PART I: SOLITON ON A BEACH AND RELATED PROBLEMS

PART II: MODULATED CAPILLARY WAVES

Thesis by

Noel Frederick Smyth

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ABSTRACT

I. SOLITON ON A SLOPING BEACH AND RELATED PROBLEMS

The problem of the behaviour of a soliton on a slowly varying beach is considered. It is shown that for a correct description, the full Boussinesq equations rather than a Korteweg-de Vries type approximation must be used. Using both energy conservation and two-timing expansions, the behaviour of the soliton is analysed. The slowly varying soliton is found not to conserve mass and momentum and it has been suggested that to conserve these quantities, both forward and reflected waves must be added behind the soliton, these waves being solutions of the linear shallow water equations. It is shown that to the order of approximation of the Boussinesq equations, only a forward wave (or tail) behind the soliton is necessary to fulfill mass and momentum conservation.

A perturbed Korteweg-de Vries equation for which the perturbation adds energy to the soliton is considered. It is found that a tail is formed behind the soliton. The development of this tail into new solitons is analysed.

II. MODULATED CAPILLARY WAVES

An exact hodograph solution for symmetric and antisymmetric capillary waves on a fluid sheet (of possibly infinite thickness) has been previously found. Using this solution, an exact averaged Lagrangian for slowly varying capillary waves is calculated. Modulation equations can be found from this averaged Lagrangian, but due to the algebraic complexity of the equations, the limit of waves on a thin fluid sheet is considered. From the modulation equations, the stability of symmetric and antisymmetric capillary waves on a thin fluid sheet is found. The modulation equations for antisymmetric waves form a hyperbolic system and the simple wave solutions for this system are calculated. These simple wave solutions are interpreted physically.

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SOLITON ON A SLOPING BEACH AND RELATED PROBLEMS

PART I

SOLITON ON A SLOPING BEACH AND RELATED PROBLEMS

The problem of the behaviour of a soliton under perturbations due to damping or variation in the medium has received much attention in recent literature. One of the most interesting examples of such motion concerns the problem of a soliton moving up or down a sloping beach and the resulting variations in such quantities as the amplitude. In this and other problems, the main result is that the amplitude of the soliton is determined by energy conservation. For a soliton on a beach of varying depth $h(x)$, for example, this energy principle gives almost immediately that the amplitude a of the soliton has the variation

$$a \propto h^{-1}$$

It is found, however, that this energy law cannot simultaneously conserve mass. To compensate for this, it is argued that a tail is formed following the soliton and a second order theory is needed to complete the discussion. This is the central part of the present work.

Most of the existing work focuses on a Korteweg-de Vries type of equation in either the form

$$u_t + \nu(t)uu_x + \lambda(t)u_{xxx} = 0,$$

or the equivalent form

$$u_t + 6uu_x + u_{xxx} = -\Gamma(t)u$$

Here $\Gamma(t)$ is small and Γ, ν and λ are slowly varying functions of t. These equations can be used as an approximation to the beach problem if the roles of t and x are reversed in that t is now interpreted as distance up the beach and x is a retarded time.

The questions concerning the tail and the related mass balance can be studied in the context of these equations. While for some physical problems they may be valid, it is claimed here that they are inadequate for a description of the tail region formed when a soliton moves on a beach. The derivation of the Korteweg-de Vries approximation makes the equation valid only to first order in the region occupied by the soliton. A Boussinesq type equation or some equivalent system is required for the discussion. The Boussinesq system

$$\eta_t + (h_0 u + \eta u)_x + (1/3\ gh_0{}^3 u_x)_{xx} = 0$$
$$u_t + uu_x + g\eta_x = 0$$

will be used in the present work, where h_0 is the undisturbed depth, η is the surface elevation and u is the fluid velocity.

These equations and the results obtained from them will be used to investigate a suggestion put forward by Miles (1979) that for mass to be conserved, a reflected wave as well as a tail are required. This is an intriguing suggestion since for the usual problem of a linear wave moving on a slowly varying beach, any reflection is exponentially small and is not important in the conservation of mass. It is an interesting general question of whether this is true for nonlinear waves. It is found in the present work that when the Boussinesq equations are used, the mass conservation is accommodated in the forward propagating tail and no reflected wave is necessary to the main orders of approximation.

The perturbation problems for the Korteweg-de Vries equation serve as useful background and a detailed consideration of these problems is presented in Chapter One. However, the main contribution of Part I is on the Boussinesq equations and these are dealt with in Chapter Two.

CHAPTER ONE

PERTURBED KORTEWEG-DE VRIES EQUATIONS

1.1 INTRODUCTION

Many physical problems are described by Korteweg-de Vries type equations in which the usual Korteweg-de Vries equation has extra terms representing such effects as change in the medium added. In this chapter, we shall consider three types of such equations, two of which are approximations to solitons moving up and down a beach and the third describing a soliton upon which viscosity acts. The techniques used to solve these equations are instructive for our central problem of the behaviour of a soliton on a beach, this being the topic of Chapter Two.

The perturbed Korteweg-de Vries equation describing motion on a slowly varying bottom has been derived by Ostrovsky and Pelinovsky (1970), Kakutani (1971) and Johnson (1973a) using the usual assumptions needed to derive the Korteweg-de Vries equation. Two distinct methods have been employed to solve this equation for a soliton initial condition; two-timing and inverse scattering. In the present work, the method of two-timing will be used.

The first attempts at the analysis of the problem of a soliton on a beach (Ott and Sudan (1970), Johnson (1973a) and Ostrovsky (1976)) assumed, to first order, a slowly varying soliton and used essentially energy conservation to find how the amplitude of the soliton varied. Ablowitz (1971), Johnson (1973b) and Ko and Kuehl (1978) then extended the perturbation analysis to second order and found that the second order term approaches a "constant" value immediately behing the soliton, which indicates that the slowly varying soliton expansion is non-uniform. The presence of this secular term was also found by perturbations on the inverse scattering solution of the Korteweg-de Vries equation done by Kaup and Newell (1978) and Karpman (1979).

Numerical work by Leibovich and Randall (1973) and Ko and Kuehl (1978) indicated the presence of a tail, called a "shelf" by these authors, behind the soliton. Karpman (1979) called the secular term in the soliton expansion a "plateau." The term shelf or plateau for this region is misleading as it, in fact, cannot be flat and obeys a linear, non-dispersive form of the perturbed Korteweg-de Vries equation. In the present work, it will be referred to as the near tail.

It was noted by Johnson (1973b) that while the slowly varying soliton satisfies energy conservation, it does not satisfy mass conservation. Johnson tried to introduce a region behind the soliton to account for the mass deficit, but the presence of distinct near and far tail regions behind the soliton was not indicated in his work.

Using the earlier inverse scattering work of Kaup and Newell (1978), Knickerbocker and Newell (1980) deduced that the near tail, which they called a "shelf," must satisfy a linear, non-dispersive form of the perturbed Korteweg-de Vries equation. They then demonstrated that the slowly varying soliton plus near tail then satisfied mass conservation within the Korteweg-de Vries framework. Confirmation of their analytical results was obtained using a numerical solution. Miles (1979) showed that while the Korteweg-de Vries mass is conserved, the actual mass is not. This question of conserving the true mass forms the central part of Chapter Two.

In the present work, we shall use two-timing and matched asymptotic expansions to find the behaviour of the slowly varying soliton and its tails. Knickerbocker and Newell (1980) implied in their paper that inverse scattering was necessary for the solution of the problem. It will be found that this is not the case and, indeed, using asymptotic expansions simplifies the details and clarifies the physical processes involved. The main part of the solution will be found to consist of a slowly varying soliton, which is first determined by energy conservation and then confirmed in a detailed perturbation analysis. This slowly varying soliton does not satisfy mass conservation and acts as a mass source which creates a tail behind it and the combined soliton and tail then satisfy energy and mass conservation within the Korteweg-de Vries framework. Grimshaw (1979) used the approach of asymptotic expansions and two-timing, but while agreeing with his slowly varying soliton, the present work differs with his formulation of the tail. Also, Grimshaw's approach to the problem seems to be more complicated than is necessary and obscures the basic ideas involved.

The asymptotic expansion for the near tail is found to become invalid at a finite time as all the terms in the expansion become of the same order. After this time, the near tail is found to steepen and move towards breaking. The near tail is prevented from breaking by the dispersion becoming important, which causes the near tail to break up into new solitons plus oscillations.

The behaviour of a Korteweg-de Vries soliton upon which a small viscosity acts will also be considered. It is again found that the solution consists of a slowly varying soliton, a near tail and a far tail. Karpman (1979) found the behaviour of the slowly varying soliton using inverse scattering and found the far tail, but did not discuss the behaviour of the near tail.

1.2 ENERGY AND MASS ARGUMENTS

A simple basic perturbation problem is

$$u_t + 6uu_x + u_{xxx} = \varepsilon u \qquad (1.1)$$

where $0<\varepsilon<<1$.

This equation has the energy conservation equation

$$d/dt\int_{-\infty}^{\infty}\tfrac{1}{2}u^2dx = \varepsilon\int_{-\infty}^{\infty}u^2dx \qquad (1.2)$$

on assuming that $u\to 0$ as $x\to\pm\infty$. The most obvious attempt at an asymptotic solution would be to propose the slowly varying soliton solution

$$u = 2\eta^2 \operatorname{sech}^2 \eta\theta \tag{1.3}$$

where

$$\begin{aligned} \theta &= x - \frac{\xi(T)}{\varepsilon} \\ T &= \varepsilon t \\ \xi'(T) &= 4\eta^2 \\ \eta &= \eta(T) \end{aligned} \tag{1.4}$$

Using this form of solution, we find from the energy equation (1.2) that

$$\eta_T = 2/3\eta \tag{1.5}$$

so that

$$\eta = \eta_0 e^{2/3T}, \tag{1.6}$$

where η_0 is the initial value of η.

The perturbed Korteweg-de Vries equation (1.1) also has a mass conservation equation

$$d/dt \int_{-\infty}^{\infty} u dx = \varepsilon \int_{-\infty}^{\infty} u dx \tag{1.7}$$

on again assuming that $u \to o$ as $x \to \pm\infty$. The slowly varying soliton solution given by (1.3) and (1.6) gives

$$d/dt \int_{-\infty}^{\infty} u dx = 8/3\varepsilon\eta$$

and

$$\varepsilon \int_{-\infty}^{\infty} u dx = 4\varepsilon\eta \tag{1.8}$$

We thus see that the slowly varying soliton conserves energy, but does not conserve mass. It is supposed, therefore, that there is some further tail behind the soliton to make the mass adjustments. We shall take up a detailed asymptotic analysis in the next section, but we can obtain an overall picture by the following more qualitative argument. Let us suppose that the soliton merges into this new region around some point x_0 behind it. Since mass balance is violated to terms of $0(\varepsilon)$, we expect that the region behind the soliton is of $0(\varepsilon)$. The perturbed Korteweg-de Vries equation (1.1) gives the exact mass conservation expression

$$d/dt \int_{x_0}^{\infty} u dx + u_{xx}|_{x_0} + u|_{x_0} \, dx_0/dt = \varepsilon \int_{x_0}^{\infty} u dx \qquad (1.9)$$

We shall now assume that u_{xx} is negligible around x_0, which can be verified later, and that x_0 moves with the soliton speed $4\eta^2$. Using the soliton expression (1.3) as an approximation for u in the integrals as in (1.8) then gives

$$4\eta_t + u|_{x_0} \, dx_0/dt = 4\varepsilon\eta$$

Therefore

$$u|_{x_0} \sim \frac{\varepsilon}{3\eta} = \frac{\varepsilon e^{-2/3T}}{3\eta_0} \qquad (1.10)$$

The soliton is acting as a mass source and this mass flux creates a region of $0(\varepsilon)$ behind the soliton. The expression for u in (1.10) acts as a boundary condition to generate a tail. The main purpose of this chapter is to analyse the development of this region.

The tail is called a "shelf" by Leibovich and Randall (1973), Ko and Kuehl (1978) and Kaup and Newell (1978) and a "plateau" by Karpman (1979).

1.3 FORMAL ASYMPTOTIC ANALYSIS

The results obtained in the previous section from energy and mass conservation arguments will now be verified by using a formal two-timing expansion for the solution to the perturbed Korteweg-de Vries equation (1.1). It is useful to show that both the conservation arguments and formal asymptotics yield the same results; while both methods are easily carried out for the Korteweg-de Vries equation, the conservation arguments require much less work for the Boussinesq equations of Chapter Two.

In the main region of the soliton we shall seek an asymptotic solution of the form

$$u = u_0(\theta,T) + \varepsilon u_1(\theta,T) + \varepsilon^2 u_2(\theta,T) + \ldots \qquad (1.11)$$

where

$$T = \varepsilon t$$
$$\theta = x - \xi(T)/\varepsilon$$
$$\xi_T = \omega_0(T) + \varepsilon^2\omega_2(T) + \ldots \qquad (1.12)$$

We can now proceed to substitute the expansion (1.11) into the perturbed Korteweg-de Vries equation (1.1) and by equating coefficients of powers of ε equal to zero, find differential equations for the u_n.

The solution for u_0 is just the slowly varying soliton

$$u_0 = 2\eta^2 \operatorname{sech}^2 \eta\theta$$
$$\omega_0 = 4\eta^2 \tag{1.13}$$

which can be easily verified from the zeroth order equation obtained by substituting the expansion (1.11) for u into the equation (1.1).

The equation for u_1 is

$$-4\eta^2 u_{1\theta} + 6u_0 u_{1\theta} + 6u_{0\theta} u_1 + u_{1\theta\theta\theta} = u_0 - u_{0T} \tag{1.14}$$

The complete details of the solution for u_1 will not be needed and all the required properties of u_1 can be obtained from the equation for u_1 and its adjoint. The adjoint to the homogeneous equation for u_1 is

$$-4\eta^2 w_\theta + 6u_0 w_\theta + w_{\theta\theta\theta} = 0 \tag{1.15}$$

Multiplying equation (1.14) for u_1 by w and adding this to the adjoint equation (1.15) multiplied by u_1 gives

$$[-4\eta^2 u_1 w + u_{1\theta\theta} w - u_{1\theta} w_\theta + u_1 w_{\theta\theta}]_{-\infty}^{\infty} = \int_{-\infty}^{\infty} (u_0 - u_{0T}) w d\theta \tag{1.16}$$

We require $u_1 \to 0$ as $\theta \to \infty$ as the soliton will have no effect on the region far ahead of it and we require u_1 to be bounded as $\theta \to -\infty$. The bounded solutions of the adjoint equation are u_0 and 1. Let us first consider the case when $w = u_0$.

When $w = u_0$, the left hand side of equation (1.16) is zero, so we require

$$\int_{-\infty}^{\infty} (u_0 - u_{0T}) u_0 \, d\theta = 0, \qquad (1.17)$$

which gives

$$\eta_T = 2/3\eta \qquad (1.18)$$

This is the same result as we obtained previously from energy conservation.

Let us now consider the case when $w = 1$. If $u_1 \to 0$ as $\theta \to -\infty$, then we would obtain the requirement

$$\int_{-\infty}^{\infty} (u_0 - u_{0T}) \, d\theta = 0, \qquad (1.19)$$

which results in a different expression for $\eta(T)$ than (1.18). We therefore see that u_1 does not tend to zero as $\theta \to -\infty$, but approaches the constant value given by

$$\begin{aligned} 4\eta^2 u_1 &= \int_{-\infty}^{\infty} (u_0 - u_{0T}) \, d\theta \\ &= 4\eta - 4\eta_T \\ &= 4/3\eta \end{aligned} \qquad (1.20)$$

We have now obtained a more precise derivation of the mass balance condition (1.10) found in Section 2. The fact that u_1 is non-zero as $\theta \to -\infty$ while $u_0 \to 0$ as $\theta \to -\infty$ shows that the asymptotic series (1.11) for u is not uniformly valid as $x \to -\infty$ and needs to be matched to an outer layer.

The mass balance result can also be obtained directly from the equation (1.14) for u_1. Integrating this equation once and choosing the "constant" of integration such that u_1 vanishes as $\theta\to\infty$ leads to the equation

$$-4\eta^2 u_1 + 6u_0 u_1 + u_{1\theta\theta} = -2/3\eta(1-\tanh\eta\theta) - 4/3\theta\eta^2 \mathrm{sech}^2\eta\theta \tag{1.21}$$

As $\theta\to-\infty$, $u_{1\theta\theta}\to 0$ and $u_0\to 0$, so that we again obtain

$$u_1 \sim \frac{1}{3\eta} \quad \text{as} \quad \theta\to-\infty \tag{1.22}$$

To the next order, the equation for u_2 is

$$-4\eta^2 u_{2\theta} + 6(u_0 u_2)_\theta + u_{2\theta\theta\theta} = -u_{1T} - 6u_1 u_{1\theta} + u_1 + \omega_2 u_{0\theta} \tag{1.23}$$

We can easily see from this equation that

$$u_2 \sim \frac{\theta}{36\eta^3} \quad \text{as } \theta\to-\infty \tag{1.24}$$

It is clear that in general, the solution for u_n will have a secular term of the form θ^{n-1} as $\theta\to-\infty$. We therefore see that the perturbation expansion (1.11) for u is not uniformly valid as $\theta\to-\infty$. We shall interpret the expansion (1.11) for u as an inner expansion and we shall match it to a suitable outer expansion. This outer expansion will be derived in the Sections below.

Johnson (1973b) in an extension of an earlier paper (Johnson (1973a)) used a two-timing expansion for u of the form (1.11), but due to algebraic errors, found that u was exponentially non-uniform as $\theta\to\infty$.

The expansion for u is also not uniformly valid as $\theta\to\infty$ as $|\varepsilon u_1| \nless\nless |u_0|$ as $\theta\to\infty$, even though u_1 decays exponentially as $\theta\to\infty$. A uniformly valid outer expansion for $\theta\to\infty$ has been derived by Grimshaw (1979) and the derivation of this expansion will not be repeated here. The region $\theta \to \infty$ is not of much importance and it suffices to say that the uniform outer expansion for $\theta \to \infty$ decays exponentially to zero.

1.4 THE NEAR-TAIL REGION

We shall now determine what form the outer solution must take. As the bounded term in u_1 as $\theta \to -\infty$ is $0(\varepsilon)$ and a function of T, we expect that the region behind the soliton will be of $0(\varepsilon)$ in height and a slowly varying function of X and T, where $X = \varepsilon x$. So we propose an expansion of the form

$$u = \varepsilon v_1(X,T) + \varepsilon^2 v_2(X,T) + \ldots \qquad (1.24)$$

for the near-tail region behind the soliton.

From the perturbed Korteweg-de Vries equation (1.1), we obtain the following equation for v_1 upon using the above expansion:

$$v_{1T} = v_1, \qquad (1.25)$$

so that

$$v_1 = A(X)e^T \qquad (1.26)$$

The function A(X) is determined by matching with the inner solution. This matching will differ from the usual matching in that it will be done with a moving inner solution.

Physically what is happening is the soliton is casting off waves. As these waves have amplitude $0(\varepsilon)$ and vary on the slow space and time scales, they have phase speed zero, so that the corresponding characteristics are parallel to the t axis. For fixed X, the expression (1.26) for v_1 gives the variation of v_1 along a characteristic. The function A(X) is determined by the value of u_1 at the point at which the characteristic starts. In figure 1, the characteristics for the near-tail have been sketched.

As the soliton speed is $4\eta^2$, we see from the equation (1.10) giving η as a function of T that the soliton position is given by

$$\varepsilon x_s = 3\eta_0{}^2(e^{4/3T} - 1), \qquad (1.27)$$

so that the soliton is at x_s when

$$T = 3/4 \log\,(1 + \frac{X_s}{3\eta_0{}^2}) \qquad (1.28)$$

Then for matching with the inner solution, we see from the behaviour of u_1 as $\theta\to-\infty$ given by (1.20) and the outer solution (1.26) for v_1 that we require

$$A(X)e^T = \frac{1}{3\eta} \quad \text{when } T = 3/4 \log\,(1 + X/3\eta^2) \qquad (1.29)$$

We thus find that

$$A(X) = \frac{1}{3\eta_0\,(1 + \frac{X}{3\eta_0{}^2})^{5/4}}\,, \qquad (1.30)$$

The slowly varying soliton and its near- and far-tails have been sketched in figure 2.

We notice that the region behind the soliton depends

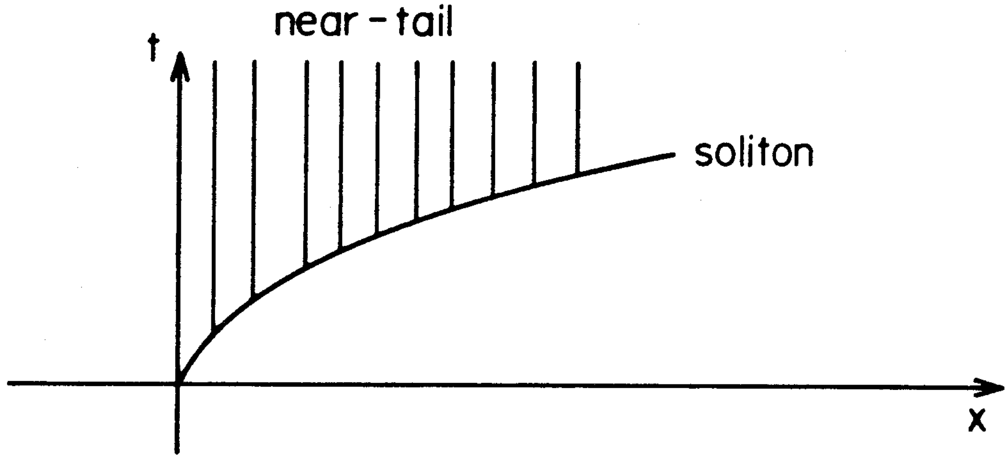

Figure 1: Boundary value problem for the near-tail

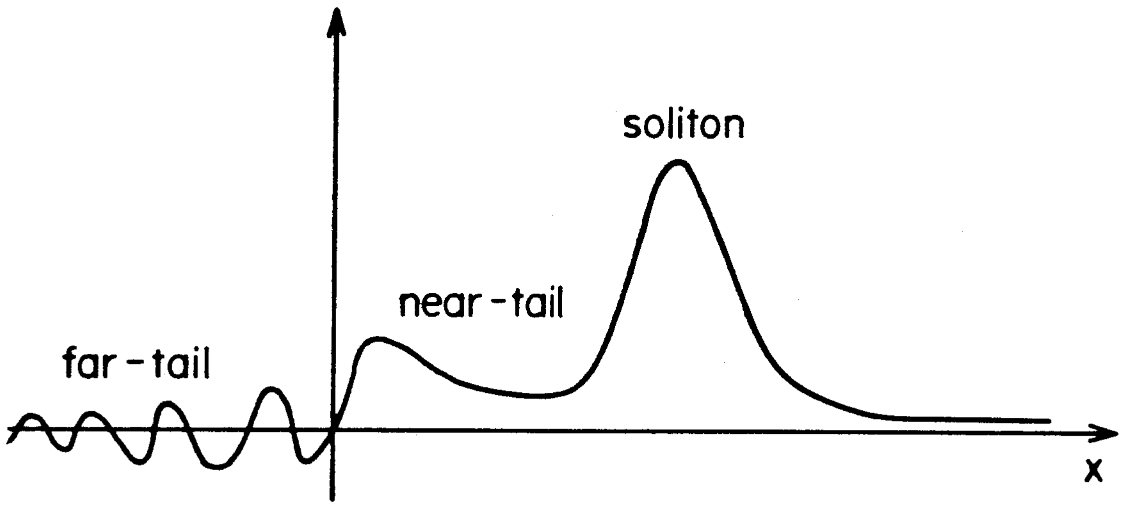

Figure 2: Sketch of the soliton and tails

on X and so is not a flat shelf as implied by Leibovich and Randall (1973), Ko and Kuehl (1978), Kaup and Newell (1978) and Karpman (1979). In the work of Grimshaw (1979) and Knickerbocker and Newell (1980), it was found that the near-tail must have a spatial dependence. Knickerbocker and Newell found that to first order, the behaviour of the near-tail is given by the linearised, non-dispersive form of the modified Korteweg-de Vries equation (1.25). Grimshaw tried to use a form of expansion which was valid for both the regions ahead of and behind the soliton, which tended to obscure the simplicity of the equations governing the behaviour of the near-tail. Grimshaw's method kept exponentially small terms for the near-tail, even though these terms are not the important ones describing the behaviour of the near-tail.

Kaup and Newell (1978), Karpman (1979) and Knickerbocker and Newell (1980) used perturbations on the inverse scattering solution of the Korteweg-de Vries equation to analyse the present problem. This method yields the slowly varying soliton of the previous section plus the secular behaviour of the inner solution, which is a non-decaying component of the continuous spectrum due to the interaction of the soliton with the perturbation. Karpman and Kaup and Newell proposed that the secular behaviour of u_1 formed a flat near-tail, while as mentioned above, Knickerbocker and Newell found that the near-tail must depend on X. The straight asymptotic approach used here yields all the results of the inverse scattering method, while having the advantage of being much simpler, especially if higher order terms are required.

Knickerbocker and Newell also obtained a numerical solution for $\Gamma(t) = \varepsilon$ and $\Gamma(t) = \frac{-\varepsilon}{\varepsilon t-1}$ and found good agreement with the analytical results for the slowly varying soliton and the near tail.

As the soliton moves, it forces up the near-tail, which then develops according to equation (1.25). The matching determines the initial height from which each point of the near-tail starts. We also note that the near-tail is growing exponentially in time, so the perturbation scheme outlined so far cannot remain valid as $t\to\infty$.

Using the outer expansion (1.24) for u, the $0(\varepsilon^2)$ equation which determines v_2 is

$$v_{2T} - v_2 = -6v_1 v_{1X}, \tag{1.31}$$

which has solution

$$v_2 = \frac{5e^{2T}}{18\eta_0^{\,4}(1 + \frac{X}{3\eta_0^{\,2}})^{7/2}} + D(X)e^T \tag{1.32}$$

where D(X) is determined from matching with the $0(\varepsilon^2)$ terms in the inner soliton solution.

We shall be interested in later sections in the term with the fastest growth in T in the solution for v_n. The form of this term will be found by induction. For the induction hypothesis, let us assume that the $v_j, j=1,\ldots,n-1$

grow like e^{jT}. The $O(\varepsilon^n)$ equation is, from the outer expansion (1.24) for u and the perturbed Korteweg-de Vries equation (1.1),

$$v_{nT} - v_n = e^T \, d/dT(v_n e^{-T})$$

$$= -6 \sum_{m=1}^{n-1} v_m (v_{n-m})_X + g_n(X,T) \qquad (1.33)$$

where g_n grows like $e^{(n-3)T}$.

By the induction hypothesis,

$$\sum_{m=1}^{n-1} v_m (v_{n-m})_X$$

has T dependence of the form e^{nT} and so we see that the fastest growing term in v_n has T dependence of the form e^{nT}. Therefore as v_1 grows like e^T, by induction, we have that v_n grows like e^{nT}.

Furthermore, we can also see from the solution for v_1 given by (1.26) and (1.30), the solution (1.32) for v_2 and the general differential equation (1.33) for v_n that the X dependence of the fastest growing term in v_n is of the form

$$\frac{Ae^{nT}}{(1 + \frac{X}{3\eta_0^{\,2}})^p}$$

where A is a constant and $p>1$. The fastest growing term in v_n is thus a monotonic decreasing function of X.

Since the soliton started at $x = 0$, we expect that the near-tail will extend from $x = 0$ to $x = x_s$, the soliton position. It can be easily verified that the mass conservation equation (1.7) is satisfied to first order by the combination of the soliton plus the near-tail extending from $x = 0$ to $x = x_s$. This is to be expected from the straight mass conservation argument of Section 1.3. We have assumed that the region $x<0$ makes no significant contribution to the total mass. That this is so will be shown in the next section.

We have now considered the ideas necessary for the study of the Boussinesq equations in Chapter 2. In the remainder of this chapter, the further development of the near-tail into new solitons and other perturbed Korteweg-de Vries equations to which our present methods can be applied will be dealt with.

1.5 THE FAR-TAIL REGION

We shall now deal with the region $x<0$. In this region, we expect u to be essentially the solution of the linearised perturbed Korteweg-de Vries equation with dependence on x, t and T. Dependence on the slow space variable X is not expected as the near-tail plus soliton and the far-tail are separate entities. Including any slow space variation in the far-tail results in secular terms. So for $x<0$, we assume the expansion

$$u = \varepsilon U_1(x,t,T) + \varepsilon^2 U_2(x,t,T) + \ldots \qquad (1.34)$$

The perturbed Korteweg-de Vries equation (1.1) then yields the $O(\varepsilon)$ equation

$$U_{1t} + U_{1xxx} = 0, \tag{1.35}$$

which has the similarity solution

$$U_1 = B(T) \int_{-\infty}^{\frac{x}{(3t)^{1/3}}} Ai(s)ds \tag{1.36}$$

where $B(T)$ is a constant of integration to be determined by matching with the near-tail.

From the solution for v_1 given by (1.26) and (1.30), we see that matching gives

$$B(T) = \frac{e^T}{3\eta_0}$$

Our assumption in the previous section that the far-tail makes a negligible contribution to the mass balance equation is seen to be valid due to the oscillations in the far-tail.

The solution for the far-tail given by (1.36) and (1.37) has also been obtained using inverse scattering by Knickerbocker and Newell (1980).

Karpman (1979) obtained an expression for the far-tail which differs from that above in the slow time dependence. This difference is due to the far-tail being calculated on the assumption that the near-tail is flat and is just the secular term in the soliton expansion. This leads to his slow time dependence being of the form $e^{-2/3T}$ rather than e^T as the secular term in u_1 has time dependence of the form $e^{-2/3T}$ rather than e^{T}. In effect, he matched the far-tail directly onto the soliton rather than the near-tail.

Johnson (1973b) noted that the slowly varying soliton expansion is non-uniform and deduced that there must be a region $0(\varepsilon)$ in height behind the soliton to account for the associated mass flux from the soliton. He then introduced an expansion which was assumed to hold in the entire region behind the soliton. Johnson found that his expansion gave the same asymptotic behaviour as $x \to -\infty$ as the Airy function solution (1.36) for the far-tail, but didn't note that his equations did in fact give this Airy function solution.

Grimshaw (1979) also tried to describe both the near- and far-tails by the same perturbation expansion. These two regions have different roles as the near-tail is due to a mass flux from the soliton and the far-tail is due to the initial conditions. It would then appear that the near- and far-tails should be described by separate expansions. Furthermore, using the same straightforward expansion for both regions gives rise to secular terms at second order, as while the Airy function Ai(x) decays as $x \to -\infty$, its derivative grows algebraically as $x \to -\infty$. It is not clear how these secular terms can be eliminated. These secular terms were not noted by Grimshaw as he did not calculate quantities to second order.

1.6 LONG TERM BEHAVIOUR

It has been noted that the near-tail grows with T and the perturbation scheme for the near-tail will cease to be valid after a certain time. We shall now deal with the long term behaviour of the near-tail.

In Section 1.4, it was shown that the near-tail is given by the expansion (1.24) for u where the fastest growing term in v_n has the form

$$\frac{Ae^{nT}}{(1 + \frac{X}{3\eta_0^{\ 2}})^p} \tag{1.38}$$

where A is a constant and p>1. We therefore see that when

$$t = 0(- \frac{\log \varepsilon}{\varepsilon}) \ , \tag{1.39}$$

all of the terms in the perturbation expansion (1.24) for the near-tail become of the same order and the perturbation scheme breaks down. When t is of this order,

$$0(\varepsilon v_1) = 0(\varepsilon v_2) = \ldots = 1$$

This implies that we should try a new expansion

$$u = w_0(X,T) + \varepsilon w_1(X,T) + \ldots \tag{1.40}$$

for the near-tail after the time is of order given by (1.39). The perturbed Korteweg-de Vries equation (1.1) gives the 0(1) equation

$$w_{0T} + 6w_0 w_{0X} = 0, \tag{1.41}$$

which has the solution

$$w_0 = f(\xi) \quad \text{on} \quad X = 6f(\xi)(T - T_0) + \xi \tag{1.42}$$

where

$$T_0 = -\log \varepsilon \tag{1.43}$$

If we use the values for v_1 and v_2 given by (1.26), (1.30) and (1.32) at $T = T_0$ as the initial condition, we have

$$f(\xi) = \frac{1}{3\eta_0 (1 + \frac{\xi}{3\eta_0{}^2})^{5/4}} + \frac{5}{18\eta_0{}^4 (1 + \frac{\xi}{3\eta_0{}^2})^{7/2}} + \ldots \tag{1.44}$$

$$\xi = X - 6f(\xi)(T - T_0)$$

as the (implicit) solution for w_0.

The solution for w_0 will in general break and the time for breaking is given by

$$T_c = T_0 + \min_{\xi} \frac{1}{|6f'(\xi)|} \tag{1.45}$$

if $f'(\xi) < 0$ for some ξ. Now in Section 1.4, it was shown that the fastest growing term in v_n is monotonic decreasing in X. We therefore have that $f'(\xi) < 0$ and

$$\min_{\xi} |f'(\xi)| = |f'(0)| = \frac{5}{36\eta_0{}^3} + \frac{35}{108\eta_0{}^7} + \ldots \tag{1.46}$$

The time for breaking is then given by

$$T_c = T_0 + \frac{6\eta_0{}^3}{5 + \frac{35}{3\eta_0{}^4} + \ldots} \tag{1.47}$$

Of course, the near-tail cannot break as as the near-tail steepens, the dispersion term u_{xxx} in the perturbed Korteweg-de Vries equation will eventually become as important as the other terms in the description of the near-tail.

When the near-tail expansion breaks down, we see from the expression (1.27) for the soliton position that the near-tail has area $0(\varepsilon^{-7/3})$, as at this stage, the near-tail is of 0(1) in height. The near-tail takes a time of $0(\varepsilon^{-1})$ to break and in that time, the soliton produces a mass deficit of $0(\varepsilon^{-2/3})$. So we see that while the near-tail is breaking, we can ignore the mass being added to the near-tail by the soliton.

The solution of the Korteweg-de Vries equation with a small dispersion term and a general initial profile has been dealt with by Gurevich and Pitaevshii (1974) using the modulation theory developed by Whitham (Whitham, Chapter 16) for the Korteweg-de Vries equation. They matched a similiarity solution of the form $\tilde{t}^{1/2} \underset{\sim}{1}(\frac{\tilde{x}}{\tilde{t}^{3/2}})$ for the modulation equations in the region of the breaking front to the profile away from the breaking front obtained from the equation

$$u_t + 6uu_x = 0, \qquad (1.48)$$

where $\tilde{t}$ is a normalized time measured from the time of breaking and $\tilde{x}$ is a normalized coordinate measured from the breaking front.

Applying their results to our case, we find that as the near-tail begins to break, the front of the near-tail begins to break up into oscillations which grow in amplitude like $(T - T_c)^{1/2}$ and spread like $(T - T_c)^{3/2}$. The modulus of the cnoidal wavetrain formed in this way varies from one at the front to zero at the trailing edge of the wavetrain. So at the front of the breaking near-tail, new solitons are formed with oscillations forming behind them. Unfortunately, the details of this break-up have to be found numerically.

Gurevich and Pitaevskii's results apply only when the region of oscillation is small compared with the near-tail. We expect that a new near-tail will form behind the new solitons. Of course, the leading (and original) soliton is continually increasing in height and decreasing in width, so that physically, it must topple at some time, which leads to the formation of a turbulent bore.

The perturbation scheme for the leading soliton remains valid even while the near-tail is breaking up into new solitons, so we expect it not to undergo any change due to the breaking near-tail. When the near-tail is breaking, $u_1 = 0(\varepsilon^{2/3})$ as $\theta \to -\infty$, so the soliton has in fact nearly detached itself from the near-tail.

1.7 OTHER EQUATIONS

We shall now consider two other perturbed Korteweg-de Vries equations which can be analysed with the methods of the previous sections.

1.71 KORTEWEG-DE VRIES EQUATION WITH DAMPING

First, we shall consider the equation

$$u_t + 6uu_x + u_{xxx} = -\varepsilon u \qquad (1.49)$$

The solution to this equation can be basically obtained by changing the sign of ε in the previous sections. The solution again consists of three regions; a slowly varying soliton, a near-tail and a far-tail.

For the slowly varying soliton, we use the expansion

$$u = u_0(\theta,T) + \varepsilon u_1(\theta,T) + \varepsilon^2 u_2(\theta,T) + \ldots \qquad (1.50)$$

where

$$\theta = x - \frac{\xi(T)}{\varepsilon} \qquad (1.51)$$

$$\xi'(T) = \omega_0(T) + \varepsilon^2\omega_2(T) + \ldots$$

As before, we can find by using either the energy and mass conservation equations

$$d/dT \int_{-\infty}^{\infty} 1/2\, u^2 dx = -\int_{-\infty}^{\infty} u^2 dx \qquad (1.52)$$

$$d/dt \int_{x_0}^{\infty} u dx + u\Big|_{x_0} \frac{dx_0}{dt} = -\varepsilon \int_{x_0}^{\infty} u dx$$

or by substituting the expansion (1.50) for u into the perturbed Korteweg-de Vries equation (1.49) that

$$u_0 = 2\eta^2 \operatorname{sech}^2 \eta\theta$$

$$\eta = \eta_0 e^{-2/3T}$$

$$\varepsilon x_s(t) = 3\eta_0{}^2(1 - e^{-4/3T}) \tag{1.53}$$

$$u_1 \sim \frac{-1}{3\eta} \quad \text{as} \quad \theta \to -\infty$$

$$u_2 \sim \frac{-5\theta}{36\eta^3} \quad \text{as} \quad \theta \to -\infty$$

In this case, the amplitude of the soliton is decreasing.

We again find a mass deficit for the slowly varying soliton (which is determined by conservation of energy, as before), which we correct by adding a near-tail behind the soliton. In this case, the near-tail is a dip behind the soliton, which deepens with increasing time.

The near-tail expansion is

$$u = \varepsilon v_1(X,T) + \varepsilon^2 v_2(X,T) + \ldots, \tag{1.54}$$

which is valid for $0<x<x_s(t)$.

We find by substituting the expansion (1.54) into the perturbed Korteweg-de Vries equation (1.49) and matching with the soliton expansion that

$$v_1 = \frac{-e^{-T}}{3\eta_0\left(1 - \frac{X}{3\eta_0{}^2}\right)^{5/4}}, \tag{1.55}$$

Behind the near-tail, we again have a far-tail in the region $x<0$, given by

$$u = \varepsilon U_1(x,t,T) + \varepsilon^2 U_2(x,t,T) + \ldots, \tag{1.56}$$

where we find that

$$U_1 = \frac{-e^{-T}}{3\eta_0}\int_{-\infty}^{\frac{x}{(3t)^{1/3}}} Ai(s)\,ds \qquad (1.57)$$

This expression again differs in the slow time variation from that of Karpman (1979), as he assumed that the near-tail was flat.

The slowly varying soliton forces down a dip behind the soliton, which then decreases in depth. As the soliton forces down the dip faster than it decays, the expansions will break down when the near-tail is of the same order as the height of the soliton. This occurs when $t = 0(-\frac{\log \varepsilon}{2\varepsilon})$. When t is of this order, the width of the soliton is $0(\varepsilon^{-1/3})$ and the height of the soliton is $0(\varepsilon^{2/3})$, so we then have the new expansion

$$u = \varepsilon^{2/3} w_1(\bar{X},T) + \ldots \qquad (1.58)$$

for the soliton and near-tail, where

$$\bar{X} = \varepsilon^{1/3} x \qquad (1.59)$$

The equation for w_1 is then

$$w_{1T} + 6w_1 w_{1\bar{X}} + w_{1\bar{X}\bar{X}\bar{X}} = -w_1 \qquad (1.60)$$

This equation cannot be solved, so we cannot determine the long term behaviour of the soliton and near-tail. We expect that the soliton and near-tail will "rapidly" (on the time scale T) decay to zero as, in a sense, the damping in equation (1.60) for w_1 is of order one. This expectation is borne out by the numerical results of Knickerbocker and Newell (1980).

1.72 KORTEWEG-DE VRIES EQUATION WITH DIFFUSION

Finally, we shall consider the perturbed Korteweg-de Vries equation

$$u_t + 6uu_x + u_{xxx} = \varepsilon u_{xx} \qquad (1.61)$$

This equation describes a soliton upon which a small damping or heat conduction ε, $0<\varepsilon<<1$, acts (see, for example, Karpman (1975)). Karpman (1979) discussed this equation using inverse scattering. We again find that we need a slowly varying soliton, a near-tail and a far-tail for a complete description of the problem.

For the slowly varying soliton, we propose that

$$u = u_0(\theta,T) + \varepsilon u_1(\theta,T) +\ldots \qquad (1.62)$$

where

$$\theta = x - \frac{\xi(T)}{\varepsilon} \qquad (1.63)$$

$$\xi'(T) = \omega_0(T) + \varepsilon^2\omega_2(T) +\ldots$$

The energy and mass conservation equations for the present equation are

$$d/dT\int_{-\infty}^{\infty} 1/2u^2dx = -\int_{-\infty}^{\infty} u_x^2\, dx \qquad (1.64)$$

$$d/dt\int_{-\infty}^{\infty} udx = 0 \qquad (1.65)$$

respectively. The slowly varying soliton satisfies energy conservation, but again does not satisfy mass conservation Using either the energy and mass conversation arguments or by substituting the expansion (1.62) into the equation (1.61) and eliminating secular terms, we find that

$$\begin{aligned} u_0 &= 2\eta^2 \operatorname{sech}^2 \eta\theta \\ \omega_0 &= 4\eta^2 \\ \eta &= \eta_0 (1 + 16/15\eta_0^2 T)^{-1/2} \\ \varepsilon x_s &= 15/4 \log(1 + 16/15\eta_0^2 T) \\ u_1 &\sim 8\eta/15 \quad \text{as } \theta \to -\infty \end{aligned} \qquad (1.66)$$

As expected, the amplitude of the soliton decreases with time.

The near-tail region cannot be flat as stated by Karpman (1979) and we find that we must use the expansion

$$u = \varepsilon v_1(X) + \varepsilon^2 v_2(X) + \ldots \qquad (1.67)$$

for the near-tail. The near-tail occurs in the region $0<x<x_s(t)$. It is interesting to note that the near-tail is independent of time in this case. This is due to the mass conservation equation having no dissipation term. As before, we find that

$$v_1 = 8/15\eta_0 e^{-2X/15} \qquad (1.68)$$

The far-tail, which occurs in the region $x<0$, is given by

$$u = \varepsilon U_1(x,t) + \varepsilon^2 U_2(x,t) + \ldots \qquad (1.69)$$

and we find that

$$U_1 = \frac{8\eta_0}{15}\int_{-\infty}^{\frac{x}{(3t)^{1/3}}} Ai(s)ds \qquad (1.70)$$

The slow time dependence of this solution for the far-tail differs from that of Karpman (1979). This is again due to his taking the near-tail as being flat.

The near-tail is a positive elevation behind the soliton which decreases in height as X increases. As the soliton is decreasing in height, the expansion above will break down when the soliton amplitude is of the same order as the near-tail. This occurs when $t = 0(\varepsilon^{-3})$ and for times greater than this, we find that we must introduce the new expansion

$$u = \varepsilon^2 w_1(\bar{T},X) + \ldots \qquad (1.71)$$

for the soliton and the near-tail, where

$$\bar{T} = \varepsilon^3 t \qquad (1.72)$$

The equation for w_1 is then

$$w_{1\bar{T}} + 6w_1 w_{1X} + w_{1XXX} = w_{1XX} \qquad (1.73)$$

We again expect that the solution of this equation for w_1 will "rapidly" (on the time scale $\bar{T}$) decay to zero as the damping for w_1 is now of order one.

It is interesting to note that the long term behaviour of the soliton for the perturbed Korteweg-de Vries equations where the soliton amplitude decays is similar and very different from that where the soliton amplitude grows. This is because for the soliton going up a beach, new solitons are being formed, whereas in the other two cases, the soliton is decaying to zero.

CHAPTER TWO

SOLITON ON A BEACH

2.1 INTRODUCTION

The behaviour of a soliton on a beach (or in a channel) cannot be fully described by the perturbed Korteweg-de Vries equation of the previous chapter. The perturbed Korteweg-de Vries equation applies for motion in one direction only and so cannot answer the important question of whether or not the shoaling of the beach produces a reflected wave. This question is also crucially affected by higher order terms in the Boussinesq equations which are incorrectly neglected in the Korteweg-de Vries approximation, this approximation being similar to a wavefront expansion, so that it is only valid in the region of the soliton. The Boussinesq equations also have the advantage of being in a fixed coordinate system with the (x,t) coordinates of the equations being the physical space and time coordinates. This makes physical interpretation of the results easier. The main problem of this chapter is the motion of a soliton up a beach, but variations in the breadth can also be included with no extra complication, so that the behaviour of a soliton in a channel will be considered as well.

In the present chapter, we shall find an asymptotic solution of the Boussinesq equations for a soliton going up a slowly varying beach (or channel). It would be expected that this solution would bear similarities to the asymptotic solution of the perturbed Korteweg-de Vries equation and indeed it is found that the solution consists of two regions; a slowly varying soliton and a tail created by a mass flux from the soliton.

Grimshaw (1970, 1971) derived the Boussinesq equations for varying depth from the Euler equations and found a two-timing expansion for the slowly varying soliton. As for the perturbed Korteweg-de Vries equation, it was found that the slowly varying soliton is determined by energy conservation with the expansion being non-uniform behind the soliton. This non-uniformity is associated with the fact that the soliton does not conserve mass. Miles (1979) suggested that this mass problem could be solved by adding a forward and reflected wave behind the soliton. These waves were then calculated on the assumption that they are given by Green's law for linear shallow water waves. This is an interesting general point. Miles suggested that in all such situations, a reflected wave is necessary to account for the mass deficit. He also added that this is so in linear theory, in which case there are strong arguments to show that any reflected wave is exponentially small in the slowly varying parameter and could not carry any appreciable mass. It is an interesting general question of whether this is still true for nonlinear waves. One of the main aims of the present investigation was to resolve this question. It was found that the mass deficit is accounted for in forward moving waves in the tail and no reflected wave is generated (this refers to approximations taken as far as second order, but Miles referred to this same level of approximation).

It will be found by a suitable rescaling of the Boussinesq equations that the tail is governed by the linear shallow water equations. The tail is then the solution to these equations with the moving boundary conditions determined by the mass flux from the soliton. As the solution to this boundary value problem could not be found for arbitrary soliton amplitude, an asymptotic solution to second order in amplitude for small soliton amplitude was found.

2.2 BOUSSINESQ EQUATIONS

We shall consider a soliton moving in a channel whose depth h_0 and breadth b are slowly varying functions of the distance x along the channel. We shall denote by ε the length scale of the variation, so that $0<\varepsilon<<1$. The surface displacement will be denoted by η, so that

$$y = h_0 + \eta = h \tag{2.1}$$

is the equation of the free surface.

Whitham (1967) has shown that the appropriate Lagrangian for obtaining Boussinesq type equations for shallow water waves for constant breadth is

$$L_1 = h(F_t + 1/2F_x^2) + 1/2g\eta^2 - 1/6h^3F_{xx}^2 , \tag{2.2}$$

where F is the velocity potential. It is clear that the Lagrangian for motion when breadth variations are included is

$$L = bh(F_t + 1/2F_x^2) + 1/2gb\eta^2 - 1/6bh^3F_{xx}^2 \tag{2.3}$$

The variational equations from the Lagrangian are:

$$\delta F: \quad (bh)_t + (bhF_x)_x + (1/3bh^3F_{xx})_{xx} = 0 \tag{2.4}$$

$$\delta\eta: \quad F_t + 1/2F_x^2 + g\eta - 1/2bh^2F_{xx}^2 = 0 \tag{2.5}$$

To obtain equations correct to second order in amplitude, it is sufficient to linearise the dispersive terms in the above equations as both the amplitude and dispersion are assumed to be small and of the same order in the derivation of the Boussinesq equations. It is also sufficient to neglect the dispersive term in the second equation (2.5) as the amplitude is assumed to be small. The equations correct to second order in amplitude are then

$$(bh)_t + (bhF_x)_x + (b\nu F_{xx})_{xx} = 0 \tag{2.6}$$

$$F_t + 1/2F_x^2 + g\eta = 0 \tag{2.7}$$

where we have set

$$\nu = 1/3h_0^3 \tag{2.8}$$

The corresponding Lagrangian for these equations is

$$L = bh(F_t + 1/2F_x^2) + 1/2gb\eta^2 - 1/2\nu bF_{xx}^2 \tag{2.9}$$

If we denote the fluid velocity by u, an alternative form for the equations is

$$(bh)_t + (bhu)_x + (b\nu u_x)_{xx} = 0 \qquad (2.10)$$

$$u_t + uu_x + g\eta_x = 0 \qquad (2.11)$$

We shall refer to these equations as the Boussinesq equations In the course of the analysis, it will be seen how the Korteweg-de Vries approximation of Chapter One arises and what its limitations are. These equations for constant breadth are the same equations as obtained by Grimshaw (1970) directly from the Euler equations when the breadth is constant. In a subsequent paper (Grimshaw (1971)), Grimshaw systematically continued the approximation for constant breadth to higher orders in amplitude and considered three dimensional motion. He then calculated quantities such as the soliton speed to third order in amplitude. Peregrine (1967) also obtained an equivalent set of equations for varying depth only from the Euler equations.

2.3 THE SOLITON SOLUTION

We shall now find the soliton solution of the Boussinesq equations (2.10) and (2.11) for constant breadth and depth. We thus look for a travelling wave solution

$$\eta = \eta(\theta) \qquad (2.12)$$

$$u = u(\theta),$$

where

$$\theta = \frac{x}{V} - t, \tag{2.13}$$

V being the soliton's speed. Using these forms for the solution, the Boussinesq equations give, upon integrating once,

$$\eta = \frac{Vu}{g} - \frac{u^2}{2g} \tag{2.14}$$

$$-\eta + \frac{1}{V}(hu) + \frac{\nu}{V^3} u_{\theta\theta} = 0, \tag{2.15}$$

as for a soliton, both u and η approach zero as $x \to \pm\infty$. We can now obtain an equation for u upon eliminating η, integrating once and noting that $u \to 0$ as $x \to \pm\infty$. This equation is

$$\frac{g\nu}{V^4} u_\theta^2 = (1 - \frac{c^2}{V^2})u^2 - \frac{u^3}{V} + \frac{u^4}{4V^2} , \tag{2.16}$$

where, for convenience, we have denoted by c the linear wave speed

$$c = \sqrt{gh_0} \tag{2.17}$$

We can non-dimensionalize this equation by setting

$$u = 2Vw, \tag{2.18}$$

so that we obtain the simple form

$$\frac{\nu}{V^4} w_\theta{}^2 = w^2 (w_1 - w)(w_2 - w) \tag{2.19}$$

$$w_1 = 1 - \frac{c}{V}$$

$$w_2 = 1 + \frac{c}{V} \tag{2.20}$$

This equation has the soliton solution

$$w = \frac{1 - \frac{c}{V}}{(1 - d)\cosh^2 p\theta + d} \quad , \tag{2.21}$$

so that

$$u = \frac{A}{(1 - d)\cosh^2 p\theta + d} = \frac{A \operatorname{sech}^2 p\theta}{1 - d \tanh^2 p\theta} \tag{2.22}$$

Where

$$A = 2(V - c)$$

$$d = \frac{V - c}{V + c} \tag{2.23}$$

$$p = 1/2(V^2 - c^2)^{1/2} (g\nu)^{-1/2} V$$

The velocity V will be used as the main parameter describing the soliton.

The amplitude of the soliton is A and for small amplitude, the profile differs from the usual $\operatorname{sech}^2 p\theta$ profile by a term of $0(A^2)$. This difference is due to the u^4 term in the differential equation (2.16), this term not appearing in the standard differential equation for the Boussinesq soliton. If we had expressed u in terms of the velocity potential F, so that $u = F_x$, eliminated between the Boussinesq equations (2.10) and (2.11) to obtain a single equation for F and ignored terms of $0(F^3)$, then the resulting equation has the standard Boussinesq soliton solution. It is the ignored $0(F^3)$ terms which give rise to the different profile at $0(A^2)$ as we are retaining them by using our Boussinesq system.

2.4 CONSERVATION EQUATIONS

We shall now turn to the case where both the depth and the breadth are slowly varying functions of x. As for the slowly varying Korteweg-de Vries soliton of the previous chapter, we shall use conservation arguments to determine the behaviour of a slowly varying Boussinesq soliton. For a slowly varying soliton, the phase function is now

$$\theta = \frac{\xi(X)}{\varepsilon} - t \tag{2.24}$$

Where

$$X = \varepsilon x \tag{2.25}$$
$$\xi'(X) = \frac{1}{V}$$

Noether's Theorem (Gelfand and Fomin, p. 177) gives that the energy conservation equation is

$$\frac{\partial}{\partial t}(L - F_t L_{F_t}) + \frac{\partial}{\partial x}(-F_t L_{F_x} + F_t (L_{F_{xx}})_x - F_{xt} L_{F_{xx}}) = 0 \tag{2.26}$$

From (2.9), we then have that the energy conservation equation for the Boussinesq equations is

$$\frac{\partial}{\partial t}(1/2bg\eta^2 + 1/2bhu^2 - 1/2b\nu u_x) + \frac{\partial}{\partial x}((g\eta + 1/2u^2)bhu$$
$$+ (g\eta + 1/2u^2)(\nu b u_x)_x - (g\eta + 1/2u^2)_x b\nu u_x) = 0 \tag{2.27}$$

In analogy with the slowly varying Korteweg-de Vries soliton of the previous chapter, we expect that the slowly varying Boussinesq soliton will be determined by energy conservation. The energy conservation equation then gives that

$$E = \int_{-\infty}^{\infty}[(g\eta + 1/2u^2)bhu + \frac{1}{V}(g\eta + 1/2u^2)(\frac{\nu b}{V}u_\theta)_\theta - \frac{1}{V^2}(g\eta + 1/2u^2)_\theta b\nu u_\theta]d\theta \ , \tag{2.28}$$

where E is a constant determined by the initial conditions. We now use the Boussinesq soliton found above and allow its parameters to be slowly varying functions of x. Using the relations (2.14) and (2.15) between u and η, we obtain

$$E = gbV^2\int_{-\infty}^{\infty}\eta d\theta - \frac{c^2v^2b}{g}\int_{-\infty}^{\infty}ud\theta - \frac{\nu b}{V}\int_{-\infty}^{\infty}u_\theta^2 d\theta \tag{2.29}$$

The integrals in this energy conservation relation are calculated using the first order slowly varying soliton solution for η and u. This gives

$$\int_{-\infty}^{\infty}\eta d\theta = 4\frac{\sqrt{\nu}}{\sqrt{g}}\sqrt{1 - \frac{c^2}{V^2}} \tag{2.30}$$

$$\int_{-\infty}^{\infty} u\, d\theta = 8V^{-1}\sqrt{g\nu}\ \log[\frac{1}{\sqrt{2c}}(\sqrt{V + c} + \sqrt{V - c})] \tag{2.31}$$

$$\int_{-\infty}^{\infty}u_\theta^2 d\theta = 4/3V^4(g\nu)^{-1/2}(1 + \frac{c}{V})^{-1}(1- \frac{c}{V})^{3/2}(1 + \frac{2c^2}{V^2}) - 8c^2V^2(g\nu)^{-1/2}\log[\frac{1}{\sqrt{2c}}(\sqrt{V + c} + \sqrt{V - c})] \tag{2.32}$$

The energy conservation integral (2.29) then gives the extremely simple result

$$E = 8/3\ V^3b\frac{\sqrt{\nu}}{\sqrt{g}}(1 - \frac{c^2}{V^2})^{3/2} \ , \tag{2.33}$$

from which we can see that the soliton speed is

$$V^2 = c^2 + E'b^{-2/3}h_0^{-1} \tag{2.34}$$

For convenience, we have set

$$E' = \left(\frac{3\sqrt{3g}E}{8}\right)^{2/3} \tag{2.35}$$

The amplitude of the velocity profile can now be found from (2.23) and it is

$$A = 2\left(\sqrt{c^2 + E'b^{-2/3}h_0^{-1}} - c\right) , \tag{2.36}$$

which for small amplitude is

$$A = \frac{E'}{\sqrt{g}b^{2/3}h_0^{3/2}} \tag{2.37}$$

This corresponds to an amplitude a in η of

$$a = \frac{E'}{gb^{2/3}h_0} \tag{2.38}$$

This is the classical result for the amplitude variation of a soliton in a channel whose depth and breadth are slowly varying. It has been found for varying depth only by Boussinesq (1872) and Grimshaw (1970, 1971). Boussinesq used the energy conservation argument outlined here and Grimshaw used both energy conservation and a formal perturbation expansion, from which he obtained the result above by a Fredholm condition. This amplitude relation for both varying depth and breadth has been found by Saeki, Takagi and

Ozaki (1971). It has also been found from the perturbed Korteweg-de Vries equation by Ostrovsky (1976) and Miles (1979) for both varying depth and breadth and for varying depth only by Johnson (1973b).

We shall now consider the central topic of this investigation, this being the nature of the region behind the soliton. The tail is caused by a mass and momentum flux from the soliton and the boundary conditions for the tail region are most easily determined by considering the mass and momentum conservation equations for the Boussinesq equations. Consideration of these equations also raises the question of mass and momentum balance and shows how these quantities are conserved by the soliton plus tail, without any reflected wave being necessary.

Two-timing the Boussinesq equations (2.10) and (2.11) gives the equations

$$-b\eta_\theta + \frac{1}{V}(bhu)_\theta + \varepsilon(bhu)_X + \left(\frac{\partial}{\partial\theta} + \varepsilon\frac{\partial}{\partial X}\right)^2(\nu bu_\theta + \varepsilon\nu bu_X) = 0 \tag{2.39}$$

$$-u_\theta + \frac{1}{V}uu_\theta + \varepsilon uu_X + \frac{1}{V}g\eta_\theta + \varepsilon g\eta_X = 0 \tag{2.40}$$

Two conservation equations, one corresponding to mass and the other related to momentum, are now obtained by integrating these equations with respect to θ from $-\infty$ to ∞ and assuming that u and η approach zero as $\theta\to\infty$ and $0(\varepsilon)$ "constants" (i.e., slowly varying functions of x) as $\theta\to-\infty$. Doing this, we obtain

$$\eta - \frac{c^2}{gV}u = -\frac{\varepsilon}{b}\frac{d}{dX}\int_{-\infty}^{\infty} bhud\theta \tag{2.41}$$

$$u - \frac{g}{V}\eta = -\varepsilon\frac{d}{dX}\int_{-\infty}^{\infty}(g\eta + 1/2u^2)d\theta \tag{2.42}$$

at $\theta = -\infty$.

The first equation can be simplified upon using the differential equation (2.15) relating η and u and the second equation can be simplified upon using the relation (2.14) between u and η. The final conservation relations are then

$$\eta - \frac{c^2}{gV}u = -\frac{\varepsilon}{b}\frac{d}{dX}[Vb\int_{-\infty}^{\infty}\eta d\theta] \tag{2.43}$$

$$u - \frac{g}{V}\eta = -\varepsilon\frac{d}{dX}[V\int_{-\infty}^{\infty}ud\theta] \tag{2.44}$$

at $\theta = -\infty$. The values of u and η as $\theta \to -\infty$ are obtained by substituting the soliton solutions given by (2.14) and (2.22) into the right hand side of these expressions. These values are interpreted as values immediately behind the soliton and must be matched to a tail soliton.

We again have that the soliton is acting as a mass and momentum source and this mass and momentum flux will create a tail behind the soliton. The structure of the tail will be examined in detail after we first make a few comments about a formal two-timing expansion for the slowly varying soliton. The Korteweg-de Vries approximation will also be examined and it will be shown that it cannot adequately describe the behaviour of a soliton on a slowly varying beach.

2.5 FORMAL ASYMPTOTIC EXPANSION AND THE KORTEWEG-DE VRIES APPROXIMATION

The details of the slowly varying soliton found in the previous section by energy, mass and momentum conservation can also be found from a formal two-timing expansion. The conservation relations are then Fredholm conditions. The formal two-timing expansion is

$$u = u_0(\theta,X) + \varepsilon u_1(\theta,X) + \ldots \tag{2.45}$$

$$\eta = \eta_0(\theta,X) + \varepsilon \eta_1(\theta,X) + \ldots, \tag{2.46}$$

where

$$\theta = \frac{\xi(X)}{\varepsilon} - t$$

$$\xi'(X) = \frac{1}{V} + \varepsilon^2 k_2 + \ldots \tag{2.47}$$

The zeroth order equations will have the soliton solution for u_0 and η_0 given by (2.14) and (2.22). From the first order equations for u_1 and η_1, the energy conservation relation (2.29) (with η_0 substituted for η and u_0 substituted for u) is obtained as an orthogonality condition. The mass and momentum relations (2.43) and (2.44) are obtained from the behaviour of the differential equations for u_1 and η_1 as $\theta\to-\infty$. To determine the tail, we do not need to know the complete details of u_1 and η_1, only their behaviour as $\theta\to-\infty$.

We shall now see how the Korteweg-de Vries approximation of Chapter 1 comes out of the two-timed form of the Boussinesq equations. To do this, it is most convenient to express the Boussinesq equations in terms of the velocity potential F, where $u = F_X$, and η.

The Boussinesq equation (2.11) can be integrated once to give

$$\eta = -\frac{1}{g}F_t - \frac{1}{2g}F_x{}^2 \qquad (2.48)$$

We can now eliminate η from the Boussinesq equation (2.10) to obtain the single equation

$$-\frac{b}{g}F_{tt} - \frac{b}{g}F_xF_{xt} + [b(h_0 - \frac{1}{g}F_t - \frac{1}{2g}F_x{}^2)F_x]_x + (b\nu F_{xx})_{xx} = 0 \quad (2.49)$$

for F. The two timed form of the equation for F is obtained by letting

$$F = F(\theta,X) \qquad (2.50)$$

This gives the equation

$$[(\frac{gbh_0}{V} - b)F_{\theta\theta} + \frac{2\varepsilon gbh_0}{V}F_{\theta X} + \frac{\varepsilon(bh_0)_X}{V}F_\theta + \frac{3b}{V^2}F_\theta F_{\theta\theta} + \frac{b\nu}{V^4}F_{\theta\theta\theta\theta}]$$

$$- \frac{3b}{2V^4}F_\theta{}^2F_{\theta\theta} + \varepsilon[\frac{4b}{V}F_\theta F_{\theta X} + \frac{2b}{V}F_{\theta\theta}F_X + \frac{b_X}{V}F_\theta{}^2 - \frac{b_X}{2V^3}F_\theta{}^3 - \frac{3b}{V^3}F_\theta F_{\theta\theta}F_X$$

$$- \frac{3b}{2V^3}F_\theta{}^2F_{\theta X} + \frac{4b\nu}{V^3}F_{\theta\theta\theta X} + \frac{2(b\nu)_X}{V^3}F_{\theta\theta\theta}] = 0 \qquad (2.51)$$

If the soliton is described relative to a frame moving with the linear wave speed, so that V = c, then the terms in the first brackets give the perturbed Korteweg-de Vries equation

$$2\varepsilon F_{\theta X} + \frac{\varepsilon(bc^2)_X}{gbc}F_\theta + \frac{3}{c^3}F_\theta F_{\theta\theta} + \frac{\nu}{c^5}F_{\theta\theta\theta\theta} = 0 \qquad (2.52)$$

for F_θ. It is this equation which was used as an approximate description of a soliton on a slowly varying beach in Chapter One. We now see that it incorrectly ignores higher order terms which are significant in the description of the behaviour of the soliton. The terms neglected are either terms of $0(a^3)$ or terms which are significant away from the soliton and may be neglected near the soliton. The Korteweg-de Vries approximation then is a type of wavefront expansion. The neglected terms affect the behaviour of F at $\theta = -\infty$ and so are important in determining the tail and the question of whether there is a reflected wave.

2.6 TAIL REGION

We have now found all the properties of the slowly varying soliton needed for the determination of the tail. In this section, we shall turn to our central problem of the structure of the tail and the question of the presence of any reflected wave.

From the behaviour of u_1 and η_1 at $\theta = -\infty$, we expect that the tail region will be of $0(\varepsilon)$ and depend on the slow space and time scales X and $T(= \varepsilon t)$. We hence propose that the appropriate expansion for the tail is

$$u = \varepsilon U_1(X,T) + \varepsilon^2 U_2(X,T) + \ldots \tag{2.53}$$

$$\eta = \varepsilon N_1(X,T) + \varepsilon^2 N_2(X,T) + \ldots \tag{2.54}$$

where

$$X = \varepsilon x$$
$$T = \varepsilon t \tag{2.55}$$

The Boussinesq equations (2.10) and (2.11) then give that to first order, the tail is determined by the equations

$$bN_{1T} + \left(\frac{bc^2}{g}U_1\right)_X = 0 \tag{2.56}$$

$$U_{1T} + gN_{1X} = 0 \tag{2.57}$$

These equations are just the linear shallow water equations. It is expected that we should obtain these equations as the scales used in the tail expansion (2.53) and (2.54) eliminate the nonlinear terms since u and η are $0(\varepsilon)$ and the dispersive term is eliminated due to the slow space variation. We notice that in the present scaling, the variations of breadth and depth are not slow relative to the variations of the tail. This point will be further discussed later.

The displacement of the surface and the fluid velocity will be continuous in the transition region between the soliton and the tail. The mass and momentum conservation relations (2.43) and (2.44) give the values of these quantities behind the soliton, so that the tail is determined as the solution of the moving boundary value problem consisting of the linear shallow water equations (2.56) and (2.57) together with the boundary conditions,

$$N_1 - \frac{c^2}{gV}U_1 = -\frac{1}{b}\frac{d}{dX}\left(Vb\int_{-\infty}^{\infty}\eta_0\,d\theta\right) \tag{2.58}$$

$$U_1 - \frac{g}{V}N_1 = -\frac{d}{dX}\left(V\int_{-\infty}^{\infty}u_0\,d\theta\right) \tag{2.59}$$

at the soliton position $x_s(t)$, where x_s is given by

$$T = \int_0^{X_s}\frac{dX}{V} \tag{2.60}$$

The solution of the shallow water equations consists of both a forward and a reflected wave, unless the boundary conditions are so related that a single wave can satisfy both boundary conditions simultaneously. We shall show that only a forward wave is necessary to second order in amplitude for the present problem.

For convenience, let us define

$$I = \int_{-\infty}^{\infty} \eta_0 \, d\theta \tag{2.61}$$

$$J = \int_{-\infty}^{\infty} u_0 \, d\theta \tag{2.62}$$

The boundary conditions then are that

$$N_1 - \frac{c^2}{gV} U_1 = - \frac{1}{b} \frac{d}{dX}(VbI) \tag{2.63}$$

$$U_1 - \frac{g}{V} N_1 = - \frac{d}{dX}(VJ) \tag{2.64}$$

at $x = x_s(t)$.

Let us set

$$P = - \frac{1}{b} \frac{d}{dX}(VbI) \tag{2.65}$$

and

$$Q = - \frac{d}{dX}(VJ) \tag{2.66}$$

The boundary conditions (2.63) and (2.64) then give that

$$N_1 = \frac{V^2 P + \frac{c^2 V}{g} Q}{V^2 - c^2} \tag{2.67}$$

$$U_1 = \frac{gVP + V^2Q}{V^2 - c^2} \tag{2.68}$$

at $x = x_s(t)$. Let us assume that the soliton amplitude is small. It can be seen from the expression (2.34) for V and the expression (2.36) for A that the denominator of these expressions for N_1 and U_1 is $0(A)$. Furthermore, it can be seen from the values (2.30) and (2.31) for I and J that the numerators of these expressions are $0(A^{1/2})$. Hence we have that N_1 and U_1 are $0(A^{-1/2})$ or, equivalently $0(a^{-1/2})$.

General moving boundary value problems are difficult to solve analytically. To obtain an analytical solution of the moving boundary value problem posed by the linear shallow water equations (2.56) and (2.57) together with the boundary conditions (2.58) and (2.59), the case of small soliton amplitude was considered. This is consistent with the Boussinesq equations as the Boussinesq equations are an approximation to the Euler equations which are valid to second order in amplitude. The small amplitude approximation means that the slope of the characteristics for the shallow water equations and the soliton speed differ by a small amount (of $0(a)$). We therefore see that the tail region behind the soliton will be small. In figure 3, the characteristics for the tail in the small amplitude limit have been sketched.

For small soliton amplitude, we shall seek a geometric optics type expansion for the solution to our boundary value problem

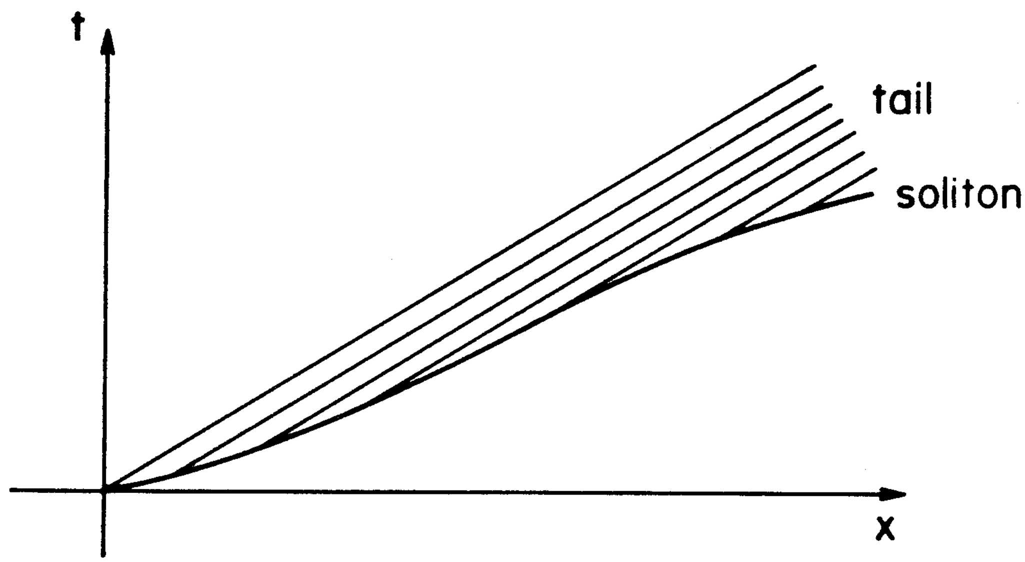

Figure 3: Tail characteristics for small soliton amplitude

$$N_1 = \frac{f'(T - \sigma(X))}{g\sqrt{bc}} + \zeta_1(X) f_1(T - \sigma(X)) + 0(a^{3/2}) \tag{2.69}$$

$$U_1 = \frac{f'(T - \sigma(X))}{\sqrt{b}\, c^{3/2}} + w_1(X) g_1(T - \sigma(X)) + 0(a^{3/2}) \ , \tag{2.70}$$

where

$$\sigma(X) = \int \frac{dX}{c} \tag{2.71}$$

The expansionsconsist of a forward wave only as it will be shown that this is all that is necessary to satisfy the boundary conditions to second order in amplitude. Successive terms in these expansions are small in $T - \sigma$ as the linear characteristic speed and the soliton speed differ by $0(a)$. The phase $T - \sigma$ is $0(a)$. The expansions for N_1 and U_1 are then expansions in $a^{1/2}$ with the lowest order terms being $0(a^{-1/2})$ and each term increasing in order by a.

The first term in these series is the classical Green's law (see Lamb, §185). Miles (1979) proposed that the tail is described by just this term. Green's law is derived on the assumption that the variations of the depth and breadth are much slower than the variations of the wave, which is not true for the tail as the tail and the depth and breadth vary on the same scale.

Let us define

$$\rho(X) = T_s(X) - \sigma(X) \ , \tag{2.72}$$

this being the value of the phase function at the soliton, where $T_s(X)$ is the time the soliton is at the position X.

$T_s(X)$ is defined by the integral (2.60).

If we substitute the expansions (2.69) and (2.70) for N_1 and U_1 into the shallow water equations (2.56) and (2.57), we find that at second order, we determine that

$$f_1 = g_1 = f \tag{2.73}$$

and

$$g\zeta_1 - cw_1 = -\frac{\sqrt{c}\, b_X}{2b^{3/2}} - \frac{c_X}{2\sqrt{bc}} \tag{2.74}$$

At this order, we find only one relation between ζ_1 and w_1 and to determine these functions separately, we must go to the next order. We shall find that to show there is no reflected wave, this is all the information we need to know about ζ_1 and w_1.

If we use the expansions (2.69) and (2.70) for N_1 and U_1 and the first boundary condition (2.63), we obtain that

$$(1 - \frac{c}{V})\frac{f'(\rho)}{\sqrt{bc}} + (g\zeta_1 - \frac{c^2}{V}w_1)f(\rho) = -\frac{g}{b}\frac{d}{dX}(bVI) \tag{2.75}$$

We can now use the relationship (2.74) between ζ_1 and w_1 derived above to replace the term involving ζ_1 and w_1, which gives

$$\frac{\sqrt{b}}{\sqrt{c}}(1 - \frac{c}{V})f'(\rho) - \frac{d}{dX}(\sqrt{bc})f(\rho) = -g(bVI)_X \tag{2.76}$$

To reduce this to a differential equation in X for f, we use the definition (2.72) for ρ, from whence we have that

$$\sqrt{bc}\,\frac{df}{dX} + \frac{d}{dX}(\sqrt{bc})\,f = g(bVI)_X \ , \qquad (2.77)$$

which has the solution

$$f = \frac{g\sqrt{b}\ VI}{\sqrt{c}} \qquad (2.78)$$

We have now evaluated the tail to first order in amplitude and in doing so, we have made the expansions (2.69) and (2.70) for N_1 and U_1 to second order in amplitude satisfy the first boundary condition (2.63). The expression we have found for the tail agrees to first order in amplitude with the forward wave calculated by Miles (1979).

To show that there is no reflected wave to second order in amplitude, we must now verify that the second boundary condition (2.64) is satisfied to second order in amplitude by our expansions for N_1 and U_1.

If we substitute the expansions (2.69) and (2.70) for N_1 and U_1 into the second boundary condition (2.64), we obtain

$$(1 - \frac{c}{V})\,\frac{f'(\rho)}{\sqrt{b}\ c^{3/2}} + (w_1 - \frac{g}{V}\zeta_1)\,f = -(VJ)_X \qquad (2.79)$$

We can again eliminate ζ_1 and w_1 from this equation by using the relationship (2.74) between them. This gives

$$-\sqrt{bc}\ f'(\rho) + \frac{d}{dX}(\sqrt{bc})\,f = -(VJ)_X \ , \qquad (2.80)$$

From the differential equation (2.76) which we obtained from the first boundary condition, we can eliminate $f'(\rho)$. Doing this, we have

$$2\frac{d}{dX}(\sqrt{bc})f = g(bVI)_X - bc(VJ)_X \tag{2.81}$$

The previously obtained expression (2.78) for f can be substituted into this equation to give the differential equation

$$\frac{g}{bc}(VbI)_X - g\left(\frac{b_X}{b^2c} + \frac{c_X}{bc^2}\right)(VbI) = (VJ)_X \,, \tag{2.82}$$

which can be integrated to give

$$I = \frac{c}{g}J \tag{2.83}$$

This is the condition that to second order in amplitude, there is no reflected wave. We note that this condition need only be satisfied to first order in amplitude for there to be no reflected wave at second order. It can be seen from the relation (2.14) between η_0 and u_0 that this condition does indeed hold to first order in amplitude as $V = c$ to first order. Therefore, to second order in amplitude there is no reflected wave.

Miles (1979) used the perturbed Korteweg-de Vries equation to describe the slowly varying soliton and found that while conserving energy, it does not conserve both mass for the perturbed Korteweg-de Vries equation and the actual mass. To conserve mass, he proposed adding both forward and reflected waves behind the soliton, these waves being solutions of the linear shallow water equations. He further proposed that these waves are given by Green's law. The forward wave was constructed so that the combination of it and the soliton satisfied mass conservation for the perturbed Korteweg-de Vries equation. The reflected wave was constructed so that by its addition, the actual mass would also be conserved. Miles found that the forward wave has amplitude $O(a^{-1/2})$ and the reflected wave has amplitude $O(a^{1/2})$. His expression for the forward wave agrees with the first term of our expansion (the Green's law term). We have seen that when the expansions for U_1 and N_1 are continued to $O(a^{1/2})$, these terms exactly account for the mass Miles included in a reflected wave, so that no reflected wave is in fact necessary.

Peregrine (1967) derived an equivalent set of Boussinesq equations to those used here and obtained a numerical solution for the case of a soliton propagating up a beach which is linear for $x>0$ and flat for $x<0$. A number of beach slopes around 0.03 were used, so it would be expected that the solution could be described by a slowly varying approximation. He claimed that his results indicated the presence of a very small reflected wave behind the soliton. As he started the soliton with its maximum at the start of the beach's slope, it is not clear that the reflected wave is in fact due to the change in the slope, rather than being due to the beach as a whole. Furthermore, Peregrine gave no indication of the size of the numerical error, so that as his reflected wave was of the order of

2.5% of the height of the soliton, it could be due to numerical error rather than any true reflection.

2.7 "SPHERICAL" WAVE

The linear shallow water equations can be solved exactly for particular depth profiles when the breadth is constant. In these cases, the linear shallow water equations can be transformed to the spherically symmetric wave equation in n dimensions, for which there is a known exact solution. We shall determine the tail for the n = 3 case and verify independently that there is no reflected wave to second order in amplitude.

From the linear shallow water equations (2.56) and (2.67) for constant breadth, we can obtain the single equation

$$N_{1TT} - N_{1\sigma\sigma} - c_X N_{1\sigma} = 0 \ , \tag{2.84}$$

where, as in the previous section,

$$\sigma = \int \frac{dX}{c} \tag{2.85}$$

If this equation is compared with the spherically symmetric wave equation in n dimensions

$$\phi_{TT} - \phi_{\sigma\sigma} - \frac{n-1}{\sigma}\phi_{\sigma} = 0, \tag{2.86}$$

we see that equation (2.84) for N_1 will be of this form if

$$c = \sqrt{g}(AX + B)^{\frac{2n-1}{2n}} \tag{2.87}$$

or

$$h_0 = (AX + B)^{\frac{2n-1}{n}} \tag{2.88}$$

where A and B are constants. For n odd, we have that

$$N_1 = \left(\frac{1}{\sigma}\frac{\partial}{\partial\sigma}\right)^{\frac{n-3}{2}} \frac{f'(T - \sigma)}{\sigma} \tag{2.89}$$

for some function f if we have a forward wave only.

The simplest case we can consider is the spherical case n = 3, which we shall now consider. Without loss of generality, for a soliton going up a beach, we can take A = -1 and B = 1. We then have

$$h_0 = (1 - X)^{4/3} \tag{2.90}$$

and

$$N_1 = \frac{f'(T - \sigma)}{(1 - X)^{1/3}} , \tag{2.91}$$

if there is a forward wave only.

The spherical case also has a depth profile closest to a linear profile. The velocity U_1 can be found from the shallow water equation (2.57) and it is

$$U_1 = \frac{\sqrt{g}\ f'(T - \sigma)}{(1 - X)} - \frac{gf(T - \sigma)}{3(1 - X)^{4/3}} \tag{2.92}$$

Unfortunately, the differential equation for f obtained by using the boundary conditions (2.65) and (2.66) cannot be integrated for arbitrary soliton amplitude and we must again use an amplitude expansion for small soliton amplitude.

The boundary conditions (2.67) and (2.68) for N_1 and U_1 when expanded out to second order in amplitude for constant breadth are

$$N_1 = \frac{-6}{\sqrt{3}} E^{-1/3} g^{1/2} h_0^2 h_{0X} - \frac{13}{2\sqrt{3}} E^{1/3} g^{-1/2} h_{0X} \qquad (2.93)$$

$$U_1 = \frac{-6}{\sqrt{3}} E^{-1/3} g h_0^{3/2} h_{0X} - \frac{11}{2\sqrt{3}} E^{1/3} h_0^{-1/2} h_{0X} \qquad (2.94)$$

at $X = X_s(T)$.

Two differential equations for f may now be obtained by using these boundary conditions and the expressions for N_1 and U_1 above. These differential equations are simultaneously satisfied by

$$f(T-\sigma) = \frac{-56}{3\sqrt{3}}\left(\frac{3}{14}\right)^{10/7} E^{13/21} g^{-9/14} (T-\sigma+\phi)^{-3/7} \qquad (2.95)$$

$$+ \frac{98}{15\sqrt{3}}\left(\frac{3}{14}\right)^{2/7} E^{11/21} g^{1/14} (T-\sigma+\phi)^{5/7} \qquad (2.95)$$

where

$$\phi = 3/14\ E^{2/3} g^{-3/2} - 3/40\ E^{4/3} g^{-5/2} \qquad (2.96)$$

The constant ϕ results from choosing $x_s = 0$ at $t = 0$. The forward wave occurs in the region

$$1 - \left(1 - \frac{\sqrt{g}\ T}{3}\right)^3 < X < X_s(T) \qquad (2.97)$$

This result is a direct confirmation for the "spherical" case that no reflected wave to second order in amplitude occurs in the region behind the soliton. We could similarily obtain expressions for f for higher "dimensions" n, but the algebra becomes more tedious as the dimension n increases.

2.8 EXPERIMENTAL RESULTS

The behaviour of a soliton travelling up a beach has also been the subject of experimental work. We shall now compare some of the experimental work with the results we have obtained.

Ippen and Kulin (1954) measured the behaviour of a soliton travelling up a channel with linearly varying depth and constant breadth. They fitted their experimental results for the amplitude variation of the soliton to curves of the form Ah_0^{-n} where A is a constant. For slopes of 0.023, 0.050 and 0.065, n was found to be 0.49, 0.26 and 0.19 respectively.

Camfield and Street (1969) also did experiments on a soliton travelling up a linearly varying channel of constant breadth with slopes of 0.01, 0.02, 0.03 and 0.045. Grimshaw (1971) extended the Boussinesq approximation to obtain equations valid to third order in amplitude and used these equations to obtain the amplitude variation of a soliton travelling up a beach whose depth is slowly varying. Camfield and Street's experimental results for the amplitude variation for a channel of slope 0.01 were plotted together with the theoretical amplitude variation and fairly good agreement was found.

Recent unpublished experiments at the California Institute of Technology* have studied a soliton travelling up a linear channel of constant breadth. Channels of various slopes around 0.0075 were used. Their results on the amplitude variation of the soliton indicate that the amplitude behaviour varies continuously from an initial Green's law $h_0^{-1/4}$ dependence to our theoretical h_0^{-1} dependence, which the soliton follows until breaking. Their results also show the presence of a tail behind the soliton and do not indicate the presence of any reflected wave.

It would seem that our theoretical h_0^{-1} amplitude dependence and slowly varying assumption are valid for sufficiently small beach slopes in a region away from where the soliton starts. Initially, there is a region where the soliton adjusts to the changing conditions. This view is supported by numerical work of Peregrine (1967) and Madsen and Mei (1969) in which the development of a soliton moving onto a linear beach was studied. Peregrine started his calculations with the soliton maximum at the start of the beach, whereas Madsen and Mei started the soliton far away from the start of the beach. Madsen and Mei fitted the computed amplitude behaviour with curves of the form Ah_0^{-n}, A being a constant, and found n to be 0.30, 0.18 and 0.15

*Private communication, Mr. James Skjelbreia for beach slopes of 0.023, 0.05 and 0.065 respectively. These results may be compared with the experimental results of Ippen and Kulin given above, but quantative comparison may not be valid as Madsen and Mei used initial amplitude to depth ratios of 0.1 and Ippen and Kulin used ratios between 0.25 and 0.68. Peregrine's results differ from those of Madsen and Mei due to the different initial condition used, but the results are essentially the same.

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PART II

MODULATED CAPILLARY WAVES

PART TWO

MODULATED CAPILLARY WAVES

The motion of slowly varying wavetrains has been studied using the modulation theory developed by Whitham (Whitham, 1974, Chapter 14), but for most of the examples dealt with, there is no known exact solution for the unmodulated periodic wave. Consequently, the discussion of the modulated waves relied on amplitude expansions for small amplitude.

Crapper (1957) found that for pure capillary waves, an exact hodograph solution can be found for arbitrary amplitude. This solution is in terms of elliptic functions in general and Crapper considered the special case in which the elliptic functions reduced to trigonometric and hyperbolic functions, this case corresponding to waves on fluid of infinite depth. Kinnersley (1976) found the solutions for general modulus and showed that these solutions corresponded to symmetric and antisymmetric waves on a fluid sheet. The symmetric waves have the surface elevation symmetric about a straight centreline (or bottom) and the antisymmetric waves are symmetric about a centreline which is a curve. In Chapter 3, an account will be given of the derivation of the hodograph solution and the limiting cases of waves on fluid of infinite depth and waves on a thin sheet will be considered.

The major contribution of Part II is the calculation of the averaged Lagrangian for the symmetric and antisymmetric waves and the derivation of the associated modulation equations for the special case of a thin fluid sheet. While the averaged Lagrangians could be calculated for waves on fluid of general depth, the algebraic complexity of the modulation equations required that when dealing with these equations, the limiting case of waves on a thin fluid sheet be considered. It is found that the modulation equations for symmetric waves on a thin fluid sheet form an elliptic system and those for antisymmetric waves on a thin fluid sheet a hyperbolic system. The symmetric waves are then unstable and the antisymmetric waves are stable.

The modulation equations for antisymmetric waves on a thin fluid sheet, while forming a hyperbolic system, have a double characteristic. In particular, this has the consequence that the simple wave solution formed by letting the double characteristic form a fan is non-unique. It is speculated that by going to next order in the modulations, the double characteristic will split, although due to the algebraic complexity, this could not be directly verified. The simple wave solutions for the two single characteristics are found to correspond to the smoothing out of an initial discontinuity in the thickness of the fluid sheet. In one case, the wavelength increases and in the other, the wavelength decreases across the simple wave.

For antisymmetric waves on a thin fluid sheet, the fluid sheet is of constant thickness to first order, so we expect the waves to bear some similarity to nonlinear waves on a string. Therefore a direct approach was tried and in view of the free surfaces, it was convenient to use Lagrangian coordinates. The first order equations in Lagrangian coordinates were found to be the same as the non-linear string equations for a string of constant tension. It was surprising to find that in the equations for the lowest order approximation, a wave travelling with permanent form could have any shape. Only at third order did a restriction that gave Kinnersley's solution appear.

CHAPTER THREE

EXACT SOLUTION FOR CAPILLARY WAVES

3.1 HODOGRAPH SOLUTION

We shall now review the solution of the inviscid water wave equations for capillary waves by using Crapper's (1957) hodograph method as extended by Kinnersley (1976). The method involves transforming the water wave equations to the hodograph plane in which it is found by separation of variables that the equations possess certain exact solutions in terms of elliptic functions. These solutions correspond to symmetric and antisymmetric waves on a sheet of fluid (of possibly infinite thickness).

Consider a capillary wavetrain travelling to the right on the surface of an inviscid, irrotational fluid of density ρ and surface tension T. We shall use a cartesian coordinate system (x,y) with y measured vertically upwards. The dimensional velocity potential and streamfunction will be denoted by ϕ^* and ψ^* respectively, the dimensional curvature of the surface will be denoted by K^* and the surface will be defined by $\psi^* = 0$.

The water wave equations for motion with surface tension alone as a restoring force are (Whitham, 1974, Chapter 13),

$$\nabla^2\phi^* = 0 \tag{3.1}$$

with the boundary conditions

$$\rho\phi_t^* + 1/2\rho(\phi_x^{*2} + \phi_y^{*2}) + TK^* = 0 \tag{3.2}$$

$$\phi_y^* = \eta_t + \eta_x\phi_x^*$$

on the free surface $y = \eta(x)$.

We shall seek a solution of these equations in the form

$$\phi^* = \beta x - \gamma t + \Phi^*(\theta,y), \tag{3.4}$$

which is the most general form for a periodic wavetrain. The mean horizontal velocity of the fluid is β and γ is a constant related to the Bernouilli constant. The phase function θ is

$$\theta = kx - \omega t \tag{3.5}$$

and Φ^* is assumed to be 2π periodic in θ with zero mean, where the mean is defined by

$$\bar{\Phi}^* = \frac{1}{2\pi}\int_0^{2\pi}\Phi^*(\theta,y)\,d\theta \tag{3.6}$$

If we substitute the general velocity potential (3.4) into the Bernouilli boundary condition (3.2), we obtain

$$\rho(1/2\beta^2 - \gamma) + \rho(\beta k - \omega)\Phi_\theta^* + 1/2\rho(k^2\Phi_\theta^{*2} + \Phi_y^{*2}) + TK^* = 0 \tag{3.7}$$

on $\psi^* = 0$.

The hodograph solution is most easily found by defining a new velocity potential which has the constant velocity $\frac{\beta k - \omega}{k}$ taken out, so that the linear term in Φ^*_θ in the above equation is eliminated. We shall also non-dimensionalize the y coordinate by k and the velocity and streamfunction by $\frac{\beta k - \omega}{k^2}$. The non-dimensional streamfunction will be defined such that the surface is $\psi = B$. We therefore set

$$Y = ky \tag{3.8}$$

$$\Phi^* = \frac{\beta k - \omega}{k^2}\frac{\phi}{A} - \frac{\beta k - \omega}{k^2}\theta \tag{3.9}$$

$$\psi^* = \frac{\beta k - \omega}{k^2}\frac{\psi - B}{A} \tag{3.10}$$

$$q^2 = \phi_\theta^2 + \phi_Y^2 \tag{3.11}$$

We have also scaled the non-dimensional velocity potential and streamfunction by the constant A. This constant will be chosen so that Φ^* is 2π periodic in θ.

In terms of complex variables, the non-dimensional complex potential is

$$w = \phi + i\psi \tag{3.12}$$

$$\frac{dw}{dz} = u - iv = qe^{-i\chi} \tag{3.13}$$

We shall work in the hodograph plane, in which q and χ are functions of ϕ and ψ. The Cauchy-Riemann equations are then

$$\chi_\phi = \sigma_\psi$$

$$\chi_\psi = -\sigma_\phi \quad , \tag{3.14}$$

where

$$\sigma = \log q \tag{3.15}$$

To determine the curvature of the surface in terms of hodograph coordinates, we note that the Cauchy-Riemann equations give

$$\begin{aligned} K^* &= kK \\ &= k\frac{d\chi}{ds} \\ &= k\chi_\phi\frac{d\phi}{ds} \\ &= kq_\psi \end{aligned} \tag{3.16}$$

Substituting the above non-dimensional forms (3.8) to (3.11) for the velocity potential and streamfunction and the curvature expression (3.16) into the Bernouilli boundary condition (3.7), we have that this boundary condition in hodograph coordinates is

$$\frac{Tk^3A^2q_\psi}{(\omega - \beta k)^2} = 1/2\rho(L^2A^2 - q^2) \text{ on } \psi = B, \tag{3.17}$$

where

$$L^2 = 1 - \frac{k^2(\beta^2 - 2\gamma)}{(\omega - \beta k)^2} \tag{3.18}$$

The constant L is related to the Bernouilli constant. Kinnesley (1976) considered the wave motion relative to cartesian coordinates moving with the wave and then set the Bernouilli constant in the boundary condition corresponding to (3.2) equal to zero, which cannot be done in a moving frame. The choice of this frame resulted in Kinnersley effectively taking L=1, which implies that $\gamma = 1/2\beta^2$. This is not true in general and holds only for linear waves and waves on fluid of infinite depth.

Following Crapper (1957) and Kinnersley (1976), we now propose that

$$q_\psi = 1/2(L^2A^2 - q^2)f(\psi) \qquad (3.19)$$

holds throughout the fluid for some function $f(\psi)$. Then from the boundary condition (3.17),

$$f(B) = \frac{\rho(\omega - \beta k)^2}{Tk^3A^2} \qquad (3.20)$$

Physically, (3.19) means that any streamline in the fluid has potentially all the properties for a free surface, except that for the boundary condition analogous to (3.20) to be satisfied, we would require a new value of the surface tension at this new boundary.

For convenience, let us define $R(\psi)$ by

$$R(\psi) = e^{LA\int f(\psi)d\psi}, \qquad (3.21)$$

so that equation (3.19) can be integrated to

$$q = LA\,\frac{R(\psi) - S(\phi)}{R(\psi) + S(\phi)}, \qquad (3.22)$$

where $S(\phi)$ is an arbitrary function of ϕ. From the boundary condition (3.20), we obtain the dispersion relation

$$\frac{(\omega - \beta k)^2}{k^3} = \frac{TAR'(B)}{\rho LR(B)} \tag{3.23}$$

We now need to find differential equations for R and S. These we obtain from the Cauchy-Riemann equations (3.14). In terms of q alone, these equations give

$$q_{\phi\phi} + q_{\psi\psi} - 1/q(q_\phi^2 + q_\psi^2) = 0 \ , \tag{3.24}$$

which, upon using the separated form (3.22) for q gives

$$(R^2 - S^2)\left(\frac{S''}{S} - \frac{R''}{R}\right) + 2S'^2 + 2R'^2 = 0 \tag{3.25}$$

Crapper (1957) now shows by separation of variables that

$$R'^2 = aR^4 + bR^2 + c \tag{3.26}$$

$$S'^2 = -aS^4 - bS^2 - c$$

is a possible solution of these equations, where a, b and c are constants. As noted in the introduction, the general solution of these equations is in terms of elliptic functions. The properties and formulae for elliptic functions to be used here are taken from Byrd and Friedman (1971).

Equations (3.26) have two general types of solution, depending on the roots of the quadratic form on the right hand side of the equations (3.26). As shown by Kinnersley (1976), these correspond to symmetric and antisymmetric waves on a fluid sheet. The special case of waves on fluid of infinite depth solved by Crapper is obtained by setting $c = 0$. This case can also by obtained as the limit as the depth becomes infinite of both the symmetric and antisymmetric waves.

The solution for capillary waves on fluid of infinite depth is

$$R = \cosh \psi$$

$$S = \cos \phi$$

$$\theta = \phi + \frac{2 \sin \phi}{\cosh \psi - \cos \phi} \qquad (3.27)$$

$$Y = \psi - \frac{2 \sinh \psi}{\cosh \psi - \cos \phi}$$

$$\frac{(\omega - \beta k)^2}{k^3} = \frac{T}{\rho} \tanh B$$

Kinnersley found the solution for waves on fluid of finite depth.

The solutions for R and S for the symmetric and antisymmetric waves on fluid of finite depth are listed in table 1.

We have thus determined q, but we still need to determine the phase χ. Kinnersley now proposes that

$$\chi = 2 \arctan \frac{U(\phi)}{V(\psi)} \qquad (3.28)$$

for some functions $U(\phi)$ and $V(\psi)$. These functions are found from the Cauchy-Riemann equations (3.14). Separation of variables gives that

$$\frac{-R'V'}{RV} = \frac{S'U'}{SU} = \alpha \ , \qquad (3.29)$$

The functions θ and Y can now be found from the Cauchy-Riemann equations

$$\theta_\psi = -Y_\phi = \frac{-v}{q^2}$$

$$\theta_\phi = Y_\psi = \frac{u}{q^2} \tag{3.30}$$

This results in

$$\theta = \frac{1}{AL\alpha}\left[\frac{2S'}{R - S} + \int \frac{S''}{S} d\phi\right] \tag{3.31}$$

$$Y = \frac{1}{AL\alpha}\left[\frac{2R'}{R - S} - \int \frac{R''}{R} d\psi\right] \tag{3.32}$$

The values of the various functions and integrals are tabulated in tables 1 and 2. As in Abramowitz and Stegun (1965), we shall use the notation m for the square of the modulus of the elliptic functions and m_1 for the square of the complementary modulus. Furthermore, as in Kinnersley, to make the notation concise, we shall not write out the dependence of R and S on the modulus. In what follows, it is understood that functions of ϕ have parameter m (modulus k) and functions of ψ parameter m_1 (modulus k').

That the solutions obtained here are symmetric and antisymmetric waves can easily be seen from the expression (3.32) for Y and the results in the tables. The solutions called symmetric waves have the property that the surface streamlines $\psi = B$ and $\psi = -B$ are symmetric about the straight centre streamline $\psi = 0$ (which can also be interpreted as a bottom surface). The antisymmetric waves have the surface streamlines $\psi = B$ and $\psi = -B$ symmetric about the centre streamline $\psi = 0$, which in this case is a curve.

We see that if we wish q to be 2π periodic, we must set

$$A = \frac{2K}{\pi} \tag{3.33}$$

since the functions of ϕ have period 4K.

To determine the constant L in terms of elliptic functions, we use the fact that we require Φ^* to have zero mean. This gives

$$L = \frac{2E - m_1 K}{m_1 K} \tag{3.34}$$

for symmetric waves and

$$L = \frac{2E - K}{K} \tag{3.35}$$

for antisymmetric waves.

We can now determine the dispersion relations for the symmetric and antisymmetric capillary waves. From the Bernouilli boundary condition (3.23), we have for symmetric waves, on noting that $R = \mathrm{dn}\psi$ and using the value of A from (3.33),

$$\frac{(\omega - \beta k)^2}{k^3} = \frac{2Tm_1{}^2K^2\mathrm{sn}B\mathrm{cd}B}{\rho\pi(2E - m_1 K)} \tag{3.36}$$

Also from the definition (3.18) for L and the value of L given by (3.34), we have for symmetric waves the connection relation

	$R(\psi)$	$S(\phi)$	$U(\phi)$	$V(\psi)$	α
symmetric waves	$\mathrm{dn}\psi$	$m^{\frac{1}{2}}\ \mathrm{cd}\phi$	$m^{\frac{1}{2}}\ \mathrm{sn}\phi$	$\mathrm{cs}\psi$	$-m_1$
antisymmetric waves	$\mathrm{ds}\psi$	$m^{\frac{1}{2}}\ \mathrm{cn}\phi$	$m^{\frac{1}{2}}\ \mathrm{sd}\phi$	$\mathrm{cn}\psi$	-1

TABLE 1: values of R, S, U, V and α

	$\int \frac{R''}{R} d\psi$	$\int \frac{S''}{S} d\phi$
symmetric waves	$(1 + m)\psi - 2E(\psi)$	$m_1\phi - 2E(\phi) + 2m\ \mathrm{sn}\phi\mathrm{cd}\phi$
antisymmetric waves	$\psi - 2E(\psi) - 2\mathrm{dn}\psi\mathrm{cs}\psi$	$\phi - 2E(\phi)$

TABLE 2: values of the integrals $\int \frac{R''}{R} d\psi$, $\int \frac{S''}{S} d\phi$

$$\gamma - 1/2\beta^2 + \frac{(\omega - \beta k)^2}{2k^2} = \frac{(2E - m_1 K)^2}{m_1^2 K^2} \frac{(\omega - \beta k)^2}{2k^2} \tag{3.37}$$

Similarily, for antisymmetric waves, we find the dispersion relation

$$\frac{(\omega - \beta k)^2}{k^3} = \frac{2TK^2 csBndB}{\rho\pi(2E - K)} \tag{3.38}$$

and the connection relation

$$\gamma - 1/2\beta^2 + \frac{(\omega - \beta k)^2}{2k^2} = \frac{(2E - K)^2}{K^2} \frac{(\omega - \beta k)^2}{2k^2} \tag{3.39}$$

The dispersion relations (3.36) and (3.38) differ from those found by Kinnersley (1976). Kinnersley's dispersion relations hold in a frame of reference in which $\omega = 0$ and β is the mean fluid velocity. To obtain the correct dispersion relations, his dispersion relations must be corrected for this. The correct dispersion relations (3.36) (3.38) can be obtained from Kinnersley's dispersion relations by dividing by L^2, which can be seen from equation (3.18) for L and the boundary condition (3.23).

The parameter B is a measure of the flux of fluid due to the wave and so may be regarded as a measure of the thickness of the fluid sheet. The parameter m may be regarded as a measure of the amplitude of the wave. From euation (3.32) for Y and the results in tables 1 and 2, we note that $\phi = 0$, $\psi = B$ corresponds to a crest of the wave and $\phi = 2K$, $\psi = B$ corresponds to a trough of the wave. So if we denote the amplitude of the wave by a, we have that for symmetric waves,

$$ka = \frac{2\pi m^{\frac{1}{2}} scB}{2E - m_1 K} \qquad (3.40)$$

and for antisymmetric waves,

$$ka = \frac{2\pi m^{\frac{1}{2}} ncB}{2E - K} \qquad (3.41)$$

We shall now summarize the hodograph solution for the symmetric and antisymmetric capillary waves. For the symmetric capillary waves, we have

$$\theta = \frac{\pi}{2(2E - m_1 K)}[2E(\phi) - m_1\phi - 2msn\phi cd\phi + \frac{2m_1 m^{\frac{1}{2}} sd\phi nd\phi}{dn\psi - m^{\frac{1}{2}} cd\phi}]$$

$$Y = \frac{\pi}{2(2E - m_1 K)}[(1 + m)\psi - 2E(\psi) + \frac{2msn\phi cn\psi}{dn\psi - m^{\frac{1}{2}} cd\phi}]$$

$$ka = \frac{2\pi}{2E - m_1 K} m^{\frac{1}{2}} scB$$

$$\chi = 2 \text{ arc } \tan(m^{\frac{1}{2}} sn\phi sc\psi) \qquad (3.42)$$

$$q = \frac{2(2E - m_1 K)}{m_1 \pi} \frac{dn\psi - m^{\frac{1}{2}} cd\phi}{dn\psi + m^{\frac{1}{2}} cd\phi}$$

$$\frac{(\omega - \beta k)^2}{k^3} = \frac{2Tm_1{}^2 K^2 snBcdB}{\rho\pi(2E - m_1 K)} ,$$

where $-B \le \psi \le B$.

For antisymmetric capillary waves, we have

$$\theta = \frac{\pi}{2(2E - K)}[2E(\phi) - \phi + \frac{2m^{\frac{1}{2}} sn\phi dn\phi}{ds\psi - m^{\frac{1}{2}} cn\phi}]$$

$$Y = \frac{\pi}{2(2E - K)}[\psi - 2E(\psi) - 2dn\psi cs\psi + \frac{2cs\psi ns\psi}{ds\psi - m^{\frac{1}{2}} cn\phi}]$$

$$ka = \frac{2\pi}{2E - K} m^{\frac{1}{2}} ncB$$

$$\chi = 2 \text{ arc } \tan(m^{\frac{1}{2}} sd\phi nc\psi) \tag{3.43}$$

$$q = \frac{2(2E - K)}{\pi} \frac{ds\psi - m^{\frac{1}{2}} cn\phi}{ds\psi + m^{\frac{1}{2}} cn\phi}$$

$$\frac{(\omega - \beta k)^2}{k^3} = \frac{2TK^2 csBndB}{\rho\pi(2E - K)},$$

where $-B \leq \psi \leq B$.

The limiting case of linear waves is obtained by letting $m \to 0$. From the results (3.42) and (3.43), we find that linear, symmetric capillary waves are given by

$$\theta = \phi$$

$$Y = \psi$$

$$ka = 4m^{\frac{1}{2}} \sinh B$$

$$\chi = 2m^{\frac{1}{2}} \sin \phi \sinh \psi$$

$$= \tfrac{1}{2} ka \operatorname{cosech} B \sin \phi \sinh \psi \tag{3.44}$$

$$q = 1 - 2m^{\frac{1}{2}} \cos \phi \cosh \psi$$

$$= 1 - \tfrac{1}{2} ka \operatorname{cosech} B \cos \phi \cosh \psi$$

$$\frac{(\omega - \beta k)^2}{k^3} = \frac{T}{\rho} \tanh B$$

Linear, antisymmetric capillary waves are given by

$$\theta = \phi$$

$$Y = \psi$$

$$ka = 4m^{\frac{1}{2}} \cosh B$$

$$\chi = 2m^{\frac{1}{2}} \sin\phi \cosh\psi$$

$$= \tfrac{1}{2} ka \operatorname{sech} B \sin\phi \cosh\psi \qquad (3.45)$$

$$q = 1 - 2m^{\frac{1}{2}} \cos\phi \sinh\psi$$

$$= 1 - \tfrac{1}{2} ka \operatorname{sech} B \cos\phi \sinh\psi$$

$$\frac{(\omega - \beta k)^2}{k^3} = \frac{T}{\rho} \coth B$$

These expressions for the linear symmetric and anti-symmetric capillary waves agree with those found by Taylor (1959) from the linearized water wave equations.

Crapper (1957) and Kinnersley (1976) found that, in general, the transformation from the hodograph plane back to the physical plane has a singularity corresponding to the surface touching itself. As the amplitude of the wave increases, a critical amplitude is reached at which the surface becomes vertical at certain points. Further increase in the amplitude results in these points moving together until they touch. After this amplitude, the solution gives that the surface intersects itself, so that the solution ceases to be physically valid. From the expression (3.31) for θ, we see that the maximum amplitude occurs when

$$\frac{2S'(\phi)}{R(B) - S(\phi)} + \int \frac{S''}{S} d\phi = 0 \qquad (3.46)$$

first has a solution $\phi \neq 0$. This is the condition for a point (ϕ,ψ) with $\phi \neq 0$ to map to a point with x coordinate zero. It is clear that the origins of the hodograph and physical planes map into each other. The minimum value of m for the equation to have a non-zero solution ϕ defines the maximum amplitude through the appropriate amplitude expression (3.40) or (3.41).

3.2 UNIQUENESS OF THE HODOGRAPH SOLUTION

The solution of the nonlinear water wave equations for pure capillary waves outlined in the previous section was based on assuming a particular form for the solution. It will become important in Chapter 5 to show that this is the only periodic solution in a certain sense of the water wave equations with surface tension alone as a restoring force. The following theorem will now be proved.

THEOREM

For given values of the wavespeed c, the wavenumber k, the amplitude a and the flux constant B, the hodograph solution of Section 3.1 is the unique solution to the water wave equations.

Proof

The Bernouilli boundary condition (3.17) can be written in the equivalent form

$$\frac{Tk^3A^2\sigma_\psi}{(\omega - \beta k)^2} = -\rho LA \sinh (\sigma - \log LA) \text{ on } \psi = B \tag{3.47}$$

Let us define the region V to be the region $(-\infty,\infty)\times[-B,B]$ of the (ϕ,ψ) plane. For conciseness, we shall set

$$\nabla = (\frac{\partial}{\partial\phi}, \frac{\partial}{\partial\psi})$$

The capillary waves are then the solution of the boundary value problem

$$\nabla^2\sigma = 0 \quad \text{in V} \tag{3.49}$$

$$\frac{Tk^3A^2\sigma_\psi}{(\omega - \beta k)^2} = -\rho LA \sinh (\sigma - \log LA) \text{ on } \psi = B \tag{3.50}$$

$$\frac{Tk^3A^2\sigma_\psi}{(\omega - \beta k)^2} = \rho LA \sinh (\sigma - \log LA) \text{ on } \psi = -B \tag{3.51}$$

Let σ_1 and σ_2 be two solutions of the boundary value problem (3.50) plus (3.51) and (3.52). Let us further set

$$\sigma = \sigma_1 \div \sigma_2 \tag{3.52}$$

This function is then the solution of the boundary value problem

$$\frac{Tk^3A^2}{(\omega - \beta k)^2}\sigma_\psi = -2\rho LA \sinh \tfrac{1}{2}\sigma \cosh \tfrac{1}{2}(\sigma_1 + \sigma_2 - 2 \log LA) \text{ on } \psi = B \tag{3.54}$$

$$\frac{Tk^3A^2}{(\omega - \beta k)^2}\sigma_\psi = 2\rho LA \sinh \tfrac{1}{2}\sigma \cosh \tfrac{1}{2}(\sigma_1 + \sigma_2 - 2 \log LA) \text{ on } \psi = -B \tag{3.55}$$

We shall now show that $\sigma = 0$. Gauss' Theorem and the fact that σ is harmonic give that

$$\int_V (\nabla\sigma)^2\, dV$$

$$= \int_{\psi = B} \sigma \frac{\partial\sigma}{\partial\psi} d\phi - \int_{\psi = -B} \sigma \frac{\partial\sigma}{\partial\psi} d\phi$$

$$= \frac{-2\rho L(\omega - \beta k)^2}{Tk^3 A}\Big[\int_{\psi = B} \sigma \sinh \tfrac{1}{2}\sigma \cosh \tfrac{1}{2}(\sigma_1 + \sigma_2 - 2 \log LA)\, d\phi$$

$$+ \int_{\psi = -B} \sigma \sinh \tfrac{1}{2}\sigma \cosh \tfrac{1}{2}(\sigma_1 + \sigma_2 - 2 \log LA)\, d\phi\Big] \qquad (3.56)$$

As $\sigma \sinh \frac{1}{2}\sigma \geq 0$, we have

$$\nabla\sigma = 0 \text{ in } V$$

and

$$\sigma = 0 \text{ on } \partial V \qquad (3.57)$$

We therefore see that $\sigma = 0$ and the solution is unique. As L and A are determined by the amplitude, we have proved that for given values of $\frac{\omega - \beta k}{k}$, k, a and B, the solution to the boundary value problem is unique. Specifying B is equivalent to specifying the fluid flux. The solution to the water wave equations cannot be unique in the usual sense as there exists both symmetric and antisymmetric waves. Specifying B as well as the wavespeed, wavenumber and amplitude determines whether the waves are symmetric or antisymmetric.

3.3 LIMITING CASE OF THE MODULUS EQUALS ONE

The hodograph solution found in Section One has four limiting cases of interest. These limiting cases are linear waves, waves on fluid of infinite depth and the limiting cases given by $m\to 1$ and $B\to 0$. The first two limiting cases were discussed in Section One. The limit $m\to 1$ corresponds to the "essentially nonlinear" waves of Kinnersley (1976) and the limit $B\to 0$ gives waves on a thin fluid sheet. We shall now consider the $m\to 1$ limit. The $B\to 0$ limit forms the major subject of Part II and the details of the form of the waves in this limit will be discussed in the next section.

The limit $m\to 1$ is possible only for symmetric waves, as for antisymmetric waves, the surface touches itself for all amplitudes before $m = 1$ is reached.

From Byrd and Friedman (1971), we have that as $m\to 1$,

$$\begin{aligned}
\mathrm{nd}\phi &\sim \cosh\phi - 1/4m_1(\sinh^2\phi\cosh\phi + \phi\sinh\phi) \\
\mathrm{cd}\phi &\sim 1 - 1/2m_1 \sinh^2\phi \\
E(\phi) &\sim \tanh\phi \\
\mathrm{sn}\, B &\sim \sin B \\
\mathrm{cn}\, B &\sim \cos B \\
\mathrm{dn}\, B &\sim 1 \\
E(B) &\sim B \\
2E - m_1K &\sim 2
\end{aligned} \tag{3.58}$$

Thus we have from the expression (3.42) for θ and Y that as $m\to 1$,

$$\theta = \frac{\pi \sinh\phi \cosh\phi}{\cosh^2\phi - \sin^2 B} \tag{3.59}$$

and

$$ky = \frac{\pi \sin B \cos B}{\cosh^2\phi - \sin^2 B} \qquad (3.60)$$

for symmetric waves. Eliminating ϕ between these expressions, we find that the surface is given by

$$\theta^2 + \pi k\eta \cot 2B + k^2\eta^2 = \pi^2 \ , \qquad (3.61)$$

which is an arc of a circle, centre $(0, -\pi\cot 2B)$, radius $\pi\operatorname{cosec} 2B$, between the points $(-\pi, 0)$ and $(\pi, 0)$.

Neighbouring arcs of the wave surface intersect when $B > \frac{\pi}{4}$. We see from the expression (3.40) for the amplitude and the asymptotic expressions (3.58) that the amplitude of the waves is given by

$$ka = \pi \tan B \ , \qquad (3.62)$$

so that the solution is physically valid for $ka < \pi$.

Byrd and Friedman (1971) give that

$$K \sim \log \frac{4}{m_1^{\frac{1}{2}}} \quad \text{as } m \to 1 \qquad (3.63)$$

It can then be seen from the expression (3.22) for q that the speed of the fluid relative to the background flow β is zero in the limit $m \to 1$. We expect this to be so as as $m \to 1$, the fluid flux $\frac{\pi B}{2K}$ relative to the background flow approaches zero. Also the dispersion relation (3.36) becomes, in the limit $m \to 1$,

$$\frac{(\omega - \beta k)^2}{k^3} = 0 \qquad (3.64)$$

The limit $m \to 1$ gives a wavetrain which consists of a series of circular arcs stationing relative to the background flow β. Kinnersley (1976) incorrectly stated that the $m \to 1$ limit gives a series of elliptical arcs. While his formulae for the surface profile for the $m \to 1$ limit are correct, he failed to notice that his equation for the surface, corresponding to (3.61), described the special case of a circle. For the flow to be quiescent relative to the background flow, the surface has to be circular for the surface tension to be constant.

3.4 THIN FILM LIMIT

As we shall be dealing with the thin film case in more detail later, we shall only briefly note here the form the solution takes in the thin film limit. The limit $B \to 0$ corresponds to a thin film as $B \to 0$ means that the total flux approaches zero.

From Byrd and Friedman (1971), we have that as, $B \to 0$,

$$E(B) \sim B$$

$$\text{sn } B \sim B$$

$$\text{cn } B \sim 1$$

$$\text{dn } B \sim 1 \tag{3.65}$$

Using these relations, we obtain from the expressions (3.42) for θ and Y that in the limit $B \to 0$, the surface is given by

$$\theta = \frac{\pi}{2(2E - m_1 K)}[2E(\phi) - m_1\phi + 2m^{\frac{1}{2}}\text{sn}\phi] \tag{3.66}$$

$$k\eta = \frac{\pi}{2(2E - m_1 K)}B[\text{dn}\phi + m^{\frac{1}{2}}\text{cn}\phi]^2 \tag{3.67}$$

for symmetric waves. That the limit $B \to 0$ corresponds to a thin film can also be seen from the expression for η.

The expression (3.40) for the amplitude of the symmetric waves gives that

$$ka = \frac{2\pi}{2E - m_1 K}m^{\frac{1}{2}}B \tag{3.68}$$

in the thin film limit and the dispersion relation (3.36) gives that

$$\frac{(\omega - \beta k)^2}{k^3} = \frac{2Tm_1{}^2K^2B}{\rho\pi(2E - m_1 K)} \tag{3.69}$$

On physical grounds, we expect that as the thickness of the sheet approaches zero, the amplitude of the symmetric waves must approach zero. This expectation is born out by the amplitude expression (3.68).

If we denote by d the distance from the centreline to a trough of the surface, then from the expression (3.67) for η, we have that for thin film, symmetric waves

$$kd = \frac{\pi B}{2(2E - m_1 K)}(1 - m^{\frac{1}{2}})^2 \tag{3.70}$$

The ratio $\frac{a}{d}$ then remains constant in the limit $B \to 0$ for symmetric waves, so this limit corresponds to shallow water theory.

If we let

$$f(\phi) = 2E(\phi) - m_1\phi + 2m^{\frac{1}{2}}\text{sn}\phi\ , \tag{3.71}$$

so that f is the function of ϕ occurring in the expression for θ, then

$$f'(\phi) = (\mathrm{dn}\phi + m^{\frac{1}{2}}\mathrm{cn}\phi)^2 \qquad (3.72)$$

As f(0) = 0, we thus see from the expression (3.66) for θ that the surface never intersects itself and the solution is valid for all amplitudes.

Similarily, we can find that in the thin film limit for antisymmetric waves, we have that

$$\theta = \frac{\pi}{2(2E - K)}(2E(\phi) - \phi) \qquad (3.73)$$

$$ky = \frac{\pi}{2E - K}m^{\frac{1}{2}}\mathrm{cn}\phi \qquad (3.74)$$

$$k\eta = \frac{\pi}{2E - K}m^{\frac{1}{2}}\mathrm{cn}\phi \qquad (3.75)$$

$$ka = \frac{2\pi}{2E - K}m^{\frac{1}{2}} \qquad (3.76)$$

$$\frac{(\omega - \beta k)^2}{k^3} = \frac{2TK^2}{\rho\pi B(2E - K)} \qquad (3.77)$$

It can again be noted that the limit $B \to 0$ indeed gives a thin sheet.

The surface intersects itself for $m \geq 0.73$ and the maximum amplitude is given by

$$ka = 16.5 \qquad (3.78)$$

As noted in Kinnersley (1976), we see from the expression (3.43) that q = LA in the thin film limit, so that the streamlines must be parallel. So for antisymmetric waves, the sheet is of constant thickness in the thin film limit. This constant thickness H can be calculated from the expression (3.43) for θ and Y and is found to be

$$kH = \frac{\pi B}{2(2E - K)} \tag{3.79}$$

Eliminating B for H in the dispersion relation (3.77) gives

$$\frac{(\omega - \beta k)^2}{k^2} = \frac{T\zeta^2}{\rho H} , \tag{3.80}$$

where we have defined

$$\zeta = \frac{K}{2E - K} = L^{-1} \tag{3.81}$$

Kinnersley (1976) obtained the dispersion relation

$$\frac{(\omega - \beta k)^2}{k^2} = \frac{T}{\rho H} \tag{3.82}$$

for antisymmetric, thin film waves, as he had in effect set L = 1. He therefore found that the waves were non-dispersive, whereas they are actually dispersive with ζ depending on a and k through the amplitude relation (3.76). The linear limit is given by $m \to 0$, in which case $\zeta \to 1$ and the waves are non-dispersive. The antisymmetric, thin film waves then have the interesting property of being non-dispersive in the linear limit and dispersive for finite amplitude.

The dispersion relations for the symmetric and antisymmetric waves in the linear limit $m \to 0$ agree with those given by Taylor (1959). Taylor found these dispersion relations directly from the linearized water wave equations as part of his theoretical and experimental investigation of capillary wave patterns on thin sheets of fluid.

From formulae in Byrd and Friedman (1971), we find that

$$\frac{d\zeta}{dm} = \frac{1}{mm_1(2E - K)^2}[(E - m_1K)^2 + mm_1K^2] \tag{3.83}$$

so that as ζ is a monotonic function of m, it is a suitable amplitude parameter for antisymmetric waves, with increasing ζ corresponding to increasing amplitude. Also as $\zeta = 1$, when $m = 0$, $\zeta \geq 1$, $\zeta = 1$ being the limit of linear waves. At the maximum amplitude, $\zeta = 6.53$.

CHAPTER FOUR

STABILITY OF CAPILLARY WAVES

4.1 AVERAGED LAGRANGIANS

In this chapter we shall calculate the averaged Lagrangians for the symmetric and antisymmetric capillary waves found in the previous chapter. Due to the complex nature of these Lagrangians, we will specialize to the case of a thin film and find the modulation equations for this case. The modulation equations for the symmetric waves will be found to be elliptic and those for the antisymmetric waves hyperbolic. Hence the symmetric waves are unstable and the antisymmetric waves are stable.

We shall now derive in detail the averaged Lagrangian for symmetric waves. The averaged Lagrangian for antisymmetric waves will just be quoted as the details for its determination are analogous to those for symmetric waves.

Luke (1967) has shown that the Lagrangian for water waves is

$$L = -\rho\int_{-h_0}^{\eta}[\phi_t^* + \tfrac{1}{2}(\underline{\nabla}\phi^*)^2]dy - V \ , \tag{4.1}$$

where V is the potential energy, η is the surface displacement and h_0 is the depth of the bottom. Let us define $\mathcal{L}$, the averaged Lagrangian, by

$$\mathcal{L} = \frac{1}{2\pi}\int_0^{2\pi} L d\theta = \bar{L} \tag{4.2}$$

On using the most general form (3.4) for the velocity potential, we have

$$\mathcal{L} = \rho(\gamma - \tfrac{1}{2}\beta^2)h + \rho(\omega - \beta k)\overline{\int_{-h_0}^{\eta} \Phi_\theta^* dy}$$

$$-\rho\overline{\int_{-h_0}^{\eta} (\tfrac{1}{2}k^2\Phi_\theta^{*2} + \tfrac{1}{2}\Phi^{*2}_y) dy} - \bar{V} \quad , \tag{4.3}$$

where

$$h = \bar{\eta} = \frac{1}{2\pi}\int_0^{2\pi} \eta d\theta \tag{4.4}$$

is the mean depth in the y direction.

Firstly, let us consider

$$I_1 = \overline{\int_{-h_0}^{\eta} \Phi_\theta^* dy} \tag{4.5}$$

From the definition (3.9) for the velocity potential ϕ in terms of Φ^*, we have that

$$I_1 = \frac{\beta k - \omega}{k^2 A}\overline{\int_{-h_0}^{\eta} \phi_\theta dy} - \frac{\beta k - \omega}{k^2} h \ , \tag{4.6}$$

on using the definition (4.4) for h.

In Section 3.1, it was noted that the surface of the fluid is given by $\psi = B$ and the centreline is given by $\psi = 0$. Therefore, upon using the Cauchy-Riemann equation

$$\phi_\theta = \psi_Y \quad , \tag{4.7}$$

we have that

$$I_1 = \frac{\beta k - \omega}{k^2}\frac{B}{A} - \frac{\beta k - \omega}{k^2} h \tag{4.8}$$

We shall now turn our attention to calculating the averaged potential energy as some of the integrals involved in its calculation also occur in the averaged kinetic energy. We have that

$$\bar{V} = \frac{1}{2\pi}\int_0^{2\pi} T\sqrt{1 + \eta'^2}\, d\theta - T \tag{4.9}$$

From the definition (3.9) for ϕ in terms of ϕ^*, we have, on $\psi = B$,

$$d\Phi^* = \frac{\beta k - \omega}{Ak^2}\, d\phi - \frac{\beta k - \omega}{k^2}\, d\theta$$

$$= \frac{\beta k - \omega}{Ak^2}\, q\big|_{\psi = B}\sqrt{1 + \eta'^2}\, d\theta - \frac{\beta k - \omega}{k^2}\, d\theta \tag{4.10}$$

Therefore, on using the separated from (3.22) for q in terms of R and S,

$$\bar{V} = \frac{T}{2\pi}\int_0^{4K} \frac{d\phi}{q\big|_{\psi = B}} - T$$

$$= \frac{T}{2\pi L}\int_0^{4K} \frac{R(B) + S(\phi)}{R(B) - S(\phi)}\,\frac{d\phi}{A} - T \tag{4.11}$$

For convenience, let us now define

$$r_1 = R(B) = \text{dn}B \tag{4.12}$$

Then

$$\bar{V} = \frac{-T}{L} + \frac{Tr_1}{\pi LA}\oint \frac{d\phi}{r_1 - S(\phi)} - T\ , \tag{4.13}$$

where the loop integral refers to integrals over one period from $\phi = 0$ to $\phi = 4K$.

We shall now consider the loop integral

$$I_2 = \oint \frac{d\phi}{r_1 - S(\phi)} \tag{4.14}$$

Since $\theta + iY$ is an analytic function of $\phi + i\psi$, we have by the Cauchy-Riemann equations that

$$\int\int \frac{\partial Y}{\partial \psi} \, d\phi d\psi = \int\int \frac{\partial \theta}{\partial \phi} d\phi d\psi \ , \tag{4.15}$$

where the integrals are evaluated over one wavelength. So we see that

$$k \int_0^{4K} \eta d\phi \ = 2\pi B \ , \tag{4.16}$$

as as noted in Section 3.1, B is the nett change in ψ between the surface and the centreline. Then using the hodograph solution (3.32) for Y, we have that

$$2\pi B = \frac{1}{AL\alpha} \int_0^{4K} \frac{2R'(B)}{r_1 - S} d\phi - \frac{4K}{AL\alpha} \left[\int \frac{R''}{R} d\psi \right]_{\psi = B}$$

$$= \frac{-R'(B)\pi}{2E - m_1 K} I_2 - \frac{2\pi K}{2E - m_1 K} W_1(B) \tag{4.17}$$

on using the value (3.33) for A, the value (3.34) for L and the definition (4.14) for I_2. We have here defined

$$W_1(B) = 2E(B) - (1 + m)B$$

$$= -\left[\int \frac{R''}{R} d\psi \right]_{\psi = B} \tag{4.18}$$

We therefore see that

$$I_2 = \frac{2K}{P_1} Z_1 \tag{4.19}$$

where

$$Z_1 = W_1(B) + \frac{B(2E - m_1 K)}{K} \tag{4.20}$$

and

$$P_1 = \sqrt{(r_1^2 - m)(1 - r_1^2)} = -R'(B) , \tag{4.21}$$

which completes the determination of $\bar{V}$.

Lastly, we shall determine h. By definition

$$kh = \frac{1}{2\pi} \int_0^{2\pi} k\eta d\theta$$

$$= \frac{R'(B)}{\pi AL\alpha} \int_0^{2\pi} \frac{d\theta}{r_1 - S(\phi)} - \frac{1}{AL\alpha}\left[\int \frac{R''}{R}\right]_{\psi = B} \tag{4.22}$$

on using the hodograph solution (3.32) for Y and the definition (4.12) for r_1. Again using the velocity potential result (4.10), we see that

$$\int_0^{2\pi} \frac{d\theta}{r_1 - S(\phi)} = \int_0^{4K} \frac{1}{r_1 - S} [q|_{\psi = B} \sqrt{1 + \eta'^2}]^{-1} d\phi$$

$$= \int_0^{4K} \frac{1}{r_1 - S} \frac{u}{q^2}\Big|_{\psi = B} d\phi \tag{4.23}$$

If we differentiate the solution (3.31) for θ with respect to ϕ, we obtain

$$\frac{u}{q^2}\Big|_{\psi = B} = \frac{1}{AL\alpha}\left[\frac{d}{d\phi}\left(\frac{2S'}{r_1 - S}\right) + \frac{S''}{S}\right] \tag{4.24}$$

Using this expression in (4.23) and integrating by parts twice results in, on noting that $S = m^{\frac{1}{2}} cd\phi$,

$$\int_0^{2\pi} \frac{d\theta}{r_1 - S} = \frac{r_1}{LA\alpha} \int_0^{4K} \frac{S'' d\phi}{S(r_1 - S)^2}$$

$$= \frac{r_1}{LA\alpha}\left[-(1 + m)\oint \frac{d\phi}{(r_1 - S)^2} + 2m\oint \frac{S^2 d\phi}{(r_1 - S)^2}\right] \tag{4.25}$$

The integrals in this expression can be expressed in terms of the known integral I_2, which was defined in (4.14). This yields

$$\int_0^{2\pi} \frac{d\theta}{r_1 - S} = \frac{r_1}{LA\alpha}\left[(1 + m) I_{2r_1} + 2m(4K - 2r_1 I_2 - r_1^2 I_{2r_1})\right] \tag{4.26}$$

We have now obtained all the integrals needed for the determination of h. Using the solution (3.32) for Y, the values of A and L given by (3.33) and (3.34) respectively and the value of I_2 already obtained in (4.19), expression (4.22) for h and the integral (4.26) give

$$kh = \frac{\pi m_1^2 K}{2(2E - m_1 K)^2}\left[\frac{r_1^2 Z_1}{P_1^2} - \frac{r_1}{P_1}\right] + \frac{\pi}{2(2E - m_1 K)}\left[-W_1 + \frac{q_1 r_1}{P_1}\right], \tag{4.27}$$

where

$$q_1 = 1 + m - 2r_1^2 \tag{4.28}$$

As it is well known that

$$\rho \overline{\int_{-h_0}^{\eta} (\tfrac{1}{2}k^2 \Phi_\theta^{*2} + \tfrac{1}{2}\Phi_Y^{*2})\, dy} = \tfrac{1}{2}(\omega - \beta k) \overline{\int_{-h_0}^{\eta} \Phi_\theta^* dy}\ , \tag{4.29}$$

we see that we have evaluated all the necessary integrals for the averaged Lagrangian. The averaged Lagrangian is now obtained by substituting the expression (4.27) for h and the expression for $\bar{V}$ obtained from (4.13) and (4.19) into the general expression (4.3) for the averaged Lagrangian, upon noting the result (4.29) above. Doing this, we obtain

$$L = \rho\left(\gamma - \tfrac{1}{2}\beta^2 + \frac{(\omega - \beta k)^2}{2k^2}\right) h - \frac{\rho(\omega - \beta k)^2 \pi B}{4k^3 K} + \frac{Tm_1 K}{2E - m_1 K}$$

$$- \frac{Tm_1 K r_1 Z_1}{(2E - m_1 K)P_1} + T \qquad (4.30)$$

For convenience, we list the following definitions and results

$$r_1 = \mathrm{dn}B \qquad (4.12)$$

$$W_1 = 2E(B) - (1 + m)B \qquad (4.18)$$

$$Z_1 = W_1(B) + \frac{B(2E - m_1 K)}{K} \qquad (4.20)$$

$$q_1 = 1 + m - 2r_1^2 \qquad (4.28)$$

$$P_1 = \sqrt{(r_1^2 - m)(1 - r_1^2)} = -\frac{d}{dB}\mathrm{dn}B \qquad (4.21)$$

$$kh = \frac{\pi m_1^2 K}{2(2E - m_1 K)^2}\left[\frac{r_1^2 Z_1}{P_1^2} - \frac{r_1}{P_1}\right] + \frac{\pi}{2(2E - m_1 K)}\left[-W_1 + \frac{q_1 r_1}{P_1}\right] \qquad (4.27)$$

The averaged Lagrangian will give the dispersion relation upon taking variations with respect to r_1 and m. To do this, we need the following derivatives, which can be found from the definitions of the various quantities listed above.

$$W_{1r_1} = \frac{q_1}{P_1} \qquad (4.31)$$

$$Z_{1r_1} = \frac{q_1}{P_1} - \frac{2E - m_1K}{KP_1} \qquad (4.32)$$

$$P_{1r_1} = \frac{r_1q_1}{P_1} \qquad (4.33)$$

$$q_{1r_1} = -4r_1 \qquad (4.34)$$

$$W_{1m} = -\frac{1}{4m}[W_1 + m_1B - \frac{2r_1(1 - r_1^2)}{P_1}] \qquad (4.35)$$

$$Z_{1m} = \frac{-E}{2mm_1K} Z_1 + \frac{r_1(1 - r_1^2)E}{mm_1KP_1} \qquad (4.36)$$

$$B_m = \frac{W_{1m}}{m_1} \qquad (4.37)$$

$$P_{1m} = \frac{-(1 - r_1^2)^2}{2P_1^2} \qquad (4.38)$$

$$q_{1m} = 1 \qquad (4.39)$$

$$K_m = \frac{E - m_1K}{2mm_1} \qquad (4.40)$$

$$E_m = \frac{E - K}{2m} \quad , \qquad (4.41)$$

Using these derivatives, we find that the variations

$$\delta r_1 : L_{r_1} = 0$$

$$\delta m: \; L_m = 0$$

give the dispersion relation

$$\frac{(\omega - \beta k)^2}{k^3} = \frac{2Tm_1 K^2 P_1}{\rho\pi r_1 (2E - m_1 K)} \qquad (4.42)$$

and

$$\gamma - \tfrac{1}{2}\beta^2 + \frac{(\omega - \beta k)^2}{2k^2} = \frac{(2E - m_1 K)^2}{m_1^2 K^2} \frac{(\omega - \beta k)^2}{2k^2} \qquad (4.43)$$

These formulae agree with the dispersion relation (3.36) and the connection relation (3.37) found directly from the hodograph solution in Section 3.1.

Similarly, we can show that the averaged Lagrangian for antisymmetric waves is

$$L = \rho(\gamma - \tfrac{1}{2}\beta^2 + \frac{(\omega - \beta k)^2}{2k^2})h - \frac{\rho(\omega - \beta k)^2 \pi B}{4k^3 K} + \frac{TK}{2E - K}$$

$$- \frac{Tr_2 KZ_2}{(2E - K)P_2} + T \qquad (4.44)$$

where

$$kh = \frac{\pi K}{2(2E - K)^2}[\frac{r_2^{\,2} Z_2}{P_2^{\,2}} - \frac{r_2}{P_2}] + \frac{\pi}{2(2E - K)}[-W_2 + \frac{q_2 r_2}{P_2}] \qquad (4.45)$$

$$Z_2 = W_2 + \frac{(2E - K)B}{K} \qquad (4.46)$$

$$W_2 = 2E(B) - B + \frac{2r_2(r_2^{\,2} - m)}{P_2} \qquad (4.47)$$

$$r_2 = dsB \qquad (4.48)$$

$$q_2 = m_1 - m + 2r_2^{\,2} \tag{4.49}$$

$$P_2 = \sqrt{(r_2^{\,2} - m)(m_1 + r_2^{\,2})} = -\frac{d}{dB}dsB \tag{4.50}$$

It can again be found that

$$W_{2r_2} = \frac{q_2}{P_2} \tag{4.51}$$

$$Z_{2r_2} = \frac{q_2}{P_2} - \frac{2E - K}{KP_2} \tag{4.52}$$

$$P_{2r_2} = \frac{r_2 q_2}{P_2} \tag{4.53}$$

$$B_{r_2} = \frac{-1}{P_2} \tag{4.54}$$

$$q_{2r_2} = 4r_2 \tag{4.55}$$

$$B_m = \frac{-1}{4mm_1}[W_2 + 2m_1 B - \frac{2r_2(r_2^{\,2} - m + m_1)}{P_2}] \tag{4.56}$$

$$W_{2m} = \frac{m_1 - m}{4mm_1} W_2 + \frac{B}{2m_1} \tag{4.57}$$

$$Z_{2m} = \frac{-(E - m_1 K)}{2mm_1 K} Z_2 + \frac{(mK - E)B}{2mm_1 K} + \frac{(2E - K)r_2(r_2^{\,2} - m + m_1)}{2mm_1 KP_2} \tag{4.58}$$

$$P_{2m} = \frac{-q_2}{2P_2} \tag{4.59}$$

$$K_m = \frac{E - m_1 K}{2mm_1} \tag{4.40}$$

$$E_m = \frac{E - K}{2m} \tag{4.41}$$

$$q_{2m} = -2 \tag{4.60}$$

The variations

$$\delta r_2: \quad L_{r_2} = 0$$

$$\delta m: \quad L_m = 0$$

can now be evaluated and the dispersion relation

$$\frac{(\omega - \beta k)^2}{k^3} = \frac{2TK^2P_2}{\rho\pi r_2(2E - K)} \tag{4.61}$$

and the connection relation

$$\gamma - \tfrac{1}{2}\beta^2 + \frac{(\omega - \beta k)^2}{2k^2} = \frac{(2E - K)^2}{K^2}\frac{(\omega - \beta k)^2}{2k^2} \tag{4.62}$$

are obtained. These formulae agree with the relations (3.38) and (3.39) found directly from the equations of motion in Section 3.1.

It is interesting to note that neither the averaged Lagrangian for symmetric nor antisymmetric waves exhibits any type of singularity at the amplitude at which the surface touches itself. This surprising result indicates that the averaging process smoothly follows the full surface given by the hodograph solution even when overlap occurs. We then need to add the side condition that the averaged Lagrangian and its associated modulation equations are valid only for values of B, m and k for which no overlap occurs, this condition being found independent of the averaged Lagrangian.

The averaged Lagrangian for the infinite depth limit can be obtained by letting $m \to 0$ in the averaged Lagrangian for either the symmetric or antisymmetric waves. Letting $m \to 0$ in the connection relation (4.43) for symmetric waves gives

$$\gamma = \tfrac{1}{2}\beta^2 \tag{4.63}$$

For the infinite depth case, we may set $\gamma = 0$ and $\beta = 0$ without loss of generality. If we do this and take the infinite depth limit in the averaged Lagrangian (4.30), we obtain

$$L = \frac{\omega^2}{k^3}\,\mathrm{cosech}^2 B - \frac{2T}{\rho}(\coth B - 1) \tag{4.64}$$

This expression for the averaged Lagrangian is equivalent to those obtained directly from Crapper's (1957) solution by Lighthill (1965) and Crapper (1970, 1979). The expressions for the averaged kinetic and potential energies in this averaged Lagrangian were also obtained from Crapper's solution by Hogan (1979). Lighthill noted that the modulation equations found from this averaged Lagrangian are elliptic, so that capillary waves on fluid of infinite depth are unstable.

4.2 MODULATION EQUATIONS FOR THIN FILM, SYMMETRIC WAVES

We shall now find the modulation equations for thin film, symmetric waves. We need to specialize to the thin film case as the modulation equations in general are too involved to be dealt with effectively.

The averaged Lagrangian for thin film, symmetric waves can be obtained by taking the limit $B \to 0$ in the averaged Lagrangian (4.30) for symmetric waves. This results in

$$L = \frac{\rho}{k}(\gamma - \tfrac{1}{2}\beta^2 + \frac{(\omega - \beta k)^2}{2k^2})H - \frac{3\rho(\omega - \beta k)^2 H}{8\,Kk^3\Gamma}$$

$$+ T[m_1^2 K - (2 - m_1)(2E - m_1 K)]\frac{3H^2}{4\pi^2\Gamma^2(2E - m_1 K)^2} \qquad (4.65)$$

being the averaged Lagrangian for thin film, symmetric capillary waves. In this expression,

$$H = kh = \frac{2\pi}{3}\Gamma B \qquad (4.66)$$

where

$$\Gamma = \frac{-m_1{}^2 K}{4(2E - m_1 K)^2} + \frac{2 - m_1}{2E - m_1 K} \qquad (4.67)$$

H is the non-dimensional mean thickness in the y direction.

The dispersion relation (3.69) is, in terms of H,

$$\omega = \beta k + \Omega H^{\frac{1}{2}} k^{3/2} \qquad (4.68)$$

where

$$\Omega = \left(\frac{3Tm_1^2 K^2}{\rho\pi^2\Gamma(2E - m_1 K)}\right)^{\frac{1}{2}} \qquad (4.69)$$

As shown by Whitham (1974, Chapter 14), the modulation equations are now found from the consistency relations

$$k_t + \omega_x = 0$$

$$\beta_t + \gamma_x = 0 \qquad (4.71)$$

and the variational equations

$$\frac{\partial L_\gamma}{\partial t} - \frac{\partial L_\beta}{\partial x} = 0 \tag{4.72}$$

$$\frac{\partial L\omega}{\partial t} - \frac{\partial Lk}{\partial x} = 0 \tag{4.73}$$

The averaged Lagrangian (4.65), the dispersion relation (4.68) and the connection relation (3.37) then give the modulation equations for thin film, symmetric waves as

$$k_t + (\beta + 2\Omega h^{\frac{1}{2}}k)k_x + k\beta_x + \tfrac{1}{2}\Omega h^{-\frac{1}{2}}k^2 h_x + \Omega_m h^{\frac{1}{2}}k^2 m_x = 0$$

$$\beta_t + \beta\beta_x + \chi\Omega^2 hkk_x + \tfrac{1}{2}\chi\Omega^2 k^2 h_x + \tfrac{1}{2}(\Omega^2\chi)_m hk^2 m_x = 0$$

$$h_t + h\beta_x + \Omega\Sigma h^{3/2}k_x + (\beta + \tfrac{3}{2}\,\Omega\Sigma h^{\frac{1}{2}}k)h_x + (\Omega\Sigma)_m h^{3/2}km_x = 0 \tag{4.74}$$

$$h^{3/2}(\Omega\Sigma)_m m_t + \tfrac{3}{2}\,h^{\frac{1}{2}}\Omega\Sigma h_t + \Sigma\Omega h^{3/2}\beta_x + (\tfrac{1}{2}\chi\Omega^2 h^2 + \tfrac{3}{2}\,\Omega^2 h^2\Sigma)k_x$$
$$+ (\tfrac{3}{2}\beta\Sigma\Omega h^{\frac{1}{2}} + hk\chi\Omega^2 + 3\Omega^2 h\Sigma k)h_x + (\beta h^{3/2}(\Sigma\Omega)_m$$
$$+ \tfrac{1}{2}(\chi\Omega^2)_m h^2 k + \tfrac{3}{2}\,(\Omega^2\Sigma)_m h^2 k)m_x = 0\ ,$$

where we have for conciseness set

$$\Sigma = 1 - \frac{3}{4K\Gamma} \tag{4.75}$$

$$\chi = \frac{(2E - m_1 K)^2}{m_1{}^2 K^2} - 1 \tag{4.76}$$

To determine the characteristic directions for the modulation equations (4.74), we need to find the zeroes of a quartic polynomial. These zeroes could not be found analytically for arbitrary amplitude, so special cases had to be considered. Firstly we shall consider the case of near-linear waves, which correspond to the limit $m \to 0$. From Abramowitz and Stegun (1965), we have that as $m \to 0$,

$$K = \frac{\pi}{2}\left(1 + \frac{m}{4} + \frac{9m^2}{64} + \ldots\right) \tag{4.77}$$

$$E = \frac{\pi}{2}\left(1 - \frac{m}{4} - \frac{3m^2}{64} - \ldots\right) \tag{4.78}$$

Using these Taylor series, we find that the characteristic directions for the modulation equations are, in the near-linear limit,

$$\frac{dx}{dt} = \beta \pm i m^{\frac{1}{2}} \frac{\sqrt{5}}{2} h^{\frac{1}{2}} \tau^{\frac{1}{2}} k, \quad \beta + \left(2 \pm i m^{\frac{1}{2}} \frac{\sqrt{5}}{2}\right) h^{\frac{1}{2}} \tau^{\frac{1}{2}} k, \tag{4.79}$$

where

$$\tau = \frac{T}{\rho} \tag{4.80}$$

We therefore have that the modulation equations form an elliptic system and that symmetric thin film capillary waves are unstable in the near-linear limit.

We shall now consider the modulation equations in the $m \to 1$ limit, which was discussed in Section 3.3. From Byrd and Friedman (1971), we have that as $m \to 1$,

$$K \sim \Lambda + \frac{1}{4}(\Lambda - 1)m_1 + \frac{9}{64}\left(\Lambda - \frac{7}{4}\right)m_1^{\,2} + \ldots \tag{4.81}$$

$$E \sim 1 + \frac{1}{2}\left(\Lambda - \frac{1}{4}\right)m_1 + \frac{3}{16}\left(\Lambda - \frac{13}{12}\right)m_1^{\,2} + \ldots, \tag{4.82}$$

where

$$\Lambda = \log \frac{4}{m_1^{\frac{1}{2}}} \tag{4.83}$$

Using these relations, we find that the modulation equations have the characteristics

$$\frac{dx}{dt} = \begin{cases} \beta \pm \dfrac{3i}{2\pi\sqrt{2\Lambda}} h^{\frac{1}{2}} \tau^{\frac{1}{2}} k \\ \beta + \left(\pm \dfrac{3\sqrt{2}}{\pi} + \dfrac{15}{16\pi\Lambda}\sqrt{\dfrac{2}{3}} \mp \dfrac{9}{16\pi\Lambda\sqrt{2}}\right) h^{\frac{1}{2}} \tau^{\frac{1}{2}} k \end{cases} \tag{4.84}$$

in the limit $m \to 1$, so that the waves are unstable in this limit.

The waves are unstable in both the modulus approaches zero and the modulus approaches one limits. This leads us to speculate that symmetric, thin film capillary waves are unstable for arbitrary amplitude.

4.3 MODULATION EQUATIONS FOR THIN FILM, ANTISYMMETRIC WAVES

As in the case of symmetric waves, we can find the averaged Lagrangian for thin film, antisymmetric waves by taking the limit $B \to 0$ in the averaged Lagrangian (4.44) for antisymmetric waves on fluid of arbitrary depth. This gives

$$L = \frac{\rho}{k}\left(\gamma - \tfrac{1}{2}\beta^2 + \frac{(\omega - \beta k)^2}{2k^2}\right) h^* - \frac{\rho(\omega - \beta k)^2 h^*}{2k^3 \zeta^2} + T - T\zeta \tag{4.85}$$

as the averaged Lagrangian for these waves. The amplitude parameter ζ has been defined before in (3.81) and

$$h^* = \frac{\pi K B}{2(2E - K)^2} \tag{4.86}$$

is the non-dimensional average thickness in the y-direction.

The modulation equations obtained from this averaged Lagrangian are

$$k_t + (\beta + \tau^{\frac{1}{2}}h^{-\frac{1}{2}}\zeta^{3/2})k_x + k\beta_x - \tfrac{1}{2}\tau^{\frac{1}{2}}h^{-3/2}k\zeta^{3/2}h_x$$

$$+ \tfrac{3}{2}\tau^{\frac{1}{2}}h^{-\frac{1}{2}}k\zeta^{\frac{1}{2}}\zeta_x = 0$$

$$\beta_t + \beta\beta_x - \tfrac{1}{2}\tau h^{-2}(\zeta - \zeta^3)h_x + \tfrac{1}{2}\tau h^{-1}(1 - 3\zeta^2)\zeta_x = 0 \qquad (4.87)$$

$$h_t + h\beta_x + (\beta + \tfrac{1}{2}\tau^{\frac{1}{2}}h^{-\frac{1}{2}}(\zeta^{3/2} - \zeta^{-\frac{1}{2}}))h_x + \tfrac{1}{2}\tau^{\frac{1}{2}}h^{\frac{1}{2}}(3\zeta^{\frac{1}{2}} + \zeta^{-3/2})\zeta_x = 0$$

$$k_t - \tfrac{1}{2}h^{-1}kh_t - \frac{3\zeta^2 + 1}{2(\zeta^3 - \zeta)}k\zeta_t + (\beta + \tau^{\frac{1}{2}}h^{-\frac{1}{2}}\zeta^{3/2})k_x$$

$$-\tfrac{1}{2}kh^{-1}\beta k_x - k\beta_x - \left(\beta k\frac{3\zeta^2 + 1}{2(\zeta^3 - \zeta)} + \tau^{\frac{1}{2}}kh^{-\frac{1}{2}}\frac{\zeta^{\frac{1}{2}}(3\zeta^2 - 1)}{\zeta^2 - 1}\right)\zeta_x = 0$$

In characteristic form, the modulation equations are

$$\frac{-2}{k}\frac{dk}{dt} + \frac{1}{2h}\frac{dh}{dt} + \frac{3\zeta^2 + 1}{2\zeta(\zeta^2 - 1)}\frac{d\zeta}{dt} = 0$$

$$\frac{-2}{k}\frac{dk}{dt} + \frac{1}{h}\frac{dh}{dt} - \zeta^{-3/2}h^{\frac{1}{2}}\tau^{-\frac{1}{2}}\frac{d\beta}{dt} = 0$$

on

$$\frac{dx}{dt} = \beta + \tau^{\frac{1}{2}}h^{-\frac{1}{2}}\zeta^{3/2}$$

$$\frac{2 - 3\zeta^2 + \zeta^{-2} \pm (3\zeta^2 + 4 + \zeta^{-2})\sqrt{1 - \zeta^{-2}}}{h(3\zeta^{\frac{1}{2}} + \zeta^{-3/2})(1 \mp \sqrt{1 - \zeta^{-2}})}\frac{dh}{dt} \qquad (4.88)$$

$$+ \frac{2 + 2\zeta^{-2} \mp 2\sqrt{1 - \zeta^{-2}}}{1 \mp \sqrt{1 - \zeta^{-2}}}h^{\frac{1}{2}}\tau^{-\frac{1}{2}}\frac{d\beta}{dt} + (3\zeta^{\frac{1}{2}} + \zeta^{-3/2})\frac{d\zeta}{dt} = 0$$

on

$$\frac{dx}{dt} = \beta + \frac{3\zeta^2 - 3 \pm 2\sqrt{1 - \zeta^{-2}}}{3\zeta^{\frac{1}{2}} + \zeta^{-3/2}}\tau^{\frac{1}{2}}h^{-\frac{1}{2}}$$

It was noted in Section 3.4 that $\zeta \geq 1$, so we see that the modulation equations for antisymmetric capillary waves on a thin fluid sheet form a hyperbolic system and that these waves are stable. We also note that even though the system is hyperbolic, it has a double characteristic and the speed of this double characteristic is the phase speed.

The dispersion and amplitude relations are, from (3.80) and (3.76),

$$\frac{(\omega - \beta k)^2}{k^3} = \frac{\tau\zeta^3}{H} \tag{3.80}$$

$$ka = \frac{2\pi m^{\frac{1}{2}}}{2E - K} \tag{3.76}$$

The importance of the interaction of the wave motion and the mean flow is shown by the characteristic equations (4.88). The antisymmetric thin film waves are "just" dispersive in the sense that the phase speed depends on ka through (3.76) and (3.80), not on both ka and k separately. This shows up in the double characteristic with characteristic speed the phase speed. The characteristics corresponding to the mean flow split from their linear double value. So we have a dual behaviour in the characteristics, with behaviour of both fully dispersive and fully non-dispersive waves showing up. Kinnersley (1976) effectively ignored any interaction between the mean flow and the waves by setting L = 1 and thus found that the waves are non-dispersive. The interaction with the mean flow causes the waves to be dispersive.

4.4 SIMPLE WAVE SOLUTIONS FOR ANTISYMMETRIC WAVES

We shall now find simple wave solutions of the modulation equations for antisymmetric waves on a thin film found in the previous section. Physical interpretations for these solutions will also be found. While the simple wave solutions can be easily found for the Riemann invariants on the single characteristics, the simple wave solution on the double characteristic will be found to be non-unique to within an arbitrary function $k(\frac{x}{t})$, where k is the wavenumber. The simple wave solutions on the single characteristics will be dealt with first. Let us consider the simple wave solution generated by allowing the characteristic

$$\frac{dx}{dt} = \beta + \frac{3\zeta^2 - 3 + 2\sqrt{1-\zeta^{-2}}}{3\zeta^{\frac{1}{2}} + \zeta^{-3/2}}\tau^{\frac{1}{2}}h^{-\frac{1}{2}} \tag{4.89}$$

to form a fan. We see from the modulation equations (4.88) that the simple wave in the expansion fan is the solution of the equations

$$\frac{-2}{k}\frac{dk}{dt} + \frac{1}{2h}\frac{dh}{dt} + \frac{3\zeta^2+1}{2\zeta(\zeta^2-1)}\frac{d\zeta}{dt} = 0$$

$$\frac{-2}{k}\frac{dk}{dt} + \frac{1}{h}\frac{dh}{dt} - \zeta^{-3/2}\tau^{-\frac{1}{2}}h^{\frac{1}{2}}\frac{d\theta}{dt} = 0$$

$$\frac{2 - 3\zeta^2 + \zeta^{-2} - (3\zeta^2 + 4 + \zeta^{-2})\sqrt{1-\zeta^{-2}}}{h(3\zeta^{\frac{1}{2}} + \zeta^{-3/2})(1+\sqrt{1-\zeta^{-2}})}\frac{dh}{dt}$$

$$+ \frac{2 + 2\zeta^{-2} + 2\sqrt{1-\zeta^{-2}}}{1+\sqrt{1-\zeta^{-2}}}h^{\frac{1}{2}}\tau^{-\frac{1}{2}}\frac{d\beta}{dt} \tag{4.90}$$

$$+ \ (3\zeta^{\frac{1}{2}} + \zeta^{-3/2})\frac{d\zeta}{dt} = 0$$

and the characteristic equation

$$\frac{2 - 3\zeta^2 + \zeta^{-2} + (3\zeta^2 + 4 + \zeta^{-2})\sqrt{1 - \zeta^{-2}}}{h(3\zeta^{\frac{1}{2}} + \zeta^{-3/2})(1 - \sqrt{1 - \zeta^{-2}})} \frac{dh}{dt}$$

$$+ \frac{2 + 2\zeta^{-2} - 2\sqrt{1 - \zeta^{-2}}}{1 - \sqrt{1 - \zeta^{-2}}} h^{\frac{1}{2}}\tau^{-\frac{1}{2}} \frac{d\beta}{dt} + (3\zeta^{\frac{1}{2}} + \zeta^{-3/2})\frac{d\zeta}{dt} = 0$$

on

$$\frac{x}{t} = \beta + \frac{3\zeta^2 - 3 + 2\sqrt{1 - \zeta^{-2}}}{3\zeta^{\frac{1}{2}} + \zeta^{-3/2}} \tau^{\frac{1}{2}} h^{-\frac{1}{2}} \qquad (4.91)$$

The first of the equations (4.90) has the solution

$$k^2 = \frac{(\zeta^2 - 1)h^{\frac{1}{2}}}{\zeta^{\frac{1}{2}}C} \qquad (4.92)$$

where C is a constant.

From the second and third of the equations (4.90), we obtain

$$h = Ae^{-\int Q(\zeta)d\zeta} \qquad (4.93)$$

$$\beta = \tau^{\frac{1}{2}}\int\left(\frac{-\zeta^{3/2}Q}{2\sqrt{A}} e^{\frac{1}{2}\int Qd\zeta} - \frac{(3\zeta^2 + 1)\zeta^{3/2}e^{\frac{1}{2}\int Qd\zeta}}{2\zeta(\zeta^2 - 1)\sqrt{A}}\right)d\zeta + B \ , \qquad (4.94)$$

where

$$Q(\zeta) = \frac{(3\zeta^2 + 1)^2(2 + (2 - \zeta^{-2})\sqrt{1 - \zeta^{-2}})}{\zeta(\zeta^2 - 1)(3 + 3\zeta + 2\zeta^{-2} - (3 + \zeta^{-2})\sqrt{1 - \zeta^{-2}})} \qquad (4.95)$$

and A and B are constants.

The characteristic equation (4.91) finally gives

$$\zeta = \text{constant on } \frac{x}{t} = \beta + \frac{3\zeta^2 - 3 + 2\sqrt{1 - \zeta^{-2}}}{3\zeta^{\frac{1}{2}} + \zeta^{-3/2}}\tau^{\frac{1}{2}}h^{-\frac{1}{2}} \qquad (4.96)$$

The simple wave solution is given by equations (4.92) to (4.96). Equations (4.93), (4.94) and (4.96) determine ζ as a function of $\frac{x}{t}$ and then equations (4.92) to (4.94) give k, h and β as functions of $\frac{x}{t}$. To determine what this solution corresponds to physically, let us consider the small amplitude limit $m \to 0$. For small m, we see from the expansions (4.77) and (4.78) for K and E that

$$\zeta = \frac{K}{2E - K} = 1 + m + \frac{9m^2}{8} + \ldots \text{ as } m \to 0 \qquad (4.97)$$

Our simple wave solution (4.92) to (4.96) then becomes in the small amplitude limit

$$\begin{aligned} m &= \tfrac{1}{2}\sqrt{\frac{A}{\tau}}\left(B - \frac{x}{t}\right) \\ h &= \frac{A}{m^2} - \frac{4\sqrt{2}A}{m^{3/2}} \\ \beta &= B - 2m\sqrt{\frac{\tau}{A}} \\ k^2 &= \frac{2\sqrt{A}}{C}\left(1 - 2\sqrt{2}m^{\frac{1}{2}}\right) \end{aligned} \qquad (4.98)$$

The thickness H for the fluid sheet was shown in Section 3.4 to be given by $h = \zeta H$. We then have that the thickness of the sheet for the simple wave in the small amplitude limit varies as

$$H = \frac{A}{m^2}\left(1 - 4\sqrt{2}m^{\frac{1}{2}}\right) \qquad (4.99)$$

The expression (3.76) for the amplitude of the antisymmetric waves in the thin film limit gives

$$a = \frac{\sqrt{C}}{\sqrt{2}A^{1/4}}(4m^{\frac{1}{2}} + 4\sqrt{2}m) \text{ as } m \to 0 \qquad (4.100)$$

We can now see that our simple wave solution corresponds to a wavetrain moving from a region in which the sheet is of a constant thickness to a region in which the sheet is of a larger thickness. As the wavetrain travels between these regions, its wavelength and amplitude are decreasing.

Similarily, we can find the simple wave solution when the characteristic

$$\frac{dx}{dt} = \beta + \frac{3\zeta^2 - 3 - 2\sqrt{1 - \zeta^{-2}}}{3\zeta^{\frac{1}{2}} + \zeta^{-3/2}}\tau^{\frac{1}{2}}h^{-\frac{1}{2}} \qquad (4.101)$$

forms a fan. The simple wave solution is given by

$$\zeta = \text{constant on } \frac{x}{t} = \beta + \frac{3\zeta^2 - 3 - 2\sqrt{1 - \zeta^{-2}}}{3\zeta^{\frac{1}{2}} + \zeta^{-3/2}}\tau^{\frac{1}{2}}h^{-\frac{1}{2}}$$

$$k^2 = \frac{(\zeta^2 - 1)h^{\frac{1}{2}}}{\zeta^{\frac{1}{2}}C}$$

$$h = Ae^{-\int R(\zeta)d\zeta}$$

$$\beta = \tau^{\frac{1}{2}}\int\left(\frac{-\zeta^{3/2}R}{2\sqrt{A}}e^{\frac{1}{2}\int Rd\zeta} - \frac{(3\zeta^2 + 1)\zeta^{3/2}e^{\frac{1}{2}\int Rd\zeta}}{2\zeta(\zeta^2 - 1)\sqrt{A}}\right)d\zeta + B \qquad (4.102)$$

where

$$R(\zeta) = \frac{(3\zeta^2 + 1)^2(2 - (2 - \zeta^{-2})\sqrt{1 - \zeta^{-2}})}{\zeta(\zeta^2 - 1)(3 + 3\zeta + 2\zeta^{-2} + (3 + \zeta^{-2})\sqrt{1 - \zeta^{-2}})}$$

In the small amplitude limit, $m \to 0$, this simple wave solution becomes

$$m = \tfrac{1}{2}\sqrt{\frac{A}{\tau}}\left(B - \frac{x}{t}\right)$$

$$H = \frac{A}{m^2}\left(1 + 4\sqrt{2}m^{\frac{1}{2}}\right)$$

$$k^2 = \frac{2\sqrt{A}}{C}\left(1 + 2\sqrt{2}m^{\frac{1}{2}}\right) \qquad (4.103)$$

$$\beta = B - 2m\sqrt{\frac{\tau}{A}}$$

$$a = \frac{\sqrt{C}}{\sqrt{2}A^{1/4}}\left(4m^{\frac{1}{2}} - 4\sqrt{2}m\right)$$

The simple wave solution again corresponds to a wavetrain moving to a region in which the sheet is of greater thickness, but the wavelength now increases and the amplitude decreases.

The final simple wave solution is for the fan generated by the double characteristic. This simple wave is the solution of the equations

$$\frac{2 - 3\zeta^2 + \zeta^{-2} \pm (3\zeta^2 + 4 + \zeta^{-2})\sqrt{1 - \zeta^{-2}}}{h(3\zeta^{\frac{1}{2}} + \zeta^{-3/2})(1 \mp \sqrt{1 - \zeta^{-2}})}\frac{dh}{dt} \qquad (4.104)$$

$$+ \frac{2 + 2\zeta^{-2} \mp 2\sqrt{1 - \zeta^{-2}}}{1 \mp \sqrt{1 - \zeta^{-2}}}h^{\frac{1}{2}}\tau^{-\frac{1}{2}}\frac{d\beta}{dt} + (3\zeta^{\frac{1}{2}} + \zeta^{-3/2})\frac{d\zeta}{dt} = 0$$

and

$$\frac{-2}{k}\frac{dk}{dt} + \frac{1}{2h}\frac{dh}{dt} + \frac{3\zeta^2 + 1}{2\zeta(\zeta^2 - 1)}\frac{d\zeta}{dt} = 0$$

$$\frac{-2}{k}\frac{dk}{dt} + \frac{1}{h}\frac{dh}{dt} - \zeta^{-3/2}h^{\frac{1}{2}}\tau^{-\frac{1}{2}}\frac{d\beta}{dt} = 0$$

on (4.105)

$$\frac{x}{t} = \beta + \tau^{\frac{1}{2}}h^{-\frac{1}{2}}\zeta^{3/2}$$

The equations (4.104) can be solved to give

$$h = D\zeta \qquad (4.106)$$

$$\frac{D^{\frac{1}{2}}}{\tau^{\frac{1}{2}}}\beta + \frac{(\frac{3}{\sqrt{2}} - 1)\sin\frac{\theta}{2}}{8^{3/4}\sin\theta\cos\theta}\log\frac{\sqrt{2}\zeta^2 + 1 - 2^{5/4}\zeta\cos\frac{\theta}{2}}{\sqrt{2}\zeta^2 + 1 + 2^{5/4}\zeta\cos\frac{\theta}{2}}$$

$$+ \frac{2(\frac{3}{\sqrt{2}} + 1)\cos\frac{\theta}{2}}{8^{3/4}\sin\theta\cos\theta}\arctan\frac{2^{5/4}\sin\frac{\theta}{2}}{\sqrt{2}\zeta^2 + 1} = A\,, \qquad (4.107)$$

where A and D are constants and

$$\theta = \pi - \arcsin\sqrt{\frac{7}{8}} \qquad (4.108)$$

As $h = \zeta H$ where H is the thickness of the sheet, we see that in this simple wave solution, the sheet is of constant thickness. The logarithmic term in the expression for β introduces no singularities as both of the quadratic expressions in its argument are strictly positive.

The characteristic equations (4.105) now give

$$k = \text{constant}$$

and (4.109)

$$\zeta = \text{constant}$$

on

$$\frac{x}{t} = \beta + \tau^{\frac{1}{2}}h^{-\frac{1}{2}}\zeta^{3/2} \tag{4.110}$$

The expressions (4.106), (4.107) and (4.110) determine ζ as a function of $\frac{x}{t}$, so that ζ, h and β are determined as functions of $\frac{x}{t}$, but k is an arbitrary function of $\frac{x}{t}$. The expression (3.76) for the amplitude of antisymmetric waves on a thin film gives

$$ka = \frac{2\pi}{2E - K}m^{\frac{1}{2}} \tag{4.111}$$

We see that ka is determined as a function of $\frac{x}{t}$, but k and a cannot be separately determined as functions of $\frac{x}{t}$. This non-uniqueness is due to the double nature of the characteristic.

The present simple wave solution corresponds to a wavetrain on a sheet of constant thickness moving from a region in which it has a given amplitude and wavenumber to a region in which these quantities have new values.

A theorem for the Cauchy problem for hyperbolic systems with multiple characteristics proved by Lax (1956) gives that the modulation equations (4.87) for antisymmetric waves on a thin fluid sheet have an unique solution for C^4 initial data. Simple wave solutions are the limit of solutions in which the region in which the initial data are non-constant approaches zero. For our non-unique simple wave solution, there exists a continuum of solutions which in the limit give this simple wave solution. These solutions are such that in the limit, their wavenumber and amplitude are related by (4.111)

In general, the modulation equations (4.74) for symmetric waves on a thin sheet and (4.87) for antisymmetric waves on a thin sheet can be solved numerically for general initial and/or boundary conditions.

4.5 HIGHER ORDER DISPERSION FOR ANTISYMMETRIC WAVES ON A THIN FILM

We shall now investigate further the nature of the double characteristic for antisymmetric waves on a thin film by considering the modulation equations in the next order of approximation. We wish to ascertain whether higher order dispersion will in some sense "split" the double characteristic.

The averaged Lagrangian will now be calculated to second order in the modulations. To second order, the exact Lagrangian (4.1) gives the averaged Lagrangian

$$L = -\rho\overline{\int_{h_0}^{\eta} \{-\gamma - \omega\Phi_\theta^* + \varepsilon\Phi_T^* + \tfrac{1}{2}[(\beta + k\Phi_\theta^* + \varepsilon\Phi_X^*)^2}$$

$$\overline{+ \Phi_y^{*2}\}dy} - \bar{V} \quad , \tag{4.112}$$

where $T = \varepsilon t$ and $X = \varepsilon x$, ε being a measure of the slow variation of the wavetrain. The lower limit of integration h_0 is the y coordinate of the centreline. So we need to evaluate the integrals in

$$L = \rho(\gamma - \tfrac{1}{2}\beta^2)h + \rho(\omega - \beta k)\overline{\int_{h_0}^{\eta} \Phi_\theta^* dy}$$

$$-\rho\overline{\int_{h_0}^{\eta} (\tfrac{1}{2}k^2\Phi_\theta^{*2} + \tfrac{1}{2}\Phi_Y^{*2})dy} - \rho\varepsilon\overline{\int_{h_0}^{\eta} \Phi_T^* dy}$$

$$- \rho\varepsilon\beta\overline{\int_{h_0}^{\eta} \Phi_X^* dy} - \rho\varepsilon k\overline{\int_{h_0}^{\eta} \Phi_X^*\Phi_\theta^* dy} - \bar{V} \qquad (4.113)$$

We shall first consider the integral

$$\overline{\int_{h_0}^{\eta} \Phi_T^* dy} \qquad (4.114)$$

The Jacobean of the transformation from the (θ, Y) coordinates to the hodograph coordinates (ϕ, ψ) is

$$J = \frac{\partial\phi}{\partial\theta}\frac{\partial\psi}{\partial Y} - \frac{\partial\phi}{\partial Y}\frac{\partial\psi}{\partial\theta}$$

$$= q^2$$

$$= L^2A^2$$

$$= \frac{4(2E - K)^2}{\pi^2} \qquad (4.115)$$

as $q = LA$ to first order in B, as was noted in Section 3.4. We then have

$$\overline{\int_{h_0}^{\eta} \Phi_T^* dy}$$

$$= \frac{1}{2\pi k}\int_0^B\int_0^{4K} \Phi_T^* \frac{\pi^2}{4K^2L^2} d\phi d\psi \ , \qquad (4.116)$$

If we now use the definition (3.9) for Φ^* in terms of ϕ, we find that

$$\int_h^\eta \overline{\Phi_T^* dy} = \frac{\pi}{8kL^2K^2}\int_0^B\int_0^{4K}[(\frac{\beta k - \omega}{k^2})_T(\frac{\pi\phi}{2K} - \theta)$$

$$+ \frac{\beta k - \omega}{k^2}(\frac{\pi\phi}{2K})_T]d\phi d\psi$$

$$= \frac{(\omega - \beta k)\pi^2 BK_T}{2k^3(2E - K)^2}, \qquad (4.117)$$

Similarily, we can find

$$\int_h^\eta \overline{\Phi_X^* dy} = \frac{(\omega - \beta k)\pi^2 BK_X}{2k^3(2E - K)^2} \qquad (4.118)$$

$$\int_h^\eta \overline{\Phi_X^*\Phi_\theta^* dy} = \frac{(\omega - \beta k)^2\pi^3 BK_X}{k^5(2E - K)^2} \qquad (4.119)$$

$$\bar{V} = T\eta - T - \frac{TK}{2(2E - K)}(2m_1 - 1 - \frac{2E - K}{K})B^2 + 0(\varepsilon^2) \qquad (4.120)$$

Using these results and the first order averaged Lagrangian (4.85), we have the second order averaged Lagrangian

$$L = \frac{\rho}{k}(\gamma - \tfrac{1}{2}\beta^2 + \frac{(\omega - \beta k)^2}{2k^2})h^* - \frac{\rho(\omega - \beta k)^2 h^*}{2k^3\zeta^2}$$

$$- \frac{\rho\varepsilon(\omega - \beta k)\pi K_m h^* m_T}{k^3 K} - \frac{\rho\varepsilon\beta(\omega - \beta k)\pi h^* K_m m_X}{k^3 K}$$

$$- \frac{2\rho\varepsilon(\omega - \beta k)^2\pi^2 h^* K_m m_X}{k^4 K}$$

$$+ T - T\zeta + T(2m_1 - 1 - \zeta)\frac{4(2E - K)^2 h^{*2}}{\zeta\pi^2} \qquad (4.121)$$

for antisymmetric waves on a thin fluid sheet. The parameter B has been replaced by h^* via the relation (4.86) between them.

The second order modulation equations are obtained from the Euler equations

$$\delta h^*: \quad L_{h^*} = 0$$

$$\delta m: \quad \frac{\partial L_{m_T}}{\partial T} + \frac{\partial L_{m_X}}{\partial X} - L_m = 0$$

$$\frac{\partial L_\gamma}{\partial T} - \frac{\partial L_\beta}{\partial X} = 0 \qquad (4.122)$$

$$\frac{\partial L_\omega}{\partial T} - \frac{\partial L_k}{\partial X} = 0$$

and the consistency relations

$$\frac{\partial \beta}{\partial T} + \frac{\partial \gamma}{\partial X} = 0$$

$$\frac{\partial k}{\partial T} + \frac{\partial \omega}{\partial X} = 0 \qquad (4.123)$$

The modulation equations can be found upon using the averaged Lagrangian (4.121), but as no detailed calculations were made using these equations due to their algebraic complexity, they will not be noted here.

To consider the influence of the higher order terms on the double characteristic, we shall consider small perturbations of the form

$$\gamma = \gamma_0 + \gamma_1 e^{i\xi(X - cT)}$$

$$\beta = \beta_0 + \beta_1 e^{i\xi(X - cT)}$$

$$\omega = \omega_0 + \omega_1 e^{i\xi(X - cT)} \tag{4.124}$$

$$k = k_0 + k_1 e^{i\xi(X - cT)}$$

$$m = m_0 + Me^{i\xi(X - cT)}$$

$$h^* = h_0 + h_1 e^{i\xi(X - cT)}$$

about the uniform wavetrain solution (γ_0, β_0, ω_0, k_0, m_0, h_0^*), where γ_1, β_1, ω_1, k_1, M and h_1^* are all constants and $\gamma_1 << \gamma_0$, $\beta_1 << \beta_0$ etc.

The consistency relations (4.123) give

$$\omega_1 = ck_1$$

$$\gamma_1 = c\beta_1 \tag{4.125}$$

The four variational equations (4.122) then give a set of four homogeneous, linear equations for k_1, β_1, M and h_1^*, which form an eigenvalue problem for c. The eigenvalues c will be determined as the roots of a quartic polynomial. If we let c_1 be the speed of the double characteristic, then the eigenvalue equation is

$$0 = (c - c_1)^2 P_1(c) + \varepsilon P_2(c) + 0(\varepsilon^2) \tag{4.126}$$

where P_1 is a (real) quartic polynomial, $P_1(c_1) \neq 0$, and P_2 is a quartic polynomial whose coefficients are complex. To second order, the double eigenvalue c_1 becomes

$$c = c_1 + \varepsilon^{\frac{1}{2}} c_2 \ , \tag{4.127}$$

where

$$c_2{}^2 = \frac{-P_2(c_1)}{P_1(c_1)} \tag{4.128}$$

If $P_2(c_1) \neq 0$, then the double characteristic "splits" at second order in the modulations. There are, of course, no characteristics for the second order modulation equations as these equations are dispersive. It seems extremely unlikely that $P_2(c_1) = 0$, so it therefore appears that the non-uniqueness at first order in the modulations is resolved at second order.

CHAPTER FIVE

DIRECT APPROACH TO ANTISYMMETRIC WAVE SOLUTIONS FOR THIN FILMS

5.1 NONLINEAR STRING EQUATIONS

The antisymmetric capillary waves propagate on a sheet of constant thickness in the thin film limit, so we expect that the waves will bear some similarity with non-linear waves on a string. In this section, we shall derive the equations for nonlinear waves on a string with constant tension.

Let us consider a perfectly elastic string with constant tension 2T, thickness 2h and density ρ. The string is assumed to have tension 2T as the fluid sheet has two surfaces with surface tension T acting on each of them. The current displacement of the string will have coordinates (x,y) and a will denote the Lagrangian x- coordinate of the string. Let us consider an arbitrary portion $a_1 \le a \le a_2$ of the string. Momentum balance in the x direction gives

$$\frac{2Tx_a(a_2,t)}{\sqrt{x_a^2 + y_a^2}}\Bigg|_{a = a_2} - \frac{2Tx_a(a_1,\ t)}{\sqrt{x_a^2 + y_a^2}}\Bigg|_{a = a_1}$$

$$= \int_{a_1}^{a_2} 2\rho h\frac{\partial^2 x}{\partial t^2}da \tag{5.1}$$

and momentum balance in the y direction gives

$$\frac{2Ty_a(a_2, t)}{\sqrt{x_a^2 + y_a^2}}\bigg|_{a = a_2} - \frac{2Ty_a(a_1, t)}{\sqrt{x_a^2 + y_a^2}}\bigg|_{a = a_1}$$

$$= \int_{a_1}^{a_2} 2\rho h \frac{\partial^2 y}{\partial t^2} da \tag{5.2}$$

Therefore in the limit of $a_2 \to a_1$, we obtain the nonlinear string equations

$$\rho h \frac{\partial^2 x}{\partial t^2} = T \frac{x_{aa} y_a^2 - x_a y_a y_{aa}}{(x_a^2 + y_a^2)^{3/2}} \tag{5.3}$$

$$\rho h \frac{\partial^2 y}{\partial t^2} = T \frac{x_a^2 y_{aa} - x_a y_a x_{aa}}{(x_a^2 + y_a^2)^{3/2}} \tag{5.4}$$

In the next section, it will be seen how these equations correspond to the equations for antisymmetric capillary waves on a thin fluid sheet.

5.2 LAGRANGIAN COORDINATES

We shall now consider the motion of antisymmetric capillary waves on a thin fluid sheet using Lagrangian coordinates (a, b). Stoker (1957, Chapter 12) gives the water wave equations in Lagrangian coordinates as

$$x_{tt} x_a + y_{tt} y_a + \frac{1}{\rho} p_a = 0 \tag{5.5}$$

$$x_{tt} x_b + y_{tt} y_b + \frac{1}{\rho} p_b = 0 \tag{5.6}$$

$$x_a y_b - x_b y_a = 1 \;, \tag{5.7}$$

where x and y are the particle displacements and p is the pressure. The first two equations are momentum conservation equations and the third equation is a mass conservation equation.

For antisymmetric waves on a thin fluid sheet, we can expand the solution in a series in b as b is small. We then seek a solution of the form

$$x = x_0(a,\ t) + bx_1(a,\ t) + \ldots \tag{5.8}$$

$$y = y_0(a,\ t) + by_1(a,\ t) + \ldots \tag{5.9}$$

$$p = \frac{-TK_1 b}{h} + b^2 p_2(a,\ t) + \ldots, \tag{5.10}$$

where K_1 is the first order curvature

$$K_1 = \frac{x_{0a} y_{0aa} - x_{0aa} y_{0a}}{(x_{0a}^2 + y_{0a}^2)^{3/2}} \tag{5.11}$$

The first order pressure is $\frac{-TK_1 b}{h}$ as the sheet is thin.

Substitution of these series into the water wave equations (5.5) to (5.7) gives the 0(1) and 0(b) equations

$$x_{0tt} x_{0a} + y_{0tt} y_{0a} = 0 \tag{5.12}$$

$$x_{0tt} x_1 + y_{0tt} y_1 - \frac{TK_1}{\rho h} = 0 \tag{5.13}$$

$$x_{0a} y_1 - x_1 y_{0a} = 1 \tag{5.14}$$

The nonlinear string equations will now be obtained from these equations.

Multiplying the curvature expression (5.11) by the mass conservation relation (5.14) and then substituting the result into the momentum equation (5.13) yields

$$x_{0tt}x_1 + y_{0tt}y_1$$
$$= \frac{T}{\rho h} \frac{x_{0aa}y_{0a}^2 y_1 - x_{0a}y_{0a}y_{0aa}x_1 - x_{0a}y_{0a}x_{0aa}y_1 + x_{0a}^2 y_{0aa}y_1}{(x_{0a}^2 + y_{0a}^2)^{3/2}} \tag{5.15}$$

If we now multiply this equation by x_1 and subtract the result from the momentum equation (5.12) multiplied by x_{0a}, we obtain

$$x_{0tt} = \frac{T}{\rho h} \frac{x_{0aa}y_{0a}^2 - x_{0a}y_{0a}y_{0aa}}{(x_{0a}^2 + y_{0a}^2)^{3/2}} \tag{5.16}$$

Alternatively, multiplying equation (5.15) by y_1 and subtracting the result from the momentum equation (5.12) multiplied by y_{0a} results in

$$y_{0tt} = \frac{T}{\rho h} \frac{x_{0a}^2 y_{0aa} - x_{0a}y_{0a}y_{0aa}}{(x_{0a}^2 + y_{0a}^2)^{3/2}} \tag{5.17}$$

These first order equations for the antisymmetric waves are the same as the nonlinear string equations (5.3) and (5.4) of Section 5.1. When antisymmetric waves propagate on a thin fluid sheet, we see that the sheet behaves as a string to first order. This is expected since, in the case of antisymmetric waves, the exact solution shows that the sheet is of constant thickness to first order.

Let us now consider a travelling wave solution of the water wave equations (5.5) to (5.7), so that

$$x = x(\theta,b)$$

$$y = y(\theta,b) \qquad (5.18)$$

$$p = p(\theta,b) \quad ,$$

where

$$\theta = ka - \omega t \qquad (5.19)$$

The water wave equation then become

$$x_{\theta\theta}x_{\theta} + y_{\theta\theta}y_{\theta} + \frac{p_{\theta}}{\rho\omega^2} = 0 \qquad (5.20)$$

$$x_{\theta\theta}x_{b} + y_{\theta\theta}y_{b} + \frac{p_{b}}{\rho\omega^2} = 0 \qquad (5.21)$$

$$x_{\theta}y_{b} - x_{b}y_{\theta} = \frac{1}{k} \qquad (5.22)$$

The boundary conditions are

$$p = -TK \quad \text{at} \quad b = h \qquad (5.23)$$

$$p = TK \quad \text{at} \quad b = -h$$

where the curvature K is

$$K = \frac{x_a y_{aa} - x_{aa} y_a}{(x_a^2 + y_a^2)^{3/2}} \qquad (5.24)$$

The first of the equations may be immediately integrated to give

$$\tfrac{1}{2}x_\theta{}^2 + \tfrac{1}{2}y_\theta{}^2 - \frac{p}{\rho\omega^2} = f(b) \;, \qquad (5.25)$$

where f is an arbitrary function of b. To determine this function f, the condition is imposed that we require the vorticity to be zero. The vorticity conservation equation for zero initial vorticity is

$$x_{\theta b}x_\theta + y_{\theta b}y_\theta - x_{\theta\theta}x_b - y_{\theta\theta}y_b = 0 \qquad (5.26)$$

from Stoker (1957, Chapter 12). Using the momentum equation (5.21), this becomes

$$x_{\theta b}x_\theta + y_{\theta b}y_\theta + \frac{p_b}{\rho\omega^2} = 0 \qquad (5.27)$$

We then see from equation (5.25) that

$$f'(b) = 0$$

or (5.28)

$$f = \tfrac{1}{2}B = \text{constant}$$

We therefore have

$$x_\theta^2 + y_\theta^2 + \frac{2p}{\rho\omega^2} = B \qquad (5.29)$$

The equations determining the travelling wave are (5.21), (5.22) and (5.29).

We shall now turn to the special case of antisymmetric capillary waves on a thin fluid sheet. As the fluid sheet is thin, the solution will be expanded as a series in b. For a travelling wave solution, the expansions (5.8) to (5.10) for x, y and p take the special form

$$x = x_0(\theta) + bx_1(\theta) + b^2x_2(\theta) + \ldots \tag{5.30}$$

$$y = y_0(\theta) + by_1(\theta) + b^2y_2(\theta) + \ldots \tag{5.31}$$

$$p = \frac{-TK_1 b}{h} + b^2p_2(\theta) + \ldots , \tag{5.32}$$

where K_1 is the first order curvature of the surface

$$K_1 = \frac{x_0'y_0'' - y_0'x_0''}{(x_0'^2 + y_0'^2)^{3/2}} \tag{5.33}$$

Substituting these expansions into the equation (5.29) gives the 0(1) equation

$$x_0'^2 + y_0'^2 = A^2 , \tag{5.34}$$

where we have set $B = A^2$. This equation is an expression of the constant first order fluid speed in the sheet. This constant speed was noted from the hodograph solution in Section 3.4. We see that to first order, x_0 and y_0 are not uniquely determined.

The equations (5.21), (5.22) and (5.29) give the 0(b) equations

$$x_0'x_1' + y_0'y_1' - \frac{TK_1}{\rho\omega^2 h} = 0 \tag{5.35}$$

$$x_0''y_1 + y_0''y_1 - \frac{TK_1}{\rho\omega^2 h} = 0 \tag{5.36}$$

$$x_0'y_1 - y_0'x_1 = \frac{1}{k} \tag{5.37}$$

The solutions for x_1 and y_1 will now be obtained. We might have expected that in finding these solutions, an extra condition on x_0 and y_0 would have been obtained, so that these functions are uniquely determined. We shall find that this is not the case and we will have to go to $0(b^2)$ to find this extra condition.

Using the relation (5.34) between x_0 and y_0, we see that the first order curvature (5.33) becomes

$$K_1 = \frac{y_0''}{x_0'A} = \frac{-x_0''}{y_0'A} \tag{5.38}$$

The mass conservation equation (5.37) gives

$$y_1 = \frac{1}{x_0'}\left(\frac{1}{k} + x_1 \; y_0'\right) \tag{5.39}$$

Substituting this expression for y_1 into the momentum equation (5.36) results in

$$\frac{y_0''}{k} = \frac{TK_1 x_0'}{\rho\omega^2 h} \tag{5.40}$$

We see from the curvature expression (5.38) that this results in the dispersion relation

$$\omega^2 = \frac{kT}{\rho hA} \tag{5.41}$$

If we now substitute the equation (5.39) for y_1 into the momentum equation (5.36) and use the curvature expression (5.38), we obtain the differential equation

$$x_1' - \frac{x_0''}{x_0'}\, x_1 = \frac{-x_0''}{k y_0' A^5} + \frac{x_0'' y_0'}{k A^2 x_0'} \quad , \tag{5.42}$$

which has the solution

$$x_1 = \frac{2x_0'}{kA^2} \operatorname{arc\,cos} \frac{x_0'}{A} - \frac{y_0'}{kA^2} + Cx_0' \quad , \tag{5.43}$$

where C is a constant. The equation (5.39) for y_1 then gives

$$y_1 = \frac{2y_0'}{kA^2} \operatorname{arc\,cos} \frac{x_0'}{A} + \frac{x_0'}{kA^2} + Cy_0' \tag{5.44}$$

The constant C is determined from the boundary conditions (5.23). To apply these boundary conditions, we need to determine the solution for p_1 , which is obtained in part from the $0(b^2)$ equations.

The expansions (5.30) to (5.32) give the $0(b^2)$ equations

$$x_2'x_0' + \tfrac{1}{2}x_1'^2 + y_2'y_0' + \tfrac{1}{2}y_1'^2 + \frac{p_1}{\rho\omega^2} = 0 \tag{5.45}$$

$$2x_0''x_2 + x_1''x_1 + 2y_0''y_2 + y_1 y_1'' + \frac{2p_1}{\rho\omega^2} = 0 \tag{5.46}$$

$$x_1'y_1 + 2x_0'y_2 - x_1 y_1' - 2x_2 y_0' = 0 \tag{5.47}$$

upon substitution into the equations (5.21), (5.22) and

(5.29).

Eliminating x_2 between the first two equations gives

$$(x_0'x_0'' + y_0'y_0'')y_2 = -\tfrac{1}{2}x_0'x_1x_1'' - \tfrac{1}{2}y_0'y_1y_1'' - \frac{p_1y_0'}{\rho\omega^2}$$

$$+ \tfrac{1}{2}y_1'x_1x_0'' - \tfrac{1}{2}y_1x_1x_0'' \tag{5.48}$$

As the coefficient of y_2 is zero by equation (5.34), we have that

$$\frac{p_1}{\rho\omega^2} = -\tfrac{1}{2}x_1x_1'' - \tfrac{1}{2}y_1y_1'' + \frac{x_1y_1'x_0''}{2y_0'} - \frac{y_1x_1'x_0''}{2y_0'} \tag{5.49}$$

At $O(b^2)$, the boundary conditions (5.23) require that

$$\frac{-A^3h}{T}\, p_1 = x_0'y_1'' + y_0''x_1' - x_0''y_1' - y_1''x_0' - \frac{3x_0'^2x_1'y_0''}{A^2}$$

$$- \frac{3}{A^2}\, y_0'y_1'x_0'y_1'' + \frac{3}{A^2}\, y_1'y_0'x_0'x_0'' + \frac{3}{A^2}\, y_0'^2y_1'x_0'' \tag{5.50}$$

This boundary condition then gives, on using the expression (5.49) for p_1 and the expressions (5.43) for x_1 and (5.44) for y_1,

$$C = 0 \tag{5.51}$$

The solutions for x_1 and y_1 are thus

$$x_1 = \frac{2x_0'}{kA^2}\ \text{arc cos}\ \frac{x_0'}{A} - \frac{y_0'}{kA^2} \tag{5.52}$$

$$y_1 = \frac{2y_0'}{kA^2}\ \text{arc cos}\ \frac{x_0'}{A} + \frac{x_0'}{kA^2} \tag{5.53}$$

The first order solutions x_0 and y_0 are still not uniquely determined at $0(b)$. It can be easily verified that

$$kx_0 = \frac{\pi}{2E - K}[2E(\frac{2K\theta}{\pi}) - \frac{2K\theta}{\pi}] \quad (5.54)$$

$$ky_0 = \frac{\pi}{2E - K}m^{\frac{1}{2}}cn(\frac{2K\theta}{\pi}) \quad (5.55)$$

$$A = \frac{K}{(2E - K)k} \quad (5.56)$$

is a possible solution of the relation (5.34) between x_0 and y_0. This is the same as the hodograph solution for antisymmetric waves in the thin film limit, this solution being given by (3.73) and (3.74). We then have that the previous hodograph solution is a possible solution of the water wave equations in Lagrangian coordinates, as it must be. In Section 3.2, it was proved that the hodograph solution is the unique solution to the water wave equations, so we need to find an extra condition on x_0 and y_0. We shall now consider the $0(b^2)$ and $0(b^3)$ equations to see if this extra condition can be found.

If we eliminate x_2 between the $0(b^2)$ momentum equations (5.45) and (5.46), we obtain the differential equation

$$(2x_0'^2y_0' + 2y_0'^3)y_2' + (2x_0'x_0''y_0' - 2x_0'^2y_0'')y_2$$

$$= - x_1''y_1x_0'y_0' + x_1y_1''x_0'y_0' + x_1'y_1y_0''x_0' - x_1y_1'y_0''x_0'$$

$$-y_0'^2x_1'^2 - y_0'^2y_1'^2 + x_1'y_1y_0'x_0'' - x_1y_1'x_0''y_0'$$

$$+ x_1x_1''y_0'^2 + y_1y_1''y_0'^2 , \tag{5.57}$$

which can be expressed in the equivalent form

$$A^2\frac{d}{d\theta}\left(\frac{y_2}{y_0'}\right) = \frac{d}{d\theta}\left(\frac{x_1x_0'y_1'}{y_0'} - \frac{x_1'y_1x_0'}{y_0'} + x_1x_1' + y_1y_1'\right)$$

$$- \frac{4x_0''^2}{A^2k^2y_0'^2} - \frac{4x_0''^2}{A^2k^2y_0'^2}\left(\text{arc cos } \frac{x_0'}{A}\right)^2 \tag{5.58}$$

upon using the solutions (5.52) and (5.55) for x_1 and y_1 and the relation (5.34) between x_0 and y_0. Unfortunately, this differential equation for y_2 could not be integrated.

To obtain the boundary condition for the $0(b^2)$ equations, we need to find p_2, which is found from the $0(b^3)$ equations. The expansions (5.30) to (5.32) when substituted into equations (5.21), (5.22) and (5.29) yield the $0(b^3)$ equations

$$x_0'x_3' + x_1'x_2' + y_0'y_3' + y_1'y_2' + \frac{p_2}{\rho\omega^2} = 0 \tag{5.59}$$

$$3x_0''x_3 + 2x_2x_1'' + x_1x_2'' + 3y_0''y_3 + 2y_2y_1'' + y_1'y_2'' + \frac{3p_2}{\rho\omega^2} = 0 \tag{5.60}$$

$$3x_0'y_3 + 2x_1'y_2 + x_2'y_1 - x_1y_2' - 2x_2y_1' - 3x_3y_0' = 0 \tag{5.61}$$

If we eliminate x_3 between the last two equations, we obtain

$$3(x_0'x_0'' + y_0'y_0'')y_3 + \frac{3y_0'p_2}{\rho\omega^2}$$

$$= -2x_1'x_0''y_2 - x_2'y_1x_0'' + x_1x_0''y_2' + 2x_2x_0''y_1' - 2x_1''x_2y_0'$$

$$-y_0'x_1x_2'' - 2y_0'y_1''y_2 - y_1y_2''y_0' \qquad (5.62)$$

The relation (5.34) between x_0 and y_0 gives that the coefficient of y_3 is zero, so that p_2 is given by

$$\frac{3y_0'p_2}{\rho\omega^2} = -2x_1'x_0''y_2 - y_1x_0''x_2' + x_0''x_1y_2' + 2x_2y_1'x_0''$$

$$-2x_2y_0'x_1'' - y_0'x_1x_2'' - 2y_0'y_1''y_2 - y_1y_0'y_2'' \qquad (5.63)$$

Now that p_2 has been found, we can apply the boundary conditions (5.23). To $0(b^2)$, these boundary conditions give

$$\frac{-A^5p_2}{T} = \frac{-3}{2}\,\frac{A^2y_0''x_1'}{x_0'} - \frac{3}{2}\,\frac{A^2y_0''y_1'^2}{x_0'} - 3A^2y_0''x_2' + 3A^2x_0''y_2'$$

$$+ \frac{9}{2}\,y_0''x_0'x_1'^2 + 9y_0'y_0''x_1'y_1' - \frac{9}{2}\,x_0''y_0'y_1'^2$$

$$-3x_0'^2y_1''x_1' - 3y_0'x_0'y_1'y_1'' + 3x_0'y_0'x_1'x_1'' + 3y_0'^2x_1''y_1'$$

$$+ A^2(x_0'y_2'' + x_1'y_1'' + x_2'y_0'' - x_0''y_2' - x_1''y_1' - x_2''y_0')$$

$$(5.64)$$

Since x_1, y_1, x_2 and y_2 can be expressed in terms of x_0 and y_0, the two expressions (5.63) and (5.64) for p_2 should give an extra condition on x_0 and y_0 , so that these functions are determined as (5.54) and (5.55), in agreement with the hodograph solution of Chapter 3. The uniqueness theorem of Section 3.2 gives that (5.54) and (5.55) must be the solution for x_0 and y_0.

It seems that the $0(1)$ solution is fully determined by conditions at $0(b^2)$. If we consider a general packet of antisymmetric waves on a thin fluid sheet, this packet will break up into separate packets whose components satisfy the first order relation (5.34). These packets have different values of A and will take a very long time to disperse for h very small as the dispersion is determined at $0(b^2)$. In Taylor's (1959) experiments, the values of h used varied from 2·5 - 50μm. We see that, in general, antisymmetric capillary waves on a thin fluid sheet are not adequately described as fully dispersive waves. This observation is linked to our earlier observation in Section 4.3 that antisymmetric waves on a thin sheet have properties of both fully dispersive and non-dispersive waves.

5.3 ASYMPTOTIC SOLUTION FOR ANTISYMMETRIC WAVES ON A THIN FLUID SHEET

As the first order solution in Lagrangian coordinates for antisymmetric waves on a thin fluid sheet is not determined uniquely until third order, we expect that the solution in Eulerian coordinates will also have a similar behaviour. In this section, we shall show that this is the case.

The hodograph solution for antisymmetric waves on a thin fluid sheet will now be derived using an asymptotic expansion. The fluid speed q satisfies the differential equation (3.24)

$$q_{\phi\phi} + q_{\psi\psi} - \frac{1}{q}(q_\phi{}^2 + q_\psi{}^2) = 0 \qquad (5.65)$$

together with the boundary condition (3.17)

$$\frac{Tk^3A^2q\psi}{(\omega - \beta k)^2} = \tfrac{1}{2}\rho(L^2A^2 - q^2) \quad \text{on} \quad \psi = B \qquad (5.66)$$

As the sheet is thin, the fluid speed q will be expanded as a series in ψ as ψ is small. So we propose the series solution

$$q = q_0(\phi) + q_1(\phi)\psi + q_2(\phi)\psi^2 + q_3(\phi)\psi^3 + \ldots \qquad (5.67)$$

Substituting this series into the differential equation (5.65) gives the 0(1) and $0(\psi)$ equations

$$q_2 = \frac{q_0'^2}{2q_0} + \frac{q_1{}^2}{2q_0} - \frac{q_0''}{2} \qquad (5.68)$$

$$6q_3 = \frac{2q_0'q_1'}{q_0} + \frac{4q_1q_2}{q_0} - \frac{q_1q_0'^2}{q_0{}^2} - \frac{q_1{}^3}{q_0{}^2} - q_1'' \qquad (5.69)$$

Further equations are needed to determine the q_i. These are obtained from the boundary condition (5.66). Let us assume that the dispersion relation is given by the asymptotic expansion in B:

$$\frac{TA^2k^3}{(\omega - \beta k)^2} = \rho\alpha_1B + \rho\alpha_2B^2 + \rho\alpha_3B^3 + \ldots \qquad (5.70)$$

The boundary condition (5.66) then yields upon substitution of this dispersion relation and the expansion (5.67) for q the 0(1) equation

$$q_0 = LA \tag{5.71}$$

This first order solution gives that the fluid velocity in the sheet is constant to first order, as has been noted previously. To 0(B), the boundary condition gives

$$\alpha_1 q_1 = -q_0 q_1$$

so that

$$\alpha_1 = -LA \tag{5.72}$$

If we similarly obtain the $0(B^2)$ and $0(B^3)$ boundary conditions, we find further equations for q_2 and q_3, from which q_1 can be determined. At $0(B^2)$ and $0(B^3)$, the boundary condition (5.66) yields

$$LAq_2 = \alpha_2 q_1 + \frac{q_1^{\;2}}{2} \tag{5.73}$$

$$LAq_3 = \tfrac{1}{2}\alpha_3 q_1 + \tfrac{1}{2}q_1 q_2 \tag{5.74}$$

upon noting (5.71) and (5.72).

The two expressions (5.68) and (5.73) for q_2 give

$$\alpha_2 = 0 \tag{5.75}$$

and the two expressions (5.69) and (5.74) for q_3 give the differential equation

$$q_1'' = \frac{-q_1^3}{2L^2A^2} - \frac{3\alpha_3 q_1}{LA} \tag{5.76}$$

for q_1. As for the Lagrangian coordinate solution of the previous section, we see that we must go two orders higher in the series expansion to determine the solution at first order. In both cases, the equations obtained at the next higher order determine constants only and do not give an equation for the first order solution.

The equation (5.76) for q_1 can be integrated once to give

$$q_1'^2 = C - \frac{3\alpha_3 q_1^2}{LA} - \frac{q_1^4}{4L^2A^2} \quad , \tag{5.77}$$

where C is a constant. This equation has the solution

$$q_1 = -2LAm^{\frac{1}{2}}\mathrm{cn}\phi \quad , \tag{5.78}$$

in agreement with the solution (3.43) for q when this solution is expanded as a series in ψ. Requiring q to be 2π periodic and Φ^* to have zero mean gives

$$A = -\frac{2K}{\pi} \tag{5.79}$$

$$L = \frac{2E - K}{K} \tag{5.80}$$

The equations (5.68) for q_2 and (5.69) and (5.74) for q_3 now give

$$q_2 = 2LAm\mathrm{cn}^2\phi \tag{5.81}$$

$$q_3 = -\frac{1}{3}LA(m_1 - m)m^{\frac{1}{2}}\text{cn}\phi - 2LAm^{3/2}\text{cn}^3\phi \qquad (5.82)$$

$$\alpha_3 = \frac{1}{3}LA(m_1 - m) \qquad (5.83)$$

We see that the Lagrangian coordinate solution and the hodograph solution for antisymmetric waves on a thin fluid sheet bear similarities in their structure, which reinforces our belief that the two expressions (5.63) and (5.64) for p_2 in the Lagrangian coordinate solution give the extra condition needed to determine x_0 and y_0 uniquely.

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