STRESSES AT TWO-DIMENSIONAL CORNERS FOR VARIOUS FORCE DISTRIBUTIONS

Thesis by

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Abstract of Thesis

This paper deals with the stress distribution under plain strain in a corner of any angular magnitude, i.e., a plane with an angular incision or notch.

The Introduction contains a brief statement of the method employed by Dr. Theodor von Kármán in his exact treatment of a beam in bending (Aachen Abhandlungen, Heft 7, 1927).

In Part I a generalization of this method is outlined which is applicable to the corner for any force distribution over the straight boundaries. Solutions are found in the 3/4-plane for:

1. Concentrated load at any point of the straight boundaries.
2. Uniform distribution between the vertex and a point of the boundary.
3. Linear distribution in the same region.
4. Superposition of 2 and 3.

Certain stresses are determined and plotted and shown to be infinite at the vertex for partial loadings of the boundaries.

In Part II an alternate method is given to obtain a solution for case 1.

The discussion points out the very interesting paradox that stresses may be finite for certain continuous loadings, but become infinite if a portion of the load is removed.

Introduction

The knowledge of two-dimensional stress distributions is useful for the solution of many problems in mechanical and civil engineering, when the usual methods of strength of materials are not applicable or not sufficiently accurate. The stress distribution in the half-plane under the influence of a concentrated force acting on the straight boundary is of fundamental importance, and the solution of this problem is due to Bousinesq. Several extensions of his results have been made by Mitchell[1], especially by means of the so-called method of inversion, the only conformal transformation of bi-harmonic solutions which leaves the boundary stresses undisturbed. By superposition of Bousinesq's solutions the stress distribution in the half-plane is easily obtained for an arbitrary load distribution along the straight boundary.

The stress distribution in a plane strip has been discussed by several authors and special reference is made to a paper by Dr. Th. von Kármán[2] containing a general method for the determination of the stresses due to an arbitrary load along the boundaries. The paper is not yet available in the English language, therefore a brief description shall be given of von Kármán's formulation of the problem.

A two-dimensional elastic system[3] is in equilibrium if the stresses σ_x, σ_y and τ are the second partial derivatives of an arbitrary function F(xy):

$$\sigma_x = \frac{d^2F}{dy^2} \qquad \sigma_y = \frac{d^2F}{dx^2} \qquad \tau_{xy} = \frac{d^2F}{dx\,dy} \qquad (1)$$

1) A. E. H. Love, Elasticity, p. 216. 2) Aachen Abhandlungen 1927, Heft 7.
3) See Appendix I.

In order to be compatible with the stress-strain relations in accordance with Hooke's law, F must be a solution of the equation

$$\frac{d^4F}{dx^4} + \frac{2 \cdot d^4F}{dx^2\, dy^2} + \frac{d^4F}{dy^4} = 0$$

or in Gibb's notation

$$\nabla^4 F = 0 \qquad (2)$$

F, known as Airy's function, and its first partial derivatives can be interpreted in terms of the quantities usually employed and defined in "strength of materials", namely, shear and moment of the external forces. By integrating (1) along the boundary between points A and B it is seen that the differences $\left[\frac{dF}{dx}\right]_B - \left[\frac{dF}{dx}\right]_A$ and $\left[\frac{dF}{dy}\right]_B - \left[\frac{dF}{dy}\right]_A$ represent respectively the resultant Y- and X- components of the load on the boundary between A and B and, integrating once more, it is seen that the difference $F_B - F_A$ represents the increase in the bending moment, in going from A to B.

Hence, if all loads are assumed applied at the upper surface of the strip x-axis, see Fig. 1, the usual moment curve becomes identical with F(x,o) for the strip in question and the problem of stress distribution becomes simply the determination of F(x,y) for given boundary values.

If the loads are perpendicular to the upper surface and if the depth of the strip is h, the boundary conditions are

$$\frac{dF}{dy} = 0, \quad F = M(x) \text{ at the upper boundary, } y = 0$$
$$\frac{dF}{dy} = 0, \quad F = 0 \text{ at the lower boundary, } y = -h$$

It is now easy to determine a solution $F(x,y)$, which is periodical in x and satisfies the conditions:

$$\frac{dF}{dy} = 0, \quad F = a(m)\cos(mx) + b(m)\cos(my); \text{ when } y = 0$$
$$\frac{dF}{dy} = 0, \quad F = 0 \qquad \text{when } y = -h$$

Such solution is

$$F(xy) = a(m).\cos(mx) . f(y) + b(m).\sin(mx) . g(y)$$

where $f(o) = g(o) = 1$ and $f(-h) = g(-h) = 0$.

$f'(o) = g'(o) = 0$ and $f'(-h) = g'(-h) = 0$.

$f(y)$ and $g(y)$ are composed of terms of the form $e^{\pm my}$ and $ye^{\pm my}$ with certain given numerical coefficients that will make F satisfy $\nabla^4 F = 0$.

It is obvious that

$$F = \int_0^\infty a(m) . \cos(mx) . f(y)\, dm + \int_0^\infty b(m) . \sin(mx) . g(y)\, dm,$$

is the desired stress function if the moment curve is

$$M(x) = \int_0^\infty \left[a(m) . \cos(mx) + b(m) . \sin(mx)\right] \cdot dm \qquad (3)$$

In other words, if it is possible to represent the moment distribution M(x) as a Fourier's integral the solution of the stress problem is immediately obtained.

von Kármán and F. Seewald have applied this method to several important cases and obtained interesting results, among others the "corrected" relation between the bending moment and shear at a point of a beam and the curvature of the central line

$$K = \frac{M}{EI} + \left[\frac{6}{5G} - \frac{9\mu}{10E}\right]\frac{P}{A} \qquad (4)$$

where K denotes the curvature, E and G Young's modulus and the corresponding modulus in shear, μ Poisson's ratio, A area of cross section, I moment of inertia, M and P the bending moment and shear at the point in question. The second term corresponds to the influence of shear on the deflection.

A new and interesting conclusion of this theory is that the deflection of the central line of a long beam is wave-shaped outside the loaded region.

Part I.

Generalization: The method described above can be applied generally, not only to the half-plane, but to any angular portion (γ) of the plane with any distribution of both normal and tangential forces over the straight boundaries.

Introducing polar coordinates and stress function must satisfy the equation

$$\nabla^4 F(r,\theta) = \left[\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} + \frac{d^2}{r^2 d\theta^2}\right]^2 F = 0 \qquad (5)$$

with corresponding stresses, see Fig. 2:

$$\sigma_r = \frac{d^2F}{r^2 d\theta^2} + \frac{1}{r}\frac{dF}{dr} \qquad (6)$$

$$\sigma_\theta = \frac{d^2F}{dr^2} \qquad (7)$$

$$\tau = -\frac{d}{dr}\left(\frac{1}{r}\frac{dF}{d\theta}\right) \qquad (8)$$

The general solution of (5) is the real and/or imaginary parts of the complex function: $\varphi(z) = Af(z) + Bxg(z) + Cyh(z)$ (9)

where $z = x + iy = r(\cos\theta + i\sin\theta) = re^{i\theta}$.

A, B, and C are arbitrary complex constants. $F(z)$, $g(z)$ and $h(z)$ are arbitrary holomorphic functions.

For the present purpose we will employ the functions obtained by taking real or imaginary parts of

$$\varphi_n = \frac{1}{2} A_n \begin{pmatrix} x \\ \text{or} \\ y \end{pmatrix} \cdot (z^{im} \pm z^{-im}) \qquad (10)$$

where m is real and A_n are arbitrary constants to be determined by the boundary conditions.

Let $\xi = \log r$

and let $f(z)_r$ denote "Real Part" of f(z) or Rf(z)

and let $f(z)_i$ denote "Imaginary Part" of f(z) or If(z),

then we obtain as particular solutions of $\nabla^4 F = 0$:

$$\begin{aligned}
A_1\phi_1 &= \tfrac{1}{2} A_1 \cdot x(z^{im} + z^{-im})_r &&= +A_1 x \cos m\xi \cdot \cosh m\theta \\
A_2\phi_2 &= \tfrac{1}{2} A_2 \cdot x(z^{im} - z^{-im})_r &&= -A_2 x \cos m\xi \cdot \sinh m\theta \\
A_3\phi_3 &= \tfrac{1}{2} A_3 \cdot x(z^{im} + z^{-im})_i &&= -A_3 x \sin m\xi \cdot \sinh m\theta \\
A_4\phi_4 &= \tfrac{1}{2} A_4 \cdot x(z^{im} - z^{-im})_i &&= +A_4 x \sin m\xi \cdot \cosh m\theta \\
A_5\phi_5 &= \tfrac{1}{2} A_5 \cdot y(z^{im} + z^{-im})_r &&= +A_5 y \cos m\xi \cdot \cosh m\theta \\
A_6\phi_6 &= \tfrac{1}{2} A_6 \cdot y(z^{im} - z^{-im})_r &&= -A_6 y \cos m\xi \cdot \sinh m\theta \\
A_7\phi_7 &= \tfrac{1}{2} A_7 \cdot y(z^{im} + z^{-im})_i &&= -A_7 y \sin m\xi \cdot \sinh m\theta \\
A_8\phi_8 &= \tfrac{1}{2} A_8 \cdot y(z^{im} - z^{-im})_i &&= +A_8 y \sin m\xi \cdot \cosh m\theta
\end{aligned} \qquad (11)$$

It is convenient once for all to compute the first and second partial derivatives of the particular solutions (11):

$$
\begin{aligned}
A_1\frac{d\varphi_1}{dr} &= +\cos\theta\ \mathrm{Cosh}\ m\theta(\cos m\xi - m\sin m\xi)A_1 \\
A_2\frac{d\varphi_2}{dr} &= -\cos\theta\ \mathrm{Sinh}\ m\theta(\cos m\xi - m\mathrm{Sin}m\xi)A_2 \\
A_3\frac{d\varphi_3}{dr} &= -\cos\theta\ \mathrm{Sinh}\ m\theta(m\cos m\xi + \mathrm{Sin}m\xi)A_3 \\
A_4\frac{d\varphi_4}{dr} &= +\cos\theta\ \mathrm{Cosh}\ m\theta(m\cos m\xi + \mathrm{Sin}m\xi)A_4 \\
A_5\frac{d\varphi_5}{dr} &= +\sin\theta\ \mathrm{Cosh}\ m\theta(\cos m\xi - m\mathrm{Sin}m\xi)A_5 \\
A_6\frac{d\varphi_6}{dr} &= -\sin\theta\ \mathrm{Sinh}\ m\theta(\cos m\xi - m\mathrm{Sin}m\xi)A_6 \\
A_7\frac{d\varphi_7}{dr} &= -\sin\theta\ \mathrm{Sinh}\ m\theta(m\cos m\xi + \mathrm{Sin}m\xi)A_7 \\
A_8\frac{d\varphi_8}{dr} &= +\sin\theta\ \mathrm{Cosh}\ m\theta(m\cos m\xi + \mathrm{Sin}m\xi)A_8
\end{aligned}
\qquad (12)
$$

$$
\begin{aligned}
A_1\frac{d\varphi_1}{rd_\theta} &= +A_1(m\cos\theta\ \mathrm{Sinh}m\theta - \sin\theta\ \mathrm{Cosh}m\theta)\cos m\xi \\
A_2\frac{d\varphi_2}{rd_\theta} &= -A_2(m\cos\theta\ \mathrm{Cosh}m\theta - \sin\theta\ \mathrm{Sinh}m\theta)\cos m\xi \\
A_3\frac{d\varphi_3}{rd_\theta} &= -A_3(m\cos\theta\ \mathrm{Cosh}m\theta - \sin\theta\ \mathrm{Sinh}m\theta)\sin m\xi \\
A_4\frac{d\varphi_4}{rd_\theta} &= +A_4(m\cos\theta\ \mathrm{Sinh}m\theta - \sin\theta\ \mathrm{Cosh}m\theta)\sin m\xi \\
A_5\frac{d\varphi_5}{rd_\theta} &= +A_5(m\sin\theta\ \mathrm{Sinh}m\theta + \cos\theta\ \mathrm{Cosh}m\theta)\cos m\xi \\
A_6\frac{d\varphi_6}{rd_\theta} &= -A_6(m\sin\theta\ \mathrm{Cosh}m\theta + \cos\theta\ \mathrm{Sinh}m\theta)\cos m\xi \\
A_7\frac{d\varphi_7}{rd_\theta} &= -A_7(m\sin\theta\ \mathrm{Cosh}m\theta + \cos\theta\ \mathrm{Sinh}m\theta)\sin m\xi \\
A_8\frac{d\varphi_8}{rd_\theta} &= +A_8(m\sin\theta\ \mathrm{Sinh}m\theta + \cos\theta\ \mathrm{Cosh}m\theta)\sin m\xi
\end{aligned}
\qquad (13)
$$

$$A_1 \frac{d^2\varphi_1}{dr^2} = -A_1 \frac{\cos\theta \;\text{Cosh}\,m\theta}{r} \left(m^2\cos m\xi + m\sin m\xi\right)$$

$$A_2 \frac{d^2\varphi_2}{dr^2} = +A_2 \frac{\cos\theta \;\text{Sinh}\,m\theta}{r} \left(m^2\cos m\xi + m\sin m\xi\right)$$

$$A_3 \frac{d^2\varphi_3}{dr^2} = -A_3 \frac{\cos\theta \;\text{Sinh}\,m\theta}{r} \left(m\cos m\xi - m^2\sin m\xi\right)$$

$$A_4 \frac{d^2\varphi_4}{dr^2} = +A_4 \frac{\cos\theta \;\text{Cosh}\,m\theta}{r} \left(m\cos m\xi - m^2\sin m\xi\right) \qquad (14)$$

$$A_5 \frac{d^2\varphi_5}{dr^2} = -A_5 \frac{\sin\theta \;\text{Cosh}\,m\theta}{r} \left(m^2\cos m\xi + m\sin m\xi\right)$$

$$A_6 \frac{d^2\varphi_6}{dr^2} = +A_6 \frac{\sin\theta \;\text{Sinh}\,m\theta}{r} \left(m^2\cos m\xi + m\sin m\xi\right)$$

$$A_7 \frac{d^2\varphi_7}{dr^2} = -A_7 \frac{\sin\theta \;\text{Sinh}\,m\theta}{r} \left(m\cos m\xi - m^2\sin m\xi\right)$$

$$A_8 \frac{d^2\varphi_8}{dr^2} = +A_8 \frac{\sin\theta \;\text{Cosh}\,m\theta}{r} \left(m\cos m\xi - m^2\sin m\xi\right)$$

$$A_1 \frac{d^2\varphi_1}{r^2\, d\,\theta^2} = +A_1 \left[(m^2-1)\cos\theta \;\text{Cosh}\,m\theta - 2m\sin\theta \;\text{Sinh}\,m\theta\right] \cos m\xi$$

$$A_2 \frac{d^2\varphi_2}{r^2\, d\,\theta^2} = -A_2 \left[(m^2-1)\cos\theta \;\text{Sinh}\,m\theta - 2m\sin\theta \;\text{Cosh}\,m\theta\right] \cos m\xi$$

$$A_3 \frac{d^2\varphi_3}{r^2\, d\,\theta^2} = -A_3 \left[(m^2-1)\cos\theta \;\text{Sinh}\,m\theta - 2m\sin\theta \;\text{Cosh}\,m\theta\right] \sin m\xi$$

$$A_4 \frac{d^2 \varphi_4}{r^2\, d\,\theta^2} = + A_4 \left[(m^2 - 1)\cos\theta\ \mathrm{Cosh} m\theta - 2m\sin\theta\ \mathrm{Sinh}\ m\theta \right] \sin m\xi$$

$$A_5 \frac{d^2 \varphi_5}{r^2\, d\,\theta^2} = + A_5 \left[(m^2 - 1)\sin\theta\ \mathrm{Cosh}\ m\theta + 2m\cos\theta\ \mathrm{Sinh}\ m\theta \right] \cos m\xi \quad (15)$$

$$A_6 \frac{d^2 \varphi_6}{r^2\, d\,\theta^2} = -A_6 \left[(m^2 - 1)\sin\theta\ \mathrm{Sinh}\ m\theta + 2m\cos\theta\ \mathrm{Cosh}\ m\theta \right] \cos m\xi$$

$$A_7 \frac{d^2 \varphi_7}{r^2\, d\,\theta^2} = -A_7 \left[(m^2 - 1)\ \sin\theta\ \mathrm{Sinh}\ m\theta + 2m\cos\theta\ \mathrm{Cosh}\ m\theta \right] \sin m\xi$$

$$A_8 \frac{d^2 \varphi_8}{r^2\, d\,\theta^2} = + A_8 \left[(m^2 - 1)\sin\theta\ \mathrm{Cosh}\ m\theta + 2m\cos\theta\ \mathrm{Sinh}\ m\theta \right] \sin m\xi$$

$$-A_1 \frac{d}{dr}\left(\frac{d\varphi_1}{r d\theta}\right) = + A_1 \frac{m}{r} (m\cos\theta\ \mathrm{Sinh}\ m\theta - \sin\theta\ \mathrm{Cosh}\ m\theta)\ \sin m\xi$$

$$-A_2 \frac{d}{dr}\left(\frac{d\varphi_2}{r d\theta}\right) = -A_2 \frac{m}{r} (m\cos\theta\ \mathrm{Cosh}\ m\theta - \sin\theta\ \mathrm{Sinh}\ m\theta)\ \sin m\xi$$

$$-A_3 \frac{d}{dr}\left(\frac{d\varphi_3}{r d\theta}\right) = + A_3 \frac{m}{r} (m\cos\theta\ \mathrm{Cosh}\ m\theta - \sin\theta\ \mathrm{Sinh}\ m\theta)\ \cos m\xi$$

$$-A_4 \frac{d}{dr}\left(\frac{d\varphi_4}{r d\theta}\right) = -A_4 \frac{m}{r} (m\cos\theta\ \mathrm{Sinh}\ m\theta - \sin\theta\ \mathrm{Cosh}\ m\theta)\ \cos m\xi \quad (16)$$

$$-A_5 \frac{d}{dr}\left(\frac{d\varphi_5}{r d\theta}\right) = + A_5 \frac{m}{r} (m\sin\theta\ \mathrm{Sinh}\ m\theta + \cos\theta\ \mathrm{Cosh}\ m\theta)\ \sin m\xi$$

$$-A_6 \frac{d}{dr}\left(\frac{d\varphi_6}{r d\theta}\right) = -A_6 \frac{m}{r} (m\sin\theta\ \mathrm{Cosh}\ m\theta + \cos\theta\ \mathrm{Sinh}\ m\theta)\ \sin m\xi$$

$$-A_7\frac{d}{dr}\left(\frac{d\varphi_7}{rd\theta}\right) = + A_7 \frac{m}{r} (m\sin\theta \operatorname{Cosh} m\theta + \cos\theta \operatorname{Sinh} m\theta) \cos m\xi$$

$$-A_8\frac{d}{dr}\left(\frac{d\varphi_8}{rd\theta}\right) = -A_8 \frac{m}{r} (m\sin\theta \operatorname{Sinh} m\theta + \operatorname{Cos}\theta \operatorname{Cosh} m\theta) \cos m\xi$$

The coefficients A_n are now defined as functions of the parameter m. The stress function may then be expressed as

$$F = \sum_{n=1}^{n=8}\int_0^\infty A_n(m)\,\varphi_n\,dm = \int_0^\infty\left[A_1\varphi_1 + A_2\cdot\varphi_2 + \cdots + A_8\varphi_8\right]dm \quad (17)$$

from which the stresses are obtained by (6) etc.:

$$\sigma_r = +\sum_{n=1}^{n=8}\int_0^\infty A_n\left[\frac{d^2\varphi_n}{r^2 d\theta^2} + \frac{1}{r}\frac{d\varphi_n}{dr}\right] dm \quad (18)$$

$$\sigma_\theta = +\sum_{n=1}^{n=8}\int_0^\infty A_n \cdot \frac{d^2\varphi_n}{dr^2}\, dm \quad (19)$$

$$\tau = -\sum_{n=1}^{n=8}\int_0^\infty A_n \frac{d}{dr}\left(\frac{1}{r}\frac{d\varphi_n}{d\theta}\right) dm \quad (20)$$

Two typical cases of boundary loads must be considered:

a) The load may be continuous (not necessarily uniform) over portions of the straight boundaries.

Let it be assumed that the load is distributed over finite portions of the boundaries or, if the load extends to infinity, that the load per unit length becomes zero at infinity. Let the boundaries be the lines $\theta = 0$ and $\theta = \gamma$, and let the normal loads be respectively p_o and p_γ and the tangential loads be q_o and q_γ. These quantities are functions of r, measured from the intersection of the boundaries, or functions of $\xi = \log r$. Using Fourier's double integrals we can write $p_o(\xi)$ in the following form

$$p_o(\xi) = \frac{1}{\pi}\int_0^\infty \left[\cos m\xi \int_{\xi_1}^{\xi_2} p_o(\alpha)\cos m\alpha \, d\alpha + \sin m\xi \int_{\xi_1}^{\xi_2} p_o(\alpha)\sin m\alpha \, d\alpha\right] dm \quad \text{--- (21)*}$$

where the load $p_o(\xi)$ is assumed to exist only between the points ξ_1 and ξ_2. Similar expressions are obtained for p_γ, q_o, and q_γ.

Therefore, the boundary conditions are:

$$(\sigma_\theta)_{\theta=0} = p_o(\xi) \quad \text{expressed as in (21)} \qquad (22)$$

$$(\tau)_{\theta=0} = q_o \qquad (23)$$

$$(\sigma_\theta)_{\theta=\gamma} = p_\gamma \qquad (24)$$

$$(\tau)_{\theta=\gamma} = q_\gamma \qquad (25)$$

where σ and τ are given by (19) and (20).

By now equating coefficients of $\cos m\xi$ and $\sin m\xi$ in the above equations, eight other equations are obtained which determine the eight functions A(m) and consequently the stress function F by substituting $A_n(m)$ in (17).

b) In the case of concentrated loads the boundary conditions are obtained in the following manner:

It was already shown that the differences $\left[\frac{dF}{dx}\right]_B - \left[\frac{dF}{dx}\right]_A$ and $\left[\frac{dF}{dy}\right]_B - \left[\frac{dF}{dy}\right]_A$ represent respectively the Y and X-components of all the the boundary forces between A and B. Therefore, in the case of concen-

*See Fourier Series by W. E. Byerly.

trated forces, these differences become discontinuous (similar to the shear polygon for concentrated forces) and the jump in these values at a certain point is equal to the respective components of the concentrated force at that point. Consider, for instance, unit normal force acting at point $r = a$ of the boundary $\theta = 0$;

Obviously

$$\left[\frac{dF}{dr}\right]_{a+\varepsilon} - \left[\frac{dF}{dr}\right]_{a-\varepsilon} = 1$$

where ε is a small positive quantity.

If there is no continuously distributed load on the boundary $\theta = 0$, then by (22)

$$(\sigma_\theta)_{\theta=0} = \left[\frac{d^2F}{dr^2}\right]_{\theta=0} = 0$$

Hence $\frac{dF}{dr}$ = constant on either side of $r = a$. These conditions can all be satisfied by writing

$$\frac{dF}{dr} = \frac{1}{\pi}\int_0^\infty \frac{\sin(m \log \frac{r}{a})}{m}\, dm \qquad (26)^{1)}$$

This integral has the value $-\frac{1}{2}$ for $\xi = \log \frac{r}{a} < 0$ or $r < a$

$+\frac{1}{2}$ for $\xi > 0$ or $r > a$

The integral:

$$\frac{1}{\pi}\int_0^\infty \frac{n \cos m\xi - m \sin m\xi}{m^2 + n^2} = \begin{cases} e^{n\xi} & \text{when } \xi < 0 \\ \frac{1}{2} & \text{" } \xi = 0 \\ 0 & \text{" } \xi > 0 \end{cases} \qquad (27)^{2)}$$

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1) See Peirce, Short Tables of Integrals, #484.
2) Riemann-Weber, Differentialgleichungen der Physik I, page 157.

with its ramifications

is often very useful in expressing boundary functions on Fourier integral form.

Application:

1. As the first application of the foregoing, the stress function F corresponding to a unit normal traction at point r = a of boundary $\theta = 0$ will be found for a reentrant right corner, i.e. $\gamma = \frac{3\pi}{2}$, see Fig. 3.

The boundary conditions are:

$$\theta = 0 \begin{cases} \frac{dF}{dr} = 0 & \text{when } \xi = \log\left(\frac{r}{a}\right) < 0 \\ \frac{dF}{dr} = 1 & \text{when } \xi > 0 \,; \\ \frac{dF}{d\theta} = 0 & \end{cases} \qquad \theta = \gamma \begin{cases} F = 0 \\ \frac{dF}{d\theta} = 0 \end{cases}$$

Let $F = F_o + F_1$

F_o is determined by the boundary conditions

$$\left(\frac{dF_o}{dr}\right)_{\theta = 0} = \begin{cases} -\frac{1}{2} \text{ for } \xi < 0 \\ +\frac{1}{2} \quad " \quad \xi > 0 \end{cases} \qquad \text{Hence, } \frac{dF_o}{dr} = \frac{1}{\pi}\int_0^\infty \frac{\sin m\xi}{m} dm, \text{ by (26)}$$

$$\left(\frac{dF_o}{d\theta}\right)_{\theta = 0} = 0$$

$$(F_o)_{\theta = \gamma} = 0$$

$$\left(\frac{dF_o}{d\theta}\right)_{\theta = \gamma} = 0$$

In order to remove the concentrated force introduced at the origin (due to the discontinuity in the first derivative) F_1 is determined so that

$$\left(\frac{dF_1}{dr}\right)_{\theta = 0} = \frac{1}{2} \quad \text{for} \quad -\infty < \xi < +\infty$$

$$\left(\frac{dF_1}{d\theta}\right)_{\theta = 0} = 0$$

$$(F_1)_{\theta = \gamma} = 0$$

$$\left(\frac{dF_1}{d\theta}\right)_{\theta = \gamma} = 0$$

F_1 is easily found by employing the simple solution to $\nabla^4 F_1 = 0$:

$$F_1 = A\theta r\cos\theta + B\theta r\sin\theta + Cr\cos\theta + Dr\sin\theta \qquad (28a)$$

Hence, by the boundary conditions for F_1:

$$\left(\frac{dF_1}{dr}\right)_{\theta = 0} = C = \frac{1}{2}$$

$$\left(\frac{dF_1}{d\theta}\right)_{\theta = 0} = Ar + Dr = 0$$

$$(F_1)_{\theta = \frac{3\pi}{2}} = -\frac{3\pi}{2} Br - Dr = 0$$

$$\left(\frac{dF_1}{d\theta}\right)_{\theta = \frac{3\pi}{2}} = \frac{3\pi}{2} Ar - Br + Cr = 0$$

Solving these and substituting in (28a) we obtain

$$F_1 = \frac{3\pi}{4-9\pi^2}\,\theta r\cos\theta + \frac{2}{4-9\pi^2}\,\theta r\sin\theta \qquad (28)$$

The first degree terms have been omitted as they contribute no stress.

In order to determine F we employ equation (17) and formulate the given boundary conditions by differentiating the functions given by (11) and obtain:

$$\left(\frac{dF_o}{dr}\right)_{\theta=0} = \int_0^\infty \left[A_1(\cos m\xi - m\sin m\xi) + A_4(m\cos m\xi + \sin m\xi)\right] dm = \frac{1}{\pi}\int_0^\infty \frac{\sin m\xi}{m}\,dm$$

$$\left(\frac{dF_o}{rd\theta}\right)_{\theta=0} = \int_0^\infty \left[-m A_2\cos m\xi - m A_3\sin m\xi + A_5\cos m\xi + A_8\sin m\xi\right] dm = 0$$

$$(F_o)_{\theta=\gamma} = \int_0^\infty \left[A_5\cosh\gamma m\cos m\xi + A_6\sinh\gamma m\cos m\xi + A_7\sinh\gamma m\sin m\xi - A_8\cosh\gamma m\sin m\xi\right]\cdot dm = 0$$

$$\left(\frac{dF_o}{rd\theta}\right)_{\theta=\gamma} = \int_0^\infty \left[A_1\cosh\gamma m\cos m\xi - A_2\sinh\gamma m\cos m\xi - A_3\sinh\gamma m\sin m\xi + A_4\cosh\gamma m\sin m\xi - mA_5\sinh\gamma m\cos m\xi + mA_6\cosh\gamma m\cos m\xi + mA_7\cosh\gamma m\sin m\xi - mA_8\sinh\gamma m\sin m\xi\right]\cdot dm = 0$$

where $\xi = \log\left(\frac{r}{a}\right)$ and $\gamma = \frac{3\pi}{2}$

By equating coefficients of $\cos m\xi$ and $\sin m\xi$:

$$A_1 + mA_4 = 0$$

$$-mA_1 + A_4 = \frac{1}{m\pi}$$

$$-mA_2 + A_5 = 0$$

$$-mA_3 + A_8 = 0$$

$$A_5\cosh\gamma m - A_6\sinh\gamma m = 0$$

$$A_7\sinh\gamma m - A_8\cosh\gamma m = 0$$

$$A_1 \text{ Cosh}\gamma m - A_2 \text{ Sinh}\gamma m - mA_5 \text{ Sinh}\gamma m + mA_6 \text{ Cosh}\gamma m = 0$$

$$-A_3 \text{ Sinh}\gamma m + A_4 \text{ Cosh}\gamma m + mA_7 \text{ Cosh}\gamma m - mA_8 \text{ Sinh}\gamma m = 0$$

and solving for A_n:

$$A_1 = -\frac{1}{\pi}\frac{1}{1+m^2}$$

$$A_2 = +\frac{1}{\pi}\frac{\text{Cosh}\gamma m \ \text{Sinh}\gamma m}{(1+m^2)\ (m^2 - \text{Sinh}^2\gamma m)}$$

$$A_3 = -\frac{1}{\pi}\frac{\text{Sinh}\gamma m \ \text{Cosh}\gamma m}{m(1+m^2)\ (m^2 - \text{Sinh}^2\gamma m)}$$

$$A_4 = +\frac{1}{\pi}\frac{1}{(1+m^2)m}$$

$$A_5 = +\frac{1}{\pi}\frac{m\ \text{Cosh}\gamma m \ \text{Sinh}\gamma m}{(1+m^2)\ (m^2 - \text{Sinh}^2\gamma m)}$$

$$A_6 = +\frac{1}{\pi}\frac{m\ \text{Cosh}^2\gamma m}{(1+m^2)\ (m^2 - \text{Sinh}^2\gamma m)}$$

$$A_7 = -\frac{1}{\pi}\frac{\text{Cosh}^2\gamma m}{(1+m^2)\ (m^2 - \text{Sinh}^2\gamma m)}$$

$$A_8 = -\frac{1}{\pi}\frac{\text{Sinh}\gamma m \ \text{Cosh}\gamma m}{(1+m^2)\ (m^2 - \text{Sinh}^2\gamma m)}$$

The stress function is then found simply by substituting the A's back into equation (17) and the stresses are found from equations (18) (19) (20).

The stress function in the 3/4-plane for unit tensile force concentrated at point $(r,\theta) = (a,0)$ after some reduction becomes:

$$F = F_o + F_1 =$$

$$F = \frac{1}{\pi}\int_0^\infty \frac{\left[xm^2 \text{Cosh}m\theta - (x\text{Sinh}\gamma m + ym\text{Cosh}\gamma m)\ \text{Sinh}m(\gamma - \theta)\right]\cdot(m\cos m\xi - \sin m\xi)}{m(1+m^2)\ (\text{Sinh}^2\gamma m - m^2)}\,dm$$

$$+ \frac{3\pi}{4 - 9\pi^2}\,\theta r\cos\theta + \frac{2}{4 - 9\pi^2}\,\theta r\sin\theta \qquad (29)$$

where $\gamma = \frac{3\pi}{2}$; $\xi = \log \frac{r}{a}$; $x = r\cos\theta$; $y = r\sin\theta$

The integral F may be evaluated by the method of contour integration in the complex plane*, making use of Cauchy's theorem of residues as outlined below.

Evaluation by Contour Integration

Cauchy's theorem of residues

$$\oint \Psi(m)\cdot dm = 2\pi i \sum \text{Residues}$$

enables us to compute the definite integral in (29)

$$m \text{ is complex} = \alpha + i\beta$$

The residues are found by integrating around each pole in the positive direction.

F_o may be written as

$$F_o = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \frac{m.f(m).\ \cos m\xi\ dm}{m(1+m^2)\ (\text{Sinh}^2\gamma m - m^2)} - \frac{1}{2\pi}\int_{-\infty}^{+\infty} \frac{f(m).\ \sin m\xi\ dm}{m(1+m^2)(\text{Sinh}^2\gamma m - m^2)} = M_1 - M_2 \qquad (30)$$

where the meaning of f(m) is evident by comparing with equation (29).

Consider the complex integrals:

$$G + iH = \frac{1}{2\pi}\int_{(S)} \frac{e^{im\xi}.\ f(m)\ dm}{m(1+m^2)\ (\text{Sinh}^2\gamma m - m^2)} \qquad (31)$$

and

$$K + iL = \frac{1}{2\pi} \frac{m.e^{im\xi}.\ f(m)\ dm}{m(1+m^2)\ (\text{Sinh}^2\gamma m - m^2)} \qquad (32)$$

taken along the contours S consisting of the real axis and the infinite half circle above or below the real axis, see Fig. 7.

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* See f. expl. MacRobert, Functions of a Complex Variable.

If $\xi = \log \frac{r}{a} < 0$ the half circle below the real axis must be chosen because in this case the integral along this part of the path converges to zero as $|m| \to \infty$. For the same reason the upper infinite half circle is chosen when $\xi > 0$, i.e. $r > a$. Consider for example the lower half-plane:

$$m = \alpha - i\beta \quad \text{where } \beta \text{ is real and positive}$$

therefore, if $0 > \xi = -c$ where c is real and positive

$$e^{im\xi} = e^{i(\alpha - i\beta)(-c)} = e^{-i\alpha c} \cdot e^{-c\beta} = \frac{\cos\alpha c - i \sin\alpha c}{e^{c\beta}}$$

which becomes zero in the limit, $\beta \to \infty$

The singularities of the integrand are all simple poles and are the roots of:

$$\mathrm{Sinh}^2 \gamma m - m^2 = 0 \tag{33}$$

or if m is pure imaginary = $i\beta$

$$\beta = \pm \sin(\gamma\beta) \tag{34}$$

The complex roots may be found with very good approximation (except n = 2) by:

$$m = \alpha_n + i\beta_n = \pm\frac{1}{\gamma} \log\left(\frac{\pi(2n+1) - \varepsilon}{\gamma}\right) \pm i\,\frac{\pi(2n+1) - \varepsilon}{2} \tag{35}$$

where

$$\varepsilon = \frac{4 \log\left(\frac{\pi(2n+1)}{\gamma}\right)}{(2n+1)} \tag{36}$$

and n = 2, 3, 4, 5 ....

$$\gamma = \frac{3\pi}{2}$$

For large values of n:

$$m = \alpha_n + i\beta_n = \pm \frac{2}{3\pi} \log \frac{4n+2}{3} \pm i\,\frac{2n+1}{3} \tag{37}$$

The lower roots are:

α:	β:
0	0
0	$\pm i.539$
0	$\pm i.909$
0	$\pm i$
$\pm .231$	± 1.629, corresponds to n = 2 in (35)

$\alpha_n = 3$ etc., use equations (35) and (36).

It will be seen that for $|m| \leq 1$ the roots are located on the imaginary axis, i.e. pure imaginary. For $|m| > 1$ all the roots are located near the imaginary axis.

Now returning to (31) and 32), integrating in the positive directions and observing (30) we find for: $\xi > 0$, i.e. $r > a$:

$$M_2 = H = \mathcal{I}\frac{1}{2\pi}\int_{(S_1)} \frac{e^{im\xi} \cdot f(m)\ dm}{m(1 + m^2)(\mathrm{Sinh}^2\gamma m - m^2)} = \left[i \sum \text{Residues}\right]_i \equiv \left[+\sum R_2\right]_r \qquad (38)$$

$$M_1 = K = \mathcal{R}\frac{1}{2\pi}\int_{(S_1)} \frac{m e^{im\xi} \cdot f(m)\ dm}{m(1 + m^2)(\mathrm{Sinh}^2\gamma m - m^2)} = \left[i \sum R_1\right]_r \equiv \left[-\sum R_1\right]_i \qquad (39)$$

Due to m = 0 being a simple pole of the integrand in (38) the contour must be indented and half the residue at the origin must be included in $\sum R_2$. In (39) m = 0 is not a pole so no indentation is necessary.

It should be noted that $R_1 = mR_2$ which facilitates the computations greatly.

For $\xi < 0$, i.e. $r < a$, and now integrating in negative direction:

$$M(-2) = H = \left[- i \sum \text{Residues} \right]_i = \left[- \sum R_{(-2)} \right]_r \qquad (40)$$

$$M(-1) = K = \left[- i \sum \text{Residues} \right]_r = \left[+ \sum R_{(-1)} \right]_i \qquad (41)$$

The same remarks as above hold here and $R(-1) = mR_{(-2)}$ where the subscript indicates integration in the negative half-plane.

The convergence of the power series is rapid and even for values of r near a, only a few of the poles need be considered outside the unit circle $|m| = 1$. For very large $|\xi|$ only the poles inside the circle $|m| = 1$ contribute.

The stress functions evaluated in this manner valid for small and large values of $\frac{r}{a}$ become: $F = F_0 + F_1 =$

$$F = a \left[.101 \left(\frac{r}{a}\right)^{1.545} \cdot g(\theta) + .125 \left(\frac{r}{a}\right)^{1.909} \cdot h(\theta) \right] \quad \text{When } \frac{r}{a} \ll 1 \qquad (42)$$

$$F = a \left[-.340 \left(\frac{r}{a}\right)^{.455} \cdot g(\theta) - 2.629 \left(\frac{r}{a}\right)^{.091} \cdot h(\theta) + \frac{r}{a} \frac{6\pi}{4 - 9\pi^2} \theta \cos\theta + \frac{r}{a} \frac{4}{4 - 9\pi^2} \theta \sin\theta - \cos^2\theta \right] \quad \text{when } \frac{r}{a} \gg 1 \qquad (43)$$

where

$$g(\theta) = \sin .545\,\theta \cdot \cos\theta - .545 \cos .545\,\theta \cdot \sin\theta - .839 \sin .545\,\theta \cdot \sin\theta; \qquad (44)$$

$$h(\theta) = \sin .909\,\theta \cdot \cos\theta - .909 \cos .909\,\theta \cdot \sin\theta + .416 \sin .909\,\theta \cdot \sin\theta; \qquad (45)$$

It is interesting to notice that the terms in (42) including those of higher order are the Biharmonics (solutions of $\nabla^4 F = 0$) which correspond to boundaries free of forces. This can also be seen by writing

$$F = Ar^n \cos n\theta + Br^n \cos(n-2)\theta + Cr^n \sin n\theta + Dr^n \sin(n-2)\theta \qquad (46)$$

where n may be complex and finding the values of n which give possible solutions in case F and $\frac{dF}{d\theta}$ are both to be zero along both boundaries. The determinant of the four boundary equations gives exactly the roots corresponding to the exponents in (42). It is, however, only possible to determine the ratios $\frac{B}{A}$ etc., in this way. The absolute values of the constants naturally depend upon the location and magnitude of the concentrated forces, and these can only be specified by the use of Fourier integrals as above.

The writer is informed that Dr. H. M. Westergaard, Professor of Theoretical and Applied Mechanics, University of Illinois, in a communication to Dr. Th. von Kármán, has pointed out the existence of these solutions, presumably obtained in the manner just described. Dr. Westergaard applies these solutions to the case of a triangle representing a masonry dam, in order to investigate the character of stress distributions near the base.

Check on boundary conditions:

By (42) and (43) it is easily seen that

$$\left.\begin{aligned} F &= 0 \\ \frac{dF}{d\theta} &= 0 \end{aligned}\right\} \quad \begin{aligned} &\text{when } \theta = 0 \\ &\text{and } \theta = \frac{3\pi}{2} \end{aligned}$$

are satisfied.

In order to investigate if all boundary forces are in equilibrium we will determine the stresses on a circle of large radius r = R, see Fig. 4. By differentiating (43) and only taking such terms that contribute:

$$\sigma_r = -\frac{6\pi}{4-9\pi^2}\cdot\frac{2\sin\theta}{R} + \frac{4}{4-9\pi^2}\cdot\frac{2\cos\theta}{R}$$

$$\tau = \frac{a\sin 2\theta}{R^2}$$

Projecting on x-axis and letting $R \to \infty$:

$$\Sigma X = \int_0^{\frac{3\pi}{2}} \sigma_r \cos\theta r d\theta = \left[\frac{3\pi\cos 2\theta}{4-9\pi^2} + \frac{4}{4-9\pi^2}\left(\theta + \frac{1}{2}\sin 2\theta\right)\right]_0^{\frac{3\pi}{2}} = 0$$

Projecting on y-axis:

$$\Sigma Y = \int_0^{\frac{3\pi}{2}} \sigma_r \sin\theta r d\theta = \left[-\frac{6\pi}{4-9\pi^2}\left(\theta - \frac{1}{2}\sin 2\theta\right) - \frac{2}{4-9\pi^2}\cos 2\theta\right]_0^{\frac{3\pi}{2}} = 1.$$

Moments about the origin:

$$M_o = \int_0^{\frac{3\pi}{2}} \tau r^2 d\theta = -\left[\frac{a\cos 2\theta}{2}\right]_0^{\frac{3\pi}{2}} = +a.$$

It will be seen that all equilibrium conditions are satisfied, the only other force being unit tensile force at a. The stresses for large and small radii are now obtained from (42) and (43) by application of formulae (6), (7), and (8). It will be noticed that stresses become infinite at the origin.

Evaluation of Stresses

If it is only desired to know the stresses at certain points or the distribution on a line, it is often more simple to specialize F in (29) before evaluating the integral.

For example, to obtain the distribution of normal stress on the Y-axis, place $\theta = \frac{\pi}{2}$ in (29):

$$F_{\theta=\frac{\pi}{2}} = \frac{r}{\pi}\int_0^\infty \frac{\cosh\gamma m \;\sinh\pi m}{(1+m^2)(\sinh^2\gamma m - m^2)}\,(m\cos m\xi - \sin m\xi)\,dm + F_1 \qquad (47)$$

$$\sigma_{\theta=\frac{\pi}{2}} = \frac{d^2F}{dr^2} = \frac{1}{\pi r}\int_0^\infty \frac{m\cosh\gamma m \;\sinh\pi m}{\sinh^2\gamma m - m^2}\cos m\xi \; dm \qquad (48)$$

Notice that F_1 contributes nothing to $\sigma_{\theta=\frac{\pi}{2}}$.

By (42) and (43) we obtain the correct stresses for small and large radii which usually are the most interesting, so that only a few additional terms of higher powers in r would be needed to give the stresses in the region r = a. These can be obtained by the method of residues applied to (48) as explained above. However, if only a plot of the stresses is desired, it is simpler to evaluate (48) for values of r in the neighborhood of a, by more primitive methods. The simplest of these would perhaps be to plot the integrand for various values of m and r and apply Simpson's rule. This method need only be done for $m \leq 1$ because for larger m, (48) can be further simplified by placing $\cosh\gamma m = \sinh m\gamma = e^{\frac{3\pi m}{2}}$ and ignoring m^2 in the denominator.

The integral then becomes $\frac{1}{\pi r}\int_0^\infty m e^{\frac{-\pi m}{2}} \cos m\xi \; dm$.

The normal stresses on the line $\theta = \frac{\pi}{2}$ for large and small $\frac{r}{a}$, are easily found by (42) and (43):

$$\sigma_{\theta=\frac{\pi}{2}} = \frac{d^2F}{dr^2} = \frac{1}{a}\left[-.084\left(\frac{r}{a}\right)^{-.455} + .062\left(\frac{r}{a}\right)^{-.091}\right]; \quad r \ll a \qquad (49)$$

$$\sigma_{\theta=\frac{\pi}{2}} = \frac{d^2F}{dr^2} = \frac{1}{a}\left[-.084\left(\frac{r}{a}\right)^{+.455} + .062\left(\frac{r}{a}\right)^{+.091}\right]\left(\frac{r}{a}\right)^{-2}; \quad r \gg a \qquad (50)$$

The stresses $\sigma_{\theta = \frac{\pi}{2}}$ obtained by evaluating (48), (49) and (50) are plotted in Fig. 5. The corresponding stresses for the half-plane, $\gamma = \pi$, are also shown. It is interesting to notice the effect of the additional quarter plane by comparing the two stress distributions, I and II. Bousinesq's solution for a unit traction at x = a on the x-axis gives

$$\sigma_x = \frac{2}{\pi a} \frac{\frac{y}{a}}{\left[1 + \left(\frac{y}{a}\right)^2\right]^2}$$

along the Y-axis, which is plotted as the curve II of Fig. 5.

If the several concentrated forces are applied on the boundary $\theta = 0$, the stress functions and stresses are obtained by superposing the functions computed for the various values of a. For a concentrated force acting on the boundary $\theta = \frac{3\pi}{2}$, the stresses are obtained from the present formulae by substituting $\frac{3\pi}{2} - \theta$ for θ, maintaining the same coordinate system.

It is of interest to note that the function $g(\theta)$ given in (44) is symmetric and $h(\theta)$ given in (45) is antisymmetric about the bisector of the angle. This holds also for the corresponding functions for higher powers in $\frac{r}{a}$. This would mean that concentrated forces of equal magnitude located on each boundary at equal distance from the vertex would have stress functions (valid near the vertex):

$$F = .202\, a \left(\frac{r}{a}\right)^{1.545} \cdot g(\theta) \qquad (42a)$$

if the unit forces are both tension

and

$$F = .250\, a \left(\frac{r}{a}\right)^{1.909} \cdot h(\theta) \qquad (43a)$$

with traction on the boundary $\theta = 0$ and pressure on $\theta = \frac{3\pi}{2}$. This shows that the stress at the vertex is always infinite for partial loading of the boundaries. It is now also clear why there should be two terms producing infinite stresses, because if there were only one term it would have to be symmetric so that equation (43a) would have to be indentically zero.

A discussion of the stresses will be given later.

2. In the second application of the general method the stress function will be determined for a uniform traction p on the region $r = 0 \rightarrow a$ of the boundary $\theta = 0$ of the 3/4-plane, in Fig. 8.

Let $F = F_0 + F_1$

We will determine a stress function F_0 which satisfies the boundary conditions:

$$\left(\frac{d^2F_0}{dr^2}\right)_{\theta = 0} = p \text{ when } -\infty \leq \xi < 0$$

$$\left(\frac{dF_0}{rd\theta}\right)_{\theta = 0} = 0$$

$$\left(F_0\right)_{\theta = \gamma} = 0$$

$$\left(\frac{dF_0}{rd\theta}\right)_{\theta = \gamma} = 0$$

Where $\xi = \log\left(\frac{r}{a}\right)$ and $\gamma = \frac{3\pi}{2}$ as before

It will be seen that discontinuities in F and its first derivative exist at the origin so that concentrated forces will have been introduced.

These must be removed by superposing a function F_1 which will be determined later.

We now express F_o as in (17) by use of the particular solutions given in (11) and from the boundary equations by means of (12) to (15):

$$\left(\frac{d^2F_o}{dr^2}\right)_{\theta = 0} = \frac{1}{r}\int_0^\infty \left[-A_1(m^2 \cos m\xi + m \sin m\xi) + A_4 (m \cos m\xi - m^2 \sin m\xi)\right] dm$$

$$= p \text{ when } \xi < 0 \qquad\qquad ------ (a)$$

$$= 0 \text{ when } \xi > 0$$

We now express $pr = ap\left(\frac{r}{a}\right) = ap\,e^{\xi}$ on Fourier integral form by Dirichlet's integral given in (27), taking $n = 1$ in this application.

Hence we obtain

$$\frac{a\,p}{\pi}\int_0^\infty \frac{\cos m\xi - m \sin m\xi}{1 + m^2}\, dm = \begin{matrix} a\,p\,e^{\xi} & \text{when } \xi < 0 \\ 0 & \text{when } \xi > 0 \end{matrix} \qquad (27a)$$

Equation (a) then becomes

$$-A_1(m^2 \cos m\xi = m \sin m\xi) + A_4 (m \cos m\xi - m^2 \sin m\xi) = \frac{ap}{\pi}\,\frac{\cos m\xi - m \sin m\xi}{1 + m^2}$$

and by equating coefficients of $\cos m\xi$ and $\sin m\xi$, respectively, we obtain

$$- m^2 A_1 + m A_4 = + \frac{1}{1 + m^2}\,\frac{a\,p}{\pi}$$

$$- m A_1 - m^2 A_4 = - \frac{m}{1 + m^2}\,\frac{a\,p}{\pi}$$

Solving these we find

$$A_1 = 0$$

$$A_4 = + \frac{1}{m(1 + m^2)}\,\frac{a\,p}{\pi}$$

The remaining boundary conditions give:

$$\left(\frac{dF_o}{rd\theta}\right)_{\theta = 0} : - m\, A_2 \text{Cos}\, m\xi - m\, A_3 \,\text{Sin}\, m\xi + A_5\, \text{Cos} m\xi + A_8\, \text{Sin}\, m\xi = 0$$

$$\left(F_o\right)_{\theta = \gamma} : - A_5 \cos m\xi\, \text{Cosh}\gamma m + A_6 \cos m\xi \cdot \text{Sinh}\gamma m + A_7 \sin m\xi\, \text{Sinh}\gamma m$$
$$- A_8 \sin m\xi\, \text{Cosh}\gamma m = 0$$

$$\left(\frac{dF_o}{rd\theta}\right)_{\theta = \gamma} : A_1 \cos m\xi\, \text{Cosh}\gamma m - A_2 \cos m\xi\, \text{Sinh}\gamma m - A_3 \sin m\xi\, \text{Sinh}\gamma m -$$
$$+ A_4 \sin m\xi\, \text{Cosh}\gamma m - m\, A_5 \cos m\xi\, \text{Sinh}\gamma m + m\, A_6 \cos m\xi\, \text{Cosh}\gamma m$$
$$+ m\, A_7 \sin m\xi\, \text{Cosh}\gamma m - m A_8 \sin m\xi\, \text{Sinh}\gamma m = 0$$

From these we obtain, by equating to zero coefficients of $\cos m\xi$ and $\sin m\xi$:

$$- m\, A_2 + A_5 = 0$$

$$- m\, A_3 = A_8 = 0$$

$$A_5\, \text{Cosh}\gamma m - A_6\, \text{Sinh}\gamma m = 0$$

$$A_7\, \text{Sinh}\gamma m - A_8\, \text{Cosh}\gamma m = 0$$

$$A_1\, \text{Cosh}\gamma m - A_2\, \text{Sinh}\gamma m - m\, A_5\, \text{Sinh}\gamma m + m\, A_6\, \text{Cosh}\gamma m = 0$$

$$-A_3\, \text{Sinh}\gamma m + A_4\, \text{Cosh}\gamma m + m\, A_7\, \text{Cosh}\gamma m - m\, A_8\, \text{Sinh}\gamma m = 0$$

By substitution of A_1 and A_4 we find:

$$A_1 = A_2 = A_5 = A_6 = 0$$

$$A_3 = + \frac{\text{Sinh}\gamma m \;\; \text{Cosh}\gamma m}{m(1 + m^2)(\text{Sinh}^2\gamma m - m^2)} \frac{a\, p}{\pi}$$

$$A_7 = + \frac{\text{Cosh}^2\gamma m}{(1 + m^2)(\text{Sinh}^2\gamma m - m^2)} \frac{a\, p}{\pi}$$

$$A_8 = + \frac{\text{Sinh}\gamma m \;\; \text{Cosh}\gamma m}{(1 + m^2)(\text{Sinh}^2\gamma m - m^2)} \frac{a\, p}{\pi}$$

Finally:

$$F_o = \sum_{n=1}^{n=8} A_n \varphi_n$$

$$F_o = \frac{a\,p\,r}{\pi}\int_0^{\infty}\Big[-\frac{\text{Sinh}\,\gamma m\ \text{Cosh}\,\gamma m\ \sin m\xi\ \text{Sinh}(m\,\theta)\cos\theta}{m(1+m^2)(\text{Sinh}^2\gamma m - m^2)}$$

$$+\ \frac{\text{Cosh}(m\,\theta)\sin m\xi\ \cos\theta}{m(1+m^2)}$$

$$-\ \frac{\text{Cosh}^2\gamma m\ \text{Sinh}(m\,\theta)\sin m\xi\ \sin\theta}{(1+m^2)(\text{Sinh}^2\ m - m^2)}$$

$$+\ \frac{\text{Sinh}\,\gamma m\ \text{Cosh}\,\gamma m\ \ \text{Cosh}(m\,\theta)\sin m\xi\ \sin\theta}{(1+m^2)(\text{Sinh}^2\gamma m - m^2)}\Big]\,dm$$

$$F_o = \frac{a\,p}{\pi}\int_0^{\infty} \frac{-\,x\,m^2\ \text{Cosh}(m\,\theta) + (x\ \text{Sinh}\,\gamma m + m\,y\ \text{Cosh}\,\gamma m)\ \text{Sinh}\ m\,(\gamma-\theta)}{m(1+m^2)(\text{Sinh}^2\gamma m - m^2)}\sin m\xi\,dm \quad ----(51)$$

where $x = \gamma\cos\theta$, $y = \gamma\sin\theta$, $\xi = \log\left(\frac{r}{a}\right)$ and $\gamma = \frac{3\pi}{2}$.

This integral can be evaluated as a power series in $\frac{r}{a}$ as indicated above, the poles being the same as for (29).

For $\xi << 0$ i.e. $r << a$, we find

$$F_o = a^2 p\left[.221\left(\frac{r}{a}\right)^{1.545}.\ g(\theta) + 1.376\left(\frac{r}{a}\right)^{1.909}.\ h(\theta) + \frac{1}{2}\left(\frac{r}{a}\right)^2\cos^2\theta\right.$$

$$\left. + \frac{1}{9\pi^2-4}\ \frac{r}{a}\ \theta\ (3\pi\cos\theta + \ 2\sin\theta)\right] \qquad (52)$$

and for $\xi >> 0$ or $r >> a$

$$F_o = -\,a^2 p\left[.221\left(\frac{r}{a}\right)^{.455}.\ g(\theta) + 1.376\left(\frac{r}{a}\right)^{.091} h\,(\theta) + \frac{1}{2}\cos^2\theta\right.$$

$$\left. + \frac{1}{9\pi^2-4}\ \frac{r}{a}\,.\ \theta\ (3\pi\cos\theta \ +\ 2\sin\theta\right] \qquad (53)$$

where $g(\theta)$ and $h(\theta)$ are given by (44) and (45).

It must be remembered that in order to eliminate the concentrated forces contained in F_o we must superpose F_1 and by (52) it is seen that

$$F_1 = \frac{p\,a}{4-9\pi^2}\,r\,\theta\left[3\pi\cos\theta + \ 2\sin\theta\right] \qquad (54)$$

The stress function for the 3/4-plane ($\gamma = \frac{3\pi}{2}$) with a uniform traction p on the boundary $\theta = 0$ between $r = 0$ and $r = a$ then becomes

$$F = F_0 + F_1$$

$$F = \frac{a\,p}{\pi}\int_0^{\infty} \frac{- x\,m^2 \operatorname{Cosh}(m\,\theta) + (x \operatorname{Sinh}\gamma m + m\,y \operatorname{Cosh}\gamma m) \operatorname{Sinh} m\,(\gamma - \theta)}{m(1 + m^2)(\operatorname{Sinh}^2\gamma m - m^2)} \operatorname{Sin} m\xi \, dm$$

$$+ \frac{p\,a}{4 - 9\,\pi^2}\; r\,\theta \left[3\,\pi \cos\theta + 2 \operatorname{Sin}\theta\right] \qquad ---(55)$$

For very small $\frac{r}{a}$:

$$F = a^2 p \left[.221 \left(\frac{r}{a}\right)^{1.545} g\,(\theta) + 1.376 \left(\frac{r}{a}\right)^{1.909} h\,(\theta) + \frac{1}{2}\left(\frac{r}{a}\right)^{2} \cos^2\theta\right] \qquad ---(56)$$

For very large $\frac{r}{a}$:

$$F = -a^2 p \left[.221 \left(\frac{r}{a}\right)^{.455} g\,(\theta) + 1.376 \left(\frac{r}{a}\right)^{.091} h(\theta) + \frac{1}{2} \cos^2\theta\right.$$

$$\left. + \frac{2}{9\,\pi^2 - 4} \cdot \frac{r}{a}\,(3\,\pi \cos\theta + 2 \operatorname{Sin}\theta)\right] \qquad (57)$$

<u>Check on Boundary Forces</u>

It is easily seen that the boundary conditions are satisfied on the straight boundaries. The next thing to make certain is that the forces on the boundaries, including the infinite boundary, are in equilibrium.

As before, we determine the stresses on a circle of large radius from (57), only taking powers which contribute when r becomes infinite, see Fig. 4.

$$\sigma_r = \frac{2\,a\,p}{R(9\,\pi^2 - 4)} \left[6\,\pi \sin\theta - 4\cos\theta \right]$$

$$\tau = \frac{a^2 p}{2R^2} \sin 2\theta$$

Projection on X-axis:

$$\sum X = \int_0^{\frac{3\pi}{2}} \sigma_r \cos\theta \, . \, r\, d\theta = \frac{a\,p}{4 - 9\,\pi^2} \Big[3\,\pi \cos 2\theta + 4(\theta + \tfrac{1}{2} \operatorname{Sin} 2\theta \Big]_0^{\frac{3\pi}{2}} = 0$$

Projection on Y-axis:

$$\sum Y = \int_0^{\frac{3\pi}{2}} \sigma_r \sin\theta \, r\, d\theta = \frac{a\,p}{9\,\pi^2 - 4} \Big[6\,\pi\,(\theta - \tfrac{1}{2} \operatorname{Sin} 2\theta) + 2\cos 2\theta \Big]_0^{\frac{3\pi}{2}} = +\,ap$$

Moment about origin:

$$\sum M_o = \int_0^{\frac{3\pi}{2}} \tau \cdot r^2 \, d\theta = -\,\frac{a^2 p}{4} \Big[\cos 2\theta \Big]_0^{\frac{3\pi}{2}} = +\,\frac{a^2 p}{2}$$

This shows that equilibrium exists with the given load on the X-axis.

Evaluation of Stresses

The stresses are now obtained from (55) by use of (6), (7), and (8).

If we are only interested in the stresses near the vertex these can be obtained by (56), for instance:

$$\sigma_\theta = p \left[.186 \left(\frac{r}{a}\right)^{-.455} g(\theta) + 2.380 \left(\frac{r}{a}\right)^{-.091} h(\theta) + \cos^2\theta \right]; \quad (58)$$

Hence, for small values of r: $(\sigma_\theta)_{\theta = 0} = +\,p$

$(\sigma_\theta)_{\theta = \frac{3\pi}{2}} = 0$.

For all other values $0 < \theta > \frac{3\pi}{2}$, σ_θ becomes infinite when $\frac{r}{a} \to 0$.

For large values of $\frac{r}{a}$ the stresses are obtained by (57) for instance

$$\sigma_\theta = p\left[.055\left(\frac{r}{a}\right)^{-1.545} g(\theta) + .114\left(\frac{r}{a}\right)^{-1.909} h(\theta)\right] \qquad (59)$$

If we desire the stresses for all values of $\frac{r}{a}$, it is necessary to compute a few more terms of higher powers into (56), (57), or else proceed as explained on pages 23 and 24. For example if we are interested in the distribution along a certain line $\theta = \theta_1$ it would be more simple to specialize F and then attempt to evaluate the integral by more elementary means than contour integration.

In the case $\theta = \frac{\pi}{2}$:

$$F_{\left(\frac{\pi}{2}\right)} = \frac{a\,p\,r}{\pi}\int_0^\infty \frac{\operatorname{Cosh}\gamma m \ \operatorname{Sinh}\pi m}{(1+m^2)(\operatorname{Sinh}^2\gamma m - m^2)} \operatorname{Sin} m\xi \, dm \qquad (60)$$

and

$$\sigma_\theta = \frac{a\,p}{\pi\,r}\int_0^\infty \frac{m\operatorname{Cosh}\gamma m \operatorname{Sinh} m\pi}{(1+m^2)(\operatorname{Sinh}^2\gamma m - m^2)} (\operatorname{Cos} m\xi - m\operatorname{Sin} m\xi)\, dm \qquad (61)$$

which could be evaluated in the manner outlined for (48).

The character of $g(\theta)$ and $h(\theta)$ have already been discussed and the stress function for uniform load on the boundary alone or on both boundaries are obtained in the same manner as described for concentrated force.

Later the stress function (55) will be superposed on the function for linear distribution so as to produce a function for hydrostatic (triangular) loading and the stresses will be computed and plotted for $\theta = \frac{\pi}{2}$, see Fig. 6.

3. In the third application of the method the stress function for the 3/4-plane will be determined for a linear normal load $p(\frac{r}{a})$ between $r = 0$ and $r = a$ on the bo undary $\theta = 0$, See Fig. 9.

We must now determine a stress function F_o satisfying the following boundary conditions:

$$\left(\frac{d^2F_o}{dr^2}\right)_{\theta = 0} = p\left(\frac{r}{a}\right) \quad \text{when } -\infty \leq \xi \leq 0$$

$$\left(\frac{dF_o}{rd\theta}\right)_{\theta = 0} = 0$$

$$\left(F_o\right)_{\theta = \gamma} = 0$$

$$\left(\frac{dF}{rd\theta}\right)_{\theta = \gamma} = 0$$

where $\xi = \log\left(\frac{r}{a}\right)$ and $\gamma = \frac{3\pi}{2}$

again we notice the discontinuities in the first derivative of the origin so that a second function F_1 will have to be superposed in order to remove the extraneous concentrated forces. This function can best be determined upon evaluation of F_o near the vertex.

As before we express F_o as in (17) by use of (11) and form the boundary equations by means of (12) to (15):

Hence:

$$\left(\frac{d^2F_o}{dr^2}\right)_{\theta = 0} = \frac{1}{r}\int_0^\infty \left[- A_1(m^2 \cos m\xi + m \sin m\xi) + A_4(\cos m\xi - m^2 \sin m\xi)\right] dm$$

$$= \begin{cases} p\left(\frac{r}{a}\right) & \text{when } \xi < 0 \\ 0 & \text{when } \xi > 0 \end{cases} \qquad (62)$$

By Dirichlet's integral (27) we express $ap\left(\frac{r}{a}\right)^2 = a\,p\,e^{2\xi}$ an Fourier integral form, taking n = 2 in this application:

$$\frac{ap}{\pi}\int_0^\infty \frac{2\cos m\xi - m\sin m\xi}{m^2 + 2^2}\,dm = \begin{matrix} a\,p\,e^{2\xi} & \text{when } \xi < 0 \\ 0 & \text{when } \xi > 0 \end{matrix} \qquad (27b)$$

Equation (62) then becomes:

$$-A_1\,(m^2\cos m\xi + m\sin m\xi) + A_4(m\cos m\xi - m^2\sin m\xi) = \frac{ap}{\pi}\,\frac{2\cos m\xi - m\sin m\xi}{m^2 + 4}$$

and by equating coefficients of $\cos m\xi$ and $\sin m\xi$ respectively we obtain:

$$-m^2A_1 + m\,A_4 = \frac{2}{m^2 + 4}\,\frac{ap}{\pi}$$

$$-m\,A_1 - m^2A_4 = -\frac{m}{m^2 + 4}\,\frac{ap}{\pi}$$

and solving these:

$$A_1 = -\frac{1}{(m^2 + 1)(m^2 + 4)}\cdot\frac{ap}{\pi}$$

$$A_4 = +\frac{m^2 + 2}{m(m^2 + 1)(m^2 + 4)}\cdot\frac{ap}{\pi}$$

The remaining boundary conditions yield the equations:

$$\left(\frac{dF_o}{rd\theta}\right)_{\theta = 0} : \; -m\,A_2\cos m\xi - m\,A_3\sin m\xi + A_5\cos m\xi + A_8\sin m\xi = 0$$

$$\left(F_o\right)_{\theta = \gamma} : \; -A_5\operatorname{Cosh}\gamma m\cos m\xi + A_6\operatorname{Sinh}\gamma m\cos m\xi + A_7\operatorname{Sinh}\gamma m\operatorname{Sin} m\xi - A_8\operatorname{Cosh}\gamma m\sin m\xi = 0$$

$$\left(\frac{dF_o}{rd\theta}\right)_{\theta=\gamma} : A_1 \operatorname{Cosh}\gamma m \cos m\xi - A_2 \operatorname{Sinh}\gamma m \cos m\xi - A_3 \operatorname{Sinh}\gamma m \sin m\xi$$

$$+ A_4 \operatorname{Cosh}\gamma m \sin m\xi - m A_5 \operatorname{Sinh}\gamma m \cos m\xi + m A_6 \operatorname{Cosh}\gamma m \cos m\xi$$

$$+ m A_7 \operatorname{Cosh}\gamma m \sin m\xi - m A_8 \operatorname{Sinh}\gamma m \sin m\xi = 0$$

By equating to zero the coefficients of $\cos m\xi$ and $\sin m\xi$ respectively we obtain:

$$- m A_2 + A_5 = 0$$

$$- m A_3 + A_8 = 0$$

$$A_5 \operatorname{Cosh}\gamma m - A_6 \operatorname{Sinh}\gamma m = 0$$

$$A_7 \operatorname{Sinh}\gamma m - A_8 \operatorname{Cosh}\gamma m = 0$$

$$A_1 \operatorname{Cosh}\gamma m - A_2 \operatorname{Sinh}\gamma m - m A_5 \operatorname{Sinh}\gamma m + m A_6 \operatorname{Cosh}\gamma m = 0$$

$$-A_3 \operatorname{Sinh}\gamma m + A_4 \operatorname{Cosh}\gamma m + m A_7 \operatorname{Cosh}\gamma m - m A_8 \operatorname{Sinh}\gamma m = 0$$

Solving these and substituting the values already determined for A_1 and A_4 we find:

$$A_2 = - \frac{\operatorname{Cosh}\gamma m \operatorname{Sinh}\gamma m}{(m^2 + 1)(m^2 + 4)(\operatorname{Sinh}^2\gamma m - m^2)} \frac{ap}{\pi}$$

$$A_3 = + \frac{(m^2 + 2) \operatorname{Cosh}\gamma m \operatorname{Sinh}\gamma m}{m(m^2 + 1)(m^2 + 4)(\operatorname{Sinh}^2\gamma m - m^2)} \frac{ap}{\pi}$$

$$A_5 = - \frac{m \operatorname{Cosh}\gamma m \operatorname{Sinh}\gamma m}{(m^2 + 1)(m^2 + 4)(\operatorname{Sinh}^2\gamma m - m^2)} \frac{ap}{\pi}$$

$$A_6 = - \frac{m \operatorname{Cosh}^2\gamma m}{(m^2 + 1)(m^2 + 4)(\operatorname{Sinh}^2\gamma m - m^2)} \frac{ap}{\pi}$$

$$A_7 = + \frac{(m^2 + 2) \operatorname{Cosh}^2\gamma m}{(m^2 + 1)(m^2 + 4)(\operatorname{Sinh}^2\gamma m - m^2)} \frac{ap}{\pi}$$

$$A_8 = + \frac{(m^2+2)\cosh\gamma m \sinh\gamma m}{(m^2+1)(m^2+4)(\sinh^2\gamma m - m^2)} \quad \frac{ap}{\pi}$$

Finally,

$$F_o = \sum_{n=1}^{n=8} A_n \varphi_n$$

$$= \frac{ap}{\pi}\int_0^\infty \Bigg[x \Bigg(- \frac{\cosh(m\,\theta)\cos m\xi}{(m^2+1)(m^2+4)} + \frac{\cosh\gamma m \sinh\gamma m \sinh(m\,\theta)}{(m^2+1)(m^2+4)(\sinh^2\gamma m - m^2)} \cos m\xi$$

$$- \frac{(m^2+2)\cosh\gamma m \sinh\gamma m \ \sinh(m\,\theta)}{m(m^2+1)(m^2+4)(\sinh^2\gamma m - m^2)} \sin m\xi$$

$$+ \frac{(m^2+2)\ \cosh(m\,\theta)}{m(m^2+1)(m^2+4)} \sin m\xi \Bigg)$$

$$+ y \Bigg(- \frac{m\cosh\gamma m \sinh\gamma m \cosh(m\,\theta)}{(m^2+1)(m^2+4)(\sinh^2\gamma m - m^2)} \cos m\xi$$

$$+ \frac{m\cosh^2\gamma m \sinh m\,\theta}{(m^2+1)(m^2+4)(\sinh^2\gamma m - m^2)} \cos m\xi$$

$$- \frac{(m^2+2)\cosh^2\gamma m \sinh m\,\theta}{(m^2+1)(m^2+4)(\sinh^2\gamma m - m^2)} \sin m\xi$$

$$+ \frac{(m^2+2)\cosh\gamma m \sinh\gamma m \cosh m\,\theta}{(m^2+1)(m^2+4)(\sinh^2 \ m - m^2)} \sin m\xi \Bigg) \Bigg] \, dm$$

$$F_o = \frac{ap}{\pi}\int_0^\infty \frac{m^2 x \cosh m\,\theta - (x\sinh\gamma m + y m\cosh\gamma m)\sinh m(\gamma-\theta)}{m(1+m^2)(4+m^2)(\sinh^2\gamma m - m^2)} \cdot \Big[$$

$$m \cos m\xi - (m^2+2)\sin m\xi \Big] \, dm \qquad (63)$$

The integral can be evaluated as explained in the case of concentrated force. The singularities in the contour integral are the same as before in addition to simple poles at $m = \pm 2\,i$.

For $\xi \ll 0$ i.e. $r \ll a$ is found:

$$F_o = a^2 p \Big[.0695 \left(\frac{r}{a}\right)^{1.545} g(\theta) + .115 \left(\frac{r}{a}\right)^{1.909} h(\theta) + \frac{1}{6} \left(\frac{r}{a}\right)^3 \cos^3 \theta$$

$$+ \frac{1}{9\pi^2 - 4} \; \frac{r}{a} \; \theta \left(\frac{3}{2}\pi \cos\theta + \sin\theta\right) \Big] \qquad (64)$$

For $\xi \gg 0$ i.e. $r \gg a$:

$$F_o = - a\, p \Big[.1344 \left(\frac{r}{a}\right)^{1.455} g(\theta) + .905 \left(\frac{r}{a}\right)^{.091} h(\theta) + \frac{1}{3} \cos^2 \theta$$

$$+ \frac{1}{9\pi^2 - 4} \; \frac{r}{a} \; \theta \left(\frac{3\pi}{2} \cos\theta + \sin\theta\right) \Big] \qquad (65)$$

where $g(\theta)$ and $h(\theta)$ are given by (44) and (45).

It can now be seen by (65) that in order to remove the extraneous concentrated forces at the origin the function

$$F_1 = \frac{a\, p}{4 - 9\pi^2} \; r\, \theta \left(\frac{3\pi}{2} \cos\theta + \sin\theta \right) \qquad (66)$$

must be superposed in order to properly satisfy the boundary conditions.

The stress function for the 3/4-plane ($\gamma = \frac{3\pi}{2}$) with a linear traction $p. \frac{r}{a}$ on the boundary $\theta = 0$ between $r = 0$ and $r = a$ then finally becomes:

$$F = F_o + F_1$$

$$F = \frac{ap}{\pi} \int_0^\infty \frac{m^2 x \operatorname{Cosh} m\theta - (x \operatorname{Sinh}\gamma m + m y \operatorname{Cosh}\gamma m) \operatorname{Sinh} m(\gamma - \theta)}{m(1 = m^2)(4 + m^2)(\operatorname{Sinh}^2 \gamma m - m^2)} \Big[m \operatorname{Cos} m\xi - (m^2+2) \operatorname{Sin} m\xi$$

$$dm$$

$$+ \frac{a\, p}{4 - 9\pi^2} \; r\, \theta \left(\frac{3\pi}{2} \cos\theta + \sin\theta\right) \qquad (67)$$

Where $\xi = \log \frac{r}{a}$; $\gamma = \frac{3\pi}{2}$; $x = r \cos\theta$; $y = r \sin\theta$.

For very small $\frac{r}{a}$:

$$F = a^2 p \left[.0695 \left(\frac{r}{a}\right)^{1.545} g(\theta) + .115 \left(\frac{r}{a}\right)^{1.909} h(\theta) + \frac{1}{6} \left(\frac{r}{a}\right)^3 \cos^3\theta \right] \quad (68)$$

For very large $\frac{r}{a}$:

$$F = - a^2 p \left[.1344 \left(\frac{r}{a}\right)^{.455} g(\theta) + .905 \left(\frac{r}{a}\right)^{.091} h(\theta) + \frac{1}{3} \cos^2\theta + \frac{1}{9\pi^2 - 4} \cdot \frac{r}{a} (3\pi \cos\theta + 2 \sin\theta) \right] \quad (69)$$

Check on boundary forces:

In order to see if the forces on the infinite boundary are such as to be in equilibrium with the forces on the straight boundary (X-axis), we compute the stresses σ_r and τ from (69) only including terms of such order which will contribute in the integration when $r \to \infty$, See Fig. 4.

$$\sigma_r = \frac{a p}{R(9\pi^2 - 4)} \left[6\pi \sin\theta - 4 \cos\theta \right]$$

$$\tau = \frac{a^2 p}{3 R^2} \sin 2\theta$$

Projection on X-axis:

$$\sum X = \int_0^{\frac{3\pi}{2}} \sigma_r \cos\theta \cdot r \, d\theta = \frac{a p}{4 - 9\pi^2} \left[\frac{3\pi}{2} \cos 2\theta + 2(\theta + \frac{1}{2} \sin 2\theta \right]_0^{\frac{3\pi}{2}} = 0$$

Projection on Y-axis:

$$\sum Y = \int_0^{\frac{3\pi}{2}} \sigma_r \sin\theta \cdot r \, d\theta = \frac{a p}{4 - 9\pi^2} \left[3\pi (\theta - \frac{1}{2} \sin 2\theta) + \cos 2\theta \right]_0^{\frac{3\pi}{2}} = + ap$$

Moment about origin:

$$\sum M_o = \int_0^{\frac{3\pi}{2}} \tau \cdot r^2 \, d\theta = - \frac{a^2 p}{6} \Big[\cos 2\theta \Big]_0^{\frac{3\pi}{2}} = + \frac{1}{3} a^2 p$$

This shows that equilibrium exists, so that the function is now in order.

Evaluation of Stresses

The stresses are obtained by partial differentiation of (67) in accordance with (6), (7), and (8).

Near the vertex we obtain by (69):

$$\sigma_\theta = p \Big[.059 \left(\frac{r}{a}\right)^{-.455} g(\theta) + .199 \left(\frac{r}{a}\right)^{-.091} h(\theta) + \frac{r}{a} \cos^3 \theta \Big] \qquad (70)$$

Hence for small values of r:

$$\left(\sigma_\theta\right)_{\theta = 0} = + p . \frac{r}{a}$$

$$\left(\sigma_\theta\right)_{\theta = \frac{3\pi}{2}} = 0$$

For all other values $0 < \theta < \frac{3\pi}{2}$, σ_θ becomes infinite when $\frac{r}{a} \rightarrow 0$.

For large values of $\frac{r}{a}$ the stresses are obtained by (68), for example:

$$\sigma_\theta = p \Big[.033 \left(\frac{r}{a}\right)^{-1.545} g(\theta) + .075 \left(\frac{r}{a}\right)^{-1.909} h(\theta) \Big] \qquad (71)$$

If the stresses are also desired in the neighborhood $\frac{r}{a} = 1$, it is either necessary to include a few more terms of higher order in (68), (69), or else proceed as on pages 23 and 24.

In case we are only interested in the stresses at a certain point or along a line it would be simpler first to specialize F and then evaluate by other methods than contour integration. If for example, the normal stress is desired on the Y-axis, we place $\theta = \frac{\pi}{2}$ in (67) and obtain:

$$\left(F\right)_{\frac{\pi}{2}} = \frac{apr}{\pi}\int_0^\infty \frac{\operatorname{Cosh}\gamma m \ \operatorname{Sinh}\pi m}{(1+m^2)(4+m^2)(\operatorname{Sinh}^2\gamma m - m^2)}\left[(m^2+2)\operatorname{Sin} m\xi - m\cdot\cos m\xi\right]dm \qquad (72)$$

and

$$\sigma_\theta = \frac{a\ p}{\pi^2}\int_0^\infty \frac{m\operatorname{Cos}\gamma m \ \operatorname{Sinh}\pi m}{(4+m^2)(\operatorname{Sinh}^2\gamma m - m^2)}(2\cos m\xi - m\sin m\xi)\,dm \qquad (73)$$

which can be evaluated in the manner outlined for (48).

The remarks made on pages 25 and 32 with regard to loads on the boundary $\theta = \frac{3\pi}{2}$ also hold in this case.

4. Stress Function for Hydrostatic Distribution

The stresses in the 3/4-plane due to normal traction as shown in Fig. 6 can be found simply as difference of the functions given in (55) and (67).

We obtain for all values of r:

$$F = \frac{ap}{\pi}\int_0^\infty \frac{-m^2\ x\operatorname{Cosh} m\theta + (x\operatorname{Sinh}\gamma m + m\,y\operatorname{Cosh}\gamma m)\operatorname{Sinh} m(\gamma-\theta)}{m(1+m^2)(4+m^2)(\operatorname{Sinh}^2 m - m^2)}(m\cos m\xi$$

$$+ 2\sin m\xi)\ dm$$

$$+\frac{ap}{4-9\pi^2}\, r\theta\left(\frac{3\pi}{2}\cos\theta + \operatorname{Sin}\theta\right) \qquad (74)$$

Where $\xi = \log\frac{r}{a}$; $\gamma = \frac{3\pi}{2}$; $x = r\cos\theta$; $y = r\operatorname{Sin}\theta$

For very small values of $\frac{r}{a}$:

$$F = a^2p\left[.151\left(\frac{r}{a}\right)^{1.545} g(\theta) + 1.261\left(\frac{r}{a}\right)^{1.909} h(\theta) + \frac{1}{2}\left(\frac{r}{a}\right)^2\cos^2\theta - \frac{1}{6}\left(\frac{r}{a}\right)^3\cos^3\theta\right] \quad \text{---}(75)$$

For very large value of $\frac{r}{a}$:

$$F = - a^2p\left[.087\left(\frac{r}{a}\right)^{.455} g(\theta) + .471\left(\frac{r}{a}\right)^{.091} h(\theta) + \frac{1}{6}\cos^2\theta + \frac{1}{9\pi^2-4}\frac{r}{a}\theta(3\pi\cos\theta + 2\sin\theta)\right] \quad \text{---}(76)$$

<u>Evaluation of Stresses</u>

The hydrostatic loading is of special interest in connection with stress analysis in masonry dams. Therefore, the formulae for certain stresses will be given and plotted in Fig. 6 for further discussion later.

For small values of $\frac{r}{a}$:

$$\sigma_\theta = p\left[.127\left(\frac{r}{a}\right)^{-.455} g(\theta) + 2.181\left(\frac{r}{a}\right)^{-.091} h(\theta) + \cos^2\theta - \frac{r}{a}\cos^3\theta\right] \quad (77)$$

For large values of $\frac{r}{a}$:

$$\sigma_\theta = p\left[.022\left(\frac{r}{a}\right)^{-1.545} g(\theta) + .039\left(\frac{r}{a}\right)^{-1.909} h(\theta)\right] \quad \text{----}(78)$$

Of special interest are the stresses on the Y-axis and substituting in (44) and (45) we obtain:

$$\theta = \frac{\pi}{2}$$

$$g(\tfrac{\pi}{2}) = -.990$$

$$h(\tfrac{\pi}{2}) = +.282$$

By inserting these values in (77) and (78) we find:

$$\left(\sigma_\theta\right)_{\theta=\frac{\pi}{2}} = p\left[-.126\left(\frac{r}{a}\right)^{-.455} + .615\left(\frac{r}{a}\right)^{-.091}\right] \quad \text{when } r \ll a \qquad (79)$$

$$\left(\sigma_\theta\right)_{\theta=\frac{\pi}{2}} = p\left[-.022\left(\frac{r}{a}\right)^{-1.545} + .011\left(\frac{r}{a}\right)^{-1.909}\right] \quad \text{when } r \gg a \qquad (80)$$

For values of $\frac{r}{a} \sim 1$, a few positive powers must be computed, but it is simpler in this case to specialize F.

Placing $\theta = \frac{\pi}{2}$ in (74) we obtain:

$$\left(F\right)_{\frac{\pi}{2}} = \frac{apr}{\pi}\int_0^\infty \frac{\cosh\gamma m \sinh\pi m}{(1+m^2)(4+m^2)(\sinh^2\gamma m - m^2)}\,(m\cos m\xi + 2\sin m\xi)\,dm\,; \qquad (81)$$

$$\left(\sigma_\theta\right)_{\frac{\pi}{2}} = \frac{ap}{\pi}\int_0^\infty \frac{m\cosh\gamma m \sinh\pi m}{(1+m^2)(4+m^2)(\sinh^2\gamma m - m^2)}\left[(2-m^2)\cos m\xi - 3m\sin m\xi\right]dm \qquad (82)$$

The stresses computed by (79), (80), and (82) are plotted in Fig. 6, Curve I. The dotted curve II represents the normal stresses due to some loading on the half plane, $\gamma = \pi$. These are obtained by integrating the Bousinesq's solution for a concentrated force, as indicated in Fig. 10.

$$\sigma_x = \frac{2}{\pi}\int_0^a \frac{x^2 y\, p(x)\, dx}{r^2}$$

where $p(x) = p.\frac{x-a}{a}$

$$\sigma_x = \frac{2yp}{\pi a}\int_0^a \frac{x^2(x-a)}{(x^2+y^2)^2}\,dx$$

$$\sigma_x = \frac{p}{\pi}\left[\tan^{-1}\frac{a}{y} + \frac{y}{a}\log\frac{y^2}{a^2+y^2}\right]$$

It will be noticed that at the origin

$\sigma_x = \frac{1}{2}p$, independent of a.

It will be noticed that equation (34) has no real roots, except $m = 0$, when γ is less than π. This means that in general a partial loading of the boundaries of a corner will produce finite stresses at the origin (vertex) if $\gamma > \pi$, but infinite stresses if $\gamma < \pi$, and the greater γ is the sharper the infinity. The half plane, $\gamma = \pi$, thus is the critical case and the stresses at the origin are zero if it is chosen outside the loaded region.

Special cases of stress distribution in an infinite wedge have been treated by a number of investigators. We find J. H. Mitchell[1] who generalizes Bousinesq's solution for a concentrated force on the half-plane to a concentrated force at the vertex of an infinite wedge , M. Levy[2] who finds a solution for a linear load distribution over the boundaries, S. D. Carother[2] who gives solutions for a couple at the vertex and uniform loads on the boundaries.

The stress function for continuous uniform pressure on the boundaries is easily determined by equation (46):

$$F = -\frac{p}{2} r^2 \cos^2 \theta \tag{83}$$

corresponds to uniform pressure on the boundary $\theta = 0$. From this it will be seen that the stresses are finite at the vertex for any value of γ. We have seen that a partial loading produces infinite stress

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1) A. E. H. Love, Elasticity, page 212.
2) Proc. Royal Society, Edinburgh, Vol. 53, 1913, page 292.

at the vertex if $\gamma > \pi$, so that we have the curious result that in this case the removal of a portion of the load raises the stress to infinity.

If uniform pressure acts on the boundary $\theta = \frac{3\pi}{2}$ (as in the case of a dam with a full reservoir), the corresponding stress function is

$$F = - \frac{p}{2} r^2 \sin^2 \theta \qquad (84)$$

This gives the result, important in the design of dams, that the base will be under compression due to this load. In Fig. 6 is plotted the normal stresses on the base, $\theta = \frac{\pi}{2}$, due to reversed hydrostatic pressure distribution on the boundary, $\theta = 0$. It is important to notice that near the vertex the base is under high tension, i.e. opposite to the stress due to the uniform pressure on the boundary, $\theta = \frac{3\pi}{2}$.

Levy's solution for linear normal pressure on the boundary of an infinite wedge (or triangle) shows linear distribution of all stresses along any straight line through the wedge. In practice this result is conveniently applied to masonry dams. However, the application is dubious because the height of a dam is of the same order of magnitude as the base width, so that great deviations from the linear stress distribution can be exprected (at any rate near the base) due to the influence of the unyielding, or rather, infinite foundation.

Engineers are well aware of this fact, and the present study grew out of an attempt to determine the stress distribution in triangular dams on infinite, rigid and elastic foundations by the Ritz' method. However, it was found that the stresses in the up- and downstream faces at the base did not converge when more parameters were included in the solution. At points only slightly distant from the up- and downstream corners Ritz' method gave reasonable results.

The present method enables us to compute the stresses in dams of triangular and other shapes for any load and far more accurate solutions can be reached in this way.

APPENDIX I.

With the usual notation

$$\varepsilon_x = \frac{du}{dx} \qquad \text{etc.} \tag{85}$$

$$\gamma_{xy} = \frac{du}{dy} + \frac{dv}{dx} = \frac{1}{G}\,\tau_{xy} \qquad \text{etc.} \tag{86}$$

and the analogous, we will define a two-dimensional elastic system such that

$$\frac{du}{dz} = \frac{dv}{dz} = 0 \quad \text{and} \quad \frac{dw}{dx} = \frac{dw}{dy} = 0 \tag{87}$$

From (87) it immediately follows that

$$\tau_{xz} = \tau_{yz} = 0$$

and it can easily be shown that

$$\frac{d^2w}{dz^2} = 0 \qquad \text{- - - —} \quad (88)$$

Hence

$$E\,\varepsilon_z = \frac{dw}{dz} = \text{constant} = \sigma_z - \frac{1}{m}(\sigma_x + \sigma_y)$$

and $$\sigma_z = \frac{1}{m}(\sigma_x + \sigma_y) + \text{constant} \tag{89}$$

The constant is unimportant and may be placed equal to zero.

By (85) and (86)

$$E\,\varepsilon_x = \sigma_x - \frac{1}{m}(\sigma_y + \sigma_z) = \sigma_x - \frac{1}{m}\left[\sigma_y + \frac{1}{m}(\sigma_x + \sigma_y)\right].$$

$$E\,\varepsilon_x = \frac{m^2-1}{m^2}\sigma_x - \frac{m+1}{m^2}\sigma_y = \frac{m^2-1}{m^2}\frac{d^2F}{dy^2} - \frac{m+1}{m^2}\frac{d^2F}{dx^2}\,; \tag{90}$$

Similarly

$$E\,\varepsilon_y = \frac{m^2-1}{m^2}\sigma_y - \frac{m+1}{m^2}\sigma_x = \frac{m^2-1}{m^2}\frac{d^2F}{dx^2} - \frac{m+1}{m^2}\frac{d^2F}{dy^2}\,; \qquad (91)$$

By (2):

$$\frac{d^2\gamma}{d\,x\,dy} = \frac{d^3u}{d\,x\,d\,y^2} + \frac{d^2v}{d\,x^2d\,y} = \frac{d^2\varepsilon_x}{d\,y^2} + \frac{d^2\varepsilon_y}{d\,x^2} = \frac{1}{G}\frac{d^2\tau}{d\,x\,d\,y}$$

Substituting (90) and (91) into (92):

$$= -\frac{1}{G}\frac{d^4F}{d\,x^2d\,y^2} \qquad \ldots(92)$$

$$-\frac{1}{G}\frac{d^4F}{dx^2\,dy^2} = \frac{1}{E}\,\frac{m+1}{m^2}\left[(m-1)\frac{d^4F}{dy^4} - \frac{d^4F}{dx^2\,dy^2} + (m-1)\frac{d^4F}{dx^4} - \frac{d^4F}{dx^2\,dy^2}\right]$$

and remembering the relation

$$\frac{E}{G} = 2\,\frac{m+1}{m}$$

$$\frac{m^2-1}{m^2}\left[\frac{d^4F}{dx^4} + \frac{d^4F}{dy^4}\right] + 2\left[\frac{m+1}{m} - \frac{m+1}{m^2}\right]\frac{d^4F}{dx^2\,dy^2} = 0$$

Hence

$$\frac{d^4F}{dx^4} + \frac{d^4F}{dy^4} + 2\frac{d^4F}{dx^2\,dy^2} = 0$$

or $\nabla^4 F = 0.$

The state of strain defined by (87) is exactly the case we deal with in the infinite half plane or any portion of the plane for that matter acted upon by a "line-load" as any distribution of "line-loads" and in this case $\varepsilon_z = 0$.

and $\sigma_z = \frac{1}{m}(\sigma_x + \sigma_y)$ (89a)

Part II.

1. In this chapter an alternate method of procedure shall be used in obtaining the solution in the case of a concentrated force acting at $r = a$ on the boundary $\theta = 0$ of the 3/4-plane, see Fig. 2.

The analogous solution in the 1/2-plane, given by Bousinesq is

$$F^1 = \frac{1}{\pi}(u - a)\tan^{-1}\frac{v}{u - a}$$

and if we define

$$\Phi^1 = E^1 + iF^1 = \frac{1}{\pi}(u - a)\Big[\log(w - a)\Big]_i \quad , \; w = u + i\,v$$

It will be seen that F' is the imaginary part of Φ'

In the conformal transformation $w = z^{2/3}$, the half-plane goes into the 3/4-plane so we are led to consider the function

$$\Phi = E + iF = C(x - a)\log(a^{2/3} - z^{2/3}) \equiv C(x - a)\cdot f(z) \qquad (1)$$

where $z = x + i\,y = r\,e^{i\theta}$ and C is a real constant. The real and imaginary parts are both biharmonics. We will only employ the imaginary part

$$F = C(x - a)\Big[\log(a^{2/3} - z^{2/3})\Big]_i = C(x - a)\Big[f(z)\Big]_i \qquad (2)$$

In order to determine the forces on the boundaries corresponding to F we must compute the first derivatives:

$$\frac{dF}{dx} = C\left[\log(a^{2/3} - z^{2/3}) - 2/3(x - a)\frac{z^{-1/3}}{a^{2/3} - z^{2/3}}\right]_i \qquad (3)$$

$$\frac{dF}{dy} = C\left[- \; 2/3\,(x - a)\;\frac{2^{-1/3}}{a^{2/3} - z^{2/3}}\right]_r = C(x - a)\left[\frac{d f(z)}{dz}\right]_r \qquad (4)$$

On the boundary $\theta = 0$:

$$\frac{dF}{dx} = \begin{cases} 0 & \text{when } x < a \\ \pi C & \text{when } x > a \end{cases}$$

We conclude that the only normal force on this boundary is πC at $x = a$ acting in the direction of the Y-axis. We wish this force to be unity traction, i.e. along the negative Y-axis

$$\pi C = -1$$

$$C = -\frac{1}{\pi}$$

$$\left(\frac{dF}{dy}\right)_{\theta = 0} = \frac{2}{3\pi}\;\frac{x - a}{x^{1/3}(a^{2/3} - x^{2/3})} \neq 0 \qquad (4a)$$

which means that tangential forces exist on this boundary.

It is our aim to clear the boundaries $\theta = 0$ and $\theta = \gamma = \frac{3\pi}{2}$ of all forces except $p = 1$ at $z = a$. Therefore, we superpose a function H defined as the imaginary part of ψ or

$$\psi = G + i\,H \qquad (5)$$

and determine ψ such that the following conditions are satisfied when $\theta = 0$.

$$\frac{dH}{dy} = -\frac{dF}{dy}$$

and $H = 0$

We have

$$\frac{dF}{dy} = C(x - a)\left[\frac{df}{d\,z}\right]_r \qquad \text{by} \quad (4)$$

and $\frac{dH}{dy} = \left[\frac{d\Psi}{d\,z}\right]_r$ by (5)

hence $\left[\frac{d\Psi}{d\,z}\right]_r = -\ C(x - a)\left[\frac{d f}{d\,z}\right]_r$ when $\theta = 0$ or $z = x$.

or $\frac{d\Psi}{d\,z} = +\ \frac{1}{\pi}(z - a)\ \frac{d f}{d\,z}$

$$\Psi = -\ \frac{2}{3\pi}\int \frac{z - a}{z^{1/3}(a^{2/3} - z^{2/3})}\,dz$$

This is easily integrated by the substitution $z = t^3$, and we obtain:

$$\Psi(z) = \frac{2}{\pi}\left[\frac{2}{3} + a^{2/3}\,z^{1/3} - a\log(a^{1/3} + z^{1/3})\right] = G + i\,H \qquad (7)$$

The stress function

$$L = F + H = (\phi + \Psi)_i \qquad (8)$$

will correspond to unit normal traction at z = a in addition perhaps to forces on the boundary $\theta = \gamma$. It is easily verified that H = 0 when $\theta = 0$, so that H has added no normal force on this boundary. We must also determine and remove the forces on boundary $\theta = \gamma$ and therefore compute the value of L when $z = r\,e^{i\cdot\frac{3\pi}{2}} = -\,i\,r$:

Therefore $x = 0,\ y = -\,r$

$z^{1/3} = i\,r^{1/3}$ (choosing this branch)

$z^{2/3} = -\,r^{2/3}$

By (2) and (7) we have,

$$L = \frac{1}{\pi}\left\{-\ (x - a)\log(a^{2/3} - z^{2/3}) + 2/3z + 2\,a^{2/3}\,z^{1/3} - 2a\log(a^{1/3} + z^{1/3})\right\}_i \qquad (9)$$

and by substituting the above values we obtain when $\theta = \gamma$:

$$L_{\theta=\gamma} = \frac{1}{\pi}\left\{a \log(a^{2/3} + r^{2/3}) - 2/3\, i\, r + 2a^{2/3} i\, r^{1/3} - 2a \log(a^{1/3} + i\, r^{1/3})\right\}_i$$

and taking the imaginary part:

$$L_{\theta=\gamma} = \frac{2a}{\pi}\left[\left(\frac{r}{a}\right)^{1/3} - \tan^{-1}\left(\frac{r}{a}\right)^{1/3} - \frac{1}{3}\frac{r}{a}\right] \qquad ----(10)$$

$$\left(\frac{dL}{dr}\right)_{\theta=\gamma} = \frac{2}{3\pi}\left[\frac{1}{\left(\frac{r}{a}\right)^{2/3}} - \frac{1}{\left[1 + \frac{r}{a}^{2/3}\right]\left(\frac{r}{a}\right)^{2/3}} - 1 = -\frac{1}{3\pi}\left[\frac{\left(\frac{r}{a}\right)^{1/3} - \left(\frac{r}{a}\right)^{-1/3}}{\left(\frac{r}{a}\right)^{1/3} + \left(\frac{r}{a}\right)^{-1/3}} + 1\right]\right. ;(11)$$

Again by differentiation of (9):

$$\frac{dL}{dx} = \left[\frac{d(\phi+\psi)}{d\, z}\right]_i = \frac{2}{3\pi}\left[-\frac{3}{2}\log(a^{2/3} - z^{2/3}) + (x - a)\frac{z^{-1/3}}{a^{2/3} - z^{2/3}} + 1 + a^{2/3} z^{-2/3} - a\frac{z^{-2/3}}{a^{1/3} + z^{1/3}}\right]_i$$

Hence

$$\left(\frac{dL}{dx}\right)_{\theta=\gamma} = \frac{2}{3\pi}\left[-\frac{3}{2}\log(a^{2/3} + r^{2/3}) - \frac{a}{(a^{2/3}+r^{2/3})\, i\, r^{1/3}} - \frac{a^{2/3}}{r^{2/3}} + \frac{a}{(a^{1/3} + i r^{1/3})\, r^{2/3}} + 1\right]_i$$

and taking the imaginary part we find

$$\left(\frac{dL}{dx}\right)_{\theta=\gamma} = \frac{2}{3\pi}\left(\frac{a}{(a^{2/3} + r^{2/3})\, r^{1/3}} - \frac{a}{(a^{2/3} + r^{2/3})\, r^{1/3}}\right) = 0$$

It is now seen that if a function K is determined such that

$$(K)_{\theta=0} = 0$$

$$\left(\frac{dK}{d\theta}\right)_{\theta=0} = 0$$

$$\left(\frac{dK}{dr}\right)_{\theta=\gamma} = -\left(\frac{dL}{dr}\right)_{\theta=\gamma} \text{ given by (11)}$$

$$\left(\frac{dK}{d\theta}\right)_{\theta=\gamma} = 0$$

and if K is superposed on L:

$$A = L + K$$

A will be the stress function corresponding to the unit normal traction at z = a being the only load. From here on the problem follows the general method explained in Part I.

We place $\xi = \log \frac{r}{a}$, therefore, $\frac{r}{a} = e^{\xi}$

The variable part of (11):

$$f(\xi) = -\frac{1}{3\pi}\frac{\left(\frac{r}{a}\right)^{1/3}-\left(\frac{r}{a}\right)^{-1/3}}{\left(\frac{r}{a}\right)^{1/3}+\left(\frac{r}{a}\right)^{-1/3}} = -\frac{1}{3\pi}\frac{e^{1/3\xi}-e^{-1/3\xi}}{e^{1/3\xi}+e^{-1/3\xi}} = -\frac{1}{3\pi}\,\mathrm{Tanh}\,\frac{\xi}{3} \quad (12)$$

Note: The constant part of (11) corresponds to concentrated force at the origin and will be considered later.

By Fourier's Theorem

$$f(\xi) = \frac{1}{\pi}\int_0^{\infty} dm \int_{-\infty}^{+\infty} f(t)\cos m(\xi - t)\, dt \quad (13)$$

In this application $f(\xi)$ is odd and (13) reduces to

$$-f(\xi) = \frac{1}{3\pi}\,\mathrm{Tanh}\,\frac{\xi}{3} = \frac{1}{3\pi^2}\int_0^{\infty}\mathrm{Sin}\,m\xi\, dm\int_{-\infty}^{+\infty}\mathrm{Sin}(m\,t)\,\mathrm{Tanh}\,\frac{t}{3}\,dt = \frac{1}{3\pi^2}\int_0^{\infty} g(m)\,\sin m\xi\, dm \quad (14)$$

$$\text{where } g(m) = \int_{-\infty}^{+\infty}\sin m\,t\,\mathrm{Tanh}\,\frac{t}{3}\,dt \quad (15)$$

The latter integral is evaluated in Bierens de Haan's table 265, page 388:

$$\int_0^{\infty}\frac{e^{qt}-e^{-qt}}{e^{qt}+e^{-qt}}\,\mathrm{Sin}\,m\,t\,dt = \frac{\pi}{q}\,\frac{1}{e^{\frac{\pi m}{2q}}-e^{\frac{-\pi m}{2q}}}.$$

which applied to (15), with $q = 1/3$, gives

$$g(m) = 6\pi\,\frac{1}{e^{\frac{3\pi m}{2}}-e^{\frac{-3\pi m}{2}}} = \frac{3\pi}{\mathrm{Sinh}\,\gamma m} \quad \text{where } \gamma = \frac{3\pi}{2}$$

Finally, by (14):

$$-f(\xi) = \frac{1}{3\pi}\text{ Tanh } \frac{\xi}{3} = \frac{1}{\pi}\int_0^\infty \frac{1}{\text{Sinh}\gamma m} \cdot \text{Sin }(m\xi) \; dm \qquad \cdots (16)$$

We now write K = M + N where N will be determined so as to remove the concentrated forces, at the origin and M must satisfy the following conditions:

When $\theta = 0$:

$M = 0$ and $\frac{dM}{rd\theta} = 0$

When $\theta = \gamma$

$\frac{dM}{dr} = -f(\xi)$ (by 16) and $\frac{dM}{rd\theta} = 0$.

The boundary equations are now formed by using (12) and (13) of Part I:

$$(M)_{\theta = 0} = r\int_0^\infty \left[A_1 \cos m + A_4 \text{ Sin } m\right] dm = 0$$

Therefore $A_1 = 0$ and $A_4 = 0$

$$\left(\frac{dM}{rd\theta}\right)_{\theta = 0} = \int_0^\infty \left[- m A_2 \cos m\xi - m A_3 \text{ Sin} m\xi + A_5 \cos m\xi + A_8 \sin m\xi\right] dm = 0$$

$$\left(\frac{dM}{dr}\right)_{\theta = \gamma} = \int_0^\infty \Big[- A_5 (\text{Cos} m\xi - m \text{ Sin } m\xi) \text{ Cosh } m\gamma + A_6 (\cos m\xi - m \sin m\xi) \text{ Sinh } m\gamma$$

$$+ A_7 (m \cos m\xi + \sin m\xi) \text{ Sinh } m\gamma - A_8 (m \cos m\xi + \sin m\xi) \text{ Cosh } m\gamma\Big] dm$$

$$= + \frac{1}{\pi}\int_0^\infty \frac{\sin m\xi}{\text{Sinh } m\gamma} dm$$

$$\left(\frac{dM}{rd\theta}\right)_{\theta = \gamma} = \int_0^\infty \Big[-A_2 \text{ Sinh } m\gamma \cos m\xi - A_3 \text{ Sinh } m\gamma \sin m\xi - m A_5 \text{Sinh } m\gamma \cos m\xi$$

$$+ m A_6 \text{Cosh } m\gamma \cos m\xi + m A_7 \text{ Cosh } m\gamma \sin m\xi - m A_8 \text{Sinh } m\gamma \sin m\xi \Big] dm$$

By comparison of the coefficients to $\cos m\xi$ and $\sin m\xi$ respectively we obtain:

$$-m A_2 + A_5 = 0$$

$$-m A_3 + A_8 = 0$$

$$m A_5 \operatorname{Cosh} m\gamma - m A_6 \operatorname{Sinh} m\gamma + A_7 \operatorname{Sinh} m\gamma - A_8 \operatorname{Cosh} m\gamma = + \frac{1}{\pi \operatorname{Sinh} m\gamma}$$

$$-A_5 \operatorname{Cosh} m\gamma + A_6 \operatorname{Sinh} m\gamma + m A_7 \operatorname{Sinh} m\gamma - m A_8 \operatorname{Cosh} m\gamma = 0$$

$$-A_2 \operatorname{Sinh} m\gamma - m A_5 \operatorname{Sinh} m\gamma + m A_6 \operatorname{Cosh} m\gamma = 0$$

$$-A_3 \operatorname{Sinh} m\gamma + m A_7 \operatorname{Cosh} m\gamma - m A_8 \operatorname{Sinh} m\gamma = 0$$

Solving these, we obtain the coefficient A_1 etc.:

$$A_1 = 0$$

$$A_2 = - \frac{m^2 \operatorname{Cosh} m\gamma}{\pi (1 + m^2)(\operatorname{Sinh}^2 \gamma m - m^2) \operatorname{Sinh} m\gamma}$$

$$A_3 = + \frac{m \operatorname{Cosh} m\gamma}{\pi (1 + m^2)(\operatorname{Sinh}^2 m\gamma - m^2) \operatorname{Sinh} m\gamma}$$

$$A_4 = 0$$

$$A_5 = - \frac{m^3 \operatorname{Cosh} m\gamma}{\pi (1 + m^2)(\operatorname{Sinh}^2 m\gamma - m^2) \operatorname{Sinh} m\gamma}$$

$$A_6 = - \frac{m}{\pi (\operatorname{Sinh}^2 m\gamma - m^2)}$$

$$A_7 = + \frac{1}{\pi (\operatorname{Sinh}^2 m\gamma - m^2)}$$

$$A_8 = + \frac{m^2 \operatorname{Cosh} m\gamma}{\pi (1 + m^2)(\operatorname{Sinh}^2 m\gamma - m^2) \operatorname{Sinh} m\gamma}$$

The function M is now found by substitution of A, etc. into (17) and after reduction it becomes:

$$M = \frac{1}{\pi}\int_0^{\infty} \frac{(x\, m \operatorname{Cosh} m\gamma + y \operatorname{Sinh} m\gamma)\operatorname{Sinh}(m\,\theta) - y m^2 \operatorname{Cosh} m\,(\gamma - \theta)}{(1 + m^2)(\operatorname{Sinh}^2 m\gamma - m^2)\operatorname{Sinh} m\gamma}(m \cos m\xi - \sin m\xi)\, dm \quad \text{- - - (18)}$$

The constant part of (11) which produces concentrated forces at the origin must now be removed by superposing the function N determined by the following boundary conditions:

When $\theta = 0$

$N = 0$ and $\frac{dN}{rd\theta} = 0$

When $\theta = \gamma$

$\frac{dN}{dr} = + \frac{1}{3\pi}$ and $\frac{dN}{rd\theta} = 0$

We employ the simple solution:

$$N = C_1\, r \cos\theta + C_2\, r \sin\theta + C_3\, r\,\theta \cos\theta + C_4\, r\,\theta \sin\theta$$

and form the boundary equations:

$(N)_{\theta = 0} = C_1\, r = 0$, therefore $C_1 = 0$

$$\left(\frac{dN}{rd\theta}\right)_{\theta = 0} = C_2 + C_3 = 0$$

$$\left(\frac{dN}{dr}\right)_{\theta = \gamma} = - C_2 - \frac{3\pi}{2}\, C_4 = \frac{1}{3\pi}$$

$$\left(\frac{dN}{rd\theta}\right)_{\theta = \gamma} = \frac{3\pi}{2}\, C_2 - C_4 = 0$$

By solving, we find:

$$C_3 = + \frac{4}{3\pi\,(4 - 9\pi^2)}$$

$$C_4 = + \frac{2}{4 - 9\pi^2}$$

Omitting the terms of first degree, which contribute no stress the function N becomes:

$$N = \frac{2}{4 - 9\pi^2}\, r\,\theta\,\left(\frac{2}{3\pi}\cos\theta + \sin\theta\right) \tag{19}$$

The stress function corresponding to unit traction at z = a finally is

$$A = L + M + N \tag{20}$$

where

$$L = \frac{1}{\pi}\left[(a - x)\log\left(a^{2/3} + z^{2/3}\right) + 2a^{2/3} z^{1/3} - 2a\log\left(a^{1/3} + Z^{1/3}\right)\right]_i \quad \text{by (9)}$$

First degree terms have been omitted, being unimportant.

M and N are given by (18) and (19).

Evaluation

The integral in (18) is evaluated as outlined in article 1, Part I. The poles are the same as in that case in addition to the roots of

$$\text{Sinh}\ \frac{3\pi}{2}\ m = 0$$

which are $m = \pm\, i\ \frac{2K}{3}$ where $K = 0, 1, 2, 3,$ ----

The details of the residue computations being again omitted, we obtain:

For small values of $\frac{r}{a}$:

$$M = a\left[.101\left(\frac{r}{a}\right)^{1.545} g(\theta) + .125\left(\frac{r}{a}\right)^{1.909} h(\theta) +\right.$$

$$+ \frac{1}{5\pi} \left(\frac{r}{a}\right)^{5/3} (2 \cos \tfrac{2}{3} \theta \operatorname{Sin} \theta - 3 \operatorname{Sin} \tfrac{2}{3} \theta \cos \theta) \Big]$$

$$- \frac{2}{4 - 9\pi^2} \; r \, \theta (\frac{2}{3\pi} \cos \theta + \sin \theta) \qquad \text{----(18a)}$$

For large values of $\frac{r}{a}$:

$$M = - a \Big[.340 \left(\frac{r}{a}\right)^{.455} g(\theta) + 2.629 \left(\frac{r}{a}\right)^{.091} h(\theta)$$

$$+ \frac{1}{\pi} \left(\frac{r}{a}\right)^{1/3} (2 \cos \tfrac{2}{3} \theta \sin \theta - 3 \sin \tfrac{2}{3} \theta \cos \;) \Big] + a \sin^2 \theta$$

$$+ \frac{2}{4 - 9\pi^2} \; r \, \theta \, (\frac{2}{3\pi} \cos \theta + \operatorname{Sin} \theta) \qquad \text{----(18b)}$$

$g(\theta) = \sin .545\,\theta \cos \theta - .545 \cos .545\,\theta \sin \theta - .839 \sin .545\,\theta \sin \theta$; (21)

$h(\theta) = \sin .909\,\theta \cos \theta - .909 \cos .909\,\theta \sin \theta + .416 \sin .909\,\theta \sin \theta$; (22)

identical with (44) and (45) of Part I.

In order to compute L for the same region we expand (9) in power series and obtain:

For $\left|\frac{z}{a}\right| << 1$:

$$L = \frac{a}{\pi} \Big[\frac{x}{a} \cdot \left(\frac{z}{a}\right)^{\frac{2}{3}} - \frac{2}{5} \left(\frac{z}{a}\right)^{5/3} \Big]_i \pm \frac{a}{\pi} \Big[\sin \tfrac{2}{3} \theta \cos \theta - \tfrac{8}{5} \sin \tfrac{5}{3} \theta \Big] \left(\frac{r}{a}\right)^{5/3}$$

$$= - \frac{a}{5\pi} \left(\frac{r}{a}\right)^{5/3} (2 \cos \tfrac{2}{3} \theta \sin \theta - 3 \sin \tfrac{2}{3} \theta \cos \theta \;) \qquad \text{----(9a)}$$

For $\left|\frac{z}{a}\right| >> 1$:

$$L = + \frac{1}{\pi} \Big[x \left(\frac{z}{a}\right)^{-2/3} + 2a^{2/3} z^{1/3} - \frac{2}{3} x \log (-z) \Big]_i$$

$$= \frac{a}{\pi} \left(\frac{r}{a}\right)^{1/3} (2 \cos \tfrac{2}{3} \theta \sin \theta - 3 \sin \tfrac{2}{3} \theta \cos \theta \;) - \frac{2}{3\pi} \; r \, \theta \cos \theta \; ; \text{(9b)}$$

By now superposing the functions, we obtain:

For small $\frac{r}{a}$ by adding (18a), (9a) and (19):

$$A = a\left[.101\left(\frac{r}{a}\right)^{1.545} g(\theta) + .125\left(\frac{r}{a}\right)^{1.909} h(\theta)\right] \quad \text{------} (23)$$

For large $\frac{r}{a}$, by adding (18b), (9b) and (19):

$$A = -a\left[.340\left(\frac{r}{a}\right)^{.455} g(\theta) + 2.629\left(\frac{r}{a}\right)^{.091} h(\theta)\right] + a \sin^2\theta$$

$$+ \frac{2}{4 - 9\pi^2} r\,\theta\,(3\pi\cos\theta + 2\sin\theta) \quad \text{----} (24)$$

If we compare these with the solutions (42) and (43) in Part I, it will be seen that they are identical except for the terms $+ a \sin^2\theta$ and $- a \cos^2\theta$, however, these terms contribute exactly the same stresses.

It is interesting to notice that (9b) contributes exactly the term which is needed to combine with (18b) and (19) in order to make the last term in (24) identical with the Bousinesq term in (43) Part I., namely:

$$-\frac{2}{3\pi} r\,\theta\cos\theta + \frac{2 \cdot 2}{4 - 9\pi^2} r\,\theta\left(\frac{2}{3\pi}\cos\theta + \sin\theta\right)$$

$$= \frac{2\,r\,\theta}{(4 - 9\pi^2)3\pi}(4\cos\theta - 4\cos\theta + 9\pi^2\cos\theta + 6\pi\sin\theta)$$

$$= \frac{2\,r\,\theta}{4 - 9\pi^2}(3\pi\cos\theta + 2\sin\theta)$$

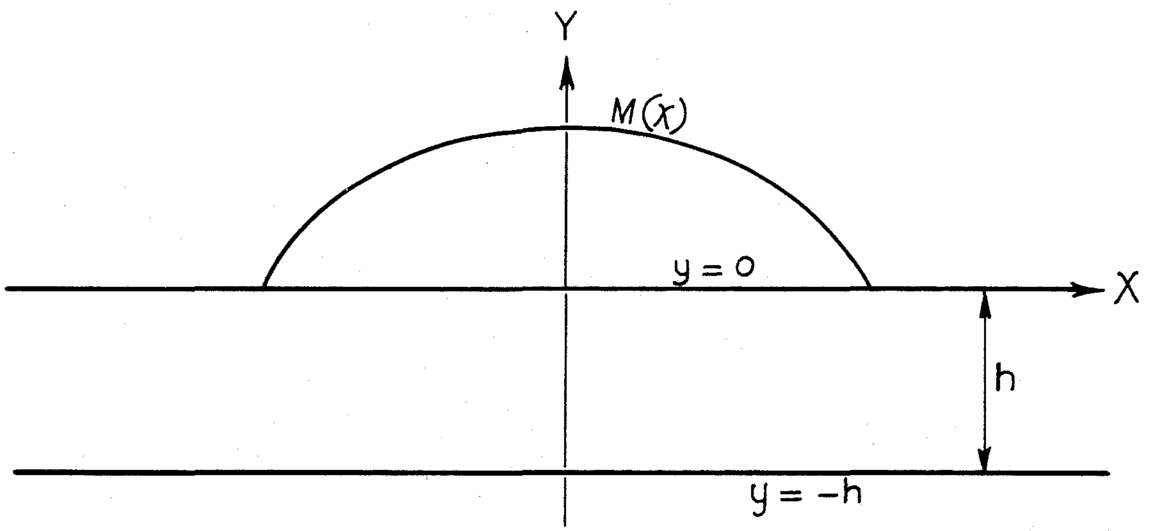

Y
M(x)
y = 0
X
h
y = −h

FIG. 1

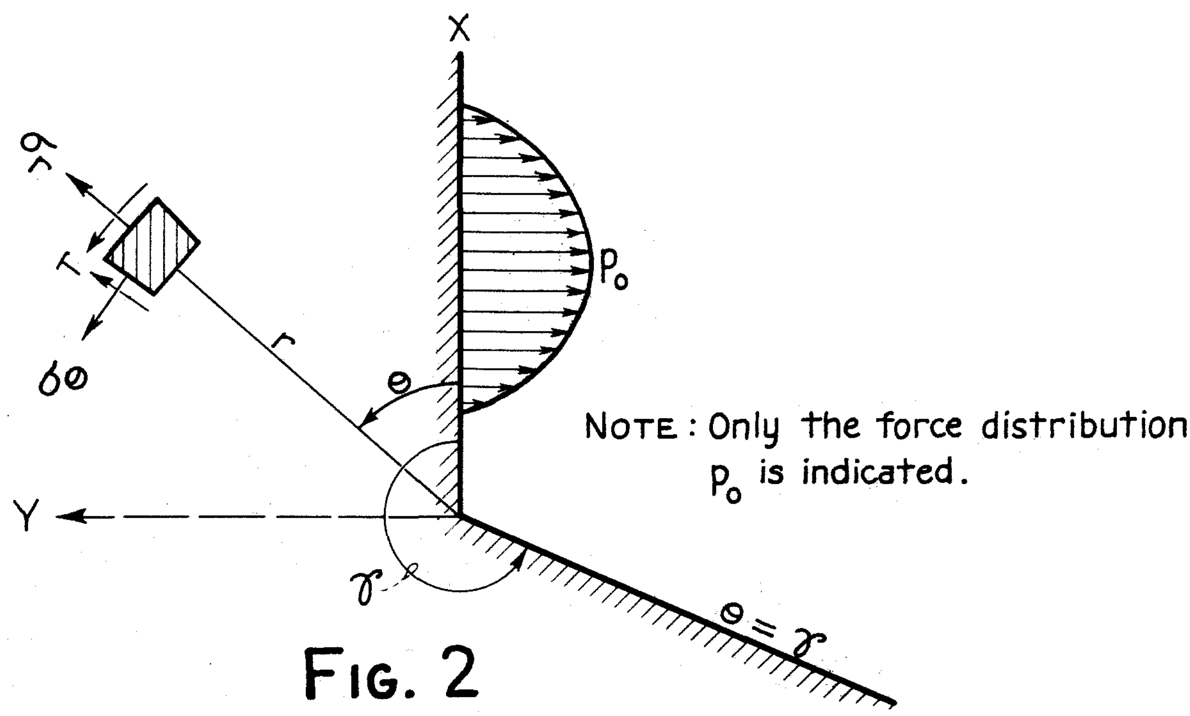

X
σr
τ
σθ
r
θ
p0
Y
γ
θ = γ
NOTE: Only the force distribution
p0 is indicated.

FIG. 2

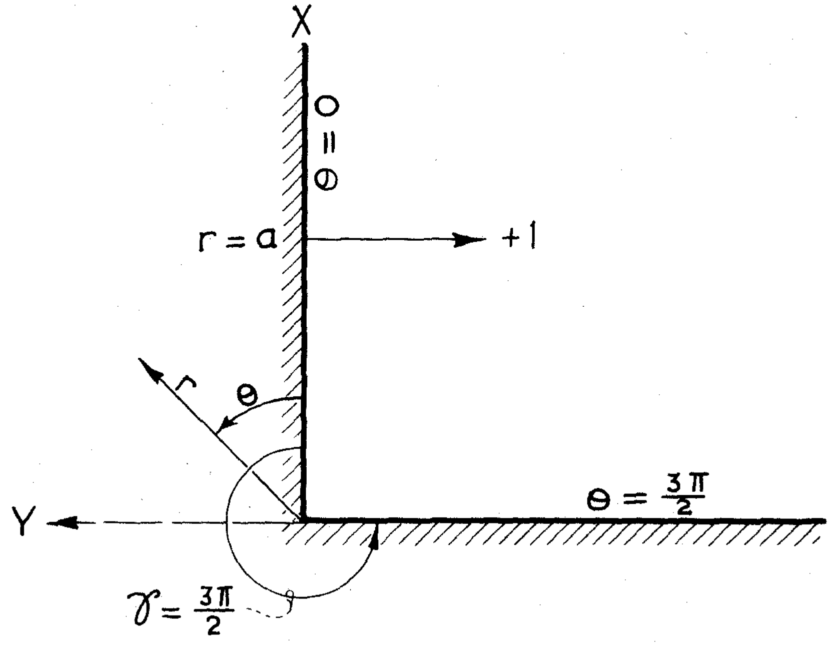


FIG. 3

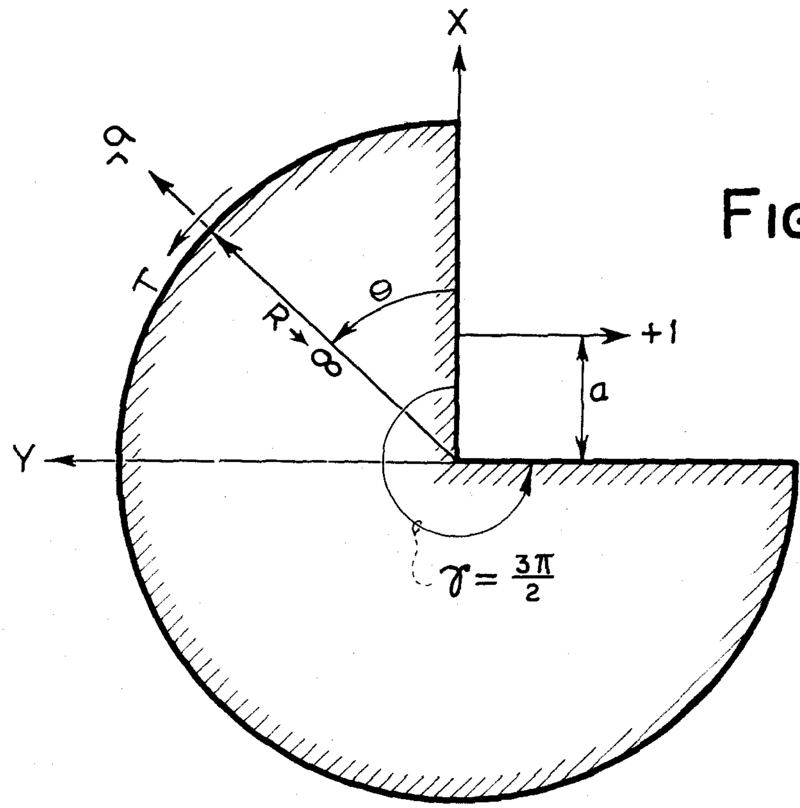
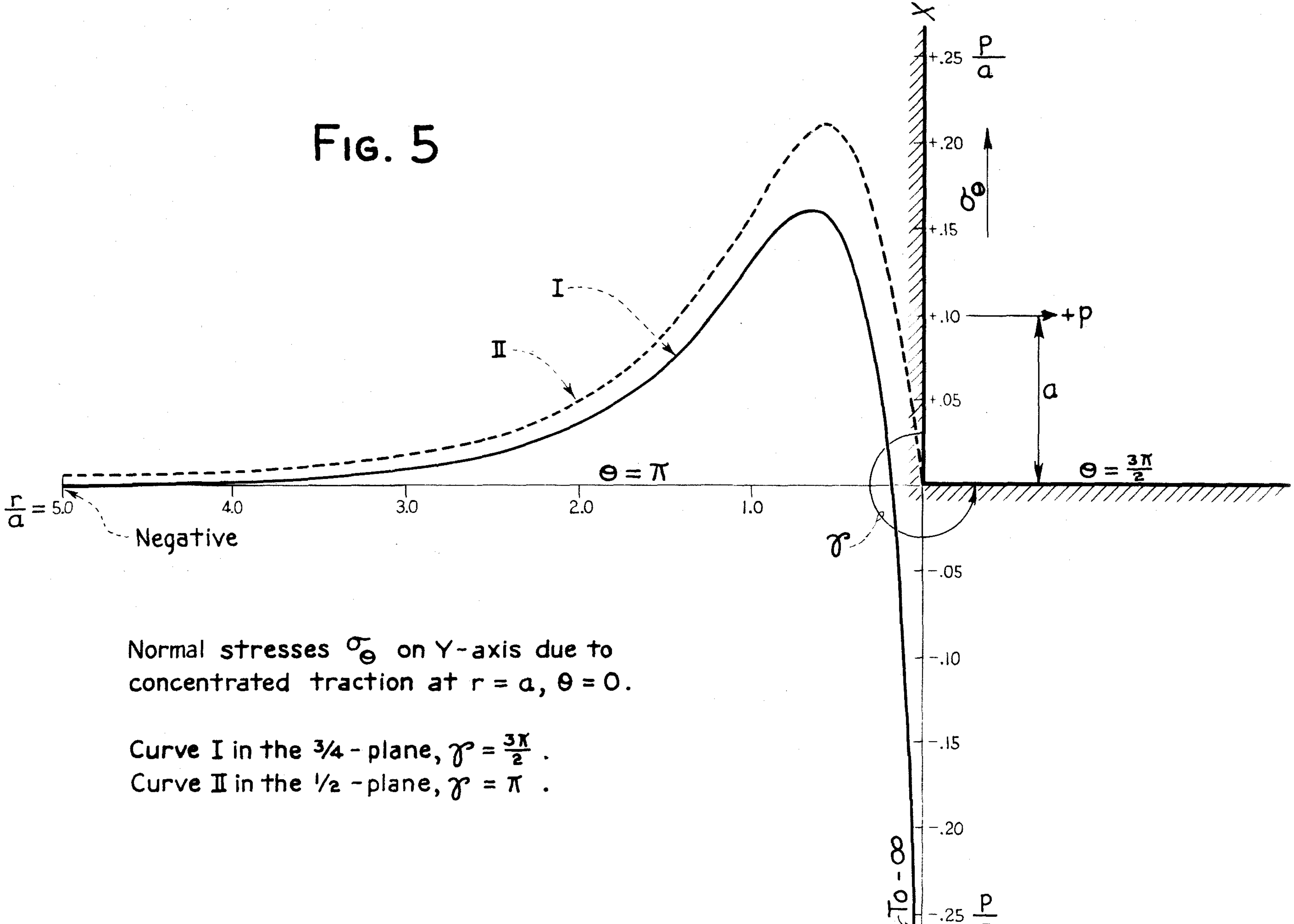


FIG. 4

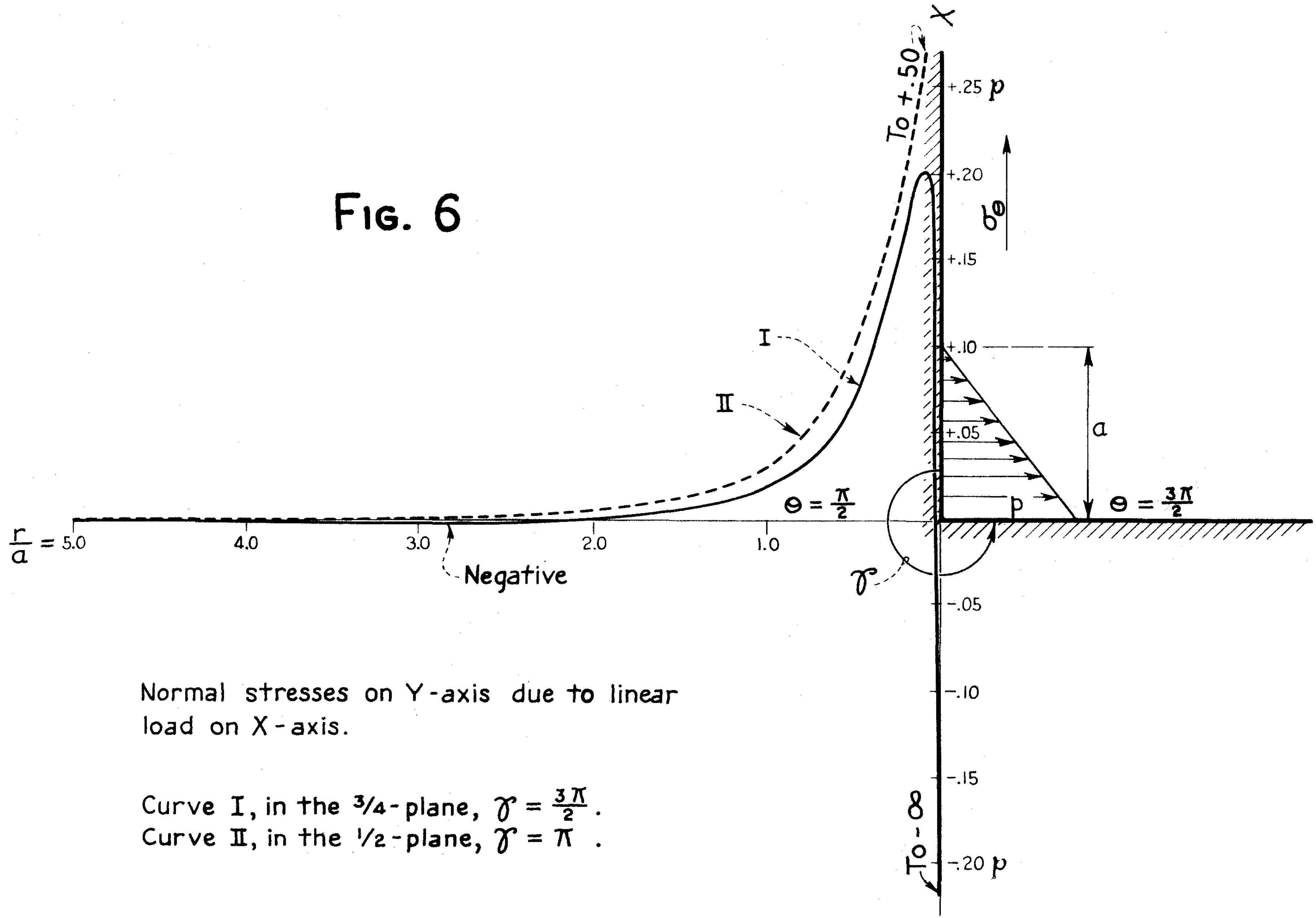
FIG. 5



Normal stresses σ_θ on Y-axis due to concentrated traction at $r = a$, $\theta = 0$.

Curve I in the 3/4-plane, $\gamma = \frac{3\pi}{2}$.
Curve II in the 1/2-plane, $\gamma = \pi$.

Fig. 6



Normal stresses on Y-axis due to linear load on X-axis.

Curve I, in the 3/4-plane, $\gamma = \frac{3\pi}{2}$.
Curve II, in the 1/2-plane, $\gamma = \pi$.

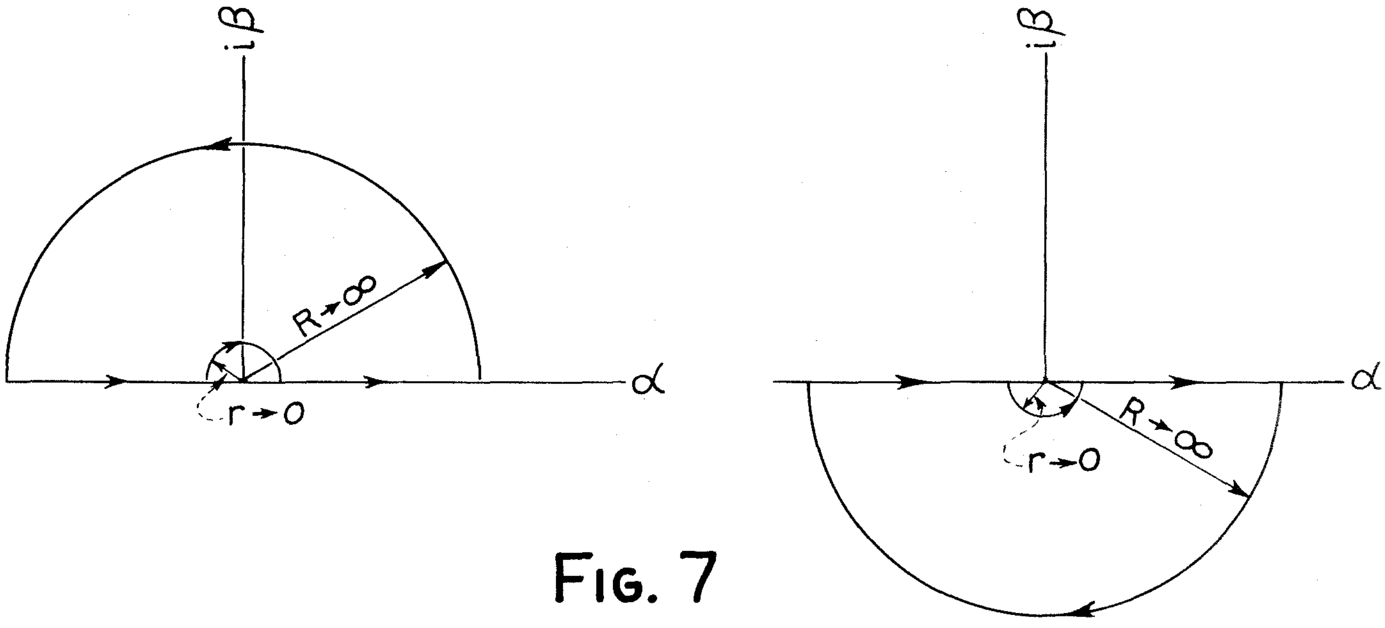


FIG. 7

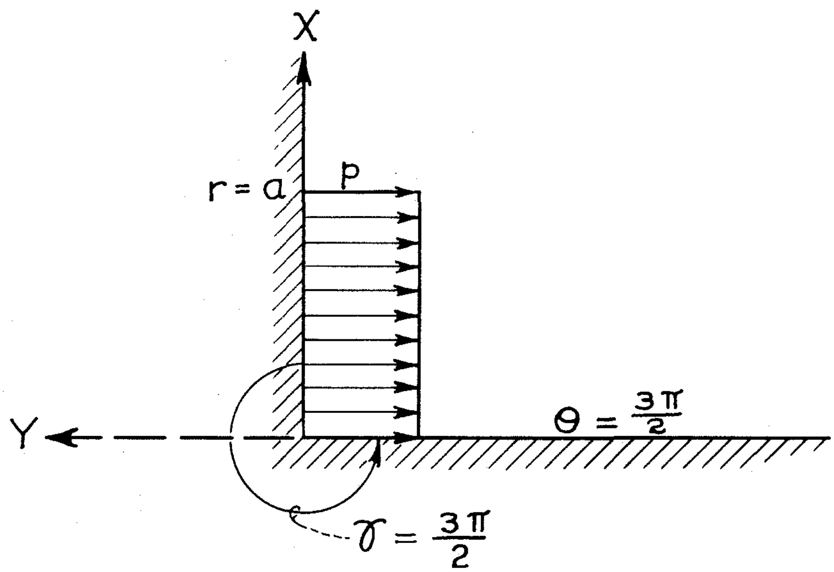


Fig. 8

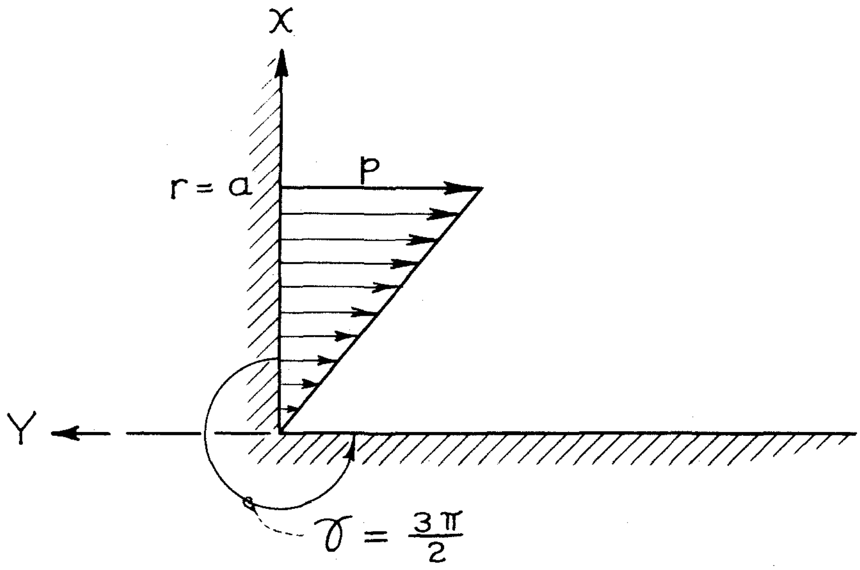


Fig. 9

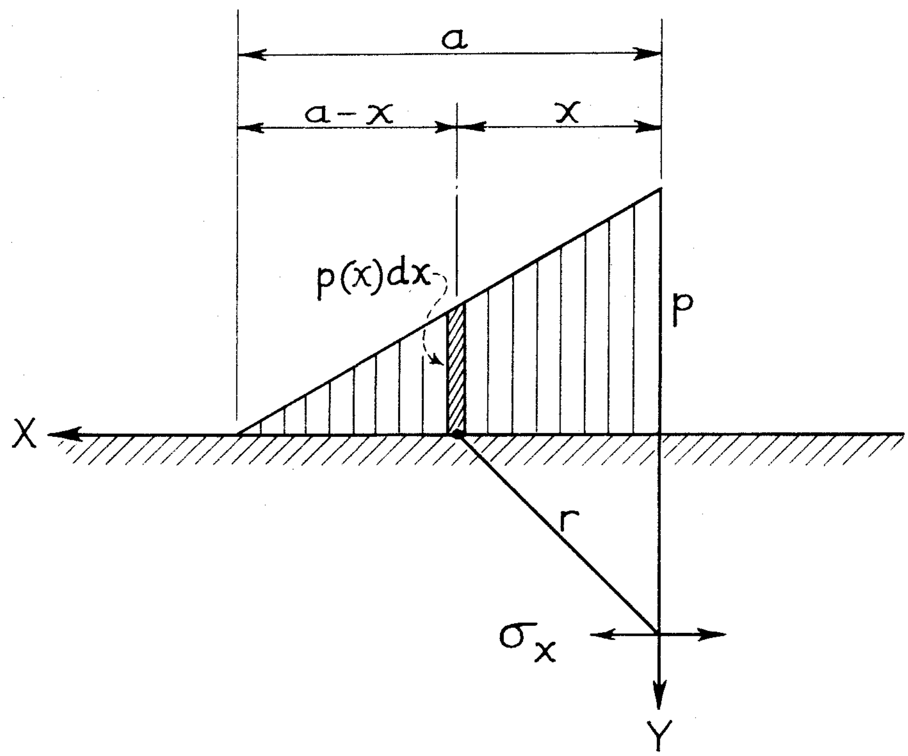


Fig. 10