Absolutely Continuous Spectrum of One-Dimensional Schrödinger Operators and Jacobi Matrices with Slowly Decreasing Potentials

Thesis by

ALEXANDER A. KISELEV

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Abstract

We show that for one-dimensional Schrödinger operators with potentials $V(x)$ satisfying the decay condition $|V(x)| \leq C(1+x)^{-\frac{3}{4}-\epsilon}$, the absolutely continuous spectrum fills the whole positive semi-axis. We also give the description of a set of zero Lebesgue measure on which the embedded singular part of the spectral measure may be supported. Under additional conditions on the integrability of the potential, we show that potentials decaying as $C(1+x)^{-\frac{2}{3}-\epsilon}$ also lead to the absolutely continuous spectrum of the Hamiltonian.

An analog of the short-range Jost functions is introduced for the square integrable potentials. The formula for the projection on the absolutely continuous component of the spectrum is derived for a certain class of power decaying potentials.

Some further applications of the introduced technique are given. We also show that similar results hold for Jacobi matrices.

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Introduction

The main theme of the thesis is the study of the spectrum of Schrödinger operators with decaying potentials. The potential decaying at infinity constitutes a relatively compact perturbation of the Laplace operator, and hence, the essential spectrum coincides with the positive semi-axis. It has been known for a long time that for a short-range potential $V(x)$, by which we mean the potential satisfying the decay condition $V(x) \leq C(1+|x|)^{-1-\epsilon}$ with some $\epsilon > 0$, the absolutely continuous spectrum fills the whole positive semi-axis. Moreover, the wave operators exist and are complete for Schrödinger operators with short range potentials [1], so that a much stronger dynamical similarity with the free Hamiltonian is present. Apart from the short-range results, there exist many works devoted to the study of some special classes of decaying potentials. We will mention only a few of them which have a closer relation with our work. For one-dimensional Schrödinger operators, it is generally known that if $V(x)$ belongs to $L^1(0, \infty)$, then the spectrum on the positive semi-axis is purely absolutely continuous (see, e.g., [37]). The situation is not so clear for decreasing potentials which are not absolutely integrable. There are many results on the absolute continuity of the spectrum on the positive semi-axis (except perhaps for a finite number of resonances in some cases) for certain classes of decaying potentials, such as potentials of bounded variation [37] or specific oscillating potentials (see, e.g., [3], [16], [38], [21] for further references). But no general relations between the rate of decay and spectral properties, apart from the absolutely integrable class, seem to be known.

The results concerning the spectral properties of Schrödinger operators with random potentials, however, suggest that there may be a general relation between the rate of decay of the potential and the preservation of the absolutely continuous spectrum on $R^+ = (0, \infty)$. Namely, Kotani-Ushiroya [20] show that when $q(x) = a(x)F(Y_x(\omega))$, where $a(x)$ is a smooth power decaying non-random factor, $Y_x(\omega)$ is a Brownian motion on a compact Riemannian manifold M with the volume element μ, and $F : M \to R$ is a non-flattening C^∞ function satisfying $\int_M F d\mu = 0$, then the question of whether the rate of decay of $a(x)$ is faster or slower than $x^{-\frac{1}{2}}$ is crucial for the spectral properties of the corresponding random Schrödinger operator. When $a(x) = (1 + |x|)^{-\alpha}$ with $0 < \alpha < \frac{1}{2}$, the spectrum on R^+ is pure point with probability one; when $\alpha > \frac{1}{2}$, then the spectrum on the positive semi-axis is a.e. purely absolutely continuous. For the exact rate of decay $x^{-\frac{1}{2}}$, one may have a mixture of singular continuous and pure point spectrum for different regions of energy.

The methods of [20] are probabilistic in nature and cannot provide information on what happens in general for potentials satisfying $|V(x)| \leq C(1+|x|)^{-\alpha}$, $\alpha > \frac{1}{2}$. Although the set of potentials leading to purely absolutely continuous spectrum is "big" in a certain sense [20], examples with eigenvalues on R^+ show there may be exceptions. Moreover, if one could find at least one potential satisfying $|V(x)| \leq C(1+|x|)^{-\alpha}$, for certain $\alpha > \frac{1}{2}$ and C, which gives rise to purely singular spectrum on the positive semi-axis, then by general principles of the genericity of singular continuous spectrum [30], there would exist another "big" (in a topological sense) set of potentials obeying the same decay condition and yielding purely singular continuous spectrum on R^+. Namely, this set would be a dense G_δ in the space of all potentials satisfying the power decay estimate $|V(x)| \leq C(1+|x|)^{-\alpha}$, equipped with the L^∞ norm. An analogous situation is exactly the case for $\alpha < \frac{1}{2}$, when the spectrum on the positive semi-axis is dense pure point with probability one by [20], but, at the same time, by the recent result of Simon [30], there exists a dense

G_δ set of potentials leading to purely singular continuous spectrum on R^+.

To further illustrate the difficulty of the passage from random to deterministic results, we note that [20] implies that there exist "many" potentials with power decay (slower than $x^{-\frac{1}{2}}$) yielding dense pure point spectrum on R^+. But, nevertheless, there are no deterministic examples of potentials with power decay even leading to just purely singular spectrum. (To construct an explicit example of a potential having dense pure point spectrum should be much harder since an arbitrarily small change in the boundary condition may change the spectrum to purely singular continuous [8], [14].) In fact, the only known explicit examples of decaying potentials yielding purely singular spectrum on R^+ are due to Pearson [26] and these potentials exhibit slower than power-rate decay.

It is an interesting problem to determine the critical rate of decay which can lead to the complete or partial destruction of the absolutely continuous spectrum on the positive half-axis, and, correspondingly, to find out which classes of potentials are not strong enough to seriously affect the absolutely continuous spectrum inherent for the free Hamiltonian. In this work, we focus on the study of the spectrum of one-dimensional Schrödinger operators. One of the results we prove says that all potentials $V(x)$ satisfying $|V(x)| \leq C(1+x)^{-\frac{3}{4}-\epsilon}$, with no additional conditions, preserve absolutely continuous spectrum on the positive semi-axis, although, of course, embedded singular spectrum may appear. This result provides a new general class of decaying potentials preserving absolutely continuous spectrum of the free Hamiltonian. It also shows that there is indeed a deterministic analog of the random potential results, at least in the range of power decay $\alpha \in (\frac{3}{4}, 1]$. The main new idea we use in the proof is a combination of a certain ODE asymptotic technique which has been commonly used for the treatment of oscillating potentials, with some results from harmonic analysis related to the almost everywhere convergence of Fourier integrals.

Another interesting aspect of the spectral behavior of Schrödinger operators with decreasing potentials is a phenomena of positive eigenvalues. Eastham-Kalf [11] show that if $V(x) = o(\frac{1}{x})$ as $x \to \infty$, then H_V does not have eigenvalues above zero. If $V(x) = O(\frac{1}{x})$, there are no eigenvalues above a certain constant. On the other hand, Eastham-McLeod [12], with further developments by Thurlow [35], show how to construct potentials $V(x)$ of the type $V(x) = \frac{C(x)}{1+x}$, with $C(x)$ converging to infinity as x tends to infinity, such that a prescribed countable set of isolated points represents embedded positive eigenvalues of H_V. These authors use the Gel'fand-Levitan approach. Later, Naboko [23] described a construction which allows for an arbitrary countable set T of rationally independent numbers in $(0, \infty)$ (and so possibly a dense set) to find a potential $V(x)$ satisfying $|V(x)| \leq \frac{C(x)}{1+x}$ with $C(x) \stackrel{x\to\infty}{\longrightarrow} \infty$ monotonously at an arbitrarily slow given rate, such that the corresponding Schrödinger operator has the set T among its eigenvalues. Recently, Simon [31] has found a different construction that does away with the rational independence assumption. The constructions of Naboko and Simon do not give information about other kinds of spectrum on R^+ in such a situation. In particular, it was not clear whether there is any other spectrum but pure point in the case when the set T of prescribed eigenvalues is dense in R^+. This thesis settles the questions arising from Naboko's and Simon's constructions. Moreover, together with these works, it provides explicit examples of decaying potentials yielding an arbitrary dense (countable) set of eigenvalues embedded in the absolutely continuous spectrum.

We should also mention that the results for random decaying potentials for discrete Schrödinger operators (Jacobi matrices) [9] raise parallel questions in the discrete case. There is also a discrete analog to the continuous case of Naboko's construction by Naboko and Yakovlev [24] which allows one to find a potential decaying arbitrarily slower than $\frac{1}{n}$, such that the corresponding discrete Schrödinger operator has eigenvalues dense in the essential spectrum $[-2, 2]$.

The thesis is organized as follows. In Section 1.1 we prove our main result for power decaying potentials. In Section 1.2 we apply developed techniques to prove that certain simple conditions on the Fourier transform of a potential from the class we study ensure the absence of embedded singular spectrum. In Section 1.3 we prove absolute continuity of the spectrum for certain bump potentials. In Section 1.4 we study asymptotics of the solutions for the complex energies. In Section 1.5 we derive an explicit formula for the projection on the absolutely continuous part of the spectrum. In Section 1.6 we consider conditionally integrable potentials and show that a certain condition on the integrability allows one to prove presence of the absolutely continuous spectrum for the slower rates of decay.

The second part of the thesis is devoted to the study of Jacobi matrices with slowly decaying potentials.

1. The absolutely continuous spectrum for Schrödinger operators

1.1. Main results for power decaying potentials

We consider the one-dimensional Schrödinger operator $H_V = -\frac{d^2}{dx^2} + V(x)$ acting on $L^2(0, \infty)$. We assume that $V(x)$ is a real-valued locally integrable function which goes to zero at infinity. It is a well-known fact that if we fix some self-adjoint boundary condition at zero, the expression H_V has a unique self-adjoint realization in $L^2(0,\infty)$. The essential spectrum of the operator H_V, $\sigma_{\text{ess}}(H_V)$, coincides with the positive semi-axis since the potential vanishing at infinity constitutes a relatively compact perturbation of the free Hamiltonian. We associate the spectral measure $d\rho$ with the operator H_V in a usual way (see, e.g., [7] or [36]).

Let us set up some notation we will need. Suppose the function $f(x)$ belongs to $L^2(0,\infty)$. Then we denote by $\Phi(f)(k)$ the Fourier transform of the function f,

$$\Phi(f)(k) = L^2 - \lim_{N\to\infty} \int_{-N}^{N} \exp(ikt) f(t)\, dt.$$

We also use the notation $M^+(g)$ for the following function corresponding to the function $g \in L^p(R)$, $1 \le p \le \infty$:

$$M^+(g)(x) = \sup_{h>0} \frac{1}{2h} \int_0^h |g(x+t) + g(x-t)|\, dt$$

and the notation $\mathcal{M}^+(g)$ for the set

$$\mathcal{M}^+(g) = \left\{ x \mid M^+(g)(x) < \infty \right\}.$$

The function $M^+(g)$ is "almost" a maximal function of the function g; in particular, $M^+(g)$ is finite whenever the maximal function of g is finite. By well-known properties of the maximal function (see, e.g., [29]) we have then that $M^+(g)$ is finite a.e. and therefore the complement of the set $\mathcal{M}^+(g)$ has measure zero.

Henceforth, we will also assume the potential $V(x)$ to be locally square integrable. We can always allow for stronger, only locally integrable singularities. We adapt this additional assumption only to simplify the formulation of results and avoid inessential technicalities.

The main result of this section is the following theorem:

Theorem 1.1.1. *Suppose that the potential* $V(x)$ *satisfies* $|V(x)| \leq Cx^{-\frac{3}{4}-\epsilon}$ *for* $x \in (a, \infty)$ *with some positive constants* ϵ, a, C. *Then the absolutely continuous spectrum of the operator* H_V *fills the whole positive semi-axis, in the sense that the absolutely continuous component* ρ_{ac} *of the spectral measure* ρ *satisfies* $\rho_{\text{ac}}(T) > 0$ *for any measurable set* $T \subset (0, \infty)$ *with* $|T| > 0$ *(where* $|\cdot| =$ *Lebesgue measure). The singular spectrum on* $(0, \infty)$ *may be located only on the complement of the set*

$$S = \frac{1}{4}(\mathcal{M}^+(\Phi(V(x)x^{\frac{1}{4}})))^2 \setminus \{0\}$$

(i.e., quarters of squares of the points from $\mathcal{M}^+(\Phi(V(x)x^{\frac{1}{4}})))$, *so that* $\rho_{\text{sing}}(S) = 0$. *Moreover, for every energy* $\lambda \neq 0$ *from the set* S, *we have two linearly independent solutions* $\phi_+(x, \lambda)$, $\overline{\phi_+(x, \lambda)}$ *(=complex conjugation of* $\phi_+(x, \lambda)$*) of the equation* $H_V\phi - \lambda\phi = 0$ *with the following asymptotics as* x *goes to infinity:*

$$\phi_+(x, \lambda) = \exp\left(i\sqrt{\lambda}x - \frac{i}{2\sqrt{\lambda}}\int_0^x V(s)\,ds\right)\left(1 + O(x^{-\epsilon}\log x)\right) \tag{1}$$

(which is exactly the WKB formula).

The asymptotic formula (1) *is one time differentiable in* x.

The main idea behind the proof is a combination of the following three ingredients:

(i) The recent studies on the connection between asymptotic behavior of solutions of the Schrödinger equation and spectral properties, which allow one to conclude the absolute continuity of the spectrum on a certain set from the boundedness of all solutions corresponding to the energies from this set;

(ii) The methods of studying the asymptotics of solutions, namely the "$\mathcal{I} + \mathcal{Q}$" transformation technique introduced by Harris and Lutz [15] and later used by many authors for treating Schrödinger operators with oscillating potentials;

(iii) The results from the theory of Fourier integrals; in particular, the question of a.e. convergence of the partial integral $\int_{-N}^{N} \exp(ikt) f(t)\, dt$ to the Fourier transform of f under certain conditions and an estimation of the rate of convergence.

As a preparation for the proof, we need several lemmas. The first lemma allows us to reduce the proof of Theorem 1.1.1 to the study of generalized eigenfunction asymptotics.

Lemma 1.1.2. *Suppose that for every λ from the set B, all solutions of the equation $H_V\phi - \lambda\phi = 0$ are bounded. Then on the set B, the spectral measure ρ of the operator H_V is purely absolutely continuous in the following sense:*

(i) $\rho_{\text{ac}}(A) > 0$ for any $A \subseteq B$ with $|A| > 0$,

(ii) $\rho_{\text{sing}}(B) = 0$.

Proof. For a large class of potentials, including those we consider here, this lemma follows from the Gilbert and Pearson subordinacy theory [13], as shown by Stolz [34]. Also, in a recent paper, Jitomirskaya and Last [17] obtained a rather transparent proof of more general results. For a direct simple proof of the lemma, we refer to a paper of Simon [32]. □

The complement of the set S in the statement of Theorem 1.1.1 has Lebesgue measure zero (which of course follows from the fact that the complement of the set $\mathcal{M}^+(\Phi(x^{\frac{1}{4}}\, V(x)))$ has measure zero). Therefore, we see that (assuming Lemma 1.1.2) for the proof of Theorem 1.1.1, it suffices to prove the stated asymptotics of generalized eigenfunctions for the energies from the set S.

The second lemma we need deals with certain properties of the Fourier integral.

Lemma 1.1.3. *Consider the function $f(x) \in L^2(R)$. Then for every $k_0 \in \mathcal{M}^+(\Phi(f))$, we have*

$$\int_{-N}^{N} f(x)\exp(ik_0x)\,dx = O(\log N).$$

Before giving the proof, let us point out the relation between the question we study and one of the subtle problems of harmonic analysis. The Fourier transform of the square integrable function $f(x)$ is usually defined as a limit in L^2-norm as $N \to \infty$ of the functions $\int_{-N}^{N} f(x)\exp(-ikx)\,dx$. The question of whether these integrals converge to the Fourier transform of f in an ordinary sense for almost all values of k is, roughly speaking, equivalent to Lusin's hypothesis that the Fourier series of square integrable functions converges almost everywhere, resolved positively by Carleson [5] in 1966. All that our simple lemma says is that we have an estimate from above on the speed of divergence of partial integrals, but for a rather explicitly described set of values of the parameter k of full measure. In the next section, which treats certain non-power decaying potentials, we will need more refined results on the a.e. convergence of Fourier integrals.

Proof of Lemma 1.1.3. The proof uses the Parseval equality:

$$\begin{aligned}\int_{-N}^{N} f(x)\exp(ik_0x)\,dx &= \frac{1}{\pi}\int_R \Phi(f)(k)\frac{\sin N(k_0-k)}{k_0-k}\,dk \\ &= \frac{1}{\pi}\int_0^{\infty} \frac{\sin Nk}{k}\left(\Phi(f)(k_0-k)+\Phi(f)(k_0+k)\right)dk.\end{aligned}$$

We split the last integral into three parts and estimate them separately:

$$\left|\int_1^{\infty} \frac{\sin Nk}{k}\left(\Phi(f)(k_0-k)+\Phi(f)(k_0+k)\right)dk\right| \le 4\pi\left\|\frac{\sin Nk}{k}\right\|_{L^2(1,\infty)} \| f \|_{L^2(-\infty,\infty)},$$

and so this part is bounded when $N \to \infty$;

$$\left| \int_0^1 \frac{\sin Nk}{k} \left(\Phi(f)(k_0 - k) + \Phi(f)(k_0 + k) \right) dk \right| \leq$$

$$\leq N \int_0^{\frac{1}{N}} |\Phi(f)(k_0 + k) + \Phi(f)(k_0 - k)| \, dk + \int_{\frac{1}{N}}^1 \frac{1}{k} |\Phi(f)(k_0 + k) + \Phi(f)(k_0 - k)| \, dk.$$

In the last expression the first summand is bounded by $M^+(\Phi(f))(k_0)$, while in the second we perform integration by parts:

$$\int_{\frac{1}{N}}^1 \frac{1}{k} |\Phi(f)(k_0 + k) + \Phi(f)(k_0 - k)| \, dk = \int_{\frac{1}{N}}^1 |\Phi(f)(k_0 + k) + \Phi(f)(k_0 - k)| \, dk +$$

$$+ \int_{\frac{1}{N}}^1 \frac{1}{k^2} \int_{\frac{1}{N}}^k |\Phi(f)(k_0 + t) + \Phi(f)(k_0 - t)| \, dt dk \leq$$

$$\leq M^+(\Phi(f))(k_0) + \int_{\frac{1}{N}}^1 \frac{1}{k} M^+(\Phi(f))(k_0) \, dk = O(\log N). \quad \square$$

To begin with the proof of the theorem, we rewrite equation $H_V \phi - \lambda \phi = 0$ as a system of first-order equations:

$$w'(x) = \begin{pmatrix} 0 & 1 \\ V(x) - \lambda & 0 \end{pmatrix} w(x), \tag{2}$$

where w and ϕ are clearly related by $w(x) = \begin{pmatrix} \phi(x) \\ \phi'(x) \end{pmatrix}$. We perform two transformations with the system (2), the first of which is the variation of the parameter formula,

$$w(x) = \begin{pmatrix} \psi_1(x) & \psi_2(x) \\ \psi_1'(x) & \psi_2'(x) \end{pmatrix} y(x), \tag{3}$$

where it is convenient for our purpose to choose $\psi_1(x) = \exp(i\sqrt{\lambda}x)$, $\psi_2(x) = \exp(-i\sqrt{\lambda}x)$. Substituting (3) into (2), we get for $y(x)$:

$$y'(x) = \frac{i}{2\sqrt{\lambda}} \begin{pmatrix} -V(x) & -V(x)\exp(-2i\sqrt{\lambda}x) \\ V(x)\exp(2i\sqrt{\lambda}x) & V(x) \end{pmatrix} y(x). \tag{4}$$

We can also write this system as

$$y' = (\mathcal{D} + \mathcal{W})y, \tag{5}$$

where $\mathcal{D}$ stays for the diagonal part of the system and $\mathcal{W}$ for the non-diagonal part which we would like to consider as a perturbation. The matrices $\mathcal{D}$ and $\mathcal{W}$ have the form

$$\mathcal{D} = \begin{pmatrix} D(x) & 0 \\ 0 & \overline{D}(x) \end{pmatrix}, \quad \mathcal{W} = \begin{pmatrix} 0 & W(x) \\ \overline{W}(x) & 0 \end{pmatrix}$$

with $D(x) = -\frac{i}{2\sqrt{\lambda}}V(x)$ and $W(x) = -\frac{i}{2\sqrt{\lambda}}V(x)\exp(-2i\sqrt{\lambda}x)$ in our case.

The main approach to the study of the asymptotics of solutions for systems similar to (5) is to attempt to find some transformation which will reduce the off-diagonal terms so that they will become absolutely integrable and then try to apply Levinson's theorem [7] on the L^1-perturbations of the systems of linear differential equations. It was discovered by Harris and Lutz [15] that when $W(x)$ is a conditionally integrable function, the following simple transformation of the system (5) works in some cases. We let

$$y(x) = (I + \mathcal{Q})z(x), \tag{6}$$

where I is an identity matrix, while $\mathcal{Q}$ satisfies $\mathcal{Q}' = \mathcal{W}$, that is,

$$\mathcal{Q}(x) = \begin{pmatrix} 0 & q(x) \\ \overline{q}(x) & 0 \end{pmatrix}$$

with $q(x) = -\int_x^\infty W(x)\,dx$. In this case, $q(x) \overset{x\to\infty}{\longrightarrow} 0$ so that for large enough x the transformation (6) is non-singular and preserves the asymptotics of the solutions. For the new variable $z(x)$ we have:

$$z' = (I + \mathcal{Q})^{-1}(\mathcal{D} + \mathcal{D}\mathcal{Q} + \mathcal{W}\mathcal{Q})z,$$

which after calculation leads to

$$z' = \left(\begin{pmatrix} D & 0 \\ 0 & \overline{D} \end{pmatrix} + (1-|q|^2)^{-1} \begin{pmatrix} \overline{W}q + 2|q|^2\overline{D} & 2\overline{q}\overline{D} - \overline{q}^2 W \\ 2qD - q^2\overline{W} & 2|q|^2 D + \overline{q}W \end{pmatrix} \right) z. \quad (7)$$

Since $q(x)$ decays at infinity, there is hope that $q(x)D(x)$ and $q(x)^2W(x)$ may be both absolutely integrable, even if initially $W(x)$ was not.

We now return to a particular case of the system (5) we consider. Our $W(x)$ is equal to $-\frac{i}{2\sqrt{\lambda}}V(x)\exp(-2i\sqrt{\lambda}x)$, depending not only on x but also on the energy λ, and we are seeking to define $q(x,\lambda) = \frac{i}{2\sqrt{\lambda}}\int_x^\infty V(s)\exp(-2i\sqrt{\lambda}s)$. The next technical lemma shows that under our assumption on the decay of the potential, we can do it and, in fact, rather successfully for every energy λ which belongs to the set S in the statement of Theorem 1.1.1.

Lemma 1.1.4. *Suppose that the potential $V(x)$ satisfies $|V(x)| \le Cx^{-\frac{3}{4}-\epsilon}$ for $x \in (a,\infty)$ with some positive constants C, a, ϵ. Then for every $k \in \mathcal{M}^+(\Phi(V(x)x^{\frac{1}{4}}))$, the integral $\int_x^\infty \exp(-iks)\,V(s)\,ds$ converges and moreover,*

$$\int_x^\infty \exp(-iks)V(s)\,ds = O(x^{-\frac{1}{4}}\log x)$$

as $x \to \infty$.

Proof. Note that $V(x)x^{\frac{1}{4}}$ is square integrable and therefore by Lemma 1.1.3, for every $k \in \mathcal{M}^+(\Phi(V(x)x^{\frac{1}{4}}))$ we have as $x \to \infty$,

$$\int_0^x V(s)s^{\frac{1}{4}} \exp(-iks)\, ds = O(\log x)$$

(we change ik in Lemma 1.1.3 to $-ik$, but since V is real, it does not change the set S). Writing $V(s) = \left(V(s)s^{\frac{1}{4}}\right) s^{-\frac{1}{4}}$ and integrating by parts we get

$$\int_0^x V(s) \exp(-iks)\, ds = x^{-\frac{1}{4}} \int_0^x V(s)s^{\frac{1}{4}} \exp(-iks)\, ds +$$

$$+\frac{1}{4} \int_0^x t^{-\frac{5}{4}} \int_0^t V(s)s^{\frac{1}{4}} \exp(-iks)\, dsdt.$$

The first summand clearly behaves at infinity like $O(x^{-\frac{1}{4}} \log x)$, while the second is absolutely convergent, since

$$\left| t^{-\frac{5}{4}} \int_0^t V(s)s^{\frac{1}{4}} \exp(-iks)\, ds \right| \le C_1 t^{-\frac{5}{4}} \log t \tag{8}$$

by Lemma 1.1.3 for all $t > 2$ with some constant C_1. The integral over $(0, 2)$ is finite since $V(x) \in L^{1,\text{loc}}$. Therefore, the integral $\int_0^x V(s) \exp(-iks)\, ds$ is conditionally convergent and its "tail" is equal to

$$\int_x^\infty V(s) \exp(-iks)\, ds = -x^{-\frac{1}{4}} \int_0^x V(s)s^{\frac{1}{4}} \exp(-iks)\, ds +$$

$$+\frac{1}{4} \int_x^\infty t^{-\frac{5}{4}} \int_0^t V(s)s^{\frac{1}{4}} \exp(-iks)\, dsdt,$$

which we can estimate for x large enough using Lemma 1.3 and (8):

$$\left| \int_x^\infty V(s) \exp(-iks)\, ds \right| \le$$

$$\le C_1 x^{-\frac{1}{4}} \log x + C_1 \int_x^\infty t^{-\frac{5}{4}} \log t\, dt = O(x^{-\frac{1}{4}} \log x). \quad \square$$

Now we note that the condition $\lambda \in S$ is equivalent to $2\sqrt{\lambda} \in \mathcal{M}^+(\Phi(V(x)x^{\frac{1}{4}}))$ by the definition of the set S. Therefore, for every $\lambda \in S$ there is a number a_λ such that for any $x > a_\lambda$ the function $q(x, \lambda)$ is less than $\frac{1}{2}$. Applying the "$I + Q$" transformation for $x > a_\lambda$ for each $\lambda \in S$, we get a system (7) for $x > a_\lambda$. The shift on the finite distance from the origin certainly does not affect asymptotics since the evolution, corresponding to such a shift, is just multiplication by some constant (for each λ) matrix. Lemma 1.1.4 allows us to see that the non-diagonal part and, in fact, the whole second summand of the matrix in the system (7) is now absolutely integrable. Indeed, every element of this matrix is equal to the product of some bounded function and the function $V(x)q(x, \lambda)$, the latter being absolutely integrable and moreover, by our assumption on V and Lemma 1.1.4, satisfying $|V(x)q(x, \lambda)| < C_2(\lambda)x^{-1-\epsilon} \log x$ for every $\lambda \in S$ with the constant C_2 depending on λ. We can now apply Levinson's theorem, but in our situation we do not need the whole power of this result. Rewriting the system (7) for every $\lambda \in S$ as

$$z'(x) = \left(\frac{i}{2\sqrt{\lambda}} \begin{pmatrix} -V(x) & 0 \\ 0 & V(x) \end{pmatrix} + R(x, \lambda) \right) z(x)$$

with $V(x)$ real and $||R(x, \lambda)|| \in L^1$ with $\int_x^\infty ||R(s, \lambda)||\, ds = O(x^{-\epsilon} \log x)$, we can, in a standard way, transfer this system into the system of integral equations, apply the Gronwall lemma, and prove (see [28] for the details) that for each $\lambda \in S$ there exist solutions $z_1(\lambda, x)$, $\overline{z}_1(\lambda, x)$ with the asymptotics

$$z_1(\lambda, x) = \exp\left(-\frac{i}{2\sqrt{\lambda}} \int_0^x V(x)dx \right) (1 + O(x^{-\epsilon} \log x)).$$

Applying now transformations (6) and (3) to obtain the asymptotics of the solution and its derivative of the initial problem, we conclude the proof of Theorem 1.1.1.

Remarks. 1. With very little effort, the introduced method yields results for the whole-line (or, more generally, two singular point) problem for the Schrödinger operator H_V with potential $V \in L^{1,\mathrm{loc}}$ satisfying $|V(x)| \leq C(1+|x|)^{-\frac{3}{4}-\epsilon}$ for $|x|$ large enough. The substitution of Lemma 1.1.2 for the whole line can be easily recovered from the remark in [32] and says that on the set $S_+ \cup S_-$, where S_+ and S_- are the sets of energies for which all solutions are bounded as x approaches correspondingly plus or minus infinity, the spectrum is purely absolutely continuous of multiplicity two (in the sense of Lemma 1.1.2). Of course, Lemmas 1.1.3 and 1.1.4 can be used for studying the asymptotics of solutions at $-\infty$ as well as at $+\infty$. What we get in this case is that the whole positive half-axis is filled by the absolutely continuous spectrum of multiplicity two and the singular spectrum may only be supported on the complement of $S_+ \cup S_-$. Moreover, it is a known fact [18] that the multiplicity of the singular spectrum may only be one for the whole-line Sturm-Liouville operators.

2. One may apply the proven results to the study of the absolutely continuous spectrum of Schrödinger operators with spherically symmetric potentials in R^n, satisfying $|V(r)| \leq Cr^{-\frac{3}{4}-\epsilon}$. In a standard way, one decomposes the Schrödinger operator H_V into a direct sum of one-dimensional operators $H_{V,l} = -\frac{d^2}{dr^2} + (f_n(l)r^{-2} + V(r))$ acting on different moment subspaces (see, e.g., [27]). It is easy to see that the set S_l of energies for which all solutions of the equation $H_{V,l}\phi - \lambda\phi$ are bounded in fact will be independent of l, since the term $f_n(l)r^{-2}$ decays fast at infinity. Correspondingly, the singular spectrum of H_V on R^+ may only be supported on the complement of the set S.

3. In fact, Theorem 1.1.1 is more than a deterministic analog of the Kotani-Ushiroya theorem in the power range $\alpha \in (\frac{3}{4}, 1]$. Indeed, one can check that from the assumption $\int_M F\,d\mu = 0$ in their random model, it follows that the a.e. potential

is conditionally integrable and satisfies

$$\int_x^\infty V(t,\omega)\,dt \leq C(\omega)(1+|x|)^{-\beta}$$

for every $\beta < \alpha - \frac{1}{2}$ with probability one. Assuming conditional integrability of V and certain power-decay estimate on the "tail" of the potential, we can extend our result about the presence of the absolutely continuous spectrum on potentials satisfying only $|V(x)| \leq Cx^{-\frac{2}{3}-\epsilon}$. We treat this case in Section 1.6.

As a byproduct of the computations we performed, let us formulate the following proposition, which is in fact a slight variation of Theorem 2.1 from Harris and Lutz [15]:

Proposition 1.1.5. *Suppose that for given energy $\lambda > 0$, the function*

$$V(x)\int_x^\infty \exp(-2i\sqrt{\lambda}t)V(t)\,dt$$

is well-defined and belongs to $L^1(0,\infty)$. Then there exist two linearly independent solutions ϕ_λ, $\overline{\phi}_\lambda$ of the equation $H_V\phi - \lambda\phi = 0$ with the following asymptotics as $x \longrightarrow \infty$:

$$\begin{aligned}\phi_\lambda(x) &= \exp\left(i\sqrt{\lambda}x - \frac{i}{2\sqrt{\lambda}}\int_0^x V(s)\,ds\right)\times\\ &\times\left(1+O\left(\int_x^\infty \left|V(s)\int_s^\infty V(t)\exp(-2i\sqrt{\lambda}t)\,dt\right|ds\right)\right).\end{aligned}$$

In particular, all solutions are bounded.

1.2. A criterion for the absence of embedded singular spectrum

Based on the technique introduced in the proof of our main theorem, we now prove the result showing that certain conditions on the Fourier transform of potentials decaying faster than $x^{-\frac{3}{4}-\epsilon}$ are sufficient to ensure the absence of the singular component of the spectrum on the positive semi-axis.

Theorem 1.2.1. *Suppose the potential $V(x)$ satisfies $|V(x)| < Cx^{-\frac{3}{4}-\epsilon}$ for all $x > a$ and the Fourier transform $\Phi(x^{\frac{1}{4}}V(x))(k)$ belongs to $L^{p,\mathrm{loc}}$ for some $p > \frac{1}{\epsilon}$. Then the spectrum of the operator H_V on the positive semi-axis is purely absolutely continuous, and for every energy $\lambda \in (0, \infty)$ there exist two solutions ϕ_λ, $\overline{\phi}_\lambda$ with the asymptotics as $x \to \infty$,*

$$\phi_\lambda(x) = \exp\left(i\sqrt{\lambda}x - \frac{i}{2\sqrt{\lambda}}\int_0^x V(s)\,ds\right)\left(1 + O(x^{-\epsilon+\frac{1}{p}})\right).$$

It is clear that we can concentrate on proving the stated asymptotics for every λ in the positive half-axis. A modification of Lemma 1.1.3 is needed:

Lemma 1.2.2. *Suppose that the Fourier transform $\Phi(f)(k)$ of the function $f(x) \in L^2$ belongs to $L^{p,\mathrm{loc}}$, $p > 2$. Then for every value of k,*

$$\int_{-N}^{N} f(x)\exp(ikx)\,dx = O(N^{\frac{1}{p}}).$$

Proof. As in the proof of Lemma 1.3 making use of the Parseval equality, we get

$$\int_{-N}^{N} f(x)\exp(ikx)\,dx = \int_0^\infty \frac{\sin Nt}{t}\left(\Phi(f)(k-t) + \Phi(f)(k+t)\right)dt.$$

Again, the integral from 1 to ∞ is bounded uniformly in N by the product of L^2-norms of the functions under the integral. The remaining part we split into two integrals and estimate them using Hölder's inequality:

$$\left| \int_{\frac{1}{N}}^{1} \frac{\sin Nt}{t} (\Phi(f)(k-t) + \Phi(f)(k+t))\, dt \right| \leq$$

$$\leq \left(\int_{\frac{1}{N}}^{1} (1/t)^{p'}\, dt \right)^{\frac{1}{p'}} \left(\int_{k-1}^{k+1} |\Phi(f)(t)|^p\, dt \right)^{\frac{1}{p}} = O(N^{\frac{1}{p}}),$$

where p' is a conjugate exponent for $p : p' = \frac{p}{p-1}$. The second integral is estimated in a similar way:

$$\left| \int_{0}^{\frac{1}{N}} \frac{\sin Nt}{t} (\Phi(f)(k-t) + \Phi(f)(k+t))\, dt \right| \leq$$

$$\leq \left(\int_{0}^{\frac{1}{N}} N^{p'}\, dt \right)^{1/p'} \left(\int_{k-\frac{1}{N}}^{k+\frac{1}{N}} |\Phi(f)(t)|^p\, dt \right)^{\frac{1}{p}} = O(N^{\frac{1}{p}}). \quad \square$$

Proof of Theorem 1.2.1. The same calculation which we performed proving Lemma 1.1.4 (integration by parts) shows that under the conditions of Theorem 1.2.1 for every positive λ, we have

$$q(x, \lambda) = \int_{x}^{\infty} V(x) \exp(-i\sqrt{\lambda}x)\, dx = O(x^{-\frac{1}{4}+\frac{1}{p}})$$

as $x \longrightarrow \infty$. This implies that for all energies, the function $V(x)q(x, \lambda)$ is absolutely integrable and moreover satisfies the estimate for large enough x,

$$|V(x)q(x, \lambda)| < C(\lambda)x^{-1-\epsilon+\frac{1}{p}}.$$

By Proposition 1.1.5, the proof is complete. □

Remark. It is easy to modify the proof of Lemma 1.2.2 and Theorem 1.2.1 to obtain a local criterion for the absence of singular spectrum. That is, if V satisfies the conditions of Theorem 1.2.1 and $\Phi(x^{\frac{1}{4}}V(x))(k)$ belongs to $L^p(a,b)$, $b > a > 0$, then the spectrum of the operator H_V is purely absolutely continuous in the energy interval $(\frac{a^2}{4}, \frac{b^2}{4})$.

We note that the conditions stated in the theorem are rather precise. For example, in the celebrated Wigner-von Neumann example (historically the first example of the decaying potential having positive eigenvalue embedded in the absolutely continuous spectrum), the asymptotic behavior of the potential at infinity is $V(x) = -\frac{8(\sin 2x)}{x} + O(x^{-2})$ (see, e.g., [28]) so that $\epsilon = \frac{1}{4}$, while the singularity of the Fourier transform of $x^{\frac{1}{4}}V(x)$ is easily seen to be of the order $(k-2)^{-\frac{1}{4}}$ which belongs to $L^{p,\mathrm{loc}}$ with $p < 4$. It is an open question whether one can replace the condition $\Phi(x^{\frac{1}{4}}V(x)) \in L^{p,\mathrm{loc}}$, $p > \frac{1}{\epsilon}$ with the simpler one $\Phi(x^{\frac{1}{4}}V(x)) \in L^{\frac{1}{\epsilon},\mathrm{loc}}$ so that the last theorem still remains true.

1.3. Non-power decreasing potentials

In this section we apply the method described in the preceding section to a wider class of potentials. This class will include, in particular, certain potentials of the bump type, which are "mostly" zero but have bumps decaying at infinity.

Let us introduce the class of potentials we will treat.

Definition. We say that the potential $\tilde{V}(x) \in L^\infty(0,\infty)$ belongs to the class $\mathcal{P}_{-\alpha}(0,\infty)$ if there exists a potential $V(x) \in L^\infty(0,\infty)$ satisfying $|V(x)| \le Cx^{-\alpha}$ for x large enough and a countable collection of disjoint intervals in $(0,\infty)$ $\{(a_j,b_j)\}_{j=1}^\infty$,

$b_j \leq a_{j+1}$ $\forall j$, such that

$$\tilde{V}(x) = \begin{cases} 0, & x \in (a_n, b_n) \\ V(x - \sum\limits_{j=1}^{n} (b_j - a_j)), & x \in (b_n, a_{n+1}) \end{cases}.$$

Roughly, the potential $\tilde{V}(x)$ is obtained from $V(x)$ by inserting a countable number of intervals on which $\tilde{V}(x)$ vanishes; while on the rest of the axis, it is $V(x)$ shifted on the distance which is equal to the sum of the lengths of the intervals inserted so far. Of course, $\tilde{V}(x) \in \mathcal{P}_{-\alpha}$ need not decay faster than any power at infinity. However, if we "compress" $\tilde{V}(x)$ by collapsing all intervals on which it vanishes, we get a potential which is bounded by $Cx^{-\alpha}$ for large x.

The following theorem holds for potentials from the class $\mathcal{P}_{-\frac{3}{4}-\epsilon}$:

Theorem 1.3.1. *Suppose $\tilde{V}(x) \in \mathcal{P}_{-\frac{3}{4}-\epsilon}$, $\epsilon > 0$. Then the absolutely continuous part of the spectral measure fills the whole positive semi-axis, in the sense that $\rho_{\text{ac}}(T) > 0$ for any measurable set $T \subset (0, \infty)$ with $|T| > 0$. For almost every energy $\lambda \in (0, \infty)$, there exist two solutions ϕ_λ, $\overline{\phi}_\lambda$ with the asymptotics as $x \to \infty$,*

$$\phi_\lambda(x) = \exp\left(i\sqrt{\lambda}x - \frac{i}{2\sqrt{\lambda}} \int\limits_0^x \tilde{V}(s)\, ds \right) (1 + o(1)).$$

Proposition 1.1.5 implies that to prove the stated result, we need only to show that the function $R(t) = \tilde{V}(x) \int_x^\infty \tilde{V}(t) \exp(-2i\sqrt{\lambda}t)\, dt$ is well-defined and belongs to $L^1(0, \infty)$ for a.e. $\lambda \in R^+$. To proceed with the proof, we need some further facts from the theory of Fourier integrals.

The following result is due to Zygmund [39].

Theorem (Zygmund). *If $f \in L^p(-\infty, \infty)$, where $1 \leq p < 2$, then the integral*

$$F(f)(k, N) = \frac{1}{\sqrt{2\pi}} \int\limits_{-N}^{N} f(x) \exp(-ikx)\, dx$$

converges as $N \to \infty$, in an ordinary sense for almost every value of k.

This will serve us as an analog of Lemma 1.1.3. However, it is a more sophisticated result by itself. One of the consequences is that we do not have an explicit description of the exceptional set on which convergence fails (and correspondingly, where the singular spectrum may be supported). For future reference, let us denote by $A(f)$ the set of full measure for which the integral $F(f)(k, N)$ does converge.

The main idea now is the same as before: to perform in some "clever" way integration by parts to get estimates on the tail

$$\tilde{q}(x, \lambda) = \int_x^\infty \tilde{V}(t) \exp(-2i\sqrt{\lambda} t)\, dt$$

for a.e. λ. Of course, there is no longer any hope that $q(x, \lambda)$ will, in general, decay even as some power for potentials we now consider. However, the special structure of the potentials allows us to overcome this problem.

Proof of Theorem 1.3.1. Let us factorize $\tilde{V}(x) = \tilde{V}_1(x)\tilde{V}_2(x)$ in the following way: if $\tilde{V}(x) = V(x, \{(a_j, b_j)\}_{j=1}^{j=\infty})$, then

$$\tilde{V}_1(x) = \begin{cases} 0, & x \in (a_n, b_n) \\ \left(x - \sum\limits_{j=1}^{n}(b_j - a_j)\right)^{\frac{1}{4}} V\left(x - \sum\limits_{j=1}^{n}(b_j - a_j)\right), & x \in (b_n, a_{n+1}) \end{cases}$$

and

$$\tilde{V}_2(x) = \begin{cases} \left(a_n - \sum\limits_{j=1}^{n-1}(b_j - a_j)\right)^{-\frac{1}{4}}, & x \in (a_n, b_n) \\ \left(x - \sum\limits_{j=1}^{n}(b_j - a_j)\right)^{-\frac{1}{4}}, & x \in (b_n, a_{n+1}) \end{cases}.$$

Therefore, $\tilde{V}_1(x)$ is obtained from the function $x^{\frac{1}{4}}V(x)$ in the same way as $\tilde{V}(x)$ is obtained from $V(x)$, while the quotient $\frac{\tilde{V}(x)}{\tilde{V}_1(x)} = \tilde{V}_2(x)$ is a continuous piecewise differentiable non-increasing function. Since $|V(x)| < Cx^{-\frac{3}{4}-\epsilon}$ for large enough x, we have that $x^{\frac{1}{4}}V(x)$, and therefore also $\tilde{V}_1(x)$ belong to $L^{2-\epsilon}(0,\infty)$. Then by Zygmund's theorem, for all λ from the set $\frac{1}{4}(A(F(\tilde{V}_1(x))))^2$ (quarters of the squares of the points from the set $A(F(\tilde{V}_1(x)))$) of full measure in the positive semi-axis, the limit $\lim_{N\to\infty}\int_0^N \tilde{V}_1(t)\exp(-2i\sqrt{\lambda}t)\,dt$ exists, so that we can consider for these λ the conditionally convergent integral $\int_x^\infty \tilde{V}_1(t)\exp(-2i\sqrt{\lambda}t)\,dt$. Let us integrate by parts the expression

$$\begin{aligned}\tilde{q}(x,\lambda) &= \int_x^\infty \tilde{V}_1(t)\tilde{V}_2(t)\exp(-2i\sqrt{\lambda}t)\,dt = \\ &= \tilde{V}_2(x)\int_x^\infty \tilde{V}_1(t)\exp(-2i\sqrt{\lambda}t)\,dt + \int_x^\infty \tilde{V}_2'(t)\int_t^\infty \tilde{V}_1(s)\exp(-2i\sqrt{\lambda}s)\,dsdt.\end{aligned}$$

For the values of λ that we consider, the value of the integral $\int_x^\infty \tilde{V}_1(t)\exp(2i\sqrt{\lambda}t)\,dt$ goes to zero at infinity and therefore is bounded by some constant C (depending on λ) for all values of x. Hence, we can estimate the right-hand side in the last equation by

$$C\left(\tilde{V}_2(x) + \int_x^\infty |\tilde{V}_2'(t)|\,dt\right) \le 2C\tilde{V}_2(x),$$

since $\tilde{V}_2(x)$ is a non-increasing positive continuous piecewise differentiable function. Thus, we get that for a.e. λ,

$$\left|\tilde{V}(x)\int_x^\infty \tilde{V}(t)\exp(2i\sqrt{\lambda}t)\,dt\right| \le C(\lambda)|\tilde{V}(x)\tilde{V}_2(x)|.$$

To conclude the proof, we notice that the function $\tilde{V}(x)\tilde{V}_2(x)$ is absolutely integrable by the way we constructed the functions $\tilde{V}(x)$ and $\tilde{V}_2(x)$; the L^1-norm of their product is equal to the L^1-norm of the function $x^{-\frac{1}{4}}V(x)$. On the intervals (a_n, b_n), where $\tilde{V}_2(x)$ is defined to be constant, $\tilde{V}(x)$ vanishes; and on the intervals where $\tilde{V}(x)$ is equal to shifted $V(x)$, $\tilde{V}_2(x)$ is just shifted $x^{-\frac{1}{4}}$. □

One of the situations to which Theorem 1.3.1 applies is when we have a sequence of repeating bumps of the same shape but with decreasing magnitude. Fix $U(x) \in L^\infty(0,a)$ and let

$$\tilde{V}(x) = \sum_{n=1}^{\infty} g_n U(x - a_n), \ a_n - a_{n-1} > a.$$

For potentials of this type, Pearson [26] has shown that if one chooses the distances between bumps to be big enough, then if $\sum_{n=1}^{\infty} g_n^2 = \infty$, the corresponding Schrödinger operator has purely singular continuous spectrum on R^+. Otherwise, there was essentially nothing known about the possible spectral behavior for Schrödinger operators with bump potentials which are not absolutely integrable and not power decaying. From the last theorem it follows that if $|g_n| < Cn^{-\frac{3}{4}-\epsilon}$, the absolutely continuous spectrum remains on the positive semi-axis, no matter how $U(x)$ looks and which distances between bumps we take.

1.4. Generalized Jost functions

In this section we are studying the asymptotics of solutions of the Schrödinger equation

$$\left(-\frac{d^2}{dx^2} + V(x)\right) u = \lambda u \tag{9}$$

with decaying potentials for the complex values of energy. Using the results we derive here, in the following section we obtain more information about the absolutely continuous spectrum on R^+ for a certain class of potentials which decrease fast enough (but not short-range in the usual sense of this term). In particular, we obtain an explicit formula for the projection on the absolutely continuous part of the spectrum.

The solutions for the complex energies, which we investigate here, carry in principle all information needed for the study of the spectrum for the general square integrable potentials, with no further conditions on the decay. The subtle point is tracing the limiting behavior of certain functions as the energy approaches the real axis and we are able to perform this analysis only under additional assumptions. Let us denote by C^+ the upper half-plane of the complex plane, not including the real axis. Let us also fix the branch of the square root function $\sqrt{\lambda}$ so that $\Im\lambda > 0$ for every $\lambda \in C^+$.

The main theorem we prove in this section is the following:

Theorem 1.4.1. *Suppose that $V(x) \in L^2(0,\infty)$. Then for every value of the spectral parameter $\lambda \in C^+$, there is a solution $\phi_+(x,\lambda)$ of the equation* (9) *satisfying the following asymptotics:*

$$\begin{aligned} \phi_+(x,\lambda) &= \exp\left(i\sqrt{\lambda}x - \tfrac{i}{2\sqrt{\lambda}}\int_0^x V(s)\,ds \right)\left(1 + \tfrac{1}{|\lambda|(\Im\sqrt{\lambda})}O(\|V\|_{L^2(\frac{x}{2},\infty)})\right), \\ \phi_+'(x,\lambda) &= i\sqrt{\lambda}\exp\left(i\sqrt{\lambda}x - \tfrac{i}{2\sqrt{\lambda}}\int_0^x V(s)\,ds \right)\left(1 + \tfrac{1}{|\lambda|(\Im\sqrt{\lambda})}O(\|V\|_{L^2(\frac{x}{2},\infty)})\right). \end{aligned} \tag{10}$$

The O is uniform in λ after taking into account the factor $\frac{1}{|\lambda|(\Im\sqrt{\lambda})}$. Moreover, for any fixed x, $\phi_+(x,\lambda)$ and $\phi_+'(x,\lambda)$ are analytic in C^+.

Let us breifly explain why we are interested in the solution $\phi_+(x,\lambda)$. Suppose that we want to study the spectrum of the Schrödinger operator H_V given by the left-hand side of (9) together with the Dirichlet boundary condition at zero. Because of our choice of the branch of $\sqrt{\lambda}$, we can check that $\phi_+(x,\lambda)$ belongs to L^2 for every $\lambda \in C^+$. Indeed, it is easy to see that the integral term under exponent in the expression (10) for $\phi_+(x,\lambda)$ cannot "spoil" the negative real part of the $i\sqrt{\lambda}x$ term if $V \in L^2$.

Let us denote by $\theta(x,\lambda)$ and $\chi(x,\lambda)$ solutions of equation (9) satisfying $\theta'(0) = 1$, $\theta(0) = 0$, and $\chi'(0) = 0$, $\chi(0) = 1$. By one of the equivalent definitions of the

Weyl m-function corresponding to the Dirichlet boundary condition at zero (see e.g., [36], [33]), we have that

$$\phi_+(x,\lambda) = h(\chi(x,\lambda) + m(\lambda)\theta(x,\lambda))$$

and

$$\phi'_+(x,\lambda) = h(\chi'(x,\lambda) + m(\lambda)\theta'(x,\lambda)),$$

where h is some constant, which may depend on λ.

This allows us to compute $m(\lambda)$:

$$m(\lambda) = \frac{\phi'_+(0,\lambda)}{\phi_+(0,\lambda)}. \tag{11}$$

Hence, if we have sufficient information about the function $\phi_+(x,\lambda)$, we can study $m(\lambda)$. The relation between the Weyl function $m(\lambda)$ and the spectral measure is well-known (see, e.g., [36], [33]) and will be discussed in more detail in Section 1.5. In particular, if we can compute the limits of $\phi'_+(0,\lambda)$ and $\phi_+(0,\lambda)$ on the real axis, we may be able to determine the spectrum of H_V, and also to derive some additional information, such as a rather explicit eigenfunction expansion. This program is carried out in Section 1.5.

The proof of the theorem relies on the complex version of the Harris-Lutz method. A similar technique has been applied by Ben-Artzi and Devinatz [4], Devinatz and Rejto [10], and White [38] to the study of the absolutely continuous spectrum of Schrödinger operators with some particular oscillating potentials.

First, we prove the existence of the solution with the stated asymptotics and then we show its analyticity. Since the "boundary conditions" for $\phi_+(x,\lambda)$ are given by the asymptotics at infinity, proving analyticity involves more work than the standard (see, e.g., [36]) proof of the analyticity of solutions with the given boundary condition at some finite point. The proof is similar to a usual proof of the analyticity of Jost functions for absolutely integrable potentials (see, e.g., [6]) but is more technical.

For the complex energies, it is convenient to modify slightly the transformations we used to derive real-energy asymptotics in Section 1.1. As before, we rewrite equation (9) as a system:

$$u' = \begin{pmatrix} 0 & 1 \\ V-\lambda & 0 \end{pmatrix} u \tag{12}$$

where $u = \begin{pmatrix} u \\ u' \end{pmatrix}$. Let

$$u = \begin{pmatrix} 1 & 1 \\ -i\sqrt{\lambda} & i\sqrt{\lambda} \end{pmatrix} y, \tag{13}$$

then

$$y' = \begin{pmatrix} -i\sqrt{\lambda} + \frac{i}{2\sqrt{\lambda}}V & \frac{i}{2\sqrt{\lambda}}V \\ -\frac{i}{2\sqrt{\lambda}}V & -i\sqrt{\lambda} - \frac{i}{2\sqrt{\lambda}}V \end{pmatrix} y. \tag{14}$$

We note that the transformation (13) is quite natural since it diagonalizes the matrix $\begin{pmatrix} 0 & 1 \\ -\lambda & 0 \end{pmatrix}$. Write the system (14) as

$$y = (\mathcal{F} + \mathcal{D} + \mathcal{W})y,$$

where $\mathcal{F}$ is a diagonal matrix with constant $\mp i\sqrt{\lambda}$ entries, $\mathcal{D}$ is a diagonal matrix with the entries $\pm\frac{i}{2\sqrt{\lambda}}V$, and $\mathcal{W}$ is the off-diagonal part. As before, we aim at finding another transformation which will help us reduce the off-diagonal part. Putting $y = (\mathcal{I} + \mathcal{Q})z$, with $\mathcal{Q} = \begin{pmatrix} 0 & q_{12} \\ q_{21} & 0 \end{pmatrix}$, we obtain

$$(\mathcal{I} + \mathcal{Q})z' + \mathcal{Q}'z = (\mathcal{F} + \mathcal{F}\mathcal{Q} + \mathcal{D} + \mathcal{W} + \mathcal{D}\mathcal{Q} + \mathcal{W}\mathcal{Q})z,$$

and after a simple calculation,

$$z' = ((\mathcal{F}+\mathcal{D})+(\mathcal{I}+\mathcal{Q})^{-1}(\mathcal{F}\mathcal{Q}-\mathcal{Q}\mathcal{F}+\mathcal{W}-\mathcal{Q}'+\mathcal{D}\mathcal{Q}-\mathcal{Q}\mathcal{D}+\mathcal{W}\mathcal{Q}))z. \quad (15)$$

In order to be able to study the asymptotics of the system (15) with the aid of the Levinson-type technique, we should choose $\mathcal{Q}$ so that:

$$\mathcal{F}\mathcal{Q} - \mathcal{Q}\mathcal{F} + \mathcal{W} - \mathcal{Q}' = 0, \quad (16)$$

$$\mathcal{D}\mathcal{Q} - \mathcal{Q}\mathcal{D} + \mathcal{W}\mathcal{Q} \in L^1((0,\infty)), \quad (17)$$

$\mathcal{I} + \mathcal{Q}$ is non-singular for x large enough uniformly in $\lambda \in U_\epsilon = \{\lambda | \Im\lambda > \epsilon > 0\}$. (18)

The last condition is needed for proving analyticity of the function $\varphi_+(x,\lambda)$.

From the condition (16) it is easy to derive differential equations for the elements q_{12}, q_{21} of the matrix $\mathcal{Q}$:

$$q_{12}'(x,\lambda) = -2i\sqrt{\lambda}q_{12}(x,\lambda) + \frac{i}{2\sqrt{\lambda}}V(x),$$

$$q_{21}'(x,\lambda) = 2i\sqrt{\lambda}q_{21}(x,\lambda) - \frac{i}{2\sqrt{\lambda}}V(x).$$

Solving these equations, we obtain one-parametric (for each λ) families of functions which may serve as q_{ij}, $i,j = 1,2$,

$$q_{12}(x,\lambda) = \exp(-2i\sqrt{\lambda}x)\left(C_1(\lambda) - \frac{i}{2\sqrt{\lambda}}\int_x^\infty \exp(2i\sqrt{\lambda}t)V(t)\,dt\right), \quad (19)$$

$$q_{21}(x,\lambda) = \exp(2i\sqrt{\lambda}x)\left(C_2(\lambda) - \frac{i}{2\sqrt{\lambda}}\int_0^x \exp(-2i\sqrt{\lambda}t)V(t)\,dt\right). \quad (20)$$

Since we have to meet condition (17), we should choose C_1 and C_2 so that q_{ij} is decaying at infinity. This forces us to put $C_1(\lambda) = 0$.

In the choice of $C_2(\lambda)$ we have some degree of freedom, which will be essential in the next section when we consider limits on the real axis. For now, to simplify computations, we also let $C_2(\lambda) = 0$.

Lemma 1.4.2. *Suppose that $V(x) \in L^2(0,\infty)$. Then for every $\lambda \in C^+$, the functions $q_{12}(x,\lambda)$ and $q_{21}(x,\lambda)$ satisfy:*

$$\|q_{12}(y,\lambda)\|_{L^2(x,\infty)} \leq \frac{1}{4|\sqrt{\lambda}|\Im\sqrt{\lambda}} \|V(y)\|_{L^2(x,\infty)}, \tag{21}$$

$$\|q_{12}(y,\lambda)\|_{L^\infty(x,\infty)} \leq \frac{1}{4|\sqrt{\lambda}|(\Im\sqrt{\lambda})^{\frac{1}{2}}} \|V(y)\|_{L^2(x,\infty)}, \tag{22}$$

$$\|q_{21}(y,\lambda)\|_{L^\infty(x,\infty)} \leq \frac{1}{4|\sqrt{\lambda}|(\Im\sqrt{\lambda})^{\frac{1}{2}}} \left(\|V(y)\|_{L^2(\frac{x}{2},\infty)} + \exp(-\Im\sqrt{\lambda}x)\|V\|_{L^2(0,\infty)}\right), \tag{23}$$

$$\|q_{21}(y,\lambda)\|_{L^2(x,\infty)} \leq \frac{1}{4|\sqrt{\lambda}|\Im\sqrt{\lambda}} \left(\|V(y)\|_{L^2(\frac{x}{2},\infty)} + \exp(-\Im\sqrt{\lambda}x)\|V\|_{L^2(0,\infty)}\right). \tag{24}$$

Proof. The functions q_{ij} look like not normalized averages of a certain type of the function V. Therefore, it is not at all suprising that one may basically estimate the L^p-norms of these functions by the L^p norms of V, but that this estimate diverges as $\Im\sqrt{\lambda}$ goes to zero. Let us first consider q_{12}. In all computations, we will omit the unessential $\frac{i}{2\sqrt{\lambda}}$ factor in front of the expressions for q_{ij}.

$$\|q_{12}(y,\lambda)\|_{L^\infty(x,\infty)} = \sup_{y\geq x}\left|\exp(-2i\sqrt{\lambda}y)\int_y^\infty \exp(2i\sqrt{\lambda}t)V(t)\,dt\right| =$$

$$= \sup_{y\geq x}\left|\int_0^\infty \exp(2i\sqrt{\lambda}s)V(s+y)\,ds\right| \leq \frac{1}{2(\Im\sqrt{\lambda})^{\frac{1}{2}}}\|V\|_{L^2(x,\infty)}.$$

We used the Hölder inequality in the third step. For the L^2-norm estimate, we have:

$$\|q_{12}(y,\lambda)\|_{L^2(x,\lambda)} = \left(\int_x^\infty dy\left|\int_0^\infty \exp(2i\sqrt{\lambda}t)V(t+y)\,dt\right|^2\right)^{\frac{1}{2}} \leq$$

$$\leq \int_0^\infty dt \left(\int_x^\infty dy\, |V(t+y)\exp(2i\sqrt{\lambda}t)|^2 \right)^{\frac{1}{2}} \leq \int_0^\infty dt\, e^{-2\Im\sqrt{\lambda}t} \|V\|_{L^2(x,\infty)} =$$

$$= \frac{1}{2\Im\sqrt{\lambda}} \|V\|_{L^2(x,\infty)}.$$

Here in the second step we applied the Minkowski inequality. Estimates for q_{21} are similar but slightly more technical.

Let us split the integral for $q_{21}(x,\lambda)$ into two parts:

$$q_{21}(x,\lambda) = \exp(2i\sqrt{\lambda}x) \int_0^x \exp(-2i\sqrt{\lambda}t) V(t)\, dt =$$

$$= \exp(2i\sqrt{\lambda}x) \left(\int_0^{\frac{x}{2}} \exp(-2i\sqrt{\lambda}t) V(t)\, dt + \int_{\frac{x}{2}}^x \exp(-2i\sqrt{\lambda}t) V(t)\, dt \right).$$

The L^∞-norms of both summands are estimated by the Hölder inequality. For the first one we have

$$\left| \exp(2i\sqrt{\lambda}x) \int_0^{\frac{x}{2}} \exp(-2i\sqrt{\lambda}t) V(t)\, dt \right| \leq$$

$$\leq \exp(-\Im\sqrt{\lambda}x) \left| \int_0^{\frac{x}{2}} \exp\left(2i\sqrt{\lambda}\left(\frac{x}{2} - t\right)\right) V(t)\, dt \right| \leq$$

$$\leq \frac{1}{2(\Im\sqrt{\lambda})^{\frac{1}{2}}} \exp(-\Im\sqrt{\lambda}x) \|V\|_{L^2(0,\infty)}.$$

For the second we obtain

$$\left| \exp(2i\sqrt{\lambda}x) \int_{\frac{x}{2}}^x \exp(-2i\sqrt{\lambda}t) V(t)\, dt \right| \leq$$

$$\leq \left(\int_{\frac{x}{2}}^x \exp(-4\Im\sqrt{\lambda}(x-t))\, dt \right)^{\frac{1}{2}} \left(\int_{\frac{x}{2}}^x |V(t)|^2 dt \right)^{\frac{1}{2}} \leq \frac{1}{2(\Im\sqrt{\lambda})^{\frac{1}{2}}} \|V\|_{L^2(\frac{x}{2},\infty)}.$$

The L^2-norm of the first summand is estimated as follows:

$$\left(\int_x^\infty \left| \int_0^{\frac{y}{2}} \exp(2i\sqrt{\lambda}(y-t)) V(t)\, dt \right|^2 dy \right)^{\frac{1}{2}} =$$

$$= \left(\int_x^\infty \left| \int_{\frac{y}{2}}^{y} \exp(2i\sqrt{\lambda}z) V(y-z)\, dz \right|^2 dy \right)^{\frac{1}{2}} \leq$$

$$\leq \int_{\frac{x}{2}}^{\infty} dz \left(\int_{\max(x,z)}^{2z} |\exp(-4\Im\sqrt{\lambda}z) V(y-z)|^2\, dy \right)^{\frac{1}{2}} \leq$$

$$\leq \frac{1}{2\Im\sqrt{\lambda}} \exp(-\Im\sqrt{\lambda}x) \|V(y)\|_{L^2(0,\infty)}.$$

Here we applied the Minkowski inequality in the second step. Similarly, for the second summand we write:

$$\left(\int_x^\infty dy \left| \int_{\frac{y}{2}}^{y} \exp(2i\sqrt{\lambda}(y - t)) V(t)\, dt \right|^2 \right)^{\frac{1}{2}} =$$

$$= \left(\int_x^\infty dy \left| \int_0^{\frac{y}{2}} \exp(2i\sqrt{\lambda}z) V(y - z)\, dz \right|^2 \right)^{\frac{1}{2}} \leq$$

$$\leq \int_0^\infty dz \left(\int_{\max(2z,x)}^{\infty} |\exp(2i\sqrt{\lambda}z) V(y - z)|^2\, dy \right)^{\frac{1}{2}} =$$

$$= \int_0^\infty \exp(-2\Im\sqrt{\lambda}z) \left(\int_{\max(2z,x)}^{\infty} |V(y - z)|^2\, dy \right)^{\frac{1}{2}} dz \leq \frac{1}{2\Im\sqrt{\lambda}} \|V\|_{L^2(\frac{x}{2},\infty)}.$$

We applied the Minkowski inequality in the second step, while in the last inequality we took into account the fact that on the region over which we perform the integration we have $y - z \leq \frac{x}{2}$. This completes the proof of Lemma 1.4.2. □

Now we are in a position to prove the result given in Theorem 1.4.1. The proof of the asymptotic behavior for the function $\phi_+(x, \lambda)$ follows the general pattern of treatment of the ODE systems with L^1-perturbations (see, e.g., [7]). We can write the system (15) for z in the following form:

$$z'(x,\lambda) = \left(\begin{pmatrix} -i\sqrt{\lambda}x + \frac{i}{2\sqrt{\lambda}}V(x) & 0 \\ 0 & i\sqrt{\lambda}x - \frac{i}{2\sqrt{\lambda}}V(x) \end{pmatrix} + R(x,\lambda) \right) z(x,\lambda), \quad (25)$$

where

$$\int\limits_x^\infty \|R(y,\lambda)\| \le \frac{1}{|2\sqrt{\lambda}|}\int\limits_x^\infty |V(y)||q_{ij}(y,\lambda)|\,dy \le \frac{1}{8|\lambda|(\Im\sqrt{\lambda})}$$

$$\left(\|V\|^2_{L^2(\frac{x}{2},\infty)} + + \exp(-\Im\sqrt{\lambda}x)\|V\|_{L^2(\frac{x}{2},\infty)}\|V\|_{L^2(0,\infty)}\right) = \frac{1}{|\lambda|(\Im\sqrt{\lambda})}O(\|V\|_{L^2(\frac{x}{2},\infty)}). \tag{26}$$

Let us denote the diagonal matrix in (25) (which is equal to $\mathcal{F}+\mathcal{D}$ in the old notation) by $\tilde{\mathcal{F}}(x)$.

The usual iteration method for finding the solutions may be applied to (25). In particular, one of the solutions, which we denote $z_+(x,\lambda)$, is represented as follows:

$$z_+(x,\lambda) = \exp\left(\int\limits_0^x \tilde{\mathcal{F}}(t)\,dt\right)\begin{pmatrix}0\\1\end{pmatrix} + \sum_{n=1}^\infty (-1)^n g_n(x,\lambda), \tag{27}$$

where $g_n(x,\lambda)$ is equal to

$$\int\limits_x^\infty e^{-\int\limits_x^{t_1}\tilde{\mathcal{F}}(s_1)ds_1} R(t_1,\lambda)\int\limits_{t_1}^\infty \ldots \int\limits_{t_{n-1}}^\infty e^{-\int\limits_{t_{n-1}}^{t_n}\tilde{\mathcal{F}}(s_n)ds_n} R(t_n,\lambda)e^{\int\limits_0^{t_n}\tilde{\mathcal{F}}(s_n)ds_n}\begin{pmatrix}0\\1\end{pmatrix} dt_1\ldots dt_n.$$

Note that

$$\int\limits_{t_1}^\infty e^{-\int\limits_{t_1}^{t_2}\tilde{\mathcal{F}}(s_2)ds_2} R(t_2,\lambda)e^{\int\limits_0^{t_2}\tilde{\mathcal{F}}(s_2)ds_2}\begin{pmatrix}0\\1\end{pmatrix} dt_2 = e^{i\sqrt{\lambda}t_1 - \frac{i}{2\sqrt{\lambda}}\int\limits_0^{t_1}V(s)ds} f(t_1,\lambda), \tag{28}$$

where

$$|f(t_1,\lambda)| \le \int\limits_{t_1}^\infty \|R(t_2,\lambda)\|\,dt_2 \tag{29}$$

This representation follows from the fact that the vector $\begin{pmatrix}0\\1\end{pmatrix}$ is multiplied by a fast decaying element $\exp(i\sqrt{\lambda}t_2 - \frac{i}{2\sqrt{\lambda}}\int_0^{t_2}V(s_2)\,ds_2)$, and if we take $\exp(i\sqrt{\lambda}t_1 - \frac{i}{2\sqrt{\lambda}}\int_0^{t_1}V(s_2)\,ds_2)$ out of the integral in (28), all the elements of the matrix under the integral sign are still majorized by the elements of the matrix R. By induction, we can show that the n^{th} summand in the series for $z_+(x,\lambda)$ may be represented as

$$g_n(x,\lambda) = \exp\left(i\sqrt{\lambda}x - \frac{i}{2\sqrt{\lambda}}\int_0^x V(s)\,ds\right) f_n(x,\lambda), \tag{30}$$

with

$$|f_n(x,\lambda)| \le \frac{\left(\int_x^\infty \|R(t,\lambda)\|\,dt\right)^n}{n!}. \tag{31}$$

This gives us a solution $z_+(x,\lambda)$ with the following asymptotics for large x:

$$z_+(x,\lambda) = \exp\left(i\sqrt{\lambda}x - \frac{i}{2\sqrt{\lambda}}\int_0^x V(s)\,ds\right)\left(\begin{pmatrix} 0 \\ 1 \end{pmatrix} + O\left(\int_x^\infty \|R(t,\lambda)\|\,dt\right)\right). \tag{32}$$

Transforming back to the initial system (12), we get a solution $\phi_+(x,\lambda)$ with the required asymptotics.

Now we turn to proving analyticity of the $\phi_+(x,\lambda)$ and $\phi'_+(x,\lambda)$. Consider an open set $U_\epsilon = \{\lambda | \Im\lambda > \epsilon\}$ in C^+. First, we will prove that $\phi_+(x,\lambda)$ and $\phi'_+(x,\lambda)$ are analytic in U_ϵ for every $\epsilon > 0$ if we choose x to be large enough (may be dependent on ϵ). To prove this fact for certain x, it suffices to show the analyticity of $z_+(x,\lambda)$ and $\mathcal{I} + \mathcal{Q}(x,\lambda)$ in U_ϵ. The analyticity of the matrix $\mathcal{I} + \mathcal{Q}(x,\lambda)$ is obvious from the expressions (19), (20) (where we let $C_{1,2} = 0$) for the elements of Q. To show the analyticity of $z_+(x,\lambda)$, we need the following standard lemma:

Lemma 1.4.3. *Let $f(x,\lambda)$ be a measurable function of (x,λ) on $S \times U$, where U is an open set in a complex plane and S is a metric space with measure $d\mu$. Suppose that $f(x,\lambda)$ is an analytic function in U for every fixed $x \in S$ and that $\int_S |f(x,\lambda)|\,d\mu < C$ for every λ. Then $\int_S f(x,\lambda)\,d\mu(x)$ is analytic in U.*

Proof. The proof of this lemma may be found, for example, in [29]. □

Now consider the expression for the n^{th} term $g_n(x,\lambda)$ in the series (19) for

$z_+(x,\lambda)$:

$$\int_x^\infty e^{-\int_x^{t_1}\tilde{\mathcal{F}}(s_1)ds_1} R(t_1,\lambda)\int_{t_1}^\infty \dots \int_{t_{n-1}}^\infty e^{-\int_{t_{n-1}}^{t_n}\tilde{\mathcal{F}}(s_n)ds_n} R(t_n,\lambda)e^{\int_0^{t_n}\tilde{\mathcal{F}}(s_n)ds_n}\begin{pmatrix}0\\1\end{pmatrix} dt_1 \dots dt_n. \tag{33}$$

All functions which stay in the expression above are analytic in λ in U_ϵ for x large enough. Indeed, this is obviously true for the terms involving $\tilde{\mathcal{F}}$, while the entries of R are the products of the elements of $\mathcal{Q}$, $(\mathcal{I}+\mathcal{Q})^{-1}$, $\mathcal{D}$, and $\mathcal{W}$. All these elements are easily seen to be analytic in U_ϵ for x large enough. For the elements of $(\mathcal{I}+\mathcal{Q})^{-1}$, it follows from the estimates (22), (23). Using (27), (28), (29), and Lemma 1.4.3, together with the uniform in U_ϵ estimate (26) on the $\int_x^\infty \|R(t,\lambda)\|\,dt$, we see that after every integration in (33) we get an analytic function. Hence, $g_n(x,\lambda)$ is analytic. The analyticity of $z_+(x,\lambda)$ follows, since the series (27) converges uniformly in U_ϵ by (30) and (31).

Therefore, we showed analyticity of $\phi'_+(x,\lambda)$ and $\phi_+(x,\lambda)$ in U_ϵ for every positive ϵ if x is large enough. Now let $\gamma_1(x,\lambda)$ and $\gamma_2(x,\lambda)$ be solutions of the equation $H_V u - \lambda u = 0$ which satisfy $\gamma_1(x,\lambda)=0$, $\gamma_1'(x,\lambda)=1$, $\gamma_2(x,\lambda)=1$, $\gamma_2'(x,\lambda)=0$. $\gamma_1(y,\lambda)$ and $\gamma_2(y,\lambda)$ are analytic for every fixed y by the standard estimate on the finite interval. We can expand ϕ_+ in a basis given by solutions $\gamma_{1,2}$:

$$\phi'_+(y,\lambda) \;=\; \phi'_+(x,\lambda)\gamma_1'(y,\lambda) \;+\; \phi_+(x,\lambda)\gamma_2'(y,\lambda) \tag{34}$$

and

$$\phi_+(y,\lambda) \;=\; \phi'_+(x,\lambda)\gamma_1(y,\lambda) \;+\; \phi_+(x,\lambda)\gamma_2(y,\lambda). \tag{35}$$

This shows analyticity of $\phi'_+(y,\lambda)$ and $\phi_+(y,\lambda)$ in U_ϵ for every $y<x$ as well, as far as we know that, for some x, $\phi'_+(x,\lambda)$ and $\phi_+(x,\lambda)$ are analytic there. This completes the proof of Theorem 1.4.1. □

In conclusion, we make one obvious but important remark. In the proof of Lemma 1.4.2 we assumed that $C_2(\lambda) = 0$ in the expression (20) for the function $q_{12}(x, \lambda)$. But certainly all the estimates in the proof of Lemma 1.4.2 can be carried out for fixed $\lambda \in C^+$ for an arbitrary choice of C_2. Indeed, C_2 is a coefficient in front of the exponentially decaying term and hence, by triangle inequality, we are just acquiring an exponentially decaying term on the right-hand side of the estimate. Therefore, all of the formulas (26)–(31) remain valid if we use in them the function q_{21} with some other choice of C_2. Since the function $\phi_+(x, \lambda)$ is a unique solution of the equation (9) with the asymptotics (10), every choice of q_{21} leads eventually to the same function $\phi_+(x, \lambda)$. This gives us certain freedom in representing this solution, which will be useful in the next section.

1.5. Formula for the projection on the absolutely continuous spectrum

In this section we are going to use the functions $\varphi_+(x, \lambda)$ to derive an explicit formula for a projection on the absolutely continuous spectrum. This formula may serve as a basis for further study of the subtler properties of the spectrum. While giving more information about the structure of the absolutely continuous spectrum, the methods we apply here fail to fully recover the best power of decay which we were able to treat in the Section 1.1.

To begin with, we set up some notation and state an auxiliary lemma, which is a straightforward generalization of Lemma 1.1.4.

Let potential $V(x)$ satisfy the decay condition $|V(x)| \leq C(x+1)^{-\beta-\epsilon}$, where $\beta > \frac{1}{2}$ and ϵ is an arbitrary positive number. Then we denote by $S_\beta(V)$ the set of energies,

$$S_\beta(V) = \frac{1}{4}\left(\mathcal{M}^+(\Phi(x^{\beta-\frac{1}{2}}V(x)))\right)^2 \setminus \{0\}.$$

We remark that $S_\beta(V)$ is clearly a set of full measure.

We have the following extension of Lemma 1.1.4:

Lemma 1.5.1. *For every energy $\lambda_0 \in S_\beta(V)$, we have*

$$\left| \int_x^\infty V(t)\exp(-2i\sqrt{\lambda_0}t)\,dt \right| \le C(\lambda) x^{-\beta+\frac{1}{2}} \log x.$$

Proof. The proof essentially repeats the argument given in the proof of Lemma 1.1.3. We have

$$\left| \int_x^\infty \exp(i\sqrt{\lambda_0}t)V(t)\,dt \right| = \left| \int_x^\infty \exp(i\sqrt{\lambda_0}t)(V(t)t^{\beta-\frac{1}{2}})t^{-\beta+\frac{1}{2}}\,dt \right| \le$$

$$\le \left| x^{-\beta+\frac{1}{2}} \int_x^\infty \exp(i\sqrt{\lambda_0}t)(V(t)t^{\beta-\frac{1}{2}})\,dt \right| +$$

$$+ \left| \left(\frac{1}{2}-\beta\right) \int_x^\infty t^{-\beta-\frac{1}{2}} \int_0^t \exp(i\sqrt{\lambda_0}s)(V(s)s^{\beta-\frac{1}{2}})\,ds dt \right|,$$

which by Lemma 1.1.3 is estimated by

$$C_1(\lambda_0)\log x\, x^{-\beta+\frac{1}{2}} + \left(\beta-\frac{1}{2}\right)\left| \int_x^\infty C_1(\lambda_0)\log t\, t^{-\beta-\frac{1}{2}}\,dt \right| \le C(\lambda_0)x^{-\beta+\frac{1}{2}}\log x. \ \square$$

The first fact we need to establish in order to derive the formula for P_{ac} is the existence of the appropriate limits of $\phi_+(x,\lambda)$ and its derivative as λ tends to the real axis.

Theorem 1.5.2. *Suppose that potential V satisfies the decay condition $|V(x)| \le C(1+x)^{-\omega-\epsilon}$, where $\omega = \frac{\sqrt{17}-1}{4}$ and ϵ is positive. Then for every $\lambda_0 \in S_\omega(V)$ and so a.e., $\phi_+(x,\lambda)$ and $\phi'_+(x,\lambda)$ converge to $\phi_+(x,\lambda_0)$ and $\phi'_+(x,\lambda_0)$ correspondingly as $\lambda \in C^+$ tends to λ_0 in a non-tangential direction.*

Remark. We note that the existence of the function $\phi_+(x,\lambda_0)$ for $\lambda_0 \in S_\omega(V)$ is a direct consequence of Theorem 1.1.1 since $\omega > \frac{3}{4}$ and $S_\omega(V)$ belongs to the set S

from the statement of Theorem 1.1.1.

To proceed with the proof, we first note that the formula for the solution $z_+(x,\lambda)$, given by (27) for the complex energies, is also valid a.e. for real energies if the potential satisfies a power decay estimate with exponent $\beta > \frac{3}{4}$. Namely, using Lemma 1.5.1, one can see that the formula (27) for the solution $z_+(x,\lambda_0)$ with the asymptotics (32) remains valid for the energies $\lambda_0 \in S_\beta(V)$, if we take

$$q_{12}(x,\lambda_0) = \frac{i}{2\sqrt{\lambda_0}} \exp(-2i\sqrt{\lambda_0}x) \int_x^\infty \exp(2i\sqrt{\lambda_0}t)\, dt$$

and

$$q_{21}(x,\lambda_0) = \frac{i}{2\sqrt{\lambda_0}} \exp(2i\sqrt{\lambda_0}x) \int_x^\infty \exp(-2i\sqrt{\lambda_0}t)\, dt. \tag{36}$$

This corresponds to the choice $C_2(\lambda) = \int_0^\infty \exp(-2i\sqrt{\lambda_0}t)V(t)\, dt$. Then, by Lemma 1.5.1, q_{12} and q_{21} are well-defined and are dominated by $C(\lambda_0)(1+x)^{-\beta+\frac{1}{2}} \log x$. This allows us to perform the $\mathcal{I}+\mathcal{Q}$ transformation and then to iterate in the system (25), proving the existence of the function $z_+(x,\lambda_0)$ given by the formula (27) and satisfying (32). The solution $z_+(x,\lambda_0)$ of the system (15) for $\lambda_0 \in S_\beta(V)$ corresponds to a solution $\phi_+(x,\lambda_0)$ of the Schrödinger equation with asymptotics (1).

We will show that, under the conditions of Theorem 1.5.2, for every $\lambda_0 \in S_\beta(V)$, the solution $z_+(x,\lambda)$ converges to the solution $z_+(x,\lambda_0)$ as λ tends to λ_0 non-tangentially in C^+. The convergence of $\phi_+(x,\lambda)$ and $\phi'_+(x,\lambda)$ to $\phi_+(x,\lambda_0)$ and $\phi'_+(x,\lambda_0)$ will follow from an explicit connection of these functions with $z_+(x,\lambda)$ and $z_+(x,\lambda_0)$. Moreover, it suffices to prove the convergence of $\phi_+(x,\lambda)$ and $\phi'_+(x,\lambda)$ to $\phi_+(x,\lambda_0)$ and $\phi'_+(x,\lambda_0)$ for x large enough. The convergence for any x would then follow by the formulas (34), (35).

Fix $\lambda_0 \in S_\beta(V)$. We are going to use the freedom we had so far in choosing the function $q_{21}(x,\lambda)$. Let

$$q_{21}(x,\lambda) = \frac{i}{2\sqrt{\lambda}}\exp(2i\sqrt{\lambda}x)\left(\int\limits_0^\infty \exp(-2i\sqrt{\lambda_0}t)V(t)\,dt - \int\limits_0^x \exp(-2i\sqrt{\lambda}t)V(t)\,dt\right).$$

Thus, $q_{21}(x,\lambda)$ is designed so that when $\lambda \to \lambda_0$, we expect $q_{21}(x,\lambda)$ to converge to $q_{21}(x,\lambda_0)$ given by (29). We remark that $\int_0^\infty \exp(-2i\sqrt{\lambda_0})V(t)\,dt$ is well-defined by Lemma 1.5.1.

Fix an angle $W(\lambda_0)$ with vertex at λ_0, symmetric with respect to the line $\Re\lambda = \lambda_0$, with the radial measure less than π. We are going to prove that $z_+(x,\lambda)$ given by (27) converges to $z_+(x,\lambda_0)$ (given by the same formula (27) with $\lambda = \lambda_0$) as λ tends to λ_0 inside $W(\lambda_0)$. In order to prove the convergence of $z_+(x,\lambda)$, it suffices to show that:

(A) $R(x,\lambda) \to R(x,\lambda_0)$ for every x as $\lambda \to \lambda_0$ inside $W(\lambda_0)$;

(B) $\|R(x,\lambda)\| \le T(x)$ with $\int_x^\infty |T(y)|\,dy < \infty$ for every λ from $W_\delta(\lambda_0) = W(\lambda_0) \cap N_\delta$, where N_δ is a small enough circular neigborhood of λ_0.

If (A) and (B) hold, we would use the Lebesgue dominated convergence theorem to conclude that the n^{th} term in the series (27) for $z_+(x,\lambda)$ converges to the n^{th} term of the series for $z_+(x,\lambda_0)$. Also, since the series (27) is easily seen to be uniformly convergent in $W_\delta(\lambda_0)$ by condition (B) and (30), (31), it would follow that $z_+(x,\lambda)$ converges to $z_+(x,\lambda_0)$ as λ tends to λ_0 in $W_\delta(\lambda_0)$.

Therefore, we reduced the proof of Theorem 1.5.2 to the proof of (A) and (B). Furthermore, by (15) and (16), we have that

$$R(x,\lambda) = (\mathcal{I} + \mathcal{Q})^{-1}(\mathcal{D}\mathcal{Q} - \mathcal{Q}\mathcal{D} + \mathcal{W}\mathcal{Q}).$$

Therefore, conditions (A), (B) are equivalent to the following statments about the functions q_{ij}:

($\tilde{\text{A}}$) $q_{ij}(x,\lambda)$ converges to $q_{ij}(x,\lambda_0)$ for every x as λ tends to λ_0 in $W(\lambda_0)$;

($\tilde{\text{B}}$) $|V(x)q_{ij}(x,\lambda)| \leq T(\lambda_0,x)$ with $\int_0^\infty T(\lambda_0,x)\,dx < \infty$ for every $\lambda \in W_\delta(\lambda_0)$. Indeed, all other functions appearing in $R(x,\lambda)$ have explicit dependence on λ and obvious limits as λ goes to the real axis. Condition ($\tilde{\text{B}}$) turns out to be harder to satisfy. The main problem why we cannot recover the weaker power condition $\beta > \frac{3}{4}$ in Theorem 1.5.2 is in establishing ($\tilde{\text{B}}$) for the function q_{21}. Otherwise, all estimates would work for $\beta > \frac{3}{4}$. The validity of ($\tilde{\text{A}}$) and ($\tilde{\text{B}}$) under the assumptions of Theorem 1.5.2 is verified in Lemmas 1.5.3 and 1.5.5. We first treat the simpler case of q_{12}.

Lemma 1.5.3 *Suppose that $V(x)$ satisfies $|V(x)| \leq C(1+x)^{-\beta-\epsilon}$, $\beta > \frac{1}{2}$. Then for every $\lambda_0 \in S_\beta(V)$, we have that $q_{12}(x,\lambda) \to q_{12}(x,\lambda_0)$ for every x as λ tends to λ_0 in W_{λ_0} and*

$$|q_{12}(x,\lambda)| \leq C(\lambda_0, W)x^{-\beta+\frac{1}{2}}\log x$$

for all $\lambda \in W_\delta$.

Proof. Consider

$$\int_x^\infty \exp(2i\sqrt{\lambda}(t-x))V(t)\,dt - \int_x^\infty \exp(2i\sqrt{\lambda_0}(t-x))V(t)\,dt =$$

$$= \int_x^\infty \left(\exp\left(2i(\sqrt{\lambda_0}-\sqrt{\lambda})(t-x)\right) - 1\right)\exp(2i\sqrt{\lambda_0}(t-x))V(t)\,dt =$$

$$= -\int_s^\infty \exp(2i\sqrt{\lambda_0}(t-x))V(t)\,dt\left(\exp\left(2i(\sqrt{\lambda_0}-\sqrt{\lambda})(t-x)\right) - 1\right)\Big|_x^\infty +$$

$$+\int_x^\infty 2i(\sqrt{\lambda_0}-\sqrt{\lambda})\exp\left(2i(\sqrt{\lambda_0}-\sqrt{\lambda})(t-x)\right)\left(\int_t^\infty \exp(2i\sqrt{\lambda_0}(s-x))V(s)\,ds\right)dt.$$

In the above calculation we used the fact that $\lambda_0 \in S_\beta(V)$ and Lemma 1.5.1, which implies that all integrals are well-defined. Now note that the first term in the sum is identically zero, while the second is estimated by

$$2\left|\sqrt{\lambda}-\sqrt{\lambda_0}\right|\left|\int\limits_x^\infty \exp(-2\Im\sqrt{\lambda}(t-x))\,dt\right|\sup_{t\geq x}\left|\int\limits_t^\infty \exp(2i\sqrt{\lambda_0 s})V(s)\,ds\right| \leq$$

$$\leq \frac{2|\sqrt{\lambda}-\sqrt{\lambda_0}|}{2\Im\sqrt{\lambda}}\sup_{t\geq x}\left|\int\limits_t^\infty \exp(2i\sqrt{\lambda_0 s})V(s)\,ds\right|.$$

Lemma 1.5.1 implies that

$$\sup_{t\geq x}\left|\int\limits_t^\infty \exp(2i\sqrt{\lambda_0 s})V(s)ds\right| \leq C(\lambda_0)x^{-\beta+\frac{1}{2}}.$$

We also note that $\frac{|\sqrt{\lambda}-\sqrt{\lambda_0}|}{\Im\sqrt{\lambda}}$ is uniformly bounded in $W(\lambda_0)$. Hence we can let

$$C(\lambda_0, W) = \sup_{\lambda\in W}\frac{|\sqrt{\lambda}-\sqrt{\lambda_0}|}{\Im\sqrt{\lambda}} + C(\lambda_0),$$

and the above estimates give us the second statement of the lemma. To prove the first statement, we note that

$$\left|\int\limits_x^\infty \exp\left(2i\sqrt{\lambda}(t-x)\right)V(t)\,dt - \int\limits_x^\infty \exp\left(2i\sqrt{\lambda_0}(t-x)\right)V(t)\,dt\right| \leq$$

$$\leq \left|\int\limits_x^A \left(\exp(2i\sqrt{\lambda}(t-x)) - \exp(2i\sqrt{\lambda_0}(t-x))\right)V(t)\,dt\right| + 2C(\lambda_0, W)\,A^{-\beta+\frac{1}{2}}.$$

For every $\epsilon > 0$, we can pick A so that the second term on the right-hand side is less than $\frac{\epsilon}{2}$. Next, we can take λ close enough to λ_0, so that the first summand is also smaller than $\frac{\epsilon}{2}$. There is no problem since we now have integration over the final interval (x, A) and V is locally integrable. □

We next show that similar statements hold for $q_{21}(x,\lambda)$. However, since the integral in the expression for q_{21} is taken over the finite interval, which grows as $x \to \infty$, the convergence question becomes simple, while the uniform boundedness is not trivial. First, we need an auxiliary statement.

Lemma 1.5.4. *Suppose that* $1 > \alpha > \frac{1}{2}$. *Let* $A(\alpha)$ *be a small enough positive number such that* $\alpha A^{-\alpha+1} \leq \frac{1}{2}(1-\alpha)$. *Then*

$$a \int_0^{x^\alpha} \exp(-a(x-t))dt = \exp(-ax)(\exp(ax^\alpha - 1)) \leq C x^{\alpha-1} \tag{37}$$

holds with some constant C *uniformly in* $0 < a \leq A$.

Proof. Consider first the case $x \leq \frac{1}{a}$. Then $\exp(ax^\alpha) - 1 \leq C_1 ax^\alpha$ with the constant C_1 uniform in a, since $ax^\alpha \leq A^{1-\alpha}$. Therefore,

$$\exp(-ax)(\exp(ax^\alpha) - 1) \leq C_1 ax^\alpha \leq C_1 x^{\alpha-1}$$

for all a if $x \leq \frac{1}{a}$.

Now note that for every fixed a, (37) definitely holds with some constant C. Indeed, when $x = 0$, the left-hand side of (37) is zero and when x tends to infinity, the left-hand side is definitely majorized by the right-hand side. To show that (37) holds with some constant C uniformly in a means to show that there exists C such that the function $h(x,a) = \exp(-ax)(\exp(ax^\alpha - 1)) - Cx^{\alpha-1}$ is negative for all x, a. Since for every fixed a, (37) holds when x is small or large with every $C > 0$, and $h(x,a)$ is continuously differentiable in x in $(0,\infty)$ for every a, it suffices to show that with a proper choice of C, for every a, $h(x)$ is negative at all points x where $h'(x) = 0$. By the above, we can also assume $x > \frac{1}{a}$. Now, $h'(x) = 0$ means that

$$a \exp(-ax)(1 - \exp(ax^\alpha)(1 - \alpha x^{\alpha-1})) - C(\alpha - 1)x^{\alpha-2} = 0.$$

Hence, since $|1 - \alpha x^{\alpha-1}| \neq 0$ because $\alpha x^{\alpha-1} \leq \alpha A^{-\alpha+1} < \frac{1}{2}$, we have

$$\exp(ax^\alpha) = \frac{-C(\alpha-1)x^{\alpha-2} + a\exp(-ax)}{a\exp(-ax)(1-\alpha x^{\alpha-1})}.$$

Substituting this into the expression for $h(x)$, we get

$$h(x) = \frac{-C(\alpha-1)x^{\alpha-2} + a\alpha x^{\alpha-1}\exp(-ax)}{a(1-\alpha x^{\alpha-1})} - Cx^{\alpha-1}.$$

But

$$\left|\frac{-C(\alpha-1)x^{\alpha-2}}{a(1-\alpha x^{\alpha-1})}\right| \leq \frac{2(1-\alpha)}{\alpha+1}Cx^{\alpha-1},$$

since $x > \frac{1}{a}$ and

$$|1-\alpha x^{\alpha-1}| > 1-\alpha A^{1-\alpha} \geq 1-\frac{1}{2}(1-\alpha) = \frac{1}{2}(1+\alpha).$$

If $\frac{1}{2} < \alpha < 1$, we have $\frac{2(1-\alpha)}{\alpha+1} \leq \frac{2}{3}$. Also,

$$\left|\frac{\alpha x^{\alpha-1}\exp(-ax)}{1-\alpha x^{\alpha-1}}\right| \leq \frac{2\alpha}{1+\alpha}x^{\alpha-1} \leq \frac{C}{3}x^{\alpha-1},$$

if we choose C large enough. Since x was an arbitrary point of extremum, the lemma is proven. □

We are ready to make our final step in the proof of Theorem 1.5.2, an estimate on q_{21}:

Lemma 1.5.5. *Suppose that potential $V(x)$ satisfies $|V(x)| \leq C(1+x)^{-\beta-\epsilon}$. Consider $\lambda_0 \in S_\beta(V)$. We can pick δ small enough so that:*

1. *$q_{21}(x,\lambda) \to q_{21}(x,\lambda_0)$ for every x as $\lambda \to \lambda_0$ in $W(\lambda_0)$ and*
2. *$|q_{21}(x,\lambda)| \leq C(\lambda_0)x^{-\gamma}\log x$ for every $\lambda \in W_\delta(\lambda_0)$, where $\gamma = \frac{\beta-\frac{1}{2}}{\beta+\frac{1}{2}}$.*

Proof. We remind that up to an irrelevant coefficient, the function $q_{21}(x,\lambda)$ is given by

$$q_{21}(x,\lambda) = \exp(2i\sqrt{\lambda}x)\left(\int_0^\infty \exp(2i\sqrt{\lambda_0}t)V(t)\,dt - \int_0^x \exp(2i\sqrt{\lambda}t)V(t)\,dt\right).$$

Therefore, $q_{21}(x,\lambda)$ converges to $q_{21}(x,\lambda_0)$ as λ tends to λ_0 for every fixed value of x simply by the Lebesgue dominated convergence theorem, since $\exp(2i\sqrt{\lambda}(x-t))$ converges to $\exp(2i\sqrt{\lambda_0}(x-t))$.

To prove the second statement of the lemma, we note that

$$|q_{21}(x,\lambda)| \leq \left|e^{2i\sqrt{\lambda}x}\int_x^\infty e^{-2i\sqrt{\lambda_0}t}V(t)\,dt\right| + \left|e^{2i\sqrt{\lambda}x}\int_0^x e^{-2i\sqrt{\lambda_0}t}V(t)(e^{-2i(\sqrt{\lambda}-\sqrt{\lambda_0})t}-1)\,dt\right|.$$

The first summand on the right-hand side is well behaved because of Lemma 1.5.1 and since $\gamma < \beta - \frac{1}{2}$. Next, we integrate by parts to estimate the second summand,

$$\exp(2i\sqrt{\lambda}x) \int_0^x \exp(-2i\sqrt{\lambda_0}t) \left(\exp(-2i(\sqrt{\lambda} - \sqrt{\lambda_0})t) - 1\right) V(t)\, dt =$$

$$\exp(2i\sqrt{\lambda}x) \left(\left(- \int_x^\infty \exp(-2i\sqrt{\lambda_0}t)V(t)\, dt \right) \left(1 - \exp(-2i(\sqrt{\lambda} - \sqrt{\lambda_0})t\right) |_0^x + \right.$$

$$\left. + \int_0^x 2(\sqrt{\lambda_0} - \sqrt{\lambda}) \exp\left(-2i(\sqrt{\lambda} - \sqrt{\lambda_0})t\right) \int_t^\infty \exp(-2i\sqrt{\lambda_0}s)V(s)\, dsdt \right).$$

The first summand is again well-behaved, while the second we split into two parts which we estimate separately:

$$\left| 2i(\sqrt{\lambda} - \sqrt{\lambda_0})e^{2i\sqrt{\lambda}x} \int_0^x e^{-2i(\sqrt{\lambda}-\sqrt{\lambda_0})t} \int_t^\infty e^{-2i\sqrt{\lambda_0}s} V(s)\, dsdt \right| \leq$$

$$\leq \left| 2(\sqrt{\lambda} - \sqrt{\lambda_0})e^{2i\sqrt{\lambda}x} \int_{x^\alpha}^x e^{-2i(\sqrt{\lambda}-\sqrt{\lambda_0})t} \int_t^\infty e^{-2i\sqrt{\lambda_0}s} V(s)\, dsdt \right| +$$

$$+ \left| 2(\sqrt{\lambda} - \sqrt{\lambda_0})e^{2i\sqrt{\lambda}x} \int_0^{x^\alpha} e^{-2i(\sqrt{\lambda}-\sqrt{\lambda_0})t} \int_t^\infty e^{-2i\sqrt{\lambda_0}s} V(s)\, dsdt \right|.$$

For the first part, we have an estimate

$$\left| 2i(\sqrt{\lambda} - \sqrt{\lambda_0}) \left(\int_{x^\alpha}^x \exp\left(-2\Im\sqrt{\lambda}(x-t)\right)\, dt \right) \right| \sup_{t \geq x^\alpha} \left| \int_t^\infty \exp(-2i\sqrt{\lambda_0}s)V(s)\, ds \right| \leq$$

$$\leq \frac{|\sqrt{\lambda} - \sqrt{\lambda_0}|}{\Im\sqrt{\lambda}} C(\lambda_0) x^{-\alpha(\beta - \frac{1}{2})} \log x \leq C(\lambda_0, W) x^{-\alpha(\beta-\frac{1}{2})} \log x.$$

For the second part, we have an estimate

$$\left| 2i(\sqrt{\lambda} - \sqrt{\lambda_0}) \int_0^{x^\alpha} \exp(-2\Im\sqrt{\lambda}(x-t))\, dt \right| \sup_t \left| \int_t^\infty \exp(-2i\sqrt{\lambda_0}s)V(s)\, ds \right| \leq$$

$$\leq 2C_1(\lambda_0) \left| \sqrt{\lambda_0} - \sqrt{\lambda} \right| \frac{1}{2\Im\sqrt{\lambda}} \exp(-2\Im\sqrt{\lambda}x) \left(\exp(2\Im\sqrt{\lambda}x^\alpha) - 1\right) \leq C(\lambda_0) x^{-1+\alpha}$$

by Lemma 1.5.4. Indeed, $\frac{|(\sqrt{\lambda_0} - \sqrt{\lambda})|}{\Im\sqrt{\lambda}}$ is uniformly bounded in $W(\lambda_0)$ and we just have to apply Lemma 1.5.4 with $a = 2\Im\sqrt{\lambda}$. If we want to optimize the power decay estimate on q_{21}, we choose α so that $1-\alpha = \alpha(\beta - \frac{1}{2})$ and hence $\alpha = \frac{1}{\beta+\frac{1}{2}}$. This gives the result we claimed in the statement of the lemma. □

To complete the proof of Theorem 1.5.2, we notice that we need to ensure that our uniform in λ decay estimate on q_{21} allows us to estimate the L^1-norm of $R(x,\lambda)$. This is equivalent to estimating the L^1-norm of $q_{21}(x,\lambda)V(x)$. To successfully apply Lemma 1.5.4, we need to have $\beta + \frac{\beta-\frac{1}{2}}{\beta+\frac{1}{2}} > 1$. This leads to the inequality $\beta^2 + \frac{\beta}{2} - 1 > 0$, which together with $\beta > \frac{1}{2}$ implies that we need to pick $\beta > \frac{\sqrt{17}-1}{4}$. This is exactly what we required in the statement of Theorem 1.5.2. □

Our final goal in this section is to derive a rather explicit representation for the projection on the absolutely continuous part of the spectrum of the operator H_V. We will prove the following

Theorem 1.5.6. *Suppose that potential $V(x)$ satisfies the decay condition $|V(x)| \leq C(1+x)^{-\omega-\epsilon}$ (where $\omega = \frac{\sqrt{17}-1}{4}$). Let r, s be measurable bounded functions with compact support in R^+. Then we have the following formula for the absolutely continuous part of the spectral projection P_{ac} on the segment $I = (a,b)$:*

$$\langle P_{ac}(I)r, s\rangle = \frac{1}{\pi}\int_{\sqrt{a}}^{\sqrt{b}} dk \int_0^\infty dx \int_0^\infty dy \psi(x,k^2)\psi(y,k^2)r(x)s(y), \tag{38}$$

where $\langle\ ,\ \rangle$ denotes the usual scalar product in $L_2(0,\infty)$ and $\psi(x,k^2)$ are the "scattered waves"

$$\psi(x,\lambda) = \phi_+(x,\lambda) - \frac{\phi_+(0,\lambda)}{\overline{\phi_+(0,\lambda)}}\overline{\phi_+(x,\lambda)}.$$

Remarks. 1. From the general theory of the spectral representation of self-adjoint differential operators of the second order (see, e.g., [7]), it follows almost immediatiely that (38) holds with some measure $m(k)dk$ in the place of dk. The non-trivial part in (38) is that we are able to compute $m(k)$ explicitly.

2. The functions we have in the formula (38) are defined only almost everywhere. The exact meaning of (38) is that the integral is taken over $(\sqrt{a}, \sqrt{b}) \cap S$, where S is a measurable set of full measure for which the functions $\psi(x, k^2)$ are well-defined. For the rest of the section, we will neglect this harmless (for the absolutely continuous spectrum) ambiguity and write integrals over the whole segment I.

Proof. We begin the proof of the theorem by recalling briefly the well-known properties of the Weyl m-function and, in particular, its connection with the spectral properties of H_V. Recall that we consider a Schrödinger operator H_V defined by (9) and Dirichlet boundary condition at zero. As before, let $\theta(x, \lambda)$, $\chi(x, \lambda)$ be solutions of the equation (9) satisfying at zero $\theta'(0) = 1$, $\theta(0) = 0$ and $\chi'(0) = 0$, $\chi(0) = 1$. It is a standard simple fact [6], [36] that the functions $\theta(x, \lambda)$, $\chi(x, \lambda)$ possess the following properties:

1) θ, χ are continuous jointly in x, λ.

2) θ, χ are analytic in λ in the whole complex plane for every fixed x.

3) θ, χ are real when λ is real.

The third property is an obvious consequence of the fact that V is real-valued. The first two properties are easy to obtain considering the integral equations which θ, χ satisfy.

The Weyl m-function, corresponding to the Dirichlet boundary condition at zero, (see, e.g., [36], [33]) may be defined for every $\lambda \in C^+$ by the condition that

$$\chi(x, \lambda) + m(\lambda)\theta(x, \lambda) = h\phi_+(x, \lambda) = f(x, \lambda) \in L^2(0, \infty). \tag{39}$$

Under very general conditions on the potential (known as "limit-point case"), the relation (39) defines $m(\lambda)$ uniquely as an analytic function in C^+ function with a positive imaginary part. In particular, bounded potentials lead to the limit-point case. We refer to [36], [27] for more information and further references. Since $m(\lambda)$ is an analytic function in C^+ with a positive imaginary part, it is well-known (see, e.g., [2]) that there exists a positive measure $d\rho$ such that the integral $\int_R \frac{d\rho(\lambda)}{\lambda^2+1}$ is finite and

$$m(z) = Cz + B + \int_R \frac{(\lambda z + 1) d\rho(\lambda)}{(\lambda - z)(\lambda^2 + 1)}. \tag{40}$$

The measure $\rho(\lambda)$, appearing here, is called the spectral measure of the corresponding Schrödinger operator. The reason for this will be clear in a moment. Now we are going to derive a formula for an action of the projection on the absolutely continuous part of the spectrum in the usual way, integrating the resolvent over the contour in a complex plane encompassing part of the real axis in which we are interested. The following lemma is well-known. We provide a sketch of the proof for completeness.

Lemma 1.5.7. *Let $F(z)$ be an entire function in C and let $m(z)$ be given by (40). Fix a segment (a, b) in R and let B_δ be a rectangular contour consisting of two horizontal segments $(a \pm i\delta, b \pm i\delta)$ and two vertical segments connecting the points $a \pm i\delta$ and $b \pm i\delta$. Then*

$$\lim_{\delta \to 0} \int_{B_\delta} m(z) F(z)\, dz = \frac{1}{2}(F(a)\rho(a) + F(b)\rho(b)) + \int_a^b d\rho(\lambda) F(\lambda). \tag{41}$$

Proof.

$$\lim_{\delta \to 0} \int_{B_\delta} m(z) F(z)\, dz = \lim_{\delta \to 0} \int_{B_\delta} dz F(z) \int_R \frac{d\rho(\lambda)(\lambda z + 1)}{(\lambda - z)(\lambda^2 + 1)} =$$

$$= \lim_{\delta \to 0} \int_R \frac{d\rho\lambda}{\lambda^2 + 1} \int_{B_\delta} dz F(z) \frac{\lambda z + 1}{\lambda - z} = \int_a^b F(\lambda)\, d\rho(\lambda) + \frac{1}{2}(\rho(a)F(a) + \rho(b)F(b)),$$

by the Cauchy formula and a simple calculation for the points a, b lying on the contour. □

Note that we actually did not assume in the proof that $d\rho(\lambda)$ was positive (which is true in (40)).

Basically by definition, we have the following formula for the action of the spectral projection on the interval (a, b), $P((a, b))$:

$$\langle P((a,b))r, s\rangle = \int_a^b d\langle E_\lambda r, s\rangle,$$

where E_λ is a spectral family of orthonormal projections corresponding to an operator. Denote by $R(\lambda)$ the resolvent of this operator at point λ. We also have

$$\langle R(z)r, s\rangle = \int_R \frac{d\langle E_\lambda r, s\rangle}{\lambda - z}.$$

Lemma 1.5.7 implies that

$$\langle P((a,b))r, s\rangle = \lim_{\delta \to 0} \int_{B_\delta} \langle R(\lambda)r, s\rangle \, d\lambda - A(r, s, a, b), \tag{42}$$

where $A(r, s, a, b)$ stays for the first term on the right-hand side of (41) coming from integration along the vertical segments of B_δ which may be distinct from zero only if the measure $d\langle E_\lambda r, s\rangle$ (and hence the spectral measure) gives non-zero weight to the points a or b.

It is well-known (and easy to check directly) that the following expression gives a kernel of the Green's function G for the operator H_V:

$$G(x_1, x_2, \lambda) = \theta(x_<, \lambda) f(x_>, \lambda)$$

(here $x_<$ is the smallest and $x_>$ is the largest of x_1, x_2. Consider a segment $I = (a, b) \subset R^+$ and a corresponding family of contours B_δ. Let $r(x)$, $s(x)$ be bounded functions of compact support in R^+. By $R(\lambda)$ we now mean the resolvent of the operator H_V at the point λ. We have

$$\frac{1}{2\pi i} \lim_{\delta \to 0} \int_{B_\delta} \langle R(z)r, s\rangle \, dz =$$

$$= \frac{1}{2\pi i} \lim_{\delta\to 0} \int\limits_{B_\delta} dz \int\limits_R dx\, r(x) \left(\theta(x,z) \int\limits_x^\infty f(y,z)\overline{s}(y)\, dy + f(x,z) \int\limits_0^x \theta(y,z)\overline{s}(y)\, dy \right).$$

Using the fact that $f(x,z) = \chi(x,z) + m(z)\theta(x,z)$, analyticity in z of the functions χ, θ and properties of r, s we obtain

$$\frac{1}{2\pi i} \lim_{\delta\to 0} \int\limits_{B_\delta} \langle R(z)r, s\rangle\, dz = \frac{1}{2\pi i} \lim_{\delta\to 0} \int\limits_{B_\delta} dz\, m(z) \int\limits_R dx\, \theta(x,z) r(x) \int\limits_R \theta(y,z)\overline{s}(y)\, dy.$$

The summand which does not contain the function $m(z)$ drops out after integration because of its analyticity. Similarly, the function $F(z)$ defined by

$$F(z) = \int\limits_R dx \theta(x,z) r(x) \int\limits_R \theta(y,z)\overline{s}(y)\, dy$$

is analytic by Lemma 1.4.3. Hence, by Lemma 1.5.7 and the representation (40), we get

$$\frac{1}{2\pi i} \lim_{\delta\to 0} \int\limits_{B_\delta} \langle R(\lambda)r, s\rangle\, d\lambda = \int\limits_a^b d\rho(\lambda) \int\limits_R \theta(x,\lambda) r(x) \int\limits_R \theta(y,\lambda)\overline{s}(y)\, dy dx + A(r,s,a,b),$$

where $A(a,b,r,s)$ is the term which comes from integration over the vertical segments of the contours, which is non-zero only if measure ρ gives positive weight to at least one of the points a, b. Comparing with (42), we get

$$\langle P((a,b))r, s\rangle = \int\limits_a^b d\rho(\lambda) \int\limits_R \theta(x,\lambda) r(x) \int\limits_R \theta(y,\lambda)\overline{s}(y)\, dy dx.$$

By the well-known properties of the Borel transforms, we have that $d\rho_{\mathrm{ac}}(\lambda) = \Im m(\lambda + i0)\, d\lambda$; see, for example, [33]. Hence, for the action of the absolutely continuous part of the spectral projection we have

$$\langle P_{\mathrm{ac}}((a,b))r, s\rangle = \int\limits_a^b \Im m(\lambda + i0)\, d\lambda \int\limits_R \theta(x,\lambda) r(x) \int\limits_R \theta(y,\lambda)\overline{s}(y)\, dy dx. \tag{43}$$

Now we express all functions in the formula (43) in terms of the generalized Jost functions for the real parameter, $\phi_+(x,\lambda)$. We claim that a.e. λ,

$$\theta(x,\lambda) = \frac{1}{2i\sqrt{\lambda}} \left(\overline{\phi_+(0,\lambda)}\, \phi_+(x,\lambda) - \phi_+(0,\lambda)\, \overline{\phi_+(x,\lambda)} \right). \tag{44}$$

Indeed, the function on the right-hand side clearly solves (9), it vanishes at zero, and its derivative is equal to 1 at zero since the Wronskian of the functions ϕ_+, $\overline{\phi_+}$ is equal to $2i\sqrt{\lambda}$ by looking at the asymptotics at infinity. Also, by the discussion in the beginning of the previous section which led to the formula (11), we have that

$$m(\lambda) = \frac{\phi'_+(0,\lambda)}{\phi_+(0,\lambda)}$$

for all $\lambda \in C^+$. On the other hand, Theorem 1.5.6 ensures that if V satisfies the power decay condition with the exponent $\beta > \omega = \frac{\sqrt{17}-1}{4}$, we have that $\phi_+(0,\lambda) \to \phi_+(0,\lambda_0)$ and $\phi'_+(0,\lambda) \to \phi'_+(0,\lambda_0)$ for every $\lambda_0 \in S_\omega(V)$, as λ tends to λ_0 in any non-tangential direction. Hence, $m(\lambda_0 + i0) = \frac{\phi'_+(0,\lambda_0)}{\phi_+(0,\lambda_0)}$ for every $\lambda_0 \in S_\omega(V)$ such that $\phi_+(0,\lambda_0)$, $\phi'_+(0,\lambda_0)$ are not simultaneously 0 or ∞. But these values are finite for every $\lambda_0 \in S_\omega(V)$ by the representation (1) (and local integrability of potential) and they are never simultaneously zero because in this case we would have $\phi_+(x,\lambda) \equiv 0$. Therefore, for every $\lambda_0 \in S_\omega(V)$ (and so a.e.) we have,

$$\Im m(\lambda_0 + i0) = \frac{1}{2i}\left|\frac{\overline{\phi_+(0,\lambda_0)}\,\phi'_+(0,\lambda_0) - \phi_+(0,\lambda_0)\,\overline{\phi'_+(0,\lambda_0)}}{|\phi_+(0,\lambda_0)|^2}\right| = \frac{\sqrt{\lambda_0}}{|\phi_+(0,\lambda_0)|^2}. \tag{45}$$

Let us denote by $\psi(x,\lambda_0)$ the "scattered wave"

$$\psi(x,\lambda) = \phi_+(x,\lambda) - \frac{\phi_+(0,\lambda)}{\overline{\phi_+(0,\lambda)}}\overline{\phi_+(x,\lambda)}, \tag{46}$$

defined for every $\lambda_0 \in S_\omega(V)$. Substituting (44), (45), and (46) into (43), we get

$$\langle P_{ac}(I)r,s\rangle = \frac{1}{\pi}\int_I d\lambda \frac{\sqrt{\lambda}}{|\phi_+(0,\lambda)|^2}\int dx \int dy\, \frac{|\phi_+(0,\lambda)|^2}{4\lambda}\psi(x,\lambda)\psi(y,\lambda)r(x)s(y) =$$

$$= \frac{1}{2\pi}\int_{\sqrt{I}} dk \int dx \int dy\, \psi(x,k^2)\psi(y,k^2)r(x)s(y),$$

exactly as claimed in Theorem 1.5.6. We remark that the formula we derived is the usual and most convenient representation used for P_{ac} in the scattering theory for one-dimensional Schrödinger equation. It is well-known for the case of $V \in L^1$ and was commonly used in problems of inverse scattering theory (see, e.g., [6], [25]).

1.6. Integrable potentials

In this section we prove

Theorem 1.6.1. *Suppose that potential V satisfies $|V(x)| \leq C_1 x^{-\frac{2}{3}-\epsilon}$ and is conditionally integrable with $|\int_x^\infty V(t)\,dt| \leq C_2 x^{-\delta}$ for some positive δ. Then the absolutely continuous component of the spectral measure of the operator H_V fills the whole R^+.*

We continue to use the notation $S_\beta(V)$, which we introduced in the previous section, for the set $\frac{1}{4}(\mathcal{M}^+(\Phi(x^{\beta-\frac{1}{2}}V(x)))^2 \setminus \{0\}$ corresponding to a potential decaying at the power rate $x^{-\beta-\epsilon}$.

Proof. To make the argument simpler, it is convenient to alter slightly the $\mathcal{I}+\mathcal{Q}$ transformation we applied to the system (4):

$$y'(x) = \frac{i}{2\sqrt{\lambda}} \begin{pmatrix} -V(x) & -V(x)\exp(-2i\sqrt{\lambda}x) \\ V(x)\exp(2i\sqrt{\lambda}x) & V(x) \end{pmatrix} y(x).$$

Now we let

$$y(x) = (1-|q|^2)^{-1/2}(\mathcal{I}+\mathcal{Q})z(x),$$

where $\mathcal{Q}$ and $q = q(x,\lambda)$ are the same as before. As we did earlier in Section 1.1, we will always assume that since we are interested in the asymptotics, we perform the $\mathcal{I}+\mathcal{Q}$ transformation "far enough" so that $|q| < 1$ for the x we consider. A calculation leads us to the following system for $z(x)$:

$$z' = \left(\begin{pmatrix} D & 0 \\ 0 & \overline{D} \end{pmatrix} + (1-|q|^2)^{-1} \begin{pmatrix} \Im(\overline{W}q) + 2|q|^2\overline{D} & 2\overline{q}\overline{D} - \overline{q}^2 W \\ 2qD - q^2\overline{W} & -\Im(\overline{W}q) + 2|q|^2 D \end{pmatrix} \right) z.$$

Here, as in the first section, D denotes $-\frac{i}{2\sqrt{\lambda}}V(x)$ and W stands for the function $-\frac{i}{2\sqrt{\lambda}}V(x)\exp(-2i\sqrt{\lambda}x)$. By Lemma 1.5.1, on the set $S_{\frac{2}{3}}(V)$ of the full measure we have the estimate $|q(x,\lambda)| \leq C(\lambda)x^{-\frac{1}{6}}\log x$. Hence, for all energies $\lambda \in S_{\frac{2}{3}}(V)$, the function $q^2(x,\lambda)V(x)$ is absolutely integrable and

$$\left|\int_x^\infty q^2(x,\lambda)V(x)\,dx\right| \leq C(\lambda)x^{-1-\epsilon}\log x.$$

The advantage of the new transformation is that the diagonal terms, which do not belong to L^1, are now purely imaginary (and hence lead to bounded solutions). We can rewrite the system in the following way:

$$z' = \left(\begin{pmatrix} D + \frac{1}{2}(\overline{W}q - W\overline{q}) & 2\overline{q}\overline{D} \\ 2qD & \overline{D} - \frac{1}{2}(\overline{W}q - W\overline{q}) \end{pmatrix} + \mathcal{R}(x)\right) z, \tag{47}$$

where all entries of the matrix $\mathcal{R}$ are from L^1. The only dangerous terms are the off-diagonal terms in the matrix. The main idea now is to iterate the $\mathcal{I} + \mathcal{Q}$ transformation, improving the rate of decay of the off-diagonal terms. To apply this procedure, we need first of all to ensure that $qD = \frac{1}{4\lambda}V(x)\int_x^\infty \exp(-2i\sqrt{\lambda}s)\,ds$ is an a.e. λ integrable function. For any $\lambda \in S_{\frac{2}{3}}(V)$, we have:

$$\int_0^x V(t)\int_t^\infty V(s)\exp(-2i\sqrt{\lambda}s)\,ds = -\left(\int_0^\infty V(t)\,dt\right)\left(\int_0^\infty V(s)\exp(-2i\sqrt{\lambda}s)\,ds\right) +$$

$$+\left(\int_x^\infty V(t)\,dt\right)\left(\int_x^\infty V(s)\exp(-2i\sqrt{\lambda}s)\,ds\right) - \int_0^x \left(V(t)\int_t^\infty V(s)\,ds\right)\exp(-2i\sqrt{\lambda}t)\,dt.$$

Therefore, it is easy to see that for the energies λ which lie in the intersection of $S_{\frac{2}{3}}(V(x))$ and $S_{\frac{2}{3}+\delta}(V(x)\int_x^\infty V(t)\,dt)$ we have (recall that by our assumption $|\int_x^\infty V(t)\,dt| \leq C_1 x^{-\delta}$)

$$\left|\int_x^\infty V(t)\int_t^\infty V(s)\exp(-2i\sqrt{\lambda}s)\,dsdt\right| \leq C(\lambda)x^{-\frac{1}{6}-\delta}\log x.$$

Applying the modified $\mathcal{I} + \mathcal{Q}_1$ transformation,

$$z = (\mathcal{I} + \mathcal{Q}_1)z_1 \text{ with } \mathcal{Q}_1 = \begin{pmatrix} 0 & q_1 \\ \overline{q}_1 & 0 \end{pmatrix},$$

where $q_1 = \frac{1}{4\lambda}\int_x^\infty V(t)\int_t^\infty V(s)\exp(2iks)\,dsdt$, we get (after a computation similar to the one leading from the system (4) to (18)),

$$z_1' = \left(\begin{pmatrix} D + \frac{1}{2}(\overline{W}q - W\overline{q}) & 2\overline{q}_1\overline{D} \\ 2q_1 D & \overline{D} - \frac{1}{2}(\overline{W}q - W\overline{q}) \end{pmatrix} + \mathcal{R}_1(x) \right) z_1. \tag{48}$$

Here $\mathcal{R}_1$ is a matrix with entries from L^1. The off-diagonal terms in the system (19) have a rate of decay $|q_1(x,\lambda)V(x)| \leq C(\lambda)x^{-\frac{5}{6}-\delta}\log x$ for a.e. λ.

To complete the proof, we need to apply the $\mathcal{I} + \mathcal{Q}$ transformation several times. The following lemma shows that under the assumptions of the theorem, we can do this, and it also determines the number of necessary iterations and the set of full measure for which we can derive the asymptotics of solutions.

Lemma 1.6.2. *Under the assumptions of Theorem* 1.6.1, *the function*

$$f_n(t_1,\lambda) = V(t_1)\int_{t_1}^\infty V(t_2)\int_{t_2}^\infty V(t_3)...\int_{t_n}^\infty V(t_{n+1})\exp(-2i\sqrt{\lambda}t_{n+1})\,dt_{n+1}$$

is integrable for every $\lambda \in \tilde{S}_n = \bigcap_{j=0}^n S_j$, *where*

$$S_j = S_{\frac{2}{3}+j\delta}\left(V(t_1)\left(\int_{t_1}^\infty V(t_2)\,dt_2\right)^j \right)$$

and moreover,

$$\left| \int_x^\infty f_n(t_1,\lambda)\,dt_1 \right| \leq Cx^{-\frac{1}{6}-n\delta}\log x.$$

Proof. The proof is by induction. We have already checked that for $n = 1$, the statement is true. For the sake of simplicity, we assume integrability of $f_n(t,\lambda)$ and give an a priori estimate for the tail integral. Of course, one can easily prove integrability by essentially the same (but a longer) computation. Now, integrating by parts, we find that

$$\int_x^\infty f_n(t_1,\lambda)\,dt_1 = \left(\int_x^\infty V(t_1)\,dt_1\right)\left(\int_x^\infty f_{n-1}(t_1,\lambda)\,dt_1\right) - \\ - \int_x^\infty f_{n-1}(t_1,\lambda)\left(\int_{t_1}^\infty V(t_2)\,dt_2\right)\,dt_1.$$

According to the induction hypothesis and our assumption on V, the first summand on the right-hand side is bounded by $C(\lambda)x^{-\frac{1}{6}-n\delta}\log x$ for every $\lambda \in \tilde{S}_{n-1}$. In the second summand we perform integration by parts, integrating $V(t_1)\int_{t_1}^\infty V(t_2)\,dt_2$. As a result we get

$$-\int_x^\infty f_{n-1}(t_1,\lambda)\int_{t_1}^\infty V(t_2)\,dt_1dt_2 = -\frac{1}{2}\left(\int_x^\infty V(t_1)\,dt_1\right)^2\int_x^\infty f_{n-2}(t_1,\lambda)\,dt_1 + \\ +\frac{1}{2}\int_x^\infty\left(\int_{t_1}^\infty V(t_2)\,dt_2\right)^2 f_{n-2}(t_1,\lambda)\,dt_1.$$

As before, the first term decays as $Cx^{-\frac{1}{6}-n\delta}\log x$ for every $\lambda \in \tilde{S}_{n-2}$. We continue to integrate by parts the second term, integrating $V(t_1)(\int_{t_1}^\infty V(t_2)dt_2)^2$; we again get a sum of two terms, the first of which (off-integral) is well-behaved while the second is again integrated by parts. We perform such a procedure n times and in the end, summarizing the result of the whole calculation, we find that

$$\int_x^\infty f_n(t_1,\lambda)\,dt_1 = g(x,\lambda) + \frac{(-1)^n}{n!}\int_x^\infty V(t_1)\left(\int_{t_1}^\infty V(t_2)dt_2\right)^n \exp(2i\sqrt{\lambda}t_1)\,dt_1,$$

where $g(x,\lambda)$ satisfies the decay condition

$$|g(x,\lambda)| \le C(\lambda)x^{-\frac{1}{6}-n\delta}\log x$$

for any $\lambda \in \tilde{S}_{n-1}$. The last term obviously satisfies the same estimate for every $\lambda \in S_n$. Hence, as claimed, $\int_x^\infty f_n(t,\lambda) \le C(\lambda)x^{-\frac{1}{6}-n\delta}\log x$ for every $\lambda \in \tilde{S}_n$. □

The proven lemma justifies the iteration of the $\mathcal{I}+\mathcal{Q}$ transformation, since on the n^{th} iteration, to obtain q_n we need to integrate $q_{n-1}D$ which, up to irrelevant energy dependent constants, is exactly f_n from the statement of the lemma. After the n^{th} iteration, we arrive at a system

$$z_n' = \left(\begin{pmatrix} D + \frac{1}{2}(\overline{W}q - W\overline{q}) & 2\overline{q}_n\overline{D} \\ 2q_nD & \overline{D} - \frac{1}{2}(\overline{W}q - W\overline{q}) \end{pmatrix} + \mathcal{R}_n(x) \right) z_1,$$

where the matrix $\mathcal{R}_n$ has absolutely integrable entries. Also by the second statement of the lemma, $|q_n(x,\lambda)V(x)| \le C(\lambda)x^{-\frac{5}{6}-n\delta}\log x$ for every $\lambda \in \tilde{S}_n$ and is therefore absolutely integrable as soon as $n > \frac{1}{6\delta}$. Therefore, for the energies from the set $\tilde{S}_m$ of full measure, $m = \left[\frac{1}{6\delta}\right] + 1$ iterations are enough to bring the system to the form where we can apply Levinson's theorem (or, as was noticed in Section 1.1, just use more straightforward integral equation techniques, bearing in mind that our unperturbed eigenfunctions are bounded). We also note that for every $\lambda \in \tilde{S}_m$, transforming back, we get the solutions $\phi_\lambda(x)$ and $\overline{\phi_\lambda(x)}$ with the asymptotics

$$\phi_\lambda(x) =$$
$$= \left(\exp\left(i\sqrt{\lambda}x - \frac{i}{2\sqrt{\lambda}}\int_0^x V(t)\,dt + \frac{i}{4\lambda}\int_0^x V(t)\int_t^\infty \sin(2\sqrt{\lambda}(t-s))V(s)\,dsdt \right)\right) \times$$
$$\times \left(1 + O(x^{-\rho}\log x)\right),$$

where $\rho = \min(\epsilon, m\delta - \frac{1}{6})$. The solutions $\phi_\lambda(x)$ and $\overline{\phi_\lambda(x)}$ are bounded and clearly linearly independent. This completes the proof of the theorem. □

2. Jacobi matrices

2.1. Main result for power decaying potentials

Now we prove the analogs of the results of Sections 1.1, 1.2, and 1.3 for Jacobi matrices. We consider the self-adjoint operator h_v on $l^2(Z_+)$ (with $Z_+ = \{1, 2, \dots\}$) given by

$$\begin{aligned} h_v u(n) &= u(n+1) + u(n-1) + v(n)u(n), \\ u(0) &= 0, \end{aligned} \tag{49}$$

where $v(n)$ is a real-valued, tending to zero at infinity sequence. All the theorems we have proven for Schrödinger operators have their analogs for Jacobi matrices. Of course, we need to replace the positive semi-axis by the segment $(-2, 2)$, the interior of the essential spectrum of the free discrete Schrödinger operator. Since we consider only decaying potentials, the essential spectrum is the same for h_v. The way the argument goes in the Jacobi matrices case is very close to the continuous analog. We will still use the notation $\Phi(f)(k)$, but now for the Fourier transform of the l^2- sequence $f(n)$:

$$\Phi(f)(k) = l^2 - \lim_{N \to \infty} \sum_{l=-N}^{N} \exp(ikl) f(l).$$

All other notations introduced in the preceding sections of the paper also remain valid. Let us begin by stating our main theorem for Jacobi matrices:

Theorem 2.1.1. *Suppose that $v(n)$ satisfies $|v(n)| < Cn^{-\frac{3}{4}-\epsilon}$ for some positive constants C, ϵ. Then the absolutely continuous component $\rho_{\rm ac}$ of the spectral measure ρ of the operator h_v fills the whole segment $(-2, 2)$ in the sense that $\rho_{\rm ac}(T) > 0$ for any measurable set $T \subset (-2, 2)$ with positive Lebesgue measure. The singular component of the spectral measure may be supported only on the complement of the* *set* $S = 2\cos(\frac{1}{2}\mathcal{M}^+(\Phi(n^{\frac{1}{4}}v(n))) \cap (-2,2)$ *(values of energy such that* $2\arccos$ *of half their value belongs to the set* $\mathcal{M}^+(\Phi(n^{\frac{1}{4}}V(n))$*; we fix the range of the* arccos *to be* $[0,\pi]$*. Moreover, for every* $\lambda \in S$ *there exist two linearly independent solutions* $\psi_\lambda(n)$, $\overline{\psi}_\lambda(n)$ *with the following asymptotics as* $n \to \infty$*:*

$$\psi_\lambda(n) = \exp\left(ikn + \frac{i}{2\sin k}\sum_{l=1}^{n} V_l\right)(1 + O(n^{-\frac{1}{4}}\log n)),$$

where $k = \arccos\frac{1}{2}\lambda$.

The strategy of the proof is the same as in the Schrödinger operator case. The analogs of the three lemmas we used heavily are as follows:

Lemma 2.1.2. *Assume that for every* λ *from the set* B*, all solutions of the equation* $h_v\phi - \lambda\phi$ *are bounded. Then on the set* B*, the spectral measure* ρ *of the operator* h_v *is purely absolutely continuous in the following sense:*

(i) $\rho_{\mathrm{ac}}(A) > 0$ for any $A \subseteq B$ with $|A| > 0$,

(ii)$\rho_{\mathrm{sing}}(B) = 0$.

Proof. This lemma follows from the subordinacy theory for infinite matrices developed by Khan and Pearson [19]. Recently, Jitomirskaya and Last proved more general results for Jacobi matrices [17]. The reference for a simple direct proof of the lemma is the paper of Simon [32]. □

Lemma 2.1.3. *Consider the function* $f(n) \in l^2(Z)$*. Then for every value* k_0 *which belongs to the set* $\mathcal{M}^+(\Phi(f))$*, we have*

$$\sum_{l=-N}^{N} f(x)\exp(ik_0 l) = O(\log N).$$

Proof. The Parseval equality in this case yields

$$\sum_{l=-N}^{N} f(l)\exp(ik_0 l) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{\sin(N+1/2)(k_0-k)}{\sin\frac{1}{2}(k_0-k)}\,\Phi(v)(k)\,dk$$
$$= \frac{1}{2\pi}\int_{0}^{\pi} \frac{\sin(N+1/2)(k)}{\sin\frac{1}{2}k}\,(\Phi(v)(k_0+k)+\Phi(v)(k_0-k))\,dk.$$

The final expression may be estimated exactly as in the proof of Lemma 1.1.3. □

Lemma 2.1.4. *Suppose that the sequence $v(n)$ satisfies $|v(n)| < Cn^{-\frac{3}{4}-\epsilon}$ with some positive constants C, ϵ. Then for every value of k from the set $\mathcal{M}^+(v(n)n^{\frac{1}{4}})$, the series $\sum_{l=n}^{\infty}\exp(-ikl)v(l)$ converges and, moreover,*

$$\sum_{l=n}^{\infty}\exp(-ikl)v(l) = O(n^{-\frac{1}{4}}\log n)$$

as $n \longrightarrow \infty$.

Proof. Summation by parts gives

$$\sum_{l=1}^{n}\exp(-ikl)v(l) = n^{-\frac{1}{4}}\sum_{l=1}^{n}\exp(-ikl)(v(l)l^{\frac{1}{4}})+$$
$$+\sum_{l=1}^{n-1}((l^{-\frac{1}{4}}-(l+1)^{-\frac{1}{4}})\sum_{j=1}^{l}\exp(-ikj)v(j)j^{\frac{1}{4}})$$

and applying Lemma 2.1.3, we obtain that for the values of $k \in \mathcal{M}^+(v(n)n^{\frac{1}{4}})$, the sum converges as $n \to \infty$. For the speed of convergence we have an estimate:

$$\left|\sum_{l=n}^{\infty}\exp(-ikl)f(l)\right| \le \left|n^{-\frac{1}{4}}\sum_{l=1}^{n}\exp(-ikl)(v(l)l^{\frac{1}{4}})\right| +$$
$$+\left|\sum_{l=1}^{n-1}((l^{-\frac{1}{4}}-(l+1)^{-\frac{1}{4}})\sum_{j=1}^{l}\exp(-ikj)v(j)j^{\frac{1}{4}})\right| \le$$

$$\leq C\left(n^{-\frac{1}{4}}\log n+\sum_{l=n}^{\infty}l^{-\frac{5}{4}}\log l\right)=O(n^{-\frac{1}{4}}\log n).\ \square$$

In the discrete case, the solution ψ of the formal equation $h_v\psi=\lambda\psi$ satisfies the recursion relation

$$\begin{pmatrix}\psi(n+1)\\ \psi(n)\end{pmatrix}=\begin{pmatrix}\lambda-v(n) & -1\\ 1 & 0\end{pmatrix}\begin{pmatrix}\psi(n)\\ \psi(n-1)\end{pmatrix}. \tag{50}$$

Let $k=\arccos\frac{1}{2}\lambda$ for $\lambda\in(-2,2)$. Applying to the system (50) a discrete analog of the variation of the parameters formula,

$$\begin{pmatrix}\psi(n+1)\\ \psi(n)\end{pmatrix}=\begin{pmatrix}\exp(ik(n+1)) & \exp(-ik(n+1))\\ \exp(ikn) & \exp(-ikn)\end{pmatrix}\begin{pmatrix}A_n\\ B_n\end{pmatrix}, \tag{51}$$

we get for new variables the finite difference system

$$\begin{pmatrix}A_{n+1}\\ B_{n+1}\end{pmatrix}=\left(\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}+\frac{iv(n)}{2\sin k}\begin{pmatrix}1 & \exp(-2ikn)\\ -\exp(2ikn) & -1\end{pmatrix}\right)\begin{pmatrix}A_n\\ B_n\end{pmatrix}. \tag{52}$$

Now we are in a position to apply the discrete analog of the Harris-Lutz technique to study the asymptotics of the solutions of the system (52). For every $\lambda\in(-2,2)$ such that $2k=2\arccos\frac{1}{2}\lambda$ belongs to $\mathcal{M}^+(\Phi(n^{\frac{1}{4}}v(n)))$, by Lemma 2.1.4 we can define

$$q(n,k)=-\frac{i}{2\sin k}\sum_{l=n}^{\infty}v(l)\exp(-2ikl),$$

and moreover, $q(n,k)$ behaves as $O(n^{-\frac{1}{4}}\log n)$ as n goes to infinity. The discrete $I+Q$ transformation will be

$$\begin{pmatrix}A_n\\ B_n\end{pmatrix}=\begin{pmatrix}1 & q(n,k)\\ \overline{q}(n,k) & 1\end{pmatrix}\begin{pmatrix}C_n\\ D_n\end{pmatrix}. \tag{53}$$

This transformation is non-singular as far as n is large enough and so we can reconstruct the asymptotics of our generalized eigenfunctions from the asymptotics of the variables C_n, D_n. Substitution of (53) into the system (52) yields

$$\begin{pmatrix} C_{n+1} \\ D_{n+1} \end{pmatrix} = \left(\begin{pmatrix} 1 + \frac{i}{2\sin k} v(n) & 0 \\ 0 & 1 - \frac{i}{2\sin k} v(n) \end{pmatrix} + R(n,k) \right) \begin{pmatrix} C_n \\ D_n \end{pmatrix}. \tag{54}$$

Direct computation shows that every element of the matrix $R(n,k)$ is a product of numbers, which are uniformly bounded in n for each $k \in \frac{1}{2}\mathcal{M}^{+}(\Phi(n^{\frac{1}{4}} v(n)))$, and $q(n,k)v(n)$ or $q(n+1,k)v(n)$. Hence by Lemma 2.1.4 and our assumptions on the potential v, we have $||R(n,k(\lambda))|| = O(n^{-1-\epsilon} \log n)$ at infinity for every λ from the set S in the statement of Theorem 2.1.1. We can further simplify (54) by applying the transformation

$$\begin{pmatrix} C_n \\ D_n \end{pmatrix} = \begin{pmatrix} \exp\left(\frac{i}{2\sin k} \sum_{l=1}^{n-1} v(l)\right) & 0 \\ 0 & \exp\left(-\frac{i}{2\sin k} \sum_{l=1}^{n-1} v(l)\right) \end{pmatrix} \begin{pmatrix} E_n \\ F_n \end{pmatrix}. \tag{55}$$

For E, F variables we have

$$\begin{pmatrix} E_{n+1} \\ F_{n+1} \end{pmatrix} = \left(I + \tilde{R}(n,k)\right) \begin{pmatrix} E_n \\ F_n \end{pmatrix}, \tag{56}$$

where I is an identity matrix and $\tilde{R}(n,k)$ satisfies the same norm decaying conditions as $R(n,k)$. One can also directly check by looking at the transformations we performed with the initial system (50) that the determinant of the matrix $\tilde{R}(n,k)$ is equal to $\frac{1-|q(n,k)|^2}{1-|q(n+1,k)|^2}$. A simple argument, carried out in Lemma 2.1.5 immediately below, shows that there exists a solution of (56) with the asymptotics at infinity

$$\begin{pmatrix} E_n \\ F_n \end{pmatrix} = \left(\begin{pmatrix} 1 \\ 0 \end{pmatrix} + O(n^{-\frac{1}{4}} \log n) \right).$$

The application of transformations (55), (53), (51) allows us to compute the asymptotics of the generalized eigenfunction ψ_λ and therefore concludes the proof. □

Lemma 2.1.5. *Suppose we have a recursive relation*

$$\begin{pmatrix} E_{n+1} \\ F_{n+1} \end{pmatrix} = \left(I + \tilde{R}(n)\right) \begin{pmatrix} E_n \\ F_n \end{pmatrix} \tag{57}$$

and the matrix $\tilde{R}(n)$ satisfies $\{||\tilde{R}(n)||\}_{n=1}^{\infty} \in l^1(Z_+)$. Moreover, suppose that determinants of the matrices $\prod_{l=1}^{n}(I+R(l))$ are bounded away from zero. Then there exists a solution H_n of (56) *such that*

$$\left\| H_n - \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\| = O\left(\sum_{l=n}^{\infty} ||\tilde{R}(n)||\right)$$

as $n \longrightarrow \infty$.

Proof. A standard argument shows that the product $\prod_{l=1}^{n}(I + R(l))$ converges as n goes to infinity under the conditions of the lemma to a matrix we will denote $R_\infty = \prod_{l=1}^{\infty}(I+R(l))$. The condition on the determinants of finite products ensures that R_∞ is invertible. Pick the vector $H_1 = R_\infty^{-1} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$. Then for n large enough so that $\sum_{l=n}^{\infty} ||R(l)|| < 1$, we have

$$\left\| H_n - \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\| \leq \left\| \left(I - \prod_{l=n}^{\infty}(I + R(l))\right) \left(\prod_{l=1}^{n}(I + R(l))H_1\right) \right\| \leq$$

$$\leq \frac{\sum_{l=n}^{\infty} ||R(l)||}{1 - \sum_{l=n}^{\infty} ||R(l)||} \exp \sum_{l=1}^{n} ||R(l)|| = O(\sum_{l=n}^{\infty} ||R(l)||). \quad □$$

Let us specifically stress one consequence of the calculations we performed and formulate

Proposition 2.1.6. *For discrete Schrödinger operators similar to the continuous ones, in order to prove that for a certain energy* $\lambda \in (-2,2)$, *all solutions of the equation* $h_v\psi - \lambda\psi = 0$ *are bounded, it is enough to show that the sequences* $q(n,k)v(n)$ *and* $q(n+1,k)v(n)$ *(where* $k = \arccos\frac{\lambda}{2}$*) belong to* $l^1(Z_+)$.

2.2. Other applications

Similarly to the Schrödinger operators case, Proposition 2.1.6 leads to the following theorem, which provides conditions under which the singular component of the spectral measure of the operator h_v on $(-2,2)$ is void. It is an analog of Theorem 1.1.6:

Theorem 2.1.7. *Suppose that* $|v(n)| < Cn^{-\frac{3}{4}-\epsilon}$ *and the Fourier transform* $\Phi(n^{\frac{1}{4}}v(n))(k)$ *belongs to* $L^p(0,2\pi)$ *with* $p > \frac{1}{\epsilon}$. *Then the spectrum of the operator* h_v *on the segment* $(-2,2)$ *is purely absolutely continuous. Moreover, for every value of* $\lambda \in (-2,2)$ *there exist two solutions* ψ_λ *and* $\overline{\psi}_\lambda$ *of the equation* $h_v\psi - \lambda\psi = 0$ *with the following asymptotics as* $n \to \infty$*:*

$$\psi_\lambda = \exp\left(ikn + \frac{i}{2\sin k}\sum_{l=1}^{n} v(l)\right)(1 + O(n^{-\epsilon+\frac{1}{p}})),$$

where $k = \arccos\frac{1}{2}\lambda$.

Proof. The proof is completely analogous to the proof of Theorem 1.1.5. One only needs to replace integration by parts with Abel's transformation (summation by parts). □

Finally, we discuss Jacobi matrices with non-power decaying potentials. The class of potentials we treat is again potentials which are "mostly" zero and become power decaying after "compression."

Namely, we say that a potential $\tilde{v}(n)$ belongs to the class $\mathcal{D}_\alpha$ if there exists a potential $v(n)$, verifying $v(n) \leq Cn^{-\alpha}$, and two sequences of positive integers $\{a_i\}_{i=1}^\infty$ and $\{b_i\}_{i=1}^\infty$ satisfying $b_{i-1} < a_i < b_i$ for all i, such that

$$\tilde{v}(n) = \begin{cases} 0, & a_l \leq n < b_l \\ v(n - \sum\limits_{j=1}^{l}(b_j - a_j)), & b_l \leq n < a_{l+1} \end{cases}.$$

We have the following theorem:

Theorem 2.1.8. *Let potential $v(n)$ belong to $\mathcal{D}_{-\frac{3}{4}-\epsilon}$. Then the absolutely continuous spectrum of the operator h_v fills the whole segment $[-2,2]$ in the sense that for any measurable set $T \subseteq [-2,2]$ with positive Lebesgue measure, we have $\rho_{\mathrm{ac}}(T) > 0$. Moreover, for a.e. $\lambda \in (-2,2)$ there exist two linearly independent solutions ψ_λ, $\overline{\psi}_\lambda$ with the following asymptotics as $n \to \infty$:*

$$\psi_\lambda = \exp\left(ikn + \frac{i}{2\sin k}\sum_{l=1}^{n} v(l)\right)\left(1 + O(n^{-\epsilon+\frac{1}{p}})\right),$$

where $k = \arccos\frac{1}{2}\lambda$.

For the proof of this theorem we need an analog of Zygmund's result for the case of the Fourier series instead of the Fourier integral. We refer to the work of Menchoff [22] for the following result:

Theorem (Menchoff). *Suppose $\{\phi_n(x)\}_{n=1}^\infty$ is an orthonormal system of functions on the interval (a,b) and the sequence $\{c_n\}_{n=1}^\infty$ belongs to $l^p(Z)$ $0 < p < 2$.*

Then the series

$$\sum_{l=1}^{N} c_n \phi_n(x)$$

converges, in the ordinary sense, for almost every $x \in (a, b)$.

In particular, taking $\phi_n(x) = \exp(inx)$ and $(a, b) = (0, 2\pi)$, we obtain an analog of Zygmund's theorem.

Proof of Theorem 2.1.8. Given Menchoff's theorem, the proof essentially repeats the argument we gave to prove Theorem 1.3.1 in Section 1.3. □

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