \vee A) \cup (C(d,g_1)) \subseteq (T-A)(S-(T-A))$. It is also clear that $\delta \subseteq \mu = (T)(S-T)$. Hence $\delta \subseteq (T-A)(S-(T-A)) \cap (T)(S-T)$ $= (T-A)(T \wedge A)(S-T) = \beta$. Thus $\delta \subseteq \beta$ and, as we've already shown $\delta \supseteq \beta$, we have $\delta = \beta$. Hence $\beta \in \Gamma$.

Let us denote the blocks $T \wedge A$, $T - A$, $S - T$ of β by B_1, B_2, B_3 respectively. Then $\beta = (B_1)(B_2)(B_3)$. Observe that

$$a \in T \wedge A = B_1,$$

$$b \in T - A = B_2,$$

$$\{g_1, g_2, \ldots, g_n\} \leq S - T = B_3, \quad \text{and also}$$

$$(a, g_1) \in \Gamma \quad \text{and}$$

$$(b, g_n) \in \Gamma.$$

It follows that

$$(B_1 \vee B_3)(B_2) = \beta \cup (a, g_1) \in \Gamma,$$

$$(B_2 \vee B_3)(B_1) = \beta \cup (b, g_n) \in \Gamma, \text{ and}$$

$$(B_1 \vee B_2)(B_3) = \quad \mu \quad \in \Gamma.$$

Therefore $i/\beta \leq \Gamma$. $\alpha \cap \beta \subset \alpha$ since a and b are equivalent mod α but not mod β. Also $\beta \subseteq \mu$ whence $\alpha \cup \beta \neq i$. Hence β is of the required type and the lemma is proved.

<u>Lemma 23.3</u>: Let $\Delta = \{(A) \in \Pi : A \in \mathfrak{S}\}$. For $\alpha \in \Delta$ let $\alpha^* = \bigcup\{\sigma \in \Delta : \sigma \neq \alpha\}$. Then $i/\alpha^* \leq \Gamma_1$.

<u>Proof</u>: $\alpha \cap \alpha^* = 0$ since Δ is strongly independent. Also $\alpha \cup \alpha^* = i$. Since α is singular, say $\alpha = (A)$, A contains exactly one element from each block of α^*. By the selection of Δ by 20.1, we have $\alpha/0 \leq \Gamma$. Hence $\alpha^* \cup (a,b) \in \Gamma_1$ for each a and b in A. But every minimal partition in i/α^* is of this form. Hence $i/\alpha^* \leq \Gamma_1$.

<u>Lemma 23.4</u>: Each element of Δ_0 is a meet of partitions of the form α^*.

<u>Proof</u>: By theorem 20.1, Γ_1 is the direct union of all ideals $\alpha/0$ where $\alpha \in \Delta$. Thus Γ_1 is a direct union of partition lattices. Partition

lattices are dual point lattices. Hence Γ_1 is a dual point lattice. Moreover, every dual point (maximal partition) of Γ_1 is clearly a maximal partition in some i/α^*.

Let $\beta \in \Delta_0$. Then $\beta \in \Gamma_1$ by lemma 23.2. Hence $\beta = \bigcap \Omega'$ where Ω' is a set of dual points in Γ_1. Let $\Omega^* = \{\alpha^* : \alpha^* \subseteq \text{some } \sigma \in \Omega'\}$. For each $\alpha^* \in \Omega^*$ there is a $\gamma \in \Omega'$ such that $\alpha^* \subseteq \gamma$ and $\beta \subseteq \gamma$ whence $\alpha^* \cup \beta \neq i$. By lemma 23.3, $i/\alpha^* \leq \Gamma$. Hence $i/\alpha^* \cap \beta \leq \Gamma$ by theorem 18. But $\beta \in \Delta_0$ where Δ_0 is characteristic. Hence $\alpha^* \cap \beta \supseteq \beta$ and therefore $\alpha^* \supseteq \beta$. Hence $\bigcap \Omega^* \supseteq \beta$. But also $\beta = \bigcap \Omega' \supseteq \bigcap \Omega^*$. Hence $\beta = \bigcap \Omega^*$ and the lemma is proved.

<u>Definition 21</u>: Let A and B be in $\mathfrak{S}$. We write

$$A \sim B$$

if there is a $\gamma \in \Delta_0$ such that $(A)^* \supseteq \gamma$ and $(B)^* \supseteq \gamma$.

<u>Lemma 23.5</u>: $A \sim B$ is an equivalence relation on $\mathfrak{S}$. Different elements of Δ_0 correspond to different equivalence classes.

<u>Proof</u>: By 23.3, $i/(A)^* \leq \Gamma$ for every $A \in \mathfrak{S}$. By 18.3 and the definition of Δ_0, each $(A)^*$ contains some $\gamma \in \Delta_0$. Also by 18.3, $(A)^*$ cannot contain two distinct elements of Δ_0. It follows that the relation $A \sim B$ partitions $\mathfrak{S}$, that is, defines an equivalence relation. The latter statement is now obvious.

<u>Lemma 23.6</u>: If $A \sim B$ and $A \wedge B \neq \emptyset$, then $A = B$.

Proof: Let $\alpha = (A)$ and let $\beta = (B)$. $\alpha \cup \beta$ is singular since $A \wedge B \neq \emptyset$. $\alpha \cup \beta$ and $\alpha^* \cap \beta^*$ are a modular pair since Δ is strongly independent. Hence for each $\sigma \subseteq \alpha \cup \beta$, $\sigma = [\sigma \cup (\alpha^* \cap \beta^*)] \cap (\alpha \cup \beta)$. $1/\alpha^* \cap \beta^* \leq \Gamma$ since $A \sim B$. Hence $\sigma \cup (\alpha^* \cap \beta^*) \in \Gamma$. But $\alpha \cup \beta$ is also in Γ. Therefore $\sigma \in \Gamma$. Since σ was an arbitrary element of $\alpha \cup \beta/0$, we have $(\alpha \cup \beta)/0 \leq \Gamma$. But α and β are maximal singular partitions with the property that $\alpha/0 \leq \Gamma$ and $\beta/0 \leq \Gamma$ (theorem 19). Hence $\alpha = \alpha \cup \beta = \beta$ and $A = B$ as was to be proved.

Lemma 23.7: Suppose $A \in \mathfrak{C}(B,C)$ where $A \neq B$ and $A \neq C$. Then B and C are contained in different blocks of $(A)^*$.

Proof: By definition

$$(A)^* = \bigcup \{(H) : H \in \mathfrak{H},\ H \neq A\}.$$ If B and C are in the same block of $(A)^*$, then there is a sequence of sets different from A such that

$$B \between G_1 \between G_2 \between \cdots \between G_n \between C.$$

By theorem 11,

$$\mathfrak{C}(B,C) \leq \{B, G_1, G_2, \ldots, G_n, C\}.$$

But then A is in the set on the right, a contradiction. Hence B and C are in different blocks of $(A)^*$.

Lemma 23.8: Let α and β be in Δ_0. Let $\mathfrak{H}(\beta)$ be the equivalence class corresponding to β (see lemma 23.5). If α has a block A which overlaps every block of β, then $T \leq A$ for each $T \in \mathfrak{H}(\beta)$.

Proof: Since the block A overlaps every block of β, we have $(A) \cup \beta = i$. Let $T \in \mathfrak{H}(\beta)$. Let s and t be distinct elements of T. Then $(s,t) \subseteq (A) \cup \beta = i$. Both (A) and β are unions of appropriate partitions (H) where $H \in \mathfrak{H}$. Hence there exists a sequence of sets G_i in $\mathfrak{H}$ such that $G_1 \between G_2 \between \cdots \between G_n$ where $s \in G_1$ and $t \in G_n$ and where each (G_i) is contained in either (A) or β. By theorem 12, $\mathfrak{C}(s,t) \leq \{G_1, G_2, \ldots, G_n\}$. But $T \in \mathfrak{C}(s,t)$ and so T equals some G_i. However $\beta = \bigcap\{(U)^* : U \in \mathfrak{H}(\beta)\} \subseteq (T)^*$. Since $\Delta = \{(U) : U \in \mathfrak{H}\}$ is strongly independent, $(T) \nsubseteq (T)^*$ and therefore $(T) \nsubseteq \beta$. Hence we must have $(T) \subseteq (A)$ and $T \leq A$ proving the lemma.

Lemma 23.9: *Let $\mathfrak{H}(\alpha)$ and $\mathfrak{H}(\beta)$ be the equivalence classes corresponding* to the elements α and β of Δ_0. If $\mathfrak{H}(\alpha)$ splits $\mathfrak{H}(\beta)$ and $\mathfrak{H}(\beta)$ splits $\mathfrak{H}(\alpha)$, then $\mathfrak{H}(\alpha) = \mathfrak{H}(\beta)$.

Proof: Suppose that $\mathfrak{H}(\alpha) \neq \mathfrak{H}(\beta)$. Then $\alpha \neq \beta$. Since $\mathfrak{H}(\alpha)$ splits $\mathfrak{H}(\beta)$, there exist $A \in \mathfrak{H}(\alpha)$, $B_1 \in \mathfrak{H}(\beta)$ and $B_2 \in \mathfrak{H}(\beta)$ such that $A \in \mathfrak{C}(B_1, B_2)$ where $A \neq B_1$ and $A \neq B_2$. Then B_1 and B_2 occur in different blocks of $(A)^*$ by lemma 23.7. Since $\alpha \subseteq (A)^*$, B_1 and B_2 occur in different blocks of α. But then α cannot have a block overlapping every block of β by lemma 23.8. *Similarly β cannot have a block overlapping every* block of α. Since $i/\alpha \leq \Gamma$ and $i/\beta \leq \Gamma$, it follows that $i/\alpha \cap \beta \leq \Gamma$ by theorem 18. But α and β are minimal partitions with the property that $i/\alpha \leq \Gamma$ and $i/\beta \leq \Gamma$. Hence $\alpha = \alpha \cap \beta = \beta$, a contradiction. We have therefore proved that $\mathfrak{H}(\alpha) = \mathfrak{H}(\beta)$.

Completion of Part I: $(\mathfrak{H},\sim)$ is an equivalence structure since

(1) $\mathfrak{H}$ is a connected structure (23.1).

(2) If $A \sim B$ and $A \wedge B \neq \emptyset$, then $A = B$ (23.6).

(3) If $\mathfrak{H}'$ and $\mathfrak{H}''$ are equivalence classes such that $\mathfrak{H}'$ splits $\mathfrak{H}''$ and $\mathfrak{H}''$ splits $\mathfrak{H}'$, then $\mathfrak{H}' = \mathfrak{H}''$ (23.9).

It remains to note that every element of Δ_0 is of the form $\mu(\alpha) = \bigcap\{\sigma^* : \sigma \in \Delta, \sigma \sim \alpha\}$ for some $\alpha \in \Delta$ and that every element of this form is in Δ_0 (lemmas 23.4 and 23.5). This shows that Δ_0 is derived from $(\mathfrak{H},\sim)$ in the way claimed, and completes the proof of part I.

Part II: Let $(\mathfrak{H},\sim)$ be an equivalence structure. Set $\Sigma = \{(A) \in \Pi : A \in \mathfrak{H}\}$. For each $\alpha \in \Sigma$ let $\mu(\alpha) = \bigcap\{\sigma^* : \sigma \in \Delta, \sigma \sim \alpha\}$. Let $\Sigma_0 = \{\mu(\alpha) : \alpha \in \Delta\}$. Let Γ be the complete sublattice of Π generated by the set $\bar{\Sigma}_0 = \bigvee\{i/\sigma : \sigma \in \Sigma_0\}$. We must show that the set Σ_0 is characteristic.

Lemma 23.10: Suppose $A \in \mathfrak{S}(a,b)$. Then a and b occur in different blocks of $(A)^*$. This is, $a \not\equiv b(A)^*$.

Proof: Suppose that $a \equiv b(A)^*$. By definition $(A)^* = \bigcup\{(T) \in \Sigma : T \neq A\}$. Then there exists a sequence of sets $\mathfrak{H}$, all different from A, such that $T_1 \between T_2 \between \cdots \between T_n$ where $a \in T_1$ and $b \in T_n$. By theorem 12, $\mathfrak{S}(a,b) \leq \{T_1,T_2,\ldots,T_n\}$. Then A equals some T_i since $A \in \mathfrak{S}(a,b)$. This is a contradiction. Hence $a \not\equiv b(A)^*$.

Lemma 23.11: Let $\mathfrak{H}(\mu)$ be the equivalence class corresponding to $\mu \in \Sigma_0$. Then $a \equiv b(\text{mod } \mu)$ if and only if $\mathfrak{H}(\mu) \wedge \mathfrak{S}(a,b) = \emptyset$.

Proof: Suppose that $\mathfrak{H}(\mu) \wedge \mathfrak{C}(a,b) \neq \emptyset$. Let $T \in \mathfrak{H}(\mu) \wedge \mathfrak{C}(a,b)$. Then $a \not\equiv b(T)^*$ by 23.10. But $\mu = \bigcap\{(U)^* : U \in \mathfrak{H}(\mu)\} \subseteq (T)^*$. Hence $a \not\equiv b(\mu)$.

Now suppose $\mathfrak{H}(\mu) \wedge \mathfrak{C}(a,b) = \emptyset$.

$$\begin{aligned} \text{Then } \mu &= \bigcap\{(U)^* : U \in \mathfrak{H}(\mu)\} \\ &= \bigcup\{(T) : T \in \mathfrak{H}-\mathfrak{H}(\mu)\} \\ &\supseteq \bigcup\{(T) : T \in \mathfrak{C}(a,b)\}. \end{aligned}$$

Hence $a \equiv b(\mu)$ and this completes the proof.

Suppose that Σ_0 is a characteristic set. Then by lemma 22.2 every element of the corresponding sublattice Γ is a meet of elements in the dual ideals i/μ where $\mu \in \Sigma_0$. This implies that the smallest partition in Γ which contains a given minimal partition (a,b) is the partition

$$\bigcap\{\mu \cup (a,b) : \mu \in \Sigma_0\}.$$

It would therefore seem promising to study partitions of this type relative to the present problem.

Definition 22: For a and b in S, we define $\pi(a,b) = \bigcap\{\mu \cup (a,b) : \mu \in \Sigma_0\}$.

Lemma 23.12: $\pi(a,b) \subseteq \lambda(a,b)$ for all a and b in S.

Proof: From theorem 16 it follows that there exist $\delta(\alpha) \in i/\alpha^*$, $\alpha \in \Sigma$, such that

$$\lambda(a,b) = \bigcap\{\delta(\alpha) : \alpha \in \Sigma\}.$$

But then

$$\begin{aligned}\pi(a,b) &= \bigcap\{(a,b) \cup \mu : \mu \in \Sigma_0\}\\ &\subseteq \bigcap\{(a,b) \cup \alpha^* : \alpha \in \Sigma\}\\ &\subseteq \bigcap\{\delta(\alpha) : \alpha \in \Sigma\}\\ &= \lambda(a,b)\end{aligned}$$

which proves the lemma.

It is natural to ask when two elements of S are equivalent mod $\pi(a,b)$.

<u>Lemma 23.13</u>: Suppose that $\{s,t\} \leq C(a,b)$ and that $s \trianglelefteq t$ relative to (a,b) (defined in chapter II). Then $s \not\equiv t(\pi(a,b))$ if and only if there exist sets P and Q in $\mathfrak{C}(a,b)$ such that $P \sim Q$ and

$$P \triangleleft s \triangleleft Q \triangleleft t(a,b) \quad \text{or}$$

$$s \triangleleft P \triangleleft t \triangleleft Q(a,b).$$

<u>Proof</u>: Suppose that $s \not\equiv t(\pi(a,b))$. Then $s \not\equiv t(\mu \cup (a,b))$ for some $\mu \in \Sigma_0$. Either $s \not\equiv a(\mu)$ or $t \not\equiv b(\mu)$, since otherwise $s \equiv t(\mu \cup (a,b))$. Suppose that $s \not\equiv a(\mu)$. (The other case is similar). Also $s \not\equiv t(\mu)$. By lemma 23.11, there exist $P \in \mathfrak{H}(\mu) \wedge \mathfrak{C}(a,s)$ and $Q \in \mathfrak{H}(\mu) \wedge \mathfrak{C}(s,t)$. Then $P \triangleleft s \triangleleft Q \triangleleft t$ where $P \sim Q$ completing the proof.

Conversely, suppose $P \triangleleft s \triangleleft Q \triangleleft t(a,b)$ where $P \sim Q$. (The other case is similar). Then there exists a $\mu \in \Sigma_0$ such that $\mu \subseteq (P)^*$ and $\mu \subseteq (Q)^*$. By lemma 23.10

$$a \not\equiv s(P)^*,\ b \not\equiv s(Q)^*, \text{ and } t \not\equiv s(Q)^*$$

which imply that

$$a \not\equiv s(\mu),\ b \not\equiv s(\mu), \text{ and } t \not\equiv s(\mu).$$

Hence s belongs neither to the block of μ containing a nor to the block of μ containing b. $\mu \cup (a,b)$ is the same as μ except that the blocks of μ containing a and b are combined. Since s belongs to neither, s is equivalent to nothing more mod $\mu \cup (a,b)$ than mod μ. Hence $s \not\equiv t(\mu)$ implies $s \not\equiv t(\mu \cup (a,b))$. But $\pi(a,b) \subseteq \mu \cup (a,b)$ and so $s \not\equiv t(\pi(a,b))$ completing the proof.

$\mathfrak{H}$ is an equivalence structure. The most interesting of the three equivalence structure axioms is (3), which states that two classes which split one-another must coincide. In the next lemma, we make our first use of this property.

Lemma 23.14: Suppose that P and Q belong to the equivalence class $\mathfrak{H}'$ and that $P \trianglelefteq Q$ where we are ordering relative to (a,b).

(1) If $x \triangleleft$ (covering) $P \trianglelefteq Q \triangleleft$ (covering) y and $\mathfrak{H}' \wedge \mathfrak{C}(a,b) \leq \mathfrak{C}(P,Q)$, then $x \equiv y(\pi(a,b))$.

(2) If $P \triangleleft$ (covering) $x \trianglelefteq y \triangleleft$ (covering) Q and $\mathfrak{H}' \wedge \mathfrak{C}(P,Q) = \{P,Q\}$, then $x \equiv y(\pi(a,b))$.

Proof of (1): Suppose $x \not\equiv y(\pi(a,b))$. Then by 23.13 there exist $U_1 \sim U_2$ such that, say,

$$x \triangleleft U_1 \triangleleft y \triangleleft U_2.$$

In view of the coverings

$$P \trianglelefteq U_1 \trianglelefteq Q \triangleleft U_2.$$

Letting $\mathfrak{H}''$ denote the equivalence class containing U_1 and U_2, we see that $\mathfrak{H}'$ splits $\mathfrak{H}''$ and $\mathfrak{H}''$ splits $\mathfrak{H}'$. Then $\mathfrak{H}' = \mathfrak{H}''$ since $\mathfrak{H}$ is an equivalence structure. But then $U_2 \in \mathfrak{H}' \wedge \mathfrak{C}(a,b)$ but $U_2 \not\in \mathfrak{C}(P,Q)$ contrary to hypothesis. Therefore $x \equiv y(\pi(a,b))$.

Now suppose $x \not\equiv y(\pi(a,b))$ under the hypothesis of (2). Then by 23.13 there exists $U_1 \sim U_2$ such that, say,

$$x \lhd U_1 \lhd y \lhd U_2.$$

In view of the coverings

$$P \lhd U_1 \lhd Q \unlhd U_2.$$

Then the class $\mathfrak{H}''$ containing $\{U_1, U_2\}$ and the class $\mathfrak{H}'$ split one-another. Hence $\mathfrak{H}' = \mathfrak{H}''$. But then $U_1 \in \mathfrak{H}' \wedge \mathfrak{C}(P,Q)$ and $U_1 \neq P$ and $U_1 \neq Q$, a contradiction. Therefore $x \equiv y(\pi(a,b))$ and the lemma is proved.

Consider the elements of $C(a,b)$ and the sets of $\mathfrak{C}(a,b)$ under their natural ordering relative to (a,b). Let $\mathfrak{H}'$ be any equivalence class which is represented in $\mathfrak{C}(a,b)$. The preceding lemma then states that if we take the outermost sets P and Q of $\mathfrak{H}'$ in $\mathfrak{C}(a,b)$, then the elements x and y of $C(a,b)$ which are just external to P and Q are equivalent by $\pi(a,b)$. It may happen that $P = Q$, and this causes no difficulty. Also, if $\mathfrak{H}'$ contains at least two sets in $\mathfrak{C}(a,b)$, then it contains a distinct pair P' and Q' in $\mathfrak{C}(a,b)$ with the property that there are *no* sets in $\mathfrak{H}'$ which are *properly* between P' and Q'. The elements x' and y' of $C(a,b)$ which are just internal to P' and Q' are then equivalent by $\pi(a,b)$ according to the preceding lemma.

The following lemma establishes a property which can easily be shown to be necessary if Σ_0 is to be a characteristic set.

Lemma 23.15: $\pi(a,c) \subseteq \pi(a,b) \cup \pi(b,c)$ for all a, b, and c in S.

Proof: By theorem 15, the graph of the set $C(a,b) \vee C(b,c) \vee C(c,a)$ takes one of two forms. In order to clarify our notation we illustrate these graphs on the following page. We also designate the sets in $\mathfrak{C}(a,b) \vee \mathfrak{C}(b,c) \vee \mathfrak{C}(c,a)$ appropriately. For example

$$C(a,c) = \{a = a_0, a_1, \ldots, a_n = c\} \quad \text{and}$$
$$\mathfrak{C}(a,c) = \{A_1, A_2, \ldots, A_n\}.$$

We also set

$$\pi(a,c) = \pi_1,$$
$$\pi(a,b) = \pi_2, \text{ and}$$
$$\pi(b,c) = \pi_3.$$

We must show that $\pi_1 \subseteq \pi_2 \cup \pi_3$; in other words, that

$$x \equiv y(\pi_1) \quad \text{implies} \quad x \equiv y(\pi_2 \cup \pi_3).$$

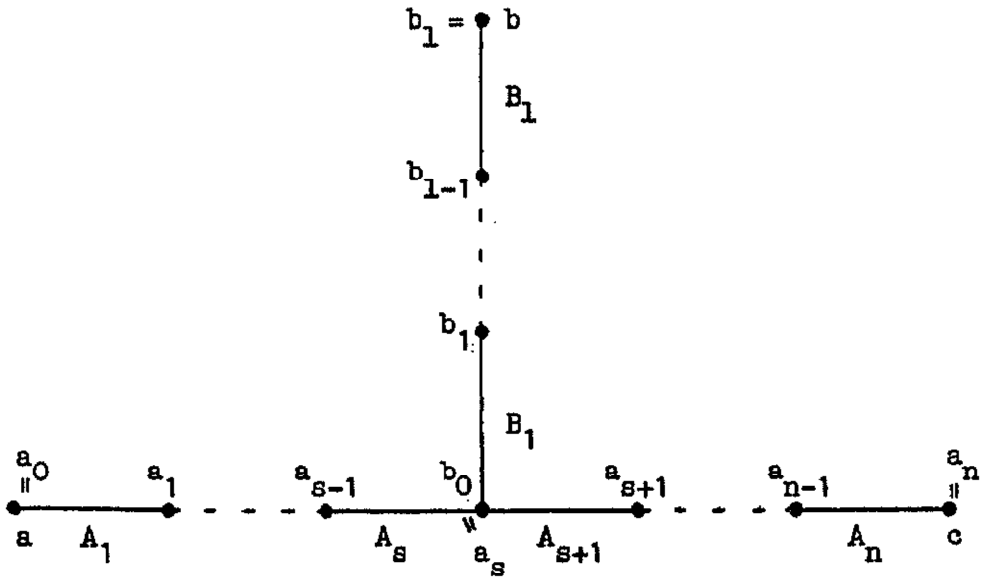
If $x \equiv y(\pi_1)$, then $x \equiv y(\lambda(a,c))$ by lemma 23.12. It follows that x and y are in $C(a,c)$. Thus x and y are of the form a_j and a_k where, say, $j \leq k$.

Suppose therefore that $a_j \equiv a_k(\pi_1)$ where $j \leq k$. We shall show that $a_j \equiv a_k(\pi_2 \cup \pi_2)$ by induction on $k \dot{-} j$, the result being trivial if $k - j = 0$. We distinguish several cases.

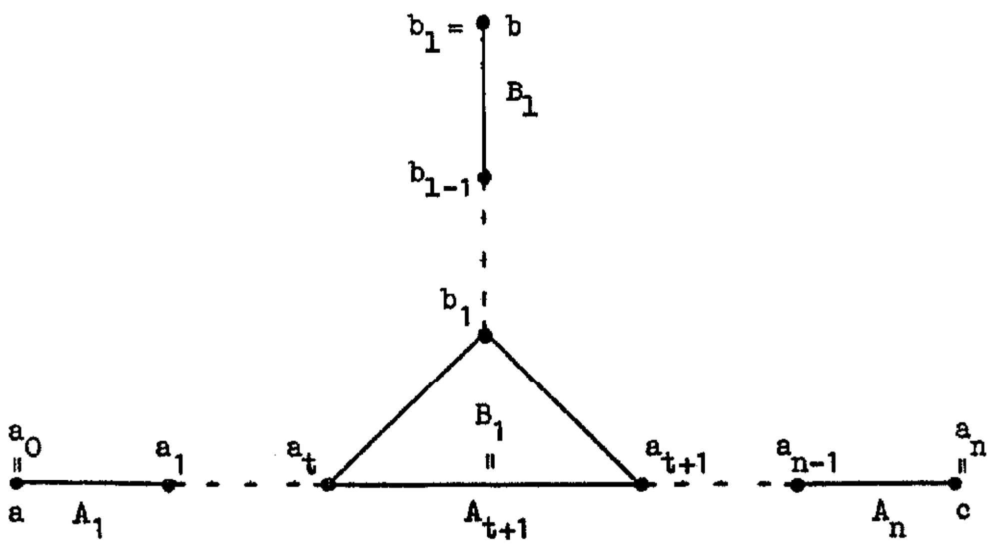
Case One: $j < s$ and $s < k$ in graph (I). By lemma 23.13, no A_i in $\mathfrak{C}(a_j,a_k)$ can be equivalent to an A_i outside $\mathfrak{C}(a_j,a_k)$ since otherwise $a_j \not\equiv a_k(\pi_1)$. $A_{j+1} \in \mathfrak{C}(a_j,a_k)$. Hence if we let r be maximal so that $A_{j+1} \sim A_r$, then 23.14 (1) is applicable and

$$a_j \equiv a_r(\pi_1).$$

(I)


$b_l = b$
B_l
b_{l-1}
b_1
B_1
a_0
a_1
a_{s-1}
b_0
a_{s+1}
a_{n-1}
a_n
a
A_1
A_s
a_s
A_{s+1}
A_n
c

(II)


$b_l = b$
B_l
b_{l-1}
b_1
B_1
a_0
a_1
a_t
a_{t+1}
a_{n-1}
a_n
a
A_1
A_{t+1}
A_n
c

This implies that $a_r \equiv a_k(\pi_1)$. If $r < k$, then $(r-j) < (k-j)$ and $(k-r) < (k-j)$. By induction

$$a_j \equiv a_r(\pi_2 \cup \pi_3) \quad \text{and} \quad a_r \equiv a_k(\pi_2 \cup \pi_3).$$

It follows that $a_j \equiv a_k(\pi_2 \cup \pi_3)$ if $r < k$. We therefore suppose that $r = k$; that is, that $A_{j+1} \sim A_k$. Now A_{j+1} is either equivalent to some B_i or it is not. Suppose first that it is. Let d be maximal so that $A_{j+1} \sim B_d$. Then

$$a_j \equiv b_d(\pi_2) \quad \text{and} \quad b_d \equiv a_k(\pi_3)$$

by 23.14 (1), and this implies that $a_j \equiv a_k(\pi_2 \cup \pi_3)$. Finally, suppose that no B_i is equivalent to A_{j+1}. Choose g maximally so that $g \leq s$ and $A_{j+1} \sim A_g$. Choose h minimally so that $s + 1 \leq h$ and $A_{j+1} \sim A_h$. Then by 23.14 (1)

$$a_j \equiv a_g(\pi_2) \quad \text{and} \quad a_{h-1} \equiv a_k(\pi_3).$$

By 23.14 (2), $a_g \equiv a_{h-1}(\pi_1)$. But clearly $[(h-1)-g] < (k-j)$. By induction

$$a_g \equiv a_{h-1}(\pi_2 \cup \pi_3).$$

Therefore, mod $\pi_2 \cup \pi_3$, we have

$$a_j \equiv a_g \equiv a_{h-1} \equiv a_k$$

completing the proof in case one.

<u>Case Two</u>: $j < k \leq s$ in graph (I). Again, we see that no A_i in $\mathfrak{C}(a_j,a_k)$ is equivalent to an A_i outside $\mathfrak{C}(a_j,a_k)$. We deal exactly as before with the case in which $A_{j+1} \not\sim A_k$. Suppose that $A_{j+1} \sim A_k$. A_{j+1} is either equivalent to some B_i or it is not. Suppose first that it is. Choose g maximally so that $A_{j+1} \sim B_g$ and choose h minimally so

that $A_{j+1} \sim B_h$. Then by 23.14 (1),

$$a_j \equiv b_g(\pi_2).$$

By 23.14 (2)

$$a_k \equiv b_{h-1}(\pi_2).$$

By 23.14 (1)

$$b_g \equiv b_{h-1}(\pi_3).$$

Hence, mod $\pi_2 \cup \pi_3$, we have

$$a_j \equiv b_g \equiv b_{h-1} \equiv a_k$$

completing the proof if A_{j+1} is equivalent to some B_i. Suppose, finally, that A_{j+1} is equivalent to no B_i. Then, by 23.14 (1), *a_j and a_k are equivalent mod π_2 and hence also mod $\pi_2 \cup \pi_3$, and this* completes the proof in case two.

<u>Remaining Cases</u>: The case in which $j \leq t$ and $t + 1 \leq k$ in graph (II) is treated as in case one. All other cases are treated as in case two. This completes the proof of the lemma.

The following lemmas will be needed in the proof that Σ_0 is a characteristic set.

<u>Lemma 23.16</u>: If x and y are equivalent mod $\pi(a,b)$, then $\pi(x,y) \subseteq \pi(a,b)$.

<u>Proof</u>: By definition $\pi(a,b) = \bigcap\{\mu \cup (a,b) : \mu \in \Sigma_0\}$. Hence $x \equiv y(\pi(a,b))$ means precisely that $(x,y) \subseteq \mu \cup (a,b)$ for every $\mu \in \Sigma_0$. But then $\mu \cup (x,y) \subseteq \mu \cup (a,b)$ for every μ and therefore $\pi(x,y) = \bigcap\{\mu \cup (x,y) : \mu \in \Sigma_0\} \subseteq \bigcap\{\mu \cup (a,b) : \mu \in \Sigma_0\} = \pi(a,b)$.

Lemmas 23.15 and 23.16 are special cases of the following lemma.

<u>Lemma 23.17</u>: Suppose $(x,y) \subseteq \bigcup\Lambda$ where Λ is a set of partitions of the form $\pi(a,b)$, then $\pi(x,y) \subset \bigcup\Lambda$.

<u>Proof</u>: If $(x,y) \subseteq \bigcup\Lambda$, then there exists a sequence of blocks of partitions in Λ such that

$$B_1 \between B_2 \between \cdots \between B_n$$

where $x \in B_1$ and $y \in B_n$. For each i, let π_i be a partition in Λ of which B_i is a block. Let $g_i \in B_i \wedge B_{i+1}$ for each i. Set $x = g_0$ and set $y = g_n$. Then for each i

$$g_{i-1} \equiv g_i(\pi_i).$$

By lemma 23.16, for each i

$$\pi(g_{i-1}, g_i) \subseteq \pi_i.$$

Hence

$$\begin{aligned} &\pi(g_0,g_1) \cup \pi(g_1,g_2) \cup \cdots \cup \pi(g_{n-1},g_n) \\ &\subseteq \pi_1 \cup \pi_2 \cup \cdots \cup \pi_n \\ &\subseteq \bigcup\Lambda. \end{aligned}$$

By lemma 23.15,

$$\pi(x,y) = \pi(g_0,g_n) \subseteq \pi(g_0,g_1) \cup \pi(g_1,g_2) \cup \cdots \cup \pi(g_{n-1},g_n).$$

Hence $\pi(x,y) \subseteq \bigcup\Lambda$ completing the proof.

We next determine the sublattice Γ generated by the set $\bar{\Sigma}_0 = \bigvee\{1/\sigma : \sigma \in \Sigma_0\}$.

<u>Lemma 23.18</u>: The complete sublattice Γ of Π generated by $\overline{\Sigma}_0 = \bigvee\{i/\sigma : \sigma \in \Sigma_0\}$ consists precisely of all unions of partitions of the form $\pi(a,b)$ where a and b are arbitrary elements of S.

<u>Proof</u>: Let Γ' denote the set of all partitions which are unions of partitions of the form $\pi(a,b)$. Since each $\pi(a,b)$ is a meet of elements of $\overline{\Sigma}_0$, we have

$$\Gamma' \leq \Gamma.$$

For each $\alpha \in \overline{\Sigma}_0$ there is a $\mu \in \Sigma_0$ such that $\mu \subseteq \alpha$. Then $(x,y) \subseteq \alpha$ implies that $(x,y) \subseteq \pi(x,y) \subseteq \mu \cup (x,y) \subseteq \alpha$. Hence

$$\alpha = \bigcup\{(x,y) : (x,y) \subseteq \alpha\} \subseteq \bigcup\{\pi(x,y) : (x,y) \subseteq \alpha\} \subseteq \alpha$$

and $\alpha = \bigcup\{\pi(x,y) : (x,y) \subseteq \alpha\} \in \Gamma'$. It follows that

$$\overline{\Sigma}_0 \leq \Gamma'.$$

Thus our proof will be complete when we have shown that Γ' is a complete sublattice of Π. Let Θ be an arbitrary subset of Γ'. It is obvious that $\bigcup\Theta \in \Gamma'$. By lemma 23.17, $(x,y) \subseteq \bigcap\Theta$ implies that $\pi(x,y) \subseteq \bigcap\Theta$. Therefore

$$\begin{aligned}\bigcap\Theta &= \bigcup\{(x,y) : (x,y) \subseteq \bigcap\Theta\} \\ &\subseteq \bigcup\{\pi(x,y) : (x,y) \subseteq \bigcap\Theta\} \\ &\subseteq \bigcap\Theta.\end{aligned}$$

Hence $\bigcap\Theta = \bigcup\{\pi(x,y) : (x,y) \subseteq \bigcap\Theta\} \in \Gamma'$. Thus $\Gamma' = \Gamma$, completing the proof.

We still have not shown that Σ_0 is a characteristic set. For this the following two lemmas will be used.

<u>Lemma 23.19</u>: If a and b are distinct elements of S, then there is at least one set T in $\mathfrak{C}(a,b)$ which is equivalent to no other set in $\mathfrak{C}(a,b)$.

<u>Proof</u>: Suppose that the lemma is false. Then every equivalence class represented in $\mathfrak{C}(a,b)$ is represented by at least two distinct sets. Let T and U be distinct equivalent sets in $\mathfrak{C}(a,b)$ chosen to minimize the number of sets in $\mathfrak{C}(T,U)$. Clearly $\mathfrak{C}(T,U) \leq \mathfrak{C}(a,b)$. By equivalence structure property (2), $T \wedge U = \emptyset$. Hence there is a set $V \in \mathfrak{C}(T,U)$ such that $V \neq T$ and $V \neq U$. $V \not\sim T$ by the minimality of the set $\mathfrak{C}(T,U)$. By assumption, V is equivalent to some set W in $\mathfrak{C}(a,b)$. $W \notin \mathfrak{C}(T,U)$ since otherwise $\mathfrak{C}(V,W) < \mathfrak{C}(T,U)$ where $V \sim W$ contrary to the assumed minimality of $\mathfrak{C}(T,U)$. Therefore, relative to (a,b) we have an ordering of the type

$$T \triangleleft V \triangleleft U \triangleleft W$$

where $T \sim U$, $V \sim W$, and $T \not\sim V$. This contradicts equivalence structure property (3) and proves the lemma.

<u>Lemma 23.20</u>: If a and b are distinct elements of S, then there exists a pair of elements s and t such that $s \mathfrak{Q} t$ and $s \equiv t(\pi(a,b))$.

<u>Proof</u>: By lemma 23.19, there is a $T \in \mathfrak{C}(a,b)$ such that T is equivalent to no other set in $\mathfrak{C}(a,b)$. The set $T \wedge C(a,b)$ is of the form $\{s,t\}$. By lemma 23.14 (1), we have $s \equiv t(\pi(a,b))$.

Next, we show that the set $\Sigma = \{(A) : A \in \mathfrak{Q}\}$ is the unique set corresponding to Γ by theorem 19.

Lemma 23.21: The set $\Sigma = \{(A) : A \in \mathfrak{S}\}$ is the unique set corresponding to Γ by theorem 19.

Proof: If $a\hat{\mathfrak{S}}b$, then clearly $\pi(a,b) = (a,b)$. Hence, by lemma 23.18, $(A)/0 \leq \Gamma$ for each $A \in \mathfrak{S}$. This verifies property (1) of theorem 19. Suppose now that $(a,b) \in \Gamma$. By lemma 23.18, $(a,b) = \pi(a,b)$. By lemma 23.20, there exist elements s and t such that $s\hat{\mathfrak{S}}t$ and $s \equiv t(\pi(a,b))$. It is immediate that $\{s,t\} = \{a,b\}$ and hence that $(a,b) \subseteq$ some (A) where $A \in \mathfrak{S}$. This verifies property (3) of theorem 19. Property (2) follows from the fact that $\mathfrak{S}$ is a structure. This completes the proof.

The proof of theorem 23 is essentially complete with the following lemma.

Lemma 23.22: Σ_0 is the characteristic set corresponding to Γ.

Proof: The fact that $\mathfrak{S}$ is connected implies that $\bigcup\{(A) : A \in \mathfrak{S}\} = i$. By the previous lemma, $(A)/0 \leq \Gamma$ for each $A \in \mathfrak{S}$. It follows that $\bigcup(\Gamma \wedge \Omega) = i$. Hence, letting Δ_0 denote the characteristic set of Γ, the lemmas of part I of the proof of theorem 23 are applicable.

By lemmas 23.21 and 23.4, every partition in Δ_0 is a meet of partitions of the form $(A)^*$ where $A \in \mathfrak{S}$. By definition, each partition in Σ_0 is also a meet of partitions $(A)^*$ where $A \in \mathfrak{S}$. Since Γ contains each ideal i/μ where $\mu \in \Sigma_0$ by definition, it follows from theorem 18 that each partition in Δ_0 is a meet of partitions in Σ_0.

Hence for the equivalence structure $(\mathfrak{H},\approx)$ corresponding to Δ_0 by the first part of the proof, we have

$$A \sim B \quad \text{implies} \quad A \approx B$$

for all A and B in $\mathfrak{H}$. Now let

$$\pi'(a,b) = \bigcap\{\mu \cup (a,b) : \mu \in \Delta_0\}$$

for all a and b in S. Since Γ is also the complete sublattice corresponding to the equivalence structure $(\mathfrak{H},\approx)$, everything we have proved about $\pi(a,b)$ also holds for $\pi'(a,b)$. In particular, $\pi'(a,b) \subseteq \pi(a,b)$ since $(a,b) \subseteq \pi(a,b)$, by lemmas 23.17 and 23.18. Similarly $\pi(a,b) \subseteq \pi'(a,b)$ and hence

$$\pi(a,b) = \pi'(a,b)$$

for all a and b in S.

Now to show that $\Sigma_0 = \Delta_0$, it will clearly suffice to show that

$$A \approx B \quad \text{implies} \quad A \sim B$$

since then the sets Σ_0 and Δ_0 are obtained in the same way from the same equivalence structure. Suppose the contrary, that

$$A \not\sim B \quad \text{and} \quad A \approx B.$$

By theorem 14, there exist elements $a \in A$ and $b \in B$ such that $\{A,B\} \leq \mathfrak{C}(a,b)$. Let

$$C(a,b) = \{a = g_0, g_1, \ldots, g_n = b\}$$

and let

$$\mathfrak{C}(a,b) = \{A = G_1, G_2, \ldots, G_n = B\}$$

where G_i is the set in $\mathfrak{Q}$ containing $\{g_{i-1}, g_i\}$. Since $A \approx B$, it follows from 23.13 that

$$a \not\equiv g_i(\pi'(a,b)) \quad \text{for } i = 1,2,\ldots,n-1.$$

Now let k be maximal so that $A \sim G_k$. $k < n$ since $A \not\sim B$. Hence, by 23.14 (1)

$$a \equiv g_k(\pi(a,b)) \text{ where } k \in \{1,2,\ldots,n-1\}.$$

We have thus obtained a contradiction to the fact that $\pi(a,b) = \pi'(a,b)$ for all a and b in S. Thus $A \approx B$ always implies that $A \sim B$. Therefore the equivalence structures $(\mathfrak{Q},\sim)$ and $(\mathfrak{Q},\approx)$ coincide, and hence $\Sigma_0 = \Delta_0$ completing the proof of the lemma.

The preceding lemma shows that Σ_0 is a characteristic set. Also contained in the proof is the fact that any two equivalence structures $(\mathfrak{Q},\sim)$ and $(\mathfrak{Q},\approx)$ corresponding to the characteristic set Σ_0 must coincide. The proof of theorem 23 is therefore complete.

CHAPTER V: PROPERTIES OF SUBLATTICES Γ WHERE $\bigcup(\Gamma \wedge \Omega) = i$.

In this final chapter we obtain two properties of sublattices Γ where $\bigcup(\Gamma \wedge \Omega) = i$. First, we show that such sublattices are always dense (defined below) in Π. Second, we show that any pair of partitions of the form $\pi(a,b)$ and $\pi(c,d)$ are a modular pair. The latter result implies that any two union irreducibles in Γ are a modular pair because of lemma 23.18.

Definition 23: The sublattice Γ of the lattice Π is called a dense sublattice of Π if every covering in Γ is also a covering in Π.

It is clear that a dense sublattice of a semi-modular lattice is always semi-modular.

Theorem 24: If Γ is a complete sublattice of Π such that $\bigcup(\Gamma \wedge \Omega) = i$, then Γ is dense in Π.

Proof: Suppose α covers β in Γ. We may suppose that $\beta = 0$ since otherwise we consider the sublattice $\Gamma \wedge (i/\beta)$ of the partition lattice i/β. Every element of Γ is a union of partitions of the form $\pi(a,b)$. Hence there is a partition $\pi(a,b)$ such that $\alpha \supseteq \pi(a,b) \supset 0$. By lemma 23.20, there is a point $(s,t) \in \Gamma$ such that $\pi(a,b) \supseteq (s,t) \supset 0$. Since α covers 0, it follows that $\alpha = (s,t)$ and hence that α covers 0 in Π. This completes the proof.

The following property is a trivial consequence of lemmas 23.17 and 23.18.

Lemma 25.1: Let $\alpha \in \Gamma$ where $\bigcup(\Gamma \wedge \Omega) = i$. Then $(x,y) \subseteq \alpha$ implies that $\pi(x,y) \subseteq \alpha$.

It follows from theorem 24 that Γ is semi-modular. We mention also that Γ is compactly generated. For each element of Γ is a union of elements of the form $\pi(a,b)$. $(a,b) \subseteq i = \bigcup(\Gamma \wedge \Omega)$ implies that there is a finite subset Λ of $\Gamma \wedge \Omega$ such that $(a,b) \subseteq \bigcup\Lambda$. But $\bigcup\Lambda \in \Gamma$. By lemma 25.1, we have $\pi(a,b) \subseteq \bigcup\Lambda$. It follows that $\pi(a,b)$ is the union of a finite number of points in Π. Hence $\pi(a,b)$ is compact in Π, and hence also in Γ. This proves compact generation.

The next theorem is of interest not only for its content, but also for its rather curious proof.

Theorem 25: Any two elements $\pi(a,b)$ and $\pi(c,d)$ in Γ form a modular pair.

Proof: Set $\pi(a,b) = \pi_1$ and set $\pi(c,d) = \pi_2$. Let blocks of π_1 be denoted by A_i, and let blocks of π_2 be denoted by B_i. We lose no generality in assuming that

$$\pi_1 \cap \pi_2 = 0.$$

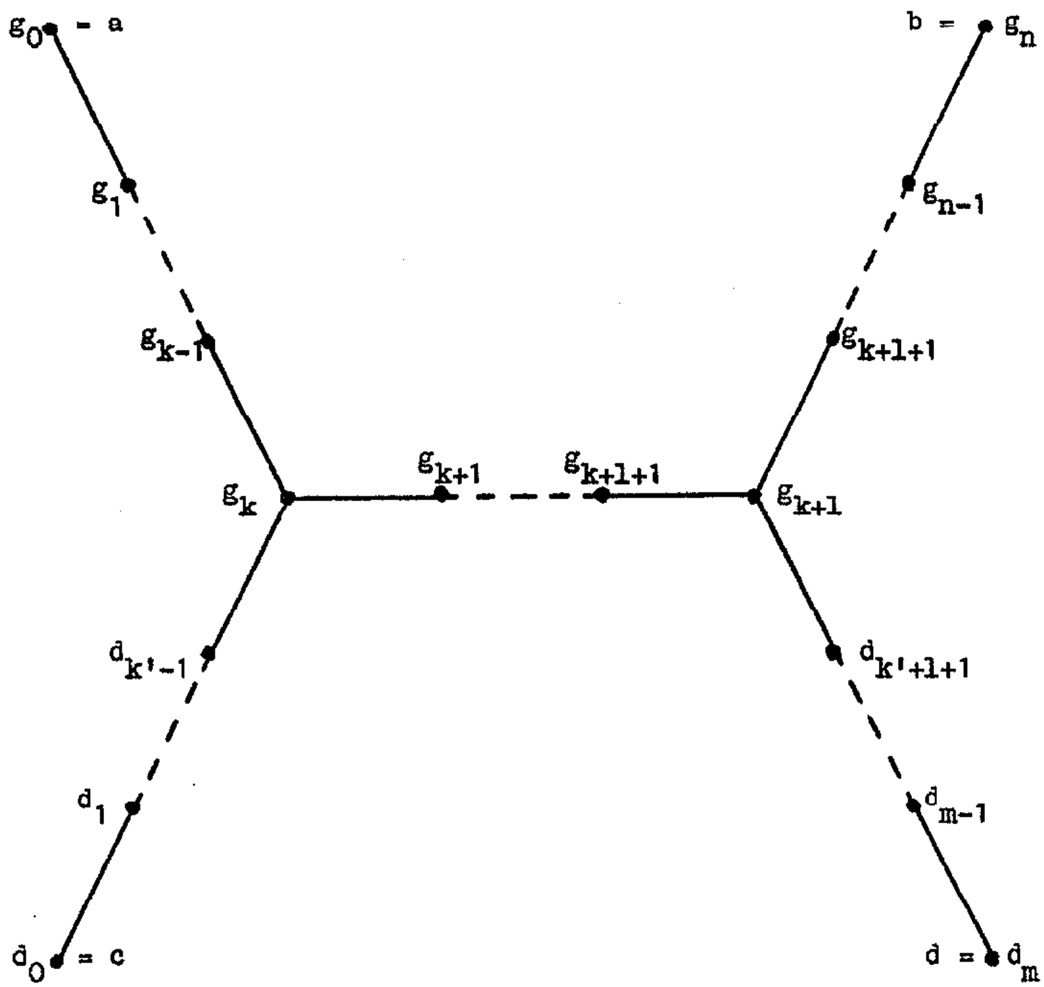
We shall prove that $\{\pi_1,\pi_2\}$ is strongly independent. Criteria are given in theorems 7 and 8. By lemma 23.12,

$$\pi_1 \subseteq \lambda(a,b) \quad \text{and} \quad \pi_2 \subseteq \lambda(c,d).$$

Hence every block A_i of π_1 is contained in $C(a,b)$ and every block B_i of π_2 is contained in $C(c,d)$. The strong independence of $\{\pi_1,\pi_2\}$ is obvious if $C(a,b) \wedge C(c,d) = \emptyset$. Suppose that

$$C(a,b) \wedge C(c,d) \neq \emptyset.$$

From theorem 15 it is clear that the graph of $C(a,b) \vee C(c,d)$ has the following form.



g_{k-1} and $d_{k'-1}$ may or may not be joined. Similarly for g_{k+l+1} and $d_{k'+l+1}$. These possibilities do not affect our proof. Set

$$C = C(a,b) \wedge C(c,d) = (g_k, g_{k+1}, \ldots, g_{k+l}).$$

We shall need the following lemma.

<u>Lemma 25.2</u>: Suppose that

$$A_1 \wedge C(g_i,g_j) \neq \emptyset \text{ where}$$
$$\{g_i,g_j\} \leq C - A_1 \quad \text{and}$$
$$g_i \equiv g_j(\pi_1).$$

Then $A_1 \wedge C(g_i,g_j)$ is an entire block both of π_1 and of π_2.

<u>Proof of lemma</u>: Set

$$A_1' = A_1 \wedge C(g_i,g_j),$$
$$\sigma = \pi_1 \cap \lambda(g_i,g_j), \text{ and}$$
$$\tau = \lambda(c,g_i) \cup \sigma \cup \lambda(g_j,d).$$

σ and τ are in Γ. A_1' is clearly a block of σ and, also, of τ. Also $(c,d) \subseteq \tau$. By lemma 25.1,

$$\pi_2 \subseteq \tau.$$

Hence A_1' contains some block B_p of π_2. Now set

$$\nu = \lambda(c,g_i) \cup \pi_2 \cup \lambda(g_i,d) \text{ and}$$
$$\mu = \lambda(a,g_i) \cup \nu \cup \lambda(g_j,b).$$

Both ν and μ are in Γ. Noting that $B_p \leq A_1' \leq \{g_{i+1},g_{i+2},\ldots,g_{j-1}\}$, we see that B_p is a block of ν and, also, of μ. We easily see that $(a,b) \subseteq \mu$. By lemma 25.1,

$$\pi_1 \subseteq \mu.$$

Hence B_p contains some block A_q of π_1. Therefore

$$A_1 \geq A_1' \geq B_p \geq A_q$$

which implies that $A_1 = A_q$ and that

$$A_1 \wedge C(g_i,g_j) = A_1' = A_1 = B_p$$

completing the proof of the lemma.

Proof of theorem: By theorems 7 and 8, we see that if the set $\{\pi_1,\pi_2\}$ is not strongly independent, then there is a sequence of blocks

$$(1) \qquad A_1,B_1,A_2,B_2,\ldots,A_r,B_r,A_1 \quad (r \geq 2)$$

such that adjacent blocks and only adjacent blocks overlap. Each adjacent intersection is a singleton since $\pi_1 \cap \pi_2 = 0$. Thus each block A_i in (1) "shares" exactly two elements $\{a_i,a_i'\}$ with other blocks in (1). Define $A_i' = C(a_i,a_i')$. Define B_j' analogously. Suppose that $x \in A_i'$ but $x \notin A_i$. Then by the preceding lemma, x belongs to a set which is a block of both π_1 and π_2. But then x can belong to no set in (1). It follows that the A_i' form disjoint subintervals of the "interval"

$$(2) \qquad g_k,g_{k+1},\ldots,g_{k+1}.$$

Similarly for the B_i'. However, since the crucial pairs have been preserved, each A_i' (resp. B_i') overlaps two distinct B_j' (resp. A_j'). We complete our proof by showing this is impossible. Let us change subscripts so that

$i < j$ implies that A_i' is to the left of A_j' in (2)

and

$i < j$ implies that B_i' is to the left of B_j' in (2).

A_1' overlaps some B_s' with $s > 1$ since it overlaps two distinct B_i'. We claim that A_t' cannot overlap B_1' if $t > 1$. This is because A_t' lies to the right of A'_1. Hence A'_t cannot overlap any B'_i to the left of B'_s; in particular, A'_t cannot overlap B'_1. But then B'_1 can overlap at most the single set A'_1. This gives the desired contradiction and proves the theorem.

Corollary 25.3: Let Γ be a complete sublattice of Π such that $\bigcup(\Gamma \wedge \Omega) = i$. Let α and β be union irreducibles in Γ. Then α and β are a modular pair.

Proof: By lemma 23.18, any element of Γ is a union of partitions of the form $\pi(x,y)$. Hence $\alpha = \pi(a,b)$ and $\beta = \pi(c,d)$ for suitable elements a, b, c, and d in S. Theorem 25 is now applicable.

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