EFFECTS OF A MAGNETIC FIELD ON THE TRACE OF THE HEAT KERNEL FOR A SCHRÖDINGER OPERATOR WITH A POTENTIAL WELL

Thesis by

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In Partial Fulfillment of the Requirements
for the Degree of
Doctor of Philosophy

California Institute of Technology
Pasadena, California
1987

(Submitted March 16, 1987)

Acknowledgements

I wish to thank my advisor, Dr. Barry Simon, for his many helpful remarks and suggestions. Further, I wish to thank Dr. Michael Taylor for some suggestions relating to the use of pseudodifferential operators for heat equations. I also want to thank Dr. W. Luxemburg for his enthusiastic discussions with me and for the motivation he gave me. I also wish to thank in general the committee members for their time.

Among my fellow graduate students, I particularly wish to thank Bart Huxtable, Askell Hardarson, Clemens Glaffig and John Lindner for various comments and suggestions as I was preparing this thesis.

Finally, I wish to thank my parents, Kirk and Annaliese, and my fiancé, Robert, for their continued support and encouragement.

Abstract

This paper looks at the effect of a uniform magnetic field on the trace of the heat kernel for a Schrödinger operator with a well type potential. Using weighted Sobolev space techniques and noticing the gauge invariance of the perturbation, I show that the magnetic field first appears at a higher term in the small time asymptotic expansion of the trace of the heat kernel than might be naively expected.

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Notation

x or y will usually refer to position space coordinates.

ξ will usually refer to momentum space coordinates.

$\partial_i = \frac{\partial}{\partial \xi_i}$, $\partial_{x_i} = \frac{\partial}{\partial x_i}$, $D_j = -i\partial_{x_j}$, $D_{\xi_j} = -i\partial_j$.

$\partial^\alpha = \prod_{i=1}^{n} \partial_i^{\alpha_i}$ for a multi index α, and similarly for D^α .

$(\text{curl } A)_{ij} \equiv \partial_i A_j - \partial_j A_i$ is the exterior derivative.

$f \sim g$ if both $f^{-1} g$ and $g^{-1} f$ are bounded.

$\langle \cdot \rangle = (1 + |\cdot|^2)^{\frac{1}{2}}$

$\Phi(x,\xi) = \langle x\rangle^k + \langle \xi \rangle$

$\mathcal{S}$: the Schwartz space, C^∞ functions of rapid decrease.

$\mathcal{D}'$: the space of distributions, the dual space to $C_0^\infty(\mathbb{R}^n)$.

H^μ: the μ^{th} Sobolev space with norm $\| \cdot \|_\mu$.

S^μ: the class of symbols of order μ .

OPS^μ: the operators corresponding to symbols in S^μ .

$||| \cdot |||_m = \| \cdot \|_{m \log \Phi}$, the norm in $H^{m \log \Phi}$.

Introduction.

Recently there has been much interest in asymptotic expansions for eigenvalue distribution functions, the heat kernel and the wave equation. The index theorem relating spectral quantities with geometric quantities from the underlying manifold has aroused this interest. This paper examines the effects a uniform magnetic field will have on the trace of the heat kernel. I start with operators of the form $-\Delta + V$ acting on $L^2(\mathbb{R}^n)$, where the potential V has polynomial growth at infinity and is bounded below so that the heat kernel of the operator is indeed trace class.

Instead of the magnetic field directly perturbing the equation, the magnetic vector potential, A, enters and one has

$$H(A) = (-i\nabla + A)^2 + V ,$$

where the curl of A is the magnetic field. Although there has been a tremendous amount of work on various properties of Schrödinger equations, the magnetic field case has often been excluded. Leinfelder (9) determined the essential spectrum of a broad class of magnetic potentials and has a comprehensive reference list of some previous work in this area. The additional complications introduced by the magnetic field lead to separate considerations of many cases, see for example (4), (8), and (10). For the more general case of a Yang-Mills potential, Schrader and Taylor (12) proved that there is a complete asymptotic expansion for the trace of the heat kernel in the parameter $\hbar$.

The magnetic field in dimensions other than three is typically defined as a 2-form or a skew-symmetric matrix. The *curl* or the exterior derivative is thus $\partial_k A_\ell - \partial_\ell A_k$ for the (k,ℓ) entry in the matrix form. I will use the matrix form of notation for the magnetic field B, denoted by B^*. For the uniform magnetic field, the standard choice for A(x) is $-\frac{1}{2} B^* x$; however, A is not uniquely determined. Leinfelder (9) proved several results that clarify this situation. He showed that if $A \in L^4_{\ell oc}$, $\nabla \cdot A \in L^4_{\ell oc}$ and $V \in L^4_{\ell oc}$ then H(A) is essentially self-adjoint on $C_0^\infty(\mathbb{R}^n)$. He also proved for two functions $A, A' \in L^4_{\ell oc}$ with $\nabla \cdot A, \nabla \cdot A' \in L^4_{\ell oc}$, related by a distribution $\lambda \in \mathcal{D}'$ with $A' = A + \nabla\lambda$ then H(A) is unitarily equivalent to H(A′) by

$e^{i\lambda}\ H(A)e^{-i\lambda} = H(A')$. This is called a gauge change and one easily sees curl A = curl A' . Leinfelder also verified that the Coulomb gauge condition, $\nabla \cdot A = 0$, can always be satisfied by some gauge change.

For the potentials considered in this paper, these results follow from the diamagnetic inequalities. Simon in (13), (14) showed that, for a large class of potentials and vector potentials, that for $t > 0$

$$\|e^{-tH(A)}\|_{p,q} \leq \|e^{-tH(0)}\|_{p,q} \quad \text{for all} \quad p,q,$$

where these are the operator norms from L^p to L^q . For convenience, H will refer to $H(A)$ and H_0 to $H(0)$ in the remainder of this thesis.

The motivation for this work originates with the simple case of a two-dimensional harmonic oscillator. In this case the eigenvalues are explicitly known in both the perturbed and unperturbed cases. For $H_0 = -\Delta + |x|^2$, the eigenvalues are at $2(n + 1)$ with multiplicities of $n + 1$. For the perturbed case, $H = -\Delta + iB^*x \cdot \nabla + \frac{1}{4}|B^*x|^2 + |x|^2$, and the eigenvalues are at $2(1 + \frac{1}{2} B^* \cdot B^*)^{\frac{1}{2}}\ (n+1) + \ell(\frac{1}{2}(B^* \cdot B^*)^{\frac{1}{2}})$ for $\ell \in \{-n, -n+2, ..., n-2, n\}$, $n \in \mathbf{Z}^+$.

Computing the trace of the heat kernel for these yields as $t \downarrow 0$ tr $e^{-tH} = 0(t^{-2})$, and tr e^{-tH} $-$ tr $e^{-tH_0} = 0(1)$. In this simple example, however, the t^{-1} term is not present in either asymptotic expansion. The question arose as to whether or not the perturbation first arising in the second term, t^0 , was a general phenominon. I show that it is such and that it is related to the gauge invariance of the magnetic vector potential.

For simplicity, I consider only those potentials with uniform polynomial growth in all directions. They must also be bounded below. In Section 6 there are some remarks about further extensions.

Let $k \geq 1$ and $\mathcal{K} \subseteq \mathbb{R}^n$ a compact set be such that V satisfies the following:

(1.1a) there is a $C > 0$ s.t.

$$C^{-1}|x|^{2k} \leq V(x) \leq C|x|^{2k} \text{ for } x \in \mathbb{R}^n \setminus \mathcal{K} .$$

(1.1b) $V \in C^m(\mathbb{R}^n)$ for some $m \geq 4 + \frac{4}{k-1}$ ($m = \infty$ if $k = 1$) and $m > n(1 + \frac{1}{k})$.

(1.1c) For each multi-index α, with $|\alpha| \equiv \sum_{i=1}^{n} |\alpha_i| \leq m$, there is a $C_\alpha > 0$ s.t $|D^\alpha V(x)| \leq C_\alpha |x|^{2k}$.

Theorem 1. For V satisfying (1.1), and all $\varepsilon > 0$ then as $t \downarrow 0$

$$\operatorname{tr}(e^{-tH} - e^{-tH_0}) = o\left(t^{\frac{3}{2} - \frac{n}{2}(1 + \frac{1}{k}) - \varepsilon}\right).$$

If one also requires that V has a homogeneous decomposition, theorem 1 can be extended. Namely, suppose there is a finite set of $V_j(x) \in C^\infty(\mathbb{R}^n)$ with the property that for

(1.2) $$x \text{ and } \rho x \in \mathbb{R}^n \setminus \mathcal{K}, \ (\rho \in \mathbb{R}^+ , \ x \in \mathbb{R}^n)$$

$$V_j(\rho x) = \rho^j \, V_j(x) \quad \text{and}$$

$$V = \sum_{0 \leq j \leq 2k} V_j(x).$$

Note: j need not be an integer in (1.2); however, in the usual examples j is taken to be an integer.

Theorem 2. If V satisfies also (1.2) then as $t \downarrow 0$

$$\operatorname{tr}(e^{-tH} - e^{-tH_0}) = \frac{B^* \cdot B^*}{4} C_0 \, t^{2 - (1 + \frac{1}{k})} + O\left(t^{\ell_0 - \frac{n}{2}(1 + \frac{1}{k})}\right)$$

where $$\ell_0 = 3 - \frac{1}{2k} \max(\{j : V_j \neq 0, \ j < 2k\} \cup \{2k-1\})$$

and $$C_0 = \frac{\operatorname{vol}(S^{n-1})}{4k} \, \Gamma(\tfrac{n}{2}(1 + \tfrac{1}{k})) \int_{S^{n-1}} \frac{d\sigma}{(V_{2k}(\sigma))^{n/2k}} .$$

Remark. It is known that (1.2) is sufficient to show that $\mathrm{tr}(e^{-tH_0}) = O\left(t^{-\frac{n}{2}(1+\frac{1}{k})}\right)$, (see Helffer and Robert (7)), and I also show in Section 3 that (1.1) implies $\mathrm{tr}(e^{-tH_0}) = o\left(t^{-\frac{n}{2}(1+\frac{1}{k})-\varepsilon}\right)$.

Section 2 has a brief outline of weighted pseudodifferential operators and weighted Sobolev spaces as developed by Beals (2) to set up notation. To actually compute the integral kernels and traces, I combine two techniques for estimating the asymptotics with gauge changes. In Section 3, I use these weighted Sobolev spaces, semigroup and operator norm estimates in a *canonical order calculus* similar to that developed by B. Simon in (3). Combining these operator norm estimates with repeated applications of Duhamel's principle and the appropriate gauge transformations leads to the proof of theorem 1.

The more powerful technique, which requires more explicit knowledge of the potential, is that of weighted pseudodifferential operators. Having the canonical order calculus, however, greatly reduces the terms necessary to evaluate for my result. Sections 4 and 5 deal with the remaining symbol computations.

For the symbol computations, I use transport equations similar to the standard method such as that in (15). An alternative choice is to use the explicit symbol construction already done by Robert (11) for H_0^z, and to use the inverse Mellin transform as in (5) or (7) to get the expansion of e^{-tH_0}. The proof of theorem 2 is completed in Section 5.

§2. Weighted Sobolev Spaces and Pseudodifferential Operators.

As I am interested in operators with growing potentials, such as $-\Delta + |x|^{2k}$, this section provides background to the theory of weighted pseudodifferential operators and weighted Sobolev spaces. The type of weights that will be used here originated from many sources; however, the notation I use mostly follows that of Beals' (2), who consolidated the notation. More details can be found in Beals' paper.

In the usual pseudodifferential operator case, a symbol a satisfies for some m an estimate of the form

$$|\partial_\xi^\alpha D_x^\beta a(x_H)| \leq C_{\alpha\beta}(1 + |\xi|)^{m-|\alpha|} .$$

Indeed, for the constant coefficient differential operator, this type of estimate clearly holds where ξ is the symbol of D.

However, the potentials with polynomial growth are excluded from the symbol class. Changing the weight $(1 + |\xi|)$ to $\langle x\rangle^k + \langle\xi\rangle$ allows a larger class of symbols to be considered. In particular, condition (1.1) implies that $\xi^2 + V$ satisfies an estimate of the form

$$|\partial_\xi^\alpha D_x^\beta(\xi^2+V)| \leq C^{\alpha\beta}(\langle x\rangle^k+\langle\xi\rangle)^{2-|\alpha|}$$

for $|\beta| \leq m$.

Throughout the remainder of this thesis, the weight Φ will be $\langle x\rangle^k + \langle\xi\rangle$ with k determined by the growth of the potential V.

Beals used two weight functions in his more general situation. For Ψ and ψ in $C(\mathbb{R}^{2n})$, there are positive constants C, c, δ, so that the following hold:

(1) $\psi \leq C$;

(2) $\Psi\psi \geq c$;

(3) $c \leq \Psi(x,\xi)\ \Psi(y,\eta)^{-1} \leq C$ and $c \leq \psi(x,\xi)\ \psi(y,\eta)^{-1} \leq C$

if $|x-y| \leq c\psi(x,\xi)$ and $|\xi - \eta| \leq c\Psi(x,\xi)$;

(4) $R(x,0) \leq C\langle x\rangle^{c}$, where $R = \Psi\psi^{-1}$;

(5) $c \leq R(x,\xi)\, R(y,\eta)^{-1} \leq C$ if $|\xi - \eta| \leq cR(x,\xi)^{\delta+\frac{1}{2}}$ and

$|x - y| \leq cR(x,\xi)^{\delta}\, R(y,\eta)^{-\frac{1}{2}}$.

These weight functions then generalize the symbol classes to functions a $\in C^{\infty}(\mathbb{R}^{2n})$, satisfying the following for all α, β :

$$|\partial_{\xi}^{\alpha} D_{x}^{\beta} a| \leq C_{\alpha\beta}\, e^{\lambda}\, \Psi^{-|\alpha|}\, \psi^{|\beta|} \tag{2.1}$$

for some *order function* λ. Although the order functions can be generalized, the common ones are $\lambda = k_1 \log \Psi + k_2 \log \psi$, for k_1 and $k_2 \in \mathbb{R}$. The symbol class corresponding to the order λ is denoted $S^{\lambda}_{\Psi,\psi}$.

From the above estimates, it is clear that equivalent weight functions, i.e., $\Psi \sim \Psi'$ and $\psi \sim \psi'$, generate the same symbol classes as $\Psi^{-1}\Psi'$, and the similar terms are bounded, affecting only the constants in the above estimates.

As I do not need the full generality of Beals' weighted ψDO (pseudodifferential operator) calculus, I take $\psi \equiv 1$ and $\Psi = \Phi$ and use the order functions $r \log \Phi$. This clearly satisfies the hypotheses. Thus, $a \in S^{\lambda}_{\Phi,1}$, which will be denoted as S^{λ} in the remainder of this paper, implies $|\partial_{\xi}^{\alpha} D_{x}^{\beta} a| \leq C_{\alpha\beta} e^{\lambda} \Phi^{-|\alpha|}$. The theorems in this section, unless otherwise explicitly stated, apply to the general case $S^{\lambda}_{\Phi,\phi}$.

From the symbols, the operators are defined in the same way as in the usual ψDO case. For the symbol $a(x,\xi)$, one associates the operator $a(x,D)$ defined on $\mathcal{S}$ by

$$(a(x,D)u)(x) = (2\pi)^{-n} \int e^{ix\cdot\xi}\, a(x,\xi)\, \hat{u}(\xi)\, d\xi\,. \tag{2.2}$$

As $a \in S^\lambda$ and $u \in \mathcal{S}$, this integral converges absolutely, and it can be shown that the integral is in $\mathcal{S}$. Further proposition 3.11 of Beals states that the mapping defined by a(x,D) is a continuous mapping from $\mathcal{S}$ to $\mathcal{S}$. This mapping can be extended to a mapping from $\mathcal{S}^*$ to $\mathcal{S}^*$. I will denote the space of operators with symbols in S^λ by OPS^λ.

Beals showed ((2) proposition 3.12) that there is a bijection between such operators and their symbols, and so there is a seminorm that can be placed on the operator space OPS^λ. This seminorm is defined by the constants $C_{\alpha\beta}$ arising from the symbol estimates (2.1). And so OPS^λ is a Frechet space.

The main computational use of pseudodifferential operators arises from the use of symbol composition and adjoint theorems. The rules of composition are the same as for the ordinary pseudodifferential operators, as the following theorem shows (Beals (2) theorems 4.1 and 4.6).

Theorem 2.3.

(a) If $A = a(x,D) \in OPS^\mu$, and $B = b(x,D) \in OPS^\lambda$, then $AB \in OPS^{\mu+\lambda}$ and the symbol $a \circ b$ of AB has the asymptotic expansion

$$a \circ b \sim \sum_\alpha \frac{1}{\alpha!} \partial_\xi^\alpha a \, D_x^\alpha b \, ;$$

that is,

$$a \circ b - \sum_{|\alpha| \leq M} \frac{1}{\alpha!} \partial_\xi^\alpha a \, D_x^\alpha b \in S^{\lambda+\mu-M \log \Phi} \, .$$

(b) A^* restricted to $\mathcal{S}$ is in OPS^μ and has symbol $a^\#$ with asymptotic expansion

$$a^\# \sim \sum_\alpha \frac{1}{\alpha!} \partial_\xi^\alpha D_s^\alpha \overline{a} \, .$$

Another useful fact is (Beals' (2) theorem 5.1) that if $A \in OPS^\lambda$ where $\lambda \equiv 0$, then $A : L^2 \to L^2$ continuously.

The weighted Sobolev spaces are defined with the help of the weighted pseudodifferential operators. The weighted Sobolev space H^λ is defined by

$$H^\lambda = \mathrm{span}\{Au : u \in L^2 \text{ and } A \in OPS^{-\lambda}\}. \tag{2.4}$$

Endowing H^λ with the finest topology so that the mapping $A : L^2 \to H^\lambda$ is continuous for any $A \in OPS^{-\lambda}$, the following can be proved (see Beals (2) theorem 6.1).

<u>Theorem</u> <u>2.5</u>. For λ, $\mu \in O(\Phi)$,

(a) $H^0 = L^2$ topologically,

(b) $\mathcal{S} \subset H^\lambda \subset \mathcal{S}^*$ densely and continuously,

(c) $(H^\lambda)^* = H^{-\lambda}$,

(d) if $A \in OPS^\mu$ then $A : H^{\mu+\lambda} \to H^\lambda$ continuously, and

(e) there is an $A \in OPS^\mu$ so that $A : H^{\lambda+\mu} \to H^\lambda$ is a topological isomorphism and in particular H^λ has the topology of L^2 .

With (e) of the above theorem, one can extend the norm of L^2 to give a natural norm to H^λ for any λ. The norm is defined by $\|\mu\|_\lambda = \|Au\|_{L^2}$, for u $\in H^\lambda$, with the operator A given by part (e) of the above theorem. The difficulty, of course, is in finding an appropriate operator A. The next proposition gives a practical way to determine a norm.

<u>Proposition</u> <u>2.6</u> (proposition 6.17 of Beals (2)). Suppose a_1 , a_2,..., $a_r \in S^\mu$ and $\sum |a_i|^2 \sim e^{2\mu}$. Suppose $\lambda \in 0(\Phi)$ and $ce^\lambda \leq e^\mu \leq C\Phi^m e^\lambda$ for some $m \geq 0$ and c, $C > 0$. Let $A_i = a_i(x,D)$. Then $u \in H^\mu$ iff $u \in H^\lambda$ and each $A_i u \in L^2$. There is an admissible norm in H^μ defined by $\|u\|_\mu^2 = \|u\|_\lambda^2 + \sum \|A_i u\|^2$.

In particular, this proposition allows easy verification of several equivalent norms.

For the Sobolev spaces with weights $r \log \Phi$, I first choose symbols $a_\alpha(x,\xi) = \langle x\rangle^{k(r-|\alpha|)}\xi^\alpha$. Then having $\sum_{|\alpha|\le r} |a_\alpha|^{2r} \sim e^{2r\log\Phi}$, and taking $\lambda = 0$, one gets

$$\|u\|^2_{r\log\Phi} = \sum_{|\alpha|\le r} \|\langle x\rangle^{k(r-|\alpha|)} u\|^2 + \| u \|^2 .$$

As $\|u\| \le \|\langle x\rangle^{kr} u\|$, another equivalent norm is

$$\|u\|_{r\log\Phi} = \sum_{|\alpha|\le r} \|\langle x\rangle^{k(r-|\alpha|)} D^\alpha u\|$$

which I will denote $|||u|||_r$. This is similar to the usual Sobolev norm. Use interpolation for non-integer values of r.

Among other norms for these weighted spaces, one final set of norms will be useful in this paper.

<u>Proposition</u> <u>2.7</u>. If V satisfies (1.1) for m, then for $0 \le s \le m$ and $-\mu < \mathrm{spec}(-\Delta + V)$, for each $|r| \le m$ there is a $c > 0$ so that

$$c^{-1}|||u|||_{r+s} \le |||(-\Delta + V + \mu)^{s/2}u|||_r \le c|||u|||_{r+s} \tag{2.8}$$

for all $u \in H^{(r+s)\log\Phi}$.

Also, since $k \ge 1$, one can prove a similar proposition for the operator with a constant magnetic field. In particular, since $B^* x \cdot D \in OPS^{2\log\Phi}$, one has that $((-i\nabla - \frac{1}{2}B^*x)^2 + V + \mu')^{s/2} \sim (-\Delta + |x|^{2k} + 1)^{s/2}$ for $0 \le s \le m$.

Proof. First note that (1.1) implies that for $0 \leq s \leq m$ there is a $c' > 0$ so that

$$(c')^{-1} \|(-\Delta + |x|^{2k} + 1)^{s/2} u\| \leq \|(-\Delta + V + \mu)u\|$$

$$\leq c' \|(-\Delta + |x|^{2k} + 1)^{s/2} u\|.$$

And similar relations hold for $<x>^{\beta} D^{\alpha}(-\Delta + |x|^{2k} + 1)^{s/2}$ for $|\alpha| \leq m$, and all β . Thus, one need only verify (2.8) for $V = |x|^{2k}$ as the Sobolev norms involve taking at most m derivatives. To establish this, one considers the symbols $b_{\alpha}(x,\xi) \equiv a_{\alpha}(x,\xi) \circ \sigma((-\Delta + |x|^{2k} + 1)^{s/2})$, where $a_{\alpha}(x,\xi) = <x>^{k(m-|\alpha|)}\xi^{\alpha}$. Expanding out the terms b_{α} shows that

$$\sum_{|\alpha| \leq r} |b_{\alpha}|^2 \sim e^{2(r+s)\log \Phi} .$$

From the spectral theorem, one has $\|u\| \leq \|(-\Delta + |x|^{2k} + 1)^{s/2} u\|$ for $s \geq 0$. Combining this with proposition 2.6, one obtains the norm

$$\|u\|^2_{(r+s)\log \Phi} = \|u\|^2 + \sum_{|\alpha| \leq r} \|<x>^{k(r-|\alpha|)} D^{\alpha}(-\Delta + |x|^{2k} + 1)^{s/2} u\|^2 ,$$

which is equivalent to the norm

$$\|u\|^2_{(r+s)\log \Phi} = \|(-\Delta + |x|^{2k} + 1)^{s/2} u\| + \sum_{|\alpha| \leq r} \|<x>^{k(r-|\alpha|)} D^{\alpha}(-\Delta + |x|^{2k} + 1)^{s/2} u\|^2 ,$$

which in turn is equivalent to

$$\|u\|_{(r+s)\log \Phi} = \||(-\Delta + |x|^{2k} + 1)^{s/2} u\||_r .$$

And so (2.8) follows. □

§3. A weighted canonical order calculus and some global results.

In this section I use operator techniques and weighted Sobolev space norms to get leading order behavior of various integral kernels. Combining these results with Duhamel's expansion, I am able to get some estimates on the significance of terms in this expansion. Gauge changes will also allow identification of many vanishing terms. As these weights have growth in x, I am able to get global results with these techniques.

The following is a generalization of the canonical order calculus developed in (3). The parameter x_0 will be used for gauge changes and $t > 0$ corresponds to the semigroup parameter for the heat kernel.

Definition 3.1. A family of operators

$\{A_{t,x_0}\}$ on $L^2(\mathbb{R}^n)$ has *weighted canonical order* $(m_1, m_2) \in \mathbb{R}^2$ with respect to Φ if

(a) $A_{t,x_0} : \mathcal{S} \to \mathcal{S}^*$

(b) for all $k \geq \ell$, there exists $c(k,\ell) > 0$ s.t. for $0 < t < 1$

$$|||A_{t,x_0} u|||_k \leq c(k,\ell) t^{m_1 - \frac{k-\ell}{2}} <x_0>^{m_2} |||u|||_\ell .$$

If V does not satisfy (1.1) for $m = \infty$, I will need a more restricted type of calculus and so will add the requirements $k - \ell \leq m$, $|k| \leq m$, $|\ell| \leq m$. For m sufficiently large, namely, $m \geq 4 + \frac{4}{k-1}$ and $m > n\left(1 + \frac{1}{k}\right)$, this will give sufficient control to prove theorem 1.

Note that for operators depending on the single parameter t, one can set $x_0 = 0$ and obtain similar estimates. In particular, if there is no parameter x_0 specified, I mean the family of operators that is constant for varying x_0 and has $m_2 = 0$ as the second coefficient in the weighted canonical order.

The motivation for this definition arises from analogs to the following proposition and lemma 3.3.

Proposition 3.2. Let V satisfy (1.1) and $H_0 = -\Delta + V$. Then e^{-tH_0} has w.c.o. (weighted canonical order) (0,0) with respect to $\Phi = \langle x\rangle^k + \langle\xi\rangle$.

Proof. This follows from proposition 2.7 and the spectral theorem. Using the fact that $|||(H_0+\mu)^{s/2} u|||_r$ is equivalent to $||| u|||_{r+s}$ when $-\mu < \inf \operatorname{spec} H_0$, one has

$$|||e^{-tH_0} u|||_{s+r} \leq C_{s,r}|||(H_0+\mu)^{s/2} e^{-tH_0} u|||_r$$

$$\leq C_{s,r}\|(H_0+\mu)^{s/2} e^{-tH_0}\| \, |||u|||_r ,$$

where the operator norm is from $H^{r \log \Phi}$ to itself. The isometry established by theorem 2.5 (e) implies that the operator norm from $H^{r \log \Phi}$ to itself is the same as the operator norm from L^2 to itself. Thus, the spectral theorem can be used to estimate the operator norm.

$$\|(H_0+\mu)^{s/2} e^{-tH_0}\| \leq \sup_{x\geq \inf \operatorname{spec} H_0} |(x+\mu)^{s/2} e^{-tx}|$$

$$\leq Ct^{-s/2} ,$$

where C depends upon $|-\mu - \inf \operatorname{spec} H_0|$ and s. Thus,

$$|||e^{-tH_0} u|||_{r+s} \leq C \cdot C_{s,r}\, t^{-s/2}|||u|||_r . \qquad \square$$

Proposition 2.7 can be extended to apply for $H = (-i\nabla + A)^2 + V$, as remarked previously since $k \geq 1$, thus a similar argument shows e^{-tH} has w.c.o. (0,0).

To estimate integral kernels, it is useful to know the norm of δ_x in various weighted Sobolev spaces. In particular, a simple extension of the standard proof (see for example Adams (1)) for the usual Sobolev space norm yields:

<u>Lemma 3.3</u>. For $u \in \mathcal{S}$, $\varepsilon > 0$,

$$|u(x)| = |\langle \delta_x , u\rangle| \le C(\varepsilon) |||u|||_{\frac{n}{2}(1+\frac{1}{k}+\varepsilon)} \langle x\rangle^{-\frac{kn}{2}(\frac{1}{k}+\varepsilon)} .$$

<u>Proof</u>. Let S_x be the unit ball centered at x, and let $r \in (0,1)$. Let $(0,\theta)$ denote x in S_x, and similarly for (s,θ). Then

$$\langle \delta_x , u\rangle = u(x) = u(0,\theta) = u(r,\theta) - \int_0^r \frac{du}{dt}(t,\theta)\, dt .$$

Thus,

$$|u(x)| \le |u(r,\theta)| + \int_0^1 |\nabla u(t,\theta)|\, dt .$$

Multiplying by the volume element and integrating over S_x, one has:

$$\text{vol } S_x \cdot |u(x)| \le \Big(\int_{S_x} (\langle y\rangle^{k\frac{n}{2}(1+\frac{1}{k}+\varepsilon)} |u(y)|)^2\, dy\Big)^{1/2} \cdot$$

$$\cdot \Big(\int_{S_x} \langle y\rangle^{-k\frac{n}{2}(1+\frac{n}{k}+\varepsilon)\cdot 2}\Big)^{1/2}$$

$$+ \int_{S_x} \int_0^1 |\nabla u(t,\theta)|\, dt\, r^{n-1}\, \omega(\theta)\, dr d\theta ,$$

where $r^{n-1}\, \omega(\theta)\, dr d\theta$ is the volume element. In the first term I have used Hölders inequality. Integrating over r in the second term and other simplifications gives:

$$\text{vol } S_x \cdot |u(x)| \le \|\langle x\rangle^{\frac{kn}{2}(1+\frac{1}{k}+\varepsilon)} u\| \cdot \langle x\rangle^{-\frac{kn}{2}(1+\frac{1}{k}+\varepsilon)} \cdot C_1 (\text{vol } S_x)^{1/2}$$

$$+ \frac{1}{n} \int_{S_x} |\nabla u(y)| \frac{dy}{|x-y|^{n-1}} ,$$

where $$C_1 = (2)^{-\frac{kn}{4}(1+\frac{1}{k}+\varepsilon)} \geq \sup_{|x-y|\leq 1} \left(\frac{<y>}{<x>}\right)^{-\frac{kn}{2}(1+\frac{1}{k}+\varepsilon)} .$$

Using Hölder's inequality with the dual indices p and p′ , one has:

$$\int_{S_X} |\nabla(y)| \frac{dy}{|x-y|^{n-1}} \leq \| <x>^{k\frac{n}{2}(\frac{1}{k}+\varepsilon)} Du \|_{p,S_X} \cdot <x>^{-k\frac{n}{2}(\frac{1}{k}+\varepsilon)} .$$

$$\cdot\, C_2 \cdot \left(\int_{S_X} \frac{dy}{|x-y|^{(n-1)p'}} \right)^{1/p'} ,$$

where $$C_2 = (2)^{-k\frac{n}{4}(\frac{1}{k}+\varepsilon)} .$$

If I choose $p' > 2$, the above integral converges for $n \geq 2$ and then $p < 2$. Using the fact that S_X is bounded, I have $\| \cdot \|_{p,S_X} \geq \| \cdot \|_{2,S_X} \geq \| \cdot \|_2$.

Thus, $$|u(x)| \leq C_1(\text{vol } S_X)^{-\frac{1}{2}} <x>^{-\frac{kn}{2}(1+\frac{1}{k}+\varepsilon)} \| <x>^{\frac{kn}{2}(1+\frac{1}{k}+\varepsilon)} u \|_2$$

$$+ \frac{C_3(\text{vol } S_X)^{-1}}{n} <x>^{-\frac{kn}{2}(\frac{1}{k}+\varepsilon)} \| <x>^{\frac{kn}{2}(\frac{1}{k}+\varepsilon)} Du \|_2 ,$$

where

$$C_3 = C_2 \cdot \left(\int_{S_0} \frac{dy}{|y|^{(n-1)p'}} \right)^{1/p'} .$$

And noting $$<x>^{-\frac{kn}{2}(1+\frac{1}{k}+\varepsilon)} \leq <x>^{k\frac{n}{2}(\frac{1}{k}+\varepsilon)} ,$$

$$|u(x)| \leq C <x>^{-\frac{kn}{2}(1/k+\varepsilon)} \left(\| <x>^{k\frac{n}{2}(1+\frac{1}{k}+\varepsilon)} u \|_2 + \| <x>^{\frac{kn}{2}(\frac{1}{k}+\varepsilon)} Du \|_2 \right)$$

$$\leq C' <x>^{-\frac{kn}{2}(\frac{1}{k}+\varepsilon)} |||u|||_{\frac{n}{2}(1+\frac{1}{k}+\varepsilon)} .$$

This last step can be seen by looking at proposition 2.6 and noting that if one adds further operators of the form $<x>^{k(\frac{n}{2}(1+\frac{1}{k}+\varepsilon)-j)} D^j$ to $<x>^{\frac{kn}{2}(1+\frac{1}{k}+\varepsilon)}$ and $<x>^{\frac{kn}{2}(\frac{1}{k}+\varepsilon)} D$, one also gets an admissible norm. And as all admissible norms must be equivalent, the conclusion follows. □

With the norm of the delta function in the weighted Sobolev space norms and the operator estimates from the weighted canonical order, the following holds:

Lemma 3.4. If $\{A_{t,x_0}\}$ has w.c.o. (m_1, m_2), then the integral kernels of A_{t,x_0}, $K_{A_{t,k_0}}(x,y)$, satisfy for all $\varepsilon > 0$

$$\lim_{t\downarrow 0} t^{\frac{n}{2}(1+\frac{1}{k})+\frac{m_2}{2k}-m_1+\varepsilon} \int_{\mathbb{R}^n} K_{A_{t,x_0}}(x_0, x_0)\, dx_0 = 0.$$

Proof. Using lemma 3.3, one has

$$|K_{A_{t,x_0}}(x,y)| = |<\delta_x, A_{t,x_0}\delta_y>|$$

$$\leq c_s\, t^{-s+m_1} <x_0>^{m_2} (<x><y>)^{-\frac{kn}{2}(\frac{1}{k}+(\frac{2s}{n}-1-\frac{1}{k}))},$$

considering $A_{t,x_0} : H^{s \log \Phi} \to H^{-s \log \Phi}$. Taking $x = y = x_0$ and integrating over $\mathbb{R}^n$, the integral converges whenever $s > \frac{n}{2}(1+\frac{1}{k}) + \frac{m_2}{2k}$. □

If $\{A_{t,x_0}\}$ is independent of x_0, note that lemma 3.4 implies

$\mathrm{tr}(A_{t,x_0}) = o\left(t^{-\frac{n}{2}(1+\frac{1}{k})-\frac{m_2}{2k}+m_1-\varepsilon}\right)$ for any $\varepsilon \geq 0$, as $t\downarrow 0$.

In order to estimate the w.c.o. of $e^{-tH} - e^{-tH_0}$, I need the following expansion, which is derived by repeated application of Duhamel's principle.

Proposition 3.5 (Proposition 12.49 (3)). Let X, Y be operators in $OPS^{r \log \Phi}$ for some $r \geq 0$, with X and X + Y self-adjoint and bounded below.

Let $$b_n = \int_{\substack{s_i \geq 0 \\ \Sigma s_i \leq t}} e^{-(t-s_1-\cdots-s_n)X}\, Ye^{-s_1 X} \cdots Ye^{-s_n X}\, ds_1 \cdots ds_n$$

and r_n be defined similarly except that in $e^{-(t-s_1-\cdots-s_n)X}$, replace X with X + Y. Then $e^{-t(X+Y)} - e^{-tX} = \sum_{j=1}^{n} (-1)^j\, b_j + (-1)^{n+1}\, r_{n+1}$ for all $n \in \mathbb{N}$.

Another similarly useful relation is

Proposition 3.6 (proposition 12.51 (3)). Let $X, Y \in OPS^{r \log \Phi}$ with X bounded below. Then

$$[Y, e^{-tX}] = -\int_0^t e^{-(t-s)X} [Y,X]\, e^{-sX}\, ds.$$

Proof of 3.5. It is enough to prove that the expansion is valid when applied to functions $\psi \in \mathcal{S}$, as $\mathcal{S}$ is densely and continuously mapped into $H^{r \log \Phi}$.

Let $U_t = e^{it(X+Y)}\, e^{-itX}$. Clearly, $U_t\, \psi \in \mathcal{S}$ and

$$\frac{\partial}{\partial t} U_t\, \psi = e^{it(X+Y)}\, iYe^{-itX}\, \psi \in \mathcal{S}.$$

By the fundamental theorem of calculus,

$$e^{it(X+Y)}\, e^{-itX}\, \psi - \psi = \int_0^t \frac{\partial}{\partial s} U_s\, \psi\, ds$$

or $$e^{-it(X+Y)}\psi - e^{-itX}\psi = -e^{-it(X+Y)}\int_0^t ie^{is(X+Y)}\,Ye^{-isX}\psi\,ds$$

$$= -\int_0^t e^{-i(t-s)(X+Y)}\,Ye^{-isX}\psi\,ids.$$

Analytically extending, one has

$$e^{-t(X+Y)}\psi - e^{-tX}\psi = -\int_0^t e^{-(t-s)(X+Y)}\,Ye^{-sX}\psi\,ds.$$

As X and X + Y are bounded below, this extension exists.

Repeating this procedure proves the proposition. □

The proof for proposition 3.6 is similar, using instead $U_t = e^{itX}\,Ye^{-itX}$.

Needing to compute the w.c.o. of terms from these expansions, we use the following:

Proposition 3.7 (propositions 12.47 and 12.50 (3)).

(a) If A_{t,x_0} has w.c.o. (m_1, m_2) and T a fixed operator satisfies for some $b \geq 0$,

$$|||Tu|||_k \leq C(k)|||u|||_{k+b}$$

for all k, then TA_{t,x_0} and $A_{t,x_0}T$ have w.c.o. $(m_1 - \frac{b}{2}, m_2)$.

(b) If $A^{(j)}_{t,x_0}$ for $j = 0, 1, \ldots, \ell$ are operators with w.c.o.'s $\left(m_1^{(j)}, m_2^{(j)}\right)$, and $m_1^{(j)} > -1$ for all j, then

$$B_{t,x_0} \equiv \int_{\substack{s_i>0 \\ \Sigma s_i \leq t}} A^{(0)}_{t-s_1\cdots s_\ell, x_0} A^{(1)}_{s_1, x_0} \cdots A^{(\ell)}_{s_\ell, x_0}\, ds_1 \cdots ds_\ell$$

converges and has w.c.o.

$$\left[\ell + \sum_{j=0}^{\ell} m_1^{(j)} \,,\ \sum_{j=0}^{\ell} m_2^{(j)}\right] .$$

Proof. Part (a) is a direct result of expanding with operator norms.

For part (b) more care is needed to choose the correct operator norms. Let $s_0 = t - s - \cdots - s_\ell$. First decompose the integral into integrals over regions R_j , where in R_j $s_t \le s_j$ for $D \le i \le \ell$. In the region R_j , one has

$$\|A^{(0)}_{s_0,x_0} A^{(1)}_{s_1,x_0} \cdots A^{(\ell)}_{s_\ell,x_0} \varphi\|_k \le C \left(\prod_{i \ne j} s_1^{m_1^{(i)}} \right) s_j^{m_1^{(j)} - \frac{1}{2}(k-p)} \langle x_0 \rangle^{\Sigma m_2^{(j)}} \|\varphi\|_p$$

by bounding the $A^{(i)}_{s_{i,x_0}}$ for $i \ne j$ as maps from either $H^{k \log \phi}$ to itself. $A^{(j)}_{s_{j,x_0}}$ is bounded by its mapping of $H^{p \log \Phi}$ to $H^{k \log \Phi}$.

Now, using the fact that $t \ge s_j \ge \frac{t}{n+1}$ on R_j , each integral can be bounded above by

$$\int_0^t \prod_{i \ne j} s_i^{m_1^{(i)}} ds_1 \cdots ds_\ell \cdot t^{m_1^{(j)} - \frac{1}{2}(k-p)} = ct^{\ell + \sum_{i=1}^{\ell} m_1^{(j)} - \frac{1}{2}(k-p)} ,$$

provided $m_1^{(i)} \ge -1$ for each $i \ne j$. Combining the results for each R_j completes the proof. □

Remark. The weights have been defined so that $\xi \in S^{\log \Phi}$ and $x \in S^{\frac{1}{k} \log \Phi}$.

Thus, the estimates $|||D\varphi|||_j \le c_j |||\varphi|||_{j+1}$ and $|||x\varphi||| \le c_j |||\varphi|||_{j+\frac{1}{k}}$ are easy to verify.

The machinery to prove theorem 1 has now been set up. Before computing any terms with Duhamel's expansion for $e^{-tH} - e^{-tH_0}$, first, let's look more closely at H. For the magnetic field B^*, I replace $-\Delta$ in H_0 with $(-i\nabla + A)^2$ where A, the magnetic vector potential, also satisfies $\nabla \cdot A = 0$ and curl $A = B^*$ (the exterior derivative is the curl). This does not uniquely determine A, and so I am left with some free parameters to determine by convenience. In particular, taking $A = A_{x_0} = -\frac{1}{2} B^*(x-x_0)$ is a valid choice for any $x_0 \in \mathbb{R}^n$. Thus, I am led to

$$H_{x_0} = -\Delta - B^*(x-x_0) \cdot D + \frac{1}{4}|B^*(x-x_0)|^2 + V$$

for any fixed x_0. So I will use Duhamel's expansion with $X = H_0$ and $Y_{x_0} = -B^*(x-x_0) \cdot D + \frac{1}{4}|B^*(x-x_0)|^2$. Further, as the H_{x_0}'s are related by a gauge transformation, one has that $e^{\frac{i}{2}(B^* x_0 \cdot x)} H e^{-\frac{i}{2}(B^* x_0 \cdot x)} = H_{x_0}$ for H the operator with $x_0 = 0$; and as these are unitarily equivalent by a multiplicative operator, the diagonal of the kernel of e^{-tH} and $e^{-tH_{x_0}}$ are the same. And so one has

$$\operatorname{tr} e^{-tH} = \operatorname{tr} e^{-tH_{x_0}} = \int K_{e^{-tH_{x_0}}}(x_0, x_0)\, dx_0 .$$

Noticing that $(Y_{x_0} u)(x_0) = 0$ for any $u \in \mathcal{S}$ leads me to use the following:

$$\operatorname{tr}(e^{-tH} - e^{-tH_0}) = \int K_{(e^{-tH_{x_0}} - e^{-tH_0})}(x_0, x_0)\, dx_0 .$$

Using the collected facts about the weighted Sobolev norms and the weighted canonical order calculus, one finds that only the first $4 + \frac{4}{k-1}$ terms are needed from proposition 3.5 to verify theorems 1 or 2.

In particular, one has to evaluate the kernel of

$$b_j = \int_{\substack{s_i>0 \\ \sum_{i=1}^{j} s_i \leq t}} e^{-(t-s_1-\cdots-s_j)H_0}\, Y_{x_0}\, e^{-s_j H_0}\, ds_1 \cdots ds_j \,.$$

These terms are not computable directly as the Y_{x_0}'s are interlaced with the e^{-tH_0} terms. With the help of proposition 3.6, we commute the Y_{x_0}'s to the left. This leaves formulas such as

$$b_{1,x_0} = -Y_{x_0}\, e^{-tH_0}\, t - \int_{\substack{s_i>0 \\ \sum s_i \leq t}} e^{-(t-s_1-s_2)H_0}\, [Y_{x_0}\, , H_0]\, e^{(-s_1-s_2)H_0}\, ds_1\, ds_2 \,.$$

Again, bringing the interlaced terms to the left, one finds from proposition 3.7 that at most, second-order commutators are needed. More precisely the terms that lead to critical behavior are:

$$Y_{x_0}^{j}\, e^{-tH_0}\, \frac{t^j}{j!} \qquad \text{for} \quad j \leq 4 + \frac{4}{k-1}\,,$$

$$[Y_{x_0}\, , H_0]\, Y_{x_0}^{j}\, e^{-tH_0}\, \frac{t^{j+2}}{(j+2)!} \qquad \text{for} \quad j \leq 2 + \frac{3}{k-1}\,,$$

$$[Y_{x_0}\, , H_0]^2\, Y_{x_0}^{j}\, e^{-tH_0}\, \frac{t^{j+4}}{(j+4)!} \qquad \text{for} \quad j \leq \frac{1}{k-1}\,, \quad \text{and}$$

$$[[Y_{x_0}\, , H_0], H_0]\, Y_{x_0}^{j}\, e^{-tH_0}\, \frac{t^{j+3}}{(j+3)!} \qquad \text{for} \quad j \leq \frac{1}{k-1}\,. \tag{3.8}$$

Remark. Some commutator terms need to be factored with one factor considered acting on the e^{-sH_0} term before it and the other after it to satisfy the hypotheses for proposition 3.7 b. However, as taking the commutator introduces extra factors of e^{-sH_0}, and I need only to apply proposition 3.4 to, at most, terms with third-order commutators, no further problems arise.

Lemma 3.9. $V \in C^m$; then the integral kernel of e^{-tH_0} is in $C^{m+2-\frac{n}{2}-\varepsilon}$ for $\varepsilon > 0$ separately in each space variable.

Proof. The integral kernel is $<\delta_y , e^{-tH_0} \delta_x> = <e^{-tH_0} \delta_y , \delta_x>$. To show continuity properties, I use ordinary Sobolev spaces on bounded subsets Ω of $\mathbb{R}^n$. As $V \in C^m$, $D^\alpha V \in L^2(\Omega)$ for $|\alpha| \leq m$, and so $V \in H^m(\Omega)$.

It is well known that e^{-tH_0} takes $L^2(\mathbb{R}^n)$ into $L^\infty(\mathbb{R}^n)$ (see theorem B.11 of (14)). For $f \in L^2$, $g \equiv e^{-tH_0} f \in L^\infty(\mathbb{R}^n)$ and $H_0 e^{-tH_0} f \in L^\infty(\mathbb{R}^n)$ by the semigroup property. As $V \in C^m$, $Vg \in L^\infty(\Omega)$. Thus, $-\Delta g \in L^\infty(\Omega)$ and by Weyl's lemma, $g \in H^2(\Omega)$. To proceed, one notices $Vg \in H^{\min(2,m)-\varepsilon}(\Omega)$ for all $\varepsilon > 0$, and thus, $g \in H^{m+2-\varepsilon}(\Omega)$. This holds for all Ω bounded and all $f \in L^2$. Thus, $e^{-tH_0} \in OPS^{-(m+2-\varepsilon)}$, and so as $\delta_x \in H^{-\frac{n}{2}-\varepsilon'}(\Omega)$, $e^{-tH_0} \delta_x \in H^{m+2\varepsilon''}(\Omega)$. The Sobolev embedding theorem then gives $e^{-tH_0} \delta_x \in C^{m+2-\frac{n}{2}-\varepsilon_0}$ for all $\varepsilon_0 > 0$. So I have $<\delta_y , e^{-tH_0} \delta_x>$ is in $C^{m+2-\frac{n}{2}-\varepsilon_0}$ with respect to x for fixed y. The symmetry also gives $<e^{-tH_0} \delta_y , \delta_x> = <\delta_y , e^{-tH_0} \delta_x>$ is in C^m w.r.t. y for fixed x. □

To complete the proof of theorem 1, integrals of the form $\int_{\mathbb{R}^n} K_{T_{x_0}e^{-tH_0}} (x_0 , x_0)\, dx_0$ need to be evaluated for various T_{x_0} . With lemma

3.9 and Fubini's theorem, an interchange of integrals produces

$$K_{T_{x_0}e^{-tH_0}}(x,y) = T_{x_0} K_{e^{-tH_0}}(x,y) ,$$

where T_{x_0} acts on the x coordinate of $K_{e^{-tH_0}}$. This holds provided T_{x_0} does not involve taking too many derivatives and is the reason m is as defined in (1.1).

As T_{x_0} is explicitly known, it is easy to identify many vanishing terms. In particular, the commutator terms are:

$$\begin{aligned}[Y_{x_0}, H_0] = & -i(B^*(x-x_0)) \cdot \nabla V + \tfrac{1}{2} B^* \cdot B^* \\ & + 2i\, B^*\nabla \cdot \nabla + B^*(x-x_0) \cdot B^*\nabla\end{aligned}$$

and

$$\begin{aligned}[[Y_{x_0}, H_0], H_0] = & -iB^*(x-x_0) \cdot \nabla(\Delta V) + 2iB^*\nabla V \cdot \nabla \\ & + B^*(x-x_0) \cdot B^*\nabla V + 2B^*\nabla \cdot B^*\nabla .\end{aligned}$$

Note that as B^* is skew-symmetric, $B^*\nabla \cdot \nabla$ is $B_{ij}\left(\frac{\partial^2}{\partial x_i \partial x_j} - \frac{\partial^2}{\partial x_i \partial x_j}\right)$, which vanishes on smooth functions. Computing, one sees that the first non-trivial term, i.e., the term not obviously vanishing with the lowest weighted canonical order, is $2iB^*\nabla V \cdot \nabla K_{e^{-tH_0}}$, which has w.c.o.

$$(3 - \tfrac{3}{2}, 0) \text{ and so } \int (B^*\nabla V \cdot \nabla K_{e^{-tH_0}})(x,x)\, dx$$

$$= o\left(t^{\frac{3}{2} - \frac{n}{2}(1 + \frac{1}{k}) - \varepsilon}\right)$$

for all $\varepsilon > 0$ as $t \downarrow 0$ by lemma 3.4. Thus theorem 1 follows.

§4. The pseudodifferential operator for the heat kernel.

There are two possible ways to proceed from this point. As Robert (11) already has explicit symbol computations for H_0^z, for Re $z < 0$, I could use the inverse Mellin transform to get asymptotics for tr $T_{x_0}e^{-tH_0}$ in the same manner as Helffer and Robert (7) did for tr e^{-tH_0}. The Mellin transform takes e^{-t} to $\Gamma(s)t^{-s}$, and thus the inverse Mellin transform allows one to compute tr $T_{x_0}e^{-tH_0}$ from tr $T_{x_0} H_0^z$, provided the integrals converge absolutely so that the order of integrations can be interchanged.

However, it is also possible to compute the symbol for the heat kernel, e^{-tH_0}, directly, in a similar manner as that of Schrader and Taylor (12). This is the method I will use. One assumes that there is a symbol for the heat kernel and one computes the transport equation from the relations such a symbol must satisfy.

Next one needs to solve this transport equation. The standard technique is to guess a first approximation and then to iterate by successive applications to the transport equations; thus, one obtains a formal series solution. The hope is that these approximations are better and better in the sense that they lie in more and more negative symbol classes. In particular, one wants the successive expansions to differ from the actual symbol for the heat kernel by symbols in these progressively smaller symbol classes. That is the difference of the formal expansion, and the symbol for the heat kernel is in $S^{-\infty} \equiv \bigcap_j S^{-j \log \Phi}$. This symbol class in particular contains smooth functions with compact support. As a final introductory remark, $\sim$ will be used in this section to mean that the symbols are equivalent up to symbols in $S^{-\infty}$.

To compute the symbol for the heat kernel, one needs the transport equation. This is the equation satisfied by such a symbol. Suppose one has a symbol $\underline{a}$, which satisfies $(e^{-tH_0} f)(x) = \int e^{ix\cdot\xi} a(t,x,\xi)\, \hat{f}(\xi)\, d\xi$ for any $f \in \mathcal{S}$. As H_0 is the generator for the semigroup e^{-tH_0} , one has $\left(\frac{\partial}{\partial t} + H_0\right) e^{-tH_0} = 0$. This implies the following transport equation:

$$\frac{\partial a}{\partial t} + \sigma(H_0) \circ a \sim 0. \tag{4.1}$$

One next assumes $a \sim \sum_j a_j$ with $a_j \in S^{-j/k \log \Phi}$. Let $\frac{\partial a_0}{\partial t} = -(\xi^2+V_{2k})a_0$ with initial condition $a_0(0;x,\xi) = 1$. This choice is taken as this leads to symbols whose trace can be easily computed. The alternative choice $\frac{\partial a_0}{\partial t} = -(\xi^2+V)a_0$ leads to evaluating $\int e^{-tV} dx$, which in general is not computable in detail.

Thus,
$$a_0(t;x,\xi) = e^{-t(\xi^2+V_{2k})} = e^{-t\Lambda(x,\xi)}\ , \tag{4.4}$$

defining Λ. This choice of initial conditions implies $a_j(0;x,\xi) = 0$ for all other j's, as $e^{-0\cdot H_0} = I$.

To continue by induction, let

$$\frac{\partial a_j}{\partial t} = -(\xi^2+V_{2k})a_j + \sum_{q<j} V_{2k-q}\, a_{j-q} + 2i\xi\cdot\nabla a_{j-k} - \Delta\, a_{j-2k}$$

Define by Ω_j the latter quantities, so

$$\Omega_j = \sum_{q>0} V_{2k-q}\, a_{j-q} - 2i\, \xi\cdot\nabla a_{j-k} - \Delta a_{j-2k}\,. \tag{4.5}$$

Thus, $$\frac{\partial a_j}{\partial t} = -\Lambda a_j + \Omega_j$$

or $$a_j(t,x,\xi) = \int_0^t e^{-(t-s)\Lambda(x,\xi)}\, \Omega_j(s,x,\xi)\, ds.$$

Remark. This choice of a recursion relation is to easily allow a proof that each $a_j \in S^{-j/k \log \Phi}$. In particular, I want to control the additional powers of ξ and x by related powers of t.

Thus, one has:

Lemma 4.6. For j,

$$a_j(t,x,\xi) = t^{j/2k}\, b_j(t,x,\xi)\, e^{-t\Lambda(x,\xi)},$$

where $$b_j(t,x,\xi) = \widetilde{b}_j(t^{1/2k}, \frac{x}{|x|}, \frac{\xi}{|\xi|}, \omega, \eta)$$

for $$\omega = t^{\frac{1}{2}}|\xi|,\ \eta = t^{1/2k}|x|,$$

and $\widetilde{b}_j$ is smooth in all arguments and a polynomial in $t^{1/2k}$, ω, and η.

Proof. For $j = 0$, it is clearly true with $b_0 \equiv 1$. One proceeds by induction.

Let $$\Omega_j = t^{\frac{j}{2k}-1}\, \widetilde{\Omega}_j(t, \frac{x}{|x|}, \frac{\xi}{|\xi|}, \omega, \eta)\, e^{-t\Lambda(x,\xi)}.$$

Then, using (4.5), one has

$$\widetilde{\Omega}_j = t^{-\frac{j}{2k}+1}\Big[\sum_{q>0} V_{2k-q}\, \widetilde{b}_{j-q}\, t^{\frac{j-k}{2k}} + 2i\, \xi \cdot \nabla b_{j-k}\, t^{\frac{j-k}{2k}}$$

$$- 2it\, \xi \cdot \nabla V_{2k}\, \widetilde{b}_{j-k}\, t^{\frac{j-k}{2k}} - \Delta \widetilde{b}_{j-2k}\, t^{\frac{j-2k}{2k}} +$$

$$+ 2t\, \nabla V_{2k} \cdot \nabla \widetilde{b}_{j-2k}\, t^{\frac{j-2k}{2k}} - t^2 |\nabla V_{2k}|^2\, \widetilde{b}_{j-2k}\, t^{\frac{j-2k}{2k}}$$

$$+ t\,\Delta V_{2k}\,\widetilde{b}_{j-2k}\,t^{\frac{j-2k}{2k}}\Big)$$

$$= \sum t^{\frac{2k-q}{2k}}\,V_{2k-q}\,\widetilde{b}_{j-q} + 2i\,t^{\frac{1}{2}}\,\xi\cdot\nabla\widetilde{b}_{j-k}$$

$$- 2i\,t^{\frac{3}{2}}\,\xi\cdot\nabla V_{2k}\,\widetilde{b}_{j-k} - \Delta\,\widetilde{b}_{j-2k} + 2t\,\nabla V_{2k}\cdot\nabla\widetilde{b}_{j-2k}$$

$$- t^2|\nabla V_{2k}|^2\,\widetilde{b}_{j-2k} + t\Delta V_{2k}\,\widetilde{b}_{j-2k}\,.$$

One sees that $\widetilde{\Omega}_j$ is smooth (except at 0) and is a polynomial in $t^{1/2k}$, ω, and η, as the V_ℓ's are radially homogeneous of degree ℓ.

Thus
$$a_j = \int_0^t e^{-(t-s)\Lambda}\,\Omega_j(s)\,ds$$

$$= \int_0^t s^{\frac{j}{2k}-1}\,\widetilde{\Omega}_j(s)\,e^{-(t-s)\Lambda}\,e^{-s\Lambda}\,ds$$

$$= t^{j/2k}\Big(t^{-j/2k}\int_0^t s^{\frac{j}{2k}-1}\,\widetilde{\Omega}_j\,ds\Big)\,e^{-t\Lambda}\,.$$

As $\widetilde{\Omega}_j$ is a polynomial in $t^{1/2k}$, $\widetilde{b}_j = t^{-j/2k}\int_0^t s^{\frac{j}{2k}-1}\,\widetilde{\Omega}_j\,ds$ is also such; and so the lemma holds. □

I still need to show that the a_j's are in $S^{-j/k\,\log\Phi}$, to prove this note that there is a $C > 0$, so that $\Lambda(x,\xi) \geq C(\langle x\rangle^k + \langle\xi\rangle)^2$. Thus terms such as $\omega^j\,e^{-t\Lambda}$ and $\eta^j\,e^{-t\Lambda}$ are bounded.

<u>Proposition</u> <u>4.7</u>. $a_j \in S^{-j/k\,\log\Phi}$, for any $t \leq T$ uniformly, for any fixed T.

Proof. Using Lemma 4.6,

$$|a_j| = |t^{j/2k} \tilde{b}_j e^{-t\Gamma}| \le C_1 t^{j/2k} e^{-t\Lambda/2},$$

as $\tilde{b}_j$ is a polynomial in ω and η and the other variables are over compact domains. Again, using the fact that $\Lambda \ge C\Phi^2$, we have

$$t^{j/2k} e^{-t\Lambda/2} \le C_2 \Phi^{-j/k}.$$

A similar argument works for all derivatives of a_j, and thus $a_j \in S^{-j/k \log \Phi}$ uniformly for $t \le T$. □

Now to see that these symbols are a good approximation to the heat kernel, let $A_\ell = \sum_{j=0}^{\ell} a_j$ and W_ℓ be the corresponding operator. It remains to show that $e^{-tH_0} - W_\ell$ are in progressively smaller classes as ℓ tends to infinity.

At $t = 0$ by construction, $e^{-0 \cdot H_0} - W_\ell(0) = 0$ for each ℓ. Further,

$$\left(\frac{\partial}{\partial t} + H_0\right) W_\ell f = \int D_\ell e^{ix \cdot \xi} \hat{f}(\xi)\, dx$$

for $f \in \mathcal{S}$, where

$$D_\ell = \sum_{\substack{q>0 \\ \ell-q<j\le\ell}} V_{2k-q} a_j +$$

$$+ \sum_{\ell-k<j\le\ell} 2i\, \xi \cdot \nabla a_j - \sum_{\ell-2k<j\le\ell} \Delta a_j .$$

So $D_\ell \in S^{(-\frac{\ell}{k}+2) \log \Phi}$, by proposition 4.7.

It remains to show that $\sigma(e^{-tH_0}) - A_\ell \in S^{(-\frac{\ell}{k}+2) \log \Phi}$, which completes the proof of the asymptotic symbol expansion for the heat kernel.

Proposition 4.8. Let $v(t,x)$ satisfy $\left(\frac{\partial}{\partial t} + H_0\right) v = g(t,x)$ with initial condition

$v(0,x) = h(x)$. Then $\sup_t |||v|||_s \leq C_1 |||h|||_s + C_2 \sup_t |||g(t,\cdot)|||_s$.

<u>Proof</u>. Duhamel's principle implies

$$v(t,x) = e^{-tH_0} h(x) + \int_0^t e^{-(t-s)H_0} g(s,x)\, ds.$$

Thus, by proposition 3.2,

$$\sup_t |||v|||_s \leq C_1 |||h|||_s + C_2 \sup_t |||g|||_s . \qquad \square$$

And so I have a symbol expansion for e^{-tH_0}.

§5. Asymptotic Estimates.

In this section the proof of theorem 2 will be completed. Recall from section 3 that it remains to evaluate $\int K_{T_{x_0}e^{-tH_0}}(x_0, x_0)\, dx_0$ for various operators T_{x_0}. In particular, let $T^{(1)}_{x_0} = \frac{t^2}{4} B^* \cdot B^*$ and

$$T^{(2,j)}_{x_0} = \frac{t^{3+j}}{(3+j)!}(2i\, B^* \nabla V \cdot \nabla + 2B^* \nabla \cdot B^*\nabla)(2i\, B^*(x-x_0) \cdot \nabla + \tfrac{1}{4}|B^*(x-x_0)|^2)^j .$$

These are the terms that do not trivially vanish in the commutators.

Using the explicit symbols for the heat kernel constructed in Section 4, these integrals can be evaluated explicitly. Some symmetry arguments show that there are more vanishing terms, but more importantly, the explicit constants can be determined. These perturbation terms are much easier to evaluate than constructing the symbol for the heat kernel for e^{-tH} and then computing the trace of the difference between the symbols.

The following proposition insures that the above integrals exist.

Proposition 5.1 (Robert, proposition 3.2 (11)). Let μ be an order function for Φ and N be a non-negative integer. Suppose that $\mu \geq N \log \Phi$ and that $\int [\langle\xi\rangle^{1-\varepsilon}\, \Phi^N\, e^{-\mu}]^2\, d\xi = 0(1)$ on compact sets of x for $0 < \varepsilon < 1$. Let $T \in OPS^{-2\mu}$. Then

(1) T is an integral operator with kernel $K_T(x,y) \in C(\mathbb{R}^n \times \mathbb{R}^n)$. Moreover for all $|\alpha| \leq N$, $|\beta| \leq N$, $D^\alpha_x D^\beta_y K_T(x,y)$ is continuous and satisfies

$$|D_x^{\alpha} D_y^{\beta} K_T(x,y)| \leq \frac{C}{\psi_N(x)\psi_N(y)} \|T\|_{H^{-\mu} \to H^{\mu}}$$

for all a $(x,y) \in \mathbb{R}^{2n}$ and

$$\psi_N(x) \equiv \left[\int_{\mathbb{R}^n} \Phi^{2N} e^{-2\mu} d\xi\right]^{-1/2} .$$

(2) If $\psi_N^{-1} \in L^2(\mathbb{R}^n)$, then T is a nuclear operator and

$$\mathrm{Tr}\, T = \int_{\mathbb{R}^n} K_T(x,x)\, dx.$$

In particular, this shows that T is trace class when $T \in OPS^{(-\frac{n}{2}(1+\frac{1}{k})-\varepsilon)\log\Phi} = OPS^{\lambda_0}$ for any $\varepsilon > 0$. Further, for a class of operators Z_{x_0}, one has $\int K_{Z_{x_0}}(x_0, x_0)\, dx_0 < \infty$ if $\left(\psi_{N,Z_{x_0}}(x_0)\right)^{-1} \in L^2$, that is, if $Z_{x_0} \in OPS^{\lambda_0}$ and if there is a positive constant M so that $\|Z_{x_0}\|_{H^{-\lambda_0} \to H^{\lambda_0}} \leq M$ independent of x_0.

<u>Remark</u>. I do not need to worry about using an asymptotic expansion for the symbol of the heat kernel. For if $d = a - \sum_{j=1}^{N} a_j$, $d \in S^{(-\frac{N}{k}+2)\log\Phi}$, for N large, and as $d(0, x, \xi) = 0$, one can find uniform constants so that $|\partial_{\xi}^{\alpha} D_x^{\beta} d| \leq Ct^m \Phi^{-M-|\alpha|}$ for $M > 2n$ and $m < -M + \frac{N}{k} - 2$. This implies $\mathrm{tr}\, d = O(t^m)$, and taking N sufficiently large implies that the results are not affected.

Lemma 5.2.

$$\int_{\mathbb{R}^n} K_{T^{(1)}_{x_0} e^{-tH_0}}(x_0, x_0)\, dx_0 = \frac{B^* \cdot B^*}{4}\, t^{2-\frac{n}{2}(1+\frac{1}{k})} \cdot C_0$$

$$+ O\left(t^{\ell_0 - \frac{n}{2}(1+\frac{1}{k})}\right) \quad \text{as} \quad t \downarrow 0 ,$$

where $$C_0 = \frac{\mathrm{vol}(S^{n-1})}{4k}\, \Gamma(\tfrac{n}{2}(1+\tfrac{1}{k})) \int_{S^{n-1}} \frac{d\sigma}{(V_{2k}(\sigma))^{n/2k}}$$

and $$\ell_0 = 3 - \frac{1}{2k} \max(\{j : v_j \neq 0,\ j < 2k\} \cup \{k\} \cup \{2k-1\}).$$

Proof. As $T^{(1)}_{x_0}$ is a constant, one needs to evaluate $\int_{\mathbb{R}^{2n}} a_j(t, x, \xi)\, dx d\xi$.

For $j = 0$, one has

$$\int_{\mathbb{R}^{2n}} e^{-t(\xi^2 + V_{2k}(x))}\, dx d\xi$$

$$= \int e^{-t\xi^2}\, d\xi \cdot \int e^{-tV_{2k}(x)}\, dx$$

If V_{2k} is homogeneous over all of $\mathbb{R}^n$ one has

$$= \mathrm{vol}(S^{n-1}) \int_{\mathbb{R}^+} \eta^{n-1} e^{-t\eta^2}\, d\eta \cdot \int_{\mathbb{R}^+ \times S^{n-1}} e^{-tV_{2k}(\frac{x}{r}) r^{2k}}\, r^{n-1}\, dr\, d\sigma$$

where $\eta = |\xi|$, $r = |x|$, and σ are the angle variables from x.

However, (1.2) requires only homogeneity outside a compact set $\mathcal{K}$. Thus assuming $\mathcal{K}$ is contained in B_R, the ball centered at 0 with radius R, one has

$$\int a_0\, dx d\xi = \mathrm{vol}(S^{n-1}) \int_{\mathbb{R}^+} \eta^{n-1} e^{-t\eta^2}\, d\eta \cdot$$

$$\left[\int_{B_R}\left(e^{-tV_{2k}(\frac{Rx}{|x|})|x|)^{2k}}\right)dx + \right.$$

$$\left.\int_{\mathbb{R}^+\times S^{n-1}} e^{-tV_{2k}(\frac{x}{r})r^{2k}}\, r^{n-1}\, drd\sigma\right].$$

Note that the integral over B_R is bounded independent of t. Integrating this, using

$$\int_0^\infty x^n\, e^{-(rx)^m}\, dx = \frac{1}{mr^{n+1}}\,\Gamma(\frac{n+1}{m}) \text{ with } n+1,\ r,\ m > 0, \text{ from (5)},$$

one has $\displaystyle\int a_0\, dxd\xi = \frac{\mathrm{vol}(S^{n-1})}{4k\ t^{n/2(1+\frac{1}{k})}}\,\Gamma\left[\frac{n}{2}\left(1+\frac{1}{k}\right)\right]\int_{S^{n-1}}\frac{d\sigma}{(V_{2k}(\sigma))^{\frac{n}{2k}}} + 0\left(t^{-\frac{n}{2}}\right).$

This gives the leading behavior and determines C_0. It remains to verify that all of the remaining terms are at least $O\left(t^{\ell_0-\frac{n}{2}(1+\frac{1}{k})}\right)$.

Recall from lemma 4.6 that $a_j = t^{j/2k}\, b_j\, e^{-t(\xi^2+V_{2k}(x))}$, where b_j is a polynomial in $t^{1/2k}$, $t^{1/2k}|x|$, and $t^{1/2}|\xi|$. $\int a_j(x,\xi)\, dxd\xi = O\left(t^{\frac{j}{2k}-\frac{n}{2}(1+\frac{1}{k})}\right)$, as each extra factor of $|x|$ or $|\xi|$ introduces an extra $t^{-1/2k}$ or $t^{-1/2}$; however, these are cancelled by the pairing of $t^{1/2k}$ with $|x|$ and $t^{1/2}$ with $|\xi|$. Thus multiplying $\int a_j(x,\xi)\, dxd\xi$ by t^2 produces terms of order $O\left(t^{2+\frac{j}{2k}-\frac{n}{2}(1+\frac{1}{k})}\right)$. Among terms other than a_0 which has already been considered, the one with the lowest order in t is $a_{(\ell_0-2)2k}$.

For the $T_{x_0}^{(2,j)}$ similar arguments are needed; however, the symbols for $T_{x_0}^{(2,j)}$ contain elements which vanish when integrated over ξ. Indeed, these terms are expected to vanish for the more general case without homogeneous decomposition. However, a proof just using operator norms is not powerful enough to identify these terms, and although there are techniques to allow for non-smoothness of symbols in the x variable, these lead to additional complications. The symbol for $T_{x_0}^{(2,j)}$ is

$$\frac{t^{3+j}}{(3+j)!}\left(-2B^*\nabla V \cdot \xi - 2B^*\xi \cdot B^*\xi\right) \circ$$

$$\underbrace{(-B^*(x-x_0) \cdot \xi + \tfrac{1}{4}|B^*(x-x_0)|^2) \circ \cdots \circ (-B^*(x-x_0) \cdot \xi + \tfrac{1}{4}|B^*(x-x_0)|^2)}_{j \text{ copies}} .$$

The terms for $T_{x_0}^{(2,j)} \circ a_\ell$ which do not have $(x - x_0)$ remaining are

$$\frac{t^{3+j}}{(3+j)!} 2i\ B^*\nabla V \cdot \nabla(\underbrace{iB^*(x-x_0)\cdot\nabla(\cdots}_{j-1 \text{ copies}}(-B^*(x-x_0) \cdot (\xi-i\nabla)a_\ell)\cdots) \tag{5.3}$$

and

$$\frac{t^{3+j}}{(3+j)!} 4i\ B^*\xi \cdot B^*\nabla(\underbrace{iB^*(x-x_0)\cdot\nabla(\cdots}_{j-1 \text{ copies}}(-B^*(x-x_0) \cdot (\xi-i\nabla)a_\ell)\cdots) \tag{5.4}$$

and

$$\frac{t^{3+j}}{(3+j)!} 2\ B^*\nabla \cdot B^*\nabla \left[2 \sum_{1\leq k\leq j} (iB^*(x-x_0) \cdot \nabla(\cdots(\overbrace{-B^*(x-x_0)\cdot(\xi-i\nabla)}^{k^{th} \text{ position}} \cdot\right.$$

$$(iB^*(x-x_0)\cdot\nabla(\cdots(\overbrace{-B^*(x-x_0)\cdot(\xi-i\nabla)a_\ell}^{j^{th}\text{ position}})\cdots)$$

$$+\ \underbrace{(iB^*(x-x_0)\cdot\nabla(\cdots}_{j-1\text{ copies}}(\tfrac{1}{4}|B^*(x-x_0)|^2\, a_\ell)\cdots)\Big]\ . \qquad (5.5)$$

Using these symbol expansion, one obtains:

Lemma 5.6.

$$\int_{\mathbb{R}^n} K_{T^{(2,j)}_{x_0}\, e_0^{-tH}}(x_0, x_0)\, dx_0 = \begin{cases} O(t^{\frac{3}{2}+\frac{1}{2k}+j-\frac{n}{2}(1+\frac{1}{k})}) & \text{for } j \geq 1 \\ C_1\, t^{2+\frac{1}{k}-\frac{n}{2}(1+\frac{1}{k})} + O(t^{\ell_0+\frac{1}{2}-\frac{n}{2}(1+\frac{1}{k})}) & \text{for } j = 0\ , \end{cases}$$

where ℓ_0 is as in lemma 5.2 and C_1 is defined in the proof.

Proof. From (5.3)-(5.5) one can read off the $|x|$ or $|\xi|$ powers in the terms composed with a_ℓ . Any x derivatives of a_ℓ will just increase the t power by $t^{1/2k}$ for that term.

(5.3) has a ∇V and a ξ remaining for $j \geq 1$, which reduces the t power by $\frac{2k-1}{2k} + \frac{1}{2} = \frac{3}{2} - \frac{1}{2k}$. Thus, these terms are $0\left(t^{3+j-(\frac{3}{2}-\frac{1}{2k})+\frac{\ell}{2k}-\frac{n}{2k}(1+\frac{1}{k})}\right)$.

When $j = 0$, the term becomes $\frac{t^3}{3!}\left(-2\, B^*\nabla V\cdot \xi a_\ell(x,\xi)d\left(\frac{\xi}{|\xi|}-i\nabla\right)a_\ell\right)$. On the surface, this appears to cause $O\left(t^{\frac{3}{2}+\frac{1}{2k}-\frac{n}{2}(1+\frac{1}{k})}\right)$ behavior. I show that these terms, in fact, vanish.

The feature that causes many of these terms to vanish is an integration over all angles of functions independent of angular change. In particular, $a_\ell(x,\xi)$ for $\ell < k$ are independent of the angular variable $\frac{\xi}{|\xi|}$. For a_0, this is clear, and for $\ell < k$, $b_\ell = t^{-\ell/2k} \int_0^t V_{2k-\ell}(x) s^{\ell/2k-1}\, ds$. Thus, the integral over the ξ angular coordinates of $\int B^*\nabla V \cdot \xi\, a_\ell(x,\xi)\, d\left(\frac{\xi}{|\xi|}\right) = 0$.

$$C_1 = -\frac{1}{3} \int B^*\nabla V_{2k} \cdot (\xi(-2i\xi \cdot \nabla a_0) - i\nabla a_0)\, dx d\xi ,$$

or
$$C_1 = -\frac{i}{12k} \operatorname{vol}(S^{n-1}) \Gamma(\tfrac{n}{2}+2-\tfrac{1}{k}) \int_{S^{n-1}} \frac{B^*\nabla V_{2k}(\sigma)\cdot\omega(DV_{2k}(\sigma)d\sigma}{(V_{2k}(\sigma))^{\frac{n}{2k}+2-\frac{1}{k}}}$$

$$+ \frac{i}{6k} \Gamma(\tfrac{n}{2}+1)\, \Gamma(\tfrac{n}{2k}+2-\tfrac{1}{k}) \int_{S^{n-1}\times S^{n-1}} \frac{B^*\nabla V_{2k}(\sigma)\cdot\omega(DV_{2k}(\sigma)\cdot\omega)d\omega d\sigma}{(V_{2k}(\sigma))^{\frac{n}{2k}+2-\frac{1}{k}}}$$

In conclusion, (5.4) and (5.5) add just $|\xi|^2$ terms, which vanish, and so the result holds. □

To complete the proof of theorem 2, one combines lemmas 5.2 and 5.6 to obtain

$$\operatorname{tr}\left(e^{-tH} - e^{-tH_0}\right) = \frac{B^*\cdot B^*}{4} C_0\, t^{2-\frac{n}{2}(1+\frac{1}{k})} + O\left(t^{\ell_0 - \frac{n}{2}(1+\frac{1}{k})}\right) \tag{5.7}$$

$$+ O\left(t^{2+\frac{1}{2k}-\frac{n}{2}(1+\frac{1}{k})}\right),$$

where $\ell_0 = 3 - \frac{1}{2k} \max(\{j : v_j \neq 0,\ j < 2k\} \cup \{2k-1\})$.

§6. Some Further Remarks.

For potentials without the same polynomial growth in all directions, similar theorems will hold. However, the growth of the potential in directions not parallel to the magnetic field must be at least quadratic in the distance; otherwise, the leading orders of tr e^{-tH} and tr e^{-tH_0} will differ. A further requirement is that H_0 must have a discrete spectrum and e^{-tH_0} must be trace class. The only change necessary for these results is to use a new weight $\Phi = 1 + \sum_{i=1}^{n} x_i^{2k_i} + |\xi|^2$, where $V \sim \sum_{i=1}^{n} x_i^{2k_i}$.

In theorem 1, one sees the weakness of the operator norm method, as the more precise pseudodifferential computations show that the term $B^*\nabla V \cdot \nabla$ does not contribute a leading term. The pseudodifferential operator computations imply that for a broad class of potentials, one should expect two orders in t faster decay in the perturbed term with a magnetic field perturbation.

The gauge invariant techniques in this paper will also extend to the non-uniform magnetic field case. These techniques imply that $[2A \cdot D + A \cdot A, H_0]$ and higher order commutators will determine the behavior of the asymptotics.

For the compact manifold case, the weighted Sobolev spaces need not be used; however, the gauge invariance will play a similar role. This gauge invariance should allow the canonical order techniques to be applied to compute the perturbative effect of a magnetic field on the asymptotics for the trace of $e^{t\Delta}$ or similar operators on a compact manifold.

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