# SPONTANEOUSLY RADIATING ATOM IN CAVITY FIELDS

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## ABSTRACT

Spontaneous emission competes with stimulated emission in many interactions of light with matter. In the usual analyses which describe the interaction of an atom with a coherent optical field, the spontaneous emission characteristics, e.g., probability and spectral distribution, are not determined. The spontaneous emission from an atom which interacts with a coherent light wave is considered. The competition between coherent photons and spontaneous photons is treated in detail for a system consisting of a stationary atom, an open cavity and spatial fields. In the model chosen, a multilevel atom which spontaneously decays by interacting with spatial fields has two nondegenerate states coupled by an interaction with a single mode of the cavity. The Laplace-transformed Schrodinger equation is solved for specified initial conditions of the system. It is found that the interaction with the coherent field modifies the spectral distribution of spontaneous radiation from the atom. For spontaneous transitions involving an atomic state which interacts with the coherent field, the spectral distributions can no longer be described by Lorentzian functions. The new distributions exhibit a broadening and splitting for strong interactions between the atom and the coherent field. It is shown that the qualitative features of these distributions can be predicted from the energy-level diagram of the atom-cavity system. The net probability of the system gaining a coherent or cavity photon is calculated by integrating over the emitted spontaneous frequencies. The equivalence of this approach to the method of computing probabilities by integrating over time is demonstrated by using Parseval's theorem.

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# INTRODUCTION

The characteristics of spectral lines are determined by the environment of the radiating atoms or molecules. The line shapes of spectral lines can serve and have served as probes to determine this environment. For example, the line shapes of atoms radiating in a plasma depend on the finite lifetimes of the radiating atoms, on the statistically varying electric field resulting from the charges in the plasma, and on collision rates. Hence, the line shapes and breadths are used to determine the temperatures and densities of the plasma constituents. Many elaborate theories have been formulated and much experimental data has been accumulated to explain these line shapes (I.1, I.2)*. Similarly, it can be expected that the interaction of atoms with coherent optical fields can be probed by studying the characteristics of their spontaneous emission spectra.

The advent of intense coherent sources of light has made the problem of determining the characteristics of spontaneous emission from an atom which interacts with a coherent optical field of timely importance. This problem is explored in detail in this thesis. In order to focus attention on this problem and to simplify the analysis, only the finite lifetimes of excited atomic states and their interaction with the coherent optical field are considered.

As an introduction, simplified derivations of lifetime broadening are presented below. The spectral distribution of spontaneous emission resulting from a very weak interaction of an excited atom

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*References follow each chapter.

with a coherent field is also given in the next section. These simple concepts are covered more rigorously in the text and extended to include strong interactions between the atom and the coherent field.

Simple Spectral Distributions

Weisskopf and Wigner (4.9) were the first to solve the detailed equations that describe the spontaneous decay of an excited atom which interacts with radiative or spatial fields. They found that, as a result of spontaneous emission, an excited atomic level decays exponentially to lower levels. They also deduced that the line shape or spectral distribution of spontaneous emission can be determined by simply considering each atomic state to be broadened in energy and to consist of a continuous distribution of states with a Lorentzian density. That is, if normalizing terms are neglected, the distribution of states for an atomic state $i$ $N_i$ is given by

$$N_i(\Omega_i) \propto \frac{1}{\left(\frac{\Gamma_i}{2}\right)^2 + (\Omega_i - \Omega_{io})^2} \quad .$$

$\Gamma_i$ is the exponential decay constant of state $i$ . $\Omega_i$ is the energy variable in angular frequency units. $\Omega_{io}$ is the unbroadened energy in angular frequency units of the state $i$ with respect to the ground state.

The spectral distribution of spontaneous emission depends on the method of excitation of an atomic state. In broadband excitation each state of the broadened atomic distribution is considered to have equal probability of becoming excited. Examples of processes which fit the

description of broadband excitation are decays from higher levels and most collision excitations. As an example, two atomic states i and j are assumed to be connected by a spontaneous decay and the upper state i is excited by some broadband means. The spectral distribution of spontaneous photons of angular frequency $\omega_a$ between states i and j is given by the sum of all the decays from the broadened state i to the broadened state j . In general, the decay constants are always assumed to be much smaller than the frequencies of spontaneous transitions and the distribution of spontaneous photons $N(\omega_a)$ can be written as an integral over the initial states. Hence the form of this distribution is given by

$$N(\omega_a) \propto \int_{-\infty}^{+\infty} \frac{1}{\left(\frac{\Gamma_i}{2}\right)^2 + (\Omega_i - \Omega_{io})^2} \cdot \frac{d\Omega_i}{\left(\frac{\Gamma_j}{2}\right)^2 + (\Omega_j - \Omega_{jo})^2}$$

with the relation

$$\omega_a = \Omega_i - \Omega_j$$

which expresses the conservation of energy. As a result of the above integration, the distribution can be expressed as

$$N(\omega_a) \propto \frac{1}{\left(\frac{\Gamma_i + \Gamma_j}{2}\right)^2 + (\omega_a - \Omega_{ij})^2}, \quad \Omega_{ij} = \Omega_{io} - \Omega_{jo}$$

where $\Omega_{ij}$ is the energy difference between states i and j in angular frequency units. It is noted that the Lorentzian half-width of the spontaneous emission is given by the sum of the decay constants of

the upper and lower state.

Narrow-band excitation is the excitation of an atomic state by the exchange of definite energy with the excitation. As an example, consider three levels in a chain of decays. The first level $i$ is considered to be excited by some broadband means. The transition to the second level $j$ takes place by means of the stimulated emission of a coherent or monochromatic photon of definite frequency $\omega$. The level $j$ then decays spontaneously to the ground state $k$ ($\Gamma_k$ equal to zero). The distribution of spontaneous radiation of angular frequency $\omega_a$ from level $j$ to $k$ can be written in the form

$$N(\omega_a) \propto \frac{1}{(\frac{\Gamma_i}{2})^2 + (\Omega_i - \Omega_{io})^2} \cdot \frac{1}{(\frac{\Gamma_j}{2})^2 + (\omega_a - \Omega_{jo})^2}$$

with the relation

$$\Omega_i = \omega + \omega_a$$

which expresses conservation of energy. This can be rewritten as

$$N(\omega_a) \propto \frac{1}{(\frac{\Gamma_i}{2})^2 + (\omega_a - \Omega_{jk} + \omega - \Omega_{ij})^2} \cdot \frac{1}{(\frac{\Gamma_j}{2})^2 + (\omega_a - \Omega_{jk})^2} .$$

This distribution of spontaneous photons is given as the product of two Lorentzian functions. It is noted that if $\omega = \Omega_{ij}$ the distribution is single peaked. However, if $\omega \neq \Omega_{ij}$ the distribution is double peaked.

The above analyses lead us, in a simple manner, to some of the results of Chapters 4, 5 and 6. These analyses show that for weak interactions between the atom and the coherent field the spectral

distributions of spontaneous emission are described by Lorentzian functions. As shown in Chapters 5 and 6, strong interactions between the atom and the coherent field modify the spectral distribution of spontaneous emission and result in distributions which are no longer described by Lorentzian functions.

Discussion of Text

The spontaneous radiation from an excited atom which interacts with a coherent optical field is investigated in this thesis. The competition between coherent photons and spontaneous photons is treated in detail for a system consisting of a stationary atom, an open cavity, and spatial fields. In the model chosen, a multilevel atom which spontaneously decays by interacting with spatial fields has two nondegenerate states coupled by an interaction with a single mode of the cavity. The analysis of this model leads to the spectral distribution of spontaneous emission from the atom and the net probability of the system gaining a coherent photon for strong interactions between the atom and the cavity fields.

In the first four chapters, the analytical methods used to treat the problem are developed. In the first chapter the expansion of the fields of a cavity in terms of the very useful creation and annihilation operators is covered. The techniques in this chapter are applied in later chapters to the interaction of an excited atom with the fields of a quantized cavity and lead to the concept of atoms interacting with coherent photons.

In Chapter 2, the Hamiltonian of the system, which consists of an atom interacting with a single-mode cavity, is diagonalized. The

results are then applied to typical problems in maser analysis. The eigenfunctions of this diagonalized Hamiltonian are used in later chapters to solve the problem of a spontaneously radiating atom which interacts with the cavity. The energy-levels found in this chapter are later used to predict the qualitative features of the line shapes of spontaneous emission.

Throughout the text the cavity is considered to be quantized in order to emphasize the photon nature of the interaction. This approach has been used by Jaynes and Cummings (2.2) to treat a maser problem in the limit that radiative decays can be neglected. The more common approach to this problem, for example, that used by Shimoda, Wang and Townes (2.3), considers the cavity field classically and neglects its photon nature. The relation between these two methods is discussed in Chapter 3. In this chapter it is shown that neglecting the non-resonant terms in the classical field theory is analogous to neglecting matrix elements between nondegenerate states in the quantized-field theory.

The theories of natural line width and the decay of an excited atom are the subjects of the fourth chapter. The simplified theory which is accurate for times short compared with decay times is used to introduce the problem. The more complete theory is introduced by solving the Laplace-transformed Schrodinger equation for an atom interacting with the radiative fields of space. This latter theory is used in the following chapters to solve the problem of a spontaneously decaying atom interacting with the fields of an open cavity.

In the fifth and sixth chapters the competition between coherent photons and spontaneous photons is treated in detail for a system consisting of a stationary atom, an open cavity and spatial fields. In the model used, a multilevel atom which spontaneously decays by interacting with spatial fields has two nondegenerate states coupled by an interaction with a single mode of the cavity. The frequency distribution of spontaneous emission is found by solving the Laplace-transformed Schrodinger equation for the spontaneously decaying atom in the highly excited cavity. The solution also yields the net probability of the emission of a coherent photon by an excited atom.

In Chapter 5, the spontaneous radiation between the states which interact with the cavity is neglected. In Chapter 6, processes that result from spontaneous decays between states which interact with the cavity are covered in detail. For spontaneous transitions involving an atomic state which interacts with the coherent field, it is found that the atom-cavity interaction modifies the spectral distribution of spontaneous radiation. Some of the spectral distributions of spontaneous emission which are plotted in Figs. 5.2 and 5.3 exhibit a broadening and splitting for strong atom-cavity interactions. The qualitative nature of these splittings can be predicted from the energy-level diagram of the atom-cavity system, as shown in Fig. 5.1.

The characteristics of spontaneous emission are found to differ greatly for different modes of excitation. Broad-band excitation gives a characteristic frequency distribution which becomes the normal Lorentzian as the atom-cavity interaction becomes very small. Narrow-band excitation, excitation by means of coherent photons, gives a

characteristic frequency distribution which for very small atom-cavity interactions becomes the product of two Lorentzians. It is found that this latter distribution can be derived by a simple application of perturbation theory.

The net probability of an excited atom emitting a coherent photon is also calculated and the saturation characteristic is shown in Fig. 6.2. This probability is calculated by integrating over the emitted spontaneous frequencies. The equivalence of this approach to the method of computing probabilities by integrating over time is demonstrated in Appendix 5 by the use of Parseval's theorem.

In Chapter 7 the results of the thesis are summarized and some limitations of the analysis are discussed. Also, possible extensions of the theory are proposed.

## REFERENCES

(I.1) R. G. Breene, Jr., "Line Shape", Rev. Mod. Phys. 29, 94 (1957).

(I.2) Shang-Yi Ch'en and Malsoto Takeo, "Broadening and Shift of Spectral Lines Due to the Presence of Foreign Gases", Rev. Mod. Phys. 29, 20 (1957).

CHAPTER I

# QUANTIZATION OF CAVITY FIELDS

The fields existing in an ideal cavity are expressed in terms of orthogonal functions by means of Maxwell's equations. The transition to quantized fields is made by using the Hamiltonian formalism, the so-called classical approach to quantum mechanics. The cavity fields are expressed in terms of the noncommuting creation and annihilation operators. The results are applied to some particular cavities.

## 1.1 Cavity Fields in Terms of Orthonormal Functions

The cavity is considered to be a source-free and lossless volume $V$ within a surface $S$. In the "empty" region within the cavity Maxwell's equations can be written in mks units as (1.1)

$$\underline{H} = \frac{1}{\mu_o} \nabla \times \underline{A} \quad , \qquad \underline{E} = - \frac{\partial \underline{A}}{\partial t} \quad ,$$

$$\nabla^2 \underline{A} - \mu_o \epsilon_o \frac{\partial^2 \underline{A}}{\partial t^2} = 0 \quad , \qquad \nabla \cdot \underline{A} = 0 \tag{1.1}$$

where $\underline{A}$ is the vector magnetic potential, $\underline{E}$ is the electric field intensity, and $\underline{H}$ is the magnetic field intensity. The surface of the cavity is considered perfectly reflecting. This boundary condition is satisfied by

$$\underline{n} \times \underline{A} = 0 \quad \text{on } S \tag{1.2}$$

where $\underline{n}$ is the unit vector normal to surface $S$.

The vector potential $\underline{A}$ and its curl can be expanded in the orthonormal functions $\underline{A}_a$ (1.2), i.e.,

$$\underline{A} = \sum_a q_a(t)\, \underline{A}_a \quad . \tag{1.3}$$

The time-independent vector functions $\underline{A}_a$ are defined by

$$\nabla^2 \underline{A}_a + k_a^2\, \underline{A}_a = 0 \qquad \text{and} \qquad \nabla \cdot \underline{A}_a = 0 \quad \text{in} \quad V \tag{1.4}$$

and

$$\underline{n} \times \underline{A}_a = 0 \quad \text{on} \quad S \; . \tag{1.5}$$

The orthogonality of the expansion functions can be shown by using the vector identity

$$\nabla \cdot (\underline{A}_b \times \nabla \times \underline{A}_a) - \nabla \cdot (\underline{A}_a \times \nabla \times \underline{A}_b) = \underline{A}_a \cdot (\nabla \times \nabla \times \underline{A}_b) - \underline{A}_b \cdot (\nabla \times \nabla \times \underline{A}_a) \; . \tag{1.6}$$

If this equation is integrated over $V$, then the left-hand side becomes a surface integral whose value vanishes as a result of equation 1.5. Using equation 1.4 one can write the result as

$$(k_a^2 - k_b^2) \int_V \underline{A}_a \cdot \underline{A}_b \, dv = 0 \quad .$$

Thus if $a \neq b$ and the cavity eigenfunctions are nondegenerate $(k_a \neq k_b)$,

$$\int_V \underline{A}_a \cdot \underline{A}_b \, dv = 0 \quad . \tag{1.7}$$

In the case of degeneracy ($k_a = k_b$), linear combinations of the cavity eigenfunctions can be chosen which satisfy this orthogonality condition.

The normalization is chosen as

$$\int_v \underline{A}_a \cdot \underline{A}_a \, dv = 1 \quad . \tag{1.8}$$

The value of

$$\int_v (\nabla \times \underline{A}_a) \cdot (\nabla \times \underline{A}_b) \, dv$$

is found by considering the vector identity

$$\nabla \cdot (\underline{A}_a \times \nabla \times \underline{A}_b) = (\nabla \times \underline{A}_a) \cdot (\nabla \times \underline{A}_b) - \underline{A}_a \cdot (\nabla \times \nabla \times \underline{A}_b) \tag{1.9}$$

If this equation is integrated over the volume $V$, the left-hand side becomes a surface integral whose value vanishes as a result of equation 1.5. Using equation 1.4, one can write the result as

$$\int_v (\nabla \times \underline{A}_a) \cdot (\nabla \times \underline{A}_b) \, dv = k_b^2 \int_v \underline{A}_a \cdot \underline{A}_b \, dv \tag{1.10}$$

Thus, the orthogonality and normalization condition can be written as

$$\int_v \underline{A}_a \cdot \underline{A}_b \, dv = \delta_{ab} \tag{1.11a}$$

$$\int_v (\nabla \times \underline{A}_a) \cdot (\nabla \times \underline{A}_b) \, db = k_a^2 \, \delta_{ab} \tag{1.11b}$$

where $\delta_{ab}$ denotes the Kronecker delta symbol.

## 1.2 Equations of Motion

Using the orthogonality of the vector eigenfunctions (equation 1.11), the definition of the vector functions (equation 1.4), and the expansion 1.3 in the equation of motion for the vector potential (equation 1.1), one can write the equation for the individual expansion coefficient as

$$\ddot{q}_a + \omega_a^2 q_a = 0 \tag{1.12}$$

where

$$\omega_a \equiv \frac{k_a}{\sqrt{\mu_o \epsilon_o}} \quad . \tag{1.13}$$

These equations of motion can be found from the Lagrangian L of the system (1.3)

$$L \equiv L(q_1 \cdots q_i \cdots q_r \; ; \; \dot{q}_1 \cdots \dot{q}_i \cdots \dot{q}_r \; ; \; t) \quad .$$

The coordinates q satisfy the following second order equations:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0 \quad , \qquad i = 1 \cdots r \quad .$$

The Lagrangian conjugate momentum $p_i$ is defined by

$$\frac{\partial L}{\partial \dot{q}_i} \equiv p_i \quad . \tag{1.14}$$

The Lagrangian for the cavity is given by

$$L = K - P$$

where $K$ , the electromagnetic "kinetic" energy, is given by

$$K \equiv \frac{1}{2} \int_v \epsilon_o \underline{E} \cdot \underline{E} \, dv \tag{1.15}$$

and $P$ , the electromagnetic "potential" energy, is given by

$$P \equiv \frac{1}{2} \int_v \mu_o \underline{H} \cdot \underline{H} \, dv \quad . \tag{1.16}$$

Expanding $\underline{E}$ and $\underline{H}$ in terms of the $\underline{A}_a$'s and using the orthogonality properties of these eigenfunctions, one can obtain:

$$K = \frac{\epsilon_o}{2} \sum_a \dot{q}_a^2 \tag{1.17a}$$

and

$$P = \frac{1}{2\mu_o} \sum_a k_a^2 q_a^2 \tag{1.17b}$$

Thus, the conjugate momentum is given as

$$p_a = \epsilon_o \dot{q}_a \quad . \tag{1.18}$$

Also, the Hamiltonian can be defined as (1.3)

$$H \equiv H(q_1 \text{ --- } q_i \text{ --- } q_r ; p_1 \text{ --- } p_i \text{ --- } p_r ; t)$$

$$= \sum_{i=1}^{r} \dot{q}_i \frac{\partial L}{\partial \dot{q}_i} - L \quad . \tag{1.19}$$

The equations of motion are:

$$\dot{q}_i \equiv \frac{\partial H}{\partial p_i} \qquad \text{and} \qquad \dot{p}_i \equiv - \frac{\partial H}{\partial q_i} , \tag{1.20}$$

where $q$ and $p$ are the canonically conjugate coordinates.

Thus, applying defining equation 1.19

$$H = \frac{1}{2\epsilon_o} \sum_a p_a^2 + \frac{1}{2\mu_o} \sum_a k_a^2 q_a^2$$

or

$$H = K + P \quad . \tag{1.21}$$

Since equations 1.20 reduce to definition 1.18 and equation 1.12, the motion of the cavity can be found from the above Hamiltonian.

## 1.3 Fields as Operators*

The canonically conjugate coordinates can be considered quantum mechanical operators satisfying the following commutation relationships (1.4):

$$\left[q_a, q_b\right] = 0 \, , \quad \left[p_a, p_b\right] = 0$$

$$\left[q_a, p_b\right] = i \not{h} \, \delta_{ab} \quad . \tag{1.22}$$

If operators $P_a$ and $Q_a$ are defined by

$$P_a \equiv \left(\frac{1}{\epsilon_o \not{h} \omega_a}\right)^{1/2} p_a \qquad \text{and} \qquad Q_a \equiv \left(\frac{\epsilon_o \omega_a}{\not{h}}\right)^{1/2} q_a \tag{1.23}$$

the Hamiltonian can be written as

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*The quantum mechanical notation used can be found in (1.3).

$$H = \frac{1}{2} \sum_a (Q_a^2 + P_a^2) \not{h}\omega_a \quad . \tag{1.24}$$

Two Hermitian conjugate operators are defined by

$$a_a \equiv \frac{1}{\sqrt{2}} (Q_a + iP_a) \quad \text{and} \quad a_a^+ \equiv \frac{1}{\sqrt{2}} (Q_a - iP_a) \; . \tag{1.25}$$

The commutation relations, equations 1.21, become

$$\left[ a_a, a_b^+ \right] = \delta_{ab} \quad . \tag{1.26}$$

The Hamiltonian becomes

$$H = \frac{1}{2} \sum_a (a_a \, a_a^+ + a_a^+ \, a_a) \not{h}\omega_a \quad . \tag{1.27}$$

If the commutation relation, equation 1.26, is used, the Hamiltonian can be written as

$$H = \sum_a (a_a^+ \, a_a + \frac{1}{2}) \not{h}\omega_a \quad . \tag{1.28}$$

Using the definition of electromagnetic field quantities in terms of the vector potential, equations 1.1, one can write the fields in terms of the "a" operators, namely,

$$\underline{A} = \sum_a \left( \frac{\not{h}}{2\epsilon_o \omega_a} \right)^{1/2} (a_a + a_a^+) \, \underline{A}_a \tag{1.29a}$$

$$\underline{E} = i \sum_a \left( \frac{\not{h}\omega_a}{2\epsilon_o} \right)^{1/2} (a_a - a_a^+) \, \underline{A}_a \tag{1.29b}$$

$$\underline{H} = \sum_a \left( \frac{\not{h}}{2\epsilon_o \mu_o \omega_a} \right)^{1/2} (a_a + a_a^+) \nabla \times \underline{A}_a \quad . \tag{1.29c}$$

## 1.4 Creation and Annihilation Operators

The Hamiltonian for the cavity is given in equation 1.21 as a sum of partial Hamiltonians. Each partial Hamiltonian corresponds to a resonant mode of the cavity. The vectors (or expansion functions) of the basis in which one of these partial Hamiltonians is diagonal are eigenvalues of the operator equation (in ket notation)

$$n_a/v_a) = v_a/v_a) \tag{1.30}$$

where

$$(v_a/v_a) > 0 \quad \text{and} \quad n_a \equiv a_a^+ a_a \quad . \tag{1.31}$$

In the following our attention is confined to a single mode and the subscript "a" is deleted. Using equation 1.26, one can derive the following:

$$na = a(n - 1) \tag{1.32a}$$

and

$$na^+ = a^+(n + 1) \quad . \tag{1.32b}$$

Thus,

$$na/v) = (v - 1)\ a/v) \tag{1.33a}$$

and

$$na^+/v) = (v + 1)\ a^+/v) \ . \tag{1.33b}$$

Therefore, $a/v)$ is the vector $/v - 1)$ and $a^+/v)$ is the vector

$/v + 1)$. It can be shown that these eigenvalues are a series of non-negative integers with lower limit 0 (1.4). Since the $/v)$'s are nondegenerate eigenfunctions of a Hermitian operator, the normalization can be chosen as

$$(n/n') = \delta_{nn'} \tag{1.34}$$

where, now, a vector is denoted by its integral eigenvalue $n$. Any vector can be found by repeated application of $a^+$ on $/0)$, i.e.,

$$/n) = (n!)^{-1/2} a^{+n}/0) \quad . \tag{1.35}$$

Also, since

$$(n/a^+a/n') = n\,\delta_{nn'} \quad , \tag{1.36}$$

one can write:

$$a/n) = n^{1/2}/n-1 \tag{1.37a}$$

and

$$a^+/n) = (n+1)^{1/2}/n+1) \quad . \tag{1.37b}$$

Therefore, in the $n$ representation, the Hamiltonian for a single mode of the cavity is a diagonal matrix with matrix elements

$$1/2\ \hbar\omega\ ,\ 3/2\ \hbar\omega\ ,\ 5/2\ \hbar\omega\ ,\ 7/2\ \hbar\omega - - - \ .$$

The energy of the cavity when in state $/n)$ can be considered as consisting of the zero-point energy $1/2\ \hbar\omega$ and the energy of $n$ photons, $n\ \hbar\omega$. It is seen from equations 1.37 that "a" can be considered as an operator that lowers the occupation number $n$ by one or destroys a photon. Hence, "a" is referred to as an

annihilation operator. Similarly, $a^+$ is a creation operator. The matrix elements for $a$ and $a^+$ are given by

$$(n'/a/n) = n^{1/2} \delta_{n',n-1} , \tag{1.38a}$$

and

$$(n'/a^+/n) = (n+1)^{1/2} \delta_{n',n+1} . \tag{1.38b}$$

When more than one mode is excited, the vectors of the representation in which $H$ is diagonal consist of all possible products of the individual vectors for each mode. When in a definite state $/\phi)$ ,

$$/\phi) = \prod_a /n_a) \equiv /n_1 \text{ --- } n_a \text{ ---}) , \tag{1.39}$$

the energy of the system is given by

$$H = \sum_a (n_a + \tfrac{1}{2}) \hbar\omega_a , \tag{1.40}$$

the energies of all the photons and the zero-point energies.

## 1.5 Typical Cavities

As shown in equation 1.29, the fields in a cavity are linear combinations of the creation and annihilation operators. The vector properties of the fields are given in terms of the cavity functions $\underline{A}_a$ . Some examples of the $\underline{A}_a$'s are:

a) The $TM_{010}$ mode in a circular-cylindrical cavity (1.5)

$$\underline{A}_a = 1_z \frac{J_o(2.405 \frac{r}{r_o})}{V^{1/2} J_1(2.405)} \quad , \quad r < r_o \tag{1.41}$$

where $1_z$ is a unit vector in the axial (z) direction, $r_o$ is the outer radius, V is the volume of the cavity, and r is the radial distance from the axis of the cavity.

b) The $TE_{011}$ mode in a circular-cylindrical cavity

$$\underline{A}_a = 1_\phi \frac{2^{1/2}}{V^{1/2} J_o(3.83)} J_1(3.83 \frac{r}{r_o}) \sin \beta z \ , \ r < r_o \tag{1.42}$$

where

$$\beta = k \left[ 1 - (\frac{3.83}{r_o k})^2 \right]^{1/2} = \frac{\pi}{L} \ .$$

The cavity extends from z equals 0 to z equals L. Both the $TM_{010}$ and the $TE_{011}$ cavities have been used in beam-type masers.

c) The TEM mode in a coaxial cylindrical cavity

$$\underline{A}_a = 1_r \frac{(r_2^2 - r_1^2)^{1/2}}{V^{1/2} \ell n^{1/2} \frac{r_2}{r_1}} \frac{\sin kz}{r} \quad , \quad r_1 < r < r_2 \tag{1.43}$$

where

$$kL = n\pi \ ; \quad n = 1,2,3, \text{ --- } .$$

The outer radius is $r_2$ ; the inner radius is $r_1$ ; and the cavity extends from z equals 0 to z equals L .

d) For many structures used at optical frequencies the modes are quasi-TEM modes (1.6). As a first approximation, these can sometimes be considered as linearly polarized TEM modes of a resonant cavity whose fields are independent of coordinates transverse to the axial direction in the "cavity" region. This is called the one-dimensional approximation. In this case,

$$\underline{A}_a = 1_x \frac{2^{1/2}}{V^{1/2}} \sin kz \tag{1.44}$$

for a mode polarized in the x direction. This is also the TEM mode for a coaxial cylindrical cavity in the limit

$$r_2 - r_1 \ll r_2$$

$$r_2 - r_1 \ll r_1 .$$

## 1.6 Matrix Elements

The electromagnetic fields in a cavity have been given in terms of creation and annihilation operators in equation 1.29. Their matrix elements in the n basis for <u>one</u> cavity mode are:

$$(n' / \underline{A} / n) = \left(\frac{\not{h}}{2\epsilon_o \omega_a}\right)^{1/2} \underline{A}_a \left[ n^{1/2} \delta_{n',n-1} + (n+1)^{1/2} \delta_{n',n+1} \right] \tag{1.45a}$$

$$(n' / \underline{E} / n) = i\left(\frac{\not{h} \omega_a}{2\epsilon_o}\right)^{1/2} \underline{A}_a \left[ n^{1/2} \delta_{n',n-1} - (n+1)^{1/2} \delta_{n',n+1} \right] \tag{1.45b}$$

$$(n' / \underline{H} / n) = \left(\frac{\not{h}}{2\epsilon_o \mu_o^2 \omega_a}\right)^{1/2} \nabla \times \underline{A}_a \left[ n^{1/2} \delta_{n',n-1} + (n+1)^{1/2} \delta_{n',n+1} \right] . \tag{1.45c}$$

## REFERENCES

(1.1) W. R. Smythe, Static and Dynamic Electricity, McGraw-Hill Book Company, Inc., New York (1950), p. 442.

(1.2) J. C. Slater, "Microwave Electronics", Rev. of Modern Physics 18, 467-483, October 1946.

(1.3) A. Messiah, Quantum Mechanics, North-Holland Publishing Company, Amsterdam (1961), pp. 31-34.

(1.4) Chapter 12 of Reference (1.3).

(1.5) S. Ramo and J. R. Whinnery, Fields and Waves in Modern Radio, John Wiley and Sons, Inc., New York (1953), Ch. 10.

(1.6) A. G. Fox and Tingye Li, "Resonant Modes in a Maser Interferometer", The Bell System Tech. Journal 40, 61 (1961).

# CHAPTER 2

## INTERACTION OF A STATIONARY ATOM WITH A LOSSLESS CAVITY

The Hamiltonian of an idealized atom interacting with cavity fields is developed. The stationary states of a system consisting of the atom and cavity operating in a single mode are found. The equations of motion of this system are discussed and applied to some typical problems in beam-type masers.

### 2.1 Hamiltonian of the System

The nonrelativistic Hamiltonian for a system of electrons in the presence of an electromagnetic field is given by (2.1)

$$H_{ef} = \sum_{1}^{z} \frac{(\underline{p}_i - e\underline{A}_i)^2}{2m} - \frac{e\hbar}{2m}\left[\underline{\sigma}_i \cdot (\nabla \times \underline{A})_i\right] , \quad e < 0 . \qquad (2.1)$$

The momentum and the Pauli spin matrices of the $i^{th}$ electron are given by $\underline{p}_i$ and $\underline{\sigma}_i$. $\underline{A}_i$ and $(\nabla \times \underline{A})_i$ are the vector potential and its curl evaluated at the position of the $i^{th}$ electron. The approximate Hamiltonian of the atom in the presence of the field is obtained by adding to $H_{ef}$ the coulomb interaction of the electrons with the nucleus and the coulomb interaction between electron pairs. The total Hamiltonian of the system is given by

$$H = H_{field} + H_{atom} + H' \quad .$$

$H_{field}$ is the Hamiltonian of the electromagnetic field alone. $H_{atom}$, the Hamiltonian of the atom itself, includes the coulomb energies of the nucleus and electrons, and the kinetic energies of the electrons.

$H'$ consists of three parts

$$H' = \sum_{1}^{Z} H'_{1i} + \sum_{1}^{Z} H'_{2i} + \sum_{1}^{Z} H'_{3i} \tag{2.2a}$$

where

$$H'_{1i} = -\frac{e}{2m} (\underline{p}_i \cdot \underline{A}_i + \underline{A}_i \cdot \underline{p}_i) \tag{2.2b}$$

$$H'_{2i} = -\frac{e\hbar}{2m} \left[ \underline{\sigma}_i \cdot (\nabla \times \underline{A})_i \right] \tag{2.2c}$$

$$H'_{3i} = \frac{e^2}{2m} \underline{A}_i \cdot \underline{A}_i \quad . \tag{2.2d}$$

The total Hamiltonian of the system of atom and cavity is written as

$$H = H_o + H' \tag{2.3a}$$

where

$$H_o \equiv H_{field} + H_{atom} \quad . \tag{2.3b}$$

## 2.2 Matrix Elements

The matrix elements of the interaction Hamiltonian $H'$ are found. The cavity is assumed to be excited in only one mode. The basis chosen is the one in which $H_o$ is diagonal. The functions or vectors for the expansion are products of the individual vectors of $H_{atom}$ and $H_{field}$. These vectors, in ket notation, are given by

$$/mn) = /m)/n) \quad .$$

$/m)$ is an eigenvalue of the operator $H_{atom}$. $/n)$ is the eigenvalue of $H_{field}$. That is,

$$H_{atom}/m) = E_m/m)$$

$$H_{field}/n) = E_n/n) \quad . \tag{2.4}$$

$\underline{p}_i$ is an operator that acts only on the atom coordinates. Its matrix element is found by using the commutation relation*

$$\left[\underline{d}_i, H_{atom}\right] = \frac{i\hbar e}{m}\,\underline{p}_i \quad , \tag{2.5}$$

where

$$\underline{d}_i = e\underline{r}_i \quad . \tag{2.6}$$

$\underline{d}_i$ is the electric dipole moment of the $i^{th}$ electron and $\underline{r}_i$ is the position of the electron with respect to the center of the atom. Thus, the matrix element for $\underline{p}_i$ is found from

$$\begin{aligned}\frac{i\hbar e}{m}\,(m'/\underline{p}_i/m) &= (m'/\underline{d}_i H_{atom} - H_{atom}\,\underline{d}_i/m) \\ &= (E_m - E'_m)(m'/\underline{d}_i/m) \quad .\end{aligned} \tag{2.7}$$

The vector potential, an operator that acts only on field coordinates, is given in terms of its matrix elements in equation 1.45a. Thus, the matrix element of $H'_{li}$ is

$$\begin{aligned}(m'n'/H'_{li}/mn) = i\,\frac{(E_m - E'_m)}{(2\epsilon_o \hbar\omega_a)^{1/2}}\,(m'/\underline{d}_i \cdot \underline{A}_{ai}/m) \\ \times \left[n^{1/2}\,\delta_{n',n-1} + (n+1)^{1/2}\,\delta_{n',n+1}\right]\end{aligned} \tag{2.8}$$

*Small terms in the $H_{atom}$ such as spin-orbit interaction can be neglected in deriving equation 2.5.

Similarly, the matrix element for $H'_{2i}$ is:

$$(m'n' / H'_{2i}/mn) =$$

$$\frac{-e\not{h}^2}{2m} \frac{(m'/\underline{\sigma}_i \cdot (\nabla \times \underline{A}_a)_i/m)}{(2\epsilon_o \not{h}\, \omega_a)^{1/2}} \left[ n^{1/2}\, \delta_{n',n-1} + (n+1)^{1/2}\, \delta_{n',n+1} \right]. \tag{2.9}$$

In the electric-dipole approximation the matrix element for $H'_{1i}$ is evaluated by considering the vector potential a constant over the atom. If the magnetic field is considered a constant over the atom, the orthogonality of /m) and /m') results in a vanishing matrix element for $H'_{2i}$.* The magnitude of $H'_{2i}$ is evaluated by considering a one-dimensional cavity as discussed in Section 1.6. To a first approximation the $(\nabla \times \underline{A}_a)_i$ is given as

$$(\nabla \times \underline{A}_a)_i = 1_y \frac{2^{1/2}}{V^{1/2}} k \left[ \cos kz_o - (z_i - z_o)\, k \sin kz_o \right]$$

where $z_o$ is the position of the center of the atom and the index a has been dropped from the frequency since only a single mode is considered. Thus,

$$\left| \frac{(m'n'/H'_{2i}/mn)}{(m'n'/H_{1i}/mn)} \right| = \frac{\not{h}k}{2mc} \left[ \frac{e(m'/\sigma_{yi}(z_i - z_o)/m)}{(m/d_{xi}/m)} \right] \left[ \frac{\not{h}\omega}{E_m - E'_m} \right]. \tag{2.10}$$

Since $\sigma_{yi}$, the Pauli spin matrix, has matrix elements of magnitude one and since for the case to be considered $\not{h}\omega \approx E_m - E'_m$,

*This is strictly true only for vanishing coupling between spin and orbital angular momentum.

the two terms in brackets are approximately equal to one. The ratio

$$\left| \frac{(m'm'/H'_{2i}/mn)}{(m'n'/H'_{1i}/mn)} \right| \approx \frac{\hbar k}{2mc} = \frac{\pi\hbar}{\lambda mc} \quad . \tag{2.11}$$

For a transition at 1 micron ($\lambda = 10^{-6}$m) this ratio becomes $10^{-6}$. Therefore, for electric-dipole transitions, the term $H'_{2i}$ can be neglected.

$H'_{3i}$ conserves the number of photons, creates two photons or annihilates two photons. Equations 2.2 and 1.29 give $H'_{3i}$ as

$$H'_{3i} = \frac{e^2\hbar}{2m\epsilon_o\omega} (a_a + a_a^+)^2 \underline{A}_{ai} \cdot \underline{A}_{ai} \quad . \tag{2.12}$$

For the one-dimensional cavity considered above, the matrix elements for $H'_{3i}$ are:

$$(m'n'/H'_{3i}/mn) = \frac{e^2\hbar}{2m\epsilon_o\omega\, AL} \sin^2 kz_i \left[ (n-1)^{1/2} n^{1/2} \delta_{n',n-2} + (2n+1)\delta_{n',n} + (n+2)^{1/2} (n+1)^{1/2} \delta_{n',n+2} \right] \delta_{m',m} \quad . \tag{2.13}$$

The off-diagonal terms of $H'_{3i}$ do not connect degenerate states and one can show by perturbation theory (Appendix 2) that they can be neglected. The diagonal terms result in an energy-level shift which splits the possible degenerate states $/mn - 1)$ and $/m'n)$. The difference in energy between these levels becomes:

$$E_{mn} - E_{m'n-1} = \hbar\omega - (E_m - E'_m) + \frac{e^2\hbar}{m\epsilon_o\omega AL} \sum_1^z \sin^2 kz_i \quad .$$

The additional energy $\hbar \frac{e^2}{m\epsilon_o\omega AL}$ is approximately $3 \cdot 10^{-8}\hbar$ for a resonance at 1 micron and a cavity volume of $10^{-4}$m. This energy is entirely negligible and can be neglected in the following analyses.

## 2.3 Stationary States

The time-independent stationary states of the system of atom and cavity are given by the matrix equation

$$H/\phi) = (H_o + H')/\phi) = E/\phi) \quad .$$

Since, for the present, all other interactions which result in normal spontaneous decay are being neglected, the atom can be considered a two-state system with states $/m)$ and energy levels $E_m$, $m = e,f$. e is chosen as the upper state and f is chosen as the lower or ground state. The interaction of the atom with the cavity is taken as

$$H' = \sum_1^z (H'_{1i} + H'_{2i})$$

where the terms are defined in equation 2.2. An interaction parameter $\alpha$ is defined by

$$\alpha(\underline{r}) \equiv -i\frac{e}{m}\frac{1}{(2\epsilon_o\hbar\omega)^{1/2}}\left(f/\sum_1^z\left[\underline{p}_i\cdot\underline{A}_{ai} + \frac{\hbar}{2}\underline{\sigma}_i\cdot(\nabla\times\underline{A}_a)_i\right]/e\right). \tag{2.14}$$

For strong electric-dipole interactions the spin term can be neglected

and the vector potential can be considered a constant over the atom. Therefore, in this case,

$$\alpha(\underline{r}) = \frac{\Omega}{(2\epsilon_o \hbar\omega_a)^{1/2}} \underline{D} \cdot \underline{A}_a(\underline{r}) \qquad \ldots \qquad (2.14a)$$

where $\underline{r}$ is the position of the atom in the cavity and where $\underline{D}$ is the electric-dipole matrix element of the atom, i.e.,

$$\underline{D} = (e/ \sum_1^z \underline{d}_i/f) = (f/ \sum_1^z \underline{d}_i/e) \qquad (2.15)$$

and

$$\Omega = \frac{E_e - E_f}{\hbar} \quad . \qquad (2.16)$$

Also, a reduced energy and a reduced Hamiltonian are defined by

$$\tilde{e} \equiv \frac{E - E_f - \frac{1}{2}\hbar\omega}{\hbar} \qquad (2.17a)$$

and

$$\hat{h} \equiv \frac{H - E_f - \frac{1}{2}\hbar\omega}{\hbar} \quad . \qquad (2.17b)$$

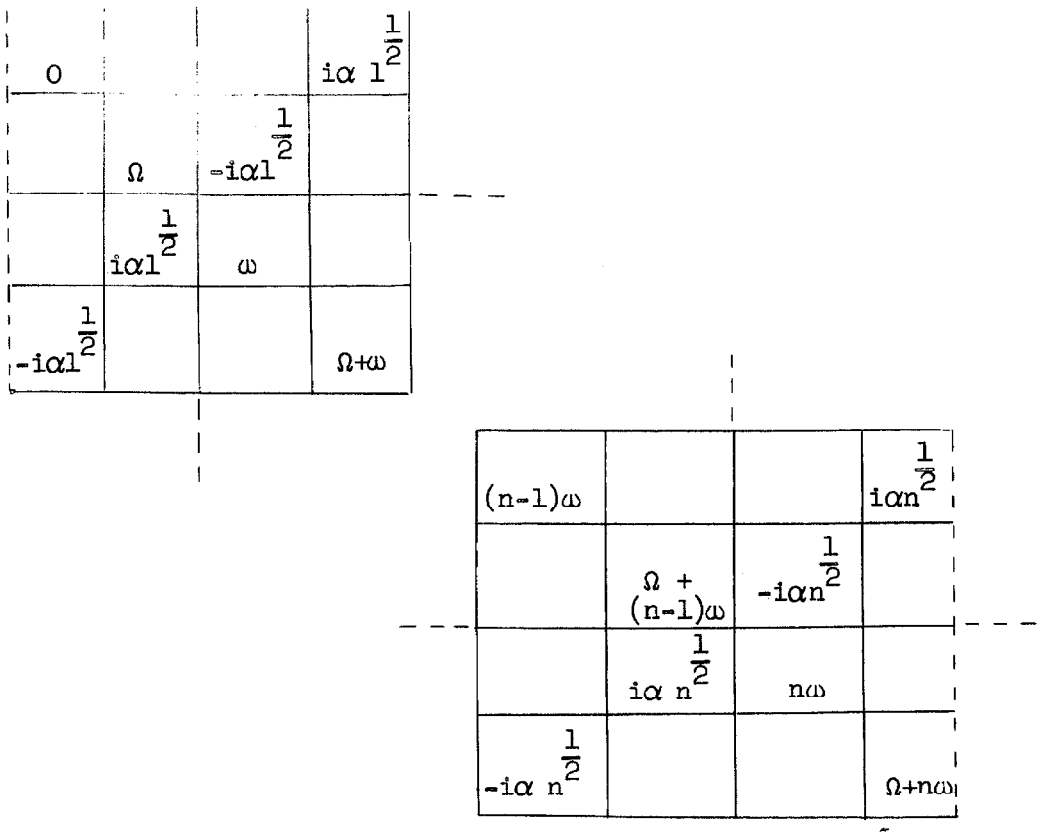
The matrix equation for the stationary states becomes

$$\hat{h}/\phi) = \tilde{e}/\phi) \quad .$$

The basis $\{H_o\}$ is chosen in the following order:

$$/fo), /eo), /f1), /e1), \text{---}, /en\text{-}1), /fn), \text{---} \; .$$

The matrix $\hat{h}$ is given in this basis as




Since $\omega \approx \Omega$ the matrix elements adjacent to the main diagonal connect states nearly degenerate. The remaining matrix elements resulting from the interaction connect nondegenerate states. Since their value is ordinarily very small compared to the energy difference of their states, perturbation theory (Appendix 2) shows that they can be disregarded. The Hamiltonian therefore consists of $2 \times 2$ matrices along the main diagonal. Each of these matrices may be individually diagonalized. A typical sub-matrix is:

$$\left( \begin{array}{c|c} \Omega + (n-1)\omega & -i\alpha\, n^{1/2} \\ \hline i\alpha\, n^{1/2} & n\omega \end{array} \right) \qquad (2.18)$$

The basis for this sub-matrix is /e n-1) and /fn). The exact

diagonalization of the matrix yields the energies:

$$e_n = n\omega + \Delta \pm e'_n \tag{2.19}$$

where

$$e'_n = (\Delta^2 + \alpha^2 n)^{1/2} \tag{2.20a}$$

and

$$\Delta \equiv \frac{\Omega - \omega}{2} \quad . \tag{2.20b}$$

The stationary states are given as

$$/\phi_n +) = \begin{pmatrix} a_n \\ ib_n \end{pmatrix} \text{ and } \quad /\phi_n -) = \begin{pmatrix} ib_n \\ a_n \end{pmatrix} \tag{2.21}$$

where

$$a_n = \frac{(\Delta + e'_n)^{1/2}}{2^{1/2}\, e'^{1/2}_n} \quad \text{and} \quad b_n = \frac{\alpha\, n^{1/2}}{2^{1/2}(\Delta + e'_n)^{1/2}\, e'^{1/2}_n} \tag{2.22}$$

The stationary states can be put into the following forms (2.2):

$$/\phi_{n+}) = \begin{pmatrix} \cos\theta \\ i \sin\theta \end{pmatrix} \text{ and } \quad /\phi_{n-}) = \begin{pmatrix} i \sin\theta \\ \cos\theta \end{pmatrix} \tag{2.23}$$

where

$$\tan 2\theta = \frac{\alpha\, n^{1/2}}{\Delta} \quad .$$

## 2.4 Equations of Motion

The solutions of the equation

$$H/\phi) - E/\phi)$$

have been found. The stationary solutions of the Schrodinger equation

$$i\hbar \frac{\partial/\phi)_t}{\partial t} = H/\phi)_t \quad , \tag{2.24}$$

where $H$ is independent of time, are given by

$$/\phi)_t = /\phi)\, e^{-i\frac{E}{\hbar}t} \quad .$$

From equations 2.16 and 2.19 the stationary solutions for the matrix 2.18 can be written as

$$/\phi_{n+})_t = \left[a_n/en-1) + ib_n/fn)\right] e^{-i(\bar{e}_n + e'_n)t} \tag{2.25a}$$

and

$$/\phi_{n-})_t = \left[ib_n/en-1) + a_n/f_n)\right] e^{-i(\bar{e}_n - e'_n)t} \tag{2.25b}$$

where

$$\bar{e}_n = \frac{E_e + E_f}{2\hbar} + n\omega \tag{2.25c}$$

corresponds to the average energy of the two states $/en-1)$ and $/fn)$. $e'_n$, the energy resulting from the atom-field interaction, splits the degeneracy.

The complete solution to Schrodinger's equation 2.24 can be written as a linear superposition of the states given in equations 2.25, i.e.,

$$/\psi)_t = \sum_n A_{n+}/\phi_{n+})_t + A_{n-}/\phi_{n-})_t \quad . \tag{2.26}$$

The values of the expansion coefficients $A$ are determined by the initial state of the system.

## 2.5 Atom and Cavity Interactions

The stationary states of the cavity and atom system are linear combinations of the states $/en-1)$ and $/fn)$. If at time $t = 0$ the atom is assumed to be in state $/e)$ and the cavity in state $/n-1)$, the initial condition for equation 2.26 is

$$/\psi)_o = /en-1) \quad .$$

This initial condition is satisfied by

$$A_{n+} = a_n \qquad A_{n-} = -ib_n$$

with all other A's zero. Therefore the time development of the system is given by

$$/\psi)_t = e^{-i\bar{e}_n t}\left\{/en-1)\left[a_n^2 e^{-ie'_n t} + b_n^2 e^{+ie'_n t}\right] + /fn)\, ia_n b_n\left[e^{-ie'_n t} - e^{+ie'_n t}\right]\right\} \quad . \tag{2.27}$$

The system is, in general, in a "mixed" state or a combination of the states $/en-1)$ and $/fn)$. The probability $p_f$ of the system being in state $/fn)$ is given by the absolute square of the coefficient of $/fn)$ in equation 2.27, i.e.,

$$p_f = 4a_n^2 b_n^2 \sin^2 e'_n t \quad . \tag{2.28}$$

From equation 2.22

$$a_n b_n = \frac{\alpha n^{1/2}}{2e'_n} \quad . \tag{2.29}$$

Thus,

$$p_f = \frac{\alpha^2 n}{e'^2_n} \sin^2 e'_n t \quad . \tag{2.30}$$

The probability $p_e$ of the system being in state /en-1) is given by

$$p_e = 1 - \frac{\alpha^2 n}{e'^2_n} \sin^2 e'_n t \quad . \tag{2.31}$$

Using definition 2.20, one can write these as

$$p_f = 1 - p_e = \frac{\alpha^2 n}{\Delta^2 + \alpha^2 n} \sin^2(\Delta^2 + \alpha^2 n)^{1/2} t \; . \tag{2.32}$$

The probability oscillates between the two states at a rate which depends on the difference between cavity and atomic frequency $\Delta$, and on the strength of the interaction $\alpha^2 n$. If $\Delta = 0$, the probability $p_f$ has a maximum of one and a time average of one-half. Alternatively, the cavity can be considered as gaining a photon with peak probability of one and a time-average probability of one-half. If $\Delta \neq 0$, the peak probability of the cavity gaining a photon is given by

$$\frac{1}{1 + \left(\frac{\Delta}{\alpha n^{1/2}}\right)^2} \quad ,$$

which is a Lorentzian factor with halfwidth $4\alpha n^{1/2}$ in units of $\omega$. Therefore, as the energy in the cavity and/or the interaction parameter $\alpha$ increases, the response of the atom to the cavity fields

becomes broader.

If the initial state of the system had been $/fn)$ , the cavity in state $/n)$ and the atom in the lower energy state, the probabilities $p_f$ and $p_e$ would be interchanged. The cavity would be losing a photon instead of gaining a photon in the discussion above.

## 2.6 Beam-Type Maser

A cavity which has been found useful for beam-type masers is the circular cylindrical cavity operating in the $TM_{010}$ mode. This cavity is mentioned in Section 1.5. The electric field is independent of axial distance. For excited molecules traveling along the length of the cavity the interaction, which is assumed to be a strong electric-dipole interaction, does not change and the results for stationary atoms can be used. In this particular case the interaction parameter $\alpha$ defined in equation 2.14a becomes:

$$\alpha = \frac{\Omega}{(2\epsilon_o \hbar \omega V)^{1/2}} \frac{D_z}{J_1(2.405)} \quad . \tag{2.33}$$

n , the number of photons in the cavity, is found from the energy in the cavity.

The molecule enters the cavity in the upper state, travels along the axis of the cavity with velocity v , and leaves the cavity after traveling a length L . The probability of the cavity gaining a photon is given by equation 2.32 with

$$t = L/v \quad .$$

This probability

$$P_1 = \frac{1}{1+\frac{\Delta^2}{\alpha n^2}} \sin^2 \left[ \frac{\Delta^2 L^2}{v^2} + \alpha^2 n \frac{L^2}{v^2} \right]^{1/2} \qquad (2.34)$$

is plotted in Figure 2.1 as a function of the "frequency" $\Delta \frac{L}{v}$ for various values of the "power-level" parameter $\alpha^2 n \frac{L^2}{v^2}$. For small values of this parameter the maximum probability occurs when the molecular frequency equals the cavity frequency. As this parameter increases the probability has significant secondary maxima for values of $\Delta$ unequal to zero. For still larger values of the parameter the probability can have its maximum value at a cavity frequency different from the molecular frequency. For very large values of this parameter the probability oscillates between 0 and a peak value. The envelope of the peak values is given by the Lorentzian factor in equation 2.34.

The above discussion gives, for a molecule with a particular velocity, the probability for giving up a photon to the cavity. In a typical ammonia beam maser the average velocity $\bar{v}$ is about $6.0 \times 10^2$ m/sec. (2.3). Assuming that a $TM_{010}$ maser cavity is loaded only by wall losses and the input, one can find that the cavity input power P is given by

$$P \approx 10^{-10} \left(\alpha^2 n \frac{L^2}{\bar{v}^2}\right) \text{ watts}$$

for the above velocity*. Therefore, the operation described previously would apply to the maser operating at very low power levels.

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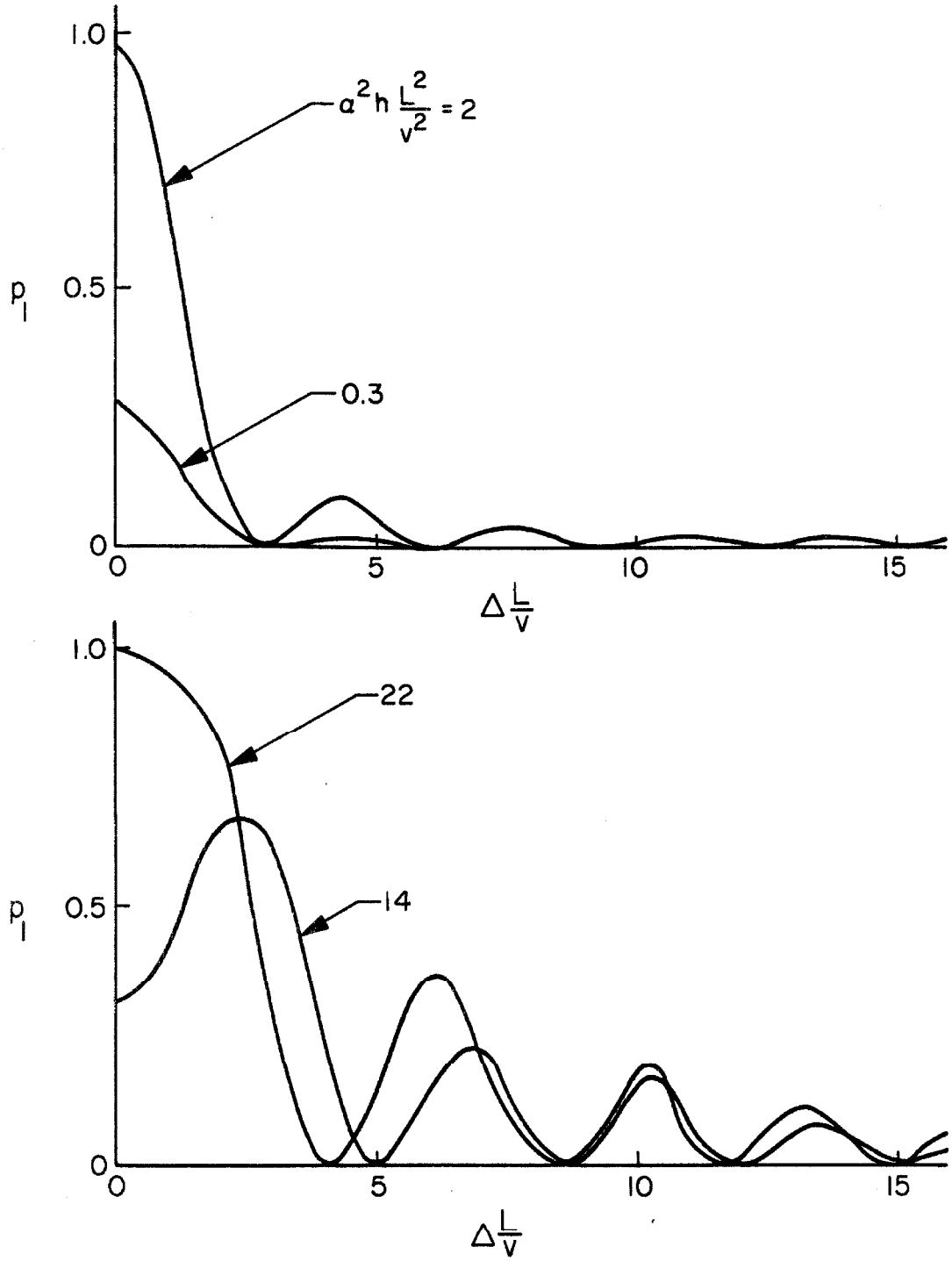
*The values of the cavity and molecule parameters are given in (2.3).




PROBABILITY vs. FREQUENCY

Figure 2.1

## REFERENCES

(2.1) A. Messiah, Quantum Mechanics, North-Holland Publishing Company, Amsterdam (1961), pp. 1037-1045.

(2.2) E. P. Jaynes and F. W. Cummings, "Comparison of Quantum and Semi-classical Radiation Theories with Application to the Beam Maser", Proc. IEEE 51, 89-110, January 1963.

(2.3) K. Shimoda, T. C. Wang and C. H. Townes, "Further Aspects of the Theory of the Maser", Phys. Review 102, 1310, June 1956.

# CHAPTER 3

## SEMICLASSICAL THEORY

The term semiclassical theory will be used to describe the formulation of the problem of excited atoms (or molecules) interacting with electromagnetic fields when the atom is quantized and the electromagnetic field is treated classically (3.1). The interaction energy in the Hamiltonian exhibits the explicit time dependence of the classically-described field. The semiclassical theory will be developed and compared with the quantized-field theory of Chapter 2.

### 3.1 Interaction of an Excited "Atom" with a Classical Field

The type of interaction between the atom and the classical field will be that discussed in Chapter 2, the so-called electric-dipole interaction. The Hamiltonian for the atom interacting with the field is:

$$H = H_{atom} + H' \quad .$$

$H_{atom}$, the Hamiltonian of the atom itself, is defined the same as in Chapter 2, Section 1. The interaction Hamiltonian in the notation of Chapter 2 is:

$$H' = \sum_{l}^{z} H'_{li} \quad ,$$

where

$$H'_{li} = -\frac{e}{2m}\left(\underline{p}_i \cdot \underline{A}_i + \underline{A}_i \cdot \underline{p}_i\right) \quad . \tag{2.2}$$

$\underline{A}_i$ is now the time-dependent field. Using the gauge relation $\nabla \cdot \underline{A} = 0$, one can write

$$H' = -\frac{e}{m}\underline{A}\cdot\sum_{1}^{Z}\underline{p}_i \quad . \tag{3.1}$$

The relationship for the momentum in terms of the commutator of the electric-dipole moment and $H_{atom}$ is found from equation 2.5 as

$$-\frac{e}{m}\sum_{1}^{Z}\underline{p}_i = \frac{i}{\hbar}\left[\sum_{1}^{Z}\underline{d}_i\ ,\ H_{atom}\right] . \tag{3.2}$$

Thus,

$$H' = \frac{i}{\hbar}\underline{A}\cdot\left[\sum_{1}^{Z}\underline{d}_i\ ,\ H_{atom}\right] . \tag{3.3}$$

The atom is considered to be a two-state system with upper state $e$ and lower state $f$. The expansion functions used to solve the problem of the atom interacting with the field are

$$/e)_t \equiv /e)\ e^{-i(E_e/\hbar)t} \tag{3.4a}$$

and

$$/f)_t \equiv /f)\ e^{-i(E_f/\hbar)t} . \tag{3.4b}$$

Thus, the Schrodinger equation of the system

$$i\hbar\frac{\partial}{\partial t}/\psi)_t = (H_{atom} + H')/\psi)_t \quad , \tag{3.5a}$$

with

$$/\psi)_t = a(t)/e)_t + b(t)/f)_t \tag{3.5b}$$

becomes

$$i\hbar\dot{a} = -i\Omega\ e^{i\Omega t}\ (\underline{D}\cdot\underline{A})\ b \tag{3.6a}$$

and

$$i\hbar\dot{b} = i\Omega e^{-i\Omega t} (\underline{D} \cdot \underline{A}) a \quad . \tag{3.6b}$$

Equation 3.3, the definition of the dipole moment $\underline{D}$, and $\Omega$ the angular frequency corresponding to the energy difference of the two atomic states, have been used to find equations 3.6. The field $\underline{A}$ is the field at the position of the atom and it is a function of spatial coordinates and time.

For a stationary atom interacting with a cavity field, $\underline{A}$ can be taken as

$$\underline{A} = \underline{A}(\underline{r}) \cos \omega t \quad . \tag{3.7}$$

The equations of motion 3.6 become the following set of nonlinear coupled differential equations:

$$i\hbar\dot{a} = - bi\Omega e^{+i\Omega t} \underline{D} \cdot \underline{A}(\underline{r}) \cos \omega t \tag{3.8a}$$

$$i\hbar\dot{b} = ai\Omega e^{-i\Omega t} \underline{D} \cdot \underline{A}(\underline{r}) \cos \omega t \quad . \tag{3.8b}$$

This set of equations is greatly simplified and easily solved if only the resonant terms are kept. The resulting equations are:

$$\dot{a} = - b \frac{\Omega}{2\hbar} e^{+i(\Omega-\omega)t} \underline{D} \cdot \underline{A}(\underline{r}) \tag{3.9a}$$

$$\dot{b} = a \frac{\Omega}{2\hbar} e^{-i(\Omega-\omega)t} \underline{D} \cdot \underline{A}(\underline{r}) \quad . \tag{3.9b}$$

Equations 3.9 are used as the differential equations for both the motion of the atom and its initial conditions. The solution of these

equations is called the Rabi solution (3.2) .

These equations can be combined to yield uncoupled equations with constant coefficients, i.e.,

$$\ddot{a} - i(\Omega-\omega)\,\dot{a} + \left[\frac{\Omega\,\underline{D}\cdot\underline{A}(\underline{r})}{2\not{h}}\right]^2 a = 0 \qquad (3.10a)$$

$$\ddot{b} + i(\Omega-\omega)\,\dot{b} + \left[\frac{\Omega\,\underline{D}\cdot\underline{A}(\underline{r})}{2\not{h}}\right]^2 b = 0 \quad . \qquad (3.10b)$$

## 3.2 Comparison of the Semiclassical and the Quantum Theory

As is shown in Appendix 1, the parameter

$$\frac{\Omega\,\underline{D}\cdot\underline{A}(\underline{r})}{2\not{h}} = \alpha\, n^{1/2} \quad . \qquad (A1.7)$$

The solutions to equations 3.10 give identical probabilities for the same initial conditions as found in Chapter 2, equation 2.32, i.e., if at $t = 0$ the atom had been in state $/f)$ the probability of its being in state $/e)$ is found to be:

$$bb^* = p_e = \frac{\alpha^2 n}{\Delta^2 + \alpha^2 n}\,\sin^2(\Delta^2 + \alpha^2 n)^{1/2}\, t$$

and the probability of being in state $/f)$ is found to be:

$$aa^* = 1 - bb^* \quad .$$

Therefore, both the semiclassical theory and the quantum theory give the same result for the probabilities of finding the atom in states e and f . Also the semiclassical theory gives the correct result for an atom interacting with a cavity containing a small number of

quanta or even with a cavity excited by only its zero-point energy, if the magnitude of the field is chosen properly*.

The correspondence between the two theories is shown more precisely by solving the atom and cavity problem of Section 2.4 by using an expansion with time-dependent coefficients. The expansion functions in this case are solutions to the Schrodinger equation with the Hamiltonian equal to $H_{atom} + H_{field}$. Thus, instead of diagonalizing the matrix for the complete Hamiltonian of Chapter 2, expansion 2.26 is written as

$$/\psi)_t = \sum_n A_{en}(t)/en)_t + A_{fn}(t)/fn)_t \quad . \tag{3.11}$$

The terms of $H_{int}$ connecting "non-degenerate" states are, as in Section 2.3, neglected. As a result, the differential equations for the time-dependent A's reduce to sets of two coupled equations, namely

$$\dot{A}_{en-1} = - A_{fn} \alpha n^{1/2} e^{i(\Omega-\omega) t} \tag{3.12a}$$

and

$$\dot{A}_{fn} = A_{en-1} \alpha n^{1/2} e^{-i(\Omega-\omega) t} \quad . \tag{3.12b}$$

Noting the equivalence stated in equation A1.7, one can see that the equations 3.12 and the equations 3.9 are identical. Therefore, for the case treated, the semiclassical theory and the quantized field theory are identical. The neglecting of the nonresonant terms in

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*See Appendix 1.

equation 3.9 can be considered somewhat analogous to the neglecting of matrix elements between "nondegenerate" states.

## 3.3 $\underline{\mu} \cdot \underline{E}$ Interaction

In both the semiclassical theory and the quantized field theory the electric-dipole interaction between an atom and the field is often added (2.2, 3.1, 3.3) to the Hamiltonian in the form

$$- \underline{\mu} \cdot \underline{E} \tag{3.13}$$

instead of the form

$$H' = - \frac{e}{2m} \sum_{1}^{z} (\underline{p}_i \cdot \underline{A}_i + \underline{A}_i \cdot \underline{p}_i) \tag{3.14}$$

used in Chapter 2. $-\underline{\mu} \cdot \underline{E}$ is analogous to the static energy of an electric dipole in an electric field. $\underline{\mu}$, the electric-dipole operator, is endowed with the characteristic that its matrix elements in the basis $H_{atom}$ for the ideal two-level atom considered in Chapter 2 consist only of off-diagonal terms given by

$$(m/\underline{\mu}/m') = \underline{D} (1 - \delta_{mm'}) \tag{3.15}$$

where $\underline{D}$, the electric dipole matrix element, has been defined in equation 2.15.

As has been shown previously in Section 2.4, the equations of motion of an atom interacting with a cavity depend on the matrix element $(en - 1/H'/fn)$. From definitions 2.14 and 1.45, the ratio of this matrix element for the two types of interaction is found to be:

$$\frac{(en - 1/ - \underline{\mu} \cdot \underline{E} / fn)}{(en - 1/ H' / fn)} = \frac{\omega}{\Omega} \qquad (3.16)$$

where $H'$ is understood to be that given in equation 3.14. In the approximation $\omega \approx \Omega$ these matrix elements are approximately equal and therefore no distinction will be made between the form of the interaction for electric-dipole interactions. However, in many cases, such as interactions with a distribution of $\omega$'s extending from 0 to $\infty$, the above approximation is not satisfied and the $\underline{\mu} \cdot \underline{E}$ interaction cannot be used.

## REFERENCES

(3.1) J. P. Gordon, H. J. Zeiger, C. H. Townes, "The Maser--New Type of Microwave Amplifier, Frequency Standard and Spectrometer", Phys. Rev. 99, 1268, (1955).

(3.2) W. E. Lamb, Jr., "Quantum Mechanical Amplifiers", in Lectures in Theoretical Physics, edited by W. E. Britten and B. D. Downs, Vol. II, Interscience Publishers, Inc., New York (1960), especially p. 444 .

(3.3) W. E. Lamb, Jr., "Theory of an Optical Maser", Phys. Rev. 134, June 15, 1964, especially pp. 1430-1431.

# CHAPTER 4

## RADIATIVE TRANSITIONS

In this chapter the decay of an excited atom to lower-lying states by means of zero-point interactions with "empty" space is considered. The subject is introduced first by considering the theory which neglects the decay of an initial excited state. The more exact theory which considers the decay of the initial state is then formulated and extended to cascaded decays of an arbitrary multilevel atom.

### 4.1 Simplified Theory of Decaying States

The concept of a spontaneous radiation is introduced by considering the simple theory which neglects the decay of an initial state. This discussion is similar to that presented in many textbooks (4.1, 4.2).

The Hamiltonian for the system of atom and radiation fields is given, as usual, by

$$H = H_{atom} + H_{rad} + H_{int} . \tag{4.1}$$

The interaction Hamiltonian $H_{int}$ is the same as that which was given previously in equations 2.2 and 2.3 as $H'$ . The $H_{rad}$ is discussed in Appendix 3. It is given in terms of creation and annihilation operators as

$$H_{rad} = \sum_a \left(a_a^+ a_a + \frac{1}{2}\right) \hbar \omega_a \tag{4.2}$$

where the $a_a^+$ and $a_a$ are the creation and annihilation operators for the traveling-wave modes described in Appendix 3. The kets forming the basis $H_{rad}$ are given, as in equation 1.39, by the product of the individual kets for each particular mode, i.e., by

$$\prod_a /n_a) = /n_1, n_2 - - -, n_a - - ) \tag{4.3}$$

where $n$ is the number of photons in each traveling wave. The kets forming the basis $\left\{H_{atom}\right\}$ are given by $/m)$ . The Schrodinger equation for the system

$$i\hbar \frac{\partial/\psi)}{\partial t} t = H/\psi)_t \quad , \tag{4.4}$$

is solved by assuming $/\psi)_t$ as an expansion in time-dependent solutions of the Schrodinger equation with the Hamiltonian $H_{atom} + H_{rad}$ . Only single-photon processes are considered.*

For an atom initially excited to level $/e)$ and none of the field oscillators excited, the initial state of the system is given by

$$/\psi)_o = /e) / 00000 - - - ) \equiv /e) \quad . \tag{4.5}$$

Furthermore, from the above discussion

$$/\psi)_t = \sum_{m,a} A_{ma}(t) / ma)_t \tag{4.6}$$

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*Multiple-photon processes can be neglected in the problems to be considered.

where

$$/ma) \equiv /m) \,/\, 0, 0, 0, \cdots, 1_a, \cdots)$$

signifies that the atom is in state $/m)$ and that one of the field oscillators is in its first excited state. Here the index $a$ replaces the more conventional use of the double indices $\underline{k}$ and $\underline{\epsilon}$ where $\underline{k}$ is the wave vector of the particular mode and $\underline{\epsilon}$ is the polarization of the transverse field.

The problem is solved for a time short enough so that the initial state of the system is considered unchanged. Thus, the single-photon states of the system are coupled to the initial state by the equation

$$i\hbar \frac{dA_{ma}(t)}{dt} = {}_t(ma/H_{int}/e)_t \qquad m \neq e \tag{4.7a}$$

with the initial condition

$$A_{ma}(0) = 0 \qquad m \neq e \ . \tag{4.7b}$$

If the matrix elements are defined by

$$(ma/H_{int}/e) \equiv H''_{ma,e} \ , \tag{4.8}$$

the probability of being in state $/ma)$ is found to be:

$$\left|A_{ma}(t)\right|^2 = \left|\frac{H''_{ma,e}}{\hbar}\right|^2 \frac{\sin^2\left[\frac{(\omega_a - \Omega_{em})t}{2}\right]}{\left(\frac{\omega_a - \Omega_{em}}{2}\right)^2} \tag{4.9}$$

where $\omega_a$ is the angular frequency of the field photon and $\Omega_{em}$ is the angular frequency corresponding to the energy difference between the state e and state m, i.e.,

$$\Omega_{em} = \frac{E_e - E_m}{\hbar}$$

The total probability of being in states other than the initial state is given by the summation of equation 4.9 over all states. The summation over a can be expressed as an integral over k using the relationship A3.22.

In terms of the angular frequencies of emitted photons, this total probability $w_e$ can be expressed for isotropic radiation as

$$w_e = \sum_{m,a} \left|A_{ma}(t)\right|^2 = \sum_m \sum_a \left|\frac{H''_{ma,e}}{\hbar}\right|^2 \frac{\sin^2\left[\frac{(\omega_a - \Omega_{em})t}{2}\right]}{\left(\frac{\omega_a - \Omega_{em}}{2}\right)^2} =$$

$$\frac{L^3}{\pi^2 c^3} \sum_m \int_0^\infty \left|\frac{H''_{ma,e}(\omega)}{\hbar}\right|^2 \frac{\sin^2\left[\frac{(\omega - \Omega_{em})t}{2}\right]}{\left(\frac{\omega - \Omega_{em}}{2}\right)^2} \omega^2 d\omega \,. \qquad (4.10)$$

The factor

$$\frac{\sin^2\left[\frac{(\omega_a - \Omega_{em})t}{2}\right]}{\left(\frac{\omega_a - \Omega_{em}}{2}\right)^2} \qquad (4.11)$$

is a function of $\omega_a$ peaked at

$$\omega_a = \Omega_{em}$$

with peak value $t^2$ and width in units of $\omega_a$ given approximately by

$$\frac{2\pi}{t} \quad .$$

If the assumption is made that the time is sufficiently long for the factor 4.11 to become very sharply peaked, this factor can be considered as a delta function in $\omega_a$. Using the results of Appendix 4, one can express this factor as

$$\frac{\sin^2\left[\frac{(\omega_a - \Omega_{em})\, t}{2}\right]}{\left(\frac{\omega_a - \Omega_{em}}{2}\right)^2} = 2\pi t\ \delta(\omega_a - \Omega_{em}) \quad . \tag{4.12}$$

Any high-frequency divergences occurring in equation 4.10 are handled by assuming an appropriate cutoff function as explained in Section A3.3. Thus, it is seen that the total probability of being in a state other than the initial state is proportional to time. Since the integration is carried out only over positive $\omega$'s, a finite probability for decay exists only for atomic states having a lower energy than the original excited state. The time proportional probability $w_e$ can be written in terms of $\Gamma_e$, the total transition probability per unit time, i.e.,

$$w_e = \Gamma_e\, t \quad . \tag{4.13}$$

Combining equations 4.10, 4.12 and 4.13 and denoting all the lower states to which the excited state $e$ decays by the symbol $f$, one finds $\Gamma_e$ to be:

$$\Gamma_e = \sum_f \Gamma_{e \to f} \tag{4.14}$$

where

$$\Gamma_{e\to f} = \sum_a \left| \frac{H''_{fa,e}}{\hbar} \right|^2 2\pi\, \delta(\omega_a - \Omega_{ef}) \quad . \tag{4.15}$$

For isotropic radiation this can be written as

$$\Gamma_{e\to f} = \frac{2L^3}{\pi c^3} \int_0^\infty \left| \frac{H''_{fa,e}}{\hbar} \right|^2 \delta(\omega_a - \Omega_{ef})\, \omega^2\, d\omega \, . \tag{4.16}$$

The general relation given by equation 4.15 is used to define the decay constant $\Gamma_{e\to f}$. For transitions involving polarized states, e.g., two states which define a magnetic sublevel, $H''_{fa,e}$ can have an angular dependence.

$\Gamma_e^{-1}$ is found to be the lifetime of the excited state by the following argument (4.1). Consider at a particular time $t$, $N_e(t)$ excited atoms. In the additional time $dt$, $\Gamma_e\, dt\, N_e(t)$ atoms will statistically decay to lower states. Thus, the number of atoms at any time is given by

$$N_e(t) = N_e(0)\, e^{-\Gamma_e t} \quad . \tag{4.17}$$

The above calculation for short time does not give the distribution of photons emitted by the excited atom. This distribution will be found in the following section.

The range of validity of the simplified theory considered in this section can be found by comparing the probability of state /ma) as given in equations 4.9 and 4.50. The latter equation results from a more exact theory which considers the decay of

the initial state and the intermediate state $m$. These two equations are found to be approximately equal when

$$t \ll \frac{1}{(\Gamma_e + \Gamma_m)}$$

where $\Gamma_e$ is the decay constant of the initial state and $\Gamma_m$ is the decay constant of an intermediate state $m$. If the decay is to a ground state $(\Gamma_m = 0)$, clearly, this is recognized as the normal restriction on the analysis of a dissipative system when dissipation is neglected. That is, the analysis is valid for time short compared to the decay time. Also, the factor given by expression 4.11 can be considered a delta function in integration 4.10 only if the function to be integrated has only a small variation over the main peak of this factor. Since for electric-dipole interactions the term $\left|H''_{ma,e}\right|^2$ is inversely proportional to $\omega$, the above condition can be stated as

$$t \gg \frac{\hbar}{E_e - E_m} .$$

This relation is also true for the magnetic and electric multipole interactions which give $\left|H''_{ma,e}\right|^2$ proportional to $k^n$ where $n$ is a relatively small integer. Thus, for decay to a ground state the probability can be considered proportional to time in the range

$$\frac{\hbar}{E_e - E_m} \ll t \ll \frac{1}{\Gamma_e}$$

or a time short compared to the decay time and long compared to the

period of the transition frequency.

Assuming that the excited state of the atom is coupled to a lower state f by a strong electric-dipole interaction, one can find that the decay constant given by equation 4.16 is

$$\Gamma_{e\to f} = \frac{D_{fe}^2 (E_e - E_f)^3}{\pi \hbar^4 \epsilon_o c^3} \quad . \tag{4.18}$$

The matrix element $H''_{fa,e}$ has been evaluated by using the definition of a strong electric-dipole interaction (equation 3.3) and by using expansion of the vector potential in terms of creation and annihilation operators (equation A3.20). Therefore, for electric-dipole transitions, the decay constant depends on the square of the dipole matrix element between the states and the cube of the energy difference between the states.

## 4.2 Damping Theory

In order to treat the problem of the decay of an excited atom the quantum mechanical equations of the motion must be solved for times long enough for the state of the system to differ appreciably from the initial state. The easily solved equations 4.7 do not adequately describe the system for longer times.

When an attempt, similar to that in Section 4.1, is made to solve the Schrodinger equation for an atom interacting with the zero-point energies of space, high frequency divergences occur from the integration over k . In addition, there occur low-frequency divergences. The high-frequency divergence or ultra-violet catastrophe may

be handled within the framework of the nonrelativistic theory by use of an appropriate cutoff function as described in Appendix 4. The low-frequency divergence or infrared catastrophe can be shown not to exist if the theory is properly reformulated (4.3, 4.4).

These divergences, which result from the interactions of the atomic electrons with virtual photons, also exist when the problem is solved in the framework of a completely relativistic theory. The low-frequency divergence does not exist if the "proper" field is used (4.3,4.4). The high-frequency divergence still remains in the relativistic theories and the concept of mass renormalization* must be used (4.5, 4.15). Any radiative correction not accounted for by mass renormalization can result in a shift of the atomic energy level, the Lamb shift (4.6) .

The problem of an atom interacting with the zero-point energies of space may be formulated within the framework of the nonrelativistic theory by introducing the concept of the counter-term (4.7). The Hamiltonian given in equation 4.1 is rewritten as

$$H = H_{atom} + H_{rad} + H_{ct} + H_{int} - H_{mn} - H_{ct} \tag{4.19}$$

where $H_{atom}$ is the atom Hamiltonian using the experimental mass of the electron, $H_{mn}$ is the mass-normalization term, and $H_{ct}$ is the counter-term introduced to account for any energy level shifts not compensated for by $H_{mn}$. The unperturbed Hamiltonian $H_o$ will be

*The divergent terms can be included as a correction to the mass of the electron. Hence, the experimental mass of the electron is assumed to include the effect of these divergent terms.

taken as

$$H_o \equiv H_{atom} + H_{rad} + H_{ct} \tag{4.20}$$

and the perturbing Hamiltonian $H''$ will be taken as

$$H'' \equiv H_{int} - H_{mn} - H_{ct} \quad . \tag{4.21}$$

It can be shown (4.8) that for hydrogen-like atoms, $H_{mn}$ the mass-renormalization term, is able to account for most of the energy level shifts resulting from $H_{int}$, and any energy-level shift remaining is negligible compared to energy-level differences for optical transitions.

With the above in mind, the problem of the decay of an excited atom will be solved with the following assumptions:

a. A counter-term and a mass renormalization term exist in the Hamiltonian to counter any energy level shifts of a state due to $H_{int}$ .

b. Any shift remaining after mass renormalization is very small compared to relevant energy level differences.

c. All corrections to the energy level of the atom resulting from $H_{int}$ will be absorbed into the definition of $E_m$ , the energy level of the atomic state.

d. To a first approximation, the wave functions of the atomic states will be given by solutions of the Schrodinger equation with the Hamiltonian $H_{atom}$.

Therefore, in all cases $E_m$ will be the experimentally determined energy. Assumption d follows from assumption c since if any energy level correction is very small, then any correction to the wave

function would be expected to be small.

Neglecting the portion of $H_{int}$ causing decay to lower states, one assumes that stationary states of the Hamiltonian of an atom interacting with the zero-point radiation fields exist and that, to a first approximation, the wavefunctions of these stationary states are given as solutions of the Hamiltonian $H_{atom}$.

The check on the validity of the above assumptions lies within the realm of relativistic quantum mechanics (4.3, 4.4, 4.8, 4.15). This is beyond the scope of the analysis presented here.

In order to illustrate the method of solution, the problem of an excited atom decaying to the ground state by means of a single-photon transition will be considered. The solution of the coupled equations resulting from the Schrodinger equation were first solved by assuming exponential decays (4.9) and more recent methods use transform methods (4.10, 4.11). The procedure used here will be similar; the convenient Laplace transform method (4.12) will be used.

The Schrodinger equation for the system is solved, as in Section 4.1, by expanding in the time-dependent solutions of $H_o$ defined in equation 4.20, with the perturbing Hamiltonian $H''$ defined in equation 4.21. Thus, equations 4.4 and 4.6 become

$$i\hbar \dot{A}_{m'a'} = \sum_{m,a} A_{ma}\, t^{(m'a' / H'' / ma)}{}_t \qquad (4.22)$$

where $m$ now refers to a "bound" atomic state of $H_o$. For the problem of an excited state coupled to one ground state $g$ by a single-photon transition, equation 4.22 becomes the following set of

equations:

$$i\hbar \dot{A}_e = \sum_a A_{ga}\, {}_t(e/H''/ga)_t \qquad (4.23a)$$

and

$$i\hbar \dot{A}_{ga} = A_e\, {}_t(ga/H''/e)_t \qquad (4.23b)$$

where, as previously, the index $e$ refers to the excited state of the atom, the index $g$ to the ground state of the atom, and the running index $a$ to excited states of the field containing one photon of a given frequency, direction and polarization. If at time $t = 0$ it is assumed that the field is in its ground state and the atom is in the excited state $e$, the initial conditions are:

$$A_e(0) = 1 \qquad (4.24a)$$

and

$$A_{ga}(0) = 0 \quad . \qquad (4.24b)$$

These differential equations are transformed from the variable $t$ to the complex variable $s$ by means of the defining equation for Laplace transforms (4.12), namely,

$$f(s) = \int_0^\infty e^{-st} f(t)\, dt \quad , \quad s \equiv \sigma + i\omega \qquad (4.25)$$

where $f(s)$ is the Laplace transform of the function $f(t)$.

$\sigma$ and $\omega$, both real, denote the real and imaginary part of $s$. Equations 4.23 and 4.24 become

$$i\not{h}\,[-1 + s\,A_e(s)] = \sum_a H''_{e,ga}\,A_{ga}(s-i[\Omega_{eg} - \omega_a]) \qquad (4.26a)$$

and

$$i\not{h}\,s\,A_{ga}(s) = H''_{ga,e}\,A_e(s+i\,[\Omega_{eg} - \omega_a]) \qquad (4.26b)$$

where $\omega_a$ is the angular frequency of the field photon and $\Omega_{eg}$ is the angular frequency corresponding to the energy difference between the excited and ground state, i.e.,

$$\Omega_{eg} = \frac{E_e - E_g}{\not{h}} \quad . \qquad (4.27)$$

The coefficient of the excited state is found, from equations 4.26 to be:

$$A_e(s) = \frac{1}{s + \sum_a \left|\frac{H''_{e,ga}}{\not{h}}\right|^2 \frac{1}{s + i(\omega_a - \Omega_{eg})}} \quad . \qquad (4.28)$$

Equation 4.28 can be rewritten as

$$A_e(s) = \frac{1}{s + \sum_a \left|\frac{H''_{e,ga}}{\not{h}}\right|^2 \frac{\sigma - i(\omega_a - \Omega_{eg} + \omega)}{\sigma^2 + (\omega_a - \Omega_{eg} + \omega)^2}} \quad . \qquad (4.29)$$

In general, $A_e(t)$ is given by the inversion theorem, i.e.,

$$A_e(t) = \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} e^{st}\,A_e(s)\,ds \quad . \qquad (4.30)$$

Since the summation in equation 4.29 is really an integration over $\omega_a$

with the limits 0 to $\infty$, $A_e(s)$ clearly has a cut along the imaginary axis for

$$-\infty < \omega < \Omega_{eg} \quad .$$

The contour indicated in equation 4.30 can be deformed to give $A_e(t)$ as

$$A_e(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{i\omega t} \left[ F_+(\omega) - F_-(\omega) \right] d\omega \qquad (4.31)$$

where

$$F_{\pm}(\omega) = \frac{1}{i\omega \pm \frac{\Gamma(\omega)}{2} - \frac{i\,\delta E(\omega)}{\hbar}} \qquad (4.32)$$

$$\frac{\Gamma(\omega)}{2} = \lim_{\sigma \to 0+} \sum_a \left| \frac{H''_{e,ga}}{\hbar} \right|^2 \frac{\sigma}{\sigma^2 + (\omega_a - \Omega_{eg} + \omega)^2} \qquad (4.33a)$$

and

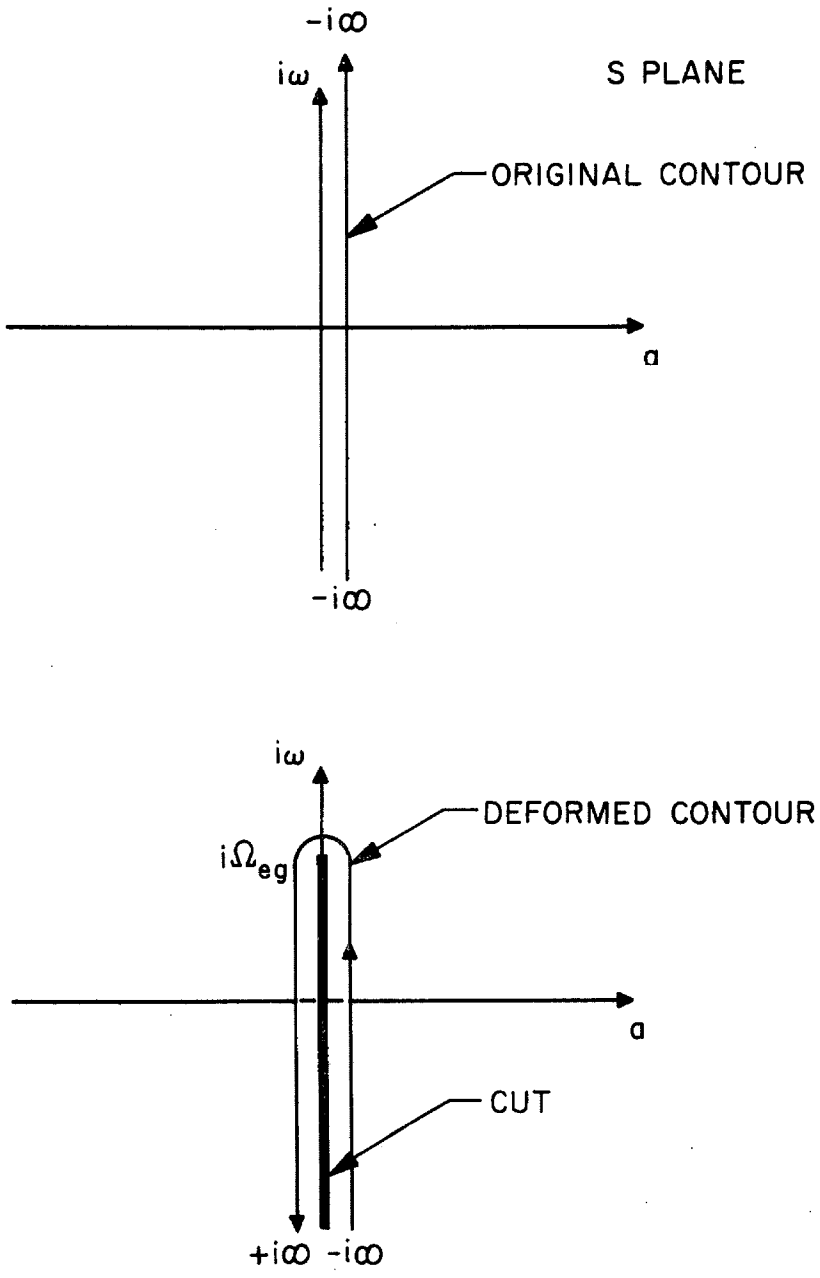
$$\frac{\delta E(\omega)}{\hbar} = \sum_a \left| \frac{H''_{e,ga}}{\hbar} \right|^2 \frac{1}{(\omega_a - \Omega_{eg} + \omega)} \quad . \qquad (4.33b)$$

The index a is retained to differentiate between the photon frequency and the imaginary part of the variable s . The deformation of the contour is shown in Figure 4.1.

If one considers

$$\lim_{\sigma \to 0+} \frac{\sigma}{\sigma^2 + (\omega_a - \Omega_{eg} + \omega)^2} = \pi \delta(\omega_a - \Omega_{eg} + \omega) \quad , \qquad (4.34)$$

equation 4.33a can be written as




CONTOUR DEFORMATION

Figure 4.1

$$\frac{\Gamma(\omega)}{2} = \pi \sum_a \left| \frac{H''_{e,ga}}{\hbar} \right|^2 \delta(\omega_a - \Omega_{eg} + \omega) \qquad (4.35)$$

where the summation over $a$ is thought of as an integration over $\omega_a$. If equation 4.35 is compared with equation 4.15, one can see that $\Gamma(0) = \Gamma_{e \to g}$. $\frac{\delta E(0)}{\hbar}$ is the principal value of the integral or the energy level correction to level $e$ as found by second-order perturbation theory. Since it has been assumed that energy-level corrections were included in the counter-term of equation 4.20,

$$\delta E(0) = 0 \quad . \qquad (4.36)$$

The term $F_+(\omega) - F_-(\omega)$ in equation 4.31 becomes

$$F_+(\omega) - F_-(\omega) = \frac{\Gamma(\omega)}{\left[\frac{\Gamma(\omega)}{2}\right]^2 + \left[\omega - \frac{\delta E(\omega)}{\hbar}\right]^2} \quad , \qquad (4.37a)$$

a sharply peaked function* about the point $\omega = 0$ if

$$\Gamma(0) \ll \Omega_{eg} \quad . \qquad (4.37b)$$

If this condition is satisfied, the term in equation 4.37a can be represented in equation 4.31 to a good approximation by

$$F_+(\omega) - F_-(\omega) \approx \frac{\Gamma(0)}{\left[\frac{\Gamma(0)}{2}\right]^2 + \omega^2} \quad . \qquad (4.38)$$

*For electric-dipole transitions $\Gamma(\omega)$ changes linearly with frequency and $\delta E(\omega)$ can be considered a constant $(0)$ in the vicinity of $\omega = 0$, cf. (4.10).

If this approximation is used, equation 4.31 can easily be integrated to yield

$$A_e(t) = e^{-\frac{\Gamma(0)}{2}t} \quad . \tag{4.39}$$

Therefore, as found in the previous section, the probability of being in the initial state decays exponentially with decay constant $\Gamma_{e\to g}$. Furthermore, the choice of function along the cut in equation 4.31 is found to satisfy the boundary condition at $t = 0$ .

If the result given by equation 4.39 is used, $A_e(s)$ can be written as

$$A_e(s) = \frac{1}{s + \frac{\Gamma_{e\to g}}{2}} \tag{4.40}$$

instead of as in equation 4.28.

The relation between equation 4.40 and the above derivation can more clearly be seen by rewriting the approximation given in equation 4.38 in the form

$$F_+(\omega) - F_-(\omega) = \frac{1}{i\omega + \frac{\Gamma(0)}{2}} - \frac{1}{-i\omega + \frac{\Gamma(0)}{2}} \quad . \tag{4.40a}$$

For $t > 0$ , $A_e(t)$ is evaluated from equation 4.31 by taking a contour encircling the upper half of the "$\omega$" plane. Since $F_-(\omega)$ has only a pole in the lower half of the "$\omega$" plane, it does not contribute to the integral in equation 4.31, hence

$$A_e(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{i\omega t} F_+(\omega)\, d\omega \qquad t > 0 \tag{4.40b}$$

or rewriting,

$$A_e(t) = \frac{1}{2\pi} \int_{-i\infty}^{+i\infty} \frac{e^{i\omega t}\, di\omega}{i\omega + \frac{\Gamma(0)}{2}} \qquad t > 0 \quad . \tag{4.40c}$$

Comparing equation 4.30 and equation 4.40c, one notes that the latter integral is just the contour for the inversion integral along the imaginary s axis and $A_e(s)$ can be given by equation 4.40. Therefore, the function given in equation 4.28 which was not analytic along the axis of imaginaries has been approximated by the analytic function given in equation 4.40. Clearly, the analytic form can be obtained from the non-analytic form by evaluating the summation in equation 4.28 as the

$$\operatorname*{Re\ lim}_{s \to 0+} \sum_a \left| \frac{H''_{e,ga}}{\hbar} \right|^2 \frac{1}{s + i(\omega_a - \Omega_{eg})} \quad . \tag{4.40d}$$

This evaluation can be justified somewhat by the physical arguments that the frequencies (real or imaginary) of any variations in $A_e(t)$ are expected to be small compared to the angular frequency $\Omega_{eg}$, that decaying rather than growing exponentials are expected, and that any energy level shift given by $H''$ has already been included in the definition of energy levels.

The coefficient of state /ga) is given by equation 4.26b. This coefficient can be found for times long compared to the decay constant by using the final value theorem for Laplace transforms (4.13). This theorem relates the behavior of $s\,f(s)$ near the origin of the complex plane to the behavior of $f(t)$ as $t$ becomes infinite. This theorem is stated as

$$\lim_{s \to 0} \; s\,F(s) \;=\; \lim_{t \to \infty} \; F(t) \tag{4.41}$$

where $F(t)$ must satisfy the same conditions as for the inversion theorem for Laplace transforms (4.14) and the function $s\,F(s)$ must be analytic on the axis of imaginaries and in the right half-plane. Therefore, from equations 4.41 and 4.26b,

$$\lim_{t \to \infty} \; i\hbar A_{ga}(t) = \lim_{s \to 0} \; H''_{ga,e}\, A_e(s + i\,[\Omega_{eg} - \omega_a]) \;, \tag{4.42}$$

where it is understood that the $A_e(s)$ is given by the approximation that is analytic along the axis of imaginaries. Using equation 4.40, one can write this as ($\Gamma_e = \Gamma_{e \to g}$ in this case)

$$i\hbar\, A_{ga}(\infty) = H''_{ga,e}\, \frac{1}{\frac{\Gamma_e}{2} + i(\Omega_{eg} - \omega_a)} \;. \tag{4.43}$$

The probability of ending in state $/ga)$ is given as

$$A_{ga}(\infty)\, A^*_{ga}(\infty) = \left| \frac{H''_{e,ga}}{\hbar} \right|^2 \frac{1}{\left(\frac{\Gamma_e}{2}\right)^2 + (\omega_a - \Omega_{eg})^2} \;. \tag{4.44}$$

Therefore the probability of giving off a photon of frequency $\omega_a$ is a Lorentzian function centered at $\Omega_{eg}$ with width $\Gamma_e$. The total probability of emitting a photon is given by summing equation 4.44 over all values of $a$. This sum is evaluated by using equations 4.34 and 4.35. The result is as expected, namely,

$$\sum_a \left| A_{ga}(\infty) \right|^2 = 1 \;.$$

## 4.3 Cascade Processes

Cascade processes have been covered in detail by Weisskopf and Wigner (4.9). In order to illustrate the method, an atom in an excited state e which decays to an intermediate state f , which finally decays to the ground state g will be considered.

Equation 4.22 can be transformed to the Laplace transform variable s

$$i\not{h}\left[-A_{m'a'}(0+) + s\,A_{m'a'}(s)\right] = \sum_{m,a} H''_{m'a',ma}\,A_{ma}\left(s - i\left[\frac{E_{m'a'} - E_{ma}}{\not{h}}\right]\right) \tag{4.45}$$

where $A(0+)$ is the initial value of $A(t)$ . For the above cascade this equation becomes

$$i\not{h}\left[-1 + s\,A_e(s)\right] = \sum_{a} H''_{e,fa}\,A_{fa}(s - i\,[\Omega_{ef} - \omega_a]) \quad , \tag{4.46a}$$

$$i\not{h}\,s\,A_{fa}(s) = H''_{fa,e}\,A_e(s + i\,[\Omega_{ef} - \omega_a]) + \sum_{a'} H''_{f,ga'}\,A_{gaa'}(s - i\,[\Omega_{fg} - \omega_{a'}]) \quad , \tag{4.46b}$$

and

$$i\not{h}s\,A_{gaa'}(s) = H''_{ga',f}\,A_{fa}(s + i\,[\Omega_{fg} - \omega_{a'}]) \tag{4.46c}$$

where the index a refers to the photon emitted in the decay from state e to state f , the index a' refers to the photon emitted in the decay from state f to state g . From equations 4.46, $A_e(s)$ is found to be:

$$A_e(s) = \frac{1}{s + \sum_a \left|\frac{H''_{e,fa}}{\hbar}\right|^2 \frac{1}{s + i(\omega_a - \Omega_{ef}) + \sum_{a'} \left|\frac{H''_{ga',f}}{\hbar}\right|^2 \frac{1}{s+i(\omega_a + \omega_{a'} - \Omega_{ef})}}} \quad . \tag{4.47}$$

As is shown in Section 4.2, the summation over $a'$ contributes a term of the order of $\frac{\Gamma_{f\to g}}{2}$ to the denominator. This, as usual, is assumed to be small compared to level differences and can be neglected in evaluating the summation over $a$. Thus $A_e(s)$ is given by equation 4.40, i.e.,

$$A_e(s) = \frac{1}{s + \frac{\Gamma_{e\to f}}{2}} \quad . \tag{4.40}$$

From equations 4.46 and 4.40 the behavior of the intermediate state $f$ can be found to be:

$$A_{fa}(s) = -i \frac{H''_{fa,e}}{\hbar} \frac{1}{\left[s + \frac{\Gamma_{e\to f}}{2} + i(\Omega_{ef} - \omega_a)\right]\left[s + \frac{\Gamma_{f\to g}}{2}\right]} \quad . \tag{4.48}$$

Therefore, the coefficient of the intermediate state is given as a function of time by

$$A_{fa}(t) = -i \frac{H''_{fa,e}}{\hbar} \frac{e^{-\left[\Gamma_e/2 + i(\Omega_{ef} - \omega_a)\right]t} - e^{-\Gamma_f/2\, t}}{\left(\frac{\Gamma_f - \Gamma_e}{2}\right) + i(\omega_a - \Omega_{ef})} \tag{4.49}$$

where it is understood that for this example

$$\Gamma_e = \Gamma_{e\to f} \qquad \text{and} \qquad \Gamma_f = \Gamma_{f\to g} \quad .$$

The probability of being in an intermediate state $f$ is given by the absolute square of equation 4.49 or

$$\left|A_{fa}(t)\right|^2 =$$

$$\left|\frac{H''_{fa,e}}{\not{h}}\right|^2 \frac{2e^{-(\frac{\Gamma_e+\Gamma_f}{2})t}}{\left(\frac{\Gamma_f-\Gamma_e}{2}\right)^2+(\omega_a-\Omega_{ef})^2}\left\{\cosh\left[\frac{\Gamma_e-\Gamma_f}{2}\,t\right]-\cos[(\omega_a-\Omega_{ef})t]\right\} . \tag{4.50}$$

The probability of ending in state $gaa'$ can be found from equations 4.46, 4.48 and 4.41, the final value theorem, to be:

$$\left|A_{gaa'}(\infty)\right|^2 =$$

$$\left|\frac{H''_{ga',f}}{\not{h}}\right|^2 \left|\frac{H''_{fa,e}}{\not{h}}\right|^2 \frac{1}{\left[\left(\frac{\Gamma_e}{2}\right)^2+(\Omega_{ef}-\omega_a+\Omega_{fg}-\omega_{a'})^2\right]\left[\left(\frac{\Gamma_f}{2}\right)^2+\left(\Omega_{fg}-\omega_{a'}\right)^2\right]} \tag{4.51}$$

This probability is localized in the region

$$\omega_{a'} \approx \Omega_{fg}$$

and

$$\omega_a \approx \Omega_{ef} \quad .$$

The probability of giving off a photon $\omega_a$ is given by summing equation 4.51 over all possible $\omega_{a'}$'s . This probability is found by

using the relation

$$\lim_{\substack{a \to 0+ \\ b \to 0+}} \frac{a}{x^2 + a^2} \cdot \frac{b}{(x+y)^2 + b^2} = \frac{\pi(a+b)}{y^2 + (a+b)^2} \delta(x) \quad . \tag{4.52}$$

This relation, which assumes that $y$ is small enough to allow the neglecting of any variation of the function to be integrated, can be found by the method of residues. Thus the desired probability is given by

$$\sum_{a'} \left| A_{gaa'}(\infty) \right|^2 = \left| \frac{H''_{fa,e}}{\hbar} \right|^2 \frac{\left( \frac{\Gamma_e + \Gamma_f}{2} \right)}{\left( \frac{\Gamma_e + \Gamma_f}{2} \right)^2 + (\omega_a - \Omega_{ef})^2} \frac{2}{\Gamma_e} \quad . \tag{4.53}$$

Equation 4.35, which relates the decay constant to an integration over the frequencies of the emitted photons, has been used to find equation 4.53. As is seen, the probability of emitting a photon of frequency $\omega_a$ is a Lorentzian function with width $\Gamma_e + \Gamma_f$. Similarly, the probability of emitting a photon of frequency $\omega_{a'}$ can be found to be a Lorentzian function with width $\Gamma_f$.

If the excited state e decays directly to a group of states f which in turn cascade through any number of states to the ground state*, the equation describing the initial state is given by

$$A_e(s) = \frac{1}{s + \frac{\Gamma_e}{2}} \tag{4.54}$$

where now $\Gamma_e$ is the total decay constant, i.e.,

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*Any of the intermediate states may be the ground state.

$$\Gamma_e \equiv \sum_f \Gamma_{e \to f} \quad . \tag{4.55}$$

As found previously, the states which follow the f states in the cascade can be neglected in finding equation 4.54. The equation describing the behavior of any state u in terms of a decay from a state t is given as

$$A_{ua_1a_2 - - - a_t a_u}(s) =$$

$$\frac{H''_{ua_u,t}}{i\hbar} \frac{1}{s + \frac{\Gamma_u}{2}} A_{ta_1a_2 - - - a_t}(s + i[\Omega_{tu} - \omega_{a_u}]) \tag{4.56}$$

where, as described in Section 4.2, the A's are given by their analytic approximation. The subscripts on the A's denote the different possible photons emitted in the cascade from the initial state. The ground state for a particular mode of decay is denoted by the coefficient

$$A_{ga_1a_2 - - - a_g}(s) \quad .$$

This coefficient can be found in terms of $A_e(s)$, the coefficient of the initial state. The final value theorem can be used to evaluate the probability of being in a certain ground state as $t \to \infty$. From the integration over all possible modes of decay, the probability of emitting a photon of angular frequency $\omega_a$ in the decay from e to f is found to be given by equation 4.53, where now $\Gamma_e$ and $\Gamma_f$ are the total decay constants for these states.

The decays of atomic states which interact with an optical field cannot be expressed in terms of simple exponentials as shown in Chapters 5 and 6. If the transformed time behavior of an atomic state e which interacts with an optical field is given by $A_e(s)$, the probability of emitting a spontaneous photon "a" in the decay from state e to state j is found to be:

$$\sum_{k,a'} \left| \frac{H''_{ka',j}}{\not{h}} \right|^2 \frac{1}{\left(\frac{\Gamma_j}{2}\right)^2 + (\omega_{a'} - \Omega_{jk})^2} \left| \frac{H''_{ja,e}}{\not{h}} \right|^2 \left| A_e\left(-i\begin{bmatrix} \omega_a - \Omega_{ej} \\ \omega'_a - \Omega_{jk} \end{bmatrix}\right) \right|^2 . \tag{4.57}$$

States j and k are assumed not to interact with the optical field. a' signifies spontaneous photons emitted in transitions from state j to lower states k . If $|A_e|^2$ is considered to have the form of a delta function in the integration over a' , expression 4.57 becomes

$$\left| \frac{H''_{ja,e}}{\not{h}} \right|^2 \frac{1}{2\pi} \int_{-\infty}^{+\infty} \frac{\Gamma_j}{\left(\frac{\Gamma_j}{2}\right)^2 + z^2} \left| A_e(-i\,[\omega_a - \Omega_{eg} + z]) \right|^2 dz \tag{4.58}$$

where the integration is along the real z axis. Expression 4.58 gives the frequency distribution of spontaneous emission from state e to state j .

## REFERENCES

(4.1) R. B. Leighton, Principles of Modern Physics, McGraw-Hill Book Company, Inc., New York (1959), Chapter 6.

(4.2) A. Messiah, Quantum Mechanics, North-Holland Publishing Company, Amsterdam (1961), pp. 991-993.

(4.3) S. S. Schweber, An Introduction to Relativistic Quantum Field Theory, Row, Peterson and Company, Elmsford, New York (1961), p. 515.

(4.4) H. Umezawa, Quantum Field Theory, North-Holland Publishing Company, Amsterdam, (1956); Chapter 1, especially pp. 16-21 and p. 280.

(4.5) Pages 524-530 of Reference (4.3).

(4.6) W. E. Lamb and R. C. Retherford, "Fine Structure of the Hydrogen Atom by a Microwave Method", Phys. Review 72, 241 (1947).

(4.7) Page 529 of Reference (4.3).

(4.8) H. A. Bethe, "The Electromagnetic Shift of Energy Levels", Phys. Review 72, 339 (1947).

(4.9) V. Weisskopf and E. Wigner, Zeitschrift für Physik 89, 27 (1934); 90, 817 (1934).

(4.10) Pages 994-1001 of Reference (4.2).

(4.11) Chapter 16 of Reference (4.4).

(4.12) H. S. Carlslaw and J. C. Jaeger, Operational Methods in Applied Mathematics, Dover Publications, Inc., New York (1963).

(4.13) M. F. Gardner and J. L. Barnes, Transients in Linear Systems, John Wiley and Sons, Inc., New York (1942), p. 265.

(4.14) Page 72 of Reference (4.12).

(4.15) W. Heitler, The Quantum Theory of Radiation, 3rd edition, Clarendon Press, Oxford (1953), Sec. 34.

# CHAPTER 5

## INTERACTION OF A STATIONARY ATOM WITH CAVITY AND RADIATIVE FIELDS

The results of Chapter 2 are extended to include spontaneous radiative transitions. The equations describing the motion of the system are found. Some distributions of emitted spontaneous radiation and the probability of an excited atom emitting a coherent photon to the cavity are calculated in an approximation which neglects the spontaneous decay between "laser" states.

### 5.1 Cavity and Radiative Fields

The cavity is considered to be an open cavity, i.e., a cavity that is transparent to spontaneous radiation. The effects of the zero-point energies of the cavity modes on spontaneous emission are neglected*. Such effects as amplification of spontaneous emission or superadiance, and radiation trapping are not considered.

A stationary atom interacting with the fields of a lossless single-mode cavity and with the zero-point energies of space which result in spontaneous radiation is considered. The Hamiltonian of the system is written as

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*It is assumed that the density of modes for spontaneous emission differs negligibly from the free-space value. These cavity conditions are satisfied in the Fabry-Perot structures ordinarily used in gas lasers. In small fiber systems (5.5) where the transverse dimensions of the cavity are of the order of a wavelength of the radiation: the free-space mode density is greatly suppressed; the spontaneous emission results mainly from interactions with the zero-point energies of the cavity; and spontaneous emission effects can be considerably changed from their free-space values.

$$H = H_{atom} + H_{ct} + H_{rad} + H' + H'' + H_{cavity} \tag{5.1}$$

where the terms $H_{atom} + H_{ct} + H_{rad}$ determine, as described in Section 4.2, the stationary states of the atom and the radiative fields. $H''$ is, as in equation 4.21, the perturbing Hamiltonian which results in radiative decay; $H_{cavity}$ is the cavity Hamiltonian; and $H'$ is the interaction energy of the atom and cavity. The representation used will be that in which the Hamiltonian

$$H_{atom} + H_{ct} + H' + H_{rad} + H_{cavity} \tag{5.2}$$

is diagonal. This representation is chosen in order to exhibit more clearly the approximations used in the analysis. The perturbing Hamiltonian is $H''$ .

The Hamiltonian of a stationary atom interacting with cavity fields has been diagonalized in Chapter 2. The stationary states as given by equations 2.25 are:

$$/\phi_{n+})_t = \left[ a_n/en-1) + ib_n/fn) \right] e^{-i(\bar{e}_n + e'_n)t} \tag{5.3a}$$

and

$$/\phi_{n-})_t = \left[ ib_n/en-1) + a_n/fn) \right] e^{-i(\bar{e}_n - e'_n)t} \tag{5.3b}$$

where the symbols in equations 5.3 have been defined previously by equations 2.14, 2.22, 2.20 and 2.25c. The upper "laser" state is denoted by $e$ , the lower "laser" state by $f$ , and the number of photons in the cavity is denoted by $n$ . States of the atom-cavity system which are not mixed by $H'$, of course, also exist as stationary

states and are denoted by

$$/mn)_t = /mn)\, e^{-i\left[\frac{E_m}{\hbar} + (n+\frac{1}{2})\omega\right]t} \qquad m \neq e,f \quad . \qquad (5.3c)$$

The stationary states of the Hamiltonian defined in equation 5.2 are, therefore, products of the wave functions of equations 5.3 and the wave functions describing the radiative fields.

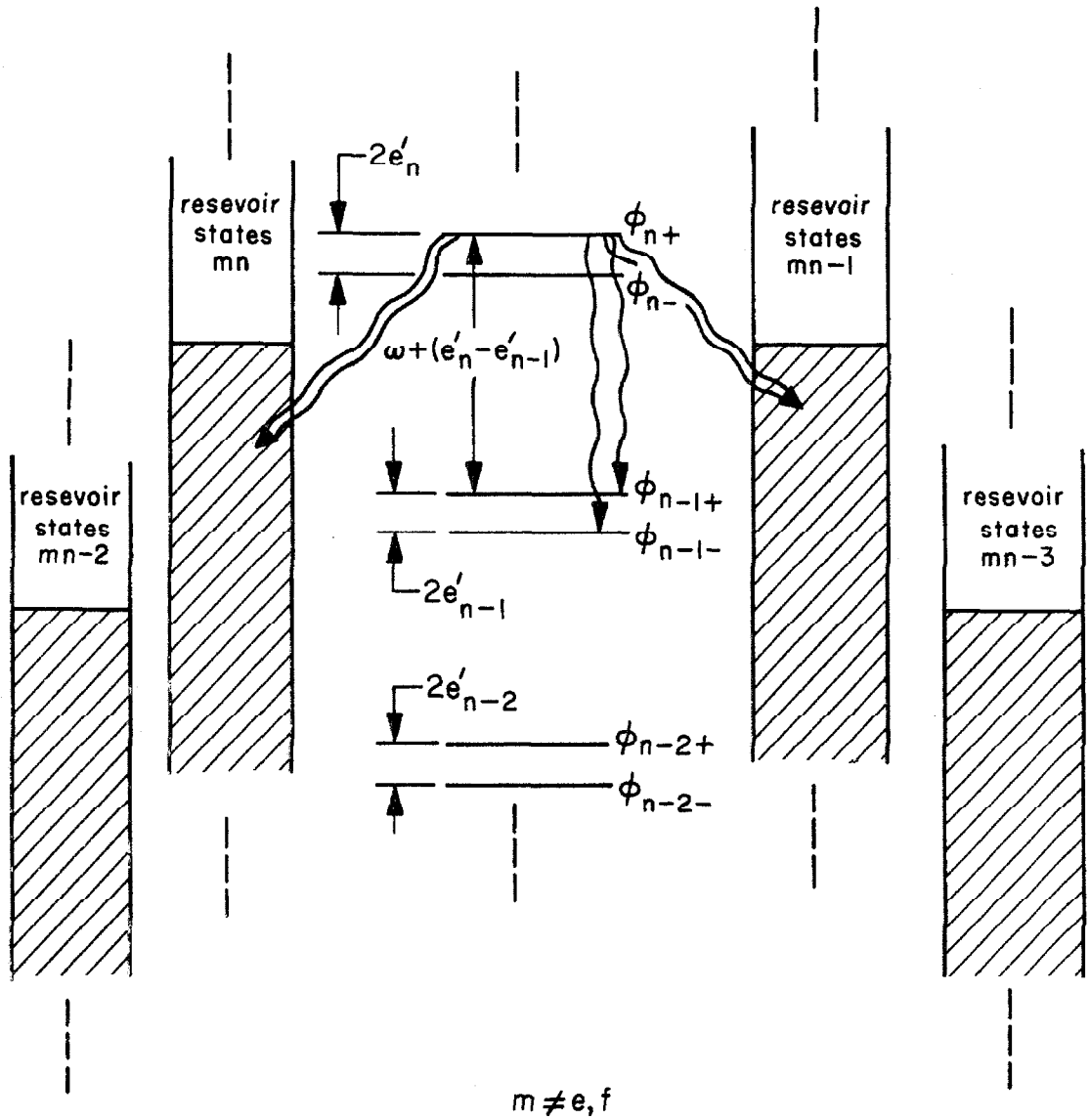
In the "interaction representation" (equation 4.45), the wave function of the system is expanded in time-dependent stationary states of the unperturbed Hamiltonian. The time-dependent coefficient of a stationary state $\gamma$ is coupled to all the coefficients in the expansion by the following equation:

$$i\hbar\left[\dot{A}_\gamma(0+) + s\, A_\gamma(s)\right] = \sum_\rho H''_{\gamma,\rho}\, A_\rho\left(s - i\left[\frac{E_\gamma - E_\rho}{\hbar}\right]\right) . \qquad (5.4)$$

The energy-level scheme is shown in Figure 5.1. The states $/\phi_{n+})$ and $/\phi_{n-})$ are coupled to lower-energy state $/\phi_{n-1+})$, $/\phi_{n-1-})$, $/mn)$ and $/mn-1)$ by means of radiative transitions. In other words H", the perturbing Hamiltonian, has for single-photon decays matrix elements between states $/\phi_{n+})$ and $/\phi_{n-})$, and states $/\phi_{n-1+}a)$, $/\phi_{n-1-}a)$, $/mna)$ and $/mn-1a)$ where a denotes a radiation photon of a given polarization, direction and frequency.

## 5.2 Decay of an Excited State

It is assumed that the system of atom, radiative fields, and single-mode cavity exists at time $t = 0$ with none of the radiation




$m \neq e, f$

ENERGY-LEVEL DIFFERENCES ARE SHOWN IN UNITS OF ANGULAR FREQUENCY

RADIATIVE DECAY IS SHOWN ONLY FOR STATE $\phi_{n+}$

ENERGY-LEVEL SCHEME

Figure 5.1

field oscillators excited* and the atom-cavity system in a linear combination of states $\phi_{n+}$ and $\phi_{n-}$ with initial values $C_{n+}$ and $C_{n-}$. The equations connecting the initial states of the system to other states become for single-photon decays

$$i\hbar\left[-C_{n\epsilon} + s\, A_{n\epsilon}(s)\right] = \sum_{m,n',a} H''_{n\epsilon,mn'a}\, A_{mn'a}\left(s - i\left[\frac{E_{n\epsilon}-E_{mn'a}}{\hbar}\right]\right) + \sum_{a,\epsilon'} H''_{n\epsilon,n-1\epsilon'a}\, A_{n-1\epsilon'a}\left(s - i\left[\frac{E_{n\epsilon}-E_{n-1\epsilon'a}}{\hbar}\right]\right). \qquad (5.5)$$

where the summation over $n'$ has values only for $n' = n$ and $n' = n-1$, and the index $\epsilon$ denotes the signs + and - .

For the present, only the evolution of the initial states will be considered. As in Section 4.3, the reaction of states not directly coupled to the initial states by single-photon decays can be neglected. Therefore, in this case the equations for states $/mn'a)$ and $/n-1\epsilon\ a)$ to be used for substitution into equation 5.5 are given by

$$i\hbar s\, A_{mn'a}(s) = \sum_{\epsilon} H''_{mn'a,n\epsilon}\, A_{n\epsilon}\left(s + i\left[\frac{E_{n\epsilon}-E_{mn'a}}{\hbar}\right]\right) \qquad (5.6a)$$

and

*This, of course, neglects the high-frequency virtual states which contribute to the mass-renormalization term and any other virtual states contributing to the energy-level shift resulting from $H_{ct}$. These are assumed to be included in the definition of the atomic states.

$$i\hbar s\, A_{n-1\epsilon' a}(s) = \sum_{\epsilon} H''_{n-1\epsilon' a,n\epsilon}\, A_{n\epsilon}\left(s + i\left[\frac{E_{n\epsilon} - E_{n-1\epsilon' a}}{\hbar}\right]\right) . \quad (5.6b)$$

Equations 5.5 and 5.6 can be combined to give

$$A_{n+}(s) = \frac{C_{n+}\left[s - i2e'_n + \bar{\gamma}_{n-}(s - i2e'_n)\right] - C_{n-}\,\beta_{n+}(s)}{\left[s + \bar{\gamma}_{n+}(s)\right]\left[s - i2e'_n + \bar{\gamma}_{n-}(s - i2e'_n)\right] - \beta_{n+}(s)\,\beta_{n-}(s - i2e'_n)} \quad (5.7a)$$

and

$$A_{n-}(s) = \frac{C_{n-}\left[s + i2e'_n + \bar{\gamma}_{n+}(s + i2e'_n)\right] - C_{n+}\,\beta_{n-}(s)}{\left[s + \bar{\gamma}_{n-}(s)\right]\left[s + i2e'_n + \bar{\gamma}_{n+}(s + i2e'_n)\right] - \beta_{n-}(s)\,\beta_{n+}(s + i2e'_n)} \quad (5.7b)$$

where

$$\bar{\gamma}_{n\epsilon}(s) = \sum_{m,n'a} \left|\frac{H''_{n\epsilon,mn'a}}{\hbar}\right|^2 \frac{1}{s - i\left(\frac{E_{n\epsilon} - E_{mn'a}}{\hbar}\right)} +$$

$$\sum_{\epsilon',a} \left|\frac{H''_{n\epsilon,n-1\epsilon' a}}{\hbar}\right|^2 \frac{1}{s - i\left(\frac{E_{n\epsilon} - E_{n-1\epsilon' a}}{\hbar}\right)} , \quad (5.8a)$$

$$\beta_{n+}(s) = \sum_{m,n',a} \frac{\hbar^{-2}\, H''_{n+,mn'a}\, H''_{mn'a,n-}}{s - i\left(\frac{E_{n+} - E_{mn'a}}{\hbar}\right)} +$$

$$\sum_{\epsilon,a} \frac{\hbar^{-2}\, H''_{n+,n-1\epsilon a}\, H''_{n-1\epsilon a,n-}}{s - i\left(\frac{E_{n+} - E_{n-1\epsilon a}}{\hbar}\right)} \quad (5.8b)$$

and $\beta_{n-}(s)$ is obtained from equation 5.8b by interchanging the indices n+ and n- .

$A_{n\epsilon}(t)$ is found from equations 5.7 by use of the inversion theorem which has been given in equation 4.30. Similar to equation 4.28, $A_{n\epsilon}(s)$ has a cut along the imaginary axis. The contour for the inversion integral can be deformed to give

$$A_{n\epsilon}(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{i\omega t} \left[ A_{n\epsilon}(\omega)_+ - A_{n\epsilon}(\omega)_- \right] d\omega \quad , \tag{5.9}$$

where $A_{n\epsilon}(\omega)_+$ is the value of $A_{n\epsilon}(s)$ as s approaches the imaginary axis from the right half-plane and $A_{n\epsilon}(\omega)_-$ is the value of $A_{n\epsilon}(s)$ as s approaches the imaginary axis from the left half-plane.

It is now assumed that at $t = 0$ the atom is in the upper laser state and the cavity is in state $n-1$, i.e., in equation 5.5

$$C_{n+} = a_n \qquad \text{and} \qquad C_{n-} = -ib_n \; . \tag{5.10}$$

With these initial conditions $A_{n+}(\omega)_\pm$ becomes

$$A_{n+}(\omega)_\pm = \frac{-ia_n(\omega - 2e'_n - i\bar{\Gamma}_{f\pm})}{(\omega + B_\pm + D_\pm)(\omega + B_\pm - D_\pm)} \tag{5.11}$$

where

$$\bar{\Gamma}_{i\pm} \equiv \pm \frac{\Gamma_i(\omega)}{2} - i \frac{\delta E_i(\omega)}{\hbar} \quad , \tag{5.12a}$$

$$B_{\pm} \equiv - e'_n - i \frac{1}{2} (\bar{\Gamma}_{f_{\pm}} + \bar{\Gamma}_{e_{\pm}}) \quad , \tag{5.12b}$$

and

$$D_{\pm} \equiv \left\{ \alpha^2 n + \left[ \Delta + \frac{i}{2} (\bar{\Gamma}_{f_{\pm}} - \bar{\Gamma}_{e_{\pm}}) \right]^2 \right\}^{1/2} \quad . \tag{5.12c}$$

Equations 2.20 and 2.22 have been used to evaluate equations 5.8 . $\Delta$ is one-half the difference between the laser transition angular frequency and the cavity angular frequency. $\alpha$ , which gives the interaction between the cavity and atom, is defined in equation 2.14. $e'_n$ is the energy-level shift, which results from the atom-cavity interaction. Similar to equations 4.33, $\Gamma_i(\omega)$ , the total decay constant for the ith level $(i = e,f)$, is defined by

$$\frac{\Gamma_i(\omega)}{2} \equiv \lim_{\sigma \to 0+} \sum_{m,a} \left| \frac{H''_{i,ma}}{\hbar} \right|^2 \frac{\sigma}{\sigma^2 + (\omega_a - \Omega_{im} + \omega)^2} \tag{5.13a}$$

and $\delta E_i(\omega)$, an energy-level shift, is defined by

$$\frac{\delta E_i(\omega)}{\hbar} = \sum_{m,a} \left| \frac{H''_{i,ma}}{\hbar} \right|^2 \frac{1}{(\omega_a - \Omega_{im} + \omega)} \tag{5.13b}$$

where now the index m includes all lower atomic states coupled to the ith state by direct radiative decay. In deriving equation 5.11 the approximation that in equations 5.13 both $e'_n$ and $\Delta$ are small compared to a frequency $\Omega_{im}$ has been used. Also, as in Chapter 4, it has been assumed that the total decay constants of any two

states connected by radiative decay are very small compared to the frequency difference between these states.

From equation 5.11 it is seen that the factor in the inversion integral, equation 5.9, is a function localized at $\omega = 0$ . Therefore, in evaluating the inversion integral the $\Gamma$'s can be approximated by their values at $\omega = 0$, i.e.,

$$\bar{\Gamma}_i \approx \frac{\Gamma_i(0)}{2} \equiv \frac{\Gamma_i}{2} \quad . \tag{5.14}$$

For $t > 0$ equation 5.9 can be evaluated as a contour integral enclosing the upper half of the complex "$\omega$" plane. The equation for $A_{n\epsilon}(s)$ can be derived in the manner explained in the discussion following equation 4.40. Since with approximation 5.14 $A_{n\epsilon}(\omega)_-$ has no poles in the upper half "$\omega$" plane, equation 5.9 can be rewritten as

$$A_{n\epsilon}(t) = \frac{1}{2\pi i} \int_{-i\infty}^{+i\infty} e^{i\omega t} A_{n\epsilon}(i\omega)_+ \, di\omega \quad . \tag{5.15a}$$

From the inversion theorem, the analytic approximation to $A_{n\epsilon}(s)$ can be given by

$$A_{n\epsilon}(s) = A_{n\epsilon}(\omega)_+ \Big|_{i\omega = s} \tag{5.15b}$$

and $A_{n\epsilon}(t)$ can be found from the $A_{n\epsilon}(s)$ in the usual manner. This analytic approximation to $A_{n\epsilon}(s)$ can, as in equation 4.40d, be found by evaluating the summations of equations 5.8 as their real parts in the limit as $s \to 0+$ . Hence, for the system initially in state /en-1), $A_{n+}(s)$ can be written as

$$A_{n+}(s) = \frac{a_n\left(s - i\,2e'_n + \frac{\Gamma_f}{2}\right)}{\left[s - ie'_n + \frac{\Gamma_f+\Gamma_e}{4} + i\,D_n\right]\left[s - ie'_n + \frac{\Gamma_f+\Gamma_e}{4} - i\,D_n\right]} \qquad (5.16a)$$

where

$$D_n = \left\{\alpha^2 n + \left[\Delta + \frac{i}{2}\left(\frac{\Gamma_f - \Gamma_e}{2}\right)\right]^2\right\}^{1/2} . \qquad (5.16b)$$

Similarly, for the system initially in state $/en-1)$, $A_{n-}(s)$ can be written as

$$A_{n-}(s) = \frac{-ib_n\left(s + i\,2e'_n + \frac{\Gamma_f}{2}\right)}{\left[s + ie'_n + \frac{\Gamma_f+\Gamma_e}{2} + iD_n\right]\left[s + ie'_n + \frac{\Gamma_f+\Gamma_e}{4} - iD_n\right]} . \qquad (5.16c)$$

The wave function of the system of cavity, atom and radiative fields can, from equation 5.3, be stated as

$$/\psi)_t = \sum A_\rho /\rho)_t =$$

$$/en-1)_t\, e^{i\Delta t}\left[a_n A_{n+}(t)\, e^{-ie'_n t} + ib_n A_{n-}(t)\, e^{ie'_n t}\right] +$$

$$/fn)_t\, e^{-i\Delta t}\left[ib_n A_{n+}(t)\, e^{-ie'_n t} + a_n A_{n-}(t)\, e^{ie'_n t}\right] +$$

$$\text{other terms} \qquad (5.17)$$

where it has been assumed that the initial state of the system is some linear combination of $/fn)$ and $/en-1)$ and where the states to which the initial states decay have not been written explicitly. The coefficient multiplying a particular state in this expansion determines the

probability of being in that state. The coefficients $A_{en-1}(s)$ and $A_{fn}(s)$ can be written from equations 5.17 as

$$A_{en-1}(s) = a_n A_{n+}(s - i\Delta + ie'_n) + ib_n A_{n-}(s - i\Delta - ie'_n) \qquad (5.18a)$$

and

$$A_{fn}(s) = ib_n A_{n+}(s + i\Delta + ie'_n) + a_n A_{n-}(s + i\Delta - ie'_n) \ . \qquad (5.18b)$$

If the initial state of the system is assumed to be /en-1) , that is, initial conditions 5.10, the coefficient of states /fn) and /en-1) are found from equations 5.16 and 5.18 to be:

$$A_{fn} = e^{-i\Delta t} e^{-\frac{(\Gamma_f + \Gamma_e)}{4}t} \frac{\alpha_n^{1/2}}{D_n} \sin D_n t \qquad (5.19a)$$

and

$$A_{en-1} = e^{i\Delta t} e^{-\frac{(\Gamma_f + \Gamma_e)t}{4}} \cos D_n t - i \frac{\left[\Delta + \frac{i}{2}\left(\frac{\Gamma_f - \Gamma_e}{2}\right)\right]}{D_n} \sin D_n t \ . \qquad (5.19b)$$

Similarly, if the initial state of the system is assumed to be /fn), i.e., initial conditions

$$C_{n+} = -ib_n \qquad \text{and} \qquad C_{n-} = a_n \ , \qquad (5.20)$$

the coefficients of states /fn) and /en-1) are found to be*:

*These, as well as the initial conditions, can be defined with an arbitrary phase factor.

$$A_{fn} = e^{-i\Delta t}\, e^{-\frac{(\Gamma_f + \Gamma_e)t}{4}} \cos D_n t + i \frac{\left[\Delta + \frac{i}{2}\left(\frac{\Gamma_f}{2} - \frac{\Gamma_e}{2}\right)\right]}{D_n} \sin D_n t \qquad (5.21a)$$

and

$$A_{en-1} = -e^{i\Delta t}\, e^{-\frac{(\Gamma_f + \Gamma_e)t}{4}} \frac{\alpha\, n^{1/2}}{D_n} \sin D_n t \quad . \qquad (5.21b)$$

The analysis in Section 2.5 which neglects radiative decay gave similar equations. For example, with cavity and atom initially in state $/en-1)$ the coefficient of state $/fn)_t$ found in expansion 2.27 can be stated as

$$e^{-i\Delta t}\, \frac{\alpha\, n^{1/2}}{e'_n} \sin e'_n t \qquad (5.22)$$

where

$$e'_n = (\Delta^2 + \alpha^2\, n)^{1/2} \quad . \qquad (2.20a)$$

Relation 5.19a may be obtained from relation 5.22 by replacing the energy $E_e$ by complex energy $E_e - i\frac{\Gamma_e}{2}$ and the energy $E_f$ by the complex energy $E_f - i\frac{\Gamma_f}{2}$. This might be considered analogous to the usual case of interaction with the zero-point energies of empty space where the exponential decay of an initial state $/m)$ can be obtained by replacing the atomic energy $E_m$ in the coefficient $\exp(-i\frac{E_m t}{\hbar})$ by the complex energy $E_m - i\frac{\Gamma_m}{2}$. Similarly, equations 5.16 and 5.21 can be obtained from the results of Chapter 2 by using this concept of complex energy. Also, these same results can be obtained from the semi-classical theory of Chapter 3 by introducing a "phenomenological"

damping term into equations 3.9 (5.1, 5.2). That is, the wave equation of the atom is given by

$$/\psi)_t = a(t)/e)_t + b(t)/f)_t \qquad (3.9a)$$

and for an electric-dipole interaction the equations for the coefficients of the upper and lower laser states can be given by

$$i\hbar\dot{a} = -ib\Omega\, e^{i\Omega t}\, \underline{D}\cdot\underline{A}(\underline{r})\cos\omega t - i\hbar(\frac{\Gamma_e}{2})a \qquad (5.23a)$$

and

$$i\hbar\dot{b} = ia\Omega\, e^{-i\Omega t}\, \underline{D}\cdot\underline{A}(\underline{r})\cos\omega t - i\hbar(\frac{\Gamma_f}{2})b \quad . \qquad (5.23b)$$

If in equations 5.23 the substitutions

$$a' = a\, e^{-\frac{\Gamma_e}{2}t} \qquad \text{and} \qquad b' = b\, e^{-\frac{\Gamma_f}{2}t}$$

are made, the resulting equations are the same as the original equations without the damping term except for the substitution of $\Omega + \frac{i}{2}(\frac{\Gamma_f - \Gamma_e}{2})$ for $\Omega$. These resulting equations are easily solved when only the resonant terms are retained and yield the same results as given by equations 5.19 and 5.21.

The probability of being in states /fn) or /en-1), which is given by the absolute square of the coefficients given by equations 5.19 and 5.21, is in general a complex combination of decaying exponentials and oscillating terms. However, these expressions become greatly simplified when the upper and lower laser states have the same lifetime, i.e., when $\Gamma_e = \Gamma_f = \Gamma$. In this case, for the system

initially in state /en-1) the probability of being in this state $p_{en-1}$ is given by

$$p_{en-1} = e^{-\Gamma t} \frac{\alpha^2 n}{\Delta^2 + \alpha^2 n} \sin^2( [\Delta^2 + \alpha^2 n]^{1/2} t) \qquad (5.24a)$$

and the probability of being in state /fn) is given by

$$p_{fn} = e^{-\Gamma t} \left\{ 1 - \frac{\alpha^2 n}{\Delta^2 + \alpha^2 n} \sin^2( [\Delta^2 + \alpha^2 n]^{1/2} t) \right\} . \qquad (5.24b)$$

These probabilities with the exception of the damping term reduce to those found in Section 2.5.

## 5.3 Spontaneous Emission in Zero Order

Zero order is defined as the approximation that neglects the atom-cavity interactions which are the result of a spontaneous decay from the upper laser state to the lower laser state*. According to Figure 5.1, the zero-order approximation neglects the radiative decays from states $\phi_{n+}$ and $\phi_{n-}$ to states $\phi_{n-1+}$ and $\phi_{n-1-}$ but does include radiation to a reservoir state /fn-1).

This zero-order analysis has been used by some authors in a semiclassical theory that obscures the degree of approximation involved (5.1, 5.2, 5.3). Higher-order processes that are the result

*This zero-order approximation is exact if the spontaneous radiation between the upper and lower laser state can be considered vanishingly small compared to spontaneous radiation from the upper and lower laser states to other states. As shown by equation 4.18, for a microwave interaction between two states decaying to lower-lying states by allowed optical transitions this condition is readily satisfied.

of the atom spontaneously decaying from the upper laser state to the lower laser state and subsequently interacting with the cavity fields are covered in the next chapter.

First, the equations for $A_{en-1}(s)$ and $A_{fn}(s)$ will be derived in a more transparent manner than the previous section. The wave function of the system is expanded in a representation where the Hamiltonian

$$H_{atom} + H_{ct} + H_{rad} + H_{cavity}$$

is diagonal and where H', the interaction of the atom and cavity, and H", the interaction of the atom with the zero-point fields of space causing radiative decay, are perturbations. The coefficients of the states /en-1) and /fn) are connected to lower states /mn-1a) and /mna) by means of single-photon decays. The equations of the initial states are found as in equations 5.23 to be:

$$i\hbar \left[ -C_e + (s + \frac{\Gamma_e}{2}) A_{en-1}(s) \right] =$$

$$H'_{en-1,fn} A_{fn}(s - i 2\Delta) \qquad (5.25a)$$

$$i\hbar \left[ -C_f + (s + \frac{\Gamma_f}{2}) A_{fn}(s) \right] =$$

$$H'_{fn,en-1} A_{en-1}(s + i 2\Delta) \qquad (5.25b)$$

where $C_e = 1$ and $C_f = 0$ if the initial state is /en-1), and where $C_e = 0$ and $C_f = 1$ if the initial state is /fn). The matrix element $H'_{en-1,fn}$ has been defined previously in expression 2.18 as

$-i\,\alpha n^{1/2}$. The conditions on the derivation of equation 5.25 have been found in Section 5.2. They are: the total decay constants of any two states connected by radiative decay are very small compared to the frequency difference between these states, and the frequencies $e'_n$ and $\Delta$ are much smaller than the frequency difference between the laser states and any states connected to the laser states by direct radiative decay. The above equations can be solved for the individual coefficients to give the following:

a) If the initial state is /en-1)

$$A_{en-1}(s) = \frac{s - i\,2\Delta + \frac{\Gamma_f}{2}}{(s - i\Delta + \frac{\Gamma_e + \Gamma_f}{4})^2 + D_n^2} \qquad (5.26a)$$

and

$$A_{fn}(s) = \frac{\alpha\, n^{1/2}}{(s + i\Delta + \frac{\Gamma_e + \Gamma_f}{4})^2 + D_n^2} \qquad (5.26b)$$

b) If the initial state is /fn)

$$A_{en-1}(s) = \frac{-\alpha\, n^{1/2}}{(s - i\Delta + \frac{\Gamma_e + \Gamma_f}{4})^2 + D_n^2} \qquad (5.26c)$$

and

$$A_{fn}(s) = \frac{s + i2\Delta + \frac{\Gamma_e}{2}}{(s + i\Delta + \frac{\Gamma_e + \Gamma_f}{4})^2 + D_n^2} \quad . \qquad (5.26d)$$

The initial conditions used, i.e., that the atom is either in its upper or lower laser state at time zero, correspond most closely in a

gas to excitation by direct electron impact or the resonant transfer of energy from a metastable atom to the lasering atom. In general, these initial conditions apply to processes in which excitation occurs very quickly compared with the decay times of the excited atomic states. For slower excitation processes such as spontaneous decay from a higher excited atomic state to the laser states, the equations for the state of the system become more complex.

In order to illustrate the method of analysis, the first problem considered will be the calculation of the distribution of spontaneous emission in a direct decay from the upper laser state to a lower ground state. The ground state in this case does not have to be the true ground state of the atom, but can be any metastable state or any state whose lifetime is very long compared with the lifetime of the upper and lower laser states.

The coefficient for this ground state /gn-1) is given by

$$i\hbar\, s\, A_{gn\text{-}1a}(s) = H''_{ga,e}\, A_{en\text{-}1}(s - i[\omega_a - \Omega_{eg}]) \tag{5.27}$$

where, as usual, the index a refers to a definite photon emitted in the decay. The cavity-atom system is considered at time $t = 0$ to be in state /en-1) . $A_{en\text{-}1}(s)$ is given by equation 5.26a. The final value theorem for Laplace transforms is used to evaluate the coefficient $A_{gn\text{-}1a}(t \to \infty)$ which determines the probability of the system ending in state /gn-1a). This probability is

$$\left|A_{gn-1a}(\infty)\right|^2 =$$

$$\left|\frac{H''_{ga,e}}{\hbar}\right|^2 \left| \frac{i(\Omega_{eg} - \omega_a - 2\Delta) + \frac{\Gamma_f}{2}}{\left[\left(\frac{\Gamma_f + \Gamma_e}{4}\right) + i(\Omega_{eg} - \omega_a - \Delta)\right]^2 + \alpha^2 n + \left[\Delta + \frac{i}{2}\left(\frac{\Gamma_f - \Gamma_e}{2}\right)\right]^2} \right|^2 . \tag{5.28}$$

Similar to the above, the coefficient of a ground state connected to the lower laser state by a direct decay is given by

$$i\hbar s\, A_{gna}(s) = H''_{ga,f}\, A_{fn}(s - i[\omega_a - \Omega_{fg}]) \quad . \tag{5.29}$$

If the system is initially in /en-1), the probability of ending in the ground state /gn) with a radiated photon a is given by

$$\left|A_{gna}(\infty)\right|^2 =$$

$$\left|\frac{H''_{ga,f}}{\hbar}\right|^2 \left| \frac{\alpha^2 n}{\left[i(\Omega_{fg} - \omega_a + \Delta) + \left(\frac{\Gamma_f + \Gamma_e}{4}\right)\right]^2 + \alpha^2 n + \left[\Delta + \frac{i}{2}\left(\frac{\Gamma_f - \Gamma_e}{2}\right)\right]^2} \right|^2 . \tag{5.30}$$

For the system initially in the state /fn) the probability distribution of radiation from the upper laser state would be given by equation 5.30 and the probability distribution of radiation from the lower laser state would be given by equation 5.28 with the interchange of $H''_{ga,e}$ and $H''_{ga,f}$, $\Gamma_e$ and $\Gamma_f$, and $-\Delta$ and $\Delta$. The interchange of $-\Delta$ and $\Delta$ changes the asymmetry of the distributions.

Decay to an arbitrary state rather than to a ground state could also have been considered. The resulting distribution would, as in Section 4.3, depend on the lifetime of the arbitrary state.

The distributions given by equations 5.28 and 5.30 can be written in terms of normalized distributions as

$$\left|A_{gn-1a}(\infty)\right|^2 = \left|\frac{H''_{ga,e}}{\hbar}\right|^2 \left(\frac{4}{\Gamma_e + \Gamma_f}\right)^2 F_1, \tag{5.31}$$

and

$$\left|A_{gna}(\infty)\right|^2 = \left|\frac{H''_{ga,f}}{\hbar}\right|^2 \left(\frac{4}{\Gamma_e + \Gamma_f}\right)^2 F_2 \quad . \tag{5.32}$$

Normalized variables are defined by

$$x \equiv \frac{4(\omega_a - \Omega_{eg})}{(\Gamma_e + \Gamma_f)} \tag{5.33a}$$

$$x' \equiv \frac{4(\omega_a - \Omega_{fg})}{\Gamma_e + \Gamma_f} \tag{5.33b}$$

$$y \equiv \frac{4\Delta}{\Gamma_e + \Gamma_f} \tag{5.33c}$$

$$\eta \equiv \frac{4\alpha n^{1/2}}{\Gamma_e + \Gamma_f} \tag{5.33d}$$

$$\gamma_f = \frac{\Gamma_f}{\Gamma_e + \Gamma_f} \tag{5.33e}$$

and

$$\gamma_e = \frac{\Gamma_e}{\Gamma_e + \Gamma_f} \tag{5.33f}$$

and the normalized frequency distributions $F_1$ and $F_2$ are given by

$$F_1 = \left| \frac{-i(x + 2y) + 2\gamma_f}{\left[-i(x + y) + 1\right]^2 + \eta^2 + \left[y + i(\gamma_f - \gamma_e)\right]^2} \right|^2 =$$

$$\frac{(x + 2y)^2 + 4\gamma_f^2}{\left[1 - (x + y)^2 + \eta^2 + y^2 - (\gamma_f - \gamma_e)^2\right]^2 + 4\left[y(\gamma_f - \gamma_e) - (x + y)\right]^2} \tag{5.34}$$

$$F_2 = \left| \frac{\eta}{\left[-i(x' - y) + 1\right]^2 + \eta^2 + \left[y + i(\gamma_f - \gamma_e)\right]^2} \right|^2 . \tag{5.35}$$

In order to illustrate the characteristics of these frequency distributions, the special case of upper state lifetime equal to lower state lifetime, namely,

$$\gamma_e = \gamma_f = \frac{1}{2} \tag{5.36}$$

is considered in the following discussion. The normalized frequency distribution $F_1$ is plotted in Figure 5.2a for $y = 0$ as a function of the normalized frequency $x$ with $\eta^2$ as a parameter. For $\eta^2 = 0$ the curve given is the normal Lorentzian distribution of emission for an atom initially in the upper laser state not interacting with the laser cavity. For increasing $\eta^2$ the interaction of the atom with the cavity becomes greater, the modulation of the laser energy levels by cavity fields increases and consequently the spontaneous emission begins to depart from a Lorentzian shape. As the laser transition approaches saturation, the spontaneous emission no longer resembles a Lorentzian and in fact becomes double-peaked with a hole "burned" in the center of the line. The spacing between peaks is given approximately

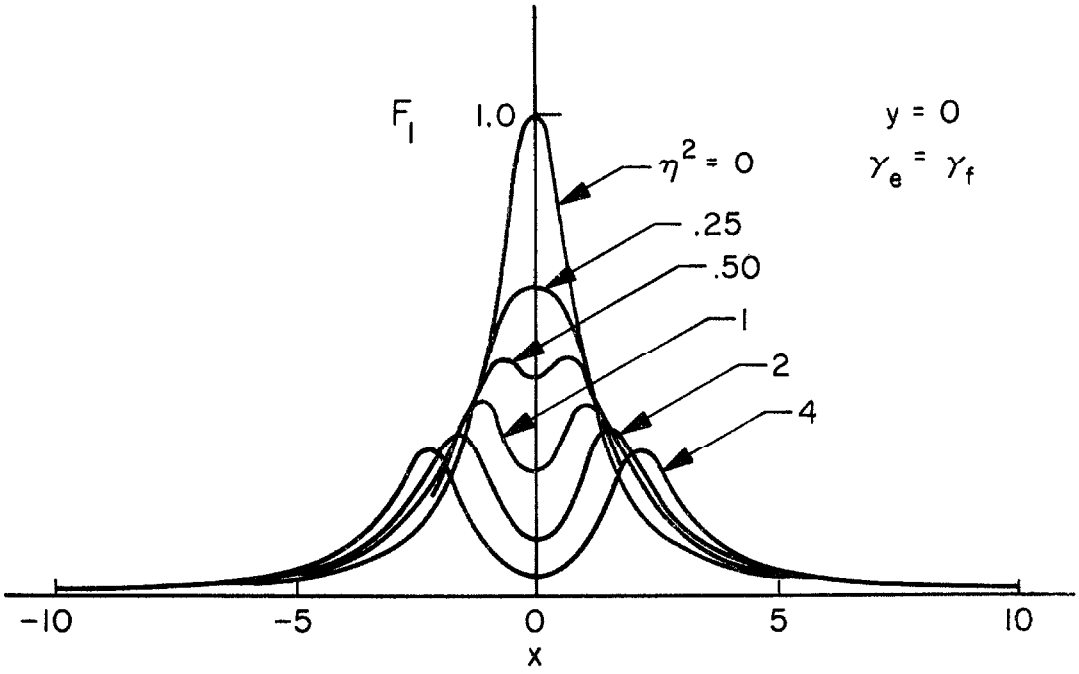



Figure 5.2a

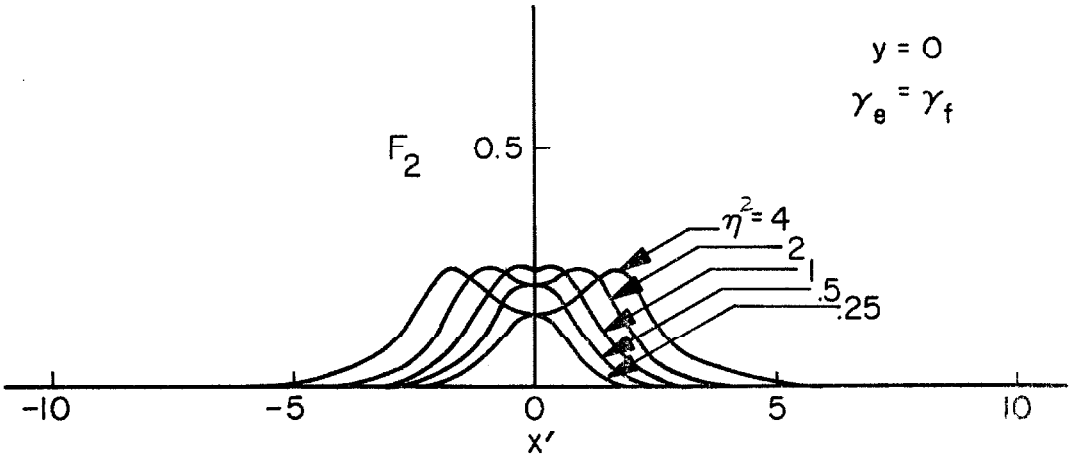



Figure 5.2b

FREQUENCY DISTRIBUTION OF SPONTANEOUS EMISSION

Probability versus frequency for atom frequency equal to cavity-field frequency with atom-cavity interactions as a parameter.

by $2\eta$ in terms of $x$ or by $2\alpha n^{1/2}$ in terms of $\omega_a$. This is the expected difference between peaks since, as illustrated in Figure 5.1, the radiation comes from two states separated in angular frequency by $2e'_n$ which is equal to $2\alpha n^{1/2}$ under condition 5.36. Moreover, as expected, the area under the curves, which represents the total probability of decay by this mode, decreases as $\eta^2$ increases.

The distribution $F_2$ for $y = 0$ is plotted in Figure 5.2b as a function of $x'$ for various values of $\eta^2$. It is noted that for small values of $\eta^2$ the distribution is single-peaked. As $\eta^2$ approaches saturation values, the distribution becomes double-peaked with spacing between peaks given by $2\alpha n^{1/2}$ in terms of $\omega_a$.

The probability distributions $F_1$ and $F_2$ are plotted in Figures 5.3a and 5.3b for a given $\eta^2$ with $y$ as a parameter. Distributions for positive y's only are plotted. The distributions for a negative $y$ is the mirror-image about the line $x = 0$ of the corresponding distribution for positive $y$. As expected for large $y$, $F_1$ approaches the Lorentzian curve found by neglecting the atom-cavity interaction. The asymmetries in Figures 5.3a and 5.3b are readily explained by referring to Figure 5.4 where the energy-level diagram for decay to the ground state is shown. It is noted that the energy-level difference between state /gn) and the states $\phi_{n+}$ and $\phi_{n-}$ increases as $\Delta$ (or y) increases and, as shown in Figure 5.3b, the spontaneous emission is shifted to higher frequencies. The energy-level difference between state gn-1 and the states $\phi_{n+}$ and $\phi_{n-}$ decreases as $y$ increases. This tendency toward lower emission frequencies is counteracted by a change in the values of $a_n$ and $b_n$;

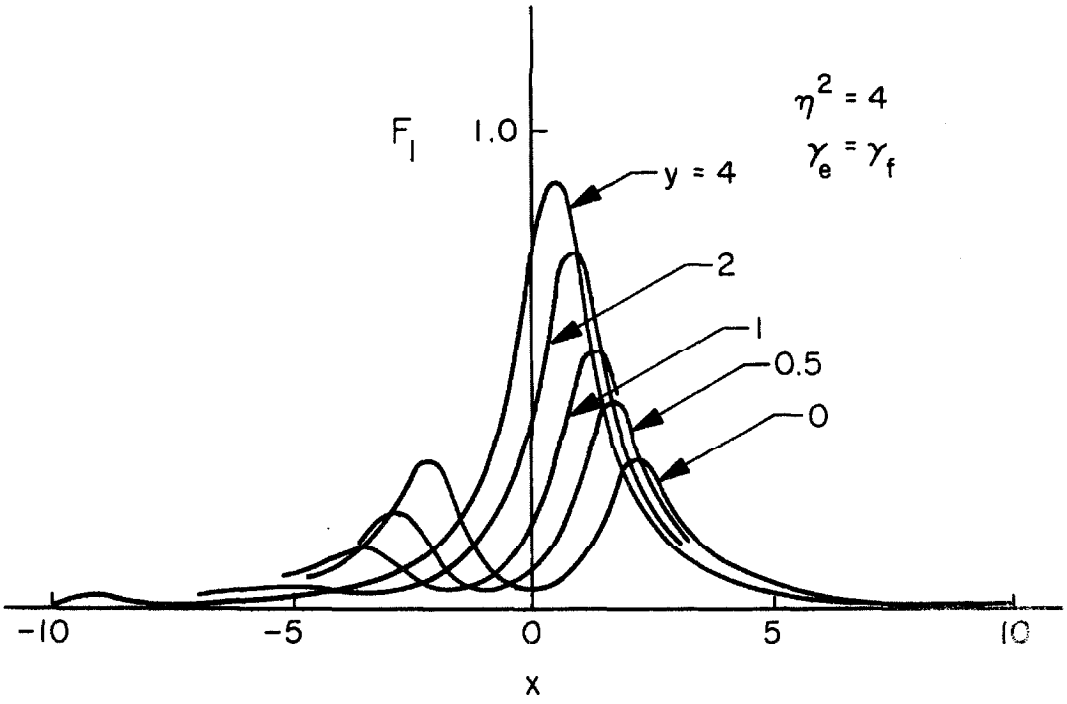



Figure 5.3a

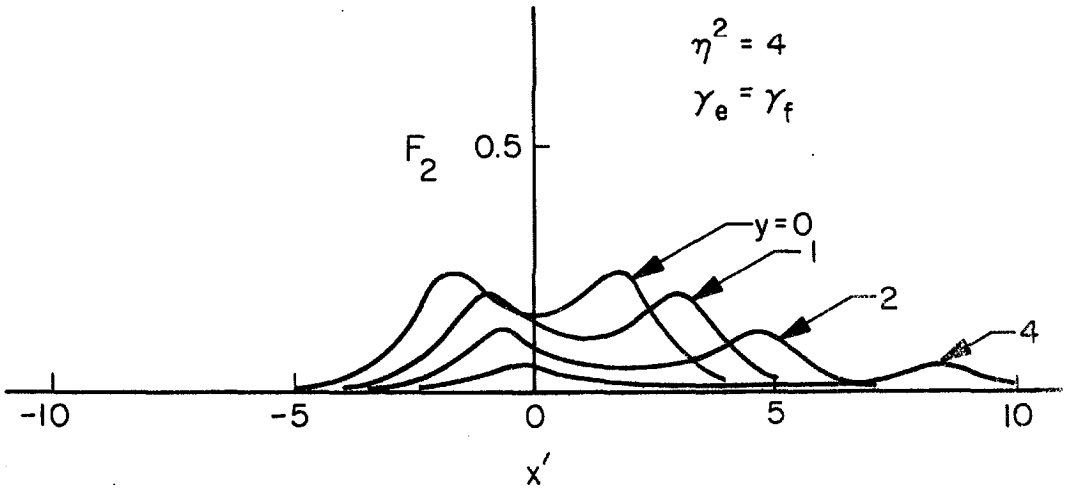
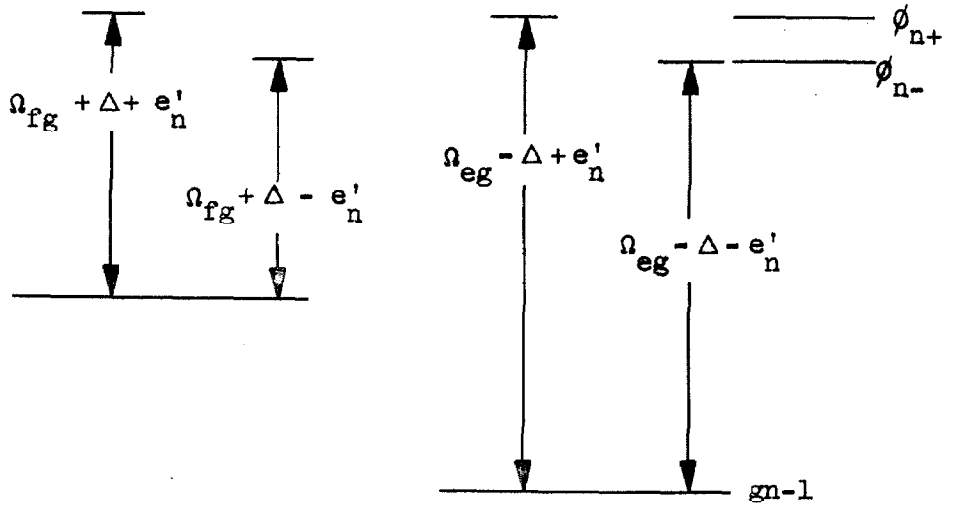



Figure 5.3b

FREQUENCY DISTRIBUTION OF SPONTANEOUS EMISSION

Probability versus frequency for a certain atom-cavity interaction with the difference between cavity-field frequency and atom frequency as a parameter.




$$/\phi_{n+}) = a_n/en-1) + ib_n/fn)$$

$$/\phi_{n-}) = ib_n/en-1) + a_n/fn)$$

$$a_n^2 = \frac{\Delta + e'_n}{2e'_n}$$

$$b_n^2 = \frac{\alpha^2 n}{2(\Delta + e'_n)\ e'_n}$$

Energy-level differences are shown in units of angular frequency.

RADIATIVE DECAY TO A GROUND STATE

Figure 5.4

as $\Delta$ increases $a_n$ increases and $b_n$ decreases. Thus, as $\Delta$ increases, the fraction of the upper state $\phi_{n+}$ which decays to gn-1 increases and the fraction of the lower state $\phi_{n-}$ which decays to gn-1 decreases. As a result, the emission is shifted toward higher frequencies. As expected, the spacing between peaks is given approximately by the energy-level difference $2e'_n$ or $2\left[\eta^2 + y^2\right]^{-1/2}$ in terms of $x$.

## 5.4 Coherent Emission and Absorption in Zero Order

In the following, the probability of the cavity gaining or losing a photon, i.e., a coherent photon, is calculated in zero order for certain initial conditions. As outlined in Section 4.3, the probability of the atom-cavity system which is initially in an excited state ending in a ground state is given by the summation over all possible modes of decay in the cascaded transitions to this ground state.

If the atom-cavity system is initially in the state /en-1) or /fn) , the probability of the system terminating in a ground state /gn) by means of a certain cascade /$aa_1 - - - a_g$) is symbolized by $p_{na - - - a_g}$ and the probability of terminating in state /gn-1) by cascade /$a - - - a_g$) is symbolized by $p_{n-1\ a - - - a_g}$. /g) is the ground state of the atom and /$a - - - a_g$) are the various spontaneous photons emitted in a certain cascade to the ground state. $P_n$, the total probability of the cavity ending in state $n$, is given by

$$P_n = \sum_{p,a,---,a_g} p_{na---a_g}$$

and $P_{n-1}$, the total probability of the cavity ending in state n-1 is given by

$$P_{n-1} = \sum_{p,a,---,a_g} p_{n-1\ a---a_g}$$

where the summations include all the various cascades. Neglecting the cavity interaction with lower laser states which are the result of cascaded as well as direct spontaneous decays from the upper laser level, one can write the above probabilities as

$$P_n = \sum_{m,a} \left| \frac{H''_{ma,f}}{\hbar} \right|^2 \left| A_{fn}(-i[\omega_a - \Omega_{fm}]) \right|^2 \qquad (5.37a)$$

and

$$P_{n-1} = \sum_{m,a} \left| \frac{H''_{ma,e}}{\hbar} \right|^2 \left| A_{en-1}(-i[\omega_a - \Omega_{em}]) \right|^2 \qquad (5.37b)$$

where the summation over m includes all states connected by direct spontaneous decays to the initial states. The A's are found from equation 5.26.

If the system is initially in the state /eh-1), $A_{fn}$ is found from equation 5.26b to be

$$\left| A_{fn}(-i[\omega_a - \Omega_{fm}]) \right|^2 = \left| \frac{\alpha\, n^{1/2}}{\left[\frac{\Gamma_f+\Gamma_e}{4} + i(\Omega_{fm}+\Delta-\omega_a)\right]^2 + D_n^2} \right|^2 . \qquad (5.38)$$

Since, as usual, $\Delta$ , $\Gamma_e$ , $\Gamma_f$ and $\alpha\, n^{1/2}$ are considered small compared with $\Omega_{fm}$ , the function in equation 5.38 is sharply localized near the point $\omega_a = \Omega_{fm}$ and can be considered in equation 5.37a as a delta function in the variable $\omega_a$ . The magnitude of the delta function is found, as explained in Appendix 4, from

$$\left|A_{fn}(-i[\omega_a - \Omega_{fm}])\right|^2 = \delta(\omega_a - \Omega_{fm}) \int_{-\infty}^{+\infty} \left|A_{fn}(-i[\omega_a - \Omega_{fm}])\right|^2 d\omega_a \ . \qquad (5.39a)$$

The integral can be evaluated by the method of residues to give

$$\frac{2\pi \alpha^2 n(\Gamma_f + \Gamma_e)}{\left(\frac{\Gamma_f + \Gamma_e}{2}\right)^4 + \left(\frac{\Gamma_f + \Gamma_e}{2}\right)^2 2(D_n^2 + D_n^{*2}) + (D_n^2 - D_n^{*2})^2} \ . \qquad (5.39b)$$

Using the defining equations 5.33b and 5.33c, and noting from equation 4.15 and A3.22 that

$$\frac{\Gamma_f}{2} = \sum_{m,a} \left|\frac{H''_{ma,f}}{\hbar}\right|^2 \pi\, \delta(\omega_a - \Omega_{fm}) \ , \qquad (5.39c)$$

one finds that the probability of the system initially in state /en-1) gaining a coherent photon is given by

$$P_n = \gamma_f \frac{\eta^2}{\eta^2 + 4\gamma_e\gamma_f(1 + y^2)} \ . \qquad (5.40)$$

The symbols used in the above equation have been defined in equations 5.33.

Similarly, for the system initially in state /en-1) the probability of the cavity state remaining unchanged can be found from equations 5.37b and 5.26a to be:

$$P_{n-1} = \gamma_e \frac{\eta^2 + 4\gamma_f(1 + y^2)}{\eta^2 + 4\gamma_e\gamma_f(1+y^2)} \quad . \tag{5.41}$$

If the system had initially started in state /fn) , $P_{n-1}$ would be the probability of losing a coherent photon and $P_n$ would be the probability of the cavity state remaining unchanged. In this case $P_{n-1}$ would be given by equation 5.40 and $P_n$ by equation 5.41 with $\gamma_e$ and $\gamma_f$ interchanged in both equations.

Clearly, in both cases probability is conserved, i.e., $P_n + P_{n-1} = 1$ .

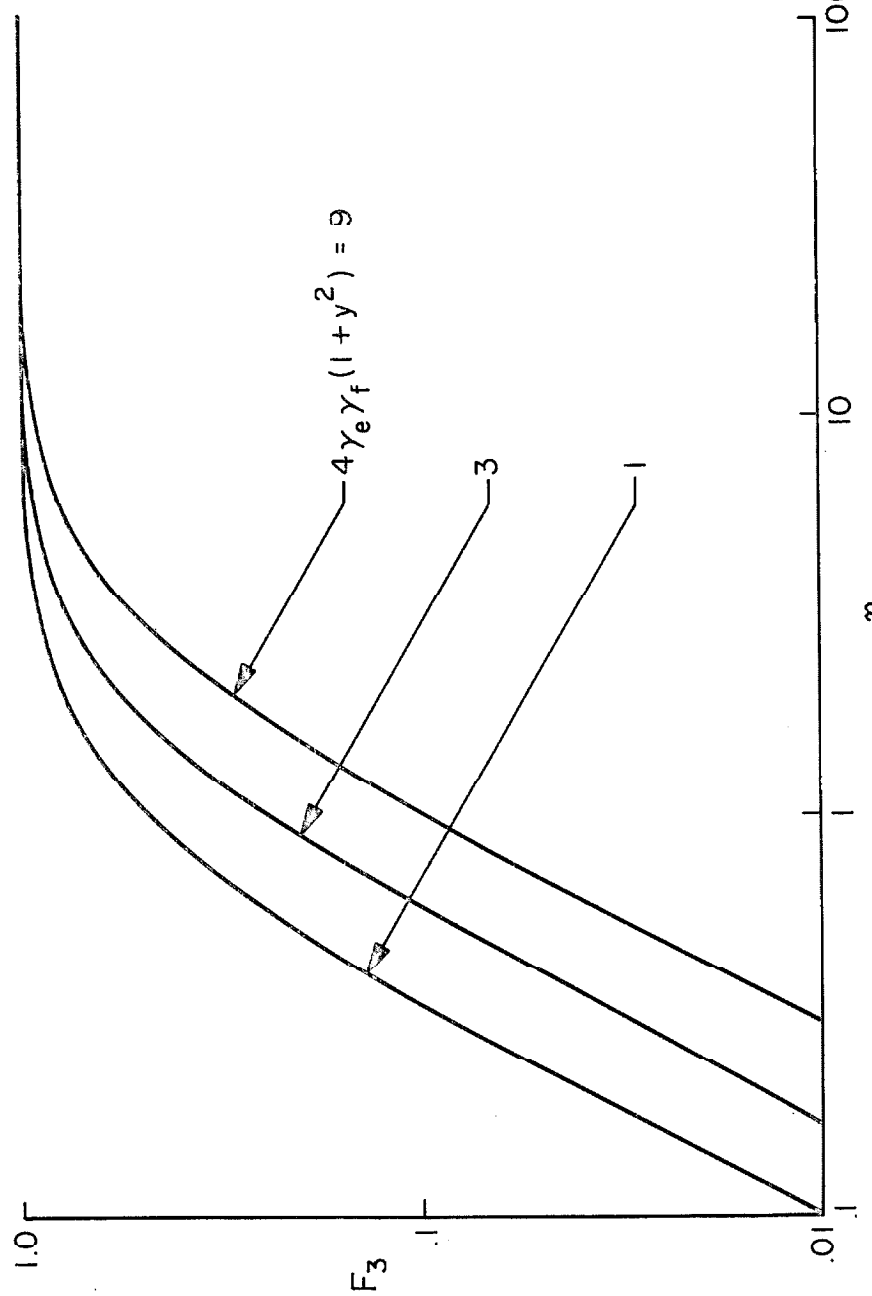
It is noted from equation 5.40 that an atom initially in the upper laser state has a probability

$$\frac{\Gamma_f}{\Gamma_e + \Gamma_f}$$

of emitting a coherent photon in the limit of very large atom-cavity interaction. The saturation factor in equation 5.40, or the relative probability of gaining a coherent photon,

$$\frac{\eta^2}{\eta^2 + 4\gamma_e\gamma_f(1+y^2)} = F_3 \tag{5.42}$$

is plotted in Figure 5.5 as a function of $\eta$ , the normalized interaction parameter, for various values of the "frequency difference"




PROBABILITY OF EMISSION OR ABSORPTION

Relative probability versus normalized interaction with the difference between cavity-field frequency and atom frequency as a parameter.

Figure 5.5

$4\gamma_e\gamma_f(1+y^2)$ . As the interaction between cavity and atom increases, the probability of emitting a coherent photon approaches the saturation value on a curve which depends on the frequency difference between the cavity photons and the laser transition. From Figures 5.2, 5.3 and 5.5, it is noted that as the spontaneous emission becomes non-Lorentzian, the probability of emission or absorption is no longer proportional to $n$ (or energy density in the cavity); hence, the atom-cavity interaction is said to be in a saturation condition. Also, the probability of emitting a coherent photon can approach one, only when the lower laser lifetime is very small compared to the upper laser lifetime, i.e., when

$$\Gamma_f \gg \Gamma_e \quad .$$

The probabilities $P_n$ and $P_{n-1}$ can also be found from the following expressions (5.1, 5.2, 5.3):

$$P_n = \Gamma_f \int_0^\infty \left|A_{fn}(t)\right|^2 dt \quad , \tag{5.42a}$$

and

$$P_{n-1} = \Gamma_e \int_0^\infty \left|A_{en-1}(t)\right|^2 dt \quad . \tag{5.42b}$$

The relation between equations 5.37 and 5.42 is discussed in Appendix 5.

Equations 5.42 lend themselves more easily to physical interpretation. For example, by defining a time-dependent probability from equation 5.42b

$$P_{n-1}(t) = \Gamma_e \int_0^t \left| A_{en-1}(t) \right|^2 dt \qquad (5.43a)$$

and differentiating this equation, a rate of spontaneous emission

$$\frac{dP_{n-1}}{dt} = \Gamma_e \left| A_{en-1}(t) \right|^2 \qquad (5.43b)$$

can be found. Therefore, this rate of probability (or flow of probability resulting from radiative decays) is proportional to the probability or "population" of an initial state.

The probability of emitting or absorbing a coherent photon is modified when higher-order processes are included. This point is covered in Chapter 6.

The methods of this chapter can be extended in a fairly straightforward manner to more complicated initial conditions, such as the atom decaying from a higher level to one or both of the laser levels, or to the problem of cascaded laser transitions.* The spontaneous radiation resulting from the decay of a higher excited state to either of the laser states would not be Lorentzian since, as seen in Figure 5.1, the decay is really to two lower states which are split in energy by the atom-cavity interaction. A further refinement on the theory presented could include the mode of excitation of the atom (5.4). Some contributions to spontaneous emission by the higher-order processes that result from the spontaneous decay of the upper laser state to the lower laser state are discussed in the next chapter.

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* Upper or lower laser level connected to another atomic level by yet another laser interaction.

## 5.5 Decay to a State with Arbitrary Lifetime

In Section 5.3 we found the spectral distributions of spontaneous radiation resulting from decays of atom-cavity states to long-lived atomic states. The frequency distribution of spontaneous radiation from the atomic states (e and f) which interact with the cavity to states (g) which have arbitrary lifetimes is given by equation 4.58. The probability of emitting a photon "a" which results from the decay of state $/en-1)$ to state $/gn-1)$ is given by

$$\left|\frac{H''_{ga,e}}{\hbar}\right|^2 \frac{1}{2\pi} \int_{-\infty}^{+\infty} \frac{\Gamma_g}{\left(\frac{\Gamma_g}{2}\right)^2 + z^2} \left|A_{en-1}(-i[\omega_a - \Omega_{eg} + z])\right|^2 dz \qquad (5.44)$$

where $\Gamma_g$ is the decay constant of atomic state $g$. Similarly, the probability of emitting a photon "a" which results from the decay of state $/fn)$ to state $/gn)$ is given by

$$\left|\frac{H''_{ga,f}}{\hbar}\right|^2 \frac{1}{2\pi} \int_{-\infty}^{+\infty} \frac{\Gamma_g}{\left(\frac{\Gamma_g}{2}\right)^2 + z^2} \left|A_{fn}(-i[\omega_a - \Omega_{fg} + z])\right|^2 dz \,. \qquad (5.45)$$

The spectral distributions of spontaneous emission between states e and f are algebraically more complex since both states interact with the cavity fields. These distributions are found in Section 6.5.

## REFERENCES

(5.1) W. R. Bennett, Jr., "Hole Burning Effects in a He-Ne Optical Maser", Phys. Review 126, especially pp. 592-593 (1961).

(5.2) W. E. Lamb, Jr., T. M. Sanders, Jr., "Fine Structure of Short-Lived States of Hydrogen by a Microwave-Optical Method, I", Phys. Review 119, especially pp. 1901-1903 (1960).

(5.3) W. E. Lamb, Jr., "Quantum Mechanical Amplifiers" in Lectures in Theoretical Physics, edited by W. E. Britten and B. D. Downs, Vol. II, Interscience Publishers, Inc., New York, 1960.

(5.4) W. Heitler, The Quantum Theory of Radiation, 3rd edition, Clarendon Press, Oxford, Sec. 34, 1953.

(5.5) E. Snitzer, "Optical Maser Action of $Nd^{+3}$ in Barium Crown Glass", Phys. Rev. Letters 7, 444 (1961).
C. J. Koester, E. Snitzer, "Amplification in a Fibre Laser", Applied Optics 3, 1182, (October 1964).

# CHAPTER 6

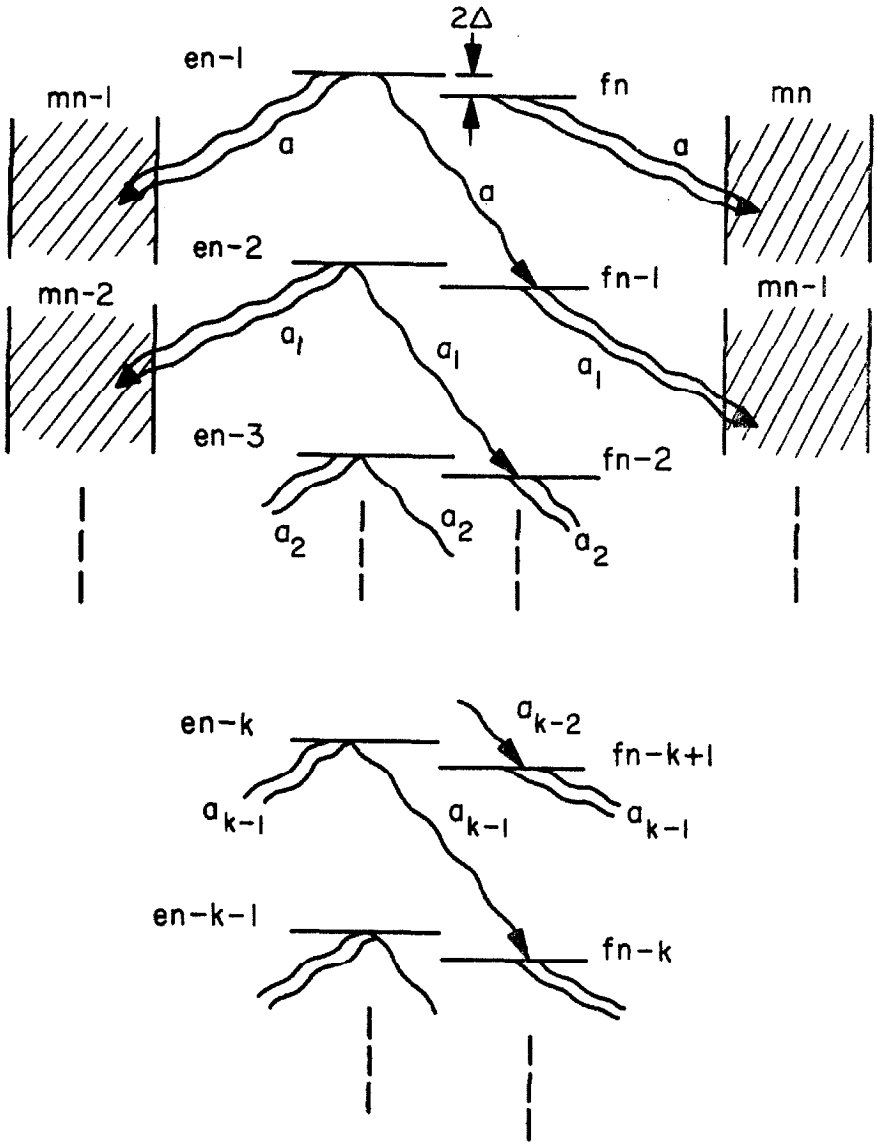
# HIGHER-ORDER PROCESSES

The results of the previous chapter are extended to include higher-order processes that are the result of an atom spontaneously decaying from the upper laser state to the lower laser state and subsequently interacting with the cavity fields. The probabilities of emitting and absorbing coherent photons and some characteristics of spontaneous emission from the laser levels are calculated.

## 6.1 Spontaneous Decay from Upper to Lower Laser State

In this chapter only direct spontaneous decays from the upper to lower laser state are considered. The theory can easily be extended to include the more general case of spontaneous decays from the upper laser state through one or more intermediate states to the lower laser state.

The energy-level diagram including the direct spontaneous decay from the upper to lower laser state is shown in Figure 6.1. Although separate indices may be used to keep track of the individual spontaneous decays, in the cases to be considered it is sufficient to use a single index for decays from the degenerate levels $e_{j-1}$ and $f_j$. That is, the index $a$ is used to signify spontaneous photons emitted from the initial states $/e_{n-1})$ or $/f_n)$ ; the index $a_1$ is used to signify spontaneous photons emitted from the states $e_{n-2}$ and $f_{n-1}$; etc. As always, only the effects of spontaneous emission are being considered; the excitation of the radiative fields is considered to be too small to cause induced effects.




RADIATIVE DECAY BETWEEN LASER STATES

Figure 6.1

The behavior of the initial states /en-1) and /fn) has been derived previously and the results are given in equations 5.26. The equations for the next two states in the chain, /en-2a) and /fn-1a), can be written in terms of the initial state /en-1) as

$$i\hbar(s + \frac{\Gamma_f}{2})\, A_{fn-1a}(s) = H''_{fa,e} A_{en-1}(s - i[\omega_a - \Omega_{ef}]) + H'_{fn-1,en-2}\, A_{en-2a}(s + i\,2\Delta) \qquad (6.1a)$$

and

$$i\hbar(s + \frac{\Gamma_e}{2})\, A_{en-2a}(s) = H'_{en-2,fn-1}\, A_{fn-1a}(s - i\,2\Delta) \,. \qquad (6.1b)$$

These equations are solved for the individual coefficients by using expressions 2.18 and 5.16b. The results can be stated as

$$A_{fn-1a}(s) = R_{n-1}(s) \frac{H''_{fa,e}}{i\hbar} A_{en-1}(s - i[\omega_a - \Omega_{ef}]) \qquad (6.2a)$$

and

$$A_{en-2a}(s) = K_{n-1}(s) \frac{H''_{fa,e}}{i\hbar} A_{en-1}(s - i[\omega_a - \omega]) \,, \qquad (6.2b)$$

where

$$R_n(s) = \frac{s + i2\Delta + \frac{\Gamma_e}{2}}{(s + i\Delta + \frac{\Gamma_e + \Gamma_f}{4})^2 + D_n^2} \qquad (6.3a)$$

and

$$K_n(s) = \frac{-\alpha n^{1/2}}{(s - i\Delta + \frac{\Gamma_e + \Gamma_f}{4})^2 + D_n^2} \,. \qquad (6.3b)$$

Equations 6.2 can easily be generalized to states further down the

chain to yield

$$A_{fn-ka---a_{k-1}}(s) =$$

$$R_{n-k}(s) \frac{H''_{fa_{k-1},e}}{i\hbar} A_{en-ka---a_{k-2}}(s - i[\omega_{a_{k-1}} - \Omega_{ef}]) . \qquad (6.4)$$

The evolution of the laser states is found by solving the above chain of equations. From the defining equations 6.3, it is noted that the atom-cavity interaction decreases as cavity photons are converted to spontaneous photons. Also, it is noted that as a result of interactions with a "dissipative" atom the cavity has a probability of terminating in any of its lower states.

## 6.2 Coherent Emission and Absorption

In the following, the probabilities of the cavity terminating in its various excited states are calculated for the system initially in the state $/en-1)$ or $/fn)$. The calculations follow the method used in Section 5.4.

With the above initial conditions, $P_n$, the probability of the system terminating with the cavity in state $n$, is the same as that given in equation 5.37a, namely,

$$P_n = \sum_{m,a} \left| \frac{H''_{ma,f}}{\hbar} \right|^2 \left| A_{fn}(-i[\omega_a - \Omega_{fm}) \right|^2 . \qquad (6.5a)$$

Since the atom can decay from the upper laser state to the lower laser state and subsequently decay to lower states, $P_{n-1}$ is no

longer given by equation 5.37b but by

$$P_{n-1} = \sum_{m,a} \left| \frac{H''_{ma,e}}{\hbar} \right|^2 \left| A_{en-1}(-i[\omega_a - \Omega_{em}]) \right|^2 +$$

$$\sum_{m,a,a_1} \left| \frac{H''_{ma_1,f}}{\hbar} \right|^2 \left| A_{fn-1a}(-i[\omega_{a_1} - \Omega_{fm}]) \right|^2 \qquad m \neq f \quad . \qquad (6.5b)$$

Similarly,

$$P_{n-2} = \sum_{m,a,a_1} \left| \frac{H''_{ma_1,e}}{\hbar} \right|^2 \left| A_{en-2a}(-i[\omega_{a_1} - \Omega_{em}]) \right|^2 +$$

$$\sum_{m,a,a_1,a_2} \left| \frac{H''_{ma_2,f}}{\hbar} \right|^2 \left| A_{fn-2aa_1}(-i[\omega_{a_2} - \Omega_{fm}]) \right|^2 \qquad m \neq f \quad . \qquad (6.5c)$$

The computation of the high-order probabilities is clearly an extension of these equations. In all the discussions to follow, the index m does not include the lower laser state f and therefore the inequality $m \neq f$ is eliminated. Using the expressions given by equations 6.2 and 6.3, one can write $P_{n-1}$ as

$$P_{n-1} = \sum_{m,a} \left| \frac{H''_{ma,e}}{\hbar} \right|^2 \left| A_{en-1}(-i[\omega_a - \Omega_{em}]) \right|^2 +$$

$$\sum_{m,a,a_1} \left| \frac{H''_{ma_1,f}}{\hbar} \right|^2 \left| R_{n-1}(-i[\omega_{a_1} - \Omega_{fm}]) \right|^2 \left| \frac{H''_{fa,e}}{\hbar} \right|^2 \left| A_{en-1}\left(-i \begin{bmatrix} \omega_a - \Omega_{ef} \\ \omega_{a_1} - \Omega_{fm} \end{bmatrix}\right) \right|^2 . \qquad (6.6a)$$

Similarly,

$$P_{n-2} = \sum_{m,a,a_1} \left|\frac{H''_{ma_1,e}}{\not{h}}\right|^2 \left|K_{n-1}(-i[\omega_{a_1} - \Omega_{em}])\right|^2 \left|\frac{H''_{fa,e}}{\not{h}}\right|^2 \left|A_{en-1}\left(-i\begin{bmatrix}\omega_a - \omega \\ \omega_{a_1} - \Omega_{em}\end{bmatrix}\right)\right|^2$$

$$+ \sum_{m,a,a_1,a_2} \left|\frac{H''_{ma_2,f}}{\not{h}}\right|^2 \left|R_{n-2}(-i[\omega_{a_2} - \Omega_{fm}])\right|^2 \left|\frac{H''_{fa_1,e}}{\not{h}}\right|^2 \left|K_{n-1}\left(-i\begin{bmatrix}\omega_{a_1} - \Omega_{ef} \\ \omega_{a_1} - \Omega_{fm}\end{bmatrix}\right)\right|^2$$

$$\left|\frac{H''_{fa,e}}{\not{h}}\right|^2 \left|A_{en-1}\left(-i\begin{bmatrix}\omega_a - \omega \\ \omega_{a_1} - \Omega_{ef} \\ \omega_{a_2} - \Omega_{fm}\end{bmatrix}\right)\right|^2 . \qquad (6.6b)$$

In the above expressions the functions $|A_{en-1}|^2$, $|K_{n-1}|^2$, $|R_{n-1}|^2$, and $|A_{fn-1}|^2$ are sharply localized functions and can be considered as delta functions in evaluating the summations (or integrations) over the spontaneous photons. The customary approximation $\omega \approx \Omega_{ef}$ is made. The frequencies of emitted spontaneous photons are localized in the region appropriate to each delta function. For example, in the last summation of equation 6.6b the spontaneous photons are localized in the region

$$\omega_a \approx \omega \quad , \qquad \omega_{a_1} \approx \Omega_{ef} \quad , \qquad \omega_{a_2} \approx \Omega_{fm} \quad .$$

The summations can be most easily accomplished in the order of $a, a_1, a_2$, etc. The functions $\tilde{K}$, $\tilde{R}$, $\tilde{A}_{en-1}$ and $\tilde{A}_{fn}$ for use in the integrations are defined by

$$|K_{n-1}(-iz)|^2 = \tilde{K}_{n-1}\, 2\pi\delta(z) \qquad (6.7a)$$

$$\left|R_{n-1}(-iz)\right|^2 = \tilde{R}_{n-1}\, 2\pi\delta(z) \tag{6.7b}$$

$$\left|A_{en-1}(-iz)\right|^2 = \tilde{A}_{en-1}\, 2\pi\delta(z) \tag{6.7c}$$

and

$$\left|A_{fn}(-iz)\right|^2 = \tilde{A}_{fn}\, 2\pi\delta(z) \quad . \tag{6.7d}$$

These delta functions have been evaluated in Chapter 5. By using equations 5.26, 5.40, 5.41, 5.42 and 6.3, one can find the following relations:

$$\Gamma_f \tilde{R}_n = 1 - \Gamma_e \tilde{K}_n \tag{6.8}$$

$$\Gamma_f \tilde{A}_{fn} = 1 - \Gamma_e \tilde{A}_{en-1} \tag{6.9}$$

If the initial state is /fn)

$$\tilde{A}_{en-1} = \tilde{K}_n \tag{6.10a}$$

and if the initial state is /en-1)

$$\Gamma_e \tilde{A}_{en-1} = 1 - \Gamma_f \tilde{K}_n \tag{6.10b}$$

where

$$\tilde{K}_n = \frac{F_3}{\Gamma_e + \Gamma_f} \quad . \tag{6.11}$$

By using the above relations and equation 5.39c the probabilities can be written as

$$P_n = \Gamma_f \tilde{A}_{fn} \tag{6.12a}$$

$$P_{n-1} = \Gamma_e \tilde{A}_{en-1} (1 - \Gamma_{e\to f} \tilde{K}_{n-1}) \tag{6.12b}$$

$$P_{n-2} = \Gamma_e \tilde{A}_{en-1} (1 - \Gamma_{e\to f} \tilde{K}_{n-2}) \Gamma_{e\to f} \tilde{K}_{n-1} \tag{6.12c}$$

where $\Gamma_{e\to f}$ is the decay constant for spontaneous emission from state e to state f . By extending these, it is found that in general

$$P_{n-j-1} = P_{n-j} \Gamma_{e\to f} \tilde{K}_{n-j} \frac{1 - \Gamma_{e\to f} \tilde{K}_{n-j-1}}{1 - \Gamma_{e\to f} \tilde{K}_{n-j}} \quad , j > 1 . \tag{6.12d}$$

The factor $\Gamma_{e\to f} \tilde{K}_{n-j}$ can, in a sense, be considered a "branching ratio" or a factor which relates the probability of a certain mode of decay as shown in Figure 6.1 to a preceding (and in this case similar) mode of decay. As seen from equation 6.12d, the successive probabilities decrease except for the special case

$$\Gamma_{e\to f} = \Gamma_e , \quad \Gamma_f = 0 \quad . \tag{6.13}$$

Since for conditions 6.13 $\Gamma_{e\to f} \tilde{K}_n$ equals one (f is now the ground state of the atom), the probabilities in equations 6.12 are not defined. To simplify the analysis this case is not considered. For a highly excited cavity, the successive probabilities can be evaluated with great accuracy by neglecting any change of $K_n$ with n . Therefore, for a highly excited cavity, the total probability can be written as

$$P = \sum_{j=0}^{\infty} P_{n-j}$$

$$P = \Gamma_f \tilde{A}_{fn} + \Gamma_e \tilde{A}_{en-1}(1 - \Gamma_{e\to f} \tilde{K}_n) \sum_{k=0}^{\infty} (\Gamma_{e\to f} K_n)^k \quad . \tag{6.14}$$

The series is readily summed to give

$$P = \Gamma_f \tilde{A}_{fn} + \Gamma_e \tilde{A}_{en-1} \tag{6.14a}$$

and from equations 6.9

$$P = 1 \quad . \tag{6.14b}$$

Therefore , probability is conserved.

The expected number of photons in the cavity after all decays have taken place is given by the usual quantum mechanical average, namely,

$$\langle n \rangle = {}_{\infty}(\psi / n / \psi)_{\infty} \quad . \tag{6.15}$$

The indicated operation is performed by using the definition of the operator $n$ as given in Section 1.4. The result can be written as

$$\langle n \rangle = P_n n + P_{n-1}(n-1) + P_{n-2}(n-2) + \text{---} \quad . \tag{6.16a}$$

If the previous result that the sum of all the probabilities is equal to one and equation 6.12d are used, $\langle n \rangle$ can be rewritten for a highly excited cavity as

$$\langle n \rangle = n - P_{n-1} \sum_{j=1}^{\infty} (j\Gamma_{e\to f} \tilde{K}_n)^{j-1} \quad . \tag{6.16b}$$

The series is easily summed and by using equations 6.12b, 6.10 and 6.11, one can find the following:

If the initial state is /en-1),

$$\langle n \rangle = n - 1 + \frac{(\gamma_f - \gamma') F_3}{1 - \gamma' F_3} \quad . \tag{6.17}$$

If the initial state is /fn) ,

$$\langle n \rangle = n - \gamma_e \frac{F_3}{1 - \gamma' F_3} . \tag{6.18}$$

The normalized decay constant $\gamma'$ is defined by

$$\gamma' \equiv \frac{\Gamma_{e \to f}}{\Gamma_e + \Gamma_f} \quad . \tag{6.19}$$

Hence, for the atom initially in the upper laser state, the average probability of the cavity gaining one coherent photon is denoted by $P_+$ . From equation 6.17 $P_+$ can be given as

$$P_+ \equiv \langle n \rangle - (n-1) = \frac{(\gamma_f - \gamma')F_3}{1 - \gamma' F_3} \quad . \tag{6.20}$$

For the atom initially in the lower laser state, the average probability of the cavity losing one coherent photon is denoted by $P_-$ . From equation 6.18 $P_-$ can be given as

$$P_- \equiv n - \langle n \rangle = \gamma_e \frac{F_3}{1 - \gamma' F_3} \tag{6.21}$$

These average probabilities $P_+$ and $P_-$ give, in the usual quantum mechanical sense, the statistical averages of the measurements performed on a large number of identically prepared systems.

In the limit

$$\gamma' \ll \gamma_f ,$$

these expressions revert to those found in Section 5.4.

Equations 6.20 and 6.21 can be written as

$$P_+ = \frac{\gamma_f - \gamma'}{1 - \gamma'} F_4 \tag{6.22}$$

and

$$P_- = \frac{\gamma_e}{1 - \gamma'} F_4 , \tag{6.23}$$

where the saturation factor $F_4$ is given by

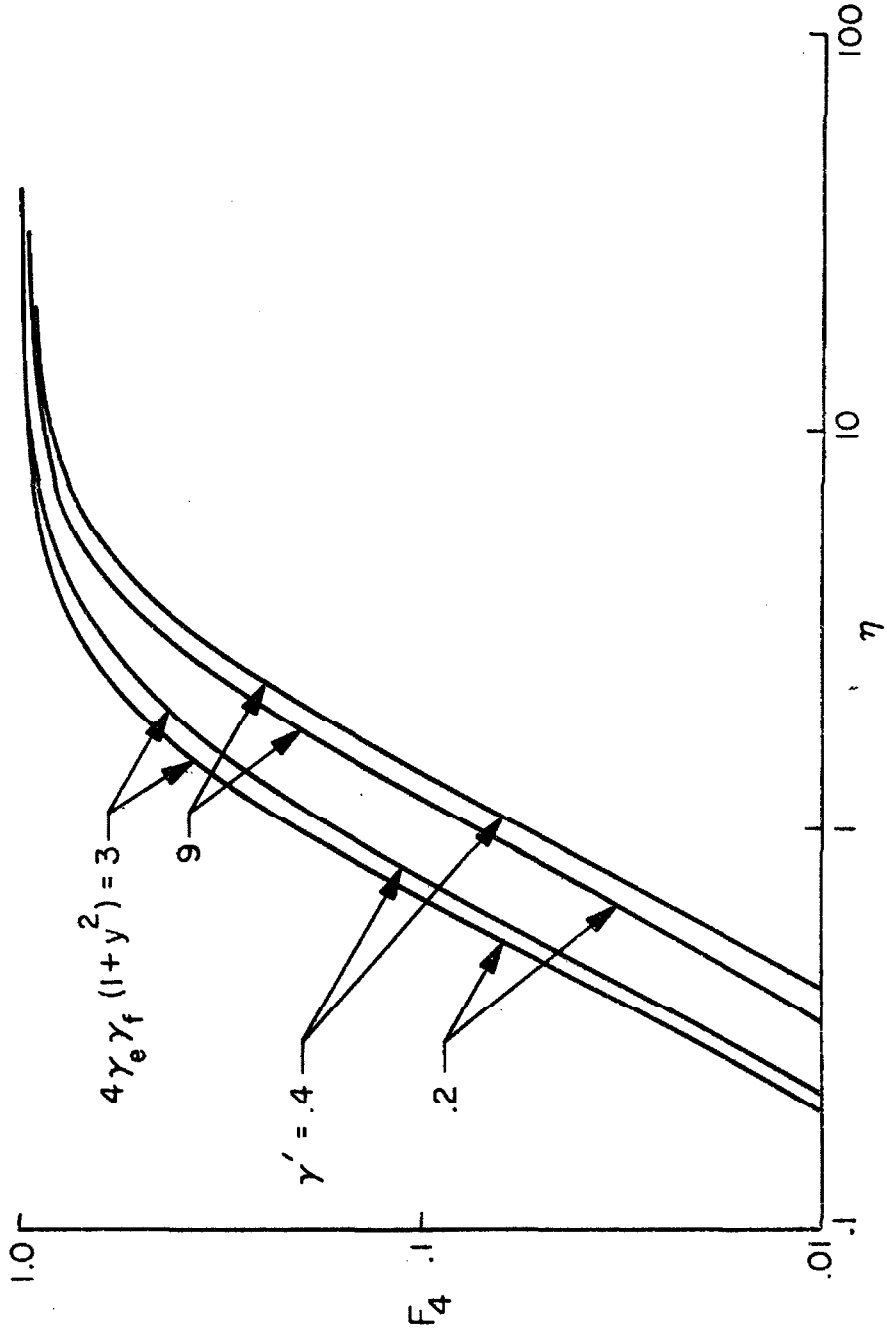
$$F_4 = \frac{(1 - \gamma')F_3}{1 - \gamma' F_3} . \tag{6.24}$$

Some of the characteristics of this factor are exhibited in Figure 6.2.

If the lower atomic state f is considered to have an extremely long lifetime, that is, if

$$\Gamma_f = 0 , \tag{6.25a}$$

and if the upper state e decays to other states in addition to state f , namely, if




PROBABILITY OF EMISSION OR ABSORPTION

Relative probability versus normalized interaction with the cavity-atom frequency difference and the decay constant between laser states as parameters.

Figure 6.2

$$\gamma_e > \gamma' \tag{6.25b}$$

then equations 6.20 and 6.21 predict

$$P_+ = \frac{-\gamma'}{1-\gamma'} \tag{6.26a}$$

and

$$P_- = \frac{1}{1-\gamma'} \quad . \tag{6.26b}$$

Hence, in this case, the cavity loses photons to the atom with an expectation which is independent of the interaction $\alpha^2 n$ and the frequency difference $\Delta$. This result is not surprising since the atomic state f can only be raised to the state e by interacting with the cavity photons. The successive decays and reexcitations of state f give the results in equations 6.23. This conclusion is true only under the conditions of the analysis, namely, that infinite time is allowed for the interactions. In an actual physical system other processes which have not been considered in this analysis, e.g., collisions, would have a probability of occurrence and would modify the above conclusions.

The time for a state $/fn)$ of the system to decay can be found from equation 5.21a. This equation, for the conditions

$$\Gamma_f \ll \Gamma_e \tag{6.27a}$$

and

$$\alpha^2 n \ll \Delta^2 \quad \text{or} \quad \alpha^2 n \ll \Gamma_e^2 \tag{6.27b}$$

gives the probability of state $/fn)$ as

$$\exp\left\{-\Gamma_e\left[\frac{\Gamma_f}{\Gamma_e}+\frac{1}{4}\frac{\alpha^2 n}{\Delta^2+\left(\frac{\Gamma_e}{4}\right)^2}\right]t\right\} \quad . \tag{6.28}$$

The decay constant is

$$\Gamma_e\left[\frac{\Gamma_f}{\Gamma_e}+\frac{1}{4}\frac{\alpha^2 n}{\Delta^2+\left(\frac{\Gamma_e}{4}\right)^2}\right] \quad . \tag{6.29}$$

Thus, for a metastable state under conditions of small atom-cavity interaction or cavity frequency much different from the atom transition frequency the decay constant can be very small. The probabilities of other states coupled to the state /fn) cannot become significant until state /fn) appreciably decays. Under conditions 6.27 the state /fn) can have a very long lifetime and processes not considered in the analysis could disturb this state and "break" the chain which gives the results in equations 6.23. In general, this same criteria could be used to check the applicability of equations 6.20 and 6.21 to an actual physical system.

## 6.3 Spontaneous Emission to a Ground State

In this section the analysis of Section 5.3 is extended to include the effects of spontaneous emission from the upper to the lower laser state. The scheme for the spontaneous decays is shown in Figure 6.1.

The equations for the successive coefficients of the ground states connected to the atomic state e by spontaneous decay can be

written as

$$s\, A_{gn-1a}(s) = \frac{H''_{ga,e}}{i\not{h}}\, A_{en-1}(s - i[\omega_a - \Omega_{eg}]) \tag{6.30a}$$

$$s\, A_{gn-2aa_1}(s) = \frac{H''_{ga_1,e}}{i\not{h}}\, A_{en-1a}(s - [\omega_{a_1} - \Omega_{eg}]) \quad . \tag{6.30b}$$

The general expression is given by

$$s\, A_{gn-k\ a - - - a_{k-1}}(s) =$$

$$\frac{H''_{ga_{k-1},e}}{i\not{h}}\, A_{en-k\ a - - - a_{k-1}}(s - i[\omega_{a_{k-1}} - \Omega_{eg}]) \quad . \tag{6.30c}$$

The probabilities of spontaneous radiation to the ground states from the upper laser state are given by the absolute square of the appropriate coefficients in the limit as $t \to \infty$. This is easily accomplished by using the final value theorem for Laplace transforms in equations 6.30. These probabilities can be stated in terms of their radiated frequencies $\omega_{a_j}$ as

$$\left|A_{gn-1a}(t \to \infty)\right|^2 = \left|\frac{H''_{ga,e}}{\not{h}}\right|^2 \left|A_{en-1}(-i[\omega_a - \Omega_{eg}])\right|^2 \tag{6.31a}$$

$$\sum_a \left|A_{gn-2aa_1}(\infty)\right|^2 = \left|\frac{H''_{ga_1,e}}{\not{h}}\right|^2 \sum_a \left|A_{en-2a}(-i[\omega_{a_1} - \Omega_{eg}])\right|^2 \tag{6.31b}$$

$$\sum_{aa_1} \left|A_{gn-3aa_1a_2}(\infty)\right|^2 = \left|\frac{H''_{ga_2,e}}{\not{h}}\right|^2 \sum_{aa_1} \left|A_{en-3aa_1}(-i[\omega_{a_2} - \Omega_{eg}])\right|^2 \tag{6.31c}$$

etc., where the summations are over-all intermediate decay quanta between laser states. The individual coefficients can be restated in terms of the initial coefficient $A_{en-1}$ by using equations 6.2b and 6.3b. The summations in equations 6.31 have been done previously in Section 6.2 in the limit of a highly excited cavity. In this limit, with the use of equations 6.7 through 6.11, one can write the probabilities in equations 6.31b and 6.31c as

$$\sum_{a} \left|A_{gn-2aa_1}(\infty)\right|^2 = \left|\frac{H''_{ga_1,e}}{\hbar}\right|^2 \left|K_n(-i[\omega_{a_1} - \Omega_{eg}])\right|^2 \Gamma_{e\to f} \tilde{A}_{en-1} \quad (6.32a)$$

and

$$\sum_{aa_1} \left|A_{gn-2aa_1a_2}(\infty)\right|^2 = \left|\frac{H''_{ga_2,e}}{\hbar}\right|^2 \left|K_n(-i[\omega_{a_2} - \Omega_{eg}])\right|^2 \Gamma_{e\to f} \tilde{K}_n \Gamma_{e\to f} \tilde{A}_{en-1}. \quad (6.32b)$$

All succeeding probabilities have an additional factor $\Gamma_{e\to f} \tilde{K}_n$. This is the same factor found in equation 6.12d which had been considered as a "branching ratio" relating successive levels. The net probability distribution is the sum of the above probabilities and can be written as

$$\left|\frac{H''_{ga,e}}{\hbar}\right|^2 \left\{\left|A_{en-1}(-i[\omega_a - \Omega_{eg}])\right|^2 + \left|K_n(-i[\omega_a - \Omega_{eg}])\right|^2 \Gamma_{e\to f} \tilde{A}_{en-1} \frac{1}{1 - \Gamma_{e\to f} \tilde{K}_n}\right\} \quad (6.33)$$

where all the probabilities have been written in terms of one frequency

variable $\omega_a$ and the resulting geometric series has been summed.

Similarly, for decay from the lower laser state the probability of emitting a photon of frequency $\omega_a$ is given by

$$\left|\frac{H''_{ga,f}}{\hbar}\right|^2 \left\{ \left|A_{fn}(-i[\omega_a - \Omega_{fg}])\right|^2 + \left|R_n(-i[\omega_a - \Omega_{fg}])\right|^2 \frac{\Gamma_{e\to f}\, \tilde{A}_{en-1}}{1 - \Gamma_{e\to f}\, \tilde{K}_n} \right\} . \tag{6.34}$$

These probability distributions can be easily compared with those of Section 5.3 if use is made of the definitions given by equations 5.33, 5.34 and 5.35, and two new functions $F'_1$ and $F'_2$ are defined in terms of $F_1$ and $F_2$ as

$$F'_1(x) = F_1 \left( \begin{array}{c} \Upsilon_f \leftrightarrow \Upsilon_e \\ y \leftrightarrow -y \end{array} \right) \tag{6.35a}$$

and

$$F'_2(x) = F_2 \left( \begin{array}{c} \Upsilon_f \leftrightarrow \Upsilon_e \\ y \leftrightarrow -y \end{array} \right) \tag{6.35b}$$

where the symbols in the brackets are interchanged to give the primed functions. From equations 6.3, it is noted that

$$\left|K_n(-i[\omega_a - \Omega_{eg}])\right|^2 = \left(\frac{4}{\Gamma_e + \Gamma_i}\right)^2 F'_2(x) \tag{6.36a}$$

and

$$\left|R_n(-i[\omega_a - \Omega_{fg}])\right|^2 = \left(\frac{4}{\Gamma_e + \Gamma_f}\right)^2 F'_1(x) \quad . \tag{6.36b}$$

Therefore, if the initial state of the system is /fn) the probability distribution of spontaneous emission from the upper laser state, expression 6.33, becomes

$$\left|\frac{H''_{ga,e}}{\not{h}}\right|^2 \left(\frac{4}{\Gamma_e + \Gamma_f}\right)^2 F'_2(x) \frac{1}{1 - r'F_3} \tag{6.37}$$

and the probability distribution of spontaneous emission from the lower laser state, expression 6.34, becomes

$$\left|\frac{H''_{ga,f}}{\not{h}}\right|^2 \left(\frac{4}{\Gamma_e + \Gamma_f}\right)^2 F'_1(x) \frac{1}{1 - r'F_3} \; . \tag{6.38}$$

If the initial state of the system is /en-1) , the probability distribution of spontaneous emission from the upper laser state is

$$\left|\frac{H''_{ga,e}}{\not{h}}\right|^2 \left(\frac{4}{\Gamma_e + \Gamma_f}\right)^2 \left[F_1(x) + F'_2(x) \frac{r'(1 - r_f F_3)}{r_e(1 - r'F_3)}\right] \tag{6.39}$$

and the probability distribution of spontaneous emission from the lower laser state is

$$\left|\frac{H''_{ga,f}}{\not{h}}\right|^2 \left(\frac{4}{\Gamma_e + \Gamma_f}\right)^2 \left[F_2(x') + F'_1(x') \frac{r'(1 - r_f F_3)}{r_e(1 - r'F_3)}\right] \; . \tag{6.40}$$

Thus the distribution of spontaneous emission from the laser states to a ground state is given by combinations of the distributions found in Section 5.3 with the probabilities modified to take into consideration the spontaneous decay from upper to lower laser states.

In this section we have found the probability distributions of spontaneous emission from the atomic states e and f to a long-lived atomic state g . These probability distributions of spontaneous emission are given by equations 6.37 through 6.40 with distribution functions $F_1$, $F_2$, $F_1'$ and $F_2'$ . If we consider that the state g has an arbitrary decay constant $\Gamma_g$ , the probability distributions of spontaneous emission from states e and f to state g can be found by the method of Section 4.3. For this case the probabilities can be found from equations 6.37 through 6.40 if each of the distribution functions $F_1$, $F_2$, $F_1'$ and $F_2'$ is replaced by the following integral function:

$$\frac{2}{\pi} \int_{-\infty}^{+\infty} \frac{\gamma_g}{(2\gamma_g)^2 + z^2} F(x+z)\, dz \quad .$$

In this integral F represents the distribution function ($F_1$, $F_2$, $F_1'$ and $F_2'$) which is being replaced in equations 6.37 through 6.40 and $\gamma_g$ is the normalized decay constant of the state g defined by

$$\gamma_g \equiv \frac{\Gamma_g}{\Gamma_e + \Gamma_f} \quad .$$

## 6.4 Method of Excitation

From equations 6.37 through 6.40 it is seen that there are two modes of spontaneous decay from the laser states. One mode is described by the distribution $F_1$ and the other by the distribution $F_2$. The distribution $F_1$ (and $F_1'$) gives the spontaneous emission from a laser state which has been excited by some broadband means such as spontaneous decay from another state or collisions (5.4). $F_2$ (and $F_2'$) gives the distribution of spontaneous emission from a laser state which has been excited as a result of an interaction with the cavity photons. A similar result was found in Section 5.3. These two modes are fundamentally different; according to Figures 5.2 and 5.3, the difference between $F_1$ and $F_2$ persists even for small atom-cavity interactions. If the atom and cavity interact only weakly, that is, if

$$\eta^2 \ll 1 \tag{6.41}$$

$F_2$, as given by equation 5.35, can be written as

$$\frac{\eta^2}{\left[(x' + 2y)^2 + 4\gamma_e^2\right]\left[x'^2 + 4\gamma_f^2\right]} \tag{6.42}$$

a product of two Lorentzian functions. This result can be justified by the following argument which was originally used by Weisskopf and Wigner (4.9).

As a result of the zero-point interactions, the upper and lower laser energy levels are considered to be broadened into Lorentzian energy distributions centered at the original energy with widths $\Gamma_e$

and $\Gamma_f$ respectively. For the atom in the upper laser state, the giving up of energy by the atom to the cavity in the form of a coherent photon $\omega$ and the decay of the lower laser state to a ground state with the emission of spontaneous photon $\omega_a$ can be considered a two-step process. The probability for this process can be written as proportional to the product of these two steps, namely

$$\frac{1}{\left(\frac{\Gamma_e}{2}\right)^2 + (\omega + \omega_a - \Omega_{eg})^2} \cdot \frac{1}{\left(\frac{\Gamma_f}{2}\right)^2 + (\omega_a - \Omega_{fg})^2} \quad . \tag{6.43}$$

If this expression is written in terms of normalized variables it is the same form as that given by expression 6.42. This argument, of course, is accurate only for small perturbations and does not give the exact answer when condition 6.41 is no longer satisfied.

## 6.5 Distribution of Spontaneous Emission between Laser States

The distribution of spontaneous emission between laser states can be found by the method used in Section 6.3. However, in this case, the result is more complex since the decay of both the initial and final state must be considered and, as shown in Figure 5.1, the decay is actually from two initial states to each of two other states. The probability of emitting a spontaneous photon of frequency $\omega_a$ is given by

$$\left|\frac{H''_{fa,e}}{\hbar}\right|^2 \frac{64}{(\Gamma_e + \Gamma_f)^2} \left[\frac{\gamma'(1 - \gamma_f F_3)}{\gamma_e(1 - \gamma' F_3)} \eta^2 F_5(x'') + F_6(x'')\right] \tag{6.44}$$

if the atom is initially excited in the upper laser state by some method of broadband excitation and by

$$\left|\frac{H''_{fa,e}}{\hbar}\right|^2 \frac{16}{(\Gamma_e + \Gamma_f)^2} \cdot \frac{4\eta^2}{1-\gamma' F_3} F_5(x'') \tag{6.45}$$

if the atom is initially excited in the lower laser state by some method of broadband excitation. The normalized frequency variable is given by

$$x'' = \frac{4(\omega_a - \Omega_{ef})}{\Gamma_e + \Gamma_f} \quad . \tag{6.46}$$

The distribution $F_5$ is given by the integral

$$\frac{1}{2\pi} \int_{-\infty}^{+\infty} \frac{\gamma_e \eta^2 + \gamma_f \left|z - 2y + i2\gamma_e\right|^2}{\left|(z-y+i)^2 - d_n^2\right|^2 \left|(z+x''+y+i)^2 - d_n^2\right|^2} dz \tag{6.47}$$

and the distribution $F_6$ is given by the integral 6.47 with an additional factor

$$\left| z + 2y + x'' + i\, 2\gamma_f \right|^2$$

multiplying the integrand, and where

$$d_n^2 = \eta^2 + \left[ y^2 + i(\gamma_f - \gamma_e) \right]^2 . \tag{6.48}$$

$F_6$ which results from the direct decay of the upper laser state becomes the normal Lorentzian as the interaction $\eta^2 \to 0$ . $F_5$ is the distribution which is the result of the excitation of the upper laser state by a coherent photon.

# CHAPTER 7

# DISCUSSION

The results of this thesis are summarized, some practical limitations of the analysis are discussed, and possible extensions of the theory are introduced.

## 7.1 Summary

The spectral distribution of spontaneous radiation from an excited atom which interacts with a coherent optical field is investigated in this thesis. In the model chosen, a stationary multilevel atom which spontaneously decays by interacting with spatial fields has two nondegenerate states coupled by an interaction with a single mode of an open cavity. The interaction between the atom and the cavity fields is treated by considering the cavity fields to be classically time-varying and also to be quantized. It is found that neglecting the non-resonant terms in the classical field theory is analogous to neglecting matrix elements between nondegenerate states in the quantized field theory. The quantized-field theory is used throughout the text in order to better emphasize the photon nature of the interaction between the cavity and the excited atom. If the classical field theory had been used the results in the text would be unchanged. However, the interpretation of the results would not be as clear since the cavity photon number does not appear in the classical field theory.

It is found that the interaction with the coherent field modifies the spectral distribution of spontaneous radiation from the atom. For

spontaneous transitions involving an atomic state which interacts with the coherent field, the spectral distributions can no longer be described by Lorentzian functions. The new distributions exhibit a broadening and splitting for strong interactions between the atom and the coherent field. It is shown that the qualitative features of these distributions can be predicted from the energy-level diagram of the atom-cavity system. The net probability of the system gaining a coherent or cavity photon is calculated by integrating over the emitted spontaneous frequencies. The equivalence of this approach to the alternate method of computing probabilities by integrating over time is demonstrated by using Parseval's theorem.

## 7.2 Distribution of Photon States

Throughout the analysis the cavity has been assumed to be initially in a definite photon eigenstate $/n)$ . For this definite state the quantum-mechanical expectation of the fields given by equation 1.29 is zero. This is not surprising since the expectation, an ensemble average, is an average over the phase of the field.

A classical cavity corresponds to a Poisson distribution over photon eigenstates (7.1)* . The characteristic of the Poisson distribution (the minimum uncertainty state) which is of interest is that the standard deviation of the distribution over the states $/n)$ is given by $(\bar{n})^{1/2}$ where $\bar{n}$ is the expected number of photons in the cavity. In the limit of large $\bar{n}$ the distribution over states $/n)$

*Suggested by D. Close.

can, in fact, be considered a Gaussian centered at $n = \bar{n}$ with width $(\bar{n})^{1/2}$. If for a highly excited cavity the initial state of the cavity is taken as a Poisson or any other narrow distribution centered at $\bar{n}$, the probabilities computed in the text would be unchanged.

## 7.3 Application to an Actual System

The conditions of the analysis are most closely satisfied in a very dilute gas system where cooperative phenomena among atoms can be disregarded and where collisions would be infrequent. However, in a gas system the atoms are moving and the stationary theory can be used only as an approximation to predict the saturation characteristics of the medium.

A typical cavity and the 1.15μ Ne transition are considered in Section A2.2. For the cavity frequency equal to the atom transition frequency, equation 6.24 predicts that for a Ne atom at the electric field maximum, saturation occurs at a power level of about 1 milliwatt. If the moving atoms in a Ne laser are considered to correspond to an inhomogeneously broadened line which has a Doppler spread of approximately 1 kmc and which interacts with the average field of the laser cavity, equation 6.24 predicts that for the above cavity in a single-mode operation, the line would begin to saturate completely at a cavity field which corresponds to an output power level of approximately 1 watt.

## 7.4 Possible Extensions of the Theory

The analysis can be extended to include the case of two sets of degenerate states coupled through an interaction with the cavity fields. In this case one would expect that the differences in the interaction with the cavity fields among the various states would cause additional splittings in the energy levels of the atom-cavity system.

The case of two or more cavity modes which simultaneously interact with the atom can also be considered. In the quantized cavity picture one would find that as a result of the interactions with the atom an exchange of photons among the cavity modes can take place.

The analysis in the text also applies to the case of an excited moving atom which interacts with a TEM traveling-wave field. This can be most easily seen by considering the field as a classical time-dependent field. The vector potential for a TEM field polarized in the x-direction and propagating in the z-direction can be written as

$$\underline{A} = 1_x A_o \cos(kz - \omega t)$$

where

$$k = \omega/c \ .$$

For an atom moving slowly compared with the speed of light

$$z = z_o + v_z t$$

where $v_z$ is the component of the atom velocity in the z-direction. Thus the field as seen by the atom is given by

$$\underline{A} = 1_x A_o \cos(kz_o - \omega\left[1 - \frac{v_z}{c}\right]t) \quad .$$

The field frequency at the atom is Doppler shifted to the frequency $\omega(1 - \frac{v_x}{c})$*. The probabilities computed in the text would give the spectral distribution of spontaneous emission in the moving frame of the atom provided the frequency of the coherent field is taken as $\omega(1 - \frac{v_z}{c})$ instead of $\omega$ .

## References

(7.1) W. H. Louisell, Radiation and Noise in Quantum Electronics, McGraw-Hill Book Company, Inc., New York (1964), p. 139.

* Any variations of the field in the transverse plane are being neglected.

## APPENDIX I

## EVALUATION OF THE PARAMETER $\alpha n^{1/2}$

The parameter $\alpha n^{1/2}$, which in Chapter 2 was found to determine the motion of the atom and single-mode cavity system, can be stated in terms of $\underline{D}$, the electric-dipole matrix element of the atom, and the strength of the cavity electric field at the atom.

For a quantum mechanical cavity operating in a particular mode and energy state as described in Chapter 1, the energy $W$ is given by

$$W = n\not{h}\,\omega_a \tag{A1.1}$$

where the zero-point energy $\frac{1}{2}\not{h}\omega$ has been deleted. The interaction parameter $\alpha$ is given by equation 2.14a as

$$\alpha = \frac{\Omega}{(2\epsilon_o \not{h}\,\omega_a)^{1/2}}\,\underline{D}\cdot\underline{A}_a \tag{2.14a}$$

and the interaction can be written as

$$\alpha n^{1/2} = \frac{\Omega}{(2\epsilon_o)^{1/2}}\,\frac{W^{1/2}}{\not{h}\omega_a}\,\underline{D}\cdot\underline{A}_a \quad . \tag{A1.2}$$

$W$ is given as the sum of the electromagnetic "kinetic" energy and the electromagnetic "potential" energy, i.e.,

$$W = \frac{1}{2}\epsilon_o \int_V \underline{E}\cdot\underline{E}\,d_v + \frac{1}{2\mu_o}\int_V \underline{B}\cdot\underline{B}\,d_v \quad . \tag{A1.3}$$

From equation 1.3 and equation 1.12 the vector potential is chosen as

$$\underline{A} = C\,\underline{A}_a \cos \omega_{at} \qquad (A1.4)$$

where $C$ is a constant which determines the strength of the cavity fields or the level of excitation of the cavity. The electromagnetic energy is given by

$$W = \frac{1}{2}\epsilon_o \omega^2 C^2 \quad . \qquad (A1.5)$$

The definition of the electric and magnetic fields in terms of the vector potential, equation 1.1, and the normalization, equation 1.11, have been used to find equation A1.5. Therefore, $\alpha n^{1/2}$ is given by

$$\alpha n^{1/2} = \frac{\Omega}{2\hbar} \underline{D} \cdot \underline{A}(\underline{r}) \qquad (A1.6)$$

where the vector potential $\underline{A}(\underline{r})$ is given by

$$\underline{A}(\underline{r}) = C\,\underline{A}_a \quad . \qquad (A1.7)$$

For an electric-dipole interaction the parameter $\alpha n^{1/2}$ depends on the strength of the electric field at the position of the atom.

For $n = 1$, $\alpha n^{1/2}$ gives the interaction of the excited atom cavity containing only zero-point energy. The zero-point interaction with the cavity may be given by using the field corresponding to one photon in the semiclassical expression given by the right-hand side of equation A1.6.

# APPENDIX 2

## PERTURBATIONS

Corrections to the eigenvalues and eigenfunctions of the atom-cavity system which is diagonalized in Chapter 2 are considered in this appendix. The corrections are estimated for a typical laser cavity.

### A 2.1 Perturbation

The eigenvalue problem, in the notation of Chapter 2, is written as

$$(H_o + H')/\rho) = E_\rho/\rho) \quad . \tag{A2.1}$$

Since no diagonal elements of the perturbation $H'$ exist in the representation given by expression 2.21, the corrected eigenvalues and eigenfunctions can be written in lowest order as (A2.1)

$$/\rho) = /\rho)^o + \sum_{\rho'} /\rho')^o \frac{{}^o(\rho'/H'/\rho)^o}{E^o_\rho - E^o_{\rho'}} \tag{A2.2a}$$

and

$$E_\rho = E^o_\rho + \sum_{\rho'} \frac{|{}^o(\rho'/H'/\rho)^o|^2}{E^o_\rho - E^o_{\rho'}} \quad , \tag{A2.2b}$$

where the superscript $^o$ signifies uncorrected terms. In the above representation $H'$ has matrix elements between state $/\phi_{n_\pm})$ and both $/\phi_{n-2_\pm})$ and $/\phi_{n+2_\pm})$. If the usual approximations

$$\Delta \ll \omega \qquad \text{and} \qquad \alpha n^{1/2} \ll \omega$$

are satisfied, the magnitudes of the coefficients in expansions A2.2

can be given as

$$\left| \frac{{}^{o}(\rho' / H' / \rho)^{o}}{E^{o}_{\rho} - E^{o}_{\rho'}} \right| < \frac{\alpha n^{1/2}}{\omega} \quad . \tag{A2.3}$$

A2.2 Typical Laser Cavity

For an idealized one-dimensional cavity operating in a TEM mode, the expected number of photons $n$ can be given by

$$n = \frac{P L}{cT \hbar\omega}$$

where $P$ is the power output, $L$ is the length, $c$ is the speed of light, $T$ is the mirror transmission coefficient at each end, and $\omega$ is the operating frequency. If one assumes the following typical values for a gas laser

$$P = 10^{-3} \text{ watts}$$

$$L = 1\text{m}$$

$$T = 10^{-2}$$

and

$$\omega = 10^{15} \text{ cps} \quad ,$$

it is found that

$$n \approx 10^{10} \quad .$$

The parameter $\alpha$ has been defined in equation 2.14a. For a typical electric-dipole matrix element $D$ of $10^{-18}$ c.g.s. (A2.2), the above energy level of the cavity, and a cavity cross section of $10^{-4}\text{m}^2$, the interaction $\alpha n^{1/2}$ is found to be:

$$\alpha\, n^{1/2} \approx 10^{7} \sin kz \ .$$

Hence, for these conditions the magnitudes of the coefficients given in A2.3 are of the order of $10^{-8}$ and the corrections to the eigenvalues and eigenfunctions of the diagonalized problem can be neglected.

## REFERENCES

(A2.1) A. Messiah, Quantum Mechanics, North-Holland Publishing Company, Amsterdam (1961), p. 686.

(A2.2) W. R. Bennett, Jr., "Gaseous Optical Masers", Applied Optics Supplement on Optical Masers, 24 (1962).

# APPENDIX 3

## SPACE FIELDS

The fields existing in space can be expanded in terms of orthogonal functions satisfying appropriate boundary conditions. Among the orthogonal functions commonly used are: the standing waves of a large cubical box (A3.1), the spherical waves of a large sphere (A3.2) or the traveling waves of a large cubical box (A3.3). The latter is particularly useful for isotropic interactions and will be developed in the following.

### A3.1 The Vector Potential in Terms of Traveling Waves

Since the discussion parallels that in Section 1.1, only the salient features and the differences with the previous analysis will be given.

The vector potential $\underline{A}$, a real function of space and time coordinates, can be expressed as

$$\underline{A} = \sum_a g_a(t)\, \underline{A}_a + g_a^*(t)\, \underline{A}_a^* \qquad (A3.1)$$

where the $\underline{A}_a$ are the traveling-wave solutions of equations 1.4. The $\underline{A}_a$ are chosen to satisfy periodic boundary conditions on the surface of a large cube of length $L$. The $\underline{A}_a$ are of the form

$$\underline{A}_a = \text{Cst}\ \underline{\epsilon}_a\, e^{i\underline{k}_a \cdot \underline{r}} \quad . \qquad (A3.2)$$

The gauge relation $\nabla \cdot \underline{A}_a = 0$ gives

$$\underline{k}_a \cdot \underline{\epsilon}_a = 0 \quad . \tag{A3.3}$$

Therefore, the polarization vector $\underline{\epsilon}_a$ is transverse to the wave vector $\underline{k}_a$. For each wave vector $\underline{k}_a$, two independent polarizations can exist. These two polarization vectors can be considered as unit vectors orthogonal to each other and to the wave vector. The wave vector, in terms of its cartesian components, is

$$\underline{k}_a = l_x k_{ax} + l_y k_{ay} + l_z k_{az} \tag{A3.4}$$

where, as a result of the periodic boundary conditions,

$$k_{ax} = n_{ax} \frac{2\pi}{L}$$

$$k_{ay} = n_{ay} \frac{2\pi}{L} \qquad n_{ax}, n_{ay}, n_{az} \text{ any positive or negative integer}$$

$$k_{az} = n_{az} \frac{2\pi}{L} \quad . \tag{A3.5}$$

The orthogonality and normalization can be chosen as

$$\int_V \underline{A}_a \cdot \underline{A}_b^* \, dv = \delta_{ab} \, . \tag{A3.6}$$

The expansion functions are given by

$$\underline{A}_a = \underline{\epsilon}_a \frac{e^{i\underline{k}_a \cdot \underline{r}}}{L^{3/2}} \quad . \tag{A3.7}$$

The expansion coefficients in equations A3.1 satisfy the equations of simple harmonic motion, i.e.,

$$\ddot{g}_a(t) + \omega_a^2 g_a(t) = 0 \quad . \tag{A3.8}$$

In order to be consistent with the description in terms of traveling waves, the $g_a$ are chosen as either proportional to $e^{i\omega_a t}$ or $e^{-i\omega_a t}$ .

The electric energy in the cavity is found to be

$$\frac{\epsilon_o}{2} \int_V \underline{E} \cdot \underline{E} \, dv = \epsilon_o \sum_a \omega_a^2 (g_a g_a^* - g_a g_{-a} - g_a^* g_{-a}^*) \tag{A3.9}$$

where $g_{-a}$ is the expansion coefficient for the mode traveling in the direction opposite to the "a" mode. The relationships between the "a" and the "-a" mode are:

$$\underline{k}_{-a} = -\underline{k}_a \tag{A3.10a}$$

$$\underline{\epsilon}_{-a} = \underline{\epsilon}_a \tag{A3.10b}$$

$$\omega_{-a} = \omega_a \quad . \tag{A3.10c}$$

From the relations:

$$\int_V (\underline{k}_a \times \underline{A}_a) \cdot (\underline{k}_a \times \underline{A}_a^*) \, dv = k_a^2 \tag{A3.11a}$$

$$\int_v (\underline{k}_a \times \underline{A}_a) \cdot (\underline{k}_{-a} \times \underline{A}_{-a})\, dv = -k_a^2 \qquad \text{(A3.11b)}$$

the magnetic energy in the cavity is found to be given by

$$\frac{1}{2\mu_o} \int_v \underline{B} \cdot \underline{B}\, dv = \frac{1}{\mu_o} \sum_a k_a^2 (g_a g_a^* + g_a g_{-a} + g_a^* g_{-a}^*) \; . \qquad \text{(A3.12)}$$

The total energy $W$, the sum of the electric and magnetic energy, is given by

$$W = \epsilon_o \sum_a \omega_a^2 (g_a g_a^* + g_a^* g_a) \; . \qquad \text{(A3.13)}$$

The oscillating terms involving two oppositely traveling waves cancel in equations A3.9 and A3.11 and the total energy is just the sum of the energy in each wave.

In order to put equation A3.13 into a more recognizable form it is written in the form*

$$W = \frac{1}{2} \sum_a \omega_a^2 Q_a^2 + P_a^2$$

or

$$W = \frac{1}{4} \sum_a (\omega_a Q_a + iP_a)(\omega_a Q_a - iP_a) + (\omega_a Q_a - iP_a)(\omega_a Q_a + iP_a) \qquad \text{(A3.14)}$$

where $Q_a$ and $P_a$ are both real.

---

*The Q's and P's defined here are not the same as in Chapter 1.

By comparing equations A3.13 and A3.14, the identity

$$\epsilon_o^{1/2}\,\omega_a g_a \equiv \frac{1}{2}\,(\omega_a Q_a + iP_a) \tag{A3.15}$$

is chosen. Therefore,

$$Q_a = \epsilon_o^{1/2}(g_a + g_a^*)$$

and

$$P_a = -i\,\epsilon_o^{1/2}\,\omega_a(g_a - g_a^*) \quad .$$

The Q's and P's satisfy Hamilton's equations (equation 1.20) with $W = H$. Therefore, $Q_a$ and $P_a$ are canonical coordinates satisfying the commutation relation

$$\left[Q_a, P_b\right] = i\not{h}\,\delta_{ab} \quad . \tag{A3.16}$$

Define $a_a$ and $a_a^+$ by

$$g_a \equiv \left(\frac{\not{h}}{2\epsilon_o\,\omega_a}\right)^{1/2} a_a \tag{A3.17}$$

and

$$g_a^* \equiv \left(\frac{\not{h}}{2\epsilon_o\,\omega_a}\right)^{1/2} a_a^+ \quad . \tag{A3.18}$$

The $a$ and $a^+$ satisfy the commutation relations for creation and annihilation operators, namely,

$$\left[a_a, a_b^+\right] = \delta_{ab} \quad , \tag{A3.19}$$

and the Hamiltonian is given in the standard form,

$$H = \sum_a (a_a^+ a_a + \frac{1}{2})\hbar\omega_a \quad . \tag{A3.20}$$

The vector potential is given in terms of creation and annihilation operators as:

$$\underline{A} = \sum_a (\frac{\hbar}{2\epsilon_o \omega_a})^{1/2} (a_a \frac{e^{i\underline{k}_a \cdot \underline{r}}}{L^{3/2}} + a_a^+ \frac{e^{-i\underline{k}_a \cdot \underline{r}}}{L^{3/2}}) \quad . \tag{A3.21}$$

## A3.2 Summation over All the Space Modes

In calculations there often occurs the summation

$$\sum_a F(k_a)$$

where the summation is over all the space modes. In the limit $L \to \infty$ this summation can be converted into an integration over the different k's.

First consider the radiation oscillators in k space as defined in equations A3.5. Each radiation oscillator occupies a volume $\frac{1}{2}(\frac{2\pi}{L})^3$. The factor $\frac{1}{2}$ results from the two independent polarizations. For wavelengths very small with respect to L the number of radiation oscillators between k and dk , $N_k dk$ (the system is considered isotropic), is given by the volume in k space between k and k+dk divided by the volume occupied by each radiation oscillator, namely

$$N_k dk = \frac{4\pi k^2 dk}{\frac{1}{2}\left(\frac{2\pi}{L}\right)^3} = \frac{L^3 k^2}{\pi^2} dk \quad . \tag{A3.22}$$

Therefore, for isotropic problems, the summation $\sum_a F(k_a)$ over all the field oscillators in the limit $L \to 0$ becomes an integral over $k$ with the kernel $\frac{L^3}{\pi^2} k^2$, i.e.,

$$\sum_a F(k_a) \quad \text{becomes} \quad \frac{L^3}{\pi^2} \int_0^\infty F(k)\, k^2 dk \quad . \tag{A3.23}$$

A3.3 Cutoff Function

In the nonrelativistic theory of electron interactions (and consequently atom interactions) used in Chapter 4, the effect of the zero-point fields of space takes the form of the integral -

$$\int_0^\infty \left|H''(k)\right|^2 k^2 dk \quad . \tag{A3.24}$$

This integral, as a result of the interactions with high frequency virtual photons*, is divergent (A3.4, A3.5). The high frequency photons contributing to the divergence are photons possessing appreciable momentum. Hence, when interacting with many of these photons, the electron would tend to be moved about and the interaction would no longer be local but would be spread out over an appreciable volume in space. The high frequency photons would therefore be ineffectual in

*In some problems low frequency divergences also appear. These are handled by reformulating the theory (A3.6).

interacting with the electrons. To be rigorous this effect would have to be treated within the framework of a completely relativistic theory. However, the postulate of the cutoff function is sufficient to salvage the nonrelativistic theory for problems involving the emission of optical frequencies by excited atoms.

The integral A3.24 is written as

$$\int_0^\infty C(k) \left|H''(k)\right|^2 k^2 dk \quad , \tag{A3.25}$$

where $C(k)$ is the cutoff function. The cutoff function may be chosen by attributing a certain spatial extent to the electron (A3.4). However, for many applications the simplest cutoff function suffices, i.e., interactions with the virtual photons possessing energy less than the rest energy of the electron are considered to interact with the electron without any correction and the virtual photons exceeding this energy are considered not to interact at all with the electron. Thus, the cutoff function in its simplest approximation takes the form:

$$\begin{aligned} C(k) &= 1 \qquad && k < \frac{mc}{\hbar} \\ C(k) &= 0 \qquad && k > \frac{mc}{\hbar} \quad . \end{aligned} \tag{A3.26}$$

## Appendix 3

## REFERENCES

(A3.1) J. C. Slater, Quantum Theory of Atomic Structure, McGraw-Hill Book Company, Inc., New York (1961), Appendix 12.

(A3.2) A. Messiah, Quantum Mechanics, North-Holland Publishing Company, Amsterdam (1961), p. 977.

(A3.3) W. Heitler, The Quantum Theory of Radiation, 3rd edition, Clarendon Press, Oxford (1953), pp. 38-42 and 54-62.

(A3.4) Reference (A3.2), pp. 980-983.

(A3.5) S. S. Schweber, An Introduction to Relativistic Quantum Field Theory, Row, Peterson and Company, Elmsford, New York (1961), pp. 524-531.

(A3.6) H. Umezawa, Quantum Field Theory, North-Holland Publishing Company, Amsterdam (1956), pp. 238-244.

# Appendix 4
## DELTA FUNCTIONS

A number of delta functions have been used in Chapters 4, 5 and 6. The magnitude or weight of these delta functions can be found by appropriate contour integrations in the complex plane. This technique is illustrated for the first delta function (a tabulated function) appearing in the text. The other delta functions are evaluated similarly but with somewhat more labor.

The factor $\frac{\sin^2 x}{x^2}$ which appears in Section 4.1 can be considered under certain conditions a delta function in $x$. Its value is evaluated by considering the definition of a delta function, namely,

$$\int_{-\infty}^{+\infty} F(x)\, \delta(x - x_o)\, dx = F(x_o) \quad . \tag{A4.1}$$

Thus, under appropriate conditions,

$$\frac{\sin^2 x}{x^2} = \delta(x) \int_{-\infty}^{+\infty} \frac{\sin^2 x}{x^2}\, dx \quad . \tag{A4.2}$$

The integral in A4.2 is evaluated by the method of residues. Since the integral is an even function,

$$\int_{-\infty}^{+\infty} \frac{\sin^2 x}{x^2}\, dx = \mathrm{Re} \int_{0}^{\infty} \frac{(1 - e^{i2x})}{x^2}\, dx = \mathrm{Re}\ I \quad . \tag{A4.3}$$

Consider the contour integral

$$\oint \frac{(1 - e^{i2z})}{z^2} dz \qquad (A4.4)$$

around a closed contour along the boundaries of the upper right half of the complex z plane. The integral I of A4.3 is just the portion of this contour along the real axis and by the method of residues

$$\text{Re } I = \pi .$$

Hence, equation A4.2 becomes

$$\frac{\sin^2 x}{x^2} = \pi \delta(x) \quad . \qquad (A4.5)$$

## Appendix 5

## PROBABILITY COMPUTATION

The probabilities related to the absorption or emission of a coherent photon, as discussed in Section 5.4, can be given in the form of equations 5.37, or alternatively by equations 5.42. The relation between these two forms is discussed in the following.

Since the discussion is similar in every respect for $P_{n-1}$ and $P_n$, the following is confined to $P_n$. $P_n$ is expressed as

$$P_n = \sum_{m,a} \left| \frac{H''_{ma,f}}{\hbar} \right|^2 \left| A_{fn}(-i[\omega_a - \Omega_{fm}]) \right|^2 \quad . \tag{5.37a}$$

In Section 5.4 it is shown that $\left|A_{fn}(-i[\omega_a - \Omega_{fm}])\right|^2$ is a delta function in the variable $\omega_a$ and from equation 5.39a and 5.39c $P_n$ can be written as

$$P_n = \Gamma_f \frac{1}{2\pi} \int_{-\infty}^{+\infty} \left| A_{fn}(-i[\omega_a - \Omega_{fm}]) \right|^2 d\omega_a \quad . \tag{A5.1}$$

Using the definitions of the Laplace transform given by equation 4.25, one can modify equation A5.1 to read

$$P_n = \Gamma_f \frac{1}{2\pi} \int_{-\infty}^{+\infty} d\omega_a \times$$

$$\int_0^\infty \int_0^\infty A_{fn}(t)\, A^*_{fn}(t')\, e^{+i(\omega_a - \Omega_{fm})(t-t')}\, dt\, dt' \quad . \tag{A5.2}$$

The integration over $\omega_a$ is carried out first by using the relation

$$\frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{i(t-t')(\omega_a-\omega_o)}\, d\omega_a = \delta(t-t') \qquad (A5.3)$$

and $P_n$ is found as

$$P_n = \Gamma_f \int_0^{\infty} \left|A_{fn}(t)\right|^2 dt \quad . \qquad (5.42a)$$

The above discussion parallels the derivation of Parseval's theorem (A5.1).

## REFERENCES

(A5.1) H.S. Carlslaw, J. C. Jaeger, Operational Methods in Applied Mathematics, Dover Publications, Inc., New York (1963), p.266.