Inverse spectral problem with partial information on the potential

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Abstract

The Schrödinger operator $-d^2/dx^2 + q(x)$ is considered on the real axis. We discuss the inverse spectral problem where discrete spectrum and the potential on the positive half-axis determine the potential completely. We do not impose any restrictions on the growth of the potential but only assume that the operator is bounded from below, has discrete spectrum, and the potential obeys $q(-|x|) \geq q(|x|)$. Under these assertions we prove that the potential for $x \geq 0$ and the spectrum of the problem uniquely determine the potential on the whole real axis. Also, we study the uniqueness under slightly different conditions on the potential. The method employed uses Weyl m-function techniques and asymptotic behavior of the Herglotz functions.

Contents

Chapter 1 Introduction

1.1 Overview

In this thesis we investigate the inverse spectral problem for the Schrödinger operator

$$H = -\frac{d^2}{dx^2} + q(x) \tag{1.1}$$

on the whole real line. We study in this work the case of *discrete* spectrum. Also, we suppose that the operator is bounded from below. We consider the inverse spectral problem when the spectrum of H and the potential on the positive half-line determine the potential everywhere. We demonstrate an example which shows that this fact is not true, in general. One needs to impose some extra assumptions on the potential. We prove the uniqueness of the potential under the following a priori bound:

$$q(-|x|) \geq q(|x|). \tag{1.2}$$

We consider the other type of condition on the potential. Namely, we ask the inequality only for sufficiently large x: $q(-|x|) > Cq(|x|)$ for $C > 1$. This result requires some additional assumptions on the potential. In particular, the result requires existence of Weyl's asymptotics for the number of the eigenvalues.

This work generalizes the result proved by Gesztesy and Simon in [GS]. In addition to the inequality (1.2), they also require the restrictive condition on the growth of the potential. We discuss their result in the next section.

1.2 Inverse problems with partial information on the potential

In this section we describe known results for the inverse spectral problems with partial information on the potential.

1.2.1 The case of the interval $[0, 1]$

In this subsection we state results concerning Schrödinger operators on the interval $[0, 1]$. The spectrum of such operators is necessarily discrete.

In 1978, Hochstadt-Lieberman [HL78] proved the following theorem:

Theorem 1.1. *Let $h_0 \in \mathbb{R}$, $h_1 \in \mathbb{R} \cup \{\infty\}$ and assume $q_1, q_2 \in L^1((0,1))$ to be real-valued. Consider the Schrödinger operators H_1, H_2 in $L^2((0,1))$ given by*

$$H_j = -\frac{d^2}{dx^2} + q_j, \quad j = 1, 2,$$

with the boundary conditions

$$\begin{cases} u'(0) + h_0 u(0) &= 0, \\ u'(1) + h_1 u(1) &= 0. \end{cases} \tag{1.3}$$

Let $\sigma(H_j) = \{\lambda_{j,n}\}$ be the (necessarily simple) spectra of $H_j, j = 1, 2$. Suppose that $q_1 = q_2$ (a.e.) on $[0, \frac{1}{2}]$ and that $\lambda_{1,n} = \lambda_{2,n}$ for all n. Then $q_1 = q_2$ (a.e.) on $[0, 1]$.

Here, in obvious notation, $h_1 = \infty$ singles out the Dirichlet boundary condition $u(1) = 0$.

We can paraphrase the results by saying "the potential on $[0, \frac{1}{2}]$ and the eigenvalues of the operator uniquely determine the potential everywhere."

For each $\varepsilon > 0$, there are simple examples where $q_1 = q_2$ on $[0, \frac{1}{2} - \varepsilon]$ and $\sigma(H_1)$ $= \sigma(H_2)$ but $q_1 \neq q_2$. (See [GS] and also Theorem I′ in the appendix of [Suz86]. We consider the similar example for the case of the whole real axis in Section 1.3.2.)

Later refinements of Theorem 1.1 in [Hal80, Suz86] (see also the summary in [Suz82]) showed that the boundary condition for H_1 and H_2 at $x = 1$ need not be assumed a priori to be the same, and that if q is continuous, then one only needs $\lambda_{1,n} = \lambda_{2,m(n)}$ for all values of n but one. The same boundary condition for H_1 and H_2 at $x = 0$, however, is crucial for Theorem 1.1 to hold (see [Hal80, dRC90]).

Moreover, analogs of Theorem 1.1 for certain Schrödinger operators are considered in [Kha84] and the interval $[0, \frac{1}{2}]$ replaced by different subsets of $[0, 1]$ was studied in [Jay] (see also [PT87], Ch. 4). Reconstruction techniques for $q(x)$ in this context are discussed in [RS94].

Gesztesy and Simon suggested in [GS] the following generalization of Theorem 1.1

Theorem 1.2. *Let* $H = -\frac{d^2}{dx^2} + q$ *in* $L^2((0,1))$ *with boundary conditions* (1.1) *and* $h_0, h_1 \in \mathbb{R}$. *Suppose* q *is* $C^{2k}((\frac{1}{2} - \varepsilon, \frac{1}{2} + \varepsilon))$ *for some* $k = 0, 1, \ldots$ *and for some* $\varepsilon > 0$. *Then* q *on* $[0, \frac{1}{2}]$, h_0, *and all the eigenvalues of* H *except for* $(k+1)$ *uniquely determine* h_1 *and* q *on all of* $[0, 1]$.

Remarks. 1. The case $k = 0$ in Theorem 1.2 is due to Hald [Hal80].

2. In the non-shorthand form of this theorem, we mean that both q_1 and q_2 are C^{2k} near $x = \frac{1}{2}$.

3. One need not know which eigenvalues are missing. Since the eigenvalues asymptotically satisfy

$$\lambda_n = (\pi n)^2 + 2(h_1 - h_0) + \int_0^1 q(x)\, dx + o(1) \quad \text{as } n \to \infty, \tag{1.4}$$

given a set of candidates for the spectrum, one can tell how many are missing.

4. For the sake of completeness we mention the precise definition of H in $L^2((0,1))$ for real-valued $q \in L^1((0,1))$ and boundary condition parameters $h_0, h_1 \in \mathbb{R} \cup \{\infty\}$:

$$\begin{aligned} H &= -\frac{d^2}{dx^2} + q, \\ D(H) &= \{g \in L^2((0,1)) \mid g, g' \in AC([0,1]);\ (-g'' + qg) \in L^2((0,1)); \\ &\qquad g'(0) + h_0 g(0) = 0,\ g'(1) + h_1 g(1) = 0\}, \end{aligned} \tag{1.5}$$

where $AC([0,1])$ denotes the set of absolutely continuous functions on $[0,1]$ and $h_{x_0} = \infty$ represents the Dirichlet boundary condition $g(x_0) = 0$ for $x_0 \in \{0,1\}$ in (1.5).

Also, in[GS] examples were constructed which show that Theorem 1.2 is optimal in the sense that if q is only assumed to be C^{2k-1} near $x = \frac{1}{2}$ for some $k \geq 1$, then it is not uniquely determined by $q \restriction [0, \frac{1}{2}]$ and all the eigenvalues but $(k+1)$.

Theorem 1.2 works because the condition that q is C^{2k} near $x = \frac{1}{2}$ gives us partial information about q on $[\frac{1}{2}, 1]$; namely, we know $q(\frac{1}{2}), q'(\frac{1}{2}), \ldots, q^{(2k)}(\frac{1}{2})$ computed on $[\frac{1}{2}, 1]$ since we can compute them on $[0, \frac{1}{2}]$. This suggests that knowing q on more than $[0, \frac{1}{2}]$ should let one dispense with a finite density of eigenvalues. That this is indeed the case is the content of the following theorem, also proved in [GS]:

Theorem 1.3. *Let $H = -\frac{d^2}{dx^2} + q$ in $L^2((0,1))$ with boundary conditions* (1.3) *and $h_0, h_1 \in \mathbb{R}$. Then q on $[0, \frac{1}{2} + \frac{\alpha}{2}]$ for some $\alpha \in (0,1)$, h_0, and a subset $S \subseteq \sigma(H)$ of all the eigenvalues $\sigma(H)$ of H satisfying*

$$\#\{\lambda \in S \mid \lambda \leq \lambda_0\} \geq (1-\alpha)\#\{\lambda \in \sigma(H) \mid \lambda \leq \lambda_0\} + \tfrac{\alpha}{2} \tag{1.6}$$

for all sufficiently large $\lambda_0 \in \mathbb{R}$, uniquely determine h_1 and q on all of $[0,1]$.

Remarks. 1. As a typical example, knowing slightly more than half the eigenvalues and knowing q on $[0, \frac{3}{4}]$ determine q uniquely on all of $[0,1]$. Theorem 1.3 solves a new type of inverse spectral problem involving fractions of the set of eigenvalues.

2. As in the case $\alpha = 0$, we have an extension of the same type as Theorem 1.2. Explicitly, if q is assumed to be C^{2k} near $x = \frac{1}{2} + \frac{\alpha}{2}$, we only need

$$\#\{\lambda \in S \mid \lambda \leq \lambda_0\} \geq (1-\alpha)\#\{\lambda \in \sigma(H) \mid \lambda \leq \lambda_0\} + \tfrac{\alpha}{2} - (k+1) \tag{1.7}$$

instead of (1.6).

In [RGS97], further generalizations were considered. Let $H(h_0)$ be a Schrödinger operator on $[0,1]$ with the boundary conditions (1.3). Fix $h_1 \in \mathbb{R}$ but think of $H(h_0)$ as a family of operators depending on h_0 as a parameter. Del Rio, Gesztesy and Simon proved that the spectrum of one $H(h_0)$ and half the spectrum of another

$H(h_0)$ and q on $[0, \frac{1}{4}]$ determine q. This extends a classical result of Borg [Bor46] (see also [Hoc73, Lev49, Lev68, Lev87, LG64, Mal94, Mal97]) that the spectra of $H(h_0)$ for two values of h_0 determine q. The other interesting result obtained in [RGS97] is that two-thirds of the spectra of three $H(h_0)$ determine q. Both these results are based on the fact that the m-function is determined by its values on the sufficiently dense set of points.

1.2.2 Jacobi matrices

Let us now discuss the application of the m-function approach to the Jacobi matrices. The finite Jacobi matrix is an N by N matrix of the form:

$$H = \begin{pmatrix} b_1 & a_1 & 0 & 0 & \cdot & \cdot & \cdot \\ a_1 & b_2 & a_2 & 0 & \cdot & \cdot & \cdot \\ 0 & a_2 & b_3 & a_3 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & a_{N-1} \\ \cdot & \cdot & \cdot & \cdot & 0 & a_{N-1} & b_N \end{pmatrix} \tag{1.8}$$

It is useful to consider the b's and a's as a single sequence $b_1, a_1, b_2, a_2, \ldots := c_1, c_2, \ldots, c_{2N-1}$ that is,

$$c_{2n-1} = b_n, \quad c_{2n} = a_n, \qquad n = 1, 2, \ldots . \tag{1.9}$$

In [Hoc79], Hochstadt proved the following theorem:

Theorem 1.4. *Let $N \in \mathbb{N}$. Suppose that $c_{N+1}, \ldots, c_{2N-1}$ are known, as well as the eigenvalues $\lambda_1, \ldots, \lambda_N$ of H. Then $c_1, \ldots, c_N$ are uniquely determined.*

In [GS97b], Gesztesy and Simon use the m-function approach to generalize this result:

Theorem 1.5. *Suppose that $1 \le j \le N$ and $c_{j+1}, \ldots, c_{2N-1}$ are known, as well as j of the eigenvalues. Then $c_1, \ldots, c_j$ are uniquely determined.*

Remark. One need *not* know which of the j eigenvalues one has.

Also, in [GS97b], a new approach to the central result of inverse analysis is presented, that is, the spectral measure uniquely determines the matrix elements. The proof is based on studying the asymptotic behavior of the m-functions.

1.2.3 The case of the whole real line

It is well known [KM56] that knowledge of the reflection coefficient at positive energies does not determine the potential V of a Schrödinger operator $-\frac{d^2}{dx^2} + V$ such that $V(x) \to 0$ sufficiently rapidly as $|x| \to \infty$, but that one also needs bound state energies and associated norming constants. This is most dramatically seen in one-soliton potentials where $R_+(\lambda) \equiv 0$, $\lambda \geq 0$, even though there is a two-parameter family of such potentials parameterized by the center and width of the soliton.

There has been a series of recent papers [Akt94, Akt96, AKvdM93, BSL95, GW95, RS94, Sac93] showing that if V is known a.e. on a half-line and vanishes sufficiently fast as $|x| \to \infty$ in the sense that at least its first moment on $\mathbb{R}$ exists, then the norming constants and even the bound state energies are not needed (some of these papers are limited to the case where V is assumed to vanish on the right half-line).

In [GS97a], it was shown that this is a special case of a very general and very elementary phenomenon: It is not required that V has a simple asymptotic as $|x| \to \infty$. Rather, all that is significant is that V be known a.e. on $(0, \infty)$ and the Schrödinger operator H_+ associated with $-\frac{d^2}{dx^2} + V$ in $L^2((0,\infty))$ and any self-adjoint boundary condition at 0 has some absolutely continuous (a.c.) component in its spectrum. Also, rather than require detailed manipulation of the machinery of inverse problems and/or trace formulas, all that is required is a uniqueness result to go from a Weyl m-function to a potential. In particular, the m-function technique used in the paper allows one to consider impurity (defect) scattering in (half) crystals, scattering off potentials with different spatial asymptotics at left and right including asymptotically periodic potentials, potential steps, and potentials diverging to $+\infty$ as $x \to -\infty$. The central result of this paper is the following theorem:

Theorem 1.6. *Assume that* $V \in L^1_{\mathrm{loc}}(\mathbb{R})$ *is real-valued and* $-\frac{d^2}{dx^2} + V(x)$ *is in the limit point case at* $\pm\infty$. *Suppose that* V *is known a.e. on* $(0, \infty)$ *and that* $R_+(\lambda)$ *is known a.e. on a set* $S \subseteq S_+$ *of positive Lebesgue measure inside the essential support* S_+ *of* $\sigma_{ac}(H_+)$. *Then* V *is uniquely determined a.e. on* $(-\infty, 0)$ *and hence a.e. on* $\mathbb{R}$.

More subtle and deep is a comparison problem concerning knowledge of the potential on a half-line where the spectrum is purely discrete rather than having an absolutely continuous component. In [GS], Gesztesy and Simon considered the inverse problem for the Schrödinger operator with *discrete spectrum* on the whole real axis. They proved the following theorem:

Theorem 1.7. *Suppose that* $q \in L^1_{\mathrm{loc}}(\mathbb{R})$ *obeys*

(i) $q(x) \geq C|x|^{2+\varepsilon} - D$ *for some* $C, \varepsilon, D > 0$, *and that*

(ii) $q(-x) \geq q(x) \quad x \geq 0$.

Then q *on* $[0, \infty)$ *and the spectrum of* $H = -\frac{d^2}{dx^2} + q$ *in* $L^2(\mathbb{R})$ *uniquely determine* q *on all of* $\mathbb{R}$.

The proof of this theorem is based on the same idea as the proof of Theorem 1.2, that is, the Weyl m-functions associated with the positive and negative half-intervals were studied. The uniqueness of the m-function associated with the negative half-line (where the potential is unknown) was shown. Then the Marchenko-Borg result [Mar52, Bor52] guarantees that the potential on the negative half-line is unique[1]. The fact of the uniqueness of the m-function was shown using the Phragmén-Lindelöf principle. The first condition of Theorem 1.7 implies that the m-functions of the problem are meromorphic functions of order less than 1. Using the second condition and the asymptotics of the m-functions at infinity, it was shown that the difference of any two m-functions vanishes as $z \to \pm i\infty$. Then, Phragmén-Lindelöf principle states that the difference is zero, that is, under conditions of the theorem, the m-function is unique.

[1]We discuss the definition of m-function and its properties (in particular, the Marchenko-Borg theorem) in Chapter 2.

1.3 Results

1.3.1 Uniqueness for the potentials of arbitrary growth

The purpose of this research is to extend Theorem 1.7 to a wider class of potentials. As was noted in the brief discussion of the proof of Theorem 1.7, the condition on the growth of the potential is technical: For such potentials the corresponding Weyl m-functions have order less than 1. The last fact is necessary for applying the Phragmén-Lindelöf principle. The importance of the second condition of Theorem 1.7 is not immediately obvious. In Section 1.3.2, we give an example demonstrating that this condition is crucial and, in some sense, optimal. Therefore, we consider the following theorem:

Theorem 1.8. *Let* $H = -\frac{d^2}{dx^2} + q(x)$ *be a Schrödinger operator in* $L^2(\mathbb{R})$ *with a potential* $q(x) \in L^1_{\text{loc}}$ *such that*

(i) *H is a bounded-from-below operator with discrete spectrum,*

(ii) $q(-x) \geq q(x), \quad x \geq 0.$

Then q on $[0, \infty)$ and the spectrum of H uniquely determine q on all of $\mathbb{R}$.

We do not require any specific conditions on the growth of the potential. The theorem works for any potential which defines a bounded-from-below operator with discrete spectrum.

We also consider the slightly different type of conditions on the potential. Instead of requiring condition (ii), we impose the condition on the counting function of the eigenvalues for the operators corresponding to the negative and positive half-lines (Lemma 2.5 and Corollary 2.2). As an application of this result, we prove the following theorem:

Theorem 1.9. *Let* $H = -\frac{d^2}{dx^2} + q(x)$ *be a Schrödinger operator in* $L^2(\mathbb{R})$ *with a potential* $q(x) \in L^1_{\text{loc}}$ *such that*

(i) *H is a bounded-from-below operator with discrete spectrum,*

(ii) *the potential q admits Weyl asymptotics for the number of the eigenvalues,*

(iii) *there exists $R > 0$ and $C > 1$ such that $q(-x) > Cq(x)$, for $x > R$,*

(iv) *q obeys $\int_{q(x)<r,\, x<0} q(x)\, dx/r \to \infty$ as $r \to \infty$.*

Then q on $[0, \infty)$ and the spectrum of H uniquely determine q on all of $\mathbb{R}$.

The last condition of the theorem is not very restrictive. In particular, it holds for any potential of the polynomial growth.

1.3.2 Example of non-uniqueness

In this section, we consider an example when violation of the second condition of Theorem 1.8 leads to the non-uniqueness of the potential. Examples of such a type were considered in [GS].

Fix some $\varepsilon > 0$. Let $q(x)$ be any function with the following properties:

1. $q(x) = q(-\varepsilon - x)$ for $x > 0$,

2. $q(x) \neq q(-\varepsilon - x)$ for $x \in (-\varepsilon, 0)$.

3. q defines a bounded-from-below Schrödinger operator with discrete spectrum.

Define the potential $q_r(x) = q(-\varepsilon - x)$ which is a reflection of q with respect to the point $-\varepsilon/2$. Then two Schrödinger operators H_1 and H_2 with the potentials q and q_r, respectively, have the same spectrum since they are unitarily equivalent: $H_1 = U^{-1} H_2 U$, where $(Uf)(x) = f(-\varepsilon - x)$ for any $f \in L_2(\mathbb{R})$. Obviously, U is a unitary operator on $L_2(\mathbb{R})$.

By construction, the potentials q and q_r coincide on the positive half-line. The first condition of Theorem 1.8 is satisfied for both q and q_r. But the theorem does not hold: $q(x) \neq q_r(x)$ on the negative half-axis. Note that $q(|x|)$ and $q(-|x|)$ have the same asymptotics at infinity; actually, $q(-|x|)$ is just "slightly" less than $q(|x|)$ (meaning that it is the same function but shifted to the left by an arbitrarily small ε). This example shows the importance of the second condition of the theorem.

1.3.3 Brief discussion of the approach to the proof

In this research work, we use the Weyl m-function approach which was used by Gesztesy and Simon in [GS, GS97a, GS97b]. In Chapter 2 we define the Weyl m-function and discuss its properties. Here we just briefly sketch the arguments. Instead of proving the uniqueness of the potential on the negative half-line we study the question of the uniqueness of the Weyl m-function associated with $-\infty$. We assume that there exist two distinct potentials $q_{1,2}(x)$ for $x < 0$. For each potential, there exists the unique Weyl m-function $m_{1,2-}(z)$, $m_1 \neq m_2$. We study the difference of these functions:

$$\Delta m(z) = m_1(z) - m_2(z).$$

Our goal is to prove that, under the conditions of Theorem 1.8, Δm must be zero. We show that Δm is a meromorphic function with zeros and poles on the real axis. The second condition of Theorem 1.8 implies that Δm has more zeros than poles on any interval $(-\infty, \lambda]$. This fact and the properties of the m-functions reduce Theorem 1.8 to the following theorem:

Theorem 1.10. *Let f be a meromorphic function on $\mathbb{C}$. Assume that*

(i) *f has some zeros and all poles in the interval $[a, \infty)$, $a > -\infty$.*

(ii) *f has the representation:*

$$f(z) = C + \sum_{k=1}^{\infty} a_k \left(\frac{1}{\nu_k - z} - \frac{1}{\nu_k} \right), \quad a_k \in \mathbb{R}, \ \nu_k \in [a, \infty), \quad \sum_{\nu_k \neq 0} \left| \frac{a_k}{\nu_k^2} \right| < \infty. \tag{1.10}$$

(iii) *$f(z) \to 0$ as $z \to \infty$, $\Re z < 0$.*

(iv) *f has at least as many zeros as poles on any interval $[a, \lambda)$ for all sufficiently large λ.*

Then f must be a zero.

Remarks. 1. The assumption that only some zeros of f lie on the interval $[a, \infty]$ comes from the fact that Δm vanishes on the spectrum of the Schrödinger operator. These are the zeros we deal with in the last condition. It could happen that Δm may have some extra zeros which are not related to the spectrum.

2. Only the last condition of Theorem 1.10 depends on the estimate on the potential (condition (ii) of Theorem 1.8). All the other conditions follow from the properties of the m-functions of the bounded-from-below operators with discrete spectrum. There could be some other condition on the potential that leads to the same bound on the number of zeros and poles in condition (iv). In such cases, Theorem 1.10 also implies the uniqueness of the potential. We discuss this issue in more detail in Section 2.4 of Chapter 2.

In Chapter 3 we discuss the asymptotic properties of the function represented by the series of simple fractions:

$$f(z) = \sum_{k=1}^{\infty} \frac{a_k}{z - \nu_k}, \qquad \sum_{\nu_k \neq 0} \left| \frac{a_k}{\nu_k} \right| < \infty.$$

We show that such functions obey

$$\int_0^{2\pi} \ln^+ |f(re^{i\theta})| \, d\theta = o(r). \tag{1.11}$$

For this result, $a_k, \nu_k \in \mathbb{C}$. Also, we prove the same asymptotics for the functions represented as the Borel transforms of signed measures.

In Chapter 4 we use the Jensen formula to obtain the bound on the counting function of zeros and poles of the functions represented by (1.10). The arguments in this chapter depend on the asymptotic formula (1.11). The bound on the counting function allows us to prove that under the conditions of Theorem 1.10, the function may have at most one extra root. For functions which are differences of Herglotz functions, this extra root must be real.

In Chapter 5 we construct an explicit representation formula for the functions with the properties (i), (ii), and (iv) of Theorem 1.10. This formula is similar to the Weierstrass product formula but it does not have any restriction on the order of the function. Theorem 1.10 immediately follows from Corollary 5.1 which establishes the contradiction between the representation formula (5.1) and the decaying property (iii) of Theorem 1.10.

Chapter 2 Weyl m-function

2.1 Definitions

In this section we briefly discuss the Weyl m-function for the Schrödinger operator on the half-line $[0,\infty)$. This function, closely related to the spectral measure of the operator, is a very powerful tool in spectral analysis. To define the m-function, we consider two independent solutions, θ and ϕ, of the differential equation:

$$\left(-\frac{d^2}{dx^2}+q(x)\right)u(x,z)=z\,u(x,z) \tag{2.1}$$

with the boundary conditions at zero:

$$\begin{cases}\theta(0,z)=1\\ \partial_x\theta(0,z)=0\end{cases}\begin{cases}\phi(0,z)=0\\ \partial_x\phi(0,z)=1\end{cases}. \tag{2.2}$$

Weyl proved that the following alternative always holds: For any z, $\Im z>0$, either any solution of the differential equation (2.1) belongs to $L_2([0,\infty))$ or there exists a unique up to normalization constant $L_2([0,\infty))$ solution. Due to Weyl's construction, the former case is called the limit circle and the latter is called the limit point case. In the limit point case, the operator defined by the equation (2.1) and a boundary condition at zero is self-adjoint. (For the limit circle case, we must also introduce a boundary condition at infinity.)

Let $\psi(x,z)$ be an L_2 solution of the differential equation (2.1) of the form:

$$\psi(x,z)=\theta(x,z)+m(z)\,\phi(x,z). \tag{2.3}$$

In the limit point case, $m(z)$ is uniquely determined by this formula since θ and ϕ form the basis of the solutions of (2.1). In the limit circle case, we also require that

ψ obeys a boundary condition at infinity. The Weyl m-function is defined as the coefficient $m(z)$ in the formula (2.3). From the boundary conditions (2.2) it follows that $m(z) = \partial_x \psi(0, z)$. Equivalently, we can consider the L_2 solution $\psi(x, z)$ with an arbitrary normalization and define the m-function as follows:

Definition 2.1. Let $\psi(x, z)$ be an L_2 solution of the differential equation (2.1) on $[0, \infty)$. The Weyl m-function is defined by the formula:

$$m(z) = \frac{\partial_x \psi(0, z)}{\psi(0, z)}. \tag{2.4}$$

Let us consider the properties of the m-function. Since the equation (2.1) analytically depends on the spectral parameter z, the solution $\psi(x, z)$ and its x-derivative are analytic functions of the spectral parameter $z, \Im z > 0$. It follows that the m-function is a meromorphic function. Moreover, poles of the m-function are the eigenvalues of the Dirichlet problem on $[0, \infty)$. Indeed, $\psi(0, z)$ vanishes at the poles of the m-function. It means that $\psi(x, z)$ is the eigenfunction of the Dirichlet problem for all poles of $m(z)$. Similarly, all zeros of $m(z)$ are the eigenvalues of the Neumann problem. Note that all zeros and poles of the m-function must lie on the real line since they are the eigenvalues of a self-adjoint operator. Therefore, $m(z)$ is analytic on the upper half-plane.

The other important property of the m-function is that it maps the open complex upper half-plane into itself. This property follows from the Weyl construction for $m(z)$. Therefore, the m-function belongs to a class of Herglotz functions.

Definition 2.2. The class of functions analytic on the upper half-plane that maps the open upper half-plane into itself is called a class of Herglotz functions.

Herglotz functions can be fully characterized by the following representation formula:

Proposition 2.1. *Let $f(z)$ be a Herglotz function. Then there exist a positive mea-*

sure μ

$$\int_{\mathbb{R}} \frac{d\mu(t)}{t^2+1} < \infty \tag{2.5}$$

and real constants $A, B,\ B > 0$ *such that*

$$f(z) = A + B\,z + \int_{\mathbb{R}} \left(\frac{1}{t-z} - \frac{t}{t^2+1}\right) d\mu(t). \tag{2.6}$$

Remark. For meromorphic Herglotz functions, the representation formula can be written as a sum over the set of poles $\{\mu_k\}_{k=1}^{\infty}$:

$$f(z) = A + B\,z + \sum_{k=1}^{\infty} a_k \left(\frac{1}{\mu_k - z} - \frac{1}{\mu_k}\right), \tag{2.7}$$

where a_k are positive, μ_k are real, and

$$\sum_{\mu_k \neq 0} \frac{a_k}{\mu_k^2} < \infty. \tag{2.8}$$

There is an important relation between the m-function and the measure μ. Namely, the measure μ is a spectral measure of the corresponding Schrödinger operator. In particular, it implies that the support of μ belongs to the spectrum of the operator and $m(z)$ can be analytically continued across any interval on the real axis free from the spectrum. In particular, if the Schrödinger operator has discrete spectrum, the m-function is a meromorphic function.

The next result deals with the asymptotic behavior of the m-function. We consider asymptotics in an angular region on the complex plane where $m(z)$ is an analytic function, that is, in any sector

$$S_{\alpha,\beta} = \{z | \alpha < \arg z < \beta\} \tag{2.9}$$

which does not contain the support of the measure μ. The following result was obtained by Everitt [Eve72]:

Proposition 2.2. *Let $m(z)$ be an analytic function in $S_{\alpha,\beta}$, $0 < \alpha \leq \beta < \pi$. Then the m-function $m(z)$ of the Schrödinger operator has the following asymptotic behavior:*

$$m(z) = i\sqrt{z} + o(1), \qquad z \in S_{\alpha,\beta}. \tag{2.10}$$

Remark. Actually, Everitt studied the Neumann m-function $m_N(z) = 1/m(z)$ and he proved that

$$m_N(z) = -\frac{i}{\sqrt{z}} + o\left(\frac{1}{|z|}\right), \qquad z \in S_{\alpha,\beta}. \tag{2.11}$$

Trivially, (2.11) and (2.10) are equivalent.

Since the leading term of the asymptotic does not depend on the potential, we have the following corollary:

Corollary 2.1. *Let H_1 and H_2 be two Schrödinger operators. Then, for any $0 < \theta < \pi$ the corresponding m-functions obey:*

$$\lim_{r\to\infty} m_1(re^{i\theta}) - m_2(re^{i\theta}) = 0. \tag{2.12}$$

Proof. For $0 < \theta < \pi$ both m_1 and m_2 have the asymptotics (2.10). Since the leading terms of these asymptotics do not depend on the potential, the difference vanishes at infinity. ☐

We finish this section with the following result:

Proposition 2.3. *The m-function uniquely determines the potential almost everywhere on $[0, \infty)$.*

This fact was independently proven by Marchenko [Mar52] and Borg [Bor52] (see also [GL51, Sim]). In the next section, we use this result to reduce the main theorem to the question of the uniqueness of the m-function.

2.2 The m-functions for the whole real line

In this section we define the m-functions for the bounded-from-below Schrödinger operator on the whole real line. Also, by the end of this section we assume that the operator has discrete spectrum.

Bounded-from-below Schrödinger operators are always of a limit point case at both $\pm\infty$. This fact was originally proven by Hartman [Har48]. Gesztesy gave a simple proof of this fact in [Ges93]. Therefore, the m-functions are well-defined without any boundary condtion at infinty.

Definition 2.3. Let $\psi_+(x, z)$ and $\psi_-(x, z)$ be L_2 solutions of the differential equation (2.1) on $[0, \infty)$ and $(-\infty, 0]$, respectively. The Weyl m-functions $m_\pm(z)$ are defined by the formulas:

$$m_+(z) = \frac{\partial_x \psi_+(0, z)}{\psi_+(0, z)}, \qquad m_-(z) = -\frac{\partial_x \psi_-(0, z)}{\psi_-(0, z)}. \tag{2.13}$$

The $m_-(z)$ function has all the same properties stated in the previous section. We defined it with a minus sign to make it a Herglotz function.

Note that the m_+ function is determined by the potential only for positive x since we need to construct the solution ψ_+ only on the positive half-line. Similarly, m_- is determined by the potential on the negative half-line. By the Marchenko-Borg result (Proposition 2.3), the inverse statement is also true, that is, the m_- function uniquely determines the potential on the negative half-line. Therefore, Theorem 1.8 can be proven by showing that the m_- function is unique. That is, we assume that under the conditions of Theorem 1.8, there exist two distinct potentials $q_{1,2}(x)$, $x < 0$. Then the corresponding m-functions m_{1-} and m_{2-} are also distinct. We study the difference $m_{1-}(z) - m_{2-}(z)$. Define

$$\Delta m(z) = m_{1-}(z) - m_{2-}(z). \tag{2.14}$$

Our goal is to show that $\Delta m(z)$ must be zero.

Since the operator has discrete spectrum, the m-functions are meromorphic functions. As mentioned in the previous section, the poles of the $m_-(z)$ function are the eigenvalues of the Dirichlet problem on the negative half-line and they lie on the real axis.

There is an important relation between m_+ and m_-. Namely, we have

$$m_+(\lambda_k) = -m_-(\lambda_k) \tag{2.15}$$

for any eigenvalue λ_k. It follows from the fact that in the limit point case, there exists a unique (up to normalization) solution of the differential equation $-\partial^2 u + qu = zu$ which is L^2 at infinity. Therefore, the eigenfunction $u_k(x)$ must coincide (up to normalization) with both $u_+(x, \lambda_k)$ and $u_-(x, \lambda_k)$. Also, note that $m_-(z)$ has poles at eigenvalues in the Dirichlet problem and zeros at eigenvalues in the Neumann problem.

2.3 Summary

The central result of this work is the following theorem:

Theorem 2.1. *Let $f(z)$ be a meromorphic function of the form* (2.7) *without linear term:*

$$f(z) = C + \sum_{k=1}^{\infty} a_k \left(\frac{1}{\mu_k - z} - \frac{1}{\mu_k} \right) \tag{2.16}$$

with any real *constants μ_k, C, a_k, $0 < \mu_k < \mu_{k+1}$ which obeys* (2.8)*:*

$$\sum_{k=1}^{\infty} \left| \frac{a_k}{\mu_k^2} \right| < \infty. \tag{2.17}$$

Let $\{\lambda_k\}_{k=1}^{\infty}$ be some zeros of $f(z)$, $0 < \lambda_k < \lambda_{k+1}$. Assume that there exist $R > 0$ such that for any $r > R$,

$$\#\{\lambda_k \,|\, \lambda_k < r\} \geq \#\{\mu_k \,|\, \mu_k < r\} \tag{2.18}$$

for any k. Also, assume there exists $\theta \in (\pi/2, \pi)$ such that

$$f(re^{i\theta}) \to 0, \qquad r \to \infty. \tag{2.19}$$

Then $f(z) = 0$.

Chapters 3–5 are devoted to proving this theorem. Note that all the conditions but (2.18) hold for the difference of m-functions, assuming only that corresponding operators have bounded-from-below discrete spectrum. In the next section we discuss some assumptions on the potential which guarantee the bound between zeros and poles (2.18).

2.4 Bound on the potential and the relation between zeros and poles of the m-function $m_-(z)$

Let $\{\lambda_k\}_{k=1}^{\infty}$ be the spectrum of the operator H and let $n(r, H)$ denote the number of the eigenvalues of the operator H which are less than r. Also, let $\{\mu_k^D\}_{k=1}^{\infty}$ denote the spectrum of the Dirichlet problem on the negative half-axis. The estimate $q(-|x|) \geq q(|x|)$ on the potential implies the following relation between these spectra [GS]:

Lemma 2.4. *Suppose the potential $q(x)$ defines the operator H bounded from below. Assume that H has discrete spectrum and $q(-|x|) \geq q(|x|)$. Then $\lambda_{2k} \leq \mu_k^D$ for $k = 1, 2, 3, \dots$.*

Proof. First, note that H is a rank two perturbation[1] of the orthogonal sum of the Dirichlet operators on the positive and negative half-axes $H_-^D \oplus H_+^D$. Therefore, the spectrum of $H_\pm^D$ must be discrete. Let us consider the operator $H_s = -\partial_x^2 + q(-|x|)$. By the same argument, its spectrum is also discrete. Let $\{\beta_k\}_{k=1}^{\infty}$ denote the set of its eigenvalues. If $u_k(x)$ is any eigenfunction of H_s, then $u_k(-x)$ must also be the eigenfunction corresponding to the same eigenvalue. Since H_s has a simple spectrum,

[1] Actually, the difference of the resolvents is one-dimensional. It is not important for this lemma but we use this fact in Lemma 2.5.

these functions must be equal (up to a sign): $u_k(x) = \pm u_k(-x)$. Therefore, any eigenfunction of H_s on the negative half-axis is either a Dirichlet or a Neumann eigenfunction. Then by the Dirichlet-Neumann alternation, it follows that $\beta_{2k} = \mu_k^D$. On the other hand, by the assumption $H_s > H$, so $\beta_k \geq \lambda_k$. □

We already defined the function $\Delta m(z)$ by (2.14). The set of poles $\{\mu_k\}_{k=1}^{\infty}$ of $\Delta m(z)$ consists of poles $\{\mu_{1,k}^D\}_{k=1}^{\infty}$ of m_{1-} and poles $\{\mu_{2,k}^D\}_{k=1}^{\infty}$ of m_{2-} though some cancellations may happen. Also, $\Delta m(\lambda_k) = 0$ for any eigenvalue λ_k of H. In general, it may happen that $\Delta m(z)$ has some other zeros. (We prove in Chapter 4 that $f(z)$ has at most one zero besides $\{\lambda_k\}_{k=1}^{\infty}$.) Lemma 2.4 implies $\lambda_k < \mu_k$, that is, in the disk of any radius r, the function $\Delta m(z)$ has at least the same number of zeros as the number of poles. Note that any possible cancellation of poles would only improve the estimate. Therefore, the condition (ii) of Theorem 1.8 implies the bound (2.18) which implies that Theorem 1.8 follows from Theorem 2.1.

Let us discuss the slightly different conditions on the potential. Let H_+^D (resp. H_-^D) denote the Schrödinger operator on the positive (resp. negative) half-axis with a Dirichlet boundary condition at zero.

Lemma 2.5. *Suppose that there exists $R > 0$ such that $n(r, H_+^D) \geq n(r, H_-^D) + 1$ for $r > R$. Then $n(r, H) \geq 2n(r, H_-^D)$ for $r > R$, that is, the condition* (2.18) *holds.*

Proof. First, let us compare the numbers of the eigenvalues $n(r, H)$ and $n(r, H_-^D \oplus H_+^D)$. Note that

$$n(r, H_-^D \oplus H_+^D) = n(r, H_-^D) + n(r, H_+^D).$$

The conclusion of the lemma follows from the estimate

$$n(r, H_-^D \oplus H_+^D) \leq n(r, H) + 1. \tag{2.20}$$

The operator H is a rank two perturbation of $H_-^D \oplus H_+^D$. It turns out that the difference of the resolvents is the rank one operator, which implies the inequality (2.20). We can write the explicit formulas for the corresponding Green functions:

$$(H-z)^{-1}(x,y) = -\frac{\psi_-(\min(x,y),z)\psi_+(\max(x,y),z)}{m_+(z)+m_-(z)}, \tag{2.21}$$

$$(H_-^D \oplus H_+^D - z)^{-1}(x,y) = \psi_-(\min(x,y),z)\theta(\max(x,y),z)\chi_{x<0,\,y<0}(x,y) \tag{2.22}$$

$$+ \psi_+(\max(x,y),z)\theta(\min(x,y),z)\chi_{x>0,\,y>0}(x,y),$$

where $\chi_A(x,y) = 1$ if $(x,y) \in A$ and zero otherwise. Computing the difference of the resolvents, we obtain:

$$(H_-^D \oplus H_+^D - z)^{-1}(x,y) - (H-z)^{-1}(x,y) = \frac{h(x,z)h(y,z)}{m_+(z)+m_-(z)}, \tag{2.23}$$

where

$$h(x,z) = \psi_-(x,z)\chi_{x<0}(x) + \psi_+(x,z)\chi_{x>0}(x). \tag{2.24}$$

This formula implies that the difference of the resolvents is the rank one operator. □

The corollary of this result deals with the behavior of the potentials only at infinity.

Corollary 2.2. *Assume that there exist $R_+ > 0$ and $R_- < 0$ such that for Dirichlet operators H_{R+}^D on the interval $[R_+,\infty)$ and $H_{R_-}^D$ on the interval $[-\infty, R_-]$, the counting functions obey*

$$\frac{n(r,H_{R+}^D) - n(r,H_{R-}^D)}{\sqrt{r}} \equiv g(r) \to \infty. \tag{2.25}$$

Then the condition (2.18) *holds.*

Proof. The number of the eigenvalues of the Schrödinger operator $H_{[a,b]}$ (with any boundary condtions) on the finite interval $[a,b]$ has asymptotic $n(r,H_{[a,b]}) = O(\sqrt{r})$. Using this fact, we can estimate

$$n(r,H_+^D) = n(r,H_{R+}^D) + O(\sqrt{r}), \tag{2.26}$$

$$n(r,H_-^D) = n(r,H_{R-}^D) + O(\sqrt{r}). \tag{2.27}$$

These asymptotics imply

$$n(r, H_+^D) - n(r, H_-^D) = \sqrt{r}\,(g(r) + C)\,.$$

Since $g(r) \to \infty$ there exist R such that $g(r) + C > 1$, which implies the condition of Lemma 2.5. □

The application of this result is Theorem 1.9. We assume that q admits the Weyl asymptotics for the number of eigenvalues. Let us denote

$$q_+(x) = q(x) \tag{2.28}$$

$$q_-(x) = q(-x) \tag{2.29}$$

for $x > 0$. Then the Weyl asymptotics are given by the formula:

$$n(r, H_\pm) = \int_{q_\pm(x)<r} \sqrt{r - q_\pm(x)}\,dx\ (1 + o(1)). \tag{2.30}$$

Using this explicit formula, we check the condition (2.25). Let us fix R and $C > 1$ such that $q_+(x) < Cq_-(x)$ for $x > R$ (the condition (iii) of Theorem 1.9). Then we have the estimate (up to lower order terms):

$$\begin{aligned} n(r, H_+) - n(r, H_-) &\geq C_1\sqrt{r} + \int_{q_-(x)<r,\, x>R} \frac{q_-(x) - q_+(x)}{\sqrt{r - q_-(x)} + \sqrt{r - q_+(x)}}\,dx \\ &\geq C_1\sqrt{r} + \frac{C-1}{\sqrt{r}} \int_{q_-(x)<r,\, x>R} q_-(x)\,dx, \end{aligned} \tag{2.31}$$

where C is a constant from the condition (iii) of Theorem 1.9 and $C_1\sqrt{r}$ is the bound for the integrals over the finite region where the estimate (iii) does not hold. Note that $g(r)$ tends to infinity by the condition (iv) of Theorem 1.9.

Chapter 3 Asymptotic properties of the Borel transforms

3.1 Overview

In this section we study the asymptotic behavior of functions represented by the series of simple fractions and Borel transforms of signed measures. The results for the first case can be found in the book [GO70]. Using similar techniques, these results are extended for the second case.

We consider two classes of functions:

Definition 3.1. Denote the class of meromorphic functions represented by the series of simple fractions as HS:

$$f \in HS \iff f = \sum_{k=1}^{\infty} \frac{a_k}{z - \nu_k}, \quad a_k, \nu_k \in \mathbb{C}, \quad \sum_{k=1,\, \nu_k \neq 0}^{\infty} \left| \frac{a_k}{\nu_k} \right| < \infty. \tag{3.1}$$

Definition 3.2. Denote the class of functions represented by the Cauchy integral with the signed measure μ on $\mathbb{R}$ as HC:

$$f \in HC \iff f(z) = \int_{\mathbb{R}} \frac{d\mu(t)}{z - t}, \quad \int_{\mathbb{R}} \frac{|d\mu(t)|}{1 + |t|} < \infty. \tag{3.2}$$

The goal of this chapter is to prove the following theorem:

Theorem 3.1. *Let $f \in HS$ or $f \in HC$. Then*

$$\int_0^{2\pi} \ln^+ |f(re^{i\theta})|\, d\theta = o(1), \quad r \to \infty \tag{3.3}$$

and also for any p, $0 < p < 1$

$$\int_0^{2\pi} |f(re^{i\theta})|^p \, d\theta = o(1), \quad r \to \infty. \tag{3.4}$$

Remark. The function $\ln^+ x$ is a positive part of the logarithm:

$$\ln^+ x = \begin{cases} \ln x & \text{if } \ln x > 0, \\ 0 & \text{otherwise.} \end{cases} \tag{3.5}$$

3.2 Lemma

To prove Theorem 3.1 of this chapter, we need the following lemma:

Lemma 3.1. *If $F(z)$ is a holomorphic function in the disk $D = \{|z| < R\}$ such that either $\Re F(z)$ or $\Im F(z)$ has the same sign for any $z \in D$, then for any p, $0 < p < 1$, the following inequality holds*

$$\int_0^{2\pi} u^p(\theta) \, d\theta \le \frac{2\pi}{\cos\frac{\pi p}{2}} |F(0)|^p,$$

where

$$u(\theta) = \liminf_{\substack{z \to Re^{i\theta} \\ |z| < R}} |F(z)|.$$

Proof. Assume that $\Re F(z) > 0$ for $z < R$. Since $F(z)$ does not have zeros in D, we may fix the branch of $\arg F(z)$ such that $|\arg F(z)| < \pi/2$ for $|z| < R$. The function

$$F^p(z) = |F(z)|^p \exp(ip \arg F(z))$$

is analytic in D. It implies that $\Re(F(z))^p$ is a harmonic function there. Since

$$\Re(F(z))^p = |F(z)|^p \cos(p \arg F(z)) \ge |F(z)|^p \cos\frac{p\pi}{2},$$

then

$$\int_0^{2\pi} |F(re^{i\theta})|^p \, d\theta \le \frac{1}{\cos \pi p/2} \int_0^{2\pi} \Re(F(re^{i\theta}))^p \, d\theta = \frac{2\pi}{\cos \pi p/2} \Re(F(0))^p,$$

which implies that

$$\int_0^{2\pi} |F(re^{i\theta})|^p \, d\theta \le \frac{2\pi}{\cos \pi p/2} |F(0)|^p.$$

The result follows from Fatou's lemma as $r \to R$. □

3.3 Proof of Theorem 3.1

Let us now prove the theorem. The formula (3.3) follows from the formula (3.4) and the inequality

$$\ln^+ a \le \frac{1}{p} a^p, \; a > 0, \tag{3.6}$$

so we only need to prove (3.4).

First, we consider the case $f \in HC$, that is, f is given by the Cauchy integral of some signed measure. It is enough to prove the theorem for positive measures, since any signed measure can be represented as the difference of two positive measures. Also, we use the fact that for $p \le 1$,

$$|a + b|^p \le |a|^p + |b|^p. \tag{3.7}$$

In addition, we may assume that the support of the measure μ does not contain zero since any measure can be split into the measure compactly supported around zero and the measure which is zero in some neigborhood of zero. The Cauchy integral corresponding to the first measure is $o(1/z)$ at infinity, so it suffices to study the second measure.

Let us fix some $R > 0$. We split the integral for $f(z)$ into two parts:

$$\int_{\mathbb{R}} \frac{d\mu(t)}{z-t} = \int_{|t|<R} \frac{d\mu(t)}{z-t} + \int_{|t|\geq R} \frac{d\mu(t)}{z-t}. \tag{3.8}$$

We estimate these integrals separately. For $|t| \geq R$ we can apply Lemma 3.1 to the function:

$$F_1(z) = \int_{|t|\geq R} \frac{d\mu(t)}{z-t}. \tag{3.9}$$

This function obeys $\Re F_1 < 0$ if $|z| < R$. It immediately gives us

$$\int_0^{2\pi} \left| \int_{|t|\geq R} \frac{d\mu(t)}{z-t} \right|^p d\theta \leq \frac{2\pi}{\cos \pi p/2} \left| \int_{|t|\geq R} \frac{d\mu(t)}{t} \right|^p = o(1). \tag{3.10}$$

To estimate the integral over $|t| < R$, we note that for $|z| = R$

$$\left| \int_{|t|<R} \frac{1}{z-t}\, d\mu(t) \right| = \left| \int_{|t|<R} \frac{R}{R^2 - zt}\, d\mu(t) \right|. \tag{3.11}$$

Applying the lemma to the function

$$F_2(z) = \int_{-R}^{R} \frac{R}{R^2 - zt}\, d\mu(t),$$

which obeys $\Re F_2(z) > 0$ for $|z| < R$, we obtain

$$\begin{aligned} \int_0^{2\pi} \left| \int_{-R}^{R} \frac{d\mu(t)}{z-t} \right|^p d\theta &= \int_0^{2\pi} \left| \int_{-R}^{R} \frac{R\, d\mu(t)}{R^2 - tz} \right|^p d\theta \\ &\leq \frac{2\pi}{\cos \pi p/2} \left| \frac{1}{R} \int_{-R}^{R} d\mu(t) \right|^p. \end{aligned}$$

The last integral tends to zero. For example, for the interval $[0, R]$ we have:

$$\frac{1}{R} \int_0^R d\mu(t) \leq \frac{1}{\sqrt{R}} \int_0^{\sqrt{R}} \frac{d\mu(t)}{t} + \int_{\sqrt{R}}^{R} \frac{d\mu(t)}{t} = o(1).$$

The same asymptotic holds for the interval $[-R, 0]$. This estimate finishes the proof for the case $f \in HC$.

Now consider the case $f \in HS$. Let

$$\theta_k = \arg \nu_k, \quad a_k = \alpha_k + i\beta_k. \tag{3.12}$$

Let us fix $R > 0$ and write $f(z)$ as a sum of eight terms

$$\begin{aligned} f(z) &= \sum_{|\nu_k|>R}' \frac{e^{i\theta_k}\Re(A_k e^{-i\theta_k})}{z-\nu_k} + \sum_{|\nu_k|>R}'' + i \sum_{|\nu_k|>R}' \frac{e^{i\theta_k}\Im(A_k e^{-i\theta_k})}{z-\nu_k} \\ &+ i \sum_{|\nu_k|>R}'' + \sum_{|\nu_k|\le R}' \frac{\alpha_k}{z-\nu_k} + \sum_{|\nu_k|\le R}'' + i \sum_{|\nu_k|\le R}' \frac{\beta_k}{z-\nu_k} + i \sum_{|h_k|\le R}'' \\ &= G_1(z) + \cdots + G_8(z), \end{aligned}$$

where the sums $\sum'$ go over terms with positive values of $\Re(A_k e^{-i\theta})$, $\Im(A_k e^{-i\theta})$, α_k, β_k and sums $\sum''$ go over negative ones. Each of the terms $G_1, \dots, G_4$ can be estimated in the same way as $F_1(z)$. The terms $G_5, \dots, G_8$ can be estimated using the same trick (3.11). This remark finishes the proof.

Chapter 4 Jensen formula and zeros of $\Delta m(z)$

4.1 Notation and overview of results

In this chapter, we study the functions represented by a series of simple fractions. Our goal is to prove some bounds on the number of zeros minus the number of poles of such functions. In the first half of the chapter, we prove some general facts which we apply in the last section to the functions with properties (2.16, 2.17, 2.18) of Theorem 2.1. We prove that under these conditions, the function may have at most one extra zero. Since all zeros and poles lie on the real line, we are able to parameterize them by the integral of the counting function. Using this fact, in the next chapter, we prove the representation formula (5.1).

We start this section with a few definitions:

Definition 4.1. Denote the class of meromorphic functions represented by the regularized series of simple fractions as HS^1:

$$f \in HS^1 \quad \Longleftrightarrow \quad f = C + \sum_{k=1}^{\infty}\left(\frac{a_k}{z-\nu_k} + \frac{a_k}{\nu_k}\right), \tag{4.1}$$

$$a_k, \nu_k \in \mathbb{C}, \quad \sum_{k=1,\,\nu_k\neq 0}^{\infty}\left|\frac{a_k}{\nu_k^2}\right| < \infty.$$

Let us introduce the counting functions for zeros and poles of meromorphic functions.

Definition 4.2. Let f be a meromorphic function. Denote

- $n(r, f)$, the number of poles of f in the closed disk of radius r counting multiplicity,

- $n_\Delta(r, f) = n(r, 1/f) - n(r, f)$, the number of zeros minus the number of poles of f in the disk of radius r.

We also define the integrated counting functions.

Definition 4.3. Let f be a meromorphic function. Denote

$$N(r, f) = \int_0^r \frac{n(t, f) - n(0, f)}{t}\, dt + n(0, f) \ln r, \tag{4.2}$$

$$N_\Delta(r, f) = N(r, 1/f) - N(r, f). \tag{4.3}$$

The last object, which we introduce, controls the behavior of a meromorphic function at infinity averaged over the argument:

Definition 4.4. Let f be a meromorphic function. Denote

$$s(r, f) = \frac{1}{2\pi} \int_0^{2\pi} \ln^+ |f(re^{i\theta})|\, d\theta, \tag{4.4}$$

where $\ln^+ f$ is a positive part of the logarithm (3.5).

The goal of this chapter is to prove the following proposition:

Proposition 4.1. *Let $f \in HS^1$. Then f obeys*

$$N_\Delta(r, f) \le \ln r + o(1). \tag{4.5}$$

In the next section we discuss the Jensen formula and show that Proposition 4.1 is equivalent to decaying of the function $s(r, f)$ for any $f \in HS$. This decaying property was proven in Chapter 3. In the last section we consider a few important corollaries for the function with properties (2.16, 2.17, 2.18) of Theorem 2.1.

4.2 Jensen formula

For any holomorphic function $h(z)$, $h(0) \neq 0$, one has the Jensen formula [Lev64]:

$$\int_0^R \frac{n(t,1/h)}{t}\, dt = \frac{1}{2\pi}\int_0^{2\pi} \ln|h(r\, e^{i\theta})|\, d\theta - \ln|h(0)|, \tag{4.6}$$

where $n(t,1/h)$ is the number of zeros of $h(z)$ (Definition 4.2). A similar formula holds for any meromorphic function f, $f \neq 0, \infty$. One can write it as a ratio of two holomorphic functions $f = g/h$ and apply the Jensen formula to the numerator and denominator:

$$\int_0^R \frac{n_\Delta(t,f)}{t}\, dt = \frac{1}{2\pi}\int_0^{2\pi} \ln|f(r\, e^{i\theta})|\, d\theta - \ln|f(0)|, \tag{4.7}$$

where $n_\Delta(t,f)$ is the number of zeros minus the number of poles of f (Definition 4.2). The formula can be generalized further to include the case when $f(0)$ is a zero or a pole. Consider $\tilde{f}(z) = f(z)\, z^n$ such that $\tilde{f}(0) \neq 0, \infty$. Applying the Jensen formula to $\tilde{f}$ and using Definition 4.3 for $N_\Delta(r,f)$, one obtains

$$N_\Delta(r,f) = \frac{1}{2\pi}\int_0^{2\pi} \ln|f(r\, e^{i\theta})|\, d\theta - \ln|c_n|, \tag{4.8}$$

where

$$c_n = \tilde{f}(0) = \frac{f^n(0)}{n!} \tag{4.9}$$

is the first non-zero coefficient in the Taylor series expansion of $f(z)$ at zero.

Let us introduce the normalized function $\hat{f}$:

$$\hat{f}(z) = f(z)/c_n. \tag{4.10}$$

Using Definition 4.4 we can rewrite the right-hand side of the Jensen formula as a difference $s(r,\hat{f}) - s(r,1/\hat{f})$. Finally, we obtain the following form of the Jensen formula:

$$N_\Delta(r,f) + s(r,1/\hat{f}) = s(r,\hat{f}). \tag{4.11}$$

Note that $s(r,f)$ is non-negative by the definition of $\ln^+$. Therefore, the Jensen formula (4.11) implies the inequality:

$$N_\Delta(r,f) \le s(r,\hat{f}). \tag{4.12}$$

Now, Proposition 4.1 follows from the following lemma:

Lemma 4.2. *Let a function $f \in HS^1$. Then*

$$s(r,f) \le \ln r + o(1). \tag{4.13}$$

Proof. We represent $f(z)$ in the form

$$f(z) = z\left(\frac{C}{z} + \int_a^\infty \frac{1}{t(z-t)}\,d\mu(t)\right) = zf_1(z).$$

Since we can apply Theorem 3.1 to the function $f_1(z)$, we easily obtain

$$s(r,f) \le s(r,f_1) + s(r,z) = \ln r + o(1), \tag{4.14}$$

where $r = |z|$. □

Remark. If $f \in HS$ instead of HS^1, we have the sharper bound

$$N_\Delta(r,f) = o(1). \tag{4.15}$$

This bound immediately follows from (4.12) and Theorem 3.1.

4.3 Applications of the bound on counting function N_Δ

In this section, we consider the meromorphic functions with a given subset of zeros S.

Definition 4.5. For a given meromorphic function f and some subset of zeros S of f, define the counting function $n^S_\Delta(r,f)$ as the number of zeros of f belonging to subset S in the disk of radius r minus the number of poles of f in the disk of radius r.

The following corollary of Proposition 4.1 states that functions of the class HS^1 cannot have more zeros than poles in the following sense:

Corollary 4.1. *Let $f \in HS^1$. Assume that S is some subset of zeros of f. Suppose that there exist $R > 0$ such that $n^S_\Delta(r,f) \geq 0$ for any $r > R$. Then f may have at most one extra zero besides the set S.*

Proof. By the assumption of the corollary, $n^S_\Delta(r,f) \geq 0$ for any $r > R$. Let $\zeta_1, \zeta_2 \notin S$ be extra roots of $f(z)$, $\zeta_1 \neq \zeta_2$. Define

$$r_0 = \max(R, |\zeta_1|, |\zeta_2|).$$

Then for $r > r_0$ we have that $n_\Delta(r,f) \geq 2$. It implies the following estimate for $N_\Delta(r,f)$, $r > r_0$:

$$N_\Delta(r,f) \geq C + \int_{r_0}^{r} \frac{n_\Delta(r,f)}{t}\,dt \geq C_1 + 2\ln r$$

for some constants C, C_1. This estimate contradicts the bound (4.5) for $N_\Delta(r,f)$. So we have proven that $f(z)$ may have at most one extra root. □

Remarks. 1. For the functions f in the class HS, the remark at the end of the previous section implies that all zeros of f are in S. Indeed, if ζ is a zero of f, $\zeta \notin S$, we have

$n_\Delta(r, f) > 1$, $r > |\zeta|$. However, $N_\Delta(r, f) = o(1)$ which implies that the integral of $n_\Delta(t, f)/t$ converges. This contradiction proves the statement.

2. If all coefficients in the simple fraction representation of a function $f \in HS^1$ are real, then all complex roots come in conjugate pairs. In such a case, the extra root of Corollary 4.1 must be real. In particular, this is a case for functions given by the differences of Herglotz functions. We will use this remark for constructing the analog of the product formula for such functions.

Corollary 4.2. *Let $f \in HS^1$. Assume that S is some subset of zeros of f. Suppose that $n^S_\Delta(r, f) \geq 0$ for any r. Then S is the set of all zeros of f.*

Proof. Assuming that $f(\zeta) = 0, \zeta \notin S$, we define the function

$$g(z) = \frac{f(z)}{(z-\zeta)}$$

which is a meromorphic function with exactly the same poles as $f(z)$. By the assertion of the corollary, we have $n_\Delta(r, g) \geq 0$. On the other hand, the function $s(r, g)$ goes to zero because $s(r, g) \leq s(r, f_1) + s(r, z/(z-\zeta))$. The function $f_1 = f/z$ was defined in Lemma 4.2. The first term of the inequality vanishes by Theorem 3.1. For the second term we have that

$$\ln^+ \left|\frac{z}{z-\zeta}\right| = O\left(\frac{1}{z}\right)$$

uniformly in $\arg z$. Therefore, we obtain $N_\Delta(r, g) \leq s(r, g) \to 0$. But $N_\Delta(r, g)$ is the integral of a positive function and it cannot tend to zero. It proves that S is the set of all roots of $f(z)$. □

The last fact we prove in this chapter gives the bound for an integral of the logarithm of a function $f \in HS^1$ with $n_\Delta > 0$.

Lemma 4.3. *Let $f \in HS^1$. Assume that there exists $R > 0$ such that $n_\Delta(r, f) > 0$*

for any $r > R$. *Then*

$$\frac{1}{2\pi}\int_0^{2\pi} \left|\ln|f(re^{i\theta})|\right| \, d\theta \leq 2\ln r + C \tag{4.16}$$

for some $C > 0$.

Proof. Note that the left-hand side of the expression (4.16) is bounded by the sum $s(r,f) + s(r,1/f)$. Since $n_\Delta(r,f) > 0$ for all $r > R$, the Jensen formula (4.8) implies

$$s(r,1/f) \leq s(r,f) + C,$$

where

$$C = \left|\int_0^R \frac{n_\Delta(t,f)}{t}\, dt\right|.$$

Therefore, the bound (4.13) implies the lemma. □

Chapter 5 Integral representation of the functions in HS^1

5.1 Overview

In this chapter, we obtain the integral representation for the functions in the class HS^1 with *positive* zeros and poles (see Definition 4.1 for the class HS^1). Namely, we prove the following theorem:

Theorem 5.1. *Let $f \in HS^1$. Assume that all zeros and poles of the function f are positive. Also, assume that there exists $R > 0$ such that $n_\Delta(r, f) \geq 0$ for $r > R$. Then*

$$f(z) = f(0) \exp\left(I(z, f)\right), \tag{5.1}$$

where $I(z, f)$ is defined for $z \notin \mathbb{R}^+$ by the formula

$$I(z, f) = \int_0^\infty \frac{z\, n_\Delta(t, f)}{t\, (z - t)}\, dt. \tag{5.2}$$

First, let us discuss the important corollary of this result:

Corollary 5.1. *Let $f \in HS^1$. Assume that all zeros and poles of f belong to the interval $[A, \infty)$, $A \in \mathbb{R}$ and there exists $R > 0$ such that for any $r > R$, the function f has at least as many zeros as poles in the disk of radius r. Then for any $\theta \in (\pi/2, 3\pi/2)$*

$$\lim_{r \to \infty} |f(re^{i\theta})| > 0.$$

Proof of Corollary 5.1. Denote $z = re^{i\theta}$. Let us introduce the function $\hat{f}(z) = f(z +$

$A+\varepsilon)$, $\varepsilon > 0$. Then

$$\lim_{r\to\infty} |\hat{f}(z)| = \lim_{r\to\infty} |f(z)|.$$

On the other hand, all zeros and poles of $\hat{f}$ belong to $\mathbb{R}^+$. Note that for $\Re z < 0$, the real part of the kernel of the integral $I(z, f)$ is positive:

$$\Re\frac{z}{z-t} = \frac{r^2 + t(-\Re z)}{(t-\Re z)^2 + \Im z^2} > 0. \tag{5.3}$$

Applying Theorem 5.1, we can estimate $\ln|\hat{f}(z)|$ for $\Re z < 0$:

$$\Re I(-r, \hat{f}) = \int_0^\infty \Re\frac{z n_\Delta(t,\hat{f})}{t(z-t)}\,dt \geq \int_0^R \Re\frac{z n_\Delta(t,\hat{f})}{t(z-t)}\,dt \longrightarrow \int_0^R \frac{n_\Delta(t,\hat{f})}{t}\,dt > -\infty.$$

We use the fact that $n_\Delta(r,\hat{f})$ is non-negative for $r > R$ so the integral over the set $[R,\infty)$ is positive. The last integral converges because the support of $n_\Delta(r,\hat{f})$ is in the interval $[\varepsilon,\infty)$ so there is no singularity at zero. Since $\ln \hat{f}(re^{i\theta})$ is bounded from below as $r\to\infty$, the function $f(re^{i\theta})$ is bounded away from zero as $r\to\infty$. □

Let us make a few remarks about Theorem 5.1.

Remarks. 1. Since all zeros and poles of f are positive, the function $n_\Delta(r,f) = 0$ in some neighborhood of zero. Therefore, the integral (5.2) does not have a singularity at zero.

2. In the case of functions of subexponential type, the formula (5.1) can be derived from the Weierstrass product formula (so that the product formula does not require correcting exponential factors). Similar arguments can be applied to Herglotz functions of finite order. Zeros and poles of Herglotz functions interlace each other. It implies that for such functions $n_\Delta(r,f)$ is bounded. Therefore, the integral (5.2) converges. Theorem 5.1 shows that the representation formula (5.1) is more general.

Let us prove the convergence of the integral (5.2) for functions in HS^1. The result follows from the lemma:

Lemma 5.1. *Let $f(z) \in HS^1$. Assume that zeros and poles of f are positive. Then* *for any* $\varepsilon > 0$, *the counting function* $n_\Delta(r, f)$ *obeys*

$$\int_a^\infty \frac{n_\Delta(t,f)}{t^{1+\varepsilon}} dt < \infty, \tag{5.4}$$

where

$$a = \inf \operatorname{supp} n_\Delta(r, f). \tag{5.5}$$

Proof. Note that $a > 0$ since all zeros and poles of f are positive. Divide the interval $[a, \infty) = [a, 2^K] \cup \left(\bigcup_{k=K}^{\infty} [2^k, 2^{k+1}]\right)$, where K is the ceiling of $\log_2 a$ (minimal possible integer which is larger than $\log_2 a$). Let us consider the integral over the segment $[2^k,\ 2^{k+1}]$, for some k. Using Theorem 4.1, we obtain:

$$\int_{2^k}^{2^{k+1}} \frac{n_\Delta(t,f)}{t^{1+\varepsilon}}\, dt \le \frac{1}{2^{k\varepsilon}} \int_{2^k}^{2^{k+1}} \frac{n_\Delta(t,f)}{t}\, dt \le \frac{\ln 2^{k+1} + C}{2^{k\varepsilon}} = 2^{-k\varepsilon}((k+1)\ln 2 + C).$$

It implies the convergence of the integral (5.4)

$$\int_a^\infty \frac{n_\Delta(t,f)}{t^{1+\varepsilon}}\, dt \le C_1 + \ln 2 \sum_{k=K}^{\infty} 2^{-k\varepsilon}(k + C_2) < \infty.$$

□

In the next section, we show that $\exp(I(z, f))$ is a meromorphic function with exactly the same zeros and poles as $f(z)$. Therefore, the ratio is the exponent of the analytic function $\phi(z)$ which implies the formula:

$$f(z) = \exp\left(I(z, f) + \phi(z)\right). \tag{5.6}$$

In the last section, we compare the asymptotics of the logarithms of both sides of the equation (5.6) at infinity. We study the asymptotical behavior of $I(z, f)$. On the other hand, the bound on the counting function n_Δ allows us control $\ln f(z)$. Finally, using these asymptotics we show that the function ϕ must be a constant.

5.2 Zeros and poles of $\exp I(z)$

The purpose of this section is to prove the following lemma:

Lemma 5.2. *Assume that $f(z)$ is a meromorphic function on the entire complex plane, whose zeros and poles lie on the positive half-axis and the integral (5.4) converges. Then $\exp(I(z,f))$ is a meromorphic function. It has exactly the same zeros and poles as $f(z)$ so $f(z)\exp(-I(z,f))$ is an entire function without zeros.*

Proof. By Lemma 5.1, the integral $I(z,f)$ converges absolutely and uniformly on every compact set K such that $K \cap \operatorname{supp} n_\Delta(r,f) = \emptyset$. Therefore, this integral is an analytic function on K so that $\exp(I(z,f)$ is also an analytic function on K.

Let us fix some $R > 0$ and divide the integral $I(z,f)$ into two parts:

$$I(z,f) = \int_a^{2R} \frac{z\,n_\Delta(t,f)}{t(z-t)}\,dt + \int_{2R}^{\infty} \frac{z\,n_\Delta(t,f)}{t(z-t)}\,dt = I_1(z,f) + I_2(z,f). \tag{5.7}$$

By the previous remark, the integral $I_2(z,f)$ defines a holomorphic function in the disk $|z| < R$. We study the integral $I_1(z,f)$. Integrating by parts, we can rewrite it as follows:

$$\int_0^{2R} \frac{z\,n_\Delta(t,f)}{t(z-t)}\,dt = \int_0^{2R} \ln\left(1-\frac{z}{t}\right)\,dn_\Delta(t,f) + \ln\left(1-\frac{z}{2R}\right) n_\Delta(2R,f). \tag{5.8}$$

Note that $n_\Delta(x,f) = 0$ for $x < a$ so the substitution at zero vanishes. The integral with respect to the measure $dn_\Delta(t,f)$ is equivalent to the sum over the jumps of $n_\Delta(t,f)$ that are zeros and poles of f. Now consider the exponent of $I(z,f)$:

$$I(z,f) = \prod_{\substack{\lambda_k:\ f(\lambda_k)=0,\\ |\lambda_k|<2R}} \left(1-\frac{z}{\lambda_k}\right) \prod_{\substack{\mu_k:\ f(\mu_k)=0,\\ |\mu_k|<2R}} \left(1-\frac{z}{\mu_k}\right)^{-1} e^{\psi(z)}, \tag{5.9}$$

where $\psi(z)$ is a holomorphic function in the disk $|z| < R$. Therefore, the function $\exp I(z,f)$ has the same zeros and poles in the disk $|z| < R$ as the function f. Since R is an arbitrary number, the lemma is proven. $\square$

5.3 Asymptotic behavior of $I(z, f)$

Lemma 5.2 in the previous section implies that the function f can be represented as

$$f(z) = \exp(I(z, f) + \phi(z))$$

with some entire function $\phi(z)$. The next step is to show that $\phi(z)$ must be a constant. Let us express the real part of ϕ:

$$\Re\phi(z) = \Re I(z, f) - \ln|f(z)|. \tag{5.10}$$

Averaging over the ring $R/2 < |z| < R$, we obtain:

$$\begin{aligned} \frac{1}{3/4\pi R^2}\int_{\frac{R}{2}<|z|<R} |\Re\phi(z)|\, d^2z \;&\leq\; \frac{1}{3/4\pi R^2}\int_{\frac{R}{2}<|z|<R} |\Re I(z, f)|\, d^2z \\ &+\; \frac{1}{3/4\pi R^2}\int_{\frac{R}{2}<|z|<R} |\ln|f(z)||\, d^2z. \end{aligned} \tag{5.11}$$

Lemma 4.3 implies the logarithmic bound on the last integral. The goal of this section is to obtain the asymptotics for the average of $|\Re I(z, f)|$:

Lemma 5.3. *For any $\varepsilon > 0$, the integral $I(z, f)$ obeys*

$$\frac{1}{R^2}\int_{R/2<|z|<R} |I(z, f)|\, d^2z = o(R^{\varepsilon}). \tag{5.12}$$

Remark. Note that $I(z, f)$ is a singular integral with a Cauchy kernel $1/(t - z)$. Therefore, it is easier to estimate the two-dimensional average of $I(z, f)$ over the ring $R/2 < |z| < R$ rather than the one-dimensional average over the argument θ (as in Lemma 4.3).

Proof. We estimate $I(z, f)$ by the sum of two integrals over the sets $[0, 2R]$ and $[2R, \infty]$, where $R \geq |z|$:

$$|I(z, f)| \leq \int_0^{2R} \frac{|z|\, n_\Delta(t, f)}{t\,|z - t|}\, dt + \int_{2R}^{\infty} \frac{|z|\, n_\Delta(t, f)}{t\,|z - t|}\, dt. \tag{5.13}$$

First, let us estimate the second integral. For $|z| \leq R \leq t/2$, the denominator of this integral is bounded from below:

$$t|z-t| \geq \frac{t^2}{2}.$$

Let us fix arbitrary $\varepsilon \in (0,1)$. Using Lemma 5.1 this integral can be estimated by R^ε:

$$\begin{aligned} \int_{2R}^{\infty} \frac{|z|\, n_\Delta(t,f)}{t\,|z-t|}\, dt &\leq 2R \int_{2R}^{\infty} \frac{n_\Delta(t,f)}{t^{1-\varepsilon}\cdot t^{1+\varepsilon}}\, dt \\ &\leq 2\frac{R}{(2R)^{1-\varepsilon}} \int_{2R}^{\infty} \frac{n_\Delta(t,f)}{t^{1+\varepsilon}}\, dt = o(R^\varepsilon). \end{aligned} \tag{5.14}$$

Finally, we average the expression (5.14) over the ring $R/2 < |z| < R$:

$$\frac{1}{3/4\pi R^2} \int_{\frac{R}{2}<|z|<R} \int_{2R}^{\infty} \frac{|z|\, n_\Delta(t,f)}{t\,|z-t|}\, dt\, d^2z = o(R^\varepsilon). \tag{5.15}$$

Now we consider the integral over the set $[0,2R]$ in the formula (5.13). Averaging over the ring and changing the order of integration, we obtain:

$$\begin{aligned} \frac{1}{3/4\pi R^2} \int_{\frac{R}{2}<|z|<R} \int_{0}^{2R} \frac{|z|\, n_\Delta(t,f)}{t\,|z-t|}\, dt\, d^2z &= \int_0^{2R} \frac{n_\Delta(t,f)}{t} \times \\ &\quad \left(\frac{1}{3/4\pi} \int_{\frac{1}{2}<|z|<1} \frac{|z|\, d^2z}{|z|-t/R} \right) dt. \end{aligned} \tag{5.16}$$

Let us consider the inside integral on the right-hand side of (5.16):

$$F(u) = \frac{1}{3/4\pi} \int_{\frac{1}{2}<|z|<1} \frac{|z|}{|z|-u}\, d^2z. \tag{5.17}$$

Since the singularity $1/(|z|-u)$ is integrable with respect to d^2z, the integral (5.17) converges for every u. Also, F is a decreasing function of u:

$$u_1 > u_2 \quad \Longrightarrow \quad F(u_1) < F(u_2).$$

Therefore, the function $F(u)$ is uniformly bounded: $F(u) < C$ for any u. Then the whole expression (5.16) is bounded by:

$$\frac{1}{3/4\pi R^2}\int_{\frac{R}{2}<|z|<R}\int_0^{2R}\frac{|z|\,n_\Delta(t,f)}{t\,|z-t|}\,dt\,d^2z \leq C\ln 2R + o(1). \tag{5.18}$$

The estimates (5.15), (5.18) imply the asymptotics (5.12). The lemma is proven. □

Lemmas 5.3, 4.3, and the formula (5.11) imply the following asymptotics for the real part of the function ϕ:

$$\frac{1}{R^2}\int_{R/2<|z|<R}|\Re\phi(z)|\,d^2z = o(R^\varepsilon). \tag{5.19}$$

On the other hand, ϕ is an analytic function. The following lemma shows that, under the condition (5.19), the function ϕ must be a constant.

Lemma 5.4. *If, for some $\varepsilon < 1$, the entire function $\phi(z)$ obeys*

$$\frac{1}{R^2}\int_{R/2<|z|<R}|\Re\phi(z)|\,d^2z = o(R^\varepsilon),$$

then $\phi(z)$ must be a constant.

Proof. Any entire function $\phi(z)$ has the Taylor series expansion

$$\phi(z) = \sum_{k=0}^{\infty} a_k z^k. \tag{5.20}$$

For every $r > 0$, the coefficients $a_k, k > 0$ are given by the Cauchy formula

$$a_k = \frac{1}{2\pi i}\oint_{|z|=r}\frac{\phi(z)}{z^{k+1}}\,dz = \frac{1}{4\pi i}\oint_{|z|=r}\frac{\Re\phi(z)}{z^{k+1}}\,dz. \tag{5.21}$$

The last equality in (5.21) follows from the fact:

$$\oint_{|z|=r}\frac{\overline{\phi(z)}}{z^k}\,dz = 0, \quad k \geq 0.$$

Integrating $|a_k|$ over the interval $R/2 < r < R$, we obtain

$$|a_k| \leq \frac{1}{2\pi R} \int_{\frac{R}{2}}^{R} \oint_{|z|=r} \frac{|\Re\phi(z)|}{r^{k+1}} \, |dz| dr \leq \frac{C}{R^{2+k}} \int_{\frac{R}{2}<|z|<R} |\Re\phi(z)| \, d^2 z = o(R^{\varepsilon-k}).$$

It follows that $a_k = 0$ for any $k \neq 0$ so $\phi(z)$ is a constant. $\square$

Equation (2.16), Lemma 5.3, and Lemma 5.4 imply that $f(z) = C\exp(I(z,f))$. Since $I(0,f) = 0$ we have $C = f(0)$. This finishes the proof of Theorem 5.1.

Bibliography

[Akt94] T. Aktosun, *Bound states and inverse scattering for the Schrödinger equation in one dimension*, J. Math. Phys. **35** (1994), 6231–6236.

[Akt96] T. Aktosun, *Inverse Schrödinger scattering on the line with partial knowledge of the potential*, SIAM J. Appl. Math. **56** (1996), 219–231.

[AKvdM93] T. Aktosun, M. Klaus, and C. van der Mee, *On the Riemann-Hilbert problem for the one-dimensional Schrödinger equation*, J. Math. Phys. **34** (1993), 2651–2690.

[Bor46] G. Borg, *Eine Umkehrung der Sturm-Liouvilleschen Eigenwertaufgabe*, Acta Math. **78** (1946), 1–96.

[Bor52] G. Borg, *Uniqueness theorems in the spectral theory of* $y'' + (\lambda - q(x))y = 0$, Proc. 11th Scandinavian Congress of Mathematicians, Johan Grundt Tanums Forlag, Oslo (1952), 276–287.

[BSL95] M. Braun, S. Sofianos, and R. Lipperheide, *One-dimensional Marchenko inversion in the presence of bound states*, Inverse Problems **11** (1995), L1–L3.

[dRC90] R. del Rio Castillo, *On boundary conditions of an inverse Sturm-Liouville problem*, SIAM J. Appl. Math **50** (1990), 1745–1751.

[Eve72] W. Everitt, *On a property of the m-coefficient of a second-order linear differential equation*, J. London Math. Soc. **4** (1972), no. 2, 443 – 457.

[Ges93] F. Gesztesy, *Double commutation method*, J. Funct. Anal. **117** (1993), 401–446.

[GL51] I.M. Gel'fand and B.M. Levitan, *On the determination of a differential equation from its special function*, vol. 15, Izv. Akad. Nauk SSR. Ser. Mat., 1951, (Russian) English transl. in Amer. Math. Soc. Transl. Ser. (2) **1** (1955), 253–304.

[GO70] A.A. Goldberg and I.V. Ostrovsky, *Raspredelenie znachenii meromorphnih funkcii*, Nauka, Moskva, 1970, (Russian).

[GS] F. Gesztesy and B. Simon, *Inverse spectral analysis with partial information on the potential, II. The case of discrete spectrum*, Trans. Amer. Math. Soc., to appear.

[GS97a] F. Gesztesy and B. Simon, *Inverse spectral analysis with partial information on the potential, I. The case of an a.c. component in the spectrum*, Helv. Phys. Acta **70** (1997), 66–71.

[GS97b] F. Gesztesy and B. Simon, *m-functions and inverse spectral analysis for finite and semi-infinite Jacobi matrices*, J. d'Analyse Math. **73** (1997), 267–324.

[GW95] B. Grebert and R. Weder, *Reconstruction of a potential on the line that is a priori known on the half-line*, SIAM J. Appl. Math. **55** (1995), 242–254.

[Hal80] O.H. Hald, *Inverse eigenvalue problem for the mantle*, Geophys. J. R. Astr. Soc. **62** (1980), 41–48.

[Har48] P. Hartman, *Differential equations with non-oscillatory eigenfunctions*, Duke Math. J. **15** (1948), 697–709.

[HL78] H. Hochstadt and B. Lieberman, *An inverse Sturm-Liouville problem with mixed given data*, SIAM J. Appl. Math. **34** (1978), 676–680.

[Hoc73] H. Hochstadt, *The inverse Sturm-Liouville problem*, Commun. Pure Appl. Math. **26** (1973), 715–729.

[Hoc79] H. Hochstadt, *On the construction of a Jacobi matrix from mixed given data*, Lin. Algebra Appl. **28** (1979), 113–115.

[Jay] S. Jayawardena, *The reconstruction of symmetric, anti-symmetric and partially known potentials from spectral data*, preprint.

[Kha84] A.B. Khaled, *Problème inverse de Sturm-Liouville associé à un opérateur différentiel singulier*, C. R. Acad. Sc. Paris **299** (1984), 221–224.

[KM56] I. Kay and H.E. Moses, *Reflectionless transmission through dielectrics and scattering potentials*, J. Appl. Phys. **27** (1956), 1503–1508.

[Lev49] N. Levinson, *The inverse Sturm-Liouville problem*, Mat. Tidskr. **B** (1949), 25–30.

[Lev64] B.Ja. Levin, *Distribution of Zeros of Entire Functions*, Amer. Math. Soc., Providence, RI, 1964.

[Lev68] B. Levitan, *On the determination of a Sturm-Liouville equation by two spectra*, Amer. Math. Soc. Transl. **68** (1968), 1–20.

[Lev87] B. Levitan, *Inverse Sturm-Liouville Problems*, VNU Science Press, Utrecht, 1987.

[LG64] B.M. Levitan and M.G. Gasymov, *Determination of a differential equation by two of its spectra*, Russ. Math. Surv. **19:2** (1964), 1–63.

[Mal94] M.M. Malamud, *Similarity of Volterra operators and related questions of the theory of differential equations of fractional order*, Trans. Moscow Math. Soc. **55** (1994), 57–122.

[Mal97] M.M. Malamud, *Spectral analysis of Volterra operators and inverse problems for systems of differential equations*, preprint (1997).

[Mar52] V. Marchenko, *Some questions in the theory of one-dimensional linear differential operators of the second order, I*, Trudy

Moskov. Mat. Obšč. (Russian) (1952), no. 1, 327–420, (English transl. in Amer. Math. Soc. Transl. (2) **101** (1973), 1–104).

[PT87] J. Pöschel and E. Trubowitz, *Inverse Scattering Theory*, Academic Press, Boston, 1987.

[RGS97] R. Del Rio, F. Gesztesy, and B. Simon, *Inverse spectral analysis with partial information on the potential, III. Updating boundary conditions*, Intl. Math. Res. Notices (1997), no. 15, 751–758.

[RS94] W. Rundell and P. Sacks, *On the determination of potentials without bound state data*, J. Comp. Appl. Math. **55** (1994), 325–347.

[Sac93] P. Sacks, *Reconstruction of steplike potentials*, Wave Motion **18** (1993), 21–30.

[Sim] B. Simon, *A new approach to inverse spectral theory, I. The basic formalism*, Annals of Math., to appear.

[Suz82] T. Suzuki, *Deformation formulas and their applications to spectral and evolutional inverse problems*, Lecture Notes in Num. Appl. Anal. **5** (1982), 289–311.

[Suz86] T. Suzuki, *Inverse problems for heat equations on compact intervals and on circles, I*, J. Math. Soc. Japan **38** (1986), 39–65.