ON THE VISCOUS HYPERSONIC BLUNT-BODY PROBLEM

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ABSTRACT

The viscous hypersonic flow past an axisymmetric blunt-body is analyzed based upon the Navier-Stokes equations for a perfect gas having constant specific heats, a constant Prandtl number, P, whose numerical value is of order one, and a viscosity varying as a power, ω, of the absolute temperature, as the free-stream Mach number, M, and the free-stream Reynolds number based on the body nose radius, R, go to infinity, and $\epsilon = [(\gamma-1)/(\gamma+1)]$ (where γ is the ratio of the specific heats) and $\delta = [1/(\gamma-1)M^2]$ go to zero.

Through the use of strict asymptotic expansions, the behavior of the flow in the three distinct regions that comprise the interior of the "shock structure" is found, as well as for the one, two, or three regions that make up the "shock layer" depending on whether the quantity $R\delta^{\omega}$ is equal to $O(1/\epsilon)$, equal to $O(1/\epsilon^{5/2})$, or greater than $O(1/\epsilon^{5/2})$, respectively.

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I. INTRODUCTION

This paper is an analysis of the viscous flow past an axisymmetric blunt body in a steady supersonic uniform stream (as illustrated in Figures 1 and 2), with special emphasis on the flow in the stagnation region. The analysis is based upon the Navier-Stokes equations for a perfect gas with constant specific heats, a constant Prandtl number, P, whose numerical value is of order one, and the viscosity varying as a power, ω, of the absolute temperature, when the free-stream Mach number, $M = U_\infty / \sqrt{\gamma \mathcal{R} T_\infty}$, and free-stream Reynolds number, $R = \rho_\infty U_\infty a / \mu_\infty$, go to infinity and the Newtonian parameter, $\epsilon = (\gamma - 1)/(\gamma + 1)$, and $\delta = (1 - \epsilon)/2\epsilon \; M^2$ go to zero.

This paper is the second by the author on the subject of the solution of viscous hypersonic flow problems by means of singular perturbation techniques. The first (BUSH, [1]) deals with the structure of a one-dimensional steady shock wave when $M \to \infty$ for the same conditions as prescribed in the above paragraph except that ε was fixed, P was 3/4, and the viscosity obeyed the Sutherland law. Further, the ideas of this previous analysis can be carried over intact to obtain the structure of the detached shock wave that is supported by an axisymmetric blunt body in a steady supersonic uniform stream when M and $R \to \infty$ such that $M^{2\omega}/R \to 0$ for the same conditions except that ε is fixed and P is 3/4. For this limit, the flow in the shock layer, the region between the shock wave itself and the body, whose thickness is a sizable fraction of the body nose radius, a, must be found by solving the full inviscid equations of motion on a computer (cf., e.g., VAN DYKE, [2]). In this paper, since $\epsilon \to 0$, the shock layer is thin (i.e., the shock layer thickness divided by the nose radius is $O(\epsilon) \to 0$), so that, in addition to the flow in the shock itself, the flow in the shock layer, at least in part, is determined analytically. As far as its treatment of the shock layer is concerned, this paper, then, is an amplification on the work done by CHENG, [3].

In the hypersonic blunt-body shock structure problem it was found that, as M and $R \to \infty$ with ϵ fixed, there are two regions to the shock structure where the behavior of the flow quantities is described by two distinct sets of asymptotic expansions. The first of these two regions is the very thin outer region, whose ratio of thickness to body nose radius is $O(1/R) \to 0$. The orders of magnitude of the flow quantities in this region are those for the free-stream. The second region is the relatively thicker inner, or principal, region. The thickness ratio of this region is $O(M^{2\omega}/R) \to 0$. The velocity components and the density here are of the same order of magnitude as in the free-stream, but the temperature and pressure in this region divided by their free-stream values are $O(M^2) \to \infty$. These two sets of asymptotic expansions for the shock structure are shown to be the correct ones by proving that the expansions for the outer and inner regions and the expansions for the inner region and the shock layer match in intermediate regions of common validity.

For the problem posed in this paper, it is found that with $\epsilon \to 0$ there are not two but three regions to the shock structure and, hence, three distinct sets of asymptotic expansions are needed to describe the behavior of the flow quantities in the shock structure. The first of these three regions is the outer region. It is essentially the same as the outer region in the ϵ -fixed problem in that the region's thickness is $O(1/R) \to 0$ and the orders of magnitude of the flow quantities are the same as in the free-stream. There must be two shock structure regions interior to the outer region in the $\epsilon \to 0$ problem rather than just one, as in ϵ -fixed problem, because, while there is no single distinguished region whose set of asymptotic expansions will match to both the expansions of the outer region and those of the shock layer as $\epsilon \to 0$, there are two distinguished regions, called in this paper the middle and inner regions, whose sets of expansions permit complete matching (i. e., outer region - middle region, middle region - inner region, and inner region - shock layer matching) in the limit as $\epsilon \to 0$. The middle region has a thickness ratio, $\{d/a\}$, which is $O(1/R\delta^{\omega}) \to 0$, and in this region the velocity component ratios, $\{u/U_\infty\}$ and $\{v/U_\infty\}$, and the density ratio, $\{\rho/\rho_\infty\}$, are all O(1), but the temperature and pressure ratios, $\{T/T_\infty\}$ and $\{p/p_\infty\}$, are $O(1/\delta) \to \infty$. For the inner region, $\{d/a\}$, the thickness ratio is $O(\epsilon / R\delta^{\omega}) \to 0$, and, as to the flow quantities in this region, $\{u/U_{\infty}\}$ is $O(1)$, $\{v/U_{\infty}\}$ is $O(\epsilon) \to 0$, while $\{\rho/\rho_{\infty}\}$ is $O(1/\epsilon) \to \infty$, $\{T/T_{\infty}\}$ is $O(1/\delta) \to \infty$ and $\{p/p_{\infty}\}$ is $O(1/\epsilon\,\delta) \to \infty$.

It should be pointed out that, in solving for the flows in the different regions of the shock structure, there are quantities in each region which cannot be found until the solution of the flow in the region just interior to the given region is known. This means that the flow in the shock structure is not completely known until the flow in the shock layer itself is known.

The shock layer has a thickness ratio that is $O(\epsilon) \to 0$ and the magnitudes of the flow quantities in this layer are the same as those in the inner region of the shock structure. In the shock layer equations of motion, the ratio of the viscosity and heat conduction contributions to the inviscid contributions is K, which is $O(1/\epsilon\, R\delta^{\omega})$. Thus, if $K \to 0$, then the shock layer is an inviscid one, where the (inviscid) Rankine-Hugoniot shock relations as $M \to \infty$ and $\epsilon \to 0$ ($\epsilon M^2 \to \infty$) are the proper boundary conditions at the outer edge of the layer (cf., CHESTER, [4] and FREEMAN, [5]). On the other hand, if K is O(1) (since the third alternative of $K \to \infty$ is ruled out as not being physically realistic), the shock layer is a viscous shock layer and the matching of this viscous shock layer with the inner region of the shock structure shows that the proper boundary conditions at the outer edge of the viscous shock layer are not the Rankine-Hugoniot shock relations but, rather, are the ones given by CHENG, [3], in which the heat conduction and viscosity terms right behind the shock are important. In the terminology of HAYES and PROBSTEIN, [7], this viscous flow regime just described is the "incipient merged layer" regime. It cannot be emphasized too strongly that the inviscid shock layer equations must be solved using the inviscid outer edge boundary conditions, and the viscous shock layer equations must be solved using the viscous outer edge boundary conditions. This rules out the "viscous layer" regime, introduced in Ref. [7], in which the viscous shock layer equations are solved subject to the Rankine-Hugoniot relations at the outer edge.

It should be noted that, since the ratio of the shock wave thickness to the shock layer thickness is O(K), the shock wave is much thinner than the inviscid shock layer but has the same order of magnitude of thickness as the viscous shock layer.

The complete solution for the flow in an inviscid shock layer is already known; it was found by CHESTER, [4], and FREEMAN, [5], in terms of modified von Mises variables. The complete solution is also presented in this paper, but in terms of modified Crocco variables, because these variables are found to be the most suitable for treating the shock layer and the sub-regions of the shock layer.

The complete solution for the flow in a viscous shock layer is another story, however. Due to the complexity of the partial differential equations of motion for such a layer, their solution can be brought about only after a large amount of time on a computer (and this was not felt to be advisable at this time). However, due to the geometrical symmetry of the problem and the fact that the partial differential equations are parabolic in type, the flow in the vicinity of the axis of symmetry is found by solving a set of ordinary differential equations in modified Crocco variables. For the special case of $\omega = 1$, these ordinary differential equations uncouple to such an extent that their solutions can be found in terms of tabulated functions, as CHENG, [3], was the first to show. For a general value of ω (e.g., $\omega = \frac{1}{2}$ is the value predicted by kinetic theory for hard sphere molecules), one must resort again to the computer. The results of such a computer solution are presented for $\omega = \frac{1}{2}$, $P = 3/4$, and a wall temperature that is zero. In addition, an approximate method, which turns out to be quite accurate, is also presented.

When the inviscid shock layer is solved (with $\{u/U_\infty\} = O(1)$), it is found that the solution is non-uniform at the body surface. To remove this non-uniformity it is necessary to introduce a correction sublayer imbedded in this inviscid shock layer in which the order of magnitude of u is some small fraction of U_∞. The proper correction layer is the one for which the thickness ratio is $O(\epsilon^{3/2}) \to 0$, the velocity components, $\{u/U_\infty\}$ and $\{v/U_\infty\}$, are $O(\epsilon^{\frac{1}{2}})$ and $O(\epsilon^2)$, respectively, and $\{\rho/\rho_\infty\}$ is $O(1/\epsilon) \to \infty$, $\{T/T_\infty\}$ is $O(1/\delta) \to \infty$, and $\{p/p_\infty\}$ is $O(1/\epsilon\,\delta) \to \infty$. The existence of such a correction layer is verified by showing that this layer matches with the inviscid shock layer. The ratio of the viscosity and heat conduction contributions to the inviscid contributions in this correction layer is D, a quantity that is $O(\epsilon^{5/2} R \delta^{\omega^{-1}})$. Therefore, if $D \to 0$, this correction layer is inviscid, but if $D = O(1)$, then the correction layer is viscous. In the terminology of HAYES and PROBSTEIN, [7], this viscous correction layer is the "vorticity interaction" layer. The reason for this designation is the following: the equations for the viscous correction layer are the same as those for the classical boundary layer, but the boundary conditions differ in that, at the outer edge of the viscous correction layer, the vorticity (or shear), which is a vanishing quantity in the boundary layer, is a nonvanishing quantity.

The complete solution for the flow in the inviscid correction layer is presented in terms of the proper modified Crocco variables. Again, invoking the geometrical symmetry and parabolicity of the partial differential equations arguments, the ordinary differential equations and proper boundary conditions for the flow in the vicinity of the axis of symmetry for the viscous correction layer are derived. No exact solution to these ordinary differential equations is presented, although the expansions of the solution for small and large values of the independent variable (i. e. expansions for near the wall and near the outer edge of the layer) are given.

The classical boundary layer that is imbedded within the inviscid correction layer is the last region discussed. For such a boundary layer the thickness ratio is $O(\sqrt{\epsilon^{\frac{1}{2}}/R\delta^{\omega}}) \to 0$, and the flow quantities are $\{u/U_{\infty}\} = O(\epsilon^{\frac{1}{2}}) \to 0$, $\{v/U_{\infty}\} = O(\sqrt{\epsilon^{3/2}/R\delta^{\omega}}) \to 0$, and $\{\rho/\rho_{\infty}\} = O(1/\epsilon) \to \infty$, $\{T/T_{\infty}\} = O(1/\delta) \to \infty$, and $\{p/p_{\infty}\} = O(1/\epsilon\,\delta) \to \infty$. The stagnation line ordinary differential equations are derived, but again no exact solution of the equations is presented, since solutions of the hypersonic boundary layer equations have been covered quite thoroughly in the literature.

The orders of magnitude of the shear and heat conduction at the nose of the body in the viscous shock layer, vorticity interaction layer, and the boundary layer are examined. In the viscous shock layer and the vorticity interaction layer the shear is of the same magnitude but less than the shear in the boundary layer. As far as the heat conduction is concerned, it is least in the viscous shock layer and most in the boundary layer and between these two extremes in the vorticity interaction layer.

Since there is some doubt as to the range of applicability of the Navier-Stokes (N-S) equations, especially when it comes to describing the structure of the shock wave, there is some question as to the meaningfulness of the answers obtained by an analysis such as the one performed in this paper. A sufficient (but not a necessary) condition that the N-S approximation be valid, is that the ratio of the Burnett to N-S terms in the Chapman-Enskog expansion of the Boltzmann equation should go to zero. This test is applied to the results of the analysis and it is found that all three regions of the shock structure and the viscous shock layer do not satisfy the condition. The outer region fails to satisfy the condition due to $M \to \infty$ while the middle and inner regions and viscous shock layer fail to meet the condition due to $\epsilon \to 0$. However, LIEPMANN, NARASIMHA, and CHAHINE, [9], solving the Bhatnagar, Gross, and Krook (B-G-K) model for the Boltzmann equation for the plane shock structure (ϵ fixed), have shown that this method reproduces the N-S solution in the downstream region of the shock which includes the middle and inner regions introduced in this paper but tends to a different solution in the upstream (or outer) region. Therefore, since the Burnett to N-S ratio going to zero is only a sufficient condition and the Burnett equations are, themselves, open to some doubt, and since the B-G-K kinetic theory results for a similar problem indicate that the N-S equations may be valid over a wider range than was, at first, thought, it is felt that the results presented in this paper may, indeed, be meaningful.

II. EQUATIONS OF MOTION

Consider an axisymmetric three-dimensional body as shown in Figure 2. The orthogonal curvilinear coordinates, or "boundary layer" coordinates, with the body surface as the reference surface, are: x = the distance along the surface of the body measured from the forward stagnation point, y = the distance normal to the body surface, and ϕ = the azimuthal, or circumferential, angle. The length element is

$$d\vec{\ell} = \vec{e}_x [\{1+\kappa(x)y\} dx] + \vec{e}_y [dy] + \vec{e}_\phi [\{B(x)+y \cos\Phi(x)\} d\phi] , \quad (2.01)$$

where $\kappa(x)$ = the curvature of the body surface in the meridian plane, positive for a convex body, $B(x)$ = the body meridian radius, i.e., the distance between a point on the body surface and the axis of symmetry, and $\Phi(x)$ = the angle between the body surface and the free stream.

However, since, in this analysis, four distinct regions are considered, each region farther away from the body than its predecessor, and it is necessary to discuss each region separately, it is convenient to introduce the coordinate transformation:

$$\xi = (x/a) = \bar{x}, \quad \eta = (y/a) - \bar{Y}(x/a) = \bar{y} - \bar{Y}(\bar{x}) = \bar{y} - \bar{Y}(\xi) , \quad (2.02a)$$

$$\frac{\partial}{\partial x} = \frac{1}{a} \left[\frac{\partial}{\partial \xi} - \bar{Y}'(\xi) \frac{\partial}{\partial \eta} \right] , \quad \frac{\partial}{\partial y} = \frac{1}{a} \frac{\partial}{\partial \eta} , \quad (2.02b)$$

where a = the nose radius of the body, and $\overline{Y}(\xi)$ = the non-dimensional "measure" of the thickness of the regions between the body and the region under discussion. At the same time, it is useful to introduce the non-dimensional curvature of the body surface and body meridian radius, defined by:

$$\kappa(x) = \overline{\kappa}(\xi)/a \quad ; \quad B(x) = a\overline{B}(\xi). \tag{2.03}$$

If the velocity, pressure, density and temperature are

$$\vec{q} = \vec{e}_x\, U_\infty \overline{u} + \vec{e}_y\, U_\infty \overline{v},\ p = p_\infty \overline{p},\ \rho = \rho_\infty \overline{\rho},\ T = T_\infty \overline{T}\,, \tag{2.04}$$

where U_∞, p_∞, ρ_∞, T_∞ = the free stream speed, pressure, density, and temperature, and $\overline{u}$, $\overline{v}$, $\overline{p}$, $\overline{\rho}$, $\overline{T}$ = the corresponding non-dimensional velocity components, pressure, density, and temperature, then the non-dimensional equations of motion in the coordinate system described may be written. The continuity equation is

$$\frac{\partial(\overline{\rho}\overline{v})}{\partial\eta} + \frac{1}{h}\frac{\partial(\overline{\rho}\overline{u})}{\partial\overline{x}} + \frac{\overline{\kappa}\overline{\rho}\overline{v}}{h} + \frac{\overline{\rho}(\overline{u}\sin\Phi + \overline{v}\cos\Phi)}{\overline{r}} = 0\,. \tag{2.05}$$

The tangential Navier-Stokes momentum equation is

$$\bar{\rho}\left(\frac{\bar{u}}{h}\frac{\partial\bar{u}}{\partial\bar{x}}+\bar{v}\frac{\partial\bar{u}}{\partial\eta}+\frac{\bar{\kappa}\bar{u}\bar{v}}{h}\right)+\frac{1-\epsilon}{1+\epsilon}\frac{1}{M^2}\frac{1}{h}\frac{\partial\bar{p}}{\partial\bar{x}}$$

$$=\frac{1}{R}\left[\!\!\left[\left(\frac{\partial}{\partial\eta}+\frac{2\bar{\kappa}}{h}+\frac{\cos\Phi}{\bar{r}}\right)\left(\bar{\mu}\left[\frac{\partial\bar{u}}{\partial\eta}-\frac{\bar{\kappa}\bar{u}}{h}+\frac{1}{h}\frac{\partial\bar{v}}{\partial\bar{x}}\right]\right)\right.\right.$$

$$+\frac{2}{3}\frac{1}{h}\frac{\partial}{\partial\bar{x}}\left(\bar{\mu}\left[\frac{2}{h}\left\{\frac{\partial\bar{u}}{\partial\bar{x}}-\bar{\kappa}\bar{v}\right\}-\frac{\partial\bar{v}}{\partial\eta}-\frac{\bar{u}\sin\Phi-\bar{v}\cos\Phi}{\bar{r}}\right]\right)$$

$$\left.\left.+\frac{2\sin\Phi}{\bar{r}}\left(\bar{\mu}\left[\frac{1}{h}\left\{\frac{\partial\bar{u}}{\partial\bar{x}}+\bar{\kappa}\bar{v}\right\}-\frac{\bar{u}\sin\Phi+\bar{v}\cos\Phi}{\bar{r}}\right]\right)\right]\!\!\right]. \qquad (2.06)$$

The normal Navier-Stokes momentum equation is

$$\bar{\rho}\left(\frac{\bar{u}}{h}\frac{\partial\bar{v}}{\partial\bar{x}}+\bar{v}\frac{\partial\bar{v}}{\partial\eta}-\frac{\bar{\kappa}\bar{u}^2}{h}\right)+\frac{1-\epsilon}{1+\epsilon}\frac{1}{M^2}\frac{\partial\bar{p}}{\partial\eta}$$

$$=\frac{1}{R}\left[\!\!\left[\left(\frac{4}{3}\frac{\partial}{\partial\eta}+\frac{2\bar{\kappa}}{h}+\frac{2\cos\Phi}{\bar{r}}\right)\left(\bar{\mu}\frac{\partial\bar{v}}{\partial\eta}\right)\right.\right.$$

$$+\left(\frac{1}{h}\frac{\partial}{\partial\bar{x}}+\frac{\sin\Phi}{\bar{r}}\right)\left(\bar{\mu}\left[\frac{\partial\bar{u}}{\partial\eta}-\frac{\bar{\kappa}\bar{u}}{h}+\frac{1}{h}\frac{\partial\bar{v}}{\partial\bar{x}}\right]\right)$$

$$-\left(\frac{2}{3}\frac{\partial}{\partial\eta}+\frac{2\bar{\kappa}}{h}\right)\left(\bar{\mu}\left[\frac{1}{h}\left\{\frac{\partial\bar{u}}{\partial\bar{x}}+\bar{\kappa}\bar{v}\right\}\right]\right)$$

$$\left.\left.-\left(\frac{2}{3}\frac{\partial}{\partial\eta}+\frac{2\cos\Phi}{\bar{r}}\right)\left(\bar{\mu}\left[\frac{\bar{u}\sin\Phi+\bar{v}\cos\Phi}{\bar{r}}\right]\right)\right]\!\!\right]. \qquad (2.07)$$

The energy equation is

$$\bar{\rho}\left(\frac{\bar{u}}{h}\ \frac{\partial\bar{T}}{\partial\bar{x}}+\bar{v}\ \frac{\partial\bar{T}}{\partial\eta}\right)-\frac{2\epsilon}{1+\epsilon}\left(\frac{\bar{u}}{h}\ \frac{\partial\bar{p}}{\partial\bar{x}}+\bar{v}\ \frac{\partial\bar{p}}{\partial\eta}\right)$$

$$=\frac{1}{PR}\left[\frac{\partial}{\partial\eta}\left(\bar{\mu}\ \frac{\partial\bar{T}}{\partial\eta}\right)+\left(\frac{\kappa}{h}+\frac{\cos\Phi}{r}\right)\bar{\mu}\frac{\partial\bar{T}}{\partial\eta}\right.$$

$$\left.+\frac{1}{h}\frac{\partial}{\partial\bar{x}}\ \left(\frac{\bar{\mu}}{h}\frac{\partial\bar{T}}{\partial\bar{x}}\right)+\ \frac{\sin\Phi}{\bar{r}}\ \ \frac{\bar{\mu}}{h}\ \ \frac{\partial\bar{T}}{\partial\bar{x}}\right]$$

$$+\ \frac{2\epsilon\ M^2}{1-\epsilon}\ \ \frac{\bar{\mu}}{R}\ \left[2\left(\frac{\partial\bar{v}}{\partial\eta}\right)^2+\ 2\left(\frac{1}{h}\ \left\{\frac{\partial\bar{u}}{\partial\bar{x}}\ +\bar{\kappa}\bar{v}\right\}\right)^2\right.$$

$$+\ 2\left(\frac{\bar{u}\ \sin\Phi+\bar{v}\ \cos\Phi}{\bar{r}}\right)^2+\left(\frac{\partial\bar{u}}{\partial\eta}\ -\frac{\bar{\kappa}\bar{u}}{h}\ +\frac{1}{h}\ \frac{\partial\bar{v}}{\partial\bar{x}}\right)^2$$

$$\left.-\ \frac{2}{3}\ \left(\frac{1}{h}\ \ \frac{\partial\bar{u}}{\partial\bar{x}}+\frac{\partial\bar{v}}{\partial\eta}+\frac{\bar{\kappa}\bar{v}}{h}+\frac{\bar{u}\ \sin\Phi+\bar{v}\ \cos\Phi}{\bar{r}}\right)^2\right]. \qquad (2.08)$$

The equation of state, considering the gas to be a perfect one, is

$$\bar{p}=\bar{\rho}\,\bar{T}. \qquad (2.09)$$

The viscosity law for the gas is taken to be

$$\mu=\mu_{\infty}\bar{\mu}=\mu_{\infty}\bar{T}^{\omega}. \qquad (2.10)$$

To shorten the writing of the continuity, momentum, and energy equations, the quantities h, $\bar{r}$ and $\partial/\partial\bar{x}$ have been introduced. To complete the system of equations, it is necessary to define these quantities. They are

$$h=1+\bar{\kappa}(\bar{Y}+\eta),\ \ \bar{r}=\bar{B}+(\bar{Y}+\eta)\cos\Phi,\ \frac{\partial}{\partial\bar{x}}\ =\frac{\partial}{\partial\xi}\ -\ \bar{Y}'(\xi)\frac{\partial}{\partial\eta}\ . \qquad (2.11)$$

The parameters in the equations, R, the Reynolds number, M, the Mach number, P, the Prandtl number, and ϵ, the Newtonian parameter, are given by

$$R = \rho_\infty U_\infty a/\mu_\infty \to \infty,$$
$$M^2 = \rho_\infty U_\infty^2/\gamma p_\infty \to \infty,$$
$$P = c_p \mu_\infty / k_\infty = O(1),$$
$$\epsilon = (\gamma-1)/(\gamma+1) \to 0 \qquad (2.12)$$

In order to obtain the above six equations, Eqs. (2.05)-(2.10), for the six variables, $\overline{u}, \overline{v}, \overline{p}, \overline{\rho}, \overline{T}, \overline{\mu}$, it is assumed that: the bulk viscosity coefficient, μ^1, is zero; the specific heats of the gas, c_p and c_v, and, hence, $\gamma = \frac{c_p}{c_v}$ are constants; and the heat conduction coefficient, k, is proportional to the shear viscosity coefficient.

III. THE OUTER REGION OF THE SHOCK STRUCTURE

Upstream of the "shock wave", where the flow is uniform, the flow quantities are

$$\bar{p}=\bar{\rho}=\bar{T}=1, \quad \bar{u}=\cos\Phi, \quad \bar{v}=-\sin\Phi . \tag{3.01}$$

With the hypersonic normal shock structure as a guide post (cf., Ref. [1]), it is natural to postulate that a region should exist in an outer portion of the shock structure adjacent to the uniform upstream region which may be thought of as acting as a very thin transition zone between the relatively cool free stream and the hot major (middle) region of the shock structure. In this outer region one expects the magnitude of the flow quantities to be characterized by their magnitude in the uniform upstream and hence, of order one. Thus, denoting the quantities in the outer region by an asterisk, the velocity, density, pressure, and temperature in this transition zone are taken to be

$$\bar{u}=\frac{u}{U_{\infty}} = \cos\Phi + \alpha^{*}u^{*} + \dots , \tag{3.02a}$$

$$\bar{v}=\frac{v}{U_{\infty}} = -\sin\Phi + \beta^{*}v^{*} + \dots , \tag{3.02b}$$

$$\bar{\rho}=\frac{\rho}{\rho_{\infty}} = 1 + \sigma^{*}\rho^{*} + \dots , \tag{3.02c}$$

$$\bar{p} = \frac{p}{p_\infty} = p^* + \ldots , \qquad (3.02d)$$

$$\bar{T} = \frac{T}{T_\infty} = T^* + \ldots , \qquad (3.02e)$$

where α^*, β^* and σ^* are functions of the parameters ϵ, M, and R, and their orders of magnitude are less than or equal to one.

The coordinates $\bar{x}$ and $\bar{y}$ for this region are

$$\bar{x} = \xi^* , \quad y = \lambda^* \eta^* + \Lambda^* Y^*(\xi^*) , \qquad (3.03)$$

where λ^* and Λ^* are also functions of ϵ, M, and R. The thickness of the outer region is considered to be quite small which means that $\lambda^* \to 0$. In addition, the thickness of the region between this outer region and the body is also taken to be small ($\Lambda^* \to 0$) but, nevertheless, greater than the outer region thickness so that $(\lambda^*/\Lambda^*) \to 0$.

Thus, taking the above into account, as well as remembering that ϵ and $\delta = (1-\epsilon)/2\epsilon M^2 \to 0$ while M and $R \to \infty$, the leading terms in the equations of motion for the outer region of the shock structure (integrated, where possible, using the upstream boundary conditions) are

$$p^* = (1 + \sigma^* \rho^*) T^* , \qquad (3.04a)$$

$$v^*(1+\sigma^*\rho^*) - \{\frac{\sigma^*}{\beta^*}\}\rho^* \sin\Phi - \{\frac{\Lambda^* \alpha^*}{\beta^*}\}\frac{dY^*}{d\xi^*} u^*(1+\sigma^*\rho^*) = 0, \qquad (3.04b)$$

$$(1 + \sigma^* \rho^*) \left[(-\sin\Phi + \beta^* v^*) \frac{\partial v^*}{\partial \eta^*} - \{\frac{\lambda^* \alpha^*}{\beta^*}\} \bar{\kappa} (\cos\Phi + \alpha^* u^*) u^* \right]$$

$$+ \{\frac{1}{\beta^* M^2}\} \frac{\partial p^*}{\partial \eta^*} = \{\frac{1}{\lambda^* R}\} \left[\frac{4}{3} \frac{\partial}{\partial \eta^*} (T^{*\omega} \frac{\partial v^*}{\partial \eta^*}) \right.$$

$$\left. - \{\frac{\Lambda^* \alpha^*}{\beta^*}\} \frac{1}{3} \frac{dY^*}{d\xi^*} \frac{\partial}{\partial \eta^*} (T^{*\omega} \frac{\partial u^*}{\partial \eta^*}) \right] , \tag{3.04c}$$

$$(1 + \sigma^* \rho^*) \left[(-\sin\Phi + \beta^* v^*) \frac{\partial u^*}{\partial \eta^*} + \{\frac{\lambda^* \beta^*}{\alpha^*}\} \bar{\kappa} (\cos\Phi + \alpha^* u^*) v^* \right]$$

$$- \{\frac{1}{\beta^* M^2}\} \{\frac{\Lambda^* \beta^*}{\alpha^*}\} \frac{dY^*}{d\xi^*} \frac{\partial p^*}{\partial \eta^*} = \{\frac{1}{\lambda^* R}\} \left[\frac{\partial}{\partial \eta^*} (T^{*\omega} \frac{\partial u^*}{\partial \eta^*}) \right.$$

$$\left. - \{\frac{\Lambda^* \beta^*}{\alpha^*}\} \frac{1}{3} \frac{dY^*}{d\xi^*} \frac{\partial}{\partial \eta^*} (T^{*\omega} \frac{\partial v^*}{\partial \eta^*}) \right] , \tag{3.04d}$$

$$(1 + \sigma^* \rho^*) (-\sin\Phi + \beta^* v^*) \frac{\partial T^*}{\partial \eta^*} = \{\frac{1}{\lambda^* R}\} \frac{1}{P} \frac{\partial}{\partial \eta^*} (T^{*\omega} \frac{\partial T^*}{\partial \eta^*})$$

$$+ \{\frac{\beta^{*2}}{\delta}\} \{\frac{1}{\lambda^* R}\} T^{*\omega} \left[\frac{4}{3} (\frac{\partial v^*}{\partial \eta^*})^2 + \{\frac{\alpha^{*2}}{\beta^{*2}}\} (\frac{\partial u^*}{\partial \eta^*})^2 \right.$$

$$\left. - \{\frac{\Lambda^* \alpha^*}{\beta^*}\} \frac{2}{3} \frac{dY^*}{d\xi^*} \frac{\partial u^*}{\partial \eta^*} \frac{\partial v^*}{\partial \eta^*} \right] . \tag{3.04e}$$

In order to keep the terms with the highest derivatives in the momentum and energy equations in addition to the inertial terms, as was demonstrated in the normal shock analysis, it is necessary that $\lambda^* R = O(1) \equiv 1$ and also that (β^{*2}/δ), (α^{*2}/δ), $(\Lambda^* \beta^*/\alpha^*)$, and $(\Lambda^* \alpha^*/\beta^*)$ be less than or equal to $O(1)$ and $\beta^* M^2 \geqslant O(1)$. As a consequence of these requirements and $\delta \to 0$, it follows that α^* and $\beta^* \to 0$. Thus, in order to retain the density term in the continuity equation, σ^* must go to zero so that $(\sigma^*/\beta^*) = O(1) \equiv 1$.

As a result of the orderings introduced in the above paragraph, Eq. (3. 04) simplifies to

$$p^* = T^* \ , \tag{3. 05a}$$

$$v^* - \rho^* \sin \Phi - \{\frac{\Lambda^* \alpha^*}{\beta^*}\} \frac{dY^*}{d\xi^*} u^* = 0, \tag{3. 05b}$$

$$-v^* \sin \Phi + \{\frac{1}{\beta^* M^2}\} (T^* - 1)$$

$$= T^{*\omega} [\frac{4}{3} \frac{\partial v^*}{\partial \eta^*} - \{\frac{\Lambda^* \alpha^*}{\beta^*}\} \frac{1}{3} \frac{dY^*}{d\xi^*} \frac{\partial u^*}{\partial \eta^*}] \ , \tag{3. 05c}$$

$$-u^* \sin \Phi - \{\frac{1}{\beta^* M^2}\} \{\frac{\Lambda^* \beta^*}{\alpha^*}\} \frac{dY^*}{d\xi^*} (T^* - 1)$$

$$= T^{*\omega} [\frac{\partial u^*}{\partial \eta^*} - \{\frac{\Lambda^* \beta^*}{\alpha^*}\} \frac{1}{3} \frac{dY^*}{d\xi^*} \frac{\partial v^*}{\partial \eta^*}] \ , \tag{3. 05d}$$

$$-\frac{\partial T^*}{\partial \eta^*} \sin \Phi = \frac{1}{P} \frac{\partial}{\partial \eta^*} (T^{*\omega} \frac{\partial T^*}{\partial \eta^*})$$

$$+ \{\frac{\beta^{*2}}{\delta}\} T^{*\omega} [\frac{4}{3} (\frac{\partial v^*}{\partial \eta^*})^2 + \{\frac{\alpha^{*2}}{\beta^{*2}}\} (\frac{\partial u^*}{\partial \eta^*})^2$$

$$- \{\frac{\Lambda^* \alpha^*}{\beta^*}\} \frac{2}{3} \frac{dY^*}{d\xi^*} \frac{\partial u^*}{\partial r^*} \frac{\partial v^*}{\partial \eta^*}] \ . \tag{3. 05e}$$

Again using the normal shock (where, for P=3/4, $\beta^* \sim \delta$) as a partial basis, it is postulated that (α^*/β^*), (β^{*2}/δ), and $(1/\beta^* M^2) \to 0$, so that the pressure and dissipation effects in the region are relatively negligible and the effect of the shock wave shape's differing from the body shape, i. e., $dY^*/d\xi^*$, is also negligible, except, possibly in the x-momentum equation. (These postulates are proven to be true in the section on the matching between this outer region and the middle region of the shock structure.) With these postulates and with a change of independent variables from (ξ^*, η^*) to (ξ^*, v^*), the system of partial differential equations of motion is simplified further to become

$$p^* = T^* , \tag{3.06a}$$

$$\rho^* = v^*/\sin\Phi , \tag{3.06b}$$

$$\frac{\partial \eta^*}{\partial v^*} = -\frac{4}{3} \frac{T^{*\omega}}{v^* \sin\Phi} , \tag{3.06c}$$

$$\frac{\partial u^*}{\partial v^*} - \frac{4}{3} \frac{u^*}{v^*} = \left\{ \frac{\Lambda^* \beta^*}{\alpha^*} \right\} \frac{1}{3} \frac{dY^*}{d\xi^*} , \tag{3.06d}$$

$$\frac{\partial (T^*-1)}{\partial v^*} - \frac{4P}{3} \frac{(T^*-1)}{v^*} = 0. \tag{3.06e}$$

Solving the partial differential equations, Eqs. (3.06c), (3.06d) and (3.06e), the flow quantities in the outer region may be written as

$$p^*(\xi^*, v^*) = \frac{v^*}{\sin \Phi(\xi^*)} \quad , \tag{3.07}$$

$$T^*(\xi^*, v^*) = p^*(\xi^*, v^*) = 1 + T_o^*(\xi^*) v^{*\,4P/3} \quad , \tag{3.08,.09}$$

$$u^*(\xi^*, v^*) = u_o^*(\xi^*) v^{*\,4/3} - \left\{\frac{\Lambda^* \beta^*}{\alpha^*}\right\} \frac{dY^*}{d\xi^*} \; v^* \; , \tag{3.10}$$

$$\eta^*(\xi^*, v^*) = - \frac{4}{3 \sin \Phi(\xi^*)} \int_{v_o^*(\xi^*)}^{v^*} \frac{\left[1 + T_o^* \nu^{4P/3}\right]^{\omega} d\nu}{\nu} \quad , \tag{3.11}$$

where T_o^*, u_o^* and v_o^* are functions of ξ^* that are determined through the matching procedures that are discussed in succeeding sections of the paper.

IV. THE MIDDLE REGION OF THE SHOCK STRUCTURE

The next region downstream from the outer region of the shock structure is the middle, or dissipation, region of the shock structure. This region is a thin zone (but a thicker one than the outer transition zone) where the magnitudes of the velocity components are still characterized by their magnitude in the uniform upstream region, but where the temperature and pressure and, possibly, the density are considerably greater than they are in the freestream. Therefore, choosing the symbol "tilde" over a quantity to set off this middle region, the velocity, density, temperature, and pressure are

$$\bar{u}=\tilde{u}+\dots \quad , \tag{4.01a}$$

$$\bar{v}=\tilde{v}+\dots \quad , \tag{4.01b}$$

$$\bar{\rho}=\tilde{\sigma}\tilde{\rho}+\dots \quad , \tag{4.01c}$$

$$\bar{T}=\tilde{\theta}\tilde{T}+\dots \quad , \tag{4.01d}$$

$$\bar{p}=\tilde{\pi}\tilde{p}+\dots \tag{4.01e}$$

where $\tilde{\sigma}$, $\tilde{\theta}$, and $\tilde{\pi}$ are functions of ϵ, M, and R. The quantities are greater than or equal to O(1). The coordinates of this region are

$$\bar{x}=\tilde{\xi},\ \ \bar{y}=\tilde{\lambda}\tilde{\eta}+\tilde{\Lambda}\,\tilde{Y}(\tilde{\xi}), \tag{4.02}$$

where $\tilde{\lambda}(\epsilon\ , M, R)$ and $\tilde{\Lambda}(\epsilon\ , M, R)$ both approach zero. No statement is made as to their relative magnitudes.

From the above, it follows that the leading terms in the equations of motion are

$$\tilde{p}=\{\frac{\tilde{\sigma}\tilde{\theta}}{\tilde{\pi}}\}\ \tilde{\rho}\ \tilde{T}\ , \qquad (4.03a)$$

$$\tilde{\rho}\ \tilde{v} = \tilde{m}(\tilde{\xi})\ , \qquad (4.03b)$$

$$\tilde{m}\frac{\partial\tilde{v}}{\partial\tilde{\eta}} + \{\frac{\tilde{\pi}}{\tilde{\sigma}M^2}\}\ \frac{\partial\tilde{p}}{\partial\tilde{\eta}} = \{\frac{\tilde{\theta}^{\omega}}{\tilde{\lambda}R\tilde{\sigma}}\}\frac{4}{3}\frac{\partial}{\partial\tilde{\eta}}(\tilde{T}^{\omega}\frac{\partial\tilde{v}}{\partial\tilde{\eta}})\ , \qquad (4.03c)$$

$$\tilde{m}\ \frac{\partial\tilde{u}}{\partial\tilde{\eta}} - \{\frac{\tilde{\Lambda}\tilde{\pi}}{\tilde{\sigma}M^2}\}\ \frac{d\tilde{Y}}{d\tilde{\xi}}\ \frac{\partial\tilde{p}}{\partial\tilde{\eta}} + \{\frac{\tilde{\lambda}\tilde{\pi}}{\tilde{\sigma}M^2}\}\ \frac{\partial\tilde{p}}{\partial\tilde{\xi}}$$

$$= \{\frac{\tilde{\theta}^{\omega}}{\tilde{\lambda}R\tilde{\sigma}}\}\ \frac{\partial}{\partial\tilde{\eta}}\ (\tilde{T}^{\omega}\ \frac{\partial\tilde{u}}{\partial\tilde{\eta}})\ , \qquad (4.03d)$$

$$[\tilde{m}\frac{\partial\tilde{T}}{\partial\tilde{\eta}} - \{\frac{\epsilon\tilde{\pi}}{\tilde{\sigma}\tilde{\theta}}\}\ 2\tilde{v}\frac{\partial\tilde{p}}{\partial\tilde{\eta}}] = \{\frac{\tilde{\theta}^{\omega}}{\tilde{\lambda}R\tilde{\sigma}}\}\ \frac{1}{P}\frac{\partial}{\partial\tilde{\eta}}(\tilde{T}^{\omega}\frac{\partial\tilde{T}}{\partial\tilde{\eta}})$$

$$+ \{\frac{\tilde{\theta}^{\omega}}{\tilde{\lambda}R\tilde{\sigma}}\}\ \{\frac{1}{\tilde{\theta}\delta}\}\ \tilde{T}^{\omega}\ [\frac{4}{3}\ (\frac{\partial\tilde{v}}{\partial\tilde{\eta}})^2 + (\frac{\partial\tilde{u}}{\partial\tilde{\eta}})^2]\ , \qquad (4.03e)$$

where $\tilde{m}(\tilde{\xi})$ is the mass flow function introduced by the integration of the differential equation of continuity.

In order to keep all three terms in the energy equation, where the last term is the dissipation term, it is necessary that $\tilde{\theta}=O(1/\delta)\equiv 1/\delta\rightarrow\infty$ and $\tilde{\lambda} = O(1/\tilde{\sigma}R\delta^{\omega})\equiv 1/\tilde{\sigma}R\delta^{\omega}\rightarrow 0$.

Further, in order to retain the complete equation of state, $\tilde{\pi}/\tilde{\sigma}\,\tilde{\theta} = O(1)\equiv 1$. This means that the pressure terms in the momentum and energy equations drop out and the system of equations simplifies to

$$\tilde{p}=\tilde{\rho}\tilde{T}\ , \qquad (4.04a)$$

$$\tilde{\rho}\tilde{v}=\tilde{m}(\tilde{\xi}), \qquad (4.04b)$$

$$\tilde{m}\tilde{v}-\frac{4}{3}\tilde{T}^{\omega}\frac{\partial\tilde{v}}{\partial\tilde{\eta}}=\tilde{P}_n(\tilde{\xi})\ , \qquad (4.04c)$$

$$\tilde{m}\tilde{u}-\tilde{T}^{\omega}\frac{\partial\tilde{u}}{\partial\tilde{\eta}}=\tilde{P}_t(\tilde{\xi})\ , \qquad (4.04d)$$

$$\tilde{m}\frac{\partial\tilde{T}}{\partial\tilde{\eta}}=\frac{1}{P}\frac{\partial}{\partial\tilde{\eta}}\left(\tilde{T}^{\omega}\frac{\partial\tilde{T}}{\partial\tilde{\eta}}\right)+\tilde{T}^{\omega}\left[\frac{4}{3}\left(\frac{\partial\tilde{v}}{\partial\tilde{\eta}}\right)^2+\left(\frac{\partial\tilde{u}}{\partial\tilde{\eta}}\right)^2\right], \qquad (4.04e)$$

where $\tilde{P}_n(\tilde{\xi})$ and $\tilde{P}_t(\tilde{\xi})$ are momentum functions that result from the integration of Equations (4.03c) and (4.03d).

As in the previous section, these equations are most easily solved by employing a modified Crocco transformation where the independent variables are changed from $(\tilde{\xi}, \tilde{\eta})$ to $(\tilde{\xi}, \tilde{V})$, where

$$\tilde{V}=\tilde{v}-(\tilde{P}_n/\tilde{m}). \qquad (4.05)$$

The partial differential equations to be solved in terms of these new variables are

$$\frac{\partial \tilde{U}}{\partial \tilde{V}} - \frac{4}{3}\,\frac{\tilde{U}}{\tilde{V}} = 0, \quad \tilde{U} = \tilde{u} - (\tilde{P}_t/\tilde{m}), \tag{4.06a}$$

$$\frac{\partial^2 \tilde{T}}{\partial \tilde{V}^2} - \frac{(\frac{4P}{3} - 1)}{\tilde{V}}\,\frac{\partial \tilde{T}}{\partial \tilde{V}} + P\left[\frac{4}{3} + \left(\frac{\partial \tilde{U}}{\partial \tilde{V}}\right)^2\right] = 0, \tag{4.06b}$$

$$\frac{\partial \tilde{\eta}}{\partial \tilde{V}} - \frac{4}{3\tilde{m}}\,\frac{\tilde{T}^{\omega}}{\tilde{V}} = 0\,. \tag{4.06c}$$

Solving these equations, the flow quantities in the middle region of the shock structure are:

$$\tilde{u}(\tilde{\xi}, \tilde{v}) = (\tilde{P}_t/\tilde{m}) + \tilde{u}_o(\tilde{\xi})\,[\tilde{v} - (\tilde{P}_n/\tilde{m})]^{4/3}, \tag{4.07}$$

$$\tilde{T}(\tilde{\xi}, \tilde{v}) = \tilde{T}_o(\tilde{\xi}) + \tilde{T}_1(\tilde{\xi})\,[\tilde{v} - (\tilde{P}_n/\tilde{m})]^{4P/3}$$

$$- [P/(3-2P)]\,[\tilde{v} - (\tilde{P}_n/\tilde{m})]^2 - [\tilde{P}/2(2-P)]\,\tilde{u}_o^2\,[\tilde{v} - (\tilde{P}_n/\tilde{m})]^{8/3}, \tag{4.08}$$

$$\tilde{\rho}(\tilde{\xi}, \tilde{v}) = \tilde{m}/\tilde{v}, \tag{4.09}$$

$$\tilde{p}(\tilde{\xi}, \tilde{v}) = (\tilde{m}/\tilde{v})\,\tilde{T}(\tilde{\xi}, \tilde{v}), \tag{4.10}$$

$$\tilde{\eta}(\tilde{\xi}, \tilde{v}) = \tilde{L}(\tilde{\xi}) + \frac{4}{3\tilde{m}} \int_{-\sin\Phi}^{\tilde{v}} \frac{\tilde{T}^{\omega}\, d\nu}{[\nu - (\tilde{P}_n/\tilde{m})]}\,. \tag{4.11}$$

V. THE MATCHING BETWEEN THE OUTER AND MIDDLE REGIONS OF THE SHOCK STRUCTURE

In the preceding two sections, the postulated leading terms in the outer and middle expansions for $\overline{\rho}$, $\overline{u}$, $\overline{T}$, $\overline{p}$, and $\overline{y}$ were found in terms of the outer region and middle region variables, (ξ^*, v^*) and $(\tilde{\xi}, \tilde{v})$, respectively. However, if these outer and middle expansions are indeed valid, they must match in a region of the shock structure that is intermediate to the outer and middle regions.

The outer region expansions are of the "boundary-layer" type. They have a certain behavior as the outer region or "boundary-layer" variable $v^* \rightarrow \infty$ which must match to the middle region expansions as $\tilde{v} \rightarrow -\sin \Phi$. The matching can be carried out by taking an intermediate limit of each set of expansions. An intermediate variable is one which is not quite in either the outer or middle regions. Let

$$\overleftrightarrow{v}_1 = \frac{\sin \Phi + \overline{v}}{\overleftrightarrow{\beta}_1} , \tag{5.01}$$

where

$$\overleftrightarrow{\beta}_1 \rightarrow 0 \quad \text{and} \quad \overleftrightarrow{\beta}_1/\beta^* \rightarrow \infty \ \text{ as } \beta^* \rightarrow 0 , \tag{5.02}$$

define the class of intermediate variables. An intermediate limit is performed if $\beta^* \to 0$ keeping v_1 fixed. Thus,

$$v^* = \frac{\overleftrightarrow{\beta}_1 \overleftrightarrow{v}_1}{\beta^*} \to \infty, \tag{5.03a}$$

$$\tilde{v} = -\sin\Phi + \overleftrightarrow{\beta}_1 \overleftrightarrow{v}_1 \to -\sin\Phi \tag{5.03b}$$

in the intermediate limit. Matching to O(1) means that

$$\lim_{\beta^* \to 0,\ \overleftrightarrow{v}_1 \text{ fixed}} \{ \bar{f}_{\text{outer}} - \bar{f}_{\text{middle}} \} \equiv \langle \bar{f} \rangle_1 = 0 . \tag{5.04}$$

Consider first the matching to O(1) for the density, $\bar{\rho}$. The intermediate limits of the outer and middle expansions are

$$\bar{\rho} = 1 + \sigma^* \rho^* + \ldots = 1 + \frac{\beta^* v^*}{\sin\Phi} + \ldots = 1 + \frac{\overleftrightarrow{\beta}_1 \overleftrightarrow{v}_1}{\sin\Phi} + \ldots \tag{5.05a}$$

$$\bar{\rho} = \tilde{\sigma}\tilde{\rho} + \ldots = \frac{\tilde{\sigma}\,\tilde{m}}{\tilde{v}} + \ldots = \frac{\tilde{\sigma}\,\tilde{m}}{(-\sin\Phi + \overleftrightarrow{\beta}_1 \overleftrightarrow{v}_1 + \ldots)} + \ldots$$

$$= -\frac{\tilde{\sigma}\,\tilde{m}}{\sin\Phi} - \frac{\tilde{\sigma}\,\tilde{m}\,\overleftrightarrow{\beta}_1 \overleftrightarrow{v}_1}{\sin^2\Phi} + \ldots \tag{5.05b}$$

Therefore,

$$\langle \bar{\rho} \rangle_1 = 1 + \frac{\tilde{\sigma}\,\tilde{m}}{\sin\Phi} + \ldots \tag{5.06}$$

Matching requires that

$$\tilde{\sigma}=O(1)\equiv 1\ ,\quad \tilde{m}(\tilde{\xi}) = -\sin\Phi\ . \tag{5.07}$$

A further consequence is that $\tilde{\lambda}=(1/R\delta^{\omega})$ and $\tilde{\pi}=\tilde{\theta}=1/\delta$.

The intermediate limits for $\overline{u}$ are

$$\overline{u} = \cos\Phi + \alpha^* u^* + \ldots$$

$$= \cos\Phi + \alpha^*\left[u_o^*\left(\frac{\overleftrightarrow{\beta}_1\,\overleftrightarrow{v}_1}{\beta^*}\right)^{4/3} - \left\{\frac{\Lambda^*\beta^*}{\alpha^*}\right\}\frac{dY^*}{d\xi^*}\left(\frac{\overleftrightarrow{\beta}_1\,\overleftrightarrow{v}_1}{\beta^*}\right)\right]+\ldots \tag{5.08a}$$

$$= \cos\Phi + \frac{\alpha^*}{\beta^{*4/3}}\, u_o^*\,(\overleftrightarrow{\beta}_1\,\overleftrightarrow{v}_1)^{4/3}+\ldots\ ,$$

$$\overline{u}= \frac{\tilde{P}_t}{\tilde{m}} + \tilde{u}_o\left[\tilde{v} - \frac{\tilde{P}_n}{\tilde{m}}\right]^{4/3} + \ldots$$

$$= \frac{\tilde{P}_t}{\tilde{m}} + \tilde{u}_o\left[-\sin\Phi - \frac{\tilde{P}_n}{\tilde{m}} + \overleftrightarrow{\beta}_1\,\overleftrightarrow{v}_1\right]^{4/3} + \ldots \tag{5.08b}$$

The matching equation is

$$\langle\overline{u}\rangle_1 = \cos\Phi - \frac{\tilde{P}_t}{\tilde{m}} + \frac{\alpha^*}{\beta^{*4/3}}\, u_o^*\,(\overleftrightarrow{\beta}_1\,\overleftrightarrow{v}_1)^{4/3}$$

$$-\tilde{u}_o\left[-\sin\Phi - \frac{\tilde{P}_n}{\tilde{m}} + \overleftrightarrow{\beta}_1\,\overleftrightarrow{v}_1\right]^{4/3} + \ldots = 0\ . \tag{5.09}$$

The matching requirements can be seen to be

$$\frac{\alpha^*}{\beta^{*4/3}} = O(1) \equiv 1, \quad \tilde{P}_t = -\sin\Phi\cos\Phi, \quad \tilde{P}_n = \sin^2\Phi,$$

$$u_o^*(\xi^*) = \tilde{u}_o(\tilde{\xi}). \tag{5.10}$$

Since $\alpha^*/\beta^{*4/3} = 1$, $\alpha^*/\beta^* = \beta^{*1/3} \to 0$ so that also $\Lambda^*\alpha^*/\beta^* \to 0$. This verifies one of the postulates of the outer region section.

Looking at the temperature $\overline{T}$, the intermediate limits are

$$\overline{T} = T^* + \ldots = 1 + T_o^* v^{*4P/3} + \ldots$$

$$= \frac{T_o^*}{(\beta^*)^{4P/3}} (\overleftrightarrow{\beta}_1 \overleftrightarrow{v}_1)^{4P/3} + \ldots \tag{5.11a}$$

$$\overline{T} = \frac{\tilde{T}}{\delta} + \ldots = \frac{1}{\delta}\left[\tilde{T}_o + \tilde{T}_1 \tilde{V}^{4P/3} - \frac{P\tilde{V}^2}{3-2P} - \frac{P\tilde{u}_o^2 \tilde{V}^{8/3}}{2(2-P)}\right] + \ldots$$

$$= \frac{1}{\delta}\left[\tilde{T}_o + \tilde{T}_1 (\overleftrightarrow{\beta}_1 \overleftrightarrow{v}_1)^{4P/3} - \frac{P(\overleftrightarrow{\beta}_1 \overleftrightarrow{v}_1)^2}{3-2P}\right] + \ldots \tag{5.11b}$$

Inspection shows that matching to $O(1)$ is accomplished if

$$P < \frac{3}{2}, \quad \beta^* = O(\delta^{3/4P}) \equiv \delta^{3/4P}, \quad \tilde{T}_o(\tilde{\xi}) = 0, \; T_o^*(\xi^*) = \tilde{T}_1(\tilde{\xi}). \tag{5.12}$$

This means that, in the outer region, $\alpha^* = \delta^{1/P}$.

Finally, consider the matching for $\bar{y}$. For the middle and outer regions, the coordinate $\bar{y}$ is written as

$$\bar{y}=\tilde{\lambda}\tilde{\eta}(\tilde{\xi},\tilde{v})+\tilde{\Lambda}\tilde{Y}(\tilde{\xi})+\ldots=\tilde{\lambda}\tilde{\eta}(\xi,\tilde{v})+\left[\hat{\lambda}\hat{\mathcal{Y}}(\xi)+\lambda_L\mathcal{Y}_L(\xi)\right]+\ldots, \qquad (5.13a)$$

$$\bar{y}=\lambda^*\eta^*(\xi^*,v^*)+\Lambda^*Y^*(\xi^*)+\ldots$$

$$=\lambda^*\eta^*(\xi,v^*)+\left[\tilde{\lambda}\tilde{\mathcal{Y}}(\xi)+\hat{\lambda}\hat{\mathcal{Y}}(\xi)+\lambda_L\mathcal{Y}_L(\xi)\right]+\ldots, \qquad (5.13b)$$

where $\tilde{\lambda}\tilde{\mathcal{Y}}$, $\hat{\lambda}\hat{\mathcal{Y}}$, and $\lambda_L\mathcal{Y}_L$ are the contributions to the normal coordinate of the middle region of the shock structure, the inner region of the shock structure (which is discussed next) and the shock layer, the region behind the "shock wave" and adjacent to the body, respectively.

Using Equation (5.13), the intermediate limits of outer and middle region expansions are

$$\bar{y}-\hat{\lambda}\hat{\mathcal{Y}}-\lambda_L\mathcal{Y}_L=\frac{\tilde{\mathcal{Y}}}{R\delta^{\omega}}+\frac{\eta^*}{R}+\ldots$$

$$=\frac{\tilde{\mathcal{Y}}}{R\delta^{\omega}}+\frac{1}{R}\left[-\frac{4}{3\sin\Phi}\int_{v_o^*}^{v^*}\frac{(1+T_o^*\nu^{4P/3})^{\omega}d\nu}{\nu}\right]+\ldots$$

$$=\frac{\tilde{\mathcal{Y}}}{R\delta^{\omega}}+\frac{1}{R}\left[-\frac{4}{3\sin\Phi}\int_{\frac{\beta^* v_o^*}{\overleftrightarrow{\beta}_1}}^{\overleftrightarrow{v}_1}\frac{\left[1+T_o^*\left(\frac{\overleftrightarrow{\beta}_1\nu}{\beta^*}\right)^{4P/3}\right]^{\omega}d\nu}{\nu}\right]+\ldots$$

$$=\frac{1}{R\delta^{\omega}}\left[\tilde{\mathcal{Y}}-\frac{\tilde{T}_1^{\omega}}{P\,\omega\sin\Phi}(\overleftrightarrow{\beta}_1\overleftrightarrow{v}_1)^{4P\omega/3}+\ldots\right], \qquad (5.14a)$$

$$\bar{y} - \hat{\lambda}\hat{\mathcal{Y}} - \lambda_L \mathcal{Y}_L = \frac{\tilde{\eta}}{R\delta^{\omega}} + \ldots$$

$$= \frac{1}{R\delta^{\omega}} \left[\tilde{L} - \frac{4}{3\sin\Phi} \int_{-\sin\Phi}^{\tilde{v}} \frac{\tilde{T}^{\omega} d\nu}{\nu + \sin\Phi} \right] + \ldots$$

$$= \frac{1}{R\delta^{\omega}} \left[\tilde{L} - \frac{\tilde{T}_1^{\ \omega}}{P\omega \sin\Phi} (\overleftrightarrow{\beta}_1 \overleftrightarrow{v}_1)^{4P\omega/3} + \ldots \right] . \qquad (5.14b)$$

Thus, the expansions for $\bar{y}$ match to O(1) if

$$\tilde{L}(\xi) = \tilde{\mathcal{Y}}(\xi) \ . \qquad (5.15)$$

With the information gained through the matchings just completed, the flow quantities in the outer region may be rewritten as

$$\bar{u} = \cos\Phi + \delta^{1/P} [\tilde{u}_o v^{*4/3} - \{\epsilon/\delta^{1/4P}\} \frac{dY^*}{d\xi^*} v^*] + \ldots , \qquad (5.16a)$$

$$\bar{T} = 1 + \tilde{T}_1 v^{*4P/3} + \ldots , \quad \bar{p} = 1 + \tilde{T}_1 v^{*4P/3} + \ldots , \qquad (5.16b,c)$$

$$\bar{\rho} = 1 + \delta^{3/4P} (v^*/\sin\Phi) + \ldots , \qquad (5.16d)$$

$$\bar{y} - (\lambda_L \mathcal{Y}_L + \hat{\lambda}\hat{\mathcal{Y}} + \frac{\tilde{\mathcal{Y}}}{R\delta^{\omega}})$$

$$= \frac{1}{R} \left[- \frac{4}{3\sin\Phi} \int_{v_o^*}^{v^*} \frac{(1 + \tilde{T}_1 \nu^{4P/3})^{\omega} d\nu}{\nu} \right] + \ldots , \qquad (5.16e)$$

where the result that $\Lambda^* = \epsilon$ has been anticipated. Further, it should be noted that, since Eq. (3.04) is invariant under the transformation $\eta^* \rightarrow (\eta^* + \text{constant})$, one should not expect to determine v_o^* .

The flow quantities in the middle region are

$$\bar{u} = [\cos\Phi + \tilde{u}_o(\sin\Phi + \tilde{v})^{4/3} + \dots , \qquad (5.17a)$$

$$\bar{T} = \frac{1}{\delta}[\tilde{T}_1(\sin\Phi + \tilde{v})^{4P/3} - \frac{P}{3-2P}(\sin\Phi + \tilde{v})^2$$

$$- \frac{P\tilde{u}_o^2}{2(2-P)}(\sin\Phi + \tilde{v})^{8/3}] + \dots , \qquad (5.17b)$$

$$\bar{\rho} = (-\sin\Phi/\tilde{v}) + \dots , \qquad (5.17c)$$

$$\bar{p} = \frac{1}{\delta}[(-\sin\Phi/\tilde{v})\,\tilde{T}] + \dots , \qquad (5.17d)$$

$$\bar{y} - (\lambda_L \mathscr{Y}_L + \hat{\lambda}\hat{\mathscr{Y}}) = \frac{1}{R\delta^{\omega}}[\tilde{\mathscr{Y}} - \frac{4}{3\sin\Phi}\int_{-\sin\Phi}^{\tilde{v}^*} \frac{\tilde{T}^{\omega} d\nu}{\sin\Phi + \nu}] + \dots \qquad (5.17e)$$

A few loose ends concerning the orders of magnitudes of parameters in the outer region may now be taken care of by showing that the assumed orders are the correct ones. It has already been shown that $(\alpha^*/\beta^{*4/3}) = 1$ so that (α^*/β^*) and $(\Lambda^*\alpha^*/\beta^*)$ go to zero. The quantity (β^{*2}/δ) is equal to $O(\delta^a)$ where $a = [(\frac{3}{2} - P)/P)] > 0$ since $P < \frac{3}{2}$ and, hence, (β^{*2}/δ) and (α^*/δ) go to zero.

The two remaining parameters and their assumed behavior are

$$\beta^* M^2 = \delta^{3/4P} M^2 = O(\delta^{(3-4P)/4P}/\epsilon) >> 1, \qquad (5.18a)$$

$$\frac{\Lambda^*\beta^*}{\alpha^*} = \epsilon/\delta^{1/4P} \leq O(1). \qquad (5.18b)$$

These relations imply that

$$\epsilon << O(\delta^{(3-4P)/4P}) , \tag{5.19a}$$

$$\epsilon \leqslant O(\delta^{1/4P}) . \tag{5.19b}$$

In order that Eq. (5.19) be satisfied it follows that

$$\delta^{1/4P} \lesseqgtr \delta^{(3-4P)/4P} \tag{5.20a}$$

or, since δ is going to zero,

$$P \gtreqless \frac{1}{2} . \tag{5.20b}$$

Therefore, in order that the matching may be accomplished it follows that the condition on the (constant) Prandtl number is

$$\frac{1}{2} \lesseqgtr P < 3/2 \tag{5.21}$$

However, if $(\Lambda^*\beta^*/\alpha^*) = O(1)$, the upper bound, then the restriction on the Prandtl number is

$$\frac{1}{2} < P < \frac{3}{2} . \tag{5.22}$$

The question now arises: Since the outer region is of the "boundary-layer" type, can the outer and middle regions problem, as formulated, where the uniform upstream conditions are reached as $\eta^* \to \infty$, be replaced by a middle region problem where the uniform upstream conditions are actually reached at a finite value of the space coordinate $\tilde{\eta}$, and where, upstream of this point, the flow quantities are constant, leading to discontinuities in derivatives at this point ? This question can be answered in the affirmative.

As $\tilde{\eta} \rightarrow \tilde{\mathcal{Y}}$, the outer edge of the middle region, $\tilde{u} \rightarrow \cos \Phi$, $\tilde{v} \rightarrow -\sin \Phi$, $\tilde{\rho} \rightarrow 1$, which are the uniform upstream conditions. Further, as $\tilde{\eta} \rightarrow \tilde{\mathcal{Y}}$, $\tilde{T}$ and $\tilde{p} \rightarrow 0$, which are the proper uniform upstream values, since the middle region is a high temperature and pressure region in comparison with the uniform upstream region. For a more complete description of this viewpoint the reader is referred to the work of SYCHEV, [8].

VI. THE INNER REGION OF THE SHOCK STRUCTURE

The inner region is the thin, dissipationless transition zone between the middle region of the shock structure and the shock layer, the region behind the "shock wave". In this region there is a decrease in the normal velocity, $\overline{v}$, and a corresponding increase in the density; the temperature remains high; and the pressure gets even higher. Denoting the quantities in this region by a circumflex, one has

$$\overline{u} = W(\hat{\xi}) + \hat{\alpha}\,\hat{u} + \ldots, \quad \hat{\alpha} = \hat{\alpha}(\epsilon, M, R) \leqslant O(1) \,, \tag{6.01a}$$

$$\overline{v} = \hat{\beta}\,\hat{v} + \ldots, \quad \hat{\beta}(\epsilon, M, R) << O(1), \tag{6.01b}$$

$$\overline{\rho} = \hat{\sigma}\,\hat{\rho} + \ldots, \quad \hat{\sigma}(\epsilon, M, R) >> O(1), \tag{6.01c}$$

$$\overline{T} = \frac{1}{\delta}\left[\Theta(\hat{\xi}) + \hat{\theta}\,\hat{T} + \ldots\right] \,, \quad \hat{\theta}(\epsilon, M, R) \leqslant O(1), \tag{6.01d}$$

$$\overline{p} = \hat{\pi}\,\hat{p} + \ldots, \quad \hat{\pi}(\epsilon, M, R) >> O(1) \,, \tag{6.01e}$$

$$\overline{y} = \hat{\lambda}\hat{\eta} + \hat{\Lambda}\hat{Y}(\hat{\xi}), \ \hat{\lambda}(\epsilon, M, R) << O(1),$$

$$\hat{\Lambda}(\epsilon, M, R) << O(1), \ \hat{\lambda}/\hat{\Lambda} \ll O(1) \,. \tag{6.01f}$$

The leading terms in the equations of motion are

$$\hat{p} = \{\hat{\sigma}/\hat{\pi}\delta\}\,\hat{\rho}\,\hat{\vartheta} \,, \tag{6.02a}$$

$$\hat{\rho}\left[\hat{v} - \{\frac{\hat{\Lambda}}{\hat{\beta}}\}\frac{d\hat{Y}}{d\hat{\xi}}(W + \hat{\alpha}\,\hat{u})\right] \equiv \hat{\rho}\,\hat{V} = \hat{m}(\hat{\xi}), \tag{6.02b}$$

$$\begin{aligned}
&\hat{m}\,\frac{\partial \hat{v}}{\partial \hat{\eta}} - \{\frac{\hat{\lambda}}{\hat{\beta}^2}\}\,\hat{\rho}\bar{\kappa}(W + \hat{\alpha}\,\hat{u})^2 + 2\,\{\frac{\epsilon}{\hat{\beta}^2}\}\,\{\frac{\pi\hat{\delta}}{\hat{\sigma}}\}\,\frac{\partial \hat{p}}{\partial \hat{\eta}} \\
&= \frac{1}{\hat{\sigma}\hat{\beta}\hat{\lambda}R\,\delta^{\omega}}\left[\!\left[\frac{4}{3}\,\frac{\partial}{\partial\hat{\eta}}\,(\hat{\vartheta}^{\omega}\frac{\partial\hat{v}}{\partial\hat{\eta}}) - \{\frac{\hat{\Lambda}\hat{\alpha}}{\hat{\beta}}\}\,\frac{1}{3}\,\frac{d\hat{Y}}{d\hat{\xi}}\,\frac{\partial}{d\hat{\eta}}\,(\hat{\vartheta}^{\omega}\frac{\partial\hat{u}}{\partial\hat{\eta}})\right.\right. \\
&\left.\left. - \{\frac{\hat{\lambda}}{\hat{\beta}}\}\,\frac{2}{3}\,\frac{\partial}{\partial\hat{\eta}}\,(\hat{\vartheta}^{\omega}[\,\frac{dW}{d\hat{\xi}} + \frac{(W + \hat{\alpha}\,\hat{u})\sin\Phi}{\overline{B}}\,])\right]\!\right] \;,
\end{aligned} \tag{6.02c}$$

$$\begin{aligned}
&\hat{m}\,\frac{\partial \hat{u}}{\partial \hat{\eta}} + \{\frac{\hat{\lambda}}{\hat{\alpha}\hat{\beta}}\}\,\hat{\rho}\,(W + \hat{\alpha}\,\hat{u})\,\frac{dW}{d\hat{\xi}} - 2\{\frac{\hat{\Lambda}\epsilon}{\hat{\alpha}\hat{\beta}}\}\{\frac{\pi\hat{\delta}}{\hat{\sigma}}\}\,\frac{d\hat{Y}}{d\hat{\xi}}\,\frac{\partial\hat{p}}{\partial\hat{\eta}} \\
&= \frac{1}{\hat{\sigma}\hat{\beta}\hat{\lambda}R\,\delta^{\omega}}\left[\!\left[\frac{\partial}{\partial\hat{\eta}}\,(\hat{\vartheta}^{\omega}\frac{\partial\hat{u}}{\partial\hat{\eta}}) - \{\frac{\hat{\Lambda}\hat{\beta}}{\hat{\alpha}}\}\,\frac{1}{3}\,\frac{d\hat{Y}}{d\hat{\xi}}\,\frac{\partial}{\partial\hat{\eta}}\,(\hat{\vartheta}^{\omega}\frac{\partial\hat{v}}{\partial\hat{\eta}})\right.\right. \\
&\left.\left. - \{\frac{\hat{\lambda}}{\hat{\alpha}}\}\,\frac{\partial}{\partial\hat{\eta}}\,(\hat{\vartheta}^{\omega}\bar{\kappa}\;[\,W + \hat{\alpha}\,\hat{u}\,]\,)\right]\!\right] \;,
\end{aligned} \tag{6.02d}$$

$$\begin{aligned}
&\hat{m}\,\frac{\partial \hat{T}}{\partial \hat{\eta}} + \{\frac{\hat{\lambda}}{\hat{\theta}\hat{\beta}}\}\;\hat{\rho}\,(W + \hat{\alpha}\hat{u})\,\frac{d\Theta}{d\hat{\xi}} - 2\{\frac{\epsilon}{\hat{\theta}}\}\{\frac{\pi\hat{\delta}}{\hat{\sigma}}\}\,\hat{V}\,\frac{\partial\hat{p}}{\partial\hat{\eta}} \\
&\frac{1}{\hat{\sigma}\hat{\beta}\hat{\lambda}R\,\delta^{\omega}}\,\frac{1}{P}\left[\frac{\partial}{\partial\hat{\eta}}\,(\hat{\vartheta}^{\omega}\frac{\partial\hat{T}}{\partial\hat{\eta}}) - \{\frac{\hat{\Lambda}\hat{\lambda}}{\hat{\theta}}\}\,\frac{d\hat{Y}}{d\hat{\xi}}\,\frac{\partial}{\partial\hat{\eta}}\,(\hat{\vartheta}^{\omega}\,\frac{d\Theta}{d\hat{\xi}})\right] \\
&+ \frac{1}{\hat{\sigma}\hat{\beta}\hat{\lambda}R\,\delta^{\omega}}\;\{\frac{\hat{\beta}^2}{\hat{\theta}}\}\,\hat{\vartheta}^{\omega}\left[2(\frac{\partial\hat{v}}{\partial\hat{\eta}})^2 + 2\{\frac{\hat{\lambda}^2}{\hat{\beta}^2}\}(\,[\frac{dW}{d\hat{\xi}}]^2 + [\,\frac{(W+\hat{\alpha}\hat{u})\sin\Phi}{\overline{B}}\,]^2\,)\right. \\
&+ (\{\frac{\hat{\alpha}}{\hat{\beta}}\}\,\frac{\partial\hat{u}}{\partial\hat{\eta}} - \{\frac{\hat{\lambda}}{\hat{\beta}}\}\,\bar{\kappa}\,W)^2 \\
&\left. - \frac{2}{3}\,(\frac{\partial\hat{v}}{\partial\hat{\eta}} + \{\frac{\hat{\lambda}}{\hat{\beta}}\}[\,\frac{dW}{d\hat{\xi}} + \frac{(W + \hat{\alpha}\hat{u})\sin\Phi}{\overline{B}}\,]\;)^2\right] \;,
\end{aligned} \tag{6.02e}$$

where the shorthand notation $\hat{\vartheta} = \Theta + \hat{\theta}\hat{T}$ has been employed.

In order to retain the complete equation of state, $(\hat{\pi}\delta/\hat{\sigma}) = O(1) \equiv 1$. If the magnitude of the dissipation term is to be negligible compared to that of the heat conduction term in the energy equation, then $\hat{\beta}^2, \hat{\alpha}^2, \hat{\lambda}^2 << \delta \to 0$. It is postulated now (and verified in the matching later) that $\hat{\beta} = \hat{\alpha} = \hat{\theta} \ll \epsilon^{\frac{1}{2}}$, $(\hat{\lambda}/\hat{\beta}) \ll O(1)$, $(\hat{\lambda}/\epsilon) \ll 1$, $\hat{\lambda} = O(\hat{\beta}/\hat{\sigma}R\delta^{\omega}) \equiv \hat{\beta}/\hat{\sigma}R\delta^{\omega}$. With these postulates, just the normal pressure gradient term remains of the left hand sides of Eqs. (6.02c), (6.02d), and (6.02e) and the equations of motion become

$$\hat{p} = \hat{\rho}\,\Theta \,, \tag{6.03a}$$

$$\hat{\rho}\left[\hat{v} - \{\frac{\hat{\Lambda}}{\hat{\beta}}\}\frac{d\hat{Y}}{d\hat{\xi}}W\right] \equiv \hat{\rho}\hat{V} = \hat{m}(\hat{\xi}), \; \hat{\Lambda}/\hat{\beta} \ll O(1) \,, \tag{6.03b}$$

$$\frac{2\hat{m}\,\Theta}{\hat{V}} - \frac{4}{3}\Theta^{\omega}\frac{\partial\hat{V}}{\partial\hat{\eta}} = \hat{P}_n(\hat{\xi}) \,, \tag{6.03c}$$

$$-\Theta^{\omega}\frac{\partial\hat{u}}{\partial\hat{\eta}} = \hat{P}_t(\hat{\xi}) \,, \tag{6.03d}$$

$$-\frac{\Theta^{\omega}}{P}\frac{\partial\hat{T}}{\partial\hat{\eta}} = \hat{E}(\hat{\xi}) \;. \tag{6.03e}$$

Changing the independent variables from $(\hat{\xi}, \hat{\eta})$ to $(\hat{\xi}, \hat{V})$, the solutions of these equations are

$$\hat{\rho}(\hat{\xi}, \hat{V}) = \frac{\hat{m}(\hat{\xi})}{\hat{V}} \,, \tag{6.04a}$$

$$\hat{p}(\hat{\xi}, \hat{V}) = \frac{\hat{m}(\hat{\xi})\Theta(\hat{\xi})}{\hat{V}} \,, \tag{6.04b}$$

$$\hat{u}(\hat{\xi},\hat{V}) = \hat{u}_o(\hat{\xi})$$

$$- \frac{4\hat{P}_t(\hat{\xi})}{3\hat{P}_n^2(\hat{\xi})} \left[\!\left[2\hat{m}(\hat{\xi})\Theta(\hat{\xi}) - \hat{P}_n(\hat{\xi})\hat{V} - 2\hat{m}(\hat{\xi})\Theta(\hat{\xi})\log_e \left| 2\hat{m}(\hat{\xi})\Theta(\hat{\xi}) - \hat{P}_n(\hat{\xi})\hat{V} \right| \right]\!\right]$$

$$= \hat{u}_o(\hat{\xi}) - \frac{4\hat{P}_t(\hat{\xi})}{3\hat{P}_n^2(\hat{\xi})} [\hat{F}(\hat{\xi},\hat{V})], \tag{6.04c}$$

$$\hat{T}(\hat{\xi},\hat{V}) = \hat{T}_o(\hat{\xi}) - \frac{4P\hat{E}(\hat{\xi})}{3\hat{P}_n^2(\hat{\xi})} [\hat{F}(\hat{\xi},\hat{V})] , \tag{6.04d}$$

$$\hat{\eta}(\hat{\xi},\hat{V}) = \hat{L}(\hat{\xi}) + \frac{4\Theta^{\omega}(\hat{\xi})}{3\hat{P}_n^2(\hat{\xi})} [\hat{F}(\hat{\xi},\hat{V})] . \tag{6.04e}$$

VII. THE MATCHING OF THE MIDDLE AND INNER REGIONS OF THE SHOCK STRUCTURE

In the previous section on matching, it was shown that the outer and middle expansions, to be valid, must match in a region that is intermediate to both the outer and middle regions with respect to $\overline{v}$. The matching of the middle and inner regions is, again, done with respect to $\overline{v}$. The middle region expansions, as $\tilde{v} \to 0$, must match to the inner region expansions, as $|\hat{v}| \to \infty$. The appropriate class of intermediate variables for this matching procedure is

$$\overleftrightarrow{v}_2 = \frac{\overline{v}}{\overleftrightarrow{\beta}_2} \quad \text{where} \quad \overleftrightarrow{\beta}_2 \to 0, \ \frac{\overleftrightarrow{\beta}_2}{\hat{\beta}} \to \infty \ \text{as} \ \hat{\beta} \to 0 . \tag{7.01}$$

In the intermediate limit, (i.e., $\hat{\beta} \to 0$, $\overleftrightarrow{v}_2$ (fixed), $\tilde{v}$ and $\hat{v}$ are

$$\tilde{v} = \overleftrightarrow{\beta}_2 \ \overleftrightarrow{v}_2 + \dots \to 0, \tag{7.02a}$$

$$\hat{v} = \frac{\overleftrightarrow{\beta}_2 \overleftrightarrow{v}_2}{\hat{\beta}} + \dots \to \infty . \tag{7.02b}$$

To perform the matching to O(1) means that it is necessary to find

$$\lim_{\hat{\beta} \to 0, \overleftrightarrow{v}_2 \text{fixed}} \{\overline{f}_{\text{middle}} - \overline{f}_{\text{inner}}\} \equiv \langle \overline{f} \rangle_2 = 0. \tag{7.03}$$

The first and simplest matching is that for $\overline{\rho}$, the density. The quantity $\langle \overline{\rho} \rangle_2$ is

$$\langle \overline{\rho} \rangle_2 = [-\frac{\sin\Phi}{\overleftrightarrow{\beta}_2 \overleftrightarrow{v}_2} + \dots] - [\hat{\sigma} \hat{\beta} \frac{\overline{m}}{\overleftrightarrow{\beta}_2 \overleftrightarrow{v}_2} + \dots] = 0. \tag{7.04}$$

Density matching thus requires

$$\hat{\sigma} = O(1/\hat{\beta}) \equiv 1/\hat{\beta}, \quad \hat{m} = -\sin\Phi . \tag{7.05}$$

The $\overline{u}$ matching equation is

$$\langle \overline{u} \rangle_2 = [\cos\Phi + \tilde{u}_o (\sin\Phi)^{4/3} + \frac{4}{3} \tilde{u}_o (\sin\Phi)^{1/3} \overleftrightarrow{\beta}_2 \overleftrightarrow{v}_2 + \ldots]$$

$$- [W + \frac{4}{3} \frac{\hat{P}_t}{\hat{P}_n} \overleftrightarrow{\beta}_2 \overleftrightarrow{v}_2 + \ldots] = 0 \tag{7.06}$$

Therefore,

$$W = \cos\Phi + \tilde{u}_o (\sin\Phi)^{4/3} , \quad \hat{P}_t/\hat{P}_n = \tilde{u}_o (\sin\Phi)^{1/3} . \tag{7.07a}$$

A more useful form of this equation is

$$\tilde{u}_o = -\frac{\cos\Phi - W}{(\sin\Phi)^{4/3}} , \quad \frac{\hat{P}_t}{\hat{P}_n} = -\frac{\cos\Phi - W}{\sin\Phi} . \tag{7.07b}$$

For the temperature $\overline{T}$, one has

$$\langle \overline{T} \rangle_2 = \frac{1}{\delta} [\{\tilde{T}_1 (\sin\Phi)^{4P/3} - \frac{P(\sin\Phi)^2}{3-2P} - \frac{P\tilde{u}_o^2 (\sin\Phi)^{8/3}}{2(2-P)}$$

$$+ \frac{\overleftrightarrow{\beta}_2 \overleftrightarrow{v}_2}{\sin\Phi} \{\frac{4P}{3} \tilde{T}_1 (\sin\Phi)^{4P/3} - \frac{2P(\sin\Phi)^2}{3-2P} - \frac{4P \tilde{u}_o^2 (\sin\Phi)^{8/3}}{3(2-P)} \} + \ldots$$

$$- \frac{1}{\delta} [\Theta + \overleftrightarrow{\beta}_2 \overleftrightarrow{v}_2 \frac{4P}{3} \frac{\hat{E}}{\hat{P}_n} + \ldots] = 0 . \tag{7.08}$$

Thus, there is matching if

$$\Theta = \tilde{T}_1 (\sin\Phi)^{4P/3} - \frac{P(\sin\Phi)^2}{3-2P} - \frac{P\tilde{u}_o^{\,2}(\sin\Phi)^{8/3}}{2(2-P)} ,$$

$$\frac{\hat{E}}{\hat{P}_n} = - \frac{1}{\sin\Phi} \left[\left(\frac{\sin^2\Phi}{2} - \Theta \right) + \frac{1}{2} (\cos\Phi - W)^2 \right] . \qquad (7.09)$$

The final matching is for $\bar{y}$. The $\bar{y}$ matching equation is

$$\langle \bar{y} \rangle_2 = \frac{1}{R\delta^\omega} \left[\left\{ \tilde{\mathcal{Y}} - \frac{4}{3\sin\Phi} \int_{-\sin\Phi}^{0} \frac{\tilde{T}^\omega d\tilde{v}}{\sin\Phi + \tilde{v}} \right\} - \frac{4}{3} \frac{\Theta^\omega}{\sin^2\Phi} \overleftrightarrow{\beta}_2 \overleftrightarrow{v}_2 + \frac{\hat{\beta}^2}{\epsilon} \hat{\mathcal{Y}} + \ldots \right]$$

$$- \frac{1}{R\delta^\omega} \left[- \frac{\hat{\beta}}{\epsilon} \frac{4}{3} \frac{\Theta^\omega}{\hat{P}_n} \overleftrightarrow{\beta}_2 \overleftrightarrow{v}_2 + \frac{\hat{\beta}^2}{\epsilon} \hat{L} + \ldots \right] = 0. \qquad (7.10)$$

Therefore,

$$\hat{\beta} = O(\epsilon) \equiv \epsilon, \quad \hat{P}_n = \sin^2\Phi, \quad \tilde{\mathcal{Y}} = \frac{4}{3\sin\Phi} \int_{-\sin\Phi}^{0} \frac{\tilde{T}^\omega d\tilde{v}}{\sin\Phi + \tilde{v}} , \quad \hat{L} = \hat{\mathcal{Y}} \qquad (7.11)$$

From the above conditions, it can also be seen that

$$\hat{\alpha} = \hat{\theta} = \frac{1}{\hat{\sigma}} = \hat{\beta} = \epsilon , \quad \frac{\epsilon}{\hat{\beta}^2} = \frac{1}{\epsilon} \to \infty , \quad \hat{\pi} = \frac{1}{\epsilon\delta} ,$$

$$\hat{\lambda} = \frac{\hat{\beta}}{\hat{\sigma}\epsilon R\delta^\omega} = \frac{\epsilon}{R\delta^\omega} = \epsilon\tilde{\lambda} \to 0 , \quad \frac{\hat{\lambda}}{\epsilon} = \tilde{\lambda} \to 0 ,$$

$$\hat{P}_t = -\sin\Phi (\cos\Phi - W), \quad \hat{E} = - \sin\Phi \left[\left(\frac{\sin^2\Phi}{2} - \Theta \right) + \frac{1}{2} (\cos\Phi - W)^2 \right] . \qquad (7.12)$$

Using the results of this matching, the equations for the inner region can be written as

$$\hat{u} = \hat{u}_o - \frac{\sin\Phi(\cos\Phi - W)}{\Theta^\omega} (\hat{\mathcal{Y}} - \hat{\eta}) , \tag{7.13a}$$

$$\hat{T} = \hat{T}_o - \frac{P\sin\Phi}{\Theta^\omega} \left[(\frac{\sin^2\Phi}{2} - \Theta) + \frac{1}{2} (\cos\Phi - W)^2 \right] (\hat{\mathcal{Y}} - \hat{\eta}) , \tag{7.13b}$$

$$\left| \sin^2\Phi (\hat{V} + \frac{2\Theta}{\sin\Phi}) \right| = \exp\left[\frac{\sin\Phi}{2\Theta} (\hat{V} + \frac{2\Theta}{\sin\Phi}) - \frac{3}{8} \frac{\sin^3\Phi}{\Theta^{1+\omega}} (\hat{\mathcal{Y}} - \hat{\eta}) \right] , \tag{7.13c}$$

$$\hat{\rho} = - \frac{\sin\Phi}{\hat{V}} , \quad \hat{p} = - \frac{\Theta\sin\Phi}{\hat{V}} . \tag{7.13d,e}$$

This is the form of the equations that is the most useful in the sections which follow.

VIII. THE SHOCK LAYER

The region between the "shock wave" and the body, called the shock layer, both viscous and inviscid, has been the subject of much discussion and the orders of magnitude of the flow quantities have been correctly determined by CHESTER, [4], and FREEMAN, [5], to be

$$\overline{u} = u_L + \dots , \tag{8.01a}$$

$$\overline{v} = \epsilon v_L + \dots , \tag{8.01b}$$

$$\overline{\rho} = \frac{1}{\epsilon} \rho_L + \dots , \tag{8.01c}$$

$$\overline{T} = \frac{1}{\delta} T_L + \dots , \tag{8.01d}$$

$$\overline{p} = \frac{1}{\epsilon\delta} p_L + \dots , \tag{8.01e}$$

The normal space coordinate is

$$\overline{y} = \epsilon \eta_L . \tag{8.02}$$

The leading terms in the equations of motion for the shock layer are

$$p_L = \rho_L T_L , \tag{8.03a}$$

$$\frac{\partial}{\partial\xi}(\overline{B}\rho_L u_L) + \frac{\partial}{\partial\eta_L}(\overline{B}\rho_L v_L) = 0 , \tag{8.03b}$$

$$2\frac{\partial p_L}{\partial\eta_L} - \overline{\kappa}\rho_L u_L^2 = \epsilon\left\{\frac{1}{\epsilon R\delta^\omega}\right\}\left[\!\!\left[\frac{2}{3}\frac{\partial}{\partial\eta_L}\left(T_L^\omega\left[2\frac{\partial v_L}{\partial\eta_L} - \left(\frac{\partial}{\partial\xi}+\frac{\sin\Phi}{\overline{B}}\right)u_L\right]\right)\right.\right.$$

$$\left.\left. + \left(\frac{\partial}{\partial\xi}+\frac{\sin\Phi}{\overline{B}}\right)\left(T_L^\omega\frac{\partial u_L}{\partial\eta_L}\right)\right]\!\!\right] , \tag{8.03c}$$

$$\rho_L(u_L \frac{\partial u_L}{\partial \xi} + v_L \frac{\partial u_L}{\partial \eta_L}) = \{\frac{1}{\epsilon R \delta^\omega}\} \frac{\partial}{\partial \eta_L} (T_L^\omega \frac{\partial u_L}{\partial \eta_L}) \,, \tag{8.03d}$$

$$\rho_L(u_L \frac{\partial T_L}{\partial \xi} + v_L \frac{\partial T_L}{\partial \eta_L}) = \{\frac{1}{\epsilon R \delta^\omega}\} [\frac{1}{P} \frac{\partial}{\partial \eta_L} (T_L^\omega \frac{\partial T_L}{\partial \eta_L}) + T_L^\omega (\frac{\partial u_L}{\partial \eta_L})^2] \,. \tag{8.03e}$$

The shock layer is inviscid for $K = (1/\epsilon R\delta^\omega) \ll O(1)$ and viscous for $K = O(1)$. Since the inertial terms in the energy and tangential momentum equations must be retained and, thus, K cannot be greater than O(1), and the right-hand side of Eq. (8.03c) must go to zero. This means that the equations to be solved are

$$p_L = \rho_L T_L \,, \tag{8.04a}$$

$$\frac{\partial}{\partial \xi} (\overline{B} \rho_L u_L) + \frac{\partial}{\partial \eta_L} (\overline{B} \rho_L v_L) = 0 \,, \tag{8.04b}$$

$$2 \frac{\partial p_L}{\partial \eta_L} - \overline{\kappa} \rho_L u_L^2 = 0 \,, \tag{8.04c}$$

$$\rho_L (u_L \frac{\partial u_L}{\partial \xi} + v_L \frac{\partial u_L}{\partial \eta_L}) - K \frac{\partial}{\partial \eta_L} (T_L^\omega \frac{\partial u_L}{\partial \eta_L}) \,, \tag{8.04d}$$

$$\rho_L (u_L \frac{\partial T_L}{\partial \xi} + v_L \frac{\partial T_L}{\partial \eta_L}) = K [\frac{1}{P} \frac{\partial}{\partial \eta_L} (T_L^\omega \frac{\partial T_L}{\partial \eta_L}) + T_L^\omega (\frac{\partial u_L}{\partial \eta_L})^2] \,. \tag{8.04e}$$

It should be noted the ratio of the thicknesses of the middle region of the shock structure and the shock layer is

$$\frac{\tilde{\lambda}}{\lambda_L} = \frac{1/R\delta^\omega}{\epsilon} = \frac{1}{\epsilon R \delta^\omega} = K \,. \tag{8.05}$$

Therefore, if the shock layer is inviscid, then the thickness of the "shock wave" is much less than $O(\epsilon a)$. However, if the shock layer is viscous, then thicknesses of the "shock wave" and the shock layer are of the same magnitude, $O(\epsilon a)$.

The boundary conditions for the shock layer, in part, must be determined by matching between the inner region of the shock structure and the shock layer and, hence, this matching must be examined. The matching in this case is with respect to $\bar{y}$. The intermediate region is the zone that is just interior to the outer edge of the shock layer so that the appropriate class of intermediate variables is

$$\overleftrightarrow{\eta} = \frac{\epsilon \mathcal{Y}_L - \bar{y}}{\overleftrightarrow{\lambda}}, \quad \overleftrightarrow{\eta} > 0,$$

$$\text{where } \frac{\overleftrightarrow{\lambda}}{\lambda_L} = \frac{\overleftrightarrow{\lambda}}{\epsilon} \rightarrow 0, \quad \frac{\overleftrightarrow{\lambda}}{\hat{\lambda}} = \frac{\overleftrightarrow{\lambda} R\delta^{\omega}}{\epsilon} \rightarrow \infty \quad \text{as } \epsilon, \ \epsilon/R\delta^{\omega} \rightarrow 0. \tag{8.06}$$

In the intermediate limit (i.e., $\overleftrightarrow{\lambda} \rightarrow 0$, $\overleftrightarrow{\eta}$ fixed), $\hat{\eta}$ and η_L are

$$\hat{\eta} = -\frac{\overleftrightarrow{\lambda} R\delta^{\omega}}{\epsilon} \overleftrightarrow{\eta} + \ldots \rightarrow -\infty, \tag{8.07a}$$

$$\eta_L = \mathcal{Y}_L - \frac{\overleftrightarrow{\lambda}\overleftrightarrow{\eta}}{\epsilon} + \ldots \rightarrow \mathcal{Y}_L . \tag{8.07b}$$

The equation for matching to O(1) is

$$\lim_{\overleftrightarrow{\lambda} \rightarrow 0, \overleftrightarrow{\eta} \text{ fixed}} \{\bar{f}_{inner} - \bar{f}_{layer}\} \equiv \langle \bar{f} \rangle_3 = 0. \tag{8.08}$$

The $\bar{u}$ matching equation is

$$\langle \bar{u} \rangle_3 = [W - \frac{1}{K} \frac{\overleftrightarrow{\lambda}\overleftrightarrow{\eta}}{\epsilon} \frac{\sin\Phi(\cos\Phi - W)}{\Theta^{\omega}} + \ldots] - [u_L + \ldots] = 0. \tag{8.09}$$

For the viscous shock layer where K = O(1), since $\overleftrightarrow{\lambda}/\epsilon \to 0$, it is clear that

$$[u_L(\xi, \mathcal{Y}_L)]_{visc} = [W(\xi)]_{visc} \; . \tag{8.10a}$$

where $[W(\xi)]_{visc}$ is yet to be determined. For the inviscid shock layer with $K \to 0$, the requirements for matching are not quite so clear. However, since K can go to zero at an arbitrary rate, in order that u_L be finite, it is necessary that

$$[u_L(\xi, \mathcal{Y}_L)]_{inv} = [W(\xi)]_{inv} = \cos\Phi(\xi) \; . \tag{8.10b}$$

The temperature matching yields

$$\langle \overline{T} \rangle_3 = \frac{1}{\delta}[\Theta - \frac{1}{K}\frac{\overleftrightarrow{\lambda}\,\overleftrightarrow{\eta}}{\epsilon}\frac{P\sin\Phi}{\Theta^{\omega}} \{(\frac{\sin^2\Phi}{2} - \Theta) + \frac{1}{2}(\cos\Phi - W)^2\} + \ldots]$$

$$- \frac{1}{\delta}[T_L + \ldots] = 0. \tag{8.11}$$

Employing arguments that parallel those for $\overline{u}$, the boundary conditions for T_L at the outer edge of the shock layer are found to be

$$[T_L(\xi, \mathcal{Y}_L)]_{inv} = [\Theta(\xi)]_{inv} = \frac{\sin^2\Phi}{2} \; , \tag{8.12a}$$

$$[T_L(\xi, \mathcal{Y}_L)]_{visc} = [\Theta(\xi)]_{visc} \; , \text{ to be determined.} \tag{8.12b}$$

The matching for $\overline{v}$ is interesting. As was seen in the previous section, the expression for $\hat{v}(\hat{\xi}, \hat{\eta})$ is

$$|\sin^2\Phi\,(\hat{V}+\frac{2\Theta}{\sin\Phi})| = \exp\left[\frac{\sin\Phi}{2\Theta}(\hat{V}+\frac{2\Theta}{\sin\Phi}) - \frac{3}{8}\frac{\sin^3\Phi}{\Theta^{1+\omega}}(\hat{\mathcal{Y}}-\hat{\eta})\right]$$

$$(\hat{V} = \hat{v} - \{\frac{\hat{\Lambda}}{\hat{\beta}}\}\frac{d\hat{Y}}{d\xi}W) . \tag{8.13}$$

The intermediate limit of this equation (since $\hat{\Lambda} = \lambda_L = \hat{\beta} = \epsilon$, $\hat{Y} = \mathcal{Y}_L$) is

$$|\sin^2\Phi\,(\hat{V}+\frac{2\Theta}{\sin\Phi})| = \exp\left[-\frac{3}{8}\frac{\sin^3\Phi}{\Theta^{1+\omega}}\frac{\overleftrightarrow{\lambda} R\delta^{\omega}}{\epsilon}\overleftrightarrow{\eta} + \ldots\right] \rightarrow 0. \tag{8.14a}$$

This means that

$$\hat{V} = -\frac{2\Theta}{\sin\Phi} + \ldots , \quad \hat{v} = \frac{d\mathcal{Y}_L}{d\xi}W - \frac{2\Theta}{\sin\Phi} + \ldots \quad . \tag{8.14b}$$

The condition for matching between the inner region and the shock layer is given by

$$\langle \bar{v} \rangle_3 = \epsilon\left[\frac{d\mathcal{Y}_L}{d}W - \frac{2\Theta}{\sin\Phi} + \ldots\right] - \epsilon\left[v_L + \ldots\right] = 0. \tag{8.15}$$

Hence,

$$[v_L(\xi,\mathcal{Y}_L)]_{inv} = -\sin\Phi + \frac{d\mathcal{Y}_L}{d\xi}\cos\Phi , \tag{8.16a}$$

$$[v_L(\xi,\mathcal{Y}_L)]_{visc} = -\frac{2\Theta}{\sin\Phi} + \frac{d\mathcal{Y}_L}{d\xi}W . \tag{8.16b}$$

Since the matchings for $\bar{\rho}$ and $\bar{p}$ follow directly from those for $\bar{T}$ and $\bar{v}$, the boundary conditions at the outer edge of the shock layer for ρ_L and p_L are presented now, omitting the intermediate steps. They are

$$[\rho_L(\xi, \mathscr{Y}_L)]_{inv} = 1, \quad [\rho_L(\xi, \mathscr{Y}_L)]_{visc} = \frac{\sin^2\Phi}{2\Theta} , \tag{8.17}$$

$$[p_L(\xi, \mathscr{Y}_L)]_{inv} = [p_L(\xi, \mathscr{Y}_L)]_{visc} = \frac{\sin^2\Phi}{2} . \tag{8.18}$$

The last step in determining the boundary conditions at the outer edge of the shock layer is performing the matchings $\langle \partial\bar{u}/\partial\bar{y} \rangle_3 = \langle \partial\bar{T}/\partial\bar{y} \rangle_3 = 0$. In the inner region of the shock structure these gradients are

$$\frac{\partial\bar{u}}{\partial\bar{y}} = \frac{\hat{\alpha}}{\hat{\lambda}} \frac{\partial\hat{u}}{\partial\hat{\eta}} + \ldots = \frac{1}{\epsilon K} \frac{\sin\Phi(\cos\Phi - W)}{\Theta^{\omega}} + \ldots , \tag{8.19a}$$

$$\frac{\partial\bar{T}}{\partial\bar{y}} = \frac{\hat{\theta}/\delta}{\hat{\lambda}} \frac{\partial\hat{T}}{\partial\hat{\eta}} + \ldots$$

$$= \frac{1}{\epsilon\delta K} \frac{P\sin\Phi}{\Theta^{\omega}} \left[\left(\frac{\sin^2\Phi}{2} - \Theta\right) + \frac{1}{2}(\cos\Phi - W)^2\right] + \ldots \tag{8.19b}$$

In the shock layer the gradients are

$$\frac{\partial\bar{u}}{\partial\bar{y}} = \frac{1}{\epsilon} \frac{\partial u_L}{\partial\eta_L} + \ldots , \quad \frac{\partial\bar{T}}{\partial\bar{y}} = \frac{1}{\epsilon\delta} \frac{\partial T_L}{\partial\eta_L} + \ldots \tag{8.20}$$

Therefore,

$$K\Theta^{\omega} \left[\frac{\partial u_L}{\partial\eta_L}(\xi, \mathscr{Y}_L)\right] = \sin\Phi(\cos\Phi - W) , \tag{8.21a}$$

$$\frac{K\Theta^{\omega}}{P} \left[\frac{\partial T_L}{\partial\eta_L}(\xi, \mathscr{Y}_L)\right] = \sin\Phi\left[\left(\frac{\sin^2\Phi}{2} - \Theta\right) + \frac{1}{2}(\cos\Phi - W)^2\right] . \tag{8.21b}$$

It must be noted that the inviscid boundary conditions are exactly the leading terms of the Rankine-Hugoniot relations as ϵ and δ both approach zero. The quantities $\mathcal{Y}_L$ (for both the viscous and inviscid cases) and W_{visc} and Θ_{visc} are found by solving the equations of motion in the shock layer.

The inviscid boundary condition at the body for the shock layer is

$$v_L(\xi, 0) = 0 \ . \tag{8.22}$$

The viscous boundary conditions at the body are

$$u_L(\xi, 0) = v_L(\xi, 0) = 0 \ , \tag{8.23a}$$

$$T_L(\xi, 0) = T_{L,w}(\xi), \text{ a given function.} \tag{8.23b}$$

It is convenient to introduce a modified Crocco transformation for the equations of motion, for the reason that the solving of the equations is made simpler by introducing the new independent variables. The independent variables are transformed from (ξ, η_L) to (s, t_L) where

$$s = \xi \ , \quad t_L = u_L/W(\xi) = u_L/W(s) \ , \tag{8.24a}$$

$$\frac{\partial}{\partial \xi} = \frac{\partial}{\partial s} - \frac{(\partial \eta_L/\partial s)}{(\partial \eta_L/\partial t_L)} \frac{\partial}{\partial t_L} \ , \quad \frac{\partial}{\partial \eta_L} = \frac{\overline{B}}{W} \frac{\tau_L}{T_L^{\omega}} \frac{\partial}{\partial t_L} \ , \tag{8.24b}$$

if $\tau_L = (T_L^{\omega}/\overline{B})(\partial u_L/u\eta_L)$. The equations in the new variables are

$$p_L = \rho_L T_L \ , \tag{8.25a}$$

$$\rho_L v_L = K\left(\frac{\overline{B}}{W}\frac{\partial \tau_L}{\partial t_L}\right) + \frac{\partial \eta_L}{\partial s}\frac{W t_L p_L}{T_L}$$

$$-\frac{W}{\overline{B}}\frac{dW}{ds}t_L^2\left(\frac{p_L}{\tau_L T_L^{1-\omega}}\right), \tag{8.25b}$$

$$\frac{\partial p_L}{\partial t_L} = \frac{\overline{\kappa} W^3 t_L^2}{2\overline{B}}\left(\frac{p_L}{\tau_L T_L^{1-\omega}}\right), \tag{8.25c}$$

$$K\frac{\partial^2 \tau_L}{\partial t_L^2} = -\frac{W^3 t_L}{\overline{B}^2}\frac{\partial}{\partial s}\left(\frac{p_L}{\tau_L T_L^{1-\omega}}\right)$$

$$+\frac{W^2}{\overline{B}^2}\frac{dW}{ds}t_L^2\frac{\partial}{\partial t_L}\left(\frac{p_L}{\tau_L T_L^{1-\omega}}\right), \tag{8.25d}$$

$$\frac{K}{P}\left(\frac{\partial^2 T_L}{\partial t_L^2} + \{1-P\}\frac{1}{\tau_L}\frac{\partial \tau_L}{\partial t_L}\frac{\partial T_L}{\partial t_L} + PW^2\right)$$

$$= \frac{W^2 t_L}{\overline{B}^2}\left(\frac{p_L}{\tau_L^2 T_L^{1-\omega}}\right)\left(W\frac{\partial T_L}{\partial s} - \frac{dW}{ds}t_L\frac{\partial T_L}{\partial t_L}\right). \tag{8.25e}$$

The boundary conditions at the outer edge, written so as to be suitable whether the layer is viscous or inviscid, become, with $f(s,1) = (f)_e$,

$$(p_L)_e = \frac{\sin^2\Phi}{2}, \tag{8.26a}$$

$$K(\tau_L)_e = \frac{\sin\Phi(\cos\Phi - W)}{\overline{B}},$$

$$K\left(\frac{\partial \tau_L}{\partial t_L}\right)_e = \frac{-W\sin\Phi}{\overline{B}} + \frac{W^2}{\overline{B}^2}\frac{dW}{ds}\left(\frac{p_L}{\tau_L T_L^{1-\omega}}\right)_e, \tag{8.26b}$$

$$(T_L)_e = \Theta,$$

$$K\,(\tau_L)_e\left(\frac{\partial T_L}{d\xi}\right)_e = \frac{PW\sin\Phi}{\overline{B}}\left[\left(\frac{\sin^2\Phi}{2} - \Theta\right) + \frac{1}{2}(\cos\Phi - W)^2\right] . \qquad (8.26c)$$

The inviscid boundary condition at the body is written most simply as

$$v_L(s, t_{L,w}) = 0 . \qquad (8.27)$$

The viscous boundary conditions at the body are

$$\frac{\partial \tau_L}{\partial t_L}(s,0) = 0, \quad T_L(s,0) = T_{L,w}(s) . \qquad (8.28)*$$

* From all the above, it should be noted that the viscous shock layer exists only in the limit of $\epsilon \rightarrow 0$ and not for ϵ fixed, a fact previously pointed out by VAN DYKE, [6], among others.

IX. THE INVISCID SHOCK LAYER

For $K \to 0$, it has been shown in Section 8 that the inviscid equations of motion in the shock layer are

$$\frac{\partial p}{\partial t} - \frac{\overline{\kappa}(s)\cos^3\Phi(s)\, t^2}{2\,\overline{B}(s)} \left(\frac{p}{\tau\, T^{1-\omega}}\right) = 0 , \tag{9.01a}$$

$$\cos\Phi(s)\frac{\partial}{\partial s}\left(\frac{p}{\tau\, T^{1-\omega}}\right) - \overline{\kappa}(s)\sin\Phi(s)\, t\,\frac{\partial}{\partial t}\left(\frac{p}{\tau\, T^{1-\omega}}\right) = 0 , \tag{9.01b}$$

$$\cos\Phi(s)\frac{\partial T}{\partial s} - \overline{\kappa}(s)\sin\Phi(s)\, t\,\frac{\partial T}{\partial t} = 0 , \tag{9.01c}$$

$$v - \frac{\partial \eta}{\partial s}\cos\Phi(s)\, t + \frac{\overline{\kappa}(s)\sin\Phi(s)\cos\Phi(s)}{\overline{B}(s)}\, t^2\, \frac{T}{p}\left(\frac{p}{\tau\, T^{1-\omega}}\right) = 0 , \tag{9.01d}$$

$$\frac{\partial \eta}{\partial t} - \frac{\cos\Phi(s)}{\overline{B}(s)}\,\frac{T}{p}\left(\frac{p}{\tau\, T^{1-\omega}}\right) = 0 , \tag{9.01e}$$

where the subscript L has been dropped since this should introduce no ambiguity. The boundary conditions for these equations are

$$p(s,1) = \frac{\sin^2\Phi(s)}{2},\quad T(s,1) = \frac{\sin^2\Phi(s)}{2} ,$$

$$\frac{p}{\tau\, T^{1-\omega}}(s,1) = \frac{\overline{B}(s)}{\overline{\kappa}(s)\cos\Phi(s)} , \tag{9.02a}$$

$$v(s,t_w(s)) = 0,\quad \eta(s,t_w(s)) = 0 . \tag{9.02b}$$

The solutions of Eqs. (9.01b) and (9.01c) that satisfy the boundary conditions are

$$\frac{p}{\tau\, T^{1-\omega}} = \frac{\bar{B}(t\cos\Phi)}{t\cos\Phi\, \bar{K}(t\cos\Phi)} = \frac{\bar{B}(u)}{u\, \bar{K}(u)} , \tag{9.03}$$

$$T = \frac{1}{2}\,(1 - \{t\cos\Phi\}^2) = \frac{1}{2}\,(1-u^2) . \tag{9.04}$$

Knowing the result of Eq. (9.03), Eq. (9.01a) integrates directly to give

$$p = \frac{1}{2}\left[\sin^2\Phi(s) - \frac{\bar{K}(s)}{\bar{B}(s)} \int_{t\cos\Phi(s)}^{\cos\Phi(s)} \frac{\nu \bar{B}(\nu)\, d\nu}{\bar{K}(\nu)}\right] . \tag{9.05}$$

The coordinate η (using the shorthand notation $c = \cos\Phi(s)$) is

$$\eta = \frac{1}{\bar{B}(c)(1-c^2)} \int_{t_w(s)}^{t} \frac{(1-c^2\nu^2)\bar{B}(c\nu)\, d\nu}{\nu\, \bar{K}(c\nu)\left[1 - \frac{\bar{K}(c)}{\bar{B}(c)(1-c^2)} \int_{c\nu}^{c} \frac{h\, \bar{B}(h)\, dh}{\bar{K}(h)}\right]} . \tag{9.06}$$

The boundary condition of no normal flow at the body, making use of Eqs. (9.03)-(9.06) becomes

$$v_w = 0 = \left(\frac{\partial \eta}{\partial s}\right)_w t_w c - \frac{\bar{K}(c)\sqrt{1-c^2}\; t_w^2\, c}{\bar{B}(c)} \frac{T_w}{p_w} \left(\frac{p}{\tau\, T^{1-\omega}}\right)_w$$

$$= - \frac{\bar{K}(c)}{\bar{K}(t_w c)} \frac{\bar{B}(t_w c)}{\bar{B}(c)} \sqrt{1-c^2}\; t_w \left(1 + \frac{c}{t_w} \frac{dt_w}{dc}\right) \frac{T_w}{p_w} . \tag{9.07}$$

This boundary condition is satisfied only if

$$t_w = 0 . \tag{9.08}$$

X. THE VISCOUS SHOCK LAYER

A. The Equations Near the Axis of Symmetry

It has been shown in Section 8 that the equations for pressure, shear, and temperature in the viscous shock layer, where K is fixed, are, dropping the subscript L for convenience, given by

$$\frac{\partial p}{\partial t} - \frac{\bar{\kappa} W^3 t^2}{2\overline{B}} \frac{p}{\tau T^{1-\omega}} = 0 , \tag{10.01a}$$

$$\frac{\partial^2 \tau}{\partial t^2} - \frac{1}{K} \frac{W^2}{\overline{B}^2} \frac{dW}{ds} t^2 \frac{\partial}{\partial t} \left(\frac{p}{\tau T^{1-\omega}}\right)$$

$$+ \frac{1}{K} \frac{W^3}{\overline{B}^2} t \frac{\partial}{\partial s} \left(\frac{p}{\tau T^{1-\omega}}\right) = 0 , \tag{10.01b}$$

$$\frac{\partial^2 T}{\partial t^2} + \frac{(1-P)}{\tau} \frac{\partial \tau}{\partial t} \frac{\partial T}{\partial t} + PW^2 + \frac{P}{K} \frac{W^2}{\overline{B}^2} \frac{dW}{ds} t^2 \left(\frac{p}{\tau^2 T^{1-\omega}}\right) \frac{\partial T}{\partial t}$$

$$- \frac{P}{K} \frac{W^3}{\overline{B}^2} t \left(\frac{p}{\tau^2 T^{1-\omega}}\right) \frac{\partial T}{\partial s} = 0 . \tag{10.01c}$$

The boundary conditions for the pressure, shear, and temperature in the viscous shock layer are

$$p(s,1) = \frac{\sin^2\Phi}{2} , \tag{10.02a}$$

$$\frac{\partial \tau}{\partial t}(s,0) = 0,$$

$$\tau(s,1) = \frac{\sin\Phi(\cos\Phi - W)}{K\overline{B}} ,$$

$$\frac{\partial \tau}{\partial t}(s,1) = - \frac{W\sin\Phi}{K\overline{B}} + \frac{W^2 \sin\Phi}{2\overline{B}\,\Theta^{1-\omega}(\cos\Phi - W)} \frac{dW}{ds} , \tag{10.02b}$$

$$T(s,0) = T_w(s) \ ,$$

$$T(s,1) = \Theta(s) \ ,$$

$$\frac{\partial T}{\partial t}(s,1) = \frac{PW}{(\cos\Phi - W)}\left[\left(\frac{\sin^2\Phi}{2} - \Theta\right) + \frac{1}{2}(\cos\Phi - W)^2\right] \ , \qquad (10.02c)$$

where $T_w(s)$ is considered to be a known even function of s. It is important to remember that, in addition to determining $\tau(s,0)$ and $\partial T/\partial t(s,0)$ from the solution of the above equations, the functions W(s) and $\Theta(s)$ must also be determined from such a solution.

Consider now Eq. (10.01) as the following system of five first-order partial differential equations for the variables p, τ, T, $Z_1 = (\partial\tau/\partial t)$, and $Z_2 = (\partial T/\partial t)$:

$$\frac{\partial p}{\partial t} - \frac{a_1(s)t^2 p}{\tau T^{1-\omega}} = 0 \ , \qquad (10.03a)$$

$$\frac{\partial Z_1}{\partial t} - \frac{a_2(s)t^2}{\tau T^{1-\omega}}\frac{\partial p}{\partial t} + \frac{a_2(s)t^2 p Z_1}{\tau^2 T^{1-\omega}}$$

$$+ \frac{(1-\omega)a_2(s)t^2 p \, Z_2}{\tau \, T^{2-\omega}}$$

$$+ \frac{a_3(s)t}{\tau T^{1-\omega}}\frac{\partial p}{\partial s} - \frac{a_3(s)\,t\,p}{\tau^2 T^{1-\omega}}\frac{\partial \tau}{\partial s} - \frac{(1-\omega)a_3(s)t\,p}{\tau \; T^{2-\omega}}\frac{\partial T}{\partial s} = 0, \qquad (10.03b)$$

$$\frac{\partial Z_2}{\partial t} + \frac{(1-P)}{\tau} Z_1 Z_2 + PW^2$$

$$+ \frac{P\,a_2(s)t^2 p\,Z_2}{\tau^2 T^{1-\omega}} - \frac{Pa_3(s)\,t\,p}{\tau^2 \, T^{1-\omega}}\frac{\partial T}{\partial s} = 0 \ , \qquad (10.03c)$$

$$\frac{\partial \tau}{\partial t} - Z_1 = 0 \ , \qquad (10.03d)$$

$$\frac{\partial T}{\partial t} - Z_2 = 0 \ , \qquad (10.03e)$$

where

$$a_1(s) = \frac{\overline{\kappa}(s)\, W^3(s)}{2\ \overline{B}(s)} \ , \qquad (10.04a)$$

$$a_2(s) = \frac{1}{K}\ \frac{W^2(s)}{\overline{B}^2(s)}\ \frac{dW(s)}{ds} \ , \qquad (10.04b)$$

$$a_3(s) = \frac{1}{K}\ \frac{W^3(s)}{\overline{B}^2(s)} \ . \qquad (10.04c)$$

If the equation of the characteristics is

$$\Psi(s,t) = \text{constant}, \qquad (10.05)$$

the characteristic condition, thus, for $\Psi_t = (\partial\Psi/\partial t)$ and $\Psi_s = (\partial\Psi/\partial s)$, becomes

$$\begin{vmatrix} \Psi_t & 0 & 0 & 0 & 0 \\ A_{21} & A_{22} & A_{23} & \Psi_t & 0 \\ 0 & 0 & A_{33} & 0 & \Psi_t \\ 0 & \Psi_t & 0 & 0 & 0 \\ 0 & 0 & \Psi_t & 0 & 0 \end{vmatrix} = 0, \qquad (10.06a)$$

if

$$A_{21} = \frac{-a_2 t^2 \Psi_t}{\tau\ T^{1-\omega}} + \frac{a_3 t \Psi_s}{\tau\ T^{1-\omega}} \ ,$$

$$A_{22} = \frac{-a_3 tp\Psi_s}{\tau^2 T^{1-\omega}} ,$$

$$A_{23} = \frac{-(1-\omega)a_3 tp\Psi_s}{\tau T^{2-\omega}} ,$$

$$A_{33} = \frac{-Pa_3 tp\Psi_s}{\tau^2 T^{1-\omega}} .$$

This determinant simplifies to

$$(\Psi_t)^5 = 0 . \tag{10.06b}$$

This means that the system of equations given in Eq. (10.01) or Eq. (10.03) is of the parabolic type and that the characteristics are the lines s = constant. For such a system, one needs to be given initial data on the characteristic s = 0 as well as along the lines t = 0 and t = 1, as given in Eq. (10.02).

Now, from the symmetry of the problem, near the stagnation line where s = 0, the pressure, shear, and temperature must have expansions of the form

$$p(s,t) = p_o(t) + p_2(t)s^2 + \ldots , \quad \lim_{s\to 0} p(s,t) = p_o(t) , \tag{10.07a}$$

$$\tau(s,t) = \tau_o(t) + \tau_2(t)s^2 + \ldots , \quad \lim_{s\to 0} \tau(s,t) = \tau_o(t) , \tag{10.07b}$$

$$T(s,t) = T_o(t) + T_2(t)s^2 + \ldots , \quad \lim_{s\to 0} T(s,t) = T_o(t) . \tag{10.07c}$$

Also, from symmetry considerations, the expansions of the other pertinent quantities near the axis of symmetry must be

$$W(s) = W_1 s + O(s^3) \quad , \quad \lim_{s \to 0} \frac{W(s)}{s} = W_1, \text{ constant,}$$

$$\Theta(s) = \Theta_o + O(s^2) \; , \qquad \lim_{s \to 0} \Theta(s) = \Theta_o, \text{ constant .} \tag{10.08}$$

$$\lim_{s \to 0} \bar{\kappa}(s) = 1, \quad \lim_{s \to 0} \frac{\overline{B}(s)}{s} = 1 \; , \tag{10.09a}$$

$$\lim_{s \to 0} \sin\Phi(s) = 1 \; , \quad \lim_{s \to 0} \frac{\cos\Phi(s)}{s} = 1 \; . \tag{10.09b}$$

Therefore, in Eq. (10.03), as $s \to 0$, the coefficients of $(\partial p/\partial s)$, $(\partial \tau/\partial s)$ and $(\partial T/\partial s)$ go to zero. Hence, at $s = 0$, the system of partial differential equations of Eq. (10.03) becomes a system of ordinary differential equations with two-point boundary conditions.

The solution of this system of ordinary differential equations, subject to its boundary conditions, generates the initial data on $s = 0$ that are necessary for completely specifying the parabolic type system. However, this system of ordinary differential equations is important in its own right since it describes the behavior of the flow in the vicinity of the stagnation line, a region of great interest to the aerodynamicist for both practical and theoretical reasons. With this in mind, as well as the greater degree of difficulty involved in solving the more general problem, the rest of this section is devoted to solving the system of ordinary differential equations near the axis of symmetry.

Substitution of the expansions of Eqs. (10.07), (10.08) and (10.09) into Eq. (10.01) or Eq. (10.03) yields the desired ordinary differential equations along the stagnation line which are

$$\frac{dp_o}{dt} = 0 , \tag{10.10}$$

$$\frac{d^2\tau_o}{dt^2} = \frac{W_1^3 t^2}{K} \frac{d}{dt} \left(\frac{p_o}{\tau_o T_o^{1-\omega}}\right) = 0 , \tag{10.11}$$

$$\frac{d^2 T_o}{dt^2} + \frac{1-P}{\tau_o} \frac{d\tau_o}{dt} \frac{dT_o}{dt} + \frac{P W_1^3 t^2 p_o}{K \tau_o^2 T_o^{1-\omega}} \frac{dT_o}{dt} = 0 . \tag{10.12}$$

The boundary conditions associated with these equations are

$$p_o(1) = \frac{1}{2} , \tag{10.13}$$

$$\frac{d\tau_o}{dt}(0) = 0 ,$$

$$\tau_o(1) = \frac{1-W_1}{K} , \frac{d\tau_o}{dt}(1) = -\frac{W_1}{K} + \frac{W_1^3}{2\Theta_o^{1-\omega}(1-W_1)} , \tag{10.14}$$

$$T_o(0) = T_{o,w} , T_o(1) = \Theta_o , \frac{dT_o}{dt}(1) = \frac{P W_1 (\frac{1}{2} - \Theta_o)}{1-W_1} , \tag{10.15}$$

where $T_{o,w}$ is a specified constant. From Eq. (10.10) and Eq. (10.13) it is clear that the pressure along the axis of symmetry is constant and is

$$p_o(t) = \frac{1}{2} . \tag{10.16}$$

Consider new shear and temperature variables, F(t) and G(t), defined by

$$\tau_o(t) = \left[\frac{W_1^{\,3}}{2K\Theta_o^{\,1-\omega}}\right]^{\frac{1}{2}} F(t), \quad T_o(t) = \Theta_o\, G(t). \tag{10.17}$$

Using Eqs. (10.16) and (10.17) the momentum and energy equations become

$$\frac{d^2F}{dt^2} + \frac{t^2}{F^2G^{1-\omega}}\frac{dF}{dt} + \frac{(1-\omega)t^2}{F\,G^{2-\omega}}\frac{dG}{dt} = 0\ , \tag{10.18}$$

$$\frac{d^2G}{dt^2} + \frac{(1-P)}{F}\frac{dF}{dt}\frac{dG}{dt} + \frac{Pt^2}{F^2G^{1-\omega}}\frac{dG}{dt} = 0. \tag{10.19}$$

The boundary conditions for the new variables are

$$G(0) = \frac{T_{o,w}}{\Theta_o}\ , \quad G(1) = 1, \quad G'(1) = P\left(\frac{\frac{1}{2}-\Theta_o}{\Theta_o}\right)\left(\frac{W_1}{1-W_1}\right)\ , \tag{10.20}$$

$$F'(0) = 0, \qquad F(1) = (1-W_1)\left(\frac{2\Theta_o^{\,1-\omega}}{KW_1^{\,3}}\right)^{\frac{1}{2}},$$

$$F'(1) = -W_1\left(\frac{2\Theta_o^{\,1-\omega}}{KW_1^{\,3}}\right)^{\frac{1}{2}} + \frac{1}{1-W_1}\left(\frac{KW_1^3}{2\Theta_o^{\,1-\omega}}\right)^{\frac{1}{2}}\ . \tag{10.21}$$

From these equations, it can be seen that

$$W_1 = \frac{1-F(1)F'(1)}{1-F(1)F'(1)+F^2(1)}\ , \tag{10.22}$$

$$\Theta_o = \frac{P}{2}\,\frac{W_1}{G'(1)+W_1[\,P-G'(1)]}\ , \tag{10.23}$$

$$K = \frac{2(1-W_1)^2\Theta_o^{\,1-\omega}}{W_1^{\,3}F^2(1)}\ . \tag{10.24}$$

Defining F(0) as F_o and G'(0) as Q_o and taking the temperature at the wall as zero, i.e., G(0) = 0, the power series solutions for F(t) and G(t), valid near the wall, are

$$F(t) = F_o [1 - \frac{(1-\omega)t^{2+\omega}}{(1+\omega)(2+\omega)F_o^2 Q_o^{1-\omega}} + O(t^{4+2\omega})] , \tag{10.25}$$

$$G(t) = Q_o t [1 - \frac{P}{(2+\omega)(3+\omega)} \{1 - \frac{1-P}{P} \frac{1-\omega}{1+\omega}\} \frac{t^{2+\omega}}{F_o^2 Q_o^{1-\omega}} + O(t^{4+2\omega})] . \tag{10.26}$$

Consider now the quantities

$$\phi = \frac{t}{F}\frac{dF}{dt} , \quad \psi = \frac{t}{G}\frac{dG}{dt} , \quad Q = \frac{G}{t} , \quad \zeta = \frac{t^3}{F^2 G^{1-\omega}} , \quad \nu = t^{2+\omega} . \tag{10.27}$$

If these quantities are substituted into Eqs. (10.18) and (10.19), four first order ordinary differential equations for ϕ, ψ, Q, and ν as functions of ζ are obtained. These four equations are

$$\frac{d\phi}{d\zeta} = - \frac{\phi(\phi-1) + \zeta[\phi + (1-\omega)\psi]}{\zeta[3-2\phi-(1-\omega)\psi]} , \tag{10.28}$$

$$\frac{d\psi}{d\zeta} = - \frac{\psi[(\psi-1)+(1-P)\phi+P\zeta]}{\zeta[3-2\phi-(1-\omega)\psi]} , \tag{10.29}$$

$$\frac{d\nu}{d\zeta} = \frac{(2+\omega)\nu}{\zeta[3-2\phi-(1-\omega)\psi]} , \tag{10.30}$$

$$\frac{dQ}{d\zeta} = \frac{Q(\psi-1)}{\zeta[3-2\phi-(1-\omega)\psi]} . \tag{10.31}$$

Making use of Eqs. (10.25) and (10.26), it can be seen that, if the temperature at the body is zero, the case that shall be considered, the boundary conditions at the wall are

$$\zeta_w = 0, \quad \phi_w = 0, \quad \psi_w = 1, \quad Q_w = Q_o, \quad \nu_w = 0 , \tag{10.32a}$$

$$(\frac{d\phi}{d\zeta})_w = - (\frac{1-\omega}{1+\omega}) , \quad (\frac{d\psi}{d\zeta})_w = - \frac{P}{3+\omega} (1 - \frac{1-P}{P} \frac{1-\omega}{1+\omega}) ,$$

$$(\frac{d\nu}{d\zeta})_w = F_o^{\,2} Q_o^{\,1-\omega}, \quad (\frac{dQ}{d\zeta})_w = - \frac{PQ_o}{(2+\omega)(3+\omega)} (1 - \frac{1-P}{P} \frac{1-\omega}{1+\omega}) . \tag{10.32b}$$

The boundary conditions at the outer edge of the shock layer (t = 1) are

$$\zeta_e = \frac{K\, W_1^3}{2\Theta_o^{1-\omega}(1-W_1)^2} ,$$

$$\phi_e = \frac{-W_1}{1-W_1} + \frac{KW_1^3}{2\Theta_o^{1-\omega}(1-W_1)^2} ,$$

$$\psi_e = P\, (\frac{1-2\Theta_o}{2\Theta_o})\, (\frac{W_1}{1-W_1}) ,$$

$$Q_e = \nu_e = 1 . \tag{10.33}$$

It should be noted that the quantities W_1, Θ_o, and K, in terms of the boundary conditions at the outer edge of the shock layer, ζ_e, ϕ_e, and ψ_e, are

$$W_1 = \frac{(\zeta_e - \phi_e)}{1+(\zeta_e - \phi_e)} ,$$

$$\Theta_o = \frac{P(\zeta_e - \phi_e)}{2[\psi_e + P(\zeta_e - \phi_e)]} ,$$

$$K = \frac{2\, P^{1-\omega} \zeta_e\, [1 + (\zeta_e - \phi_e)]}{(\zeta_e - \phi_e)^{2+\omega} [\psi_e + P(\zeta_e - \phi_e)]^{1-\omega}} . \tag{10.34}$$

If, instead of Q and ν, one considers the variables $\overline{Q}$ and $\overline{\nu}$, defined by

$$\overline{Q} = Q/Q_o, \; \overline{\nu} = \nu/(F_o^2 Q_o^{1-\omega}) , \tag{10.35}$$

then Eqs. (10.28) and (10.29) are unchanged and Eqs. (10.30) and (10.31) are changed (just slightly) to

$$\frac{d\overline{\nu}}{d\zeta} = \frac{(2+\omega)\,\overline{\nu}}{\zeta[3-2\phi-(1-\omega)\psi]} , \quad \frac{d\overline{Q}}{d\zeta} = \frac{(\psi-1)\,\overline{Q}}{\zeta[3-2\phi-(1-\omega)\psi]} . \tag{10.36,.37}$$

The boundary conditions for $\overline{Q}$ and $\overline{\nu}$ are

$$\overline{Q}_w = 1, \; \left(\frac{d\overline{Q}}{d\zeta}\right)_w = -\frac{P}{(2+\omega)(3+\omega)}\left(1 - \frac{1-P}{P}\,\frac{1-\omega}{1+\omega}\right), \; \overline{Q}_e = \frac{1}{Q_o} , \tag{10.38a}$$

$$\overline{\nu}_w = 0, \; \left(\frac{d\overline{\nu}}{d\zeta}\right)_w = 1, \quad \overline{\nu}_e = \frac{1}{F_o^2 Q_o^{1-\omega}} . \tag{10.38b}$$

B. The General Procedure for Solution of the Equations

From the differential equations, Eqs. (10.28), (10.29), (10.36) and (10.37), and the boundary conditions at the wall, Eqs. (10.32) and (10.38), it can be seen that solution of these differential equations may be initiated at the wall ($\zeta = \zeta_w = 0$) without prior knowledge as to the values of K, F_o, Q_o, W_1, or Θ_o. These quantities are then determined by the choice of ζ_e which terminates the solution of the equations.

The preceding means that the general procedure for computing the solution of the equations is the following:

(i) Integrate Eqs. (10.28), (10.29), (10.36) and (10.37) starting at $\zeta = 0$.

(ii) Choose the value of ζ_e and, thus, determine $\phi_e(\zeta_e)$, $\psi_e(\zeta_e)$, $\overline{Q}_e(\zeta_e)$ and $\overline{\nu}_e(\zeta_e)$ from the integrations.

(iii) Calculate Q_o and F_o from

$$Q_o = \frac{1}{\overline{Q}_e(\zeta_e)} \quad , \quad F_o = [Q_o^{1-\omega}\overline{\nu}_e(\zeta_e)]^{-\frac{1}{2}} . \tag{10.39}$$

(iv) Knowing ζ_e and $\phi_e(\zeta_e)$ and $\psi_e(\zeta_e)$, calculate W_1, Θ_o, and K using Eq. (10.34).

In general, a numerical solution of the equations is required. However, the special case of $\omega = 1$ should be discussed because it is the one case for which the solutions can be expressed in terms of tabulated functions. For $\omega = 1$, Eqs. (10.18) and (10.19) reduce to

$$\frac{d^2F}{dt^2} + \frac{t^2}{F^2}\frac{dF}{dt} = 0 \quad ,$$

$$\frac{d^2G}{dt^2} + \frac{(1-P)}{F}\frac{dF}{dt}\frac{dG}{dt} + \frac{Pt^2}{F^2}\frac{dG}{dt} = 0 \, . \tag{10.40}$$

The only solution for F consistent with boundary condition $F'(0) = 0$ is

$$F = F_o = \text{constant}. \tag{10.41}$$

Making use of Eqs. (10.21) and (10.41), a little algebraic manipulation gives

$$F_o = \left[\frac{(1+K)-(1+2K)^{\frac{1}{2}}}{(1+2K)^{\frac{1}{2}} - 1}\right]^{\frac{1}{2}} , \quad W_1 = \frac{(1+2K)^{\frac{1}{2}}-1}{K} . \tag{10.42}$$

With $F = F_o$ = constant, the energy equation simplifies to

$$\frac{d^2G}{dt^2} + \frac{Pt^2}{F_o^{\,2}} \frac{dG}{dt} = 0 . \tag{10.43}$$

The solution of this equation is

$$G = Q_o \left(\frac{F_o^{\,2}}{9P}\right)^{1/3} \Gamma\left(\frac{1}{3} , \frac{Pt^3}{3F_o^{\,2}}\right) , \tag{10.44}$$

where Γ is the incomplete gamma function. Q_o and Θ_o are determined from

$$Q_o = \frac{(F_o^{\,2}/9P)^{-1/3}}{\Gamma(\frac{1}{3}, \frac{P}{3F_o^{\,2}})} ,$$

$$\Theta_o = \frac{PW_1}{2(1-W_1)[Q_o \exp(-P/3F_o^{\,2})+P(W_1/1-W_1)]} . \tag{10.45}$$

The quantities F_o and W_1 in this equation are those determined by Eq. (10.42).

The numerical calculations for the stagnation region of the viscous shock layer were performed in the manner described above for the case of $\omega = 1/2$ and $P = 3/4$. Under the supervision of Dr. M. T. Chahine, these calculations were carried on the Jet Propulsion Laboratory's IBM 7090 digital computer using the fourth-order Runge-Kutta method.

C. Approximate Solution of the Equations

In Part B of this section the method for obtaining numerical solutions of the equations is presented. This procedure also offers a starting point for an iterative process, the first step of which will be considered to give approximate solutions of the equations.

From a consideration of Eqs. (10.28), (10.29), (10.36), and (10.37) the equations relating the successive approximations are taken to be

$$\frac{d\phi_{(j+1)}}{\phi_{(j+1)}} = \frac{[(1-\phi_{(j)}) + \frac{\zeta}{\phi_{(j)}}\{\phi_{(j)} + (1-\omega)\psi_{(j)}\}]}{\zeta[3-2\phi_{(j)} - (1-\omega)\psi_{(j)}]}\, d\zeta \; , \tag{10.46a}$$

$$\frac{d\psi_{(j+1)}}{\psi_{(j+1)}} = -\frac{[(\psi_{(j)}-1) + (1-P)\phi_{(j)} + P\zeta]}{\zeta[3-2\phi_{(j)} - (1-\omega)\psi_{(j)}]}\, d\zeta \; , \tag{10.46b}$$

$$\frac{d\bar{\nu}_{(j+1)}}{\bar{\nu}_{(j+1)}} = \frac{2+\omega}{\zeta[3-2\phi_{(j)} - (1-\omega)\psi_{(j)}]}\, d\zeta \; , \tag{10.46c}$$

$$\frac{d\overline{Q}_{(j+1)}}{Q_{(j+1)}} = \frac{\psi_{(j)}-1}{\zeta[3-2\phi_{(j)} - (1-\omega)\psi_{(j)}]}\, d\zeta \; , \tag{10.46d}$$

Once the forms of $\phi_{(o)}$ and $\psi_{(o)}$, the zeroth approximations, have been assumed to initiate the iteration process, $\phi_{(1)}$, $\psi_{(1)}$, $\bar{\nu}_{(1)}$, and $\overline{Q}_{(1)}$ can be determined from the above equations, using the boundary conditions of Eqs. (10.32), (10.33), and (10.38). Knowing the first approximations for ϕ, ψ, $\overline{\nu}$, and $\overline{Q}$, the second approximation may be calculated, and so forth.

The forms of the zeroth approximations are chosen so as to satisfy the ϕ and ψ boundary conditions at $\zeta = 0$ and are

$$\phi_{(o)} = -\left(\frac{1-\omega}{1+\omega}\right)\zeta \;, \quad \psi_{(o)} = 1 - \left[\frac{P}{3+\omega}\left(1 - \frac{1-P}{P}\,\frac{1-\omega}{1+\omega}\right)\right]\zeta \equiv 1 - b\zeta \;. \qquad (10.47)$$

Substituting Eq. (10.47) into Eq. (10.46), one gets

$$\begin{aligned} \phi_{(1)} &= 0 \quad \text{for } \omega = 1 \\ &= -\left(\frac{1-\omega}{1+\omega}\right)\frac{\zeta}{(1+k_2\zeta)^{k_1}} \quad \text{for } \omega \neq 1 \;, \end{aligned} \qquad (10.48a)$$

$$\begin{aligned} \psi_{(1)} &= \exp(-b\zeta) \quad \text{for } \omega = 1 \\ &= \frac{1}{(1+k_2\zeta)^{k_3}} \quad \text{for } \omega \neq 1 \;, \end{aligned} \qquad (10.48b)$$

$$\overline{\nu}_{(1)} = \frac{\zeta}{1+k_2\zeta} \;, \qquad (10.48c)$$

$$\begin{aligned} \overline{Q}_{(1)} &= \exp(-b\zeta/3) \quad \text{for } \omega = 1 \\ &= \frac{1}{(1+k_2\zeta)^{k_4}} \quad \text{for } \omega \neq 1 \;, \end{aligned} \qquad (10.48d)$$

where k_1, k_2, k_3, and k_4 are constants given by

$$k_1 = 1 + \frac{2\omega + b\ (1+\omega)^2}{(1-\omega)[2+b(1+\omega)]} \quad , \tag{10.49a}$$

$$k_2 = \frac{(1-\omega)[2+b(1+\omega)]}{(1+\omega)(2+\omega)} \quad , \tag{10.49b}$$

$$k_3 = b/k_2 \ , \qquad k_4 = k_3/(2+\omega) \quad . \tag{10.49c,d}$$

Thus, taking ζ_e as the parameter, the solution of the equations gives

$$\phi_{e(1)} = 0 \qquad \text{for } \omega = 1$$

$$= -\left(\frac{1-\omega}{1+\omega}\right)\frac{\zeta_e}{(1+k_2\zeta_e)^{k_1}} \quad \text{for } \omega \neq 1 \ , \tag{10.50a}$$

$$\psi_{e(1)} = \exp\ (-b\zeta_e) \qquad \text{for } \omega = 1 \ ,$$

$$= \frac{1}{(1+k_2\zeta_e)^{k_3}} \qquad \text{for } \omega \neq 1 \ , \tag{10.50b}$$

$$\bar{\nu}_{e(1)} = \frac{\zeta_e}{1+k_2\zeta_e} \quad , \tag{10.50c}$$

$$\overline{Q}_{e(1)} = \exp\ (-b\zeta_e/3) \qquad \text{for } \omega = 1$$

$$= \frac{1}{(1+k_2\zeta_e)^{k_4}} \qquad \text{for } \omega \neq 1 \ . \tag{10.50d}$$

From the boundary conditions at the outer edge of the shock layer, it follows that

$$Q_o = \frac{1}{\overline{Q}_{e(1)}} = \exp (b\zeta_e/3) \quad \text{for } \omega = 1$$

$$= (1 + k_2\zeta_e)^{k_4} \text{ for } \omega \neq 1 \ , \qquad (10.51a)$$

$$F_o = [Q_o^{1-\omega} \ \overline{\nu}_{e(1)}]^{-1/2} = (1+k_2\zeta_e)^{k_5}/\zeta_e^{1/2} \ , \qquad (10.51b)$$

where $k_5 = [1/\{2+b(1+\omega)\}]$. In addition, W_1, Θ_o and K are determined by substituting Eqs. (10.50a) and (10.50b) into the right hand side of Eq. (10.34).

In Table 1 and Fig. 3 are shown C_f and C_h as functions of K for $\omega = 1/2$ and $P = 3/4$, where

$$C_f = \left(\frac{W_1^{\ 3}}{2K\Theta_o^{1-\omega}}\right)^{1/2} F_o = \tau_o \ (t = 0) \qquad (10.52a)$$

and

$$C_h = \left(\frac{\Theta_o Q_o}{W_1}\right) C_f = \frac{\tau_o(t=0)}{W_1} \ \frac{dT_o}{dt} \ (t = 0) \qquad (10.52b)$$

are the shear and heat transfer at the nose of the body. For given values of ϵ and δ, it can be seen that the shear and heat transfer at the nose decrease as the Reynolds number decreases (K increases). The reason for this is that the velocity gradient at the nose, W_1, and temperature at the outer edge, Θ_o, decrease much more rapidly than

F_o and Q_o increase as K increases. From the table and figure, the approximate results are seen to agree quite well with the exact ones.

XI. THE CORRECTION (OR VORTICITY INTERACTION) LAYER

The inviscid shock layer solution presented is non-uniform at the body surface (cf. CHESTER, [4]). This suggests that there should exist a correction layer to the inviscid shock layer which will remove this non-uniformity. Subject to verification by matching with the inviscid shock layer, the quantities in this correction layer are taken to be

$$\overline{u} = \epsilon^{\frac{1}{2}} u_c + \dots \quad , \tag{11.01a}$$

$$\overline{v} = \epsilon^{2} v_c + \dots \quad , \tag{11.01b}$$

$$\overline{\rho} = \frac{1}{\epsilon} \rho_c + \dots \quad , \tag{11.01c}$$

$$\overline{T} = \frac{1}{\delta} T_c + \dots \quad , \tag{11.01d}$$

$$\overline{p} = \frac{1}{\epsilon\delta} p_c + \dots \quad , \tag{11.01e}$$

$$\overline{y} = \epsilon^{3/2} \eta_c \quad . \tag{11.01f}$$

The leading terms of the equations of motion for this region are

$$p_c = \rho_c T_c \ , \tag{11.02a}$$

$$\frac{\partial}{\partial \xi}(\overline{B}\, \rho_c u_c) + \frac{\partial}{\partial \eta_c}(\overline{B}\, \rho_c v_c) = 0 \ , \tag{11.02b}$$

$$\frac{\partial p_c}{\partial \eta_c} = 0 \ , \tag{11.02c}$$

$$\rho_c\left(u_c \frac{\partial u_c}{\partial \xi} + v_c \frac{\partial u_c}{\partial \eta_c}\right) + 2\frac{\partial p_c}{\partial \xi} = D \frac{\partial}{\partial \eta_c}\left(T_c^{\omega} \frac{\partial u_c}{\partial \eta_c}\right) \ , \tag{11.02d}$$

$$\rho_c\left(u_c \frac{\partial T_c}{\partial \xi} + v_c \frac{\partial T_c}{\partial \eta_c}\right) = \frac{D}{P} \frac{\partial}{\partial \eta_c}\left(T_c^{\omega} \frac{\partial T_c}{\partial \eta_c}\right) \ , \tag{11.02e}$$

where $D \equiv (1/\epsilon^{5/2} R\delta^{\omega})$. The quantity D goes to zero for an inviscid correction layer and is O(1) for a viscous correction layer. Note that there is no dissipation term in the viscous correction layer's energy equation, due to the low velocities in the layer.

Since the inviscid shock layer was solved after the introduction of a modified Crocco transformation, in order to perform a matching of the correction layer with this layer, it is convenient to employ a similar transformation on the correction layer. This transformation is defined by

$$(\xi, \eta_c) \rightarrow (s, t_c), \; s = \xi, \; t_c = u_c/\cos\Phi \,, \tag{II.03a}$$

$$\tau_c = \frac{T_c^{\omega}}{\overline{B}} \frac{\partial u_c}{\partial \eta_c} \,. \tag{II.03b}$$

The pertinent equations after this transformation are

$$D \frac{\partial^2 \tau_c}{\partial t_c^2} + \frac{\cos^2\Phi}{\overline{B}^2}\left[t_c \cos\Phi \frac{\partial}{\partial s}\left(\frac{p_c}{\tau_c T_c^{1-\omega}}\right) - \overline{\kappa}\, t_c^2 \sin\Phi \frac{\partial}{\partial t_c}\left(\frac{p_c}{\tau_c T_c^{1-\omega}}\right) - \frac{2}{\cos\Phi}\frac{1}{p_c}\frac{dp_c}{ds}\frac{\partial}{\partial t_c}\left(T_c \frac{p_c}{\tau_c T_c^{1-\omega}}\right)\right] = 0, \tag{II.04a}$$

$$\frac{D}{P}\tau_c \left[\frac{\partial^2 T_c}{\partial t_c^2} + \frac{(1-P)}{\tau_c}\frac{\partial \tau_c}{\partial t_c}\frac{\partial T_c}{\partial t_c}\right] + \frac{\cos^2\Phi}{\overline{B}^2}\left(\frac{p_c}{\tau_c T_c^{1-\omega}}\right) \times \left[-t_c \cos\Phi \frac{\partial T_c}{\partial s} + \overline{\kappa}\, t_c^2 \sin\Phi \frac{\partial T_c}{\partial t_c} + \frac{2}{\cos\Phi}\frac{T_c}{p_c}\frac{dp_c}{ds}\frac{\partial T_c}{\partial t_c}\right] = 0. \tag{II.04b}$$

$$\rho_c v_c = D \frac{\overline{B}}{\cos\Phi} \frac{\partial \tau_c}{\partial t_c}$$

$$+\left[\frac{p_c \cos\Phi}{T_c} t_c \frac{\partial \eta_c}{\partial s} - \frac{\overline{\kappa}\sin\Phi\cos\Phi}{\overline{B}} t_c^2 \left(\frac{p_c}{\tau_c T_c^{1-\omega}}\right) - \frac{2}{\overline{B}} \frac{T_c}{p_c} \frac{dp_c}{ds} \left(\frac{p_c}{\tau_c T_c^{1-\omega}}\right)\right].$$

(11.04c)

The matching between the correction layer and inviscid shock layer takes place with respect to t. The correction layer expansions, as $t_c \to \infty$, must match with the shock layer expansions, as $t_L \to 0$. The class of intermediate variables is

$$\overleftrightarrow{t} = \frac{\overline{u}}{\overleftrightarrow{\alpha} \cos\Phi} + \ldots, \text{ where } \overleftrightarrow{\alpha} \to 0, \frac{\overleftrightarrow{\alpha}}{\epsilon^{1/2}} \to \infty \text{ as } \epsilon \to 0 , \tag{11.05a}$$

$$t_L = \overleftrightarrow{\alpha}\,\overleftrightarrow{t} + \ldots \to 0,\ t_c = \frac{\overleftrightarrow{\alpha}\,\overleftrightarrow{t}}{\epsilon^{1/2}} + \ldots \to \infty \text{ for } \overleftrightarrow{t} \text{ fixed} . \tag{11.05b}$$

The intermediate expansions for $\overline{p}$ are

$$\overline{p} = \frac{1}{\epsilon\delta}\left[\frac{\sin^2\Phi(s)}{2} - \frac{\overline{\kappa}(s)}{2\overline{B}(s)} \int_0^{\cos\Phi(s)} \frac{\nu \overline{B}(\nu) d\nu}{\overline{\kappa}(\nu)} + \ldots\right]$$

$$= \frac{1}{\epsilon\delta}\left[p_c(s) + \ldots\right] \quad , \tag{11.06}$$

so that

$$p_c(s) = \frac{\sin^2\Phi(s)}{2} - \frac{\overline{\kappa}(s)}{2\overline{B}(s)} \int_0^{\cos\Phi(s)} \frac{\nu\, \overline{B}(\nu) d\nu}{\overline{\kappa}(\nu)} \quad . \tag{11.07}$$

For the temperature the intermediate expansions are

$$\overline{T} = \frac{1}{\delta}\left[\frac{1 - \overleftrightarrow{\alpha}^2\, \overleftrightarrow{t}^2 \cos^2\Phi(s)}{2} + \ldots\right]$$

$$= \frac{1}{\delta}\left[T_c(s,\infty) + \ldots\right] \tag{11.08}$$

and, hence,

$$T_c(s,\infty) = \frac{1}{2} \tag{11.09}$$

The matching for the shear is a little more difficult. First of all,

$$\frac{\overline{T}^\omega}{\overline{B}}\frac{\partial \overline{u}}{\partial \overline{y}} = \frac{1}{\epsilon\,\delta^\omega}\;\frac{T_L{}^\omega}{\overline{B}}\frac{\partial u_L}{\partial \eta_L} + \ldots = \frac{1}{\epsilon\delta^\omega}\,\tau_L + \ldots$$

$$= \frac{1}{\epsilon\delta^\omega}\;\frac{T_c{}^\omega}{\overline{B}}\frac{\partial u_c}{\partial \eta_c} + \ldots = \frac{1}{\epsilon\,\delta^\omega}\,\tau_c + \ldots \tag{11.10}$$

For the inviscid shock layer

$$\tau_L(s,t_L) = \frac{p_L(s,t_L)}{T_L^{1-\omega}(s,t_L)}\quad\frac{t_L\cos\Phi\,\overline{\kappa}\,(t_L\cos\Phi)}{\overline{B}(t_L\cos\Phi)}\ , \tag{11.11}$$

and, therefore, the intermediate expansion for τ_L is

$$\tau_L(s,\ \overleftrightarrow{\alpha}\,\overleftrightarrow{t} \to 0) = \frac{p_c(s)}{(1/2)^{1-\omega}} + \ldots\ . \tag{11.12}$$

Therefore, whether the correction layer is viscous or inviscid, from Eq. (11.10),

$$\tau_c(s,\infty) = 2^{1-\omega}\, p_c(s) \;. \tag{II.13}$$

The inviscid boundary condition at the body is

$$v_c(s,t_{c,w}) = 0 \;. \tag{II.14}$$

The viscous boundary conditions at the body are

$T_c(s,0) = T_{c,w}(s)$, a known function,

$$\frac{\partial \tau_c}{\partial t_c}(s,0) = \frac{2}{\bar{D}}\,\frac{\cos\Phi}{\bar{B}^2}\,\frac{dp_c}{ds}\,\frac{T_c^{\omega}(s,0)}{\tau_c(s,0)} \;. \tag{II.15}$$

XII. INVISCID VORTICITY LAYER

For the inviscid vorticity layer the equations of motion are

$$t \cos\Phi \frac{\partial}{\partial s}\left(\frac{p}{\tau T^{1-\omega}}\right) - \bar{\kappa}\, t^2 \sin\Phi \frac{\partial}{\partial t}\left(\frac{p}{\tau T^{1-\omega}}\right)$$

$$- \frac{2}{\cos\Phi}\, \frac{1}{p}\frac{dp}{ds}\, \frac{\partial}{\partial t}\left(T \frac{p}{\tau T^{1-\omega}}\right) = 0 , \qquad (12.01a)$$

$$t \cos\Phi \frac{\partial T}{\partial s} - \bar{\kappa}\, t^2 \sin\Phi \frac{\partial T}{\partial t} - \frac{2}{\cos\Phi}\, \frac{T}{p}\frac{dp}{ds}\frac{\partial T}{\partial t} = 0 , \qquad (12.01b)$$

$$\frac{\partial \eta}{\partial t} - \frac{\cos\Phi}{\bar{B}}\, \frac{T}{p}\left(\frac{p}{\tau T^{1-\omega}}\right) = 0 , \qquad (12.01c)$$

$$v - t \cos\Phi \frac{\partial \eta}{\partial s} + \frac{\bar{\kappa} \sin\Phi \cos\Phi}{\bar{B}}\, t^2 \frac{T}{p}\left(\frac{p}{\tau T^{1-\omega}}\right)$$

$$+ \frac{2}{\bar{B}}\, \frac{T^2}{p^2}\, \frac{dp}{ds}\left(\frac{p}{\tau\, T^{1-\omega}}\right) = 0 , \qquad (12.01d)$$

where the subscript c has been dropped and where

$$p(s) = \frac{1}{2}\left[\sin^2\Phi(s) - \frac{\bar{\kappa}(s)}{\bar{B}(s)} \int_0^{\cos\Phi(s)} \frac{\nu \bar{B}(\nu) d\nu}{\bar{\kappa}(\nu)}\right] . \qquad (12.02)$$

The inviscid vorticity layer boundary conditions are

$$\frac{p}{\tau T^{1-\omega}}(s,\infty) = 1 , \quad T(s,\infty) = \frac{1}{2} , \qquad (12.03a)$$

$$\eta(s,t_w) = v(s,t_w) = 0 . \qquad (12.03b)$$

The only solutions of Eqs. (12.01a) and (12.01b) that satisfy the boundary conditions of Eq. (12.03a) are the constant solutions,

$$\frac{p}{\tau T^{1-\omega}} = 1, \tag{12.04}$$

$$T = \frac{1}{2} . \tag{12.05}$$

In turn this means that the normal coordinate, η, is given by

$$\eta = \frac{\cos\Phi(s)}{2\overline{B}(s)p(s)} [t - t_w(s)] . \tag{12.06}$$

Finally, the substitution of Eqs. (12.04), (12.05), and (12.06) gives the normal velocity as

$$v = t \cos\Phi \left[\frac{d}{ds}\left(\frac{\cos\Phi}{2\overline{B}\,p}\right) \cdot \{t - t_w\} - \frac{\cos\Phi}{2\overline{B}\,p}\frac{dt_w}{ds}\right]$$

$$- \frac{t^2 \kappa \sin\Phi\cos\Phi}{2\overline{B}p} - \frac{1}{2\overline{B}p^2}\frac{dp}{ds} . \tag{12.07}$$

Applying the boundary condition at the wall, it is found that the normal velocity here is

$$v(s,t_w) = 0 = - \frac{1}{2\overline{B}p}\frac{d}{ds}\left(\frac{t_w^2\cos^2\Phi}{2} + \log_e p\right) . \tag{12.08}$$

To satisfy this boundary condition it is necessary that

$$u_w(s) = t_w(s)\cos\Phi(s) = \left[(\text{constant}) + 2 \log_e \left\{\frac{1}{p(s)}\right\}\right]^{\frac{1}{2}}$$

$$= \left[2 \log_e \left\{\frac{1}{2p(s)}\right\}\right]^{\frac{1}{2}} , \tag{12.09}$$

where the constant of integration is chosen so that the requirement that both sides of the above equation be zero when s = 0 is satisfied.

XIII. VISCOUS VORTICITY LAYER

In Section 11, when D is fixed, it was shown that the vorticity layer is a viscous one. Dropping the subscript c for convenience, the equations that define it are

$$\frac{\partial^2\tau}{\partial t^2}+\frac{1}{D}\,\frac{\cos^2\Phi}{\overline{B}^2}\Big[t\cos\Phi\,\frac{\partial}{\partial s}\Big(\frac{p}{\tau T^{1-\omega}}\Big)-\overline{\kappa}t^2\sin\Phi\,\frac{\partial}{\partial t}\Big(\frac{p}{\tau T^{1-\omega}}\Big)$$

$$-\frac{2}{\cos\Phi}\,\frac{1}{p}\,\frac{dp}{ds}\,\frac{\partial}{\partial t}\Big(T\,\frac{p}{\tau T^{1-\omega}}\Big)\Big]=0\ , \tag{13.01}$$

$$\frac{\partial^2 T}{\partial t^2}+\frac{(1-P)}{\tau}\,\frac{\partial\tau}{\partial t}\,\frac{\partial T}{\partial t}$$

$$+\frac{P}{D}\,\frac{\cos^2\Phi}{\overline{B}^2}\Big(\frac{p}{\tau^2T^{1-\omega}}\Big)\Big[-t\cos\Phi\,\frac{\partial T}{\partial s}+\overline{\kappa}\,t^2\sin\Phi\,\frac{\partial T}{\partial t}+\frac{2}{\cos\Phi}\,\frac{T}{p}\,\frac{dp}{ds}\,\frac{\partial T}{\partial t}\Big]=0. \tag{13.02}$$

The boundary conditions for this system of equations are

$T(s,0)=T_w(s)$, a known even function,

$$\frac{\partial\tau}{\partial t}(s,0)=\frac{2}{D}\,\frac{\cos^2\Phi}{\overline{B}^2}\Big[\frac{1}{\cos\Phi}\,\frac{dp}{ds}\,\frac{T^{\omega}(s,0)}{\tau(s,0)}\Big]\ , \tag{13.03}$$

$$T(s,\infty)=\frac{1}{2}\ ,\quad \tau(s,\infty)=2^{1-\omega}\,p(s)\ , \tag{13.04}$$

where

$$p(s)=\frac{1}{2}\Big[\sin^2\Phi(s)-\frac{\overline{\kappa}(s)}{\overline{B}(s)}\int_0^s\overline{B}(\nu)\sin\Phi(\nu)\cos\Phi(\nu)\,d\nu\Big]\ . \tag{13.05}$$

As in the case of the viscous shock layer, the complexity of these partial differential equations makes their solution quite difficult. However, using arguments that are essentially the same as those for the viscous shock layer concerning the system of equations near the stagnation line, one can find the solution of a system of ordinary differential equations that describes the flow in this region. Therefore, this tack is followed. The expansions of τ, T, and p near the axis of symmetry are

$$\tau = \tau_o(t) + O(s^2) , \tag{13.06a}$$

$$T = T_o(t) + O(s^2) , \tag{13.06b}$$

$$p = \frac{1}{2} [1 - \frac{p_2}{2} s^2 + O(s^4)] , \quad p_2 = \text{constant} > 0 . \tag{13.06c}$$

Similarly, the quantities $\overline{\kappa}$, $\overline{B}$, $\sin\Phi$ and $\cos\Phi$ near the stagnation line may be expanded as

$$\overline{\kappa} = 1 + O(s^2) , \tag{13.07a}$$

$$\overline{B} = s + O(s^3) , \tag{13.07b}$$

$$\sin\Phi = 1 + O(s^2) , \tag{13.07c}$$

$$\cos\Phi = s + O(s^3) . \tag{13.07d}$$

Hence, substitution of these expansions into Eqs. (13.01) and (13.02) gives the following ordinary differential equations for the flow near the axis of symmetry:

$$\frac{d^2\tau_o}{dt^2} + \frac{1}{2D}\left[2p_2 \frac{d}{dt}\left(\frac{T_o^{\omega}}{\tau_o}\right) - t^2 \frac{d}{dt}\left(\frac{1}{\tau_o T_o^{1-\omega}}\right)\right] = 0 , \tag{13.08}$$

$$\frac{d^2T_o}{dt^2} + \frac{(1-P)}{\tau_o}\frac{d\tau_o}{dt}\frac{dT_o}{dt} + \frac{P}{2D}\left[t^2 \frac{dT_o}{dt} - 2p_2 T_o \frac{dT_o}{dt}\right] = 0 . \tag{13.09}$$

The boundary conditions for τ_o and T_o are

$T_o(0) = T_{o,w}$, a known constant,

$$\frac{d\tau_o}{dt}(0) = -\frac{1}{2D}\left[\frac{2p_2 T_o^{\omega}(0)}{\tau_o(0)}\right] , \tag{13.10}$$

$$T_o(\infty) = \frac{1}{2} , \quad \tau_o(\infty) = \frac{1}{2^{\omega}} . \tag{13.11}$$

Consider the new variables

$$z = \left(\frac{2^{\omega}}{D}\right)^{1/3} t, \; F(z) = 2^{\omega}\,\tau_o(t), \quad G(z) = 2T_o(t) . \tag{13.12}$$

The equations become

$$\frac{d^2G}{dz^2} + \frac{(1-P)}{F}\frac{dF}{dz}\frac{dG}{dz} + \frac{P(z^2-AG)}{F^2G^{1-\omega}}\frac{dG}{dz} = 0 , \tag{13.13}$$

$$\frac{d^2F}{dz^2} + \frac{z^2-AG}{F^2G^{1-\omega}}\frac{dF}{dz} = -\frac{(\{1-\omega\}z^2 + \omega AG)}{F\,G^{2-\omega}}\frac{dG}{dz} , \tag{13.14}$$

where

$$A = \left(\frac{2^{\omega}}{D}\right)^{2/3} p_2 . \tag{13.15}$$

The boundary conditions associated with these equations are

$$G(0) = 2T_{o,w}\ , \quad \frac{dF}{dz}(0) = -A\,\frac{G^{\omega}(0)}{F(0)}\ , \tag{13.16}$$

$$G(\infty) = 1, \quad F(\infty) = 1\ . \tag{13.17}$$

Hence, the four parameters of the original problem, D and p_2 and P and ω are reduced to just three, A and P and ω in the system of Eqs. (13.12) - (13.17).

Taking as a special case the case where the temperature at the wall is zero, the four boundary conditions at the wall are

$$F(0) = F_o, \quad G(0) = 0, \quad \frac{dF}{dz}(0)=0, \frac{dG}{dz}(0) = Q_o\ , \tag{13.18}$$

where F_o and Q_o must be chosen so that when Eqs. (13.13) and (13.14) are solved, the outer edge boundary conditions of Eq. (13.17) are satisfied. Expanding F(z) and G(z) in power series of z and substituting these series in Eqs. (13.13) and (13.14), one finds that the behavior of the solution near the wall is given by

$$F(z) = F_o\left[1 - \frac{A}{1+\omega}\,\frac{Q_o^{\omega} z^{1+\omega}}{F_o^{\,2}} - \frac{(1-\omega)z^{2+\omega}}{(1+\omega)(2+\omega)F_o^2 Q_o^{1-\omega}} + O(z^{2+2\omega})\right], \tag{13.19}$$

$$G(z)=Q_o z\left[1+ \frac{AQ_o^{\omega} z^{1+\omega}}{(1+\omega)(2+\omega)F_o^2} - \frac{P\{1-\frac{1-P}{P}\frac{1-\omega}{1+\omega}\}\, z^{2+\omega}}{(2+\omega)(3+\omega)F_o^2 Q_o^{1-\omega}} + O(z^{2+2\omega})\right]\ . \tag{13.20}$$

At the outer edge of the vorticity layer, where z goes to infinity and F and G both approach one, asymptotic solutions of Eqs. (13.13) and (13.14) may be found. Consider

$$F = 1 + f_1 + f_2 + \dots , \quad 1 \gg f_1 \gg f_2 \dots \quad \text{as } z \to \infty , \tag{13.21a}$$

$$G = 1 + g_1 + g_2 + \dots , \quad 1 \gg g_1 \gg g_2 \dots \text{ as } z \to \infty . \tag{13.21b}$$

Then the equations that determine g_1 and f_1 are

$$\frac{d^2 g_1}{dz^2} + Pz^2 \frac{dg_1}{dz} = 0, \quad g_1, \frac{dg_1}{dz} \to 0 \text{ as } z \to \infty, \tag{13.22a}$$

$$\frac{d^2 f_1}{dz^2} + z^2 \frac{df_1}{dz} = -(1-\omega)\, z^2 \frac{dg_1}{dz} \quad , \quad f_1, \frac{df_1}{dz} \to 0 \text{ as } z \to \infty . \tag{13.22b}$$

Solving for g_1, one gets

$$\frac{dg_1}{dz} = C_1 \exp\left(\frac{-Pz^3}{3}\right) , \quad C_1 = \text{constant}, \tag{13.23a}$$

$$g_1 = - \frac{C_1}{(9P)^{1/3}} \left[\Gamma\left(\tfrac{1}{3}\right) - \Gamma\left(\tfrac{1}{3}, \tfrac{Pz^3}{3}\right) \right] \quad , \tag{13.23b}$$

where $\Gamma(m,x)$ is the incomplete gamma function and C_1 is an undetermined constant.

Using Eq. (13.23), the equation for f_1 becomes

$$\frac{d^2 f_1}{dz^2} + z^2 \frac{df_1}{dz} = - (1-\omega)\, C_1 z^2 \exp\left(- \frac{Pz^3}{3}\right) . \tag{13.24}$$

Solving this equation for f_1 for the case of $(1-P)$ of order one, one gets

$$\frac{df_1}{dz} = C_2 \exp\left(-\frac{z^3}{3}\right) - \frac{1-\omega}{1-P} C_1 \exp\left(-\frac{Pz^3}{3}\right), \quad (13.25a)$$

$$f_1 = -\frac{C_2}{9^{1/3}}\left[\Gamma(\tfrac{1}{3}) - \Gamma(\tfrac{1}{3}, \tfrac{z^3}{3})\right] + \frac{1-\omega}{1-P}\frac{C_1}{(9P)^{1/3}}\left[\Gamma(\tfrac{1}{3}) - \Gamma(\tfrac{1}{3}, \tfrac{Pz^3}{3})\right], \quad (13.25b)$$

where C_2 is a second undetermined constant.

Thus, the leading terms in the asymptotic expansions for G and F for $(1-\omega)$ and $(1-P)$ of O(1) are

$$G = 1 - \{k_1 \left[\Gamma(\tfrac{1}{3}) - \Gamma(\tfrac{1}{3}, \tfrac{Pz^3}{3})\right]\} + \ldots \quad , \quad (13.26a)$$

$$F = 1 + \{k_2\left[\Gamma(\tfrac{1}{3}) - \Gamma(\tfrac{1}{3}, \tfrac{z^3}{3})\right] + \frac{1-\omega}{1-P} k_1 \left[\Gamma(\tfrac{1}{3}) - \Gamma(\tfrac{1}{3}, \tfrac{Pz^3}{3})\right]\} + \ldots \quad (13.26b)$$

where k_1 and k_2 are undetermined constants.

XIV. THE BOUNDARY LAYER

The "classical" boundary layer is imbedded within the inviscid correction layer. The quantities in the boundary layer are

$$\overline{u} = \epsilon^{\frac{1}{2}} u_{BL} + \dots , \qquad (14.01a)$$

$$\overline{v} = (\epsilon^{3/2}/R\delta^{\omega})^{\frac{1}{2}} v_{BL} + \dots , \qquad (14.01b)$$

$$\overline{\rho} = \frac{1}{\epsilon} \rho_{BL} + \dots , \qquad (14.01c)$$

$$\overline{T} = \frac{1}{\delta} T_{BL} + \dots , \qquad (14.01d)$$

$$\overline{p} = \frac{1}{\epsilon\delta} p_{BL} + \dots , \qquad (14.01e)$$

$$\overline{y} = (\epsilon^{\frac{1}{2}}/R\delta^{\omega})^{\frac{1}{2}} \eta_{BL} . \qquad (14.01f)$$

Note that the ratio of the boundary layer thickness to the inviscid correction layer thickness is $\lambda_{BL}/\lambda_c = [(\epsilon^{\frac{1}{2}}/R\delta^{\omega})^{\frac{1}{2}}/\epsilon^{3/2}] = D^{\frac{1}{2}}$ and, of course, $D \to 0$ for the inviscid correction layer.

The leading terms of the equations of motion are

$$p_{BL} = \rho_{BL} T_{BL}, \qquad (14.02a)$$

$$\frac{\partial}{\partial\xi} (\overline{B} \rho_{BL} u_{BL}) + \frac{\partial}{\partial\eta_{BL}} (\overline{B} \rho_{BL} v_{BL}) = 0 , \qquad (14.02b)$$

$$\frac{\partial p_{BL}}{\partial\eta_{BL}} = 0 , \qquad (14.02c)$$

$$\rho_{BL}(u_{BL}\frac{\partial u_{BL}}{\partial \xi} + v_{BL}\frac{\partial u_{BL}}{\partial \eta_{BL}}) + Z\frac{\partial p_{BL}}{\partial \xi} = \frac{\partial}{\partial \eta_{BL}}(T^{\omega}_{BL}\frac{\partial u_{BL}}{\partial \eta_{BL}}) \ , \tag{14.02d}$$

$$\rho_{BL}(u_{BL}\frac{\partial T_{BL}}{\partial \xi} + v_{BL}\frac{\partial T_{BL}}{\partial \eta_{BL}}) = \frac{1}{P}\frac{\partial}{\partial \eta_{BL}}(T^{\omega}_{BL}\frac{\partial T_{BL}}{\partial \eta_{BL}}) \ . \tag{14.02e}$$

It should be noted here that the dissipation is also negligible in the boundary layer.

The matching between the boundary layer and the inviscid correction layer takes place with respect to $\overline{y}$. The boundary layer expansions, as $\eta_{BL} \to \infty$ must match with the inviscid correction layer as $\eta_c \to 0$. The class of intermediate variables is

$$\overleftrightarrow{\eta} = \frac{y}{\overleftrightarrow{\lambda}} \ , \quad \frac{\overleftrightarrow{\lambda}}{\epsilon^{3/2}} \to 0 \ , \quad \frac{\overleftrightarrow{\lambda}}{(\epsilon^{\frac{1}{2}}/R\delta^{\omega})^{\frac{1}{2}}} \to \infty \ , \tag{14.03a}$$

$$\eta_c = \frac{\overleftrightarrow{\lambda}\,\overleftrightarrow{\eta}}{\epsilon^{3/2}} \to 0 \ , \quad \eta_{BL} = \frac{\overleftrightarrow{\lambda}\,\overleftrightarrow{\eta}}{(\epsilon^{\frac{1}{2}}/R\delta^{\omega})^{\frac{1}{2}}} \to \infty \ . \tag{14.03b}$$

The pertinent quantities in the inviscid correction layer are

$$\overline{p} = \frac{p_c(s)}{\epsilon\delta} + \dots \ , \quad p_c(s) = \tfrac{1}{2}[\sin^2\Phi - \frac{\overline{\kappa}(s)}{\overline{B}(s)}\int_0^{\cos\Phi(s)} \frac{\nu\overline{B}(\nu)d\nu}{\overline{\kappa}(\nu)}] \ , \tag{14.04a}$$

$$\overline{T} = \frac{1}{\delta}T_c + \dots \ , \quad T_c = \tfrac{1}{2} \ , \tag{14.04b}$$

$$\overline{u} = \epsilon^{\frac{1}{2}}[(2\log_e\{\frac{1}{2p_c(s)}\})^{\frac{1}{2}} + 2\overline{B}(s)\,p_c(s)\eta_c] + \dots \ . \tag{14.04c}$$

Therefore, the velocity and temperature boundary conditions at the outer edge of the boundary layer ($\eta_{BL} \rightarrow \infty$) are

$$u_{BL}(\xi,\infty) = (-2\log_e\{2p_c(s)\})^{\frac{1}{2}}\ ,\ T_{BL}(\xi,\infty) = \tfrac{1}{2}\ , \qquad (14.05)$$

and the pressure across the boundary layer, p_{BL}, is $p_c(s)$.

The boundary conditions at the wall are

$$u_{BL}(\xi,0) = v_{BL}(\xi,0) = 0,$$

$$T_{BL}(\xi,0) = T_{BL,w}(\xi),\ \text{ a known function.} \qquad (14.06)$$

In order to be able to compare the heat transfer and shear for the boundary layer with that for the viscous shock and correction layers, it is convenient to introduce again a modified Crocco transformation. This transformation is defined by:

$$(\xi,\eta_{BL}) \rightarrow (s,t_{BL})\ ,$$

$$s = \xi,\ t_{BL} = u_{BL}/u_{BL}(\xi,\infty) = u_{BL}/u_{BL,e}, \qquad (14.07a)$$

$$\tau_{BL} = \frac{T_{BL}^{\ \omega}}{\overline{B}p_c}\frac{\partial u_{BL}}{\partial\eta_{BL}}\ . \qquad (14.07b)$$

The pertinent equations after the transformation are (upon dropping the subscript BL for simplicity)

$$\frac{\partial^2 \tau}{\partial t^2} + \left(\frac{u_e}{\overline{B}}\right)^2 \frac{2}{p_c} \frac{du_e}{ds} \left[\left(1 - \frac{t^2}{2T}\right) \frac{\partial}{\partial t} \left(\frac{T^\omega}{\tau}\right) + \frac{t^2}{2\tau\, T^{2-\omega}} \frac{\partial T}{\partial t} \right]$$

$$+ \left(\frac{u_e}{\overline{B}}\right)^2 \frac{t u_e}{p_c} \frac{\partial}{\partial s} \left(\frac{1}{\tau\, T^{1-\omega}}\right) = 0 \; . \qquad (14.08a)$$

$$\frac{\partial^2 T}{\partial t^2} + \frac{1-P}{\tau} \frac{\partial \tau}{\partial t} \frac{\partial T}{\partial t} - P\left(\frac{u_e}{\overline{B}}\right)^2 \frac{2}{p_c} \frac{du_e}{ds} \frac{T^\omega}{\tau^2} \left(1 - \frac{t^2}{2T}\right) \frac{\partial T}{\partial t}$$

$$- P \left(\frac{u_e}{\overline{B}}\right)^2 \frac{t u_e}{p_c} \frac{1}{\tau^2 T^{1-\omega}} \frac{\partial T}{\partial s} = 0 \; . \qquad (14.08b)$$

The boundary conditions for these equations are

$$\left(\frac{\partial \tau}{\partial t}\right)_w = - \left(\frac{u_e}{\overline{B}}\right)^2 \frac{2}{p_c} \frac{du_e}{ds} \frac{(T^\omega)_w}{(\tau)_w} \; , \quad (\tau)_e = 0 \; , \qquad (14.09a)$$

$$(T)_w = T_w(s), \text{ a known function}, \quad (T)_e = \tfrac{1}{2} \; . \qquad (14.09b)$$

Taking the temperature at the wall to be zero, as was done in the other cases, the boundary conditions at the wall reduce to

$$\left(\frac{\partial \tau}{\partial t}\right)_w = (T)_w = 0 \; . \qquad (14.10)$$

Again looking for the solution near the stagnation line, since, for $s \to 0$,

$$\tau(s,t) = \tau_o(t) + \ldots \; , \quad T(s,t) = T_o(t) + \ldots \; , \qquad (14.11a)$$

$$p_c = \tfrac{1}{2} + \ldots \, . \quad \frac{u_e}{B} = w_1 + \ldots \, . \quad \frac{du_e}{ds} = w_1 + \ldots \; (w_1 = \text{constant}). \tag{14.11b}$$

The ordinary differential equations for the flow in this region are

$$\frac{d^2\tau_o}{dt^2} + 4w_1^3 \left[\left(1 - \frac{t^2}{2T_o}\right) \frac{d}{dt}\left(\frac{T_o^\omega}{\tau_o}\right) + \frac{t^2}{2\tau_o T_o^{2-\omega}} \frac{dT_o}{dt} \right] = 0, \tag{14.12a}$$

$$\frac{d^2T_o}{dt^2} + \frac{1-P}{\tau_o} \frac{d\tau_o}{dt} \frac{dT_o}{dt} - 4Pw_1^3 \left(1 - \frac{t^2}{2T_o}\right) \frac{T_o^\omega}{\tau_o^2} \frac{dT_o}{dt} = 0. \tag{14.12b}$$

The boundary conditions for these ordinary differential equations are

$$\frac{d\tau_o}{dt}(0) = 0 \,, \qquad \tau_o(1) = 0 \,, \tag{14.13a}$$

$$T_o(0) = 0, \quad T_o(1) = \tfrac{1}{2} \, . \tag{14.13b}$$

Consider the following new variables:

$$z = t, \quad F(z) = \frac{\tau_o(t)}{(2^{2-\omega} w_1^3)^{\frac{1}{2}}} \,, \quad G(z) = 2T_o(t) \, . \tag{14.14}$$

In terms of these new variables, the equations to be solved are

$$\frac{d^2F}{dz^2} + \frac{z^2 - G}{F^2 G^{1-\omega}} \frac{dF}{dz} + \frac{z^2 - \omega(z^2 - G)}{F\,G^{2-\omega}} \frac{dG}{dz} = 0 \,, \tag{14.15a}$$

$$\frac{d^2G}{dz^2} + \frac{1-P}{F} \frac{dF}{dz} \frac{dG}{dz} + \frac{P(z^2 - G)}{F^2 G^{1-\omega}} \frac{dG}{dz} = 0 \, . \tag{14.15b}$$

The boundary conditions associated with these equations are

$$\frac{dF}{dz}(0) = F(1) = 0, \quad G(0) = 0, \quad G(1) = 1 \, . \tag{14.16}$$

It should be noted that the equations to be solved, Eqs. (14.15a) and (14.15b) have exactly the same form as the equations for the viscous correction layer, Eqs. (13.13) and (13.14), but that these sets of equations are subject to far different boundary conditions for each regime.

XV. THE SHEAR AND HEAT TRANSFER IN THE VISCOUS LAYERS

Having derived the equations for the three regimes of flow in the shock layer, it is of interest to compare the shear and heat conduction for these regimes.

A. THE VISCOUS SHOCK LAYER. (The notation is that of Section 10.)

In the viscous shock layer, the shear and the heat transfer are

$$\frac{1}{\overline{B}} \frac{\mu(\partial u/\partial y)}{\mu_\infty U_\infty/a} = \frac{\overline{T}^\omega}{\overline{B}} \frac{\partial \overline{u}}{\partial \overline{y}} = \frac{1}{\epsilon\delta^\omega} \frac{T_L^\omega}{\overline{B}} \frac{\partial u_L}{\partial \eta_L} + \dots , \tag{15.01a}$$

$$\frac{k(\partial T/\partial y)}{k_\infty T_\infty/a} = \overline{T}^\omega \frac{\partial \overline{T}}{\partial \overline{y}} = \frac{1}{\epsilon\delta^{1+\omega}} T_L^\omega \frac{\partial T_L}{\partial \eta_L} + \dots . \tag{15.01b}$$

In terms of variables of the modified Crocco transformation,

$$\frac{T_L^\omega}{\overline{B}} \frac{\partial u_L}{\partial \eta_L} = \tau_L = \tau_{L,o}(t_L) + \dots$$

$$= (W_1^3/2K\Theta_o^{1-\omega})^{\frac{1}{2}} F_{L,o}(t_L;K) + \dots , \tag{15.02a}$$

$$T_L^\omega \frac{\partial T_L}{\partial \eta_L} = \frac{\overline{B}}{W} \tau_L \frac{\partial T_L}{\partial t_L} = \frac{1}{W_1} \tau_{L,o} \frac{dT_{L,o}}{dt_L} + \dots$$

$$= \left(\frac{\Theta_o^{1+\omega} W_1}{2K}\right)^{\frac{1}{2}} F_{L,o}(t_L;K) \frac{dG_{L,o}}{dt_L}(t_L;K) + \dots . \tag{15.02b}$$

B. THE VISCOUS VORTICITY LAYER. (The notation is that of Section 13.)

For the viscous vorticity layer,

$$\frac{\overline{T}^{\omega}}{\overline{B}} \frac{\partial \overline{u}}{\partial \overline{y}} = \frac{1}{\epsilon \delta^{\omega}} \frac{T_c^{\omega}}{\overline{B}} \frac{\partial u_c}{\partial \eta_c} + \ldots , \tag{15.03a}$$

$$\overline{T}^{\omega} \frac{\partial \overline{T}}{\partial \overline{y}} = \frac{1}{\epsilon^{3/2} \delta^{1+\omega}} T_c^{\omega} \frac{\partial T_c}{\partial \eta_c} + \ldots , \tag{15.03b}$$

where

$$\frac{T_c^{\omega}}{\overline{B}} \frac{\partial u_c}{\partial \eta_c} = \tau_c(s, t_c) = \tau_{c,o}(t_c) + \ldots$$

$$= \frac{1}{2^{\omega}} F_c(z_c; A_c) + \ldots , \tag{15.04a}$$

$$T_c^{\omega} \frac{\partial T_c}{\partial \eta_c} = \frac{\overline{B}}{\cos \Phi} \tau_c(s, t_c) \frac{\partial T_c}{\partial t_c}(s, t_c) = \tau_{c,o} \frac{dT_{c,o}}{dt_c} + \ldots$$

$$= \frac{1}{2^{1+\omega}} \left(\frac{2^{\omega}}{D}\right)^{1/3} F_c(z_c; A_c) \frac{dG_c}{dz_c}(z_c; A_c) + \ldots . \tag{15.04b}$$

C. THE BOUNDARY LAYER. (The notation is that of Section 14.)

The shear and heat conduction in the boundary layer are

$$\frac{\overline{T}^{\omega}}{\overline{B}} \frac{\partial \overline{u}}{\partial \overline{y}} = \frac{1}{D^{\frac{1}{2}} \epsilon \delta^{\omega}} \frac{T_{BL}^{\omega}}{\overline{B}} \frac{\partial u_{BL}}{\partial \eta_{BL}} + \ldots , \tag{15.05a}$$

$$\overline{T}^{\omega} \frac{\partial \overline{T}}{\partial \overline{y}} = \frac{1}{D^{\frac{1}{2}} \epsilon^{3/2} \delta^{1+\omega}} \; T_{BL}^{\omega} \frac{\partial T_{BL}}{\partial \eta_{BL}} + \ldots \quad , \tag{15.05b}$$

where, for the boundary layer, $D \rightarrow 0$ and

$$\frac{T_{BL}^{\omega}}{\overline{B}} \frac{\partial u_{BL}}{\partial \eta_{BL}} = p_c(s)\, \tau_{BL}(s, t_{BL}) = \tfrac{1}{2}\, \tau_{BL,o}(t_{BL}) + \ldots$$

$$= (w_1^{3}/2^{\omega})^{\frac{1}{2}} \; F_{BL}\,(z_{BL}) + \ldots \quad , \tag{15.06a}$$

$$T_{BL}^{\omega} \frac{\partial T_{BL}}{\partial \eta_{BL}} = \frac{\overline{B} p_c}{u_{BL,e}} \tau_{BL} \frac{\partial T_{BL}}{\partial \eta_{BL}} = \frac{1}{2w_1} \tau_{BL,o} \frac{dT_{BL,o}}{dt_{BL}} + \ldots$$

$$= (w_1/2^{2+\omega})^{\frac{1}{2}} \; F_{BL}\,(z_{BL}) \frac{dG_{BL}}{dz_{BL}}\,(z_{BL}) + \ldots \tag{15.06b}$$

In writing the above, the dependence of the solutions upon P and ω has been suppressed because these parameters are chosen once the gas to be considered has been chosen.

From the above, it can be seen that the shears in the viscous shock layer and the viscous vorticity interaction layer are of the same order of magnitude and less than the shear in the boundary layer. As for the heat conduction, it is equally clear that $\{\overline{T}^{\omega}\,(\partial\overline{T}/\partial\overline{y})\}_L \ll \{\overline{T}^{\omega}(\partial\overline{T}/\partial\overline{y})\}_c \ll \{\overline{T}^{\omega}\,(\partial\overline{T}/\partial\overline{y})\}_{BL}$.

APPENDIX

THE KINETIC THEORY ASPECTS OF THE PROBLEM

The question has been raised as to the validity of the use of the Navier-Stokes (N-S) equations to describe the structure of a shock wave because of the large gradients in the variables that occur within the shock. (See, for example, LIEPMANN, NARASIMHA, and CHAHINE, [9].) The question, in part, can be answered by considering the ratio $\Omega = \mu(\partial v/\partial y)/p$. This ratio is a characteristic parameter of the problem and a dimensionless measure of the gradients mentioned. Ω is essentially the expansion parameter in the Chapman-Enskog expansion procedure for solving the Boltzmann equation (CHAPMAN and COWLING, [10]) and it can be shown that the N-S equations must apply (at least in a monatonic gas) if $\Omega \ll 1$. Writing Ω in terms of the quantities introduced in the analysis, one has

$$\Omega = \frac{1+\epsilon}{1-\epsilon}\,\frac{M^2}{R}\,\frac{\bar{\mu}}{\bar{p}}\,\frac{\partial\bar{v}}{\partial\bar{y}} = \frac{1+\epsilon}{1-\epsilon}\,\frac{M^2}{R}\,\frac{\bar{T}^{\omega}}{\bar{p}}\,\frac{\partial\bar{v}}{\partial\bar{y}} \,. \tag{A.01}$$

For the outer region of the shock structure,

$$\Omega^* = \frac{M^2\delta^{3/4P}}{T^{*1-\omega}}\,\frac{\partial v^*}{\partial\eta^*} + \ldots \sim \frac{M^{(4P-3)/2P}}{\epsilon^{3/4P}} \,. \tag{A.02a}$$

Taking $\epsilon \sim M^{-2/4P+1}$, so that Eq. (5.18b) is an equality,

$$\Omega^* \sim (M^2)^{(4P-2)/(4P+1)} \,. \tag{A.02b}$$

Therefore, since for this case $P > \frac{1}{2}$, Ω^{*} will tend to infinity and the results for the outer region, based on the N-S equations, should be suspect. This unsuitability of the N-S equations in the outer region is predicted by LIEPMANN et al., [9], where they compare the normal shock structure based on the N-S equations with that found by solving the Bhatnagar-Gross-Krook (B-G-K) model for the Boltzmann equation, which can be shown to be correct in both the N-S and free molecule limits.

In the middle and inner regions of the shock structure,

$$\tilde{\Omega} = \frac{\tilde{T}^{\omega}}{2\epsilon\tilde{p}} \frac{\partial\tilde{v}}{\partial\tilde{\eta}} + \ldots \sim \frac{1}{\epsilon} \rightarrow \infty \;, \qquad \text{(A.03)}$$

$$\hat{\Omega} = \frac{\Theta^{\omega}}{2\epsilon\hat{p}} \frac{\partial\hat{v}}{\partial\hat{\eta}} + \ldots \sim \frac{1}{\epsilon} \rightarrow \infty \;. \qquad \text{(A.04)}$$

This means that the shock structure solutions for the middle and inner regions, based upon the N-S equations, are also questionable since ϵ has been taken to be a parameter that is going to zero. For ϵ a quantity of O(1), the middle and inner regions become just one inner region (BUSH, [1]), and the ratio Ω for this region is of O(1) and the results should still be questionable. However, LIEPMANN et al., [9] have shown that results for this region, based upon the N-S equations for large values of M, in this limit, are more than adequate when compared with results using the B-G-K method. Therefore, before saying that the N-S equations for the middle and inner regions are not suitable as $\epsilon \rightarrow 0$, it would be interesting to have results based upon a B-G-K approach for $\epsilon \rightarrow 0$.

For the shock layer and the vorticity layer,

$$\Omega_L = \frac{\epsilon K T_L^{\omega}}{2 p_L} \frac{\partial v_L}{\partial \eta_L} + \ldots \sim \epsilon K \to 0 , \tag{A.05}$$

$$\Omega_c = \frac{\epsilon^3 D T_c^{\omega}}{2 p_c} \frac{\partial v_c}{\partial \eta_c} + \ldots \sim \epsilon^3 D \to 0 . \tag{A.06}$$

However, in the shock layer and the vorticity interaction, the more pertinent parameter is the ratio $\Omega' = \mu(\partial u/\partial y)/p$. Therefore,

$$\Omega'_L = \frac{K T_L^{\omega}}{2 p_L} \frac{\partial u_L}{\partial \eta_L} + \ldots \sim K \leqslant O(1) , \tag{A.07}$$

$$\Omega'_c = \frac{\epsilon^{3/2} D T_c^{\omega}}{2 p_c} \frac{\partial u_c}{\partial \eta_c} + \ldots \sim \epsilon^{3/2} D \to 0 . \tag{A.08}$$

Thus, the N-S equations should be valid for the inviscid shock layer and the vorticity interaction layer (viscous or inviscid), but the validity of the viscous shock layer solutions, based on the N-S equations, should be suspect. Hopefully, since the N-S equations were shown to be adequate when the gradient parameter was of O(1) above, this may also be true for the case of the viscous shock layer.

Finally, it should be pointed out that the flow regime is definitely not the free molecule flow regime. To demonstrate this, it must be shown that the ratio of mean free path in the free-stream to body nose radius goes to zero. In terms of quantities introduced in the analysis, this ratio is

$$\frac{\ell_\infty}{a} = \sqrt{\frac{\pi\gamma}{8}}\ \frac{M}{R} \sim \frac{K\epsilon^{1-\omega}}{M^{2\omega-1}} \ . \tag{A.09}$$

Since K is less than or equal to order one and the viscosity exponent, ω, must be $\frac{1}{2} \leqslant \omega \leqslant 1$, based upon physical reasoning, the ratio does, indeed, go to zero.

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Table 1

$$C_f = \left(\frac{W_1^3}{2K\Theta_o^{1-\omega}}\right)^{\frac{1}{2}} F_o, \quad C_h = \left(\frac{Q_o \Theta_o}{W_1}\right) C_f \text{ vs. } K$$

$$\omega = \tfrac{1}{2}, \quad P = \tfrac{3}{4}$$

K	C_f (exact)	C_f (approx.)	C_h (exact)	C_h (approx.)
.403	.616		.411	
.419	.610		.404	
.435	.603		.396	
.453	.596		.388	
.472	.590		.380	
.495	.583		.372	
.518	.574		.364	
.545	.565		.354	
.576	.555		.343	
.610	.545		.334	
.647	.534		.323	
.692	.522	.512	.311	.307
.746	.506		.297	
.808	.492		.285	
.878	.476		.271	
.984	.461		.254	
1.09	.432		.236	
1.24	.405		.217	
1.32		.398		.214
1.44	.376		.197	
1.73	.340		.173	
2.08		.306		.152
2.20	.294		.144	
2.83		.251		.119
3.02	.238		.112	
3.12		.234		.110
3.59		.213		.099
3.95		.199		.092
4.82	.171		.076	
5.15		.163		.076
11.6	.0827		.034	
14.6		.0709		.032

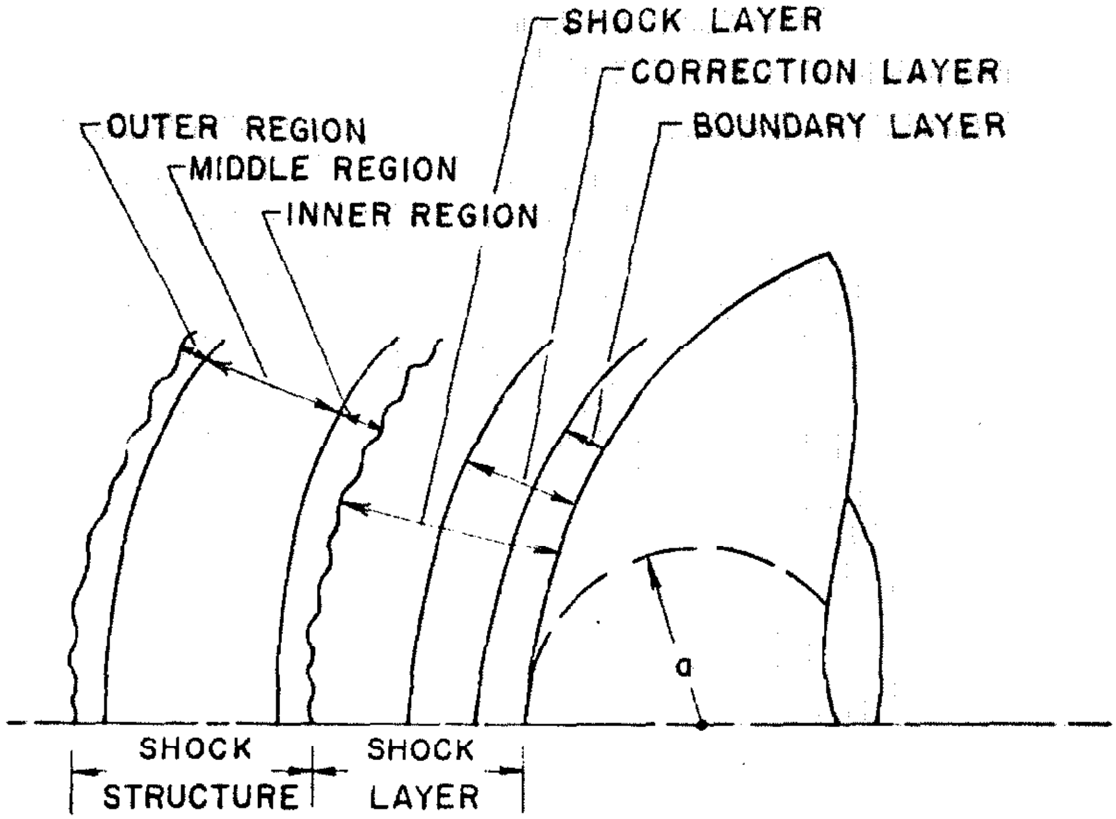

FIGURE 1-REGIONS OF THE FLOW

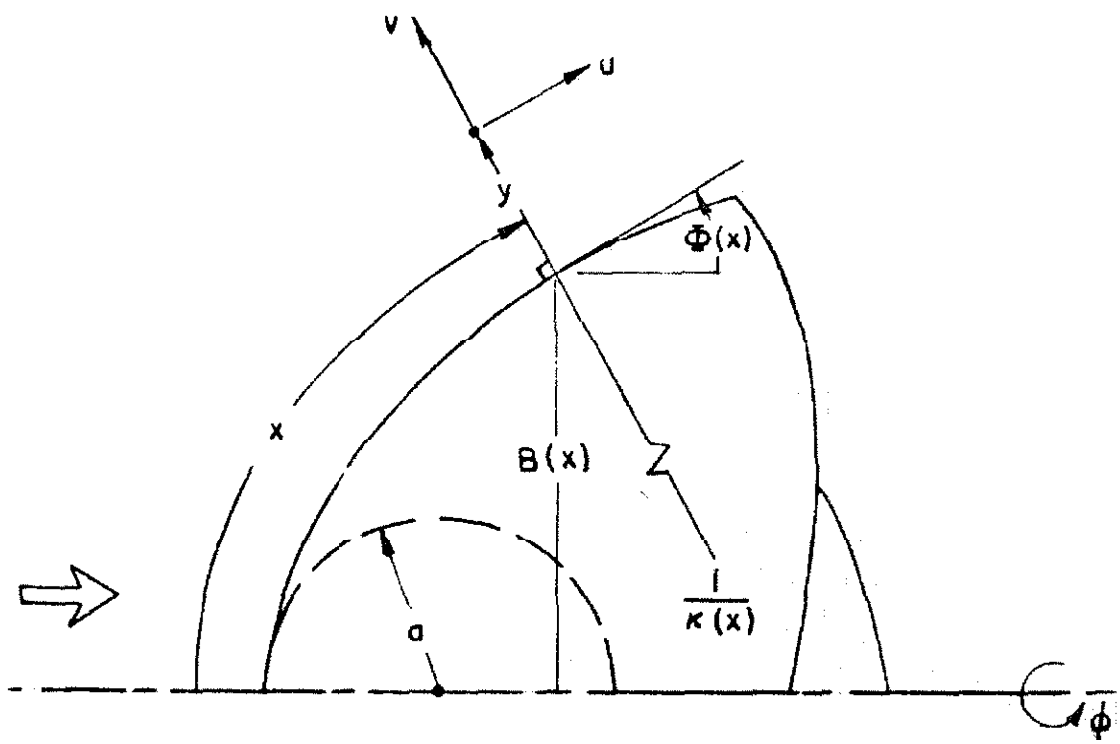

FIGURE 2-NOTATION

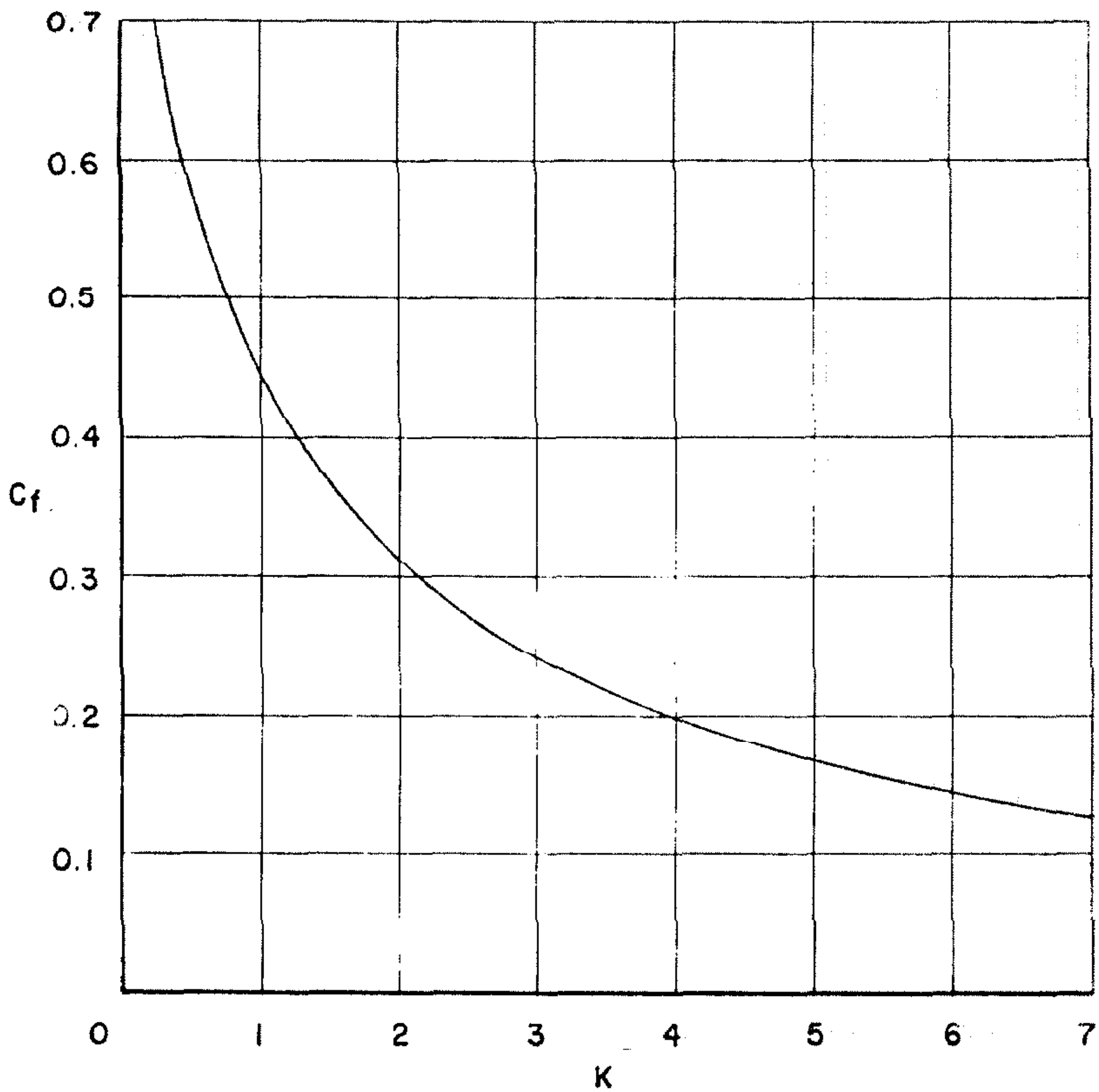

FIGURE 3a - C_f vs K

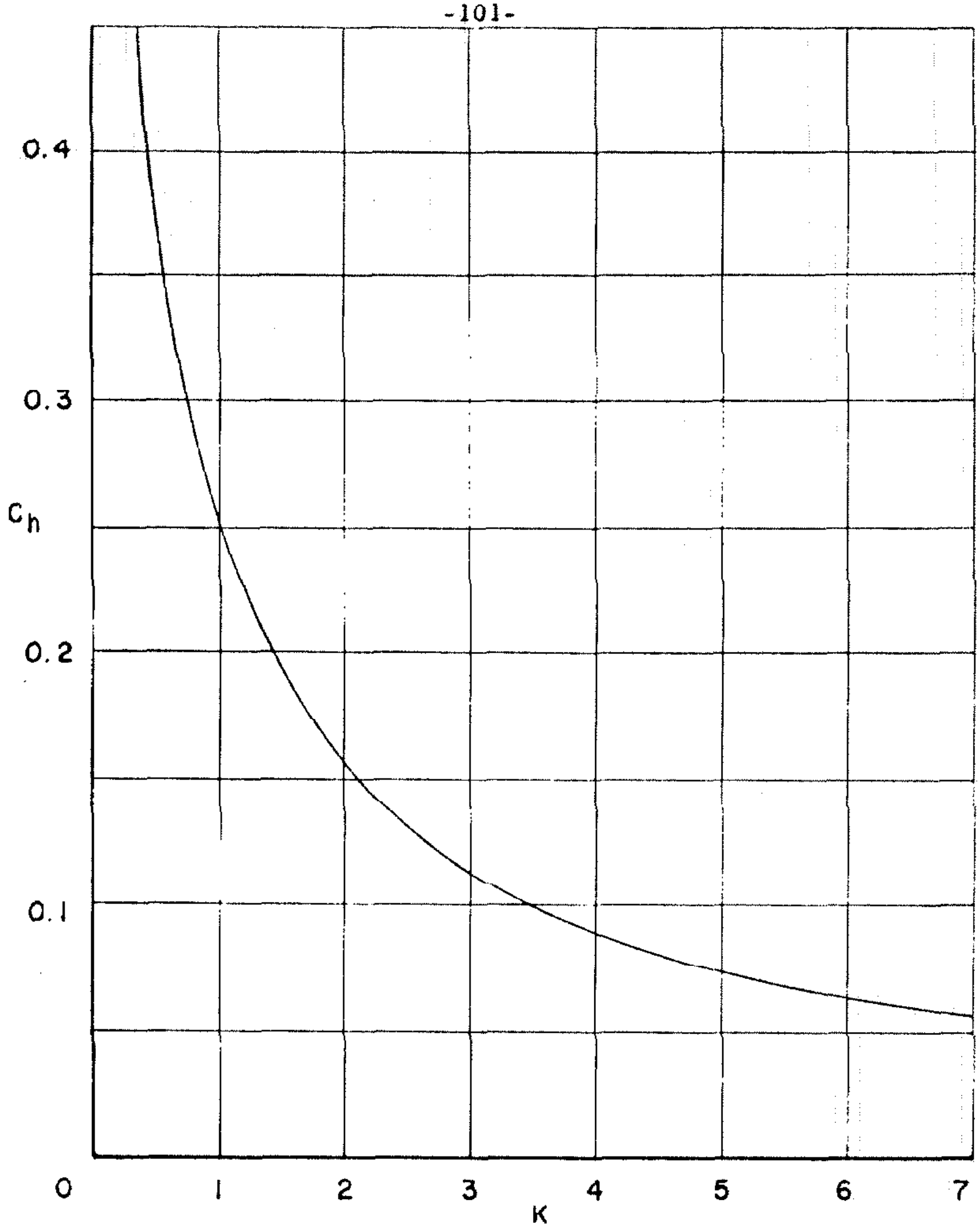

FIGURE 3b – C_h vs K