THE TRANSVERSE FORCE DISTRIBUTION ON ELLIPSOIDAL AND NEARLY ELLIPSOIDAL BODIES MOVING IN AN ARBITRARY POTENTIAL FLOW

by

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I. INTRODUCTION

The forces acting on an airship moving in a certain field of flow and the resulting path of motion are the result of numerous aerodynamic factors. These factors are principally skin friction drag, form drag, induced drag, aerodynamic transverse force distribution, lateral and longitudinal inertial and static force distribution, lift, and moments arising from all above factors. With the exception of aerodynamic transverse force distribution, lift, and induced drag, these factors are either small or may be determined by methods now available.

The transverse force distribution is of primary importance in the development of an airship of sufficient structural strength. An airship designed to withstand the bending moment arising from all possible transverse forces will usually have sufficient hull strength to withstand other smaller moments arising. Local forces, however, must be investigated, such as the high pressure at bow.

While certain methods for finding the transverse force distribution on airship hulls are now available, all either make basic assumptions at the start of the analysis that make the results of quite dubious value, or the analysis is extended only to flows represented by potential functions of first degree in x and y, and rotation. These methods apply only to ellipsoids or similar shapes in steady potential flows. In an article yet unpublished Dr. W. Tollmien has recently extended the analysis to potential functions of second degree in x and y, and has indicated the extention of his method to include the unsteady state.

In the present paper general relations are developed for the transverse force distribution on an ellipsoid moving in an arbitrary field of flow, whose potential function is expressed as a polynomial in x and y, or the coordinate system used. While exact only for the ellipsoid, the expressions result in forms applicable to bodies nearly ellipsoidal, hence may be applied with some degree of accuracy to such shapes. For certain potential fields and velocities of motion of an airship that occur in practice, it becomes apparent that the higher order terms are quite important if an accuracy of better than about 50% is desired, and that the motion of the airship may cause changes of force of the order of 100% over the steady state transverse force.

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II. DEVELOPMENT OF THE GENERAL TRANSVERSE FORCE DISTRIBUTION EQUATIONS.

The development of the equations for transverse force distribution is based on the assumptions that we have:

1) Prolate ellipsoid body shape;
2) Perfect fluid (viscosity and compressibility forces are not considered);
3) Potential flow (no fluid rotation).

The extension to airship hull body shapes must be regarded as an approximation of the exact ellipsoidal analysis. Since the flow about an airship is of good streamline form, the flow may be considered that of a perfect fluid potential flow. Viscous forces can be accounted for separately, while compressibility effects may be neglected for present airship velocities.

The integration of Euler's equations of motion for a perfect fluid potential flow gives for the pressure relation,

$$p = \frac{\rho}{2}\left(-2\frac{\partial\Phi}{\partial t} - q^2 + V + F(t)\right) \tag{2.1}$$

where: p = pressure

Φ = velocity potential, defined such that; $\frac{\partial\Phi}{\partial x} = u$, $\frac{\partial\Phi}{\partial y} = v$, $\frac{\partial\Phi}{\partial z} = w$

t = time

ρ = mass density

q = resultant velocity

V = external force potential

$F(t)$= integration constant, a function of time.

All units are compatable.

In order to satisfy boundary conditions on the surface of an ellipsoid it is convenient to express the potential functions of the undisturbed flow and the additional superimposed flow in terms of elliptical coordinates. Elliptical coordinates are orthogonal, being defined by, (Reference, Lamb: Hydrodynamics),

$$(2.2) \qquad \begin{aligned} x &= k\xi\mu \\ y &= k(\xi^2-1)^{1/2}(1-\mu^2)^{1/2}\cos\omega \\ z &= k(\xi^2-1)^{1/2}(1-\mu^2)^{1/2}\sin\omega \end{aligned}$$

The elements of length are,

$$(2.3) \qquad ds_\mu = k\left(\frac{\xi^2-\mu^2}{1-\mu^2}\right)^{1/2} d\mu;\; ds_\xi = k\left(\frac{\xi^2-\mu^2}{\xi^2-1}\right)^{1/2} d\xi \;;\quad ds_\omega = k(1-\mu^2)^{1/2}(\xi^2-1)^{1/2}\,d\omega$$

where:

$$(2.4) \qquad k = \sqrt{a^2-b^2} \qquad k\xi = a = \text{major semi-axis}$$
$$\xi = \sqrt{\frac{a^2}{a^2-b^2}} \qquad k(\xi^2-1)^{1/2} = b = \text{minor semi-axis}$$

The coordinate surfaces μ = constant are a series of confocal ellipsoids, the surfaces ξ = constant are confocal hyperboloids, and the surfaces ω = constant are coaxial meridianal planes. The ranges of variation of the coordinates are:

$$(2.5) \qquad -1 \leqq \mu \leqq 1 \quad ; \quad 1 \leqq \xi \leqq \infty \quad ; \quad 0 \leqq \omega \leqq 2\pi$$

In elliptical coordinates the solution of Laplace's equation for potential flow is given by the two forms,

$$(2.6) \qquad \phi_n^s = P_n^s(\mu)\, P_n^s(\xi)\cos s\omega$$
$$(2.6a) \qquad \phi_n^s = P_n^s(\mu)\, Q_n^s(\xi)\cos s\omega$$

$P_n^s(\mu)$ is the well known Legendre polynomial expression, and $Q_n^s(\xi)$ is the associated Legendre polynomial. The first form of the solution gives finite values inside the ellipsoid but goes to

∞ as ξ goes to $-\infty$. Hence for the solution of the problem outside the ellipsoid, the second form must be used, in which $Q_n^s(\xi)$ goes to 0 as ξ goes to ∞.

Let the ellipsoid be immersed in a flow whose potential function is arbitrarily expressed as a sum of superimposed potential flows,

(2.7) $$\phi = \phi_1 + \phi_2 + \phi_3 + \cdots\cdots + \phi_n$$

This undisturbed potential function can now be expressed in elliptical coordinates and expanded as a sum of terms containing the Legendre polynomials. Hence we may write,

(2.8) $$\phi_n^s = \sum_n \sum_s A_n^s(t)\, P_n^s(\mu)\, R_n^s(\xi) \cos s\omega$$

where n and s are indices denoting the particular Legendre polynomial. $R_n^s(\xi)$ is a function of ξ only and cos $s\omega$ represents the asymmetry of the flow. $A_n^s(t)$ is a constant with regard to the coordinates but in general may be a function of the time.

When an ellipsoid is placed in a velocity field, the total potential function must satisfy the boundary condition $\frac{\partial \Phi_n^s}{\partial s_\xi} = 0$ at $\xi = \xi_0$ for no flow across the boundary of the ellipsoid ξ_0. The resulting velocities at the surface will necessarily be tangential. Superimposing the additional flow given by the second type solution of Laplace's equation (2.6a) so that the additional flow vanishes at infinity, we get for the total potential function,

(2.9) $$\Phi_n^s = \sum_n \sum_s \left(A_n^s(t)\, P_n^s(\mu)\, R_n^s(\xi) \cos s\omega + B_n^s(t)\, P_n^s(\mu)\, Q_n^s(\xi) \cos s\omega \right)$$

which upon solving for $B_n^s(t)$ to satisfy boundary conditions gives, (2.10) $$\Phi_m^s = \sum_m \sum_s A_m^s(t)\, P_m^s(\mu)\, R_m^s(\xi) \cos s\omega \left[1 - \frac{\frac{\partial R_m^s(\xi_0)}{\partial \xi}}{R_m^s(\xi_0)} \frac{Q_m^s(\xi_0)}{\frac{\partial Q_m^s(\xi_0)}{\partial \xi}}\right]$$

Since ξ_0 is a constant for any given ellipsoid, we may write,

(2.11) $$\Phi_m^s = \sum_m \sum_s A_m^s(t)\, P_m^s(\mu)\, R_m^s(\xi)\, \mathcal{H}_m^s(\xi_0) \cos s\omega$$

The interesting fact here is that placing the ellipsoid in the stream has the effect of increasing each term of the undisturbed potential function by a constant factor $\mathcal{H}_m^s(\xi_0)$ which is a function of the ellipsoid size parameter ξ_0 only. For the simple linear flows these $\mathcal{H}_m^s$ factors are the well known apparent masses. For terms of higher order the same conception carries thru. For each Legendre polynomial term in the potential function expression, there corresponds one apparent mass factor.

The pressure on an annular element of the ellipsoid is determined by integrating the pressure times the cosine of the angle around the annulus, hence,

(2.12) $$\frac{dF}{dx} = r\int_0^{2\pi} p \cos\omega \, d\omega$$

Now introducing μ and the pressure the expression becomes,

(2.13) $$\frac{dF}{d\mu} = a\frac{dF}{dx} = ab(1-\mu^2)^{1/2}\int_0^{2\pi} \frac{\rho}{2}\left[F(t)+V-2\frac{\partial \Phi_m^s}{\partial t} - \left(\frac{\partial \Phi_m^s}{\partial s_\mu}\right)^2 - \left(\frac{\partial \Phi_m^s}{\partial s_\omega}\right)^2\right]\cos\omega \, d\omega$$

$\frac{\partial \Phi_m^s}{\partial s_\xi} = 0$ by the boundary conditions. It is apparent for homogenous matter that the integral of $[F(t)+V]$ around the circuit vanishes. Also the circuit integral of $\frac{\partial \Phi_m^s}{\partial s_\omega}$ vanishes. Hence there remains only,

(2.14) $$\frac{dF}{d\mu} = -\frac{\rho}{2}ab(1-\mu^2)^{1/2}\int_0^{2\pi}\left[\frac{\partial \Phi_m^s}{\partial t} + \frac{1}{k^2}\left(\frac{1-\mu^2}{\xi^2-\mu^2}\right)\left(\frac{\partial \Phi_m^s}{\partial \mu}\right)^2\right]\cos\omega \, d\omega$$

Substituting for the total potential function and integrating, the result reduces to,

$$\frac{dF}{d\mu} = -\pi\rho ab(1-\mu^2)^{\frac{1}{2}}\left\{\frac{1}{K^2}\frac{1-\mu^2}{\varepsilon^2-\mu^2}\sum_n A_n^0(t)\mathcal{K}_n^0 R_n^0(\varepsilon)\frac{\partial P_n^0(\mu)}{\partial\mu}\sum_n A_n^1(t)\mathcal{K}_n^1 R_n^1(\varepsilon)\frac{\partial P_n^1(\mu)}{\partial\mu} + \sum_n \mathcal{K}_n^1 P_n^1(\mu) R_n^1(\varepsilon)\frac{\partial A_n^1(t)}{\partial t}\right\} \tag{2.15}$$

This general expression is exact for the ellipsoid and yet has been reduced to very simple form. It will be noted that in the integration all terms of $\left(\sum_n\sum_s A_n^s(t)\mathcal{K}_n^s R_n^s(\varepsilon)\frac{\partial P_n^s(\mu)}{\partial\mu}\cos s\omega\right)^2$ vanished except the cross product given above.

The first term within the bracket represents the force on the ellipsoid in a steady potential flow. The second term represents the force due to the change in potential function with time, and here only the terms with s = 1 remain, the other terms integrating to zero.

Hence we write for the force distribution,

<u>Steady State</u>

$$\frac{dF}{d\mu} = -\pi\rho\frac{b}{a}(1-\mu^2)^{\frac{1}{2}}\cos^2\tau\sum_n A_n^0(t)\mathcal{K}_n^0 R_n^0(\varepsilon)\frac{\partial P_n^0(\mu)}{\partial\mu}\sum_n A_n^1(t)\mathcal{K}_n^1 R_n^1(\varepsilon)\frac{\partial P_n^1(\mu)}{\partial\mu} \tag{2.16}$$

where τ is the angle between the longitudinal axis and the tangent to the surface of the ellipsoid. The fineness ratio and $\cos^2\tau$ are multiplicative factors for the general relation. All terms of the initial potential function except those with s = 0 and s = 1 give no contribution to the force, and need not be considered.

<u>Unsteady State</u>

$$\frac{dF}{d\mu} = -\pi\rho ab(1-\mu^2)^{\frac{1}{2}}\sum_n \mathcal{K}_n^1 P_n^1(\mu) R_n^1(\varepsilon)\frac{\partial A_n^1(t)}{\partial t} \tag{2.17}$$

which for translation and lateral motion becomes,

$$\frac{dF}{d\mu} = -\pi\rho ab(1-\mu^2)^{\frac{1}{2}} \sum_n \mathcal{K}_m' P_n'(\mu) R_n'(\xi) \left\{ u \frac{\partial A_n'(x_0 y_0)}{\partial x_0} + v \frac{\partial A_n'(x_0 y_0)}{\partial y_0} \right\} \tag{2.18}$$

where $x = x_0 + \mu a \qquad \mu = \frac{x_{ell.}}{a}$

$y = y_0 + \psi b \qquad \psi = \frac{y_{ell.}}{b}$

x_0 and y_0 give the coordinates of the potential function origin with reference to the center of the ellipsoid. Only the terms with s = 1 contribute to this force.

When the ellipsoid is moving thru a potential field the total force is actually represented by the sum of three component parts, steady state force, superimposed velocity -u force, and unsteady state force. This is readily visualized if we consider the ellipsoid stationary and the origin of the disturbance such as vortex to move with -u velocity. Since the disturbance must move with the fluid it is necessary to superimpose the velocity -u. The three components of force resulting then simulate exactly the motion of the ellipsoid.

III. FORCE DISTRIBUTION EQUATIONS FOR ANY POTENTIAL FLOW EXPRESSED AS A POLYNOMIAL IN CARTESIAN COORDINATES.

The preceding general relation for force distribution makes possible the development of a similar expanded expression where the general undisturbed potential function is expressed as a polynomial in x and y,

(3.1)
$$\varphi_{nm} = f(x,y) = \sum_n \sum_m C_{nm} x^n y^m$$

Substituting for x and y in terms of elliptical coordinates,

(3.1a)
$$\varphi_{nm} = \sum_n \sum_m C_{nm} k^{n+m} \xi^n (\xi^2-1)^{\frac{m}{2}} \mu^n (1-\mu^2)^{\frac{m}{2}} \cos^m \omega$$

The only terms which contribute to the transverse force are those containing $\cos 0\cdot\omega$ and $\cos 1\cdot\omega$ (i.e. s = 0 and s = 1 in notation of Section II). Expanding $\cos^m \omega$ it is found that each value of m contributes one important term in either $M_m \cos 0\cdot\omega$ or $M'_m \cos 1\cdot\omega$, since for,

(3.2) m even
$$\cos^m\omega = \frac{1}{2^{m-1}} \cos m\omega + \cdots\cdots + M_m$$

m odd
$$\cos^m\omega = \frac{1}{2^{m-1}} \cos m\omega + \cdots\cdots + M'_m \cos\omega$$

where

(3.3)
$$M_m = \frac{m(m-1)\cdots\cdots\left(\frac{m+2}{2}\right)}{2^{m-1}\left(\frac{m}{2}\right)!}$$

$$M_m = \frac{m(m-1)\cdots\cdots\left(\frac{m+1}{2}\right)}{2^{m-1}\left(\frac{m-1}{2}\right)!}$$

Discarding then all terms except those contributing a force there results the modified potential function,

$$\phi_{nm} = \phi_{nm} + \phi'_{nm} = \sum_{n}\sum_{m(even)} C_{nm} M_m R_{nm}(\xi) \mu^n (1-\mu^2)^{m/2} \tag{3.4}$$
$$+ \sum_{n}\sum_{m(odd)} C'_{nm} M'_m R_{nm}(\xi) \mu^n (1-\mu^2)^{\frac{m-1}{2}} (1-\mu^2)^{1/2} \cos\omega$$

It is apparent from the general relation developed in Section II that the staedy state force will be a function of the cross product of the even and odd portion, when the proper apparent mass factors are included. Likewise the unsteady state force will be a function of the odd term only when the apparent mass factors are included.

Considering each portion individually since they are finally used thusly, and noting that the higher value of m raise the power of μ in the expression when expanded, we get.

$$\phi_{nm\,(even)} = \sum_{n=0}^{n+m} \left\{ M_0 C_{n0} R_{n0} + M_2 \left(C_{n2} R_{n2} - C_{(n-2)2} R_{(n-2)2} \right) + M_4 \left(C_{n4} R_{n4} - 2 C_{(n-2)4} R_{(n-2)4} + C_{(n-4)4} R_{(n-4)4} \right) + \cdots\cdots \right\} \mu^n \tag{3.5}$$
$$= \sum_{n=0}^{n+m} \sum_{m} D_{nm} R_{nm} \mu^n$$

where the series cuts off at the highest value of m appearing in the polynomial.

It was demonstrated in Section II that when the undisturbed potential function is expressed as a series of Legendre polynomials, the total potential function giving no flow across the boundary of the ellipsoid differs only in the introduction of the corresponding apparent mass factors $\mathcal{H}_m^s$. Hence the total potential function must be found by expanding in a series of Legendre polynomials. Performing this expansion, introducing the apparent mass factors and reversing the process to obtain the total potential in terms of a simple power series in μ^n, D_{nm}, R_{nm}, and $\mathcal{H}_{nmp}$, we get for the total potential function,

$$\Phi_{nm\,(even)} = \sum_m \Big\{ \big[D_{0m} R_{0m} \mathcal{H}_{0m0} + D_{2m} R_{2m} (\tfrac{1}{3}\mathcal{H}_{2m0} - \tfrac{1}{3}\mathcal{H}_{2m2}) + \cdots\cdots \big] \tag{3.6}$$
$$+ \mu \big[D_{1m} R_{1m} \mathcal{H}_{1m1} + D_{3m} R_{3m} (\tfrac{3}{5}\mathcal{H}_{3m1} - \tfrac{3}{5}\mathcal{H}_{3m3}) + \cdots \big]$$
$$+ \mu^2 \big[D_{2m} R_{2m} \mathcal{H}_{2m2} + \cdots\cdots \big] + \mu^3 \big[D_{3m} R_{3m} \mathcal{H}_{3m3} + \cdots \big]$$
$$+ \cdots\cdots\cdots\cdots \mu^{n+m} \big[D_{(n+m)m} R_{(n+m)m} \mathcal{H}_{(n+m)m(n+m)} \big] \Big\}$$
$$= \sum_{n=0}^{n+m} \sum_m \big[DR\mathcal{H} \big]_{nm} \mu^n$$

where,

$$\mathcal{H}_{nmp} = 1 - \frac{\frac{d}{d\xi} R_{nm}(\xi_0)}{R_{nm}(\xi_0)} \frac{Q_p(\xi_0)}{\frac{d}{d\xi} Q_p(\xi_0)} = 1 - \frac{(n+m)\xi_0^2 - m}{\xi_0(\xi_0^2 - 1)} \frac{Q_p(\xi_0)}{\frac{d}{d\xi} Q_p(\xi)} \tag{3.7}$$

Hence the derivative desired for the general transverse force equation becomes,

$$\frac{\partial \Phi_{nm}}{\partial \mu} = \sum_{n=0}^{n+m} \sum_m n \big[DR\mathcal{H} \big]_{nm} \mu^n \tag{3.8}$$

In a similar manner the portion of the potential function containing odd values of m is treated. The term from the undisturbed potential function is,

$$\phi'_{nm} = \sum_n \sum_m C'_{nm} M'_m R_{nm}(\xi) \mu^n (1-\mu^2)^{\frac{m-1}{2}} (1-\mu^2)^{1/2} \cos\omega \tag{3.9}$$

which becomes upon expanding,

$$\phi'_{nm} = \sum_{n=0}^{n+m-1} \Big\{ M'_1 C'_{n1} R_m + M'_3 (C'_{n3} R_{n3} - C'_{(n-2)3} R_{(n-2)3}) + \cdots \Big\} \mu^n \tag{3.10}$$
$$= \sum_{n=0}^{n+m-1} \sum_m D'_{nm} R_{nm} \mu^n$$

The total potential function is now found by expanding in terms of the Legendre polynomials $P'_n(\mu)$ and introducing the corresponding apparent mass factors. Expanding thusly and retransposing in terms of the simple polynomial in μ we obtain,

$$\Phi_{nm}^{(odd)} = (1-\mu^2)^{1/2} \cos\omega \sum_m \Big\{ \Big[D'_{0m} R_{0m} \mathcal{H}'_{0m1} + D'_{2m} R_{2m} \big(\tfrac{1}{3}\mathcal{H}'_{2m1} - \tfrac{1}{5}\mathcal{H}'_{2m3}\big) + \cdots \Big] \tag{3.11}$$
$$+ \mu \Big[D'_{1m} R_{1m} \mathcal{H}'_{1m2} + D'_{3m} R_{3m} \big(\tfrac{3}{7}\mathcal{H}'_{3m2} - \tfrac{3}{7}\mathcal{H}'_{3m4}\big) + \cdots \Big]$$
$$+ \mu^2 \Big[D'_{2m} R_{2m} \mathcal{H}'_{2m3} + \cdots \Big] + \mu^3 \Big[D'_{3m} R_{3m} \mathcal{H}'_{3m4} + \cdots \Big]$$
$$+ \cdots\cdots + \mu^{n+m-1} \Big[\cdots\cdots \Big] \Big\}$$
$$= (1-\mu^2)^{1/2} \cos\omega \sum_{n=0}^{n+m-1} \sum_m [DR\mathcal{H}]'_{nm} \mu^n$$

where,

$$\mathcal{H}'_{nmp} = 1 - \frac{\frac{d}{d\xi} R_{nm}(\xi_0)}{R_{nm}(\xi_0)} \frac{Q'_p(\xi_0)}{\frac{d}{d\xi} Q'_p(\xi_0)} = 1 - \frac{(n+m)\xi_0^2 - n}{\xi_0(\xi_0^2 - 1)} \frac{Q'_p(\xi_0)}{\frac{d}{d\xi} Q'_p(\xi_0)} \tag{3.12}$$

Equation (3.11) is the expression for the total potential function of the terms with m odd from the modified potential function. For use in the transverse force equation we need the first derivative with respect to μ,

$$\frac{\partial}{\partial\mu} \Phi'_{nm} = \frac{\cos\omega}{(1-\mu^2)^{1/2}} \sum_{n=0}^{n+m-1} \sum_m \Big\{ n [DR\mathcal{H}]'_{nm} (1-\mu^2) \mu^{n-1} - [DR\mathcal{H}]'_{nm} \mu^{n+1} \Big\} \tag{3.13}$$

or,

$$\frac{d}{d\mu} \Phi'_{nm} = \frac{1}{2} \frac{\cos\omega}{(1-\mu^2)^{1/2}} \sum_{n=0}^{n+m-1} \sum_m \Big\{ 2\big(\tfrac{r}{b}\big)^2 n [DR\mathcal{H}]'_{nm} \mu^{n-1} + \frac{d\big(\frac{r}{b}\big)^2}{d\mu} [DR\mathcal{H}]'_{nm} \mu^n \Big\} \tag{3.13a}$$

Steady State

The transverse force expression of Section II for the steady state becomes upon substitution of the even and odd total potential functions corresponding to the $P_n^0(\mu)$ and $P_n^1(\mu)$ terms in Section II,

$$\frac{dF}{d\mu} = -\pi\rho \frac{b}{a} \cos^2\tau \, (1-\mu^2)^{1/2} \frac{\partial \Phi_{nm}}{\partial\mu} \frac{\partial \Phi'_{nm}}{\partial\mu} \tag{3.14}$$

$$\frac{dF}{d\mu} = -\pi\rho \frac{b}{a} \cos^2\tau \Big\{ \sum_{n=0}^{n+m} \sum_m n [DR\mathcal{H}]_{nm} \mu^{n-1} \Big\} \Big\{ \sum_{n=0}^{n+m-1} \sum_m \Big(n [DR\mathcal{H}]'_{nm} (1-\mu^2) \mu^{n-1} - [DR\mathcal{H}]'_{nm} \mu^{n+1} \Big) \Big\} \tag{3.15}$$

or in terms of physical demensions of the ellipsoid and the derivatives of the final potential flows,

$$\frac{dF}{d\mu} = -\pi\frac{\rho}{2}\frac{b}{a}\cos^2\tau\left\{\sum_{n=0}^{n+m}\sum_{m} n[DR\chi]_{nm}\mu^{n-1}\right\}\left\{\sum_{n=0}^{n+m-1}\sum_{m}\left(2\left(\frac{r}{b}\right)^2 n[DR\chi]'_{nm}\mu^{n-1} + \frac{d\left(\frac{r}{b}\right)^2}{d\mu}[DR\chi]'_{nm}\mu^{n}\right)\right\} \tag{3.16}$$

In Cartesian coordinates the latter equation is of the physical form,

$$\frac{dF}{dx} = -\frac{\rho}{2}\cos^2\tau\left\{u_{Total}\right\}\left\{2S\frac{dv_T}{dx} + v_T\frac{dS}{dx}\right\} \tag{3.17}$$

This form lends an excellent picture of the nature of the exact solution for the transverse force on an ellipsoid or nearly ellipsoidal body in a steady potential flow. The factor $\cos^2\tau$ is an exact end effect correction for all potential flows, The force at any point is proportional to the product of the forward velocity derived from the derivative of the total potential of the even terms of the modified potential, and the sum of twice the area of the annulus times the derivative of the total potential transverse velocity plus the transverse velocity times the derivative of the area. Equation (3.17) might be written alternatively,

$$\frac{dF}{dx} = -\frac{\rho}{2}\cos^2\tau\left\{u_T\right\}\left\{\frac{d(Sv_T)}{dx} + S\frac{dv_T}{dx}\right\} \tag{3.17a}$$

The physical expressions cannot be applied directly in the general case because of the implicit form of the velocities indicated, which are derived from the total potentials. For bodies of nearly elliptical shapes, such as present airship hulls, equation (3.16) should produce good results if the apparent mass factors are chosen for the ellipsoid of the equivalent fineness ratio and if the physical dimensions of the airship hull are substituted for $\left(\frac{r}{b}\right)^2$. Any error arising from such calculations must be attributed to the shape only because the exact undisturbed potential flow can be used.

For the ellipsoid we can expand the force expression into the polynomial,

$$\frac{dF}{d\mu} = -\pi\rho\frac{b}{a}\cos^2\tau\sum_{m}\Big\{[D R \mathcal{H}]_{1m}[D R \mathcal{H}]'_{1m} + \mu\big(2[D R \mathcal{H}]_{1m}[D R \mathcal{H}]'_{2m} - [D R \mathcal{H}]_{1m}[D R \mathcal{H}]'_{0m} + 2[D R \mathcal{H}]_{2m}[D R \mathcal{H}]'_{1m}\big)$$
$$+\mu^2\big(3[\,]_{1m}[\,]'_{3m} - 2[\,]_{1m}[\,]'_{1m} + 4[\,]_{2m}[\,]'_{2m} - 2[\,]_{2m}[\,]'_{0m} + 3[\,]_{3m}[\,]'_{1m}\big)$$
$$+\mu^3\big(4_{1m4m} - 3_{1m2m} + 6_{2m3m} - 4_{2m1m} + 6_{3m2m} - 3_{3m0m} + 4_{4m1m}\big)$$
$$+\mu^4\big(5_{1m5m} - 4_{1m3m} + 8_{2m4m} - 6_{2m2m} + 9_{3m3m} - 6_{3m1m} + 8_{4m2m} - 4_{4m0m} + 5_{5m1m}\big)$$
$$+\mu^5\big(6_{1m6m} - 5_{1m4m} + 10_{2m5m} - 8_{2m3m} + 12_{3m4m} - 9_{3m2m} + 12_{4m3m} - 8_{4m1m} + 10_{5m2m}$$
$$- 5_{5m0m} + 6_{6m1m}\big) + \mu^6\big(7_{1m7m} - 6_{1m5m} + 12_{2m6m} - 10_{2m4m} + 15_{3m5m} - 12_{3m3m}$$
$$+ 16_{4m4m} - 12_{4m2m} + 15_{5m3m} - 10_{5m1m} + 12_{6m2m} - 6_{6m0m} + 7_{7m1m}\big) + \cdots\cdots\Big\} \tag{3.19}$$

This expression is used in later examples in which are calculated the force distribution on an ellipsoid.

<u>Non Steady Flow</u>.

From the general equation for transverse force distribution in Section II, we obtain upon substitution of the odd total potential function,

$$\frac{dF}{d\mu} = -\pi\rho ab(1-\mu^2)^{1/2}\frac{\partial}{\partial t}\Phi'_{nm} \tag{3.20}$$

$$\frac{dF}{d\mu} = -\pi\rho ab(1-\mu^2)\sum_{n=0}^{n+m-1}\sum\mu^n\frac{\partial}{\partial t}[D R \mathcal{H}]'_{nm} \tag{3.21}$$

For translation and lateral motion we have as in Section II,

$$\frac{dF}{d\mu} = -\pi\rho ab(1-\mu^2)\sum_{n=0}^{n+m-1}\sum_{m}\mu^n\left(u\frac{\partial}{\partial x_0} + v\frac{\partial}{\partial y_0}\right)[D R \mathcal{H}]'_{nm} \tag{3.22}$$

Only $\mathbf{D}$ in the bracket is a function of time when x_0 and y_0 are changing, since R and $\mathcal{H}$ are constants for a given ellipsoid. For application to body shapes nearly elliptical the equation may be written,

$$(3.23) \qquad \frac{dF}{d\mu} = -\pi\rho\, ab\left(\frac{r}{b}\right)^2 \sum_{n=0}^{n+m-1}\sum_{m} \mu^n \frac{\partial}{\partial t}\left[DRX\right]'_{nm}$$

Equation (3.23) may be written with x_o and y_o derivatives, as was equation (3.22).

These above equations provide a method for calculating the transverse force distribution that is exact for the ellipsoid, and probably applicable to within good accuracy for nearly ellipsoidal shapes. The exact undisturbed flow can be used.

<u>Remarks on Evaluation of Factors.</u>

The values of the apparent mass factors for terms up to the order of $n = 3$ and $m = 1$ have been computed and are plotted in Figures 1 and 2. General analytical expressions for the apparent mass factors were given in this Section. The value of the coefficients $C_{nm}\, R_{nm}$ may be immediately determined from dimensionless Cartesian coordinates, since from the undisturbed potential function,

$$(3.24) \qquad \begin{aligned} \varphi_{nm} &= \sum_n \sum_m C_{nm} x^n y^m \\ &= \sum_n \sum_m C_{nm}\, a^n b^m \mu^n \psi^m \\ &= \sum_n \sum_m C_{nm}\, R_{nm}(\xi_o) \mu^n \psi^m \end{aligned}$$

where

$$\mu = \frac{x}{a} \quad ; \quad \psi = \frac{y}{b}$$

The modified potential is then found as in the original development in Section III. This introduction of the dimensionless parameter ψ takes the place of $(1-\mu^2)^{1/2}\cos\omega$. The modified potential functions become,

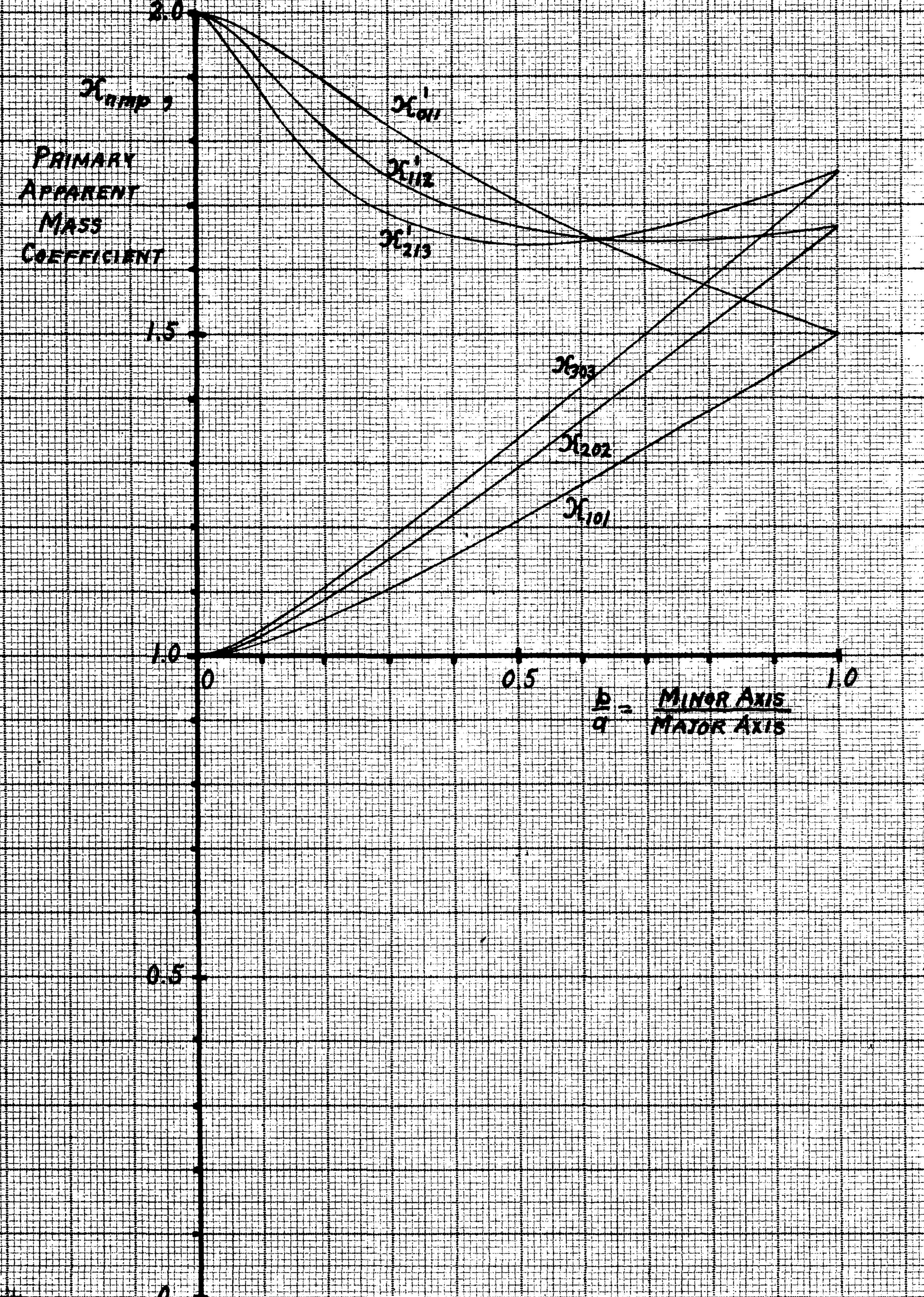

Figure 1 . Primary Apparent Mass Coefficients for Prolate Ellipsoids. Lower Order Coefficients only Plotted.

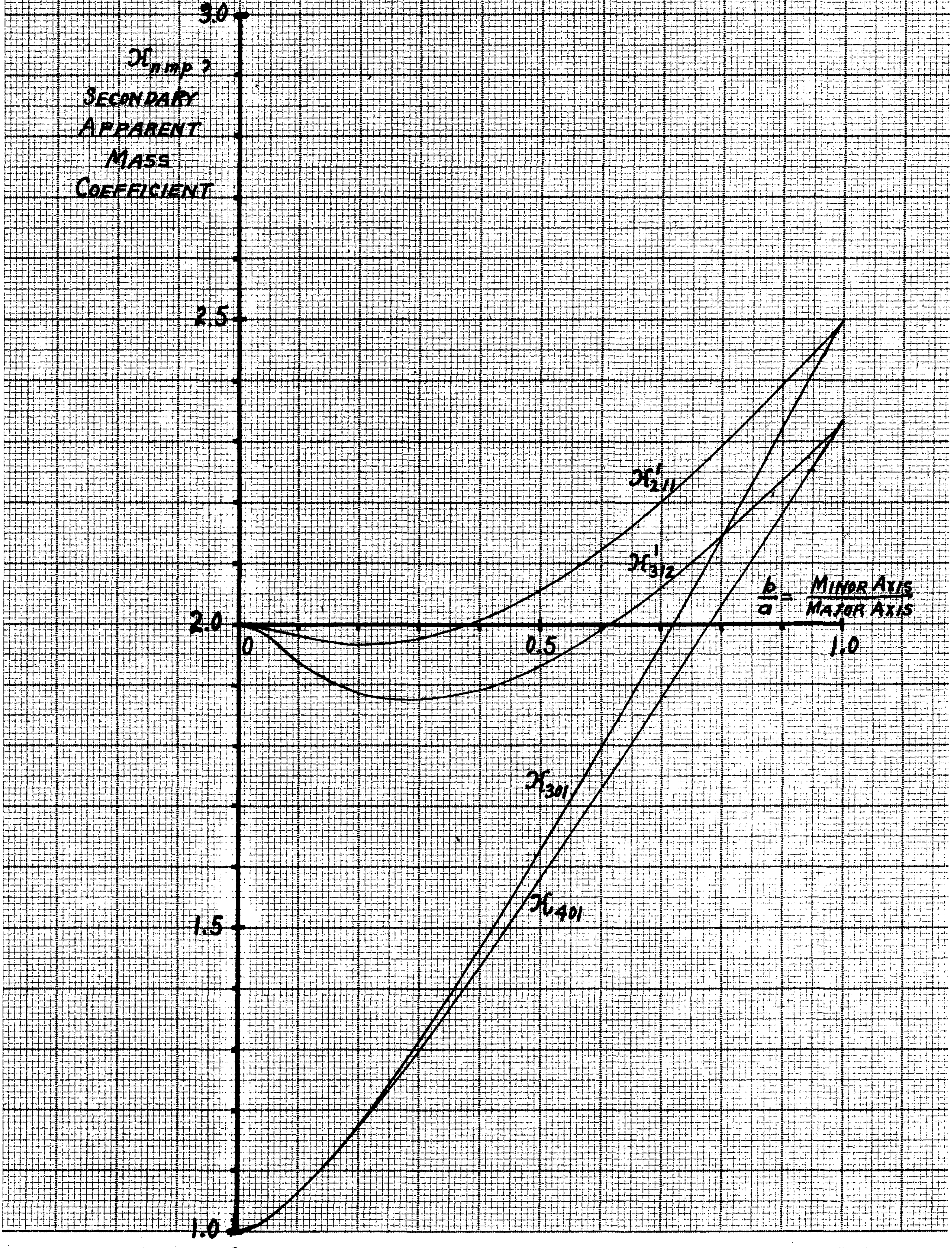

Figure 2 . Secondary Apparent Mass Coefficients for Prolate Ellipsoids. Lower Order Coefficients only Plotted.

(3.25)
$$\phi_{\substack{nm\\(even)}} = \sum_{n=0}^{n+m}\sum_{m} C_{nm} M_m R_{nm}(\xi_o)\mu^n$$

(3.26)
$$\phi'_{\substack{nm\\(odd)}} = \sum_{n=0}^{n+m-1}\sum_{m} C'_{nm} M'_m R_{nm}(\xi_o)\mu^n \psi$$

and the development is exactly analogous to that used before. The expressions for determining D_{nm} are given in equations (3.5) and (3.10).

Simple Application to Force in Steady Pitched Flight.

The transverse force distributions for the simple cases of potential flow are immediately obtained from equation (3.19). For example, in pitched flight,

(3.27)
$$\varphi = \phi = u x + v y$$
$$= u a \mu + v b \psi$$
$$[DR\mathcal{H}]_{10} = u a \mathcal{H}_{101}$$
$$[DR\mathcal{H}]_{01} = v b \mathcal{H}'_{011}$$

(3.28)
$$\frac{dF}{d\mu} = -\pi\rho \frac{b}{a} \cos^2\tau \left\{-\mu[u a \mathcal{H}_{101}][v b \mathcal{H}'_{011}]\right\}$$
$$= \pi\rho\, b^2 \cos^2\tau\, u v\, \mathcal{H}_{101} \mathcal{H}'_{011} \mu$$

or,

(3.29)
$$\frac{dF}{dx} = -\rho u v \frac{\mathcal{H}_{101}\mathcal{H}_{011}}{2} \cos^2\tau \frac{dS}{dx}$$

IV. FORCE DISTRIBUTION ON AN ELLIPSOID MOVING IN A VORTEX FIELD.

The potential function of a vortex field is expressed by,

$$\varphi = \frac{\Gamma}{2\pi}\theta = \frac{\Gamma}{2\pi}\tan^{-1}\frac{y+y_0}{x+x_0} \tag{4.1}$$

in the neighborhood of the x-axis, and

$$\varphi = \frac{\Gamma}{4} - \frac{\Gamma}{2\pi}\tan^{-1}\frac{x+x_0}{y+y_0} \tag{4.2}$$

in the vicinity of the y-axis. x_0 and y_0 are coordinates measured from the center of the vortex to the center of the ellipsoid. x and y are measured from the center of the ellipsoid, their origin moving with the ellipsoid. The vortex field potential may be represented by a polynomial of the type,

$$\varphi = \frac{\Gamma}{4} - \frac{\Gamma}{2\pi}\left(e + f\frac{x+x_0}{y+y_0} + g\left(\frac{x+x_0}{y+y_0}\right)^2 + h\left(\frac{x+x_0}{y+y_0}\right)^3\right) \tag{4.3}$$

Expanding,

$$\begin{aligned}\varphi = K - \frac{\Gamma}{2\pi}\Big\{&\left[\gamma(f+g\gamma+h\gamma^2) + \mu\epsilon(f+2g\gamma+3h\gamma^2) + \mu^2\epsilon^2(g+3h\gamma) + \mu^3\epsilon^3(h)\right] \\ &- \psi\beta\left[\gamma(f+2g\gamma+3h\gamma^2) + \mu\epsilon(f+4g\gamma+9h\gamma^2) + \mu^2\epsilon^2(2g+9h\gamma) + \mu^3\epsilon^3(3h)\right] \\ &+ \psi^2\beta^2[\cdots\cdots] + \cdots\cdots\Big\}\end{aligned} \tag{4.4}$$

where,

$$\gamma = \frac{x_0}{y_0} \qquad \mu = \frac{x}{a}$$

$$\epsilon = \frac{a}{y_0} \qquad \psi = \frac{y}{b} \tag{4.5}$$

$$\beta = \frac{b}{y_0}$$

Neglecting terms of order ψ^2 and higher, the constants are,

$$[DR\mathcal{K}]_{00} = \text{(not needed)}$$

(4.6)
$$[DR\mathcal{K}]_{10} = -\frac{\Gamma}{2\pi}\left[\epsilon(f+2g\gamma+3h\gamma^2)\mathcal{K}_{101}+\epsilon^3(h)\left(\tfrac{3}{5}\mathcal{K}_{301}-\tfrac{3}{5}\mathcal{K}_{303}\right)\right]$$

$$[DR\mathcal{K}]_{20} = -\frac{\Gamma}{2\pi}\left[\epsilon^2(g+3h\gamma)\mathcal{K}_{202}\right]$$

$$[DR\mathcal{K}]_{30} = -\frac{\Gamma}{2\pi}\left[\epsilon^3(h)\mathcal{K}_{303}\right]$$

$$[DR\mathcal{K}]'_{01} = \frac{\Gamma}{2\pi}\beta\left[\gamma(f+2g\gamma+3h\gamma^2)\mathcal{K}'_{011}+\epsilon^2(2g+9h\gamma)\left(\tfrac{1}{5}\mathcal{K}'_{211}-\tfrac{1}{5}\mathcal{K}'_{213}\right)\right]$$

$$[DR\mathcal{K}]'_{11} = \frac{\Gamma}{2\pi}\beta\left[\epsilon(f+4g\gamma+9h\gamma^2)\mathcal{K}'_{112}+\epsilon^3(3h)\left(\tfrac{3}{7}\mathcal{K}'_{312}-\tfrac{3}{7}\mathcal{K}'_{314}\right)\right]$$

$$[DR\mathcal{K}]'_{21} = \frac{\Gamma}{2\pi}\beta\left[\epsilon^2(2g+9h\gamma)\mathcal{K}'_{213}\right]$$

$$[DR\mathcal{K}]'_{31} = \frac{\Gamma}{2\pi}\beta\left[\epsilon^3(3h)\mathcal{K}'_{314}\right]$$

<u>Case 1.</u> Ellipsoid of fineness ratio 6 : 1. Vortex center on equatorial plane at a distance from the center of the ellipsoid equal to its length, 2a.

(4.7)
$$\mathcal{L}_0 = \sqrt{\frac{a^2}{a^2-b^2}} = 1.0142$$

The apparent mass factors are,

(4.8)
$$\mathcal{K}_{101} = 1.05 \qquad \mathcal{K}'_{011} = 1.91 \qquad \mathcal{K}'_{211} = 1.97$$
$$\mathcal{K}_{202} = 1.07 \qquad \mathcal{K}'_{112} = 1.85 \qquad \mathcal{K}'_{312} = 1.90$$
$$\mathcal{K}_{303} = 1.08 \qquad \mathcal{K}'_{213} = 1.79$$
$$\mathcal{K}_{301} = 1.14 \qquad \mathcal{K}'_{314} = 1.73$$

The values of the parameters corresponding to the configuration are,

(4.9)
$$\gamma = \frac{x_0}{y_0} = 0 \quad ; \quad \epsilon = \frac{a}{y_0} = \frac{1}{2} \quad ; \quad \beta = \frac{b}{y_0} = \frac{1}{12}$$

The potential function,

$$\varphi = \frac{\Gamma}{4} - \frac{\Gamma}{2\pi}\left(\frac{x+x_0}{y+y_0} - 0.24\left(\frac{x+x_0}{y+y_0}\right)^3\right) \tag{4.10}$$

where,

$$f = 1 \quad ; \quad g = 0 \quad ; \quad h = -0.24$$

represents the potential and velocities of the vortex in the neighborhood of the y - axis to within 3% for values of $\frac{x+x_0}{y+y_0}$ up to 0. 7.

(a) Steady State

Substituting in the general expressions for the constants (4.6) and then substituting these values in equation (3.19) for the transverse force on an ellipsoid in a steady field we get,

$$\frac{dF}{d\mu} = \pi\rho\left(\frac{\Gamma}{2\pi}\right)^2 \cos^2\tau \left\{.00668 - .01817\mu^2 + .00767 - .00085\mu^6\right\} \tag{4.11}$$

(b) Ellipsoid moving in axial direction with velocity u .

The force distribution due to this motion of the ellipsoid thru the field is obtained from the sum of the forces due to the unsteady potential $\frac{\partial \bar{\Phi}}{\partial t}$ and a superimposed velocity $-u$.

For the superimposed velocity $-u$,

$$\varphi = -ux = -ua\mu \tag{4.12}$$

$$[DRX]_{10} = -ua\,\mathcal{X}_{101} \tag{4.13}$$

Hence the force distribution upon substitution in the general relation in combination with the vortex field relation gives,

$$\frac{dF}{d\mu} = \pi\rho u y_0 \frac{\Gamma}{2\pi}\cos^2\tau \left\{.00670 - .01680\mu^2 + .00455\mu^4\right\} \tag{4.14}$$

For the force due to the unsteady potential we obtain,

$$(4.15) \qquad \frac{dF}{d\mu} = -\pi\rho ab \frac{u\Gamma}{2\pi y_0}(1-\mu^2) \frac{\partial}{\partial \mu_0}\left\{[DRX]_{01}' + \mu [DRX]_{11}' + \cdots \cdots\right\}$$

Substituting for the values of the derivatives indicated we obtain,

$$(4.16) \qquad \frac{dF}{d\mu} = -\pi\rho\, u y_0 \frac{\Gamma}{2\pi}(1-\mu^2)\left\{.00657 - .00336\mu^2\right\}$$

These components of the transverse force distribution have been plotted in Figure 3 . The force distribution due to the vortex field on the stationary ellipsoid shows the large buoyancy force at the center directed toward the vortex and small reversed force at the ends. The superimposed velocity $-u$ tends primarily to increase this same general distribution. The striking fact of this problem is that the unsteady potential force is of the same magnitude as the superimposed flow force and of opposite sign. The resultant of these two latter forces gives the actual additional force due to motion thru the vortex. It is apparent that force distributions derived from steady potentials are far from representing the actual force on a moving body, because the $\frac{\partial \Phi}{\partial t}$ force is of the same order of magnitude as the steady state forces. For comparison of the accuracy of the force distribution due to a potential function of only two terms, $\varphi = u x + \beta x y$, the resulting force distribution for this potential function has been plotted for the stationary ellipsoid. The error reaches approximately 35%.

<u>Case 2.</u> Vortex center on longitudinal axis at a distance from the center of the illipsoid equal to twice its length, 4a.

(a) <u>Steady State</u>

In an exactly similar manner the force distribution has been computed for the ellipsoid moving along the x-axis away from the vortex. The resulting force distributions have been plotted in Figure 4. There results no transverse force for the ellipsoid stationary. The buoyancy force is here axial and not transverse. This result of no transverse force for the stationary ellipsoid would hardly be valid in a real fluid, however, as a cylinder is known to have a high form drag. Nevertheless in combination with forward velocity this steady component of force may be assumed zero provided the resulting angle of attack is not too great.

(b) Ellipsoid moving in axial direction with velocity u .

The resultant force due to the axial motion of the ellipsoid in the vortex field is again the sum of the forces arising from the superimposed velocity $-u$, and the non-steady potential $\frac{\partial \Phi}{\partial t}$. The unsteady potential gives rise to a force entirely on one side of the body whereas the superimposed flow $-u$ gives rise to the familiar pitching moment type of distribution. The force is larger toward the center of the vortex as would be expected. The hull alone is unstable in this motion and will tend to turn broadside. In case controls were operated to hold the ellipsoid in the axial direction it would move laterally off the path.

Numerical Evaluation

In order to get a picture of the actual probable forces and distribution on a full scale airship these forces have been computed and plotted for an ellipsoid of length 240 meters and fineness ratio 1 : 6 as in the preceding cases. These are approximately the physical dimensions of the Akron. For Case 1

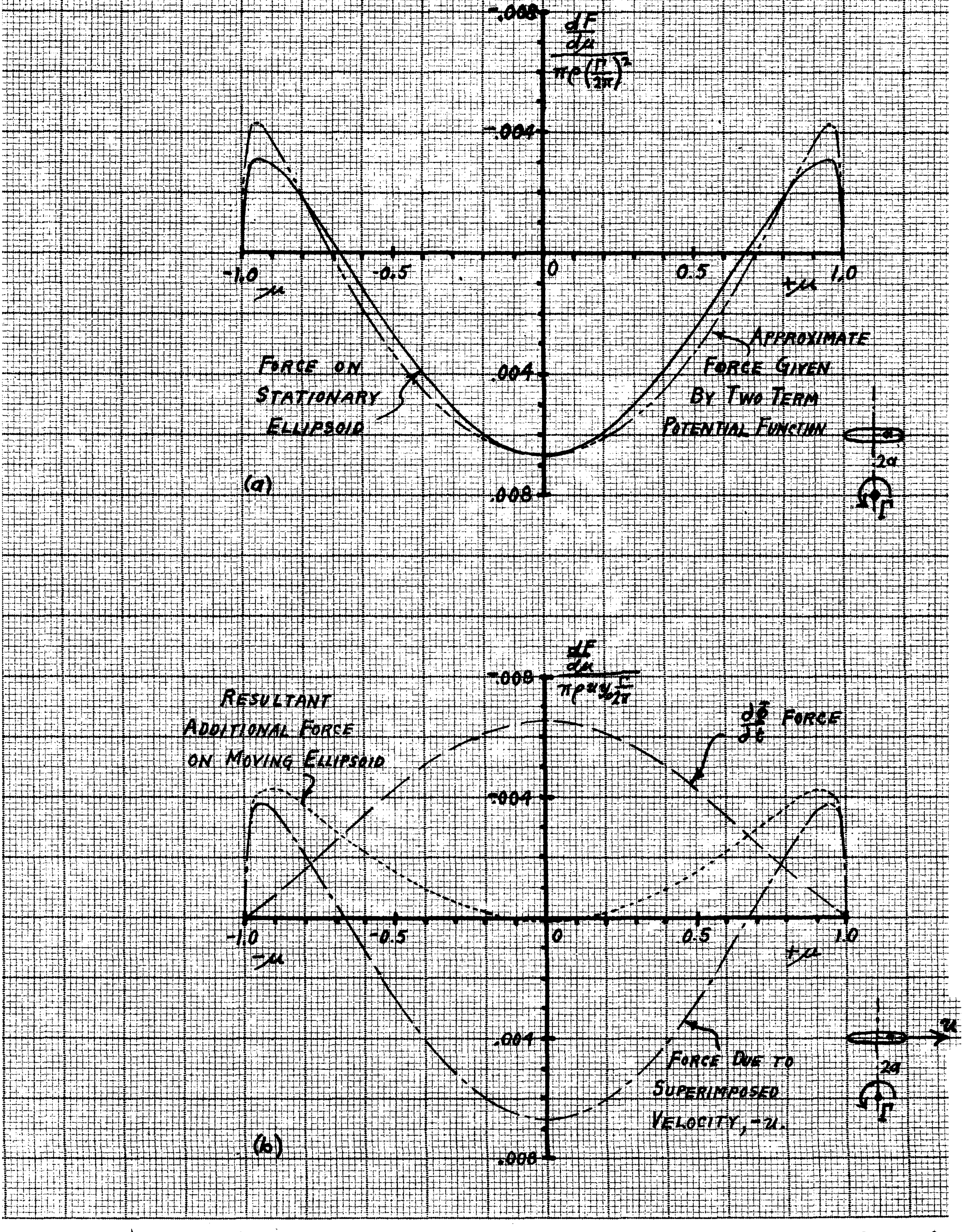

Figure 3 . Components of Force Distribution on an Ellipsoid Moving in a Vortex Field. Vortex on Equatorial Plane.

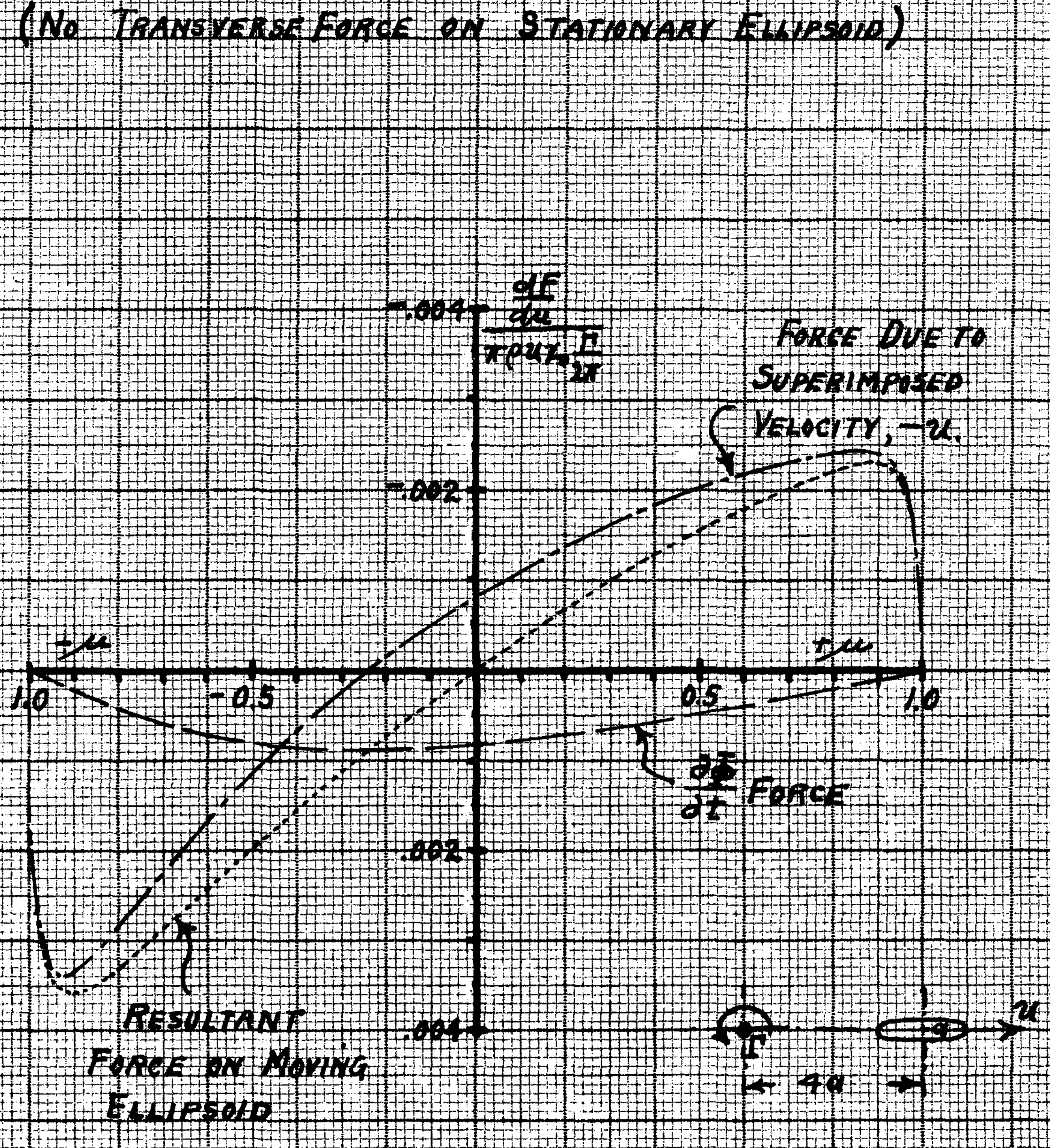

Figure 4 . Components of Force Distribution on an Ellipsoid Moving in a Vortex Field. Vortex on Longitudinal Axis.

the distribution has been computed for several values of forward velocity and vortex velocity at the ellipsoid and has been plotted in Figure 5. The dotted curve shows the distribution for the stationary ellipsoid in a vortex of velocity 20 m/s. at the ellipsoid. It is interesting to note that flying thru a vortex field against the direction of the vortex field velocity increases the force at the ends, while flying with the vortex may reverse the end force giving a forced distribution along the ellipsoid of the same sign. Control might be difficult in such a state as the force might easily reach about 500 kg./m. along the entire length. Doubling the strength of the vortex practically quadruples the values of the forces acting.

When the ellipsoid is moving along the x-axis, the force is directly proportional to the product of $u\frac{\Gamma}{2\pi x_o}$. The curves for numerical values have been plotted in Figure 6 for this product $\frac{u\Gamma}{2\pi x_o} = 400$ m/s. . Reversing either the direction of the motion or vortex velocity reverses the force. The force at the end of the ellipsoid toward the vortex is approximately 5/3 that at the far end, which corresponds approximately to the ratio of transverse velocities. The moment is large and forces might reach magnitudes of 1000 kg./m. or more. It should be noted that the moment and forces will increase rapidly if flying into the vortex, possibly becoming prohibitive near the center of the vortex.

A rough prediction of the motion of the airship in a vortex field is possible. Assuming that controls are operated such that the nose of the airship is kept pointed in the x direction, and assuming that it enters the field along the x-axis the probable motion would be as follows: The airship would move laterally off its course passing the vortex at a certain distance which would be a function of the vortex strength and airship velocity. Upon passing the vortex the buoyancy force would move the airship back toward the x-axis, which the airship would approach again asymptotically. Such motion would be fortunate in that it would prevent the airship from flying directly into the vortex.

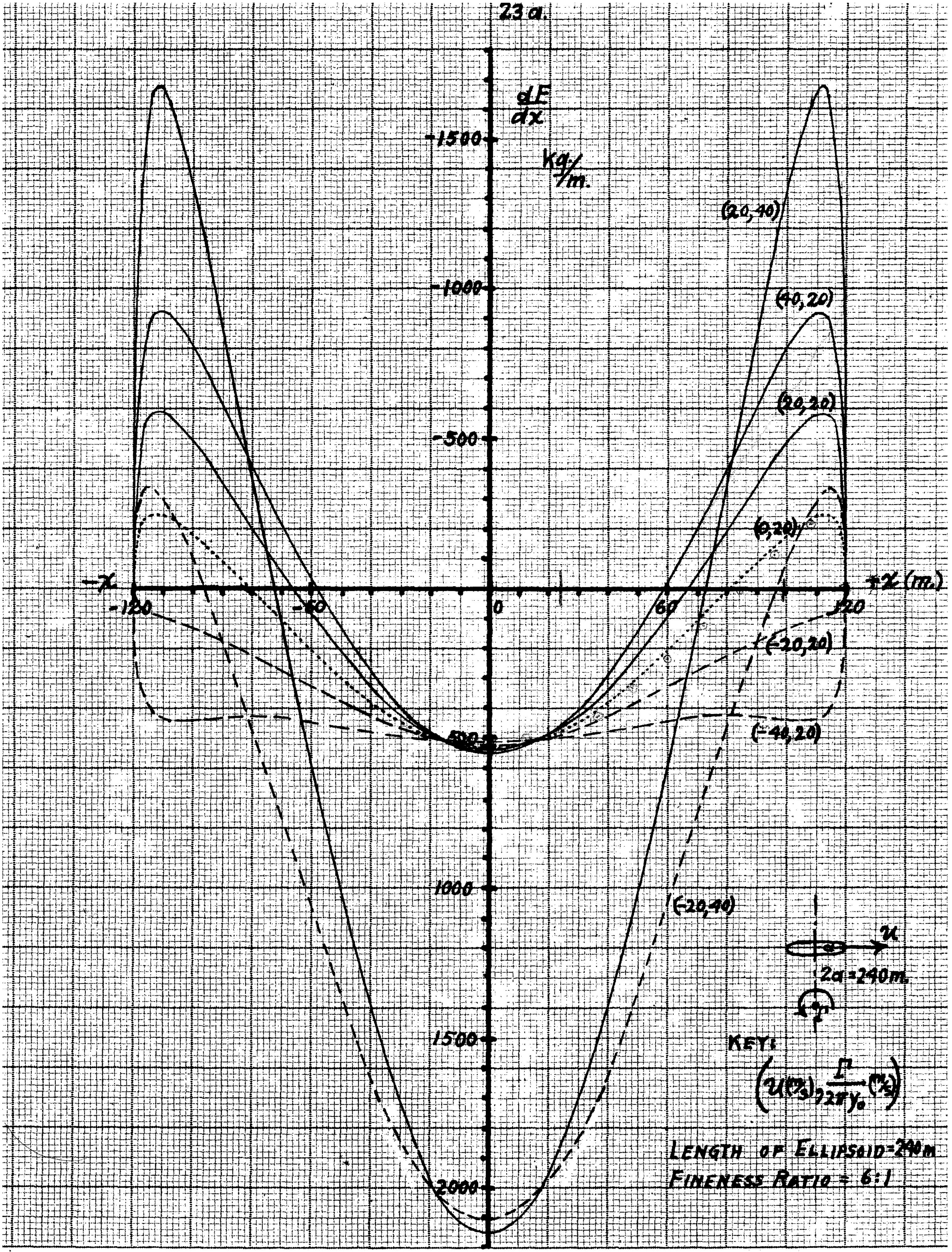

Figure 5 . Force Distribution on an Ellipsoid Moving at Various Velocities in a Vortex Field. Vortex on Equatorial Plane

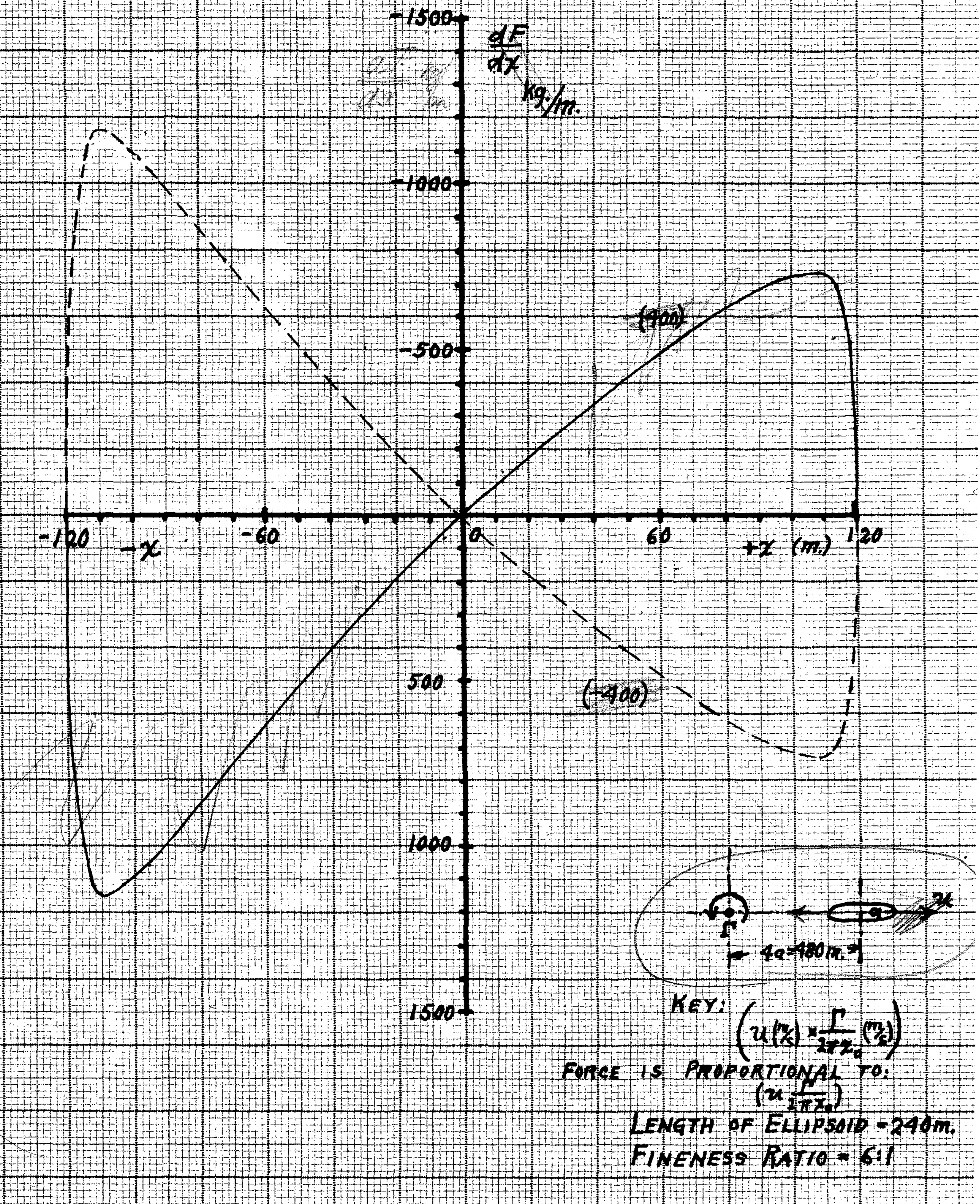

Figure 6 . Force Distribution on an Ellipsoid Moving in a Vortex Field. Vortex on Longitudinal Axis.

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V. CONCLUSION

When the undisturbed potential function is expressed as a series of Legendre polynomials $P_n^s(\mu)$ the exact general equations for transverse force distribution on an ellipsoid have been derived. For the steady state the force is shown to be proportional to the cross product only of the Legendre polynomial terms with s = 0 and s = 1. The force due to an unsteady potential is a function of the terms with s = 1 only.

The total potential function is derived from the undisturbed potential function by multiplying each term by a constant, which is the apparent mass factor.

The general relation for transverse force distribution shows that $\cos^2 \tau$ is an exact end effect correction for all flows.

When the potential function is expressed as a simple polynomial in x and y, or μ and ψ, the exact transverse force equations for the ellipsoid can be expressed as polynomials in x, or μ, whose coefficients are functions of the undisturbed potential function coefficients, physical properties of the ellipsoid, and apparent mass factors.

For application to bodies of nearly ellipsoidal shape the equations for transverse force have been expressed in terms of the final potential function and the geometrical dimensions of the body. These equations should lead to good results because the exact undisturbed potential flow can be used.

General analytical expressions for the apparent mass factors have been developed. The apparent mass factors of lower order have been plotted in Figures 1 and 2.

A polynomial has been developed which well represents the potential function of a vortex field. This polynomial has been used in conjunction with the general relations of Section II to compute the force distribution on an ellipsoid moving in a vortex field. The computations show that the forces due to the unsteady potential arising from the motion of the ellipsoid are of the same order of magnitude as the steady state forces. Hence forces due to the motion of a body cannot be determined from steady state analysis only.

The forces on a body moving thru a vortex field may be of more importance when the body is moving with the vortex velocity than when moving against the vortex velocity since in the former case all forces might act toward the vortex center, whereas in the latter case the forces at the ends and center of the body are oppositely directed.